

NEW CENTURY MATHS Advanced

10⁺
10_A

NSW Stages
5.2/5.3

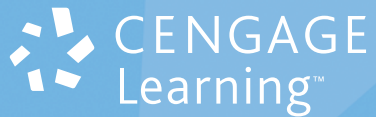
For the
**australian
curriculum**



Klaas Bootsma
David Badger
Sarah Hamper

Robert Yen Series Editor





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Klaas Bootsma
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NelsonNet print resources by Megan Boltze



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First published in 1997, *New Century Maths 10 Stages 5.2/5.3* has now been completely rewritten to reflect the spirit of the new Australian curriculum and the NSW syllabus.

For Years 9 and 10, we publish two levels of mathematics books:

- *New Century Maths Stages 5.1/5.2*
- *New Century Maths Advanced Stages 5.2/5.3*

The NSW mathematics syllabus describes Stage 5 (Years 9–10) as a continuum with content classified into three sub-stages, Stage 5.1, Stage 5.2 and Stage 5.3, with each sub-stage including the content of the previous sub-stage. This book has been designed for students and classes progressing along Stages 5.2 or 5.3 of the continuum.

We have retained those familiar features that have made *New Century Maths* a leading series: clear worked examples, graded exercises, multiple-choice questions, Investigation, Technology, Mental skills, SkillCheck pre-chapter exercise, Power Plus extension exercise, Language of Maths, Topic overview with mind map, glossary, video tutorials and worksheets.

New features include:

- *GeoGebra* activities in our Technology sections
- Chapter outlines showing coverage of the Working Mathematically proficiencies
- *NelsonNet* student and teacher websites of print and multimedia resources
- *NelsonNetBook* web version of this book with interactive features

We always aim to write textbooks to provide flexibility for teachers to plan and teach for a variety of pathways. We thank our families and teaching colleagues for their continued support and patience, and editors Anna Pang and Alan Stewart for transforming our manuscript into this fine text. Finally, we wish all teachers and students using this book every success in embracing the new Australian mathematics curriculum.

About the authors

Klaas Bootsma was head teacher of mathematics at Ambarvale High School in Campbelltown and has taught at Lurnea and Grantham high schools. He was a senior HSC examiner and has worked on the HSC Advice Line and School Certificate marking. Klaas has wide experience teaching all types of students, and his interests are in curriculum and the use of technology (ICT) in learning.

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Money and financial mathematics	2 Working with numbers	2 Interest and depreciation
Patterns and algebra	3 Products and factors 5 Indices 7 Equations	5 Products and factors 7 Equations and logarithms 14 Polynomials
Linear and non-linear relationships	7 Equations 11 Coordinate geometry and graphs	3 Coordinate geometry 5 Products and factors 7 Equations and logarithms 8 Graphs 10 Simultaneous equations 11 Quadratic equations and the parabola 14 Polynomials
MEASUREMENT AND GEOMETRY		
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Geometric reasoning	6 Geometry 13 Congruent and similar figures	13 Geometry 15 Circle geometry
Pythagoras and trigonometry	1 Pythagoras' theorem and surds 4 Trigonometry	9 Trigonometry
STATISTICS AND PROBABILITY		
Chance	12 Probability	12 Probability
Data representation and interpretation	9 Investigating data	6 Investigating data



Curriculum grid: NSW syllabus

Strand and substrand	New Century Maths Advanced 9 Stages 5.2/5.3 chapter	New Century Maths Advanced 10 + 10A Stages 5.2/5.3 chapter
NUMBER AND ALGEBRA		
Financial mathematics (Stages 5.1, 5.2)	2 Working with numbers 8 Earning money	2 Interest and depreciation
Indices (Stages 5.1, 5.2)	5 Indices	5 Products and factors
Linear relationships (Stages 5.1, 5.2, 5.3)	7 Equations 11 Coordinate geometry and graphs	3 Coordinate geometry
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Area and surface area (Stages 5.1, 5.2, 5.3)	10 Surface area and volume	4 Surface area and volume
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Properties of geometrical figures (Stages 5.1, 5.2, 5.3)	6 Geometry 13 Congruent and similar figures	4 Surface area and volume 13 Geometry
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STATISTICS AND PROBABILITY		
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Bivariate data analysis (Stages 5.2, 5.3)		6 Investigating data
Probability (Stages 5.1, 5.2)	12 Probability	12 Probability

Curriculum grid: Year 10 content descriptions

This is an extract from the Australian Curriculum, except for the additional NSW content. 

Content description	New Century Maths Advanced 10 + 10A Stages 5.2/5.3 chapter
NUMBER AND ALGEBRA	
Money and financial mathematics	
ACMNA229: Connect the compound interest formula to repeated applications of simple interest using appropriate digital technologies	2 Interest and depreciation
NSW STAGE 5.1: Solve problems involving earning income and paying income tax	2 Interest and depreciation
Patterns and algebra	
ACMNA230: Factorise algebraic expressions by taking out a common algebraic factor	5 Products and factors
ACMNA231: Simplify algebraic products and quotients using index laws	5 Products and factors
ACMNA232: Apply the four operations to simple algebraic fractions with numerical denominators	5 Products and factors
NSW STAGE 5.2: Apply the four operations to algebraic fractions with algebraic denominators	5 Products and factors
ACMNA233: Expand binomial products and factorise monic quadratic expressions using variety of strategies	5 Products and factors
ACMNA234: Substitute values into formulas to determine an unknown	7 Equations and logarithms
Linear and non-linear relationships	
NSW STAGE 5.2: Use the gradient-intercept form of the equation of a straight line	3 Coordinate geometry
ACMNA235: Solve problems involving linear equations, including those derived from formulas	7 Equations and logarithms
ACMNA236: Solve linear inequalities and graph their solutions on a number line	7 Equations and logarithms
ACMNA237: Solve linear simultaneous equations, using algebraic and graphical techniques including using digital technology	10 Simultaneous equations
ACMNA238: Solve problems involving parallel and perpendicular lines	3 Coordinate geometry
ACMNA239: Explore the connection between algebraic and graphical representations of relations such as simple quadratics, circles and exponentials using digital technology as appropriate	8 Graphs
ACMNA240: Solve linear equations involving simple algebraic fractions	7 Equations and logarithms
ACMNA241: Solve simple quadratic equations using a range of strategies	7 Equations and logarithms

Curriculum grid: Year 10 content descriptions

Content description	New Century Maths Advanced 10 + 10A Stages 5.2/5.3 chapter
MEASUREMENT AND GEOMETRY	
Using units of measurement	
ACMMG242: Solve problems involving surface area and volume for a range of prisms, cylinders and composite solids	4 Surface area and volume
Geometric reasoning	
ACMMG243: Formulate proofs involving congruent triangles and angle properties	13 Geometry
ACMMG244: Apply logical reasoning, including the use of congruence and similarity, to proofs and numerical exercises involving plane shapes	13 Geometry
NSW STAGE 5.2: Apply the interior and exterior angle sum of polygons	13 Geometry
Pythagoras and trigonometry	
ACMMG245: Solve right-angled triangle problems including those involving direction and angles of elevation and depression	9 Trigonometry
STATISTICS AND PROBABILITY	
Chance	
ACMSP246: Describe the results of two- and three-step chance experiments, both with and without replacements, assign probabilities to outcomes and determine probabilities of events. Investigate the concept of independence.	12 Probability
ACMSP247: Use the language of 'if ... then', 'given', 'of', 'knowing that' to investigate conditional statements and identify common mistakes in interpreting such language	12 Probability
Data representation and interpretation	
ACMSP248: Determine quartiles and interquartile range	6 Investigating data
ACMSP249: Construct and interpret box plots and use them to compare data sets	6 Investigating data
ACMSP250: Compare shapes of box plots to corresponding histograms and dot plots	6 Investigating data
ACMSP251: Use scatter plots to investigate and comment on relationships between two numerical variables	6 Investigating data
ACMSP252: Investigate and describe bivariate numerical data where the independent variable is time	6 Investigating data
ACMSP253: Evaluate statistical reports in the media and other places by linking claims to displays, statistics and representative data	6 Investigating data

Curriculum grid: Year 10A content descriptions

This is an extract from the Australian Curriculum, except for the additional NSW content. 

Content description	New Century Maths Advanced 10 + 10A Stages 5.2/5.3 chapter
NUMBER AND ALGEBRA	
Real numbers	
ACMNA264: Define rational and irrational numbers and perform operations with surds and fractional indices	1 Surds 5 Products and factors
ACMNA265: Use the definition of a logarithm to establish and apply the laws of logarithms	7 Equations and logarithms 16 Functions
Patterns and algebra	
NSW STAGE 5.3: Add and subtract algebraic fractions with binomial numerators and numerical denominators	5 Products and factors
ACMNA266: Investigate the concept of a polynomial and apply the factor and remainder theorems to solve problems	14 Polynomials
Linear and non-linear relationships	
ACMNA267: Describe, interpret and sketch parabolas, hyperbolas, circles and exponential functions and their transformations	8 Graphs 11 Quadratic equations and the parabola
NSW STAGE 5.3: Use different forms of the equation of a straight line	3 Coordinate geometry
NSW STAGE 5.3: Graph cubic curves and higher power curves	8 Graphs 14 Polynomials
ACMNA268: Apply understanding of polynomials to sketch a range of curves and describe the features of these curves from their equation	14 Polynomials
ACMNA269: Factorise monic and non-monic quadratic expressions and solve a wide range of quadratic equations derived from a variety of contexts	5 Products and factors 11 Quadratic equations and the parabola
ACMNA270: Solve simple exponential equations	7 Equations and logarithms
NSW STAGE 5.3: Solve linear equations involving algebraic fractions with binomial numerators and numerical denominators	7 Equations and logarithms
NSW STAGE 5.3: Solve cubic equations of the form $ax^3 = c$	7 Equations and logarithms
NSW STAGE 5.3: Change the subject of a formula	7 Equations and logarithms
NSW STAGE 5.3: Solve simultaneous equations involving linear and non-linear equations	11 Quadratic equations and the parabola
NSW STAGE 5.3: Use and graph functions	16 Functions

Curriculum grid: Year 10A content descriptions

Content description	New Century Maths Advanced 10 + 10A Stages 5.2/5.3 chapter
MEASUREMENT AND GEOMETRY	
Using units of measurement	
ACMMG271: Solve problems involving surface area and volume of right pyramids, right cones, spheres and related composite solids	4 Surface area and volume
Geometric reasoning	
ACMMG272: Prove and apply angle and chord properties of circles	15 Circle geometry
NSW STAGE 5.3: Prove and apply tangent and secant properties of circles	15 Circle geometry
Pythagoras and trigonometry	
ACMMG273: Establish the sine, cosine and area rules for any triangle and solve related problems	9 Trigonometry
ACMMG274: Use the unit circle to define trigonometric functions, and graph them with and without the use of digital technologies	9 Trigonometry
ACMMG275: Solve simple trigonometric equations	9 Trigonometry
ACMMG276: Apply Pythagoras' theorem and trigonometry to solving three-dimensional problems in right-angled triangles	9 Trigonometry
STATISTICS AND PROBABILITY	
Data representation and interpretation	
ACMSP277: Investigate reports of studies in digital media and elsewhere for information on their planning and implementation	6 Investigating data
ACMSP278: Calculate and interpret the mean and standard deviation of data and use these to compare data sets	6 Investigating data
ACMSP279: Use information technologies to investigate bivariate numerical data sets. Where appropriate use a straight line to describe the relationship allowing for variation	6 Investigating data

Coverage of the Australian curriculum and NSW syllabus

- *New Century Maths Advanced 10+10A Stages 5.2/5.3* covers both the Australian curriculum and the NSW syllabus, as shown by the table of contents and curriculum grids on the previous pages. The previous four pages lists all of the Year 10 and 10A content descriptions explicitly.
- This book contains Stages 5.2 and 5.3 content, including NSW syllabus content that is not covered by the Australian curriculum. NSW-only content is highlighted in **orange**, while Stage 5.3 content is marked with a *.
- There are three content strands in Mathematics:

NA = Number and Algebra

MG = Measurement and Geometry

SP = Statistics and Probability

- Each chapter begins with a **chapter outline** that includes the Working Mathematically proficiencies covered in each section:

U = Understanding

F = Fluency

PS = Problem solving

R = Reasoning

C = Communicating

Further references to the Working

Mathematically proficiencies can be found in the teaching program.

10ANA269	5-09 Factorising quadratic expressions of the form $ax^2 + bx + c^*$	175
10ANA269	5-10 Mixed factorisations*	178
NSW	5-11 Factorising algebraic fractions*	179
	Power plus	181

Chapter outline		Proficiency strands				
		U	F	R	C	
5-01 The index laws		U	F	R	C	
5-02 Fractional indices*		U	F	R	C	
5-03 Adding and subtracting algebraic fractions		U	F	R	C	
5-04 Multiplying and dividing algebraic fractions		U	F	R	C	
5-05 Expanding and factorising expressions		U	F	R	C	
5-06 Expanding binomial products		U	F	R	C	
5-07 Factorising special binomial products*		U	F	R	C	
5-08 Factorising quadratic expressions		U	F	R	C	
5-09 Factorising quadratic expressions of the form $ax^2 + bx + c^*$		U	F	R	C	
5-10 Mixed factorisations*		U	F	R	C	
5-11 Factorising algebraic fractions*		U	F	R	C	

*STAGE 5.3

Working Mathematically explained

While the three **content strands** are the 'nouns' of the mathematics curriculum, the five **Working Mathematically proficiencies** are the 'verbs': the doing and thinking processes that go hand-in-hand with the content being taught.

- **Understanding** is 'knowing and relating' maths. It is more than just learning facts. It's deep understanding, seeing how mathematical content is interconnected, knowing 'why' as well as 'how'.
- **Fluency** is 'applying' maths. It is being able to use mathematics competently and effectively. When you are fluent in a language, you have mastered it so that you can improvise and confidently use the correct word or phrase. Fluency in maths is choosing an appropriate skill, method or formula to use at the right place and time.
- **Problem solving** is 'modelling and investigating' with maths. It involves interpreting a rich, elaborate problem, selecting an appropriate strategy or model, solving the problem, then evaluating, communicating and justifying the solution.
- **Reasoning** is 'generalising and proving' with maths, using higher-order thinking to connect specific facts to general principles, using algebra, logic, proof and justification.
- **Communicating** is 'describing and explaining' maths, representing mathematical theory and solutions in words, algebraic symbols, special notations, diagrams, graphs and tables.

In each chapter

- **Wordbank** is a sample glossary of 6–8 terms from the chapter
- There is also a full glossary at the back of the book: in this book, all terms printed in **red** appear in the glossary
- Curriculum content descriptions are listed at the beginning of each chapter, with the Australian Curriculum descriptions printed with the **AC** symbol.
- **SkillCheck** and **StartUp assignment** review prerequisite skills and knowledge for the chapter
- Important facts and formulas are highlighted in a **Summary** box.

Summary

$a^{\frac{1}{2}} = \sqrt{a}$	Any number raised to the power of $\frac{1}{2}$ is the square root of that number
$a^{\frac{1}{3}} = \sqrt[3]{a}$	Any number raised to the power of $\frac{1}{3}$ is the cube root of that number
$a^{\frac{1}{n}} = \sqrt[n]{a}$	Any number raised to the power of $\frac{1}{n}$ is the n th root of that number
$a^{\frac{m}{n}} = (\sqrt[n]{a})^m$ or $\sqrt[n]{a^m}$	Any number raised to the power of $\frac{m}{n}$ is the n th root of that number raised to the power m . Note: Taking the root first often makes the calculation simpler.

- Pages containing **Stage 5.3** content are marked by a shaded margin.
- Graded exercises are linked to worked examples and include multiple-choice questions, exam-style problems and realistic applications.
- Worked solutions for selected exercise questions are provided.

Investigation: Squaring a number ending in 5

Study this mental shortcut for squaring a number ending in 5:

- To evaluate 35^2 , calculate $3 \times 4 = 12$, add '25' to the end: $35^2 = 1225$.
- To evaluate 75^2 , calculate $7 \times 8 = 56$, add '25' to the end: $75^2 = 5625$.
- To evaluate 105^2 , calculate $10 \times 11 = 110$, add '25' to the end: $105^2 = 11\,025$.

Let n stand for the tens digit of the number ending in 5 being squared.
Expand $(10n + 5)^2$ and investigate why the above method works.

In this chapter you will:

- apply index laws to numerical expressions with integer indices
- apply index laws to algebraic expressions with integer indices
- simplify algebraic products and quotients using index laws
- expand binomial products and factorise monic quadratic expressions using a variety of strategies
- interpret and use zero and negative indices
- (STAGE 5.3) interpret and use fractional indices
- add, subtract, multiply and divide simple algebraic fractions
- (STAGE 5.3) add and subtract algebraic fractions with binomial numerators
- expand and factorise algebraic expressions
- expand and factorise algebraic expressions involving terms with indices
- (STAGE 5.3) expand special binomial products such as $(a + b)^2$, $(a - b)^2$, $(a + b)(a - b)$
- (STAGE 5.3) factorise algebraic expressions using grouping in pairs, perfect squares, the difference of two squares and quadratic expressions of the form $ax^2 + bx + c$
- (STAGE 5.3) factorise and simplify expressions involving algebraic fractions

SkillCheck

- Simplify each expression.

a $g^4 \times g^5$	b $r^8 \div r^2$	c $(a^2)^3$	d $(-k)^2$
e $h \times h^9$	f $m^2 \div m$	g a^1	h a^0
i $3e^2 \times 2e^3$	j $18a^6 \div 6a^2$	k $(10w^3)^3$	l $25q^0$
m $(vw)^3$	n $(\frac{1}{w})^3$	o y^{-1}	p y^{-2}
- Evaluate each expression.

a $\frac{a}{5} + \frac{3a}{4}$	b $\frac{7p}{2} - \frac{10p}{3}$	c $\frac{8}{t} \times \frac{5t}{24}$	d $\frac{x}{14} \div \frac{y}{2}$
- Expand each expression.

a $6m(3m + 11)$	b $-5(3g - 8)$
- Factorise each expression.

a $4x + 24$	b $20 - 15a$	c $q^2 + q$
d $18a^2 - 12a$	e $-2y - 30$	f $-18w + 24$
- Find two numbers whose:

a product is 18 and sum is 9	b product is 8 and sum is -6
c product is -20 and sum is -1	d product is -16 and sum is 6

Stage 5.3

Exercise 5-07 Factorising special binomial products

- See Example 19** 1 Factorise each expression.
- | | |
|-----------------------------------|--|
| a $3px + 2qx + 3py + 2qy$ | b $2wh + 2wk + 3uh - 3uk$ |
| c $15mk + 20mg + 6nk + 8ng$ | d $4yx - 8ay + 7ax - 14a^2$ |
| e $2ak - 5af + 8k - 20f$ | f $cd + cy - hd - hy$ |
| g $4am + 4at + 4em + 4et$ | h $3yk - 6by + 12k - 24b$ |
| i $3mn - 6m + pn - 2p$ | j $9p^2 - 27 + qp^2 - 3q$ |
| k $fg - fh - 10g + 10h$ | l $9kl - 12ml + 9kn - 3q$ |
| m $2p - 2c - p^2 + pc$ | n $l^3 + lm^2 - 3l^2 - 3m^2$ |
| o $a(x + 1) + y(x + 1) - ka - ky$ | p $pl(a - b) - 2q(a - b) + 3qp - 6q^2$ |
- See Example 20** 2 Factorise each expression.
- | | | | |
|-------------------------|------------------------|--------------------------|---------------------|
| a $d^2 - 16$ | b $x^2 - 25$ | c $p^2 - 121$ | d $y^2 - 81$ |
| e $25 - t^2$ | f $100 - k^2$ | g $4r^2 - 9d^2$ | h $25g^2 - 4u^2$ |
| i $144 - 49m^2$ | j $81y^2 - 16k^2$ | k $1 - 81d^2$ | l $m^2 - 4n^2$ |
| m $y^2 - z^2$ | n $49 - 16m^2$ | o $b^2 - 121d^2$ | p $36c^2 - 25k^2$ |
| q $16 - 81k^2$ | r $25a^2 - 64m^2$ | s $100 - 49n^2$ | t $121p^2 - 144q^2$ |
| u $\frac{1}{4} - 25c^2$ | v $4r^2 - \frac{1}{9}$ | w $25h^2 - 2\frac{1}{4}$ | x $1 - m^2n^2$ |

- **Investigations** explore the syllabus in more detail, through group work, discovery and modelling activities.
- **Just for the record** contains interesting facts and applications of the mathematics learnt in the chapter

- **Technology** promotes ICT in the classroom, using spreadsheets, *GeoGebra* and the Internet.

Technology **The angle of inclination**

In this activity we will use GeoGebra to calculate the angle of inclination of a line.

- 1 Close the **Algebra** View so that only the graphics window is showing and select the grid option at the top left-hand corner. Click on the input bar at the bottom of the screen and enter: $y = 2x + 1$

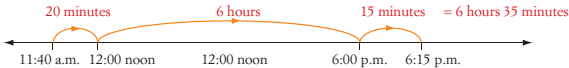


Mental skills 3 **Maths without calculators**

Time differences

- 1 Study each example.
 - a What is the time difference between 11:40 a.m. and 6:15 p.m.?

From 11:40 a.m. to 5:40 p.m. = 6 hours
 Count: '11:40, 12:40, 1:40, 2:40, 3:40, 4:40, 5:40'
 From 5:40 a.m. to 6:00 p.m. = 20 min
 From 6:00 p.m. to 6:15 p.m. = 15 min
 5 hours + 20 min + 15 min = 6 hours 35 min
 OR:



- **Mental skills** reinforce mental calculation strategies ('maths without calculators')

NelsonNet resources

Margin icons link to print (PDF) and multimedia resources found on the NelsonNet website, www.nelsonnet.com.au. These include:

Worksheet Conditional probability	Worksheets	Video tutorial Interquartile range	Video tutorials of worked examples
Puzzle sheet Coordinate geometry crossword	Puzzle sheets of matching activities, crosswords and find-a-word puzzles	Technology worksheet Graphing non-linear equations	Technology worksheets: additional technology activities
Skillsheet Percentage calculations	Skillsheets of examples and exercises of prerequisite skills and knowledge	Quiz Geometry	ExamView quizzes: interactive and self-marking

Each resource has a unique identifier code. For example, the video tutorial 'Simplifying surds' has the code **MAT10NAVT10002**, which stands for Mathematics, Year 10, Number and Algebra strand, Video Tutorial 10002.

NelsonNet is your portal to the premium digital resources for Nelson student books, which includes worksheets and tests, quizzes, videos and animations, and an online/offline NelsonNetBook. Visit www.nelsonnet.com.au to find out more, register and log in. Please note that complimentary access to NelsonNet and the NelsonNetBook is only available to teachers who use the accompanying student textbook as a core educational resource in their classroom. Contact your sales representative information about access codes and conditions.

At the end of each chapter

- **Power plus** is an extension/challenge exercise
- **Language of maths** has a chapter word list and literacy questions
- **Topic overview** has reflection questions and an incomplete mind map
- **Chapter revision** is a review exercise with links to each exercise set of the chapter
- **Mixed revision** is a review exercise after every 3–4 chapters
- **General revision** exercise
- **Instructional and Mathematical glossaries** (in this book, words printed in red also appear in the glossary)
- **Answers and Index**

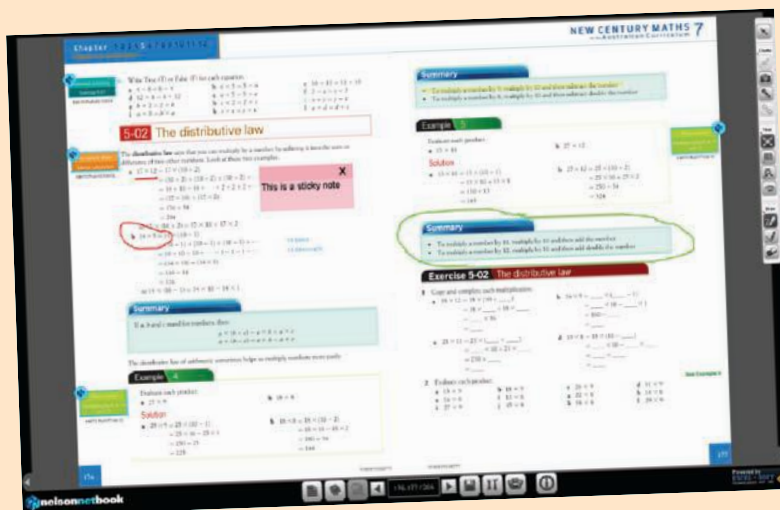
Language of maths

axes	distance	exact answer	general form
gradient	gradient-intercept form	horizontal	interval
length	linear equation	midpoint	parallel
perpendicular	point-gradient form	reciprocal	rise
run	surd	vertical	x-axis
x-intercept	y-axis	y-intercept	

- 1 What is the difference between the **y-axis** and the **y-intercept**?
- 2 When finding the length of an interval on a number plane, what is meant by an **exact answer**?
- 3 What measurement is the fraction given by the vertical rise of a line divided by the horizontal run?
- 4 What is the everyday meaning of the word **intercept**? Look it up in a dictionary.
- 5 What is the property of the gradients of perpendicular lines?
- 6 What form of the linear equation is $ax + by + c = 0$?

NelsonNetBook

- **NelsonNetBook** is the interactive digital version of this textbook found on NelsonNet, containing margin icons that link directly to NelsonNet resources.
- You can add notes, voice and sound bites, highlighting, weblinks and bookmarks to each page of the NelsonNetBook.
- **Zoom** and **Search** functions
- Chapters can be customised for different groups of students.



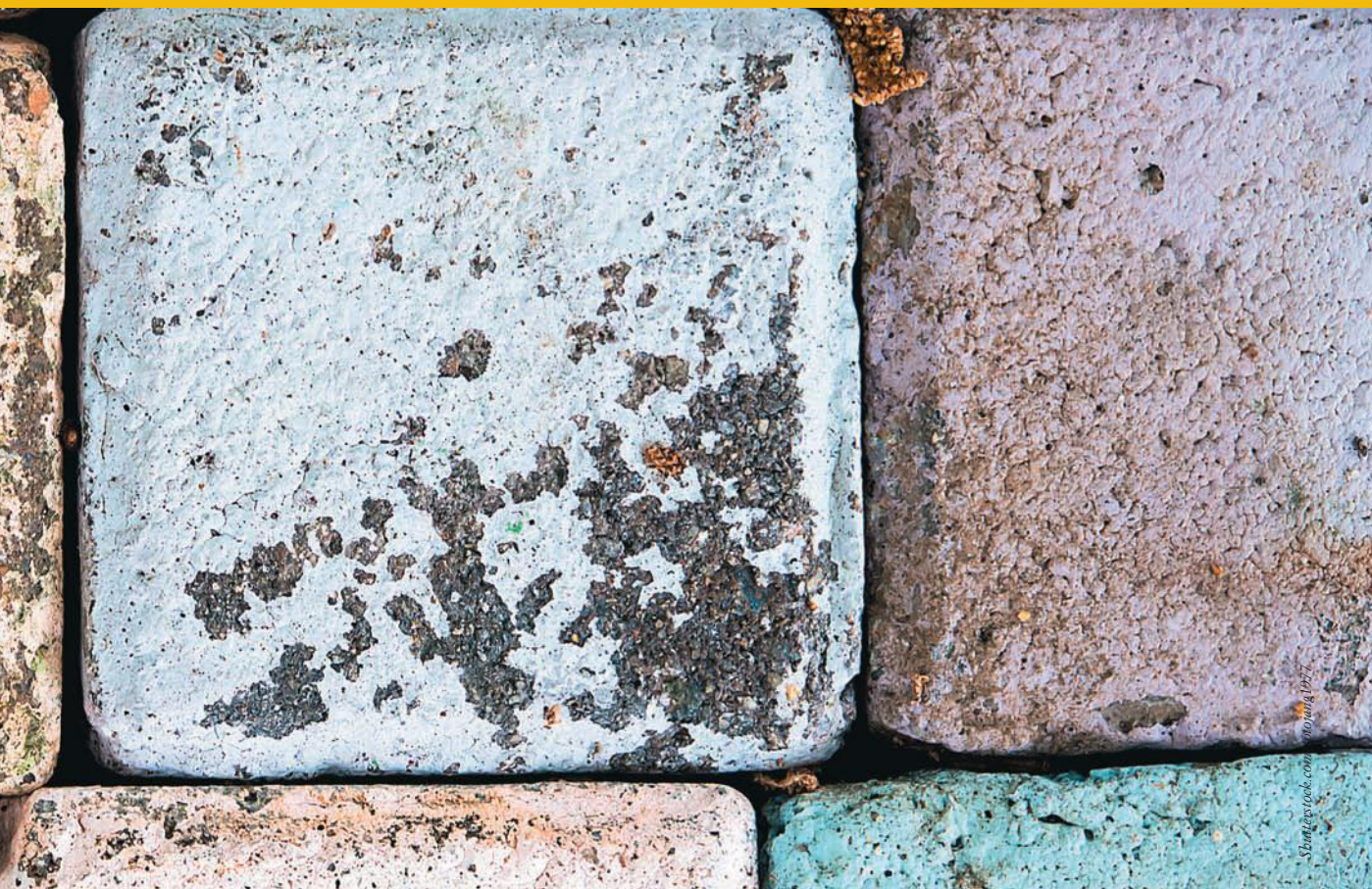


1

Number and Algebra

Surds

When applying Pythagoras' theorem, we have found lengths that cannot be expressed as an exact rational number. Pythagoras encountered this when calculating the diagonal of a square of side length 1 unit. A **surd** is a square root ($\sqrt{}$), cube root ($\sqrt[3]{}$), or any type of root whose exact decimal or fraction value cannot be found.



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Chapter outline

	Proficiency strands			
1-01 Surds and irrational numbers*	U	F	R	C
1-02 Simplifying surds*	U	F	R	
1-03 Adding and subtracting surds*	U	F	R	
1-04 Multiplying and dividing surds*	U	F	R	
1-05 Binomial products involving surds*	U	F	R	C
1-06 Rationalising the denominator*	U	F	R	C

*STAGE 5.3

Wordbank

irrational number A number such as π or $\sqrt{2}$ that cannot be expressed as a fraction $\frac{a}{b}$

rational number Any number that can be written in the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$

rationalise the denominator To simplify a fraction involving a surd by making its denominator rational (that is, not a surd)

real number A number that is either rational or irrational and whose value can be graphed on a number line

simplify a surd To write a surd $\sqrt{x}$ in its simplest form so that x has no factors that are perfect squares

surd A square root (or other root) whose exact value cannot be found

In this chapter you will:

- (STAGE 5.3) define rational and irrational numbers and perform operations with surds 
- (STAGE 5.3) describe real, rational and irrational numbers and surds
- (STAGE 5.3) add, subtract, multiply and divide surds
- (STAGE 5.3) expand and simplify binomial products involving surds
- (STAGE 5.3) rationalise the denominator of expressions of the form $\frac{a\sqrt{b}}{c\sqrt{d}}$

SkillCheck

Worksheet

StartUp assignment 13

MAT10NAWK10091

1 Simplify each expression.

a $(5y)^2$

b $(4m)^3$

c $(-3x)^2$

2 Expand each expression.

a $5(x + 2)$

b $4(y - 3)$

c $3(1 + 2w)$

d $2(5 - y)$

e $-5(2a + 3)$

f $k(1 + 2k)$

3 Select the square numbers from the following list of numbers.

44 81 25 100 75 72 16 50 64 32

4 Expand and simplify each expression.

a $(m + 3)(m + 7)$

b $(y + 1)(y - 4)$

c $(n - 2)(n - 3)$

d $(2d + 3)(1 + 3d)$

e $(1 - 5p)(4 + 3p)$

f $(3a + 2f)(a + 5f)$

g $(x + 4)^2$

h $(y - 3)^2$

i $(2k + 1)^2$

j $(a - 5)(a + 5)$

k $(t + 7)(t - 7)$

l $(3m + 4)(3m - 4)$

Stage 5.3

1-01 Surds and irrational numbers

A **surd** is a square root ($\sqrt{\quad}$), cube root ($\sqrt[3]{\quad}$), or any type of root whose exact decimal or fractional value cannot be found. As a decimal, its digits run endlessly *without repeating* (like π), so they are neither terminating nor recurring decimals.

$\sqrt{7}$ is read as 'the square root of 7' or simply 'root 7'.

Rational numbers such as fractions, terminating or recurring decimals, and percentages, can be expressed in the form $\frac{a}{b}$, where a and b are integers (and $b \neq 0$). Surds are **irrational numbers** because they cannot be expressed in this form.

Rational numbers can be expressed in the form $\frac{a}{b}$

Integers

$$\frac{4}{1} = 4, \frac{26}{1} = 26, \frac{-3}{1} = -3$$

Terminating decimals

$$0.5, 7\frac{1}{8} = 7.125,$$

$$16\% = 0.16, 1.32$$

Recurring decimals

$$\frac{2}{3} = 0.666 \dots$$

$$\frac{5}{6} = 0.833 \dots$$

$$\frac{4}{11} = 0.3636 \dots$$

Irrational numbers *cannot* be expressed in the form $\frac{a}{b}$

Surds

$$\sqrt{5}, -\sqrt{2}, \frac{\sqrt{11}}{3}, 8\sqrt{6}$$

Transcendental numbers

Non-surds whose decimal value also have no pattern and are non-recurring, for example,
 $\pi = 3.14159\dots$, $\cos 38^\circ = 0.7880\dots$,
 $0.0097542\dots$

Example 1

Select the surds from this list of square roots: $\sqrt{56}$ $\sqrt{135}$ $\sqrt{289}$ $\sqrt{99}$ $\sqrt{81}$

Solution

$$\sqrt{56} = 7.4833 \dots$$

$$\sqrt{135} = 11.6189 \dots$$

$$\sqrt{289} = 17$$

$$\sqrt{99} = 9.9498 \dots$$

$$\sqrt{81} = 9$$

So the surds are $\sqrt{56}$, $\sqrt{135}$ and $\sqrt{99}$.

Example 2

Is each number rational or irrational?

a 37.5%

b $-\sqrt[4]{216}$

c 10π

d $0.2\dot{6}$

e $\sqrt{48}$

Solution

a $37.5\% = \frac{37.5}{100} = \frac{3}{8}$

$\therefore 37.5\%$ is a rational number.

b $-\sqrt[4]{256} = -4$

$\therefore -\sqrt[4]{256}$ is a rational number.

c $10\pi = 31.415\ 926\ 54 \dots$

$\therefore 10\pi$ is an irrational number.

d $0.2\dot{6} = 0.26666 \dots$
 $= \frac{4}{15}$

$\therefore 0.2\dot{6}$ is a rational number.

e $\sqrt{48} = 6.928\ 203\ 23 \dots$

$\therefore \sqrt{48}$ is an irrational number.

which is in the form of a fraction $\frac{a}{b}$

which can be written as $\frac{-4}{1}$

The digits run endlessly without repeating.

which is a recurring decimal

which is a fraction

The digits run endlessly without repeating.

Stage 5.3

Square roots

The symbol $\sqrt{\quad}$ stands for the positive square root of a number. For example, $\sqrt{4} = 2$ (not -2). Furthermore, it is not possible to find the square root of a negative number. It is only possible to find the square root of a positive number or zero, because the square of any real number is positive or zero.

Summary

For $x > 0$, $\sqrt{x}$ is the positive square root of x .
 For $x = 0$, $\sqrt{x}$ is 0.
 For $x < 0$, $\sqrt{x}$ is undefined.

Your calculator will tell you that there is a mathematical error if you enter, for example, $\sqrt{-5}$.

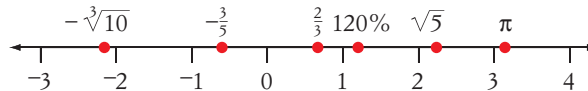
Worksheet

Surd

Surd on the number line

MAT10NAWK10092

The rational and irrational numbers together make up the **real numbers**. Any real number can be represented by a point on the number line.



$$-\sqrt[3]{10} \approx -2.1544\dots$$

irrational (surd)

$$120\% = 1.2$$

rational (percentage)

$$-\frac{3}{5} = -0.6$$

rational (fraction)

$$\sqrt{5} \approx 2.2360\dots$$

irrational (surd)

$$\frac{2}{3} \approx 0.6666\dots$$

rational (fraction)

$$\pi \approx 3.1415\dots$$

irrational (pi)

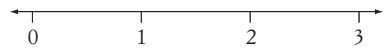
Example 3

Use a pair of compasses and Pythagoras' theorem to estimate the value of $\sqrt{2}$ on a number line.

Solution

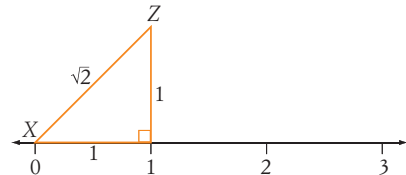
Step 1

Using a scale of 1 unit to 2 cm, draw a number line as shown.



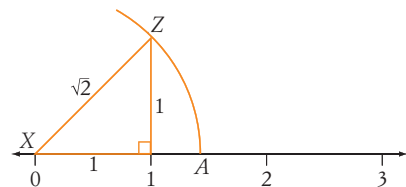
Step 2

Construct a right-angled triangle on the number line with base length and height 1 unit as shown. By Pythagoras' theorem, show that $XZ = \sqrt{2}$ units.



Step 3

With 0 as the centre, use compasses with radius $XZ(\sqrt{2})$ to draw an arc to meet the number line at A as shown. The point A represents the value of $\sqrt{2}$ and should be approximately 1.4142...



Exercise 1-01 Surds and irrational numbers

- Which one of the following is a surd? Select the correct answer A, B, C or D.
A $\sqrt{64}$ B $\sqrt{100}$ C $\sqrt{250}$ D $\sqrt{400}$
- Which one of the following is NOT a surd? Select the correct answer A, B, C or D.
A $\sqrt{84}$ B $\sqrt{196}$ C $\sqrt{27}$ D $\sqrt{160}$
- Select the surds from the following list of square roots.
 $\sqrt{32}$ $\sqrt{125}$ $\sqrt{625}$ $\sqrt{400}$ $\sqrt{4.9}$
 $\sqrt{52}$ $\sqrt{169}$ $\sqrt{0.0009}$ $\sqrt{5625}$ $\sqrt{288}$
- Is each number rational (R) or irrational (I)?
a $5.\dot{6}$ b $\sqrt{8}$ c $\sqrt{4}$ d $3\frac{1}{7}$ e $\sqrt[3]{27}$ f $1.3\dot{5}$
g $\sqrt[3]{-64}$ h $27\frac{1}{2}\%$ i $\sqrt{5 \times 10^3}$ j $\frac{3}{11}$ k $\frac{\sqrt{50}}{3}$ l $\sqrt{\sqrt{4}}$
- Arrange each set of numbers in descending order.
a $1\frac{4}{7}, \sqrt{2}, \frac{\pi}{2}$ b $\sqrt[3]{20}, 2.\dot{6}, 2\frac{7}{9}$
- Express each real number correct to one decimal place and graph them on a number line.
a $-1\frac{4}{5}$ b 74% c $\frac{4}{11}$ d $-\sqrt{12}$
e $-\sqrt[3]{15}$ f $2\frac{5}{9}$ g $\frac{\pi}{2}$ h 187%
- Use the method from Example 3 to estimate the value of $\sqrt{2}$ on a number line.
- a Use the method from Example 3 to estimate the value of $\sqrt{5}$ on a number line by constructing a right-angled triangle with base length 2 units and height 1 unit.
b Use a similar method to estimate the following surds on a number line.
i $\sqrt{10}$ ii $\sqrt{17}$

Investigation: Proof that $\sqrt{2}$ is irrational

A method of proof sometimes used in mathematics is to assume the opposite of what is being proved, and show that it is false. This is called a **proof by contradiction**.

We will use this method to prove that $\sqrt{2}$ is irrational.

Firstly, assume that $\sqrt{2}$ is *rational*. This means we assume that $\sqrt{2}$ can be written in the form $\frac{a}{b}$, where $b \neq 0$, and a and b are integers with no common factor.

$$\sqrt{2} = \frac{a}{b}$$

$$2 = \frac{a^2}{b^2} \quad \text{Squaring both sides}$$

$$a^2 = 2b^2$$

$2b^2$ is an even number because it is divisible by 2.

$\therefore a^2$ is even.

Stage 5.3

If a^2 is even, then a is also even because any odd number squared gives another odd number.

If a is even, then it is divisible by 2 and can be expressed in the form $2m$, where m is an integer.

$$\therefore a^2 = (2m)^2 = 2b^2$$

$$4m^2 = 2b^2$$

$$2m^2 = b^2$$

$$b^2 = 2m^2$$

$\therefore b^2$ is even

$\therefore b$ is even

$\therefore a$ and b are both even.

This contradicts the assumption that a and b have no common factor. Therefore, the assumption that $\sqrt{2}$ is rational is false.

$\therefore \sqrt{2}$ must be irrational.

1 Use the method of proof just described to show that these surds are irrational.

a $\sqrt{3}$

b $\sqrt{5}$

2 Compare your proofs with those of other students.

Puzzle sheet

Simplifying surds

MAT10NAPS10093

Technology worksheet

Excel worksheet:
Simplifying surds quiz

MAT10NACT00019

Technology worksheet

Excel spreadsheet:
Simplifying surds

MAT10NACT00049

1-02 Simplifying surds

Summary

For $x > 0$ (positive):

$$(\sqrt{x})^2 = \sqrt{x} \times \sqrt{x} = x$$

$$\sqrt{x^2} = x$$

Example 4

Simplify each expression.

a $(\sqrt{12})^2$

b $(4\sqrt{7})^2$

c $(-5\sqrt{2})^2$

Solution

a $(\sqrt{12})^2 = 12$

b $(4\sqrt{7})^2 = 4\sqrt{7} \times 4\sqrt{7}$
 $= 4^2 \times (\sqrt{7})^2$
 $= 16 \times 7$
 $= 112$

$4\sqrt{7}$ means $4 \times \sqrt{7}$

c $(-5\sqrt{2})^2 = (-5)^2 \times (\sqrt{2})^2$
 $= 25 \times 2$
 $= 50$

Summary

The square root of a product

For $x > 0$ and $y > 0$:

$$\sqrt{xy} = \sqrt{x} \times \sqrt{y}$$

A surd $\sqrt{n}$ can be simplified if n can be divided into two factors, where one of them is a square number such as 4, 9, 16, 25, 36, 49, ...

Example 5

Simplify each surd.

a $\sqrt{8}$

b $\sqrt{108}$

c $4\sqrt{45}$

d $\frac{\sqrt{288}}{3}$

Solution

$$\begin{aligned} \text{a } \sqrt{8} &= \sqrt{4} \times \sqrt{2} \\ &= 2 \times \sqrt{2} \\ &= 2\sqrt{2} \end{aligned}$$

4 is a square number.

b Method 1

$$\begin{aligned} \sqrt{108} &= \sqrt{36} \times \sqrt{3} \\ &= 6 \times \sqrt{3} \\ &= 6\sqrt{3} \end{aligned}$$

Method 2

$$\begin{aligned} \sqrt{108} &= \sqrt{4} \times \sqrt{27} \\ &= 2 \times \sqrt{9} \times \sqrt{3} \\ &= 2 \times 3 \times \sqrt{3} \\ &= 6\sqrt{3} \end{aligned}$$

Method 2 involves simplifying surds *twice* ($\sqrt{108}$ and $\sqrt{27}$). Method 1 shows that when simplifying surds, you should look for the highest square factor possible.

$$\begin{aligned} \text{c } 4\sqrt{45} &= 4 \times \sqrt{9} \times \sqrt{5} \\ &= 4 \times 3 \times \sqrt{5} \\ &= 12\sqrt{5} \end{aligned}$$

$$\begin{aligned} \text{d } \frac{\sqrt{288}}{3} &= \frac{\sqrt{144} \times \sqrt{2}}{3} \\ &= \frac{12\sqrt{2}}{3} \\ &= \frac{\cancel{12}^4 \sqrt{2}}{\cancel{3}^1} \\ &= 4\sqrt{2} \end{aligned}$$

Video tutorial

Simplifying surds

MAT10NAVT10002

Stage 5.3

Exercise 1-02 Simplifying surds

See Example 4

1 Simplify each expression.

a $(\sqrt{2})^2$

b $(\sqrt{5})^2$

c $(3\sqrt{3})^2$

d $(5\sqrt{10})^2$

e $(\sqrt{0.09})^2$

f $(-2\sqrt{7})^2$

g $(-3\sqrt{5})^2$

h $(-5\sqrt{2})^2$

See Example 5

2 Simplify each surd.

a $\sqrt{50}$

b $\sqrt{12}$

c $\sqrt{28}$

d $\sqrt{150}$

e $\sqrt{700}$

f $\sqrt{45}$

g $\sqrt{48}$

h $\sqrt{200}$

i $\sqrt{96}$

j $\sqrt{63}$

k $\sqrt{288}$

l $\sqrt{108}$

m $\sqrt{75}$

n $\sqrt{147}$

o $\sqrt{32}$

p $\sqrt{242}$

q $\sqrt{162}$

r $\sqrt{245}$

s $\sqrt{125}$

t $\sqrt{512}$

3 Simplify each expression.

a $3\sqrt{20}$

b $4\sqrt{32}$

c $8\sqrt{72}$

d $\frac{\sqrt{40}}{2}$

e $\frac{\sqrt{243}}{9}$

f $\frac{\sqrt{28}}{6}$

g $3\sqrt{24}$

h $9\sqrt{68}$

i $\frac{\sqrt{3125}}{10}$

j $\frac{1}{2}\sqrt{72}$

k $\frac{3}{4}\sqrt{48}$

l $10\sqrt{160}$

m $3\sqrt{75}$

n $7\sqrt{68}$

o $\frac{\sqrt{52}}{6}$

4 Which one of the following is equivalent to $4\sqrt{50}$? Select A, B, C or D.

A $8\sqrt{5}$

B $20\sqrt{2}$

C $8\sqrt{2}$

D $20\sqrt{5}$

5 Which one of the following is equivalent to $\frac{\sqrt{250}}{10}$? Select A, B, C or D.

A $\frac{\sqrt{5}}{10}$

B $\frac{\sqrt{10}}{2}$

C $2\sqrt{10}$

D $5\sqrt{10}$

6 Decide whether each statement is true (T) or false (F).

a $3\sqrt{5} = \sqrt{15}$

b $\sqrt{18} = 9$

c $(\sqrt{9.4})^2 = 9.4$

d $\sqrt{75} = 5\sqrt{3}$

e $\sqrt{3} \approx 1.7$

f The exact value of $\sqrt{10}$ is 3.162 277 8

Just for the record

Unreal numbers are imaginary!

There exist numbers that are neither **rational** nor **irrational**, so they are also not **real numbers**. For example, $\sqrt{-2}$ is not a real number, because there is no real number which, if squared, equals -2 . Numbers such as $\sqrt{-2}$, $\sqrt{-10}$ and $\sqrt[4]{-17}$ are called **unreal** or **imaginary numbers** and cannot be graphed on a number line (that is, their values cannot be ordered).

Imaginary numbers were first noticed by Hero of Alexandria in the 1st century CE. In 1545, the Italian mathematician Girolamo Cardano wrote about them, but believed negative numbers did not have a square root. Imaginary numbers were largely ignored until the 18th century when they were studied by Leonhard Euler and the Carl Friedrich Gauss.

$\sqrt{-1}$ is defined to be the imaginary number i , so $\sqrt{-1} = i$.

$$\therefore \sqrt{-36} = \sqrt{36 \times (-1)} = \sqrt{36} \times \sqrt{-1} = 6i.$$

Imaginary numbers are useful for solving physics and engineering problems involving heat conduction, elasticity, hydrodynamics and the flow of electric current.

Simplify each imaginary number.

a $\sqrt{-100}$

b $\sqrt{-25}$

c $\sqrt[4]{-16}$

d $\sqrt[6]{-64}$

1-03 Adding and subtracting surds

Just as you can only add or subtract 'like terms' in algebra, you can only add or subtract 'like surds'. You may first need to express all the surds in their simplest forms.

Example 6

Simplify each expression.

- a $5\sqrt{11} + 7\sqrt{11}$ b $8\sqrt{5} - 3\sqrt{5}$ c $3\sqrt{6} - 4\sqrt{2} + 5\sqrt{6}$
d $\sqrt{80} + \sqrt{20}$ e $\sqrt{8} - \sqrt{27} + \sqrt{18}$ f $5\sqrt{20} - 3\sqrt{125}$

Solution

- a $5\sqrt{11} + 7\sqrt{11} = 12\sqrt{11}$ b $8\sqrt{5} - 3\sqrt{5} = 5\sqrt{5}$
c $3\sqrt{6} - 4\sqrt{2} + 5\sqrt{6} = 8\sqrt{6} - 4\sqrt{2}$
d $\sqrt{80} + \sqrt{20} = \sqrt{16}\sqrt{5} + \sqrt{4}\sqrt{5}$ **Simplifying each surd.**
 $= 4\sqrt{5} + 2\sqrt{5}$
 $= 6\sqrt{5}$
e $\sqrt{8} - \sqrt{27} + \sqrt{18} = \sqrt{4}\sqrt{2} - \sqrt{9}\sqrt{3} + \sqrt{9}\sqrt{2}$ f $5\sqrt{20} - 3\sqrt{125} = 5\sqrt{4}\sqrt{5} - 3\sqrt{25}\sqrt{5}$
 $= 2\sqrt{2} - 3\sqrt{3} + 3\sqrt{2}$ $= 5 \times 2\sqrt{5} - 3 \times 5\sqrt{5}$
 $= 5\sqrt{2} - 3\sqrt{3}$ $= 10\sqrt{5} - 15\sqrt{5}$
 $= -5\sqrt{5}$

Exercise 1-03 Adding and subtracting surds

1 Simplify each expression.

- a $9\sqrt{3} + 2\sqrt{3}$ b $11\sqrt{2} - 8\sqrt{2}$ c $5\sqrt{6} - \sqrt{6}$
d $\sqrt{5} + 3\sqrt{5}$ e $5\sqrt{17} - 5\sqrt{17}$ f $3\sqrt{10} - 2\sqrt{10}$
g $4\sqrt{15} - 3\sqrt{15} + 7\sqrt{15}$ h $5\sqrt{6} - 2\sqrt{6} - 4\sqrt{6}$ i $3\sqrt{3} + 4\sqrt{3} - 5\sqrt{3}$
j $4\sqrt{5} + 7\sqrt{5} - \sqrt{5}$ k $8\sqrt{10} - 5\sqrt{10} + 3\sqrt{10}$ l $10\sqrt{3} - 3\sqrt{3} - 12\sqrt{3}$

2 Simplify each expression.

- a $5\sqrt{3} - 9 + 2\sqrt{3}$ b $11\sqrt{10} - 7\sqrt{2} - 4\sqrt{10}$
c $-4\sqrt{3} + 5\sqrt{2} - 5\sqrt{3}$ d $3\sqrt{15} + 3\sqrt{2} + 4\sqrt{15} + 5\sqrt{2}$
e $\sqrt{7} - 3\sqrt{5} - 4\sqrt{7} + \sqrt{5}$ f $4\sqrt{6} - 3\sqrt{3} - 2\sqrt{6} - 5\sqrt{3}$
g $10\sqrt{11} - 5\sqrt{3} + 3\sqrt{11} + 4\sqrt{3}$ h $\sqrt{13} + 8\sqrt{7} - 7\sqrt{13} + 3\sqrt{7}$
i $2\sqrt{5} - 3\sqrt{7} - 2\sqrt{5} - 3\sqrt{7}$ j $4\sqrt{10} - 3\sqrt{5} - 4\sqrt{10}$

3 For each expression, select the correct simplified answer **A**, **B**, **C** or **D**.

- a $\sqrt{3} + \sqrt{12}$
A $5\sqrt{3}$ B $\sqrt{15}$ C $2\sqrt{6}$ D $3\sqrt{3}$
b $4\sqrt{5} - 2\sqrt{125}$
A $-6\sqrt{5}$ B $\sqrt{5}$ C $-\sqrt{45}$ D $-46\sqrt{5}$

Stage 5.3

Puzzle sheet

Surds code puzzle

MAT10NAPS10094

See Example 6

Stage 5.3

4 Simplify each expression.

a $\sqrt{8} + \sqrt{32}$

b $\sqrt{108} - \sqrt{27}$

c $\sqrt{20} - \sqrt{80}$

d $\sqrt{28} - \sqrt{63}$

e $3\sqrt{6} + \sqrt{24}$

f $2\sqrt{5} + \sqrt{125}$

g $\sqrt{40} - \sqrt{90}$

h $5\sqrt{11} + \sqrt{99}$

i $3\sqrt{2} + \sqrt{18}$

j $\sqrt{27} + 5\sqrt{3}$

k $\sqrt{200} - 7\sqrt{2}$

l $\sqrt{50} + \sqrt{32}$

m $5\sqrt{3} + 2\sqrt{27}$

n $3\sqrt{20} - \sqrt{245}$

o $7\sqrt{12} - 5\sqrt{48}$

p $4\sqrt{27} + 2\sqrt{243}$

q $3\sqrt{63} - 2\sqrt{28}$

r $2\sqrt{98} + 3\sqrt{162}$

s $-5\sqrt{6} + 2\sqrt{150}$

t $4\sqrt{50} + 3\sqrt{18}$

u $5\sqrt{27} - 6\sqrt{75}$

v $3\sqrt{112} - 2\sqrt{252}$

w $\sqrt{32} + \sqrt{8} + \sqrt{12}$

x $\sqrt{27} + \sqrt{54} + \sqrt{243}$

y $\sqrt{98} - 3\sqrt{20} - 2\sqrt{8}$

z $3\sqrt{96} - 2\sqrt{150} + \sqrt{24}$

Worksheet

Multiplying and dividing
surds

MAT10NAWK10095

Puzzle sheet

Surds

MAT10NAPS00043

Technology worksheet

Excel worksheet:
Simplifying surds quiz

MAT10NACT00019

Technology worksheet

Excel spreadsheet:
Simplifying surds

MAT10NACT00049

1-04 Multiplying and dividing surds

Summary

The square root of products and quotients

For $x > 0$ and $y > 0$:

$$\sqrt{xy} = \sqrt{x} \times \sqrt{y}$$

$$\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$$

Example 7

Simplify each expression.

a $\sqrt{7} \times \sqrt{5}$

b $\sqrt{6} \times \sqrt{14}$

c $4\sqrt{3} \times 10\sqrt{3}$

d $\frac{12\sqrt{90}}{3\sqrt{10}}$

e $\sqrt{54} \div (-\sqrt{2})$

f $5\sqrt{27} \times 3\sqrt{6}$

Solution

a $\sqrt{7} \times \sqrt{5} = \sqrt{35}$

$$\begin{aligned} \text{b } \sqrt{6} \times \sqrt{14} &= \sqrt{84} \\ &= \sqrt{4} \times \sqrt{21} \\ &= 2\sqrt{21} \end{aligned}$$

$$\begin{aligned} \text{c } 4\sqrt{3} \times 10\sqrt{3} &= 4 \times 10 \times \sqrt{3} \times \sqrt{3} \\ &= 40 \times 3 \\ &= 120 \end{aligned}$$

$$\begin{aligned} \text{d } \frac{12\sqrt{90}}{3\sqrt{10}} &= 4 \times \sqrt{9} \\ &= 4 \times 3 \\ &= 12 \end{aligned}$$

$$\begin{aligned} \text{e } \sqrt{54} \div (-\sqrt{2}) &= -\frac{\sqrt{54}}{\sqrt{2}} \\ &= -\sqrt{27} \\ &= -\sqrt{9} \times \sqrt{3} \\ &= -3\sqrt{3} \end{aligned}$$

$$\begin{aligned} \text{f } 5\sqrt{27} \times 3\sqrt{6} &= 5 \times 3 \times \sqrt{27} \times \sqrt{6} \\ &= 15\sqrt{162} \\ &= 15 \times \sqrt{81} \times \sqrt{2} \\ &= 15 \times 9\sqrt{2} \\ &= 135\sqrt{2} \end{aligned}$$

Example 8

Simplify $\frac{5\sqrt{2} \times 4\sqrt{12}}{10\sqrt{8}}$.

Solution

$$\frac{5\sqrt{2} \times 4\sqrt{12}}{10\sqrt{8}} = \frac{20\sqrt{24}}{10\sqrt{8}} = 2\sqrt{3}$$

See Example 7

Exercise 1-04 Multiplying and dividing surds

1 Simplify each expression.

a $\sqrt{10} \times \sqrt{3}$

b $-\sqrt{6} \times \sqrt{11}$

c $\sqrt{5} \times \sqrt{8}$

d $\sqrt{2} \times \sqrt{18}$

e $\sqrt{5} \times (-\sqrt{5})$

f $8\sqrt{3} \times 6\sqrt{3}$

g $5\sqrt{10} \times 3\sqrt{3}$

h $-2\sqrt{7} \times 5\sqrt{3}$

i $7\sqrt{5} \times 4\sqrt{5}$

j $2\sqrt{3} \times (-5\sqrt{6})$

k $4\sqrt{3} \times \sqrt{27}$

l $-3\sqrt{5} \times 4\sqrt{10}$

m $-7\sqrt{2} \times 4\sqrt{8}$

n $\sqrt{18} \times 8\sqrt{3}$

o $10\sqrt{2} \times 2\sqrt{8}$

p $3\sqrt{18} \times 5\sqrt{12}$

q $3\sqrt{44} \times (-2\sqrt{99})$

r $5\sqrt{8} \times 4\sqrt{40}$

s $8\sqrt{3} \times 3\sqrt{54}$

t $-8\sqrt{32} \times \sqrt{27}$

u $\sqrt{90} \times \sqrt{72}$

v $-5\sqrt{20} \times 3\sqrt{8}$

w $7\sqrt{18} \times 3\sqrt{24}$

x $3\sqrt{48} \times 2\sqrt{12}$

2 Simplify each expression.

a $\sqrt{24} \div \sqrt{8}$

b $\sqrt{30} \div (-\sqrt{5})$

c $\frac{40\sqrt{14}}{5\sqrt{2}}$

d $10\sqrt{54} \div 5\sqrt{27}$

e $-3\sqrt{98} \div 6\sqrt{14}$

f $\frac{7\sqrt{18}}{\sqrt{2}}$

g $2\sqrt{24} \div 4\sqrt{6}$

h $\frac{\sqrt{128}}{\sqrt{2}}$

i $15\sqrt{18} \div 3\sqrt{6}$

j $\frac{-20\sqrt{10}}{-4\sqrt{5}}$

k $36\sqrt{24} \div 9\sqrt{8}$

l $16\sqrt{30} \div 8\sqrt{5}$

m $12\sqrt{14} \div 6$

n $\frac{3\sqrt{2}}{-12}$

o $\sqrt{80} \div 4\sqrt{5}$

p $5\sqrt{60} \div \sqrt{15}$

q $6\sqrt{8} \div 3\sqrt{2}$

r $\frac{-42\sqrt{54}}{6\sqrt{3}}$

s $12\sqrt{63} \div 3\sqrt{7}$

t $\frac{8\sqrt{50}}{2\sqrt{200}}$

u $6\sqrt{3} \div \sqrt{243}$

3 Simplify the expressions below.

a $\sqrt{6} \times \sqrt{6}$

b $\sqrt{7} \times \sqrt{7}$

c $2\sqrt{3} \times \sqrt{3}$

d $5\sqrt{y} \times 3\sqrt{y}$

e $\sqrt{x} \times \sqrt{x}$

f $\sqrt{a^2} \times \sqrt{a}$

4 Simplify $3\sqrt{2} \times \sqrt{6}$. Select the correct answer A, B, C or D.

A 6

B $6\sqrt{2}$

C $6\sqrt{3}$

D $12\sqrt{2}$

Stage 5.3

- 5 Simplify
- $20\sqrt{10} \div 5\sqrt{2}$
- . Select
- A**
- ,
- B**
- ,
- C**
- or
- D**
- .

A $4\sqrt{5}$

B $15\sqrt{5}$

C 10

D 20

See Example 8

- 6 Simplify each expression.

a $\frac{3\sqrt{5} \times 4\sqrt{2}}{3\sqrt{40}}$

b $\frac{3\sqrt{12} \times 8\sqrt{6}}{4\sqrt{27}}$

c $\frac{5\sqrt{8} \times 2\sqrt{90}}{10\sqrt{24}}$

d $\frac{4\sqrt{5}}{2\sqrt{15} \times 5\sqrt{27}}$

e $\frac{10\sqrt{686} \times 3\sqrt{12}}{5\sqrt{28} \times \sqrt{18}}$

f $\frac{8\sqrt{80} \times 3\sqrt{2}}{4\sqrt{5} \times 6\sqrt{8}}$

Mental skills 1

Maths without calculators

Percentage of a quantity

Learn these commonly-used percentages and their fraction equivalents.

Percentage	50%	25%	12.5%	75%	20%	10%	$33\frac{1}{3}\%$	$66\frac{2}{3}\%$
Fraction	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{3}{4}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{3}$	$\frac{2}{3}$

Now we will use them to find a percentage of a quantity.

- 1 Study each example.

$$\begin{array}{lll} \text{a} & 20\% \times 25 = \frac{1}{5} \times 25 & \text{b} \quad 50\% \times 120 = \frac{1}{2} \times 120 \\ & = 5 & \quad \quad \quad = 60 \end{array}$$

$$\text{c} \quad 12.5\% \times 32 = \frac{1}{8} \times 32$$

$$= 4$$

$$\text{d} \quad 75\% \times 56 = \frac{3}{4} \times 60 \quad \text{e} \quad 33\frac{1}{3}\% \times 27 = \frac{1}{3} \times 27 \quad \text{f} \quad 66\frac{2}{3}\% \times 60 = \frac{2}{3} \times 60$$

$$\begin{array}{lll} = \left(\frac{1}{4} \times 60\right) \times 3 & = 9 & = \left(\frac{1}{3} \times 60\right) \times 2 \\ = 15 \times 3 & & = 20 \times 2 \\ = 45 & & = 40 \end{array}$$

- 2 Now simplify each expression.

$$\begin{array}{llll} \text{a} & 25\% \times 44 & \text{b} & 33\frac{1}{3}\% \times 120 & \text{c} & 20\% \times 35 & \text{d} & 66\frac{2}{3}\% \times 36 \\ \text{e} & 10\% \times 230 & \text{f} & 12\frac{1}{2}\% \times 48 & \text{g} & 50\% \times 86 & \text{h} & 20\% \times 400 \\ \text{i} & 75\% \times 24 & \text{j} & 33\frac{1}{3}\% \times 45 & \text{k} & 25\% \times 160 & \text{l} & 10\% \times 650 \\ \text{m} & 12.5\% \times 88 & \text{n} & 66\frac{2}{3}\% \times 21 & \text{o} & 20\% \times 60 & \text{p} & 75\% \times 180 \end{array}$$

1-05 Binomial products involving surds

Stage 5.3

Surd expressions involving brackets can be expanded in the same way as algebraic expressions of the form $a(b + c)$ and $(a + b)(c + d)$.

Example 9

Expand and simplify each expression.

a $\sqrt{3}(\sqrt{5} + \sqrt{7})$

b $2\sqrt{11}(3\sqrt{11} - 5\sqrt{2})$

Solution

a $\sqrt{3}(\sqrt{5} + \sqrt{7})$
 $= \sqrt{3} \times \sqrt{5} + 3 \times \sqrt{7}$
 $= \sqrt{15} + \sqrt{21}$

b $2\sqrt{11}(3\sqrt{11} - 5\sqrt{2})$
 $= 2\sqrt{11} \times 3\sqrt{11} - 2\sqrt{11} \times 5\sqrt{2}$
 $= 6 \times 11 - 10 \times \sqrt{22}$
 $= 66 - 10\sqrt{22}$

Example 10

Expand and simplify each expression.

a $(\sqrt{7} + \sqrt{5})(3\sqrt{2} - \sqrt{3})$

b $(3 - 2\sqrt{10})(\sqrt{5} - 3\sqrt{2})$

Solution

a $(\sqrt{7} + \sqrt{5})(3\sqrt{2} - \sqrt{3}) = \sqrt{7}(3\sqrt{2} - \sqrt{3}) + \sqrt{5}(3\sqrt{2} - \sqrt{3})$
 $= \sqrt{7} \times 3\sqrt{2} - \sqrt{7} \times \sqrt{3} + \sqrt{5} \times 3\sqrt{2} - \sqrt{5} \times \sqrt{3}$
 $= 3\sqrt{14} - \sqrt{21} + 3\sqrt{10} - \sqrt{15}$

b $(3 - 2\sqrt{10})(\sqrt{5} - 3\sqrt{2}) = 3(\sqrt{5} - 3\sqrt{2}) - 2\sqrt{10}(\sqrt{5} - 3\sqrt{2})$
 $= 3 \times \sqrt{5} - 3 \times 3\sqrt{2} - 2\sqrt{10} \times \sqrt{5} + 2\sqrt{10} \times 3\sqrt{2}$
 $= 3\sqrt{5} - 9\sqrt{2} - 2\sqrt{50} + 6\sqrt{20}$
 $= 3\sqrt{5} - 9\sqrt{2} - 2(5\sqrt{2}) + 6(2\sqrt{5})$
 $= 3\sqrt{5} - 9\sqrt{2} - 10\sqrt{2} + 12\sqrt{5}$
 $= 15\sqrt{5} - 19\sqrt{2}$

Summary

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a - b)^2 = a^2 - 2ab + b^2$$

$$(a + b)(a - b) = a^2 - b^2$$

Stage 5.3

Example 11

Expand and simplify each expression.

a $(\sqrt{7} - \sqrt{5})^2$ b $(2\sqrt{3} + 3\sqrt{5})^2$
 c $(\sqrt{5} - \sqrt{2})(\sqrt{5} + \sqrt{2})$ d $(3\sqrt{11} + 4)(3\sqrt{11} - 4)$

Solution

a $(\sqrt{7} - \sqrt{5})^2 = (\sqrt{7})^2 - 2 \times \sqrt{7} \times \sqrt{5} + (\sqrt{5})^2$ Using $(a - b)^2 = a^2 - 2ab + b^2$
 $= 7 - 2\sqrt{35} + 5$
 $= 12 - 2\sqrt{35}$

b $(2\sqrt{3} + 3\sqrt{5})^2 = (2\sqrt{3})^2 + 2 \times 2\sqrt{3} \times 3\sqrt{5} + (3\sqrt{5})^2$ Using $(a + b)^2 = a^2 + 2ab + b^2$
 $= (4 \times 3) + 12\sqrt{15} + (9 \times 5)$
 $= 12 + 12\sqrt{15} + 45$
 $= 57 + 12\sqrt{15}$

c $(\sqrt{5} - \sqrt{2})(\sqrt{5} + \sqrt{2}) = (\sqrt{5})^2 - (\sqrt{2})^2$ Using $(a + b)(a - b) = a^2 - b^2$
 $= 5 - 2$
 $= 3$

Note that because of the 'difference of two squares', the answer is not a surd but a rational number.

d $(3\sqrt{11} + 4)(3\sqrt{11} - 4) = (3\sqrt{11})^2 - 4^2$ Using $(a + b)(a - b) = a^2 - b^2$
 $= (9 \times 11) - 16$
 $= 83$

Exercise 1-05 Binomial products involving surds

See Example 9

1 Expand and simplify each expression.

a $\sqrt{5}(\sqrt{3} + \sqrt{2})$ b $\sqrt{6}(\sqrt{2} - 1)$ c $\sqrt{2}(\sqrt{3} + \sqrt{7})$
 d $\sqrt{5}(3\sqrt{2} - \sqrt{5})$ e $3\sqrt{2}(\sqrt{2} + 2\sqrt{3})$ f $-\sqrt{11}(4 - \sqrt{5})$
 g $2\sqrt{7}(3\sqrt{7} - 4)$ h $5\sqrt{5}(1 + 3\sqrt{5})$ i $3\sqrt{2}(4\sqrt{2} + \sqrt{3})$

See Example 10

2 Which expression is equivalent to $(\sqrt{3} + 2\sqrt{5})(5\sqrt{2} + \sqrt{3})$? Select the correct answer **A**, **B**, **C** or **D**.

A $20\sqrt{10}$ B $2\sqrt{15} + 5\sqrt{6}$
 C $5\sqrt{6} + 3 + 10\sqrt{10} + 2\sqrt{15}$ D $5\sqrt{5} + \sqrt{3} + 7\sqrt{7} + 4\sqrt{2}$

3 Expand and simplify each expression.

a $(\sqrt{5} - 3)(2\sqrt{5} + \sqrt{2})$ b $(\sqrt{7} - \sqrt{3})(\sqrt{7} + 2)$
 c $(7\sqrt{3} + 2)(4\sqrt{2} + \sqrt{3})$ d $(3\sqrt{2} - \sqrt{5})(5\sqrt{2} + 2\sqrt{5})$
 e $(\sqrt{7} + 2\sqrt{11})(3\sqrt{7} + 4\sqrt{11})$ f $(5\sqrt{3} - 2\sqrt{2})(4\sqrt{3} - 3\sqrt{2})$
 g $(6 + 2\sqrt{10})(3\sqrt{10} - 1)$ h $(\sqrt{7} - 2\sqrt{5})(3\sqrt{5} + 2\sqrt{7})$

- 4 Which expression is equivalent to $(5 + \sqrt{7})^2$? Select **A**, **B**, **C** or **D**.
A 12 **B** 32 **C** $32 + 10\sqrt{7}$ **D** $32 + 5\sqrt{7}$
- 5 Expand and simplify each expression.
a $(\sqrt{5} - \sqrt{3})^2$ **b** $(\sqrt{7} + \sqrt{2})^2$ **c** $(\sqrt{5} - 2)^2$ **d** $(3 + \sqrt{10})^2$
e $(5\sqrt{2} + 3\sqrt{3})^2$ **f** $(5\sqrt{7} - 2)^2$ **g** $(3\sqrt{2} + 2\sqrt{5})^2$ **h** $(2\sqrt{5} + \sqrt{3})^2$
- 6 Expand and simplify each expression.
a $(\sqrt{3} - \sqrt{2})(\sqrt{3} + \sqrt{2})$ **b** $(5 + \sqrt{3})(5 - \sqrt{3})$
c $(6 + 2\sqrt{7})(6 - 2\sqrt{7})$ **d** $(\sqrt{5} - \sqrt{3})(\sqrt{5} + \sqrt{3})$
e $(\sqrt{11} - \sqrt{10})(\sqrt{11} + \sqrt{10})$ **f** $(5\sqrt{7} + 3)(5\sqrt{7} - 3)$
g $(3\sqrt{2} + \sqrt{5})(3\sqrt{2} - \sqrt{5})$ **h** $(4\sqrt{2} - 5\sqrt{3})(4\sqrt{2} + 5\sqrt{3})$
- 7 Which expression is equivalent to $(5\sqrt{2} - 4\sqrt{3})(5\sqrt{2} + 4\sqrt{3})$? Select **A**, **B**, **C** or **D**.
A $25\sqrt{2} - 16\sqrt{3}$ **B** $10\sqrt{2} + 10\sqrt{6}$ **C** 2 **D** 26
- 8 Expand and simplify each expression.
a $(3\sqrt{7} - 5)^2$ **b** $(5\sqrt{2} - 4)(\sqrt{2} + 5)$
c $(2\sqrt{7} + 3\sqrt{5})(\sqrt{5} + \sqrt{7})$ **d** $(4\sqrt{3} + 5)^2$
e $(4\sqrt{2} + \sqrt{3})(4\sqrt{2} - \sqrt{3})$ **f** $(3\sqrt{10} - \sqrt{2})^2$

Investigation: Making the denominator rational

If $\sqrt{2} \approx 1.4142$, what is the value of $\frac{1}{\sqrt{2}}$? Fractions containing surds in the denominator are difficult to work with. When approximating the value of $\frac{1}{\sqrt{2}}$, it is difficult to mentally divide by 1.4142. We can overcome this by making the denominator **rational** (that is, **not** a surd).

- 1 What happens when we multiply the numerator and denominator of a fraction by the same number?

a Simplify $\frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$.

b Mentally approximate the value of $\frac{\sqrt{2}}{2}$, given that $\sqrt{2} \approx 1.4142$.

c Check, using a calculator, that $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$. Why is this true?

2 **a** Is it true that $\frac{3}{\sqrt{7}} = \frac{3}{\sqrt{7}} \times \frac{\sqrt{7}}{\sqrt{7}}$? Why?

b Simplify $\frac{3}{\sqrt{7}} \times \frac{\sqrt{7}}{\sqrt{7}}$. Compare your answer with those of other students.

c Check, using a calculator, that $\frac{3}{\sqrt{7}} = \frac{3\sqrt{7}}{7}$.

d Explain why $\frac{3\sqrt{5}}{\sqrt{2}} = \frac{3\sqrt{5}}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$.

e Show that $\frac{3\sqrt{5}}{\sqrt{2}} = \frac{3\sqrt{10}}{\sqrt{2}}$.

Stage 5.3

Worksheet

Rationalising the denominator

MAT10NAWK10201

1-06 Rationalising the denominator

Surds of the form $\frac{1}{\sqrt{5}}, \frac{3}{2\sqrt{7}}, \frac{\sqrt{3}}{\sqrt{2}}, \frac{5\sqrt{7}}{\sqrt{3}}, \dots$ have denominators that are irrational. These expressions may be rewritten with a rational denominator by multiplying both the numerator and denominator by the surd that appears in the denominator. This method is called **rationalising the denominator**.

Example 12

Rationalise the denominator of each surd.

a $\frac{3}{\sqrt{2}}$

b $\frac{5}{4\sqrt{3}}$

c $\frac{8\sqrt{2}}{3\sqrt{5}}$

d $\frac{\sqrt{2}+1}{\sqrt{3}}$

Solution

$$\begin{aligned} \text{a } \frac{3}{\sqrt{2}} &= \frac{3}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{3\sqrt{2}}{2} \end{aligned}$$

$$\text{because } \frac{\sqrt{2}}{\sqrt{2}} = 1$$

Because $\frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$, it is easier to approximate $\frac{3}{\sqrt{2}}$ by mentally multiplying $\frac{3}{2}$ by 1.4142 than by dividing 3 by 1.4142.

$$\begin{aligned} \text{b } \frac{5}{4\sqrt{3}} &= \frac{5}{4\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} \\ &= \frac{5\sqrt{3}}{4 \times 3} \\ &= \frac{5\sqrt{3}}{12} \end{aligned}$$

$$\begin{aligned} \text{c } \frac{8\sqrt{2}}{3\sqrt{5}} &= \frac{8\sqrt{2}}{3\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} \\ &= \frac{8\sqrt{10}}{3 \times 5} \\ &= \frac{8\sqrt{10}}{15} \end{aligned}$$

$$\begin{aligned} \text{d } \frac{\sqrt{2}+1}{\sqrt{3}} &= \frac{\sqrt{2}+1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} \\ &= \frac{\sqrt{6}+\sqrt{3}}{3} \end{aligned}$$

Exercise 1-06 Rationalising the denominator

See Example 12

- 1 By rationalising the denominator, which surd is equivalent to $\frac{2}{\sqrt{6}}$? Select the correct answer A, B, C or D.

A $2\sqrt{6}$

B $\frac{\sqrt{6}}{3}$

C $\frac{\sqrt{6}}{6}$

D $\frac{2\sqrt{6}}{3}$

- 2 Rationalise the denominator of each surd.

a $\frac{1}{\sqrt{2}}$

b $\frac{1}{\sqrt{7}}$

c $\frac{1}{\sqrt{3}}$

d $\frac{3}{\sqrt{2}}$

e $\frac{2}{\sqrt{7}}$

f $\frac{1}{3\sqrt{2}}$

g $\frac{1}{2\sqrt{3}}$

h $\frac{1}{4\sqrt{7}}$

i $\frac{7}{3\sqrt{5}}$

j $\frac{\sqrt{2}}{3\sqrt{5}}$

k $\frac{3\sqrt{2}}{2\sqrt{6}}$

l $\frac{5\sqrt{3}}{4\sqrt{5}}$

- 3 Which surd is equivalent to $\frac{\sqrt{3}}{2\sqrt{5}}$? Select A, B, C or D.

A $\frac{\sqrt{15}}{10}$

B $2\sqrt{15}$

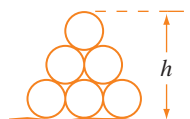
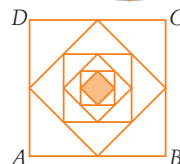
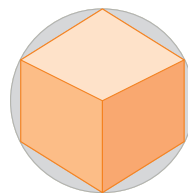
C $\frac{\sqrt{15}}{3}$

D $\sqrt{5}$

- 4 Which surd is equivalent to $\frac{\sqrt{27}}{3\sqrt{18}}$? Select **A**, **B**, **C** or **D**.
- A** $\frac{1}{2}$ **B** $\frac{\sqrt{2}}{2}$ **C** $\frac{\sqrt{5}}{6}$ **D** $\frac{\sqrt{6}}{6}$
- 5 Rationalise the denominator of each expression.
- a** $\frac{\sqrt{2}-1}{\sqrt{2}}$ **b** $\frac{1-\sqrt{5}}{\sqrt{5}}$ **c** $\frac{5+\sqrt{3}}{2\sqrt{2}}$ **d** $\frac{\sqrt{2}-\sqrt{3}}{3\sqrt{6}}$
- 6 Simplify each expression, giving the answer with a rational denominator.
- a** $\frac{1}{\sqrt{7}} + \frac{1}{\sqrt{2}}$ **b** $\frac{\sqrt{2}}{\sqrt{5}} + \frac{3}{\sqrt{3}}$ **c** $\frac{3}{2\sqrt{3}} - \frac{1}{\sqrt{2}}$

Power plus

- 1 **a** Is it true that $\frac{1}{3+\sqrt{2}} = \frac{1}{3+\sqrt{2}} \times \frac{3-\sqrt{2}}{3-\sqrt{2}}$? Explain.
- b** Simplify $\frac{1}{3+\sqrt{2}} \times \frac{3-\sqrt{2}}{3-\sqrt{2}}$. Is the denominator rational?
- c** Use a calculator to check that the value of your answer to part **b** is equal to the value of $\frac{1}{3+\sqrt{3}}$.
- 2 The conjugate of $3 + \sqrt{2}$ is $3 - \sqrt{2}$. Find the conjugate of:
- a** $5 + 2\sqrt{3}$ **b** $2 - \sqrt{3}$ **c** $\sqrt{5} + 1$ **d** $\sqrt{7} - \sqrt{3}$
- 3 The process shown in question 1 involves rationalising a surd with a binomial denominator. By first finding the conjugate of the denominator, rationalise the denominator of each expression below.
- a** $\frac{1}{2-\sqrt{3}}$ **b** $\frac{\sqrt{2}}{\sqrt{5}+1}$ **c** $\frac{1}{\sqrt{7}-\sqrt{3}}$ **d** $\frac{2}{5+2\sqrt{3}}$
- 4 The largest cube that can fit into a sphere must have its eight vertices touching the surface of the sphere. Express the side length, s , of the cube in terms of the diameter, D , of the sphere.
- 5 Squares are formed inside squares by joining the midpoints of the sides of the squares as shown. If $AB = 4$ cm, find the exact length of the side of the shaded square.
- 6 Six stormwater pipes, each 2 mm in diameter, are stacked as shown in the diagram. Find the exact height, h , of this stacking.



Chapter 1 review

Language of maths

Puzzle sheet

Surds crossword

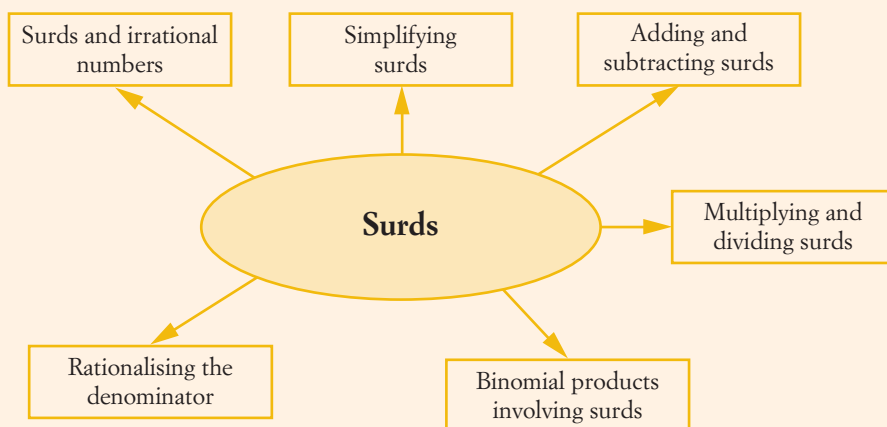
MAT10NAPS10202

approximate	binomial	denominator	difference of two squares
expand	irrational number	perfect square	product
Pythagoras' theorem	quotient	rational number	rationalise
real number	root	simplify	square number
square root	surd	undefined	

- 1 Why do you think a **rational number** has that name?
- 2 What is the difference between a **rational number** and a **real number**?
- 3 What is a **surd**?
- 4 How do you **simplify** a surd?
- 5 Why is π an example of an **irrational number** that is not a **surd**?
- 6 How do you **rationalise the denominator** of a surd expression?

Topic overview

Copy and complete this mind map of the topic, adding detail to its branches and using pictures, symbols and colour where needed. Ask your teacher to check your work.



- 1 Which one of the following is a rational number? Select the correct answer **A, B, C** or **D**. See Exercise 1-01
- A** $1\frac{2}{5}$ **B** $\sqrt[3]{9}$ **C** $2\sqrt{5}$ **D** 2π
- 2 Is each number rational (R) or irrational (I)? See Exercise 1-01
- a** $\sqrt{8}$ **b** $\frac{22}{7}$ **c** $0.5\dot{6}$ **d** $3\sqrt{5}$
e $\sqrt{81}$ **f** $\sqrt[3]{125}$ **g** $\sqrt[3]{-8}$ **h** $5 + \sqrt{3}$
- 3 Simplify each surd. See Exercise 1-02
- a** $\sqrt{72}$ **b** $\sqrt{98}$ **c** $\sqrt{275}$ **d** $\sqrt{128}$
e $3\sqrt{150}$ **f** $7\sqrt{28}$ **g** $4\sqrt{288}$ **h** $5\sqrt{45}$
i $7\sqrt{48}$ **j** $\frac{1}{2}\sqrt{24}$ **k** $\frac{2}{3}\sqrt{3125}$ **l** $\frac{1}{2}\sqrt{32}$
m $2\sqrt{44}$ **n** $\frac{2}{5}\sqrt{225}$ **o** $\frac{2}{3}\sqrt{162}$
- 4 Simplify each expression. See Exercise 1-03
- a** $\sqrt{200} + \sqrt{18}$ **b** $3\sqrt{5} + \sqrt{50} - 2\sqrt{125}$
c $7\sqrt{32} - \sqrt{27} - 2\sqrt{98} + 4\sqrt{75}$ **d** $4\sqrt{45} - 3\sqrt{63} + 5\sqrt{80}$
e $\sqrt{800} - 2\sqrt{243} + 3\sqrt{72} - 2\sqrt{27}$ **f** $7\sqrt{44} - 2\sqrt{99}$
- 5 Simplify each expression. See Exercise 1-04
- a** $\sqrt{3} \times \sqrt{7}$ **b** $\sqrt{8} \times \sqrt{5}$ **c** $\sqrt{6} \times \sqrt{8}$
d $\sqrt{5} \times \sqrt{11}$ **e** $\sqrt{72} \div \sqrt{12}$ **f** $\sqrt{98} \times \sqrt{7}$
g $8\sqrt{42} \div 2\sqrt{7}$ **h** $\sqrt{125} \div 5\sqrt{5}$ **i** $\frac{\sqrt{75}}{3\sqrt{3}}$
j $\frac{\sqrt{18} \times \sqrt{3}}{\sqrt{12}}$ **k** $\frac{\sqrt{6} \times \sqrt{24}}{\sqrt{27} \times 2\sqrt{3}}$ **l** $\frac{4\sqrt{90} \times 7\sqrt{8}}{5\sqrt{32} \times 6\sqrt{10}}$
- 6 Expand and simplify each expression. See Exercise 1-05
- a** $-3\sqrt{2}(2\sqrt{2} - 3)$ **b** $\sqrt{10}(1 - 5\sqrt{2})$
c $(3\sqrt{5} - 2\sqrt{7})(3\sqrt{7} + \sqrt{5})$ **d** $(\sqrt{7} - 4)^2$
e $(5\sqrt{3} + \sqrt{2})^2$ **f** $(3\sqrt{7} - 2\sqrt{5})(3\sqrt{7} + 2\sqrt{5})$
g $(3 + 4\sqrt{7})(4\sqrt{7} - 3)$ **h** $(5\sqrt{10} - 3\sqrt{2})(2\sqrt{10} + 5\sqrt{2})$
- 7 Rationalise the denominator of each surd. See Exercise 1-06
- a** $\frac{1}{\sqrt{10}}$ **b** $\frac{3}{\sqrt{2}}$ **c** $\frac{1}{5\sqrt{7}}$
d $\frac{3}{4\sqrt{3}}$ **e** $\frac{5\sqrt{3}}{3\sqrt{2}}$ **f** $\frac{4 + \sqrt{2}}{3\sqrt{2}}$



2

Number and Algebra

Interest and depreciation

The value of an investment increases over time as a result of interest being added to it, whether it be simple or compound interest. On the other side of the coin, the value of assets and items such as cars and office equipment decreases over time due to age and wear-and-tear. Compound interest and depreciation use formulas that involve repeated percentage increase and decrease respectively.



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Chapter outline

	Proficiency strands				
2-01 Earning an income	U	F	PS		C
2-02 Income tax	U	F			C
2-03 Simple interest	U	F	PS		C
2-04 Compound interest	U	F	PS		C
2-05 The compound interest formula	U	F	PS	R	C
2-06 Term payments	U	F	PS		C
2-07 Depreciation	U	F	PS	R	C

Wordbank

allowable deduction A part of a person's yearly income that is not taxed, such as work-related expenses and donations to charities

compound interest Interest calculated on the principal invested as well as on any accumulated interest

depreciation The decrease in the value of items over time due to ageing

instalment or repayment The amount of money paid at regular time periods (weekly, fortnightly, monthly) to pay off a loan

net pay Pay received after deductions from gross pay; 'take-home' pay

per annum (p.a.) Per year

principal The original amount of money invested or borrowed, for the purpose of earning interest

simple interest Interest calculated on the original principal invested only

In this chapter you will:

- solve problems involving simple interest
- connect the compound interest formula to repeated applications of simple interest using appropriate digital technologies 
- calculate weekly, fortnightly, monthly and yearly incomes
- calculate wages, salaries, overtime, commission, piecework and annual leave loading
- use tables to calculate income tax and PAYG tax
- calculate compound interest for two to three years by repeated percentage increase
- calculate compound interest using the compound interest formula $A = P(1 + R)^n$
- solve problems involving term payments
- calculate depreciation using the compound interest formula

SkillCheck

Worksheet

StartUp assignment 1

MAT10NAWK10001

Worksheet

Percentage shortcuts

MAT10NAWK10002

Skillsheet

Mental percentages

MAT10NASS10001

Skillsheet

Percentage calculations

MAT10NASS10002

1 Convert each percentage to a decimal.

- a** 4% **b** 22% **c** 18.3% **d** 4.7%
e $9\frac{10}{2}\%$ **f** 6.75% **g** $15\frac{10}{4}\%$ **h** 20%

2 Find:

- a** 6% of \$1200 **b** 2.5% of \$4650 **c** 12% of \$37 450

3 Increase:

- a** \$7000 by 5% **b** \$3955 by 2% **c** \$8600 by 1.6%

4 How many months are there in:

- a** 3 years? **b** 2 years? **c** 5 years?

5 Copy and complete:

- a** One year = _____ weeks **b** One year = _____ fortnights
c One year = _____ days **d** 48 months = _____ years
e 84 days = _____ weeks **f** 100 months = _____ years _____ months

6 If $P = mvt$, find:

- a** P when $m = 1600$, $v = 0.072$, $t = 10$
b m when $P = 120$, $v = 0.3$, $t = 8$
c v when $P = 18$, $m = 60$, $t = 5$

7 Evaluate, correct to the nearest cent:

- a** $\$5000 \times (1.045)^4$ **b** $\$28\,000 \times (1.03)^6$
c $\$15\,300 \times (1.065)^3$ **d** $\$32\,400 \times (1.072)^{10}$

2-01 Earning an income

NSW

Wages, salaries and overtime

A **wage** is calculated from the number of hours worked and is usually paid weekly. Wage earners can make more income by working extra hours (overtime).

A **salary** is a fixed annual amount, paid weekly, fortnightly or monthly. Salary earners do not earn overtime pay but can receive benefits such as a computer, company car, expense account, shares in the company or paid medical expenses.



Alamy/Kentil Morris

Summary

- 1 year = 12 months
- 1 fortnight = 2 weeks
- 1 year = 52 weeks for wage earners
- 1 year = 52.18 weeks for salary earners

The two most common rates of overtime pay are:

- **time-and-a-half** = $1.5 \times$ normal hourly rate
- **double time** = $2 \times$ normal hourly rate

Example 1

Thomas earns a salary of \$70 400 p.a.
How much does he earn:

p.a. = per annum = per year

- a each week? b each fortnight? c each month?

Solution

a Weekly income = $\$70\,400 \div 52.18$
 $= \$1349.1759 \dots$
 $\approx \$1349.18$

Rounded to the nearest cent.

b Fortnightly income = $2 \times \$1349.18$
 $= \$2698.36$

1 fortnight = 2 weeks

c Monthly income = $\$70\,400 \div 12$
 $= \$5866.6666 \dots$
 $\approx \$5866.67$

1 year = 12 months

Rounded to the nearest cent.

Example 2

Noor earns \$22.65 per hour at normal rates. Last week, she worked 38 hours at normal rates, 6 hours at time-and-a-half and 3 hours at double time. Calculate Noor's total earnings for the week.

Solution

$$\begin{aligned}\text{Normal pay} &= \$22.65 \times 38 \\ &= \$860.70\end{aligned}$$

$$\begin{aligned}\text{Time-and-a-half pay} &= 6 \times \$22.65 \times 1.5 && \text{6 hours} \\ &= \$203.85\end{aligned}$$

$$\begin{aligned}\text{Double time pay} &= 3 \times \$22.65 \times 2 && \text{3 hours} \\ &= \$135.90\end{aligned}$$

$$\begin{aligned}\text{Total earnings} &= \$860.70 + \$203.85 + \$135.90 \\ &= \$1200.45\end{aligned}$$

Commission, piecework and annual leave loading

Commission is earned by salespeople and agents, and is a percentage of the total value of items sold.

Piecework is earned according to the number of items made or tasks completed.

Annual leave loading or **holiday loading** is extra pay given during annual leave (holidays), and is 17.5% of four weeks' normal pay.



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Example 3

Georgia is a real estate agent and is paid a commission of 2.5% on the value of the apartments she sells. She also receives a weekly retainer of \$750. How much will Georgia earn if she sells an apartment for \$152 500?

A **retainer** is a fixed amount paid regardless of how many items are sold.

Solution

$$\begin{aligned}\text{Commission} &= 2.5\% \text{ of } \$152\,500 \\ &= \$3812.50\end{aligned}$$

$$\begin{aligned}\text{Total earnings} &= \text{commission} + \text{retainer} \\ &= \$3812.50 + \$750 \\ &= \$4562.50\end{aligned}$$

∴ Georgia earns \$4562.50.

Example 4

Emad is a jewellery designer. He makes handmade jewellery and is paid at the following rates.

- \$278 per necklace
- \$62 per pair of earrings
- \$95 per bracelet

This month, Emad made 23 necklaces, 7 pairs of earrings and 19 bracelets. How much did he earn?

Solution

$$\begin{aligned}\text{Monthly earnings} &= 23 \times \$278 + 7 \times \$62 + 19 \times \$95 \\ &= \$8633\end{aligned}$$

Example 5

Sanjay's annual salary is \$70 590. For his Christmas holidays, he received 4 weeks' normal pay plus 17.5% annual leave loading for the 4 weeks. Calculate Sanjay's:

- normal weekly pay
- annual leave loading
- total pay for the Christmas holiday.

Solution

$$\begin{aligned}\text{a Weekly pay} &= \$70\,590 \div 52.18 & \text{b Annual leave loading} &= 17.5\% \times \$1352.82 \times 4 \\ &= \$1352.8171 \dots & &= \$946.974 \\ &\approx \$1352.82 & &\approx \$946.97 \\ \text{c Total holiday pay} &= (4 \times \$1352.82) + \$946.97 \\ &= \$6358.25\end{aligned}$$

Exercise 2-01 Earning an income

Express all answers correct to the nearest cent where necessary.

- Find the weekly wage for each person.
 - Mary earns \$21.85 per hour and works for 40 hours.
 - Connor works 8 hours a day, Monday to Friday, and is paid \$23.47 per hour.
 - Yoshe works on Monday and Tuesday from 8:30 a.m. until 4:00 p.m. and Thursday from midday until 9:00 p.m., and earns \$15.30 per hour.
- Greta earns \$19.56 an hour and works for 31 hours each week. Chandler earns \$21.44 per hour for his 27 hours of work. Who earns more per week and by how much?
- Maggie earns a salary of \$180 640 p.a. How much does she earn:
 - each week?
 - each fortnight?
 - each month?
- Rakitu considers two jobs, one locally with an annual salary of \$57 640 p.a. and the other one in the city with a fortnightly pay of \$2320. Calculate the weekly income for each job, determine which one pays more per week, and by how much.

See Example 1

See Example 2

- 5 Anan works 38 hours at normal rates, 7 hours at time-and-a-half and 4 hours at double-time. Calculate Anan's total earnings if he earns \$19.40 per hour at normal rates.
- 6 Jacqui works 8.5 hours per day from Tuesday to Friday. She is paid \$18.78 per hour. She also works on Saturday for 4.5 hours at a special rate of \$21.49 per hour. How much did Jacqui earn for the week?
- 7 Idra works the following hours in a week at the clothing chain *Shop til U Drop*.

Day	Hours worked
Monday	9:00 a.m. – 5:00 p.m.
Tuesday	9:00 a.m. – 4:00 p.m.
Thursday	11:00 a.m. – 7:30 p.m.
Friday	10:00 a.m. – 5:00 p.m.
Saturday	10:30 a.m. – 5:00 p.m.

She is paid at the following rates.

Day	Rate of pay
Monday to Friday	\$19.62 per hour
Saturday Thursday after 4:00 p.m.	\$23.15 per hour

What is Idra's total income for the week?

See Example 3

- 8 Fatimah is paid a commission of 2.5% on the value of the goods she sells. She also receives a weekly retainer of \$875. How much will Fatimah earn if she sells goods to the value of \$41 600 in one week? Select the correct answer **A**, **B**, **C** or **D**.
A \$1915 **B** \$1061.88 **C** \$2187.50 **D** \$1018.13

- 9 Nathan is a real estate agent whose commission is calculated on the value of the properties he sells:
 - 3% paid on first \$300 000
 - 1.5% paid on next \$250 000
 - 0.75% paid on any value thereafter

How much commission did Nathan earn for selling a house for \$625 000?

- 10 Brandon designed an app, *iCalc*, that is selling for \$2.49. If he makes 70% profit on the sale price of each app sold, how much would he make from selling 800 units of this app?

See Example 4

- 11 Matt charges \$60 for each lawn he mows and \$45 for trimming hedges in each yard. In a week, he mows 24 lawns and trims 15 hedges. How much does he earn for the week?
- 12 *Clean 2 Swim* charges \$86 to clean backyard pools. If this business earned \$4644 in the first week of summer, how many pools were cleaned?
- 13 Jade makes homemade eco-friendly soaps, shampoos and cleaning products. A customer purchases 3 homemade soaps, 2 bottles of shampoo and 3 of the cleaning sprays. How much does Jade receive for these purchases?

	Homemade soaps	\$5.60
	Eco-friendly shampoo	\$12.70
	Natural cleaning spray	\$7.25

- 14 Calculate the annual leave loading for each person if it is 17.5% of 4 weeks' pay.
- a Peter earns \$610 per week
 - b Jamilla earns \$2000 per fortnight
 - c Samir earns \$5944 per month
 - d Ellie earns \$46 630 p.a.
- 15 For his annual holidays, Jake received 4 weeks' normal pay plus 17.5% annual leave loading for the 4 weeks. If Jake's annual salary is \$50 725, find his:
- a normal weekly pay
 - b annual leave loading
 - c total pay for the four-week holiday.

See Example 5

Investigation: Workers' entitlements

The Australian Government sets the minimum standards for pay and conditions for all Australian workers. Different industries can have different needs from employees in terms of:

- normal and overtime hours worked, breaks allowed
- allowances
- dress codes, such as uniforms
- working conditions



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- 1 Visit the Fair Work Ombudsman website www.fairwork.gov.au and select **Industries**.
- 2 Select two industries and identify any similarities and differences in the requirements of those industries.
- 3 Write a summary of your findings.
- 4 Give a report in class.

2-02 Income tax

NSW

Not all of a person's income is taxed. If we use some of our income for work-related expenses or donate money to charities, these amounts are called **allowable deductions** (or **tax deductions**) and are not taxed. Examples of allowable deductions are tools of trade, uniforms, car-related expenses, subscriptions to professional organisations and journals.

Summary

Income tax is calculated on a person's **taxable income**, which is the **gross income** (total earnings) less all allowable deductions, **rounded down to the nearest dollar**.

Taxable income = gross income – allowable deductions

The more a person earns, the higher the rate of tax to be paid. The table below shows how income tax is calculated.

Tax rates for Australian residents	
Taxable income	Tax on this income
0 – \$18 200	Nil
\$18 201 – \$37 000	19c for each \$1 over \$18 200
\$37 001 – \$80 000	\$3572 plus 32.5c for each \$1 over \$37 000
\$80 001 – \$180 000	\$17 547 plus 37c for each \$1 over \$80 000
\$180 001 and over	\$54 547 plus 45c for each \$1 over \$180 000

Source: © Australian Taxation Office for the Commonwealth of Australia

Example 6

Sophia earned \$62 348 last financial year and collected bank interest of \$440.81. She had allowable deductions of \$427.52 in work expenses and \$110 in donations to charities.

- Calculate her taxable income.
- Use the tax table to calculate the income tax that Sophie must pay.

Solution

- $$\begin{aligned}\text{Taxable income} &= \$62\,348 + \$440.81 - \$427.52 - \$110 \\ &= \$62\,251.29 \\ &\approx \$62\,251 \quad \text{Rounded down to the nearest dollar.}\end{aligned}$$
- According to the table, a taxable income of \$62 251 is in the \$37 001 – \$80 000 tax bracket.

$$\begin{aligned}\text{Income tax} &= \$3572 + 0.325 \times (\$62\,251 - \$37\,000) \\ &= \$11\,778.575 \\ &\approx \$11\,778.58\end{aligned}$$

← '32.5c for each \$1' means 32.5% or 0.325

PAYG tax and net pay

Income tax deducted from your pay by your employer every payday is called **PAYG (Pay As You Go) tax**. The total amount of PAYG tax paid over the year is usually more than the actual income tax payable, so at the end of the financial year you will receive the difference as a **tax refund**.

Gross pay is the total amount a person earns or receives, but most workers have a variety of deductions taken from their pay before they receive it, including PAYG tax, superannuation contributions, union fees and health fund payments. The amount of income left after the deductions is called **net pay**.

Summary

Net pay = gross pay – tax – other deductions

Example 7

Jay earns a gross pay of \$2290.33 per fortnight. His deductions are for PAYG tax, \$44.10 for private health insurance and \$55.82 for superannuation.

- Use this PAYG tax table to find Jay's PAYG tax per fortnight.
- Calculate Jay's net pay.
- Calculate Jay's total deductions as a percentage of his gross income (correct to one decimal place).

Fortnightly earnings (\$)	PAYG tax withheld (\$)
2274–2279	450
2280–2285	452
2286–2291	454
2292–2297	456
2298–2303	458
2304–2309	460

Solution

- In the table, \$2290.33 falls in the \$2286 – \$2291 range.

Fortnightly PAYG tax = \$454

- Net pay = $\$2290.33 - (\$454 + \$44.10 + \$55.82)$
 $= \$2290.33 - \$553.92$
 $= \$1736.41$

Net pay = gross pay – total deductions

- Total deductions = \$553.92

$$\begin{aligned}\text{Deductions percentage} &= \frac{\$553.92}{\$2290.33} \times 100\% \\ &= 24.1851 \dots \% \\ &\approx 24.2\%\end{aligned}$$

$$\frac{\text{Total deductions}}{\text{Gross pay}} \times 100$$

Exercise 2-02 Income tax

- Mrs Shepherd earns \$47 628 in a year and has allowable deductions of \$1930.46.
 - Calculate her taxable income, rounded down to the nearest dollar.
 - Use the tax table on the previous page to calculate the income tax that Mrs Shepherd must pay.
- Adam is an environmental engineer who had a gross income of \$118 742 this year and work-related expenses totalling \$4022.80, which are tax-deductible. Calculate:
 - Adam's taxable income, rounded down to the nearest dollar
 - Adam's payable income tax.
- Erin is a graphic designer who earns an annual salary of \$90 541 and has collected \$1029.45 in bank interest. She has allowable deductions of \$379 for tools related to her work and \$287 in donations to charity. Calculate:
 - Erin's taxable income
 - the amount of tax payable.
- Riley the builder had a gross income of \$56 922 this year. He is entitled to these tax deductions: tools \$1538, training courses \$445 and outdoor protective clothing \$506. How much should Riley pay in tax? Select the correct answer **A**, **B**, **C** or **D**.

A \$13 046.65 **B** \$10 855.58 **C** \$9237.73 **D** \$6884.27

See Example 6

- 5 Michaela is a nurse earning \$87 996 per year. Her allowable deductions are the cost of non-slip footwear –\$225, the cost of laundering uniforms –\$1046, and union fees –\$297.60. How much should Michaela pay in tax?
- 6 William owns a photography business and earned \$196 000 last year. His allowable deductions were Internet costs for his website –\$968, photographic equipment –\$23 672, and travel to photo locations –\$15 930. Calculate the amount that William should pay in tax.
- 7 Jackson earns a gross weekly income of \$1075.26. His weekly deductions are \$309.11 PAYG tax, \$44.55 for private health insurance and \$25.18 for superannuation. Calculate Jackson's net weekly pay.
- 8 Isha earns a gross income of \$788.20 per week. Her deductions are \$132.44 tax and \$32.24 for private health insurance. Calculate Isha's net income.

Use the PAYG table from Example 7 on the previous page to answer questions 9 to 12.

See Example 7

- 9 Every fortnight, Mr Singh earns \$2280 and pays \$22.80 in union fees and \$94.10 in superannuation.
 - a Find how much PAYG tax he pays per fortnight.
 - b Calculate Mr Singh's fortnightly net pay.
 - c What percentage (correct to one decimal place) of his gross pay do the deductions make up?
- 10 Holly earns a gross pay of \$2300 per fortnight. Her deductions are PAYG tax, \$64.35 for superannuation and \$30 for life insurance. Find Holly's:
 - a PAYG tax
 - b net pay
 - c total deductions as a percentage of her gross income (correct to one decimal place).
- 11 Stefan earns \$1148 per week.
 - a If he is paid fortnightly, what is his fortnightly gross pay?
 - b Find the PAYG tax that is taken out of his gross pay.
 - c Stefan's deductions are \$141.94 for his health fund and \$51.33 for superannuation. Calculate Stefan's net pay.
- 12 Anne earns a salary of \$59 944 p.a. Each fortnight she has deductions of \$256.20 for family health insurance and \$35 for superannuation taken from her gross income.
 - a Calculate Anne's fortnightly gross income.
 - b How much PAYG tax does she pay per fortnight?
 - c Calculate Anne's fortnightly net income.
- 13 Copy and complete this pay slip.

Employee: Ziad Chaker		Hourly pay rate: \$19.45	
Hours worked		Deductions	
Normal	39	Tax: \$200.72	Other: \$168.38
Time-and-a-half	2	Gross weekly income	
Double time	0	Total deductions	
		Net weekly income	

Technology Online income tax calculators

The Australian Taxation Office (ATO) website www.ato.gov.au has online calculators for income tax and PAYG tax. Visit the website and search 'Simple Tax Calculator' to find the income tax calculator for individuals.

- 1 Enter the taxable income \$63 000 as '63000' (no spaces).
- 2 Select the current financial year.
- 3 Select 'Resident for full year' and click 'Next'.
- 4 The estimated tax payable will be shown on a new screen.
- 5 Repeat for at least two more taxable incomes.
- 6 Find the PAYG tax calculator and use it to find the PAYG tax payable and net pay for each of the following gross pays.
 - a \$1408 weekly b \$2870 fortnightly c \$5610 monthly

2-03 Simple interest

- When you invest money, you **receive interest** from your investment.
 - When you borrow money, you **pay interest** on your loan.
 - The original amount of money invested or borrowed is called the **principal**.
 - This interest rate is a percentage of the principal, usually written as a rate **per annum** ('per year'), abbreviated 'p.a.'
 - **Simple interest** (or **flat rate interest**) is interest calculated simply on the original principal.
- Simple interest is calculated using the following formula.

Summary

$I = PRN$, where:

I is the simple interest,

P is the principal,

R is the interest rate per time period, expressed as a fraction or decimal, and

N is the number of time periods

Example 8

Find the simple interest on:

- a \$4000 at 3.5% p.a. for 6 years
- b \$13 500 at 5.5% p.a. for 7 months
- c \$75 640 at 18.3% p.a. for 210 days.

Solution

a $P = \$4000$, $R = 3.5\% = 0.035$, $N = 6$ years

$$I = PRN$$

$$= \$4000 \times 0.035 \times 6$$

$$= \$840$$

$$\text{b } P = \$13\,500, R = 5.5\% = 0.055, N = \frac{7}{12} \text{ years}$$

$$I = PRN$$

$$= \$13\,500 \times 0.055 \times \frac{7}{12}$$

$$= \$433.125$$

$$\approx \$433.13$$

Rounded up to the nearest cent.

$$\text{c } P = \$75\,640, R = 18.3\% = 0.183, N = \frac{210}{365} \text{ years}$$

$$I = PRN$$

$$= \$75\,640 \times 0.183 \times \frac{210}{365}$$

$$= \$7963.9594 \dots$$

$$\approx \$7963.96$$

Rounded up to the nearest cent.

Example 9

Petra invests \$17 400 for 2 years at 3.75% p.a. flat rate interest. To what final value will her investment grow?

Solution

$$P = \$17\,400, R = 3.75\% = 0.0375, N = 2 \text{ years}$$

$$I = PRN$$

$$= \$17\,400 \times 0.0375 \times 2$$

$$= \$1305$$

$$\begin{aligned} \text{Value of investment} &= \$17\,400 + \$1305 \\ &= \$18\,705 \end{aligned}$$

Principal + interest

Example 10

After 4 years, an investment of \$13 000 has earned \$2808 in simple interest. What is the annual interest rate?

Solution

$$I = \$2808, P = \$13\,000, N = 4 \text{ years}$$

$$I = PRN$$

$$\$2808 = \$13\,000 \times R \times 4$$

$$\$2808 = \$52\,000R$$

$$R = \frac{\$2808}{\$52\,000}$$

$$= 0.054$$

$$= 5.4\%$$

$$\therefore \text{Annual interest rate} = 5.4\%$$

Example 11

For how many months will \$10 000 need to be invested to earn \$250 in simple interest at 3.25% p.a.?

Solution

$$I = \$250, P = \$10\,000, R = 3.25\% = 0.0325$$

$$I = PRN$$

$$\$250 = \$10\,000 \times 0.0325 \times N$$

$$\$250 = \$325N$$

$$N = \frac{\$250}{\$325}$$

$$= 0.7692 \dots \text{ years}$$

$$= 0.7692 \dots \times 12 \text{ months}$$

$$= 9.230 \dots \text{ months}$$

$$\approx 10 \text{ months}$$

Rounded up to the nearest month.

Exercise 2-03 Simple interest

In this exercise, give all money answers correct to the nearest cent.

- 1 Calculate the simple interest earned on each investment.

a \$35 000 for 4 years at 3.6% p.a.	b \$26 850 at 5.5% p.a for 2 years
c \$8200 invested for 5 months at 3% p.a.	d \$6590 invested for 17 weeks at 4.5% p.a.
e \$5250 invested for 250 days at 2% p.a.	f \$18 400 invested for 153 days at 5.85% p.a.

- 2 Calculate the flat-rate interest charged on each loan.

a \$1250 for 2 years at 3.5% p.a.	b \$18 900 for $5\frac{1}{2}$ years at 5.7% p.a.
c \$1.15 million at 8.5% p.a. for 4 years.	d \$12 000 for 10 months at 8.2% p.a.
e \$9750 for 16 weeks at 9.3% p.a.	f \$24 720 for 136 days at 7.85% p.a.

- 3 Harry owed \$783.26 on his credit card. The credit card company charged him one month's simple interest at 18% p.a. How much interest was he charged? Select the correct answer **A**, **B**, **C** or **D**.

A \$11.75	B \$14.10	C \$27.39	D \$43.51

- 4 Find the final value of each investment using simple interest.

a \$10 000 invested for 3 years at 4% p.a.	b \$1500 invested for 2 years at 2.3% p.a.
c \$8500 invested for 3.5 years at 3% p.a.	d \$9250 invested for 6 years at 3.75% p.a.

- 5 Liong borrowed \$6000 to go on an overseas holiday, at 12% p.a. flat-rate interest for 2 years. Calculate:

a the total interest	b the total amount Liong must repay.

- 6 The interest on a loan of \$2500 over 4 years is \$450. Calculate the flat rate of interest p.a.

See Example 8

See Example 9

See Example 10

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Solution

- a The interest for each year is calculated separately.

After the first year:

$$I = \$23\,000 \times 0.04 = \$920$$

$$\text{Investment} = \$23\,000 + \$920$$

$$= \$23\,920$$

Principal + interest

After the second year:

$$I = \$23\,920 \times 0.04 = \$956.80$$

$$\text{Investment} = \$23\,920 + \$956.80$$

$$= \$24\,876.80$$

New principal + interest

- b Compound interest earned = final investment – principal
 $= \$24\,876.80 - \$23\,000$
 $= \$1876.80$

Compound interest involves repeated percentage increase

In Example 12, to calculate compound interest on a principal of \$23 000 over two years at 4% p.a., we are actually increasing \$23 000 by 4% twice. Adding 4% to the principal is the same as increasing the principal by 4%, which is the same as multiplying the principal by 104% or 1.04.

$$\text{Investment after first year} = \$23\,000 \times 1.04 = \$23\,920$$

$$\text{Investment after second year} = \$23\,920 \times 1.04 = \$24\,876.80$$

We can even combine these two steps into one step by repeated percentage increases on the original principal of \$23 000:

$$\text{Investment after second year} = \$23\,000 \times 1.04 \times 1.04 = \$24\,876.80$$

Using repeated percentage increases can simplify our compound interest calculations.

Example 13

A principal of \$9000 is invested at 3.7% p.a. compounded yearly over 3 years. What is:

- a the value of the investment after 3 years?
 b the compound interest earned?

Solution

- a Adding 3.7% interest to the principal is the same as multiplying the principal by 1.037.

$$\therefore \text{Investment after 3 years} = \$9000 \times 1.037 \times 1.037 \times 1.037$$

$$= \$9000 \times (1.037)^3$$

$$= \$10\,036.4188 \dots$$

$$\approx \$10\,036.42$$

Rounded to the nearest cent.

- b Compound interest earned = final investment – original principal
 $= \$10\,036.42 - \$9000$
 $= \$1036.42$

Exercise 2-04

In this exercise, give all money answers correct to the nearest cent.

See Example 12

- 1 A principal of \$23 000 is invested at 5% p.a. interest, compounded yearly over 2 years.
- a Copy and complete the following working to calculate the value of the investment after 2 years.

After the first year:

$$I = \$23\,000 \times 0.05 = \$\_\_\_$$

Investment = \$23 000 + \$_____
= \$_____

After the second year:

$$I = \$\_\_\_\_\_ \times 0.05 = \$\_\_\_\_\_$$

Investment = \$_____ + \$_____
= \$_____

- b** Copy and complete the following working to calculate the compound interest earned.

Compound interest earned = final investment – principal
 = \$_____ – \$23 000
 = \$_____

- 2 Finn invests \$15 000 at 2.5% p.a. compounded yearly over 3 years. Show all working (as in question 1) to find:

- a** the value of the investment after 3 years
b the total amount of compound interest earned.

- 3 Selina invests \$34 100 at 6.2% p.a. interest compounded yearly over 2 years. Calculate:

- a** the final value of the investment **b** the compound interest earned.

See Example 13

- 4 Use repeated percentage increases to calculate the final value of each investment compounded annually, then calculate the compound interest earned.

- a** \$5000 for 2 years at 4% p.a.
b \$27 800 for 3 years at 2.85% p.a.
c \$9600 for 3 years at 5% p.a.
d \$39 500 for 2 years at 3% p.a.
e \$18 400 for 4 years at 1.25% p.a.

- 5** For each investment, calculate the compound interest earned.

- a \$30 400 at 5% p.a. interest for 3 years. b \$19 150 at 4.2% p.a. interest for 2 years.
c \$8750 at 1.75% p.a. interest for 2 years. d \$36 000 at 3.5% p.a. interest for 3 years.
e \$18 960 at 6.35% p.a. interest for 5 years.

Mental skills 2

Maths without calculators

Finding 15%, $2\frac{1}{2}\%$, 25% and $12\frac{1}{2}\%$

- To find 10% or $\frac{1}{10}$ of a number, divide by 10
- To find 5% of a number, find 10% first, then halve it (since 5% is half of 10%)
- So to find 15% of a number, find 10% and 5% of the number separately, then add the answers together.

1 Study each example.

a $15\% \times 80 = (10\% \times 80) + (5\% \times 80) = 8 + 4 = 12$

b $15\% \times \$170 = (10\% \times \$170) + (5\% \times \$170) = \$17 + \$8.50 = \$25.50$

c $15\% \times 3600 = (10\% \times 3600) + (5\% \times 3600) = 360 + 180 = 540$

d $15\% \times \$28 = (10\% \times \$28) + (5\% \times \$28) = \$2.80 + \$1.40 = \$4.20$

2 Now find 15% of each amount.

a 120 **b** \$840 **c** 260 **d** \$202 **e** \$50 **f** 72

g \$180 **h** 400 **i** \$1600 **j** \$22 **k** 6000 **l** \$350

To find $2\frac{1}{2}\%$ of a number, first find 5%, then halve it.

3 Study each example.

a $2\frac{1}{2}\% \times 600$

$10\% \times 600 = 60$

$5\% \times 600 = \frac{1}{2} \times 60 = 30$

$2\frac{1}{2}\% \times 600 = \frac{1}{2} \times 30 = 15$

b $2\frac{1}{2}\% \times \$820$

$10\% \times \$820 = \$82$

$5\% \times \$820 = \frac{1}{2} \times 82 = \$41$

$2\frac{1}{2}\% \times \$820 = \frac{1}{2} \times \$41 = \$20.50$

4 Now find $2\frac{1}{2}\%$ of each amount.

a 400 **b** 6640 **c** \$2000 **d** \$880

e 1500 **f** \$232 **g** 5400 **h** \$904

To find 25% of a number, halve the number twice as $25\% = \frac{1}{4}$.

5 Study each example.

a $25\% \times 700$

$50\% \times 700 = \frac{1}{2} \times 700 = 350$

$25\% \times 700 = \frac{1}{2} \times 350 = 175$

b $25\% \times \$86$

$50\% \times \$86 = \frac{1}{2} \times \$86 = \$43$

$\therefore 25\% \times \$86 = \frac{1}{2} \times \$43 = \$21.50$

6 Now find 25% of each amount.

a 2000 **b** \$80 **c** 18 **d** \$25 **e** \$324 **f** \$140

g 66 **h** 298 **i** \$780 **j** \$1700 **k** \$126 **l** 1160

To find $12\frac{1}{2}\%$ of a number, find 25% first, then halve it. In other words, halve three times because $12\frac{1}{2}\% = \frac{1}{8}$.

7 Study each example.

a $12\frac{1}{2}\% \times 400$

$50\% \times 400 = \frac{1}{2} \times 400 = 200$

$25\% \times 400 = \frac{1}{2} \times 200 = 100$

$12\frac{1}{2}\% \times 400 = \frac{1}{2}\% \times 100 = 50$

b $12\frac{1}{2}\% \times \$144$

$50\% \times \$144 = \frac{1}{2} \times \$144 = \$72$

$25\% \times \$144 = \frac{1}{2} \times \$72 = \$36$

$12\frac{1}{2}\% \times \$144 = \frac{1}{2}\% \times \$36 = \$18$

8 Now find $12\frac{1}{2}\%$ of each amount.

a 1280 **b** \$12 **c** 60 **d** \$260

e \$540 **f** \$250 **g** 304 **h** 1360

Worksheet

Compound interest

MAT10NAWK10004

Technology worksheet

Excel worksheet:
Comparing interest
rates

MAT10NACT00014

Technology worksheet

Excel spreadsheet:
Interesting facts

MAT10NACT00044

Technology worksheet

Excel worksheet:
Simple and compound
interest calculator

MAT10NACT00015

Technology worksheet

Excel spreadsheet:
Simple and compound
interest

MAT10NACT00045

Puzzle sheet

Compound interest
with annual rests

MAT10NAPS00028

Puzzle sheet

Compound interest
with non-annual rests

MAT10NAPS00029

2-05 The compound interest formula

There is a formula for calculating the final amount of an investment earning compound interest. Note the following pattern.

- Final amount of \$23 000 at 4% p.a. interest for 2 years = $\$23\,000 \times (1.04)^2$
- Final amount of \$9000 at 3.7% p.a. interest for 3 years = $\$9000 \times (1.037)^3$
- Final amount of \$18 960 at 6.35% p.a. interest for 5 years = $\$18\,960 \times (1.0635)^5$

Summary

Compound interest

$A = P(1 + R)^n$, where:

A is the total (final) amount of the investment

P is the principal

R is the interest rate per compounding period, expressed as a decimal

n is the number of compounding periods

The compound interest is then calculated using this formula:

Compound interest = total amount – principal

$$I = A - P$$

Example 14

For each of the following investments, calculate:

- the total amount of the investment
- the compound interest earned.
 - \$26 750 is invested at 4% p.a. for 3 years with interest compounded annually.
 - \$52 000 is invested at 3.8% p.a. for 5 years with interest compounded annually.

Solution

- a i** $P = \$26\,750$, $R = 4\% = 0.04$, $n = 3$

$$\begin{aligned} A &= P(1 + R)^n \\ &= \$26\,750(1 + 0.04)^3 \\ &= \$26\,750(1.04)^3 \\ &= \$30\,090.112\dots \\ &\approx \$30\,090.11 \end{aligned}$$

The total amount of the investment is \$30 090.11.

- ii** Compound interest = $\$30\,090.11 - \$26\,750$
= \$3340.11

$$I = A - P$$

b i $P = \$52\,000$, $R = 3.8\% = 0.038$, $n = 5$

$$A = P(1 + R)^n$$

$$= \$52\,000(1 + 0.038)^5$$

$$= \$52\,000(1.038)^5$$

$$= \$62\,659.9597 \dots$$

$$\approx \$62\,659.96$$

ii Compound interest = $\$62\,659.96 - \$52\,000$
= $\$10\,659.96$

Example 15

Calculate the compound interest when \$24 500 is invested at 6.3% p.a. for 5 years

a compounded annually

b compounded monthly.

Solution

a $P = \$24\,500$, $R = 0.063$, $n = 5$

$$A = \$24\,500(1 + 0.063)^5$$

$$= \$24\,500(1.063)^5$$

$$= \$33\,253.1205 \dots$$

$$\approx \$33\,253.12$$

$$I = \$33\,253.12 - \$24\,500$$

$$= \$8753.12$$

b Because interest is compounded monthly, R and n must be expressed in months, not years.

$$P = \$24\,500, R = \frac{0.063}{12} = 0.00525 \text{ per month}, n = 5 \times 12 = 60 \text{ months}$$

$$A = \$24\,500(1 + 0.00525)^{60}$$

$$= \$24\,500(1.00525)^{60}$$

$$= \$33\,543.70198 \dots$$

$$\approx \$33\,543.70$$

$$I = \$33\,543.70 - \$24\,500$$

$$= \$9043.70$$

Note: More interest is earned when it is compounded monthly rather than yearly. Why do you think this is so?

Video tutorial

Compound interest

MAT10NAVT10021

Exercise 2-05 The compound interest formula

In this exercise, give all money answers correct to the nearest cent.

See Example 14

- 1 An amount of \$13 000 is invested at 5% p.a. interest, compounded over 2 years. What is the total value of the investment?
- 2 For each of the following investments where interest is compounded yearly, calculate:
 - i the total amount of the investment
 - ii the compound interest earned.
 - a \$6500 invested at 7% p.a. for 6 years
 - b \$10 000 invested at 8.5% p.a. for 4 years
 - c \$12 240 invested at 1.6% p.a. for 2 years
 - d \$34 600 invested at 4.9% p.a. for 5 years
 - e \$8000 invested at 1.75% p.a. for 3 years
- 3 Calculate the amount of interest earned on an investment of \$6500 if it is invested at 2.5% p.a. compounded annually for 8 years. Select the correct answer **A, B, C** or **D**.
A \$131.14 **B** \$832.81 **C** \$1300 **D** \$1419.62
- 4 Find the amount of interest earned on one million dollars invested at 14.9% p.a. for 6 years.
- 5 Find the amount of interest charged on a loan of \$25 000 if it is borrowed over 10 years at 8% p.a. compounded annually. Select the correct answer **A, B, C** or **D**.
A \$31 250 **B** \$28 973.12 **C** \$28 589.72 **D** \$20 000

See Example 15

- 6 For each investment, calculate:
 - i the total amount
 - ii the interest earned.
 - a \$10 000 for 5 years at 6% p.a., compounded monthly
 - b \$35 500 for 10 years at 4% p.a., compounded twice a year
 - c \$8900 for 2 years at 3% p.a., compounded quarterly
 - d \$42 000 for 5 years at 4.8% p.a., compounded monthly
 - e \$16 500 for 3 years at 5.6% p.a., compounded every 6 months
 - f \$4900 for 1 year at 5.5% p.a., compounded daily
- 7 Find the total value of an investment of \$4300 over 5 years at 4.6% p.a. interest, compounded every 6 months. Select the correct answer **A, B, C** or **D**.
A \$4817.78 **B** \$5384.27 **C** \$5397.90 **D** \$8506.24
- 8
 - a Reese invested \$6000 for 2 years at a flat rate of 5% p.a. Calculate the interest earned.
 - b Tegan invested \$6000 for 2 years at an interest rate of 5% p.a. compounded annually. Calculate the amount of interest earned.
 - c Whose investment earned more interest? How much more?
- 9 Lisa is setting up a trust account for her new grandson Stefan. In 18 years' time, she wants the investment to be worth \$30 000, to help with the cost of university fees or the purchase of a car. Suppose the interest rate for the account is 6% p.a. compounding yearly.
 - a How much should Lisa invest now to achieve the \$30 000 target?
 - b If Lisa opened a trust account that earns 7% p.a. compounding monthly instead, how much less would she need to invest?
- 10 Zoe is 5 years old and about to start school. Her parents want to invest \$15 000, for her high school education expenses, in an account that earns 6% p.a. over 7 years.
 - a Calculate the total interest earned if interest is compounded:
 - i yearly
 - ii every six months
 - iii quarterly
 - iv monthly
 - b Which compounding period should Zoe's parents choose? Why?

Quarterly means '4 times per year' or 'every three months'.

Technology

Comparing simple with compound interest

Skillsheet

Spreadsheets

MAT10NASS10003

Technology

GeoGebra: Compound and simple interest investments

MAT10NATC00009

In this activity you will compare the interest earned on an investment of \$1000 for 10 years at 8% p.a. simple interest and 8% p.a. compound interest, compounded annually.

- 1 Create this spreadsheet. The principal (P) is entered in cell A1 and the annual interest rates (in decimal form) in cells B1 and B2.

	A	B	C
1	\$ 1,000.00	0.08	0.08
2			
3	Years	Simple Interest	Compound Interest
4	1		
5	2		
6	3		
7	4		
8	5		
9	6		
10	7		
11	8		
12	9		
13	10		
14			

- 2 To calculate the simple interest in column B, in cell B4 enter the formula $=A\$1*\$B\$1*A4$. Now **Fill Down** from cell B4 to B13.
- 3 To calculate the compound interest in column C, in cell C4 enter the formula $=A\$1*(1+\$C\$1)^A4-A\$1$. Now **Fill Down** from cell C4 to C13.
- 4 Highlight cells A3 to C13. **Insert 'Scatter with Smooth lines and markers'**.
- 5 When the interest rate is the same, which account pays better interest – simple or compound interest? (Type your answer in cell A15)
- 6 Now compare the interest earned on an investment of \$1000 for 10 years at 9% p.a. simple interest and 7% p.a. compound interest, compounded annually. Change the interest rates in cells B1 (0.09) and B2 (0.07) respectively.

Answer the following questions in the spreadsheet cells indicated in brackets.

- 7 After how many years did the compound interest rate pay more than the simple interest rate? (A16)
- 8 How much extra interest did the compound interest rate pay at the end of the 10 years? (A17)
- 9 Change the interest rate in B1 to 10% (0.1) and B2 to 9% (0.09). How does the change in interest rate affect the amount of interest paid? Include calculations to justify your answer. (A18)
- 10 Change the interest rate in B1 to 12% (0.12) and B2 to 8.5% (0.085). After how many years did the amount of compound interest earned overtake the amount of simple interest earned? (A19)
- 11 What is the difference in the amount of compound interest earned for the 10-year period compared to the simple interest investment? Is it a significant amount? Justify your answer. (A20)

NSW

2-06 Term payments

Worksheet

Term payments:
Spreadsheet

MAT10NAWK10005

Many customers buy expensive household items **on terms**, which means ‘paying off’ the item by regular instalments over time, after paying a deposit. A **term payments** plan is also called **hire purchase** because the purchaser actually hires the item until it is completely paid off. Special offers can include interest-free periods, but there may be other conditions such as establishment fees and extra charges if the regular repayments are not paid on time. Also, if the purchaser fails to keep up with the payments, higher interest may be charged or the item may be **repossessed** (taken back).

Example 16

Sonia purchases a new fridge and dishwasher package valued at \$4925. She pays a 10% deposit and repays the balance in monthly instalments over 3 years. Interest on the balance is charged at a flat rate of 12% p.a. Find:

- the deposit paid
- the balance owing
- the interest charged on the balance
- the total to be repaid
- the amount of each instalment
- the total price paid for the package.



Stockphoto/gabryalbert

Solution

$$\begin{aligned}\text{a Deposit} &= 10\% \times \$4925 \\ &= \$492.50\end{aligned}$$

$$\begin{aligned}\text{b Balance owing} &= \$4925 - \$492.50 && \text{or } 90\% \times \$4925 \\ &= \$4432.50\end{aligned}$$

- c Interest charged on the balance is flat or simple interest.

$$\begin{aligned}P &= \$4432.50 \times R = 0.12, N = 3 \\ I &= PRN\end{aligned}$$

$$\begin{aligned}&= \$4432.50 \times 0.12 \times 3 \\ &= \$1595.70\end{aligned}$$

$$\begin{aligned}\text{d Total to be repaid} &= \text{balance} + \text{interest} \\ &= \$4432.50 + \$1595.70 \\ &= \$6028.20\end{aligned}$$

$$\begin{aligned}\text{e Monthly instalment} &= \$6028.20 \div 36 && 3 \text{ years} = 3 \times 12 \text{ months} \\ &= \$167.45 && = 36 \text{ months}\end{aligned}$$

$$\begin{aligned}\text{f Total price paid} &= \text{deposit} + \text{total repayments} && \text{or cash price} + \text{interest} \\ &= \$492.50 + \$6028.20 && \text{or } \$4925 + \$1595.70 \\ &= \$6520.70\end{aligned}$$

Deferred payment plan

With a **deferred payment plan**, the customer does not make any repayments until a later date, such as after three years.

Deferred means 'delayed'.

Example 17

Ilhea and Robert purchase a \$1600 home theatre system on a deferred payment plan over 2 years. They make no repayments for the first 3 months, then pay \$105 per month.

- a Calculate the total repayments.
- b How much interest did they pay?
- c Ilhea and Robert were also charged the following fees for the plan.
 - One-off establishment fee of \$25
 - Account service fee of \$4.95 per month
- i How much was paid in fees over the 2-year period?
- ii What percentage (correct to one decimal place) of the purchase price was paid in fees?

Solution

- a Total cost = $\$105 \times 21$ $2 \text{ years} - 3 \text{ months} = 21 \text{ months}$
 $= \$2205$
- b Interest = total cost – cash price
 $= \$2205 - \$1600$
 $= \$605$
- c i Total fees paid = $\$25 + \$4.95 \times 24$
 $= \$143.80$
- ii Percentage = $\frac{\$143.80}{\$1600} \times 100\%$
 $= 8.9875\%$
 $\approx 9.0\%$



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Exercise 2-06 Term payments

In this exercise, give all money answers correct to the nearest cent.

See Example 16

- 1 Gaspard purchases a laundry package of washing machine and dryer valued at \$1755. He pays a 10% deposit and repays the balance in 18 months. Interest on the balance is charged at a flat rate of 13.85% p.a. Find:
 - a the deposit Gaspard paid
 - b the balance owing
 - c the interest charged on the balance
 - d the total to be repaid
 - e the amount of each monthly instalment
 - f the total price paid for the package
- 2 Maree bought a backyard swimming pool for \$25 500. She paid a 5% deposit and will repay the balance over 5 years, with flat-rate interest charged 8.6% p.a. Calculate:
 - a the deposit
 - b the balance owing
 - c the interest charged
 - d the total to be repaid
 - e the monthly instalment
 - f the total price paid.
- 3 Sarah wants to buy a new 3-piece lounge suite that she saw advertised at the lounge sale pictured, costing \$6895.
 - a How much deposit will she have to pay?
 - b What will be the flat-rate interest on the balance?
 - c Find the size of each monthly repayment.
- 4 Peter pays a \$1200 deposit on his holiday and then \$185 per month for a year. The cash price of the holiday is \$2700.
 - a How much did Peter pay for his holiday?
 - b Calculate the interest paid.
 - c Calculate the balance owing after the deposit was paid.
 - d Calculate, as a percentage correct to one decimal place, the flat interest rate charged p.a.
- 5 Mrs Allan buys a used car for her daughter by paying a 20% deposit and \$275 per month for 4 years. If the car has a cash price of \$10 400, find:
 - a the deposit Mrs Allan paid
 - b the balance owing after the deposit is paid
 - c the total amount Mrs Allan repaid in instalments
 - d the interest charged
 - e the flat interest rate (p.a.), correct to one decimal place.
- 6 Ming wanted to buy a new smartphone costing \$329.
 - a How much deposit did she pay?
 - b How much did she pay altogether for the smartphone?
 - c What was the total interest paid for the year?
 - d What was the annual simple interest rate for this purchase? Answer correct to 2 decimal places.



- 7 Derek bought a laptop computer priced at \$800 on a deferred payment plan: nothing to pay for 3 months and then 9 monthly payments of \$110. However, a monthly account fee of \$6.95 was added to the plan.
- Find the total cost of the laptop computer to Derek.
 - How much in excess of the cash price was paid?
 - What is the flat rate of interest p.a. (correct to one decimal place) charged?
- 8 Tahlia bought a new outdoor setting and BBQ for her backyard that retails for \$2899. She paid \$300 deposit, no payments for 6 months and then fortnightly payments of \$63 for 2 years.
- Find the balance owing after Tahlia had paid the deposit.
 - Calculate the total cost of the outdoor setting and BBQ.
 - How much in excess of the cash price did Tahlia pay under this plan?
 - What is the annual flat rate of interest charged, correct to one decimal place?
- 9 Sophie bought a home cinema system priced at \$2100 on interest-free terms for one year with no repayments for the first 4 months.
- If Sophie makes 8 equal monthly repayments, what is the amount of each payment?
 - There is a service charge of \$12.95 every month for this deferred payment plan. What percentage (correct to one decimal place) of the purchase price was paid in service charges?

See Example 17

2-07 Depreciation

NSW

Worksheet

Depreciation

MAT10NAWK10006

Depreciation is the decrease in value of an item over time. When items we buy lose value because of age or frequency of use, they are said to **depreciate**.

The compound interest formula can be adapted to find the depreciated value of an item. While compound interest involves repeated percentage increases, depreciation involves repeated percentage *decreases*, so its formula has a minus sign.

Summary

Depreciation formula

$A = P(1 - R)^n$, where:

A is the final value of the item

P is the original value of the item

R is the rate of depreciation per period, expressed as a decimal

n is the number of periods of depreciation

The amount of depreciation is then calculated using this formula:

$$\text{Depreciation} = \text{original value} - \text{final value}$$

Example 18

An accountant's computer and printer depreciates by 15% each year.

- a** If the computer and printer are currently valued at \$2600, what will its value be in 5 years?
b What is the depreciation over this time?

Solution

a $P = \$2600, R = 15\% = 0.15, n = 5$

$$\begin{aligned} A &= P(1 - R)^n \\ &= \$2600(1 - 0.15)^5 \\ &= \$2600(0.85)^5 \\ &= \$1153.633 \dots \\ &\approx \$1153.63 \end{aligned}$$

The value of the items after 5 years is \$1153.63.

b Depreciation = $\$2600 - \$1153.63$ original value – final value
 $= \$1446.37$

Example 19

An industrial oven in a restaurant originally costs \$19 800, then depreciates at a rate of 12% p.a.

- a** Find the value of the oven after 6 years, correct to the nearest dollar.
b Express the depreciated value as a percentage of the cost price, correct to one decimal place.

Solution

a $P = \$19\,800, R = 0.12, n = 6$

$$\begin{aligned} A &= P(1 - R)^n \\ &= \$19\,800(1 - 0.12)^6 \\ &= \$19\,800(0.88)^6 \\ &= \$9195.200 \dots \\ &\approx \$9195 \end{aligned}$$

b Percentage of cost price = $\frac{\$9195}{\$19800} \times 100\%$
 $= 46.4393 \dots \%$
 $\approx 46.4\%$

This means that after 6 years, the oven is worth approximately 46% of its original price (or has lost 54% of its original value).

Exercise 2-07 Depreciation

In this exercise, give all money answers correct to the nearest cent.

- 1 Find the value of a photocopier after 5 years if its purchase price was \$2850 and the annual depreciation rate is 20%.
- 2 a Find the value of a car after 7 years if it is purchased new for \$49 990 and it depreciates at 12% p.a.
b Find the amount of depreciation over this time.
- 3 For each item shown in the table, calculate:
 - i its value after 4 years of depreciation
 - ii its value after 4 years as a percentage of its original value, correct to one decimal place.

	Item	Original value	Depreciation rate (p.a.)
a	Stove	\$1100	12%
b	Fishing boat	\$5440	18%
c	Library	\$8460	12%
d	Computer	\$1900	20%
e	Furniture	\$27 500	15.5%
f	Bike	\$2300	22%
g	Electrical tools	\$870	17.5%
h	Air conditioner	\$1600	9%



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- 4 A DVD recorder originally valued at \$225 depreciates at 10% p.a.
 - a What percentage (to the nearest whole number) of the original value remains after:
 - i 1 year? ii 3 years? iii 6 years? iv 7 years?
 - b Approximately how long would it take the DVD recorder to halve its original value?
- 5 A security system costs a company \$12 500 to buy new. It depreciates at a yearly rate of 20%.
 - a Find the value of the system after:
 - i 1 year ii 2 years iii 5 years
 - b Find the value of the system after 5 years as a percentage of its original value. Answer correct to one decimal place.
- 6 Paul pays \$15 800 for a new car. The car will depreciate in value by an average of 11% p.a.
 - a Find the market value of the car in 3 years.
 - b Calculate the amount of depreciation in the car after 3 years.
- 7 Asha has spent \$12 400 on equipment to set up his painting business. The equipment depreciates at 15% per year.
 - a Find the value of the equipment after 4 years.
 - b Find the amount of depreciation in the equipment after 2 years.
 - c Find, by trial and error, how long it will take for the value to be under \$3000. Answer in years and months.
 - d Find the value of Asha's equipment after 9 years as a percentage of its original value, correct to one decimal place.

See Example 18

- 8 A salesperson claims that, at 10% p.a. depreciation, a car will lose half its value after 7 years. Is the salesperson correct? Show all working to justify your answer.

- 9 Office equipment worth \$12 000 when new, depreciates at 15% p.a. as shown in the table below.

- How much did the office equipment lose in value in the first year?
- After how many years did the office equipment fall below half its original value?
- By how much did the office equipment depreciate between the 5th and 6th years?
- Will the value of the office equipment ever fall below \$100?
- Will the value of the office equipment ever be zero?

Year	Depreciated value
0	\$12 000
1	\$10 200
2	\$8670
3	\$7369.50
4	\$6264.08
5	\$5324.46
6	\$4525.79
7	\$3846.93
8	\$3269.89
9	\$2779.40
10	\$2362.49
11	\$2008.12
12	\$1706.90
13	\$1450.87
14	\$1233.24

Power plus

- How long, in years and days, will it take an investment of \$3000 to earn \$500 in simple interest at 4% p.a.?
- What amount should I invest to earn \$100 in simple interest if the investment will last for 9 months and the interest rate is 3% p.a.?
- A principal of \$10 000 is invested for 5 years at an interest rate of 5% p.a., with interest compounded weekly. Calculate the final value of the investment.
- Ali needs \$80 000 in 4 years time. What amount should she invest now at an interest rate of 6% p.a., with interest compounded annually, to reach her target?
- A painting appreciates in value at a rate of 3% p.a. while a computer depreciates in value at a rate of 10% p.a. If I bought the painting for \$1200 and the computer for \$1500 new, what would be their combined value in 5 years time?
- A bacteria colony is growing at a rate of 20% per hour. If there are 10 000 bacteria now, use the compound interest formula to calculate how many there will be after 1 day. (Give your answer correct to the nearest 10 000.)
- You invest \$2000 in a bank account at an interest rate of 4% p.a. with interest compounded annually. How long will it take for your investment to double in value?
 - If you invested \$4000 instead of \$2000 at the same interest rate, how long will it take to double in value?
 - Does the size of the principal make any difference to the time taken for it to double?

Language of maths

allowable deductions	annual leave loading	compound interest	deposit
depreciation	double time	flat rate	fortnightly
gross pay	income tax	interest	net pay
overtime	PAYG tax	per annum (p.a.)	principal
quarterly	repayment	salary	simple interest
taxable income	term payments	time-and-a-half	wage

Puzzle sheet

Interest and
depreciation crossword

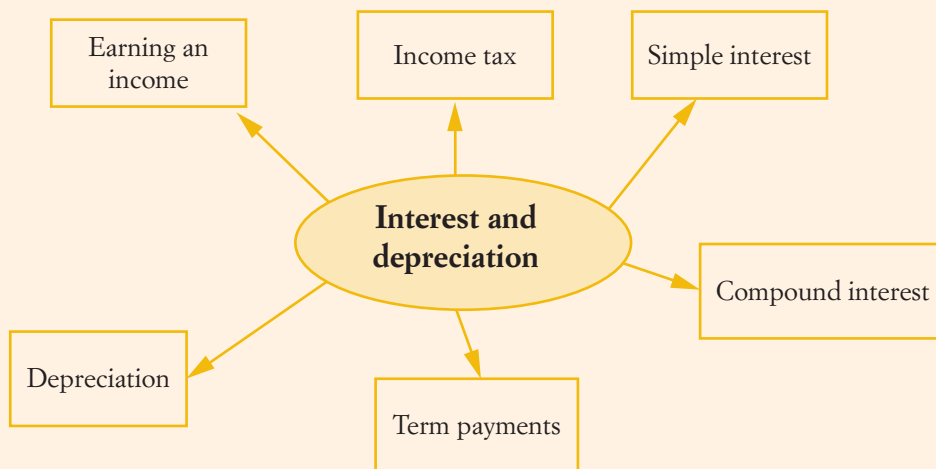
MAT10NAPS10007

- 1 When investing, why is **compound interest** better than **simple interest**?
- 2 What do the P and R stand for in the formulas $I = PRN$ and $A = P(1 + R)^n$?
- 3 What is another name for **flat-rate interest**?
- 4 What word means a decrease in the value of an item over time?
- 5 Why is **gross pay** higher than **net pay**?
- 6 Use a dictionary to find at least two different meanings of **principal**.

Topic overview

- Which parts of this chapter were revision of Year 9 knowledge and skills?
- Which parts of this chapter were new to you?
- Do you know how to use the simple interest and compound interest formulas?
- How is income tax calculated?
- How is the depreciation formula similar to the compound interest formula?

Copy and complete this mind map of the topic, adding detail to its branches and using pictures, symbols and colour where needed. Ask your teacher to check your work.



Quiz

Money and finance

MAT10NAQZ00009

Chapter 2 revision

In this exercise, give all money answers correct to the nearest cent.

See Exercise 2-01

- 1** Hayley is paid a commission of 2.5% on the value of the properties she sells. She also receives a weekly retainer of \$1150. How much will Hayley earn if she sells a house for \$475 830?

See Exercise 2-01

- 2** Thomas earns a salary of \$70 400 p.a. How much is he paid each week?

See Exercise 2-01

- 3 A supermarket cashier is employed under the following award.

Normal rate: \$19.45 per hour	
Normal rate	For 0 to 38 hours worked
Time-and-a-half	For the next 4 hours worked
Double time	For each hour worked after that

Calculate the wage for working:

- a** 40 hours

See Exercise 2-01

- 4 For his Christmas holidays, Jacan received 4 weeks normal pay plus 17.5% annual leave loading for the 4 weeks. If Jacan's annual salary is \$54 920, find:

- a** his normal weekly pay
- b** his leave loading
- c** his total pay for the four-week holiday.

See Exercise 2-02

- 5 Alia earns a salary of \$68 650 p.a. Her allowable deductions are donations to charities of \$540 and work-related expenses of \$385.

- Calculate Alia's taxable income.
- Use the tax table on page 30 to calculate the income tax that Alia should pay.

See Exercise 2-03

- 6 Calculate the simple interest earned on each investment.

- \$20 000 invested for 3 years at 4% p.a.
- \$7850 invested at 2.5% p.a for 2 years
- \$4500 invested for 6 months at 3.5% p.a.
- \$25 200 invested for 100 days at 9% p.a.

See Exercise 2-04

- 7 An amount of \$5000 is invested at 6% p.a. interest, compounded over 3 years.

- What is the total value of the investment after 3 years?
- What is the amount of compound interest earned?

See Exercise 2-05

- 8 Calculate the value of the investment when \$34 200 is invested at 3% p.a. for 2 years, with interest compounded annually.

See Exercise 2-05

- 9 Find the final value if \$11 000 is invested for 4 years at 8.4% p.a., with interest compounded monthly.

See Exercise 2-05

- 10 Find the interest earned when \$4895 is invested at 5.5% p.a. for 3 years, with interest compounded annually.

See Exercise 2-05

- 11 Calculate the interest earned when \$46 230 is invested for 9 years at 7.8% p.a., with interest compounded twice a year.

- 12** Yang purchases a furniture package valued at \$4875. She pays a 10% deposit and repays the balance in 36 monthly instalments. Interest on the balance is charged at a flat rate of 14.5% p.a. Find: See Exercise 2-06
- | | |
|--|---|
| a the deposit Yang paid | b the balance owing |
| c the interest charged | d the total to be repaid |
| e the amount of each instalment | f the total price Yang paid for the package. |
- 13** Caroline bought a new car for \$14 990, which depreciates by 10% p.a. See Exercise 2-07
- a** Find the depreciated value of the car after 5 years.
 - b** What is the depreciation over this time?
 - c** Express the depreciated value as a percentage of the original price (correct to one decimal place).



3

Number and Algebra

Coordinate geometry

Straight lines are an important part of our environment. We play sport on courts with parallel and perpendicular lines, and skyscrapers would not be standing without straight lines. We can also use straight lines to model different types of data and predict future outcomes.



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Chapter outline

	Proficiency strands			
3-01 Length, midpoint and gradient of an interval	U	F	R	C
3-02 Parallel and perpendicular lines	U	F	R	C
3-03 Graphing linear equations	U	F	R	C
3-04 The gradient–intercept equation $y = mx + b$	U	F	R	C
3-05 The general form of a linear equation $ax + by + c = 0$	U	F	R	C
3-06 The point–gradient form of a linear equation*	U	F	R	C
3-07 Finding the equation of a line	U	F	R	C
3-08 Equations of parallel and perpendicular lines	U	F	R	C

*STAGE 5.3

Wordbank

general form Any linear equation expressed as $ax + by + c = 0$, where a , b and c are integers and a is positive

gradient The steepness of a line or interval, measured by the fraction $\frac{\text{rise}}{\text{run}}$

gradient–intercept form Any linear equation expressed as $y = mx + b$, where m is the gradient and b is the y -intercept

linear equation An equation whose graph is a straight line

parallel lines Lines that point in the same direction and have the same gradient

perpendicular lines Lines that cross at right angles (90°) and have gradients whose product is -1

x -intercept The x -value at which a graph cuts the x -axis

y -intercept The y -value at which a graph cuts the y -axis

In this chapter you will:

- find the distance between two points located on the Cartesian plane using a range of strategies, including graphing software
- find the midpoint and gradient of a line segment (interval) on the Cartesian plane using a range of strategies, including graphing software
- sketch linear graphs using the coordinates of two points and solve linear equations
- solve problems involving parallel and perpendicular lines 
- (STAGE 5.3) use coordinate geometry formulas to calculate the length, midpoint and gradient of an interval
- (STAGE 5.3) find the angle of inclination of a line using the formula $m = \tan \theta$
- (STAGE 5.3) graph a line by finding its x - and y -intercepts
- test whether a point lies on a line
- use the gradient–intercept equation of a straight line $y = mx + b$
- find the equation of a line from its graph
- recognise the general form of the equation of a straight line and convert it to the gradient–intercept equation
- (STAGE 5.3) find the equation of a line given its gradient and a point on the line, or given two points, by using the point–gradient formula
- find the equation of a line that is parallel or perpendicular to a given line
- (STAGE 5.3) use coordinate geometry methods to prove geometrical properties

SkillCheck

Worksheet

StartUp assignment 2

MAT10NAWK10008

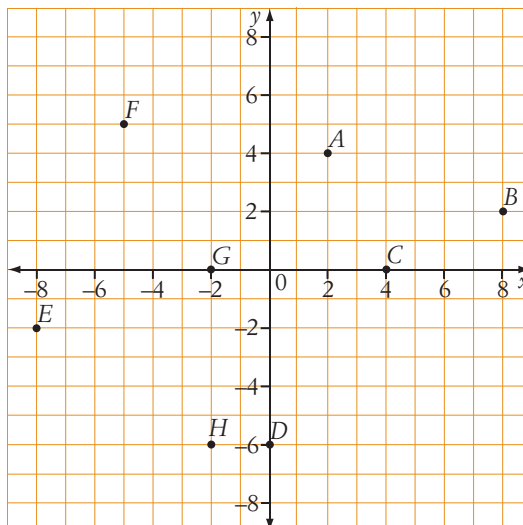
Skillsheet

Pythagoras' theorem

MAT10MGSS10004

1 For this number plane, find:

a the midpoint of interval BC	b the midpoint of interval HE
c the length of interval GC	d the length of interval GH
e the lengths of AC and BC , correct to one decimal place	f the type of triangle $\triangle ABC$ is
g the gradient of GE	h the gradient of EH



2 For each linear equation, copy and complete the table of values and graph the equation.

a $y = x - 3$

x	0	1	2	3
y				

b $y = 3x + 2$

x	-2	-1	0	1
y				

c $y = 1 - 2x$

x	-1	0	1	2
y				

3 If $x_1 = 3$, $y_1 = 4$, $x_2 = -5$ and $y_2 = 6$, then evaluate each expression.

a $x_1 + x_2$

b $x_2 - x_1$

c $\frac{y_1 + y_2}{2}$

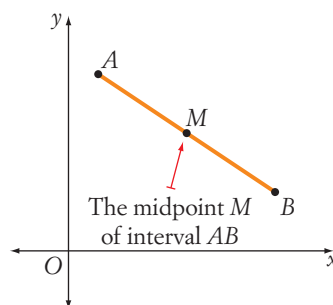
d $(y_2 - y_1)^2$

e $\frac{y_2 - y_1}{x_2 - x_1}$

f $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

3-01 Length, midpoint and gradient of an interval

The **length** of an interval AB (or the **distance** between A and B) can be calculated using Pythagoras' theorem if we know the coordinates of A and B .

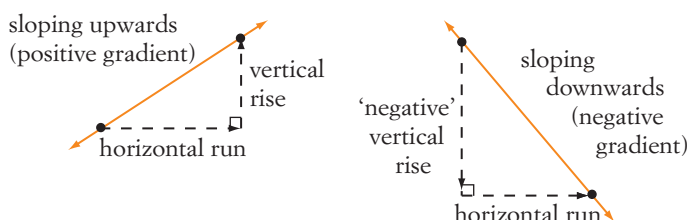


The **midpoint** of an interval AB is the point in the middle of AB or halfway between A and B .

- Its x -coordinate is the average of the x -coordinates of A and B .
- Its y -coordinate is the average of the y -coordinates of A and B .

The **gradient** of an interval measures its steepness. It is given by the formula:

$$m = \frac{\text{vertical rise}}{\text{horizontal run}} = \frac{\text{rise}}{\text{run}}$$



- A line **sloping upwards** has a **positive rise** and a **positive gradient**.
- A line **sloping downwards** has a **negative rise** and a **negative gradient**.
- The **run** is always **positive**.

Worksheet

Gradient, midpoint, distance

MAT10NAWK00014

Puzzle sheet

Intervals match-up

MAT10NAPS10009

Technology worksheet

Excel worksheet:
Midpoint and distance
between two points

MAT10NACT00008

Technology worksheet

Excel spreadsheet:
Midpoint and distance

MAT10NACT00038

Example 1

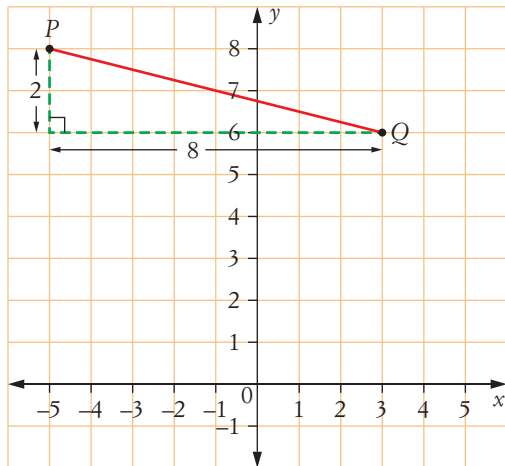
For the interval joining the pair of points $P(-5, 8)$ and $Q(3, 6)$, find:

- the length of the interval, correct to one decimal place
- the midpoint of the interval
- the gradient of the interval

Solution

- Draw a right-angled triangle on the number plane with PQ as the hypotenuse.

The height of the triangle is 2 units.
The base of the triangle is 8 units.



$$PQ^2 = 2^2 + 8^2$$

$$= 68$$

$$PQ = \sqrt{68}$$

$$= 8.2462 \dots$$

$$\approx 8.2 \text{ units}$$

by Pythagoras' theorem

- For $P(-5, 8)$ and $Q(3, 6)$, the average of the x -coordinates is $\frac{-5+3}{2} = -1$.

The average of the y -coordinates is $\frac{8+6}{2} = 7$.

$\therefore$ The midpoint of PQ is $(-1, 7)$.

From the diagram above, a midpoint at $(-1, 7)$ looks reasonable.

- The rise is -2 units.
The run is 8 units.

$$m = \frac{\text{rise}}{\text{run}}$$

$$= \frac{-2}{8}$$

$$= -\frac{1}{4}$$

Line slopes downwards.

The distance, midpoint and gradient formulas

The methods for finding the length, midpoint and gradient of an interval can each be summarised by a formula.

The **distance formula** is used to calculate the distance (d) between any two points $P(x_1, y_1)$ and $Q(x_2, y_2)$, in other words, the length of the interval PQ .

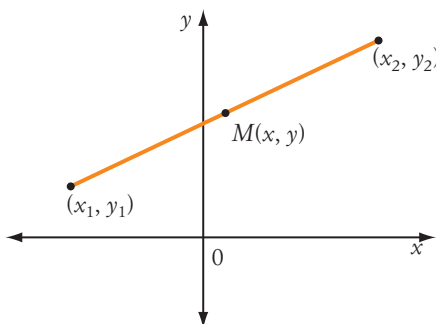
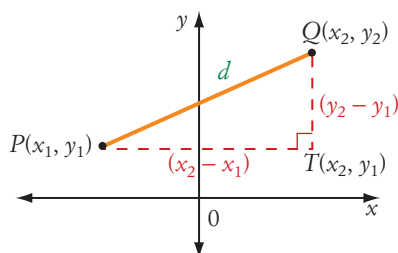
$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$\therefore d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

by Pythagoras' theorem

The **midpoint formula** gives the coordinates of the point M , the midpoint of the interval joining $P(x_1, y_1)$ and $Q(x_2, y_2)$:

$$M(x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$



The **gradient formula** gives the gradient of the interval or line joining P and Q .

$$\text{Gradient, } m = \frac{\text{rise}}{\text{run}} = \frac{\text{difference in } y}{\text{difference in } x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Summary

For an interval PQ with endpoints $P(x_1, y_1)$ and $Q(x_2, y_2)$, the formulas for distance (length), midpoint and gradient are:

Distance $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Midpoint $M(x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$

Gradient $m = \frac{y_2 - y_1}{x_2 - x_1}$

Example 2

For the interval joining $P(-5, 8)$ and $Q(3, 6)$ from Example 1b, use a formula to find:

- the length of the interval, correct to one decimal place
- the midpoint of the interval
- the gradient of the interval.

Stage 5.3

Video tutorial

Coordinate geometry

MAT10NAVT00005

Video tutorial

Distance, midpoint
and gradient
formulas

MAT10NAVT10010

Puzzle sheet

Finding coordinates for
given segment lengths

MAT10NAPS00048

Stage 5.3

Solution

For $P(-5, 8)$ and $Q(3, 6)$:

$$\begin{array}{cc} \uparrow & \uparrow \\ (x_1, y_1) & (x_2, y_2) \end{array}$$

$$\begin{aligned} \text{a } d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(3 - (-5))^2 + (6 - 8)^2} \\ &= \sqrt{68} = 8.2462 \dots \approx 8.2 \text{ units} \end{aligned}$$

$$\begin{aligned} \text{b } M(x, y) &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) \\ &= \left(\frac{-5 + 3}{2}, \frac{8 + 6}{2} \right) \\ &= (-1, 7) \end{aligned}$$

$$\begin{aligned} \text{c } m &= \frac{\text{difference in } y}{\text{difference in } x} \\ &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{6 - 8}{3 - (-5)} = \frac{-2}{8} = -\frac{1}{4} \end{aligned}$$

$$x_1 = -5, y_1 = 8, x_2 = 3, y_2 = 6$$

Apply the distance formula.

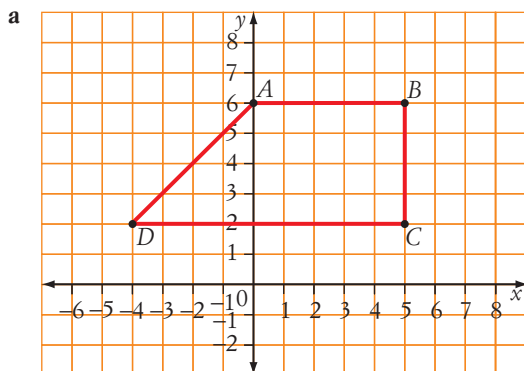
Apply the midpoint formula.

Apply the gradient formula.

Example 3

- Plot the points $A(0, 6)$, $B(5, 6)$, $C(5, 2)$ and $D(-4, 2)$ on a number plane and join them to make the quadrilateral $ABCD$.
- What type of quadrilateral is $ABCD$?
- Find the exact length of AD .
- Hence find the perimeter of $ABCD$ correct to two decimal places.

Solution



Join the points in the correct order.

b Since $AB \parallel CD$, the quadrilateral is a trapezium.

c $AD^2 = 4^2 + 4^2 = 32$

$AD = \sqrt{32}$ units

In exact surd form.

d By counting grid squares, $AB = 5$, $BC = 4$, $CD = 9$.

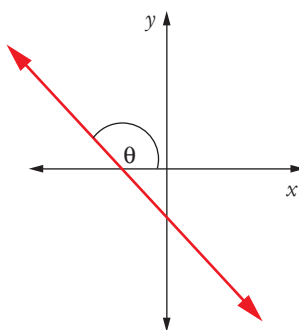
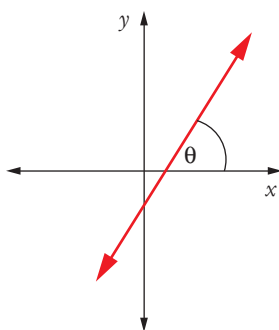
Perimeter of $ABCD = 5 + 4 + 9 + \sqrt{32} = 23.656 \dots \approx 23.66$ units

The angle of inclination of a line

The **angle of inclination**, θ , of a line is the angle it makes with the x -axis in the positive direction.

acute angle = positive gradient

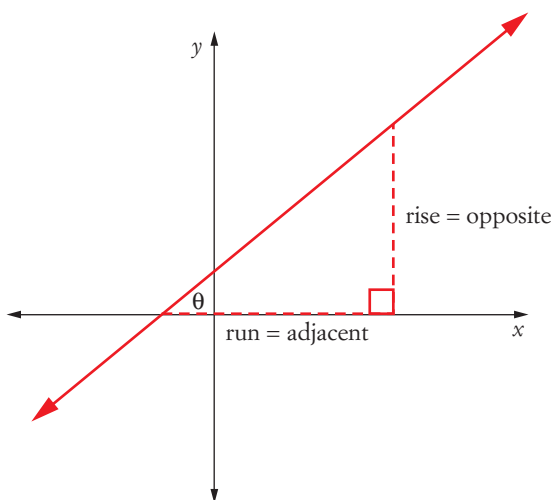
obtuse angle = negative gradient



Note from the above diagrams that θ is **acute** when the line has a **positive gradient**, and **obtuse** when the line has a **negative gradient**.

We can use trigonometry to calculate the angle of inclination of a line using its gradient, m .

The diagram below shows that $m = \frac{\text{rise}}{\text{run}}$, but in trigonometry, $\tan \theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{\text{rise}}{\text{run}}$.
 $\therefore m = \tan \theta$.



Summary

The **angle of inclination**, θ , of a line is related to the gradient, m , of the line by the formula:

$$m = \tan \theta$$

Stage 5.3

Example 4

Find, correct to the nearest degree, the angle of inclination of a line with gradient:

a $\frac{1}{3}$ b -4

Solution

a $m = \tan \theta$

$$\frac{1}{3} = \tan \theta$$

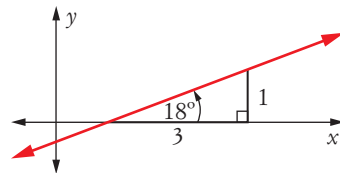
$$\tan \theta = \frac{1}{3}$$

$$\theta = 18.4349 \dots$$

$$\approx 18^\circ$$

The positive gradient means that it is an acute angle

On a calculator: **SHIFT** **tan** **1** **a^{b/c}** **3** **=**



b $m = \tan \theta$

$$-4 = \tan \theta$$

$$\tan \theta = -4$$

$$\theta = -75.9637 \dots$$

$$\approx -76^\circ$$

But this negative angle is the angle below the x -axis.

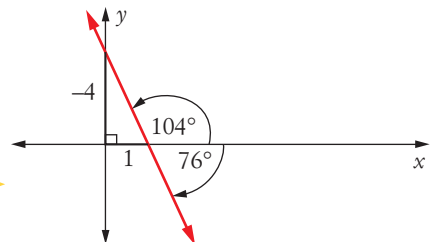
To find the angle of inclination,

$$\theta \approx 180^\circ - 76^\circ$$

$$\approx 104^\circ$$

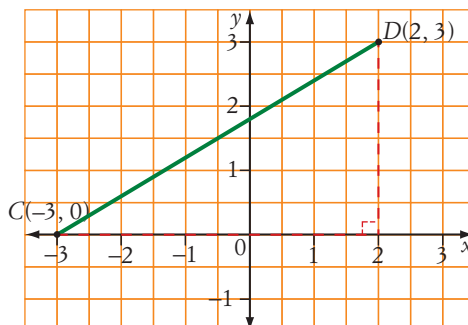
The negative gradient means that it is an obtuse angle

On a calculator: **SHIFT** **tan** **(-)** **4** **=**



Exercise 3-01 Length, midpoint and gradient of an interval

Questions 1, 2 and 3 refer to this diagram of interval CD .



- 1 What is the length of interval CD ? Select the correct answer **A**, **B**, **C** or **D**.

A 2 units **B** 5.8 units **C** 3.2 units **D** 8 units

- 2 What is the midpoint of CD ? Select **A**, **B**, **C** or **D**.

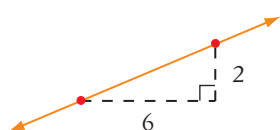
A $(-1, 3)$ **B** $(-5, 3)$ **C** $(-0.5, 1.5)$ **D** $(-2.5, 1.5)$

- 3 What is the gradient of CD ? Select **A**, **B**, **C** or **D**.

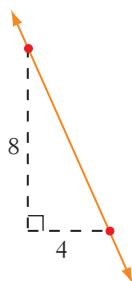
A $\frac{3}{5}$ **B** -3 **C** $-\frac{5}{3}$ **D** 2

- 4 Calculate the gradient of each line.

a



b



c



- 5 For the interval joining each pair of points given, find:

i the length of the interval correct to one decimal place

ii the midpoint of the interval

iii the gradient of the interval.

a $A(5, 3)$ and $B(7, 2)$

b $J(-1, 0)$ and $K(8, 6)$

c $M(0, -3)$ and $N(-5, 2)$

d $R(-3, -6)$ and $S(4, -9)$

e $A(-7, 2)$ and $B(-5, -8)$

f $U(3, -2)$ and $V(7, 2)$

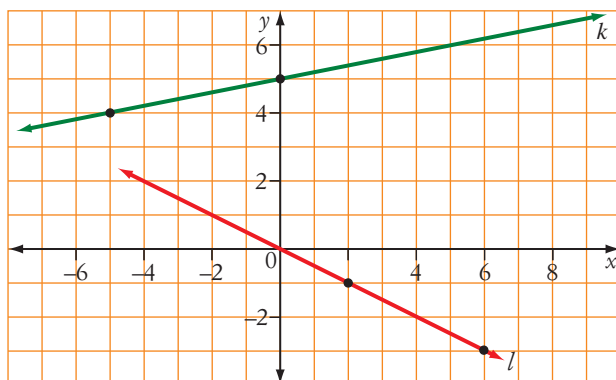
- 6 Calculate, in exact (surd) form, the distance between each pair of points.

a $(-8, -1)$ and $(0, 4)$

b $(12, -6)$ and $(-1, -1)$

c $(7, -2)$ and $(-2, -3)$

- 7 Find the gradient of the lines labelled k and l .



- 8 Which expression gives the y -coordinate of the midpoint of the interval joining points $(3, 8)$ and $(-1, 5)$? Select the correct answer **A**, **B**, **C** or **D**.

A $\frac{-1+5}{2}$

B $\frac{8+5}{2}$

C $\frac{8-5}{2}$

D $\frac{5-8}{2}$

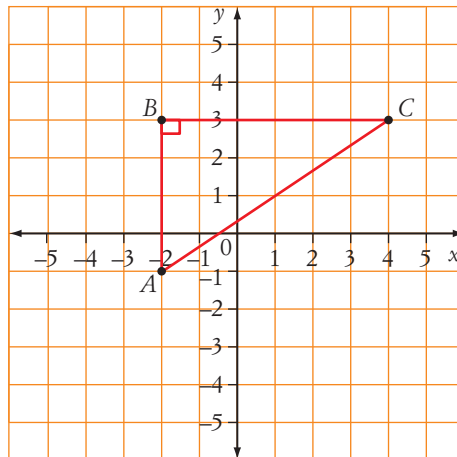
Stage 5.3

See Example 3

- 9 The vertices of triangle ABC are $A(-1, -1)$, $B(1, 3)$ and $C(3, 1)$.
- Draw $\triangle ABC$ on a number plane.
 - Find the exact length of each side of the triangle.
 - Are any sides of the triangle equal in length?
 - What type of triangle is ABC ?
 - Find the perimeter of $\triangle ABC$, correct to one decimal place.
- 10 The vertices of quadrilateral $KLMP$ are $K(1, 6)$, $L(7, 2)$, $M(3, -4)$ and $P(-3, 0)$.
- Draw the quadrilateral on a number plane.
 - What type of quadrilateral is $KLMP$?
 - Find the gradients of sides KL and PM .
 - Find the gradients of sides KP and LM .
 - What do you notice about the gradients of opposite sides of this quadrilateral? What does that mean about those sides?
 - Find the exact length of each side of $KLMP$.
 - Find the perimeter of $KLMP$, correct to one decimal place.
 - Find the area of $KLMP$.

- 11 This diagram shows a right-angled triangle with vertices $A(-2, -1)$, $B(-2, 3)$ and $C(4, 3)$.

- Copy the diagram and find the coordinates of P and Q , the midpoints of BA and BC respectively. Mark P and Q on your diagram.
- Calculate, correct to one decimal place, the lengths of PQ and AC . What do you notice about your answers?
- Find the gradients of PQ and AC . What do you notice about your answers?



See Example 4

- 12 Find, correct to the nearest degree, the angle of inclination of a line with gradient:
- | | | | |
|------|-----------------|-------------------|------------------|
| a 3 | b $\frac{1}{2}$ | c 1 | d 2.5 |
| e -2 | f $\frac{3}{4}$ | g $-\frac{1}{10}$ | h $-\frac{2}{3}$ |
- 13 Find, correct to two decimal places, the gradient of a line with angle of inclination:
- | | | | |
|--------------|---------------|---------------|--------------|
| a 60° | b 158° | c 42° | d 94° |
| e 8° | f 135° | g 177° | h 0° |

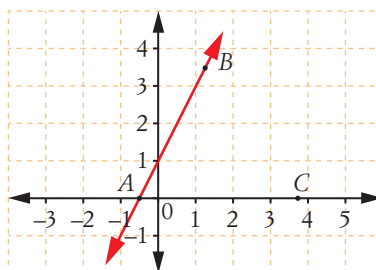
Technology The angle of inclination

In this activity we will use GeoGebra to calculate the angle of inclination of a line.

- 1 Close the **Algebra** View so that only the graphics window is showing and select the grid option at the top left-hand corner. Click on the input bar at the bottom of the screen and enter: $y = 2x + 1$



- 2 Click New Point. Click on the x-intercept of the line $y = 2x + 1$ (where it meets the x-axis). Also place New Points on the straight line (shown below as B) and the x-axis (shown below as C).



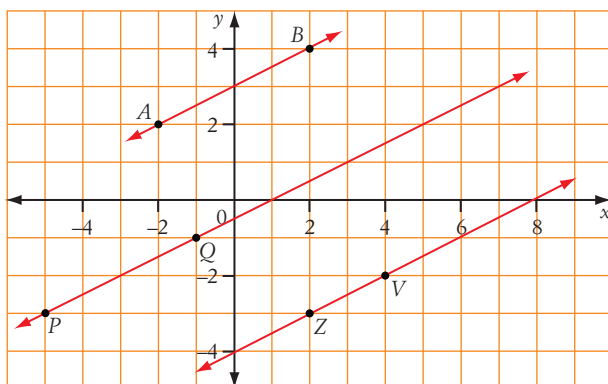
- 3 Click Angle and select in a clockwise direction the points C , A and B in order.
- 4 What is the angle of inclination of the line? Answer to the nearest degree.
- 5 Use GeoGebra to measure the angle of inclination of the line with equation:

a $y = -3x - 5$	b $y = x + 2$	c $y = x - 6$	d $y = -2x + 3$
e $y = -5x - 7$	f $y = -8x + 1$	g $y = 3x - 12$	h $y = \frac{2}{3}x + 4$

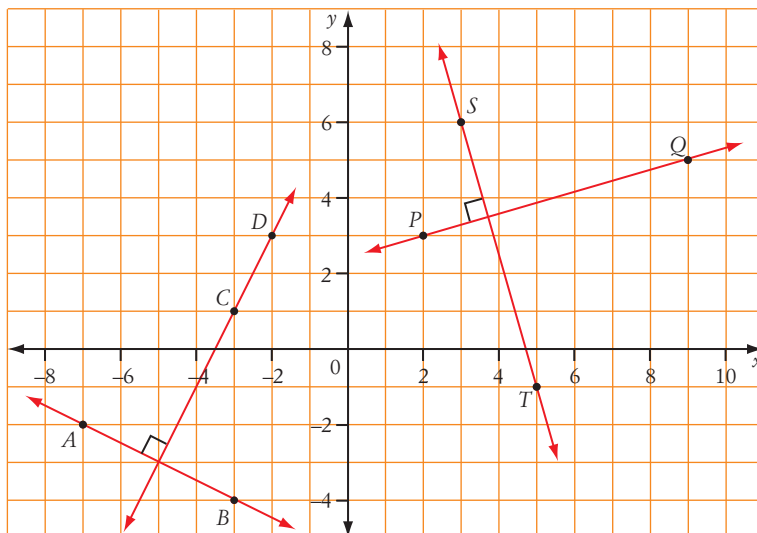
Investigation: Parallel and perpendicular lines

- 1 These three lines are parallel. Calculate the gradient of:

a AB	b PQ	c ZV



- 2 What can you conclude about the gradients of parallel lines?
 3 This diagram shows two pairs of perpendicular lines. $AB \perp CD$ and $PQ \perp ST$.



Calculate the gradient of:

- a AB b CD c PQ d ST
 4 Is there a relationship between:
 a the gradients of AB and CD ? b the gradients of PQ and ST ?
 5 Calculate the product of (multiply):
 a the gradients of AB and CD b the gradients of PQ and ST
 6 What can you conclude about the gradients of perpendicular lines?

Puzzle sheet

Gradients of parallel
and perpendicular lines

MAT10NAPS00012

Technology

GeoGebra:
Perpendicular lines

MAT10NATC00005

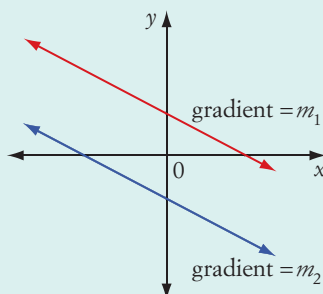
3-02 Parallel and perpendicular lines

Parallel lines

Summary

Parallel lines have the same gradient.

If two lines with gradients m_1 and m_2 are parallel, then $m_1 = m_2$

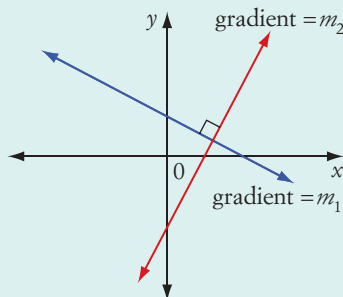


Perpendicular lines

Summary

Perpendicular lines have gradients whose product is -1 .

If two lines with gradients m_1 and m_2 are perpendicular, then $m_1 \times m_2 = -1$ or $m_2 = -\frac{1}{m_1}$.



Note that m_2 is the **negative reciprocal** of m_1 .

Example 5

State whether each pair of gradients represent parallel lines, perpendicular lines or neither.

a $m_1 = \frac{1}{2}$, $m_2 = 2$

b $m_1 = 0.4$, $m_2 = \frac{2}{5}$

c $m_1 = 1\frac{3}{5}$, $m_2 = -\frac{5}{8}$

Solution

a $m_1 \neq m_2$ so the lines are not parallel.

$$\begin{aligned} m_1 \times m_2 &= \frac{1}{2} \times 2 \\ &= 1 \\ &\neq -1 \end{aligned}$$

so the lines are not perpendicular.

$\therefore$ The lines are neither parallel nor perpendicular.

c $m_1 = 1\frac{3}{5} = \frac{8}{5}$

$$\begin{aligned} m_1 \times m_2 &= \frac{8}{5} \times \left(-\frac{5}{8}\right) \\ &= -1 \end{aligned}$$

$\therefore$ The lines are perpendicular.

b $m_2 = \frac{2}{5} = 0.4$

$$m_1 = m_2$$

$\therefore$ The lines are parallel.

Example 6

Find the gradient of a line that is perpendicular to a line with gradient:

- a** 2 **b** -3 **c** $\frac{3}{4}$ **d** -0.6

Solution

a $m_1 = 2$

$$\begin{aligned} m_2 &= \frac{-1}{m_1} \text{ for perpendicular lines} \\ &= \frac{-1}{2} \\ &= -\frac{1}{2} \end{aligned}$$

The gradient is $-\frac{1}{2}$.

c $m_1 = \frac{3}{4}$

$$\begin{aligned} m_2 &= \frac{-1}{m_1} \\ &= \frac{-1}{(\frac{3}{4})} \\ &= -\frac{4}{3} \end{aligned}$$

The gradient is $-\frac{4}{3}$.

b $m_1 = -3$

$$\begin{aligned} m_2 &= \frac{-1}{m_1} \\ &= \frac{-1}{-3} \\ &= \frac{1}{3} \end{aligned}$$

The gradient is $\frac{1}{3}$.

d $m_1 = -0.6 = -\frac{3}{5}$

$$\begin{aligned} m_2 &= \frac{-1}{(-\frac{3}{5})} \\ &= \frac{5}{3} \end{aligned}$$

The gradient is $\frac{5}{3}$.

The negative reciprocal of m_1 .

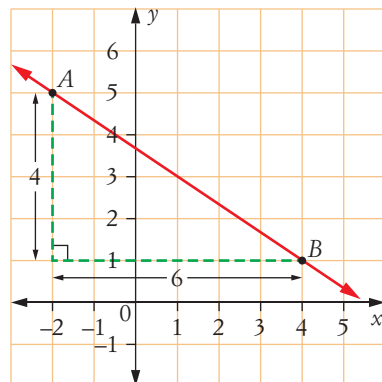
Example 7

A line passes through the points $A(-2, 5)$ and $B(4, 1)$. What is the gradient of a line:

- a** parallel to AB ? **b** perpendicular to AB ?

Solution

Find the gradient of AB by calculating the rise and run.



$$\text{Rise} = 1 - 5 = -4$$

$$\text{Run} = 4 - (-2) = 6$$

$$\text{Gradient } AB = \frac{-4}{6} = -\frac{2}{3}$$

Difference between y -coordinates.

Difference between x -coordinates.

$\frac{\text{rise}}{\text{run}}$

- a Any line parallel to AB will have the same gradient as AB .

$$\therefore m = -\frac{2}{3}$$

- b The gradient of a line perpendicular to AB will be given by:

$$m = \frac{-1}{(-\frac{2}{3})} = \frac{3}{2}$$

Exercise 3-02 Parallel and perpendicular lines

- 1 State whether each pair of gradients represent parallel lines, perpendicular lines or neither.

a $m_1 = \frac{1}{4}, m_2 = 4$ b $m_1 = 3, m_2 = -\frac{1}{3}$ c $m_1 = 0.5, m_2 = \frac{1}{2}$

d $m_1 = \frac{2}{7}, m_2 = \frac{7}{2}$ e $m_1 = \frac{3}{10}, m_2 = 0.3$ f $m_1 = 1\frac{1}{5}, m_2 = -\frac{6}{5}$

- 2 Find the gradient of a line that is parallel to a line with gradient:

a 4 b -2 c $\frac{1}{3}$ d -0.2

- 3 Find the gradient of a line that is perpendicular to a line with gradient:

a 1 b -6 c -1.5 d $\frac{5}{2}$

- 4 What is the gradient of a line that is perpendicular to a line with a gradient of 0.8? Select the correct answer **A**, **B**, **C** or **D**.

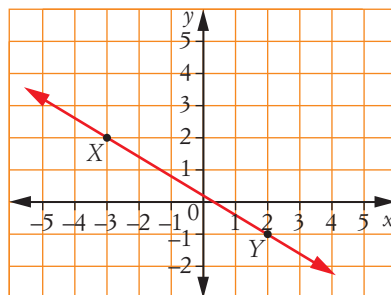
A 0.2 B -0.2 C 1.25 D -1.25

- 5 What is the gradient of a line that is parallel to a line that goes through $P(0, 3)$ and $Q(5, -2)$? Select **A**, **B**, **C** or **D**.

A 1 B -1 C $\frac{1}{5}$ D $-\frac{1}{5}$

- 6 What is the gradient of a line perpendicular to line XY shown on the right? Select **A**, **B**, **C** or **D**.

A $\frac{5}{3}$ B -5 C $\frac{3}{5}$ D $\frac{1}{5}$



See Example 5

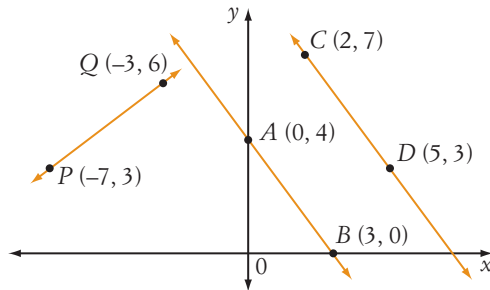
See Example 6

See Example 7

- 7 Calculate the gradient of each line shown below and test whether:

a $AB \parallel CD$

b $PQ \perp CD$.



- 8 A line passes through the points $R(-5, 2)$ and $S(1, 4)$. What is the gradient of a line:

a parallel to RS ?

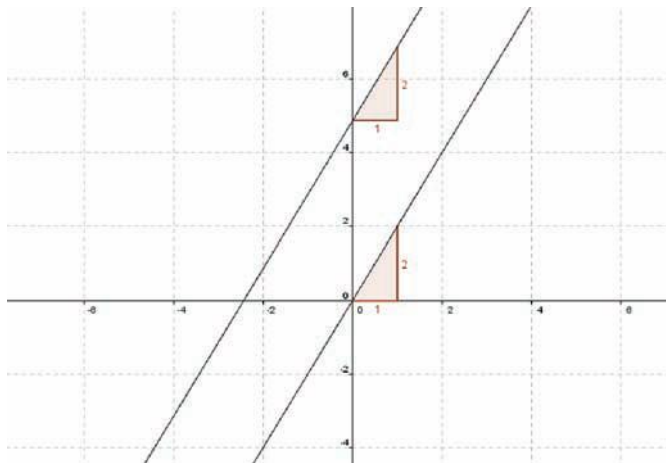
b perpendicular to RS ?

Technology Parallel and perpendicular lines

This activity uses GeoGebra to find out if sets of linear equations are parallel or perpendicular.

Parallel lines

- 1 Show the **Axes and Grid**.
- 2 Use the **Input bar** to enter the pair of linear equations $y = 2x + 5$ and $y = 2x$.
- 3 Use **Move Graphics View**  and **Zoom In**  to enlarge the axes if required.
- 4 Find the **Slope** (gradient) of each line.  **Slope**
- 5 Check if the two lines are parallel, using $m_1 = m_2$

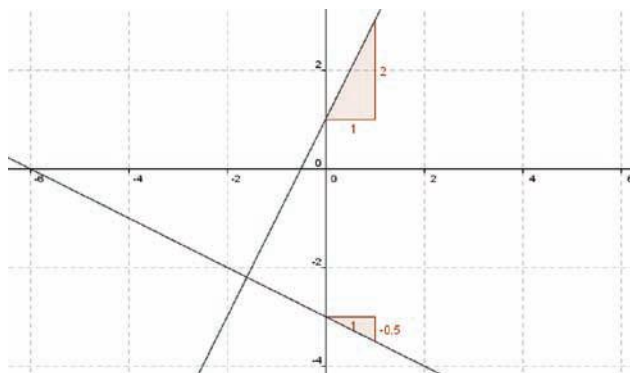


Since $m_1 = m_2 = 2$, this pair of lines is parallel.

- 6 Repeat steps 1 to 5 for the pairs of equations below. Decide if the lines are parallel or not.
 - a $5x - 3y = 0$ and $y = \frac{5x}{3}$
 - b $x + y + 4 = 0$ and $x + y - 6 = 0$
 - c $x - 2y = 0$ and $y = 0.5x$
 - d $y = 5x - 9$ and $5x - y - 1 = 0$

Perpendicular lines

- 1 Show the **Axes and Grid**.
- 2 Use the **Input bar** to enter the pair of linear equations $y = 2x + 1$ and $y = -0.5x - 3$.
- 3 Use **Move Graphics View** and **Zoom In** to enlarge the axes if required.
- 4 Find the **Slope** (gradient) of each line.
- 5 Check if the two lines are perpendicular, using $m_1 \times m_2 = -1$



Since $2 \times (-0.5) = -1$, the two lines are perpendicular.

- 6 Repeat steps 1 to 5 for the pairs of equations below. Decide if the lines are perpendicular or not.
 - a $y = 0.6x + 2$ and $y = \frac{5}{3}x$
 - b $x - 4y + 1 = 0$ and $y = -4x - 3$
 - c $3x - 2y = 0$ and $y = -\frac{2x}{3}$
 - d $y = 2x + 4$ and $x - 2y - 1 = 0$

3-03 Graphing linear equations

A relationship between two variables, x and y , whose graph is a straight line is called a **linear relationship**. The expression of that relationship as an algebraic formula, such as $y = 3x + 2$, is called a **linear equation**.

Example 8

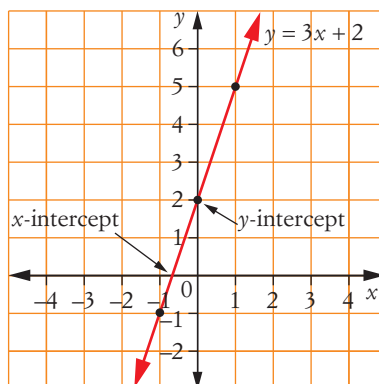
Graph $y = 3x + 2$ on a number plane.

Solution

Complete a table of values. Choose x -values close to 0 for easy calculation and graphing.

x	-1	0	1
y	-1	2	5

Graph $(-1, -1)$, $(0, 2)$ and $(1, 5)$ on a number plane. Rule a straight line through the points, place arrows at each end, and label the line with its equation.



Worksheet

Graphing linear equations

MAT10NAWK10010

Worksheet

Graphing linear equations (Advanced)

MAT10NAWK10203

Skillsheet

Graphing linear equations

MAT10NASS10005

Note:

- the **x-intercept** of the line is $-\frac{2}{3}$: this is the x value where the line cuts the x -axis
- the **y-intercept** of the line is 2: this is the y value where the line cuts the y -axis
- every point on the line follows the linear equation $y = 3x + 2$. For example, $(-1, 1)$, $(0, 2)$ and $(1, 5)$ lie on the line and follow the rule $y = 3x + 2$
- there are an infinite number of points that follow the rule. Arrows on both ends of the line indicate that it has infinite length.

Stage 5.3

Using x- and y-intercepts to graph lines

We can also graph a linear equation by finding its x - and y -intercepts first.

Since any point on the x -axis has a y -coordinate of 0, we can substitute $y = 0$ into the equation to find the x -intercept.

Similarly, any point on the y -axis has an x -coordinate of 0, so we can substitute $x = 0$ into the equation to find the y -intercept.

Summary

- To find the x -intercept, substitute $y = 0$ and solve the equation.
- To find the y -intercept, substitute $x = 0$ and solve the equation.

Example 9

Find the x - and y -intercepts of the line $2x - 3y = 6$ and draw its graph.

Solution

For the x -intercept, $y = 0$.

$$2x - 3 \times 0 = 6$$

$$2x = 6$$

$$x = 3$$

The x -intercept is 3.

For the y -intercept, $x = 0$.

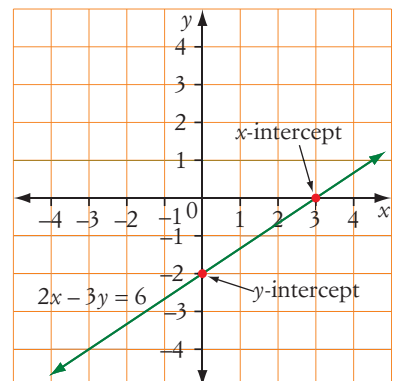
$$2 \times 0 - 3y = 6$$

$$-3y = 6$$

$$y = -2$$

The y -intercept is -2 .

Plot both intercepts on the axes, draw a line through the two points and label the line with its equation.



Testing if a point lies on a line

Summary

A point lies on a line if its (x, y) coordinates satisfy the equation of the line.

Example 10

Which of the following points lie on the line $x - 2y = 5$?

a $(17, 6)$

b $(8, -4)$

Solution

- Separate the equation into its left-hand side (LHS) and right-hand side (RHS)
- Substitute the coordinates of the point into both sides
- If $\text{LHS} = \text{RHS}$, the point satisfies the equation and so lies on the line
- If $\text{LHS} \neq \text{RHS}$, the point does **not** lie on the line.

a Substitute $x = 17, y = 6$ into $x - 2y = 5$.

$$\begin{aligned} \text{LHS} &= x - 2y & \text{RHS} &= 5 \\ &= 17 - 2 \times 6 \\ &= 5 \end{aligned}$$

$\text{LHS} = \text{RHS}$, so $(17, 6)$ lies on the line.

b Substitute $x = 8, y = -4$ into $x - 2y = 5$.

$$\begin{aligned} \text{LHS} &= x - 2y & \text{RHS} &= 5 \\ &= 8 - 2 \times (-4) \\ &= 16 \end{aligned}$$

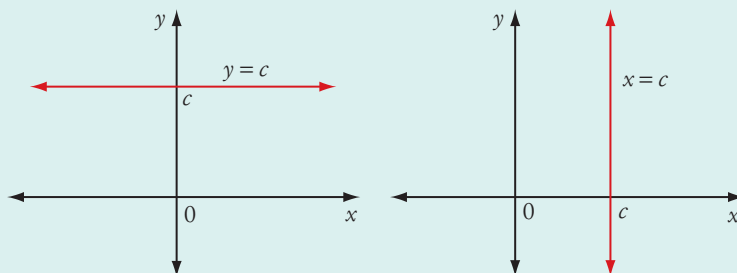
$\text{LHS} \neq \text{RHS}$, so $(8, -4)$ does not lie on the line.

Horizontal and vertical lines

Summary

The **equation of a horizontal line** is of the form $y = c$ (where c is a constant number).

The **equation of a vertical line** is of the form $x = c$ (where c is a constant number).



Technology worksheet

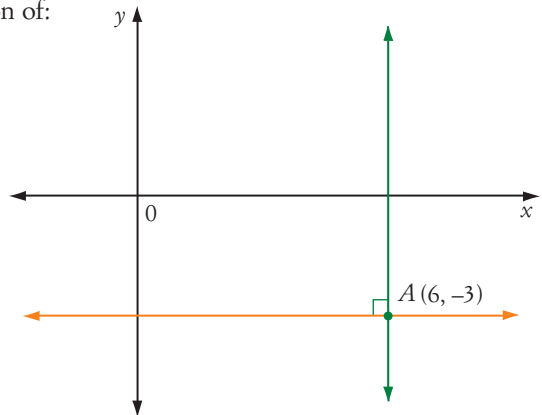
Horizontal and vertical lines

MAT10NACT10001

Example 11

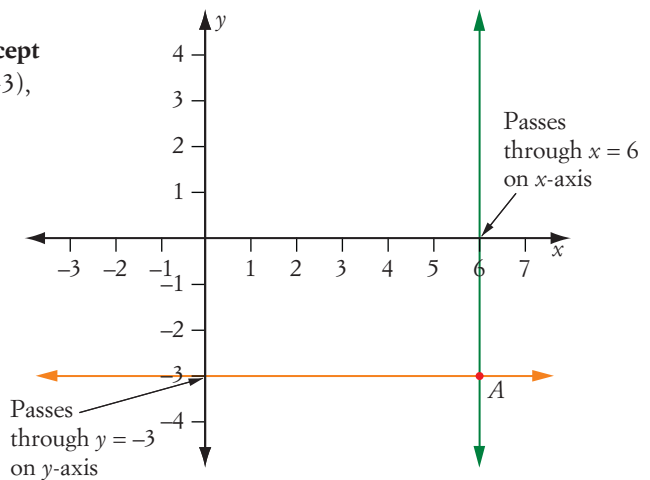
For the graph on the right, find the equation of:

- a the vertical line
b the horizontal line



Solution

- a The vertical line has an **x-intercept of 6** and passes through $A(6, -3)$, so its equation is $x = 6$.
b The horizontal line has a **y-intercept of -3** and passes through $A(6, -3)$, so its equation is $y = -3$.



Exercise 3-03 Graphing linear equations

See Example 7

- 1 Graph each linear equation on a number plane, and write:

i its x-intercept	ii its y-intercept.	
a $y = 3x - 1$	b $y = 2x + 5$	c $y = -x + 4$
d $y = -2x - 2$	e $y = 4x$	f $y = \frac{x}{2} + 3$

Stage 5.3

See Example 9

- 2 Graph each linear equation after finding its x- and y-intercepts.

a $y = 4 - 2x$	b $2x = 4y - 8$	c $y - x = 6$
d $3x - 2y = 12$	e $2x + 2y = 5$	f $6 - x = 2y$
g $y = 4 + 2x$	h $5x + 3y - 15 = 0$	i $3x - y = 6$
j $2x - 5y - 20 = 0$	k $4x + 2y - 8 = 0$	l $x - 4y - 2 = 0$

See Example 10

- 3 Test whether the point $(3, -1)$ lies on each line.

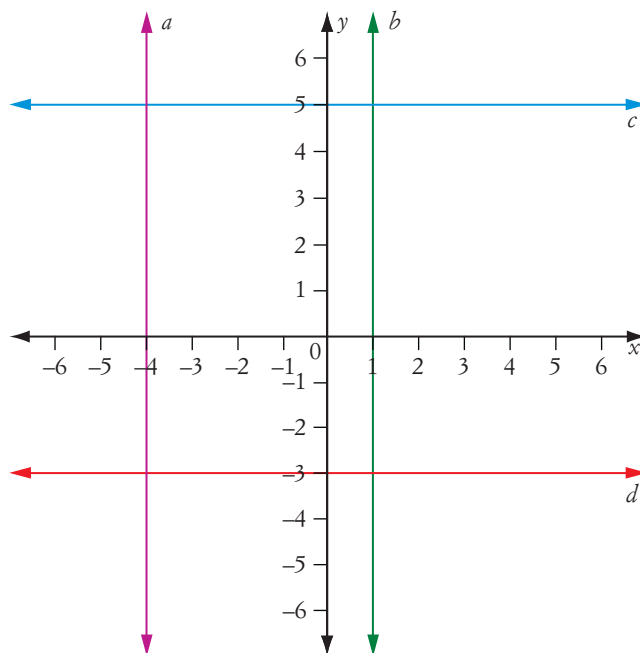
a $y = 2x - 5$	b $x - y = 4$	c $y + 2x = 5$
d $y = x - 4$	e $x + y = 5$	f $3x + y + 8 = 0$

4 Which of these points lies on the line $y = 6x - 5$? Select the correct answer **A**, **B**, **C** or **D**.

- A** $(-1, 11)$ **B** $(3, -13)$ **C** $(-2, -17)$ **D** $(-5, 25)$

5 Find the equation of each line shown below.

See Example 11



6 Graph each set of lines on a number plane.

- a** $x = 2\frac{1}{2}$, $y = -3$, $y = 1$ **b** $x = 6$, $y = -2$, $x = -\frac{1}{2}$

7 Find the equation of the line that is:

- a** horizontal and passes through the y -axis at 2
- b** vertical with an x -intercept of 4
- c** parallel to the y -axis and passes through the point $(-1, 4)$
- d** parallel to the x -axis and passes through the point $(0, -2)$
- e** 3 units above the x -axis
- f** 1 unit to the left of the y -axis
- g** drawn through the points $(-1, 6)$ and $(2, 6)$
- h** drawn through the points $(-1, 8)$ and $(-1, 2)$.

8 Which of these points lies on the line $4x + y = 1$? Select **A**, **B**, **C** or **D**.

- A** $(-1, 5)$ **B** $(-2, 7)$ **C** $(6, 9)$ **D** $(-\frac{1}{2}, 1)$

9 Which equation represents a line that is horizontal and passes through the point $(8, -2)$? Select **A**, **B**, **C** or **D**.

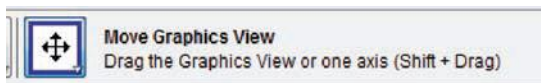
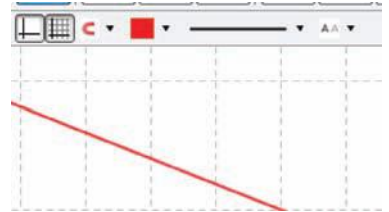
- A** $y = 8$ **B** $x = 8$ **C** $y = -2$ **D** $x = -2$

10 **a** What is another name for the line $y = 0$?

b What is another name for the line $x = 0$?

Technology Graphing $y = mx + b$

- 1 Show the **Axes and Grid**.
- 2 Enter the four lines $y = 3x + 2$, $y = 5x + 2$, $y = -2x + 2$, $y = -0.1x + 2$, using **Input** at the bottom of the screen.
- 3 Each straight line can be a different colour. Right-click on a line and choose a colour.
- 4 Find the **Slope** of each line.
- 5 Find the **y-intercept** of each line. Click on the right drop-down menu and use the mouse to zoom in on the y-intercept. Read off the value.



- 6 Save your GeoGebra file.
- 7 Record your results in a table as shown.

	Equation	Gradient	y-intercept
a			
b			
c			
d			

- 8 What do you notice about your results?
- 9 Repeat the steps above for each set of equations.

a $y = -4x$
 $4x + y + 1 = 0$
 $y = -4x - 10$

b $y = 2x + 3$
 $7x + y - 3 = 0$
 $0.2x - y + 3 = 0$

c $x + y + 1 = 0$
 $y = -x - 1$
 $x + y = -1$

- 10 For each set of lines drawn in question 9, complete a table as shown in Step 7 above.
- 11 What do you notice about each set of lines? Identify any key features of each set of graphs, such as gradients and y-intercepts.

NSW

Puzzle sheet

Equations in gradient form

MAT10NAPS00011

Technology worksheet

Excel spreadsheet:
Drawing linear graphs:
gradient and
y-intercept

MAT10NACT00039

3-04 The gradient–intercept equation $y = mx + b$

Summary

The equation of a straight line is $y = mx + b$, where m is the **gradient** and b is the **y-intercept**.

For this reason, $y = mx + b$ is also called the **gradient–intercept form** of a linear equation.

Example 12

Find the gradient and y -intercept of the line with equation:

a $y = -4x + 9$ b $y = 10 - 6x$ c $y = \frac{5x+4}{2}$ d $3x + 2y - 6 = 0$

Solution

a $y = -4x + 9$ is in the form $y = mx + b$.

$\therefore$ Gradient $m = -4$ and y -intercept $b = 9$.

b $y = 10 - 6x$ can be rewritten as $y = -6x + 10$.

$\therefore$ Gradient $m = -6$ and y -intercept $b = 10$.

c For $y = \frac{5x+4}{2} = \frac{5x}{2} + \frac{4}{2} = \frac{5x}{2} + 2$, gradient $m = \frac{5}{2}$ and y -intercept $b = 2$

d $3x + 2y - 6 = 0$ can be rearranged in the form $y = mx + b$.

$$3x + 2y - 6 - 3x = 0 - 3x$$

$$2y - 6 = -3x$$

$$2y - 6 + 6 = -3x + 6$$

$$2y = -3x + 6$$

$$y = \frac{-3x + 6}{2}$$

$$y = \frac{-3x}{2} + 3$$

$\therefore 3x + 2y - 6 = 0$ has gradient $m = -\frac{3}{2}$ and y -intercept $b = 3$.

Video tutorial

The gradient-intercept formula

MAT10NAVT10011

Example 13

Graph each linear equation by finding the gradient and y -intercept first.

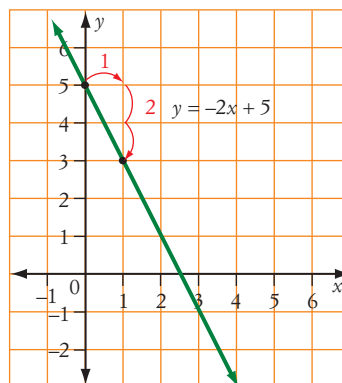
a $y = -2x + 5$ b $y = \frac{3}{4}x - 2$

Solution

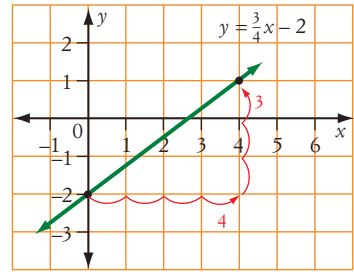
a $y = -2x + 5$ has a gradient of -2 and a y -intercept of 5 .

- Plot the y -intercept 5 on the y -axis.
- Make a gradient of -2 by moving **across** 1 unit (**run**) and **down** 2 units (**'negative' rise**) and marking the point at $(1, 3)$.
- Rule a line through this point and the y -intercept.

Don't forget to label the line with its equation ' $y = -2x + 5$ '



- b $y = \frac{3}{4}x - 2$ has a gradient of $\frac{3}{4}$ and a y-intercept of -2 .
- Plot the y-intercept -2 on the y-axis.
 - Make a gradient of $\frac{3}{4}$ by moving **across** 4 units (**run**) and **up** 3 units (**rise**) and marking the point at $(4, 1)$.
 - Rule a line through this point and the y-intercept.



Example 14

Which of the following lines is parallel to $y = -2x + 3$?

- A $y = 2x + 3$ B $y = -2x + 1$ C $y = -2x$ D $y = 5x + 3$

Solution

Parallel lines have the **same gradient**. The line $y = -2x + 3$ has the gradient $m = -2$.

- A $y = 2x + 3$ has gradient 2.
- B $y = -2x + 1$ has gradient -2 .
- C $y = -2x$ has gradient -2 .
- D $y = 5x + 3$ has gradient 5.

$\therefore$ The lines **B** ($y = -2x + 1$) and **C** ($y = -2x$) are parallel to $y = -2x + 3$.

Exercise 3-04 The gradient–intercept formula $y = mx + b$

See Example 12

- Find the gradient and y-intercept of each line below.

a $y = 3x - 2$	b $y = -2x + 7$	c $y = x + 4$	d $y = 9 - x$
e $y = \frac{3x}{4} + 6$	f $y = x$	g $y = \frac{x}{2} - 11$	h $y = \frac{2x + 18}{3}$
i $y = \frac{-24 - x}{3}$	j $y = 2(x - 3)$	k $11 - 3x = y$	l $\frac{2x - 7}{2} = y$
- Find the equation of a line with:

a a gradient of 2 and a y-intercept of 1	b a gradient of $\frac{3}{4}$ and a y-intercept of 2
c a gradient of -7 and a y-intercept of 5	d a gradient of $-\frac{2}{5}$ and a y-intercept of 3
e $m = -2, b = -3$	f $m = -3, b = \frac{1}{2}$

See Example 13

- Graph each linear equation by finding the gradient and y-intercept first.

a $y = 2x + 1$	b $y = 3x - 2$	c $y = 2x$	d $y = \frac{x}{2} - 1$
e $y = -2x + 3$	f $y = -\frac{3x}{4}$	g $y = \frac{-5x + 2}{2}$	h $y = \frac{3x - 20}{5}$
- Write the equation of a line with a gradient of 2 and a y-intercept of 0.

- 5 Select the lines that are parallel to the given line each time. There may be more than one answer.

See Example 14

a $y = x + 6$

A $y = 6x$

B $y = 6 - x$

C $y = x + 1$

D $y = 2x$

b $y = 3x + 10$

A $y = 10x + 3$

B $y = 3x - 1$

C $y = 1 - 3x$

D $y = 4 + 3x$

c $y = \frac{x}{2} + 5$

A $y = 2x - 1$

B $y = \frac{x+6}{2}$

C $y = 1 - \frac{x}{2}$

D $y = x + 2$

d $y = 6$

A $y = 2x + 6$

B $y = 6x$

C $y = -1$

D $y = 10$

e $y = 4x$

A $y = 4x - 2$

B $y = 4x + 3$

C $y = 4$

D $y = 1 - 4x$

f $x = 10$

A $y = 10$

B $y = 10x$

C $x = 2y$

D $x = -6$

- 6 For each set of linear equations, find a pair of equations whose graphs are parallel lines.

a $y = 4x + 3$

$y = x + 2$

$y = 4x - 6$

$y = 2x$

b $y = 5x + 1$

$3x - y + 7 = 0$

$y = 3x - 2$

$y = -5x + 2$

Mental skills 3

Maths without calculators

Time differences

- 1 Study each example.

- a What is the time difference between 11:40 a.m. and 6:15 p.m.?

From 11:40 a.m. to 5:40 p.m. = 6 hours

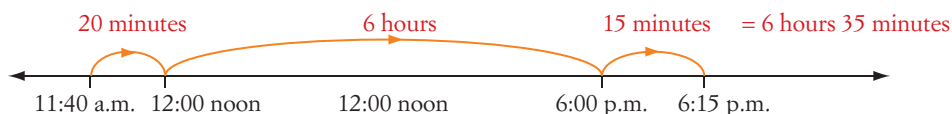
Count: '11:40, 12:40, 1:40, 2:40, 3:40, 4:40, 5:40'

From 5:40 a.m. to 6:00 p.m. = 20 min

From 6:00 p.m. to 6:15 p.m. = 15 min

5 hours + 20 min + 15 min = 6 hours 35 min

OR:



- b What is the time difference between 2030 and 0120?

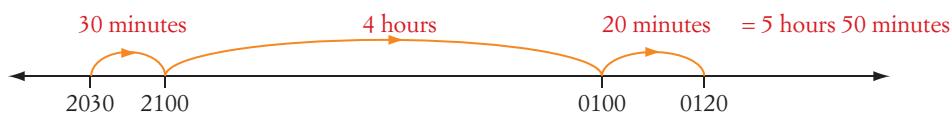
From 2030 to 0030 = 4 hours ($24 - 20 = 4$)

From 0030 to 0100 = 30 min

From 0100 to 0120 = 20 min

4 hours + 30 minutes + 20 minutes = 5 hours 50 minutes

OR:



2 Now find the time difference between:

a 11:10 a.m. and 7:40 p.m.

c 4:45 p.m. and 8:10 p.m.

e 1:05 p.m. and 12:30 a.m.

g 0425 and 0935

i 7:55 a.m. and 3:50 p.m.

b 6:20 p.m. and 12:00 midnight

d 2:35 a.m. and 10:50 a.m.

f 9:35 a.m. and 11:15 a.m.

h 1440 and 2025

j 2:40 p.m. and 10:20 p.m.

NSW

3-05

The general form of a linear equation $ax + by + c = 0$

Puzzle sheet

Linear equations code
puzzle

MAT10NAPS10011

Worksheet

Parallel and
perpendicular lines

MAT10NAWK00015

A linear equation written in **gradient–intercept form**, such as $y = -\frac{3}{4}x + 2$, can also be written in **general form** $3x + 4y - 8 = 0$. Note that, for the general form $ax + by + c = 0$, all of the terms on the left-hand side of the equation are written with no fractions, and only 0 is on the right-hand side. Sometimes the general form is neater and more convenient.

Example 15

Write each linear equation in general form.

a $y = 6x + 2$

b $y = -\frac{2}{3}x + 2$

c $y = 2x - \frac{3}{5}$

Solution

a $y = 6x + 2$

$$0 = 6x - y + 2$$

$$6x - y + 2 = 0$$

Subtracting y from both sides.

Swapping sides so that zero appears on the RHS.

b $y = -\frac{2}{3}x + 2$

$$3y = 3\left(-\frac{2}{3}x + 2\right)$$

$$= -2x + 6$$

$$2x + 3y = 6$$

$$2x + 3y - 6 = 0$$

Multiplying both sides by 3 to remove the fraction.

Adding $2x$ to both sides.

Subtracting 6 from both sides.

c $y = 2x - \frac{3}{5}$

$$5y = 5\left(2x - \frac{3}{5}\right)$$

$$= 10x - 3$$

$$0 = 10x - 5y - 3$$

$$10x - 5y - 3 = 0$$

Multiplying both sides by 5 to remove the fraction.

Subtracting $5y$ from both sides.

Swapping sides so that zero appears on the RHS.

Summary

The **general form of a linear equation** is written as $ax + by + c = 0$, where a , b and c are integers and a is positive.

Example 16

Find the gradient and y -intercept of the line whose equation is $5x + 2y - 10 = 0$.

Solution

Rewrite $5x + 2y - 10$ in the form $y = mx + b$.

$$5x + 2y - 10 = 0$$

$$2y - 10 = -5x$$

$$2y = -5x + 10$$

$$\frac{2y}{2} = \frac{-5x + 10}{2}$$

$$y = \frac{-5x}{2} + 5$$

Aim to have y on its own on the LHS of the equation.

Subtracting $5x$ from both sides.

Adding 10 to both sides.

Dividing both sides by 2.

$\therefore$ Gradient: $m = -\frac{5}{2}$, y -intercept: $b = 5$

Exercise 3-05 The general form of a linear equation $ax + by + c = 0$

1 Write each linear equation in general form.

a $y = x + 2$

b $y = 3x - 1$

c $y = 8 + 5x$

d $x + 2y = 3$

e $x - 2y = 6$

f $y = 8x + 2$

g $y + 3 = 6x$

h $2y = x - 6$

i $y = \frac{3}{5}x + 2$

2 Find the gradient and y -intercept of the line with each equation.

a $2x + y = 6$

b $8x - 2y = 10$

c $3x - 2y + 4 = 0$

d $y + 2x - 1 = 0$

e $2x + y + 5 = 0$

f $4x + 3y - 12 = 0$

3 Find the gradient, m , and the y -intercept, b , of the line with equation $x - 3y + 5 = 0$.

Select the correct answer **A**, **B**, **C** or **D**.

A $m = -1$, $b = 5$

B $m = \frac{1}{3}$, $b = \frac{5}{3}$

C $m = 1$, $b = -5$

D $m = \frac{1}{3}$, $b = -\frac{5}{3}$

4 Which statement is false about the line whose equation is $3x + y - 6 = 0$? Select **A**, **B**, **C** or **D**.

A The gradient is -3 .

B The y -intercept is -6 .

C The x -intercept is 2.

D It is parallel to the line $y = -3x$.

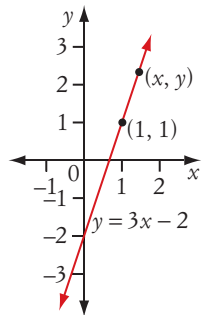
See Example 15

See Example 16

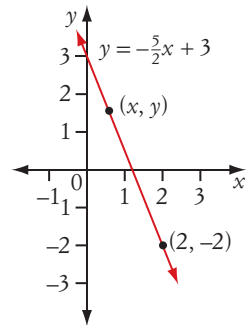
Stage 5.3

Investigation: The equation of a line given its gradient and a point

- 1 The graph shows the line $y = 3x - 2$.
- What is its gradient?
 - If (x, y) is any other point on the line, show that $m = \frac{y-1}{x-1}$.
 - Explain why $\frac{y-1}{x-1} = 3$
 - Hence show that $y - 1 = 3(x - 1)$ and simplify this equation to obtain $y = 3x - 2$.



- 2 The graph shows the line $y = -\frac{5}{2}x + 3$.
- What is its gradient?
 - If (x, y) is any other point on the line, show that $m = \frac{y+2}{x-2}$.
 - Explain why $\frac{y+2}{x-2} = -\frac{5}{2}$
 - Hence show that $y + 2 = -\frac{5}{2}(x - 2)$ and simplify this expression to obtain $y = -\frac{5}{2}x + 3$.

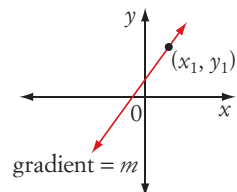


- 3 The equation of a line is given by $\frac{y-2}{x-7} = \frac{3}{4}$.
- What is the gradient of the line?
 - Can you give the coordinates of a point on this line by looking at its equation? Why?
- 4 Write the equation of a line which passes through the point $(-3, 5)$ and has a gradient equal to 2. Compare your result with other groups.
- 5 A line with gradient m passes through the point (x_1, y_1) .
- Show that $m = \frac{y-y_1}{x-x_1}$, where (x, y) is any other point on the line.
 - Explain why $y - y_1 = m(x - x_1)$.

3-06 The point–gradient form of a linear equation

There is a formula for finding the equation of a line if we know its gradient m and a point on the line (x_1, y_1) . Let (x, y) be any other point on the line.

Then $m = \frac{y-y_1}{x-x_1}$ or $y - y_1 = m(x - x_1)$.



Summary

The equation of a line with gradient m and which passes through the point (x_1, y_1) is:

$$y - y_1 = m(x - x_1)$$

It is called the **point-gradient form of a linear equation**.

Example 17

Find the equation of the line with a gradient of $\frac{2}{3}$ that passes through the point $(-2, 1)$.

Solution

$$m = \frac{2}{3}, x_1 = -2, y_1 = 1.$$

$$y - y_1 = m(x - x_1)$$

$$y - 1 = \frac{2}{3}[x - (-2)]$$

$$3(y - 1) = 2(x + 2)$$

$$3y - 3 = 2x + 4$$

$$0 = 2x - 3y + 7$$

$$2x - 3y + 7 = 0$$

In general form

Video tutorial

The point-gradient formula

MAT10NAVT10012

Example 18

Find the equation of the line passing through the points $(1, 3)$ and $(4, -3)$.

Solution

First find the gradient of the line by using the points $(1, 3)$ and $(4, -3)$.

$$m = \frac{-3 - 3}{4 - 1}$$

$$= \frac{-6}{3}$$

$$= -2$$

Now use $y - y_1 = m(x - x_1)$ with $m = -2$ and $(1, 3)$.

$$y - 3 = -2(x - 1)$$

$$= -2x + 2$$

$$y = -2x + 5 \quad \text{or} \quad 2x + y - 5 = 0 \text{ in general form}$$

OR: Using the other point $(4, -3)$ instead:

$$y - (-3) = -2(x - 4)$$

$$y + 3 = -2x + 8$$

$$y = -2x + 5 \quad \text{or} \quad 2x + y - 5 = 0 \text{ in general form}$$

Either of the points $(1, 3)$ or $(4, -3)$ can be used to find the equation of the line.

Video tutorial

The point-gradient formula

MAT10NAVT10012

Stage 5.3

Exercise 3-06

The point–gradient form of a linear equation

In this exercise, express all equations of lines in general form.

- 1 Find the equation of each line, given a point on the line and the gradient.

a $(2, 5)$, gradient 2

b $(-6, 4)$, gradient -1

c $(3, -8)$, gradient 4

d $(-1, -2)$, gradient $\frac{2}{3}$

e $(2, -8)$, gradient $-\frac{1}{5}$

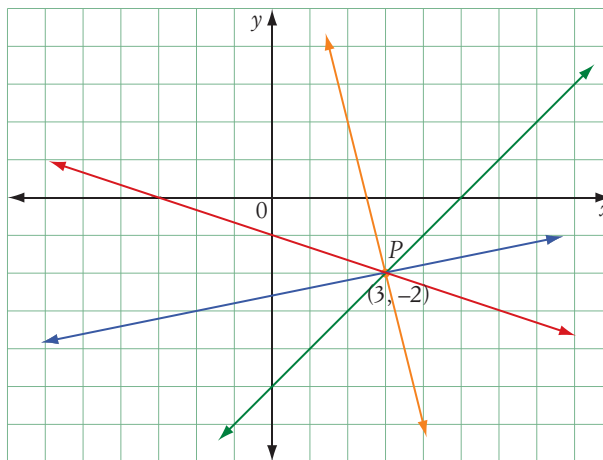
f $(-1, 7)$, gradient -3

g $(\frac{1}{2}, -3)$, gradient -4

h $(-4, -\frac{1}{2})$, gradient $\frac{3}{4}$

i $(-2, -6)$, gradient -2

- 2 Four lines a , b , c and d intersect at $P(3, -2)$. The gradients of a , b , c and d are 1 , $-\frac{1}{3}$, -4 and $\frac{1}{5}$ respectively.



- a Copy the diagram and correctly label the lines a , b , c and d .

- b Find the equation of each line.

- 3 Find the equation of the line passing through each pair of points.

a $(7, 3)$ and $(10, 6)$

b $(8, 10)$ and $(-2, 2)$

c $(-1, 3)$ and $(5, 8)$

d $(2, -2)$ and $(-1, 6)$

e $(4, -3)$ and $(6, -6)$

f $(-1, -2)$ and $(2, 3)$

g $(-10, 2)$ and $(1, -4)$

h $(-3, 6)$ and $(1, 2)$

i $(-4, -9)$ and $(-1, -5)$

- 4 Two lines, k and l , intersect at $(-1, 4)$. Line k has a gradient of $-\frac{1}{2}$, while line l has a gradient of 3. Find the equations of lines k and l .
- 5 Find the equation of a line with a gradient of -4 and an x -intercept of 5.
- 6 A line passes through the y -axis at $(0, 6)$ and has a gradient of $\frac{5}{7}$. What is its equation?
- 7 A line with a gradient of $\frac{2}{3}$ passes through the midpoint of $(5, 6)$ and $(1, 10)$. Find the equation of the line.
- 8 A line with a gradient of $-\frac{3}{5}$ passes through the midpoint of $(-8, -2)$ and $(-2, 20)$. Find its equation.

See Example 18

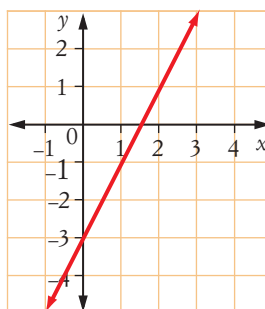
- 9 a The gradient–intercept form of a line, $y = mx + b$, can also be used to find the equation of a line given its gradient and a point on the line. Use $y = mx + b$ to find the equation of the line with gradient 2 that passes through the point (2, 5).
- b Compare your equation with your answer to question 1a.
- 10 a The point–gradient formula can be converted to a formula for finding the equation of a line passing through two points (x_1, y_1) and (x_2, y_2) . Prove that the ‘two-point formula’ is $\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$.
- b Use the two-point form to find the equation of a line passing through the points (7, 3) and (10, 6).
- c Compare your equation with your answer to question 3a.

3-07 Finding the equation of a line

NSW

Example 19

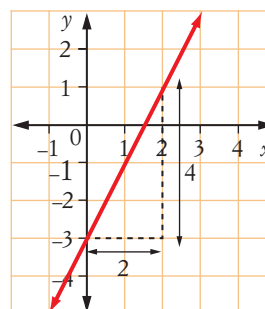
Find the equation of the line.



Solution

Select two points on the line to find the gradient, say (0, -3) and (2, 1).

$$\text{Gradient } m = \frac{\text{rise}}{\text{run}} = \frac{4}{2} = 2$$



y-intercept: $b = -3$

$\therefore$ The equation of the line is $y = 2x - 3$.

We can check that this equation is correct for any point on the line, say (3, 3).

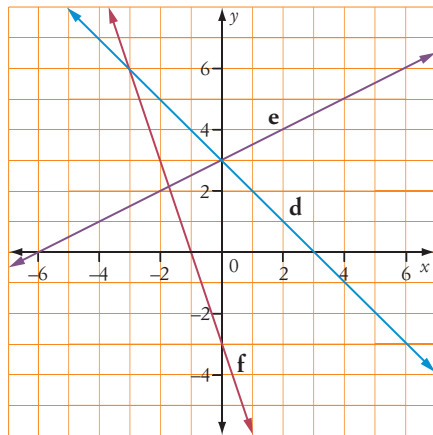
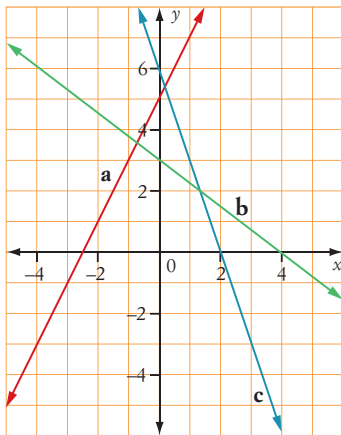
When $x = 3$, $y = 2 \times 3 - 3 = 3$.

from the graph

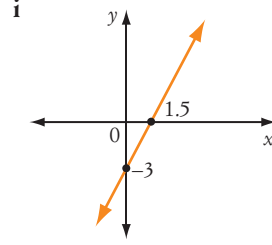
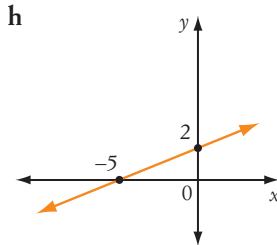
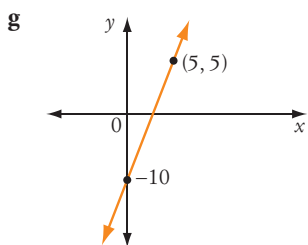
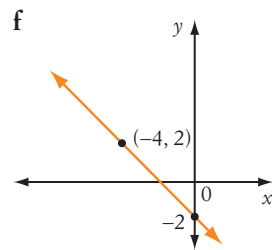
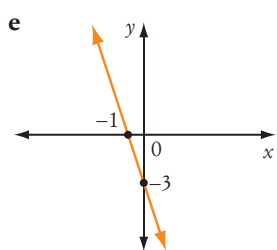
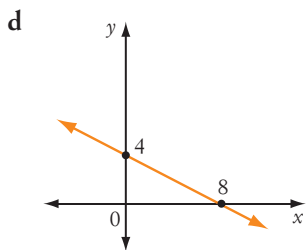
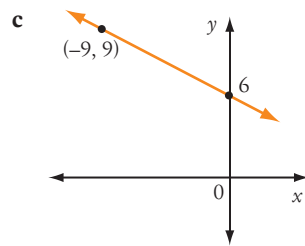
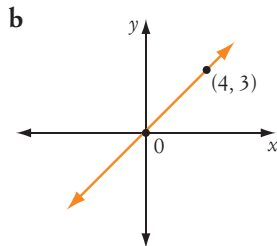
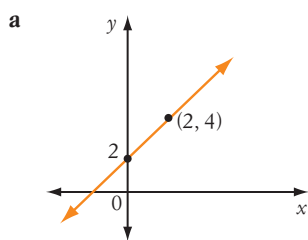
$$y = mx + b$$

Exercise 3-07 Finding the equation of a line

See Example 19 1 Find the equation of each line.



2 Find the equation of each line.



Investigation: Sausage sizzle

A local football club is organising a sausage sizzle on Saturday to raise money to buy new equipment. It costs \$25 to hire a gas bottle to run the barbecue and each sandwich costs \$0.90 to make.



Fairfax Syndication/Craig Abraham

- 1 Copy and complete this table below to show the cost of making sausage sandwiches. Include the cost of hiring the gas bottle.

No. of sandwiches (x)	0	10	20	30	40	50	60	70	80	90	100
Cost (\$ y)	25	34									

- 2 Find the linear equation (formula) for y that represents the cost of making x sausage sandwiches.
- 3 Use an appropriate scale to construct a graph that shows the cost of making from $x = 0$ to $x = 100$ sandwiches. Label your axes and give your graph an appropriate title.
- 4 How much will it cost to make 35 sausage sandwiches?
- 5 How many sandwiches can be made for \$98.80?
- 6 How much would it cost to make 120 sausage sandwiches?
- 7 **a** If the club sold 75 sausage sandwiches for \$3 each, how much money would they take?
b How much **profit** would the club make?

3-08

Equations of parallel and perpendicular lines



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Puzzle sheet

Linear equations
match-up

MAT10NAPS10012

Worksheet

Writing equations
of lines

MAT10NAWK10013

Puzzle sheet

Equations of
parallel lines

MAT10NAPS00013

Technology

GeoGebra:
Perpendicular lines

MAT10NATC00005

Video tutorial

Coordinate geometry

MAT10NAVT00005

Summary

If two lines with gradients m_1 and m_2 are parallel, then $m_1 = m_2$.

If two lines with gradients m_1 and m_2 are perpendicular, then $m_1 \times m_2 = -1$ or $m_2 = -\frac{1}{m_1}$.

Example 20

Find the equation of the line parallel to $y = 8 - 3x$ that passes through the point $(-1, 6)$.

Solution

For $y = 8 - 3x$ (or $y = -3x + 8$), the gradient is $m = -3$.

A line parallel to $y = 8 - 3x$, will also have $m = -3$.

Using the point–gradient formula $y - y_1 = m(x - x_1)$ with $m = -3$ and point $(-1, 6)$:

$$\begin{aligned} y - 6 &= -3[x - (-1)] \\ &= -3(x + 1) \\ &= -3x - 3 \\ y &= -3x + 3 \end{aligned}$$

OR: Using the gradient–intercept equation $y = mx + b$:

$$y = -3x + b$$

To find the value of b , substitute the point $(-1, 6)$ into the equation:

$$\begin{aligned} y &= -3x + b \\ 6 &= -3 \times (-1) + b & x = -1, y = 6 \\ 6 &= 3 + b \\ b &= 3 \end{aligned}$$

$\therefore$ The equation is $y = -3x + 3$.

Example 21

Find the equation of the line perpendicular to $3x - 4y + 6 = 0$, which passes through the point $(5, 4)$.

Solution

To find the gradient of $3x - 4y + 6 = 0$, first convert it to the form $y = mx + b$:

$$\begin{aligned} 3x - 4y + 6 &= 0 \\ 3x + 6 &= 4y \\ 4y &= 3x + 6 \\ y &= \frac{3x + 6}{4} \\ y &= \frac{3}{4}x + \frac{3}{2} \end{aligned}$$

$$y = mx + b$$

$$\therefore \text{Gradient} = \frac{3}{4}$$

$$\begin{aligned} \therefore \text{Gradient of perpendicular line} &= \frac{-1}{\left(\frac{3}{4}\right)} \\ &= -\frac{4}{3} \end{aligned}$$

The negative reciprocal of $\frac{3}{4}$.

Using the point–gradient formula $y - y_1 = m(x - x_1)$

with $m = -\frac{4}{3}$ and point (5, 4):

$$y - 4 = -\frac{4}{3}(x - 5)$$

In general form

$$3(y - 4) = -4(x - 5)$$

$$3y - 12 = -4x + 20$$

$$4x + 3y - 32 = 0$$

OR: Using the gradient–intercept equation $y = mx + b$:

$$y = -\frac{4x}{3} + b$$

To find the value of b , substitute the point (5, 4) into the equation.

$$x = 5, y = 4$$

$$4 = \left(-\frac{4}{3}\right) \times 5 + b$$

$$= -\frac{20}{3} + b$$

$$4 + \frac{20}{3} = b$$

$$b = \frac{32}{3}$$

$$\therefore \text{The equation is } y = -\frac{4x}{3} + \frac{32}{3} \text{ or } y = \frac{-4x + 32}{3}$$

or, converting to the neater general form:

$$3y = -4x + 32$$

$$4x + 3y - 32 = 0$$

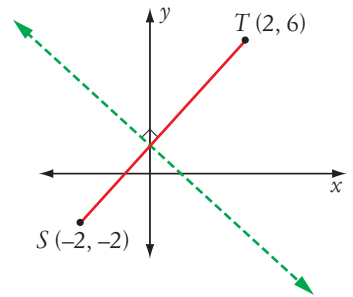
Exercise 3-08 Equations of parallel and perpendicular lines

- 1 Find the equation of the line that is parallel to:
 - a $y = 2x + 9$ and has a y -intercept of 4
 - b $y = 3x$ and has a x -intercept of -2
 - c $y = 5 - \frac{x}{2}$ and passes through $(-1, 6)$
 - d $2x - y = 6$ and passes through $(5, -2)$
 - e $y = -5x - 8$ and passes through the midpoint of $(3, -10)$ and $(-5, -6)$
 - f $2y = x - 3$ and passes through $(6, -7)$
- 2 Find the equation of a line that is perpendicular to:
 - a $y = \frac{x}{2}$ and has a y -intercept of -2
 - b $y = -5x$ and has a x -intercept of 1
 - c $y = 3x - 1$ and passes through the x -axis at 4
 - d $y = \frac{x-6}{3}$ and passes through $(1, -6)$
 - e $x + y - 6 = 0$ and passes through $(-4, 2)$
 - f $3x - y - 9 = 0$ and passes through $(-10, -7)$

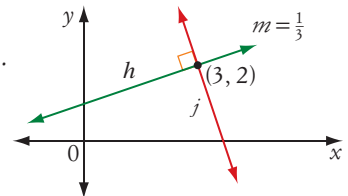
See Example 20

See Example 21

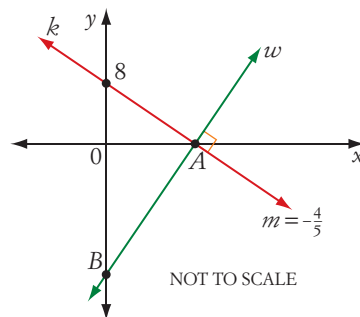
- 3 a Find the gradient of interval ST in the diagram on the right.
 b Find the midpoint of ST .
 c The dotted line is perpendicular to ST and passes through its midpoint. What is its gradient?
 d Find the equation of the dotted line, in the form $y = mx + b$.



- 4 a Find the equation of line h in the diagram.
 b Find the gradient of line j (which is perpendicular to line h).
 c Find the equation of line j .



- 5 a Find the equation of line k .
 b Find the coordinates of point A .
 c Find the gradient of line w .
 d Find the equation of line w .
 e Find the coordinates of point B .



Stage 5.3
NSW

Worksheet

Geometry problems
using coordinates

MAT10NAWK10204

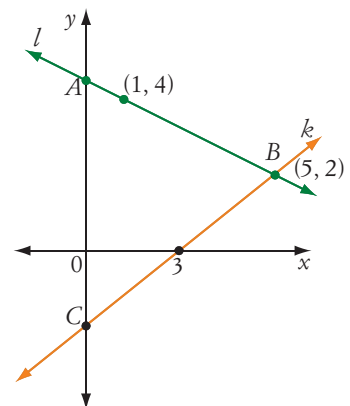
3-09 Coordinate geometry problems

A variety of problems can be solved by applying coordinate geometry methods, including proving geometric properties of triangles and quadrilaterals.

Example 22

Lines k and l are shown in the diagram. Find:

- a the equation of line k
 b the equation of line l
 c the coordinates of point A
 d the coordinates of point C
 e the area of the triangle ABC



Solution

- a** Line k passes through $(3, 0)$ and $(5, 2)$.

$$m = \frac{2 - 0}{5 - 3} = \frac{2}{2} = 1$$

Using the point–gradient formula $y - y_1 = m(x - x_1)$:

$$y - 2 = 1(x - 5)$$

$$= x - 5$$

$$y = x - 3$$

(or $x - y - 3 = 0$ in general form)

using the point $(5, 2)$

- b** Line l passes through $(1, 4)$ and $(5, 2)$.

$$\therefore m = \frac{2 - 4}{5 - 1} = \frac{-2}{4} = -\frac{1}{2}$$

$$y - 4 = -\frac{1}{2}(x - 1)$$

$$2(y - 4) = -1(x - 1)$$

$$2y - 8 = -x + 1$$

$$x + 2y - 9 = 0$$

using the point $(1, 4)$

- c** A is the y -intercept of line l .

Substitute $x = 0$ into $x + 2y - 9 = 0$.

$$0 + 2y - 9 = 0$$

$$2y = 9$$

$$y = \frac{9}{2}$$

$$= 4.5$$

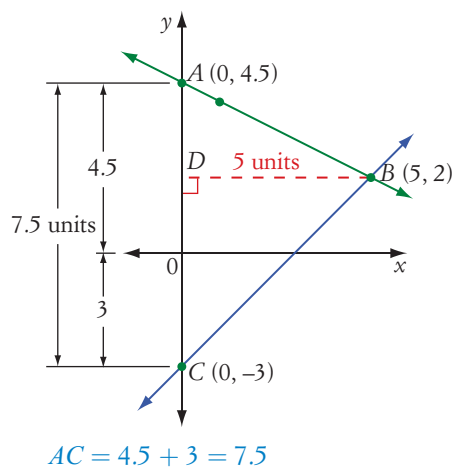
$\therefore A$ is $(0, 4.5)$

- d** C is the y -intercept of line k .

The y -intercept of $y = x - 3$ is -3 .

$\therefore C$ is $(0, -3)$

- e** Area of $\triangle ABC = \frac{1}{2} \times \text{base} \times \text{height}$
 $= \frac{1}{2} \times AC \times BD$
 $= \frac{1}{2} \times 7.5 \times 5$
 $= 18.75 \text{ units}^2$

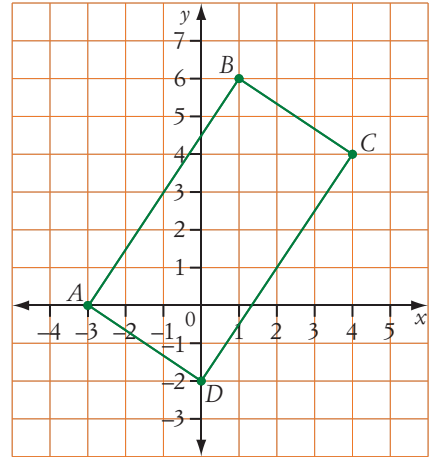


Stage 5.3

Example 23

$A(-3, 0)$, $B(1, 6)$, $C(4, 4)$ and $D(0, -2)$ are the vertices of a rectangle.

- By finding the lengths of AC and BD , show that the diagonals of the rectangle are equal.
- Find the midpoints of the diagonals AC and BD .
- Show that the diagonals of the rectangle bisect each other.



Solution

- a** $A(-3, 0)$, $C(4, 4)$

$$\begin{aligned} AC &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{[4 - (-3)]^2 + (4 - 0)^2} \\ &= \sqrt{65} \end{aligned}$$

- $B(1, 6)$, $D(0, -2)$

$$\begin{aligned} BD &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(0 - 1)^2 + (-2 - 6)^2} \\ &= \sqrt{65} \end{aligned}$$

$$\therefore AC = BD$$

$\therefore$ The diagonals are equal.

- b** Midpoint of $AC \equiv \left(\frac{-3 + 4}{2}, \frac{0 + 4}{2} \right)$
 $\equiv \left(\frac{1}{2}, 2 \right)$

$$\begin{aligned} \text{Midpoint of } BD &\equiv \left(\frac{1 + 0}{2}, \frac{6 + (-2)}{2} \right) \\ &\equiv \left(\frac{1}{2}, 2 \right) \end{aligned}$$

- c** The midpoints of both diagonals are the same, so the diagonals bisect each other.

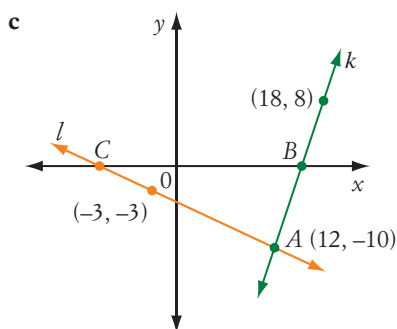
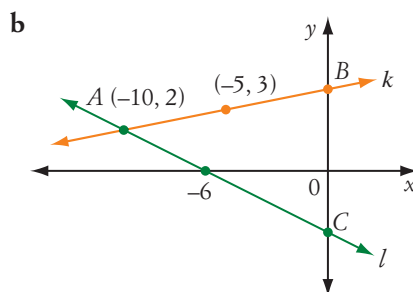
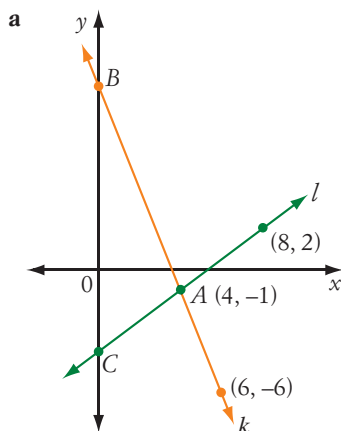
Exercise 3-09 Coordinate geometry problems

Stage 5.3

See Example 22

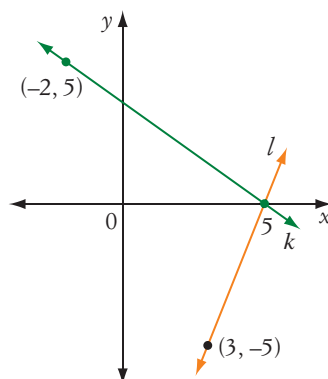
1 For each graph, find:

- i the equation of line k
- ii the equation of line l
- iii the coordinates of point B
- iv the coordinates of point C
- v the area of $\triangle ABC$



2 For this graph, find:

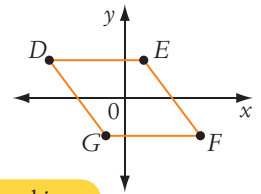
- a the equation of line l
- b the equation of line k
- c w if the point $(7, w)$ lies on l
- d t if the point $(t, 11)$ lies on k



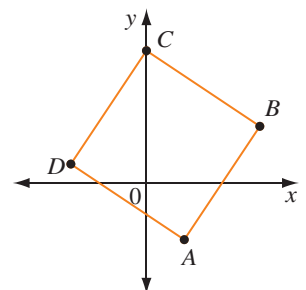
Stage 5.3

See Example 23

- 3 The vertices of a rhombus are $D(-4, 2)$, $E(1, 2)$, $F(4, -2)$ and $G(-1, -2)$.
- Show that all sides of the rhombus are equal.
 - By finding their gradients, show that the opposite sides of the rhombus are parallel.
 - Show that the diagonals DF and GE of the rhombus cross at right angles.
 - Find the midpoints of the diagonals DF and GE . Do the diagonals bisect each other? Give reasons.
 - List the properties of a rhombus that have been demonstrated in this question.
- 4 A quadrilateral has vertices $P(-7, 2)$, $Q(2, -7)$, $R(5, -4)$ and $S(-4, 5)$.
- Draw a diagram showing the given information.
 - Find the lengths of PR and QS in surd form.
 - Find the midpoints of PR and QS .
 - Is PR perpendicular to QS ? Why?
 - What type of quadrilateral is $PQRS$? Explain.
- 5 A quadrilateral has vertices $C(2, 6)$, $D(-5, 2)$, $E(-1, -5)$ and $F(6, -1)$.
- Draw a diagram showing the given information.
 - Find the length of each diagonal.
 - Find the midpoint of each diagonal.
 - Show that the diagonals are perpendicular.
 - What type of quadrilateral is $CDEF$? Explain.
- 6 A quadrilateral has vertices $B(1, 7)$, $C(-5, 2)$, $D(2, -2)$ and $E(8, 3)$.
- Find the length of each side.
 - Find the gradient of each side.
 - Find the midpoint of each diagonal.
 - What type of quadrilateral is $BCDE$?
- 7 A square has vertices $A(2, -3)$, $B(6, 3)$, $C(0, 7)$ and $D(-4, 1)$.
- Show that its diagonals are equal.
 - Show that its diagonals bisect each other at right angles.
 - Hence explain why $ABCD$ is a square.
- 8 Show that the diagonals of the rhombus with vertices $K(4, 2)$, $L(-1, 4)$, $M(1, -1)$ and $N(6, -3)$ bisect each other at right angles.
- 9 Show that $W(-5, -4)$, $X(4, -1)$, $Y(6, 6)$ and $Z(-3, 3)$ are the vertices of a parallelogram by finding the gradients of each side and showing that the opposite sides are parallel.



'Show that' means all working out must be provided to fully explain your answer

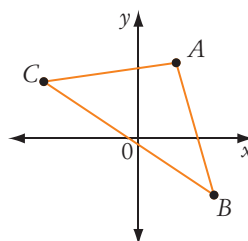


10 $J(-3, 0)$, $K(3, -2)$, $L(1, 3)$ and $M(-5, 5)$ are the vertices of a quadrilateral.

- Find the gradient of each side.
- What type of quadrilateral is $JKLM$? Explain.

11 $A(2, 4)$, $B(4, -3)$ and $C(-5, 3)$ are the vertices of a triangle.

- X and Y are the midpoints of AB and AC respectively.
Find the coordinates of X and Y .
- Find the gradients of XY and CB . Is it true that $XY \parallel CB$?
- Find the lengths of XY and CB and, hence, show that $CB = 2XY$.



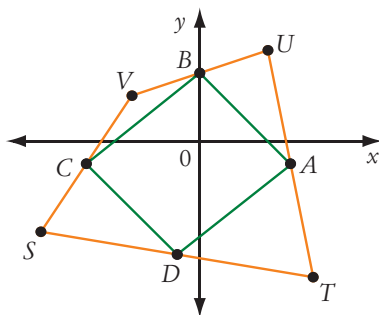
12 $C(-7, -6)$, $N(1, -3)$, $T(4, 5)$ and $W(-4, 2)$ are the vertices of a quadrilateral.

- Show that its diagonals bisect each other at right angles.
- Hence, what type of quadrilateral is $CNTW$?

13 What type of a quadrilateral is formed by the points $H(-6, 2)$, $I(6, -4)$, $J(4, 2)$ and $K(-2, 5)$?

14 Show that the points $S(5, -6)$, $T(6, 0)$, $W(-6, 2)$ and $X(-7, -4)$ are the vertices of a rectangle.

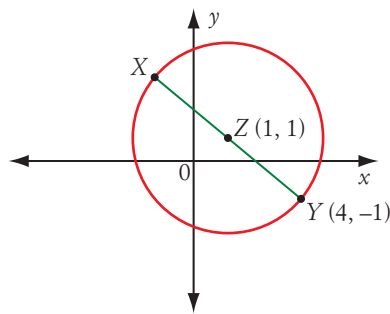
15 The points $T(5, -6)$, $U(3, 4)$, $V(-3, 2)$ and $S(-7, -4)$ are the vertices of a quadrilateral.



- Find the coordinates of the midpoint of each side.
 - What type of quadrilateral is formed when the midpoints of the sides are joined? Explain.
- 16 For the points $L(-8, -1)$, $M(1, 2)$ and $N(10, 5)$:
- Find the gradients of LM , LN and MN .
 - What can you say about the three points L , M and N ?

Power plus

- 1 A line is drawn through the points $A(0, -2)$ and $B(3, 0)$. The x -coordinate of a point C on AB is 9. Find:
 - a the gradient of AB
 - b the equation of AB
 - c the y -coordinate of C .
- 2 The point $(-1, 6)$ lies on the line $kx + 3y - 13 = 0$, where k is a constant number. Find k .
- 3 $Z(-1, 3)$ is the midpoint of the interval joining $A(-4, 7)$ and B . Find the coordinates of B .
- 4 The circle has XY as a diameter and centre Z . What are the coordinates of X ?



- 5
 - a Find the gradient of any line parallel to $3x + 2y = 4$.
 - b Find the equation of the line that passes through the point $(0, -1)$ and is parallel to $3x + 2y = 4$.
- 6 $A(-1, 4)$, $B(4, 6)$, $C(2, 7)$ and D are the vertices of a parallelogram. Find the coordinates of D .

Language of maths

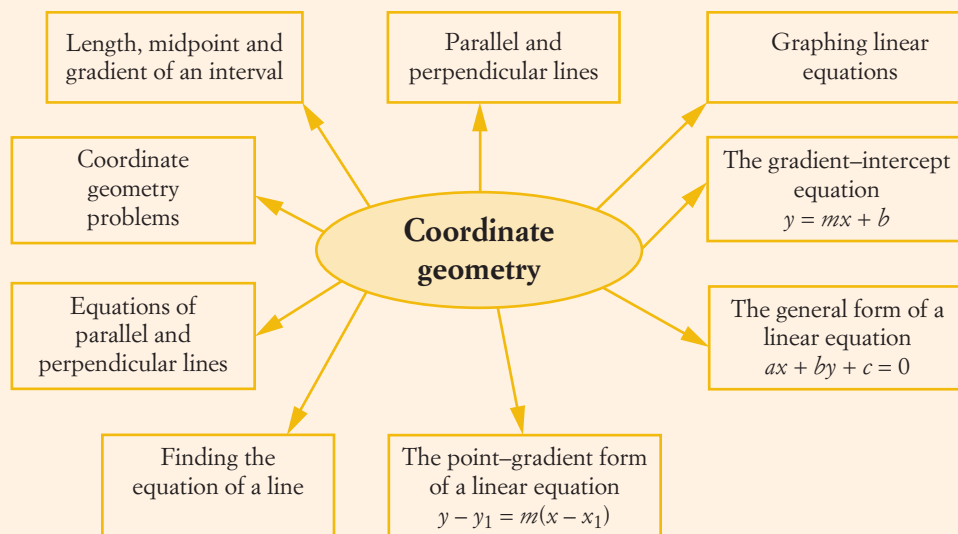
axes	distance	exact answer	general form
gradient	gradient–intercept form	horizontal	interval
length	linear equation	midpoint	parallel
perpendicular	point–gradient form	reciprocal	rise
run	surd	vertical	x-axis
x-intercept	y-axis	y-intercept	

- 1 What is the difference between the **y-axis** and the **y-intercept**?
- 2 When finding the length of an interval on a number plane, what is meant by an **exact answer**?
- 3 What measurement is the fraction given by the vertical rise of a line divided by the horizontal run?
- 4 What is the everyday meaning of the word **intercept**? Look it up in a dictionary.
- 5 What is the property of the gradients of perpendicular lines?
- 6 What form of a linear equation is $ax + by + c = 0$?

Topic overview

- How can you find the gradient of a line?
- When do you use the formula $y - y_1 = m(x - x_1)$?
- How can you test whether a pair of lines are perpendicular?
- What parts of this topic did you find difficult?

Copy and complete this mind map of the topic, adding detail to its branches and using pictures, symbols and colour where needed. Ask your teacher to check your work.



Chapter 3 revision

See Exercise 3-01

- 1 An interval is formed by joining the points $K(5, 6)$ and $L(-7, 2)$.
 - a Find, correct to one decimal place, the length of interval KL .
 - b Find the midpoint of KL .
 - c Find the gradient of KL .

See Exercise 3-01

- 2 The vertices of a quadrilateral $HJKL$ are $H(-8, -5)$ $J(-1, -2)$ $K(2, 5)$ $L(-5, 2)$.
 - a Find the exact length of the sides of the quadrilateral.
 - b Find the gradient of each side of $HJKL$.
 - c Find the exact length of the diagonals HK and JL .
 - d What type of quadrilateral is $HJKL$?

Stage 5.3

See Exercise 3-01

- 3 Find, correct to the nearest degree, the angle of inclination of a line with gradient:
 - a 3
 - b $\frac{5}{4}$
 - c -1
 - d $-\frac{2}{3}$

See Exercise 3-02

- 4 A line passes through the points $V(8, -1)$ and $W(10, -2)$. What is the gradient of a line:
 - a parallel to VW ?
 - b perpendicular to VW ?

See Exercise 3-03

- 5 Graph each linear equation on a number plane.
 - a $y = -5x - 1$
 - b $x + 2y = 16$
 - c $3x + 4y - 12 = 0$

See Exercise 3-03

- 6 Test which of the following points lie on the line of $3x + y = 2$. Select the correct answer **A, B, C** or **D**.
 - A (1, 0)
 - B (2, 4)
 - C (-1, 5)
 - D (-1, -5)

See Exercise 3-03

- 7 What is the equation of the line through $(-2, 3)$ and parallel to the x -axis? Select the correct answer **A, B, C** or **D**.
 - A $x = -2$
 - B $x = 3$
 - C $y = -2$
 - D $y = 3$

See Exercise 3-04

- 8 Write the gradient, m , and y -intercept, b , for each linear equation.
 - a $y = 2x - 10$
 - b $y = 4x + 3$
 - c $y = \frac{4 - 3x}{8}$

See Exercise 3-05

- 9 Convert each equation to general form $ax + by + c = 0$.
 - a $y = 3x + 5$
 - b $y = \frac{2x}{5} - 10$
 - c $x = 3y + 6$

See Exercise 3-05

- 10 Rewrite each equation in the form $y = mx + b$, then state the value of the gradient, m , and the y -intercept, b .
 - a $x - y + 2 = 0$
 - b $2x - 8y + 8 = 0$
 - c $3x + y - 9 = 0$

Stage 5.3

See Exercise 3-06

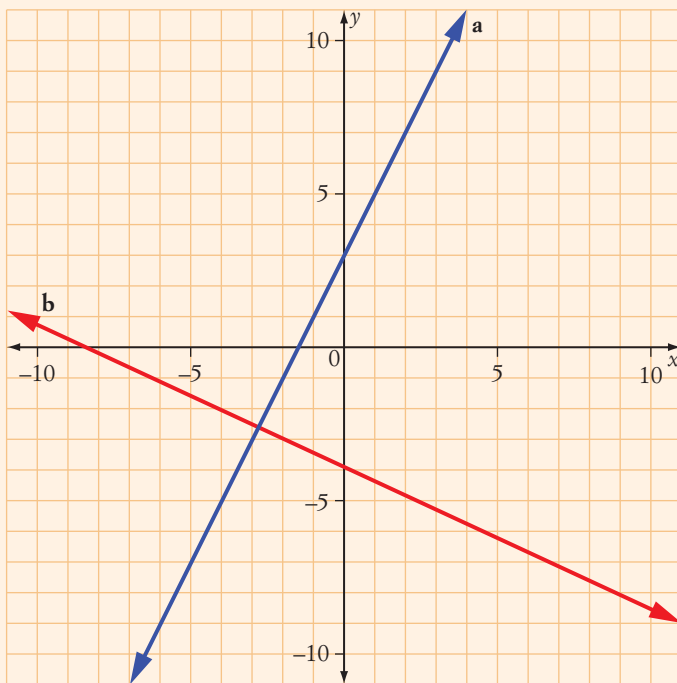
- 11 Find, in general form, the equation of a line which passes through the point:
 - a (5, 5) and has a gradient of -3
 - b (-1, 8) and has a gradient of $\frac{2}{5}$

See Exercise 3-06

- 12 Find, in general form, the equation of a line which passes through the points:
 - a (10, 2) and (5, -1)
 - b (-6, 3) and (-2, -1)

13 Find the equation of each line.

See Exercise 3-07



See Exercise 3-08

14 Find the equation of a line that is:

a parallel to $y = 3x + 1$ and passes through the x-axis at 2

b perpendicular to $y = \frac{x}{2}$ and passes through the origin.

15 The line $3x - 8y + 10 = 0$ and another line intersect at right angles at the point $(10, 5)$. Find the equation of the other line.

16 $L(-1, 2)$, $M(4, 5)$, $N(1, 10)$ and $P(-4, 7)$ are the vertices of a quadrilateral. Show that $LMNP$ is a square.

Stage 5.3

See Exercise 3-08

See Exercise 3-09



4

Measurement and geometry

Surface area and volume

Some theme parks have wave pools, which are big swimming pools that simulate the movement of the water at a beach. A large volume of water is quickly released into one end of the pool, which produces a large wave that moves across the pool to the other end. The excess water in the pool is recycled so that it can be used to produce more waves.



Shutterstock.com/CPhoto

Chapter outline

	Proficiency strands				
4-01 Surface area of a prism	U	F	PS	R	C
4-02 Surface area of a cylinder	U	F	PS	R	
4-03 Surface area of a pyramid*	U	F	PS	R	C
4-04 Surface areas of cones and spheres*	U	F	PS	R	C
4-05 Surface areas of composite solids	U	F	PS	R	C
4-06 Volumes of prisms and cylinders	U	F	PS	R	
4-07 Volumes of pyramids, cones and spheres*	U	F	PS	R	
4-08 Volumes of composite solids*	U	F	PS	R	
4-09 Areas of similar figures*	U	F	PS	R	C
4-10 Surface areas and volumes of similar solids*	U	F	PS	R	C

*STAGE 5.3

Wordbank

cross-section A 'slice' of a solid, taken across the solid rather than along it

curved surface area The area of the curved surface of a solid such as a cylinder or sphere. The curved surface of a cylinder is a rectangle when flattened.

hemisphere Half a sphere

pyramid A solid with a polygon for a base and triangular faces that meet at a point called the apex

sector A region of a circle cut off by two radii, shaped like a piece of pizza

slant height The height of a pyramid or cone from its apex (top) to its base along a side face rather than its perpendicular height

In this chapter you will:

- solve problems involving the surface areas and volumes of right prisms
- calculate the surface area and volume of cylinders and solve related problems
- (STAGE 5.3) solve problems involving surface area and volume of right pyramids, right cones, spheres and related composite solids 
- calculate the surface areas and volumes of composite solids
- (STAGE 5.3) investigate ratios of areas of similar figures
- (STAGE 5.3) investigate ratios of surface areas and volumes of similar solids

SkillCheck

Worksheet

StartUp assignment 3

MAT10MGWK10015

Skillsheet

Solid shapes

MAT10MGSS10007

Skillsheet

What is volume?

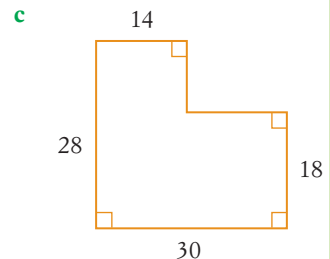
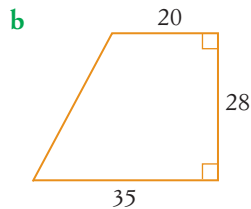
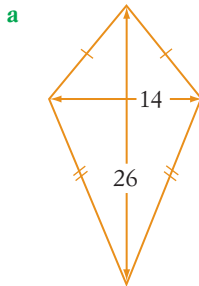
MAT10MGSS10008

Puzzle sheet

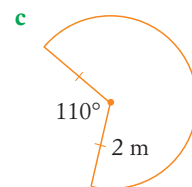
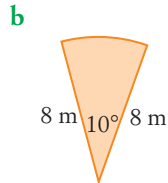
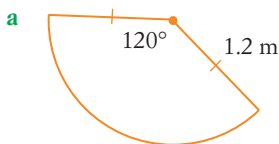
Area

MAT10MGPS00010

1 Calculate the area of each shape. All measurements are in centimetres.



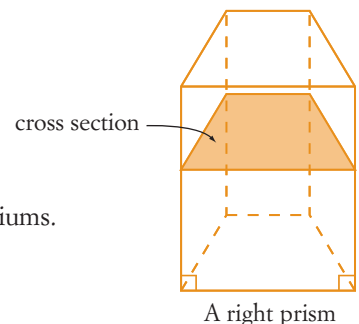
2 Find, correct to two decimal places, the area of each sector.



4-01 Surface area of a prism

A **cross-section** of a solid is a 'slice' of the solid cut *across* it, parallel to its end faces, rather than along it. A **prism** has the same (uniform) cross-section along its length, and each cross-section is a **polygon** (with straight sides).

This trapezoidal prism has identical cross-sections that are trapeziums.



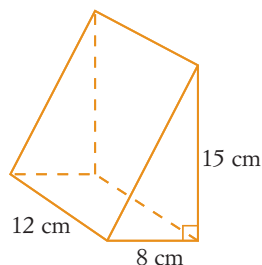
Summary

The **surface area** of a solid is the total area of all the faces of the solid. To calculate the surface area of a solid, find the area of each face and add the areas together.

Example 1

Find the surface area of the prism.

Closed triangular prism



Solution

The open prism has five faces: two identical triangles (front and back) and three different rectangles. Using Pythagoras' theorem to find m , the hypotenuse of the triangle:

$$m^2 = 8^2 + 15^2$$

$$= 289$$

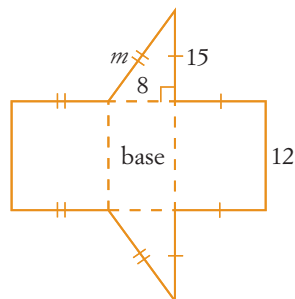
$$m = \sqrt{289}$$

$$= 17$$

Surface area = 2 triangles + 3 rectangles

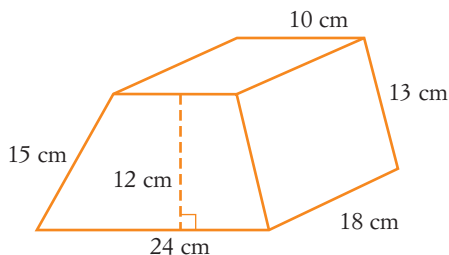
$$= 2 \times \left(\frac{1}{2} \times 8 \times 15 \right) + (17 \times 12) + (8 \times 12) + (15 \times 12)$$

$$= 600 \text{ cm}^2$$



Example 2

Calculate the surface area of this trapezoidal prism.

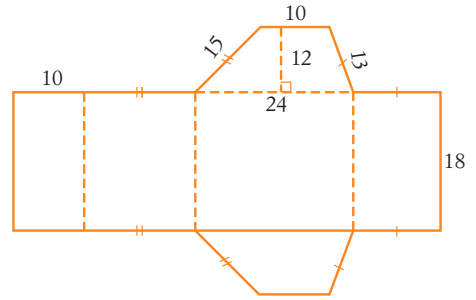


Solution

This trapezoidal prism has 6 faces: two identical trapeziums (front and back) and four different rectangles.

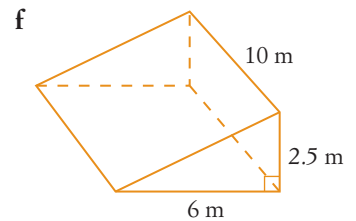
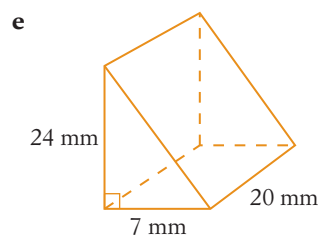
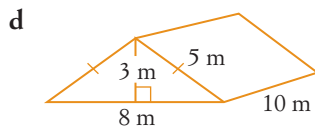
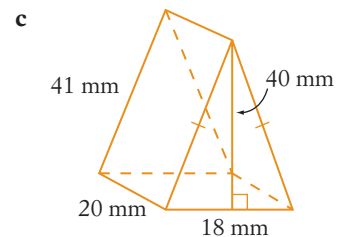
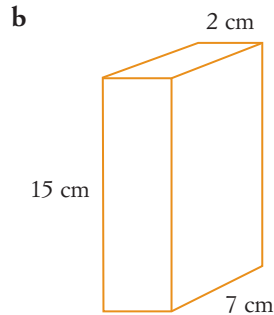
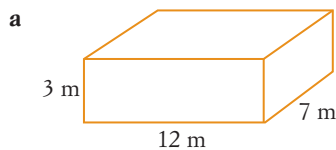
$$\begin{aligned}\text{Area of each trapezium} &= \frac{1}{2} \times (10 + 24) \times 12 \\ &= 204 \text{ cm}^2\end{aligned}$$

$$\begin{aligned}\text{Surface area} &= (2 \times 204) + (18 \times 10) + (18 \times 15) + (18 \times 24) + (18 \times 13) \\ &= 1524 \text{ cm}^2\end{aligned}$$

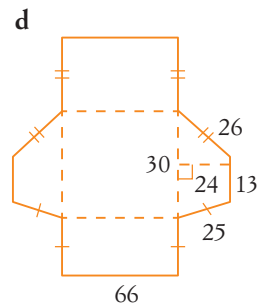
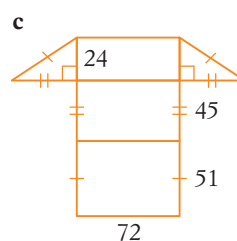
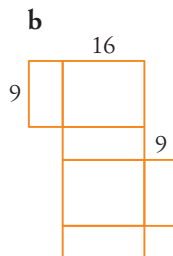
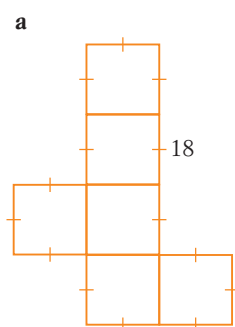


Exercise 4-01 Surface area of a prism

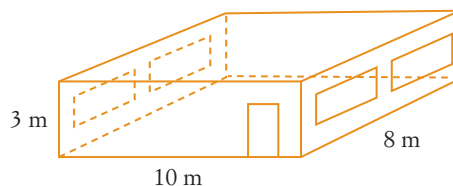
See Example 1 1 Find the surface area of each prism.



2 Name the prism that each net represents, then calculate the surface area of the prism. All lengths are in metres.

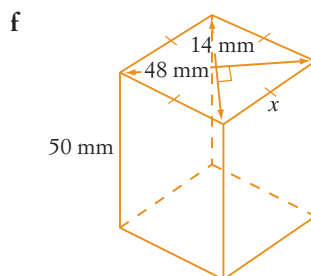
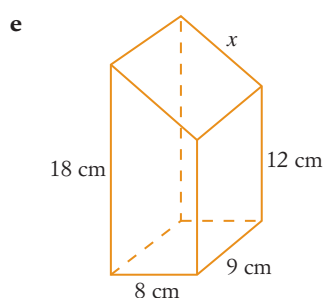
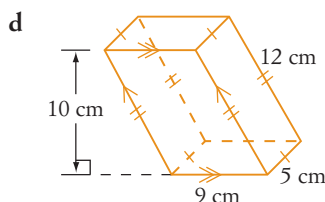
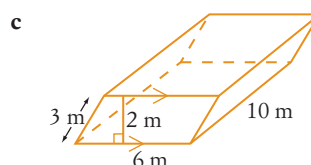
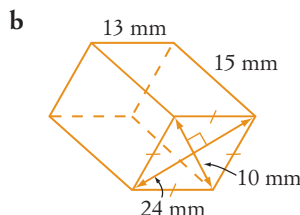
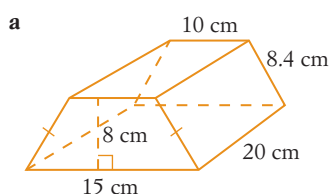


- 3 This classroom is being renovated. Find:
- the area of the floor to be carpeted and the cost, at \$105 per square metre
 - the ceiling and wall area to be painted if the room contains four windows, each 2.5 m by 1.5 m, and a doorway 2 m by 0.8 m.

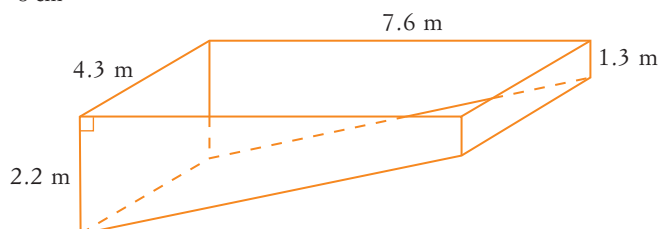


- 4 Calculate the surface area of each prism.

See Example 2



- 5 This swimming pool is 7.6 m long and 4.3 m wide. The depth of the water ranges from 1.3 m to 2.2 m. Calculate, correct to two decimal places:
- the area of the floor of the pool
 - the total surface area of the pool.



4-02 Surface area of a cylinder

Worksheet

Surface area

MAT10MGWK10016

Puzzle sheet

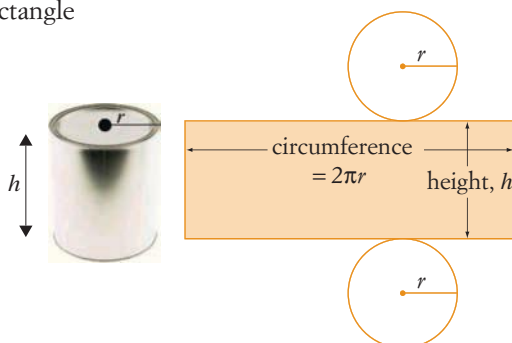
Surface area

MAT10MGPS00009

Surface area = area of two circles + area of rectangle

$$= 2 \times \pi r^2 + 2\pi r \times h$$

$$= 2\pi r^2 + 2\pi rh$$



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Summary

Surface area of a cylinder

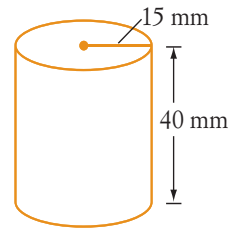
$$SA = 2\pi r^2 + 2\pi rh$$

where r = radius of circular base and h = perpendicular height

The area of the two circular ends = $2\pi r^2$ and the area of the curved surface = $2\pi rh$.

Example 3

Find, correct to the nearest mm^2 , the surface area of this cylinder.



Solution

$$\begin{aligned}\text{Surface area} &= 2\pi r^2 + 2\pi rh \\ &= 2 \times \pi \times 15^2 + 2 \times \pi \times 15 \times 40 \quad r = 15, h = 40 \\ &= 5183.627 \dots \\ &\approx 5184 \text{ mm}^2\end{aligned}$$

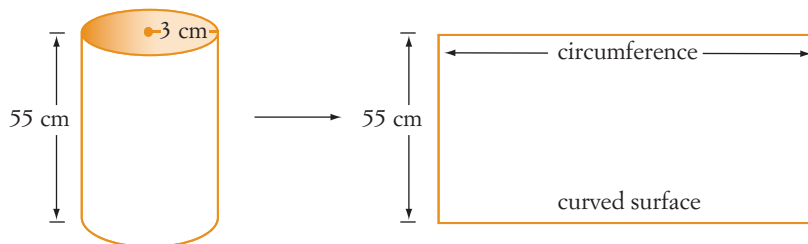
Example 4

Find, correct to two significant figures, the surface area of:

- a a cylindrical tube, open at both ends, with radius 3 cm and length 55 cm
- b an open half-cylinder with radius 0.5 m and height 3 m.

Solution

a



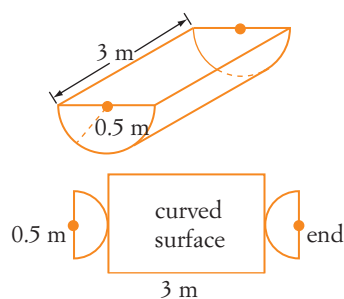
$$\begin{aligned}\text{Surface area} &= \text{curved surface} \\ &= 2\pi rh \\ &= 2 \times \pi \times 3 \times 55 \quad r = 3 \text{ and } h = 55 \\ &= 1036.725 \dots \\ &\approx 1000 \text{ cm}^2\end{aligned}$$

b Surface area = 2 semicircle ends + $\frac{1}{2} \times$ curved surface

$$= 2 \times \left(\frac{1}{2} \times \pi \times 0.5^2 \right) + \frac{1}{2} \times (2 \times \pi \times 0.5 \times 3)$$

$$= 5.49778 \dots$$

$$\approx 5.5 \text{ cm}^2$$



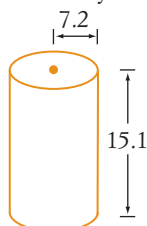
Exercise 4-02 Surface area of a cylinder

- Calculate, correct to two decimal places, the surface area of a cylinder with:
 - radius 3.4 m, height 6 m
 - diameter 35 mm, height 15 mm
 - diameter 6.2 cm, height 7.5 cm
 - radius 0.8 m, height 2.35 m
- Find, correct to the nearest whole number, the curved surface area of a cylinder with:
 - radius 1.5 m, height 3.75 m
 - diameter 27 cm, height 41 cm
- Calculate the surface area of each solid, correct to the nearest square metre. All lengths shown are in metres.

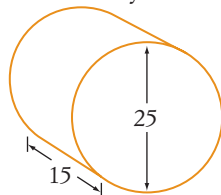
See Example 3

See Example 4

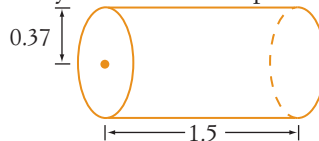
a closed cylinder



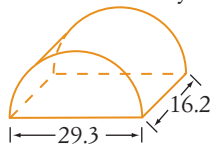
b closed cylinder



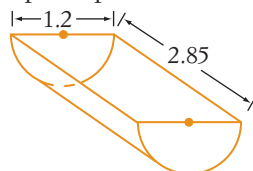
c cylinder with one open end



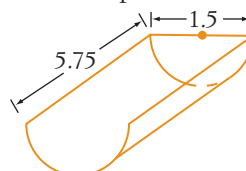
d closed half cylinder



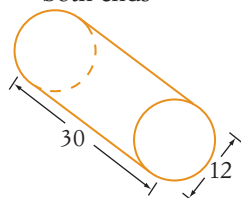
e half cylinder with open top



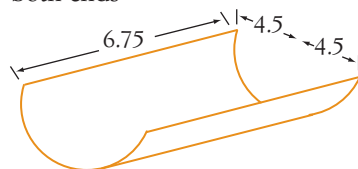
f half cylinder with open top, one end open



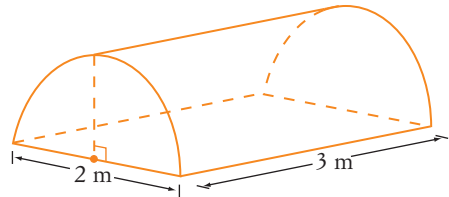
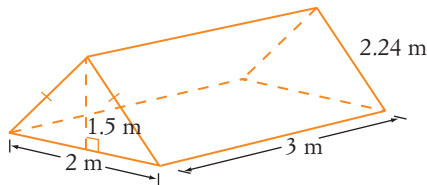
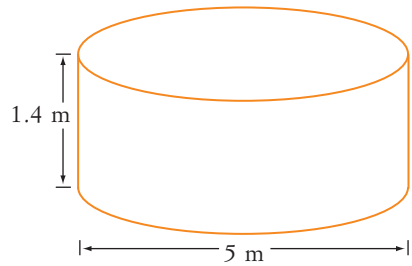
g cylinder open both ends



h half cylinder, open both ends



- 4 A swimming pool is in the shape of a cylinder 1.4 m deep and 5 m in diameter. The inside of the pool is to be repainted, including the floor.
- Find the area to be repainted, correct to one decimal place.
 - Find the number of whole litres of paint needed if coverage is 9 m^2 per litre.
- 5 Which tent has the greater surface area?



(Note: the floor is included for both tents)

Technology Surface areas and volumes of solids

Technology worksheet

Excel worksheet:
Volume calculator

MAT10MGCT00006

Technology worksheet

Excel spreadsheet:
Volume calculator

MAT10MGCT00036

Technology worksheet

Excel worksheet:
Volume of a box

MAT10MGCT00007

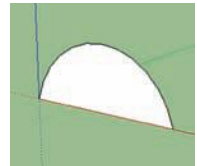
Technology worksheet

Excel spreadsheet:
Volume of a box

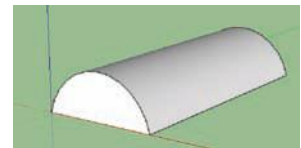
MAT10MGCT00037

In this activity, we will use Google Sketchup to construct and measure solid shapes.

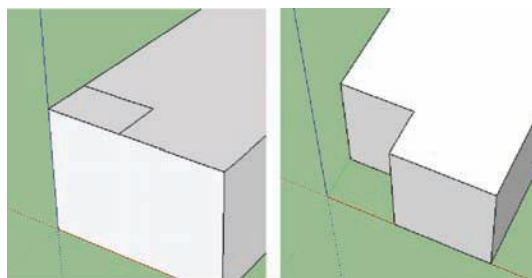
- Use the **arc** tool and the line tool to create a semicircle.



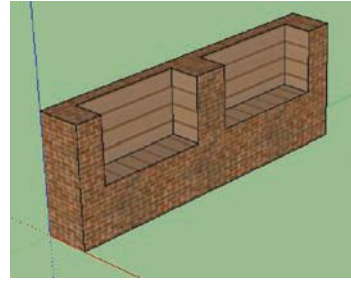
- To make a solid, select **Push/Pull**.



- Use the **Orbit** tool to reorientate your solid.
- Use the **Dimension** tool to obtain the dimensions of your half-cylinder. Calculate its surface area and volume.
- Draw a rectangular prism using the **Rectangle** tool and the **Push/Pull** tool.
- The **Push/Pull** tool can be used to cut away parts of a solid. Use the **Rectangle** tool to create rectangles on the top of the prism. Then use the **Push/Pull** tool to remove it. An example is shown below.

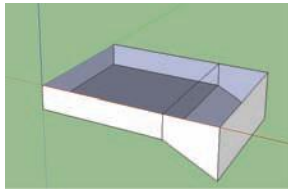


- 7 Start with a rectangular prism and cut out 2 rectangles to create a seat. Click **Window** and **Materials** to change the appearance of the seat.

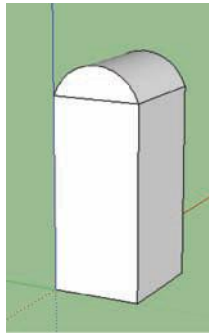


- 8 Draw each solid shown below and find its surface area and volume.

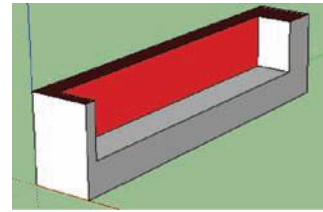
a swimming pool



b a bin

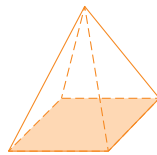


c a bench

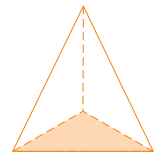


4-03 Surface area of a pyramid

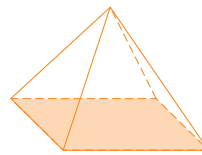
A **pyramid** is a solid shape with a polygon for its base and triangular faces that meet at a point or vertex called its **apex**. Like a prism, a pyramid is named by the shape of its base.



Square pyramid



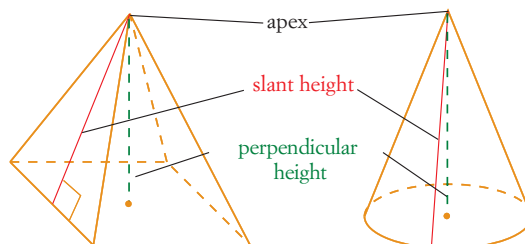
Triangular pyramid



Rectangular pyramid

A **cone** is a solid shape with a circular base and a curved surface that also has an **apex**. However, a cone is not a pyramid because its base is not a polygon (a circle does not have straight sides). The **slant height** of a pyramid or cone is the height from its apex to the base, along a side face. It is different from the **perpendicular height** of a pyramid or cone, which is the perpendicular distance from the apex to the base.

The surface area of a pyramid is calculated by adding the area of the base and the areas of the triangular faces.



Stage 5.3

Technology worksheet

Measuring pyramids

MAT10MGCT10002

Technology worksheet

Drawing pyramids and cones

MAT10MGCT10006

Stage 5.3

Video tutorial

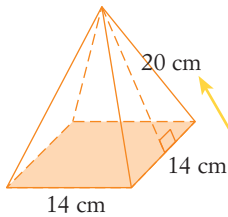
Surface area of
a pyramid

MAT10MGVT10008

Example 5

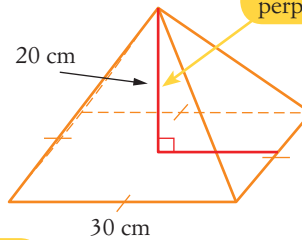
Find the surface area of each square pyramid.

a



This pyramid has a slant height of 20 cm

b



This pyramid has a perpendicular height of 20 cm

Solution

a Surface area = area of square base + area of 4 triangular faces

$$= 14 \times 14 + 4 \times \frac{1}{2} \times 14 \times 20$$

$$= 756 \text{ cm}^2$$

b First find the slant height, s , using Pythagoras' theorem.

$$s^2 = 20^2 + 15^2$$

$$= 625$$

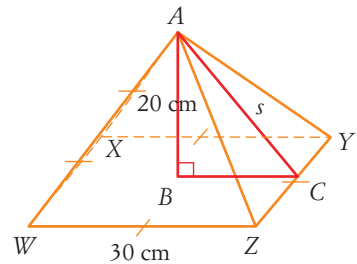
$$BC = \frac{1}{2} \times 30$$

$$AP = \sqrt{625}$$

$$= 25 \text{ cm}$$

$$\text{Surface area} = 30 \times 30 + 4 \times \frac{1}{2} \times 30 \times 25$$

$$= 2400 \text{ cm}^2$$



Example 6

A rectangular pyramid with a base measuring 10 cm by 8 cm has a perpendicular height of 15 cm. Find its surface area correct to one decimal place.

Solution

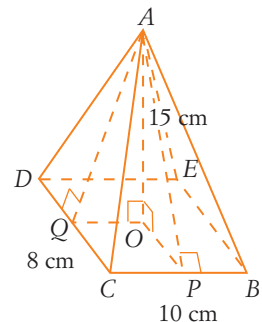
First find the slant heights AP and AQ .

$$AP^2 = AO^2 + OP^2$$

$$= 15^2 + 4^2$$

$$= 241$$

$$OP = \frac{1}{2} \times 8$$



$$AP = \sqrt{241} \text{ cm}$$

$$\begin{aligned} AQ^2 &= AO^2 + OQ^2 \\ &= 15^2 + 5^2 \\ &= 250 \end{aligned}$$

$$AQ = \sqrt{250} \text{ cm}$$

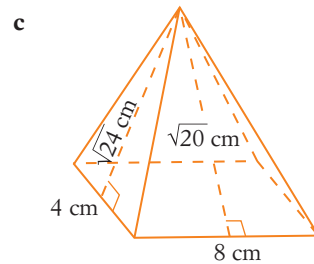
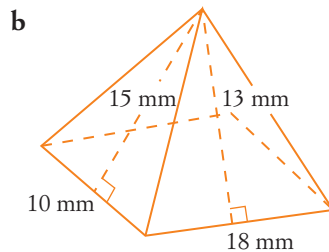
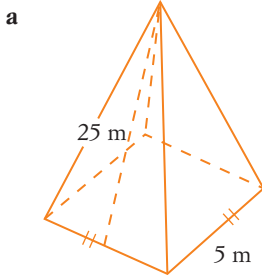
$$OQ = \frac{1}{2} \times 10$$

It is better to leave the lengths of AP and AQ in surd form rather than round them to decimals so that the final answer is accurate.

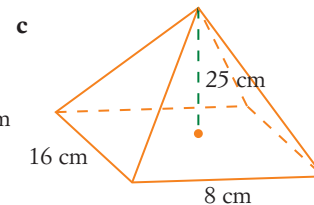
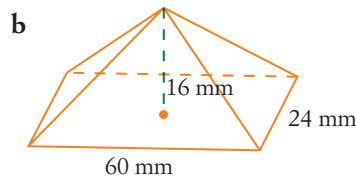
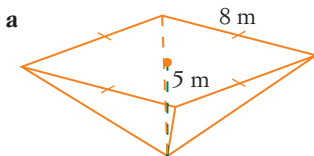
$$\begin{aligned} \text{Surface area} &= \text{area of rectangle base} + 2 \times \text{Area } \triangle ABC + 2 \times \text{Area } \triangle ADC \\ &= 10 \times 8 + 2 \times \frac{1}{2} \times 10 \times \sqrt{241} + 2 \times \frac{1}{2} \times 8 \times \sqrt{250} \\ &= 80 + 10\sqrt{241} + 8\sqrt{250} \\ &= 361.7328... \\ &\approx 361.7 \text{ cm}^2 \end{aligned}$$

Exercise 4-03 Surface area of a pyramid

- 1 Find the surface area of each pyramid. Write the answer to part c correct to one decimal place.



- 2 Calculate, correct to one decimal place, the surface area of each pyramid.

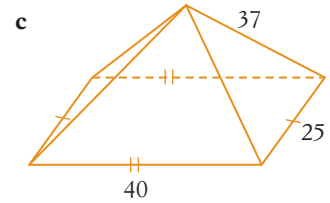
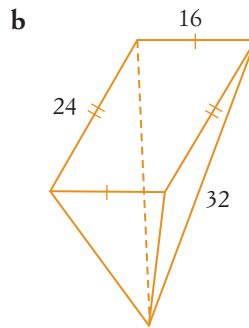
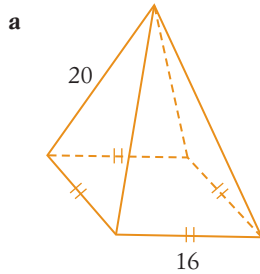


See Example 5

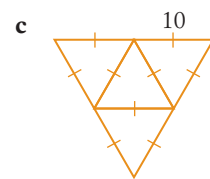
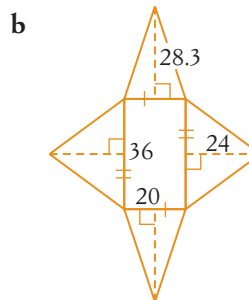
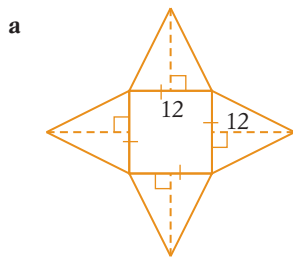
See Example 6

Stage 5.3

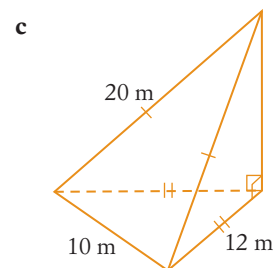
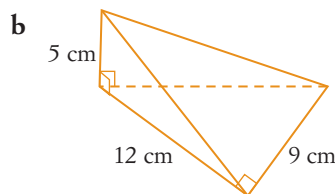
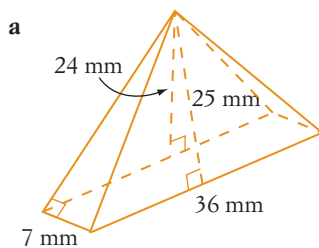
- 3 Calculate, correct to the nearest square centimetre, the surface area of each pyramid. All measurements are in centimetres.



- 4 Find the area of each net and hence the surface area of the corresponding pyramid. All measurements are in centimetres. Write the answer to part **c** correct to the nearest whole number.



- 5 The great pyramid of Khufu (or Cheops) in Egypt has a height of 147 m, and each side of its square base measures 230 m. Find its surface area (excluding the base), correct to the nearest square metre.
- 6 Calculate, correct to one decimal place, the surface area of each pyramid.

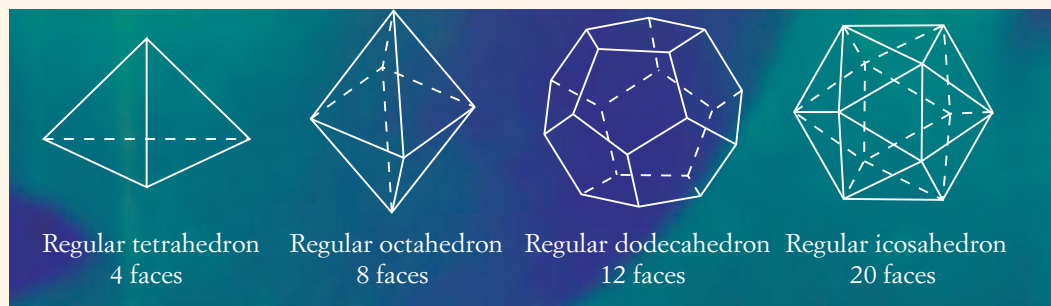


- 7 A square pyramid has a surface area of 4704 m^2 and a base area of 1764 m^2 . Find:
- the length of its base
 - the area of each triangular face
 - the slant height of each triangular face
 - the perpendicular height of the pyramid

Just for the record

The Platonic solids

The cube is an example of a **regular polyhedron** or **Platonic solid** because it has six identical faces. The more formal name for a cube is **regular hexahedron**, hex meaning 'six'. There are only six possible regular polyhedrons: the other five are shown below. Each face on a regular polyhedron is a **regular polygon**.

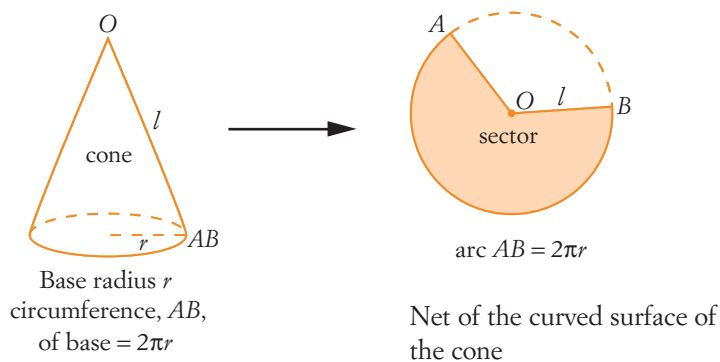


The tetrahedron, cube and octahedron occur in nature in the form of certain crystals.
What are the shapes of the faces on each Platonic solid shown?

Investigation: The surface area of a cone

Stage 5.3

The net of a cone is made up of a circle (for the base) and a sector of a circle (for the curved surface). The second diagram below shows the curved surface of a cone.



We can use this fact to find a formula for the surface area of a cone. Suppose the cone has a base radius of r and a slant height of l .

Looking at the second diagram, the major arc length AB is a fraction of the circumference of the circle and the area of the sector is a fraction of the area of the circle. They should be the same fraction, so:

$$\frac{\text{Arc length}}{\text{Circumference}} = \frac{\text{Area of sector}}{\text{Area of circle}}$$

Stage 5.3

- 1 The major arc length AB is equal to the circumference of the base of the cone in the first diagram. Write an algebraic expression for the circumference of the circle in the first diagram.
- 2 Write an algebraic expression for the circumference of the complete circle in the second diagram.
- 3 Write an algebraic expression for the area of the complete circle in the second diagram.
- 4 $\frac{\text{Arc length}}{\text{Circumference}} = \frac{\text{Area of sector}}{\text{Area of circle}}$ becomes $\frac{2\pi r}{2\pi l} = \frac{\text{Area of sector}}{\pi l^2}$
Complete:

$$\therefore \text{Area of sector} = \frac{2\pi r}{2\pi l} \times \pi l^2$$

$$= \text{-----}$$

- 5 But the area of the sector is equal to the curved surface area of the cone.
Complete the formula for the surface area of a cone.

$$\text{Surface area} = \text{area of curved surface} + \text{area of circular base}$$

$$= \text{-----} + \text{-----}$$

4-04 Surface areas of cones and spheres

Technology worksheet

Drawing pyramids and cones

MAT10MGCT10006

Surface area of a cone

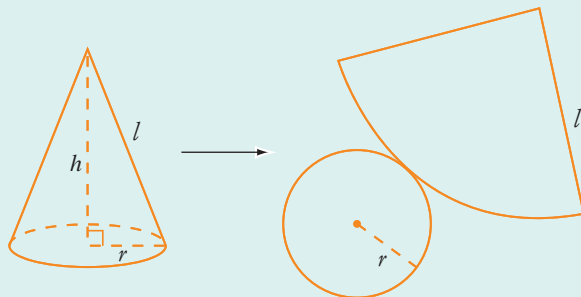
Summary

Surface area of a cone

$$\text{SA} = \text{area of curved surface} + \text{area of circular base}$$

$$= \pi r l + \pi r^2$$

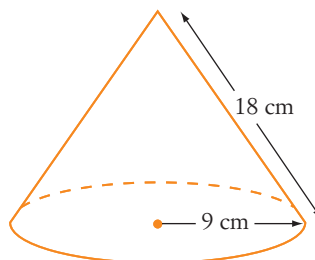
where l = slant height and r = radius of the base



Example 7

For this cone, find correct to one decimal place:

- a the curved surface area
- b the total surface area



Solution

$r = 9$ cm and $l = 18$ cm

a Curved surface area $= \pi rl$

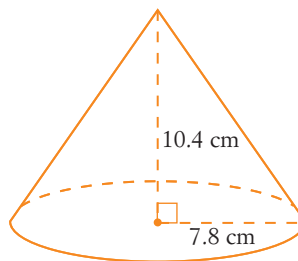
$$\begin{aligned} &= \pi \times 9 \times 18 \\ &= 508.9380 \dots \\ &\approx 508.9 \text{ cm}^2 \end{aligned}$$

b Total surface area $= \pi rl + \pi r^2$

$$\begin{aligned} &= \pi \times 9 \times 18 + \pi \times 9^2 \\ &= 763.4070 \dots \\ &\approx 763.4 \text{ cm}^2 \end{aligned}$$

Example 8

Find, correct to two decimal places, the surface area of this cone.



Solution

First calculate the slant height, l :

$$\begin{aligned} l^2 &= 7.8^2 + 10.4^2 \\ &= 169 \end{aligned}$$

$$\begin{aligned} l &= \sqrt{169} \\ &= 13 \end{aligned}$$

$$\begin{aligned} \text{Surface area} &= \pi rl + \pi r^2 \\ &= \pi \times 7.8 \times 13 + \pi \times 7.8^2 \\ &= 509.6919 \dots \\ &\approx 509.69 \text{ cm}^2 \end{aligned}$$

Video tutorial

Surface area of a cone and sphere

MAT10MGVT10009

Video tutorial

Area and volume

MAT10MGVT00004

Surface area of a sphere

A **sphere** is a ball shape, a solid that is completely round. All points on its surface lie the same distance (radius) from its centre. A **hemisphere** is half a sphere.

Stage 5.3

Summary

Surface area of a sphere

$$SA = 4\pi r^2$$



where r = radius of the sphere

Note that the surface area of a sphere is four times the area of the circle that slices through the centre of a sphere.

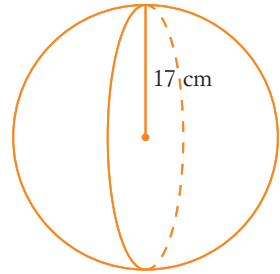
Video tutorial

Surface area of a cone
and sphere

MAT10MGVT10009

Example 9

Find, correct to two decimal places, the surface area of this sphere.

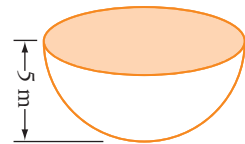


Solution

$$\begin{aligned}\text{Surface area} &= 4\pi r^2 \\ &= 4 \times \pi \times 17^2 \\ &= 3631.6811 \dots \\ &\approx 3631.7 \text{ cm}^2\end{aligned}$$

Example 10

Find the surface area of this hemisphere in exact form, in terms of π .



Solution

Surface area = Area of circular base + curved surface area

$$\begin{aligned}&= \pi r^2 + \frac{1}{2} \times 4\pi r^2 \\ &= \pi r^2 + 2\pi r^2 \\ &= 3\pi r^2 \\ &= 3 \times \pi \times 5^2 \\ &= 75\pi \text{ m}^2\end{aligned}$$

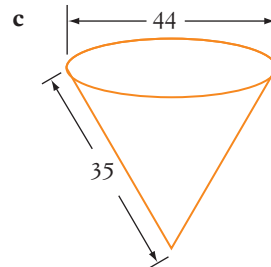
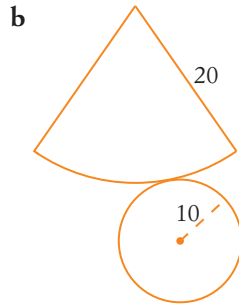
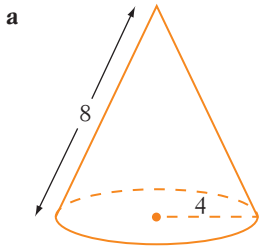
Answers written as surds or in terms of π are **exact** because they are not decimal approximations.

Exercise 4-04 Surface areas of cones and spheres

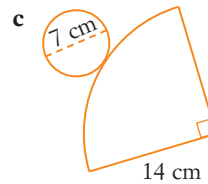
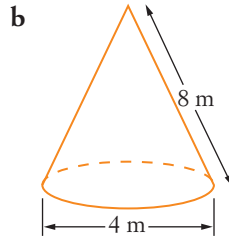
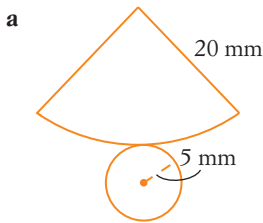
Stage 5.3

See Example 7

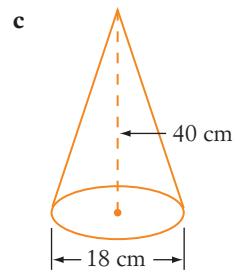
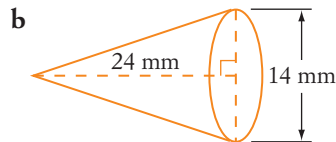
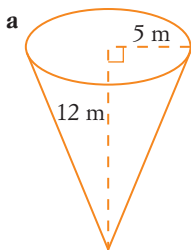
- 1 Calculate, correct to the nearest cm^2 , the curved surface area of each cone. All measurements are in centimetres.



- 2 Find, correct to one decimal place, the total surface area of each cone.

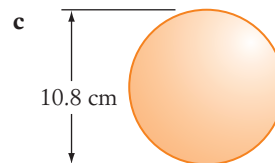
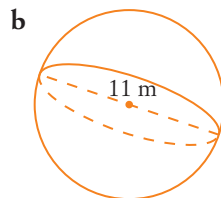
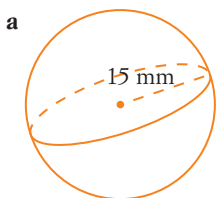


- 3 Calculate in exact form (in terms of π) the total surface area of each cone.



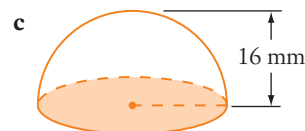
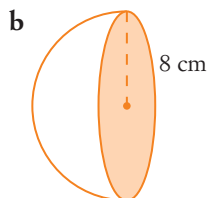
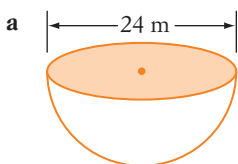
See Example 8

- 4 Calculate, correct to two decimal places, the surface area of each sphere.



See Example 9

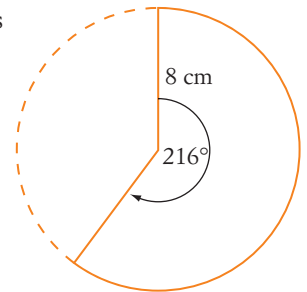
- 5 Find in exact form the surface area of each hemisphere.



See Example 10

Stage 5.3

- 6 Find, correct to the nearest square metre, the surface area of each solid.
- a sphere with diameter = 10 m
 - an open cone with base radius = 10 m, slant height = 20 m
 - an open hemisphere with radius = 10 m
 - a cone with base diameter = 10 m, perpendicular height = 20 m
- 7 The Earth has a radius of approximately 6400 km. Calculate the surface area of the Earth in scientific notation, correct to two significant figures.
- 8 Find the amount of sheet metal needed to form a conical funnel of base radius 30 cm and vertical height 50 cm, allowing for a 0.5 cm overlap at the join.
- 9 The curved surface of a cone is made from a sector of a circle with radius 8 cm and central angle 216° . Find, correct to two decimal places:
- the length of the arc of the circle that forms the circumference of the cone's base
 - the radius of the cone's base
 - the slant height of the cone
 - the total surface area of the cone, including the base
- 10 Find the radius of each solid if it has a surface area of 6000 mm^2 . Give your answer correct to three significant figures.
- a sphere
 - a closed hemisphere
 - an open hemisphere
- 11 A cone has a surface area of 2000 cm^2 . If the area of its base is 150 cm^2 , find, correct to two decimal places:
- the radius of its base
 - its slant height
 - its perpendicular height



Mental skills 4

Maths without calculators

Estimating answers

A quick way of estimating an answer is to round each number in the calculation.

1 Study each example.

$$\begin{aligned}
 \text{a } 55 + 132 - 34 + 17 - 78 &\approx 60 + 130 - 30 + 20 - 80 \\
 &= (60 + 20 - 80) + (130 - 30) \\
 &= 0 + 100 \\
 &= 100 \quad (\text{Actual answer} = 92)
 \end{aligned}$$

$$\begin{aligned}
 \text{b } 78 \times 7 &\approx 80 \times 7 \\
 &= 560 \quad (\text{Actual answer} = 546)
 \end{aligned}$$

$$\begin{aligned}
 \text{c } 510 \div 24 &\approx 500 \div 20 \\
 &= 50 \div 2 \\
 &= 25 \quad (\text{Actual answer} = 21.25)
 \end{aligned}$$

2 Now estimate each answer.

a $27 + 11 + 87 + 142 + 64$

c $684 + 903$

e $517 - 96$

g $766 - 353$

i 83×81

k $828 \div 3$

b $55 + 34 - 22 - 46 + 136$

d $35 + 81 + 110 + 22 + 7$

f $210 - 38 - 71 + 151 - 49$

h 367×2

j 984×16

l $507 \div 7$

3 Study each example involving decimals.

a $20.91 - 11.3 + 2.5 \approx 21 - 11 + 3$
 $= 13$ (Exact answer = 12.11)

b $4.78 \times 19.2 \approx 5 \times 20$
 $= 100$ (Exact answer = 91.776)

c $\frac{37.6 + 9.3}{41.2 - 12.7} \approx \frac{38 + 9}{40 - 13}$
 $= \frac{47}{27}$
 $\approx \frac{50}{30}$
 ≈ 1.6 (Exact answer = 1.645...)

4 Now estimate each answer.

a $3.75 + 9.381 + 4.6 + 10.5$

c 18.47×9.61

e $\frac{11.07 + 18.4}{12.2}$

g $54.75 - 18.6 - 14.4$

i $62.13 \div 10.7$

b $14.807 + 6.6 - 7.22$

d 4.27×97.6

f $\frac{38.18}{17.2 - 9.6}$

h $\frac{18.46 \times 4.9}{39.72 - 15.2}$

j $(4.89)^2$

4-05 Surface area of composite solids

Worksheet

A page of prisms and cylinders

MAT10MGWK10017

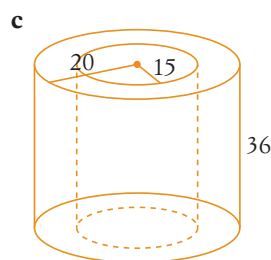
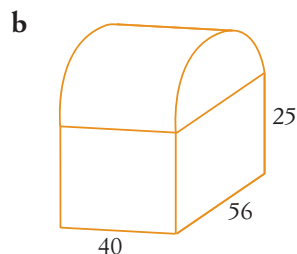
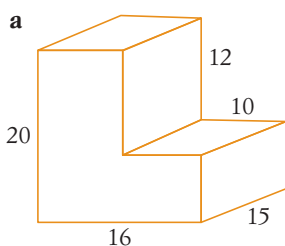
Worksheet

A page of solid shapes

MAT10MGWK10205

Example 11

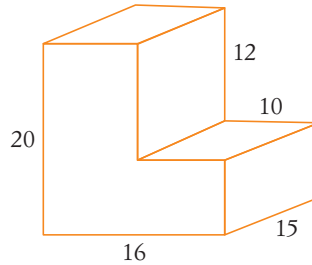
Find, correct to one decimal place, the surface area of each solid. All measurements are in centimetres.



Solution

- a This prism has 8 faces: 2 identical L-shapes (front and back) and 6 different rectangles.

$$\begin{aligned}\text{Area of L-shape} &= 16 \times 20 - 10 \times 12 \\ &= 200 \text{ cm}^2\end{aligned}$$



$$\begin{aligned}\text{Surface area} &= \text{Front and back L-faces} + \text{1st top} + \text{1st right} + \text{2nd top} \\ &\quad + \text{2nd right} + \text{bottom} + \text{left} \\ &= (2 \times 200) + (6 \times 15) + (12 \times 15) \\ &\quad + (10 \times 15) + (8 \times 15) + (16 \times 15) \\ &\quad + (20 \times 15) \\ &= 1480 \text{ cm}^2\end{aligned}$$

Length of long rectangle
= perimeter of L
= $6 + 12 + 10 + 8 + 16 + 20$
= 72

Note that the six rectangles can also be thought of as one long rectangle of width 15 cm:

$$\begin{aligned}\text{Surface area} &= (2 \times 200) + (72 \times 15) \\ &= 1480 \text{ cm}^2\end{aligned}$$

- b This solid is made up of a half-cylinder (3 faces) and a rectangular prism (5 faces).

Surface area of half-cylinder = 2 semi-circular ends + curved surface area

$$\begin{aligned}&= 2 \times \frac{1}{2} \times \pi \times 28^2 + \frac{1}{2} \times 2 \times \pi \times 28 \times 40 \\ &\approx 5981.5924 \dots \text{ cm}^2\end{aligned}$$

Radius of semicircle
= $\frac{1}{2} \times 56 = 28$

Do not round this partial answer, else the final answer will be inaccurate.

$$\begin{aligned}\text{Surface area of rectangular prism} &= \text{Front and back faces} + \text{2 side faces} + \text{bottom face} \\ &= (2 \times 40 \times 25) + (2 \times 56 \times 25) + (40 \times 56) \\ &= 7040 \text{ cm}^2\end{aligned}$$

$$\begin{aligned}\text{Total surface area} &= 5981.5924 \dots + 7040 \\ &= 13\,021.5924 \dots \\ &\approx 13\,021.6 \text{ cm}^2\end{aligned}$$

- c The hollow cylinder is made up of 2 annulus (ring) faces, an outside curved surface area and an inside curved surface area.

$$\begin{aligned}\text{Surface area of annulus faces} &= 2 \times (\pi \times 20^2 - \pi \times 15^2) \quad 2 \times \text{area between two circles} \\ &= 1099.5574 \dots\end{aligned}$$

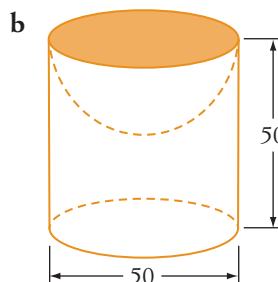
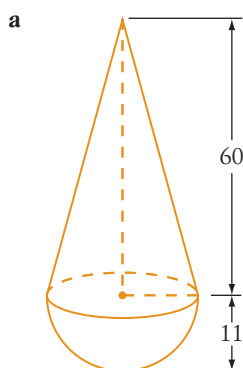
$$\begin{aligned}\text{Outside curved surface area} &= 2 \times \pi \times 20 \times 36 \\ &\approx 4523.8934 \dots\end{aligned}$$

$$\begin{aligned}\text{Inside curved surface area} &= 2 \times \pi \times 15 \times 36 \\ &\approx 3392.9200 \dots\end{aligned}$$

$$\begin{aligned}\text{Total surface area} &= 1099.5574 \dots + 4523.8934 \dots + 3392.9200 \dots \\ &= 9016.3708 \dots \\ &\approx 9016.4 \text{ cm}^2\end{aligned}$$

Example 12

Find, correct to the nearest square centimetre, the surface area of each solid. All measurements are in centimetres.



Solution

- a** Surface area = curved surface area of cone
+ curved surface area of hemisphere

$$= \pi r l + \frac{1}{2} \times 4\pi r^2$$

$$r = 11, h = 60 \text{ and } l = ?$$

$$l^2 = 11^2 + 60^2$$

$$= 3721$$

$$l = \sqrt{3721}$$

$$= 61 \text{ cm}$$

$$\text{Surface area} = \pi \times 11 \times 61 + \frac{1}{2} \times 4 \times \pi \times 11^2$$

$$= 2868.2740 \dots$$

$$\approx 2868 \text{ cm}^2$$

- b** Surface area = curved surface of cylinder + circular base
+ curved surface of hemisphere

$$= 2\pi r h + \pi r^2 + \frac{1}{2} \times 4\pi r^2$$

$$= 2\pi r h + 3\pi r^2$$

$$= 2 \times \pi \times 25 \times 50 + 3 \times \pi \times 25^2$$

$$= 13\,744.4678 \dots$$

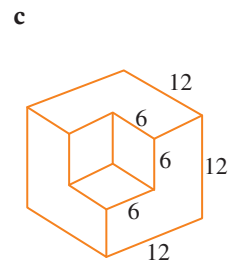
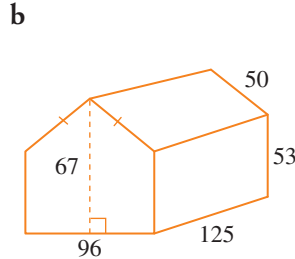
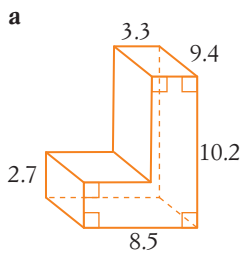
$$\approx 13\,744 \text{ cm}^2$$

$$\begin{aligned} \pi r^2 + 2\pi r^2 &= 3\pi r^2 \\ r &= \frac{1}{2} \times 50 = 25 \end{aligned}$$

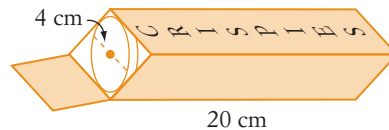
Exercise 4-05 Surface areas of composite solids

See Example 11

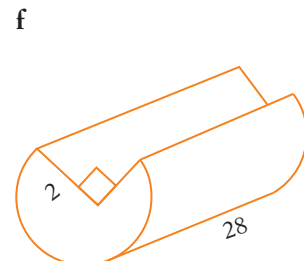
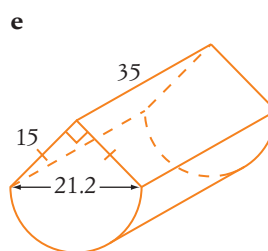
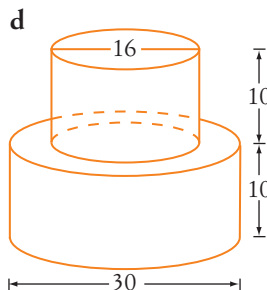
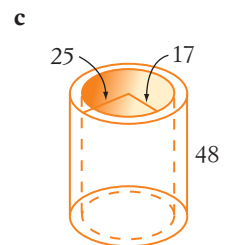
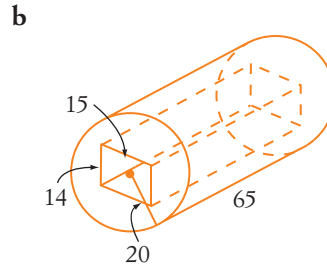
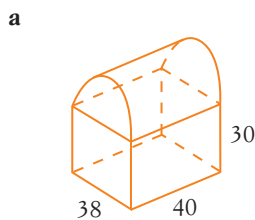
- 1 Find the surface area of each prism. All measurements are in centimetres.



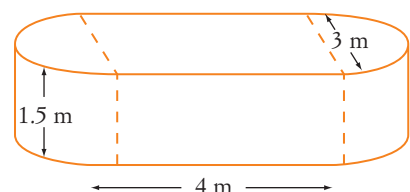
- 2 Circular cracker biscuits of diameter 4 cm are packed in a cardboard box of length 20 cm.



- a Calculate the surface area of the box.
b How much cardboard would be saved if the biscuits were packed into a cylindrical box?
3 Find, correct to one decimal place, the surface area of each solid. All measurements are in centimetres.



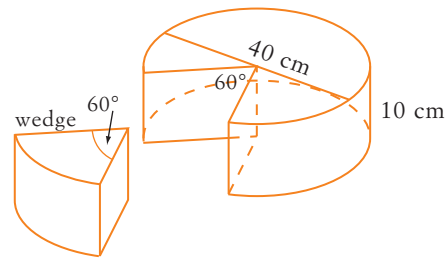
- 4 a Find, correct to two decimal places, the total external wall area of this above-ground swimming pool.
b Calculate the area of the water surface, correct to the nearest m^2 .



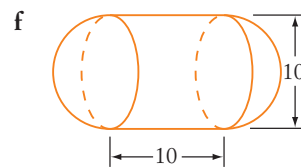
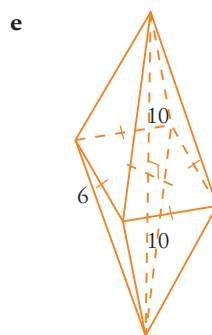
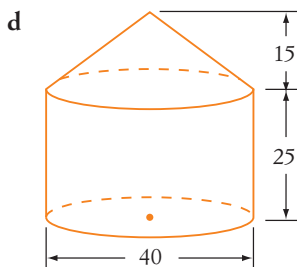
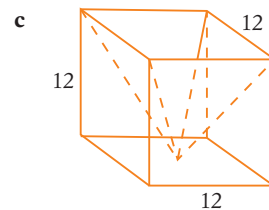
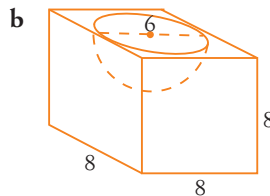
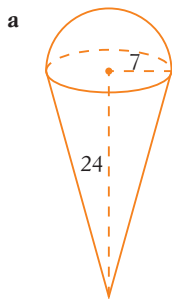
- 5 A wedding cake with three tiers rests on a table. Each tier is 6 cm high. The layers have radii of 20 cm, 15 cm and 10 cm respectively. Find the total visible surface area, correct to the nearest cm^2 .



- 6 A wedge of cheese is cut from a cylindrical block of height 10 cm and diameter 40 cm. Find the total surface area of the wedge, correct to two decimal places.



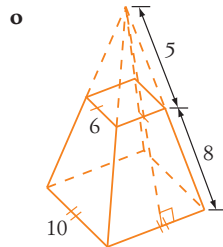
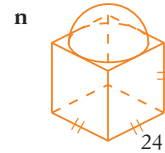
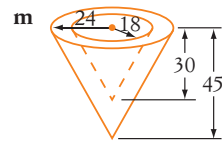
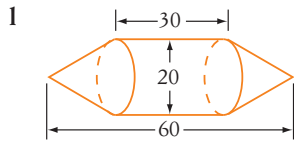
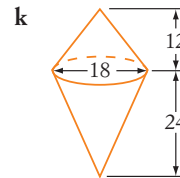
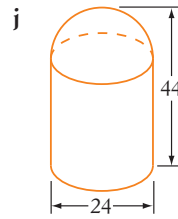
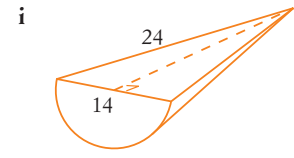
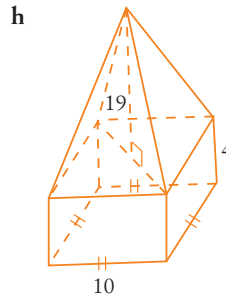
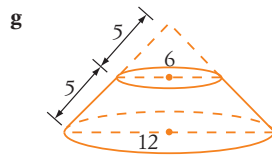
- 7 Find, correct to one decimal place, the surface area of each solid. All measurements are in centimetres.



Stage 5.3

See Example 12

Stage 5.3



Worksheet

A page of prisms and cylinders

MAT10MGWK10017

Puzzle sheet

Formula matching game

MAT10MGPS10018

Worksheet

Volumes of solids

MAT10MGWK10020

Worksheet

Back-to-front problems

MAT10MGWK10021

Worksheet

Volume and capacity

MAT10MGWK10022

Animated example

Volumes of shapes

MAT10MGAE00004

4-06 Volumes of prisms and cylinders

The **volume** of a solid is the amount of space it occupies. Volume is measured in **cubic units**, for example, cubic metres (m^3) or cubic centimetres (cm^3).

Summary

Volume of a prism

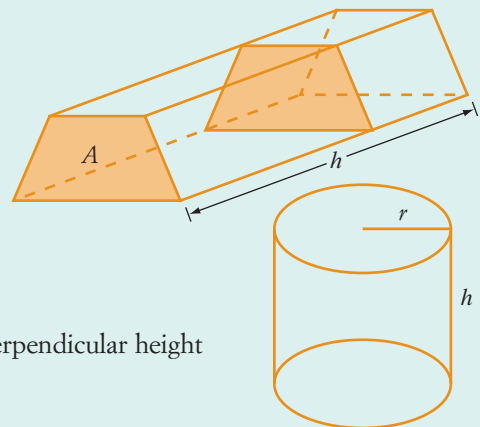
$$V = Ah$$

where A = area of base and
 h = perpendicular height

Volume of a cylinder

$$V = \pi r^2 h$$

where r = radius of circular base and h = perpendicular height



The **capacity** of a container is the amount of fluid (liquid or gas) it holds, measured in millilitres (mL), litres (L), kilolitres (kL) and megalitres (ML).

Worksheet

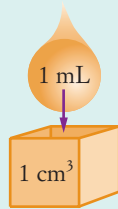
Biggest volume

MAT10MGWK10019

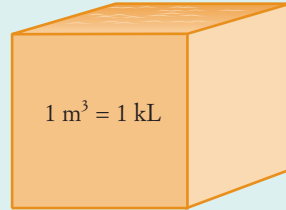
Summary

1 cm^3 contains 1 mL.

1 m^3 contains 1000 L or 1 kL.



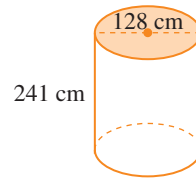
$\times 1\,000\,000 =$



Example 13

For this cylinder, calculate:

- its volume, correct to the nearest cm^3
- its capacity in kL, correct to 1 decimal place.



Solution

$$\begin{aligned} \text{a } V &= \pi \times 64^2 \times 241 \\ &= 3\,101\,179.206 \dots \\ &\approx 3\,101\,179 \text{ cm}^3 \end{aligned}$$

$$r = \frac{1}{2} \times 128 = 64$$

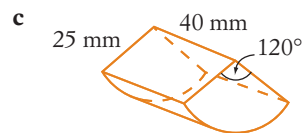
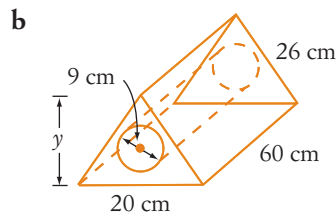
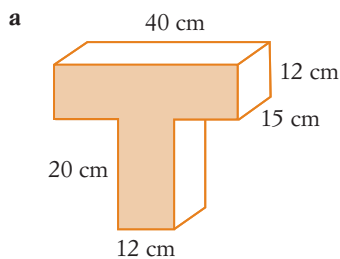
$$\begin{aligned} \text{b Capacity} &= 3\,101\,179 \text{ mL} \\ &= (3\,101\,179 \div 1000 \div 1000) \text{ kL} \\ &= 3.101\,179 \text{ kL} \\ &\approx 3.1 \text{ kL} \end{aligned}$$

$$1 \text{ cm}^3 = 1 \text{ mL}$$



Example 14

Find, correct to the nearest whole number, the volume of each solid.



Solution

a $A = 40 \times 12 + 20 \times 12$

$$= 720 \text{ cm}^2$$

$$V = Ah$$

$$= 720 \times 15$$

$$= 10\,800 \text{ cm}^3$$

- b The cross-section is a triangle minus a circle.
Use Pythagoras' theorem to find y .

$$26^2 = y^2 + 10^2$$

$$y^2 = 26^2 - 10^2$$

$$= 576$$

$$y = \sqrt{576}$$

$$= 24 \text{ cm}$$

$$\text{Radius of circle} = \frac{1}{2} \times 9 = 4.5$$

$$A = \frac{1}{2} \times 20 \times 24 - \pi \times 4.5^2$$

$$= 176.3827 \dots \text{ cm}^2$$

$$V = Ah$$

$$= 176.3827 \dots \times 60$$

$$= 10\,582.9649 \dots$$

$$\approx 10\,583 \text{ cm}^3$$

c $A = \frac{120}{360} \times \pi \times 25^2$

$$= 654.498 \dots \text{ mm}^2$$

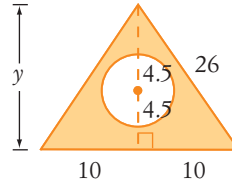
$$V = Ah$$

$$= 654.498 \dots \times 40$$

$$= 26\,179.938 \dots$$

$$\approx 26\,180 \text{ mm}^3$$

Area of T cross-section



Area of triangle – area of circle

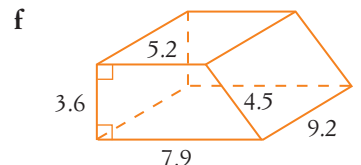
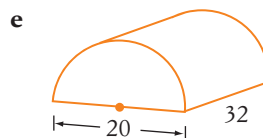
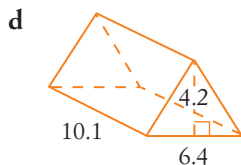
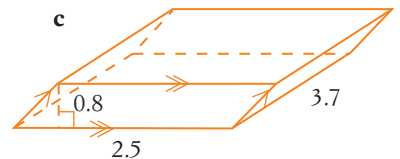
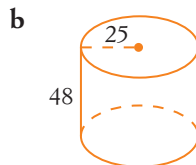
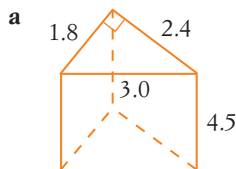
Do not round this partial answer

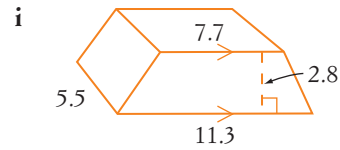
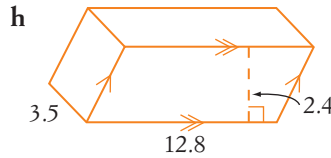
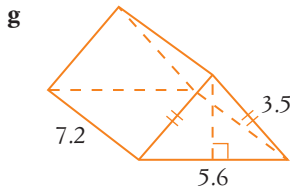
Area of sector

Do not round this partial answer

Exercise 4-06 Volumes of prisms and cylinders

See Example 13 1 Calculate, correct to one decimal place, the volume of each solid. All lengths are in metres.



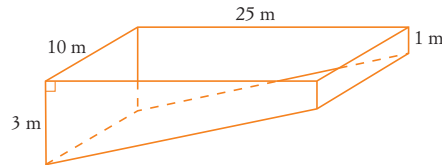


- 2** Rice crackers of diameter 4 cm are packed in a cardboard box of height 20 cm. Calculate, correct to one decimal place:

- the volume of the crackers in the box
- the volume of the box
- the percentage of the box that is empty space.



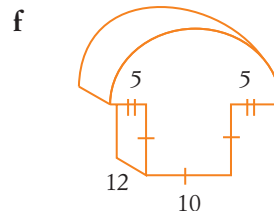
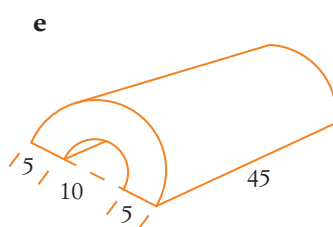
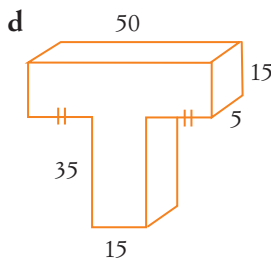
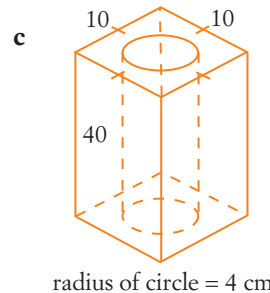
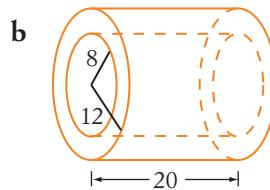
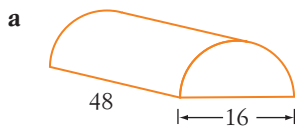
- 3** This swimming pool is 25 m long and 10 m wide. The depth of the water ranges from 1 m to 3 m. Calculate the capacity of this pool in kilolitres.

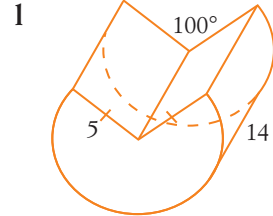
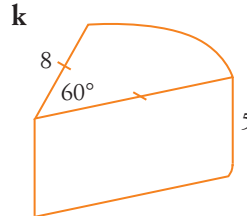
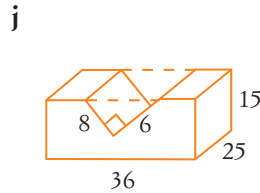
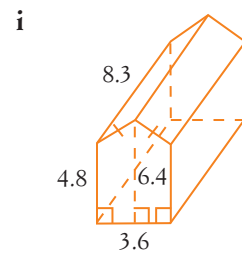
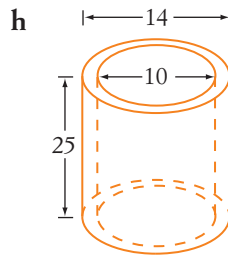
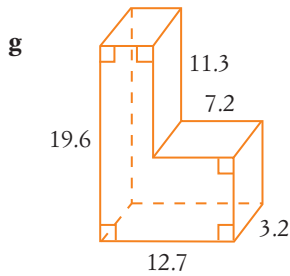


- 4** A wedding cake with three tiers rests on a table. Each tier is 6 cm high. The layers have radii of 20 cm, 15 cm and 10 cm respectively. Find the total volume of the cake, correct to the nearest cm^3 .

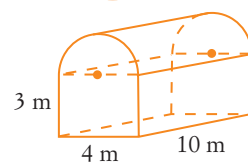


- 5** A fish tank that is 60 cm long, 30 cm wide and 40 cm high is filled with water to 5 cm below the top. Calculate the volume of the water in litres.
- 6** Find, correct to two decimal places, the volume of each solid. All lengths shown are in centimetres. See Example 14





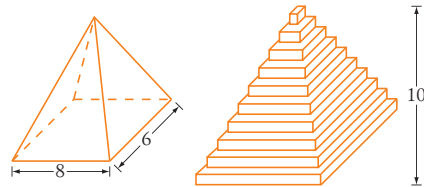
- 7 **a** Find, correct to two decimal places, the volume of this greenhouse.
- b** If this greenhouse costs 0.5c per m³ per hour to heat, how much is this per day, correct to the nearest cent?



Stage 5.3

Technology Approximating the volume of a pyramid

In this activity, we will use a spreadsheet to approximate the volume of a rectangular pyramid by slicing it into tiny layers of rectangular prisms of equal thickness.



Let $L = 8$ be the length of the prism, $W = 6$ be the width and $H = 10$ be the height.

The thickness, T , of each layer is given by the formula $T = \frac{H}{\text{number of layers}}$.

Starting at the bottom, the length and width of each layer are decreased by the amounts

$\frac{L}{\text{number of layers}}$ and $\frac{W}{\text{number of layers}}$ with each step.

- 1 Set up your spreadsheet as shown.

	A	B	C	D	E	F
1						
2			Number of layers =			
3	H	L	W	Thickness of layers	Volume of layer	Sum of volumes
4	10	8	6	=A\$4/\$D\$2	=B4*C4*D4	=E4
5		=B4-\$B\$4/\$D\$2	=C4-\$C\$4/\$D\$2			=E5+F4
⋮						
13						

- 2 Let the number of layers be 10. Enter **10** in cell D2.
- 3 Copy each formula down to row 13.
- 4 Explain the results in cells E13 and F13.
- 5 How accurate was your result in F13? Explain.
- 6 Print out your spreadsheet and paste it into your book.
- 7 Enter **40** (layers) in cell D2 and copy each formula down to row 43.
- 8 In one or two sentences compare your result in F43 with the previous result in F13 from question 4.
- 9 Enter each value in cell D2 and copy down the formulas as requested.
 - a 100 (copy down to row 103)
 - b 200 (copy down to row 203)
 - c 400 (copy down to row 403)
- 10 Use the formula $V = \frac{1}{3}Ah$ to calculate the exact volume of the pyramid.
- 11 Write a brief report about your results in questions 9 and 10.

4-07

Volumes of pyramids, cones and spheres

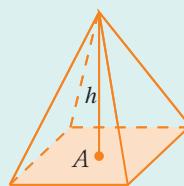
Volume of a pyramid

Summary

Volume of a pyramid

$$V = \frac{1}{3}Ah$$

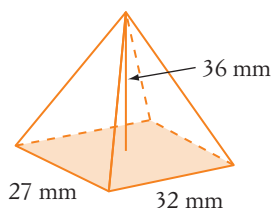
where A = area of the base and h = perpendicular height.



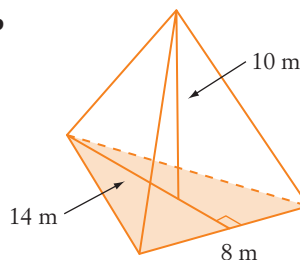
Example 15

Find the volume of each pyramid.

a



b



Stage 5.3

Technology worksheet

Drawing pyramids and cones

MAT10MGCT10006

Technology worksheet

Measuring pyramids

MAT10MGCT10002

Worksheet

Back-to-front problems (Advanced)

MAT10MGWK10206

Stage 5.3

Solution

$$\begin{aligned}\text{a } A &= 27 \times 32 \\ &= 864 \\ V &= \frac{1}{3}Ah \\ &= \frac{1}{3} \times 864 \times 36 \\ &= 10\,368 \text{ mm}^3\end{aligned}$$

$$\begin{aligned}\text{b } A &= \frac{1}{2} \times 8 \times 14 \\ &= 56 \\ V &= \frac{1}{3}Ah \\ &= \frac{1}{3} \times 56 \times 10 \\ &= 186\frac{2}{3} \text{ m}^3\end{aligned}$$

Example 16

Find the volume of a square pyramid with base length 48 mm and slant height 51 mm.

Solution

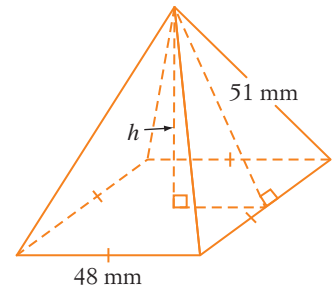
First find h , the perpendicular height of the pyramid.

$$\begin{aligned}h^2 &= 51^2 - 24^2 \\ &= 2025\end{aligned}$$

$$\begin{aligned}h &= \sqrt{2025} \\ &= 45 \text{ mm}\end{aligned}$$

$$\begin{aligned}A &= 48 \times 48 \\ &= 2304\end{aligned}$$

$$\begin{aligned}V &= \frac{1}{3} \times 2304 \times 45 \\ &= 34\,560 \text{ mm}^3\end{aligned}$$



Technology worksheet

Approximating the
volume of a cone

MAT10MGCT10003

Video tutorial

Area and volume

MAT10MGVT00004

Volume of a cone

A cone is like a 'circular pyramid' so:

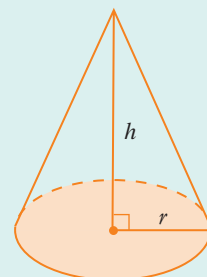
$$V = \frac{1}{3}Ah = \frac{1}{3} \times \pi r^2 \times h = \frac{1}{3}\pi r^2 h$$

Summary

Volume of a cone

$$V = \frac{1}{3}\pi r^2 h$$

where r = radius of the base and h = perpendicular height.

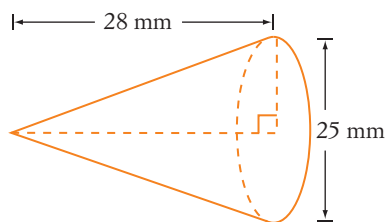


Example 17

Find, correct to the nearest cubic millimetre, the volume of this cone.

Solution

$$\begin{aligned} V &= \frac{1}{3} \pi r^2 h \\ &= \frac{1}{3} \times \pi \times 12.5^2 \times 28 \\ &= 4581.4892 \dots \\ &\approx 4581 \text{ mm}^3 \end{aligned}$$



Example 18

A cone has a base radius of 14 cm and a slant height of 50 cm. Find its volume, correct to two significant figures.

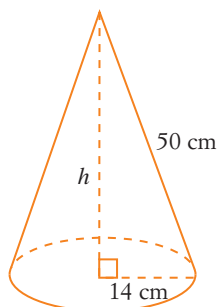
Solution

First find the height, h .

$$\begin{aligned} h^2 &= 50^2 - 14^2 \\ &= 2304 \end{aligned}$$

$$\begin{aligned} h &= \sqrt{2304} \\ &= 48 \text{ cm} \end{aligned}$$

$$\begin{aligned} V &= \frac{1}{3} \times \pi \times 14^2 \times 48 \\ &= 9852.0345 \dots \\ &\approx 9900 \text{ cm}^3 \end{aligned}$$



Volume of a sphere

Summary

Volume of a sphere

$$V = \frac{4}{3} \pi r^3$$

where r = radius of the sphere.

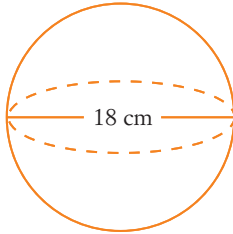


Stage 5.3

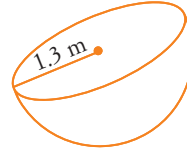
Example 19

Find, correct to two significant figures, the volume of each solid.

a



b



Solution

$$\begin{aligned} \text{a } V &= \frac{4}{3}\pi r^3 \\ &= \frac{4}{3} \times \pi \times 9^3 \\ &= 3053.6280 \dots \\ &\approx 3100 \text{ cm}^3 \end{aligned}$$

$$r = \frac{1}{2} \times 18 = 9$$

$$\begin{aligned} \text{b } V &= \frac{1}{2} \times \frac{4}{3}\pi r^3 \\ &= \frac{2}{3}\pi r^3 \\ &= \frac{2}{3} \times \pi \times 1.3^3 \\ &= 4.6013 \dots \\ &\approx 4.6 \text{ m}^3 \end{aligned}$$

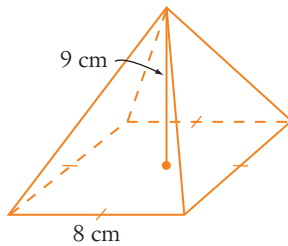
Exercise 4-07

Volumes of pyramids, cones and spheres

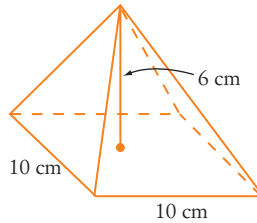
See Example 15

1 Find the volume of each pyramid.

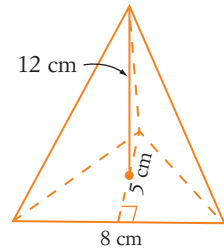
a



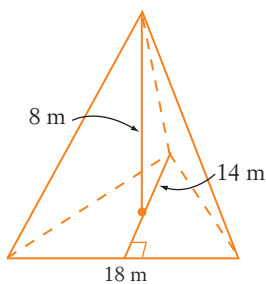
b



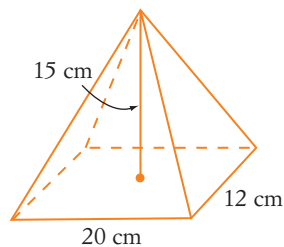
c



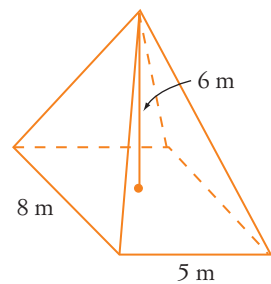
d



e



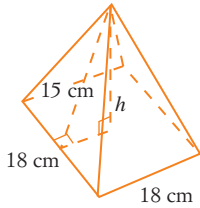
f



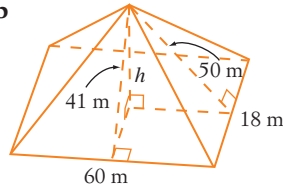
2 For each pyramid, find correct to one decimal place:

- i its perpendicular height ii its volume

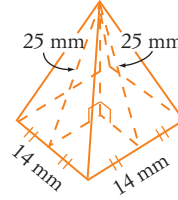
a



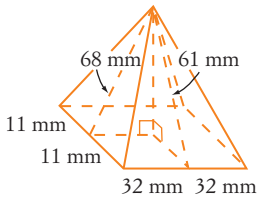
b



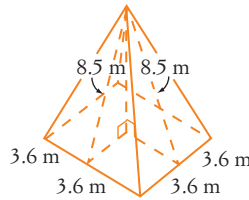
c



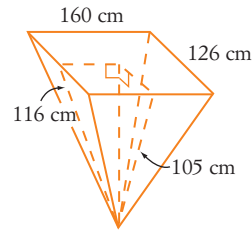
d



e

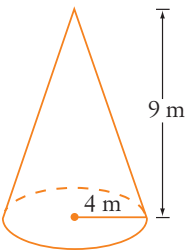


f

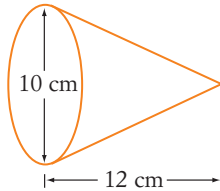


3 Find, correct to the nearest whole number, the volume of each cone.

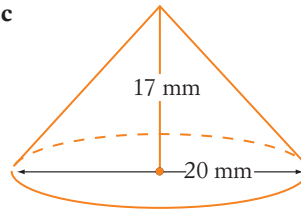
a



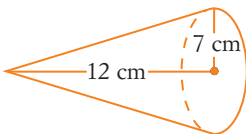
b



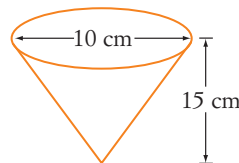
c



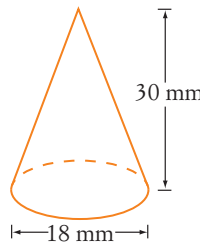
d



e



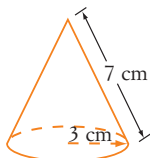
f



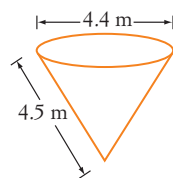
4 For each cone, find correct to one decimal place:

- i its perpendicular height ii its volume

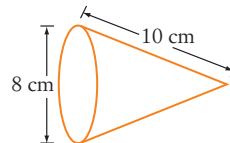
a



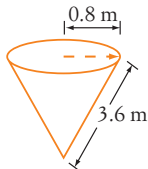
b



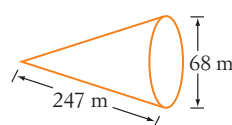
c



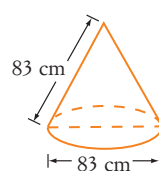
d



e



f



Stage 5.3

See Example 16

See Example 17

See Example 18

Stage 5.3

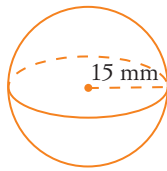
See Example 19

5 For each solid, find correct to the nearest whole number:

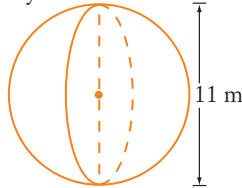
i its volume

ii its capacity

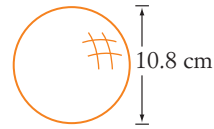
a



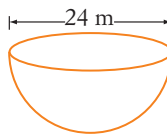
b



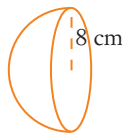
c



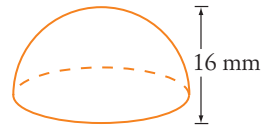
d



e



f

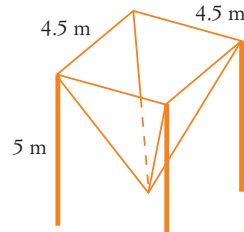


6 The Earth has a radius of approximately 6400 km. Calculate its volume in scientific notation correct to two significant figures.

7 A grain hopper is in the shape of a square pyramid.

a Find the volume of grain that it holds when full.

b If there are 750 kg of wheat per m^3 , find the mass of grain in the hopper when it is filled to three-quarters of capacity. Give your answer correct to the nearest tonne.



8 A pyramid has a volume of 360 m^3 and a base area of 48 m^2 . Calculate its perpendicular height.

9 A square pyramid has a volume of 800 cm^3 and a perpendicular height of 12 cm. Calculate, correct to one decimal place, the length of its base.

10 A cone has a volume of 600 m^3 and a base radius of 10 m. Calculate, correct to one decimal place, its perpendicular height.

11 A cone has a volume of 160 cm^3 and a perpendicular height of 20 cm. Calculate, correct to one decimal place, its radius.

12 Calculate, correct to one decimal place, the radius of a sphere with a volume of $81\,585 \text{ mm}^3$.

4-08 Volumes of composite solids

Worksheet

A page of solid shapes

MAT10MGWK10205

Worksheet

Volume and capacity

MAT10MGPS00046

Summary

Prism

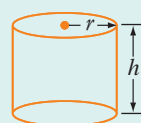
$$V = Ah$$



Cylinder

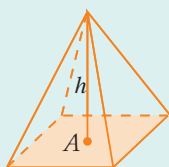
$$SA = 2\pi r^2 + 2\pi rh$$

$$V = \pi r^2 h$$



Pyramid

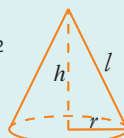
$$V = \frac{1}{3}Ah$$



Cone

$$SA = \pi rl + \pi r^2$$

$$V = \frac{1}{3}\pi r^2 h$$



Sphere

$$SA = 4\pi r^2$$

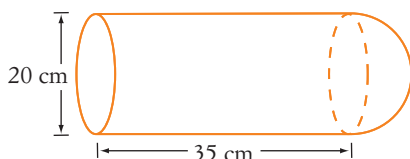
$$V = \frac{4}{3}\pi r^3$$



Note that the formulas for **surface area** involve **two dimensions**, for example, r^2 or rh , while the formulas for **volume** involve **three dimensions**, for example, lwh , r^2h or r^3 .

Example 20

- Find, correct to the nearest cubic centimetre, the volume of this solid.
- Find, correct to the nearest litre, the capacity of this solid.



Solution

- Volume = volume of cylinder + volume of hemisphere

$$= \pi r^2 h + \frac{1}{2} \times \frac{4}{3} \pi r^3$$

$$= \pi r^2 h + \frac{2}{3} \pi r^3$$

$$= \pi \times 10^2 \times 35 + \frac{2}{3} \times \pi \times 10^3$$

$$r = \frac{1}{2} \times 20 = 10$$

$$= 13\,089.9693 \dots$$

$$\approx 13\,090 \text{ cm}^3$$

- Capacity = 13 090 mL

$$= 13.09 \text{ L}$$

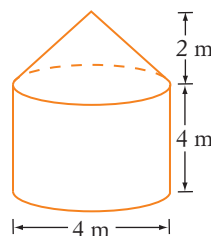
$$\approx 13 \text{ L}$$

See Example 20

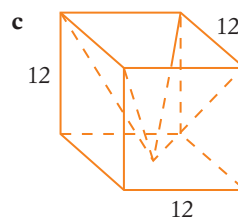
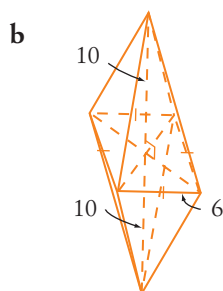
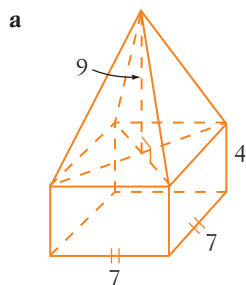
Exercise 4-08 Volumes of composite solids

- The storage tank shown is completely filled with water.

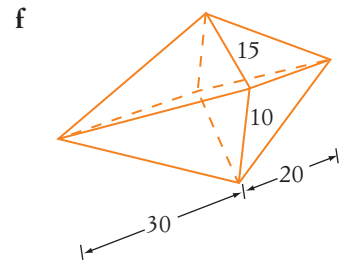
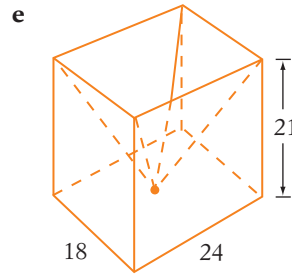
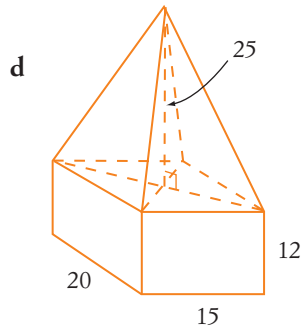
- Calculate, correct to the nearest cubic metre, the volume of the tank.
- Find the capacity of the tank, correct to the nearest kilolitre.



- Find the volume of each solid. All measurements are in centimetres.

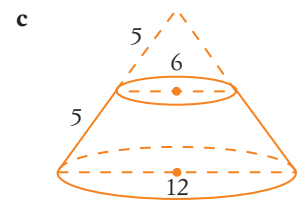
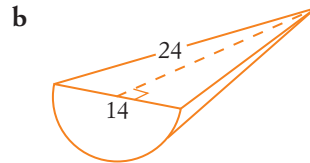
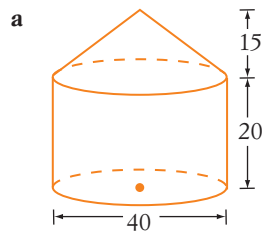


Stage 5.3

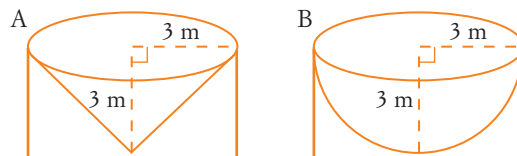


- 3 For each solid, find:
- the volume (to the nearest cm^3)
 - the capacity (in litres, correct to three decimal places).

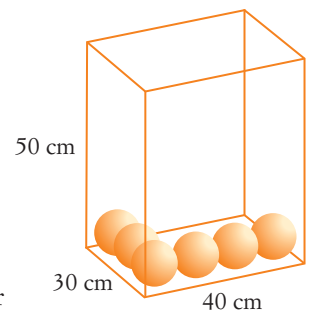
All measurements are in centimetres.



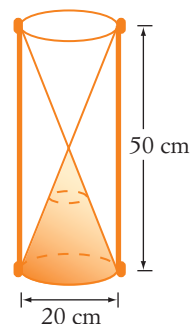
- 4 A conical tank (A) and a hemispherical tank (B) have measurements as shown. How much more does tank B hold compared to tank A? Answer correct to two decimal places.



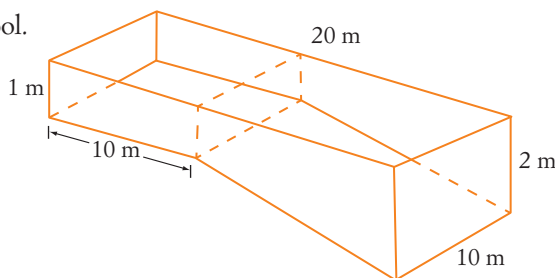
- 5 Spherical balls of diameter 10 cm are stacked inside a box in the shape of a rectangular prism, as shown.
- How many balls will fit in the bottom layer?
 - If the balls are stacked in the same manner as in the bottom layer until the box is full, how many balls will fit in the box?
 - Calculate, correct to the nearest cubic centimetre, the volume of the space occupied by the balls when the box is full.
 - What percentage of the box is empty space? Give your answer correct to the nearest whole percentage.



- 6 The sand in this hourglass takes up three-quarters of the volume of the bottom cone.
- Calculate, correct to the nearest cubic centimetre, the volume of sand in the hourglass.
 - If the sand takes one hour to fall from the top cone to the bottom cone, at what rate is it falling? Give your answer in cm^3/s , correct to two significant figures.



- 7
- Calculate the volume of this swimming pool.
 - Calculate the capacity of the pool if it is filled to a depth of 20 cm from the top.
 - If water costs \$1.98/kL, find the cost of filling the pool.



4-09 Areas of similar figures

Summary

Areas of similar figures

If the matching sides of two similar figures are in the ratio $1 : k$, then their areas are in the ratio $1 : k^2$.

If the matching sides are in the ratio $m : n$, then their areas are in the ratio $m^2 : n^2$.

$$A_1 : A_2 = m^2 : n^2 \quad \text{or} \quad \frac{A_1}{A_2} = \frac{m^2}{n^2}$$

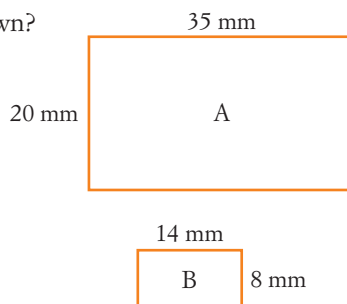
Example 21

What is the ratio of the areas of the similar rectangles shown?

Solution

Ratio of matching sides (A to B) = $35 : 14$
= $5 : 2$

Ratio of areas = $5^2 : 2^2$
= $25 : 4$



Stage 5.3

Technology worksheet

Excel worksheet: Area of similar shapes

MAT10MGCT00013

Technology worksheet

Excel spreadsheet: Area of similar shapes

MAT10MGCT00043

Stage 5.3

Example 22

Two similar pentagons have areas in the ratio $144 : 169$. Find the ratio of the lengths of their matching sides.

Solution

$$\text{Ratio of areas} = m^2 : n^2 = 144 : 169$$

$$\therefore \text{Ratio of sides} = m : n = \sqrt{144} : \sqrt{169} = 12 : 13$$

Example 23

Two similar triangles have matching sides in the ratio $3 : 5$. If the area of the larger triangle is 225 cm^2 , find the area of the smaller triangle.

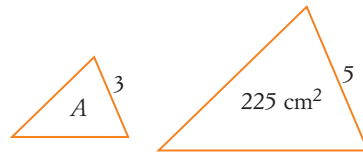
Solution

Let the area of the smaller figure be A .

$$\text{Ratio of matching sides} = 3 : 5$$

$$\text{Ratio of areas} = 3^2 : 5^2 = 9 : 25$$

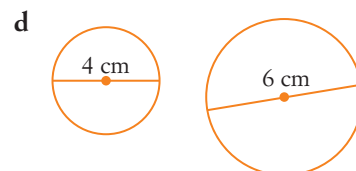
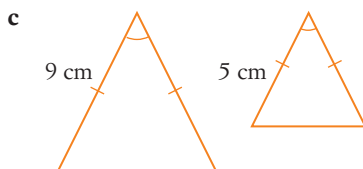
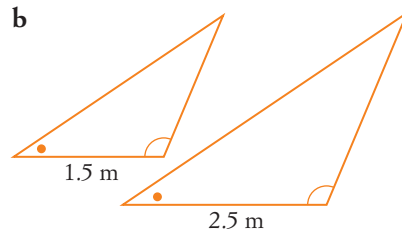
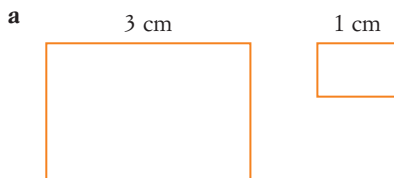
$$\begin{aligned} \therefore \frac{A}{225} &= \frac{9}{25} \\ A &= \frac{9}{25} \times 225 \\ &= 81 \text{ cm}^2 \end{aligned}$$



The area of the smaller triangle is 81 cm^2 .

Exercise 4-09 Areas of similar figures

1 For each pair of similar figures, find the ratio of their areas.

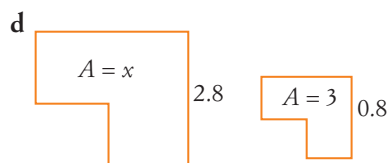
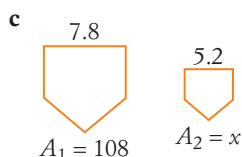
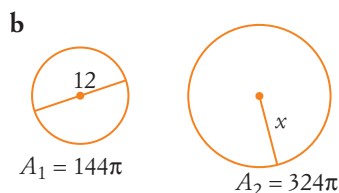
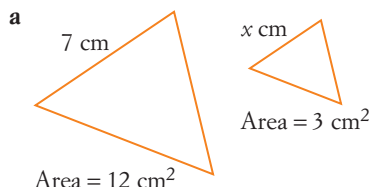


See Example 21

- 2 For each ratio of the areas of two similar figures, find the ratio of the lengths of their matching sides.

a 9 : 25 b 1 : 100 c 64 : 25 d 16 : 81

- 3 Find x if these triangles are similar.



- 4 Two circles have radii in the ratio 3 : 5. If the larger area is 150 cm², find the area of the smaller circle.
- 5 Similar squares have sides in the ratio 7 : 4. If the area of the smaller square is 14.4 cm², find the area of the larger square.
- 6 Two similar triangles have areas in the ratio 4 : 9. If the length of the base of the smaller triangle is 5 cm, find the length of the base of the larger triangle.
- 7 Two similar rectangles have their areas in the ratio 36 : 121. If the width of the smaller rectangle is 84 cm, find the width of the larger rectangle.
- 8 If the radius of a circle is doubled, how has its area changed?
- 9 If the area of a square is divided by 9, how have the sides changed?
- 10 If the sides of a triangle are increased by 2.5, how has its area changed?
- 11 If the area of a trapezium is decreased by $\frac{1}{100}$, how have the sides changed?

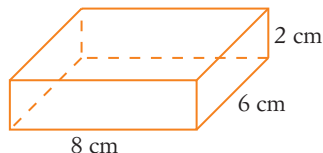
Stage 5.3

See Example 22

See Example 23

Investigation: Surface areas and volumes of similar solids

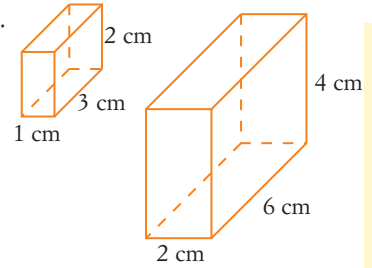
- 1 a Calculate the volume of this rectangular prism.
b Calculate the surface area of the rectangular prism.
c If the length, width and height are all **doubled**, what happens to:
i the volume? ii the surface area?
d Copy and complete:



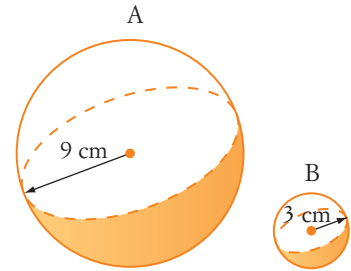
If the length, width and height are all doubled, the volume is increased _____ times and the surface area is increased _____ times.

Stage 5.3

- 2 a Explain why these rectangular prisms are similar solids.
 b What is the ratio of their matching sides?
 c What is the ratio of their surface areas?
 d What is the ratio of their volumes?



- 3 For the spheres A and B, find the ratio of:
 a their radii
 b their surface areas
 c their volumes



- 4 How is the ratio of the surface areas of similar solids related to the ratio of matching sides?
 5 How is the ratio of the volumes of similar solids related to the ratio of their matching sides?

NSW

4-10 Surface areas and volumes of similar solids

Worksheet

Areas and volumes of similar figures

MAT10MGWK10207

Summary

Surface areas and volumes of similar solids

If the matching sides of two similar solids are in the ratio $1 : k$, then their surface areas are in the ratio $1 : k^2$ and their volumes are in the ratio $1 : k^3$.

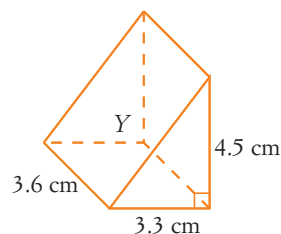
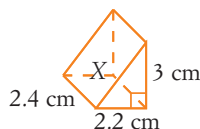
If the matching sides are in the ratio $m : n$, then their surface areas are in the ratio $m^2 : n^2$ and their volumes are in the ratio $m^3 : n^3$.

$$\frac{SA_1}{SA_2} = \frac{m^2}{n^2} \text{ and } \frac{V_1}{V_2} = \frac{m^3}{n^3}$$

Example 24

For these two similar triangular prisms, find the ratio of their:

- a surface areas
- b volumes



Solution

- a Ratio of sides (X to Y) = $3 : 4.5$ (or $2.2 : 3.3$ or $2.4 : 3.6$)
 $= 6 : 9$
 $= 2 : 3$

$$\begin{aligned}\text{Ratio of surface areas} &= 2^2 : 3^2 \\ &= 4 : 9\end{aligned}$$

- b Ratio of volumes = $2^3 : 3^3$
 $= 8 : 27$

Example 25

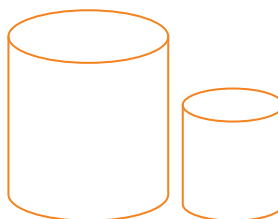
Two similar cylinders have their surface areas in the ratio $25 : 36$. If the volume of the smaller cylinder is 250 cm^3 , find the volume of the larger solid.

Solution

$$\text{Ratio of surface areas} = 25 : 36$$

$$\begin{aligned}\therefore \text{Ratio of matching sides} &= \sqrt{25} : \sqrt{36} \\ &= 5 : 6\end{aligned}$$

$$\begin{aligned}\therefore \text{Ratio of volumes} &= 5^3 : 6^3 \\ &= 125 : 216\end{aligned}$$



Let the volume of the larger cylinder be V .

$$\begin{aligned}\frac{V}{250} &= \frac{216}{125} \\ V &= \frac{216}{125} \times 250 \\ &= 432\end{aligned}$$

$\therefore$ The volume of the larger cylinder is 432 cm^3 .

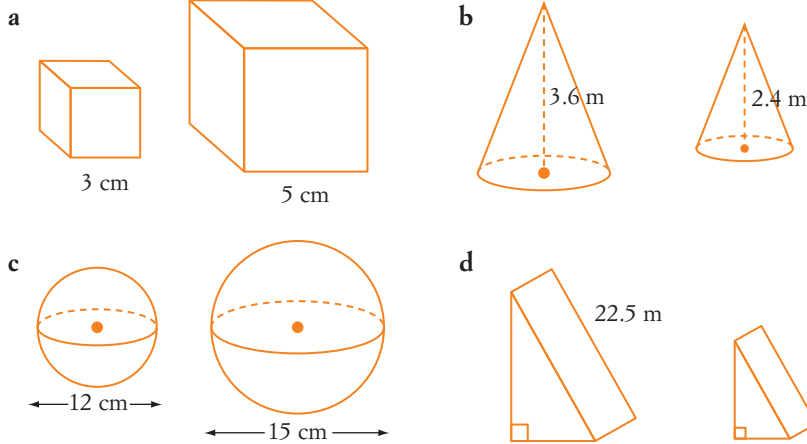
Stage 5.3

Exercise 4-10

Surface areas and volumes of similar solids

See Example 24

- 1 For each pair of similar solids, find the ratio of:
 - i the smaller surface area to the larger surface area
 - ii the smaller volume to the larger volume

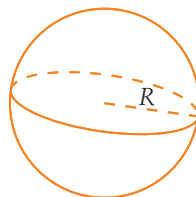
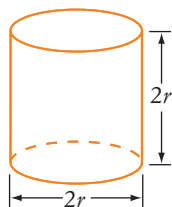


See Example 25

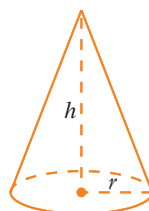
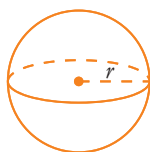
- 2 Two similar pyramids have surface areas of 81 cm^2 and 100 cm^2 . Find the ratio of their:
 - a matching side lengths
 - b volumes.
- 3 Two similar prisms have volumes of 125 cm^3 and 343 cm^3 . Find the ratio of their:
 - a matching sides
 - b surface areas.
- 4 Blocks of chocolate are sold in the shape of similar triangular prisms. The areas of the triangular faces of two prisms are 6400 mm^2 and 1600 mm^2 . If the volume of the smaller prism is 9600 mm^3 , find the volume of the larger prism.
- 5 There are two similar cylindrical drink cans. The larger can is 15 cm high and contains 350 mL of drink. If the smaller can is 9 cm high, how much drink does it contain?
- 6 A box of washing powder is 12 cm tall and contains 750 g of washing powder. A similar box is 18 cm tall. How much washing powder does it contain?
- 7 A large fish tank has a capacity of 624 L. A smaller, similar fish tank has half the length, width and depth of the large tank. Find the capacity of the smaller tank.
- 8 A cylinder has its height and radius increased 1.5 times. By what factor has its:
 - a surface area increased?
 - b volume increased?
- 9 A spherical balloon has a radius of 8 cm. By what factor is the volume decreased if the radius changes to 6 cm?

Power plus

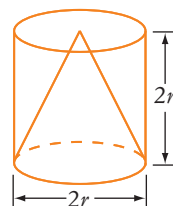
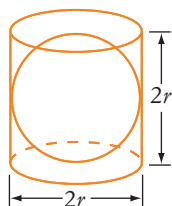
- 1 A square prism and square pyramid have the same base and the same surface area. Show that the slant height, l , of the pyramid is $l = \frac{5}{2}s$ where s is the length of the base.
- 2 A cylinder with diameter and height $2r$ has the same surface area as a sphere of radius R . Show that $R = \sqrt{\frac{3}{2}}r$.



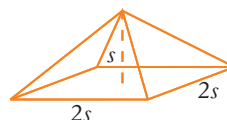
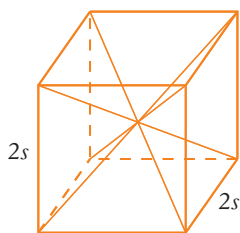
- 3 A sphere and a cone have the same radius and volume. Show that the cone's height, h , is four times the radius, r .



- 4 A sphere and a cone fit inside identical cylinders with the same base diameter and height.



- a Find the ratio 'Volume of cone : Volume of sphere : Volume of cylinder'
 - b Show that 'Volume of cone + Volume of sphere = Volume of cylinder'
- 5 A cube is divided into six identical square pyramids as shown, each with a perpendicular height that is half the length of the base edge. Show that the volume of each pyramid is one-third the volume of a square prism with the same base edge and perpendicular height.



Language of maths

Puzzle sheet
Surface area and
volume crossword
(Advanced)
MAT10MGPS10208

apex	base	capacity	circle
cone	cross-section	cubic	curved surface
cylinder	diameter	hemisphere	kilolitre
litre	perpendicular height	pyramid	radius
ratio	sector	similar figures	similar solids
slant height	sphere	surface area	volume

- Which word means a ‘slice’ of a prism or cylinder?
- Name three solids that have a **curved surface area**.
- What is the formula for the curved surface area of a cone?
- Explain the difference between the **perpendicular height** and the **slant height** of a pyramid.
- What is the formula $V = \frac{1}{3}\pi r^2 h$ used for?
- Describe the relationship between the volumes of similar solids.

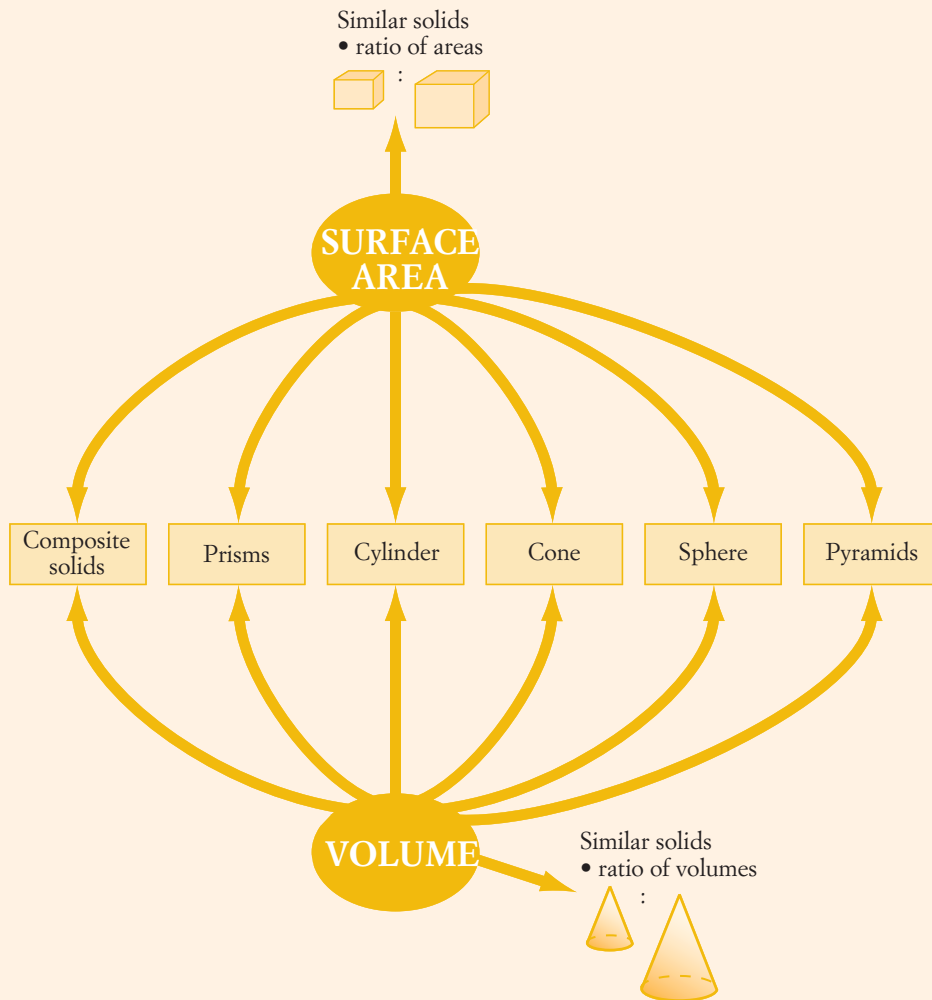
Topic overview

Quiz
Area and volume
MAT10MGQZ00004

Copy and complete the table below.

	The best part of this chapter was ...
	The worst part was ...
	New work ...
	I need help with ...

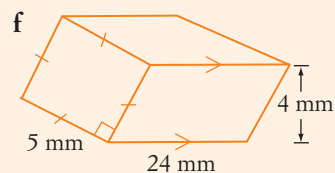
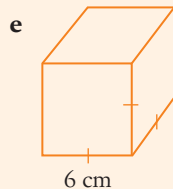
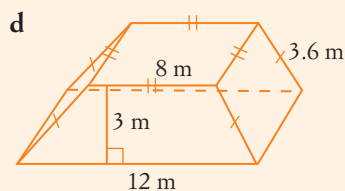
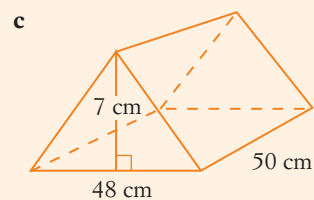
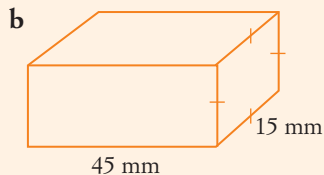
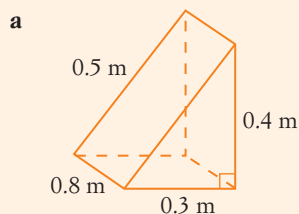
Copy and complete this mind map of the topic, adding detail to its branches and using pictures, symbols and colour where needed. Ask your teacher to check your work.



Chapter 4 revision

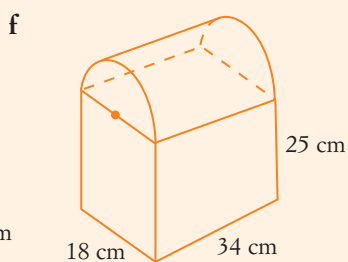
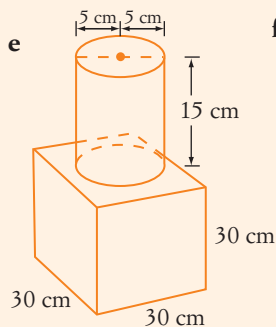
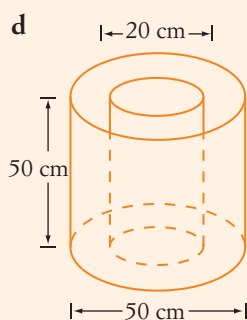
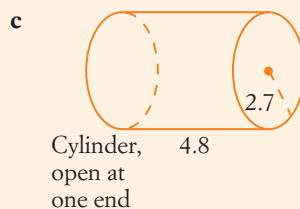
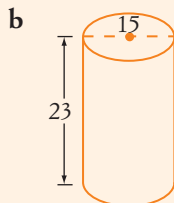
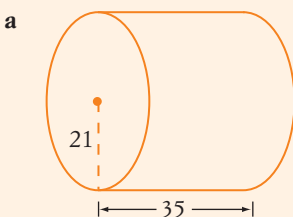
See Exercise 4-01

1 Find the surface area of each prism.



See Exercise 4-02

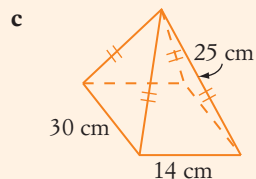
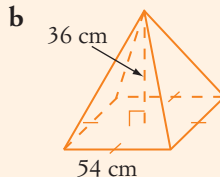
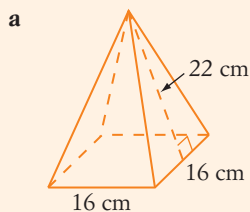
2 Calculate, correct to one decimal place, the surface area of each solid.



Stage 5.3

See Exercise 4-03

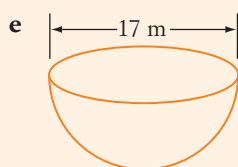
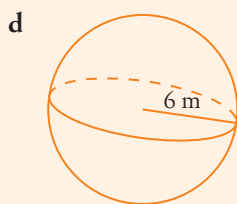
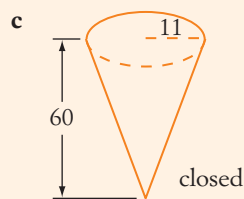
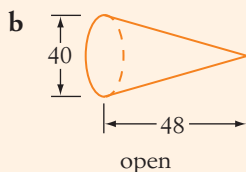
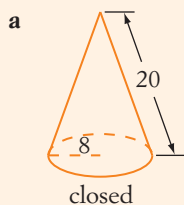
3 Find the surface area of each pyramid.



- 4 Find, correct to the nearest square metre, the surface area of each solid. All measurements are in metres.

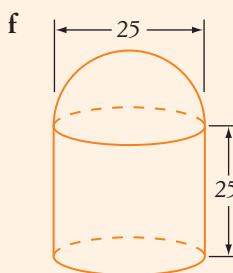
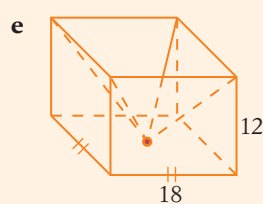
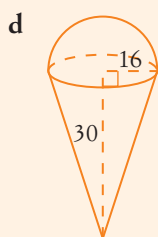
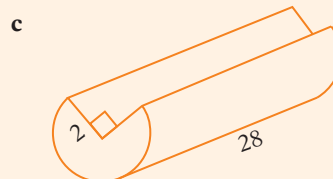
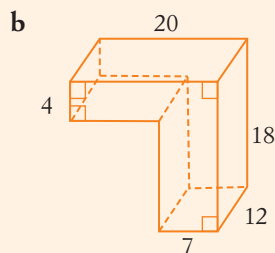
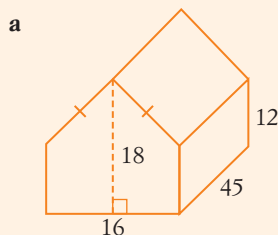
Stage 5.3

See Exercise 4-04



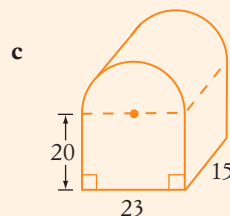
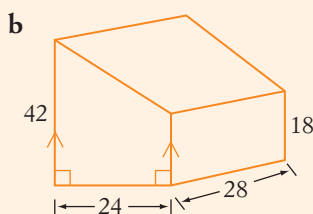
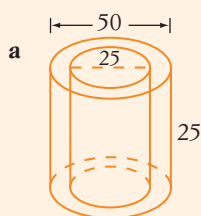
- 5 Find, correct to the nearest square centimetre, the surface area of each solid. All measurements are in centimetres.

See Exercise 4-05



- 6 Calculate, correct to nearest cubic metre, the volume of each solid. All measurements are in metres.

See Exercise 4-06



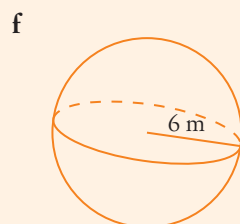
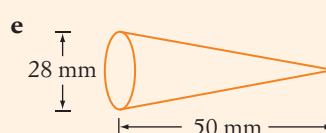
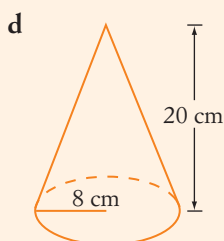
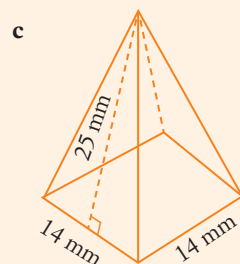
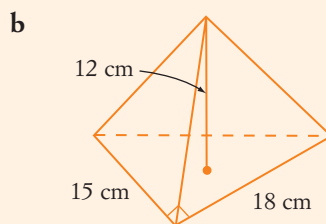
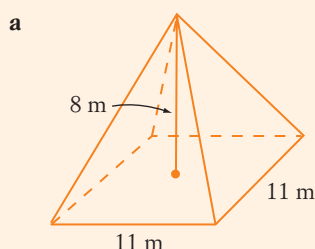
Stage 5.3

Chapter 4 revision

Stage 5.3

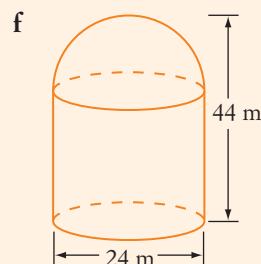
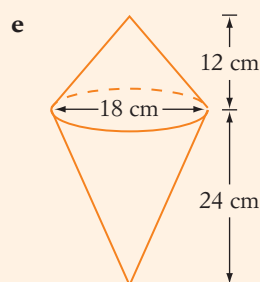
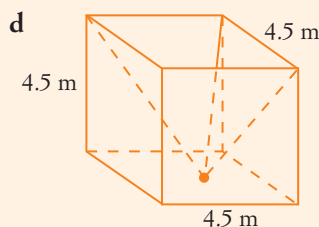
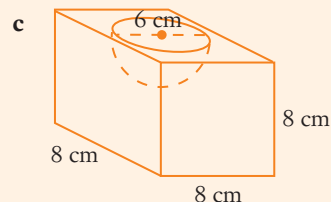
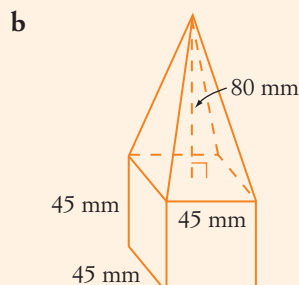
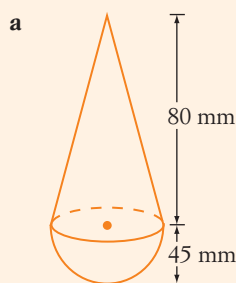
See Exercise 4-07

- 7 Find, correct to two decimal places (where necessary), the volume of each solid.



See Exercise 4-08

- 8 Find, correct to the nearest whole number, the volume of each solid.



See Exercise 4-09

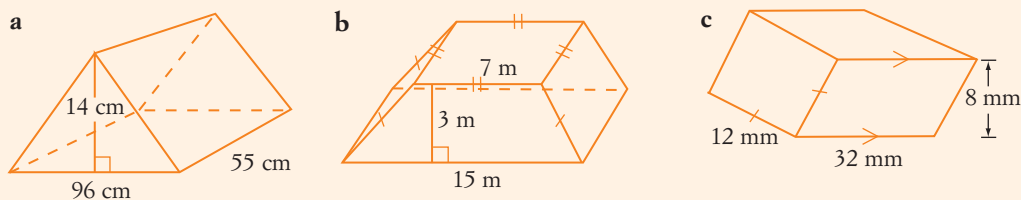
- 9 **a** Two similar circles have radii in the ratio 4 : 5. If the smaller area is 150 cm^2 , find the area of the larger circle.
b The radius of a circle is increased by a factor of $2\frac{1}{2}$. By what factor has the area increased?

See Exercise 4-10

- 10 **a** The areas of the bases of two similar rectangular prisms are in the ratio of 25 : 64. If the volume of the larger prism is 1024 cm^3 , find the volume of the smaller prism.
b Two similar pyramids have volumes of 216 cm^3 and 343 cm^3 . Find the ratio of their surface areas.

- 1 Find the surface area of each prism.

See Exercise 4-01



- 2 Find, correct to the nearest degree, the angle of inclination of a line with gradient $-\frac{3}{4}$.
- 3 Aymin is paid four weeks normal pay plus a 17.5% leave loading for his four-week annual holiday. If Aymin's salary is \$52 270, find his:
- a normal weekly pay b leave loading c total pay for the four-week holiday.
- 4 A checkout operator is employed under the following award.

See Exercise 2-01

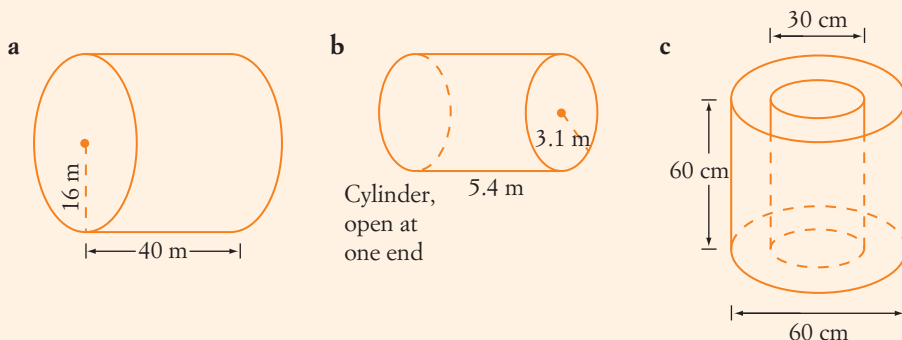
Normal rate: \$21.25 per hour	
Normal rate	For 0 to 36 hours worked
Time-and-a-half	For the next 4 hours worked
Double time	For each hour worked after that

Calculate the wage for working:

- a 20 hours b 39 hours c 45 hours
- 5 A line passes through the points $H(6, -3)$ and $K(8, -6)$. Calculate the gradient of the line:
- a parallel to HK b perpendicular to HK
- 6 Calculate, correct to one decimal place, the surface area of each solid.

See Exercise 3-02

See Exercise 4-02



- 7 Simplify each expression.
- a $\sqrt{200} + \sqrt{18}$ b $7\sqrt{44} - 6\sqrt{99}$ c $3\sqrt{6} + \sqrt{54} - 2\sqrt{27}$
- 8 Crystal earns a weekly wage of \$950. She has annual deductions made to her health fund of \$623 and work expenses totalling \$1567.
- a Calculate Crystal's taxable income.
- b Use the tax table on page 30 to calculate the income tax that Crystal should pay.

See Exercise 2-02

Stage 5.3

See Exercise 1-03

Mixed revision 1

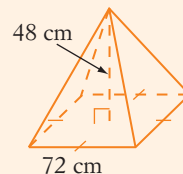
See Exercise 3-03

- 9 Graph the linear equations $y = 3x - 2$ and $y = -2x + 3$ on a number plane. Where do the lines intersect?

Stage 5.3

See Exercise 4-03

- 10 Find the surface area of this pyramid.



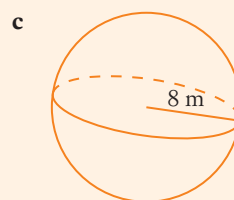
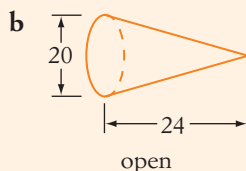
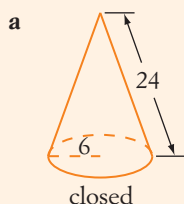
See Exercise 3-03

- 11 Which of the following points lie on the line of $2x + y = 3$? Select the correct answer **A, B, C** or **D**.
A (1, 0) **B** (2, -1) **C** (-1, -1) **D** (-1, -5)

Stage 5.3

See Exercise 4-04

- 12 Find, correct to the nearest square metre, the surface area of each solid. All measurements are in metres.



See Exercise 2-05

- 13 Calculate the value of an investment if \$4000 is invested at 6% p.a. for 3 years with interest compounded:
a annually **b** quarterly **c** monthly

See Exercise 3-05

- 14 Convert the linear equation $y = \frac{3x}{4} + 6$ to general form $ax + by + c = 0$

See Exercise 3-05

- 15 Find the gradient and y-intercept of the line with equation $5x - y + 3 = 0$.

See Exercise 2-06

- 16 Jovan purchases a home theatre system valued at \$7680. He pays a 10% deposit and repays the balance in 48 monthly instalments. Interest on the balance is charged at a flat rate of 12% p.a. Find:
a the deposit paid **b** the balance owing
c the interest charged **d** the total amount to be repaid
e the amount of each instalment **f** the total price Jovan paid for the system.

See Exercise 2-07

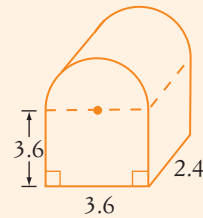
- 17 Aileen purchases a new car for \$29 990, which depreciates by 10% p.a.
a Find the depreciated value of the car after 4 years.
b What is the depreciation over this time?
c Express the depreciated value as a percentage of the cost price (correct to one decimal place).

Stage 5.3

See Exercise 3-06

- 18 Find, in general form, the equation of a line which passes through the points (4, -2) and (-1, 3).

- 19 Calculate, correct to nearest cubic metre, the volume of this solid.
All measurements are in metres.



See Exercise 4-06

- 20 The line $2x + 5y + 7 = 0$ and another line intersect at right angles at the point $(4, -5)$. Find the equation of the other line.

- 21 Rationalise the denominator of each surd.

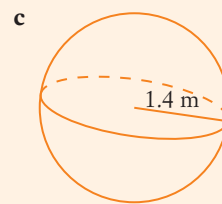
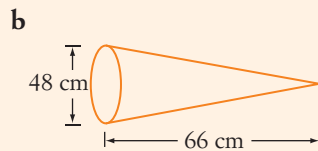
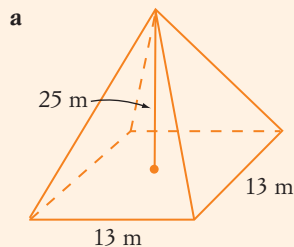
a $\frac{1}{\sqrt{10}}$

b $\frac{5\sqrt{2}}{2\sqrt{3}}$

c $\frac{4 + \sqrt{5}}{3\sqrt{5}}$

- 22 Show that the points $P(-2, -1)$, $Q(3, 2)$, $R(0, 7)$ and $S(-5, 4)$ are the vertices of a square.

- 23 Find, correct to two decimal places, the volume of each solid.



- 24 Two similar squares have sides in the ratio $3 : 5$. If the smaller area is 360 cm^2 , find the area of the larger square.

- 25 Expand and simplify each expression.

a $(5\sqrt{10} - 4\sqrt{2})(5\sqrt{10} + 4\sqrt{2})$

b $(\sqrt{6} - 7)^2$

Stage 5.3

See Exercise 3-08

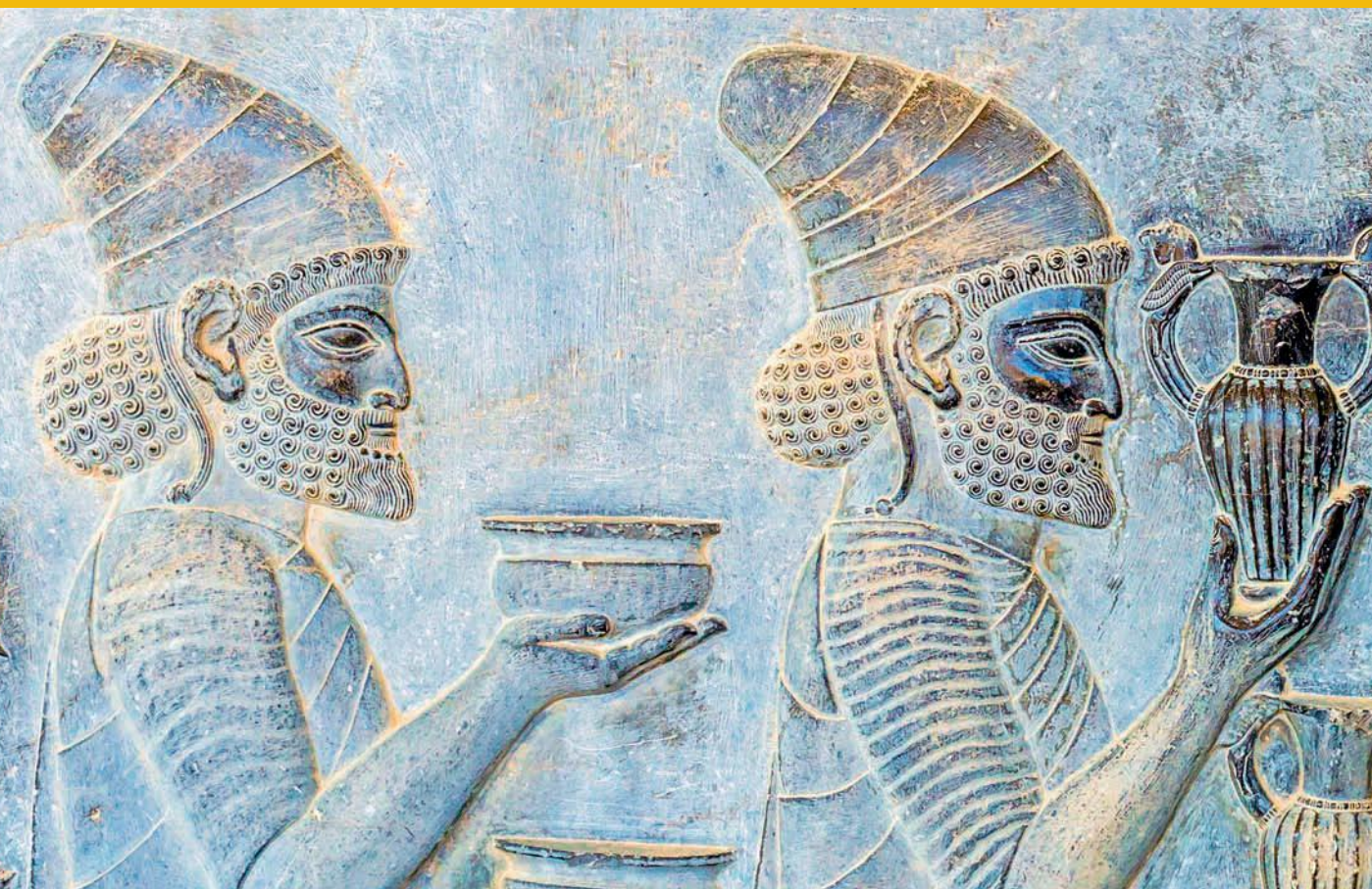
See Exercise 1-06

See Exercise 3-09

See Exercise 4-07

See Exercise 4-09

See Exercise 1-05



5

Number and Algebra

Products and factors

In 825 CE, the Persian mathematician al-Khwarizmi used the Arabic word 'al-jabr' to describe the process of adding equal quantities to both sides of an equation. When al-Khwarizmi's book was translated into Latin and introduced to Europe, 'al-jabr' became 'algebra' and the word was adopted as the name for the branch of mathematics that uses formulas to describe number patterns and relationships in our world.



Shutterstock.com/Ko Yo

Chapter outline

	Proficiency strands			
5-01 The index laws	U	F	R	C
5-02 Fractional indices*	U	F	R	C
5-03 Adding and subtracting algebraic fractions	U	F	R	C
5-04 Multiplying and dividing algebraic fractions	U	F	R	C
5-05 Expanding and factorising expressions	U	F	R	C
5-06 Expanding binomial products	U	F	R	C
5-07 Factorising special binomial products*	U	F	R	C
5-08 Factorising quadratic expressions	U	F	R	C
5-09 Factorising quadratic expressions of the form $ax^2 + bx + c$ *	U	F	R	C
5-10 Mixed factorisations*		F	R	C
5-11 Factorising algebraic fractions*	U	F	R	C

*STAGE 5.3

Wordbank

binomial An algebraic expression that consists of two terms; for example, $4a + 9$, $3 - y$, $x^2 - 4x$

binomial product Binomials multiplied together; for example, $(x + 9)(3x - 4)$.

index laws Rules for simplifying algebraic expressions involving powers of the same base, for example, $a^m \div a^n = a^{m-n}$.

perfect square A square number or an algebraic expression that represents one; for example, 64 , $(x + 9)^2$

quadratic expression An algebraic expression in which the highest power of the variable is 2; for example, $2x^2 + 5x - 3$ or $x^2 + 2$

In this chapter you will:

- apply index laws to numerical expressions with integer indices
- apply index laws to algebraic expressions with integer indices
- simplify algebraic products and quotients using index laws
- expand binomial products and factorise monic quadratic expressions using a variety of strategies 
- interpret and use zero and negative indices
- (STAGE 5.3) interpret and use fractional indices
- add, subtract, multiply and divide simple algebraic fractions
- (STAGE 5.3) add and subtract algebraic fractions with binomial numerators
- expand and factorise algebraic expressions
- expand and factorise algebraic expressions involving terms with indices
- (STAGE 5.3) expand special binomial products such as $(a + b)^2$, $(a - b)^2$, $(a + b)(a - b)$
- (STAGE 5.3) factorise algebraic expressions using grouping in pairs, perfect squares, the difference of two squares and quadratic expressions of the form $ax^2 + bx + c$
- (STAGE 5.3) factorise and simplify expressions involving algebraic fractions

SkillCheck

Worksheet

StartUp assignment 4

MAT10NAWK10024

1 Simplify each expression.

a $g^4 \times g^5$

b $r^8 \div r^2$

c $(d^5)^3$

d $(-k)^2$

e $h \times h^9$

f $m^5 \div m$

g a^1

h a^0

i $3e^2 \times 2e^5$

j $18n^6 \div 6n^2$

k $(10w^3)^3$

l $25q^0$

m $(vw)^5$

n $\left(\frac{v}{w}\right)^3$

o y^{-1}

p y^{-2}

2 Evaluate each expression.

a $\frac{a}{5} + \frac{3a}{4}$

b $\frac{7p}{2} - \frac{10p}{3}$

c $\frac{8}{t} \times \frac{5t}{24}$

d $\frac{x}{14} \div \frac{y}{2}$

3 Expand each expression.

a $6m(3m + 11)$

b $-5(3g - 8)$

4 Factorise each expression.

a $4x + 24$

b $20 - 15a$

c $q^2 + q$

d $18a^2 - 12a$

e $-2y - 30$

f $-18w + 24$

5 Find two numbers whose:

a product is 18 and sum is 9

b product is 8 and sum is -6

c product is -20 and sum is -1

d product is -16 and sum is 6

5-01 The index laws

Worksheet

Index laws review

MAT10NAWK10025

Animated example

Index laws

MAT10NAAE00001

Video tutorial

Numbers and powers

MAT10NAVT00001

Summary

When **multiplying terms with the same base**, add the powers

$$a^m \times a^n = a^{m+n}$$

When **dividing terms with the same base**, subtract the powers

$$a^m \div a^n = \frac{a^m}{a^n} = a^{m-n}$$

When **raising a term with a power to another power**, multiply the powers

$$(a^m)^n = a^{m \times n}$$

When **raising a product of terms to a power**, raise each term to that power

$$(ab)^n = a^n b^n$$

When **raising a quotient of terms to a power**, raise each term to that power

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

Any number **raised to the power of zero** is equal to 1

$$a^0 = 1$$

A number **raised to a negative power** gives a fraction (with a numerator of 1)

$$a^{-n} = \frac{1}{a^n}$$

A number **raised to a power of -1** gives its reciprocal

$$a^{-1} = \frac{1}{a}$$

$$\left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$$

A number **raised to a power of $-n$** gives its reciprocal raised to the power of n .

$$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n = \frac{b^n}{a^n}$$

Stage 5.3

Example 1

Simplify each expression.

a $\frac{24e^6n^{12}}{8en^4}$

b $(4d^4q^2r)^3$

c $\left(\frac{2c^2}{d}\right)^4$

d $31x^0 - (31x)^0$

e $9x^{-1}$

f $(4q)^{-3}$

Solution

$$\begin{aligned} \text{a } \frac{24e^6n^{12}}{8en^4} &= \frac{\overset{3}{\cancel{24}}e^6n^{12}}{\underset{1}{\cancel{8}}en^4} \\ &= 3e^5n^8 \end{aligned}$$

$$\begin{aligned} \text{b } (4d^4q^2r)^3 &= 4^3d^{4 \times 3}q^{2 \times 3}r^3 \\ &= 64d^{12}q^6r^3 \end{aligned}$$

$$\begin{aligned} \text{c } \left(\frac{2c^2}{d}\right)^4 &= \frac{(2c^2)^4}{d^4} \\ &= \frac{2^4c^{2 \times 4}}{d^4} \\ &= \frac{16c^8}{d^4} \end{aligned}$$

$$\begin{aligned} \text{d } 31x^0 - (31x)^0 &= 31 \times 1 - 1 \\ &= 30 \end{aligned}$$

$$\begin{aligned} \text{e } 9x^{-1} &= 9 \times \frac{1}{x} \\ &= \frac{9}{x} \end{aligned}$$

$$\begin{aligned} \text{f } (4q)^{-3} &= \frac{1}{(4q)^3} \\ &= \frac{1}{64q^3} \end{aligned}$$

Stage 5.3**Example 2**Simplify $\left(\frac{2}{3r}\right)^{-3}$.**Solution**

$$\begin{aligned} \left(\frac{2}{3r}\right)^{-3} &= \left(\frac{3r}{2}\right)^3 \\ &= \frac{27r^3}{8} \end{aligned}$$

Exercise 5-01 The index laws**See Example 1**

1 Simplify each expression.

a $3p^2 \times 2p^5$

b $20w^9 \div 4w^3$

c $(-m^3)^{10}$

d $(3q^3)^2$

e $5n^8t \times 6n^8t^4$

f $32x^6y^8 \div 8xy^4$

g $5e^4g^3 \times (-e^6g)$

h $\frac{45a^{15}b^5}{-5a^5}$

i $(10y^{10})^2$

j $(-4p)^3$

k $\frac{36pq^3r^6}{24qr^2}$

l $9u^3v \times 6uv^2w^8$

2 Simplify each expression.

a $(l^3m^5)^6$

b $\left(\frac{n}{2}\right)^3$

c $7x^0$

d $\left(\frac{w^2}{t^3}\right)^5$

e $\left(\frac{2}{3}\right)^0$

f $(-8ky^5)^2$

g $(16a)^0 - 16a^0$

h $\left(\frac{2b}{3d}\right)^4$

i $(-5d^3y^5)^3$

j $\left(-\frac{3k^4}{10}\right)^3$

k $-9(a^2b^3)^0$

l $(3p^2q^3r^4)^4$

3 Evaluate each expression.

a 4^0	b $(-4)^0$	c 7×2^0	d $(7 \times 2)^0$
e $(-2)^3$	f $(-3)^2$	g $(5^2)^2$	h $2^4 \times 2^3$
i $(7^2)^0$	j $4^5 \div 4^2$	k $4^2 \div 4^5$	l $10^3 \div 10^3$
m $5^2 \div 5^0$	n $10^{-2} \div 10^2$	o $(\frac{1}{2})^0$	p $10^{-2} \times 10^2$

4 Express each of the terms below as fractions.

a 5^{-2}	b 2^{-5}	c 20^{-1}	d 10^{-3}

5 Simplify each expression using a positive index.

a 8^{-7}	b 3^{-5}	c y^{-1}	d x^{-3}
e $(5b)^{-2}$	f $5b^{-2}$	g $(ab)^{-1}$	h ab^{-1}
i $11t^{-3}$	j $(11t)^{-3}$	k p^3q^{-5}	l mw^{-3}
m $8u^{-3}v^{-4}$	n $-2r^6y^{-5}$	o $10e^{-1}f^3$	p $\frac{1}{2}k^{-4}n^7$

6 Simplify each expression.

a $4h^{-2}$	b $(\frac{k}{3})^{-1}$	c $(4h)^{-2}$	d $(-\frac{m}{2})^{-1}$
e $(\frac{5}{3g^3})^{-1}$	f $(\frac{2r}{3t})^{-1}$	g m^3np^{-2}	h $(\frac{5b}{4a^2})^{-1}$

7 Simplify each expression.

a $(\frac{4}{3})^{-5}$	b $(\frac{5}{4})^{-4}$	c $(2\frac{1}{4})^{-2}$	d $(-1\frac{2}{5})^{-3}$
e $(\frac{k}{3})^{-2}$	f $(\frac{3}{x})^{-3}$	g $(-\frac{a^2}{4})^{-4}$	h $(\frac{4}{3g^3})^{-2}$
i $(-\frac{2d}{5t})^{-2}$	j $(-\frac{h^2}{m^3})^{-5}$	k $(\frac{5d^3}{3p^4})^{-2}$	l $(\frac{3c^3}{4a^2})^{-3}$

8 Simplify each expression.

a $(10x^{10}y)^3 \times 5x^{-2}y^3$	b $(10x^{10}y \div 5x^{-2}y^3)^3$	c $(3q^{-5}r^3 \div 6qr^2)^2$
d $3q^{-5}r^3 \div (6qr^2)^2$	e $(\frac{4a^5x^6}{a^6x^2})^4$	f $(\frac{4a^5x^6}{a^6x^2})^{-4}$
g $(4p^{-3}h)^2 \times (-2p^6h^9)$	h $(4p^{-3}h)^2 \div (-2p^6h^9)$	i $[4p^3h \times (-2p^2h^9)]^2$

Stage 5.3

See Example 2

Investigation: Videos going viral

Use a spreadsheet to help you with this investigation.

With the invention of social media, we can

now communicate with each other

instantly. However, what often starts

off as a simple message between friends

or an online video posting can quickly

multiply at an alarming rate and

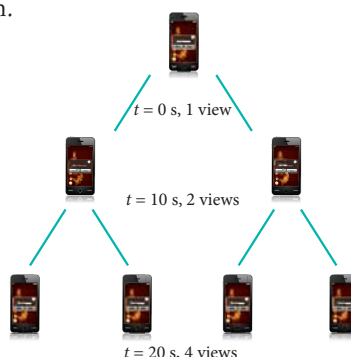
'go viral' (spread like a virus).

Suppose a video is shared with a friend

who 10 seconds later shares it with two

other friends. This occurs every 10 seconds so

that every 10 seconds the number of views doubles.



- 1 Use a spreadsheet to calculate the total number of views after:

a 30 s	b 40 s	c 1 min	d 1 min 30 s
e 2 min	f 3 min	g 4 min	h 5 min
- 2 Find how long it will take until the total number of views reaches:

a 64	b 256	c over 500	d over 1000
e over 3000	f over 10 000	g over 1 million	h over 4 million
- 3 Use the spreadsheet's **Graph Option** to graph the viral pattern. Describe the graph's shape.

Stage 5.3

Puzzle sheet

Indices squaresaw

MAT10NAPS00002

Puzzle sheet

Exponential equations

MAT10NAPS00040

5-02 Fractional indices

Summary

$$a^{\frac{1}{2}} = \sqrt{a}$$

Any number raised to the power of $\frac{1}{2}$ is the square root of that number

$$a^{\frac{1}{3}} = \sqrt[3]{a}$$

Any number raised to the power of $\frac{1}{3}$ is the cube root of that number

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

Any number raised to the power of $\frac{1}{n}$ is the n th root of that number

$$a^{\frac{m}{n}} = (\sqrt[n]{a})^m \text{ or } \sqrt[n]{a^m}$$

Any number raised to the power of $\frac{m}{n}$ is the n th root of that number raised to the power m .

Note: Taking the root first often makes the calculation simpler.

Example 3

Evaluate each expression.

a $729^{\frac{1}{2}}$

b $512^{\frac{1}{3}}$

c $6561^{\frac{1}{8}}$

Solution

$$\begin{aligned} \text{a } 729^{\frac{1}{2}} &= \sqrt{729} \\ &= 27 \end{aligned}$$

$$\begin{aligned} \text{b } 512^{\frac{1}{3}} &= \sqrt[3]{512} \\ &= 8 \end{aligned}$$

$$\begin{aligned} \text{c } 6561^{\frac{1}{8}} &= \sqrt[8]{6561} \\ &= 3 \end{aligned}$$

Enter on calculator: 8 $\sqrt[n]{}$ 6561 =
because $3^8 = 6561$

Summary

On a calculator, the n th root key is $\sqrt[n]{}$ or $\sqrt[n]{}$, found by pressing the **SHIFT** or **2ndF** key before pressing $x^{\frac{1}{n}}$ or y^x respectively.

Example 4

Evaluate each expression.

a $8^{\frac{2}{3}}$

b $64^{-\frac{1}{3}}$

c $16^{-\frac{3}{4}}$

Solution

$$\begin{aligned} \text{a } 8^{\frac{2}{3}} &= (\sqrt[3]{8})^2 \\ &= 2^2 \\ &= 4 \end{aligned}$$

$$\begin{aligned} \text{b } 64^{-\frac{1}{3}} &= \frac{1}{64^{\frac{1}{3}}} \\ &= \frac{1}{\sqrt[3]{64}} \\ &= \frac{1}{4} \end{aligned}$$

$$\begin{aligned} \text{c } 16^{-\frac{3}{4}} &= \frac{1}{16^{\frac{3}{4}}} \\ &= \frac{1}{(\sqrt[4]{16})^3} \\ &= \frac{1}{2^3} = \frac{1}{8} \end{aligned}$$

Example 5

Evaluate $420^{\frac{4}{7}}$ correct to two decimal places.

Solution

$$\begin{aligned} 420^{\frac{4}{7}} &= 31.550\,016\,78\dots \\ &\approx 31.55 \end{aligned}$$

Enter on a calculator: 420 $\times^{\frac{\Box}{\Box}}$ 4 $\frac{\Box}{\Box}$ 7 =

Example 6

Write each expression using a fractional index.

a $\sqrt{w^9}$

b $\sqrt[5]{k^2}$

c $\frac{1}{\sqrt[7]{y^5}}$

Solution

$$\begin{aligned} \text{a } \sqrt{w^9} &= (w^9)^{\frac{1}{2}} \\ &= w^{\frac{9}{2}} \end{aligned}$$

$$\begin{aligned} \text{b } \sqrt[5]{k^2} &= (k^2)^{\frac{1}{5}} \\ &= k^{\frac{2}{5}} \end{aligned}$$

$$\begin{aligned} \text{c } \frac{1}{\sqrt[7]{y^5}} &= \frac{1}{(y^5)^{\frac{1}{7}}} \\ &= \frac{1}{y^{\frac{5}{7}}} \text{ or } y^{-\frac{5}{7}} \end{aligned}$$

Example 7

Simplify each expression.

a $\sqrt[4]{(16r^2)^3}$

b $(32a^5)^{-\frac{3}{5}}$

Solution

$$\begin{aligned} \text{a } \sqrt[4]{(16r^2)^3} &= (16r^2)^{\frac{3}{4}} \\ &= 16^{\frac{3}{4}} r^{2 \times \frac{3}{4}} \\ &= 8r^{\frac{3}{2}} \end{aligned}$$

$$\begin{aligned} \text{b } (32a^5)^{-\frac{3}{5}} &= \frac{1}{(32a^5)^{\frac{3}{5}}} \\ &= \frac{1}{32^{\frac{3}{5}} a^{5 \times \frac{3}{5}}} \\ &= \frac{1}{8a^3} \end{aligned}$$

Stage 5.3

Exercise 5-02 Fractional indices

See Example 3

1 Evaluate each expression.

a $64^{\frac{1}{2}}$

b $-27^{\frac{1}{3}}$

c $625^{\frac{1}{5}}$

d $1000^{\frac{1}{3}}$

e $16^{\frac{1}{4}}$

f $(-0.000\ 32)^{\frac{1}{5}}$

g $(0.01)^{\frac{1}{2}}$

h $-512^{\frac{1}{3}}$

i $(-8)^{\frac{1}{3}}$

j $(-729)^{\frac{1}{3}}$

k $256^{\frac{1}{8}}$

l $3125^{\frac{1}{5}}$

2 Write each expression using a radical (root) sign.

a $10^{\frac{1}{2}}$

b $12^{\frac{1}{3}}$

c $g^{\frac{1}{2}}$

d $m^{\frac{1}{4}}$

e $(8r)^{\frac{1}{2}}$

f $(6h)^{\frac{1}{6}}$

g $(5j^8)^{\frac{1}{5}}$

h $(90ab)^{\frac{1}{3}}$

3 Write each expression using a fractional index.

a $\sqrt{5}$

b $\sqrt[3]{100}$

c $\sqrt[4]{20}$

d $\sqrt[5]{400}$

e $\sqrt{a}$

f $\sqrt[3]{q}$

g $\sqrt[7]{xy}$

h $\sqrt[6]{36w}$

See Example 4

4 Evaluate each expression.

a $8^{\frac{5}{3}}$

b $81^{\frac{3}{4}}$

c $256^{\frac{3}{4}}$

d $125^{\frac{2}{3}}$

e $8^{-\frac{1}{3}}$

f $64^{-\frac{1}{3}}$

g $64^{-\frac{1}{6}}$

h $81^{-\frac{1}{4}}$

i $1000^{-\frac{2}{3}}$

j $128^{-\frac{3}{7}}$

k $400^{-\frac{3}{2}}$

l $3125^{-\frac{2}{5}}$

See Example 5

5 Evaluate each expression correct to two decimal places.

a $300^{\frac{1}{3}}$

b $215^{\frac{1}{2}}$

c $\sqrt[3]{-144}$

d $\sqrt[5]{1000}$

e $15^{\frac{2}{3}}$

f $8^{-\frac{7}{3}}$

g $50^{\frac{5}{3}}$

h $6^{-\frac{2}{3}}$

See Example 6

6 Write each expression using a fractional index.

a $\sqrt[5]{n^2}$

b $\sqrt{d^5}$

c $\sqrt[6]{p^{18}}$

d $\frac{1}{\sqrt[4]{m^{16}}}$

e $\sqrt[3]{a^5}$

f $\sqrt[3]{a^3}$

g $\frac{1}{\sqrt[4]{x^3}}$

h $\frac{1}{\sqrt[3]{x^4}}$

See Example 7

7 Simplify each expression.

a $(16x^4)^{\frac{3}{4}}$

b $(8y)^{\frac{2}{3}}$

c $\sqrt[5]{(243d^{10})^3}$

d $(\sqrt[3]{64m^8})^3$

e $(\sqrt[4]{81r^4})^5$

f $\sqrt[3]{(1024h^{10})^3}$

g $\frac{1}{(\sqrt[3]{8s^6})^4}$

h $\left(\frac{1}{\sqrt[4]{256p^{12}}}\right)^2$

i $(125n^6)^{-\frac{2}{3}}$

j $(625t^2)^{-\frac{3}{4}}$

k $(49a^4b^{10})^{\frac{3}{2}}$

l $(1000x^3y^6)^{-\frac{2}{3}}$

Video tutorial

Algebraic fractions

MAT10NAVT10007

5-03

Adding and subtracting algebraic fractions

Summary

To **add or subtract fractions**, convert them (if needed) so that they will have the same denominator, then simply add or subtract the numerators.

Example 8

Simplify each expression.

a $\frac{a}{2} + \frac{a}{3}$

b $\frac{2x}{5} - \frac{x}{3}$

c $\frac{5}{y} + \frac{7}{2y}$

Solution

$$\begin{aligned} \text{a } \frac{a}{2} + \frac{a}{3} &= \frac{3 \times a}{3 \times 2} + \frac{2 \times a}{2 \times 3} \\ &= \frac{3a}{6} + \frac{2a}{6} \\ &= \frac{5a}{6} \end{aligned}$$

Common denominator = $2 \times 3 = 6$

$$\begin{aligned} \text{b } \frac{2x}{5} - \frac{x}{3} &= \frac{3 \times 2x}{3 \times 5} - \frac{5 \times x}{5 \times 3} \\ &= \frac{6x}{15} - \frac{5x}{15} \\ &= \frac{x}{15} \end{aligned}$$

$$\begin{aligned} \text{c } \frac{5}{y} + \frac{7}{2y} &= \frac{2 \times 5}{2y} + \frac{7}{2y} \\ &= \frac{10}{2y} + \frac{7}{2y} \\ &= \frac{17}{2y} \end{aligned}$$

The lowest common denominator is $2y$.

Example 9

Simplify each expression.

a $\frac{2x+5}{6} + \frac{x-4}{3}$

b $\frac{x}{3} - \frac{x+1}{5}$

Solution

$$\begin{aligned} \text{a } \frac{2x+5}{6} + \frac{x-4}{3} &= \frac{2x+5}{6} + \frac{2(x-4)}{6} \\ &= \frac{2x+5}{6} + \frac{2x-8}{6} \\ &= \frac{2x+5+2x-8}{6} \\ &= \frac{4x-3}{6} \end{aligned}$$

$$\begin{aligned} \text{b } \frac{x}{3} - \frac{x+1}{5} &= \frac{5x}{15} - \frac{3(x+1)}{15} \\ &= \frac{5x}{15} - \frac{3x+3}{15} \\ &= \frac{5x-3x-3}{15} \\ &= \frac{2x-3}{15} \end{aligned}$$

Stage 5.3

Animated example

Algebraic fractions

MAT10NAAE00003

Video tutorial

Algebraic fractions

MAT10NAVT00003

Exercise 5-03

Adding and subtracting algebraic fractions

See Example 8

1 Simplify each expression.

a $\frac{n}{2} - \frac{n}{7}$

b $\frac{3c}{2} + \frac{c}{5}$

c $\frac{4r}{7} + \frac{r}{2}$

d $\frac{7y}{8} - \frac{5y}{3}$

e $\frac{4t}{3} + \frac{t}{9}$

f $\frac{y}{16} - \frac{3y}{8}$

g $\frac{11t}{12} - \frac{5t}{9}$

h $\frac{3a}{10} + \frac{4a}{15}$

i $\frac{p}{z} + \frac{3}{z}$

j $\frac{5u}{8g} - \frac{3u}{8g}$

k $\frac{4}{9f} - \frac{1}{9f}$

l $\frac{7e}{10} - \frac{3e}{10}$

m $\frac{5}{t} + \frac{3}{r}$

n $\frac{7}{4h} - \frac{1}{h}$

o $\frac{5}{3b} + \frac{3}{4b}$

p $\frac{6}{2n} - \frac{4}{3p}$

2 Simplify each expression.

a $\frac{m}{8} - \frac{n}{5}$

b $\frac{5t}{4} - \frac{2t}{5}$

c $\frac{2a}{15} + \frac{3h}{10}$

d $\frac{d}{16} - \frac{r}{24}$

e $\frac{3c}{2} - \frac{c}{5}$

f $\frac{2d}{11} - \frac{r}{3}$

g $\frac{3h}{5} + \frac{2a}{3}$

h $\frac{5}{6} + \frac{4w}{5}$

i $\frac{3}{4} + \frac{2a}{7}$

j $\frac{7e}{8} - \frac{2e}{3}$

k $\frac{m}{2} - \frac{n}{7}$

l $\frac{2k}{5} + \frac{k}{6}$

3 Simplify each expression.

a $\frac{m+3}{5} + \frac{m-2}{4}$

b $\frac{x}{5} + \frac{x+2}{3}$

c $\frac{y+3}{4} + \frac{y-1}{3}$

d $\frac{x-1}{4} + \frac{x-2}{5}$

e $\frac{m+6}{5} - \frac{m+2}{3}$

f $\frac{k+3}{10} - \frac{k-1}{7}$

g $\frac{k-1}{2} - \frac{k}{3}$

h $\frac{3x-1}{4} + \frac{2x-9}{5}$

i $\frac{2m+5}{4} - \frac{m-1}{9}$

j $\frac{1-m}{5} - \frac{m-2}{7}$

k $\frac{9-2k}{3} - \frac{k-5}{2}$

l $\frac{8-3x}{2} - \frac{2x+5}{3}$

Stage 5.3

See Example 9

Worksheet

Algebraic fractions

MAT10NAWK10026

5-04

Multiplying and dividing algebraic fractions

Summary

- To **multiply fractions**, cancel any common factors, then multiply the numerators and denominators separately.
- To **divide by a fraction** $\frac{a}{b}$, multiply by its reciprocal $\frac{b}{a}$.

Video tutorial

Algebraic fractions

MAT10NAVT10007

Example 10

Simplify each expression.

a $\frac{3x}{4} \times \frac{2x}{9}$

b $\frac{4}{k} \times \frac{3k}{16}$

c $\frac{2}{v} \div \frac{3}{w}$

d $\frac{xy}{5} \div \frac{3x}{25}$

Solution

$$\begin{aligned} \text{a } \frac{3x}{4} \times \frac{2x}{9} &= \frac{3x \times 2x}{4 \times 9} & \text{or } \frac{3x}{4} \times \frac{2x}{9} &= \frac{\cancel{3}^1 x}{\cancel{2}^2 4} \times \frac{\cancel{2}^1 x}{\cancel{9}_3} \\ &= \frac{6x^2}{36} & &= \frac{x^2}{6} \\ &= \frac{x^2}{6} \end{aligned}$$

$$\begin{aligned} \text{b } \frac{4}{k} \times \frac{3k}{16} &= \frac{\cancel{4}^1}{\cancel{k}_1} \times \frac{3\cancel{k}}{\cancel{16}_4} \\ &= \frac{3}{4} \end{aligned}$$

$$\begin{aligned} \text{c } \frac{2}{v} \div \frac{3}{w} &= \frac{2}{v} \times \frac{w}{3} \\ &= \frac{2w}{3v} \end{aligned}$$

$$\begin{aligned} \text{d } \frac{xy}{5} \div \frac{3x}{25} &= \frac{\cancel{x}y}{\cancel{5}_1} \times \frac{\cancel{25}^5}{\cancel{3x}_1} \\ &= \frac{5y}{3} \end{aligned}$$

Exercise 5-04

Multiplying and dividing algebraic fractions

1 Simplify each product.

a $\frac{m}{5} \times \frac{3}{4}$

b $\frac{w}{6} \times \frac{k}{2}$

c $\frac{4}{p} \times \frac{7}{t}$

d $\frac{2}{5y} \times \frac{3}{q}$

e $\frac{d}{3} \times \frac{9}{e}$

f $\frac{12}{v} \times \frac{2}{3}$

g $\frac{2x}{3} \times \frac{9}{8y}$

h $\frac{6}{5b} \times \frac{10b}{18}$

i $\frac{d}{e} \times \frac{e}{g}$

j $\frac{4ad}{9} \times \frac{3d}{16a}$

k $\frac{5p}{4r} \times \frac{8r}{15p}$

l $\frac{20a}{3k} \times \frac{9a}{5k}$

2 Simplify each quotient.

a $\frac{x}{2} \div \frac{y}{5}$

b $\frac{t}{18} \div \frac{r}{3}$

c $\frac{k}{2} \div \frac{k}{8}$

d $\frac{h}{k} \div \frac{k}{h}$

e $\frac{3y}{5} \div \frac{2y}{d}$

f $\frac{q}{4} \div \frac{3q}{4}$

g $\frac{4t}{9} \div \frac{3t}{4}$

h $\frac{5m}{n} \div \frac{2m}{3n}$

i $\frac{8a}{3c} \div \frac{2a}{9c}$

j $\frac{3}{2ab} \div \frac{b}{6a}$

k $\frac{3p^2}{2} \div \frac{p}{4}$

l $\frac{8y}{5} \div \frac{32y^2}{3g}$

3 Simplify each expression.

a $\frac{xy}{z} \times \frac{5}{y}$

b $\frac{a}{b} \times \frac{c}{b} \times \frac{b}{3a}$

c $\frac{5}{2g} \div \frac{2}{g}$

d $5 \div \frac{3}{5b}$

e $\frac{4t}{9} \div 3$

f $\frac{5ty}{k} \times \frac{5ky}{t}$

g $5p \div \frac{7}{10pt}$

h $\frac{7}{h} \times \frac{1}{42h}$

i $\frac{s}{3} \div \frac{5s}{2} \times \frac{3s}{7}$

j $\frac{5mn}{2d} \times \frac{4d}{n} \times \frac{1}{15mn}$

k $\frac{8ac}{w} \div \frac{4w}{9c} \div \frac{2c}{3w}$

l $\frac{2u}{7y} \div \frac{3}{4xy} \div \frac{8x}{21y}$

See Example 10

Mental skills 5

Maths without calculators

Multiplying by 9, 11, 99 and 101

We can use expanding when we multiply by a number near 10 or near 100.

1 Study each example.

$$\begin{aligned} \text{a } 25 \times 11 &= 25 \times (10 + 1) \\ &= 25 \times 10 + 25 \times 1 \\ &= 250 + 25 \\ &= 275 \end{aligned}$$

$$\begin{aligned} \text{b } 14 \times 9 &= 14 \times (10 - 1) \\ &= 14 \times 10 - 14 \times 1 \\ &= 140 - 14 \\ &= 126 \end{aligned}$$

$$\begin{aligned} \text{c } 32 \times 12 &= 32 \times (10 + 2) \\ &= 32 \times 10 + 32 \times 2 \\ &= 320 + 64 \\ &= 384 \end{aligned}$$

$$\begin{aligned} \text{d } 7 \times 99 &= 7 \times (100 - 1) \\ &= 7 \times 100 - 7 \times 1 \\ &= 700 - 7 \\ &= 693 \end{aligned}$$

$$\begin{aligned} \text{e } 27 \times 101 &= 27 \times (100 + 1) \\ &= 27 \times 100 + 27 \times 1 \\ &= 2700 + 27 \\ &= 2727 \end{aligned}$$

$$\begin{aligned} \text{f } 18 \times 8 &= 18 \times (10 - 2) \\ &= 18 \times 10 - 18 \times 2 \\ &= 180 - 36 \\ &= 144 \end{aligned}$$

2 Now evaluate each product.

$$\text{a } 16 \times 11$$

$$\text{b } 33 \times 11$$

$$\text{c } 29 \times 9$$

$$\text{d } 45 \times 9$$

$$\text{e } 62 \times 11$$

$$\text{f } 7 \times 101$$

$$\text{g } 18 \times 101$$

$$\text{h } 36 \times 99$$

$$\text{i } 19 \times 8$$

$$\text{j } 45 \times 12$$

$$\text{k } 21 \times 102$$

$$\text{l } 6 \times 98$$

$$\text{m } 32 \times 9$$

$$\text{n } 7 \times 99$$

$$\text{o } 39 \times 101$$

$$\text{p } 71 \times 12$$

Worksheet

Expanding and
factorising

MAT10NAWK10027

Puzzle sheet

The distributive law

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Skillsheet

Algebra using diagrams

MAT10NASS10009

Skillsheet

HCF by factor trees

MAT10NASS10010

5-05 Expanding and factorising expressions

Expanding and **factorising** are inverse operations.

When $4(2y + 5)$ is **expanded**, the answer is $8y + 20$.

When $8y + 20$ is **factorised**, the answer is $4(2y + 5)$.

Summary

Expanding an expression

Multiply each term inside the brackets by the term outside.

$$a(b + c) = ab + ac$$

$$a(b - c) = ab - ac$$

Example 11

Expand each expression.

a $3x(4y + 8x)$

b $-5(3w - 6)$

Solution

a $3x(4y + 8x) = 12xy + 24x^2$

b $-5(3w - 6) = -15w + 30$

Skillsheet

Factorising using diagrams

MAT10NASS10011

Technology

GeoGebra: Simplifying expressions

MAT10NATC00003

Worksheet

Algebra 4

MAT10NAWK00009

Example 12

Expand and simplify by collecting like terms.

a $-6n^2(2 + n) + 5n^3$

b $2b(b - 5) - (3 - b)$

Solution

a $-6n^2(2 + n) + 5n^3 = -12n^2 - 6n^3 + 5n^3$
 $= -12n^2 - n^3$

Expanding
Collecting like terms to simplify

b $2b(b - 5) - (3 - b) = 2b^2 - 10b - 3 + b$
 $= 2b^2 - 9b - 3$

Expanding
Collecting like terms to simplify

Summary

Factorising an expression

- Find the HCF of the terms and write it outside the brackets
- Divide each term by the HCF and write the answers inside the brackets
 $ab + ac = a(b + c)$ $ab - ac = a(b - c)$
- To check that the factorised answer is correct, expand it

Example 13

Factorise each expression.

a $25b^2 - 20ab$

b $x(4 + y) + 2(4 + y)$

c $-b^2 + 8b$

Solution

a The HCF of $25b^2$ and $20b$ is $5b$.
 $\therefore 25b^2 - 20b = 5b \times 5b - 5b \times 4$
 $= 5b(5b - 4)$

Rewrite the expression using the HCF $5b$.
Write the HCF at the front of the brackets.

- b The HCF of $x(4 + y) + 2(4 + y)$ is $(4 + y)$.
 $\therefore x(4 + y) + 2(4 + y) = (4 + y) \times x + (4 + y) \times 2$
 $= (4 + y)(x + 2)$
- c When factorising expressions that begin with a **negative** term, we use the 'negative' HCF.
 The highest 'negative' common factor of $-b^2$ and $8b$ is $-b$.
 $\therefore -b^2 + 8b = (-b) \times b + (-b) \times (-8)$ $(-b) \times (-8) = +8b$
 $= (-b)[b + (-8)]$
 $= -b(b - 8)$

Example 14

Factorise each expression.

- a $8a^3 + 4a^2$ b $20h^3k + 25h^4k - 10h^2k$

Solution

- a The HCF of $8a^3$ and $4a^2$ is $4a^2$.
 $\therefore 8a^3 + 4a^2 = 4a^2 \times 2a + 4a^2 \times 1$ Rewrite the expression using the HCF $4a^2$.
 $= 4a^2(2a + 1)$
- b The HCF is $5h^2k$.
 $\therefore 20h^3k + 25h^4k - 10h^2k = 5h^2k \times 4h + 5h^2k \times 5h - 5h^2k \times 2$
 $= 5h^2k(4h + 5h^2 - 2)$

Exercise 5-05 Expanding and factorising expressions

See Example 11 1 Expand each expression.

a $4(h + 6)$	b $-3(r + 10)$	c $7(x - 9y)$	d $-4(a - 5z)$
e $-(2 - t^2)$	f $-10e(2e + 3)$	g $6y(1 + 7y)$	h $4xy(3xy - 1)$
i $8rt(2t - r)$	j $3ab(4b - 7a)$	k $-6h^2(1 - 3h)$	l $-5x(5x^2 + 4y)$

2 Expand $-5u(8 + 2u)$. Select the correct answer **A**, **B**, **C** or **D**.

- A** $3u - 3u^2$ **B** $-40u + 10u^2$ **C** $-40u - 10u^2$ **D** $-40u - 3u^2$

3 Use the substitution $x = 2$ to test whether each equation is correct or incorrect.

- a $4(x + 10) = 4x + 40$ b $5(x - 1) = 5x - 6$ c $x(3 - x) = 3x - x^2$

See Example 12 4 Expand and simplify by collecting like terms.

a $7m(2m + 3) + m^2$	b $-3e(1 - 5e) - 6e$	c $9w^3 - 3w(5 + 2w^2)$
d $24x^3 - 5x^2(2x^2 - 5x)$	e $t(t + 4) + 3(t + 4)$	f $4(3 - h) - h(7 + 2h)$
g $3x^2(2x + 5) + 4(2 + 5x^2)$	h $-v(2v + 3) + 6(v + 1)$	i $3(1 - 2w) - w(2 - w)$
j $3y(3y - 7) - 5(3y - 7)$	k $3m(m + 5m^2) - m^2(1 - m)$	l $4x(2y + 5) - 6y(10 - 2x)$

5 Factorise each expression.

a $24x + 30$	b $36 - 27a$	c $x^2 + x$
d $30y - 20y^2$	e $36d^2 + 24d$	f $16r^2 - 12r$
g $a(a - 3) + 6(a - 3)$	h $t(8 + t) - 3(8 + t)$	i $b(3b + 5) - 2(3b + 5)$
j $-q^2 - 36q$	k $-6t + 10t^2$	l $-3y^2 - 6xy$
m $-hn^2 + h^2n$	n $-20e^2 - 22e$	o $45m^2 - 54m$

See Example 13

6 Factorise $-10kr + 4rn$. Select the correct answer A, B, C or D.

- A $-2r(5k - 4n)$ B $-2r(5k - 2n)$ C $-5r(2k - 2n)$ D $-2r(5k + 2n)$

7 Factorise each expression.

a $12x^2y - 16xy$	b $18p^2r + 16pr$	c $36m^2n - 108mn^2$
d $36ab^2c - 144bc$	e $48v^2w + 64vw^2$	f $75g^3h^2 - 125gh$
g $-4p^3 - 8p^2 + p$	h $6mn^2 + 3mn + 48m^2n$	i $32p^3g + 8pg^2 - 8pg$
j $18a^5 - 12a^2 + 15a^4$		

See Example 14

Just for the record

CAPTCHA

For security purposes on an Internet website, have you ever been asked to enter letters, words or numbers that have been displayed in a wavy, difficult-to-read format like the one below?



This process is called **CAPTCHA**, which stands for **C**ompletely **A**utomated **P**ublic **T**uring **t**est **t**o **t**ell **C**omputers **a**nd **H**umans **A**part. It is designed to ensure that the person accessing the website is a human and not another computer that could be hacking into the site, sending spam or viruses. CAPTCHA uses optical character recognition (OCR), which is a technology that can convert images of text into editable text.

5-06 Expanding binomial products

$(k + 3)$ and $(k - 7)$ are called **binomial expressions** because each expression has exactly two terms (binomial = '2 terms'). $(k + 3)(k - 7)$ is called a **binomial product** because it is a product (multiplication answer) of two binomial expressions.

Example 15

- a $(k + 3)(k - 7)$ b $(3t - 1)(2t - 5)$

Solution

$$\begin{aligned} \text{a } (k + 3)(k - 7) &= k(k - 7) + 3(k - 7) \\ &= k^2 - 7k + 3k - 21 \\ &= k^2 - 4k - 21 \end{aligned}$$

Each term in $(k + 3)$ is multiplied by $(k - 7)$.
Expanding.
Simplifying.

Worksheet

Mixed expansions

MAT10NAWK10028

Puzzle sheet

Expanding binomials

MAT10NAPS00007

Puzzle sheet

Trinominoes

MAT10NAPS00022

Worksheet

Algebra 2

MAT10NAWK00007

One way of remembering which pairs of terms to multiply together in a binomial product is called the FOIL method, as shown below.

$$\begin{array}{c} \text{F} \quad \text{O} \\ \text{I} \quad \text{L} \end{array} \quad (k+3)(k-7) = k^2 - 7k + 3k - 21 \\ = k^2 - 4k - 21$$

- **F** means multiply the **f**irst terms: $k \times k = k^2$
- **O** means multiply the **o**utside terms: $k \times (-7) = -7k$
- **I** means multiply the **i**nside terms: $3 \times k = 3k$
- **L** means multiply the **l**ast terms: $3 \times (-7) = -21$

$$\text{b } (3t-1)(2t-5) = 6t^2 - 15t - 2t + 5 \\ = 6t^2 - 17t + 5$$

Using FOIL.

Simplifying.

Summary

Expanding a binomial product

Multiply each term in the first binomial by each term in the second binomial.

$$(a+b)(c+d) = ac + ad + bc + bd$$

Stage 5.3

Worksheet

Special products

MAT10NAWK10086

Perfect squares

The special binomial product where a binomial is multiplied by itself, for example,

$(x+4)(x+4) = (x+4)^2$ or $(2a-9)(2a-9) = (2a-9)^2$, is called a **perfect square**.

16, 49, v^2 and $(x+4)^2$ are called **perfect squares** because they are square numbers. The rule for expanding the perfect square of a binomial is:

Summary

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a-b)^2 = a^2 - 2ab + b^2$$

Video tutorial

Special binomial products

MAT10NAVT10006

Example 16

Expand each perfect square.

a $(n-5)^2$

b $(k+4)^2$

c $(3y-8)^2$

Solution

$$\begin{aligned} \text{a } (n-5)^2 &= n^2 - 2 \times n \times 5 + 5^2 \\ &= n^2 - 10n + 25 \end{aligned}$$

$$\begin{aligned} \text{b } (k+4)^2 &= k^2 + 2 \times k \times 4 + 4^2 \\ &= k^2 + 8k + 16 \end{aligned}$$

1st term squared – double product
+ 2nd term squared

$$\begin{aligned} \text{c } (3y-8)^2 &= (3y)^2 - 2 \times 3y \times 8 + 8^2 \\ &= 9y^2 - 48y + 64 \end{aligned}$$

Difference of two squares

The special binomial product where the sum of two terms is multiplied by their difference, for example, $(x+4)(x-4)$ or $(2a-9)(2a+9)$, is called a **sum by a difference** or a **difference by a sum**. The rule for expanding a sum by a difference is:

Summary

$$(a+b)(a-b) = a^2 - b^2$$

The answer is called the **difference of two squares**.

When the sum of two terms is multiplied by their difference, the answer is the square of the first term minus the square of the second term (the difference of two squares).

Example 17

Expand each expression.

$$\text{a } (d+3)(d-3) \quad \text{b } (2+r)(2-r) \quad \text{c } (7x+2)(7x-2) \quad \text{d } (4k-5p)(4k+5p)$$

Solution

$$\begin{aligned} \text{a } (d+3)(d-3) &= d^2 - 3^2 \\ &= d^2 - 9 \end{aligned}$$

$$\begin{aligned} \text{b } (2+r)(2-r) &= 2^2 - r^2 \\ &= 4 - r^2 \end{aligned}$$

$$\begin{aligned} \text{c } (7x+2)(7x-2) &= (7x)^2 - 2^2 \\ &= 49x^2 - 4 \end{aligned}$$

$$\begin{aligned} \text{d } (4k-5p)(4k+5p) &= (4k)^2 - (5p)^2 \\ &= 16k^2 - 25p^2 \end{aligned}$$

Example 18

Expand and simplify each expression.

$$\text{a } x(5y-x)^2 \quad \text{b } (m-2)^2 - (m-2)(m+2)$$

Solution

$$\begin{aligned} \text{a } x(5y-x)^2 &= x(25y^2 - 10xy + x^2) \\ &= 25xy^2 - 10x^2y + x^3 \end{aligned}$$

$$\begin{aligned} \text{b } (m-2)^2 - (m-2)(m+2) &= m^2 - 4m + 4 - (m^2 - 4) \\ &= m^2 - 4m + 4 - m^2 + 4 \\ &= -4m + 8 \end{aligned}$$

Stage 5.3

Video tutorial

Special binomial
products

MAT10NAVT10006

Exercise 5-06 Expanding binomial products

See Example 15

- 1 Expand each binomial product.

a $(m + 4)(m + 3)$

b $(w + 5)(w + 5)$

c $(y + 12)(y - 12)$

d $(a - 8)(a + 3)$

e $(b - 2)(9 + b)$

f $(u - 8)(u - 7)$

g $(15 - k)(k + 1)$

h $(r - 11)(r - 7)$

i $(6 - c)(3 - c)$

j $(t - 1)(t + 2)$

k $(x - 4)(x + 10)$

l $(n + 11)(9 - n)$

- 2 Expand $(b + 7)^2$. Select the correct answer **A**, **B**, **C** or **D**.

A $b^2 + 49$

B $b^2 + 49b$

C $b^2 + 7b + 49$

D $b^2 + 14b + 49$

- 3 Expand each binomial product.

a $(x + 3)(2x + 5)$

b $(3e + 7)(3e + 7)$

c $(10 + 3p)(p - 1)$

d $(7d - 2)(7d - 2)$

e $(2f - 2)(3f + 5)$

f $(4m - 5)(5 + 3m)$

g $(3 - h)(2 + 5h)$

h $(4p - 5)(4p - 5)$

i $(2m - 3)(4 - 5m)$

j $(6t + 1)(2t - 1)$

k $(5y - 5)(5y + 5)$

l $(6 - 7a)(7a - 6)$

- 4 Copy and complete the expansion of each perfect square.

a $(y + 4)^2 = y^2 + \underline{\hspace{1cm}} + 16$

b $(w - 10)^2 = \underline{\hspace{1cm}} - 20w + 100$

c $(g + m)^2 = g^2 + 2gm + \underline{\hspace{1cm}}$

d $(u - 7)^2 = u^2 \underline{\hspace{1cm}} + 49$

e $(k - 9)^2 = k^2 \underline{\hspace{1cm}} + 81$

f $(8 + 5f)^2 = 64 \underline{\hspace{1cm}} + 25f^2$

g $(2d + 3)^2 = \underline{\hspace{1cm}} + \underline{\hspace{1cm}} + 9$

h $(6a + 1)^2 = \underline{\hspace{1cm}} + 12a + \underline{\hspace{1cm}}$

- 5 Expand each perfect square.

a $(h + 7)^2$

b $(k - 5)^2$

c $(x - 1)^2$

d $(q + 10)^2$

e $(5 - h)^2$

f $(7 + k)^2$

g $(x - w)^2$

h $(a + g)^2$

i $(2m - 3)^2$

j $(5x - 6)^2$

k $(9a + 2)^2$

l $(5 + 7b)^2$

m $(4 - 5p)^2$

n $(11 - 2c)^2$

o $(8a - 3y)^2$

p $\left(1 + \frac{1}{y}\right)^2$

q $\left(t - \frac{1}{t}\right)^2$

r $\left(\frac{3}{w} + w\right)^2$

- 6 Expand each expression.

a $(k + 3)(k - 3)$

b $(y - 8)(y + 8)$

c $(w + 11)(w - 11)$

d $(7 - m)(7 + m)$

e $(9 - k)(9 + k)$

f $(3d + 5)(3d - 5)$

g $(5r + 4)(5r - 4)$

h $(4p - 7)(4p + 7)$

i $(3 - 8k)(3 + 8k)$

j $(2 + 9m)(2 - 9m)$

k $(9k - 4l)(9k + 4l)$

l $(7n + 8m)(7n - 8m)$

m $\left(t + \frac{1}{t}\right)\left(t - \frac{1}{t}\right)$

n $\left(\frac{w}{3} - 2\right)\left(\frac{w}{3} + 2\right)$

o $\left(1 - \frac{1}{r}\right)\left(1 + \frac{1}{r}\right)$

- 7 Expand each expression.

a $(3t - d)^2$

b $(2e - 1)(1 + 2e)$

c $(5a + 4)(5a - 4)$

d $(2 - p)(p - 2)$

e $(10 - 6y)(10 + 6y)$

f $(h - 3g)^2$

g $(2x - 3y)(y + 3x)$

h $(7a - 4b)(7a + 4b)$

i $\left(u - \frac{1}{u}\right)\left(u + \frac{1}{u}\right)$

- 8 Expand and simplify each expression.

a $(3x + 1)(2 - x) + 2x$

b $16 + (4k - 8)(4k + 8)$

c $7xy - (2x - 3)(y + 3)$

d $6y + (y - 3)^2 + 9$

e $(2m - n)^2 + (2m + n)^2$

f $(2h + 3)^2 - (2h - 3)(2h + 3)$

g $(x - 2)(x + 3) - (x - 2)(x + 2)$

h $2(b - 1)^2 - (2b - 1)^2$

Puzzle sheet

Products and factors
squaresaw

MAT10NAPS10087

Worksheet

Mixed expansions

MAT10NAWK10089

Stage 5.3

See Example 16

See Example 17

See Example 18

Investigation: Squaring a number ending in 5

Study this mental short cut for squaring a number ending in 5.

- To evaluate 35^2 , calculate $3 \times 4 = 12$, add '25' to the end: $35^2 = 1225$.
- To evaluate 75^2 , calculate $7 \times 8 = 56$, add '25' to the end: $75^2 = 5625$.
- To evaluate 105^2 , calculate $10 \times 11 = 110$, add '25' to the end: $105^2 = 11\,025$.

Let n stand for the tens digit of the number ending in 5 being squared.
Expand $(10n + 5)^2$ and investigate why the above method works.

5-07 Factorising special binomial products

Factorising by grouping in pairs

An **algebraic expression with four terms** can often be factorised in pairs, that is, two terms at a time, to make a binomial product.

Example 19

Factorise each expression.

a $3mk + 5pd + 3md + 5pk$ **b** $3dy - 2gy + 9hd - 6gh$ **c** $12aw - 8cw + 30ax - 20cx$

Solution

$$\begin{aligned} \text{a } 3mk + 5pd + 3md + 5pk &= 3mk + 3md + 5pd + 5pk \\ &= 3m(k + d) + 5p(d + k) \\ &= (k + d)(3m + 5p) \end{aligned}$$

$$\begin{aligned} \text{b } 3dy + 2gy - 9hd - 6gh &= y(3d + 2g) - 3h(3d + 2g) \\ &= (3d + 2g)(y - 3h) \end{aligned}$$

$$\begin{aligned} \text{c } 12aw + 20cx - 8cw - 30ax &= 2(6aw + 10cx - 4cw - 15ax) \\ &= 2(6aw - 15ax - 4cw + 10cx) \\ &= 2[3a(2w - 5x) - 2c(2w - 5x)] \\ &= 2(2w - 5x)(3a - 2c) \end{aligned}$$

Grouping into pairs.
Factorising each pair.
Factorising again.
Factorising each pair.
Factorising again.
Factorising all terms first.
Grouping into pairs.
Factorising each pair.
Factorising again.

Factorising the difference of two squares

You should recall the product $(a + b)(a - b) = a^2 - b^2$.

If we use this rule in reverse, then the factors of $a^2 - b^2$ are $(a - b)$ and $(a + b)$.

Summary

$$a^2 - b^2 = (a + b)(a - b)$$

Stage 5.3

Puzzle sheet

Grouping

MAT10NAPS00021

Puzzle sheet

Difference of two perfect squares

MAT10NAPS00019

Puzzle sheet

Mixed factorisations

MAT10NAPS00023

Stage 5.3

Example 20

Factorise each expression.

a $y^2 - 9$

b $25 - 36d^2$

c $48m^2 - 3x^2$

d $w^4 - w^2$

Solution

a $y^2 - 9 = y^2 - 3^2$

$$= (y + 3)(y - 3)$$

b $25 - 36d^2 = 5^2 - (6d)^2$

$$= (5 + 6d)(5 - 6d)$$

c $48m^2 - 3x^2 = 3(16m^2 - x^2)$

$$= 3[(4m)^2 - x^2]$$

$$= 3(4m + x)(4m - x)$$

d $w^4 - w^2 = w^2(w^2 - 1)$

$$= w^2(w + 1)(w - 1)$$

Exercise 5-07 Factorising special binomial products

See Example 19

1 Factorise each expression.

a $3px + 2qx + 3py + 2qy$

c $15mk + 20mg + 6nk + 8ng$

e $2ak - 5af + 8k - 20f$

g $4am + 4at + 4em + 4et$

i $3mn - 6m + pn - 2p$

k $fg - fh - 10g + 10h$

m $2p - 2c - p^2 + pc$

o $a(x + 1) + y(x + 1) - ka - ky$

b $2wh + 2wk - 3uh - 3uk$

d $4yx - 8ay + 7ax - 14a^2$

f $cd + cy - hd - hy$

h $3yk - 6by + 12k - 24b$

j $9p^2 - 27 + qp^2 - 3q$

l $9kl - 12ml + 9kn - 12mn$

n $l^3 + lm^2 - 3l^2 - 3m^2$

p $p(a - b) - 2q(a - b) + 3qp - 6q^2$

See Example 20

2 Factorise each expression.

a $d^2 - 16$

b $x^2 - 25$

e $25 - t^2$

f $100 - k^2$

i $144 - 49m^2$

j $81y^2 - 16k^2$

m $y^2 - z^2$

n $49 - 16m^2$

q $16 - 81h^2$

r $25a^2 - 64m^2$

u $\frac{1}{4} - 25c^2$

v $4t^2 - \frac{1}{9}$

c $p^2 - 121$

d $y^2 - 81$

g $4r^2 - 9d^2$

h $25g^2 - 4e^2$

k $1 - 81d^2$

l $m^2 - 4n^2$

o $b^2 - 121d^2$

p $36c^2 - 25k^2$

s $100 - 49n^2$

t $121p^2 - 144q^2$

w $25h^2 - 2\frac{1}{4}$

x $1 - m^2n^2$

3 Factorise each expression.

a $4m^2 - 16p^2$

b $3d^2 - 27$

c $y^3 - 25y$

d $18 - 50g^2$

e $k - 16k^3$

f $50q^2 - 2$

g $3d^2 - 12v^2$

h $5t^5 - 125t^3$

i $2a^2b^2 - 2$

j $x^2y^2 - x^2w^2$

k $192f^2 - 108g^2$

l $45d^2 - \frac{5}{4}$

m $2x^2 - 8a^2$

n $100 - 25w^2$

o $1\frac{1}{4} - 80e^2$

p $9c^2 - 6\frac{1}{4}$

4 Factorise each expression.

a $c^2 - \frac{1}{16}$

b $\frac{m^2}{16} - \frac{n^2}{9}$

c $\frac{w^2}{25} - \frac{9u^2}{16}$

d $k^4 - 25$

e $\frac{16a^2}{49} - \frac{25b^2}{4}$

f $t^4 - 81$

g $10\,000 - n^4$

h $(x + y)^2 - x^2$

i $(p - 2q)^2 - (2p + q)^2$

j $\frac{x^2}{4} - \frac{y^2}{36}$

k $(a + b)^2 - (a - b)^2$

5-08 Factorising quadratic expressions

A **quadratic expression** is an algebraic expression in which the highest power of the variable is 2, such as $x^2 - 5x + 7$, $x^2 - 15$, $2x^2 - 3x + 9$ and $-4x^2 + 7x$.

A quadratic expression such as $x^2 - 5x + 7$ is called a **trinomial** because it has three terms.

The expansion of $(x + 2)(x + 4)$ is $x^2 + 6x + 8$, a quadratic trinomial.

$\therefore$ The factorisation of $x^2 + 6x + 8$ is $(x + 2)(x + 4)$.

Puzzle sheet

Factorominoes

MAT10NAPS10029

Puzzle sheet

Trinominoes

MAT10NAPS00022

Worksheet

Simplifying algebraic fractions

MAT10NAWK10030

Summary

In the factorisation of a quadratic trinomial such as $x^2 + 6x + 8$:

- each factor must have an x term to give x^2

$$x^2 + 6x + 8 = (x + 2)(x + 4)$$

- $2 + 4 = 6$, which is the **coefficient** of x , the number in front of the x

$$x^2 + 6x + 8 = (x + 2)(x + 4)$$

- $2 \times 4 = 8$, which is the **constant term** with no x

$$x^2 + 6x + 8 = (x + 2)(x + 4)$$

Example 21

Factorise each quadratic expression.

a $x^2 + 7x + 12$

b $x^2 + 9x + 8$

Solution

- a** Find the two numbers that have a sum of 7 and a product of 12.

It is best to test numbers that have a **product of 12** and then check if their sums equal 7.

The correct numbers are 3 and 4.

$$\therefore x^2 + 7x + 12 = (x + 3)(x + 4)$$

- b** Find two numbers with a sum of 9 and a product of 8.

Test numbers that have a **product of 8** and check if their sums equal 9.

The correct numbers are 1 and 8.

$$\therefore x^2 + 9x + 8 = (x + 1)(x + 8)$$

Pair of numbers	Product	Sum
6, 2	$6 \times 2 = 12$	$6 + 2 = 8$
3, 4	$3 \times 4 = 12$	$3 + 4 = 7$

Pair of numbers	Product	Sum
4, 2	$4 \times 2 = 8$	$4 + 2 = 6$
1, 8	$1 \times 8 = 8$	$1 + 8 = 9$

Video tutorial

Factorising quadratic expressions

MAT10NAVT10016

Summary

Factorising quadratic expressions of the form $x^2 + bx + c$

- Find two numbers that have a sum of b and a product of c
- Use these two numbers to write a binomial product of the form $(x \text{ --- })(x \text{ --- })$

Video tutorial

Factorising quadratic expressions

MAT10NAVT10016

Example 22

Factorise each quadratic expression.

a $x^2 + x - 6$

b $a^2 - 2a - 15$

c $y^2 - 6y + 8$

Solution

a $x^2 + x - 6$

Find two numbers that have a product of -6 and a sum of 1 .

Since the product is negative, one of the numbers must be negative.

They are $+3$ and -2 .

$$\therefore x^2 + x - 6 = (x + 3)(x - 2)$$

b $a^2 - 2a - 15$

Product $= -15$, sum $= -2$.

Since the product is negative, one of the numbers must be negative.

They are -5 and $+3$.

$$\therefore a^2 - 2a - 15 = (a - 5)(a + 3)$$

c $y^2 - 6y + 8$

Product $= 8$, sum $= -6$.

Since the sum is negative, one of the numbers must be negative.

Since the product is positive, both of the numbers must be negative.

They are -4 and -2 .

$$\therefore y^2 - 6y + 8 = (y - 4)(y - 2)$$

Exercise 5-08 Factorising quadratic expressions**See Example 21** 1 Find two numbers whose:**a** product is 24 and sum is -11 **b** product is -10 and sum is -3 **c** product is -15 and sum is -2 **d** product is 12 and sum is 7 **See Example 22** 2 Factorise each quadratic expression.

a $x^2 + 8x + 15$

b $d^2 + 9d + 14$

c $m^2 + 12m + 27$

d $e^2 + 5e + 6$

e $h^2 + 4h + 4$

f $n^2 + 11n + 10$

g $n^2 - 2n - 3$

h $r^2 - 5r - 14$

i $h^2 - 3h - 4$

j $w^2 - 7w - 18$

k $f^2 - 6f - 27$

l $a^2 - 4a - 12$

m $x^2 + 3x - 4$

n $t^2 + 5t - 24$

o $m^2 + 2m - 15$

p $a^2 + a - 2$

q $k^2 + 5k - 14$

r $w^2 + 4w - 12$

s $p^2 - 10p + 24$

t $n^2 - 3n + 2$

u $r^2 - 6r + 9$

v $m^2 + 4m + 4$

w $p^2 - 20p + 100$

x $c^2 - 10c + 25$

5-09

Factorising quadratic expressions of the form $ax^2 + bx + c$

We have factorised **quadratic expressions** of the type $x^2 + bx + c$. For example, the factorisation of $x^2 + 6x + 8$ is $(x + 2)(x + 4)$.

We will now factorise quadratic expressions of the type $ax^2 + bx + c$, such as $6x^2 + 19x + 15$, where x^2 has a **coefficient**.

Example 23

Factorise each quadratic expression.

a $3g^2 + 12g - 36$

Solution

$$\begin{aligned} \text{a } 3g^2 + 12g - 36 &= 3(g^2 + 4g - 12) \\ &= 3(g - 2)(g + 6) \end{aligned}$$

b $48 - 8p - p^2 = -p^2 - 8p + 48$

$$\begin{aligned} &= -1(p^2 + 8p - 48) \\ &= -(p + 12)(p - 4) \end{aligned}$$

b $48 - 8p - p^2$

Taking out the HCF of 3 first.
Product = -12 , sum = 4

Rearranging the terms to make the p^2 term first.
Taking out a common factor of -1 .
Product = -48 , sum = 8

Example 24

Factorise $3x^2 + 8x + 4$.

Solution

There is no HCF, so we need to split up the middle term $8x$.

Find two numbers that have a product of 12 and a sum of 8.

$$3x^2 + 8x + 4 = 3x^2 + 8x + 4$$

sum of 8
product of 12 (3×4)

The two numbers are $+6$ and $+2$, so we will split $8x$ into $6x$ and $2x$.

$$\begin{aligned} \therefore 3x^2 + 8x + 4 &= 3x^2 + 6x + 2x + 4 \\ &= 3x(x + 2) + 2(x + 2) \\ &= (x + 2)(3x + 2) \end{aligned}$$

Factorising by grouping in pairs.
Factorising again.

Stage 5.3

Puzzle sheet

Products and factors
squaresaw

MAT10NAPS10087

Puzzle sheet

Factorising puzzle

MAT10NAPS10090

Technology worksheet

Excel worksheet:
Factorising trinomials

MAT10NACT00011

Technology worksheet

Excel spreadsheet:
Factorising trinomials

MAT10NACT00041

Puzzle sheet

Perfect squares

MAT10NAPS00020

Puzzle sheet

Trinominoes

MAT10NAPS00022

Video tutorial

Factorising quadratic
expressions
(Advanced)

MAT10NAVT10017

Video tutorial

Advanced algebra and
functions

MAT10NAVT00007

Stage 5.3

Summary

Factorising quadratic trinomials of the form $ax^2 + bx + c$ with no HCF

- find two numbers that have a sum of b and a product of ac
- use these two numbers to split the middle term bx into two terms
- factorise by grouping in pairs

Example 25

Factorise each quadratic expression.

a $3x^2 - 11x + 10$

b $4x^2 - 3x - 7$

c $6t^2 + t - 12$

Solution

a $3x^2 - 11x + 10$

$3 \times 10 = 30$. Find two numbers that have a product of 30 and a sum of -11 .

Since the sum is negative, one of the numbers must be negative.

Since the product is positive, **both** of the numbers must be negative.

They are -6 and -5 . Split $-11x$ into $-6x$ and $-5x$.

$$3x^2 - 11x + 10 = 3x^2 - 6x - 5x + 10$$

$$= 3x(x - 2) - 5(x - 2)$$

$$= (x - 2)(3x - 5)$$

Factorising by grouping in pairs.

b $4x^2 - 3x - 7$

$4 \times (-7) = -28$. Find two numbers with a product of -28 and a sum of -3 .

Since the product is negative, one of the numbers must be negative.

They are -7 and 4 .

$$4x^2 - 3x - 7 = 4x^2 - 7x + 4x - 7$$

$$= x(4x - 7) + 1(4x - 7)$$

$$= (4x - 7)(x + 1)$$

c $6t^2 + t - 12$

$6 \times (-12) = -72$. Find two numbers with a product of -72 and a sum of 1 .

They are -8 and $+9$.

$$6t^2 + t - 12 = 6t^2 - 8t + 9t - 12$$

$$= 2t(3t - 4) + 3(3t - 4)$$

$$= (3t - 4)(2t + 3)$$

Example 26

Factorise each quadratic expression.

a $24k^2 - 54k - 15$

b $14 + 29d - 15d^2$

Solution

a $24k^2 - 54k - 15 = 3(8k^2 - 18k - 5)$ Taking out the HCF of 3 first.
 $= 3(8k^2 - 20k + 2k - 5)$ Product = -40, sum = -18
 $= 3[4k(2k - 5) + 1(2k - 5)]$
 $= 3(2k - 5)(4k + 1)$

b $14 + 29a - 15a^2 = -15a^2 + 29a + 14$ Rearranging the terms to make the a^2 term first.
 $= -(15a^2 - 29a - 14)$ Taking out a common factor of -1.
 $= -(15a^2 + 6a - 35a - 14)$ Product = -210, sum = -29
 $= -[3a(5a + 2) - 7(5a + 2)]$
 $= -(5a + 2)(3a - 7)$

Stage 5.3

See Example 23

See Example 24

See Example 25

See Example 26

Exercise 5-09 Factorising quadratic expressions of the form $ax^2 + bx + c$

1 Factorise each quadratic expression. Look for the highest common factor first.

a $3m^2 + 9m + 6$ b $2y^2 + 2y - 4$ c $5t^2 - 10t - 400$ d $5e^4 + 25e^3 - 120e^2$
e $x^3 - x^2 - 110x$ f $4b^2 - 4b - 168$ g $4w^2 + 4w - 48$ h $3a^3 - 9a^2 - 12a$
i $2e^2 + 18e + 40$ j $24 - 5t - t^2$ k $42 + u - u^2$ l $28 + 3x - x^2$
m $12 - b - b^2$ n $7k - 12 - k^2$ o $12x - 35 - x^2$

2 Factorise each quadratic expression.

a $6d^2 + 19d + 15$ b $8m^2 + 10m + 3$ c $2y^2 + 17y + 35$ d $2d^2 + 27d + 70$
e $2w^2 + 31w + 15$ f $4e^2 + 15e + 9$ g $8f^2 + 14f + 3$ h $3d^2 + 5d + 2$
i $2b^2 + 9b + 7$ j $5y^2 + 16y + 11$ k $8g^2 + 26g + 15$ l $6a^2 + 23a + 21$

3 Factorise each quadratic expression.

a $4k^2 - 11k + 6$ b $6w^2 - 17w + 5$ c $5p^2 - 19p + 12$ d $4g^2 - 28g + 49$
e $12f^2 - 25f + 12$ f $4h^2 - 36h + 81$ g $5y^2 - 6y - 11$ h $4d^2 - d - 5$
i $2m^2 - 3m - 9$ j $8a^2 - 2a - 3$ k $15u^2 - 7u - 4$ l $9c^2 - 12c - 5$
m $5m^2 + 2m - 7$ n $6g^2 + g - 12$ o $3p^2 + 4p - 4$ p $7w^2 + 6w - 1$
q $5y^2 + 14y - 3$ r $3n^2 + 10n - 8$

4 Factorise each of the perfect squares below.

a $81w^2 - 180w + 100$ b $4y^2 + 8y + 4$ c $25h^2 - 40h + 16$

5 Factorise each quadratic expression by first taking out a common factor.

a $9m^2 + 30m - 24$ b $4y^2 - 6y - 10$ c $30k^2 + 25k - 30$ d $12w^2 - 44w - 16$
e $12t^2 + 20t - 8$ f $-25q^2 - 5q + 6$ g $-12m^2 + 14m - 4$ h $20 - h - 12h^2$
i $18 + 48c + 24c^2$ j $15 + 9z - 6z^2$ k $12d^2 + 2d - 30$ l $22x - 12 - 6x^2$

Stage 5.3

6 Factorise each quadratic expression.

a $7w^2 - 8w + 1$

b $4h^2 - 7h - 15$

c $8x^2 - 2x - 3$

d $5r^2 + 26r + 5$

e $2d^2 - 15d + 7$

f $6n^2 - 7n - 3$

g $8 - 6m - 9m^2$

h $3 - 2c - 5c^2$

i $15g^2 + 19g + 6$

j $15 + 14q - 8q^2$

k $3x^2 - 13x + 14$

l $16 - 8d - 3d^2$

Puzzle sheet

Factoronomoes

MAT10NAPS10029

Puzzle sheet

Mixed factorisations

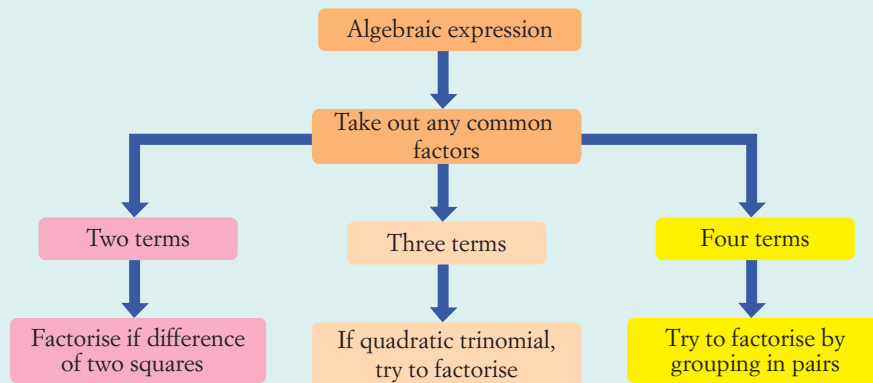
MAT10NAPS00023

5-10 Mixed factorisations

Summary

Factorisation strategies

- Look for any common factors and factorise first.
- If there are two terms, try factorising using the difference of two squares.
- If there are three terms, try factorising as a quadratic trinomial.
- If there are four terms, try factorising by grouping in pairs.



Example 27

Factorise each quadratic expression.

a $5k^2 - 125$

b $4y^2 + 32d^2$

c $24u^2 - 68u + 20$

d $a^3 - 2a^2 - 4a + 16$

Solution

$$\begin{aligned} \text{a } 5k^2 - 125 &= 5(k^2 - 25) \\ &= 5(k + 5)(k - 5) \end{aligned}$$

$$\text{b } 4y^2 + 32d^2 = 4(y^2 + 8d^2)$$

$$\text{c } 24u^2 - 68u + 20 = 4(6u^2 - 17u + 5)$$

$$= 4(6u^2 - 2u - 15u + 5)$$

$$= 4[2u(3u - 1) - 5(3u - 1)]$$

$$= 4(3u - 1)(2u - 5)$$

Taking out the HCF of 5 first.

Difference of two squares

Two terms but not a difference of two squares.

Taking out the HCF of 4 first.

Product = 30, sum = -17

Factorising each pair.

Factorising again.

$$\begin{aligned} \text{d } a^3 - 2a^2 - 4a + 8 &= a^2(a - 2) - 4(a - 2) && \text{Factorising by grouping in pairs.} \\ &= (a - 2)(a^2 - 4) \\ &= (a - 2)(a + 2)(a - 2) && \text{Difference of two squares} \\ &= (a - 2)^2(a + 2) \end{aligned}$$

See Example 27

Exercise 5-10 Mixed factorisations

1 Factorise each expression.

a $m^2 + 16m + 64$	b $3d^2 - 3$	c $3d^2 + 4d - 15$
d $3k - 15 - 5h + hk$	e $25y^2 - 64$	f $100f^2 - 64$
g $q^2 + 3q - 3pq$	h $3 + 2g - g^2$	i $24b^2 + 44b - 40$
j $25r^2 - 1$	k $b^3 + b^2 + b + 1$	l $4x^2 - 20x + 25$
m $4 - d - 5d^2$	n $b^3 - b^2 - b + 1$	o $8 - 2v^2$
p $mn^2 + mnp + 3mn + 3mp$	q $2w^2 - 24w + 72$	r $36h^2 + 12h + 1$
s $15r^2 - 31rt - 24t^2$	t $4d^2 + 4d + 1$	u $9g^2 - 36k^2$

2 Factorise each expression.

a $e^3 - 3e^2 - 10e$	b $5(p + q)^2 - 125(p - q)^2$	c $28x^2 - 7$
d $a^2 - b^2 + 4a - 4b$	e $c^3 - 2c^2 - 4c + 8$	f $6a^2 + 13a - 5$
g $t^2 - 3t + 5t - 35$	h $18p^2 + 24p + 8$	i $1 - 2a - 24a^2$
j $9x^2 - 27x + 18x - 54$	k $2a^2b - 6ab - 3a + 9$	l $2a^2 + 12a + 18$
m $25u^2 - 10u + 1$	n $4k^2 - 5k - 21$	o $48 - 3w^2$
p $3 - 27s^2$	q $k^3 + 4k^2 - 16k - 64$	r $5y^3 - 10y^2 + 15y$
s $m^3n - 4mn$	t $8 - 2a^2$	u $32c^2 - 40c - 12$

5-11 Factorising algebraic fractions

NSW

Example 28

Simplify each expression.

a $\frac{12x + 8y}{4}$	b $\frac{30m - 18}{25m^2 - 9}$	c $\frac{2h^2 + 8h}{-3h - 12}$	d $\frac{5k^2 + 9k - 2}{k^2 + 7k + 2}$

Solution

$$\begin{aligned} \text{a } \frac{12x + 8y}{4} &= \frac{\cancel{4}(3x + 2y)}{\cancel{4}} \\ &= 3x + 2y \end{aligned}$$

$$\begin{aligned} \text{b } \frac{30m - 18}{25m^2 - 9} &= \frac{6(\cancel{5m - 3})}{(5m + 3)(\cancel{5m - 3})} \\ &= \frac{6}{5m + 3} \end{aligned}$$

$$\begin{aligned} \text{c } \frac{2h^2 + 8h}{-3h - 12} &= \frac{2h(\cancel{h + 4})}{-3(\cancel{h + 4})} \\ &= -\frac{2h}{3} \end{aligned}$$

$$\begin{aligned} \text{d } \frac{5k^2 + 9k - 2}{k^2 + 7k + 2} &= \frac{(5k - 1)(\cancel{k + 2})}{(\cancel{k + 2})(k + 5)} \\ &= \frac{5k - 1}{k + 5} \end{aligned}$$

Stage 5.3

Example 29

Simplify each expression.

$$\text{a } \frac{5}{x^2+x-2} - \frac{2}{x^2-x} \quad \text{b } \frac{6m}{m^2+4m} \times \frac{m^2-16}{9} \quad \text{c } \frac{h^2+5h+6}{h^2+4h+3} \div \frac{h^2+2h}{h^2-2h-3}$$

Solution

$$\begin{aligned} \text{a } \frac{5}{x^2+x-2} - \frac{2}{x^2-x} &= \frac{5}{(x+2)(x-1)} - \frac{2}{x(x-1)} && \text{Factorising denominators} \\ &= \frac{5x}{x(x+2)(x-1)} - \frac{2(x+2)}{x(x+2)(x-1)} && \text{Using common denominators} \\ &= \frac{5x-2(x+2)}{x(x+2)(x-1)} \\ &= \frac{5x-2x-4}{x(x+2)(x-1)} \\ &= \frac{3x-4}{x(x+2)(x-1)} \end{aligned}$$

$$\begin{aligned} \text{b } \frac{6m}{m^2+4m} \times \frac{m^2-16}{9} &= \frac{6m}{m(m+4)} \times \frac{(m+4)(m-4)}{9} \\ &= \frac{\cancel{6}m}{\cancel{m}(m+4)} \times \frac{\cancel{(m+4)}(m-4)}{9} \\ &= \frac{2(m-4)}{3} \end{aligned}$$

$$\begin{aligned} \text{c } \frac{h^2+5h+6}{h^2+4h+3} \div \frac{h^2+2h}{h^2-2h-3} &= \frac{h^2+5h+6}{h^2+4h+3} \times \frac{h^2-2h-3}{h^2+2h} \\ &= \frac{(h+3)(h+2)}{(h+3)(h+1)} \times \frac{(h-3)(h+1)}{h(h+2)} \\ &= \frac{(h-3)}{h} \end{aligned}$$

Exercise 5-11 Factorising algebraic fractions

See Example 28

1 Simplify each expression.

$$\text{a } \frac{7x+7y}{7}$$

$$\text{b } \frac{3}{6t-6r}$$

$$\text{c } \frac{br-cr}{ar}$$

$$\text{d } \frac{h-2}{2-h}$$

$$\text{e } \frac{w^2-16}{w+4}$$

$$\text{f } \frac{5d-5t}{d^2-t^2}$$

$$\text{g } \frac{(k+5)^2}{k^2-25}$$

$$\text{h } \frac{6c^2-6}{2c+2}$$

$$\text{i } \frac{am-an+m-n}{m^2-n^2}$$

$$\text{j } \frac{y^2+9y+20}{2y+10}$$

$$\text{k } \frac{k^2-3k-4}{k^2-16}$$

$$\text{l } \frac{16a^2-25c^2}{4a^2-9ac+5c^2}$$

$$\text{m } \frac{s^2+4s+4}{s^2-s-6}$$

$$\text{n } \frac{1-c-2c^2}{3c^2+2c-1}$$

$$\text{o } \frac{ap+4p-2a-8}{2p^2-8}$$

2 Simplify each expression.

a $\frac{5}{m(m+1)} + \frac{2}{(m+1)(m+2)}$

c $\frac{3}{(b+2)(b-1)} + \frac{1}{(b-1)(b-3)}$

e $\frac{5}{4h+4} + \frac{3}{h^2+h}$

g $\frac{3}{r^2-36} - \frac{5}{4r+24}$

i $\frac{5}{k^2-3k-4} - \frac{k}{k^2-1}$

b $\frac{6}{(w+5)(w+3)} - \frac{4}{w(w+3)}$

d $\frac{2}{k^2+k} - \frac{3}{k^2-1}$

f $\frac{3}{d^2+3d+2} - \frac{4}{d+2}$

h $\frac{3}{d^2+2d} + \frac{d}{d^2-4}$

j $\frac{2}{q^2-1} + \frac{3}{q+1}$

3 Simplify each expression.

a $\frac{3m+9}{2} \times \frac{4m}{m+3}$

c $\frac{4}{e+2} \times \frac{e^2+2e}{8e}$

e $\frac{5h}{3h+9} \times \frac{6h+18}{h^2+h}$

g $\frac{r+t}{t^2-r^2} \times \frac{r^2-rt}{5r+5t}$

i $\frac{p^2+2p+1}{p^2-1} \times \frac{4p-4}{p^2+p}$

k $\frac{5}{x^2-4} \div \frac{15}{2x+4}$

m $\frac{d^2+d}{d+3} \div \frac{6d}{d^2-9}$

o $\frac{3f+6}{f^2+f-6} \div \frac{f^2-2f-8}{f^2-f-12}$

b $\frac{5d-10}{3d-9} \times \frac{5d-15}{8d-16}$

d $\frac{3k+6}{5} \times \frac{10k}{k+2}$

f $\frac{4}{a^2-b^2} \times \frac{3a+3b}{8}$

h $\frac{20m+16}{7m-7} \times \frac{7m}{5m+4}$

j $\frac{y+2}{5y} \div \frac{7y+14}{15y}$

l $\frac{4n+8}{n+5} \div \frac{6n+12}{5n+25}$

n $\frac{1}{f^2-6f+9} \div \frac{4}{f^2-9}$

Power plus

1 Expand and simplify each expression.

a $(x + 5)^3$

b $(y - 2)^3$

c $(a + b)^3$

d $(3d + 10)^3$

2 Use the given expansion to evaluate each square number without using a calculator.

a $21^2 = (20 + 1)^2$

b $45^2 = (40 + 5)^2$

c $29^2 = (30 - 1)^2$

d $59^2 = (60 - 1)^2$

e $102^2 = (100 + 2)^2$

f $98^2 = (100 - 2)^2$

3 By expressing 31×29 as $(30 + 1)(30 - 1)$, use the difference of two squares to find the value of 31×29 .

4 Use the method of question 3 to evaluate each expression.

a 21×19

b 51×49

c 89×91

d 78×82

5 Expand and simplify each expression.

a $(y + 1)^2 + (y + 2)^2 + (y + 3)^2$

b $(x - 3)(x + 3) + (x + 3)^2 + (x - 3)^2$

c $(5n + 3)(5n - 3) + (3n - 5)(3n + 5)$

d $2(a - b)(a + b) - (a + b)^2 - (a - b)^2$

6 Factorise each expression.

a $n^2 + 4mn + 4m^2$

b $x^2 - 2xy + y^2$

c $25x^2 - 40xy + 16y^2$

d $5a^2 - 30ab + 45b^2$

e $c^4 + 2c^2 + 1$

f $8 - 7t^2 - t^4$

g $\frac{x^2}{16} - \frac{y^2}{25}$

h $x^4 - 1$

i $25c^4 - 100$

j $(a + b)^2 - c^2$

k $\frac{16a^2}{49} - \frac{25b^2}{4}$

l $(p + q)^2 - (p - q)^2$

Language of maths

algebraic fraction	base	binomial
binomial product	coefficient	constant term
denominator	difference of two squares	expand
factorise	fractional	grouping in pairs
highest common factor (HCF)	index law	indices
perfect square	power	product
quadratic expression	reciprocal	root
sum	term	trinomial

Puzzle sheet

Algebra crossword

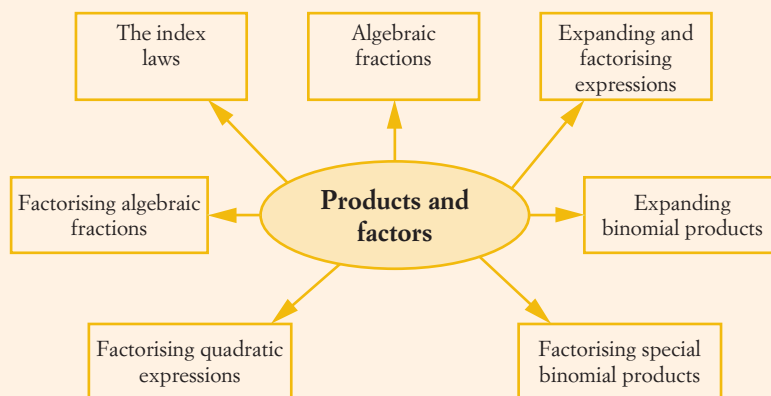
MAT10NAPS10031

- 1 Explain the meanings of **quadratic** and **trinomial**. Give an example of a **quadratic trinomial**.
- 2 Any number raised to the power of zero is equal to what?
- 3 What is the difference between **expand** and **factorise**?
- 4 What power is associated with the **reciprocal** of a term or number?
- 5 In the quadratic expression $2x^2 - 3x + 6$, what is:
 - a the constant term?
 - b the coefficient of x ?
- 6 Copy and complete: To factorise quadratic expressions of the form $x^2 + bx + c$, first find two numbers that have a _____ of b and a _____ of c .

Topic overview

- What was this topic about? What was the main theme?
- What content was new and what was revision for you?
- Write three index laws in both words and symbols.
- Write 10 questions (with solutions) that could be used in a test for this chapter. Include some questions that you have found difficult to answer.
- List the sections of work in this chapter that you did not understand. Follow up this work.

Copy and complete this mind map of the topic, adding detail to its branches and using pictures, symbols and colour where needed. Ask your teacher to check your work.



Quiz

Numbers and powers

MAT10NAQZ00001

Quiz

Advanced algebra and equations

MAT10NAQZ00007

Quiz

Algebra

MAT10NAQZ00003

Worksheet

Algebra 6

MAT10NAWK00035

Chapter 5 revision

See Exercise 5-01

1 Simplify each expression.

a $3v^4w^2 \times 2vw^5$

b $\frac{24t^8h^8}{3th^2}$

c $(5xy^2)^2$

d $\left(\frac{2p}{3}\right)^0$

e $(4k)^{-1}$

f $\left(\frac{2p}{3}\right)^3$

g $(4m)^{-2}$

h $4m^{-2}$

i $\left(\frac{5b^8y^6}{b^2y^3}\right)^4$

j $(4t^4u^5)^3 \times 8t^2u$

k $45c^6d^8 \div (-3cd^2)^2$

l $\left(\frac{45ab^4}{54a^2b^3}\right)^{-1}$

Stage 5.3

See Exercise 5-01

2 Simplify each expression.

a $\left(\frac{1}{10}\right)^{-5}$

b $\left(\frac{5t}{4}\right)^{-3}$

c $\left(\frac{9}{2d}\right)^{-2}$

See Exercise 5-02

3 Evaluate each expression.

a $400^{\frac{1}{2}}$

b $(-27)^{\frac{1}{3}}$

c $64^{-\frac{1}{2}}$

d $32^{\frac{2}{3}}$

See Exercise 5-02

4 Simplify each expression.

a $(16a^4)^{\frac{3}{2}}$

b $\sqrt[4]{(81t^{12})^5}$

c $\frac{1}{(\sqrt[3]{8p^6})^4}$

d $(125x^3y^6)^{-\frac{2}{3}}$

See Exercise 5-03

5 Simplify each expression.

a $\frac{t}{4} - \frac{2t}{5}$

b $\frac{5g}{3} + \frac{3g}{2}$

c $\frac{5x}{16} - \frac{x}{4}$

Stage 5.3

See Exercise 5-03

6 Simplify each expression.

a $\frac{r-4}{2} + \frac{r}{5}$

b $\frac{w+1}{2} - \frac{w-4}{10}$

c $\frac{2p+5}{3} - \frac{3p}{12}$

See Exercise 5-04

7 Simplify each expression.

a $\frac{4m}{5} \times \frac{3m}{8}$

b $\frac{6}{t} \times \frac{3t}{24}$

c $\frac{5}{a} \div \frac{10}{b}$

d $\frac{vw}{5} \div \frac{3v}{20}$

See Exercise 5-05

8 Expand and simplify each expression.

a $8tp(7p-5t)$

b $-5h^2(3h+7)$

c $4fg(g-6f) - 6f^2g$

d $12(9-n) - 5(2n+3)$

e $x^2(6x+x^2) + 2x(3x^3+x^2)$

f $3(7-2y) - 5y(7-2y)$

See Exercise 5-05

9 Factorise each expression.

a $16ar^2 + 24ar$

b $-24p + 18q$

c $2(5x-1) - 3x(5x-1)$

d $15xy^2 - 30x^3y^3$

e $6pt^2 + 12p^2t - 48p^3$

f $32r^2s^4 + 12r^4s^3$

g $50x^4y^3 - 75x^3y^4$

h $-8p^3q^3 + 48p^3q^6$

i $n(n^2+6) - (n^2+6)$

See Exercise 5-06

10 Expand each binomial product.

a $(b+3)(b+10)$

b $(d+8)(d-7)$

c $(t-6)(9-t)$

d $(5x+7)(4x-3)$

e $(7y-3)(7y+3)$

f $(5p-8)(5p-8)$

11 Expand each binomial product.

- a** $(n + 9)^2$ **b** $(n + 9)(n - 9)$
c $(3y - 2)^2$ **d** $(4n - 11)(4n + 11)$

12 Factorise each expression.

- a** $3pr + 2qt + 2qr + 3pt$ **b** $4ab + 6bc - 6ad - 9dc$
c $b^2 - 100$ **d** $25 - 16y^2$
e $20x^2 - 5$ **f** $3t^3 - 27t$

13 Factorise each expression.

- a** $y^2 + 10y + 25$ **b** $x^2 - 21x + 20$
c $n^2 + 8n - 33$ **d** $a^2 - 11a + 28$
e $m^2 - 5m - 84$ **f** $p^2 + 3p - 54$

14 Factorise each expression.

- a** $3w^2 + 5w + 2$ **b** $2y^2 - 3y - 9$
c $60 - 5b - 5b^2$ **d** $3p^2 + 10p - 8$
e $12x^2 - 46x + 14$ **f** $6n^2 - 13n + 6$

15 Factorise each expression.

- a** $5q^2 - 45$ **b** $20x^2 - 52x + 24$
c $t^3 - t^2 - t + 1$

16 Simplify each expression.

- a** $\frac{16n + 24t}{8}$ **b** $\frac{6m^2 - 24}{5m - 10}$
c $\frac{a^2 - a - 6}{a^2 - 2a - 3}$ **d** $\frac{4}{y^2 - 3y} - \frac{2}{y^2 - 9}$
e $\frac{3p - 15}{3} \times \frac{9p}{p^2 - 5p}$ **f** $\frac{b^2 + 3b + 2}{b^2 - 9} \div \frac{b^2 + b}{3b + 9}$

Stage 5.3

See Exercise 5-06

See Exercise 5-07

See Exercise 5-08

See Exercise 5-09

See Exercise 5-10

See Exercise 5-11



6

Statistics and probability

Investigating data

Which capital city in Australia has the highest average temperature? Does Melbourne have higher rainfall than Sydney?

To answer these questions, sets of data need to be collected and then compared by looking at the shape of their displays or by analysing their measures of location and spread.



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Chapter outline

	Proficiency strands				
6-01 The shape of a frequency distribution	U	F	PS	R	C
6-02 Quartiles and interquartile range	U	F	PS	R	C
6-03 Standard deviation*	U	F	PS		C
6-04 Comparing means and standard deviations*		F	PS	R	C
6-05 Box plots	U	F	PS	R	C
6-06 Parallel box plots	U	F	PS	R	C
6-07 Comparing data sets		F	PS	R	C
6-08 Scatter plots	U	F		R	C
6-09 Line of best fit*	U	F	PS	R	C
6-10 Bivariate data involving time	U	F		R	C
6-11 Statistics in the media	U	F	PS	R	C
6-12 Investigating statistical studies*			PS	R	C

*STAGE 5.3

Wordbank

bivariate data Data that measures two variables, represented by an ordered pair of values that can be graphed on a scatter plot

boxplot (also called **box-and-whisker plot**) A graph that shows the quartiles of a set of data and the highest and lowest scores; the box contains the middle 50% of scores while the lines or 'whiskers' extend to the two extremes

five-number summary For a set of numerical data, the lowest score, lower quartile, median, upper quartile and highest score

interquartile range (IQR) The difference between the upper quartile and lower quartiles, $IQR = Q_3 - Q_1$, representing the middle 50% of scores

scatter plot A graph consisting of dots on a number plane that represent bivariate data

standard deviation (symbol σ_n) A measure of spread that depends on every score in the data set and their mean

In this chapter you will:

- construct back-to-back stem-and-leaf plots and histograms and describe data using terms such as 'skewed', 'symmetric' and 'bi-modal'
- determine quartiles and interquartile range
- (STAGE 5.3) calculate and interpret the mean and standard deviation of data and use these to compare data sets
- construct and interpret box plots and use them to compare data sets
- compare shapes of box plots to corresponding histograms and dot plots
- use scatter plots to investigate and comment on relationships between two numerical variables
- investigate and describe bivariate numerical data where the independent variable is time
- evaluate statistical reports in the media and other places by linking claims to displays, statistics and representative data
- investigate reports of surveys in digital media and elsewhere for information on how data was obtained to estimate population means and medians
- (STAGE 5.3) investigate reports of studies in digital media and elsewhere for information on their planning and implementation 
- find the five-number summary for a set of data and use it to construct a box-and-whisker plot
- describe the strength and direction of the linear relationship of bivariate data shown on a scatter plot
- (STAGE 5.3) use technology to construct a line of best fit for bivariate data and use it to make predictions

SkillCheck

Worksheet

StartUp assignment 5

MAT10SPWK10032

Skillsheet

Statistical measures

MAT10SPSS10012

Worksheet

Statistical match-up

MAT10SPWK10033

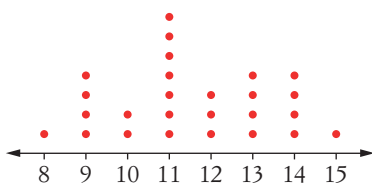
1 For each set of data, find:

i the range **ii** the mean (correct to one decimal place) **iii** the median **iv** the mode

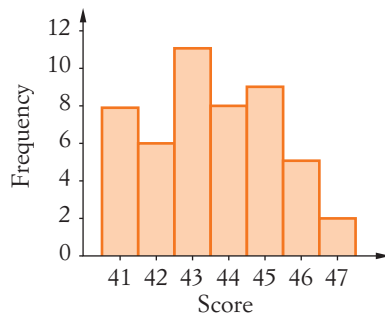
a 15 13 18 14 15 18 23 14 20 16 15

b 8°C 3°C -5°C 2°C -4°C 7°C 3°C 0°C

c



d



e

Stem	Leaf
1	0 3 6
2	1 4 4 7 8
3	2 3 4 5 5 7 9
4	0 5 7 8
5	2 6 8

f

Score	Frequency
0	2
1	5
2	8
3	4
4	3
5	1

2 A cricketer made the following scores in 10 innings.

34 21 78 30 26 19 41 36 16 32

a Find:

i the median ii the mean iii the range.

b Which score is the outlier?

c i Calculate the median, mean and range if the outlier is not included in the scores.

ii What effect does the outlier have on the mean, median and range?



123rf/Lance Bells

6-01 The shape of a frequency distribution

A **statistical distribution** is the way the scores of a data set are arranged, especially when graphed. When looking at histograms, dot plots and stem-and-leaf plots, an overall pattern can be seen from the shape of the display.

The shape of a statistical distribution shows how the data is spread and can be seen by drawing a curve around the graph or display.

A distribution is **symmetrical** if the data is evenly spread or balanced about the centre.

Technology worksheet

Excel worksheet:
Skewness

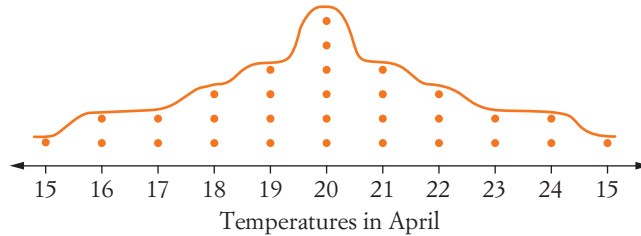
MAT10SPCT00005

Technology worksheet

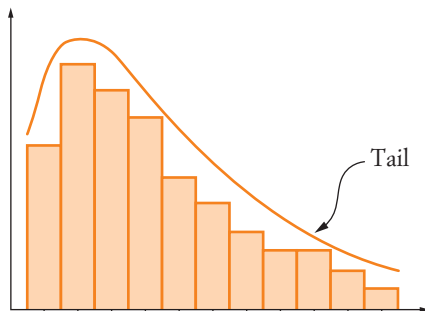
Excel spreadsheet:
Skewness

MAT10SPCT00035

Stem	Leaf
3	0 2 4
4	1 8 9 9
5	2 4 5 6 6 7 8 8
6	0 3 4 5 5 6 7 8 9 9
7	2 4 4 4 5 5 5 5
8	2 8 8 8
9	3 5 7



A distribution is **skewed** if most of the data is bunched or clustered at one 'end' of the distribution and the other 'end' has a 'tail'.



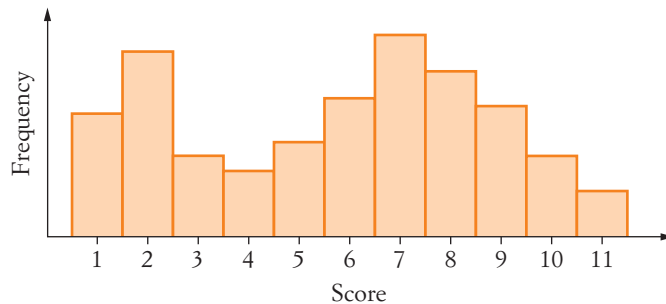
A distribution is **positively skewed** if its tail points to the right.

Stem	Leaf
0	3 5
1	0 6
2	5 7 8
3	0 3 8 9
4	1 1 2 3 4 8
5	0 0 1 1 2 2 5 5
6	3 5 7 5 6 6 7 7 9
7	0 2 2 4 5

A distribution is **negatively skewed** if its tail points to the left.

A distribution is **bimodal** if it has two peaks. The higher peak is the mode, while the other peak indicates another score that has a high frequency.

For example, this frequency histogram has two peaks at 2 and 7 so it is bimodal. The mode, however, is 7.

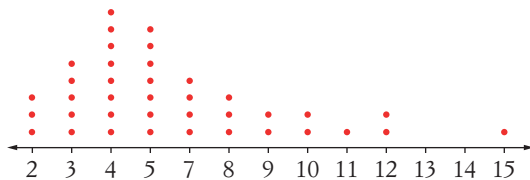


Example 1

For each statistical distribution:

- i describe the shape ii identify any outliers and clusters

a



b Stem Leaf

Stem	Leaf
10	4 5
11	3 4 4 9
12	1 2 2 6 8
13	0 1 5 5 7 9 9 9
14	4 5 6 8 8
15	0 0 1 1
16	0 2

Solution

- a i The shape is positively skewed (tail points towards the higher scores).
 ii 15 is an outlier and clustering occurs at 4 and 5.
 b i The shape is symmetrical (the data is balanced about the stem of 13).
 ii There are no outliers but clustering occurs in the 13s.

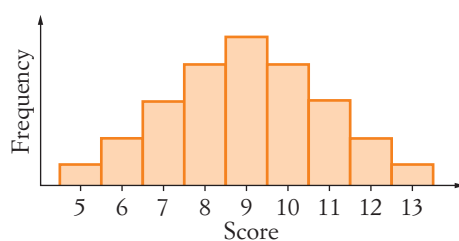
Exercise 6-01 The shape of a distribution

See Example 1

1 For each statistical distribution:

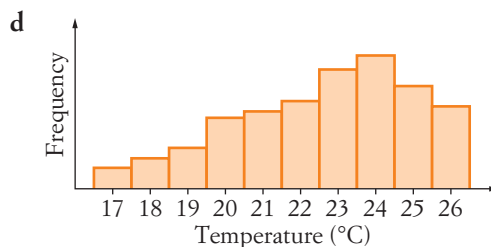
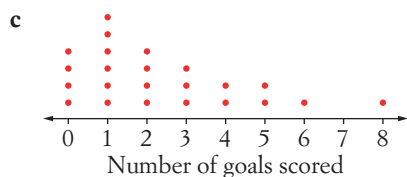
- i describe the shape ii identify any outliers and clusters.

a



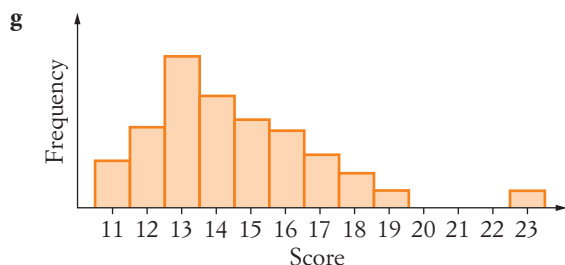
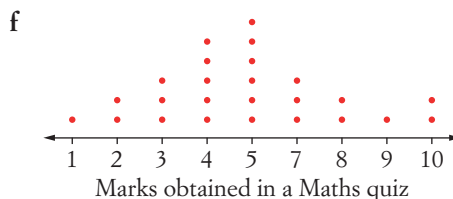
b Stem Leaf

Stem	Leaf
2	4 5 6 9
3	1 2 3 3 4 5 7 8
4	0 4 4 6 8 9
5	4 5 5 8
6	0 0 2 3 5 6 7 8 9 9
7	3 5 7 8 8 9 9
8	1 1 3 5 6
9	0 3 5 6



e

Stem	Leaf
12	0 2 4 9
13	2 4 6 7 8 8 8
14	3 3 4 4 5 5 8 9 9 9
15	0 1 1 5 7 8 9 9
16	1 1 5 6 7
17	2 4 5 8
18	0 3 9
19	5 8
20	6 8



h

Stem	Leaf
5	3 4 4 6 7 8 9
6	0 0 5 9 9
7	2 4 5 6
8	5 7 8
9	3 3 6 7 8
10	2 4 6 8 8 8 8
11	
12	
13	6

2 These are the final round scores for players in a golf tournament.

66 70 67 72 75 72 70 74 75 72 74 72 73 71 71 69 70 71 71 74
72 69 75 73 69 75 73 69 69 67 74 72 72 73 71 73 77 68 72 72

a Arrange the data into a frequency table and construct a frequency histogram.

b Are there any outliers?

c Describe the shape of the distribution.

d Give a possible reason for the shape of the distribution.

e Where does clustering occur?

f Find the mode, the mean and the median and show their position in the histogram.

- 3 The stem-and-leaf plot shows the number of hours that students spend on their computers during the week.

Stem	Leaf
0	1 1 1 1 1 2 2 2 2 3 3 3 5 6 6 7 7 7 7 9 9
1	0 1 1 2 4 4 5 6 8 8 9
2	0 5 5 5 8 8
3	0 0 0 1 5
4	0 0

- a How many students were surveyed? b Where does the clustering occur?
 c Are there any outliers? d Describe the shape of the distribution.
 e Give a possible reason for the shape of the distribution. f Find the mean, median and mode.
- 4 The following scores are the heights (in cm) of thirty Year 8 students.
 162 155 153 162 182 173 165 165 142 167 164 168 150 155 143
 153 123 163 170 169 153 162 161 170 160 162 172 151 160 171

- a Arrange the data into an ordered stem-and-leaf plot.
 b Describe the shape of the distribution. c Are there any outliers?
 d Where does clustering occur? e Find the mode, median and mean.

- 5 The daily maximum temperatures (correct to one decimal place) for July 2013 at the Sydney Observatory are shown in the stem-and-leaf plot.

Stem	Leaf
13	8
14	
15	9
16	3
17	0 2 4 4 7
18	4 4 4 7 8
19	1 2 5 6 8 9
20	1 2 3 4 4
21	5 6
22	0 6
23	4
24	0 3

- a Describe the shape of the distribution.
 b Are there any outliers?
 c What is the mode?
 d Find the mean, correct to one decimal place.
 e What is the median?
 f Find the range.
 g Is the range a good indicator of the spread of the temperatures? Give reasons.

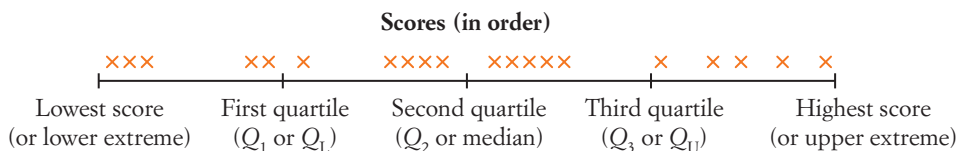
Source: © Bureau of Meteorology

6-02 Quartiles and interquartile range

Quartiles

The median, being the middle score, divides a set of data into two equal parts (halves).

Quartiles are the values Q_1 , Q_2 and Q_3 that divide the set of data into four equal parts (quarters).



The **first quartile** Q_1 , also called the **lower quartile** Q_L , is the value that divides the lower 25% of scores. $\frac{1}{4}$ of the scores lie below Q_1 .

The **second quartile** Q_2 is the value that divides the lower 50% of scores, so it is also the **median**. $\frac{1}{2}$ of the scores lie below Q_2 .

The **third quartile** Q_3 , also called the **upper quartile** Q_U , is the value that divides the lower 75% of scores from the upper 25% of scores. $\frac{3}{4}$ of the scores lie below Q_3 , $\frac{1}{4}$ of the scores lie above it.

Summary

Finding the quartiles of a data set

- sort the scores in order, find the median and call it Q_2
- find the median of the bottom half of the scores and call it Q_1 (or Q_L)
- find the median of the top half of scores and call it Q_3 (or Q_U).

Example 2

Find the quartiles for each set of data.

a 65 84 75 82 97 70 68 76 93 48 79 54 80 79 82 96 63 85 72 70

b 9 3 8 7 6 8 4 6 2 10 9

c 15 18 7 16 23 9 15 20 16 14 13 11 19

Solution

a Arranging the 20 scores in ascending order, we have:

48 54 63 65 68 70 70 72 75 76 79 79 80 82 82 84 85 93 96 97

$$Q_1 = \frac{68 + 70}{2} = 69$$

$$Q_2 \text{ (median)} = \frac{76 + 79}{2} = 77.5$$

$$Q_3 = \frac{82 + 84}{2} = 83$$

When finding the quartiles, first find the median, then the lower and upper quartiles.

Q_1 (lower quartile) = 69; Q_2 (median) = 77.5; Q_3 (upper quartile) = 83

b Arranging the 11 scores in ascending order, we have:

2 3 4 6 6 7 8 8 9 9 10

Lower quartile
 $Q_1 = 4$

Median
 $Q_2 = 7$

Upper quartile
 $Q_3 = 9$

c Arranging the 13 scores in ascending order, we have:

7 9 11 13 14 15 15 16 16 18 19 20 23

$$Q_1 = \frac{11 + 13}{2} = 12$$

Median
 $Q_2 = 15$

$$Q_3 = \frac{18 + 19}{2} = 18.5$$

Worksheet

Interquartile range

MAT10SPWK10034

Video tutorial

Interquartile range

MAT10SPVT10003

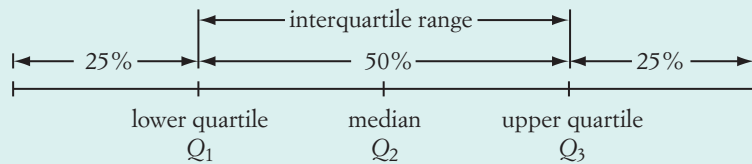
The interquartile range

The **range** is a **measure of spread** because it gives an indication of how widely the scores are spread in a set of data.

The **interquartile range** is another measure of spread. It is the difference between the upper and lower quartiles and so it is the range of the middle 50% of the data.

Summary

$$\begin{aligned}\text{Interquartile range (IQR)} &= \text{upper quartile} - \text{lower quartile} \\ &= Q_3 - Q_1\end{aligned}$$



The interquartile range ignores very low or very high scores (outliers), so sometimes it is better than the range as a measure of spread.

Example 3

The number of points scored by the NSW Waratahs per rugby match during the 2013 season were:

17 31 6 26 30 23 29 25 19 72 21 28 22 28 12

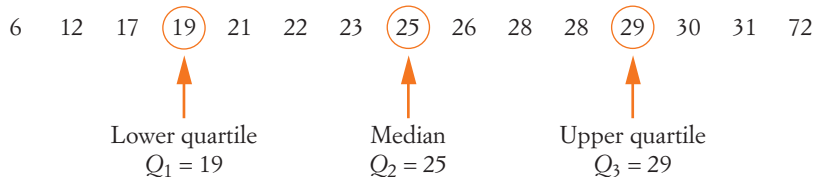
- Find the range.
- Find the interquartile range.
- Which is the better measure of spread of the points scored by the Waratahs – the range or interquartile range?



Getty Images Sport/Cameron Spencer

Solution

First arrange the scores in order:

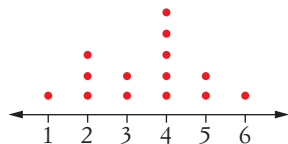


- $$\begin{aligned}\text{Range} &= 72 - 6 \\ &= 66\end{aligned}$$
- $$\begin{aligned}\text{Interquartile range} &= Q_3 - Q_1 \\ &= 29 - 19 \\ &= 10\end{aligned}$$
- The interquartile range is the better measure of spread as the outlier of 72 is excluded. The score of 72 has affected the range, making it very big.

Example 4

Find the interquartile range of each set of data.

a



b

Stem	Leaf
4	0 1 3
5	2 5 6 9
6	2 8
7	0 3 3 4 7 9
8	3 4 5 6 8
9	0 3 4 5

Solution

- a There are 14 scores, so the median is 'between' the 7th and 8th scores.

$$\text{Median, } Q_2 = \frac{4 + 4}{2} = 4$$

Q_1 is the median of the lower half of scores.

$$Q_1 = 2.$$

Q_3 is the median of the upper half of scores.

$$Q_3 = 4.$$

$$\therefore \text{IQR} = Q_3 - Q_1$$

$$= 4 - 2$$

$$= 2$$

- b There are 24 scores, so the median is 'between' the 12th and 13th scores.

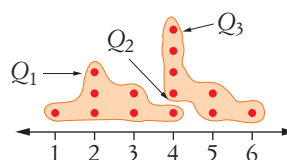
$$\text{Median, } Q_2 = \frac{73 + 74}{2} = 73.5$$

$$\text{Lower quartile, } Q_1 = \frac{56 + 59}{2} = 57.5$$

$$\text{Upper quartile, } Q_3 = \frac{85 + 86}{2} = 85.5$$

$$\therefore \text{IQR} = 85.5 - 57.5$$

$$= 28$$



Stem	Leaf
4	0 1 3
5	2 5 6 9
6	2 8
7	0 3 3 4 7 9
8	3 4 5 6 8
9	0 3 4 5

Exercise 6-02 Quartiles and interquartile range

- 1 Find the quartiles for each set of data.

a 3 7 9 5 5 6 2 8 9 7

b 15 19 18 12 20 34 28 18 28 20 23 25

c 34 45 32 38 29 40 37 33 35 30 34 35 38 37 38 31 30 34

- 2 Calculate the range and the interquartile range of each data set in question 1.

- 3 Calculate the interquartile range for each set of data below.

a 5 6 6 7 8 9 9 10 14 14 15 16

b 2 0 3 5 2 1 0 6 4 3 8 4 2

See Example 2

See Example 3

- 4 The monthly rainfall figures for Ulladulla one year were:

31 174 288 89 15 123
26 5 8 275 38 58

For this data, find:

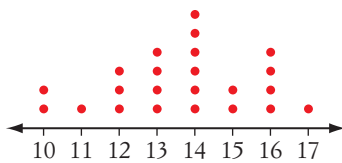
- a the range
b the interquartile range



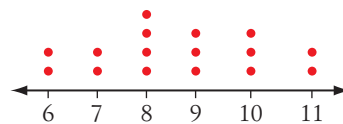
Getty Images/Peter Harrison

See Example 4 5 Find the interquartile range for each set of data.

a



b



c

Stem	Leaf
3	2 7
4	0 3 3 5
5	2 4 5 6 7 8 8
6	3 4 7
7	2

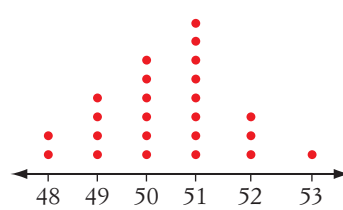
d

Stem	Leaf
1	3 5 8 9
2	0 1 3 3 4 5 6
3	5 8 9 9
4	1 3
5	4

e

Stem	Leaf
10	3 5 5 6 6
11	0 1 2
12	3 4 6 7 8
13	4 7
14	1

f



- 6 The pulse rates for a group of students are as follows.

82 81 72 58 79 77 62 66 92 78 80 67 91 75 72 68

- a Find the range.
b Find the interquartile range.
c i List the scores that lie between the lower and upper quartiles.
ii What percentage of scores lie between Q_1 and Q_3 ?
d What percentage of scores lie above the lower quartile?

- 7 The number of goals per game scored by the Sydney Swifts netball team during 2013 were:

55 35 49 53 51 55 42
48 63 43 48 48 62

- a Find:
i the range ii the interquartile range
b Which is the better measure of spread?
c List the scores that lie in the interquartile range. What percentage of the scores is this?

Just for the record

Statistics: Where did it all begin?

In prehistoric times, when the number of people and animals was recorded in pictures and symbols on the walls of caves, a simple form of statistics was being used.



Alamy/Zew Radovan

Before 3000 BCE, ancient Babylonians used clay tablets to record crop yields and trade data, and around 2650 BCE the Egyptians 'surveyed' the population and wealth of their country before building the pyramids. Forms of statistics were also used in the Bible in the 'Book of Numbers' and the 'First Book of Chronicles'. Numerical records existed in China before 2000 BCE, and the Greeks (to help collect taxes) held a census in 594 BCE. The Roman Empire was the first government to collect information about the population. In 1086 a census was conducted in England. The information obtained in this census was recorded in the Domesday Book.

Use your library or the Internet to find out more about the Domesday Book. Write a one-page report suitable for a classroom presentation.

6-03 Standard deviation

The **standard deviation** is another measure of spread. Like the mean, its value is calculated using every score in a data set.

Summary

The **standard deviation** is a measure of the spread of a set of scores.

The symbol for standard deviation is σ or σ_n .

Its value is an average of how different each score is from the mean.

σ is the lower case Greek letter 'sigma'

Standard deviation has a complex formula so it is best calculated using the calculator's statistics mode. It is a better measure of spread than the range and interquartile range because its value depends on every score in the data set.

Stage 5.3

Worksheet

Statistical calculations

MAT10SPWK10209

Stage 5.3

Example 5

Calculate, correct to two decimal places, the standard deviation of each set of data.

a The daily maximum temperature (in °C) in Campbelltown for two weeks in January.

45.0	24.5	24.8	29.1	35.0	26.9	31.8
33.8	32.9	23.6	22.1	29.2	27.1	32.7

b The scores of Year 10 students in a Science quiz.

Score	2	3	4	5	6	7	8	9	10
Frequency	2	1	3	3	2	5	6	4	2

Solution

Follow the instructions for the statistics mode (SD or STAT) of your calculator as shown in the tables below.

a	Operation	Casio scientific	Sharp scientific
	Start statistics mode.	MODE STAT 1-VAR	MODE STAT =
	Clear the statistical memory.	SHIFT 1 Edit, Del-A	2ndF DEL
	Enter data	SHIFT 1 Data to get table 45.0 = 24.5 = , etc. to enter in column AC to leave table	45.0 M+ 24.5 M+ , etc.
	Calculate the standard deviation ($\sigma_x = 5.75$)	SHIFT 1 Var σ_x =	RCL σ_x
	Return to normal (COMP) mode.	MODE COMP	MODE 0

$\sigma = 5.75$

b

Operation	Casio scientific	Sharp scientific
Start statistics mode.	MODE STAT 1-VAR	MODE STAT =
Clear the statistical memory.	SHIFT 1 Edit, Del-A	2ndF DEL
Enter data	SHIFT 1 Data to get table 2 = 3 = , etc. to enter in x column 2 = 1 = , etc. to enter in $FREQ$ column AC to leave table	2 2ndF STO 2 M+ 3 2ndF STO 1 M+
Calculate the standard deviation ($\sigma_x = 2.26$)	SHIFT 1 Var σ_x =	RCL σ_x
Return to normal (COMP) mode.	MODE COMP	MODE 0

$$\sigma = 2.26$$

Exercise 6-03 Standard deviation

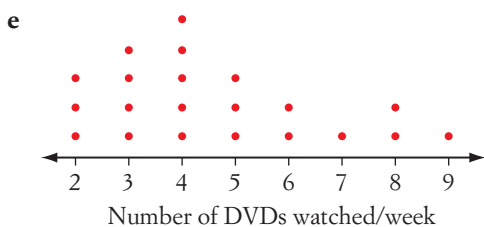
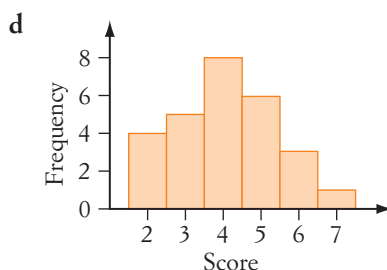
Note: In this exercise, express all means and standard deviations correct to two decimal places.

1 Calculate the standard deviation of each set of data.

- a 5 4 7 8 2 9 10
b 20 23 28 24 19 25 26 24 23

c

x	f
10	2
11	5
12	9
13	8
14	3
15	1



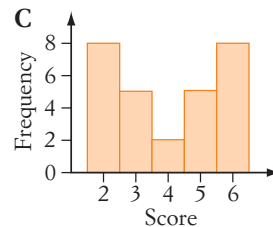
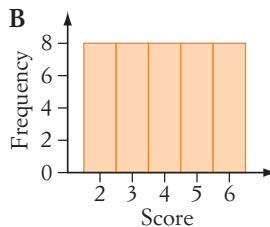
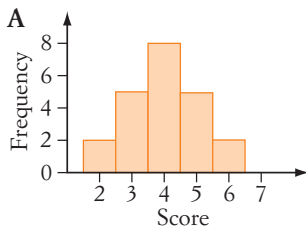
See Example 5

Stage 5.3

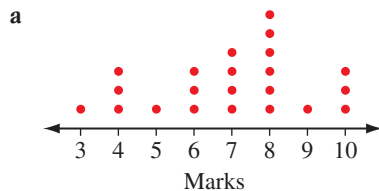
- 2 An English class of Year 10 students scored the following marks for their speeches.

12 15 14 16 16 12 11 18 7 10
15 14 13 13 18 10 12 12 14 13

- a Which score is the outlier?
b Find the standard deviation of the scores:
i with the outlier ii without the outlier.
c What effect does removing the outlier have on the standard deviation?
- 3 For the three statistical distributions **A**, **B** and **C** shown, which one has:
a the greatest standard deviation? b the smallest standard deviation?



- 4 Find the standard deviation of each data set.



b

Stem	Leaf
2	0 2 7
3	5 5 6 8 9
4	1 2 4 5 6 6 7
5	0 3 4 5 9 9
6	1 5 5
7	6

- 5 The heights of girls in a Year 9 basketball team are as follows.

151 161 171 175 176 157 175 163 164

- a Calculate the mean and standard deviation of the heights in the basketball team.
b Another girl joins the basketball team. What is the possible height of the student if the standard deviation:
i increases ii decreases?

- 6 The training times (in seconds) of a sprinter over 100 m are as follows.

11.2 11.0 10.9 12.3 11.8 11.1 11.4 11.6 11.0

- a Find the mean and standard deviation of the training times.
b What training time would the sprinter have to do to:
i increase the standard deviation? ii decrease the standard deviation?

- 7 Brooke's times (in seconds) for swimming 100 m are as follows.

55.7 59.8 58.4 56.7 60.0 55.8 57.4 58.0

An error was made in recording these times and 2 s needs to be added to each of these times.

Which of the following is true? Select the correct answer **A**, **B**, **C** or **D**.

- A** the standard deviation will increase and the mean will stay the same
B the standard deviation will decrease and the mean will increase
C the standard deviation will stay the same and the mean will increase
D the standard deviation and the mean are unchanged

Investigation: The formula for standard deviation

The formula for the standard deviation of a set of scores is $\sigma = \sqrt{\frac{\sum (x - \bar{x})^2}{n}}$ where x is each score, $\bar{x}$ is the mean and n is the number of scores.

The steps for calculating standard deviation are as follows.

- Calculate the mean $\bar{x}$
- For every score in the data set, find the difference between the score and the mean, then square this difference: $(x - \bar{x})^2$
- Calculate the average of these 'squared deviations' by adding them and dividing their sum by the number of scores: $\frac{\sum (x - \bar{x})^2}{n}$
- Calculate the square root of this average: $\sqrt{\frac{\sum (x - \bar{x})^2}{n}}$

We will now use this method to calculate the standard deviation of this set of scores.

4 5 6 7 2 8 6 5 2

- 1 Calculate the mean of these scores.
- 2 Copy and complete the table below by finding, for each score, its difference from the mean and the square of this difference.

Score, x	4	5	6	7	2	8	6	5	2
$x - \bar{x}$	-1	0							
$(x - \bar{x})^2$	1	0							

- 3 Find the mean of the 'squared deviations' calculated in the bottom row of the table.
- 4 The standard deviation is the square root of this mean. Calculate the standard deviation correct to two decimal places.
- 5 Check your answer by calculating the standard deviation using your calculator's statistics mode and comparing both answers.
- 6 Use the standard deviation formula to calculate the standard deviation of each set of scores.

a	5	4	7	8	2	9	10		
b	20	23	28	24	19	25	26	24	23

Check your results by using your calculator.
- 7 The standard deviation is never negative. Explain why.
- 8 If the scores of a set of data are all the same, what is the standard deviation? Explain.

Stage 5.3

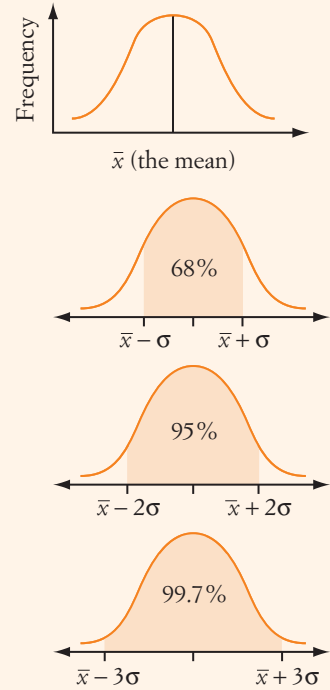
Just for the record

The normal curve

If the heights of all of the people in Australia were graphed on a frequency polygon, the graph would be a normal curve, a symmetrical bell-shaped curve that peaks in the middle.

The normal curve has the following features.

- The mean, median and mode are the same.
- About 68% of scores lie within one standard deviation of the mean.
- About 95% lie within two standard deviations of the mean.
- About 99.7% lie within three standard deviations of the mean.



Measure and analyse the heights of the students at your school. Do the data follow a normal curve?

6-04

Comparing means and standard deviations

The **mean** and **standard deviation** can be used to compare different sets of data.

Example 6

The heights (in cm) of the girls and boys in a Year 10 PE class at Baramvale High were measured.

Girls: 163 155 171 162 165 158 172 166 163 150 160 181 160 156

Boys: 174 167 164 175 189 145 165 166 165 168 167 171 169 172 168

- Calculate, correct to two decimal places, the mean and standard deviation for:
 - the girls
 - the boys
 - the class.
- Which group has the greater spread of heights?
- Is there a significant difference between the heights of girls and boys?

Solution

- a Using the calculator's statistics mode:
 - i Girls: $\bar{x} = 163$ cm, $\sigma_n \approx 7.60$
 - ii Boys: $\bar{x} \approx 168.33$ cm, $\sigma_n \approx 8.64$
 - iii Class: $\bar{x} \approx 165.76$ cm, $\sigma_n \approx 8.58$
- b The group of boys in the class has the greater spread of heights as its standard deviation is higher.
- c The mean height of boys was greater than that of the girls, but the girls had the lower spread of heights.

Comparing measures of spread

The **standard deviation** is usually the most appropriate measure of spread as it uses all of the scores in the data set.

The **range** is the easiest to calculate but its value only depends upon two scores: the highest score and the lowest score.

If there are **outliers** in the data set, then the standard deviation and range will be affected by these extreme scores. In this case, the **interquartile range** is the better measure, because it is the range of the middle 50% of scores and so is not affected by outliers.

Example 7

The ages of the children using a jumping castle and visiting a petting zoo are shown.

Jumping castle:	3	3	4	5	5	6	8	10	18
Petting zoo:	3	4	5	6	6	7	8	8	10

- a For each set of data, calculate:
 - i the range
 - ii the interquartile range
 - iii the standard deviation (to two decimal places)
- b Which is the best measure of spread for each set of data?

Solution

- a For the jumping castle:
 - i Range = $18 - 3 = 15$
 - ii IQR = $9 - 3.5 = 5.5$
 - iii $\sigma_n \approx 4.48$
- For the petting zoo:
 - i Range = $10 - 3 = 7$
 - ii IQR = $8 - 4.5 = 3.5$
 - iii $\sigma_n \approx 2.05$
- b The jumping castle data has an outlier, 18, that affects the range and standard deviation. The interquartile range is the best measure for this data set.
The petting zoo data does not have an outlier, so the standard deviation is the best measure for this data set.

Stage 5.3

Exercise 6-04

Comparing means and standard deviations

See Example 6

Note: In this exercise, express all means and standard deviations correct to two decimal places.

- 1 The pulse rates (in beats/minute) of a sample of men and women taken at a suburban shopping centre.

Men:	68	72	75	73	81	77	69	68	79	83	65	59	60	72	70
Women:	82	61	79	77	75	68	86	81	72	77	78	81	90	83	73

- a Find the mean and standard deviation of each group.
 b Is there a significant difference between the mean and standard deviation for men and women? Give reasons.
- 2 The reaction times (in seconds) for the dominant and non-dominant hands of a group of athletes were measured.

Dominant hand:	0.41	0.29	0.35	0.42	0.42	0.43	0.39	0.61	0.38
	0.34	0.75	0.34	0.38	0.47	0.34	0.32	0.29	0.30
Non-dominant hand:	0.46	0.34	0.38	0.39	0.39	0.39	0.51	0.50	0.47
	0.40	2.60	0.34	0.39	0.51	0.35	0.37	0.31	0.32

- a Find the mean and standard deviation for each data set.
 b Is there a significant difference between the results? Explain your answer.
 c i What are the outliers for the reaction time of the dominant hand?
 ii Find the mean and standard deviation without the outliers.
 iii What effect does removing the outliers have on the mean and standard deviation?
 d Find the mean and standard deviation of the reaction time for the non-dominant hand without the outlier.
 e On which group has the removal of outliers had the greater effect on the mean and standard deviation? Justify your answer.
- 3 The scores of two cricket teams were recorded on a back-to-back stem-and-leaf plot.

Western Tigers					Barrington City	
5				2	7	
					8	3
				7	9	0 8
				8	10	7
					11	4 6
				6	12	1 5
9	9	8	5	13	7	
				7	4	14 6
				5	15	6 8

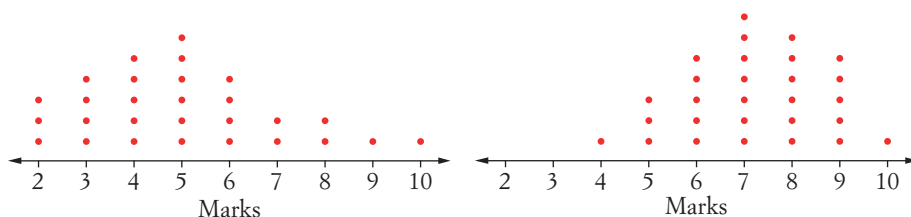
- a Find the mean and standard deviation for each team.
 b Which team was more consistent with its scores?

- 4 Vatha and Ana's times for running 100 m time trials are given below.

Vatha:	13.0	13.5	14.2	13.7	13.2	14.7	13.5	14.3
Ana:	14.2	13.2	15.1	13.8	14.2	15.2	13.9	13.5

- a Find the mean and standard deviation for each runner.
 b Which runner is more consistent? Give reasons.

- 5 The dot plots show the test results of a class before and after using a tutorial website.



Which of the following is true?

- A Both the mean and standard deviation increased
 B The mean increased and the standard deviation decreased
 C The mean decreased and the standard deviation increased
 D Both the mean and standard deviation decreased
- 6 The marks obtained by students in a Maths and Science exams are given below.

Maths:	40	72	76	74	60	64	64	59	74	84	62	84	66	64
	71	68	78	63	57	55	73	80	67	86	57	87	62	52
Science:	42	54	61	72	76	54	65	80	39	74	82	54	57	63
	64	75	68	76	81	40	37	43	58	68	67	49	54	62

- a For each subject, find:
- the range
 - the interquartile range
 - the standard deviation
- b Find the mean for each subject.
- c Determine which subject the students performed better in, giving reasons.
- 7 The points scored per match by the Roosters and the Dragons during a NRL season were:
- Roosters:** 10 16 8 50 22 38 34 30 16 12 18 38 12 20 18 36 40 28 42 28 56 22 22 24
- Dragons:** 10 6 17 25 19 13 10 18 14 32 0 14 14 16 10 0 22 18 20 26 18 18 22 19
- a For each team, find:
- the range
 - the interquartile range
 - the mean
 - the standard deviation
- b By comparing the means and the measures of spread, decide which was the better team.

Stage 5.3

See Example 7

Mental skills 6

Maths without calculators

Multiplying and dividing by 5, 15, 25 and 50

It is easier to multiply or divide a number by 10 than by 5. So whenever we multiply or divide a number by 5, we can double the 5 (to make 10) and then adjust the first number.

- 1 Study each example.

- a To multiply by 5, halve the number, then multiply by 10.

$$\begin{aligned}
 18 \times 5 &= 18 \times \frac{1}{2} \times 10 \quad (\text{or } 9 \times 2 \times 10) \\
 &= 9 \times 10 \\
 &= 90
 \end{aligned}$$

- b To multiply by 50**, halve the number, then multiply by 100.

$$\begin{aligned} 26 \times 50 &= 26 \times \frac{1}{2} \times 100 \quad (\text{or } 13 \times 2 \times 100) \\ &= 13 \times 100 \\ &= 1300 \end{aligned}$$

- c To multiply by 25**, quarter the number, then multiply by 100.

$$\begin{aligned} 44 \times 25 &= 44 \times \frac{1}{4} \times 100 \quad (\text{or } 11 \times 4 \times 25) \\ &= 11 \times 100 \\ &= 1100 \end{aligned}$$

- d To multiply by 15**, halve the number, then multiply by 30.

$$\begin{aligned} 8 \times 15 &= 8 \times \frac{1}{2} \times 30 \quad (\text{or } 4 \times 2 \times 15) \\ &= 4 \times 30 \\ &= 120 \end{aligned}$$

- e To divide by 5**, divide by 10 and double the answer. We do this because there are two 5s in every 10.

$$\begin{aligned} 140 \div 5 &= 140 \div 10 \times 2 \\ &= 14 \times 2 \\ &= 28 \end{aligned}$$

- f To divide by 50**, divide by 100 and double the answer. This is because there are two 50s in every 100.

$$\begin{aligned} 400 \div 50 &= 400 \div 100 \times 2 \\ &= 4 \times 2 \\ &= 8 \end{aligned}$$

- g To divide by 25**, divide by 100 and multiply the answer by 4. This is because there are four 25s in every 100.

$$\begin{aligned} 600 \div 25 &= 600 \div 100 \times 4 \\ &= 6 \times 4 \\ &= 24 \end{aligned}$$

- h To divide by 15**, divide by 30 and double the answer. This is because there are two 15s in every 30.

$$\begin{aligned} 240 \div 15 &= 240 \div 30 \times 2 \\ &= 8 \times 2 \\ &= 16 \end{aligned}$$

- 2** Now evaluate each expression.

a 32×5

b 14×5

c 48×5

d 18×50

e 52×50

f 36×25

g 28×5

h 12×25

i 12×15

j 22×35

k $90 \div 5$

l $170 \div 5$

m $230 \div 5$

n $1300 \div 50$

o $900 \div 50$

p $300 \div 25$

q $1000 \div 25$

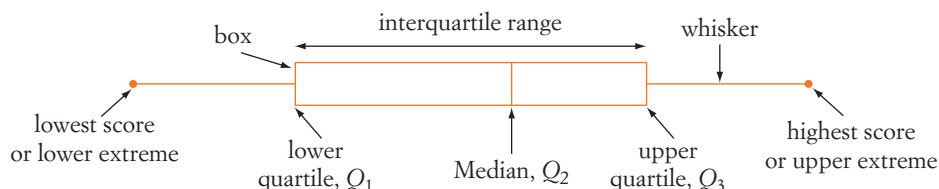
r $360 \div 45$

s $210 \div 15$

t $360 \div 15$

6-05 Boxplots

A **boxplot** (or box-and-whisker plot) displays the quartiles of a set of data and the lowest and highest scores (lower and upper extremes).



The 'box' represents the middle 50% of scores and the interquartile range, while the 'whiskers' represent the lowest and highest 25% of scores.



Summary

A boxplot gives a **five-number summary** of a data set:

- the lower extreme (or lowest score)
- the lower quartile, Q_1
- the median, Q_2
- the upper quartile, Q_3
- the upper extreme (or highest score)

Example 8

The number of hours per week that Nick worked at the Big Chicken over summer were:

5 5 4 8 10 3 12 7 7 3 8 8 15

- Find a five-number summary for this data.
- Represent this data on a box-and-whisker plot.

Solution

- First arrange the scores in order.

3 3 4 5 5 7 8 8 8 10 12 15

$\uparrow$ $\uparrow$ $\uparrow$
 Q_1 median Q_2 Q_3

Lower extreme = 3

Lower quartile = $\frac{4+5}{2} = 4.5$

Median = 7

Upper quartile = $\frac{8+10}{2} = 9$

Upper extreme = 15

Video tutorial
Box-and-whisker plots

MAT10SPVT10004

Video tutorial
Statistics

MAT10SPVT00002

Worksheet
Five number summaries

MAT10SPWK10035

Puzzle sheet
Mode, median and mean

MAT10SPPS00044

Technology
GeoGebra:
Boxplot and dot plot

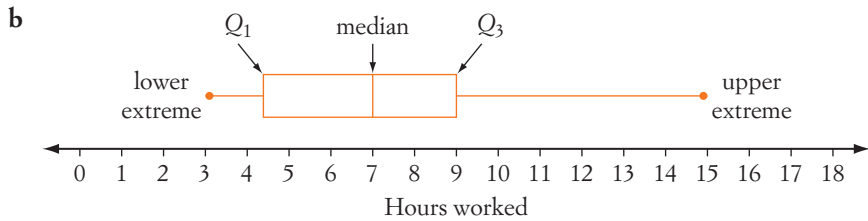
MAT10SPCT00002

Technology worksheet
Excel worksheet:
Five number summary

MAT10SPCT00002

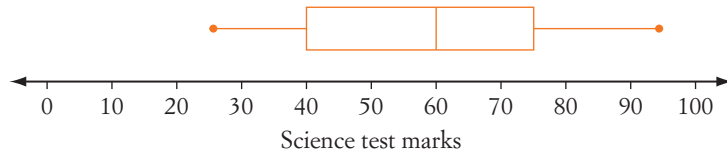
Technology worksheet
Excel spreadsheet:
Five number summary

MAT10SPCT00032



Example 9

The boxplot represents the results of 80 students in a Science test.



- Find the range of the test results.
- Find the median test score.
- What is the interquartile range?
- How many students had a test mark between:
 - 25 and 75?
 - 40 and 60?
- What percentage of students scored more than 75?

Solution

- Range = highest score – lowest score
 $= 95 - 25$
 $= 70$
- Median = 60
- Interquartile range = $Q_3 - Q_1$
 $= 75 - 40$
 $= 35$
- 25 is the lowest score and 75 is Q_3 , so $75\% \times 80 = 60$ students had a mark between 25 and 75.
 - 40 is Q_1 and 60 is the median, so $25\% \times 80 = 20$ students had a mark between 40 and 60.
- 75 is the third quartile so $25\% \times 80 = 20$ students scored more than 75.

Exercise 6-05 Boxplots

- 1 The number of orders taken per hour at Bramavale Pizza on a weekend were:

3 5 1 2 4 6 8 10 7 6
12 15 10 3 5 18 5 8 9 10

- a Find the five-number summary for this data.
b Represent this data on a box-and-whisker plot.

See Example 8

- 2 The daily amount of snow (in cm) that fell at Thredbo during one ski season was:

2 5 5 2 5 7 1 2 2 2 2 2 12
20 12 5 40 50 10 40 13 30 5 35 2 6

- a On how many days did it snow?
b Find a five-number summary for this data.
c Represent this data on a box-and-whisker plot.

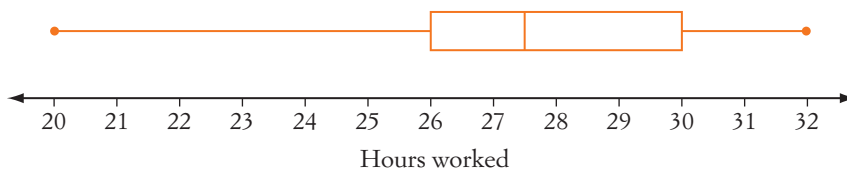
- 3 The monthly rainfall figures in millimetres for Penrith in 2012 were:

98 266 149 94 15 65 19 5 24 34 67 28

Source: Bureau of Meteorology

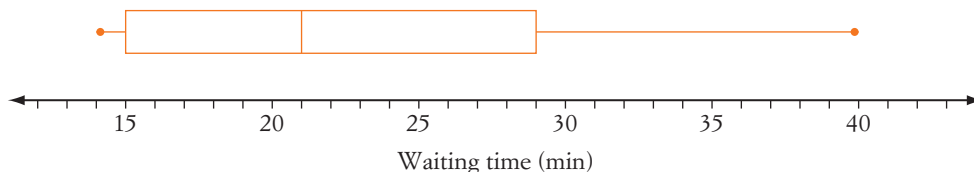
- a Find the range. b Find the five-number summary.
c Represent the data on a boxplot.

- 4 This boxplot represents the number of hours worked in one week by the staff at a supermarket.



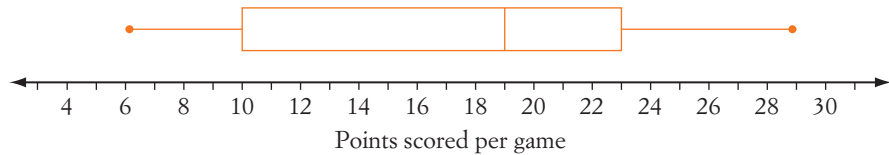
See Example 9

- a What is the median number of hours worked? b What is the lower quartile?
c What is the upper quartile? d Find the interquartile range.
e Estimate the percentage of employees that worked between 26 and 30 hours.
- 5 The ages of 16 people waiting at a bus stop are displayed by the boxplot below.



- a What is the range?
b What is the median age?
c Find the interquartile range.
d What percentage of people were aged from:
i 21 to 29? ii 15 to 40?

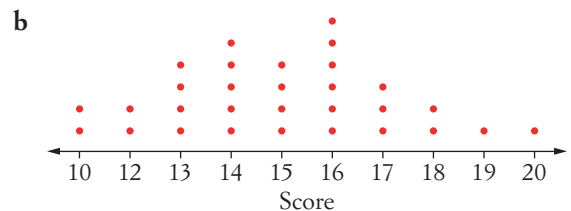
- 6 The box-and-whisker plot shows the number of points per game scored by Ben in 28 basketball games during the season.



- What is the five-number summary for the boxplot?
 - Find the interquartile range.
 - In how many games did Ben score:
 - more than 19 points?
 - between 19 and 23 points?
 - less than 10 points?
 - at least 10 points?
- 7 For each set of data, find the five-number summary and draw a boxplot.

a

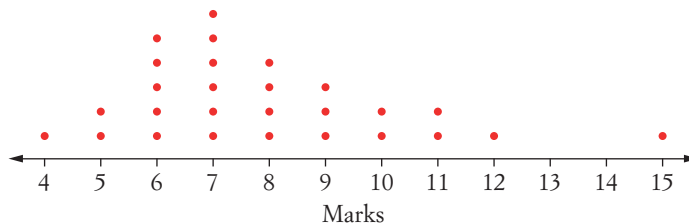
Stem	Leaf
2	0 2 3 5
3	3 7
4	4 6 7 8 8 9 9
5	0 1 1 5 6
6	0 3 3 8 8
7	2 5 6
8	5 5 7 8



c

Stem	Leaf
3	0 7
4	2 6 6
5	1 2 5 9
6	0 4 7 7 9
7	2 3 5 6 8
8	3 4
9	5

- 8 The results of a general knowledge quiz (out of 15) taken by Year 10 students are displayed by the dot plot.

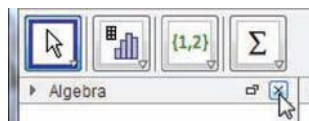


- Find the five-number summary for the dot plot and then draw a box-and-whisker plot.
- Describe the shape of the dot plot and compare it to the shape of the boxplot.
- What is the outlier?
- Find the five-number summary for the data in the dot plot without the outlier and draw a boxplot.
- Compare the two boxplots. How are they:
 - similar?
 - different?

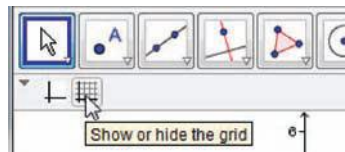
Technology Boxplots

In this activity we will use GeoGebra to draw boxplots.

- 1 Close the **Algebra** window so that only the graphics window is showing.



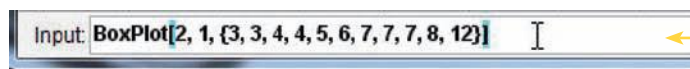
- 2 Select the grid option at the top left-hand corner.



- 3 Data for boxplots is entered in the format shown below.

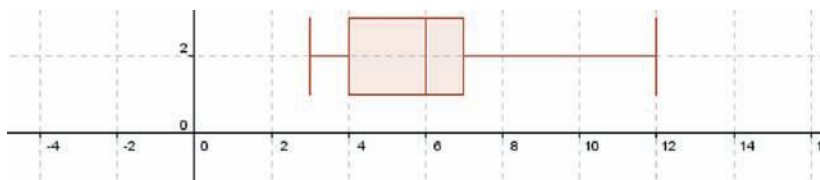
Boxplot[y-position, width of box, {data set}]

The **y-position** is where you want the boxplot to sit above the *x*-axis. In the Input panel at the bottom, type **BoxPlot[2, 1, {3, 3, 4, 4, 5, 6, 7, 7, 7, 8, 12}]**.



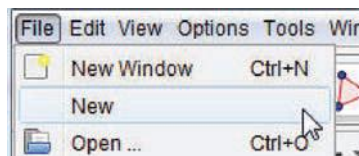
Press spacebar after each number e.g. {3, SPACE 3, SPACE 4, etc.}

- 4 To move the screen view, hold down the **Ctrl** key on your keyboard and use your mouse to drag the screen across. Your boxplot should look exactly like the one below.



- 5 Write down the five-number summary for this data set.

- 6 We will show the results of an English exam completed by classes 10A and 10B using a boxplot. To start up a new file with the same settings, select **File, New**.



In the input panel, enter the following formula for the results for 10A.

BoxPlot[4, 2, {21, 81, 33, 58, 67, 76, 64, 74, 56, 60, 54, 74, 49, 83, 66}]

- 7 Move the screen view as before. To zoom in, hold down the **Ctrl** key on your keyboard and scroll up using your mouse scroll wheel. Scroll down to zoom out. This will allow you to view the boxplot.

Worksheet

Box-and-whisker plots

MAT10SPWK10036

Worksheet

Data 1

MAT10SPWK00032

Animated example

Analysing data

MAT10SPAE00002

Technology worksheet

Excel worksheet:
Parallel box plots

MAT10SPCT00004

Technology worksheet

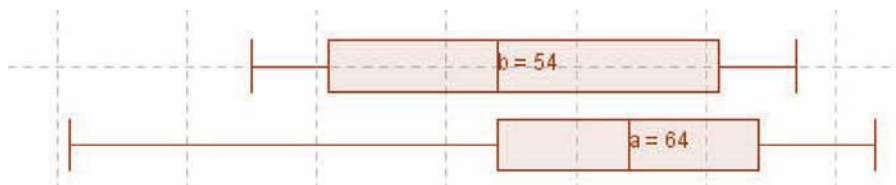
Excel spreadsheet:
Parallel box plots

MAT10SPCT00034

8 In the input panel, enter the following formula for the results for 10B.

BoxPlot[10, 2, {77, 63, 63, 35, 51, 42, 54, 55, 71, 43, 41, 41, 40, 76, 72}]

Note: '10' means the box-and-whisker plot for 10B will be above the one for 10A (i.e. not drawn on top of each other). You will now have two boxplots to compare.



9 Complete a five-number summary for each data set.

10 What is the IQR for each class?

11 Which class had the highest mark?

12 Which class had the lowest mark?

13 Which class performed better? Give reasons for your answer, including explanations using the five-number summaries you found in step 9.

6-06 Parallel boxplots

Parallel box-and-whisker plots can be used to compare two or more sets of data. They are drawn on the same scale, but above each other.

Example 10

Two sprinters run the following times (in seconds) over 100 metres.

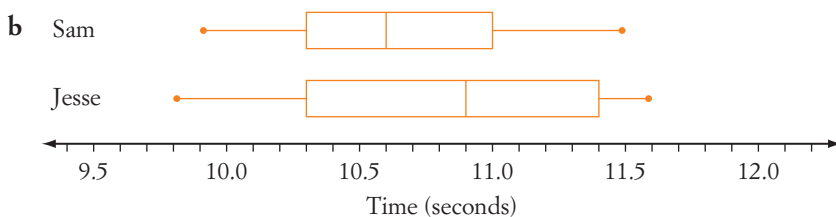
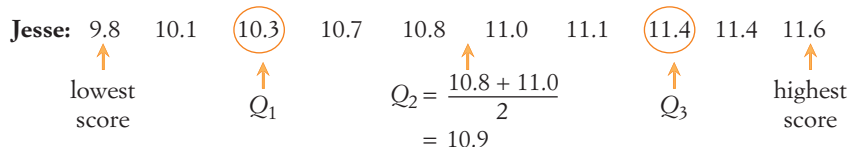
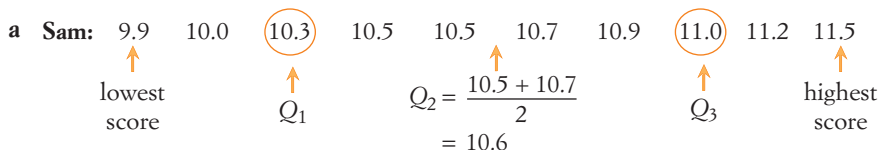
Sam	10.9	10.5	11.0	9.9	10.7	10.5	10.0	11.2	11.5	10.3
Jesse	11.0	11.4	10.1	9.8	10.8	11.4	10.7	10.3	11.1	11.6

- Find the five-number summary for each sprinter.
- Draw parallel boxplots to display the data for both sprinters.
- Find the interquartile range for each sprinter.
- Find the range for each sprinter.
- Which sprinter is more consistent? Justify your answer.



Alamy/moodboard

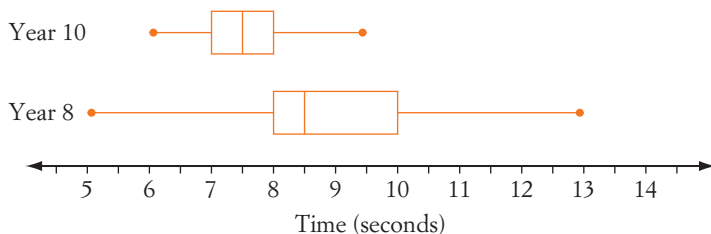
Solution



- c** Interquartile range for Sam = $11.0 - 10.3 = 0.7$
 Interquartile range for Jesse = $11.4 - 10.3 = 1.1$
- d** Range for Sam = $11.5 - 9.9 = 1.6$
 Range for Jesse = $11.6 - 9.8 = 1.8$
- e** Sam is the more consistent sprinter since both the range and interquartile of his times are lower than those of Jesse.

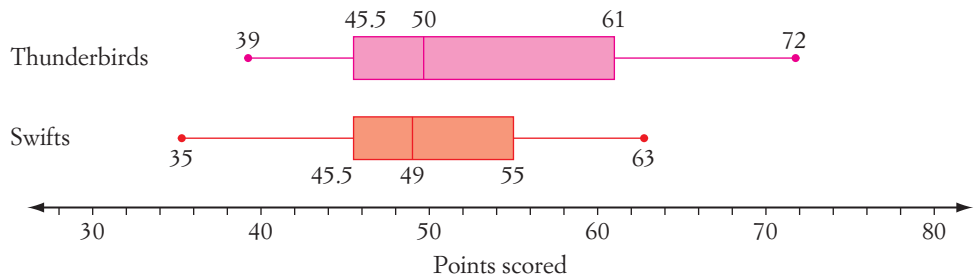
Exercise 6-06 Parallel boxplots

- 1** The parallel boxplot shows the amount of sleep that Year 8 and Year 10 students usually get on a school night.



- a** For each Year group, find:
- i the range ii the median iii the interquartile range
- b** What percentage of students usually had at most 8 hours of sleep on a school night in:
- i Year 8? ii Year 10?
- c** 40 students in both Year 8 and Year 10 were surveyed. How many students usually had at least 10 hours of sleep in:
- i Year 8? ii Year 10?

- 2 The number of points scored by the Adelaide Thunderbirds and the Sydney Swifts during the 2013 netball season are shown in the parallel box-and-whisker plot.



- a Find the range of points scored by:

i the Adelaide Thunderbirds

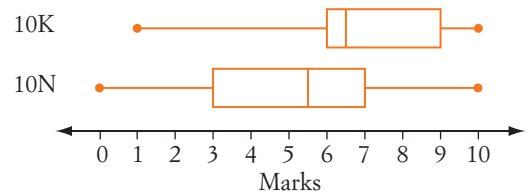
ii the Sydney Swifts

- b What is the median number of points scored for both teams?
c Find the interquartile range for both teams.
d Which team is more consistent?
e Which team performed better? Give reasons.

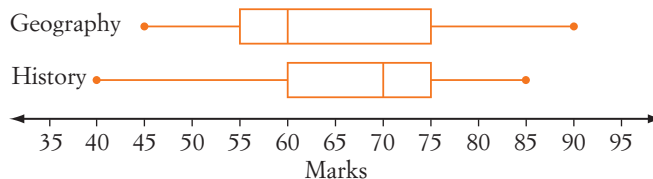


- 3 The boxplots show the test results of students from two different classes.

- a Find the range of marks for each class.
b Find the median mark for each class.
c Find the interquartile range for each class.
d Which class is more consistent?
e Find the percentage of students who scored 6 or more in 10K.



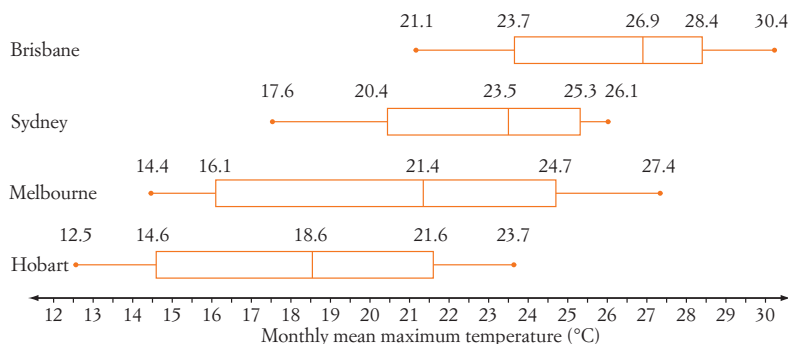
- 4 In a Year 10 class of 28 students, the marks for History and Geography tests were displayed on a double boxplot.



Which of the following statements could be true?

- A In Geography, more students scored between 60 and 75 than between 55 and 60.
B Fourteen students scored the same or more in History than the median mark in Geography.
C More students scored 60 or more in History than they did in Geography.
D The interquartile range for Geography is 5 less than the interquartile range for History.

- 5 The monthly mean maximum temperatures for four Australian capital cities are shown in the boxplots below.



- Find the median, range and interquartile range for each city.
 - Which capital city had the most spread in temperature?
 - Which capital city had the highest mean monthly temperatures? Justify your answer.
 - Which city is warmer – Sydney or Melbourne? Give reasons.
 - Which city was more consistent – Sydney or Melbourne? Give reasons.
- 6 The number of text messages received by a group of students in one hour are as follows.

See Example 10

Male: 2 0 3 0 1 2 5 6 2 1 3 2 3 7 4

Female: 4 5 6 3 7 5 8 7 4 2 4 5 10 4 3

- Find the five-number summary for each gender.
 - Draw parallel box-and-whisker plots to display the data.
 - Find the interquartile range for each gender.
 - Find the range for each gender.
 - Compare the number of text messages that males and females receive. Are there any significant differences between the spread of the two sets of data?
- 7 Students in a PE class had their heights measured in centimetres.

Male: 174 167 164 175 189 145 165 166 165 167 171 169

Female: 163 155 171 162 165 183 172 175 166 163 150 186

- Find the five-number summary for each group and draw a parallel boxplot to display the data.
 - Find the range and interquartile range for each group.
 - How does the spread of heights of male students compare with the spread of heights of female students?
- 8 Students at a university were asked whether their frequency of exercise was high or low and then had their pulse taken. The results are as follows.

Low: 90 78 80 84 70 66 92 80 80 77 64 88

High: 96 71 68 56 64 60 50 76 78 49 68 74

- Find a five-number summary for each group and then draw parallel boxplots to show the information.
- Find the range and interquartile range for each group.
- Compare the spread between the two groups. Are there significant differences between them?
- Which group had the lower pulse rates?

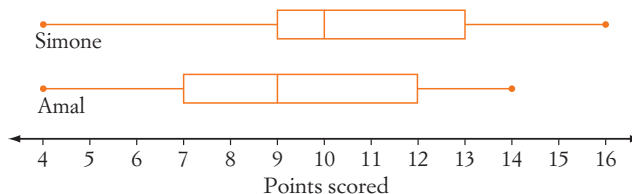
- 9 The average monthly temperatures for Sydney and Brisbane in 2012 are as follows.

Sydney: 26.1 25.8 24.7 23.6 20.9 17.7 17.6 19.9 22.5 23.3 24.1 26.0

Brisbane: 28.7 29.8 28.2 26.5 24.0 21.1 21.4 23.3 25.5 27.3 28.2 30.4

Source: Bureau of Meteorology

- Find the five-number summary for each city and draw a parallel boxplot.
 - Find the range and interquartile range for each city.
 - Which city had more consistent average monthly temperatures? Give reasons.
- 10 These box-and-whisker plots show the numbers of points scored by two basketball players during the season.



- Which player has the highest point score for a single game?
- What is the range of the points scored by each player?
- By just looking at the range, which player would seem to be more consistent? Justify your answer.
- Find the median score of each player.
- Find the interquartile range for each player.
- Which player is more consistent?
- Estimate the percentage of games in which Simone scored 9 or 10 points.

Worksheet

Comparing city temperatures

MAT10SPWK10037

6-07 Comparing data sets

Example 11

The back-to-back stem-and-leaf plot shows the results in Year 10 Maths and Science tests.

Maths						Science				
				5 2	3	6 8				
			8 6 3 0		4	4 6				
			8 7 7 4 1		5	1 5 9				
8 8 7 6 6 3 2 0					6	0 2 8 9				
		6 5 4 2 1 1			7	2 3 4 4 5 8 8				
			6 4 3		8	0 0 2 4 5 6 7 8 9				
			6 0		9	0 4 4				

- Find the mean mark (correct to one decimal place) for each subject.
- Find the median for each subject.
- Find the range and interquartile range for each subject.
- For each subject:
 - describe the shape
 - identify any outliers and clusters.
- In which subject have the students performed better? Justify your answer.

Solution

a Mean for Maths = $\frac{1919}{30}$
= 64.0

Mean for Science = $\frac{2151}{30}$
= 71.7

b Median for Maths = 66
Median for Science = 74.5

Average of the 15th and 16th scores.

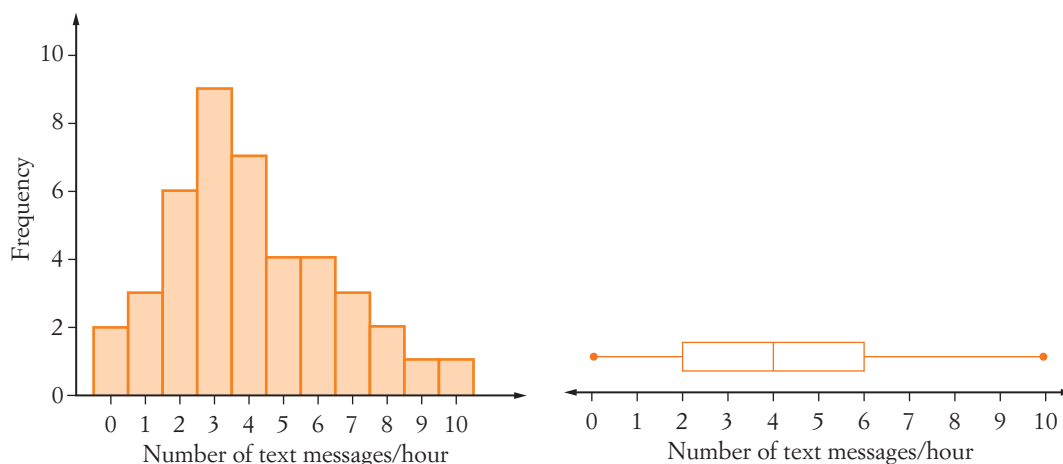
c Range for Maths = $96 - 32 = 64$
Interquartile range = $74 - 54 = 20$

Range for Science = $94 - 36 = 58$
Interquartile range = $85 - 60 = 25$

- d i The results for Maths are symmetrical, while the results for Science are negatively skewed.
ii There is some clustering for the Maths results in the 60s and in Science the clustering occurs in the 70s and 80s.
- e The students have performed better in Science as the mean and median for it are greater than the mean and median for Maths. The range for Maths is greater than the range for Science, but the interquartile range is less than that of Science.

Example 12

The number of text messages received by a group of teenagers are displayed in the frequency histogram and the boxplot below.



- a How many teenagers received more than 6 text messages per hour?
- b Find:
- the mode
 - the median
 - the range
 - the interquartile range.
- c The shape of the distribution is positively skewed. How is this shown by:
- the frequency histogram
 - the boxplot?
- d According to the boxplot, what percentage of teenagers received 2 or more text messages?
- e What information is better seen on:
- the frequency histogram
 - the boxplot?

Solution

- a Number of teenagers receiving more than 6 text messages

$$= 3 + 2 + 1 + 1$$

$$= 7$$

Using the frequency histogram.

- b i Mode = 3

Using the frequency histogram.

- ii Median = 4

Using the boxplot.

- iii Range = $10 - 0$
= 10

Using the frequency histogram or boxplot.

- iv Interquartile range = $6 - 2$
= 4

Using the boxplot.

- c i The tail of the frequency histogram leans towards the higher scores.

- ii The length of the boxplot to the right of the median (Q_2) is greater than its length to the left of the median.

- d $Q_1 = 2$, so 75% of teenagers received 2 or more text messages/hour.

- e i The mode and information regarding the number of text messages received by teenagers can be determined from the frequency histogram.

- ii The median, quartiles and interquartile range are easily determined from the boxplot.

Exercise 6-07 Comparing data sets

See Example 11

- 1 The back-to-back stem-and-leaf plot shows the amount of cash (in dollars) carried by a sample of Year 11 students at Mavbalear Senior High.

- a Find the mean amount of cash (to the nearest cent) carried by each group.

- b Find the median amount of cash carried by each group.

- c Find the range and interquartile range of each group.

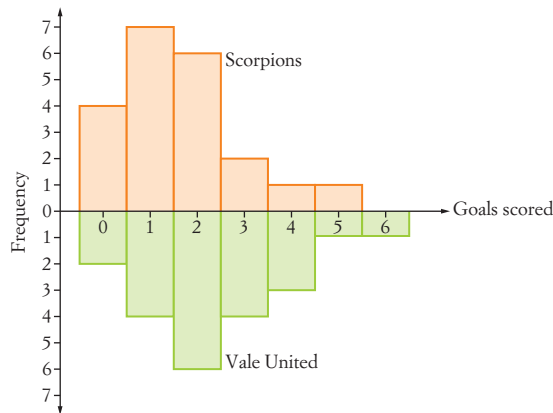
- d For each group:

- i describe the shape ii identify any outliers and clusters.

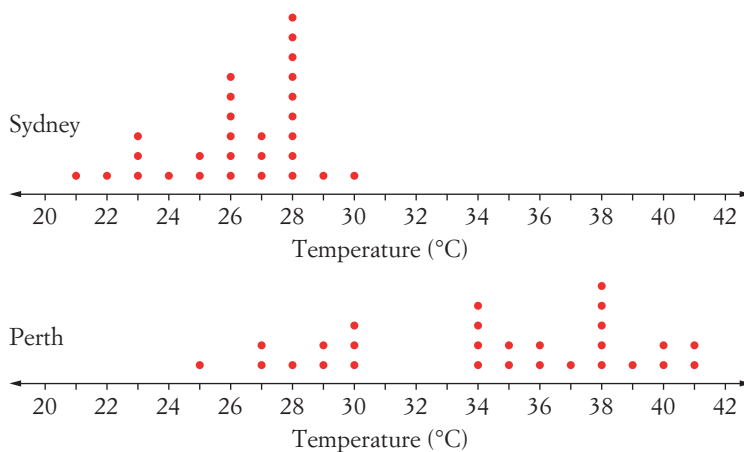
- e Who generally carries more cash – boys or girls? Justify your answer.

	Boys		Girls
	5 5 3	0	5 5 6 8 9
	8 5 5 2 0	1	0 2 2 5 5 8 8 9
	9 6 5 5 5 0 0	2	0 5 6 8 8 8
	8 5 5 4 3 2 0 0	3	0 1 4 5 6
	5 4 4 2 2 0	4	0 0 5 6
	6 6 5 4 3	5	0 3 5
	4 2 2	6	5 5 8
		7	0 4

- 2 The back-to-back histogram shows the number of goals scored by two football teams during a season.

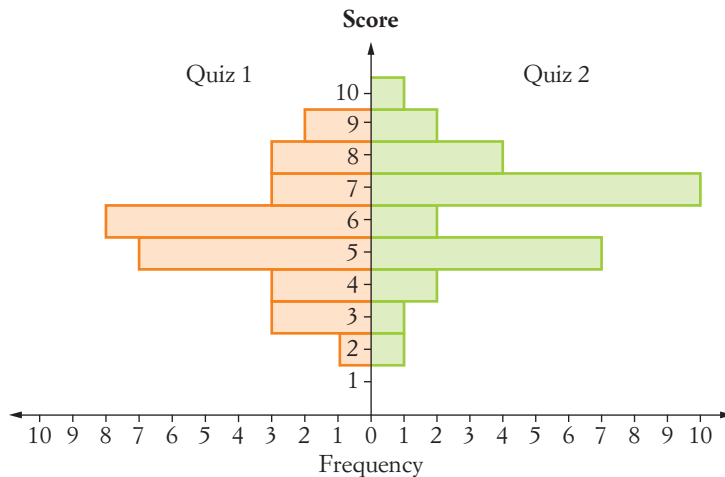


- How many games were played by each team?
 - How many goals were scored by:
 - Scorpions
 - Vale United?
 - Find the mean number of goals scored by each team.
 - What is the range for each team?
 - Describe the shape of each team's results.
 - Which team performed better? Give reasons.
- 3 The daily maximum temperatures for Sydney and Perth in February are shown below.



- Find the mean, median and modal temperatures for each city.
- Find the range and interquartile range of temperatures for each city.
- Describe the distribution shape of the temperatures for each city and identify any outliers and clusters.
- Compare the temperatures in Sydney and Perth. Comment on measures of location (the mean, median and mode), and measures of spread (range and interquartile range).

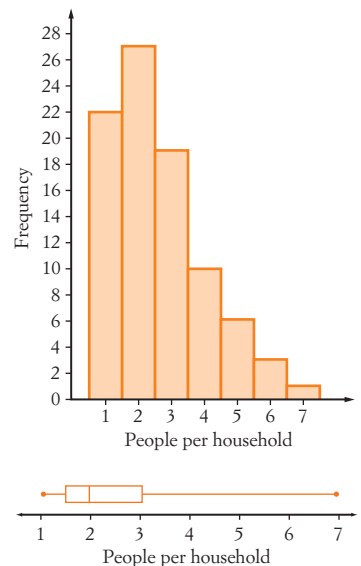
- 4 The results for two quizzes taken by a Year 10 History class are shown below.



- How many students are in the Year 10 History class?
- Find the mean and mode for each quiz.
- Find the median for each quiz.
- For each quiz, find:
 - the range
 - the interquartile range.
- Describe the distribution for each quiz, identifying any clusters and outliers.
- Are there significant differences between the results of the two quizzes? Justify your answer.

See Example 12

- 5 A survey to determine the number of people per household was conducted in several shopping centres. The results are shown in the frequency histogram and boxplot on the right.
- How many households had 3 or more people?
 - Find the:
 - mode
 - median
 - range
 - interquartile range.
 - Describe the shape of the distribution.
 - According to the boxplot, what percentage of households had 2 or more people?
 - Clustering occurs at 1 to 3 people per household. How is this shown on the:
 - frequency histogram?
 - boxplot?
 - What information is better seen on:
 - the frequency histogram?
 - the boxplot?

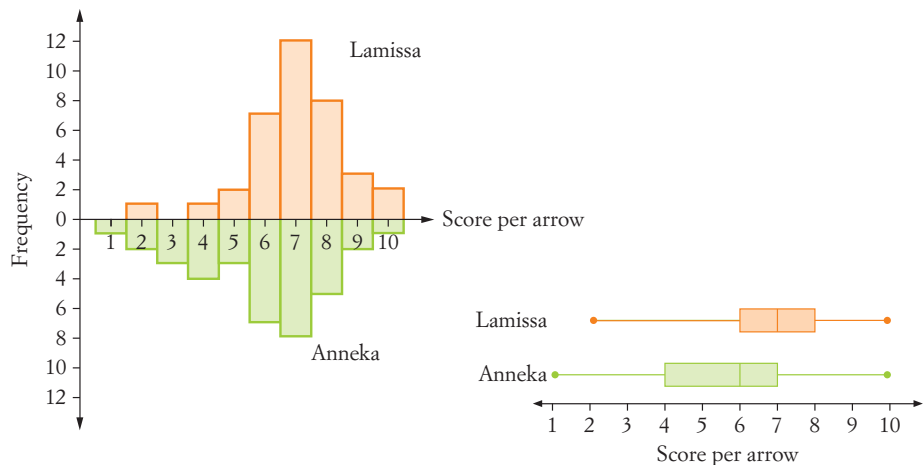


-
- The top dot plot shows the following data points (hours per week): 10, 11, 12, 13, 14, 14, 15, 15, 15, 16, 16, 17, 18, 19, 19, 20, 20, 21, 21, 22, 22, 22, 22, 23, 24, 24, 24, 25, 25, 25, 26, 27, 28.
- The bottom box plot shows the following summary statistics: Minimum = 10, Q1 = 15, Median = 21, Q3 = 24, Maximum = 28.

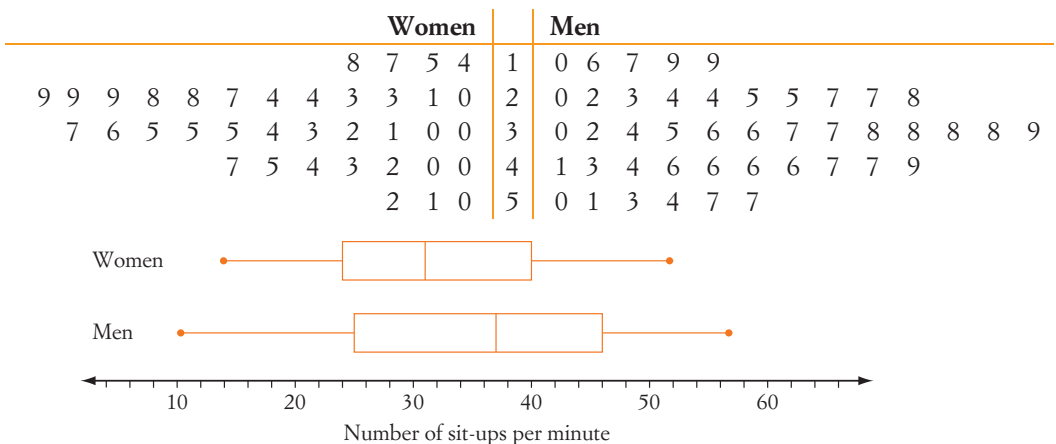
Sunbeam Valley	Bentley's Beach
8	5
9 8 8 7 4 3 3 3 2 0	6 0 0 1 2 3 5 5 7 8 9
9 9 6 5 5 4 4 3 3 2 2 1 1 0 0 0	7 0 0 2 2 3 3 5 5 5 6 6
2 0 0	8 0 2 3 4 5 5 5 8
	9 0
-
- Box plot showing the distribution of car speeds (km/h) for two locations: Sunbeam Valley and Bentley's Beach.
- The x-axis represents Speed (km/h), ranging from 50 to 90.
- Sunbeam Valley:**
- Minimum: 58
 - Q1: 67
 - Median: 71
 - Q3: 75
 - Maximum: 82
- Bentley's Beach:**
- Minimum: 60
 - Q1: 67
 - Median: 73
 - Q3: 82
 - Maximum: 90

- 221

- 8 Lamissa and Anneka each shot arrows at a target 50 m away during an archery contest. They scored 10 for a bulls-eye down to 1 for the outer ring. Their results are displayed in the back-to-back histogram and the parallel box-and-whisker plots below.

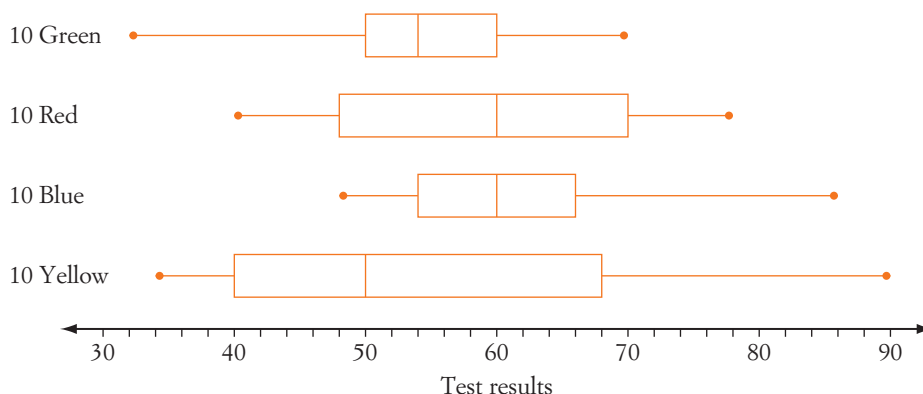


- How many arrows each did Lamissa and Anneka shoot?
 - Find the mode and median score per arrow for each contestant.
 - Find the range and interquartile range for each contestant.
 - Describe the shape of the distribution for each contestant.
 - According to the boxplots, on what percentage of the arrows shot was a score of 6 or less achieved by:
 - Lamissa?
 - Anneka?
 - Who was the better archer during this contest? Justify your answer by referring to the measures of location and spread.
- 9 The number of sit-ups per minute completed by men and women at the Full On Fitness Centre are displayed in the back-to-back histogram and parallel boxplots.



- Why would a dot plot be an inappropriate way to display the data shown above?
- What is the median number of sit-ups per minute completed by each group?
- Find the range and interquartile range for each group.
- Describe the shape of the distributions for women and for men.
- Which group has more spread in the number of sit-ups completed per minute? Give reasons for your answer.

10 The results of a Maths test given to four Year 10 classes are shown below.



- What is the range of test results for:
 - 10 Yellow?
 - 10 Blue?
- For which class are the test results:
 - positively skewed?
 - negatively skewed?
 - symmetrical?
- Which class had:
 - the lowest interquartile range?
 - the highest test score?
 - the highest median?
- Which class had the best test results overall? Give reasons.

6-08 Scatter plots

Bivariate data is data that measures two variables, such as a person's height and arm span (distance between outstretched arms). Bivariate data is represented by an ordered pair of values that can be graphed on a **scatter plot** for analysis.

A scatter plot is a graph of points on a number plane. Each point represents the values of the two different variables and the resulting graph may show a pattern that may be linear or non-linear. If there is a pattern, then a relationship may exist between the two variables.

Example 13

The heights and arm spans of a group of students are shown in the table.

Height, H cm	162	182	153	145	172	163	150	142	183	145	192	171
Arm Span, S cm	158	185	145	143	174	165	151	141	181	158	191	178

- Plot the data on a scatter plot.
- Describe the pattern of the plotted points.
- Describe the relationship between the students' heights and arm spans.

Puzzle sheet

Scatter plots matching game

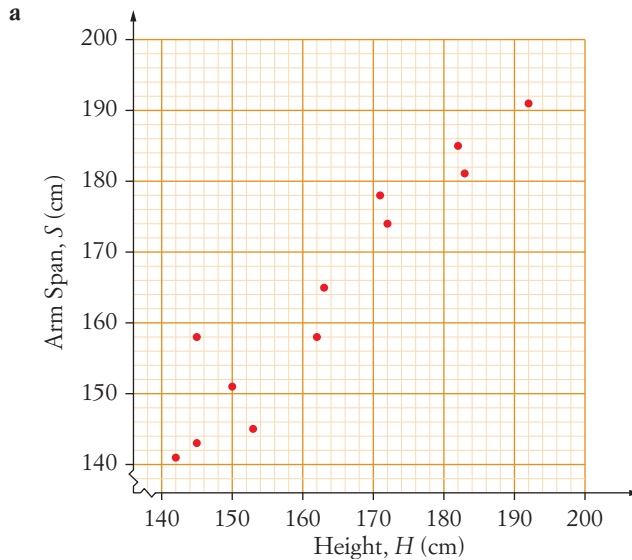
MAT10SPPS10038

Worksheet

Scatter plots

MAT10SPWK00002

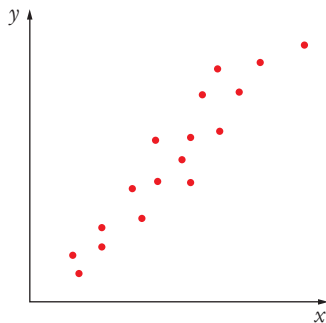
Solution



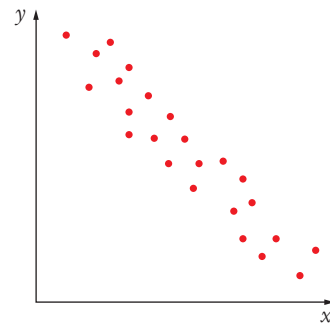
- b** The points form a linear pattern.
c As the heights of students increase, their arm spans tend to increase.

Strength and direction of linear relationships

The type of linear pattern will indicate the **strength** and **direction** of the relationship between the two variables.



Two variables x and y have a **positive relationship** if y increases as x increases.



Two variables x and y have a **negative relationship** if y decreases as x increases.

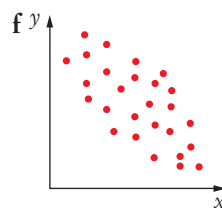
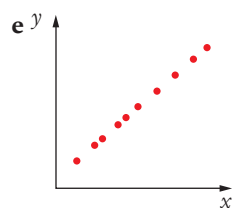
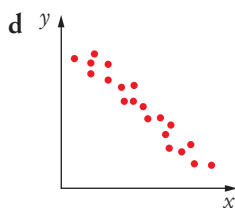
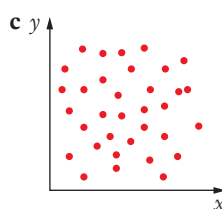
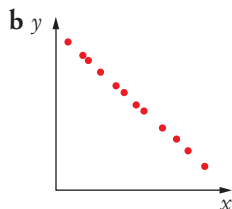
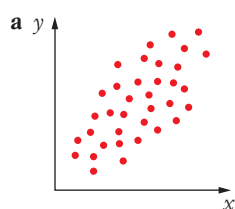
Summary

The strength of a relationship between two variables can be described as:

- **strong** if the points are close together
- **weak** if the points are more spread out
- **perfect** if all points lie on a straight line

Example 14

Describe the strength and direction of the relationship shown in each scatter plot.



Solution

- a** weak positive relationship
- b** perfect negative relationship
- c** no relationship
- d** strong negative relationship
- e** perfect positive relationship
- f** weak negative relationship

The points can be seen to form a line but they are very spread out.

The points seem to lie on a decreasing straight line.

The points are very spread out with no pattern.

The points can be seen to form a decreasing line and they are close together.

The points lie on an increasing straight line.

The points can be seen to form a decreasing line but they are very spread out.

Dependent and independent variables

If a variable y depends on the value of the variable x , y is called the **dependent variable**, and x is called the **independent variable**. For example, stride length (the length of a person's walking step or pace) depends on the person's height, so stride length is the dependent variable and height is the independent variable. When graphing, the dependent variable is shown on the vertical (y -) axis while the independent variable is shown on the horizontal (x -) axis.

Exercise 6-08 Scatter plots

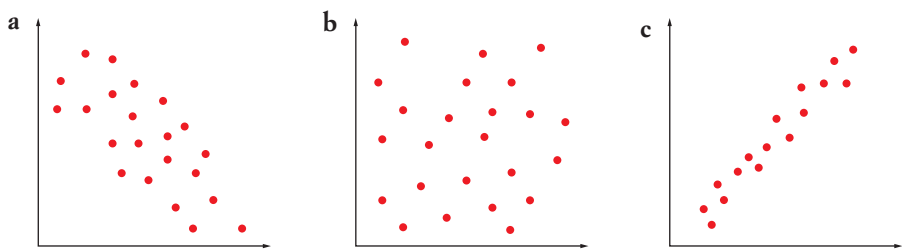
- 1 The heights and handspans of a group of students are shown in the table.

See Example 13

Height, H cm	168	175	175	156	160	173	171	180	185	175	182	180
Handspan, S cm	20.0	21.1	17.6	16.5	17.5	19.0	20.8	22.5	25.0	23.0	20.2	21.1

- a** Plot the data on a scatter plot.
- b** Describe the pattern of the plotted points.
- c** Describe the relationship between the students' heights and their handspans.

See Example 14 2 Describe the strength and direction of the relationship shown in each scatter plot.



- 3 Describe the strength and direction between the variables height, H and handspan, S in question 1.
4 The height and stride length measurements of some students are shown in the table below.

Height, H cm	174	160	158	180	169	172	171	171	148	190	166	173
Stride Length, L cm	72.2	64.0	66.4	74.7	70	71.5	70.9	71.2	61.4	78.9	68.0	71.9

- a Explain why stride length is the dependent variable.
b Graph this data on a scatter plot.
c Describe the pattern of the plotted points.
d Describe the relationship between the students' heights and stride lengths.
e Describe the strength and direction of the relationship.
f Predict the stride length of a student who is 175 cm tall.
- 5 The table lists the points scored for and against each NRL team one season.
- a Graph this data on a scatter plot.
b Is the pattern of the points linear?
c Describe the strength and direction of the relationship between points scored for and points scored against.



Dreamstime/Vsevolka

Points scored for, F	Points scored against, A
568	369
579	361
559	438
497	403
597	445
545	536
445	441
481	447
405	438
506	551
449	477
448	488
462	626
497	609
409	575
431	674

- 6 Year 10 students were surveyed on the number of hours in a week they spent doing homework and the number of hours they spent on the computer. The results are shown in the table.

Homework, H	2	15	12	5	4	2	4	15	14	5	2	5	20	4	2	11
Computer, C	25	30	18	35	6	30	20	22	6	40	8	3	20	30	5	8

- a Plot the points on a scatter plot.
b Describe the strength and direction of the relationship between the hours spent doing homework and the hours spent on the computer.

- 7 A survey was conducted to see whether there was a relationship between height and the age of students in a high school. The results are in the table below.

Age, A (years)	14	16	15	13	11	14	17	15	12	11	14	16	13	18
Height, H (cm)	162	174	182	162	132	173	187	160	154	145	165	171	151	181

- Graph the points on a scatter plot.
- Which variable could be considered as the dependent variable? Give reasons.
- Describe the strength and direction of the relationship between the age and height of students.

Technology Scatter plot patterns

Investigate one of the following pairs of bivariate data for a group of students or people. You will need instruments (measuring tapes and/or trundle wheels) and stopwatches to help you collect your data.

- Height vs arm span
 - Reaction time vs hours of sleep
 - Stride length vs 50 m sprint time
- Enter your data into a spreadsheet. Graph it using **Scatter with Smooth Lines and Markers**.
 - Analyse your graph. What type of linear relationship does it show? Positive or negative? Strong or weak?
 - Write a brief summary describing the relationship between the two variables.

6-09 Line of best fit

If two variables x and y show a strong linear relationship when graphed on a scatter plot, the linear relationship can be approximated by drawing a **line of best fit** through the points and finding its equation $y = mx + b$. This line can be done on paper but it is easier to graph it using technology such as a spreadsheet, dynamic geometry or graphing software.

Summary

A **line of best fit**:

- represents most or all of the points as closely as possible
- goes through as many points as possible
- has roughly the same number of points above and below it
- is drawn so that the distances of points from the line are as small as possible

A line of best fit can be used to predict what might happen:

- between the points on the scatter plot, within the range of data (this is called **interpolation**, pronounced 'in-terp-o-lay-shun'), or
- beyond the points on the scatter plot, outside the range of data (this is called **extrapolation**, pronounced 'ex-trap-o-lay-shun').

Stage 5.3

Worksheet

Line of best fit

MAT10SPWK10210

Worksheet

Data 2

MAT10SPWK00033

Technology worksheet

Excel spreadsheet:
Line of best fit

MAT10NACT00033

Technology worksheet

Excel worksheet: Line
of best fit

MAT10NACT00003

Stage 5.3

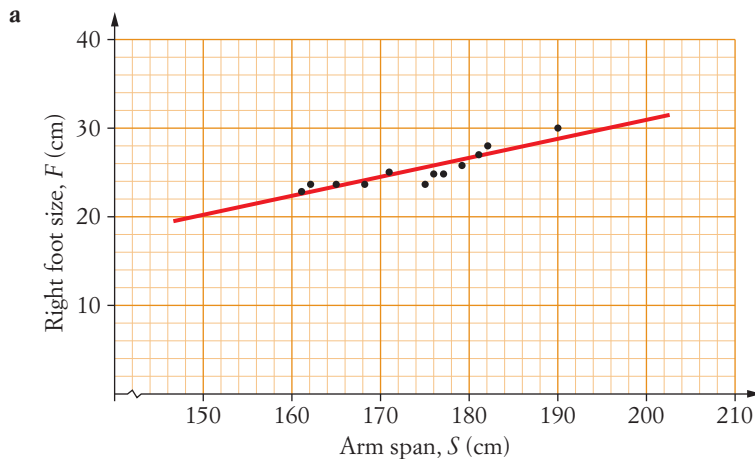
Example 15

The arm span and right foot size of 12 Year 10 students were measured.

Arm span, S (cm)	177	179	162	182	181	171	161	176	175	190	168	165
Right foot size, F (cm)	25	26	24	28	27	25	23	25	24	30	24	24

- Graph the points on a scatter plot and construct a line of best fit.
- Find the equation of the line of best fit.
- Use the equation to estimate the foot size of a student with an arm span of 173 cm.
- Use the graph to interpolate the foot size of a Year 10 student with an arm span of 185 cm.
- Use the graph to extrapolate the arm span of a Year 10 student who has a foot size of 31 cm.

Solution



- b Use the point–gradient formula $y - y_1 = m(x - x_1)$ to find the equation of the line.

$$\begin{aligned}
 m &= \frac{y_2 - y_1}{x_2 - x_1} \\
 &= \frac{27 - 20}{181 - 150} \\
 &= \frac{7}{31} \\
 &\approx 0.226
 \end{aligned}$$

Using two points on the line (150, 20) and (181, 27).

$$\begin{aligned}
 y - 20 &= 0.226(x - 150) \\
 &= 0.226x - 33.9
 \end{aligned}$$

Using the point (150, 20).

$$\begin{aligned}
 y &= 0.226x - 33.9 \\
 F &= 0.226S - 33.9
 \end{aligned}$$

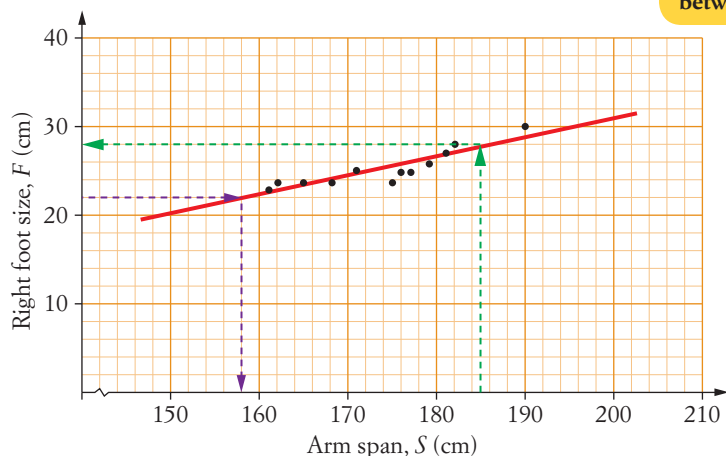
x and y replaced by S and F respectively.

- c When $S = 173$ cm,
 $F = 0.226 \times 173 - 33.9$
 $= 25.198$ cm.

A Year 10 student with an arm span of 173 cm would have a foot size of 25.198 cm.

- d From the graph, a Year 10 student with an arm span of 185 cm would have a foot size of 28 cm.

This is **interpolating** because we are reading from the graph **between** the given points.



- e From the graph, a Year 10 student with a foot size of 22 cm would have an arm span of 158 cm.

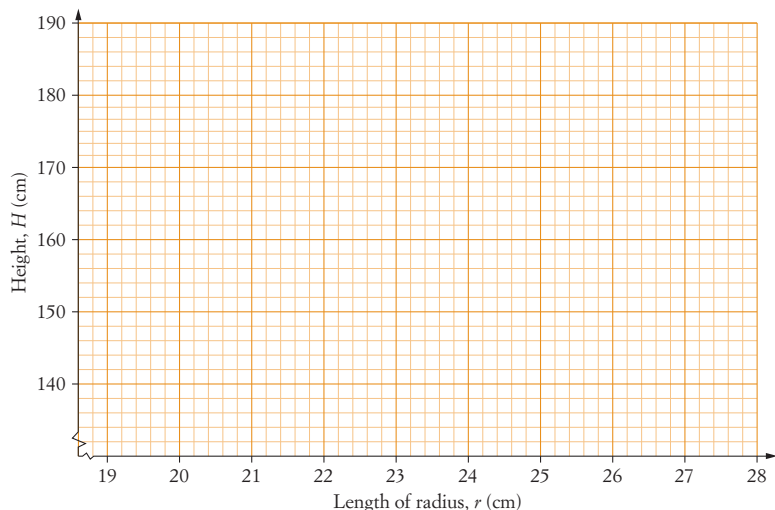
This is **extrapolating** because we are reading from the graph **outside** the given points.

Exercise 6-09 Line of best fit

- 1 Forensic scientists can estimate people's heights from the lengths of their bones such as the tibia, femur, humerus and radius. The table below gives the heights of females and the length of their radius.

See Example 15

Length of radius, r (cm)	25.2	22	23	22.5	21.8	26.2	20.4	23.5	24.3	21.4
Height, H (cm)	173	158	165	161	158	179	152	167	169	156



- Plot the points on a scatter plot as shown and construct a line of best fit.
- Find the equation of the line of best fit.
- Use your equation to find the height of a female whose radius is 25 cm long.
- If the radius is 27 cm in length, use the line of best fit to predict the height of the female.

Stage 5.3

- 2 The heights and shoe sizes of a group of Year 11s were measured and recorded below.

Height, H (cm)	175	174	177	180	179	176	170	175	179	180	178	183	178	173	179	174
Shoe size, S	10.5	10	10	12	11	9.5	7.5	9	11.5	12.5	11	12.5	12	9.5	10.5	9

- Graph the points on a scatter plot and construct a line of best fit.
- Find the equation of the line of best fit.
- Use the equation to estimate the shoe size (to the nearest 0.5) of a student whose height is 172 cm.
- Use the graph to interpolate the shoe size of a student who is 181 cm tall.
- Use the graph to extrapolate the shoe size of a student with height 185 cm.

- 3 The air temperature, T ($^{\circ}\text{C}$) was measured at various heights, h (m), above sea level.

Height, h (m)	0	500	1000	2000	2500	4000	5900	7500	10 000
Temperature, T ($^{\circ}\text{C}$)	20	14	8	3	-5	-13	-20	-35	-50

- Graph the points on a scatter plot and construct a line of best fit.
- Find the equation of the line of best fit.
- Use the equation to estimate the temperature at a height of 1500 m.
- Use the graph to find the height above sea level for a temperature of -10°C .

- 4 The results obtained by 18 Year 10 students in Maths and Science exams are shown below.

Maths	59	52	72	85	75	45	65	64	62	58	78	90	40	70	50	45	82	50
Science	65	54	67	83	75	39	59	64	60	56	80	95	38	65	48	48	85	51

- Graph the points on a scatter plot and construct a line of best fit.
- Simone missed the Science test but obtained 80 in her Maths exam. Use the line of best fit to predict Simone's Science result.
- If Mario obtained 96 in the Science exam, predict what result he might have achieved in the Maths exam.

- 5 Angela is measuring the amount by which a spring is stretched when different masses are hung from the spring for a Science experiment. Her results are as follows.

Mass, M (g)	10	20	25	30	35	40	50
Spring stretch, S (cm)	5.9	11.2	12.3	14.8	17	22.4	25.2

- Graph the points on a scatter plot and construct a line of best fit.
- Use the line of best fit to predict the length the spring stretches for a mass of 45 g.
- What mass would have to be attached to stretch the spring 28 cm?
- Are there limitations to using the line of best fit to predict the length of stretch in the spring by different masses?

- 6 The men's 100 m world record times for 1964 to 2009 are given in the table below.

Year	1964	1968	1983	1988	1991	1994	1996	1999	2005	2006	2007	2008	2009
Time (s)	10.06	9.95	9.93	9.92	9.86	9.85	9.84	9.79	9.77	9.76	9.74	9.69	9.58

- Graph the points and construct a line of best fit.
- Use the line of best fit to predict the record time taken to run the 100 m in 2020.
- What are the limitations of using the line of best fit to predict times to run 100 m?

Technology Lines of best fit

In this activity, we will use a spreadsheet to create a scatter plot and graph a line of best fit. The heights of men and the lengths of their femur bone are recorded in the table below.

Length of femur, $f(\text{cm})$	40	42.9	44.2	46.1	46.8	47	48.4	50.3	51.2	57.2
Height, H (cm)	162	165	164	173	174	178	179	182	186	200

- 1 Enter the data from the table into a spreadsheet. Type **Length of femur** in cell A1 and **Height** in B1.
- 2 To graph a scatter plot, select all the values in cells B1 to K2, and under the **Insert** menu, select **Scatter** and **Scatter with Straight Lines and Markers**.
- 3 To draw the line of best fit, select one of the points on the scatterplot and right-click. Select **Add Trendline**, **Linear** and **Display Equation on chart**, then **Close**.
- 4 Check your answers to questions 1–3 from Exercise 6-09 using a spreadsheet.

6-10 Bivariate data involving time

Bivariate data involving time, or **time series data**, is two-variable data where the independent variable is time. Examples of time series data are population changes over time, weekly share prices, daily rainfall and patients' heart rates.

Example 16

This table shows the average household size between 1961 and 2011, according to the Census.

Year	1961	1966	1971	1976	1981	1986	1991	1996	2001	2006	2011
Average household size	3.6	3.5	3.3	3.1	3.0	2.9	2.8	2.6	2.6	2.6	2.6

Source: Australian Government, Australian Institute of Family Studies

- a Graph the data on a scatter plot and join the points.
- b Use your graph to describe the change in average household size from 1961 to 2011.
- c Based on your time series graph, estimate the household size for 2021.

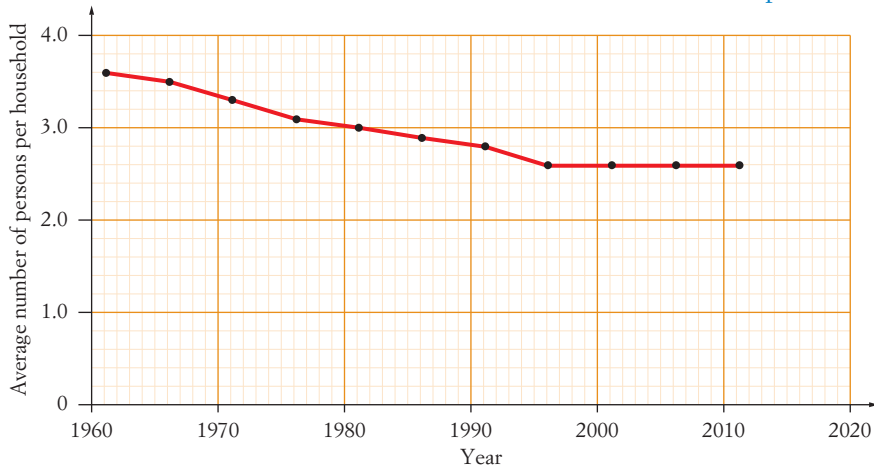


iStockphoto/Yuri

Solution

a

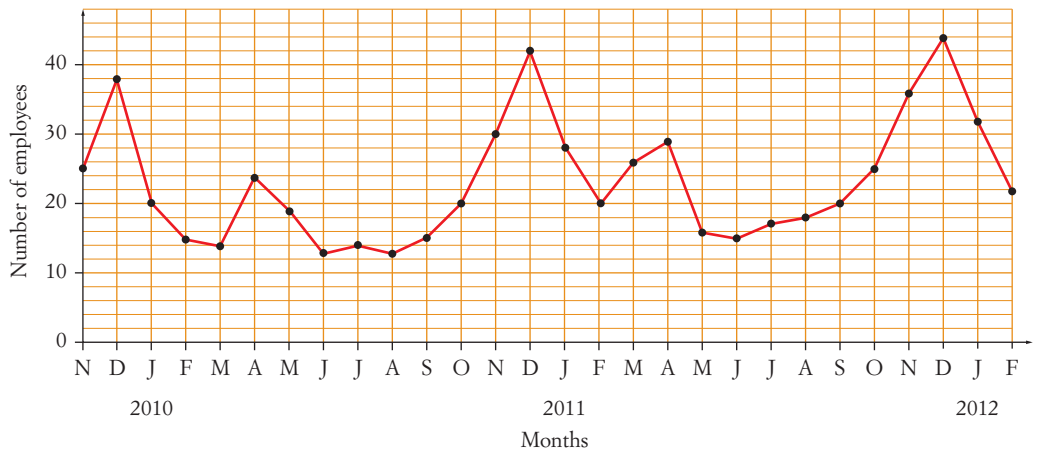
Year is the independent variable.



- b The average household size decreased from 3.6 in 1961 to 2.6 in 1996 and since then there has been little or no change.
- c $2.4 - 2.6$ people per household.

Exercise 6-10 Bivariate data involving time

- 1 The number of people employed per month at SUPA SAVE SUPERMARKET from November 2009 to February 2012 is displayed in the time series graph below.



- a How many people were employed by the supermarket in:
- November 2009?
 - December 2010?
 - June 2011?
- b In which month of the year were the most people employed by the supermarket? Suggest a reason why.
- c In which month of the year were the least number of people employed? Suggest a reason why.
- d Describe how the number of people employed by the supermarket changes from November 2009 to February 2012.

- 2 The population figures for Australia from 1960 to 2010 are given in the table below.

See Example 16

Year	1960	1965	1970	1975	1980	1985	1990	1995	2000	2005	2010
Population (millions)	10.28	11.39	12.51	13.89	14.70	15.76	17.07	18.07	19.15	20.39	22.3

- a Graph the data on a scatter plot and join the points.
 b Between which years was the greatest population increase?
 c Use your graph to describe the change in Australia's population from 1960 to 2010.
 d Based on your time series graph, estimate the population for Australia in:
 i 2020 ii 2045.

- 3 The table below shows the fatalities on NSW roads from 1950 to 2010.

Year	1950	1960	1970	1980	1990	2000	2010
Fatalities	634	978	1309	1303	797	603	405

- a Draw a time series graph for this data.
 b Describe the change in road fatalities from 1950 to 2010.
 c Give possible reasons for the reduction in road fatalities from a high of 1309 in 1970 to 405 in 2010.
- 4 The annual mean maximum temperatures for Sydney from 1990–2012 and from 2001–2012 are given in the tables below.

Year	1990	1991	1992	1993	1994	1995	1996	1997	1998	1999	2000	2001
Temperature (°C)	22.3	22.8	21.5	22.3	22.6	21.8	22.1	22.4	22.7	22.1	22.7	23.1

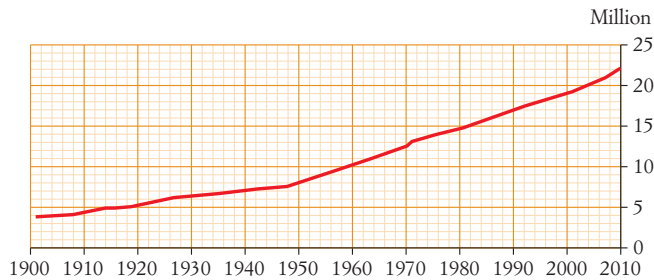
Year	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	2012
Temperature (°C)	23.1	23.1	22.7	23.4	23.4	23.1	22.7	22.1	22.1	22.6	22.6	22.7

Source: Bureau of Meteorology

- a Draw a time series graph for temperatures from:
 i 1990 to 2000 ii 2001 to 2012.
- b Has there been much change in Sydney's temperature from
 i 1990 to 2000? ii 2001 to 2012?
 Justify your answer.
- c Are there differences in temperature between the periods 1990–2000 and 2001–2012?
 Give reasons.
- 5 The table below shows the annual emissions of carbon (measured in Megatonnes, Mt) from 2002 to 2012.

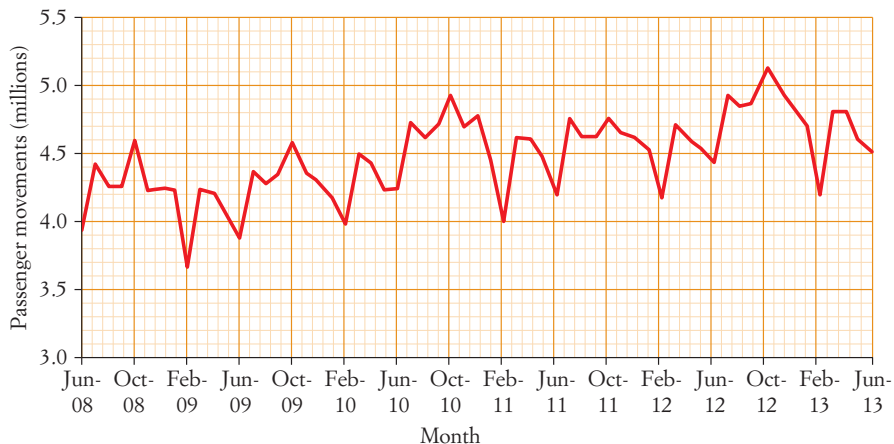
Year	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	2012
Annual emissions (Mt CO ₂ -e)	509.5	514.5	529.2	530.2	539.8	546.5	554	542.8	551.8	553.2	551.9

- a Draw a time series graph for this data.
 - b Describe the change in carbon emissions from 2002 to 2008.
 - c What happens to the carbon emissions after 2010?
 - d Give a possible reason for your answer to part c.
 - e What is your estimate of carbon emissions for:
 - i 2015?
 - ii 2025?
- 6 The graph below shows Australia's population from 1901 to 2010.



Source: Australian Historical Population Statistics (3105.0.65.001); Australian Demographic Statistics (3101.0).

- a What was Australia's population in 1901?
 - b By how much had Australia's population increased between 1901 and 2010?
 - c What was the average annual rate of increase in population between 2000 and 2010?
 - d If this trend continues, what is the expected population in 2025?
- 7 The time series graph below shows the monthly amount of passenger traffic on Australian domestic commercial airlines.



Source: Australian Government, Department of Infrastructure and Transport <http://www.bitre.gov.au/statistics/aviation/domestic.aspx#summary>

- a Describe the trend in domestic passenger traffic for June 2008 – June 2013.
- b What was the approximate amount of passenger traffic per month in:
 - i June 2008?
 - ii June 2010?
 - iii June 2011?
 - iv June 2013?
- c What was the percentage increase in domestic passenger movements from June 2008 to June 2013?

Investigation: Australian Bureau of Statistics

The Australian Bureau of Statistics (ABS) is the official organisation in charge of collecting data for government departments. The data collected covers many areas – from population, employment, weekly earnings, weight and obesity in adults, to health of children in Australia.

Visit the ABS website www.abs.gov.au to answer the following questions.

- 1 **a** What is the current population of Australia?
- b** What is the predicted population for:
 - i 2020? ii 2030? iii 2040?
- c** What is Australia's rate of population increase?
- 2 Go to 2011 Census Data by Location, and then to Data and analysis.
 - a** What was the population in NSW and its increase from 2006?
 - b** Which state had the:
 - i largest increase in population?
 - ii the smallest increase in population?

6-11 Statistics in the media

We live in a world of 24-hour news, whether it is from newspapers, TV or the Internet, which often quote results from surveys. When survey data is used in the media we need to consider:

- where the news comes from and what samples the statistics are based on
- who supplied the information
- the number of samples and what sample size was used
- the way in which the collected data has been presented



123rf/Oleksiy Mark

Example 17

What concerns could be raised about the following claim?

'The Daily Sun newspaper reports that it has an average issue readership of 1.385 million and that its Travel liftout has a readership of 1.455 million.'

Solution

The newspaper is reporting about its own readership and so may be biased. It also states that its Travel liftout has a higher readership than its issue readership.

Example 18

The weights (in kg) of a large group of 18–20-year-olds attending University are:

57	58	62	84	64	74	57	55	56	90
68	63	49	66	63	65	60	60	46	70
85	60	70	41	73	75	67	63	70	85
51	49	75	77	87	54	60	75	58	68
55	65	66	57	85	75	56	60	62	75
74	58	51	62	50	55	71	57	58	100
72	58	103	64	52	55	80	96	45	87
81	80	48	54	65	54	59	50	78	60
74	70	64	59	72	78	104	63	102	95

- How many students were in the group?
- Randomly select four groups of 10 and for each sample calculate:
 - the mean
 - the median
 - the interquartile range.
- Use your results to estimate the mean, median and interquartile range of the population from your four samples.
- Compare your estimates to the mean, median and interquartile range of the population.

Solution

- There were 90 students in the group.
- Randomly select four samples of 10 from the 'population'.

Sample 1:	90	63	75	48	74	85	51	96	60	78
Sample 2:	62	75	103	64	65	54	55	54	60	75
Sample 3:	68	70	57	52	78	74	60	63	58	87
Sample 4:	72	54	52	80	45	87	49	77	54	58

The statistics for each group are:

Sample 1: $\bar{x} = 72$ median = 74.5 interquartile range = 25

Sample 2: $\bar{x} = 66.7$ median = 63 interquartile range = 20

Sample 3: $\bar{x} = 66.7$ median = 65.5 interquartile range = 16

Sample 4: $\bar{x} = 62.8$ median = 56 interquartile range = 25

- Taking averages, population statistics estimates are:

$$\text{Mean, } \bar{x} = \frac{72 + 66.7 + 66.7 + 62.8}{4} = 67.1 \text{ (correct to 1 decimal place)}$$

$$\text{Median} = \frac{74.5 + 63 + 65.5 + 56}{4} = 64.8 \text{ (correct to 1 decimal place)}$$

$$\text{Interquartile range} = \frac{25 + 20 + 16 + 25}{4} = 21.5$$

- The statistics for the population are:
Mean, $\bar{x} = 66.9$ (correct to 1 decimal place)

Median = 64

Interquartile range = 18

The estimates for the mean, median and interquartile range compare very favourably with the population statistics.

Exercise 6-11 Statistics in the media

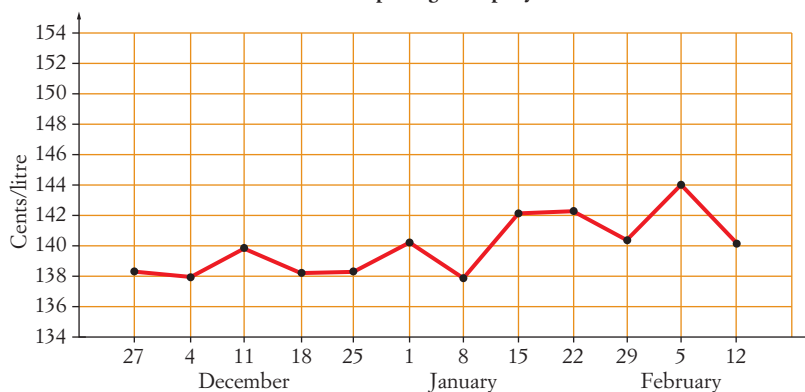
- 1 A TV network surveys 300 people in shopping centres between 9 a.m. and 11 a.m. to get feedback on its new game show.
 - a How may this survey be biased?
 - b Suggest a better method for obtaining feedback about its game show.
- 2 A report about hot-water systems recommended a heat pump system. The report stated that people in Queensland who had the heat pump hot-water system saved 30% of their electricity bill per quarter. The company is using this information in their advertising of the product in NSW and Victoria.

Should people in NSW and Victoria install this type of hot-water system? Give reasons.
- 3 A report on petrol pricing was conducted by two companies. The following graphs, showing the price of petrol for the same 12-week period, were used to present their findings on the price of petrol.

See Example 17

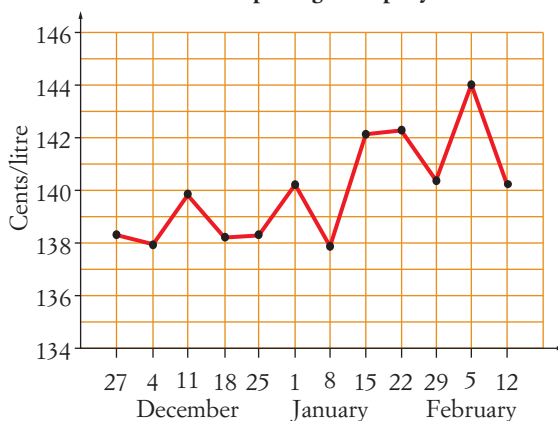
Company A

Petrol pricing: Company A



Company B

Petrol pricing: Company B

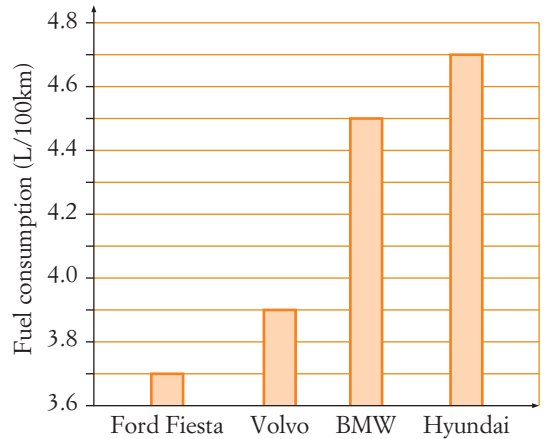


- a What is being implied about petrol prices by:
 - i Company A?
 - ii Company B?
- b How could both graphs be improved to give a true picture of changing petrol prices?

- 4 A report on the diesel fuel consumption of different cars was published in a motoring magazine.

- a What is the magazine report implying about the fuel consumption of the different cars?
- b What is the difference in fuel consumption between the:
 - i Ford Fiesta and the Volvo?
 - ii Ford Fiesta and the Hyundai?
 - iii BMW and the Hyundai?

- c How should the graph be redrawn so that it is not biased towards the Ford Fiesta and the Volvo?



- 5 A company manufactures a product. After 3 months, they conduct a survey and customers are asked to rate the product as Excellent, Good or Satisfactory. Is the survey biased? Justify your answer.
- 6 A market research company working for a car manufacturer needs to determine the most popular car colours.
 - a Give an example of a biased question for this survey.
 - b What other information should the market research company use, apart from the survey, to determine the most popular colour car?

See Example 18

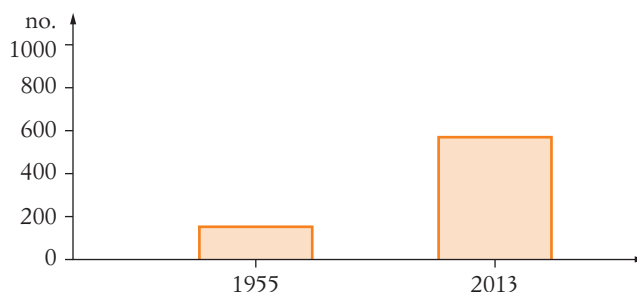
- 7 a Randomly select four samples of 10 weights from the population shown in Example 17, and for each sample calculate:
 - i the mean
 - ii the median
 - iii the interquartile range.
 - b Use your results to estimate the mean, median and interquartile range of the population from your four samples.
 - c How do the statistics of your samples compare to the mean, median and interquartile range of the population?
 - d How do the estimated statistics compare to the population statistics?
- 8 a Repeat the process of question 7 by taking two samples of size:
 - i 5
 - ii 15
 - iii 20
 - b Do the sample statistics become more accurate and move closer to the population statistics as sample size increases?

6-12 Investigating statistical studies

Stage 5.3

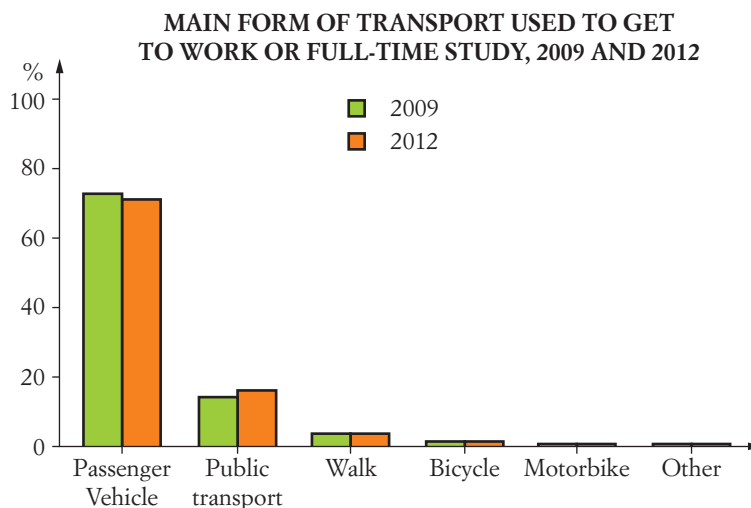
Exercise 6-12 Investigating statistical studies

- 1 The graph compares the number of passenger vehicles per 1000 people in Australia in 1955 and 2013.



Source: www.abs.gov.au

- How many passenger vehicles per 1000 people were there in 1955?
 - What was the percentage increase in the rate between 1955 and 2013?
- 2 Visit the Australian Bureau of Statistics (ABS) website www.abs.gov.au and search for Motor vehicle census.
- What was the total number of vehicles registered last year?
 - How many passenger vehicles were registered last year?
 - What was the average annual growth rate over the last five years?
- 3 This graph compares the types of commuter transport used by Australians in 2009 and 2012.

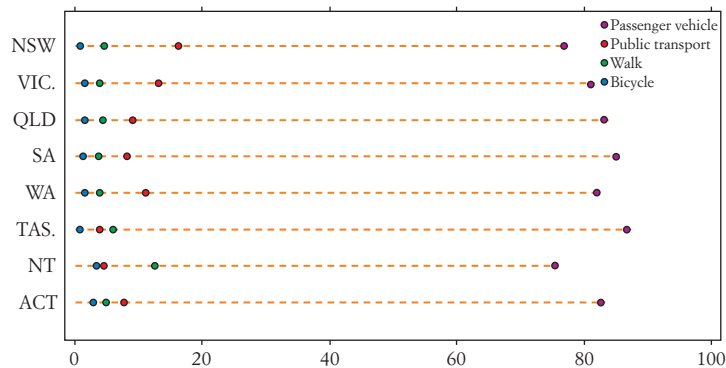


Source: www.abs.gov.au

Stage 5.3

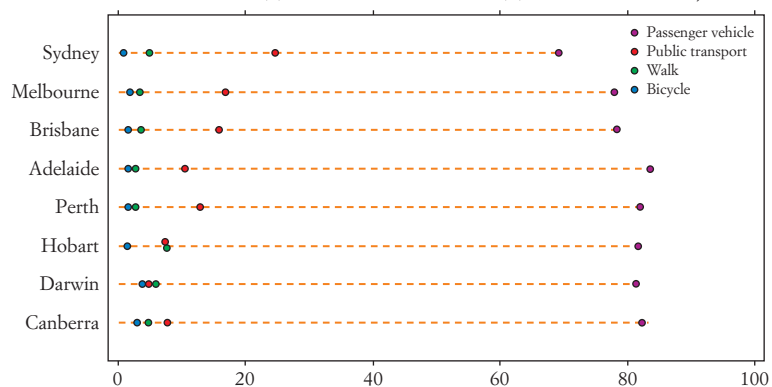
- a What percentage of people used a passenger vehicle to get to work or full-time study in 2012 and what change has occurred from 2009 to 2012?
 - b What percentage of Australians used public transport in 2012? Visit the ABS website to see whether this has changed this year.
 - c What reasons are there for Australians not using public transport? (Search Car nation at the ABS website)
 - d List three advantages of using public transport.
 - e Use the Internet to compare Australia's transport use with transport use in other countries (for example, China, Indonesia, Japan and USA).
- 4 These graphs compare the types of commuter transport used by state, territory and capital city in 2011.

ALL METHODS(a) OF TRAVEL TO WORK(b) BY STATE AND TERRITORIES, 2011



- a Which state had the highest proportion of people using passenger vehicles to travel to work?
- b What percentage of people in South Australia used public transport?
- c Which capital city had the highest public transport use and which city had the lowest? Give possible reasons for your answer.

ALL METHODS(a) OF TRAVEL TO WORK(b) BY CAPITAL CITY, 2011



- d The percentage of people using public transport in capital cities is higher than the percentage of people using public transport in the State. Give a possible reason for this.
 - e Which state and which capital city had the lowest percentage of people using passenger vehicles?
- 5 Summarise your answers to questions 1 to 4 in a brief report about passenger vehicle use in Australia. Using your results, indicate what action governments (Federal, State and local) should take in terms of building roads, accident research and consideration of the environment.

- 6 Is Australia becoming a warmer continent? Investigate this by looking at data from the Australian Bureau of Statistics and the Australian Bureau of Meteorology (www.bom.gov.au).
- 7 Investigate tobacco and alcohol use by teenagers in Australia. Include tables and graphs in your report. Refer to the National Drug Strategy Household Survey (www.nationaldrugstrategy.gov.au) and NSW Health (www.healthinsite.gov.au), and search 'alcohol and teenage statistics in Australia' on the Internet.

Power plus

- 1 The strength and direction of the relationship between two variables can be measured by the correlation coefficient (r).
 - a Between which two values does the correlation coefficient lie?
 - b What is the strength and direction of the relationship if the correlation coefficient is zero?
 - c Write a possible value for the correlation coefficient for each relationship described.
 - i perfect positive
 - ii weak negative
 - iii strong negative
- 2 Two variables may have a strong relationship, but this does not mean that a change in one variable causes a change in the other. Which of the following pairs of variables have a causal relationship?
 - a height and weight of people
 - b the time that it takes to walk to school and the distance from home to school
 - c the number of children per household and the number of mobile phones per household
 - d the age of people and their reaction time
 - e the price of petrol and the amount of petrol sold
 - f the interest rate of loans and the number of new housing loans
- 3 The following scores are the test results on a History exam for a class of 20 students.

13	14	16	12	14	16	18	13	15	10
9	15	13	14	13	10	8	14	16	14

 - a Find the mean, median and mode.
 - b Find the range and interquartile range.
 - c An error was made in recording the scores and 4 marks need to be added to each score. What effect will this have on the statistics calculated in parts a and b?

Chapter 6 review

Language of maths

Puzzle sheet

Data crossword

MAT10SPPS10039

Quiz

Statistics

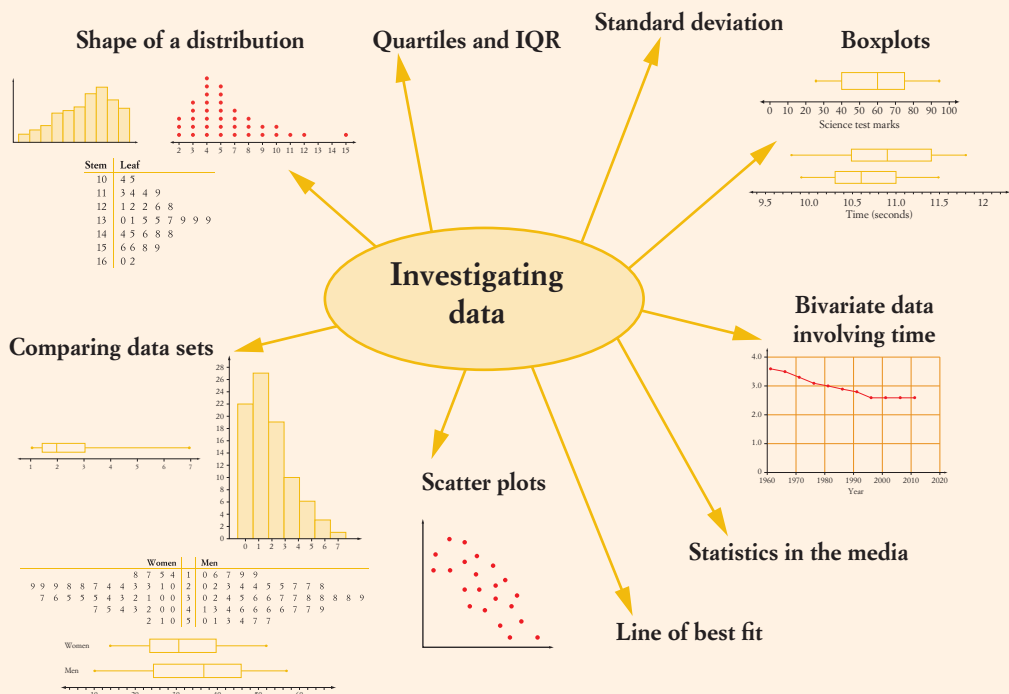
MAT10SPQZ00002

bivariate data	boxplot	cluster	dependent variable
five-number summary	independent variable	interquartile range	line of best fit
mean	measure of location	measure of spread	median
mode	negatively skewed	outlier	positively skewed
quartile	range	scatter plot	skewed
standard deviation	strong	symmetrical	weak

- 1 What is represented by the 'whiskers' on a **box-and-whisker plot**?
- 2 What are the **measures of location** and the **measures of spread**?
- 3 What are the five things found in a **five-number summary**?
- 4 Describe a statistical distribution that is **positively skewed**.
- 5 What type of graph is used to represent **bivariate data**?
- 6 Give two examples of how statistics can be misleading.

Topic overview

Copy and complete this mind map of the topic, adding detail to its branches and using pictures, symbols and colour where needed. Ask your teacher to check your work.



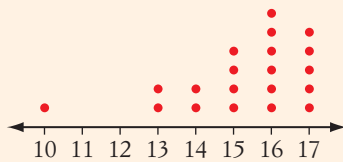
1 For each statistical distribution:

i describe the shape

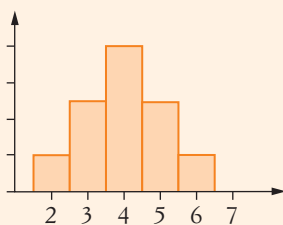
ii identify any outliers and clusters.

See Exercise 6-01

a



c



b

Stem	Leaf
3	0 1 2
4	1 3 4 4 5 6
5	0 4 5 7 8
6	3 7 8
7	0 1
8	4
9	8

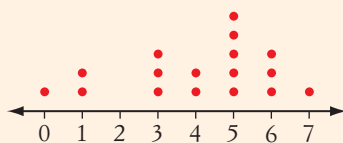
2 Find the interquartile range of each set of data.

a 5 8 8 10 12 13 14 15 18

b 24 15 23 28 20 20 18 30 21 18

See Exercise 6-02

c



d

Stem	Leaf
3	0 1 2
4	3 5 8 8 9 9 9
5	4 5 6 6 8
6	0 1 3 7
7	2

e

Score	Frequency
10	3
11	8
12	15
13	18
14	10
15	5

3 The reaction times (in seconds) of a sample of truck drivers were measured.

0.34 0.35 0.34 0.37 0.42 0.45 0.43 0.29 0.38 0.40 0.37 0.62

a Find, correct to two decimal places, the mean and standard deviation.

b Find the range and interquartile range.

c Which is the best measure of spread for this set of data? Justify your answer.

4 The Health exam results for a class of PE students are shown here.

Girls: 83 78 63 84 65 51 76 69 42 84 60 64 92 73 32

Boys: 65 34 75 68 56 63 79 55 68 52 49 85 64 58 54

a Find the mean and standard deviation of both groups.

b Which group performed better in the exam? Give reasons.

Stage 5.3

See Exercise 6-03

See Exercise 6-04

Chapter 6 revision

See Exercise 6-05

- 5 The number of goals scored by the Under-18s Vale soccer team are:

2 0 0 4 2 1 1 2 3
2 3 7 4 3 1 0 4 2

- Find the range and interquartile range of the scores.
- Find the five-number summary for the data.
- Draw a box-and-whisker plot for the data.

See Exercise 6-06

- 6 The pulse rates of students were taken before and after exercising. The results were:

Before exercise: 78 80 66 70 56 64 68 65 50 76 80 70 70 59

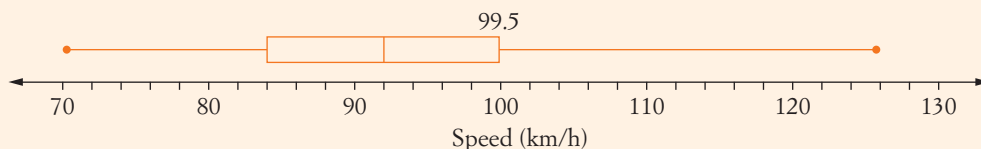
After exercise: 141 140 89 95 110 126 84 82 90 88 146 98 96 92

- Find the five-number summary for the pulse rates before and after exercise and construct a parallel boxplot to display the two sets of data.
- Find the range and interquartile for:
 - before exercising
 - after exercising.
- Compare the two sets of pulse rates. Are there significant differences between them? Justify your answer.

See Exercise 6-07

- 7 The speeds of cars (in km/h) were monitored between 1:00 and 1:30 p.m. on a main road. The results are shown in the stem- and-leaf plot and boxplot below.

Stem	Leaf
7	0 3 7 9
8	0 2 2 3 5 6 8 8 9
9	0 0 1 3 5 5 7 8 9 9 9
10	0 0 4 4 6
11	0 1
12	6



- Which display best indicates:
 - skewness?
 - clustering and outliers?
- Why would a dot plot be unsuitable for displaying the data?
- Find:
 - the median
 - the interquartile range.
- What percentage of cars were travelling at a speed of at least 92 km/h?

- 8 Eleven boxes containing 60 oranges each were placed in cold storage for different periods. After storage, the number of good oranges in each box was counted. See Exercise 6-08

Weeks in storage (W)	2	5	12	8	14	6	5	9	10	3	11
Number of good oranges (N)	58	50	33	40	28	50	52	38	35	55	33

- Which is the independent variable? Give reasons.
 - Graph this data on a scatter plot.
 - Describe the pattern of the plotted points.
 - Describe the relationship between the number of weeks in storage and the number of good oranges.
 - Describe the strength and direction of the relationship between the variables W and N .
- 9 The heights and weights of 15 people were measured.

Height, H (cm)	152	160	179	180	185	174	165	150	145	142	155	153	175	155
Weight, W (kg)	50	65	72	77	81	77	65	57	48	53	61	67	72	56

- Graph the points on a scatter plot and construct a line of best fit.
 - Find the equation of the line of best fit.
 - Use the equation to estimate the weight of a student who is 170 cm tall.
 - Use the graph to interpolate the weight of a student with height 163 cm.
 - Use the graph to extrapolate the height of a student who weighs:
 - 85 kg
 - 45 kg
- 10 The mean maximum temperatures in Blacktown, NSW for the month of January from 2004 to 2013 are shown in the table. (Source: Bureau of Meteorology) See Exercise 6-10

Year (t)	2004	2005	2006	2007	2008
Temperature (T, °C)	30.6	29.1	29.0	30.1	28.5

Year (t)	2009	2010	2011	2012	2013
Temperature (T, °C)	31.7	30.6	29.9	27.4	30.0

- What is the independent variable?
 - Graph the data on a scatter plot and join the points.
 - Has any change occurred to the temperatures in Blacktown for the month of January over the 10-year period? Give reasons for your answer.
- 11 An advertisement in a magazine states that a product is 75% fat-free. See Exercise 6-11
- What impression is the advertisement trying to make about the product?
 - What doesn't the advertisement say about the product?

Stage 5.3

See Exercise 6-09



7

Number and algebra

Equations and logarithms

Historically, algebra dates back to ancient Egypt and Babylon where linear and quadratic equations were solved. In ancient Babylon, quadratic equations were solved by very similar methods to those still relevant and taught today. Logarithms were developed in the seventeenth century and are still in use, most recognised in the pH, decibel and Richter scales.



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Chapter outline

	Proficiency strands				
7-01 Equations with algebraic fractions	U	F		R	
7-02 Quadratic equations $x^2 + bx + c = 0$	U	F		R	C
7-03 Simple cubic equations $ax^3 = c^*$	U	F		R	C
7-04 Equation problems	U	F	PS	R	C
7-05 Equations and formulas	U	F	PS	R	C
7-06 Changing the subject of a formula*	U	F		R	C
7-07 Graphing inequalities on a number line	U	F			C
7-08 Solving inequalities	U	F		R	
7-09 Logarithms*	U	F		R	C
7-10 Logarithm laws*	U	F		R	C
7-11 Exponential and logarithmic equations*	U	F		R	C

*STAGE 5.3

Wordbank

cubic equation An equation involving a variable cubed (power of 3), such as $4x^3 = 500$

exponential equation An equation where the variable is a power, such as $3^x = 243$

logarithm The power of a number, to a given base. For example, $\log_{10} 1000 = 3$, meaning that the logarithm of 1000 to base 10 is 3, because $1000 = 10^3$

inequality A mathematical statement that two quantities are not equal, involving algebraic expressions and an inequality sign ($>$, $\geq$, $<$, or $\leq$)

quadratic equation An equation involving a variable squared (power of 2), such as $3x^2 - 6 = 69$

solution The answer to an equation, inequality or problem, the correct value(s) of the variable that makes an equation or inequality true

In this chapter you will:

- solve linear equations involving simple algebraic fractions
- solve simple quadratic equations using a range of strategies
- substitute values into formulas to determine an unknown
- solve linear inequalities and graph their solutions on a number line
- (STAGE 5.3) use the definition of a logarithm to establish and apply the laws of logarithms
- (STAGE 5.3) solve simple exponential equations 
- solve linear equations and problems involving equations
- (STAGE 5.3) solve linear equations involving complex algebraic fractions
- (STAGE 5.3) solve simple cubic equations of the form $ax^3 = c$
- (STAGE 5.3) change the subject of a formula
- (STAGE 5.3) solve simple logarithmic equations

SkillCheck

Worksheet

StartUp assignment 6

MAT10NAWK10040

1 Solve each equation.

a $4a + 5 = 2a - 19$

b $\frac{3x+2}{5} = 4$

c $4(2 - x) = -24$

2 Factorise each expression.

a $k^2 + 5k + 4$

b $y^2 - 10y + 16$

c $m^2 - m - 56$

d $u^2 + 8u - 65$

e $w^2 - 10w + 21$

f $x^2 - 2x - 24$

Video tutorial

Equations with algebraic fractions

MAT10NAVT10026

Puzzle sheet

Equations code puzzle

MAT10NAPS10041

Puzzle sheet

Equations order activity

MAT10NAPS10042

Puzzle sheet

Solving linear equations 1

MAT10NAPS00035

Puzzle sheet

Solving linear equations 2

MAT10NAPS00036

7-01 Equations with algebraic equations

Example 1

Solve each equation.

a $\frac{2m}{3} - \frac{m}{2} = 2$

b $\frac{2a+4}{5} = \frac{2}{3}$

Solution

a $\frac{2m}{3} - \frac{m}{2} = 2$

Multiply both sides by a common multiple of the denominators to remove the fractions. The lowest common multiple (LCM) of 3 and 2 is 6, so multiply both sides by 6.

$$6\left(\frac{2m}{3} - \frac{m}{2}\right) = 6 \times 2$$

$$\cancel{6}^2 \times \frac{2m}{\cancel{3}_1} - \cancel{6}^3 \times \frac{m}{\cancel{2}_1} = 12$$

$$4m - 3m = 12$$

$$m = 12$$

Check by substituting that this solution is correct.

b $\frac{2a+4}{5} = \frac{2}{3}$

Multiply both sides by 15, the LCM of 5 and 3.

$$\frac{2a+4}{\cancel{5}_1} \times \cancel{15}^3 = \frac{2}{\cancel{3}_1} \times \cancel{15}^5$$

$$3(2a+4) = 10$$

$$6a+12 = 10$$

$$6a = -2$$

$$a = \frac{-2}{6}$$

$$= -\frac{1}{3}$$

Example 2

Solve $\frac{2n+1}{3} - \frac{3n-2}{2} = -5$

Solution

$$\frac{2n+1}{3} - \frac{3n-2}{2} = -5$$

$$\cancel{6}^2 \left(\frac{2n+1}{\cancel{3}_1} \right) - \cancel{6}^3 \left(\frac{3n-2}{\cancel{2}_1} \right) = 6 \times (-5)$$

The LCM of 3 and 2 is 6.

$$2(2n+1) - 3(3n-2) = -30$$

$$4n+2-9n+6 = -30$$

$$-5n+8 = -30$$

$$-5n = -38$$

$$n = \frac{-38}{-5}$$

$$= 7\frac{3}{5}$$

Stage 5.3

Video tutorial

Equations with
algebraic fractions

MAT10NAVT10026

Exercise 7-01 Equations with algebraic fractions

1 Solve each equation.

a $\frac{3y}{5} = 9$

b $\frac{2a}{9} = 2$

c $\frac{m+5}{2} = 6$

d $\frac{k-2}{5} = 11$

e $\frac{n+5}{3} = -10$

f $\frac{y-1}{4} = -2$

g $\frac{x+1}{4} + 2 = 10$

h $\frac{y-1}{5} - 6 = 3$

i $\frac{m+2}{5} - 1 = 3$

j $\frac{x-6}{5} + 7 = 0$

k $\frac{2(x+1)}{5} = 10$

l $\frac{3(m-2)}{4} = 6$

m $\frac{8(n+1)}{3} + 2 = 4$

n $\frac{5(1-n)}{2} - 1 = 3$

o $\frac{4(1+d)}{3} + 1 = 7\frac{1}{3}$

See Example 1

2 Solve each equation.

a $\frac{2k}{3} = \frac{5}{4}$

b $\frac{3w}{10} = \frac{2}{5}$

c $\frac{5x}{2} = -\frac{10}{3}$

d $\frac{x-1}{2} = \frac{x+1}{4}$

e $\frac{y+2}{5} = \frac{y-1}{2}$

f $\frac{a+5}{3} = \frac{a-1}{8}$

g $\frac{p+2}{5} = \frac{p-5}{2}$

h $\frac{2y-1}{5} = \frac{y+1}{4}$

i $\frac{3y+2}{3} = \frac{2y+1}{4}$

j $\frac{w}{5} + \frac{w}{2} = 7$

k $\frac{w}{2} - \frac{w}{5} = 15$

l $\frac{2w}{3} - \frac{w}{4} = 4$

m $\frac{3a}{2} + \frac{a}{3} = 1$

n $\frac{2y}{5} - \frac{y}{3} = 4$

o $\frac{a}{3} + \frac{3a}{4} = 2$

3 Solve each equation. Select the correct answer A, B, C or D.

a $\frac{4m}{5} - \frac{m}{3} = 2$

A $m = 10$

B $m = 12$

C $m = \frac{30}{7}$

D $m = \frac{4}{3}$

b $\frac{m+1}{2} = \frac{3+2m}{5}$

A $m = 1$

B $m = 5$

C $m = \frac{5}{3}$

D $m = \frac{2}{3}$

4 Solve each equation.

a $\frac{x-1}{4} + \frac{2x}{7} = 0$

b $\frac{p+2}{3} + \frac{p+1}{4} = 10$

c $\frac{m+2}{3} + \frac{m+1}{4} = 12$

d $\frac{x-3}{5} + \frac{x-2}{2} = 6$

e $\frac{3x-10}{3} + \frac{x-2}{2} = 11$

f $\frac{3y+1}{4} - \frac{y+2}{3} = 4$

g $\frac{7+2a}{5} - \frac{a-1}{2} = 6$

h $\frac{6a-1}{4} - \frac{a+2}{3} = 8$

i $\frac{w+3}{6} - \frac{w-1}{5} = \frac{1}{3}$

j $\frac{a-10}{5} - \frac{5-2a}{4} = \frac{1}{2}$

Stage 5.3

See Example 2

7-02 Quadratic equations $x^2 + bx + c = 0$ 

iStockphoto/Lagui

An equation in which the highest power of the variable is 2 is called a **quadratic equation**; for example, $x^2 = 5$, $3m^2 + 7 = 10$, $d^2 - d - 6 = 0$ and $4y^2 - 3y = 8$.

Solving $ax^2 = c$

Worksheet

Equations review

MAT10NAWK10043

Summary

The quadratic equation $x^2 = c$ (where c is a positive number) has two solutions,

$$x = \pm\sqrt{c} \text{ (which means } x = \sqrt{c} \text{ and } x = -\sqrt{c}\text{)}$$

Example 3

Solve each quadratic equation.

a $m^2 = 16$

b $3x^2 = 75$

c $3m^2 - 12 = 0$

Solution

a $m^2 = 16$

$$\begin{aligned} m &= \pm\sqrt{16} \\ &= \pm 4 \end{aligned}$$

Finding the square root of both sides.

b $3x^2 = 75$

$$x^2 = \frac{75}{3}$$

$$x^2 = 25$$

$$\begin{aligned} x &= \pm\sqrt{25} \\ &= \pm 5 \end{aligned}$$

c $3m^2 - 12 = 0$

$$3m^2 - 12 + 12 = 0 + 12$$

$$3m^2 = 12$$

$$m^2 = \frac{12}{3}$$

$$m^2 = 4$$

$$\begin{aligned} m &= \pm\sqrt{4} \\ &= \pm 2 \end{aligned}$$

Video tutorial

Simple quadratic equations

MAT10NAVT10028

Example 4

Solve $\frac{5x^2}{9} = 25$, writing the solution in exact (surd) form.

Solution

$$\frac{5x^2}{9} = 25$$

$$5x^2 = 25 \times 9$$

$$= 225$$

$$x^2 = \frac{225}{5}$$

$$= 45$$

$$x = \pm\sqrt{45}$$

$$= \pm\sqrt{9}\sqrt{5}$$

$$= \pm 3\sqrt{5}$$

As a surd

In simplest surd form

Example 5

Solve $7x^2 - 88 = 0$, writing the solution correct to one decimal place.

Solution

$$7x^2 - 88 = 0$$

$$7x^2 = 88$$

$$x^2 = \frac{88}{7}$$

$$x = \pm \sqrt{\frac{88}{7}}$$

$$x = \pm 3.54562 \dots$$

$$\approx \pm 3.5$$

Solving $x^2 + bx + c = 0$ by factorising

To solve quadratic equations of the form $x^2 + bx + c = 0$, we need to factorise the quadratic expression on the LHS, which we learnt in Chapter 5, Products and Factors.

Example 6

Solve $x^2 + 5x + 6 = 0$.

Solution

$$x^2 + 5x + 6 = 0$$

$$(x + 2)(x + 3) = 0$$

The LHS has been factorised into two factors, $(x + 2)$ and $(x + 3)$, whose product is 0.

If two numbers have a product of 0, then one of the numbers *must* be 0.

$$\begin{array}{ll} \therefore x + 2 = 0 & \text{or} \quad x + 3 = 0 \\ \therefore x = -2 & \text{or} \quad x = -3 \end{array}$$

$\therefore$ The solution to $x^2 + 5x + 6 = 0$ is $x = -2$ or $x = -3$.

Check:

When $x = -2$,

$$\text{LHS} = (-2)^2 + 5 \times (-2) + 6 = 0$$

$$\text{RHS} = 0$$

Therefore LHS = RHS.

When $x = -3$,

$$\text{LHS} = (-3)^2 + 5 \times (-3) + 6 = 0$$

$$\text{RHS} = 0$$

Therefore LHS = RHS.

Video tutorial

Quadratic equations
by factorising

MAT10NAVT10029

Summary

When **solving quadratic equations by factorising**, the following property is used.
If $pq = 0$, then $p = 0$ or $q = 0$.

Example 7

Solve each quadratic equation.

a $x^2 - x - 2 = 0$

b $u^2 + 3u - 28 = 0$

c $a^2 - 2a = 0$

d $p^2 = 5p + 24$

Solution

a $x^2 - x - 2 = 0$

$$(x - 2)(x + 1) = 0$$

$$\therefore x - 2 = 0 \quad \text{or} \quad x + 1 = 0$$

$$\therefore x = 2 \quad \text{or} \quad x = -1$$

$\therefore$ The solution to $x^2 - x - 2 = 0$ is $x = 2$ or $x = -1$.

b $u^2 + 3u - 28 = 0$

$$(u + 7)(u - 4) = 0$$

$$\therefore u + 7 = 0 \quad \text{or} \quad u - 4 = 0$$

$$\therefore u = -7 \quad \text{or} \quad u = 4$$

$\therefore$ The solution to $u^2 + 3u - 28 = 0$ is $u = -7$ or $u = 4$.

c $a^2 - 2a = 0$

This requires a simpler factorisation as there are only two terms, both involving a .

$$a(a - 2) = 0$$

$$\therefore a = 0 \quad \text{or} \quad a - 2 = 0$$

$$\therefore a = 0 \quad \text{or} \quad a = 2$$

$\therefore$ The solution to $a^2 - 2a = 0$ is $a = 0$ or $a = 2$.

d $p^2 = 5p + 24$

$$p^2 - 5p - 24 = 0$$

$$(p - 8)(p + 3) = 0$$

$$\therefore p - 8 = 0 \quad \text{or} \quad p + 3 = 0$$

$$\therefore p = 8 \quad \text{or} \quad p = -3$$

$\therefore$ The solution to $p^2 = 5p + 24$ is $p = 8$ or $p = -3$.

Moving all terms to the LHS
and making the RHS = 0

Video tutorial

Simple quadratic
equations

MAT10NAVT10028

Note: Quadratic equations of the form $ax^2 + bx + c = 0$ will be met in Chapter 11, Quadratic equations and the parabola.

Exercise 7-02 Quadratic equations $x^2 + bx + c = 0$

See Example 3

1 Solve each quadratic equation.

a $m^2 = 144$

b $x^2 = 400$

c $y^2 = 225$

d $k^2 - 169 = 0$

e $y^2 - 1 = 0$

f $w^2 - 16 = 0$

g $x^2 + 10 = 14$

h $t^2 - 9 = 7$

i $\frac{a^2}{2} = 8$

j $5k^2 = 180$

k $3w^2 = 300$

l $d^2 + 60 = 204$

m $\frac{k^2}{2} = 0.5$

n $\frac{w^2}{10} = 2.5$

o $5y^2 = 5$

p $\frac{m^2}{4} = 9$

q $4x^2 = 1$

r $2p^2 + 3 = 21$

s $\frac{3k^2}{4} + 5 = 8$

t $\frac{y^2}{5} - 2 = 18$

See Example 4

2 Solve each quadratic equation, writing the solution in exact (surd) form where necessary.

a $5m^2 - 20 = 0$

b $\frac{4a^2}{9} = 36$

c $m^2 = 28$

d $9k^2 + 10 = 13$

e $\frac{k^2}{16} = 6$

f $\frac{3k^2}{10} = 27$

See Example 5

3 Solve each quadratic equation, writing the solution correct to two decimal places where necessary.

a $9m^2 - 2 = 32$

b $\frac{2x^2}{5} = 23$

c $6y^2 = 0.726$

d $\frac{2w^2}{5} = 20$

e $3a^2 + 11 = 267$

f $2y^2 - 14 = 63$

See Example 6

4 Solve each quadratic equation.

a $x^2 + 3x + 2 = 0$

b $y^2 + 5y + 4 = 0$

c $y^2 + 16y + 48 = 0$

d $x^2 + x - 12 = 0$

e $x^2 + 2x - 3 = 0$

f $x^2 + 3x - 40 = 0$

See Example 7

5 Solve each quadratic equation.

a $x^2 - x - 30 = 0$

b $x^2 - 8x + 16 = 0$

c $x^2 - 5x - 66 = 0$

d $d^2 - 2d = 0$

e $x^2 - 3x - 10 = 0$

f $n^2 + 4n = 0$

g $k^2 - 7k = 0$

h $y^2 = 5y$

i $v^2 = 12v$

j $m^2 = 3m$

k $a^2 + 24a = -80$

l $n^2 = -10n$

m $u^2 + 2u = 8$

n $x^2 = x + 42$

o $p(p + 9) = -20$

6 Explain why the quadratic equation $x^2 = -25$ has no solutions.

7 State which of these quadratic equations have no solutions. Give reasons.

a $x^2 = -9$

b $2k^2 + 5 = 9$

c $3m^2 + 8 = 4$

d $\frac{9w^2}{2} - 1 = 1$

e $4 + \frac{d^2}{3} = 8$

f $\frac{5a^2}{2} + 3 = 2$

Stage 5.3

NSW

7-03 Simple cubic equations $ax^3 = c$

An equation in which the highest power of the variable is 3 is called a **cubic equation**, for example, $x^3 = 12$, $2m^3 + 1 = 25$, $d^3 - 14 = 4$ and $x^3 - 3x^2 + 5x + 4 = 0$.

Summary

The cubic equation $x^3 = c$ has one solution: $x = \sqrt[3]{c}$

Example 8

Solve each cubic equation.

a $y^3 = 64$

b $p^3 = 50$

c $-2x^3 = 2000$

Solution

a $y^3 = 64$

$$y = \sqrt[3]{64}$$

$$= 4$$

Finding the cube root of both sides.

b $p^3 = 50$

$$p = \sqrt[3]{50}$$

50 is not a cube number so leave the answer as a surd.

c $-2x^3 = 2000$

$$x^3 = \frac{2000}{-2}$$

$$= -1000$$

$$x = \sqrt[3]{-1000}$$

$$= -10$$

Dividing both sides by (-2) .

Example 9

Solve each cubic equation, writing the solution correct to one decimal place.

a $11x^3 - 102 = 0$

b $\frac{2y^3}{7} = -11$

Solution

a $11x^3 - 102 = 0$

$$11x^3 = 102$$

$$x^3 = \frac{102}{11}$$

$$= 9.272 \dots$$

$$x = \sqrt[3]{9.272 \dots}$$

$$= 2.1008 \dots$$

$$\approx 2.1$$

b $\frac{2y^3}{7} = -11$

$$2y^3 = -11 \times 7$$

$$= -77y^3$$

$$= \frac{-77}{2}$$

$$= -38.5$$

$$y = \sqrt[3]{-38.5}$$

$$= -3.3766 \dots$$

$$\approx -3.4$$

Stage 5.3

Exercise 7-03

Simple cubic equations $ax^3 = c$

See Example 8

- 1 Solve each cubic equation, writing the solution in exact form where necessary.

a $x^3 = 1$

b $m^3 = 125$

c $a^3 = 1331$

d $u^3 = -8$

e $y^3 = -729$

f $n^3 = 20$

g $h^3 = 11$

h $k^3 = -48$

i $5m^3 = -75$

j $7m^3 = 448$

k $-4x^3 = 81$

l $12x^3 = -480$

See Example 9

- 2 Solve each cubic equation, writing the solution correct to one decimal place.

a $w^3 - 16 = 0$

b $m^3 + 6 = 22$

c $5m^3 - 1080 = 0$

d $3t^3 - 10 = 87$

e $\frac{x^3}{3} = 9$

f $\frac{5x^3}{7} = -120$

g $\frac{3x^3}{4} = -10$

h $\frac{2x^3}{5} = 0.2048$

i $\frac{7a^3}{9} - 10 = 121$

j $a^3 - 0.064 = 0$

k $-\frac{7x^3}{9} = 10$

l $5t^3 + 46 = -370$

- 3 a Does a cubic equation of the form $ax^3 = c$ always have a solution?
 b When is the solution to $x^3 = c$ positive?
 c When is the solution to $x^3 = c$ negative?
 d Can $x^3 = c$ have two solutions?

7-04 Equation problems

Example 10

At a concert, an adult's ticket costs \$5 more than twice the cost of a child's ticket. The total cost for 3 adults and 7 children is \$327. Find the cost of a child's ticket and an adult's ticket.

Solution

Let the cost of a child's ticket be \$ c .

$\therefore$ Cost of an adult's ticket = $\$(2c + 5)$

$$3(2c + 5) + 7c = 327$$

$$6c + 15 + 7c = 327$$

$$13c + 15 = 327$$

$$13c = 312$$

$$c = 24$$

$\therefore$ Cost of an adult's ticket = $2 \times \$24 + 5$
 $= \$53$

$\therefore$ A child's ticket costs \$24 and an adult's ticket costs \$53.

Check: $3 \times \$53 + 7 \times \$24 = \$327$.

Using a variable to represent an unknown quantity.

Forming an equation.

Solving the equation.

A child's ticket costs \$24.

Example 11

Jennifer is 7 years older than Amy. Ten years from now, the sum of their ages will be 43. How old are they now?

Solution

Let x = Amy's age now.

$\therefore$ Jennifer's age now = $x + 7$.

In 10 years' time:

$$(x + 10) + (x + 17) = 43$$

$$2x + 27 = 43$$

$$2x = 16$$

$$x = 8$$

Jennifer's age now = $8 + 7$

$$= 15$$

Amy is 8 years old now and Jennifer is 15 years old now.

$\therefore$ Check: In 10 years' time, the sum of their ages will be $18 + 25 = 43$.

	Now	In 10 years' time
Amy	x	$x + 10$
Jennifer	$x + 7$	$x + 7 + 10 = x + 17$

Amy is 8 now.

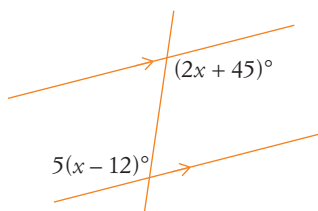
Exercise 7-04 Equation problems

For each question, write an equation and solve it to answer the problem.

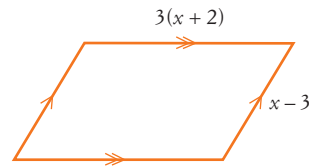
- 1 A rectangle is four times as long as it is wide. The perimeter of the rectangle is 250 cm. Find the dimensions of the rectangle.
- 2 The equal sides of an isosceles triangle are twice as long as the other side. The perimeter of the triangle is 90 mm. Find the lengths of the sides of the triangle.
- 3 At the football match, an adult's ticket costs \$6 more than twice the cost of a child's ticket. The total cost for 3 adults and 5 children is \$249. Find the cost of a child's ticket and an adult's ticket.
- 4 The sum of three consecutive numbers is 186. Find the numbers.
- 5 The sum of three consecutive even numbers is 288. Find the numbers.
- 6 Sanjay is nine times the age of his son, Anand. In 5 years he will be four times the age of Anand. How old are they now?
- 7 When 15 is subtracted from three times a certain number, the answer is 63. What is the number?
- 8 The product of 2 and a number is the same as 12 subtract the number. Find the number.
- 9 The sum of the present ages of Vatha and Chris is 36. In 4 years time, the sum of their ages will equal twice Vatha's present age. How old are they now?
- 10 Four consecutive numbers have a sum of 858. Find the numbers.
- 11 Find x .

See Example 10

See Example 11



- 12 Manori's bag has 10-cent and 20-cent coins. She has 202 coins with a total value of \$31.90. How many 20-cent coins does Manori have?
- 13 If 17 more than a number is 5 more than three times the number, what is the number?
- 14 If the perimeter of this parallelogram is 130, find x .



- 15 The sum of Scott's age and his mother's age is 45. In 5 years' time, three times Scott's age less 9 will be the same as his mother's age. Find the present ages of Scott and his mother.
- 16 One angle in a triangle is double the smallest angle, and the third angle in the triangle is 5 more than four times the smallest angle. Find the size of each angle.
- 17 A large container of water is $\frac{7}{8}$ full. After 15 L has been taken out, the container is $\frac{2}{3}$ full. When full, how many litres does the container hold?
- 18 The total cost of a school camp for Year 10 students was \$21 280. Each teacher paid \$185 to attend and each student paid \$165. There was one teacher for every 15 students. Find the numbers of teachers and students that attended the camp.

Mental skills 7

Maths without calculators

Multiplying decimals

- 1 Study each example.

a $3 \times 8 = 24$, so $3 \times 0.8 = 2.4$

$\uparrow \quad \uparrow \quad \uparrow$
 0 dp + 1 dp = 1 dp (dp = decimal places)

The number of decimal places in the answer is equal to the total number of decimal places in the question. Also, the answer sounds reasonable because, by estimation:

$3 \times 0.8 \approx 3 \times 1 = 3$ ($2.4 \approx 3$)

b $6 \times 5 = 30$, so $0.6 \times 0.5 = 0.30 = 0.3$

$\uparrow \quad \uparrow \quad \uparrow$
 1 dp + 1 dp = 2 dp

By estimation, $0.6 \times 0.5 \approx 0.5 \times 0.5 = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} = 0.25$ ($0.3 \approx 0.25$)

c $7 \times 3 = 21$, so $0.07 \times 0.3 = 0.021$

$\uparrow \quad \uparrow \quad \uparrow$
 2 dp + 1 dp = 3 dp

By estimation, $0.07 \times 0.3 \approx 0.07 \times \frac{1}{3} \approx 0.02$ ($0.021 \approx 0.02$)

2 Now evaluate each product.

a 0.7×5	b 12×0.2	c 0.4×0.3	d $(0.6)^2$
e 8×0.1	f 0.03×0.9	g 4×0.05	h 1.1×8
i 0.3×0.8	j 0.2×0.06	k 9×0.2	l 0.07×0.4

3 Study each example.

Given that $15 \times 23 = 345$, evaluate each product.

a $1.5 \times 2.3 = 3.45$

$\uparrow \quad \uparrow \quad \uparrow$
1 dp + 1 dp = 2 dp (Estimate: $1.5 \times 2.3 \approx 2 \times 2 = 4$)

b $150 \times 0.23 = 15 \times 10 \times 0.23 = 15 \times 0.23 \times 10 = 3.45 \times 10 = 34.5$

$\uparrow \quad \uparrow \quad \uparrow$
0 dp + 2 dp = 2 dp
(Estimate: $150 \times 0.23 \approx 150 \times 0.2 = 150 \times \frac{1}{5} = 30$)

c $0.15 \times 2300 = 0.15 \times 23 \times 100 = 3.45 \times 100 = 345$

$\uparrow \quad \uparrow \quad \uparrow$
2 dp + 0 dp = 2 dp
(Estimate: $0.15 \times 2300 \approx 0.2 \times 2300 = \frac{1}{5} \times 2300 = 460$)

4 Now given that $39 \times 17 = 663$, evaluate each product.

a 3.9×17	b 39×170	c 39×0.17	d 0.39×1.7
e 3.9×1.7	f 390×1.7	g 3.9×0.17	h 3.9×170
i 3900×1.7	j 39×1.7	k 39×0.017	l 0.39×0.17

7-05 Equations and formulas

Puzzle sheet

Getting the right
formula

MAT10NAPS10044

A **formula** is an equation that describes a relationship between variables. For example, the formula for the perimeter of a rectangle is $P = 2(l + w)$, where P is the **subject** of the formula and appears on the LHS of the '=' sign.

Example 12

The cost, \$ C , of a taxi trip is $C = 5 + 2.4d$, where d is the distance travelled in kilometres.

- a Find the cost of a taxi trip if the distance travelled is 15 km.
b Find the distance travelled by the taxi if the cost of the trip was \$78.20.

Solution

- a When $d = 15$:
 $C = 5 + 2.4 \times 15$
 $= 41$
The cost was \$41.

- b When $C = 78.20$:
 $78.20 = 5 + 2.4d$
 $73.20 = 2.4d$
 $d = \frac{73.20}{2.4} = 30.5$
The distance travelled was 30.5 km.

Example 13

The surface area of a sphere is $SA = 4\pi r^2$, where r is the radius. Find, correct to one decimal place, the radius of a sphere with surface area 40 m^2 .

Solution

When $r = 40$:

$$40 = 4\pi r^2$$

$$10 = \pi r^2$$

$$r^2 = \frac{10}{\pi}$$

$$= 3.183 \dots$$

$$r = \sqrt{3.183}$$

$$= 1.784 \dots$$

$$\approx 1.8 \text{ m}$$

r is positive

The radius of the sphere is 1.8 m.

Exercise 7-05 Equations and formulas

See Example 12

- The formula for the circumference of a circle is $C = 2\pi r$, where r is the radius. Find, correct to one decimal place:
 - the circumference of a circle with radius 2.4 m
 - the radius of a circle whose circumference is 200 cm
- The perimeter of a rectangle is $P = 2(l + w)$. Find the width of a rectangle whose perimeter is 58 m and length is 12 m.
- The formula for converting speed expressed in m/s to a speed expressed as km/h is $k = 3.6M$, where M is the speed in m/s. Calculate in m/s the speed of a car travelling at 110 km/h.
- Use the formula from question 3 to convert each speed to km/h.
 - 10 m/s
 - 24 m/s
 - 50 m/s
- The average of m and n is $A = \frac{m+n}{2}$. If two numbers have an average of 28 and one of them is 13, find the other number.
- The formula for converting a temperature recorded in $^{\circ}\text{F}$ to a temperature in $^{\circ}\text{C}$ is $C = \frac{5}{9}(F - 32)$. Convert each temperature to $^{\circ}\text{C}$, correct to the nearest degree.
 - 80°F
 - 32°F
 - 212°F
 - 102°F

See Example 13

- The body mass index (BMI) of an adult is $B = \frac{M}{h^2}$, where M is the mass in kilograms and h is the height in metres. Find, correct to one decimal place:
 - the BMI of Dean who is 1.85 m tall and has a mass of 72 kg
 - the mass of a person with a BMI of 24, who is 2.1 m tall.
- The volume of a sphere is $V = \frac{4}{3}\pi r^3$, where r is the radius. Find, correct to one decimal place, the radius of a sphere with a volume of 500 m^3 .

- 9 The average speed in km/h of a car is given by the formula $S = \frac{D}{T}$, where D is the distance covered in kilometres and T is the time taken in hours. Find, correct to the nearest whole number:
- the distance travelled, if a car maintains a speed of 87.2 km/h for 5 hours
 - the time taken, if a distance of 650 km is covered at a speed of 91 km/h
- 10 The cost, \$ C , of hiring a car is $C = 45 + 0.15d$, where d is the number of kilometres travelled. Calculate:
- the cost of hiring a car to travel 350 km
 - the distance travelled, if the cost is \$138.
- 11 The surface area of a cylinder is given by the formula $SA = 2\pi r^2 + 2\pi rh$. Calculate, correct to one decimal place, the height of a cylinder with surface area 1255.38 cm² and radius 9 cm.

7-06 Changing the subject of a formula

Example 14

Change the subject of the formula:

a $A = \frac{1}{2}bh$ to h

b $v^2 = u^2 + 2as$ to s

c $\frac{a+2}{a+10} = k$ to a

Solution

a $A = \frac{1}{2}bh$

$$\frac{1}{2}bh = A$$

$$bh = 2A$$

$$h = \frac{2A}{b}$$

b $v^2 = u^2 + 2as$

$$u^2 + 2as = v^2$$

$$2as = v^2 - u^2$$

$$s = \frac{v^2 - u^2}{2a}$$

c $\frac{a+2}{a+10} = k$

$$a+2 = k(a+10)$$

$$= ak + 10k$$

$$a - ak = 10k - 2$$

$$a(1 - k) = 10k - 2$$

$$a = \frac{10k - 2}{1 - k}$$

Swapping sides so that h appears on the LHS.

Multiplying both sides by 2.

Dividing both sides by b .

Swapping sides so that s appears on the LHS.

Subtracting u^2 from both sides.

Dividing both sides by $2a$.

Multiplying both sides by $a + 10$.

Expanding

Moving the a -terms to the LHS, the 2 to the RHS.

Factorising a from the LHS.

Dividing both sides by $1 - k$.

Stage 5.3

NSW

Video tutorial

Changing the subject
of a formula

MAT10NAVT10005

Worksheet

Changing the subject
of a formula

MAT10NAWK10211

Stage 5.3

Exercise 7-06 Changing the subject of a formula

See Example 14

1 Make y the subject of each formula.

a $x + 2y = 5$

b $m + py = k$

c $P - ky = 8$

d $\frac{m}{3} = \frac{y}{5}$

e $D = K - My$

f $\frac{5+8y}{d} = 4$

g $\frac{ay-k}{2} = c$

h $\frac{y+3}{5} = \frac{4m}{3}$

i $xy^2 + 5 = w$

j $x = \sqrt{\frac{y}{k}}$

k $n = \frac{d}{5-y}$

l $T = \sqrt{\frac{y+k}{c}}$

2 Change the subject of each formula to the variable indicated in brackets.

a $a^2 + b^2 = c^2$ [b]

b $s = ut + \frac{1}{2}at^2$ [a]

c $v = u + at$ [a]

d $V = \frac{4}{3}\pi r^3$ [r]

e $A = \pi(R^2 - r^2)$ [R]

f $A = \pi rl + \pi r^2$ [l]

g $S = 180(n - 2)$ [n]

h $\frac{1}{x} + \frac{1}{r} = \frac{1}{s}$ [r]

i $x = \sqrt{b^2 - 4ac}$ [b]

j $x + y = 5 - 3x$ [x]

k $m = \frac{5A}{2A+n}$ [A]

l $S = \frac{a(p-1)}{p}$ [p]

m $X(a+b) = Y(a-b)$ [a]

n $\frac{5+x}{3x+a} = 2$ [x]

o $y = \frac{u+bx}{u+ab}$ [b]

Investigation: Restricting values of variables1 Consider the formula $x^2 + y^2 = 4$.a Explain why the least value that x can take is -2 and the largest value that x can take is 2 .b Does the same restriction apply to the values that the variable y can take? Explain why.c By making y the subject, show that $y = \pm\sqrt{4-x^2}$.d Are the values that x and y can take in $y = \pm\sqrt{4-x^2}$ different from the values that they can take in $x^2 + y^2 = 4$? Give reasons.2 a If $Z = ax^2$, what range of values can Z , a and x take?b If $x = \sqrt{\frac{Z}{a}}$, what range of values can Z , a and x take?3 a In the formula $A = \pi r^2$, explain why there are no restrictions on r but $A \geq 0$.b Make r the subject of the formula given that the formula is for the area of a circle. Have the restrictions on the variables r and A changed?4 a What are the restrictions on the variables x and y in the formula $y = \sqrt{16-x^2}$?b Change the subject of the formula to x . Are the restrictions on the variables the same as for part a? Explain.5 Consider the formula $y = \frac{1}{x-3}$.a What are the restrictions on the variables x and y ? Give reasons.b Make x the subject of the formula. Are the restrictions on the variables the same as in part a or different? Explain.

c Compare your answers to the above questions with those of other students in your class.

7-07 Graphing inequalities on a number line

Worksheet
Graphing inequalities
MAT10NAWK10045

An **inequality** looks like an equation except that the equals sign ($=$) is replaced by an inequality symbol $>$, $\geq$, $<$ or $\leq$.

$2x - 7 = 15$ is an **equation**. There is only one value of x that makes it true.

$2x - 7 \leq 15$ is an **inequality**. There is a range of values of x that make it true.

Summary

$>$ means 'is greater than' $\geq$ means 'is greater than or equal to'
 $<$ means 'is less than' $\leq$ means 'is less than or equal to'

The inequality $x \geq 3$ is read 'x is greater than or equal to 3' and includes 3 and all the numbers above 3, such as 3.01, 4, 10, 20 000, etc.

The inequality $x > 3$ is read 'x is greater than 3' and means all the numbers above 3, but **not** 3.

Inequality	In words	Meaning
$x > 3$	x is greater than 3	Values above 3
$x < 3$	x is less than 3	Values below 3
$x \geq 3$	x is greater than or equal to 3	Values above and including 3
$x \leq 3$	x is less than or equal to 3	Values below and including 3

For convenience, we can represent all the values in an inequality using a number line.

Example 15

Graph each inequality on a number line.

- a** $x \geq 1$ **b** $x < 5$ **c** $x > -3$

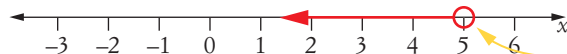
Solution

- a** $x \geq 1$ means that x can be any number greater than 1 or equal to 1.



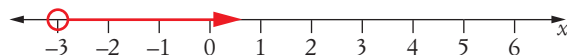
The filled circle at 1 means we include 1.

- b** $x < 5$ means that x can be any number less than 5, but not including 5.



The open circle on 5 means that 5 is not included.

- c** $x > -3$ means that x can be any number greater than -3, but not including -3.



Exercise 7-07 Graphing inequalities on a number line

See Example 15

- 1 Graph each inequality on a separate number line.

a $x \geq 2$

b $x < -3$

c $x \leq 1$

d $x > 7$

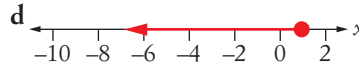
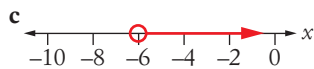
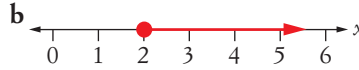
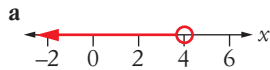
e $x \leq 4$

f $x > 0$

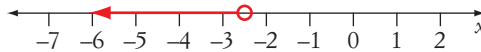
g $x \geq -2$

h $x < 10$

- 2 Write the inequality illustrated by each number line.



- 3 Which inequality is graphed below? Select the correct answer A, B, C or D.



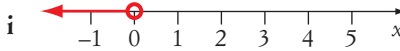
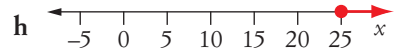
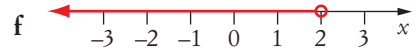
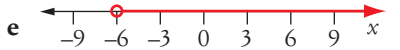
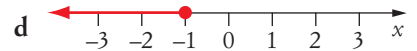
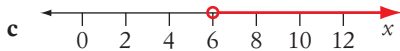
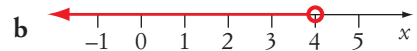
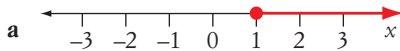
A $x > -2.5$

B $x < -2.5$

C $x < -3.5$

D $x > -3.5$

- 4 Write the inequality represented on each number line.



Investigation: The language of inequalities

Work in pairs to complete this activity.

Use inequality symbols to write each statement algebraically.

a The minimum height (H) for rides at an amusement park is 1.3 m.

b The speed limit in a school zone is 40 km/h.

c To be eligible to vote, you must be at least 18 years old ($A = \text{age}$).

d The overseas tour is only for people whose age (A) is from 18 to 35.

e The cost (A) of a tennis racquet will be at least \$95 but no more than \$360.

f A new flute (F) costs at least \$475.

g The price of units (U) in a new block start at \$240 000.

Investigation: Solving inequalities

We have solved equations by doing the same thing to both sides (keeping the equation 'balanced'). Will this method work with inequalities, such as $x + 4 > 10$ or $6x < 13$?

- 1 Start with an inequality that is true, such as $7 > 4$.
- 2 Add 5 (or any number you choose) to both sides of the inequality; for example $7 > 4$ becomes $12 > 9$. Is the new inequality true or false?
- 3 Subtract 9 (or any number you choose) from each side of the original inequality; for example $7 > 4$ becomes $-2 > -5$. Is the new inequality true or false?
- 4 Multiply both sides of the original inequality by 4 (or any positive number you choose); for example $7 > 4$ becomes $28 > 16$. Is the new inequality true or false?
- 5 Divide both sides of the original inequality by 2 (or any positive number you choose); for example $7 > 4$ becomes $3\frac{1}{2} > 2$. Is the new inequality true or false?
- 6 Multiply both sides of the original inequality by -3 (or any negative number you choose); for example $7 > 4$ becomes $-21 > -12$. Is the new inequality true or false?
- 7 Divide both sides of the original inequality by -4 (or any negative number you choose), for example $7 > 4$ becomes $-1\frac{3}{4} > -1$. Is the new inequality true or false?
- 8 Which of the six operations used in questions 2 to 7 can be used on inequalities to give a true result?
- 9 Which of the six operations used in questions 2 to 7 cannot be used with inequalities because they give a false result?
- 10 Copy and complete the following inequality statements.
 - a $6 < 8$
 $6 \times 3 < 8 \times \underline{\hspace{1cm}}$ (multiplying both sides by 3)
 $\therefore 18 \underline{\hspace{1cm}} 24$
 - b $10 > -4$
 $10 \div 2 \underline{\hspace{1cm}} -4 \div \underline{\hspace{1cm}}$ (dividing both sides by 2)
 $\therefore \underline{\hspace{2cm}}$

Does the inequality sign ($<$ or $>$) stay the same when multiplying or dividing by a positive number?
- 11
 - a Is it true that $5 < 8$?
 - b Multiply both sides by -2 . Is it true that $-10 < -16$?
 - c What needs to be reversed to change $-10 < -16$ into a true inequality statement?
 - d Copy and complete the following to make a true inequality statement: $-10 \underline{\hspace{1cm}} -16$.
- 12
 - a Is it true that $18 > -6$?
 - b Divide both sides by -3 . Is it true that $-6 > 2$?
 - c What needs to be reversed to change $-6 > 2$ into a true inequality statement?
 - d Copy and complete the following to make a true inequality statement: $-6 \underline{\hspace{1cm}} 2$.
- 13 Copy and complete:
 When multiplying or d_____ both sides of an inequality by a n_____
 number, the inequality sign must be r_____.

7-08 Solving inequalities

Worksheet

Inequalities review

MAT10NAWK10046

Example 16

Solve each inequality and graph its solution on a number line.

a $2x - 10 \leq 16$

b $2(y - 1) \geq 12$

c $\frac{w+3}{2} > -1$

Solution

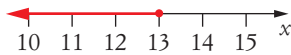
a $2x - 10 \leq 16$

$$2x - 10 + 10 \leq 16 + 10$$

$$2x \leq 26$$

$$\frac{2x}{2} \leq \frac{26}{2}$$

$$x \leq 13$$



b $2(y - 1) \geq 12$

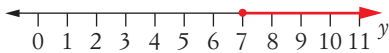
$$2y - 2 \geq 12$$

$$2y - 2 + 2 \geq 12 + 2$$

$$2y \geq 14$$

$$\frac{2y}{2} \geq \frac{14}{2}$$

$$y \geq 7$$



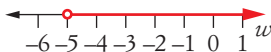
c $\frac{w+3}{2} > -1$

$$\frac{w+3}{2} \times 2 > -1 \times 2$$

$$w + 3 > -2$$

$$w + 3 - 3 > -2 - 3$$

$$w > -5$$



Summary

Inequalities can be solved algebraically in the same way as equations, by using inverse operations. However, when **multiplying** or **dividing** both sides of an inequality by a **negative** number, you must **reverse the inequality sign**.

Example 17

Solve each inequality.

a $1 - 2x \geq -11$

b $4 - r < 7$

c $\frac{a+5}{-3} > 4$

Solution

a $1 - 2x \geq -11$

$$1 - 2x - 1 \geq -11 - 1$$

$$-2x \geq -12$$

$$\frac{-2x}{-2} \leq \frac{-12}{-2}$$

$$x \leq 6$$

Dividing both sides by a negative number reverses the inequality sign.

b $4 - r < 7$

$$4 - r - 4 < 7 - 4$$

$$-r < 3$$

$$\frac{-r}{-1} > \frac{3}{-1}$$

$$r > -3$$

Dividing both sides by a negative number reverses the inequality sign.

c $\frac{a+5}{-3} > 4$

$$\frac{a+5}{-3} \times (-3) < 4 \times (-3)$$

$$a + 5 < -12$$

$$a + 5 - 5 < -12 - 5$$

$$a < -17$$

Multiplying both sides by a negative number reverses the inequality sign.

Exercise 7-08 Solving inequalities

1 Solve each inequality and graph its solution on a number line.

a $x - 1 > 6$

b $3y \geq 12$

c $m + 4 \leq 2$

d $\frac{x}{5} \geq -20$

e $12x < 60$

f $5y > -20$

g $4a \geq 2$

h $3w \leq -30$

i $8a + 5 \geq 45$

j $3a + 1 \leq 10$

k $6a + 4 \geq -2$

l $3w - 3 < -12$

2 Solve each inequality.

a $3(x + 2) \geq 9$

b $5(m - 4) \leq 10$

c $2(y + 5) \leq -6$

d $\frac{x-1}{2} \leq 2$

e $\frac{w-2}{5} > -1$

f $\frac{2a+1}{3} < 3$

g $\frac{2(m+1)}{3} \leq 3$

h $\frac{4(m-2)}{3} \geq -6$

i $3 + \frac{x}{5} < 10$

j $3 + 2x < 9 + x$

k $11 - 5y \geq 9 - 6y$

l $2(3 + 5a) \leq 5(4 + a)$

See Example 16

- 3 What is the solution to $\frac{3(x-2)}{5} \geq -1$? Select the correct answer A, B, C or D.

A $x \geq 0$ B $x \geq -1$ C $x \geq \frac{1}{3}$ D $x \geq -\frac{11}{3}$

See Example 17

- 4 Solve each inequality and graph its solution on a number line.

a $5 - x \leq 2$

b $15 > 7 - y$

c $1 - k < 12$

d $7 - m \geq 7$

e $2 - p > 8$

f $-t + 6 \geq 10$

- 5 Solve each inequality.

a $-2x < 6$

b $\frac{k}{-3} \geq 4$

c $-5t > 12$

d $\frac{-x}{3} \leq -4$

e $4 - 3w > 7$

f $-4y + 3 \leq 11$

g $8 - 5a < 3$

h $-2d - 3 > 8$

i $\frac{5+w}{-3} > 2$

j $\frac{-p+2}{-3} < -2$

k $1 - 3m < 9 - 5m$

l $3(3x + 4) \geq 6(1 - 2x)$

Just for the record

Film and game classification

In Australia, films, publications and computer games are rated by the Classification Board. Films and videos are rated G, PG, M, MA15+ or R18+, with each category containing a list of guidelines related to the film's use of violence, coarse language, adult themes, sex and nudity.

General (G) means suitable for all ages. Children can watch films classified G without adult supervision.



Parental guidance (PG) means that parental guidance is recommended for persons under 15 years of age. These films contain material that may be confusing or upsetting to children, but not harmful or disturbing. Parents should watch the film with their children or preview it to check elements such as language used or inappropriate themes.



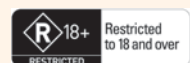
Mature (M) means recommended for mature audiences, 15 years and over. The film or computer game may contain material that is harmful or disturbing to children, but the impact is not so strong as to require restriction.



Mature accompanied (MA15+) means legal restrictions apply to persons under the age of 15. Children are not allowed to see MA15+ films unless accompanied by a parent or guardian, because they contain material that is likely to be harmful or disturbing to them.



Restricted (R18+) means legally restricted to adults, 18 years and over. It applies to films that deal with issues and scenes that require an adult perspective, and so are unsuitable for persons under 18 years of age. A person will be asked for proof of age before buying, hiring or viewing films or computer games in this category.



- Each of the classifications is represented by a logo (as shown) with the letter inside a particular shape. What shape is each logo?
- Write each classification category as an inequality.

Investigation: Power tables

- 1 Copy and complete this table of powers of 2 from 0 to 20.

x	0	1	2	...	20
2^x					

- 2 Use the table to calculate 32×128 . Explain the method you used.
 3 Use the table to calculate:
 a 16×1024 b 128×2048 c 256×64 d 4096×32
 4 Use the table to calculate $262\,114 \div 8192$. Explain the method you used.
 5 Use the table to calculate:
 a $16\,384 \div 512$ b $128 \div 8$ c $8192 \div 1024$ d $1\,048\,576 \div 65\,536$
 When powers are used this way in calculations, they may be called logarithms.

7-09 Logarithms

The **logarithm** of a number is the power of the number, to a given positive base.
 For example, the logarithm of 256 to the base 2 is 8, written $\log_2 256 = 8$, because $2^8 = 256$.

Example 18

Evaluate each expression.

a $\log_3 81$

b $\log_4 16$

c $\log_{10} 10\,000$

Solution

a $\log_3 81$ means $3^? = 81$

'3 to the power of what equals 81?'

Since $3^4 = 81$

then $\log_3 81 = 4$.

b $\log_4 16$ means $4^? = 16$

'4 to the power of what equals 16?'

Since $4^2 = 16$

then $\log_4 16 = 2$.

c $\log_{10} 10\,000$ means $10^? = 10\,000$

Since $10^4 = 10\,000$

then $\log_{10} 10\,000 = 4$.

Summary

If $y = a^x$, then $\log_a y = x$

where a is the base, $a > 0$, x is the power, and $y > 0$.

Since $a > 0$, $a^x > 0$ and $y > 0$.

Logarithms are only meaningful for positive numbers, y .

Stage 5.3

Puzzle sheet

Logarithms 1

MAT10NAPS00059

Puzzle sheet

Logarithms 2

MAT10NAPS00060

Stage 5.3

Example 19

Write each expression as a logarithm.

a $243 = 3^5$

b $0.01 = 10^{-2}$

c $2 = 8^{\frac{1}{3}}$

d $p = q^r$

Solution

a $243 = 3^5$

$\therefore \log_3 243 = 5$

b $0.01 = 10^{-2}$

$\therefore \log_{10} 0.01 = -2$

c $2 = 8^{\frac{1}{3}}$

$\therefore \log_8 2 = \frac{1}{3}$

d $p = q^r$

$\therefore \log_q p = r$

Example 20

Rewrite $\log_n m = x$ in index form.

Solution

$\log_n m = x$

$\therefore m = n^x$

n is the base, x is the power.

Exercise 7-09 Logarithms

See Example 18

1 Evaluate each expression.

a $\log_5 25$

b $\log_2 8$

c $\log_7 49$

d $\log_2 16$

e $\log_3 243$

f $\log_{10} 1000$

g $\log_5 125$

h $\log_6 36$

i $\log_2 64$

j $\log_3 6561$

k $\log_{10} 1000\ 000$

l $\log_8 512$

See Example 19

2 Write each expression in logarithmic form.

a $5^2 = 25$

b $4^3 = 64$

c $10\ 000 = 10^4$

d $25^{\frac{1}{2}} = 5$

e $\frac{1}{16} = 2^{-4}$

f $3^{-2} = \frac{1}{9}$

g $8^{\frac{2}{3}} = 4$

h $0.01 = 10^{-2}$

i $\sqrt{2} = 4^{\frac{1}{4}}$

j $16^{\frac{1}{2}} = 4$

k $9^{\frac{3}{2}} = 27$

l $\frac{1}{\sqrt{6}} = 6^{-\frac{1}{2}}$

See Example 20

3 Write each expression in index form.

a $\log_5 125 = 3$

b $\log_{10} 10 = 1$

c $\log_{\sqrt{3}} 27 = 6$

d $\log_2 8\sqrt{2} = 3.5$

e $\log_2 64 = 6$

f $\log_3 \frac{1}{81} = -4$

g $\log_5 \frac{1}{125} = -3$

h $\log_8 \sqrt{2} = \frac{1}{6}$

i $\log_{100} 10 = \frac{1}{2}$

j $\log_5 5\sqrt{5} = \frac{3}{2}$

k $\log_8 2 = \frac{1}{3}$

l $\log_{100} \frac{1}{100} = -1$

4 Why can't you find the logarithm of a negative number or zero?

7-10 Logarithm laws

The index laws from Chapter 5, Products and factors, are related to the logarithm laws.

Summary

The logarithm of a product is equal to the sum of the logarithm of each factor.

$$\log_a(xy) = \log_a x + \log_a y$$

For example, $\log_2(8 \times 4) = \log_2 8 + \log_2 4$.

This law corresponds to the index law $a^m \times a^n = a^{m+n}$.

Proof:

Let $m = \log_a x$ and $n = \log_a y$.

$$\therefore x = a^m \text{ and } y = a^n$$

$$\therefore xy = a^m \times a^n = a^{m+n}$$

$$\therefore \log_a(xy) = m + n$$

$$= \log_a x + \log_a y$$

Summary

The logarithm of a quotient is equal to the difference between the logarithm of each term.

$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$

For example, $\log_3\left(\frac{243}{27}\right) = \log_3 243 - \log_3 27$

This law corresponds to the index law $a^m \div a^n = a^{m-n}$.

Proof:

Let $m = \log_a x$ and $n = \log_a y$.

$$\therefore x = a^m \text{ and } y = a^n$$

$$\therefore \frac{x}{y} = \frac{a^m}{a^n} = a^{m-n}$$

$$\therefore \log_a\left(\frac{x}{y}\right) = m - n$$

$$= \log_a x - \log_a y$$

Summary

The logarithm of a term raised to a power is equal to the power multiplied by the logarithm of the term.

$$\log_a x^n = n \log_a x$$

Stage 5.3

For example, $\log_4 8^2 = 2 \log_4 8$

This law corresponds to the index law $(a^m)^n = a^{mn}$.

Proof:

Let $m = \log_a x$

$$\therefore x = a^m$$

$$\therefore x^n = (a^m)^n$$

$$= a^{mn}$$

$$\therefore \log_a x^n = mn$$

$$= \log_a x \times n$$

$$= n \log_a x$$

Summary

Properties of logarithms

$$\log_a a^x = x$$

$$\log_a a = 1, \text{ because } a^1 = a$$

$$\log_a 1 = 0, \text{ because } a^0 = 1$$

$$\log_a \left(\frac{1}{x} \right) = -\log_a x$$

Proof:

$$\log_a \left(\frac{1}{x} \right) = \log x^{-1}$$

$$= -\log_a x$$

Using the law $\log_a x^n = n \log_a x$

Example 21

Evaluate each expression.

a $\log_5 0.04$ **b** $\log_2 5 - \log_2 10$ **c** $2 \log_3 6 - \log_3 4$ **d** $\log_5 10 + \log_5 2 - \log_5 4$

Solution

$$\begin{aligned} \text{a } \log_5 0.04 &= \log_5 \left(\frac{4}{100} \right) \\ &= \log_5 \left(\frac{1}{25} \right) \\ &= \log_5 \left(\frac{1}{5^2} \right) \\ &= \log_5 5^{-2} \\ &= -2 \end{aligned}$$

$$\begin{aligned} \text{b } \log_2 5 - \log_2 10 &= \log_2 \left(\frac{5}{10} \right) \\ &= \log_2 \left(\frac{1}{2} \right) \\ &= \log_2 2^{-1} \\ &= -1 \end{aligned}$$

$$\begin{aligned} \text{c } 2 \log_3 6 - \log_3 4 &= \log_3 6^2 - \log_3 4 \\ &= \log_3 36 - \log_3 4 \\ &= \log_3 \left(\frac{36}{4} \right) \\ &= \log_3 9 \\ &= 2 \end{aligned}$$

$$\begin{aligned} \text{d } \log_5 10 + \log_5 2 - \log_5 4 &= \log_5 (10 \times 2) - \log_5 4 \\ &= \log_5 20 - \log_5 4 \\ &= \log_5 \left(\frac{20}{4} \right) \\ &= \log_5 5 \\ &= 1 \end{aligned}$$

Video tutorial

Logarithm laws

MAT10NAVT10001

Example 22

Simplify each expression.

a $6 \log_a a + \log_a a^4 - \log_a a^9$

b $\log_2 x + \log_2 w - 2 \log_2 y$

c $\frac{\log_3 a^3}{5 \log_3 a}$

Solution

a $6 \log_a a + \log_a a^4 - \log_a a^9 = 6 \times 1 + 4 - 9$
 $= 1$

b $\log_2 x + \log_2 w - 2 \log_2 y = \log_2(xw) - \log_2 y^2$
 $= \log_2 \left(\frac{xw}{y^2} \right)$

c $\frac{\log_3 a^3}{5 \log_3 a} = \frac{3 \log_3 a}{5 \log_3 a}$
 $= \frac{3}{5}$

Example 23

Given $\log_{10} 7 \approx 0.8451$, find the value of each expression.

a $\log_{10} 49$

b $\log_{10} 700$

c $\log_{10} (0.07)$

Solution

a $\log_{10} 49 = \log_{10} 7^2$
 $= 2 \log_{10} 7$
 $\approx 2 \times 0.8451$
 $= 1.6902$

b $\log_{10} 700 = \log_{10} (7 \times 100)$
 $= \log_{10} 7 + \log_{10} 100$
 $\approx 0.8451 + 2$
 $= 2.8451$

c $\log_{10} (0.07) = \log_{10} \left(\frac{7}{100} \right)$
 $= \log_{10} 7 - \log_{10} 100$
 $\approx 0.8451 - 2$
 $= -1.1549$

See Example 21

Exercise 7-10 Logarithm laws

1 Evaluate each expression.

a $\log_2 128$

b $\log_{10} 1000$

c $\log_8 64$

d $\log_5 \frac{1}{5}$

e $\log_2 \sqrt{2}$

f $\log_3 \frac{1}{9}$

g $\log_{10} 0.0001$

h $\log_2 \frac{1}{16}$

i $\log_8 2 + \log_8 4$

j $\log_4 32 - \log_4 2$

k $\log_{10} 4 + \log_{10} 25$

l $\log_5 200 - \log_5 8$

m $\log_2 18 - 2 \log_2 3$

n $3 \log_{10} 2 + \log_{10} 12.5$

Stage 5.3

See Example 22

See Example 23

- 2 Simplify each expression.

a $\log_x 5 + \log_x 6$	b $\log_x 10 - \log_x 2$	c $3 \log_x 2$
d $2 \log_x 4 - \log_x 8$	e $\log_x 10 + \log_x 4$	f $\frac{1}{2} \log_x 100$
g $-\log_x 4$	h $\log_x 8 - (\log_x 10 + \log_x 4)$	i $\frac{1}{2} (\log_x 8 + \log_x 18)$
- 3 If $\log_{10} 4 = 0.6021$, find the value of each expression.

a $\log_{10} 16$	b $\log_{10} 400$	c $\log_{10} 4000$	d $\log_{10} 2$
e $\log_{10} 0.4$	f $\log_{10} 160$	g $\log_{10} 2.5$	h $\log_{10} \sqrt{40}$
- 4 Evaluate each expression.

a $\log_3 4 + \log_3 15 - \log_3 20$	b $\log_3 270 - (\log_3 2 + \log_3 5)$
c $\log_4 20 + (\log_4 32 - \log_4 10)$	d $2 \log_{10} 25 - \log_{10} 6.25$
e $2 \log_{10} 2 - (\log_{10} 5 + \log_{10} 8)$	f $\log_{100} 50 - \log_{100} 5$
g $2 \log_5 10 + (\log_5 50 - \log_5 40)$	h $5 \log_8 2 + \frac{1}{2} \log_8 4$
i $\frac{1}{2} \log_4 25 - 2 \log_4 \sqrt{20}$	j $\frac{1}{3} \log_2 125 - 3 \log_2 \sqrt[3]{80}$
- 5 Simplify each expression.

a $\log_a a^2 + 3 \log_a a$	b $\log_a (a^3)$	c $5 \log_a a - \log_a a^4$
d $\frac{\log_a (x^7)}{\log_a x}$	e $\log_a y^3 - 3 \log_a y$	f $\log_a \sqrt{x} - \log_a \frac{1}{x}$

7-11 Exponential and logarithmic equations

Worksheet

Logarithms review

MAT10NAWK10212

Puzzle sheet

Exponential equations

MAT10NAPS00040

Exponential equations are equations like $3^x = 243$, where the variable is a power. Logarithms can be used to solve exponential equations rather than using a 'guess-and-check' method. The **log** key on your calculator can be used to evaluate $\log_{10} x$, that is, logarithms to the base 10.

Example 24

Solve each exponential equation.

a $3^x = 243$	b $4^{m+1} = \frac{1}{8\sqrt{2}}$

Solution

a $3^x = 243$

$$\log_{10} 3^x = \log_{10} 243$$

$$x \log_{10} 3 = \log_{10} 243$$

$$x = \frac{\log_{10} 243}{\log_{10} 3}$$

$$= 5$$

Taking $\log_{10}$ of both sides.

Enter on a calculator:

log 243 ÷ log 3 =

Note: The **log** key means $\log_{10}$, and for convenience we will write log to mean $\log_{10}$.

b $4^{m+1} = \frac{1}{8\sqrt{2}}$

$$\log(4^{m+1}) = \log\left(\frac{1}{8\sqrt{2}}\right)$$

Taking $\log_{10}$ of both sides.

$$(m+1)\log 4 = \log\left(\frac{1}{8\sqrt{2}}\right)$$

$$(m+1)\log 4 = -\log(8\sqrt{2})$$

$$m+1 = \frac{-\log 8\sqrt{2}}{\log 4}$$

$$= -1.75$$

$$m = -2.75$$

Example 25

Solve $5^x = 17$, writing the solution correct to three decimal places.

Solution

$$5^x = 17$$

$$\log 5^x = \log 17$$

$$x \log 5 = \log 17$$

$$x = \frac{\log 17}{\log 5}$$

$$= 1.7603 \dots$$

$$\approx 1.760$$

Logarithmic equations are equations like $\log_5 x = -3$, which can be solved by rewriting the equation in index form.

Example 26

Solve each logarithmic equation.

a $\log_5 x = -3$

b $\log_x 18 = 3$

Solution

a $\log_5 x = -3$

$$\therefore x = 5^{-3}$$

$$= \frac{1}{5^3}$$

$$= \frac{1}{125}$$

b $\log_x 18 = 3$

$$\therefore 18 = x^3$$

$$x = \sqrt[3]{18}$$

$$= 2.6207 \dots$$

$$\approx 2.62$$

Stage 5.3

Exercise 7-11 Exponential and logarithmic equations

See Example 24

1 Solve each exponential equation.

a $2^k = 512$

b $5^m = 78\,125$

c $3^d = 59\,049$

d $5^x = 25\sqrt{5}$

e $2^y = \frac{1}{16\sqrt{2}}$

f $4^a = 128$

g $3^{k+2} = 27\sqrt{3}$

h $6^{n-2} = \frac{1}{216\sqrt{6}}$

i $9^{1-d} = \frac{1}{27\sqrt{3}}$

See Example 25

2 Solve each exponential equation, writing the solution correct to three decimal places.

a $7^x = 16$

b $5^x = 36$

c $11^x = 420$

d $2^x = 0.52$

e $3^x = 1.6$

f $4^x = \frac{2}{5}$

g $2^{x-2} = 47$

h $3^{x+4} = 72$

i $6^{x+3} = 29$

j $8^{5-x} = 4000$

k $5^y = 4.8$

l $7^{k+5} = 300$

3 Solve each exponential equation by expressing both sides to base 2.

a $2^{x+2} = 16$

b $8^x = 32$

c $4^{x-1} = \sqrt{2}$

d $8^{1-x} = 16\sqrt{2}$

e $4^{2-x} = \frac{1}{8}$

f $8^{x+1} = \frac{1}{8\sqrt{2}}$

g $\left(\frac{1}{4}\right)^{1-x} = \sqrt{2}$

h $5\left(\frac{1}{2}\right)^x = 20$

See Example 26

4 Solve each logarithmic equation, expressing your answer correct to three decimal places where necessary.

a $\log_2 x = 3$

b $\log_{10} x = 3$

c $\log_5 x = -2$

d $\log_4 x = -3$

e $\log_{27} x = \frac{1}{3}$

f $\log_4 x = \frac{1}{2}$

g $\log_{10} x = -3$

h $\log_8 x = \frac{3}{2}$

i $\log_{10} x = -\frac{1}{2}$

j $\log_4 x = -\frac{3}{2}$

k $\log_4 x = 3\frac{1}{2}$

l $\log_{\sqrt{5}} x = -4$

m $\log_x 4 = 2$

n $\log_x 5 = -1$

o $\log_x \frac{1}{4} = 2$

p $\log_x 0.01 = 2$

q $\log_x 16 = 1$

r $\log_x 8 = 3$

s $\log_x 60 = 3$

t $\log_x 4.8 = \frac{1}{2}$

5 Use the compound interest formula $A = P(1 + r)^n$ to determine the number of years (to the nearest year) it will take an investment of \$1000 to grow to \$2000, if it earns compound interest at a rate of 6% p.a.

6 Penny invests \$12 000 at 1% per month compound interest. How many whole months will it take for Penny's investment to grow to \$15 000?

7 A radioactive substance with a mass of 150 grams decays according to the equation $A = 150 \times \left(2^{\frac{-t}{20}}\right)$, where A (grams) is the amount remaining after t days. Find, correct to the nearest whole number:

a the mass of substance remaining after 10 days

b the time taken for the substance to decay to half its original mass

c the time taken for the substance to decay to a mass of 20 g.

Power plus

1 Solve each equation.

a $\frac{2-x}{3} - \frac{x-5}{6} = \frac{5+x}{4} + \frac{x-2}{3}$

b $\frac{1}{x-1} + \frac{2}{x+1} = 0$

c $x^2(x-2) - 2(x+1) = x(x^2-3) - 2(5+x^2)$

2 Graph each inequality on a number line.

a $1 \leq x \leq 4$

b $-2 \leq x \leq 3$

c $-12 < 4x \leq 4$

3 The number of diagonals, D , in a polygon with n sides is $D = \frac{1}{2}n(n-3)$. Show that there is no polygon that has exactly 100 diagonals.

4 The two solutions of $x^2 - 8x - 11 = 0$ are in the form $x = p \pm q\sqrt{3}$, where p and q are integers. Find p and q .

5 Solve each logarithmic equation.

a $\log a + \log 3 = \log 21$

b $\log x - \log 4 = \log 5$

c $\log 7 + \log m = \log (m+12)$

d $\log (h+7) - \log 3 = \log (h-1)$

Chapter 7 review

Language of maths

Puzzle sheet

Equations and
inequalities crossword

MAT10NAPS10047

check	cubic equation	exact	expand
exponential equation	factorise	formula	fraction
greater than	inequality	LHS	less than
logarithm	logarithmic equation	lowest common multiple (LCM)	number line
quadratic equation	RHS	solution	solve
subject	surd	variable	

- 1 What type of equation has 2 as the highest power of x ? Write an example of this type of equation.
- 2 Write $\log_7 a = -3$ in index form.
- 3 What is the difference between an **equation** and an **inequality**?
- 4 Why is it possible for a **quadratic equation** to have more than one solution?
- 5 When checking the solution to an equation, you need to show that 'LHS = RHS'. What does that mean?
- 6 What does the inequality symbol ' $\leq$ ' mean?

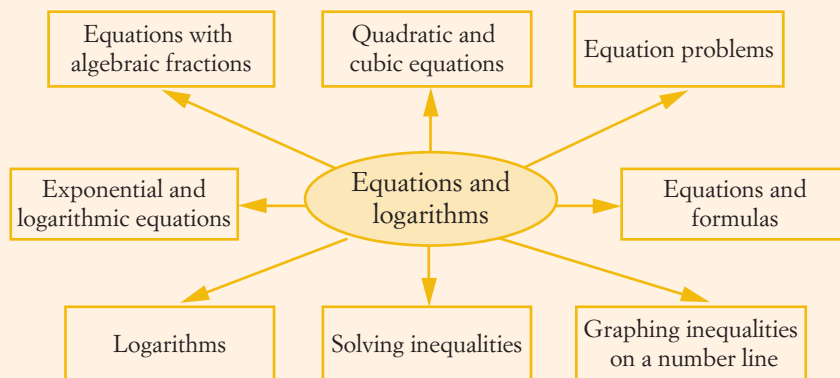
Topic overview

Quiz

Equations

MAT10NAQZ00011

Copy and complete this mind map of the topic, adding detail to its branches and using pictures, symbols and colour where needed. Ask your teacher to check your work.



1 Solve each equation.

See Exercise 7-01

a $\frac{3w+2}{5} = 4$

b $\frac{y}{5} = \frac{7}{4}$

c $\frac{2a+1}{2} = \frac{3a-1}{4}$

d $\frac{3m+5}{6} = \frac{10-m}{3}$

e $\frac{2s}{3} - \frac{s}{6} = 2$

f $\frac{x}{10} + \frac{x}{2} = 1$

2 Solve each equation.

Stage 5.3

See Exercise 7-01

a $\frac{m-1}{4} + \frac{m}{2} = 6$

b $\frac{y+1}{3} + \frac{y-1}{4} = \frac{1}{2}$

c $\frac{2m-1}{4} - \frac{m-4}{3} = \frac{4}{3}$

3 Solve each quadratic equation.

See Exercise 7-02

a $y^2 = 4$

b $p^2 - 100 = 0$

c $4x^2 = 36$

d $3m^2 - 3 = 0$

e $\frac{2w^2}{5} = 10$

f $x^2 + 8x + 7 = 0$

g $h^2 - 8h - 9 = 0$

h $u^2 + 4u - 77 = 0$

i $k^2 + 5k = 0$

4 Solve each cubic equation, correct to one decimal place.

Stage 5.3

See Exercise 7-03

a $u^3 - 7 = 0$

b $5m^3 - 125 = 0$

c $\frac{x^3}{2} = 1.5$

5 Grace is three years younger than her sister Jane. Twice the sum of their ages is 4 more than their father's age. If their father is 54, find the ages of Grace and Jane.

See Exercise 7-04

6 The braking distance (in metres) of a bicycle travelling at a speed of v metres/second is $d = \frac{v(v+1)}{2}$. Calculate the braking distance when the speed of the bicycle is 15 m/s.

See Exercise 7-05

7 Make a the subject of each formula.

See Exercise 7-06

a $y = ax + b$

b $P = \sqrt{\frac{a}{m}}$

c $M(1+a) = 1-a$

8 Graph each inequality on a number line.

See Exercise 7-07

a $x \geq 0$

b $x < 3$

c $x \leq -2$

d $x > -5$

9 Solve each inequality.

See Exercise 7-08

a $y - 6 \geq 10$

b $2y \leq -15$

c $3a + 10 > -5$

d $10 - 6x < 28$

e $\frac{a+2}{-4} > \frac{7}{2}$

f $\frac{3-5x}{2} \geq 9$

10 Write each expression in index form.

Stage 5.3

See Exercise 7-09

a $\log_6 216 = 3$

b $\log_2 \frac{1}{16} = -4$

c $\log_7 7\sqrt{7} = \frac{3}{2}$

11 Evaluate each expression.

See Exercise 7-10

a $\log_7 84 - \log_7 12$

b $\log_2 3 + \log_2 \frac{1}{3}$

c $2 \log_3 9 + \log_3 2 - \log_3 6$

12 If $\log_{10} 3 \approx 0.4771$, find the value of:

See Exercise 7-10

a $\log_{10} 9$

b $\log_{10} 300$

c $\log_{10} \frac{10}{3}$

d $\log_{10} \sqrt{90}$

13 Solve each exponential equation, correct to three decimal places.

See Exercise 7-11

a $5^x = 11$

b $2^x = 0.52$

c $3^{x+4} = 105$

d $16^{2-x} = 5$

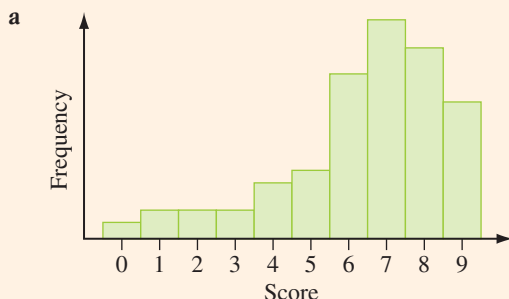
Mixed revision 2

See Exercise 6-01

1 For each statistical distribution:

i describe its shape

ii identify any outliers and clusters.



b

Stem	Leaf
3	0 5
4	3 3 3 3 5 8
5	2 5 8 9 9
6	3 7 8
7	0
8	
9	8

See Exercise 7-01

2 Solve each equation.

a $\frac{3b+1}{4} = \frac{2b-1}{3}$

b $\frac{2x}{3} - \frac{x}{6} = 5$

See Exercise 5-01

3 Simplify each expression.

a $(3nm^2)^4$

b $\frac{32a^5b^6}{8ab^3}$

c 3^{-2}

Stage 5.3

See Exercise 5-03

4 Simplify each expression.

a $\frac{a}{15} + \frac{a-4}{3}$

b $\frac{x+3}{3} - \frac{x-4}{8}$

c $\frac{6r-3}{5} - \frac{r}{2}$

See Exercise 6-02

5 The heart rates (in beats/second) of a random sample of shoppers were measured and recorded as follows.

84 85 84 87 92 95 83 79 88 90 87 112

a Find the range and interquartile range.

b Which is the better measure of spread for this set of data? Justify your answer.

See Exercise 7-02

6 Solve each quadratic equation, writing the solution in exact form.

a $\frac{x^2}{3} = 36$

b $n^2 = 64n$

c $u^2 + 11u - 60 = 0$

- 7 Simplify each expression.

See Exercise 5-04

a $\frac{21e}{10} \times \frac{15d}{7e}$

b $\frac{m}{8} \div \frac{m}{16}$

c $\frac{12p^2v}{5} \div \frac{3p}{10v}$

- 8 The number of properties sold per month by a real estate agency over a year were:

See Exercise 6-05

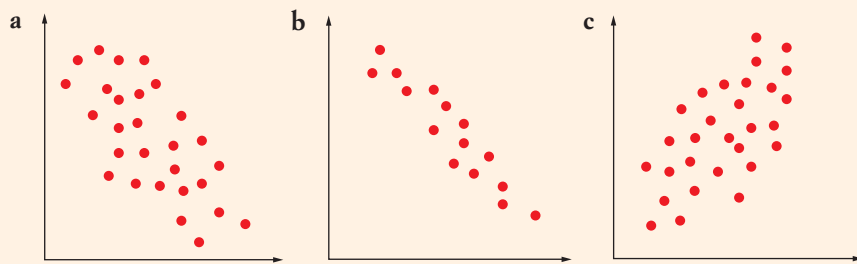
1 4 6 3 4 6 3 5 7 10 5 5

a Find the five-number summary for the data.

b Draw a box-and-whisker plot of the data.

- 9 Describe the strength and direction of the relationship between the variables shown in each scatter plot.

See Exercise 6-08



- 10 Evaluate each expression.

a $(-64)^{\frac{1}{3}}$

b $1024^{\frac{2}{5}}$

c $144^{-\frac{1}{2}}$

d $100^{-\frac{3}{2}}$

- 11 Solve $5y^3 + 1000 = 0$, correct to one decimal place.

- 12 Expand and simplify each expression.

a $9gh(6 - 5h) - gh$

b $y^2(4y + y^2) + 2y(3y^3 - y^2)$

- 13 The test results for a class of students are as follows.

Girls: 78 73 58 79 60 46 71 64 37 79 55 59 87 68 27

Boys: 70 39 80 73 61 68 84 60 73 60 54 90 69 63 59

a Find, correct to one decimal place, the mean and standard deviation of each group.

b Which group performed better in the exam? Give reasons.

- 14 Expand each binomial product.

a $(x + 7)^2$

b $(5m - 2)^2$

c $(3n - 10)(3n + 10)$

- 15 The sum of three consecutive numbers is 63. Find the three numbers by writing and solving an equation.

See Exercise 7-04

Stage 5.3

See Exercise 5-02

See Exercise 7-03

See Exercise 5-05

Stage 5.3

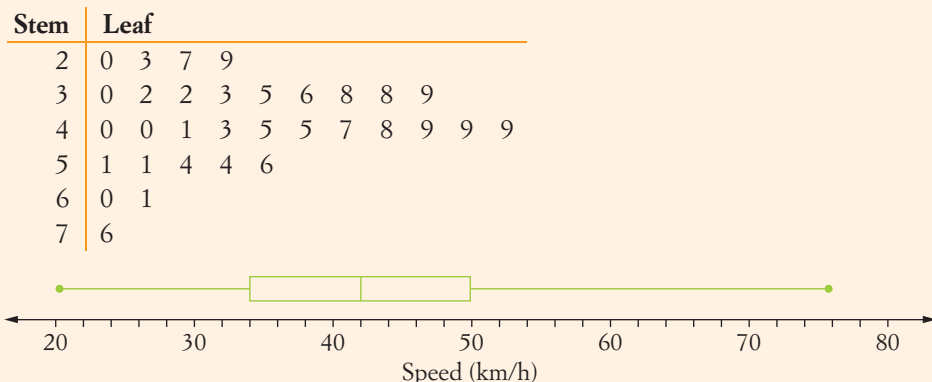
See Exercise 6-04

See Exercise 5-06

Mixed revision 2

See Exercise 6-07

- 16 The speeds of cars (in km/h) were monitored between 2:30 p.m. and 4:00 p.m. in a school zone. The results are shown in the stem-and-leaf plot and boxplot below.



- a Which plot gives the better indication of:
- skewness?
 - clustering and outliers?
- b For the data set, what is:
- the median?
 - the interquartile range?

See Exercise 5-08

- 17 Factorise each expression.

a $y^2 - 18y + 32$ b $n^2 + 2n - 48$ c $a^2 - a - 72$

See Exercise 6-08

- 18 Ten boxes containing 60 apples each were placed in cold storage for different periods. After each period of storage, a box was withdrawn and the number of good apples was counted.

Weeks in storage (W)	2	3	5	6	7	8	9	10	11	14
Number of good apples (N)	56	53	48	48	50	39	36	33	31	26

- a Which is the independent variable? Give reasons.
- b Graph this data on a scatter plot.
- c Describe the relationship between the weeks in storage and the number of good apples.
- d Describe the strength and direction of the relationship between the variables W and N .

Stage 5.3

See Exercise 5-09

- 19 Factorise each expression.

a $5n^2 + 7n + 2$ b $2a^2 - 7a - 15$ c $3x^2 + x - 10$

See Exercise 7-06

- 20 Make n the subject of the formula $t = \sqrt{\frac{n}{3}}$.

See Exercise 7-07

- 21 Graph each inequality on a number line.

a $x \geq 0$ b $x < 4$ c $x \leq 1$ d $x > -6$

See Exercise 7-08

- 22 Solve each inequality.

a $n - 5 \geq 0$ b $4k + 21 < -7$ c $\frac{2x}{3} \geq 1$

23 Simplify each expression.

a $\frac{d^2 - d - 12}{d^2 + 2d - 3}$

b $\frac{4a - 16}{3} \times \frac{12a}{a^2 - 4a}$

c $\frac{y^2 + 4y + 3}{y^2 - y} \div \frac{y^2 - 9}{3y - 3}$

24 The heights and shoe sizes of people were measured.

Weight, W (kg)	60	78	82	88	90	94	75	66	58	65
Height, H (cm)	142	168	170	184	180	188	170	158	150	146

a Graph the points on a scatter plot and construct a line of best fit.

b Find the equation of the line of best fit.

d Use the graph to extrapolate the weight of a person with a height of 130 cm.

e Use the graph to interpolate the height of a person who weighs 80 kg.

25 Simplify and evaluate each expression.

a $\log_9 108 - \log_9 12$

b $\log_6 5 + \log_6 \frac{1}{5}$

c $3 \log_5 10 - \log_5 8$

26 Solve $4^x = 120$, correct to three decimal places.

Stage 5.3

See Exercise 5-11

See Exercise 6-09

See Exercise 7-10

See Exercise 7-11



8

Number and algebra

Graphs

When an object is thrown upwards, its path is a curve called a parabola. The shape and length of the path will depend on the initial speed of the object. Furthermore, car headlights and satellite dishes use mirrors or reflectors that have the shape of a parabola.



Shutterstock.com/HSNphotography

Chapter outline

	Proficiency strands				
8-01 Direct proportion	U	F	PS	R	C
8-02 Inverse proportion	U	F	PS	R	C
8-03 Conversion graphs	U	F	PS	R	C
8-04 Distance-time graphs*	U	F	PS	R	C
8-05 Graphs of change*	U	F	PS	R	C
8-06 The parabola $y = ax^2 + c$	U	F		R	C
8-07 The parabola $y = a(x - r)^2$ *	U	F		R	C
8-08 The cubic curve $y = ax^3 + c$ *	U	F		R	C
8-09 The power curves $y = ax^n + c$ *	U	F		R	C
8-10 The hyperbola $y = \frac{k}{x}$ *	U	F		R	C
8-11 The exponential curve $y = a^x$	U	F		R	C
8-12 The circle $(x - h)^2 + (y - k)^2 = r^2$ *	U	F		R	C
8-13 Identifying graphs*		F		R	C

*STAGE 5.3

Wordbank

asymptote A line that a curve gets very close to but never touches, for example, the x -axis is an asymptote of the exponential curve

direct proportion A relationship between two variables of the form $y = kx$, where k is a constant, for example, if $y = 8.5x$, then y is directly proportional to x

exponential equation An equation involving a variable as a power, such as $y = 3^x$, whose graph is an exponential curve

hyperbola The graph of $y = \frac{k}{x}$, where k is a constant, which has two branches and two asymptotes

inverse proportion A relationship between two variables of the form $y = \frac{k}{x}$, where k is a constant, for example, if $y = \frac{50}{x}$, then y is inversely proportional to x

In this chapter you will:

- solve problems involving direct proportion and explore the relationship between graphs and equations corresponding to simple rate problems
- explore the connection between algebraic and graphical representations of relations such as simple quadratics, circles and exponentials using digital technology as appropriate
- (STAGE 5.3) describe, interpret and sketch parabolas, hyperbolas, circles and exponential functions and their transformations 
- solve problems involving inverse proportion
- read and interpret conversion graphs
- (STAGE 5.3) read and interpret distance-time (travel) graphs where the speed is variable
- (STAGE 5.3) read and interpret graphs of variables changing at different rates
- graph parabolas of the form $y = ax^2 + c$
- (STAGE 5.3) graph parabolas of the form $y = a(x - r)^2$ from the graph of $y = ax^2$
- (STAGE 5.3) graph cubic curves of the form $y = ax^3 + c$
- (STAGE 5.3) graph higher-power curves of the form $y = ax^n + c$ and $a(x - r)^n$
- (STAGE 5.3) graph hyperbolas of the form $y = \frac{k}{x}$
- graph exponential curves of the form $y = a^x$
- graph circles of the form $x^2 + y^2 = r^2$ and (STAGE 5.3) $(x - h)^2 + (y - k)^2 = r^2$
- match graphs to their equations

SkillCheck

Worksheet

StartUp assignment 7

MAT10NAWK10048

- If $A = 2x^2 - 3$, find A if:

a $x = 1$	b $x = 4$	c $x = 0$	d $x = -6$
- If $R = \frac{8}{u}$, find R if:

a $u = 2$	b $u = -16$	c $u = 5$	d $u = -2.5$
- If $y = 5^x$, find y if:

a $x = 4$	b $x = 5$	c $x = 0$	d $x = -2$

Technology worksheet

Direct proportion

MAT10NACT10004

8-01 Direct proportion

Two variables are **directly proportional** to each other if one variable is a constant multiple of the other; when one variable changes, the other one changes by the same factor.

Summary

If y is **directly proportional** to x , then $y = kx$, where k is a constant (number) called the **constant of proportionality** or **constant of variation**.

- A direct linear relationship exists between x and y
- If x increases (or decreases), y increases (or decreases)
- If x is doubled (or halved), y is doubled (or halved)
- Another way of saying 'y is directly proportional to x' is y **varies directly** with x'
- The graph of direct proportion is a straight line going through (0, 0) with gradient k

Example 1

The distance (d) in metres travelled by a car is directly proportional to the number of rotations (r) of its tyres. After 540 rotations, a distance of 950 m is travelled.

- What distance (correct to the nearest metre) will be travelled after 800 rotations?
- How many full rotations will be needed to cover 360 km?



Getty Images/John Borthwick

Solution

- d is directly proportional to r

$$\therefore d = kr$$

To find k , substitute the information given for r and d .

When $r = 540$, $d = 950$:

$$950 = k \times 540$$

$$k = \frac{950}{540}$$

$$= 1.759 \dots$$

Do not round the value of k .

$$\therefore d = 1.759 \dots \times r$$

When $r = 800$,

$$d = 1.759 \dots \times 800$$

$$= 1407.4074 \dots$$

$$\approx 1407 \text{ m}$$

After 800 rotations, the distance travelled will be 1407 m.

- When $d = 360 \text{ km} = 360\,000 \text{ m}$,

$$360\,000 = 1.759 \dots \times r$$

$$r = \frac{360\,000}{1.759 \dots}$$

$$= 204\,631.578 \dots$$

$$\approx 204\,631 \text{ rotations}$$

Rounding down for full rotations.

For a distance of 360 km, there will be 204 631 rotations.

Summary

To solve a direct proportion problem:

- 1 identify the two variables (say x and y) and form a proportion equation, $y = kx$
- 2 substitute values for x and y to find k , the constant of proportionality
- 3 rewrite $y = kx$ using the value of k
- 4 substitute a value for x or y into $y = kx$ to solve the problem.

Exercise 8-01 Direct proportion

See Example 1

- 1 The distance, D , travelled by Craig, a marathon runner, varies directly with time, T .

Time, T (min)	Distance, D (m)
1	190
2	380
3	570

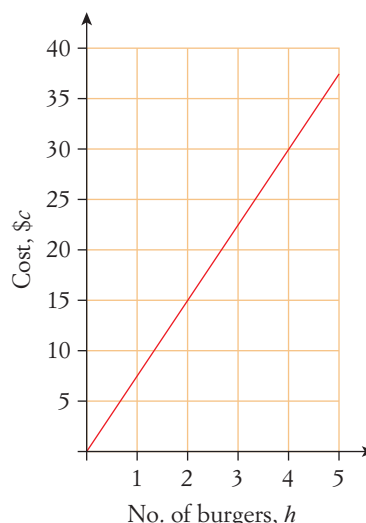
- a Write a variation equation for D .
 - b How far in kilometres will Craig run in:
 - i 20 minutes?
 - ii 45 minutes?
 - c How long would it take Craig to run 12.35 kilometres? Answer in hours and minutes.
- 2 Mehta's earnings for working a shift at the local nursery are directly proportional to the number of hours she works. Yesterday, she earned \$222.70 for working an 8.5 hour shift.
- a If Mehta's earnings are represented by E and the number of hours worked is represented by h , write an equation for E .
 - b How much will she earn for working a 7-hour shift?
 - c How many hours did she work today if she earned \$144.10 for the shift?
- 3 The amount of interest, I , earned for one year on an investment account varies directly with the size of the deposit, D .
- a If Caterina earns \$16 interest on an investment of \$425, find the variation equation for I .
 - b Hence, how much will she earn on an investment of \$900?
 - c If Caterina's uncle doubles the size of her investment in **b**, how much will she earn in interest?
- 4 S varies directly with t . If when $t = 14$, $S = 106.4$, what is the value of S when $t = 0.3$?
 Select the correct answer **A**, **B**, **C** or **D**.
- A** 2.28 **B** 27.72 **C** 36.12 **D** 446.88
- 5 Find the linear formula for b in terms of a for this table of values.

a	4	8	12	16	20
b	10	20	30	40	50

- 6 The graph on the right shows that the cost of hamburgers purchased from the local takeaway store depends directly on the number of burgers purchased.

- a Copy the table below and use the graph above to complete it.

No. of burgers, h	Cost, c (\$)
1	
2	
3	



- b Find the variation equation to represent the relationship between the cost (\$ c) and the number of burgers (h).
- c If Kim buys 6 hamburgers, what is the total cost of the hamburgers?
- d The total cost of one order of hamburgers is \$82.50. How many hamburgers were ordered?
- e Find the gradient of the line. How is it related to the constant of variation?
- 7 K varies directly with L . If $L = 9.5$ when $K = 1045$, what is the value of K when $L = 1.65$?
Select **A**, **B**, **C** or **D**.
A 0.015 **B** 93.7 **C** 181.5 **D** 1708.575
- 8 A linear relationship exists between the mass of a car (m kg) and its fuel consumption rate (F L/100 km).
a Find the variation equation for F if a 1000 kg car uses fuel at a rate of 6 L/100 km.
b Find the fuel consumption of a 2500 kg car.
- 9 For an object that is cooling, the drop in temperature varies directly with time. If the temperature drops 8°C in 5 minutes, how long would it take to drop 10°C ? Select the correct answer **A**, **B**, **C** or **D**.
A 6.25 min **B** 7 min **C** 12.8 min **D** 16 min
- 10 The weight of an astronaut on Mars is proportional to his weight on Earth. A 72 kg astronaut weighs 27.4 kg on Mars.
a Calculate how much a 60 kg astronaut weighs on Mars, correct to 1 decimal place.
b If an astronaut weighs 32 kg on Mars, calculate his weight on Earth, correct to 1 decimal place.

8-02 Inverse proportion

Two variables are **inversely proportional** to each other if, when one variable increases, the other one decreases by the same factor.

The table below shows the different speeds of a car (s km/h), and the time it takes to travel 100 km (t min). As the speed increases, the time taken decreases.

Speed (s km/h)	Time (t min)
50	120
60	100
80	75
100	60

NSW

Worksheet

Direct and inverse proportion

MAT10NAWK10049

Summary

If y is **inversely proportional** to x , then $y = \frac{k}{x}$, where k is a constant (number) called the **constant of proportionality** or **constant of variation**.

- If x increases, y decreases ('inverse' means 'opposite')
- If x decreases, y increases
- If x is doubled, y is halved
- If x is halved, y is doubled
- Another way of saying 'y is inversely proportional to x' is 'y **varies inversely** with x'

Example 2

The time (t) in minutes taken by a car to travel 100 km is inversely proportional to the speed (s km/h) of the car, as shown in the table on the previous page. At 50 km/h, the time taken is 120 minutes.

- Find the inverse variation equation for t .
- How long did the car take to travel 100 km at:
 - 40 km/h?
 - 110 km/h?
- Find the car's speed if it took 45 minutes to travel 100 km.

Solution

- t is inversely proportional to s .

$$\therefore t = \frac{k}{s}$$

To find k , substitute the information given for s and t .

When $s = 50$, $t = 120$:

$$120 = \frac{k}{50}$$

$$k = 120 \times 50$$

$$= 6000$$

$$\therefore t = \frac{6000}{s}$$

- When $s = 40$, $t = \frac{6000}{40} = 150$ min

At 40 km/h, the trip takes 150 min (or 2 h 30 min).

- When $s = 110$,

$$t = \frac{6000}{110}$$

$$= 54.5454 \dots$$

$$\approx 55 \text{ min}$$

At 110 km/h, the trip takes 55 min.

c When $t = 45$,

$$45 = \frac{6000}{s}$$

$$45s = 6000$$

$$s = \frac{6000}{45}$$

$$= 133\frac{1}{3} \text{ km/h}$$

For a travel time of 45 min, the speed must be $133\frac{1}{3}$ km/h.

Summary

To solve inverse proportion problem:

- 1 Identify the two variables (say x and y) and form a proportion equation, $y = \frac{k}{x}$
- 2 Substitute values for x and y to find k , the constant of proportionality
- 3 Rewrite $y = \frac{k}{x}$ using the value of k
- 4 Substitute a value for x or y into $y = \frac{k}{x}$ to solve the problem.

Example 3

The temperature, T (in degrees Celsius), of the air is inversely proportional to the height, h (in metres), above sea level. At 600 m above sea level, the temperature is 8°C .

- a What is the temperature at 1000 m above sea level?
- b Graph the relationship between temperature and height above sea level.

Solution

- a T is inversely proportional to h .

$$T = \frac{k}{h}$$

Substitute $h = 600$ and $T = 8$ to find k .

$$8 = \frac{k}{600}$$

$$k = 8 \times 600$$

$$= 4800$$

$$\therefore T = \frac{4800}{h}$$

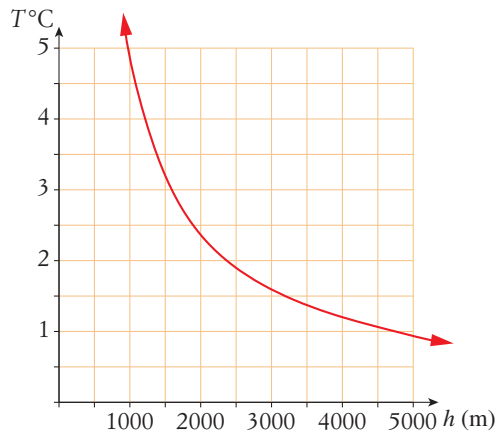
$$\text{When } h = 1000, T = \frac{4800}{1000} = 4.8^\circ\text{C}$$

The temperature at a height of 1000 metres above sea level is 4.8°C .

- b Draw a table of values for $T = \frac{4800}{h}$

h	1000	2000	3000	4000	5000
T	4.8	2.4	1.6	1.2	0.96

Note that as h increases,
 T decreases.



Exercise 8-02 Inverse proportion

See Example 2

- The time taken, T hours, to travel from Sydney to Melbourne varies inversely with the speed, s km/h.
 - If it takes 11.5 hours at an average speed of 80 km/h, find the variation equation for T .
 - If the average speed is increased to 90 km/h, how long will the journey take? Answer in hours and minutes.
 - Find the average speed needed to complete the trip in 10 hours.
- The rate of vibration of a string varies inversely as its length. A string that is 8 cm long vibrates at 9375 Hz (hertz). What length of string will vibrate at 6250 Hz? Select the correct answer **A**, **B**, **C** or **D**.
A 5 cm **B** 7 cm **C** 12 cm **D** 73 cm



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See Example 3

- The temperature, T (in degrees Celsius), of the air varies inversely with the height, h (in metres), above sea level. At 150 m above sea level, the temperature is 30°C .
 - What is the temperature at:
 - 300 m above sea level?
 - 2500 m above sea level?
 - What is the height above sea level when the temperature is:
 - 8°C ?
 - 22.5°C ?
 - Graph the relationship between temperature and height above sea level. Use T on the vertical axis and h on the horizontal axis with $h = 0, 500, 1000, 1500, \dots, 3000$.
- The number of people who attend a concert varies inversely with the amount of space allocated to each person. If 80 cm^2 is allowed per person, the ground can hold 3400 people. How many people could attend the concert if only 60 cm^2 was allocated per person?

- 5 Which equation represents the table of values shown below? Select **A**, **B**, **C** or **D**.

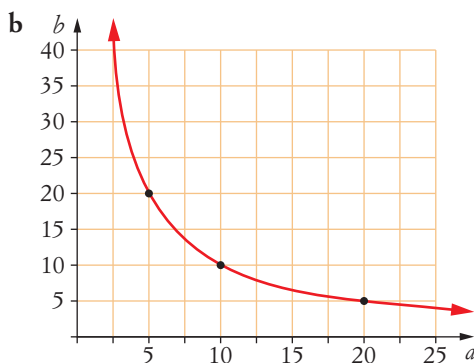
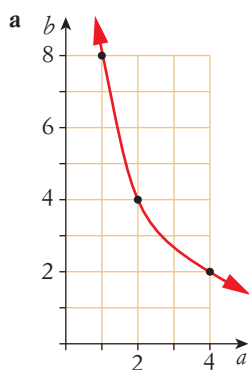
x	2	5	8	10
y	2.5	1	0.625	0.5

A $y = \frac{10}{x}$ **B** $y = \frac{5}{x}$ **C** $y = \frac{2.5}{x}$ **D** $y = \frac{1}{x}$

- 6 K is inversely proportional to L . If $L = 2$ when $K = 7$, find K when $L = 15$.
- 7 Paul believes that at a train station, the number of people waiting on the platform is inversely proportional to the time until the next train arrives. According to his model, when there are 16 people waiting, the train will arrive in 2.5 minutes.



- a** When will the train arrive if there are 5 people waiting?
- b** How many people are waiting at the station 10 minutes before the train arrives?
- 8 Each graph below shows an inverse relationship between a and b . Find each variation equation.



- 9 The frequency, F beats per second, that a bird beats its wings varies inversely as the length, L cm, of its wings. A bird with wings of length 14 cm beats them at a frequency of 8 beats per second.
- a** Find the variation equation for F in terms of L .
- b** Calculate, to the nearest whole number, the wingbeat frequency for wings of length 18 cm.
- c** A bird beats its wings with a frequency of 4.5 beats per second. What is the length of its wings, correct to the nearest centimetre?
- 10 For a certain equation, y varies inversely with x .
- a** Given $x = 0.2$ when $y = 10$, find y when $x = 32$. **b** Find x when $y = 1.6$.
- 11 The amount of time it takes Sarah to move house is inversely proportional to the number of friends she has to help her. When she has 4 friends helping, the job takes $3\frac{3}{4}$ hours.
- a** How long will it take if she has 6 friends helping?
- b** How many friends must she have to help her to move house in 3 hours?

NSW

8-03 Conversion graphs

Worksheet

Currency conversion
graph

MAT10NAWK10050

Example 4

Exchange rates change daily but suppose that the exchange rate between the Australian dollar and the UK pound sterling is $\$A1 = \pounds 0.653$, then $\$A100 = \pounds 65.30$ sterling. These values are used to draw this conversion graph.

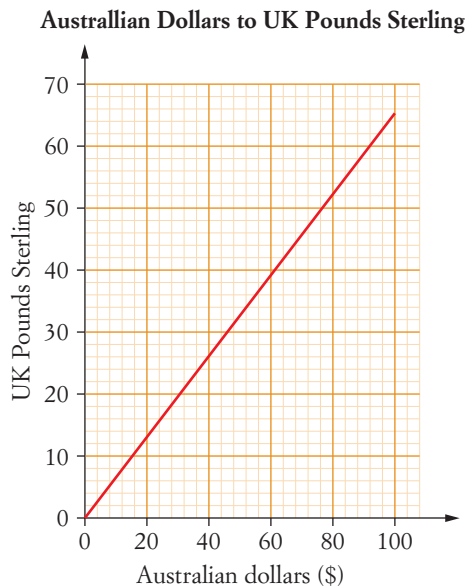
Use the graph to convert:

- a \$A50 to pounds
- b £10 to Australian dollars.

Solution

Reading from the graph:

- a $\$A50 \approx \pounds 33$
- b $\pounds 10 \approx \$A15$



Exercise 8-03 Conversion graphs

See Example 4

- 1 Use the graph in Example 4 to answer the following questions.
 - a Convert to pounds:
 - i \$A40 ii \$A88
 - b Convert to Australian dollars:
 - i £18 ii £60
 - c In June 2008, $\$A1 = \pounds 0.49$.
 - i How much **less** was \$A40 worth in UK pounds sterling in 2008 than it is using this more recent conversion graph?
 - ii How much **more money** was £60 worth in Australian dollars for visiting tourists, than it is using this more recent conversion graph?

- 2 The furlong is an imperial measure once used to measure length. This conversion graph shows distances in furlongs converted to kilometres.

a Convert to kilometres:

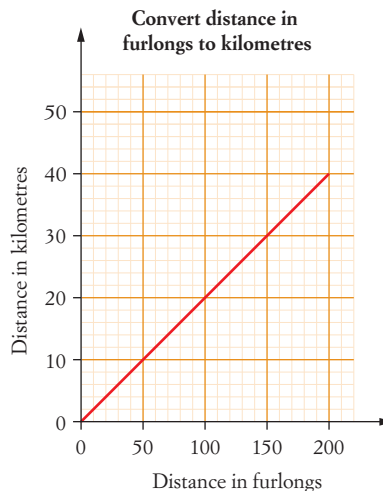
- i 10 furlongs
- ii 100 furlongs
- iii 170 furlongs

b Convert to furlongs:

- i 10 km
- ii 25 km
- iii 36 km

c Use an answer from part **a** to convert 300 furlongs to kilometres.

d Use an answer from part **b** to convert 100 kilometres to furlongs.



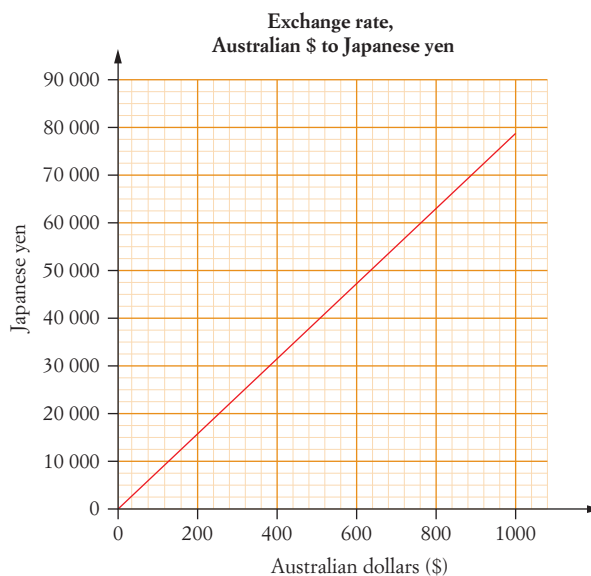
- 3 The graph on the right shows the exchange rate to convert Australian dollars to Japanese yen (¥).

a Convert to Japanese yen:

- i \$A200
- ii \$A800
- iii \$A1000

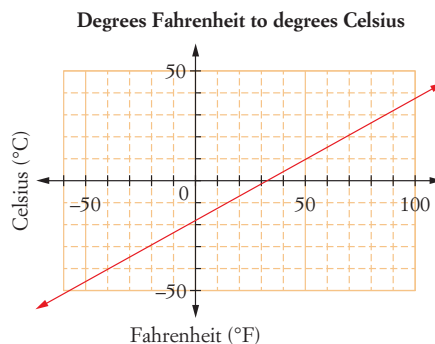
b Convert to Australian dollars:

- i ¥20 000
- ii ¥60 000
- iii ¥72 000

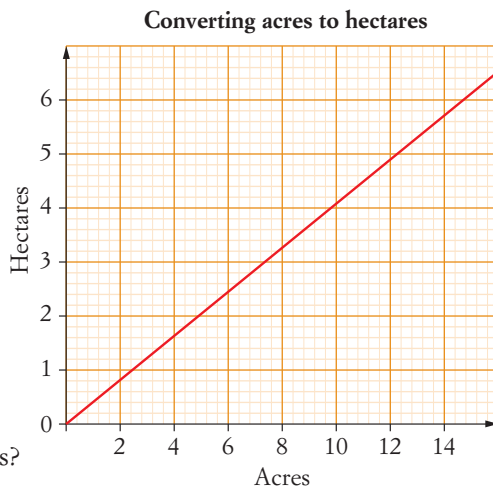


- 4 The graph on the right shows the temperature conversion from degrees Fahrenheit to degrees Celsius. Convert:

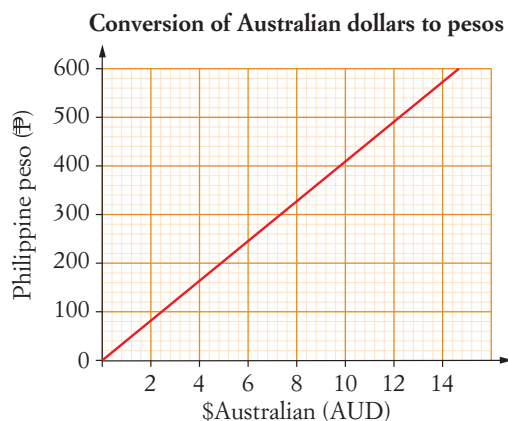
- a** 0°F to $^{\circ}\text{C}$
- b** 50°F to $^{\circ}\text{C}$
- c** 80°F to $^{\circ}\text{C}$
- d** 0°C to $^{\circ}\text{F}$
- e** -10°C to $^{\circ}\text{F}$
- f** 30°C to $^{\circ}\text{F}$



- 5 This conversion graph is used to convert acres to hectares. The acre is an Imperial measure of land area while the hectare (ha) is the metric measure.
- Use the graph to convert 12 acres to hectares.
 - A garden has an area of 5 acres. What is this area in hectares?
 - Use the graph to convert 4.4 hectares to acres.
 - Mr Ferguson has a property with an area of 5 hectares. How big is this in acres?
 - A rectangular playing field measures 250 m by 128 m.
 - What is the area of the field in square metres?
 - What is the area of the field in hectares?
 - What is the area of the field in acres?



- 6 This graph is used to convert Australian dollars (AUD) to Philippine pesos (₱).
- Change into Philippine pesos.
 - \$15
 - \$50
 - \$88
 - Change ₱500 to AUD.
 - How many Australian dollars would you receive for ₱200?
 - Calculate the number of Philippine pesos you should get for \$120.



Stage 5.3

NSW

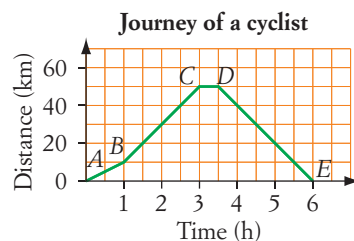
8-04 Distance–time graphs

The distance travelled by a moving object can be shown on a **distance–time graph**, also called a **travel graph**.

Example 5

This distance–time graph shows the journey of a cyclist on a training ride.

- What was the total distance covered?
- The cyclist's speed increases at *B*, after the first hour. How is this shown by the graph?
- Calculate the speed of the cyclist from:
 - B* to *C*
 - D* to *E*
 What do you notice?
- When was the cyclist stationary?



Solution

a The cyclist travelled 50 km, then returned to the starting point, so the total distance covered was 100 km.

b The graph becomes steeper after B.

c $\text{Speed} = \frac{\text{distance travelled}}{\text{time taken}}$

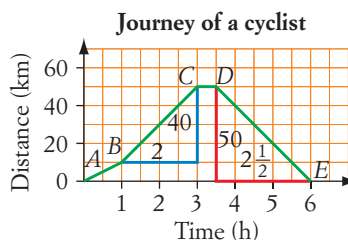
$$\begin{aligned} \text{i Speed from } B \text{ to } C &= \frac{40}{2} \\ &= 20 \text{ km/h} \end{aligned}$$

$$\begin{aligned} \text{ii Speed from } D \text{ to } E &= \frac{50}{2\frac{1}{2}} \\ &= 20 \text{ km/h} \end{aligned}$$

The speeds were the same.

d The cyclist was stationary (not moving) where the travel graph was flat, that is, between 3 and 4 hours (CD on the graph).

or the gradient of the interval on the graph



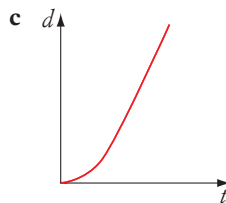
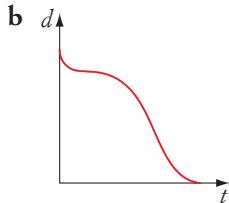
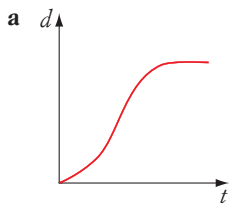
- The gradient of the line shows the speed of the object.
- The steeper the graph, the greater the speed.
- If the gradient of the line is positive, the object is moving away from a fixed point.
- If the gradient of the line is negative, the object is moving back towards the fixed point.

Distance–time graphs with variable speed

Example 5 showed a distance–time graph with straight lines, indicating that the speed of the object did not change much. However real-life situations are more complex and involve variable speed or a rate of change that is **not** constant. In these cases, the graph will be curved.

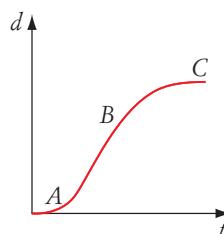
Example 6

Describe the journey represented by each distance–time graph.



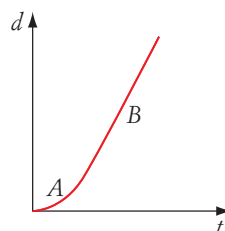
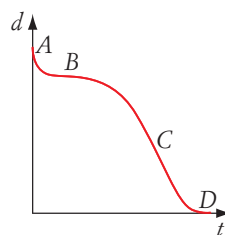
Solution

a The person starts the journey slowly (at A, the graph is not very steep), then increases his speed to a maximum (at B, the graph is the steepest), then slows down and stops (at C, graph becomes horizontal).



Stage 5.3

- b** The person going home starts the journey at high speed (at A , the graph is very steep) before slowing down and stopping briefly (at B , the graph is flat). The person then speeds up again (at C , the graph becomes very steep) before slowing down again and arriving home (at D).
- c** The person starts the journey slowly, then speeds up (at A) before reaching a constant speed (at B , the graph is a straight line) and continuing on the journey.



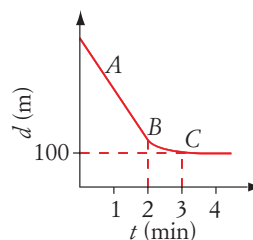
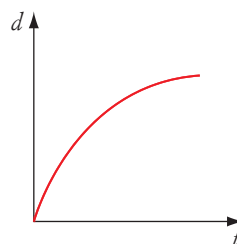
Example 7

Draw distance–time graphs to represent each situation described.

- a** Kobi was running quickly but then his speed decreased until he stopped.
- b** Susanna was running home at a constant speed for 2 minutes, before slowing down and stopping 100 m from home after a further 1 minute.

Solution

- a** Kobi's speed is gradually decreasing, so the steepness (or gradient) of the graph must be steep at first, then decreasing.
- b** The graph should be a decreasing straight line (at A) for 2 minutes, indicating a constant speed. Then for one minute (at BC), the gradient of the graph should gradually decrease, until becoming flat (at C) to show when Susanna stops, 100 m from home.

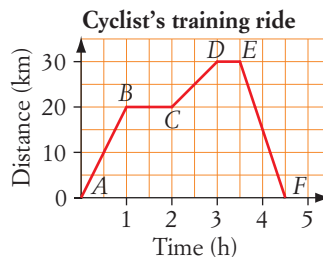


Exercise 8-04 Distance–time graphs

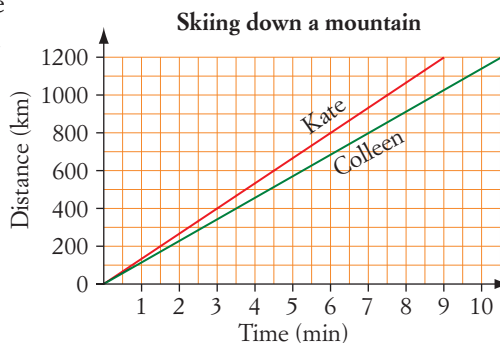
Stage 5.3

See Example 5

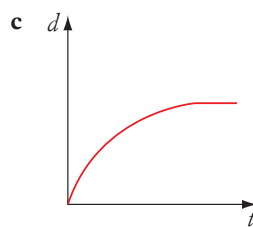
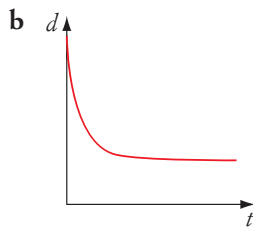
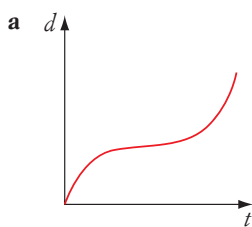
- 1 This graph shows a cyclist's journey on a training ride.
 - a Describe the journey of the cyclist, giving the speeds at each stage.
 - b Do any intervals of the graph indicate that the cyclist is travelling at the same speed? Give reasons.
 - c The gradient of the interval EF is -30 but the speed at this stage is 30 km/h . What does the negative gradient indicate?



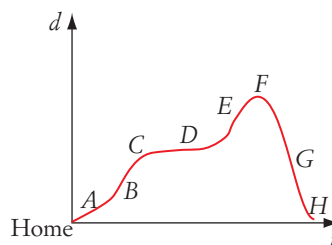
- 2 Kate and Colleen are downhill skiers. Here is the distance–time graph for their 1200 m trip down the mountain.



- a What was the speed of each skier?
 - b Who reached the base of the mountain first and how many minutes did it take her?
 - c How many minutes later did the second skier arrive?
 - d How far had Colleen gone after 6 minutes?
 - e How far *ahead* was Kate after 6 minutes?
 - f If Colleen and Kate were skiing *down* the mountain, why do the graphs go *up*?
- 3 Describe the journey represented by each distance–time graph.



- 4 Match each statement to the correct part of the distance–time graph.
 - a the person slows down because he is almost home
 - b the person is not moving
 - c after moving at a constant speed, the person increases his speed
 - d the person changes direction for the return trip
 - e after stopping, the person increases his speed
 - f the person gradually slows down



See Example 6

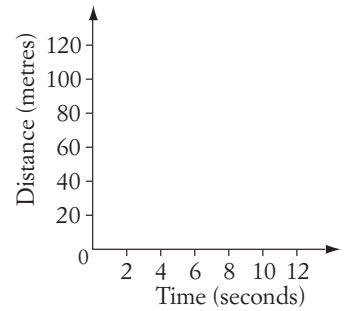
Stage 5.3

See Example 7

- 5 Copy and complete this distance–time graph, using the following information about Damien completing a 100 m race.

- Damien covered 5 m at 2 seconds and 20 m at 4 seconds as his running speed increased
- Damien then ran at a constant speed until reaching the 80 m mark at 9 seconds
- Damien slowed down over the last 20 m, completing the 100 m race in 11 seconds
- Damien finally stopped after running a further 20 m

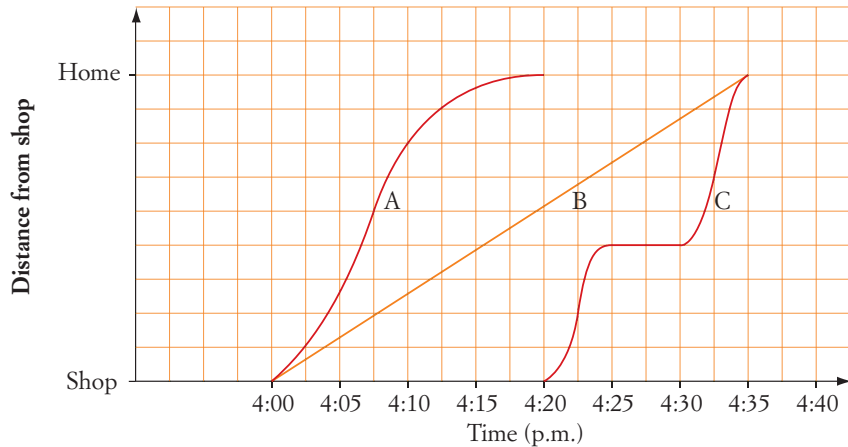
Damien's race



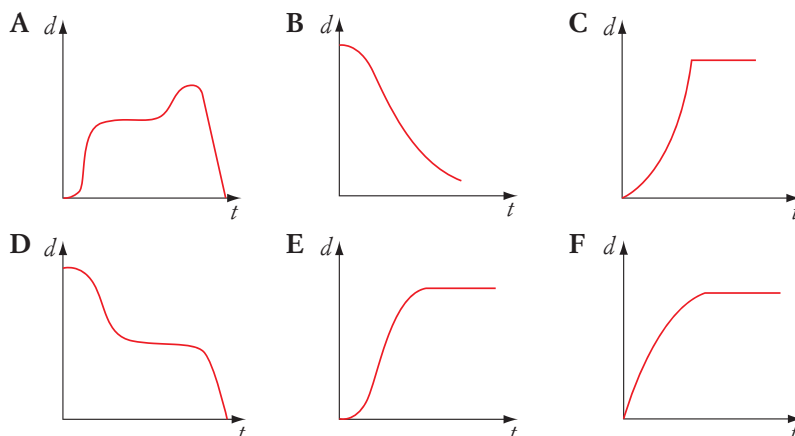
- 6 Sketch a distance–time graph for each situation described.
- Starting at home and running away from home at a decreasing speed
 - Starting at home and running away from home at an increasing speed
 - Starting away from home and running home at an increasing speed
 - Starting away from home and running home at a decreasing speed

- 7 Three stories that can be matched to the graphs below are:

- I Jade rode her bicycle home II Cameron walked home III Kiet ran home



- Match each story to its correct graph.
 - Explain in words why you made each match.
 - What could have caused the level section at about 4:25 p.m. in graph C?
 - Describe how the speed changes in graph A.
- 8 Match each description below to its correct graph.
- the speed increases at an increasing rate and then stops
 - the speed increases slowly, slows down to a stop, then increases to a constant rate
 - the speed increases at an increasing rate, slows down and stops
 - the speed starts high, decreases, then stops
 - the speed increases to a maximum, then slows down
 - the speed increases, then slows down and stops, then begins to increase again before stopping for an instant before returning to start at a constant speed



Mental skills 8

Maths without calculators

Divisibility tests

A number is divisible by:	if:
2	its last digit is 2, 4, 6, 8 or 0
3	the sum of its digits is divisible by 3
4	its last two digits form a number divisible by 4
5	its last digit is 0 or 5
6	it is even and the sum of its digits is divisible by 3
9	the sum of its digits is divisible by 9
10	its last digit is 0

1 Study each example.

a Test whether 748 is divisible by 2, 3 or 4.

- Last digit is 8 (even), $\therefore$ 748 is divisible by 2
- Sum of digits = $7 + 4 + 8 = 19$, which is not divisible by 3, $\therefore$ 748 is not divisible by 3
- 48 is divisible by 4, $\therefore$ 748 is divisible by 4 ($748 \div 4 = 187$)

b Test whether 261 is divisible by 5 or 9.

- Last digit is 1, not 0 or 5, $\therefore$ 261 is not divisible by 5
- $2 + 6 + 1 = 9$, which is divisible by 9, $\therefore$ 261 is divisible by 9. ($261 \div 9 = 29$).

c Test whether 570 is divisible by 4, 6 or 10.

- 70 is not divisible by 4, $\therefore$ 570 is not divisible by 4
- 570 is even and $5 + 7 + 0 = 12$, which is divisible by 3, $\therefore$ 570 is divisible by 6 ($570 \div 6 = 95$)
- Last digit is 0, $\therefore$ 570 is divisible by 10 ($570 \div 10 = 57$)

2 Test whether each number is divisible by 2, 3, 5 or 6.

a 250 **b** 189 **c** 78 **d** 465 **e** 1024 **f** 840 **g** 715 **h** 627

3 Test whether each number is divisible by 4, 9 or 10.

a 144 **b** 280 **c** 522 **d** 4170 **e** 936 **f** 726 **g** 342 **h** 5580

Stage 5.3

NSW

Worksheet

Graphs of change

MAT10NAWK10213

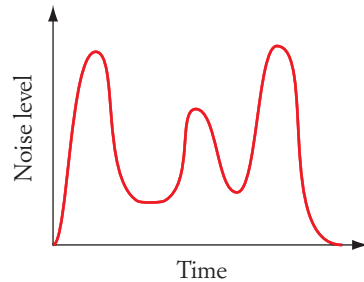
8-05 Graphs of change

Example 8

This graph shows the noise level of a classroom during a lesson. Describe what may have happened in the classroom during the lesson.

Solution

- The noise level increased as the students entered the classroom.
- The noise level became low as the students settled down to work.
- There may have been a classroom discussion (the noise level increased) and then the class settled down again.
- The noise level increased towards the end of the lesson as students became restless.
- The period ended and the noise level decreased after the students left the classroom.

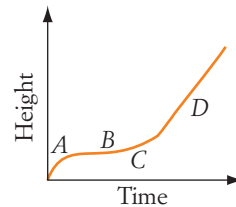
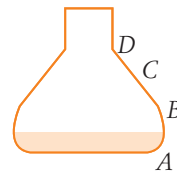


Example 9

Water is poured into the container shown at a constant rate. Draw a graph to show the height of water rising in the container over time.

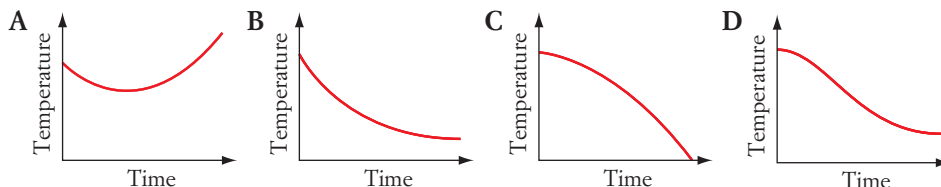
Solution

- The container is wider at B than at A , so the rate of increase in the water level will slow down.
- As the container narrows at C , the water level will increase faster (the graph becomes steeper).
- The container is cylindrical at D , so the water level will rise at a constant rate (the graph is a straight line).



Example 10

A bowl of hot soup sits on the kitchen bench cooling. At first it loses heat quickly but, as time passes, it loses heat more slowly until it is at room temperature. Which of the graphs below best illustrates this?



Solution

The correct graph is **B**. The temperature decreases rapidly initially (steep graph), but then this rate of decrease slows (graph becoming flatter, the temperature decreases at a decreasing rate).

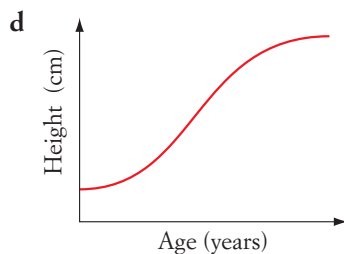
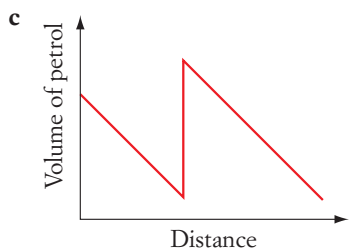
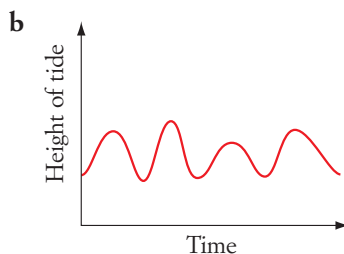
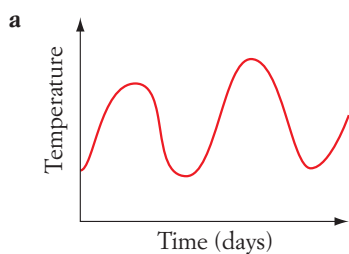
A is incorrect, because it shows the temperature decreasing, then increasing.

C is incorrect because it shows the temperature decreasing slowly initially (flat graph), then decreasing quickly to zero (steep graph, decreasing at an increasing rate).

D is incorrect because it shows the temperature decreasing slowly initially (flat graph), then quickly, then slowly again.

Exercise 8-05 Graphs of change

- 1 Describe what may be happening as represented in each graph.

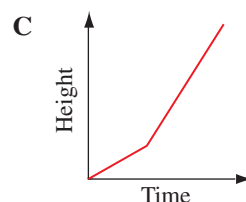
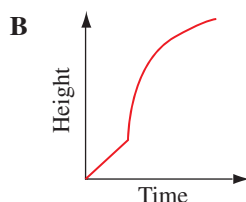
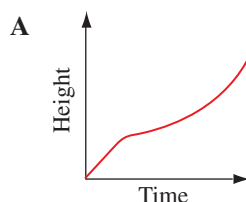
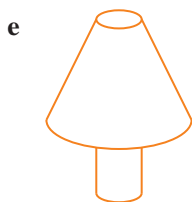
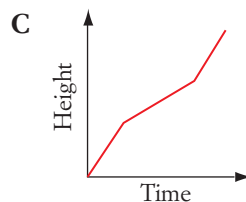
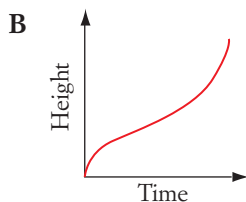
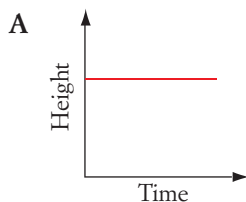
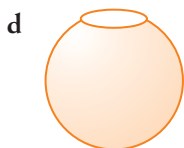
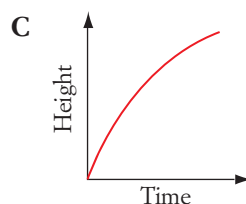
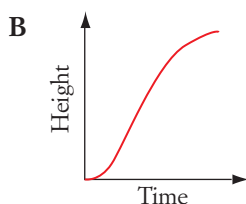
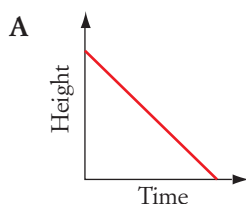
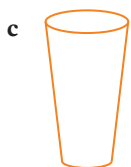
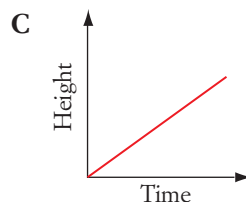
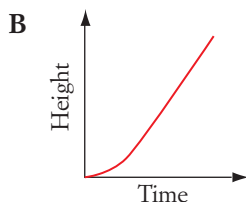
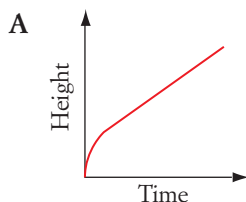
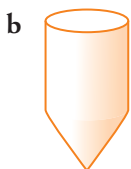
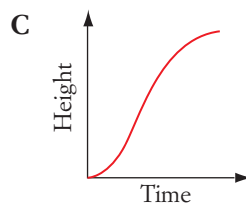
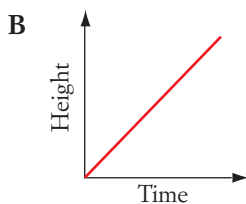
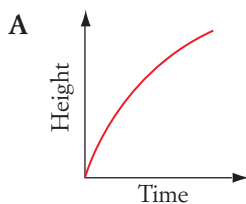
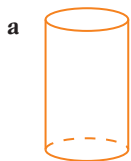


See Example 8

Stage 5.3

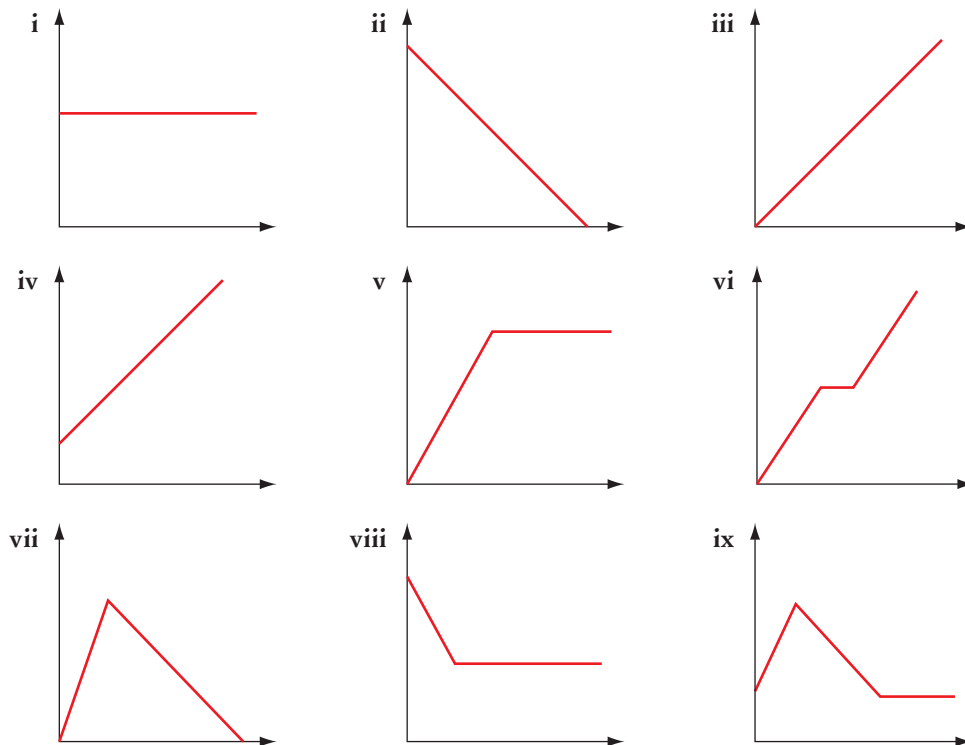
See Example 9

- 2 For each container, select the graph that best describes the height of the water as it is poured at a constant rate into the given container.

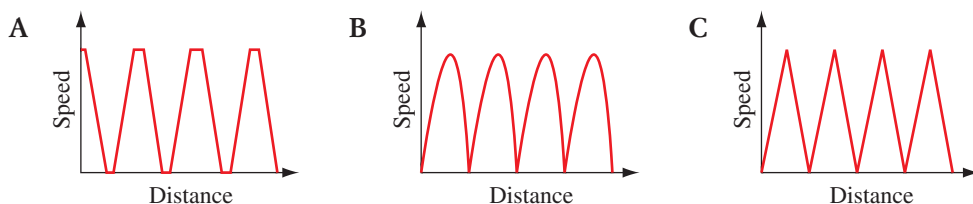


- 3 Draw a graph that models each situation described. Use the variables given in brackets.
- The water level in a pool with people swimming in it on a hot summer day. (height–time)
 - The height of a girl above the ground as she climbs up the ladder of a slide, sits and then slides down. (height–time)
 - The distance from a shop of a person walking at a steady speed up and down the street past that shop. (distance–time)

- 4 Match each story to one of the graphs below. (The variables are given in brackets.)
- a People were purchasing from the drink-vending machine until it broke down. (number of cans in machine, time)
 - b The tank was half full of water all day. (depth of water, time)
 - c The cost of a mobile phone call is charged at a constant rate. (rate, time)
 - d Jo was watching a movie DVD but stopped to take a phone call halfway through. (amount of movie seen, time)
 - e Interest rates rise rapidly at a constant rate, then steadily fall at a constant rate and stay at a constant low. (rate, time)
 - f A taxi fare includes the hire charge plus a constant amount per kilometre. (cost, kilometres travelled)
 - g The car is consuming petrol at a steady rate until it runs out of fuel. (litres in tank, time)
 - h Dean jogs at a steady rate, then stops and rests. (distance travelled, time)
 - i Jordan jogs at a steady rate, then walks home. (distance travelled, time)

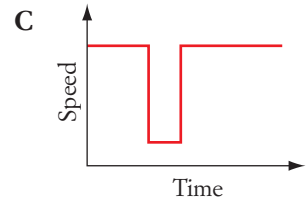
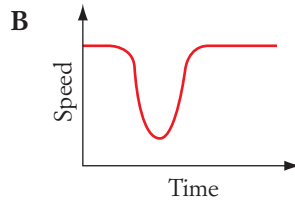
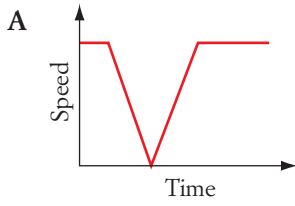


- 5 For each description, select the graph that best describes the situation. Select the correct answer **A**, **B** or **C**.
- a the speed of a bus that stops three times

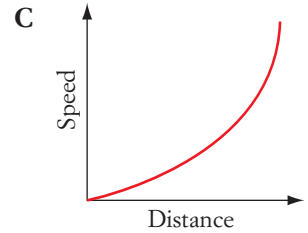
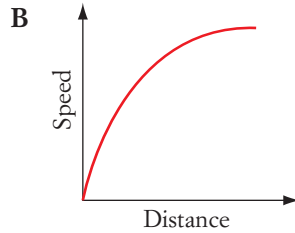
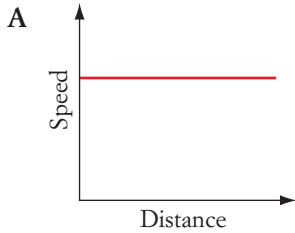


Stage 5.3

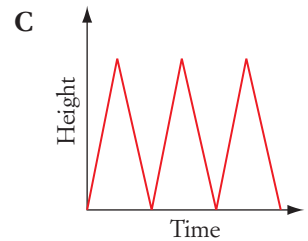
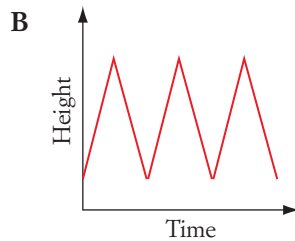
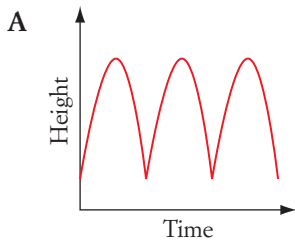
b the speed of a car as it goes around a sharp corner



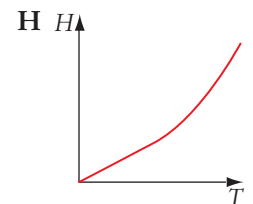
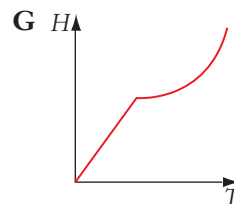
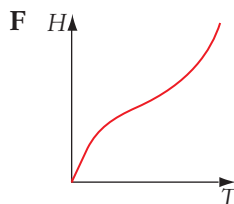
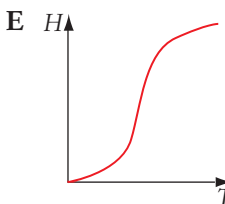
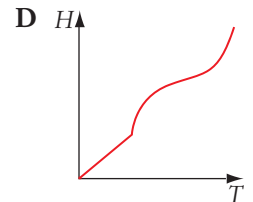
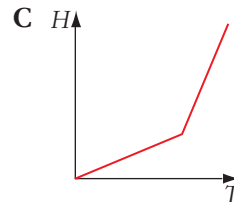
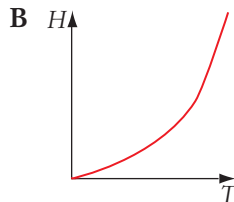
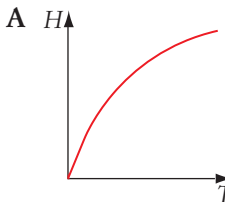
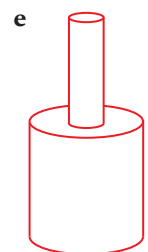
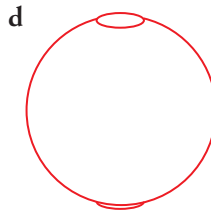
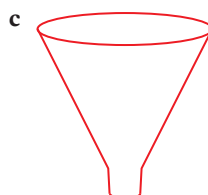
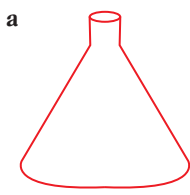
c the speed of a runner going up a steep hill from a standing start



d the height reached by a person jumping on a trampoline

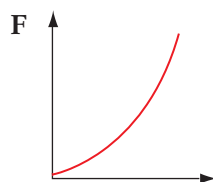
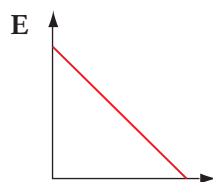
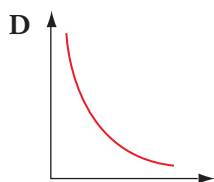
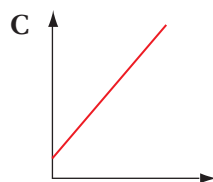
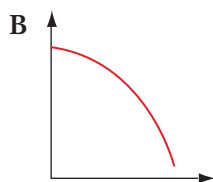
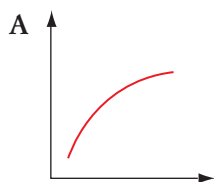


6 The five containers below are filled at a constant rate. Match a graph to each container, showing the water height against time.



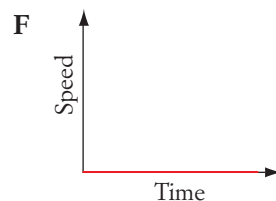
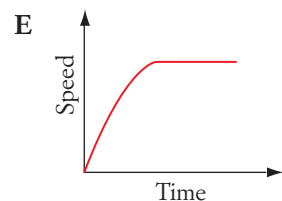
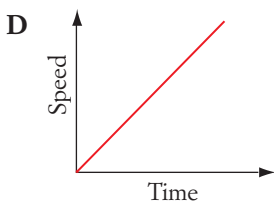
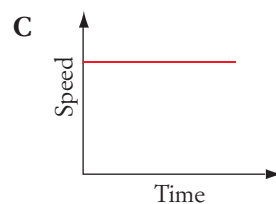
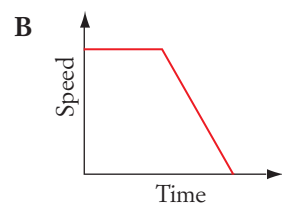
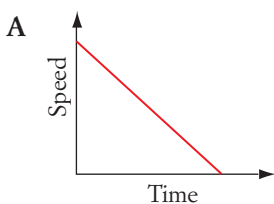
7 The graphs below describe the temperature change of an object over time. Match each description to its correct graph.

- a The temperature increases at a constant rate.
- b The temperature decreases at a constant rate.
- c The temperature increases at an increasing rate.
- d The temperature decreases at a decreasing rate.
- e The temperature increases at a decreasing rate.
- f The temperature decreases at an increasing rate.



8 Select the speed–time graph that best describes:

- a a car accelerating until it reaches a constant speed
- b a car stopped at traffic lights
- c a bus travelling at a constant speed before stopping for passengers
- d a car travelling at a constant speed
- e a train slowing down at a constant rate until it stops
- f a rocket launched into space



Worksheet

Graphing parabolas

MAT10NAWK10051

Technology worksheet

Graphing non-linear equations

MAT10NACT10005

Technology worksheet

Excel worksheet:
Investigating
parabolas 1

MAT10NACT00010

Technology worksheet

Excel spreadsheet:
Investigating
parabolas 1

MAT10NACT00040

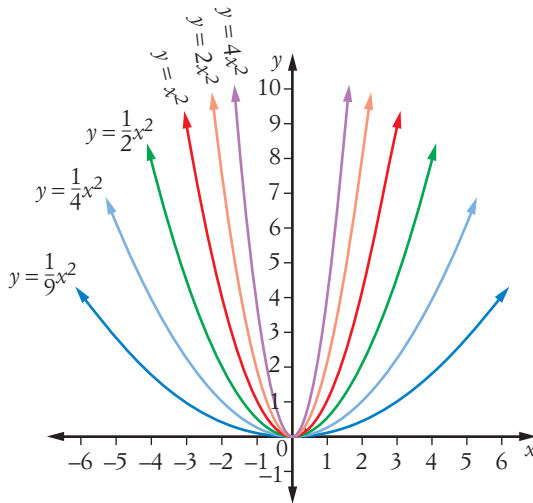
8-06 The parabola $y = ax^2 + c$

An equation in which the highest power of the variable is 2 is called a **quadratic equation**, for example, $y = 2x^2 - 5$, $y = x^2 + 7x + 12$ and $y = -5x^2$. The graph of a quadratic equation is a smooth U-shaped curve called a **parabola** (pronounced 'pa-rab-o-la').

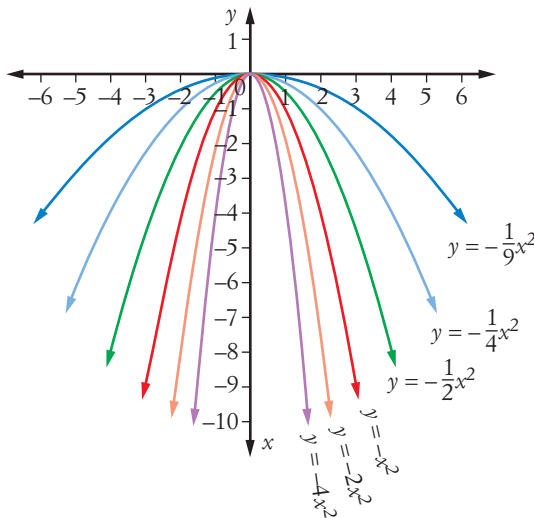
The graph of $y = ax^2$

For the graph of a quadratic equation in the form $y = ax^2$, where a is a constant (number), the size of a (the **coefficient** of x^2) affects whether the parabola is 'wide' or 'narrow'.

As the size of a increases, the parabola becomes 'narrower' and as the size of a decreases, the parabola 'widens'. If a is negative, then the parabola is concave down.



Concave up (looks like a smile ☺)
Minimum value of the parabola is 0



Concave down (looks like a frown ☹)
Maximum value of the parabola is 0

- The axis of symmetry, called the **axis of the parabola**, is the y-axis
- The **vertex** or **turning point** is (0, 0)

The graph of $y = ax^2 + c$

For the graph of a quadratic equation in the form $y = ax^2 + c$, where a and c are constants, the effect of c is to move the parabola $y = ax^2$ up or down from the origin. Also, c is the y -intercept of the parabola.

Example 11

Graph each set of quadratic equations, showing the vertex of each parabola.

a $y = x^2$, $y = x^2 - 4$, $y = x^2 + 2$

b $y = -x^2$, $y = -x^2 - 4$, $y = -x^2 + 5$

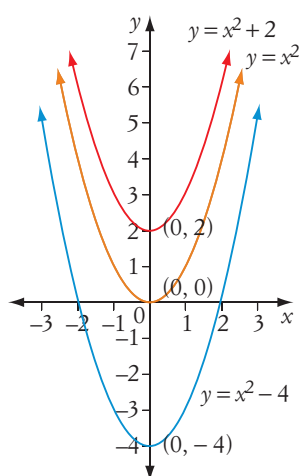
Solution

- a** First draw the graph of $y = x^2$. Its vertex is at $(0, 0)$.

The graph of $y = x^2 - 4$ is identical to that of $y = x^2$, but it is moved 4 units down.

Its vertex is at $(0, -4)$.

The graph of $y = x^2 + 2$ is identical to that of $y = x^2$, but it is moved 2 units up. Its vertex is at $(0, 2)$.



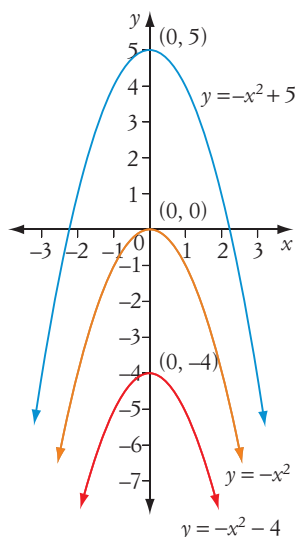
- b** The graph of $y = -x^2$ is the graph of $y = x^2$ reflected across the x -axis. Its vertex is at $(0, 0)$ as well.

The graph of $y = -x^2 - 4$ is identical to that of $y = -x^2$, but it is moved 4 units down.

Its vertex is at $(0, -4)$.

The graph of $y = -x^2 + 5$ is identical to that of $y = -x^2$, but it is moved 5 units up.

Its vertex is at $(0, 5)$.



Note:

- In part **a**, all parabolas are concave up, because of the positive coefficient of x^2
- In part **b**, all parabolas are concave down, because of the negative coefficient of x^2
- For $y = ax^2 + c$, the y -intercept of the parabola is c

Example 12

For the graph of each given quadratic equation, state:

- i whether the parabola is wider or narrower than the graph of $y = x^2$
 - ii whether the parabola has moved up or down when compared to the graph of $y = x^2$
 - iii the y -intercept.
- a** $y = 3x^2 - 1$ **b** $y = \frac{1}{3}x^2 + 2$

Solution

- a i** The coefficient of x^2 is 3, while the coefficient of x^2 in $y = x^2$ is 1.
 $\therefore$ The parabola will be narrower than $y = x^2$.
- ii** The constant term is -1 .
 $\therefore$ The parabola has moved down.
- iii** The y -intercept is -1 .
- b i** The coefficient of x^2 is $\frac{1}{3}$.
 $\therefore$ The parabola will be wider than $y = x^2$.
- ii** The constant term is 2.
 $\therefore$ The parabola has moved up.
- iii** The y -intercept is 2.

Example 13

A parabola has the equation $y = 3x^2 - 1$. Find the x -coordinate of the point on the parabola that has a y -coordinate of 191.

Solution

Substitute $y = 191$ into $y = 3x^2 - 1$

$$191 = 3x^2 - 1$$

$$192 = 3x^2$$

$$3x^2 = 192$$

$$x^2 = \frac{192}{3}$$

$$= 64$$

$$x = \pm\sqrt{64}$$

$$= \pm 8$$

This means there are two points on the parabola with a y -coordinate of 191, they are (8, 191) and $(-8, 191)$.

Exercise 8-06 The parabola $y = ax^2 + c$

Some of this exercise may also be completed using a graphics calculator or graphing software.

- 1 a Graph each quadratic equation, showing the vertex of each parabola.

$$y = x^2 \quad y = -x^2 \quad y = x^2 + 2 \quad y = -2x^2 \quad y = x^2 - 1$$

- b State which graphs you have drawn in part a:

i are concave up

ii are concave down

iii have a turning point at (0, 0)

- 2 Which statement is false about this parabola?

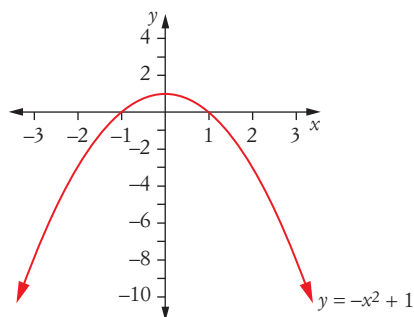
Select A, B, C or D.

A Its axis of symmetry is the x -axis.

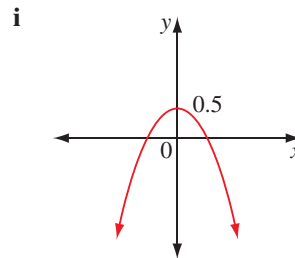
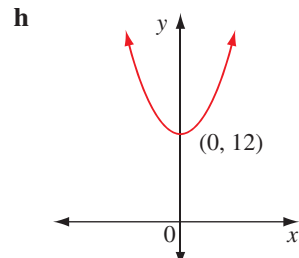
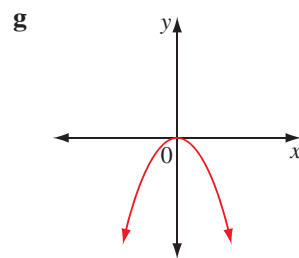
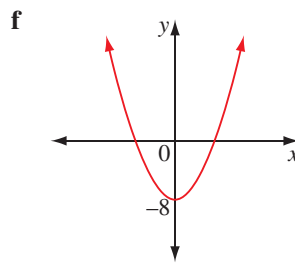
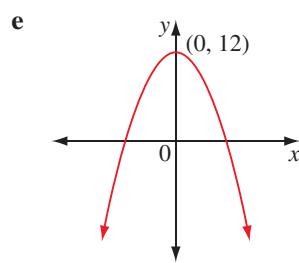
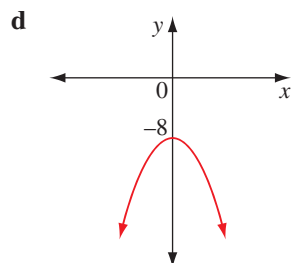
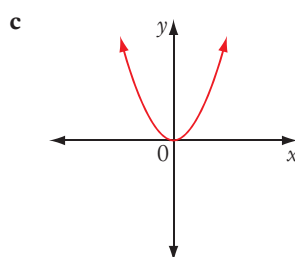
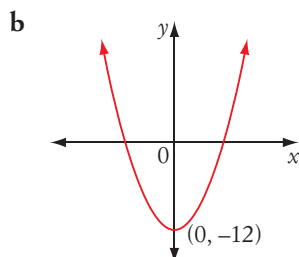
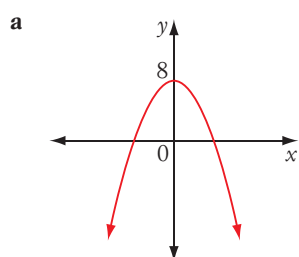
B It is concave down.

C Its vertex is (0, 1).

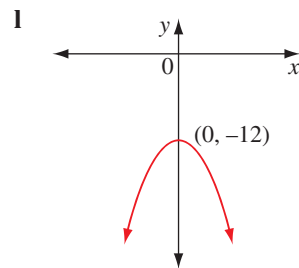
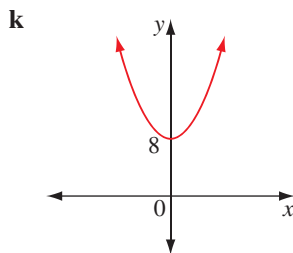
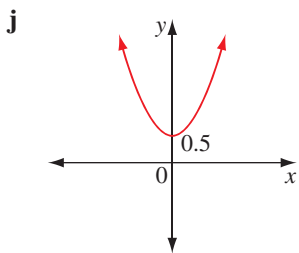
D It has a maximum value.



- 3 Match each graph with its correct quadratic equation.



See Example 11



- A** $y = x^2$ **B** $y = -x^2$ **C** $y = x^2 - 8$ **D** $y = -12 - x^2$
E $y = \frac{1}{2} + x^2$ **F** $y = 8 - x^2$ **G** $y = 8 + x^2$ **H** $y = -x^2 + \frac{1}{2}$
I $y = x^2 - 12$ **J** $y = 12 - x^2$ **K** $y = -x^2 - 8$ **L** $y = x^2 + 12$

4 Find the equation of each of the following parabolas in the form $y = x^2 + c$ or $y = -x^2 + c$ (where c is a constant), given:

- a** vertex $(0, 0)$, concave down **b** concave up, turning point $(0, 0)$
c axis of symmetry $x = 0$, maximum $y = -\frac{1}{4}$ **d** concave down, maximum $y = -9$
e turning point $(0, \frac{1}{2})$, concave down **f** axis of symmetry y -axis, minimum $y = 9$

5 **a** Graph $y = 2x^2 + 1$ after copying and completing this table.

x	-2	-1	0	1	2
y					

- b** State the turning point (vertex).
c Is the parabola concave up or concave down?
d What is its minimum value?

6 **a** Graph $y = -3x^2 + 2$ after copying and completing this table.

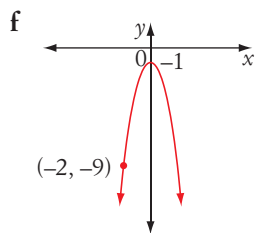
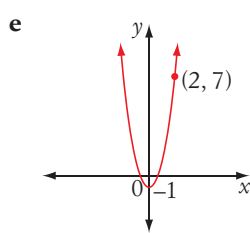
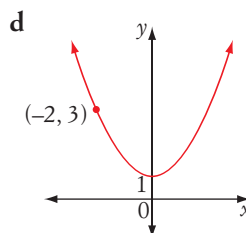
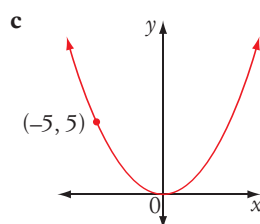
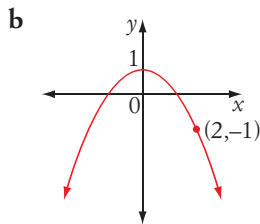
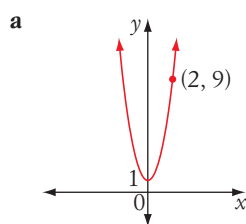
x	-2	-1	0	1	2
y					

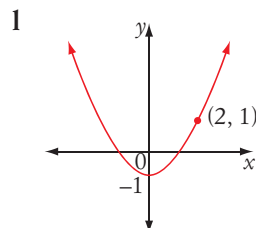
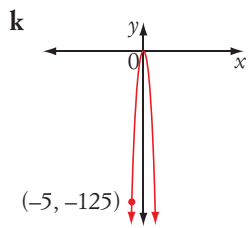
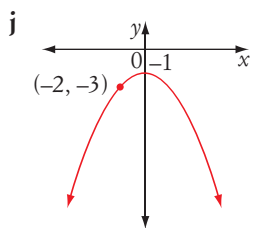
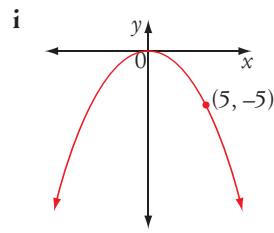
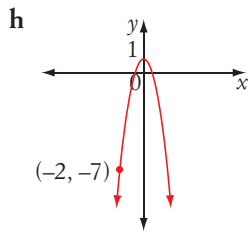
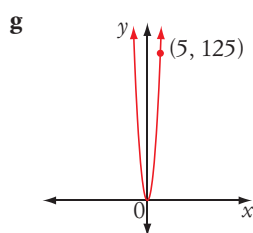
- b** Find the vertex.
c Write the equation of its axis of symmetry.
d Find its maximum value.

7 Which statement is false about the graph of $y = 4x^2 - 1$? Select **A**, **B**, **C** or **D**.

- A** Its axis of symmetry is $y = 0$. **B** It is concave up.
C The vertex is $(0, -1)$. **D** It has a minimum value of $y = -1$.

8 Match each graph with its correct quadratic equation.





A $y = 5x^2$

B $y = 2x^2 + 1$

C $y = \frac{1}{2}x^2 - 1$

D $y = \frac{1}{5}x^2$

E $y = 2x^2 - 1$

F $y = -5x^2$

G $y = -\frac{1}{2}x^2 + 1$

H $y = -\frac{1}{5}x^2$

I $y = -2x^2 - 1$

J $y = \frac{1}{2}x^2 + 1$

K $y = -2x^2 + 1$

L $y = -\frac{1}{2}x^2 - 1$

9 For the graph of each given quadratic equation, state:

i whether the parabola is wider or narrower than the graph of $y = x^2$

ii whether the parabola has moved up or down when compared to the graph of $y = x^2$

iii the y-intercept.

a $y = 2x^2 + 3$

b $y = \frac{1}{2}x^2 + 1$

c $y = 6x^2 - 5$

d $y = 0.2x^2 - 12$

10 A parabola has the equation $y = x^2 - 5$. Find the x-coordinates of the points on the parabola that have a y-coordinate of:

a 11

b 116.

11 A stone is dropped from a cliff and its height (h metres) at any time (t seconds) is given by $h = 80 - 4.9t^2$.

a Draw a graph of the equation for values of t from 0 to 5.

b What is the height of the cliff?

c What is the height of the stone after 3 seconds?

d When will the stone hit the ground?

e How long after it is dropped is the stone 5 metres above the ground? Answer correct to 2 decimal places.

12 A parabola has the equation $y = 2x^2 + 3$. Find the x-coordinates of the points on the parabola that have a y-coordinate of:

a 165

b 396.

See Example 12

See Example 13

Just for the record

Parabolas in architecture

There are many examples of parabolas in architecture and engineering. The Notre Dame Cathedral in Paris, France is almost 900 years old and has flying buttresses on the outside that have the shape of parabolas.



Bridges also often use parabolic curves in their construction. One modern application is the cables used in the suspension of the Golden Gate Bridge in San Francisco.

Find 2 different uses of parabolas in real-life constructions and create a presentation with pictures.

Technology Graphing $y = a(x - r)^2$

In this activity, we will use graphing software such as GeoGebra or Fx-Graph to compare the shapes of parabolas of the form $y = a(x - r)^2$.

- 1
 - a Use the software to graph the parabolas $y = (x - 1)^2$, $y = (x - 2)^2$, $y = (x + 2)^2$, $y = (x + 3)^2$.
 - b Compare and contrast the parabolas. What is the same and what is different?
 - c Are the parabolas concave up or down?
 - d Do they have a common vertex?
 - e What are their axes of symmetry?
 - f Given that these parabolas are of the form $y = (x - r)^2$, what does r represent?
- 2
 - a Use the software to graph the parabolas $y = 3(x - 2)^2$, $y = -(x - 4)^2$, $y = \frac{1}{2}(x + 3)^2$, $y = -0.1(x + 1)^2$.
 - b Compare and contrast the parabolas. What is the same and what is different?
 - c Which parabola is the widest? Which parabola is the narrowest?
 - d Which parabolas are concave up and concave down? Why?
 - e What is the vertex of each parabola?
 - f Given that these parabolas are of the form $y = a(x - r)^2$, what does r represent?
 - g What is the effect of the size of a on the shape of the parabola?

8-07 The parabola $y = a(x - r)^2$

Stage 5.3

NSW

Worksheet

Matching parabolas

MAT10NAWK10214

Worksheet

Matching parabolas
with their equations

MAT10NAWK00016

Technology worksheet

Excel worksheet:
Investigating
parabolas 2

MAT10NACT00012

Technology worksheet

Excel spreadsheet:
Investigating
parabolas 2

MAT10NACT00042

Summary

The graph of $y = a(x - r)^2$ is a parabola with a vertex at $(r, 0)$.

The graph of $y = a(x - r)^2$ is the graph of $y = ax^2$ translated r units to the right (or left if r is negative)

Example 14

Graph each parabola, clearly showing the vertex and y-intercept.

a $y = (x - 1)^2$ **b** $y = -(x + 2)^2$ **c** $y = 2(x + 1)^2$

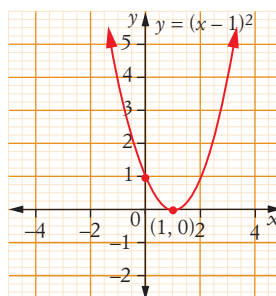
Solution

a $y = (x - 1)^2$

This parabola is the graph of $y = x^2$ translated 1 unit to the right, with a vertex at $(1, 0)$.

Substitute $x = 0$ for the y-intercept:

$$y = (0 - 1)^2 = (-1)^2 = 1$$



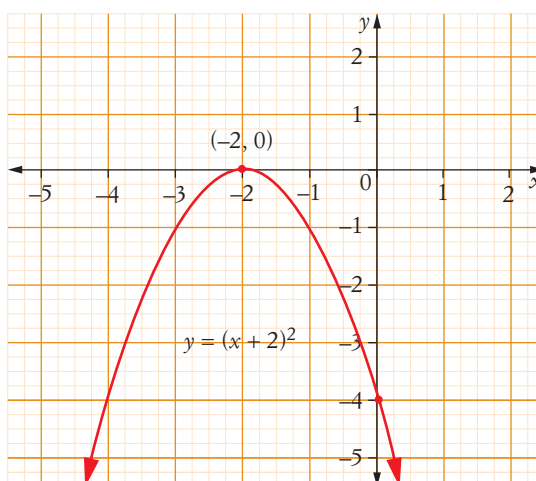
b $y = -(x + 2)^2$

This parabola is the graph of $y = -x^2$ translated 2 units to the left, with a vertex at $(-2, 0)$.

It is concave down because a is negative.

Substituting $x = 0$ for the y-intercept.

$$y = -(0 + 2)^2 = -4$$



Stage 5.3

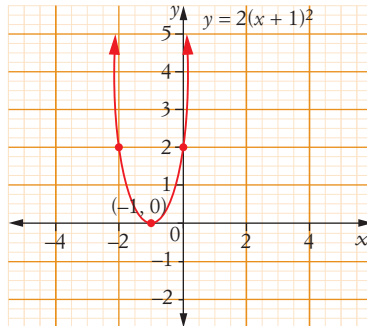
c $y = 2(x + 1)^2$

This is $y = 2x^2$ translated 1 unit to the left,
with a vertex at $(-1, 0)$.

Substituting $x = 0$ for the y -intercept.

$$y = 2(0 + 1)^2$$

$$= 2$$



Exercise 8-07 The parabola $y = a(x - r)^2$

See Example 14

- 1 Graph each parabola, clearly showing the vertex and
- y
- intercept.

a $y = (x - 3)^2$

b $y = (x - 2)^2$

c $y = (x + 1)^2$

d $y = -(x - 3)^2$

e $y = -(x + 1)^2$

f $y = -(x - 5)^2$

g $y = 3(x + 4)^2$

h $y = -2(x - 1)^2$

i $y = -\frac{1}{4}(x + 6)^2$

- 2 Which statement is true about the parabola
- $y = (x + 6)^2$
- ? Select the correct answer

A, B, C or D.

A Its axis of symmetry is $x = 6$ B It has a y -intercept at 36C Its vertex is $(6, 0)$ D It passes through the point $(-1, 5)$

- 3 Match each equation to its graph below.

a $y = (x + 4)^2$

b $y = -(x - 1)^2$

c $y = (x + 2)^2$

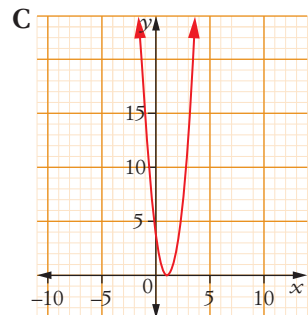
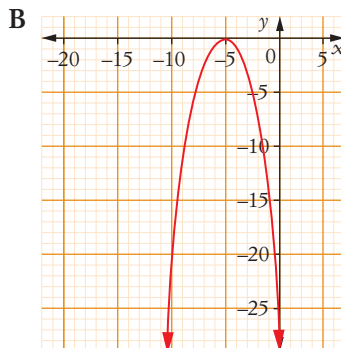
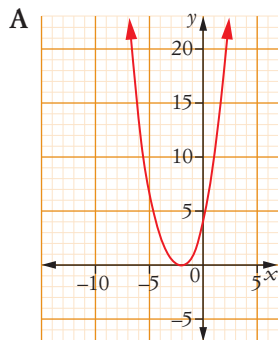
d $y = (x - 3)^2$

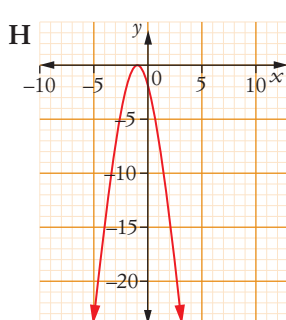
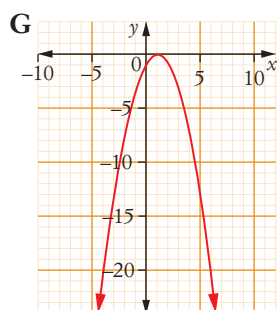
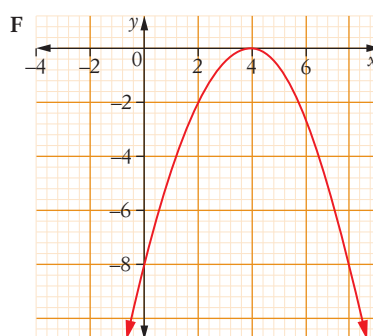
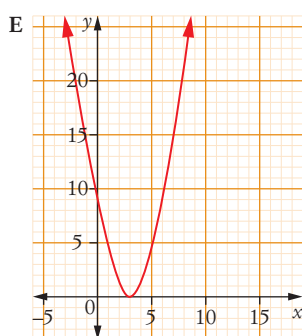
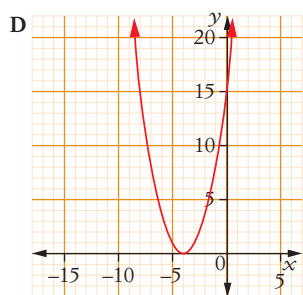
e $y = -(x + 5)^2$

f $y = -2(x + 1)^2$

g $y = -0.5(x - 4)^2$

h $y = 4(x - 1)^2$

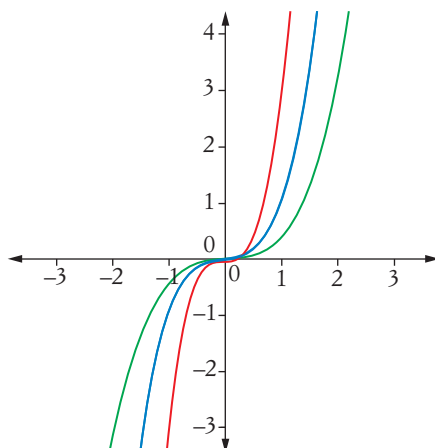




Technology Graphing $y = ax^3 + c$

Use GeoGebra or other graphing technology to graph the cubic equations $y = 0.4x^3$, $y = x^3$ and $y = 3x^3$.

- 1 Enter as $y=0.4x^3$, $y=x^3$ and $y=3x^3$.
- 2 Right-click on each graph, choose **Object properties** and **Colour** to select different colours for each cubic curve.



- 3 What is the effect of a in $y = ax^3$?
- 4 Graph $y = 2x^3$, $y = 2x^3 - 4$ and $y = 2x^3 + 1$ and compare the curves.
- 5 Graph $y = -\frac{1}{2}x^3$, $y = -\frac{1}{2}x^3 - 2$ and $y = -\frac{1}{2}x^3 + 3$ and compare the curves.
- 6 What is the effect of c in $y = ax^3 + c$?

Stage 5.3

NSW

Worksheet

Graphing cubics 1

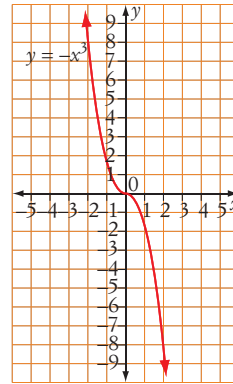
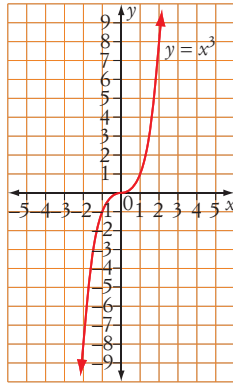
MAT10NAWK10215

8-08 The cubic curve $y = ax^3 + c$

An equation in which the highest power of the variable is 3 is called a **cubic equation**, for example, $y = 3x^3$, $y = -x^3 + 9$ and $y = x^3 + 2x^2 - 5x + 10$. The graph of a cubic equation is called a **cubic curve**.

The graph of $y = ax^3$

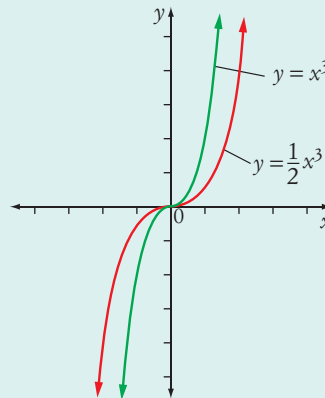
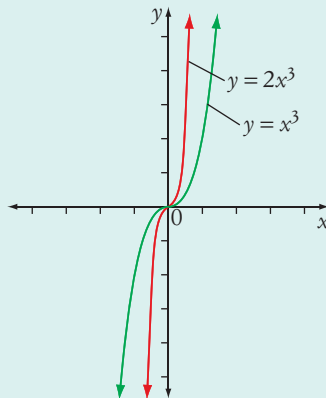
The graphs of $y = x^3$ and $y = -x^3$ are shown below.



Summary

The graph of $y = ax^3$

- The graph has no axis of symmetry.
- The graph has rotational symmetry of 180° about $(0, 0)$: if you spin the curve upside-down, it maps onto itself.
- If a is positive, the graph is always increasing except at $(0, 0)$
- If a is negative, the graph is always decreasing except at $(0, 0)$
- The size of a (the coefficient of x^3) determines whether the cubic curve is 'wider' or 'narrower' (when compared to $y = x^3$).

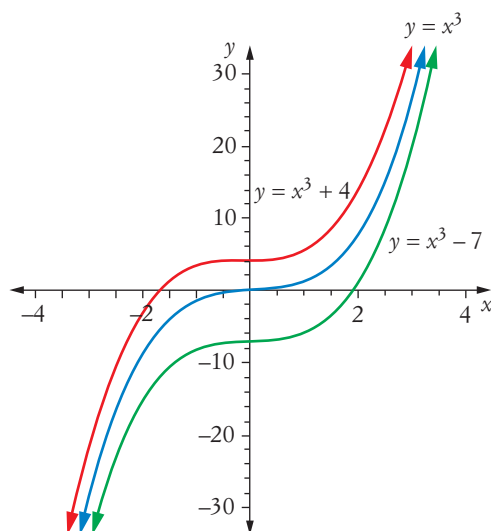


The graph of $y = ax^3 + c$

Summary

The graph of $y = ax^3 + c$

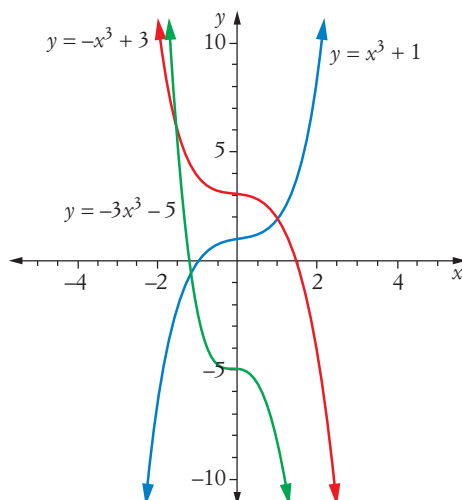
For the graph of $y = ax^3 + c$, where a and c are constants, the effect of c is to move the cubic curve $y = ax^3$ up or down from the origin. Also, c is the y -intercept of the cubic curve.



Example 15

Graph the cubic equations $y = x^3 + 1$, $y = -x^3 + 3$ and $y = -3x^3 - 5$, showing the y -intercept of each curve.

Solution



Stage 5.3

Exercise 8-08 The cubic curve $y = ax^3 + c$

See Example 15

1 Graph each cubic equation.

a $y = x^3 - 2$

b $y = -x^3$

c $y = 2x^3$

d $y = x^3 + 3$

e $y = -x^3 - 4$

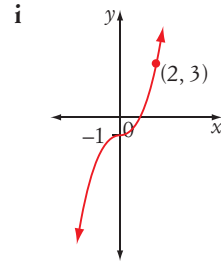
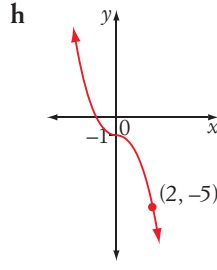
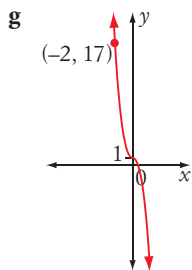
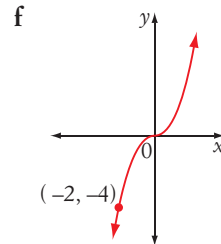
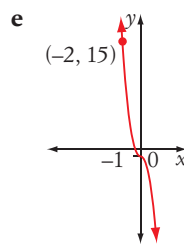
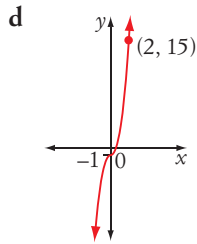
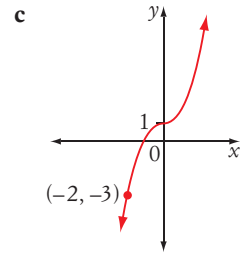
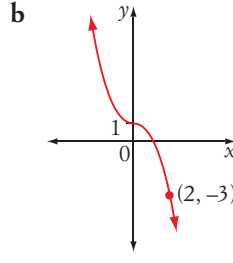
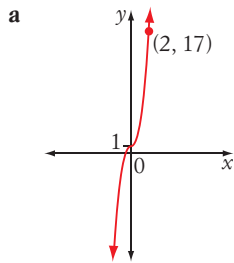
f $y = -x^3 + 2$

g $y = 2x^3 - 3$

h $y = -3x^3 - 2$

i $y = \frac{1}{2}x^3 + 4$

2 Match each cubic curve to its equation below.



A $y = 2x^3 + 1$

B $y = \frac{1}{2}x^3 - 1$

C $y = 2x^3 - 1$

D $y = \frac{1}{2}x^3$

E $y = -\frac{1}{2}x^3 + 1$

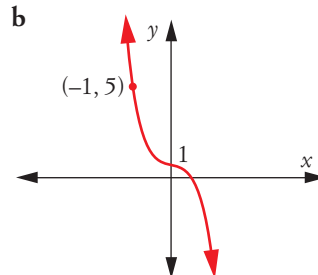
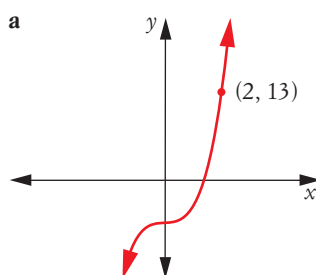
F $y = -2x^3 - 1$

G $y = \frac{1}{2}x^3 + 1$

H $y = -2x^3 + 1$

I $y = -\frac{1}{2}x^3 - 1$

3 Find the equation of each cubic curve in the form $y = ax^3 + c$.



Technology Graphing $y = ax^n$

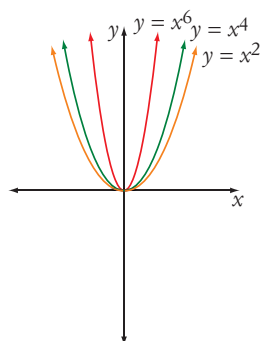
- 1 Using GeoGebra or other graphing software, graph on the same axes $y = x^2$, $y = x^4$ and $y = x^6$.
- 2 How are the graphs similar? How are the graphs different?
- 3 Graph on the same axes $y = x^3$ and $y = x^5$.
- 4 How are the graphs similar? How are the graphs different?
- 5 Describe the differences between the graphs of $y = x^n$ when n is even and when n is odd.

8-09 The power curves $y = ax^n + c$

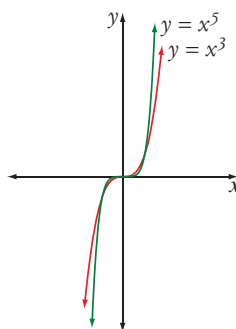
We have already graphed $y = ax^2 + c$ and $y = ax^3 + c$. Now we will graph equations containing higher powers of x , of the form $y = ax^n + c$, where n is a positive integer.

The graph of $y = ax^n$

The graphs of $y = x^2$, $y = x^4$ and $y = x^6$ are shown below.



The graphs of $y = x^3$ and $y = x^5$ are shown below.



Summary

The graph of $y = ax^n$ if n is even

- The graph is like a steeper parabola, symmetrical about the y -axis, with a vertex at $(0, 0)$
- The graph is concave up if a is positive and concave down if a is negative
- The higher the power (the value of n), the narrower the graph

The graph of $y = ax^n$ if n is odd

- The graph is like a steeper cubic curve, with rotational symmetry of 180° about $(0, 0)$
- The graph has a steep gradient, except around $(0, 0)$ where it is flat.
- The higher the power (the value of n), the narrower the graph

Stage 5.3

NSW

Technology worksheet

Excel worksheet:
Power functions

MAT10NACT00020

Technology
spreadsheet

Excel spreadsheet:
Power functions

MAT10NACT00050

Stage 5.3

The graph of $y = ax^n + c$

Summary

The graph of $y = ax^n + c$

For the graph of $y = ax^n + c$, where a and c are constants, the effect of c is to move the curve $y = ax^n$ up or down from the origin. Also, c is the y -intercept of the curve.

The graph of $y = ax^n + c$ is obtained from the graph of $y = ax^n$ by a **vertical transformation**.

The graph of $y = a(x - r)^n$

Summary

The graph of $y = a(x - r)^n$ is the graph of $y = ax^n$ translated r units to the right (or left if r is negative).

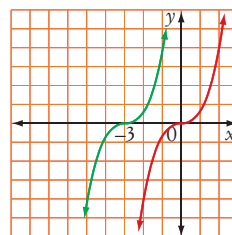
The graph of $y = a(x - r)^n$ is obtained from the graph $y = ax^n$ by a **horizontal transformation**.

Example 16

Sketch the graphs of $y = \frac{1}{2}x^3$ and $y = \frac{1}{2}(x + 3)^3$ on the same axes.

Solution

The graph of $y = \frac{1}{2}(x + 3)^3$ is obtained by moving the graph of $y = \frac{1}{2}x^3$ to the left by 3 units.

Exercise 8-09 The power curves $y = ax^n + c$

1 Match each equation to its correct graph.

a $y = x^2 - 1$

b $y = (x - 3)^3$

c $y = 2x^4 + 2$

d $y = \frac{1}{3}(x + 2)^3$

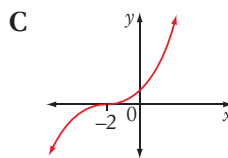
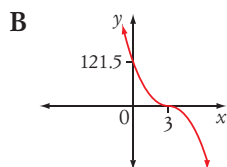
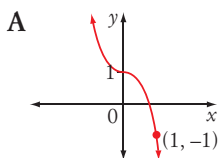
e $y = 2x^5 - 3$

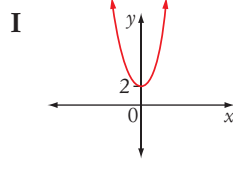
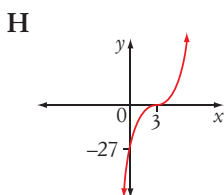
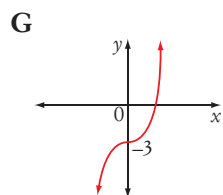
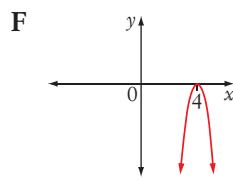
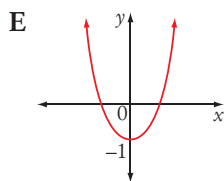
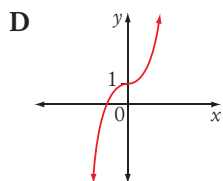
f $y = -2x^3 + 1$

g $y = -(x - 4)^4$

h $y = 3x^3 + 1$

i $y = -\frac{1}{2}(x - 3)^5$





2 Sketch the graphs of each pair of equations.

a $y = x^2$ and $y = (x - 2)^2$

b $y = 3x^2$ and $y = 3x^2 + 1$

c $y = -x^3$ and $y = -x^3 + 2$

d $y = -2x^4$ and $y = -2(x + 2)^4$

e $y = x^4$ and $y = (x - 1)^4$

f $y = -x^5$ and $y = -x^5 - 2$

3 For each pair of equations, explain how the second equation can be graphed using the graph of the first equation, for example, move left 4 units.

a $y = -x^5$ and $y = -x^5 + 4$

b $y = x^2$ and $y = (x - 5)^2$

c $y = 5x^6$ and $y = 5(x + 3)^6$

d $y = \frac{1}{4}x^3$ and $y = 4 + \frac{1}{4}x^3$

e $y = -x^4$ and $y = -(3 - x)^4$

f $y = -x^3$ and $y = -(x + 2)^3$

See Example 16

Investigation: Graphing $y = \frac{1}{x}$

1 Copy and complete this table for $y = \frac{1}{x}$. Explain why no y value exists for $x = 0$.

x	-5	-4	-3	-2	-1	-0.5	-0.2	-0.1	0	0.1	0.2	0.5	1	2	3	4	5
y																	

2 Hence graph $y = \frac{1}{x}$ on a number plane.

3 There are two parts or 'branches' to your graph. In which quadrants of the number plane are the branches?

4 Use your graph to explain what happens to the y value as x becomes very large.

5 Explain what happens to the y value as x approaches 0.

6 The graph of $y = \frac{1}{x}$ has two axes of symmetry. Draw them on your graph.

7 Copy and complete the table from question 1 for $y = -\frac{1}{x}$.

8 Hence graph $y = -\frac{1}{x}$ on a number plane.

9 How does the graph of $y = -\frac{1}{x}$ compare with that of $y = \frac{1}{x}$?

Stage 5.3

Technology Graphing $y = \frac{k}{x}$

- 1 Use GeoGebra or other graphing technology to graph each equation.

a $y = \frac{1}{x}$ **b** $y = \frac{2}{x}$ **c** $y = \frac{5}{x}$ **d** $y = \frac{10}{x}$

- 2 Compare the graphs from question 1. What happens to the graph of $y = \frac{k}{x}$ as k increases?

- 3 Graph $y = \frac{2}{x}$ and $y = \frac{-2}{x}$ and compare them.

- 4 Graph $y = \frac{4}{x}$ and use **Trace** to complete this table of values.

x	1	2	5	10	100	200	1000
y							

- 5 What happens to the y -values when the x -values become very large?

- 6 For $y = \frac{4}{x}$ use the **Trace** function to complete this table of values.

x	0.0001	0.01	0.1	0.5	1	5
y						

- 7 What happens to the y -values when the x -values become very small and close to zero?

Worksheet

Graphing hyperbolas

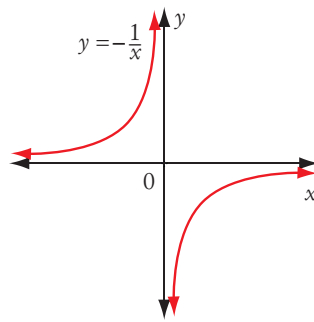
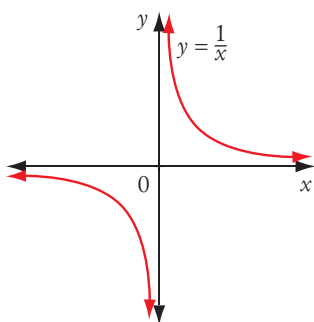
MAT10NAWK10216

8-10 The hyperbola $y = \frac{k}{x}$

The graph of $y = \frac{k}{x}$, where k is a constant, is a curve with two branches called a **hyperbola** (pronounced 'hy-perb-o-la').

The graph of $y = \frac{k}{x}$

The graphs of $y = \frac{1}{x}$ and $y = -\frac{1}{x}$ are shown below.



Summary

The graph of $y = \frac{k}{x}$

- The graph has two separate branches in different quadrants.
- If k is positive, the graph is in the 1st and 3rd quadrants.
- If k is negative, the graph is in the 2nd and 4th quadrants.
- The graph has two axes of symmetry: their equations are $y = x$ and $y = -x$.
- The graph has rotational symmetry of 180° about $(0, 0)$.
- The higher the value of k , the further the hyperbola is from the x - and y -axes.
- As x becomes very large, y gets closer to 0.
- As x becomes closer to 0, y gets very large.
- The graph gets very close to the x - and y -axes but never crosses them. The x - and y -axes are called **asymptotes** because the graph approaches them but never touches them.

Example 17

Graph each hyperbola and mark the coordinates of one point on the curve.

a $y = \frac{2}{x}$

b $y = -\frac{3}{x}$

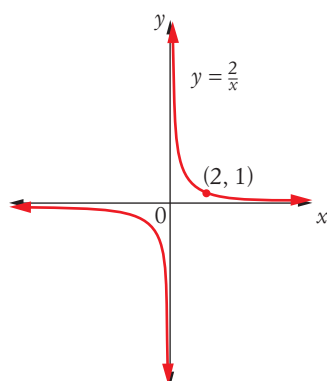
Solution

a Let $x = 2$ Choosing any value of x

$$y = \frac{2}{2}$$

$$= 1$$

A point on the curve is $(2, 1)$.



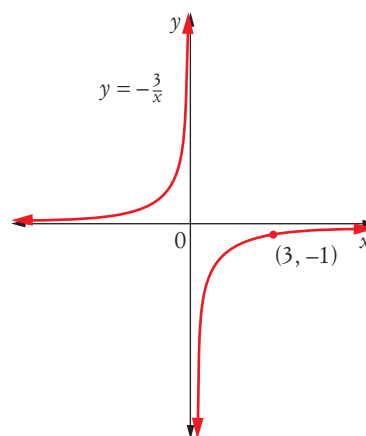
b Let $x = 3$

$$y = \frac{-3}{3}$$

$$= -1$$

A point on the curve is $(3, -1)$.

As $k = -3$ is negative, the hyperbola is in the 2nd and 4th quadrants.



Stage 5.3

The graphs of $y = \frac{k}{x} + c$ and $y = \frac{k}{x - r}$

Summary

The graph of $y = \frac{k}{x} + c$

For the graph of $y = \frac{k}{x} + c$, where k and c are constants, the effect of c is to move the hyperbola $y = \frac{k}{x}$ up or down from the origin.

The graph of $y = \frac{k}{x - r}$

The graph of $y = \frac{k}{x - r}$ is the graph of $y = \frac{k}{x}$ translated r units to the right (or left if r is negative).

Example 18

Graph each hyperbola, find any intercepts and mark the coordinates of one point on the curve.

a $y = \frac{2}{x} + 1$ **b** $y = \frac{-3}{x - 2}$

Solution

a Let $x = 2$

$$y = \frac{2}{2} + 1$$

$$= 2$$

A point on the curve is $(2, 2)$.

The graph of $y = \frac{2}{x} + 1$ is the graph of $y = \frac{2}{x}$ translated up 1 unit. This means that the horizontal asymptote is now at $y = 1$.

An x -intercept now occurs when $y = 0$.

$$0 = \frac{2}{x} + 1$$

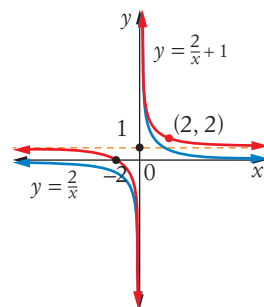
$$0 = 2 + x$$

$$x = -2$$

The x -intercept is -2 .

Choosing any value of x

Multiplying both sides by x .



b Let $x = 3$

$$y = \frac{-3}{3-2}$$

$$= -3$$

A point on the curve is $(3, -3)$.

The graph of $y = \frac{-3}{x-2}$ is the graph of $y = \frac{-3}{x}$ translated right 2 units. This means that the vertical asymptote is now at $x = 2$.

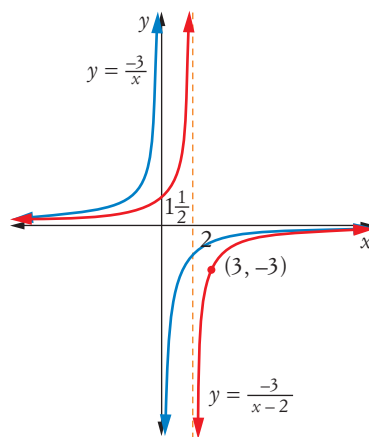
A y -intercept now occurs when $x = 0$.

$$y = \frac{-3}{0-2}$$

$$= \frac{3}{2}$$

$$= 1\frac{1}{2}$$

The y -intercept is $1\frac{1}{2}$.



Exercise 8-10 The hyperbola $y = \frac{k}{x}$

- 1 a Copy and complete this table for $y = \frac{2}{x}$.

x	-3	-2	-1	0	1	2	3
y							

- b Graph $y = \frac{2}{x}$, showing the coordinates of one point on the hyperbola.
c On your diagram, draw in the axes of symmetry for the hyperbola.
d What are the equations of these axes?

- 2 Graph each hyperbola and mark the coordinates of one point on the curve.

a $y = \frac{4}{x}$ b $y = -\frac{2}{x}$ c $y = \frac{3}{x}$

- 3 a The distance from Sydney to Melbourne is close to 1000 km. Copy and complete the following table that relates time (t hours) and speed (s km/h) for the trip. Round your answers to the nearest km/h.

$$s = \frac{1000}{t}$$

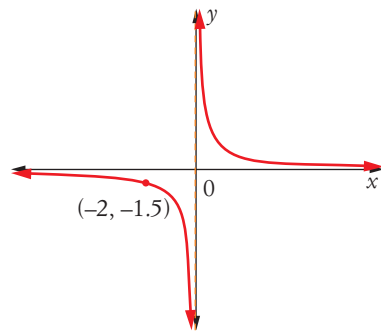
t	1	2	3	...	10
s					

- b Hence graph the equation $s = \frac{1000}{t}$.
c Why are the values for t only positive numbers? Explain why t cannot be equal to 0.
d If the time is doubled, is the speed halved? Use the information from your graph to support your answer.

See Example 17

Stage 5.3

- 4 The curve below is a hyperbola of the form $y = \frac{k}{x}$.
- Find the value of k .
 - Hence state the equation of this hyperbola.

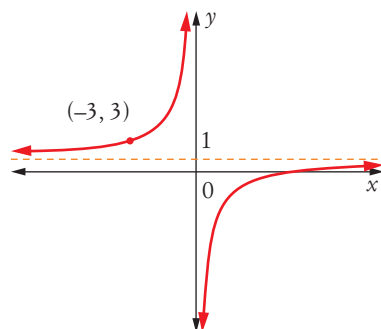


See Example 18

- 5 Graph each hyperbola and mark the coordinates of one point on the curve.

a $y = \frac{1}{x} + 2$ b $y = -\frac{2}{x} - 3$ c $y = \frac{2}{x-1}$ d $y = \frac{-3}{x+2}$

- 6 The curve below is a hyperbola of the form $y = \frac{k}{x} + c$
- Find the values of c and k .
 - Hence state the equation of this hyperbola.



- 7 Sarah and David want to buy a rectangular block of land that has an area of 800 m^2 . There are several blocks available with this area.

- a Copy and complete this table that relates the length (L metres) and width (W metres) of the block of land.

L	10	20	30	...	100
W					

- What is the formula for W ?
- Explain why the length or width cannot be equal to 0 metres.
- Graph the formula for W .
- What happens to the width as the length continues to increase? How is this shown on the graph?
- What happens to the width as the length approaches 0? How is this shown on the graph?

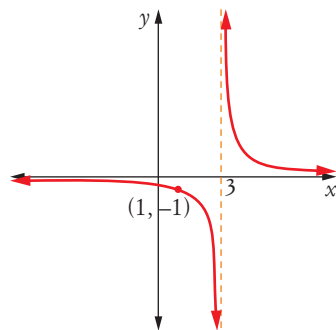
- 8 Which equation best represents the graph shown below?
Select the correct answer **A**, **B**, **C** or **D**.

A $y = \frac{2}{x-3}$

B $y = \frac{-1}{x+3}$

C $y = \frac{2}{x} + 3$

D $y = -\frac{1}{x} - 3$



Investigation: Graphing $y = 2^x$

This activity can also be completed using a graphics calculator or graphing software.

- 1 Copy and complete this table of values for $y = 2^x$.

x	-3	-2	-1	0	1	2	3	4
y								

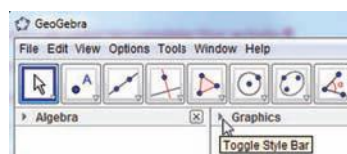
- 2 Graph the points from the table and join them with a smooth curve. The equation $y = 2^x$ is called an **exponential equation** and its graph is called an **exponential curve** (exponent means 'power').
- 3 Graph $y = 2^{-x}$ in a similar way.
- 4 Compare the graphs of $y = 2^x$ and $y = 2^{-x}$. Describe any similarities and differences.
- 5 The y -intercept of any graph with equation $y = a^x$ (where a is a positive constant) is always 1. Explain why.
- 6 The graph of $y = 2^x$ is increasing. Is the graph of $y = 2^{-x}$ increasing or decreasing? Give reasons.
- 7 Describe what happens to the graph of $y = 2^x$ when:
 - a x approaches a large positive number
 - b x approaches a large negative number.

Technology Exponential curves

Use **GeoGebra**, **Fx-Graph** or a graphics calculator to complete this activity.

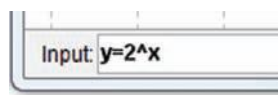
The instructions below are provided using **GeoGebra**.

- 1 Open up **GeoGebra** and click the little arrow in front of **Graphics**.

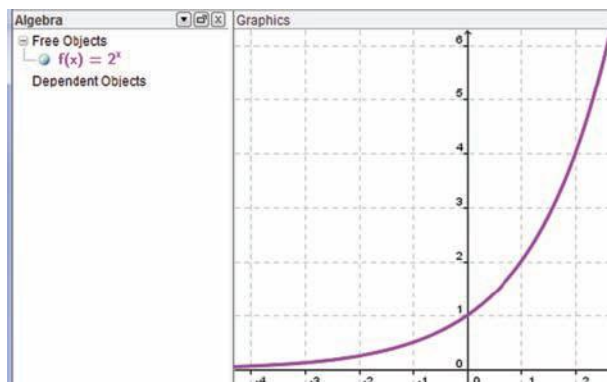


From the new panel that pops up, select the grid option at the top left-hand side.

- 2 Enter the function $y = 2^x$ into the **Input** bar, using '^'.



Press ENTER. The colour of the graph can be changed by right-clicking on the graph and choosing **Object Properties** and **Colour**. The thickness of the curve can also be changed by clicking **Object Properties** and **Style**. The **Algebra View** shows the equation of each graph in the same colour as its graph.



- 3 Repeat step 2 by entering each of the following equations. Change the colours as required.

$$\begin{array}{llll} y = 2^{-x} & (\text{enter } y=2^{-x}), & y = -2^x & (\text{enter } y=-2^x) \\ y = -2^{-x} & y = 2^x + 1 & y = 2^x - 1 & \end{array}$$

- 4 a Which graphs are similar?
 b Identify any features such as y -intercepts.
 c Which graphs are similar as:
 i x becomes larger? ii x becomes smaller?
- 5 Repeat steps 1 to 3 and answer question 4 for the exponential curves below.
 $y = 3^x, y = 3^{-x}, y = -3^x, y = -3^{-x}, y = 3^x + 1, y = 3^x - 1$

Worksheet

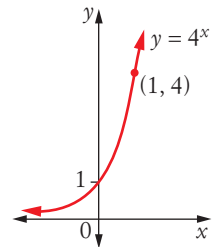
Graphing exponentials

MAT10NAWK10052

8-11 The exponential curve $y = a^x$

An equation of the form $y = a^x$, where a is a positive constant and the variable x is a power, is called an **exponential equation**, for example, $y = 5^x$, $y = 2^x$ and $y = 3^x$. The graph of an exponential equation is a smooth curve called an **exponential curve**.

The graph of $y = 4^x$ is shown.



- The y -intercept of $y = a^x$ is 1 since $a^0 = 1$.
- As x increases (to the right in the positive direction), a^x becomes very large. Graphically, this means that the graph of $y = a^x$ increases sharply with a steep gradient.
- As x decreases (to the left in the negative direction), a^x approaches zero. This means that the graph of $y = a^x$ flattens out and approaches the x -axis as x approaches a large negative number. The x -axis is an **asymptote** because the curve approaches it but never touches it.
- The exponential curve is always above the x -axis because the value of a^x is always positive.

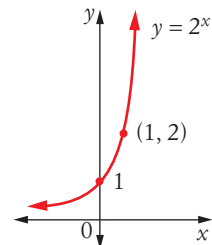
Example 19

Sketch each exponential equation and mark the y -intercept on each curve.

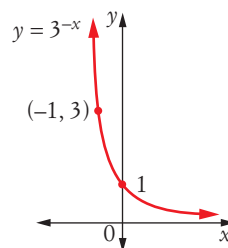
a $y = 2^x$ b $y = 3^{-x}$

Solution

- a
- The y -intercept of $y = 2^x$ is 1
 - At $x = 1$, $y = 2$
 - As x increases (to the right in the positive direction), 2^x becomes very large (steep gradient)
 - As x decreases (to the left in the negative direction), 2^x approaches zero. The x -axis is an asymptote.



- b**
- The y -intercept of $y = 3^{-x}$ is 1
 - At $x = -1$, $y = 3$
 - As x decreases (to the left in the negative direction), 3^{-x} becomes very large (steep gradient)
 - As x increases (to the right in the positive direction), 3^{-x} approaches zero. The x -axis is an asymptote.



Note that the graph of $y = 3^{-x}$ (and of $y = a^{-x}$ in general) is decreasing, and is actually a reflection of the graph of $y = 3^x$ across the y -axis.

Exercise 8-11 The exponential curve $y = a^x$

Some of this exercise may also be completed using a graphics calculator or graphing software.

- 1 a** Graph these exponential equations on the same axes.

i $y = 2^x$ **ii** $y = 3^x$ **iii** $y = 5^x$

- b** What is the y -intercept of each graph?

- c** Describe what happens to the graph $y = a^x$ as a increases.

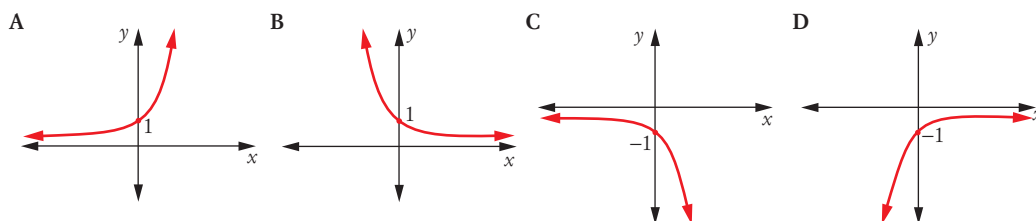
- 2 a** Graph $y = 4^x$ and $y = 4^{-x}$ on the same axes.

- b** Copy and complete:

i The reflection of $y = 4^x$ in the y -axis is ...

ii The reflection of $y = a^x$ in the y -axis is ...

- 3** Which graph represents $y = 2^{-x}$? Select the correct answer **A**, **B**, **C** or **D**.



- 4 a** Graph $y = 2^x$ and $y = -2^x$ on the same axes.

- b** How are the two graphs related?

- c** Copy and complete: The reflection of $y = a^x$ in the x -axis is ...

- 5** Graph $y = 3^x + 1$ and $y = 3^x - 1$ on the same axes and describe how they are related.

- 6** Sketch each exponential curve, showing the y -intercept.

a $y = 2^x$

b $y = 3^{-x}$

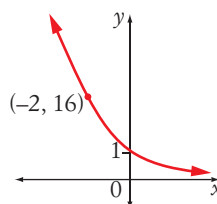
c $y = -4^x$

d $y = -2^{-x}$

e $y = 4^x + 1$

f $y = 4^x - 1$

- 7** Find an exponential equation for this graph.



See Example 19

Just for the record

Exponential growth

When an increase can be described using an exponential equation, it is called **exponential growth**. Examples include the growth of population (people and bacteria) and monetary investments. **Population growth** is monitored in different countries through the fertility (birth) and mortality (death) rates as well as migration. The data collected for these figures can often be modelled as an **exponential function**. By modelling the changes in population, predictions of future changes in population can be simulated and towns and cities can prepare for possible expansion in the numbers of schools, hospitals, housing and other necessary infrastructure.

At what rate is the population of Australia growing? What about the world's population?



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Puzzle sheet

Circle equations

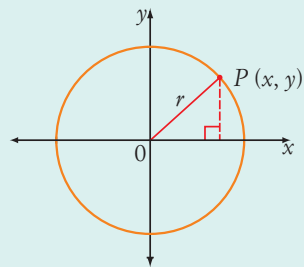
8-12 The circle $(x - h)^2 + (y - k)^2 = r^2$

MAT10NAPS00047

The circle $x^2 + y^2 = r^2$

Summary

The **equation of a circle** with centre $(0, 0)$ and radius r units is $x^2 + y^2 = r^2$



Example 20

Find the equation of a circle with centre $(0, 0)$ and diameter 14 units.

Solution

$$\text{Radius} = \frac{1}{2} \times 14 = 7 \text{ units.}$$

$$r^2 = 7^2 = 49.$$

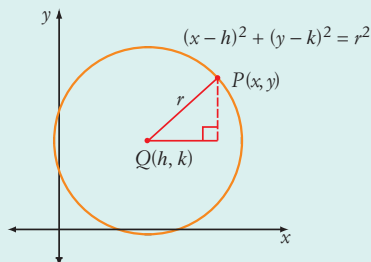
The equation of the circle is $x^2 + y^2 = 49$.

The circle $(x - h)^2 + (y - k)^2 = r^2$

If the centre of the circle is shifted from $(0, 0)$ to new coordinates (h, k) , then the equation of the circle changes from $x^2 + y^2 = r^2$ to $(x - h)^2 + (y - k)^2 = r^2$.

Summary

The **equation of a circle** with centre (h, k) and radius r units is $(x - h)^2 + (y - k)^2 = r^2$.



Proof

Let $P(x, y)$ be any point on the circle and $Q(h, k)$ be its centre so that the distance $PQ = r$ units. Using the distance formula:

$$d = \sqrt{(x - h)^2 + (y - k)^2} = r$$

$$(x - h)^2 + (y - k)^2 = r^2$$

Example 21

Find the centre and radius of the circle represented by each equation.

a $(x - 2)^2 + (y - 5)^2 = 9$

b $(x + 3)^2 + (y + 1)^2 = 16$

c $x^2 + (y - 6)^2 = 1$

d $(x + 1)^2 + (y - 4)^2 = 40$

Solution

a Centre is $(2, 5)$, radius $= \sqrt{9} = 3$ units

b Centre is $(-3, -1)$, radius $= \sqrt{16} = 4$ units

c Centre is $(0, 6)$, radius $= \sqrt{1} = 1$ unit

d Centre is $(-1, 4)$, radius $= \sqrt{40} = 2\sqrt{10}$ units

Completing the square

$(x - 2)^2 + (y - 5)^2 = 9$ can be expanded to become $x^2 - 4x + y^2 - 10y = -20$, so this is another equation of a circle.

If $x^2 + 4x + y^2 - 6y + 4 = 0$ is also an equation of a circle, to find its centre and radius we need to factorise the LHS so that it is of the form $(x - h)^2 + (y - k)^2$. To find the two perfect squares, we use a method called **completing the square**. This method depends on the following results for perfect squares.

$$(x + a)^2 = x^2 + 2ax + a^2$$

$$(x - a)^2 = x^2 - 2ax + a^2$$

We note that the last term, a^2 , is the square of 'half the coefficient of x '.

Stage 5.3

Example 22

Find the numbers that complete the square in each equation.

a $x^2 + 10x + \dots = (x + \dots)^2$

b $x^2 - 14x + \dots = (x - \dots)^2$

Solution

a The coefficient of x is 10.

Half of 10 is 5, and $5^2 = 25$.

The perfect square is $x^2 + 10x + 25 = (x + 5)^2$

b The coefficient of x is -14 .

Half of -14 is -7 , and $(-7)^2 = 49$.

The perfect square is $x^2 - 14x + 49 = (x - 7)^2$

Example 23

Graph the circle with equation $x^2 + 4x + y^2 - 6y + 4 = 0$.

Solution

We need to rewrite the equation into the form

$$(x - h)^2 + (y - k)^2 = r^2.$$

$$x^2 + 4x + y^2 - 6y + 4 = 0$$

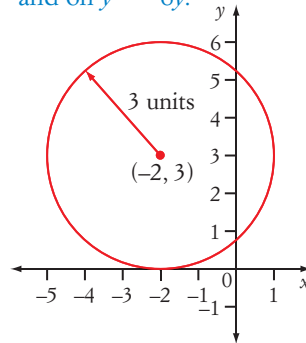
$$x^2 + 4x + y^2 - 6y = -4$$

$$(x^2 + 4x + 4) + (y^2 - 6y + 9) = -4 + 4 + 9$$

$$(x + 2)^2 + (y - 3)^2 = 9$$

which is a circle, with centre $(-2, 3)$ and radius $\sqrt{9} = 3$ units.

Moving the constant, 4, to the RHS
Completing the square on $x^2 + 4x$
and on $y^2 - 6y$.


Exercise 8-12 The circle $(x - h)^2 + (y - k)^2 = r^2$

Some of this exercise may also be completed using GeoGebra or other graphing technology.

See Example 20

1 Find the centre and radius of the circle given by each equation.

a $x^2 + y^2 = 4$

b $x^2 + y^2 = 36$

c $x^2 + y^2 = 64$

d $x^2 + y^2 = 100$

e $x^2 + y^2 = 81$

f $2x^2 + 2y^2 = 50$

2 Which equation represents a circle with centre $(0, 0)$ and radius 3 units? Select the correct answer **A**, **B**, **C** or **D**.

A $x^2 + y^2 = -9$

B $x^2 + y^2 = 3$

C $x^2 + y^2 = -3$

D $x^2 + y^2 = 9$

3 Find the centre and radius of the circle given by each equation.

a $(x+2)^2 + (y-4)^2 = 49$ **b** $(x-3)^2 + (y-1)^2 = 1$ **c** $(x-9)^2 + (y-12)^2 = 225$

d $x^2 + (y+3)^2 = 4$ **e** $(x+6)^2 + (y+1)^2 = 10$ **f** $(x+5)^2 + (y+8)^2 = 16$

g $x^2 + y^2 = 72$ **h** $(x+2)^2 + (y-1)^2 = 50$ **i** $(x-2)^2 + y^2 = 64$

j $(x-4)^2 + (y+3)^2 = \frac{25}{4}$ **k** $(x-3)^2 + (y-4)^2 = 81$ **l** $3x^2 + 3(y+1)^2 = \frac{1}{3}$

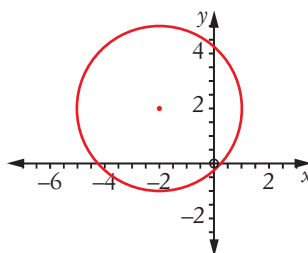
4 What is the equation of this circle? Select **A**, **B**, **C** or **D**.

A $(x-2)^2 + (y+2)^2 = 3$

B $(x+2)^2 + (y-2)^2 = 9$

C $(x+2)^2 + (y-2)^2 = 4$

D $(x-2)^2 + (y+2)^2 = 16$



5 Write the equation of each circle whose centre and radius are given.

a $(1, -2), 3$ **b** $(10, -11), 2$ **c** $(-3, 2), 10$

d $(0, -1), 1$ **e** $(-6, 2), \sqrt{5}$ **f** $(-1, 5), 2\sqrt{2}$

6 Graph each equation.

a $(x+1)^2 + (y+1)^2 = 1$

b $x^2 + (y-4)^2 = 25$

c $(x-1)^2 + y^2 = 16$

d $(x+5)^2 + (y+2)^2 = 4$

7 Find the numbers that 'complete the square' in each equation.

a $x^2 + 2x + \underline{\hspace{1cm}} = (x + \underline{\hspace{1cm}})^2$ **b** $p^2 - 6p + \underline{\hspace{1cm}} = (p - \underline{\hspace{1cm}})^2$

c $m^2 - 8m + \underline{\hspace{1cm}} = (m - \underline{\hspace{1cm}})^2$ **d** $k^2 + 4k + \underline{\hspace{1cm}} = (k + \underline{\hspace{1cm}})^2$

e $y^2 - 7y + \underline{\hspace{1cm}} = (y - \underline{\hspace{1cm}})^2$ **f** $w^2 - 3w + \underline{\hspace{1cm}} = (w - \underline{\hspace{1cm}})^2$

g $x^2 + x + \underline{\hspace{1cm}} = (x + \underline{\hspace{1cm}})^2$ **h** $h^2 - 5h + \underline{\hspace{1cm}} = (h - \underline{\hspace{1cm}})^2$

8 Find the centre and radius of the circle given by each equation.

a $x^2 + 6x + y^2 - 2y = 15$

b $x^2 - 8x + y^2 - 4y = 29$

c $x^2 + 4x + y^2 - 10y = 7$

d $x^2 + 20x + y^2 - 12y + 135 = 0$

e $x^2 + y^2 = 4x - 8y + 5$

f $x^2 - 12x + y^2 + 6y + 29 = 0$

g $x^2 + y^2 = 20y - 6x - 28$

h $x^2 + y^2 + 5 = 8x - 2y$

8-13 Identifying graphs

Summary

Straight line: $y = mx + b$ or $ax + by + c = 0$

Parabola: $y = ax^2 + c$ or $y = a(x - r)^2$

Cubic curve: $y = ax^3 + c$

Power curves: $y = ax^n + c$

Hyperbola: $y = \frac{k}{x}$

Exponential curve: $y = a^x$

Circle: $x^2 + y^2 = r^2$ or $(x - h)^2 + (y - k)^2 = r^2$

Stage 5.3

See Example 21

See Example 22

See Example 23

Puzzle sheet

Matching graphs

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Puzzle sheet

Matching graphs
(Advanced)

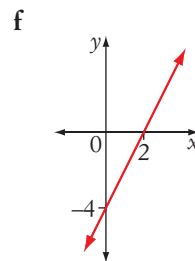
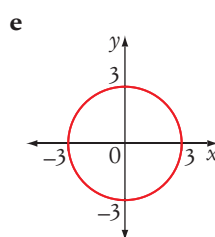
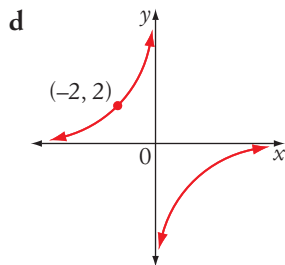
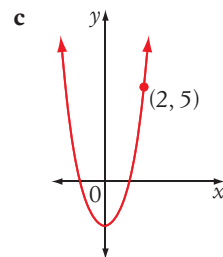
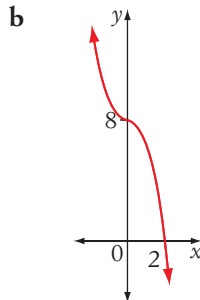
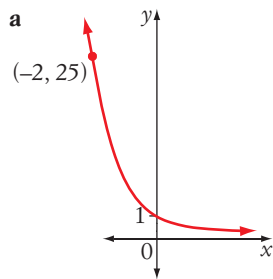
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Stage 5.3

When matching graphs with equations, the x value and y value of a point on the graph may need to be substituted into the equation to verify that the equation represents the graph.

Example 24

Match each graph to its equation.



A $y = 2x - 4$

B $x^2 + y^2 = 9$

C $y = -\frac{4}{x}$

D $y = 2x^2 - 3$

E $y = 5^{-x}$

F $y = -x^3 + 8$

Solution

- a** An exponential curve that matches with **E**, $y = 5^{-x}$

Test point: $(-2, 25)$

LHS = 25

RHS = $5^{-(-2)} = 5^2 = 25 = \text{LHS}$

- b** A decreasing cubic curve with y -intercept 8 that matches with **F**, $y = -x^3 + 8$

Test point: $(2, 0)$

LHS = 0

RHS = $-2^3 + 8 = -8 + 8 = 0 = \text{LHS}$

- c** A concave up parabola that matches with **D**, $y = 2x^2 - 3$

Test point: $(2, 5)$

LHS = 5

RHS = $2 \times 2^2 - 3 = 5 = \text{LHS}$

- d** A 'negative' hyperbola that matches with **C**, $y = -\frac{4}{x}$

Test point: $(-2, 2)$

LHS = 2

RHS = $-\frac{4}{(-2)} = 2 = \text{LHS}$

- e** A circle with centre $(0, 0)$ and radius 3 that matches with **B**, $x^2 + y^2 = 9$

- f** A straight line with gradient 2 and y -intercept -4 that matches with **A**, $y = 2x - 4$

Exercise 8-13 Identifying graphs

Stage 5.3

- 1 For each equation, state whether its graph is a straight line (L), a parabola (P), an exponential (E) or a circle (C).

a $y = 9x^2 - 4$

b $y = 9x$

c $y = 9^x$

d $y = 9$

e $x^2 + y^2 = 81$

f $y = 3x - 8$

g $y = 3x^2 - 8$

h $y = 2x + 5$

i $y = -x^2 + 6$

j $y = 10^{-x}$

k $y = 7x^2 + 2$

l $x^2 + y^2 = 36$

- 2 Match each equation to its graph.

a $x = 4$

b $y = -\frac{1}{2}x + 1$

c $y = 1 - x^2$

d $y = 5$

e $y = 3x^2 - 1$

f $y = 3^x$

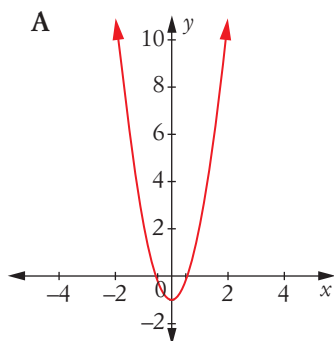
g $x^2 + y^2 = 9$

h $y = 3^{-x}$

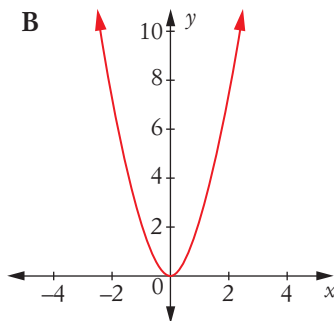
i $y = 2x^2$

j $y = 9x^2 - 4$

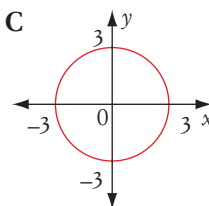
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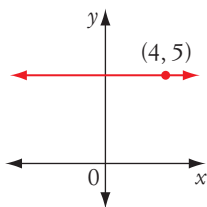
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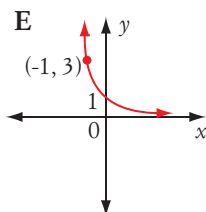
C



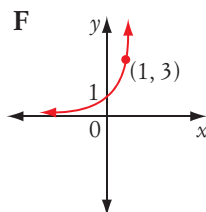
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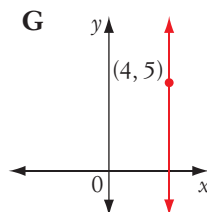
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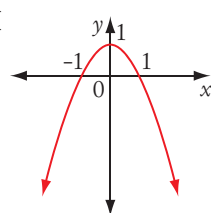
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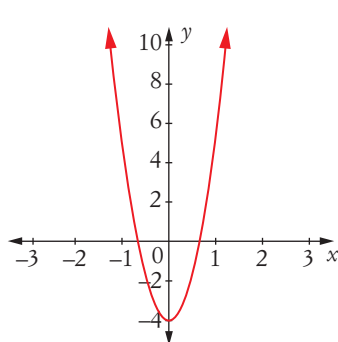
G



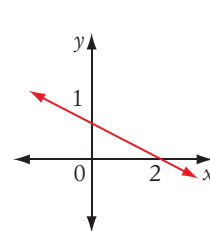
H



I



J



- 3 Sketch the graph of each equation, showing a point on the graph.

a $y = x^2 - 3$

b $y = 5^x$

c $y = -x^2 + 4$

d $x^2 + y^2 = 49$

e $y = \frac{1}{2}x^2$

f $y = -2x + 4$

g $x^2 + y^2 = 144$

- 4 Find the y-intercept of the graph of each equation.

a $y = 3^x$

b $y = 2x^2 + 3$

c $y = -7x^2 - 6$

d $y = 5^{-x}$

See Example 24

Stage 5.3

- 5 For each equation, state whether its graph is a parabola (P), a cubic (Q), a hyperbola (H), an exponential (E) or a circle (C).

a $y = \frac{9}{x}$

b $(x - 2)^2 + y^2 = 4$

c $y = 2(x - 2)^2$

d $y = \frac{-2}{x - 4}$

e $y = 2x^3 + 4$

f $(x + 4)^2 + (y - 1)^2 = 15$

g $y = 4^x + 2$

h $y = -\frac{8}{x} - 1$

- 6 Match each equation to its graph.

a $y = -\frac{4}{x}$

b $y = (x - 3)^3$

c $y = -x^3 + 1$

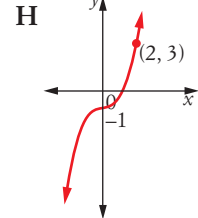
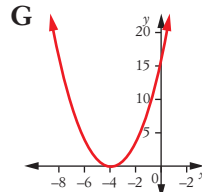
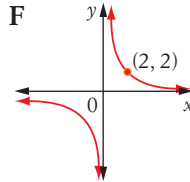
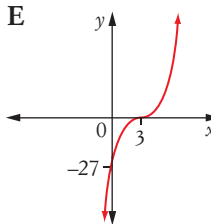
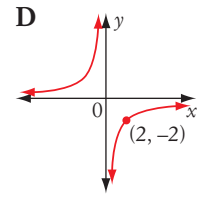
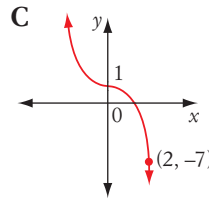
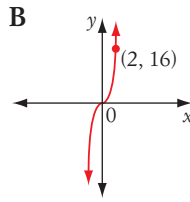
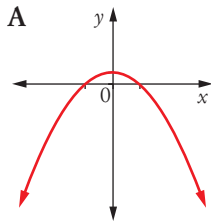
d $y = 1 - x^2$

e $y = \frac{1}{2}x^3 - 1$

f $y = 2x^3$

g $y = \frac{4}{x}$

h $y = (x + 4)^2$



- 7 For each equation,

i write the name of its graph

ii find its y-intercept.

iii find the equation of its asymptote (s).

a $y = 2^x$

b $y = 5^x + 1$

c $y = 10^x - 3$

d $y = \frac{4}{x}$

e $y = \frac{2}{x - 3}$

f $y = -\frac{1}{x} + 2$

- 8 Sketch the graph of each equation, showing all main features.

a $y = \frac{6}{x}$

b $y = 3^x + 2$

c $y = x^3 + 3$

d $y = 2(x - 5)^2$

e $y = -\frac{1}{x + 4}$

f $(x + 5)^2 + (y - 5)^2 = 4$

Power plus

- 1 Graph the equation $y = \frac{1}{x - 1} + 2$, showing all intercepts and asymptotes.

- 2 Sketch the graph of each equation and find the centre and radius of the semicircle.

a $y = \sqrt{16 - x^2}$

b $y = \sqrt{25 - x^2}$

c $y = -\sqrt{9 - x^2}$

Language of maths

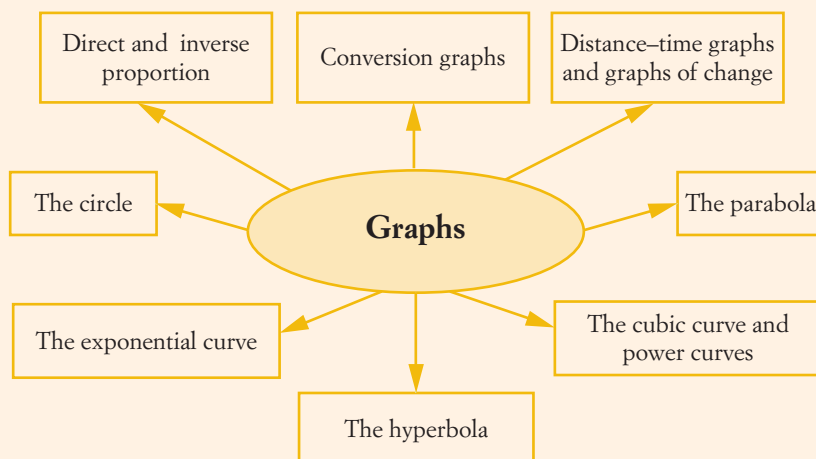
asymptote	axis	centre	circle
coefficient	concave down	concave up	constant
conversion graph	cubic	curve	direct proportion
distance–time graph	exponential	hyperbola	inverse proportion
parabola	quadratic	radius	table of values
variable	vertex	x -intercept	y -intercept

- 1 What is the **coefficient** of x^3 in the cubic equation $y = -x^3 + 10$?
- 2 What is the graph of a **quadratic equation** called?
- 3 True or false: The exponential curve $y = 2^x$ passes through the point $(0, 0)$.
- 4 In the variation equation $y = \frac{k}{x}$, which is the **constant of proportionality**?
- 5 In which quadrants of the number plane does the graph of $y = -\frac{2}{x}$ appear?
- 6 What is the **asymptote** of the exponential curve $y = a^x$?

Topic overview

- Which parts of this chapter were revision of Year 9 knowledge and skills?
- Which parts of this chapter were new to you?
- What is the difference between direct and inverse proportion?
- Do you know the equations of a parabola, cubic curve, hyperbola, exponential curve and circle, and how to graph them?
- Explain how the graph of $y = 2x^2 + 3$ is different to the graph of $y = -2x^2 + 3$. How are they similar?

Copy and complete this mind map of the topic, adding detail to its branches and using pictures, symbols and colour where needed. Ask your teacher to check your work.



Chapter 8 revision

See Exercise 8-01

- 1 H is directly proportional to t . If when $t = 12$, $H = 138$, find H when $t = 27$.

See Exercise 8-02

- 2 The temperature, T (in degrees Celsius), of the air is inversely proportional to the height, h (in metres), above sea level. At 400 m above sea level, the temperature is 15°C . What is the temperature at 600 m above sea level?

See Exercise 8-03

- 3 The graph in Example 4 on page 294 converts Australian dollars to UK pounds sterling. Use the graph to convert:

- a \$A70 to £ b £56 to Australian dollars

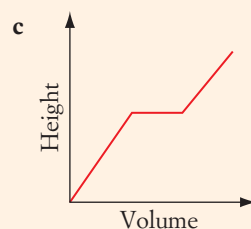
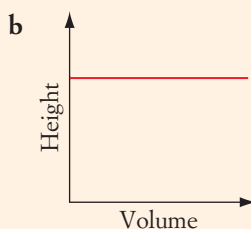
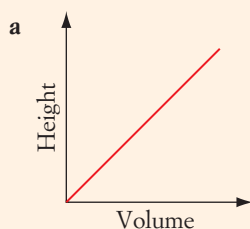
Stage 5.3

See Exercise 8-04

- 4 Draw a distance–time graph to show a person leaving home at an increasing speed and then slowing down.

See Exercise 8-05

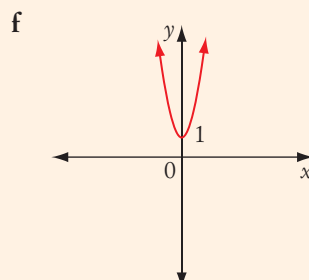
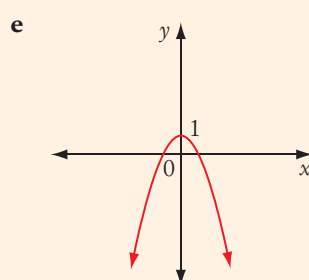
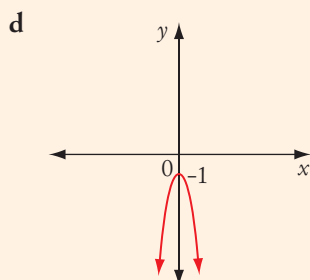
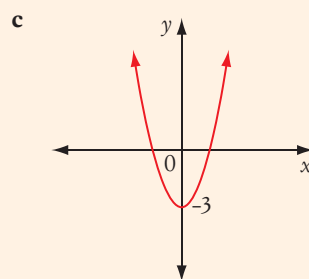
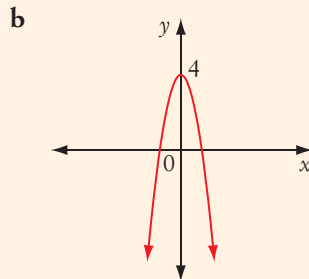
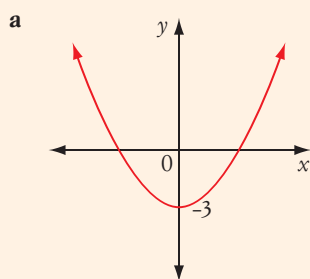
- 5 The graphs below show the height of water in a bathtub over time. Match each graph to the correct description.



- A The bath is three-quarters full of water.
 B Hot water is accidentally run into the bath, so the tap is turned off to let the bath cool. Then it is filled with cold water only.
 C The bath is filled at a steady rate.

See Exercise 8-06

- 6 Match each graph with its correct equation.



A $y = x^2 - 3$

B $y = 3x^2 + 1$

C $y = \frac{1}{2}x^2 - 3$

D $y = -x^2 + 1$

E $y = -4x^2 - 1$

F $y = 4 - 3x^2$

- 7 Graph the equation $y = 2(x + 3)^2$, showing all main features.
- 8 Graph the equation $y = x^3 + 2$, showing all main features.
- 9 Sketch the graphs of the equations $y = -3x^4$ and $y = -3(x - 2)^4$, showing all main features.
- 10 Sketch the graph of $y = \frac{2}{x+1}$, showing the x - and y -intercepts and a point on the curve.
- 11 Sketch each curve described below.

a $y = 4^x$

b $y = 4^{-x}$

c $y = -4^x$

d $y = -4^{-x}$

- 12 Find the centre and radius of each circle described below.

a $x^2 + y^2 = 100$

b $x^2 + y^2 = 36$

c $x^2 + y^2 = 49$

d $(x + 5)^2 + (y - 6)^2 = 81$

e $(x - 10)^2 + y^2 = 225$

f $(x + 7)^2 + (y + 10)^2 = 80$

- 13 Match each equation with its correct graph.

a $y = \frac{1}{4}x^2$

b $y = 3^x$

c $y = -2x^2 - 1$

d $x = -5$

e $y = -3x^2$

f $y = 2x^2 - 1$

g $y = 3^{-x}$

h $x^2 + y^2 = 25$

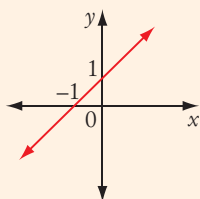
i $y = x^2$

j $y = x + 1$

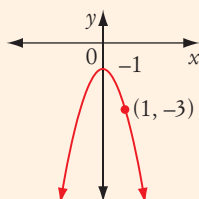
k $y = -5$

l $y = -2 - 2x$

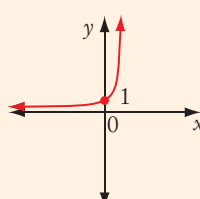
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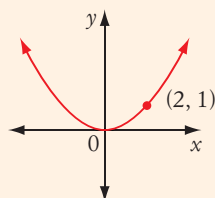
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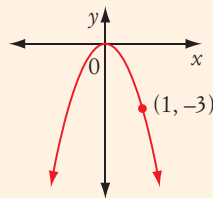
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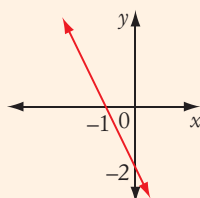
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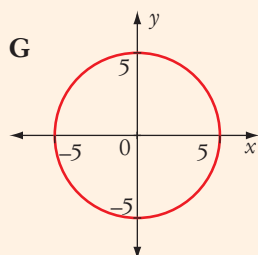
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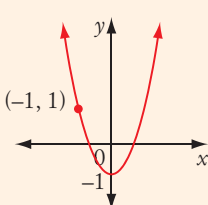
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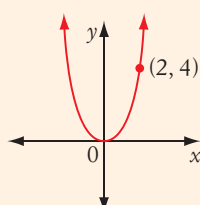
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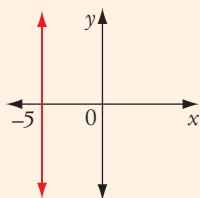
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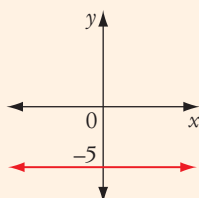
I



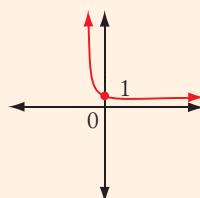
J



K



L



Stage 5.3

See Exercise 8-07

See Exercise 8-08

See Exercise 8-09

See Exercise 8-10

See Exercise 8-11

See Exercise 8-12

Stage 5.3

See Exercise 8-13



9

Measurement and geometry

Trigonometry

The word **trigonometry** comes from the Greek language: *trigonon*, meaning triangle, and *metron*, meaning measure. Trigonometry uses triangles to find unknown lengths and angles that cannot be measured physically. It has wide applications in engineering, surveying, navigation, astronomy, electronics and construction.



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Chapter outline

	Proficiency strands				
9-01 Right-angled trigonometry	U	F	PS	R	C
9-02 Bearings	U	F	PS	R	C
9-03 Pythagoras' theorem and trigonometry in 3D*		F	PS	R	C
9-04 Trigonometric relations*	U	F		R	C
9-05 The trigonometric functions*	U	F		R	C
9-06 Trigonometric equations*	U	F		R	C
9-07 The sine rule*	U	F	PS	R	C
9-08 The sine rule for angles*	U	F	PS	R	C
9-09 The cosine rule*	U	F	PS	R	C
9-10 The cosine rule for angles*	U	F	PS	R	C
9-11 The area of a triangle*	U	F	PS	R	C
9-12 Problems involving the sine and cosine rules*		F	PS	R	C

*STAGE 5.3

9780170194662

Wordbank

bearing The angle used to show the direction of one location from a given point

complementary angles Two angles that add to 90°

cosine rule A rule that relates the three sides and one of the angles of any triangle: $a^2 = b^2 + c^2 - 2bc \cos A$

exact ratio The sine, cosine and tangent of the special angles 30° , 45° and 60° , which can be expressed as exact fractions or surds rather than decimal approximations

included angle The angle between two known sides

sine rule A rule that relates the sides of any triangle to the sine of their opposite angles: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

In this chapter you will:

- solve right-angled triangle problems including those involving direction and angles of elevation and depression
- (STAGE 5.3) apply Pythagoras' theorem and trigonometry to solving three-dimensional problems in right-angled triangles
- (STAGE 5.3) use the unit circle to define trigonometric functions, and graph them with and without the use of digital technologies
- (STAGE 5.3) solve simple trigonometric equations
- (STAGE 5.3) establish the sine, cosine and area rules for any triangle and solve related problems 
- apply trigonometry to problems involving bearings
- (STAGE 5.3) understand and use the exact trigonometric ratios and relations such as $\sin \theta = \cos (90^\circ - \theta)$, $\tan \theta = \frac{\sin \theta}{\cos \theta}$ and $\cos A = -\cos (180^\circ - A)$

SkillCheck

Worksheet

StartUp assignment 9
(Advanced)

MAT10MGWK10218

Worksheet

Trigonometric
calculations

MAT10MGWK10056

Worksheet

Investigating
trigonometry ratios

MAT10MGWK00027

Puzzle sheet

Trigonometry equations

MAT10MGPS00032

Puzzle sheet

Finding an unknown
angle

MAT10MGPS00033

Worksheet

Trigonometry review

MAT10MGWK10057

Puzzle sheet

Solving triangles

MAT10MGPS00034

- Round each angle to the nearest degree.
a $64^\circ 27'$ b $25^\circ 43'$ c $12^\circ 8' 50''$
- Evaluate each expression correct to four decimal places.
a $\cos 32^\circ$ b $\sin 50.9^\circ$ c $\tan 8^\circ 45'$
d $200 \tan 18^\circ$ e $14 \sin 87^\circ 40'$ f $\frac{13}{\cos 18^\circ 27'}$
- Convert each angle to degrees and minutes, correct to the nearest minute.
a 45.80 b 33.1750 c 5.3460
- Find the size of angle A , correct to the nearest minute.
a $\cos A = \frac{3}{7}$ b $\tan A = 2.7$ c $\sin A = 0.4716$

9-01 Right-angled trigonometry

There are three **trigonometric ratios** that relate the lengths of two sides of a right-angled triangle: **sine**, **cosine** and **tangent**.

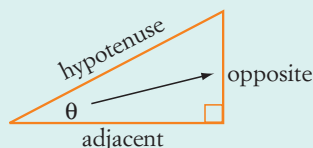
Summary

The trigonometric ratios

$$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

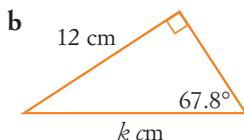
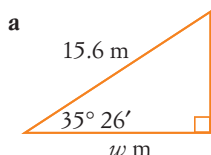
$$\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan \theta = \frac{\text{opposite}}{\text{adjacent}}$$



Example 1

Find the value of each pronumeral, correct to one decimal place.



Solution

$$\begin{aligned}\text{a } \cos 35^\circ 26' &= \frac{w}{15.6} \\ w &= 15.6 \cos 35^\circ 26' \\ &= 12.710 \dots \\ &\approx 12.7 \text{ m}\end{aligned}$$

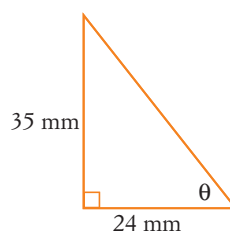
w is adjacent and 15.6 is the hypotenuse, so use $\cos$.
On a calculator: $15.6 \cos 35 \text{ O } ^\circ \text{ ' } 26 \text{ O } ^\circ \text{ ' } =$
From the diagram, 12.7 m seems to be a reasonable answer.

$$\begin{aligned}\text{b } \sin 67.8^\circ &= \frac{12}{k} \\ k &= \frac{12}{\sin 67.8^\circ} \\ &= 12.960 \dots \\ &\approx 13.0 \text{ cm}\end{aligned}$$

12 is opposite and k is the hypotenuse, so use $\sin$.
On a calculator: $12 \div \sin 67.8 =$
From the diagram, 13.0 cm seems to be a reasonable answer.

Example 2

Find the value of θ , correct to the nearest minute.



Solution

$$\begin{aligned}\tan \theta &= \frac{35}{24} \\ \theta &= 55.561 \dots \\ &= 55^\circ 33' 39.64'' \\ &\approx 55^\circ 34'\end{aligned}$$

35 is opposite and 24 mm is the adjacent side, so use $\tan$.

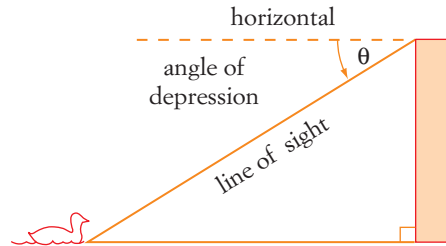
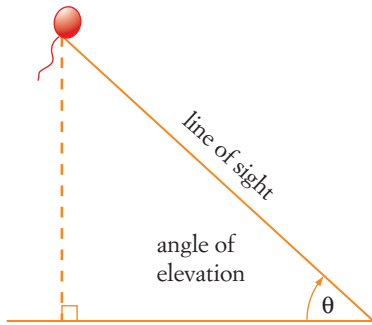
On a calculator: $\text{SHIFT tan } 35 \text{ [DMS]} 24 =$
On a calculator: press $\text{O } ^\circ \text{ ' } \text{''}$ or **DMS** to convert to degrees, minutes, seconds.
From the diagram, $55^\circ 34'$ seems to be a reasonable answer.

Video tutorial

Trigonometry

MAT10MGVT00010

Angles of elevation and depression



Problems involving angles of elevation and depression usually require the $\tan$ ratio in their solutions.

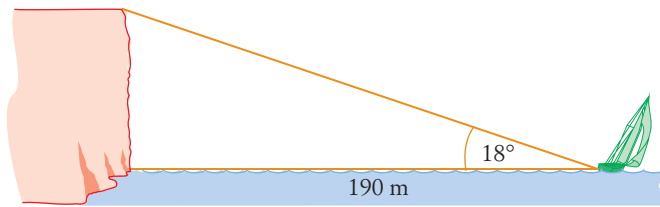
Video tutorial

Angles of elevation and depression

MAT10MGVT10023

Example 3

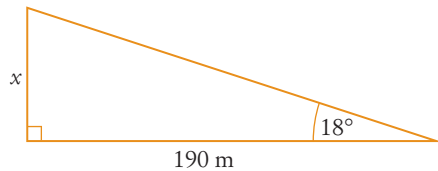
The angle of elevation from a yacht to the top of a cliff is 18° . If the yacht is 190 m from the base of the cliff, find correct to one decimal place the height of the cliff.



Solution

Let the height be x metres.

$$\begin{aligned}\tan 18^\circ &= \frac{x}{190} \\ x &= 190 \tan 18^\circ \\ &= 61.73474 \dots \\ &\approx 61.7 \text{ m}\end{aligned}$$



The height of the cliff is 61.7 m.

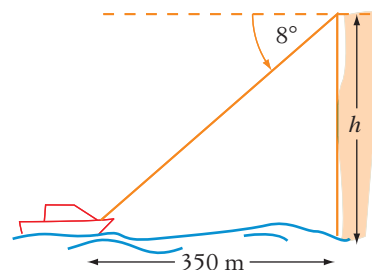
Video tutorial

Angles of elevation and depression

MAT10MGVT10023

Example 4

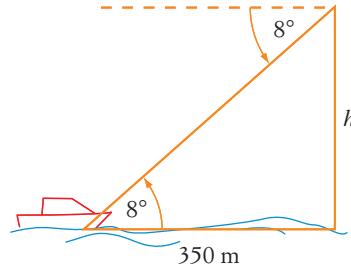
The angle of depression of a boat from the top of a cliff is 8° . If the boat is 350 m from the base of the cliff, calculate the height of the cliff, correct to the nearest metre.



Solution

By alternate angles, the angle of elevation of the top of the cliff from the boat is also 8° .

$$\begin{aligned}\tan 8^\circ &= \frac{h}{350} \\ h &= 350 \tan 8^\circ \\ &= 49.1892 \dots \\ &\approx 49\end{aligned}$$



The height of the cliff is 49 m.

Alternative method

The third angle in the triangle (adjacent to the angle of depression) = $90^\circ - 8^\circ = 82^\circ$.

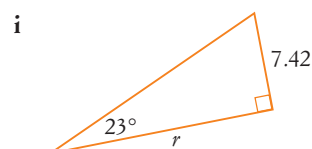
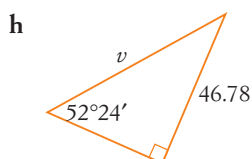
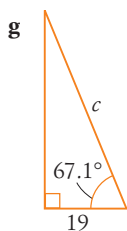
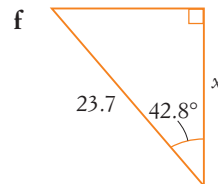
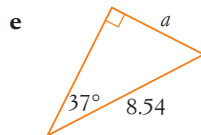
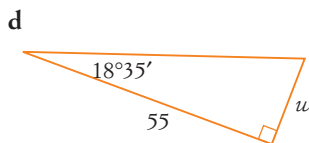
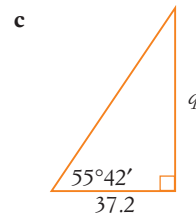
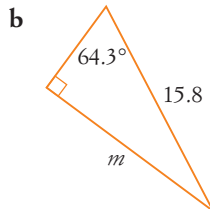
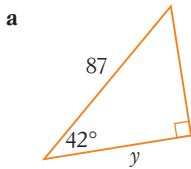
$$\begin{aligned}\tan 82^\circ &= \frac{350}{h} \\ h &= \frac{350}{\tan 82^\circ} \\ &= 49.1892 \dots \\ &\approx 49\end{aligned}$$

The height of the cliff is 49 m.

Exercise 9-01 Right-angled trigonometry

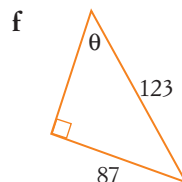
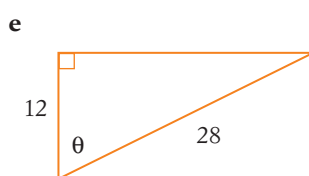
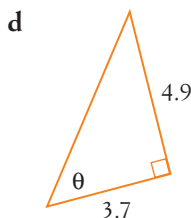
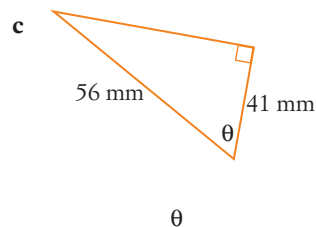
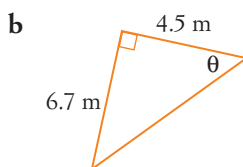
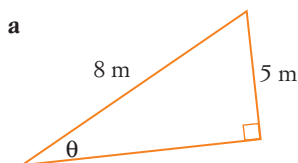
- 1 Calculate, correct to one decimal place, the value of each pronumeral. All measurements are in centimetres.

See Example 1

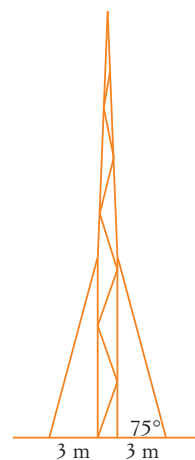


See Example 2

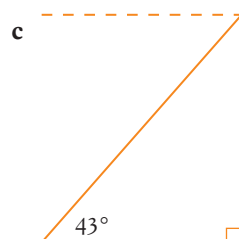
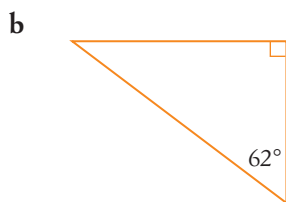
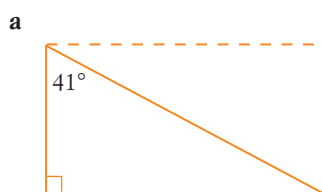
- 2 Find the value of θ , correct to the nearest minute.



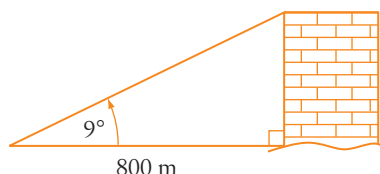
- 3 A 6-metre ladder rests against a wall. The foot of the ladder is 1.8 m from the base of the wall. Find:
- the angle (to the nearest degree) that the ladder makes with the ground
 - the distance (correct to one decimal place) that the ladder reaches up the wall.
- 4 A road rises 55 m for every 750 m travelled along it. To the nearest degree, at what angle is the road rising?
- 5 A kite attached to a string is flying at a height of 75 m. If the string makes an angle of 49° with the vertical, what is the length of the string, correct to the nearest metre?
- 6 A ramp rises at an angle of 12° to the ground. If the top of the ramp is 1.35 m above the ground, calculate the length of the ramp, correct to two decimal places.
- 7 A section on a water slide has a vertical rise of 50 m and a horizontal run of 40 m.
- At what angle is the slide descending, correct to the nearest minute?
 - What is the gradient of the section?
- 8 The training of a football team involves running up and down a sandhill 25 times. If the hill is 30 m high and inclined at an angle of 35° to the horizontal, how far does the team run during training? Answer correct to the nearest 0.1 km.
- 9 A radio tower is supported by cables as shown. The cables make an angle of 75° with the ground and are fixed 3 m from the base of the tower. Calculate, correct to one decimal place:
- the length of each cable
 - how far up the tower each cable is fastened.



10 Copy each diagram, mark the angle of depression θ and find its size.

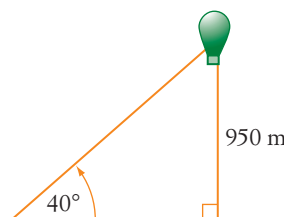


- 11 Sang Koo stands 800 m from the base of a building. His angle of elevation to the top of the building is 9° . Find the height of the building, correct to the nearest metre.

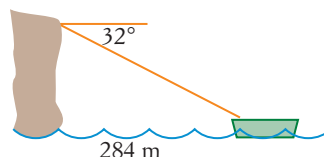


See Example 3

- 12 The angle of elevation of a weather balloon at a height of 950 m is 40° . How far (to the nearest metre) is the observer from being directly under the balloon?

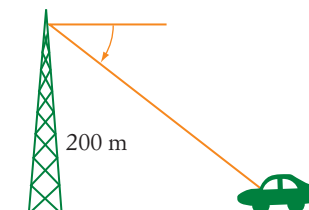


- 13 A raft is 284 m from the base of a cliff. The angle of depression of the raft from the top of the cliff is 32° . Find the height of the cliff, correct to the nearest metre.



See Example 4

- 14 From the top of a 200 m tower, the angle of depression of a car is 48° . How far is the car from the foot of the tower? Answer correct to the nearest metre.



- 15 In a concert hall, Bill is sitting 20 m from the stage by line of sight, 5 m above the level of the stage. At what angle of depression is the stage? Answer correct to the nearest minute.
- 16 A tree that is 8.45 m tall casts a shadow that is 1.53 m long. Find the angle of elevation of the Sun, correct to the nearest minute.
- 17 A glider is directly above one end of a runway at a height of 255 m. The angle of depression to the other end of the runway is 28° . How long is the runway, correct to the nearest metre?
- 18 A building casts a shadow 53 m long. The angle of elevation of the Sun is 42° . How tall is the building, correct to the nearest 0.1 metre?
- 19 A flagpole is mounted on top of a building. At a distance of 250 m from the base of the building, the angles of elevation of the bottom and top of the flagpole are 38° and 40° respectively. Calculate the height of the flagpole, correct to one decimal place.
- 20 A news helicopter hovers at a height of 500 m. The angles of depression of a fire moving in the direction of the helicopter are first 10° and then 15° . How far (to the nearest metre) has the fire moved between the two observations?

Worksheet

A page of bearings

MAT10MGWK10058

Worksheet

Bearings match-up

MAT10MGWK10061

Worksheet

NSW map bearings

MAT10MGWK10060

Worksheet

16 points of the compass

MAT10MGWK10059

Worksheet

Elevations and bearings

MAT10MGWK10062

Quiz

Trigonometry

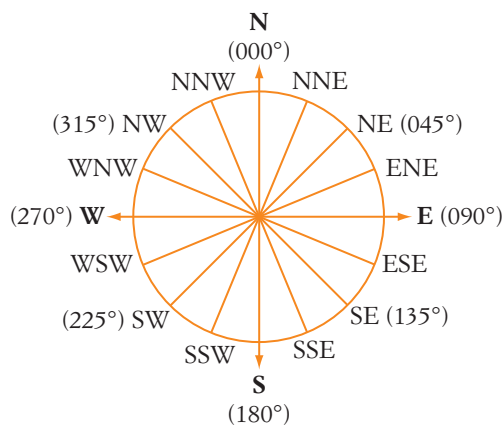
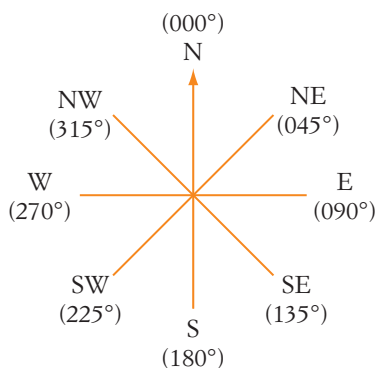
MAT10MGQZ00010

9-02 Bearings

Bearings are used in navigation. A bearing is an angle measurement used to describe precisely the direction of one location from a given reference point.

Three-figure bearings, also called true bearings, use angles from 000° to 360° to show the amount of turning measured **clockwise from north 000°** . Note that the angles are always written with three digits.

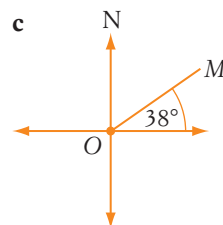
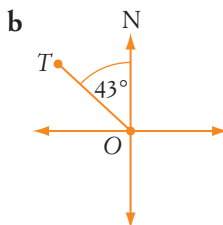
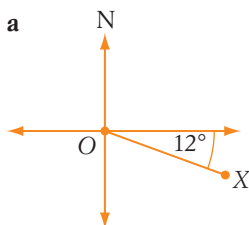
The **compass rose** (below left) shows the three-figure bearings of eight points on the compass. A bearing of due east is 090° , while a compass direction of southwest (SW) is 225° .



Compass bearings refer to sixteen points of a mariner's compass (above right).

Example 5

Write the three-figure bearing of each point from O .



Solution

a The bearing of X from O is $90^\circ + 12^\circ = 102^\circ$.

b The bearing of T from O is $360^\circ - 43^\circ = 317^\circ$.

c The bearing of M from O is $90^\circ - 38^\circ = 052^\circ$

Must be written as a three-digit angle.

Example 6

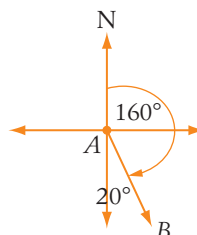
Sketch point B on a compass rose if B has a bearing of 160° from A .

Solution

Draw the compass rose on the point where the bearing is being measured from.

160° is between 90° and 180° , so B is in the southeast (SE) quadrant.

$180^\circ - 160^\circ = 20^\circ$, so B is 20° from south (S).



Example 7

A plane leaves a town and remains on a bearing of 122° for 260 km.

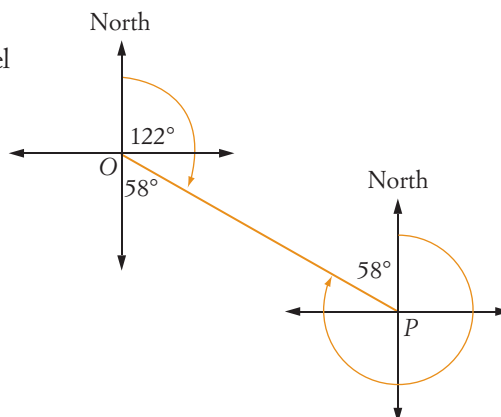
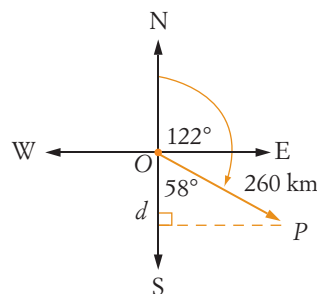
- How far south of the town is the plane, correct to one decimal place?
- What is the bearing of the town from the plane?

Solution

- Let d km = distance south
 $\angle SOP = 180^\circ - 122^\circ$ (angles on a straight line)
 $= 58^\circ$
 $\cos 58^\circ = \frac{d}{260}$
 $d = 260 \cos 58^\circ$
 $= 137.7790 \dots$
 ≈ 137.8

The plane is 137.8 km south of the town.

- Draw a compass rose with its centre at P .
 $\angle OPN = 58^\circ$ (alternate angles on parallel lines)
 Bearing of O from $P = 360^\circ - 58^\circ$
 $= 302^\circ$
 Bearing of town from plane = 302°



Video tutorial

Bearings

MAT10MGVT10022

Video tutorial

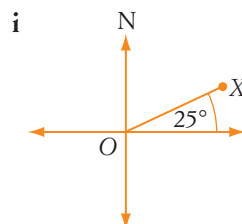
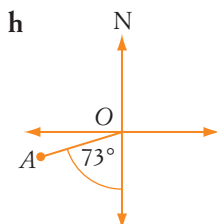
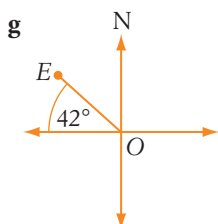
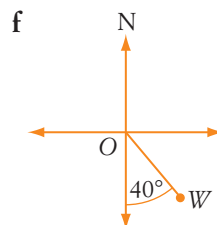
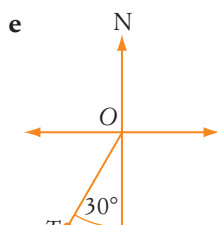
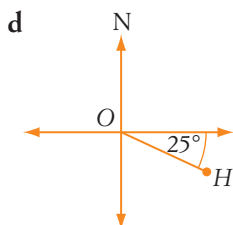
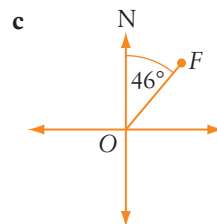
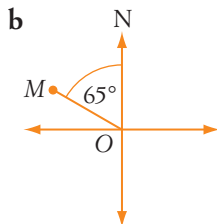
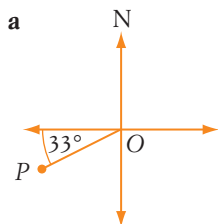
Trigonometry

MAT10MGVT00010

Exercise 9-02 Bearings

See Example 5

- 1 Write the bearing of each point from O .

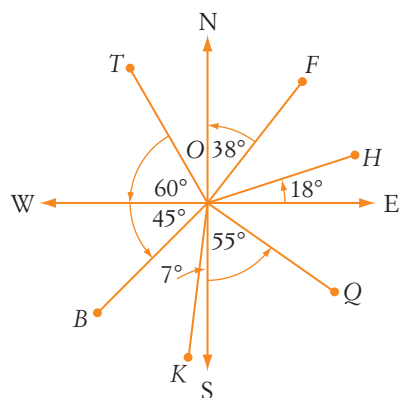


- 2 What is the bearing of each point from O ?

a N **b** E **c** S **d** W
e F **f** Q **g** T **h** B
i H **j** K

- 3 What is the compass direction shown by point B in question 2?

- 4 **a** What is the compass direction halfway between northwest and west?
b What is the three-figure bearing of this compass direction?

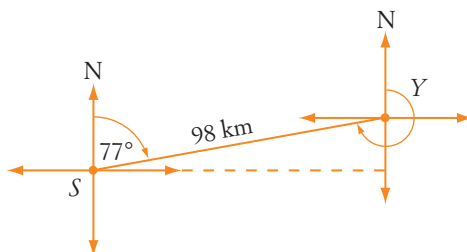


See Example 6

- 5 Sketch P on a compass rose if P has a bearing of:
a 132° from T **b** 260° from M **c** 335° from X **d** 010° from K

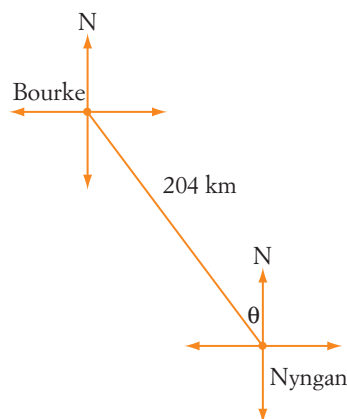
See Example 7

- 6 A yacht leaves Sydney and sails 98 km on a bearing of 077° .
a How far north of Sydney is the yacht?
b What is the bearing of Sydney from the yacht?



- 7 Colin leaves Bourke and drives 204 km to Nyngan.
The bearing of Nyngan from Bourke is 127° .

- a Find the value of θ .
- b How far south of Bourke is Nyngan?
- c What is the bearing of Bourke from Nyngan?



- 8 Jana cycles 10 km due west, then 7 km due north.
 - a How far (correct to one decimal place) is Jana from her starting point?
 - b What is her bearing from the starting point, correct to the nearest degree?
- 9 A triathlete cycles 20 km on a SSE bearing to the finish line.
 - a How far (to the nearest km) has the triathlete travelled in a southerly direction?
 - b What is the compass bearing of the starting point from the finish line?
- 10 A hiking group walks from Sandy Flats to Black Ridge (a distance of 20.9 km) in the direction 078° . They then turn and hike due south to Rivers End, then due west back to Sandy Flats. How far have they hiked altogether (to the nearest 0.1 km)?
- 11 A triangular orienteering run starts at Alpha and passes through the checkpoints of Bravo and Charlie before finishing at Alpha. Bravo is 8.5 km due east of Alpha, and Charlie is 10.5 km due south of Bravo.
 - a Calculate, correct to three decimal places, the distance from Charlie to Alpha.
 - b Find the bearing of Alpha from Charlie, correct to the nearest degree.
- 12 A plane takes off from Darwin at 10:15 a.m. and flies on a bearing of 150° at 700 km/h.
 - a How far (to the nearest km) due south of Darwin is the plane at 1:45 p.m.?
 - b What is the bearing (correct to the nearest degree) of Darwin from the plane?
- 13 A fishing trawler sails 30 km from port on a bearing of 120° until it reaches a submerged reef. How far (to the nearest km) is the port:
 - a north of the reef?
 - b west of the reef?
- 14 Two racing pigeons are set free at the same time. The first bird flies on a course of 040° while the second bird flies on a course of 130° .
 - a The first bird flies 200 km until it is due north of the second bird. Find their distance apart, correct to two decimal places.
 - b How far has the second bird flown?

Stage 5.3

Puzzle sheet

Pythagorean two-step problems

MAT10MGPS00031

Puzzle sheet

Pythagorean triads

MAT10MGPS00030

Animated example

2D and 3D applications of trigonometry

MAT10MGAE00010

9-03

Pythagoras' theorem and trigonometry in 3D

Example 8

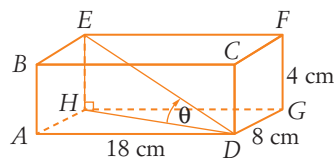
A wooden box has the shape of a rectangular prism with dimensions $18\text{ cm} \times 8\text{ cm} \times 4\text{ cm}$.



- Find, correct to one decimal place, the length of the longest pencil that can lie flat in the base of the box.
- Find, correct to one decimal place, the length of the longest diagonal of the box.
- Find, correct to the nearest degree, the angle that the longest diagonal makes with the base of the box.

Solution

Label the box as shown. HD is the length of the longest pencil that can lie flat in the base of the box, while ED is the longest diagonal of the box.



using Pythagoras' theorem

- Using the right-angled triangle DAH :

$$\begin{aligned} HD^2 &= DA^2 + AH^2 \\ &= 18^2 + 8^2 \\ &= 388 \\ HD &= \sqrt{388} \\ &= 19.6977 \dots \\ &\approx 19.7 \text{ cm} \end{aligned}$$

The longest pencil that can lie flat in the base of the box is 19.7 cm.

From the diagram, 19.7 cm seems to be a reasonable answer.

- Using the right-angled triangle EHD :

$$\begin{aligned} ED^2 &= HD^2 + HE^2 \\ ED^2 &= (\sqrt{388})^2 + 4^2 \\ &= 404 \\ ED &= \sqrt{404} \\ &= 20.0997 \dots \\ &\approx 20.1 \text{ cm} \end{aligned}$$

The longest diagonal of the box is 20.1 cm.

using Pythagoras' theorem

From the diagram, 20.1 cm seems to be a reasonable answer.

- c In the right-angled triangle EHD , θ is the angle that the longest diagonal makes with the base of the box.

$$\begin{aligned}\tan \theta &= \frac{EH}{HD} \\ &= \frac{4}{\sqrt{388}}\end{aligned}$$

$$\theta = 11.4789 \dots^\circ$$

$$\approx 11^\circ$$

The longest diagonal makes an angle of 11° with the base of the box.

or use sin or cos as ED is also known

$$\text{SHIFT} \tan 4 \left(\frac{\square}{\square} \right) \sqrt{} 388 =$$

Example 9

A 100 m high flagpole is observed from two different locations. From point A , due south of the flagpole the angle of elevation of the top of the flagpole is 35° ; from point B , due east of the flagpole the angle of elevation is 22° . Find, correct to the nearest metre, the distance between A and B .

Solution

There are three right-angled triangles in this diagram. To find AB , we must first find AC and CB using trigonometry.

In triangle ACT ,

$$\tan 35^\circ = \frac{100}{AC}$$

$$\begin{aligned}AC &= \frac{100}{\tan 35^\circ} \\ &\approx 142.8148 \dots\end{aligned}$$

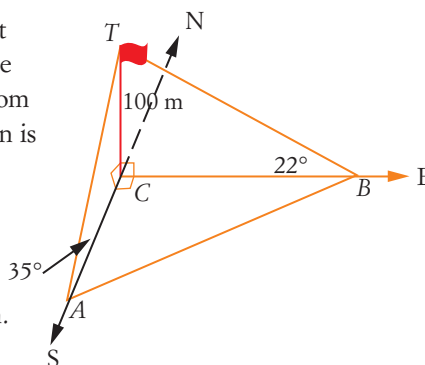
In triangle BCT ,

$$\tan 22^\circ = \frac{100}{CB}$$

$$\begin{aligned}CB &= \frac{100}{\tan 22^\circ} \\ &\approx 247.5086 \dots\end{aligned}$$

In triangle ABC ,

$$\begin{aligned}AB^2 &= AC^2 + CB^2 \\ &= (142.8148 \dots)^2 + (247.5086 \dots)^2 \\ &= 81\,656.6166 \dots \\ AB &= \sqrt{81\,656.6166 \dots} \\ &= 285.7562 \dots \\ &\approx 286 \text{ m}\end{aligned}$$



Do not round yet.

Do not round yet.

From the diagram, 286 m seems to be a reasonable answer.

Stage 5.3

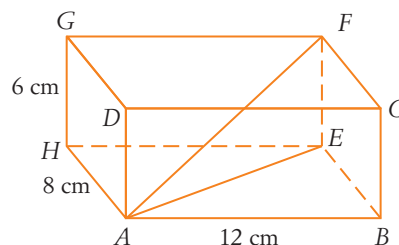
See Example 8

Exercise 9-03

Pythagoras' theorem and trigonometry problems in 3D

- 1 For this rectangular prism, calculate:

- the length of AE in surd form
- the length of AF correct to one decimal place
- the size of $\angle FAE$ correct to the nearest degree.

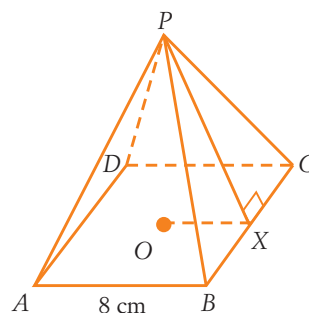


- 2 Sketch a diagram of a cube of length 20 cm, then find:

- the exact length of the longest diagonal on any face
- the length of the longest diagonal of the cube, correct to two decimal places
- the angle that the longest diagonal makes with the base, correct to the nearest minute.

- 3 The diagram shows a square pyramid with base length 8 cm and perpendicular height 20 cm. PX is the slant height of the pyramid. Calculate:

- the length of PX , correct to two decimal places
- the angle of inclination of PX , correct to the nearest degree

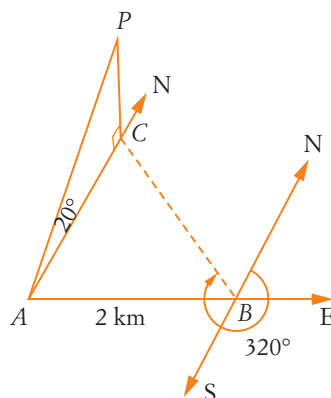


- A cone has a base diameter of 2.8 m and a slant height of 2.5 m. Find the angle that the cone makes with the vertical at the top of the cone.
- A fruit juice container has the shape of a square prism with dimensions 8 cm by 3 cm by 3 cm.
 - Find, correct to one decimal place, the length of the longest straw that fits inside the container.
 - Correct to the nearest degree, what angle does the longest straw make with the vertical?
- From a point X , 37 m from the base of a tree, the angle of elevation is 55° while the angle of elevation of the tree, from a point Y due east of the tree, is 25° . Find, correct to the nearest metre:
 - the height of the tree
 - the distance of the tree from point Y
- A 15 m flagpole stands on level ground. From point P , due west of the flagpole the angle of elevation of the top of the pole is 38° . From point Q , due north of the flagpole, the flagpole has an angle of elevation of 25° . Find the distance of PQ , correct to one decimal place.

See Example 9

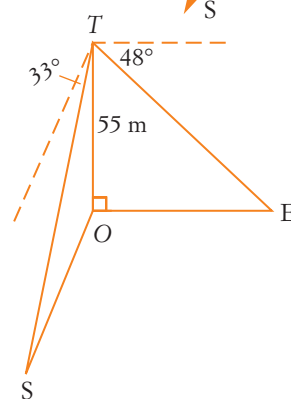
- 8 From a point, A , at the base of a mountain, the mountain peak, P , is due north and has an angle of elevation of 20° . From a point, B , 2 km due east of A on the same level, the mountain peak has a bearing of 320° .

- a What is the size of $\angle CBA$?
- b Calculate the height of the mountain PC correct to the nearest metre.



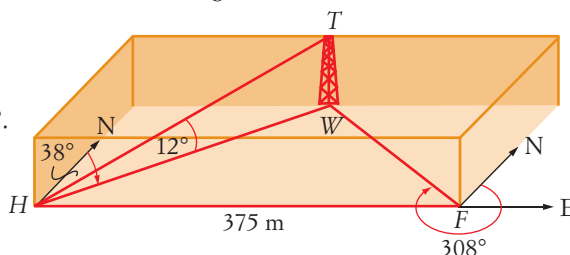
- 9 From the top of her 55 metre office building, Madison observes two cars parked at ground level. The angle of depression of the car due east of the building is 48° and the angle of depression of the car parked due south of the building is 33° . Calculate, correct to the nearest metre, how far:

- a each car is from the building
- b the cars are apart.



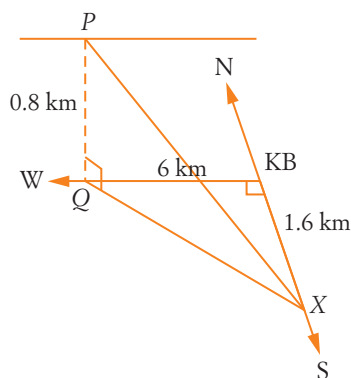
- 10 Hassan observes a transmission tower at an elevation angle of 12° and bearing 038° . Fatima stands 375 m due east of Hassan and observes the tower at a bearing of 308° .

- a Find the sizes of the angles of $\triangle FHW$.
- b Find the height of the tower TW correct to one decimal place.



- 11 A plane flies 6 km due west of Keira Bay at a constant height of 800 m. Xander sees the plane from his house 1.6 km south of Keira Bay. Find, correct to the nearest degree:

- a the bearing of the plane from Xander
- b the angle of elevation of the plane from Xander.



Animated example
2D and 3D applications
of trigonometry

MAT10MGAE00010

Stage 5.3

NSW

9-04 Trigonometric relations

Trigonometric ratios of complementary angles

Complementary angles add up to 90° and the sine and cosine ratios are called **complementary ratios**. That is why **co-sine** is named as the complement of **sine**.

In $\triangle ABC$, $\angle C = 90^\circ$ so $\angle A + \angle B = 90^\circ$ due to the angle sum of a triangle.

$\therefore A$ and B are complementary angles and $B = 90^\circ - A$.

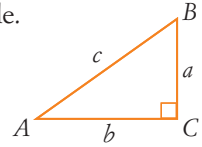
Now $\sin A = \frac{a}{c}$ and $\cos A = \frac{b}{c}$

But $\sin B = \frac{b}{c}$ and $\cos B = \frac{a}{c}$

$\therefore \sin A = \cos B$ and $\cos A = \sin B$

But $B = 90^\circ - A$

$\therefore \sin A = \cos (90^\circ - A)$ and $\cos A = \sin (90^\circ - A)$



Summary

The sine of an angle is equal to the cosine of its complement: $\sin A = \cos (90^\circ - A)$

The cosine of an angle is equal to the sine of its complement: $\cos A = \sin (90^\circ - A)$

Example 10

- a If $\sin 35^\circ = \cos \alpha$, find α .
 b If $P + Q = 90^\circ$ and $\sin Q = \frac{15}{17}$, find:
 i $\cos P$ ii $\cos Q$

Solution

a $\sin 35^\circ = \cos (90^\circ - 35^\circ)$
 $= \cos 55^\circ$

$\therefore \alpha = 55$

b i Since $P + Q = 90^\circ$
 $\cos P = \sin Q$

$\therefore \cos P = \frac{15}{17}$

- ii Since $\sin Q = \frac{15}{17}$, draw a right-angled triangle to find $\cos Q$.

$$x^2 = 17^2 - 15^2$$

$$= 64$$

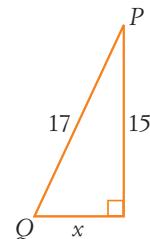
$$x = \sqrt{64}$$

$$= 8$$

$$\therefore \cos Q = \frac{8}{17}$$

$$\sin A = \cos (90^\circ - A)$$

P and Q are complementary angles.



by Pythagoras' theorem

Exact trigonometric ratios of 30° , 45° and 60°

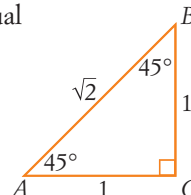
The value of most trigonometric ratios can only be approximated in decimal form. However, the trigonometric ratios of the special angles 30° , 45° and 60° can be written in exact (surd) form.

These are called the **exact ratios**.

For the angle 45° , consider the isosceles right-angled triangle ABC with equal sides 1 unit and equal angles 45° .

By Pythagoras' theorem,

$$\begin{aligned} AB^2 &= 1^2 + 1^2 \\ &= 2 \\ AB &= \sqrt{2} \text{ units} \end{aligned}$$



$$\therefore \sin 45^\circ = \frac{1}{\sqrt{2}}, \cos 45^\circ = \frac{1}{\sqrt{2}}, \tan 45^\circ = \frac{1}{1} = 1$$

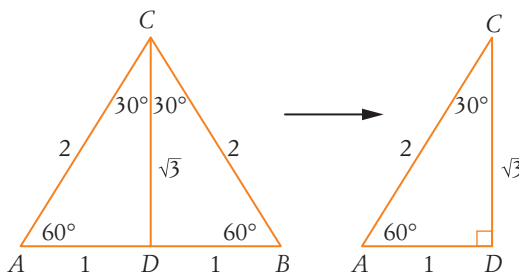
For the angles 30° and 60° , consider the equilateral triangle ABC with equal sides 2 units long and equal angles 60° , then draw an axis of symmetry AD bisecting angle A and side BC .

By Pythagoras' theorem,

$$\begin{aligned} CD^2 &= 2^2 - 1^2 \\ &= 3 \\ CD &= \sqrt{3} \text{ units} \end{aligned}$$

$$\therefore \sin 30^\circ = \frac{1}{2}, \cos 30^\circ = \frac{\sqrt{3}}{2}, \tan 30^\circ = \frac{1}{\sqrt{3}}$$

$$\sin 60^\circ = \frac{\sqrt{3}}{2}, \cos 60^\circ = \frac{1}{2}, \tan 60^\circ = \sqrt{3}$$



Stage 5.3

Worksheet

The exact ratios

MAT10MGWK10219

Summary

The exact trigonometric ratios

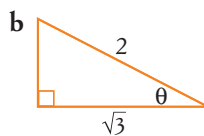
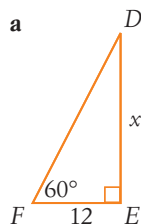
	30°	45°	60°
sin	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
cos	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
tan	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$

Note that $\sin 30^\circ = \cos 60^\circ$, $\sin 60^\circ = \cos 30^\circ$, and $\sin 45^\circ = \cos 45^\circ$, because 30° and 60° are complementary angles and 45° is the complement of itself.

Stage 5.3

Example 11

Find the exact value of each pronumeral.



Solution

$$\begin{aligned}\mathbf{a} \quad \tan 60^\circ &= \frac{x}{12} \\ \therefore x &= 12 \tan 60^\circ \\ &= 12\sqrt{3} \quad \tan 60^\circ = \sqrt{3}\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad \cos \theta &= \frac{\sqrt{3}}{2} \\ \theta &= 30^\circ\end{aligned}$$

Exercise 9-04 Trigonometric relations

See Example 10

- 1 Find the value of each pronumeral.

a $\sin 47^\circ = \cos X$

b $\cos 74^\circ = \sin Y$

c $\sin 2.55^\circ = \cos P$

d $\cos 55.2^\circ = \sin \theta$

e $\sin 38^\circ 17' = \cos M$

f $\cos 17^\circ 38' = \sin T$

- 2 Simplify each expression.

a $\cos 22^\circ - \sin 68^\circ$

b $\frac{\sin 57^\circ}{\cos 33^\circ}$

c $\frac{\sin(90^\circ - \theta)}{\cos \theta}$

- 3 If $\sin \alpha = \frac{5}{13}$ and $\alpha + \beta = 90^\circ$, find $\cos \beta$, $\cos \alpha$ and $\sin \beta$.

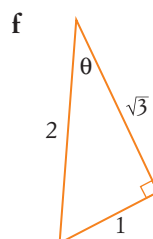
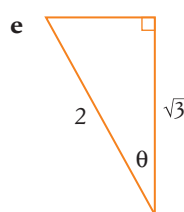
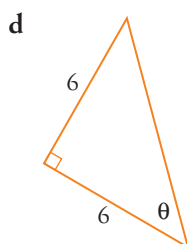
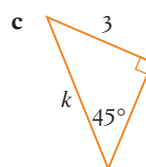
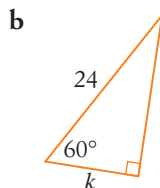
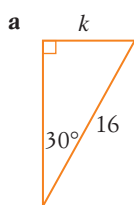
- 4 Given $\cos E = \frac{40}{41}$, and E and F are complementary angles, find $\sin F$, $\sin E$ and $\cos F$.

- 5 If $X + Y = 90^\circ$, find the exact values of $\cos Y$, $\sin Y$ and $\sin X$, given $\cos X = \frac{2}{3}$.

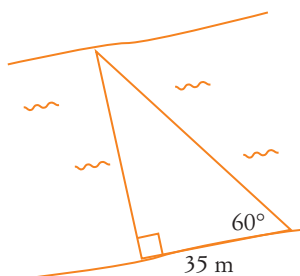
- 6 Find the exact values of $\cos \phi$, $\sin \phi$ and $\cos \theta$ if $\sin \theta = \frac{\sqrt{5}}{4}$ and $\theta = 90^\circ - \phi$.

See Example 11

- 7 In each triangle, find the exact value of the pronumeral.



- 8 Find the width of this river in exact form.



Stage 5.3

Mental skills 9

Maths without calculators

Simplifying fractions and ratios

When simplifying a fraction or a ratio, look for a common factor to divide into both the numerator and the denominator, preferably the highest common factor (HCF).

- 1 Study each example.

- a Simplify $\frac{160}{400}$.

$$\frac{\cancel{16}^{\cancel{10}}}{\cancel{40}^{\cancel{10}}} = \frac{16}{40}$$

dividing numerator and denominator by 10

$$\frac{\cancel{16}^2}{\cancel{40}_5} = \frac{2}{5}$$

dividing numerator and denominator by 8

Note: This fraction could be simplified in one step if you divided by 80, the HCF of 160 and 400.

- b Simplify $135 : 90$.

$$135 : 90 = \cancel{135}^{27} : \cancel{90}^{18} = 27 : 18$$

dividing both terms by 5

$$\cancel{27}^3 : \cancel{18}^2 = 3 : 2$$

dividing both terms by 9

- c What fraction is 36 minutes of 1 hour?

$$\frac{36}{1 \text{ h}} = \frac{36 \text{ min}}{60 \text{ min}} = \frac{3}{5}$$

- 2 Now simplify each fraction or ratio.

a $\frac{10}{15}$

b $\frac{16}{20}$

c $\frac{30}{42}$

d $\frac{8}{16}$

e $\frac{20}{80}$

f $\frac{6}{36}$

g $\frac{20}{24}$

h $\frac{12}{30}$

i $20 : 36$

j $25 : 45$

k $18 : 40$

l $28 : 35$

m $27 : 21$

n $16 : 12$

o $\frac{5}{6} \times \frac{18}{25}$

p $\frac{12}{50} \times \frac{10}{21}$

- 3 Express each as a simplified fraction.

a 425 g of 1 kg

b 8 months of 1 year

c 64 cm of 1 m

d 750 mL of 3 L

e 10 hours of 2 days

f 80c of \$10

Stage 5.3

9-05 The trigonometric functions

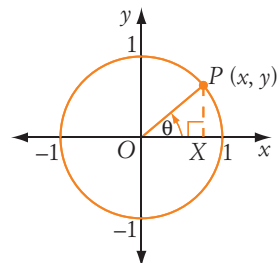
Trigonometric ratios of any angle

The sine, cosine and tangent ratios can be extended to include angles that are over 90° , that is, obtuse and reflex angles. The trigonometric ratios for angles of any size can be best explained using a **unit circle**.

A unit circle is a circle of radius 1 drawn on a number plane, with the origin as the centre of the circle. Starting from the positive direction of the x -axis, angles can be measured around this circle in an anticlockwise direction.

Let $P(x, y)$ be any point on the unit circle as shown and θ the angle that PO makes with the positive x -axis.

Let the vertical interval from P meet the x -axis at X to make the right-angled triangle OXP . Since P has coordinates (x, y) , $OX = x$ and $XP = y$.



$$\begin{aligned}\text{In } \triangle XOP, \cos \theta &= \frac{OX}{OP} \\ &= \frac{x}{1} \quad OP = 1 \text{ because it is the radius of the unit circle}\end{aligned}$$

$$\therefore \cos \theta = x \quad \text{The } x\text{-coordinate of point } P \text{ on the unit circle}$$

$$\begin{aligned}\text{Also, } \sin \theta &= \frac{XP}{OP} \\ &= \frac{y}{1}\end{aligned}$$

$$\therefore \sin \theta = y \quad \text{The } y\text{-coordinate of the point } P \text{ on a unit circle}$$

$$\begin{aligned}\text{and } \tan \theta &= \frac{XP}{OX} \\ \therefore \tan \theta &= \frac{y}{x}\end{aligned}$$

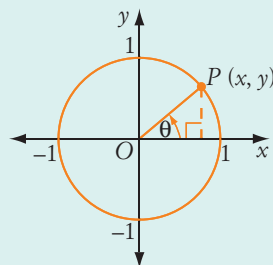
Summary

If $P(x, y)$ is any point on the unit circle, and θ is the angle that PO makes with the positive x -axis, then:

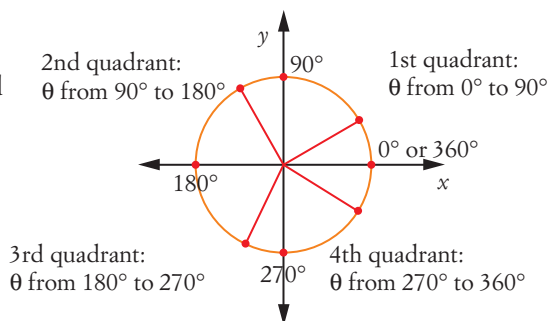
$\sin \theta = y$ -coordinate of P

$\cos \theta = x$ -coordinate of P

$\tan \theta = \frac{y\text{-coordinate of } P}{x\text{-coordinate of } P}$



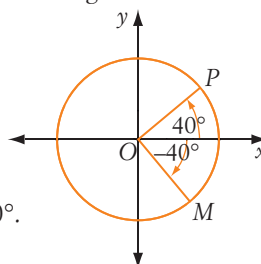
Now we can investigate the trigonometric ratios for all angles from $\theta = 0^\circ$ to 360° , by looking at $P(x, y)$ on the unit circle in the 1st, 2nd, 3rd and 4th quadrants.



The unit circle can also be used to define the trigonometric ratios for angles below 0° and above 360° .

Negative angles (below 0°) are measured in a clockwise direction on the unit circle. In this diagram, M represents -40° but it could also represent $360^\circ - 40^\circ = 320^\circ$.

Angles above 360° are measured on the unit circle by going around the circle more than once. In the diagram, P represents 40° but it could also represent $360^\circ + 40^\circ = 400^\circ$.



The tangent ratio

The tangent ratio can be expressed in terms of the sine and cosine ratios.

Since $\sin \theta = y$ and $\cos \theta = x$,

$$\frac{\sin \theta}{\cos \theta} = \frac{y}{x}$$

But $\tan \theta = \frac{y}{x}$

$$\therefore \tan \theta = \frac{\sin \theta}{\cos \theta}$$

Summary

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

Stage 5.3

Example 12

Given that $\sin \alpha = \frac{2}{\sqrt{13}}$ and $\cos \alpha = \frac{3}{\sqrt{13}}$, find $\tan \alpha$.

Solution

$$\begin{aligned}\tan \alpha &= \frac{\sin \alpha}{\cos \alpha} \\ &= \frac{2}{\sqrt{13}} \div \frac{3}{\sqrt{13}} \\ &= \frac{2}{\sqrt{13}} \times \frac{\sqrt{13}}{3} \\ &= \frac{2}{3}\end{aligned}$$

Technology

GeoGebra:
Angles of any
magnitude

MAT10MGTC00010

Worksheet

The sine and cosine
curves

MAT10MGWK10220

Worksheet

Unit circle investigation

MAT10MGWK00042

Technology worksheet

Excel worksheet
Trigonometric graphs

MAT10MGCT00016

Technology worksheet

Excel spreadsheet:
Trigonometric graphs

MAT10MGCT00046

Trigonometric graphs

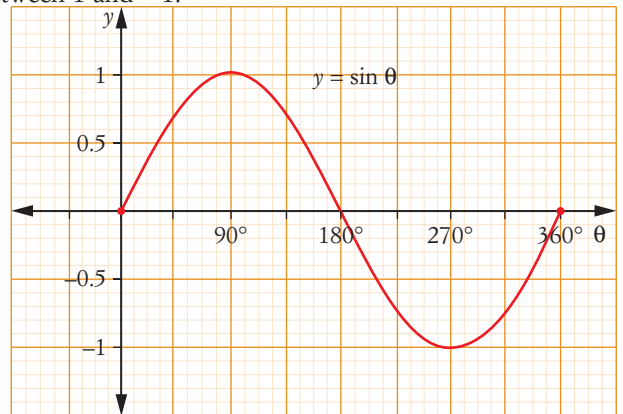
The sine curve

$\sin \theta$ = y -coordinate of P (the height of $\triangle OXP$), so note its value for θ from 0° to 360° .

θ	0°	1st quadrant 0° to 90°	90°	2nd quadrant 90° to 180°	180°	3rd quadrant 180° to 270°	270°	4th quadrant 270° to 360°	360°
$\sin \theta$	0	from 0 to 1	1	from 1 to 0	0	from 0 to -1	-1	from -1 to 0	0

Note that the value of $\sin \theta$ always lies between 1 and -1 .

The graph of $y = \sin \theta$ for θ from 0° to 360° is a 'wave curve' that repeats itself after 360° .



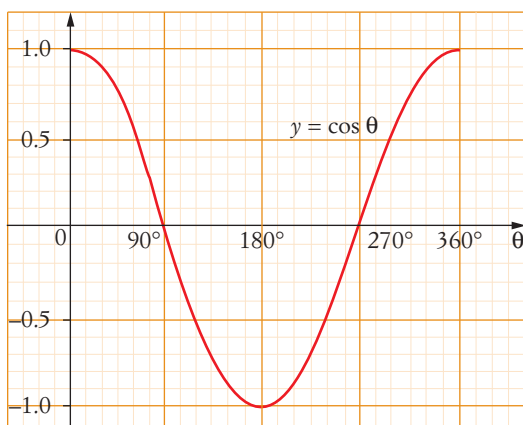
The cosine curve

$\cos \theta$ = x -coordinate of P (the length of the base of $\triangle OXP$), so note its value for θ from 0° to 360° .

θ	0°	1st quadrant 0° to 90°	90°	2nd quadrant 90° to 180°	180°	3rd quadrant 180° to 270°	270°	4th quadrant 270° to 360°	360°
$\cos \theta$	1	from 1 to 0	0	from 0 to -1	-1	from -1 to 0	0	from 0 to 1	1

Note that the value of $\cos \theta$ always lies between 1 and -1 .

The graph of $y = \cos \theta$ for θ from 0° to 360° is a 'wave curve' that repeats itself after 360° .



The tangent curve

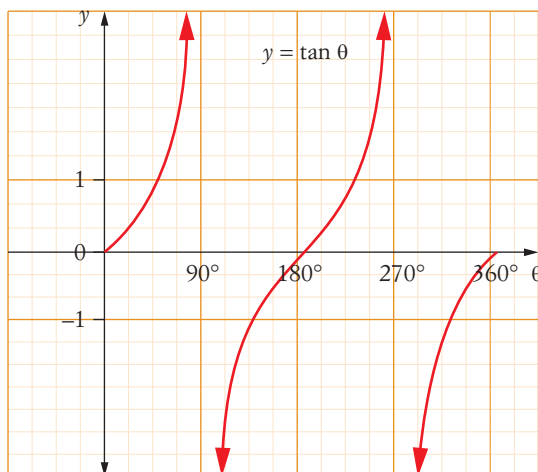
$\tan \theta = \frac{y\text{-coordinate of } P}{x\text{-coordinate of } P} = \frac{\sin \theta}{\cos \theta}$, so note its value for θ from 0° to 360° .

θ	0°	1st quadrant 0° to 90°	90°	2nd quadrant 90° to 180°	180°
$\tan \theta$	$\frac{0}{1} = 0$	$\frac{+}{+} = \text{positive}$	$\frac{1}{0} = \text{undefined}$	$\frac{+}{-} = \text{negative}$	$\frac{0}{-1} = 0$

θ	3rd quadrant 180° to 270°	270°	4th quadrant 270° to 360°	360°
$\tan \theta$	$\frac{-}{-} = \text{positive}$	$\frac{-1}{0} = \text{undefined}$	$\frac{-}{+} = \text{negative}$	$\frac{0}{1} = 0$

Note that the value of $\tan \theta$ has no value at 90° and 270° .

The graph of $y = \tan \theta$ for θ from 0° to 360° is a curve that repeats itself after 180° , with asymptotes at $\theta = 90^\circ$ and 270° .

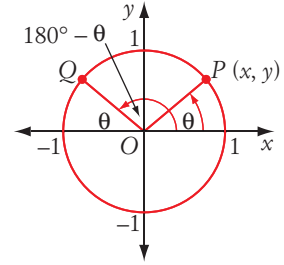


Stage 5.3

Trigonometric ratios of supplementary angles

For obtuse angles (between 90° and 180° , represented by the 2nd quadrant in the unit circle), we can use this diagram where Q is a reflection of P across the y -axis.

QO makes an angle of $180^\circ - \theta$ with the positive x -axis. In the 2nd quadrant, Q has a negative x -coordinate and a positive y -coordinate, so if the coordinates of P are (x, y) , then the coordinates of Q are $(-x, y)$.



$$\begin{aligned}\therefore \cos(180^\circ - \theta) &= -x && \text{sine} \\ &= -\cos \theta\end{aligned}$$

$$\begin{aligned}\therefore \sin(180^\circ - \theta) &= y \\ &= \sin \theta\end{aligned}$$

$$\begin{aligned}\therefore \tan(180^\circ - \theta) &= \frac{y}{-x} \\ &= -\frac{y}{x} \\ &= -\tan \theta\end{aligned}$$

Summary

For obtuse angles (in the second quadrant), sine is positive while cosine and tangent are negative.

The sine of an obtuse angle is equal to the sine of its supplement: $\sin(180^\circ - A) = \sin A$

The cosine of an obtuse angle is equal to the negative cosine of its supplement:

$$\cos(180^\circ - A) = -\cos A$$

The tangent of an obtuse angle is equal to the negative tangent of its supplement:

$$\tan(180^\circ - A) = -\tan A$$

Example 13

If θ is acute, find θ if:

a $\tan 140^\circ = -\tan \theta$

b $\sin 100^\circ = \sin \theta$

c $\cos 120^\circ = -\cos \theta$

Solution

a $\theta = 180^\circ - 140^\circ$
 $= 40^\circ$

$\therefore \tan 140^\circ = -\tan 40^\circ$

b $\theta = 180^\circ - 100^\circ$
 $= 80^\circ$

$\therefore \sin 100^\circ = \sin 80^\circ$

c $\theta = 180^\circ - 120^\circ$
 $= 60^\circ$

$\therefore \cos 120^\circ = -\cos 60^\circ$

Example 14

Find the exact value of each expression.

a $\sin 120^\circ$

b $\tan 135^\circ$

Solution

$$\begin{aligned} \text{a } \sin 120^\circ &= \sin (180^\circ - 120^\circ) \\ &= \sin 60^\circ \\ &= \frac{\sqrt{3}}{2} \end{aligned}$$

$$\begin{aligned} \text{b } \tan 135^\circ &= -\tan(180^\circ - 135^\circ) \\ &= -\tan 45^\circ \\ &= -1 \end{aligned}$$

See Example 12

Exercise 9-05 The trigonometric functions

- a If $\sin A = \frac{60}{109}$ and $\cos A = \frac{91}{109}$, find $\tan A$

b If $\sin Y = 0.2$ and $\cos Y = 0.15$, find $\tan Y$

c If $\sin X = \frac{2}{\sqrt{13}}$ and $\cos X = \frac{3}{\sqrt{13}}$, find $\tan X$

d If $\cos P = \frac{40}{41}$ and $\sin P = \frac{9}{41}$, find $\tan P$

e If $\cos Q = \frac{3}{7}$ and $\sin Q = \frac{\sqrt{40}}{7}$, find $\tan Q$

f If $\cos X = \frac{60}{61}$ and $\tan X = \frac{11}{60}$, find $\sin X$

g If $\tan X = \frac{24}{7}$ and $\sin X = \frac{24}{25}$, find $\cos X$

h If $\tan X = \frac{2}{\sqrt{5}}$ and $\cos X = \frac{\sqrt{5}}{3}$, find $\sin X$
- State whether each acute or obtuse angle is positive (P) or negative (N).

a $\sin 95^\circ$ b $\cos 46^\circ$ c $\tan 153^\circ$ d $\cos 171^\circ$
e $\sin 142^\circ$ f $\tan 91^\circ$ g $\tan 130^\circ$ h $\cos 87^\circ$
- Evaluate, correct to two decimal places, each trigonometric expression.

a $\cos 153^\circ$ b $\tan 349^\circ$ c $\sin 230^\circ$ d $\tan 173^\circ 42'$
e $\cos 300.9^\circ$ f $\sin 324.8^\circ$ g $\sin 176^\circ 54'$ h $\cos 245^\circ 23'$
i $\tan (-38^\circ)$ j $\sin (-61^\circ)$ k $\tan 370^\circ$ l $\cos 434^\circ$
- a Copy and complete this table of values for $y = \sin \theta$, evaluating y correct to two decimal places.

θ	0°	30°	60°	...	360°
y	0	0.5	0.87	...	0

- Graph $y = \sin \theta$, either by using graphing technology like GeoGebra or on paper using a scale of 1 cm = 30° on the θ -axis and a scale of 4 cm = 1 unit on the y -axis.
- Comment on the shape of the graph $y = \sin \theta$. What are the maximum and minimum values of the graph and when do they occur?
- Does the graph have an axis of symmetry? If so, what is it?
- Does the graph have rotational symmetry? If so, what is the centre of symmetry?
- For what range of values of θ is $\sin \theta$:
 - positive
 - negative?

Stage 5.3

- 5 a Copy and complete this table of values for $y = \cos \theta$, evaluating y correct to two decimal places.

θ	0°	30°	60°	...	360°
y	1	0.87	0.5	...	1

- b Graph $y = \cos \theta$, either by using graphing technology or on paper.
 c Comment on the shape of the graph $y = \cos \theta$. What are the maximum and minimum values of the graph and when do they occur?
 d Does the graph have an axis of symmetry? If so, what is it?
 e Does the graph have rotational symmetry? If so, what is the centre of symmetry?
 f For what range of values of θ is $\cos \theta$:
 i positive ii negative?
 g Comment on the similarities and differences between the graphs of $y = \sin x$ and $y = \cos x$.

See Example 13

- 6 If θ is acute, find θ if:
 a $\cos 170^\circ = -\cos \theta$ b $\sin 110^\circ = \sin \theta$ c $\tan 130^\circ = -\tan \theta$
 d $\tan 97^\circ = -\tan \theta$ e $\cos 115^\circ = -\cos \theta$ f $\sin 168^\circ = \sin \theta$
 7 Express each expression in terms of $\sin A$, $\cos A$ or $\tan A$, where A is acute.
 a $\cos 142^\circ$ b $\sin 105^\circ$ c $\cos 155^\circ$ d $\tan 102^\circ$
 e $\cos 172.7^\circ$ f $\sin 115.5^\circ$ g $\cos 139^\circ 35'$ h $\tan 170.8^\circ$
 i $\sin 120^\circ 35'$ j $\tan 160^\circ 10'$ k $\sin 95.5^\circ$ l $\tan 139.5^\circ$

See Example 14

- 8 Find the exact value of each expression.
 a $\sin 150^\circ$ b $\tan 135^\circ$ c $\sin 135^\circ$ d $\cos 120^\circ$ e $\tan 150^\circ$
 f $\sin 120^\circ$ g $\cos 150^\circ$ h $\tan 120^\circ$ i $\sin 90^\circ$ j $\cos 135^\circ$
 9 a Copy and complete this table of values for $y = \tan \theta$, evaluating y correct to two decimal places.

θ	0°	30°	60°	...	360°
y	0	0.58	1.73	...	0

- b Graph $y = \tan \theta$, either by using graphing technology or on paper.
 c Comment on the shape of the graph $y = \tan \theta$. When does the graph start to repeat itself?
 d Does the graph have an axis of symmetry? If so, what is it?
 e Does the graph have rotational symmetry? If so, what is the centre of symmetry?
 f For what range of values of θ is $\tan \theta$:
 i positive ii negative?

9-06 Trigonometric equations

Stage 5.3

Worksheet

Trigonometric
equations

MAT10MGWK10221

Example 15

Solve each trigonometric equation, giving all possible acute and obtuse solutions correct to the nearest degree.

a $\sin \theta = 0.7538$

b $\tan \theta = -2.5$

Solution

a $\sin \theta = 0.7538$

$$\theta = 48.9206 \dots$$

$$\approx 49^\circ$$

But θ could be obtuse, because $\sin \theta$ is also positive in the second quadrant.

$$\theta \approx 180^\circ - 49^\circ$$

$$= 131^\circ$$

$$\therefore \theta \approx 49^\circ \text{ or } 131^\circ.$$

On a calculator: **SHIFT** **sin** **0.7538** **=**

(Check: $\sin 49^\circ = \sin 131^\circ = 0.7547 \dots$)

b $\tan \theta = -2.5$

$$\theta = -68.1985 \dots$$

$$\approx -68^\circ$$

But θ is obtuse, because $\tan \theta$ is negative in the second quadrant.

$$\theta \approx 180^\circ - 68^\circ$$

$$= 112^\circ$$

On a calculator: **SHIFT** **tan** **(-)** **2.5** **=**

On a calculator: **180** **+** **ANS** **=**
(Check: $\tan 112^\circ = -2.4750 \dots$)

Example 16

Solve each trigonometric equation correct to the nearest minute, if x is obtuse.

a $\cos x = -0.09$

b $\sin x = 0.64$

Solution

a $\cos x = -0.09$

$$x = 95.1636 \dots$$

$$= 95^\circ 9' 48.99''$$

$$\approx 95^\circ 10'$$

On a calculator: **SHIFT** **cos** **(-)** **0.09** **=**

SHIFT **cos** automatically gives the obtuse angle when you enter a negative value

On a calculator: **° ' "** or **DMS**

Stage 5.3

b $\sin x = 0.64$
 $x = 39.7918\dots$
 But x is obtuse, so:
 $x = 180 - 39.7918\dots$
 $= 140.2081\dots$
 $= 140^\circ 12' 29.45''$
 $\approx 140^\circ 12'$

On a calculator: **SHIFT** **sin** **0.64** **=**

On a calculator: **180** **-** **ANS** **=**

On a calculator: **0' "** or **DMS**

Exercise 9-06 Trigonometric equations

See Example 15

- 1 Solve each trigonometric equation, giving all possible acute and obtuse solutions, correct to the nearest degree.

a $\sin \theta = 0.84$

b $\tan \theta = -\frac{3}{4}$

c $\cos \theta = -0.342$

d $\cos \theta = -\frac{7}{11}$

e $\sin \theta = 0.1164$

f $\tan \theta = -1$

g $\tan \theta = -5.8671$

h $\sin \theta = \frac{3}{7}$

i $\cos \theta = -0.4$

j $\sin \theta = \frac{3.8}{7}$

k $\cos \theta = \frac{-21}{80}$

l $\tan \theta = -\frac{15}{8}$

See Example 16

- 2 Solve each trigonometric equation correct to the nearest minute, if x is obtuse. Note: some equations have no solution.

a $\sin A = \frac{4}{7}$

b $\sin A = 0.7438$

c $\sin A = 0.3514$

d $\sin A = 0.108$

e $\sin A = \frac{5}{11}$

f $\sin A = 0.9$

g $\cos x = 0.6$

h $\cos x = -0.6$

i $\tan x = 0.3$

j $\tan x = -0.3$

k $\sin x = 0.8$

l $\sin x = \frac{3}{7}$

- 3 Solve each trigonometric equation correct to the nearest degree, if A is between 0° and 180° .

a $\cos x = \frac{-8}{11}$

b $\tan x = -0.95$

c $\sin x = \frac{7}{8}$

d $4 \cos x = \sqrt{2}$

e $3 \sin x = 2$

f $-4 \tan x = 3$

g $\tan x = 1$

h $\cos x = \frac{1}{2}$

i $\sin x = \frac{1}{\sqrt{2}}$

Investigation: Sides and opposite angles

- Construct three triangles of different sizes.
- Label the angles of each triangle A , B and C and the sides opposite them a , b and c respectively.
- Measure the sides of each triangle correct to the nearest millimetre and the angles to the nearest degree. Copy the table below and record your results in it.

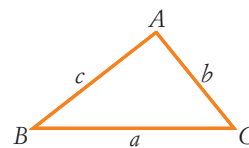
	Side a	Angle A	$\frac{a}{\sin A}$	Side b	Angle B	$\frac{b}{\sin B}$	Side c	Angle C	$\frac{c}{\sin C}$
Triangle 1									
Triangle 2									
Triangle 3									

- Complete the table by calculating $\frac{a}{\sin A}$, $\frac{b}{\sin B}$ and $\frac{c}{\sin C}$ correct to two decimal places.
- Compare the values of $\frac{a}{\sin A}$, $\frac{b}{\sin B}$ and $\frac{c}{\sin C}$ for each triangle. What do you notice?

9-07 The sine rule

Stage 5.3

The angles of a triangle are labelled with capital letters while the sides are labelled with lower case letters. By convention, we use a to label the side opposite $\angle A$, b to label the side opposite $\angle B$, and so on. There is a relationship between each angle in a triangle and its opposite side. The longest side is always opposite the largest angle, the next smallest side is opposite the next smallest angle and so on. This relationship is called the **sine rule**.

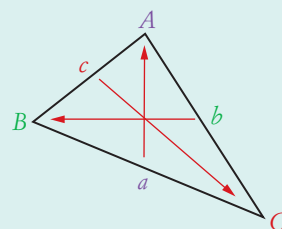


Summary

For any triangle ABC :

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

The ratios of the sides in a triangle to the sine of their opposite angles are equal.



Proof:

In $\triangle ABC$, draw CX for the perpendicular height, h , of the triangle. CX divides $\triangle ABC$ into two right-angled triangles.

$$\text{In } \triangle AXC, \sin A = \frac{h}{b}$$

$$\therefore h = b \sin A$$

$$\text{In } \triangle BXC, \sin B = \frac{h}{a}$$

$$\therefore h = a \sin B$$

$$\therefore a \sin B = b \sin A$$

$$\frac{a \sin B}{\sin B} = \frac{b \sin A}{\sin B}$$

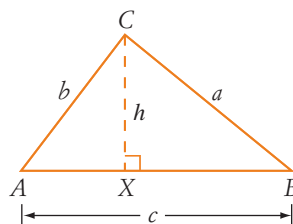
Dividing both sides by $\sin B$

$$a = \frac{b \sin A}{\sin B}$$

$$\frac{a}{\sin A} = \frac{b \sin A}{\sin B \sin A}$$

Dividing both sides by $\sin A$

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$



By drawing the perpendicular from A to BC , it can be shown that $\frac{b}{\sin B} = \frac{c}{\sin C}$

$$\therefore \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

We have used trigonometry to solve problems that, until now, have involved only right-angled triangles. The sine rule allows us to apply trigonometry to any triangle.

The sine rule can be used in problems involving two sides of a triangle and the two angles opposite them.

Stage 5.3

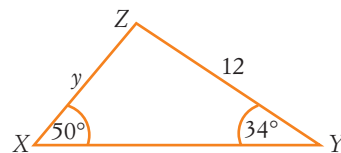
Video tutorial

The sine rule

MAT10MGVT10024

Example 17

Find y , correct to one decimal place.



Solution

$$\begin{aligned}\frac{a}{\sin A} &= \frac{b}{\sin B} \\ \frac{y}{\sin 34^\circ} &= \frac{12}{\sin 50^\circ} \\ y &= \frac{12 \sin 34^\circ}{\sin 50^\circ} \\ &= 8.7596 \dots \\ &\approx 8.8 \text{ cm}\end{aligned}$$

From the diagram, an answer of 8.8 cm looks reasonable.

Exercise 9-07 The sine rule

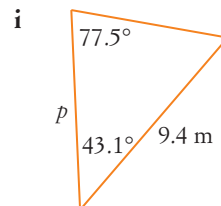
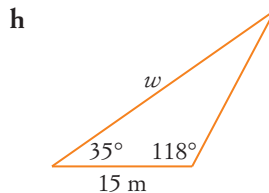
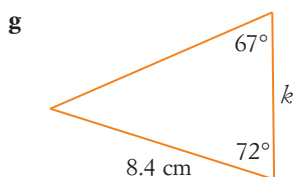
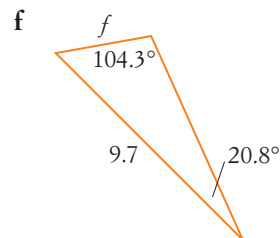
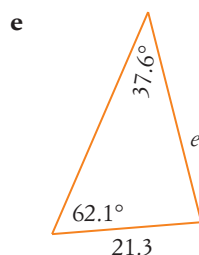
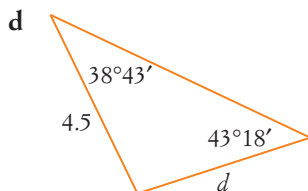
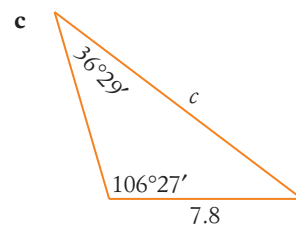
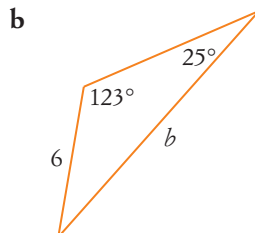
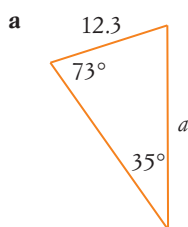
- 1 Evaluate each expression, correct to one decimal place.

a $\frac{14.7 \sin 64^\circ}{\sin 46^\circ}$

b $\frac{34.5 \sin 33.4^\circ}{\sin 115.7^\circ}$

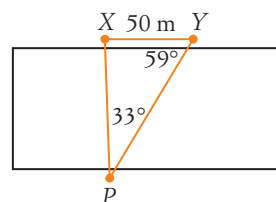
c $\frac{69 \sin 107^\circ 33'}{\sin 38^\circ 47'}$

- 2 Find the value of each pronumeral, correct to two decimal places.

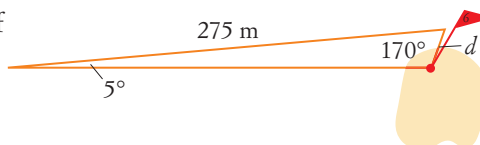


See Example 17

- 3 X and Y are two light towers 50 m apart on one side of a park. P is a light tower on the other side of the park. If $\angle Y = 59^\circ$ and $\angle P = 33^\circ$, find PX to the nearest metre.

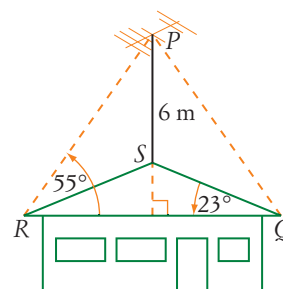


- 4 A golfer drives a ball 275 m at an angle of 5° off centre. The ball lands at an angle of 107° from the hole. Calculate the distance of the ball from the hole, correct to the nearest metre.



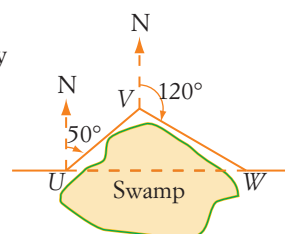
- 5 A 6 m television antenna is mounted on a roof pitched at an angle of 23° . It is supported by two wires, PQ and PR , inclined at 55° to the horizontal.

- Show that $\angle PSR = 113^\circ$.
- Calculate the length of the wire PR , correct to the nearest centimetre.

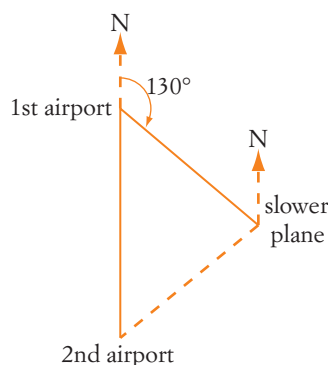


- 6 To avoid a swamp, Jesinta runs 70 m on a bearing of 050° to V . She then turns and runs to W on a bearing of 120° . If W is directly east of U :

- find $\angle UVW$
- calculate UW , correct to one decimal place.



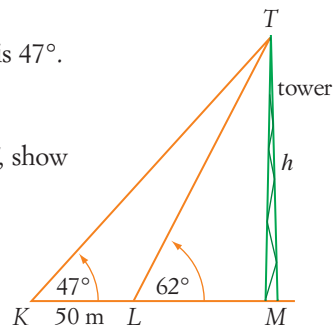
- 7 Two planes leave the airport at the same time. One flies due south at 400 km/h and lands at a second airport after $1\frac{1}{2}$ hours. The other flies on a bearing of 130° and after $1\frac{1}{2}$ hours is at a bearing of 075° from the second airport. How far (to the nearest km) is the slower plane from the second airport?



Stage 5.3

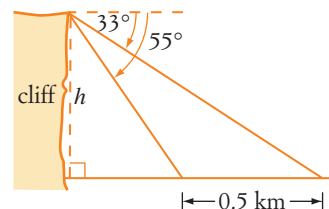
- 8 The angle of elevation of a tower from a point L is 62° . From a point K , 50 m further from the tower, the angle of elevation is 47° .

- Use the sine rule in $\triangle KTL$ to show that $TL = \frac{50 \sin 47^\circ}{\sin 15^\circ}$.
- Let the height of the tower be h . In the right-angled $\triangle LMT$, show that $TL = \frac{h}{\sin 62^\circ}$.
- Hence show that $h = \frac{50 \sin 47^\circ \sin 62^\circ}{\sin 15^\circ}$.
- Hence calculate the height of the tower, correct to one decimal place.



- 9 From the top of a cliff, the angles of depression of two boats at sea that are 0.5 km apart are 55° and 33° .

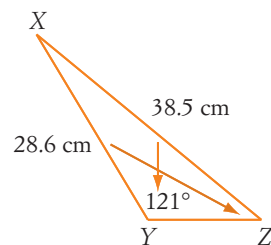
- Let the height of the cliff be h . Show that $h = \frac{0.5 \sin 33^\circ \sin 55^\circ}{\sin 22^\circ}$.
- Hence calculate the height correct to the nearest metre.



9-08 The sine rule for angles

Example 18

Find angle Z , correct to the nearest degree.



Solution

Look for two sides and the angles opposite them.

$$\frac{28.6}{\sin Z} = \frac{38.5}{\sin 121}$$

$$\frac{\sin Z}{28.6} = \frac{\sin 121^\circ}{38.5}$$

$$\sin Z = \frac{28.6 \sin 121^\circ}{38.5}$$

$$= 0.636 \dots$$

$$Z = 39.55 \dots$$

$$\approx 40^\circ$$

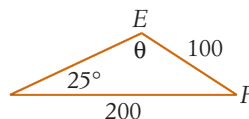
Inverting both sides so that Z is in the numerator.

On a calculator: **SHIFT** **sin** **ANS** **=**

From the diagram, an answer of 40° looks reasonable.

Example 19

Find θ correct to the nearest minute if it is an obtuse angle.



Solution

$$\frac{200}{\sin \theta} = \frac{100}{\sin 25^\circ}$$

$$\frac{\sin \theta}{200} = \frac{\sin 25^\circ}{100}$$

$$\sin \theta = \frac{200 \sin 25^\circ}{100}$$

$$= 0.845 \dots$$

$$\theta = 57.697 \dots$$

But θ is obtuse, so:

$$\theta = 180 - 57.697 \dots$$

$$= 122.3027 \dots$$

$$= 122^\circ 18' 9.77''$$

$$\approx 122^\circ 18'$$

The ambiguous case (when there are two possible answers)

When we use the sine rule to find an angle, it is possible to find both an acute angle and an obtuse angle as solutions. Likewise, there could be two possible triangles: one acute-angled, the other obtuse-angled. However, the obtuse-angled triangle may not be possible. We need to check that the sum of the angles in the triangle is not greater than 180° .

Example 20

- a In $\triangle DEF$, $\angle D = 42^\circ$, $d = 5$ cm and $f = 7$ cm. Find $\angle F$ correct to the nearest degree.
b In $\triangle LMN$, $\angle M = 130^\circ$, $LN = 15$ cm and $LM = 7$ cm. Find $\angle N$, correct to the nearest degree.

Solution

- a Draw a rough diagram.

$$\frac{7}{\sin F} = \frac{5}{\sin 42^\circ}$$

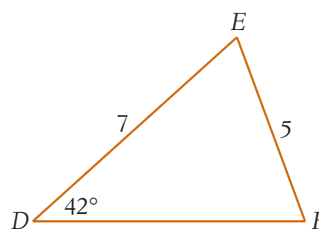
$$\frac{\sin F}{7} = \frac{\sin 42^\circ}{5}$$

$$\sin F = \frac{7 \sin 42^\circ}{5}$$

$$= 0.93678 \dots$$

$$F = 69.5181 \dots$$

$$\approx 70^\circ$$



Video tutorial

The sine rule

MAT10MGVT10024

Stage 5.3

But F could be obtuse.

$$F = 180^\circ - 70^\circ$$

$$= 110^\circ$$

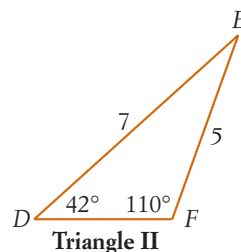
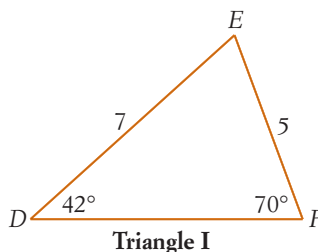
Checking the third angle of the obtuse-angled triangle:

$$\angle E = 180^\circ - 42^\circ - 110^\circ$$

$$= 28^\circ$$

$\therefore$ The obtuse-angled solution is possible.

$$\therefore \angle F = 70^\circ \text{ or } 110^\circ$$



b Draw a rough diagram.

$$\frac{\sin N}{7} = \frac{\sin 130^\circ}{15}$$

$$\sin N = \frac{7 \sin 130^\circ}{15}$$

$$= 0.3574 \dots$$

$$N = 20.9459 \dots$$

$$\approx 21^\circ$$

But N could be obtuse.

$$N = 180^\circ - 21^\circ$$

$$= 159^\circ$$

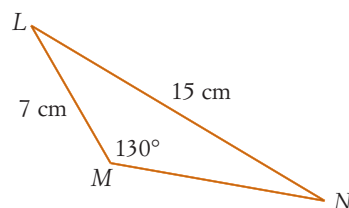
Checking the third angle of the obtuse-angled triangle:

$$\angle L = 180^\circ - 42^\circ - 159^\circ$$

$$= -21^\circ \quad \text{Impossible}$$

$\therefore$ The obtuse-angled solution is not possible.

$$\therefore \angle N = 21^\circ$$

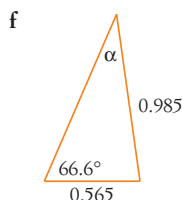
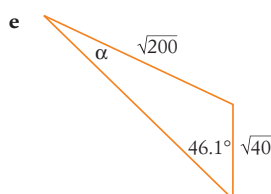
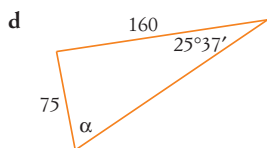
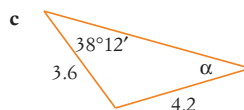
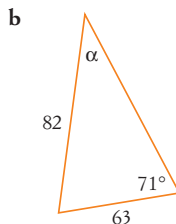
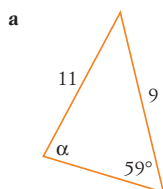


Exercise 9-08 The sine rule for angles

1 Find the acute angle X in each equation, correct to the nearest degree.

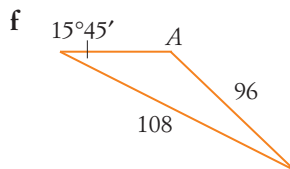
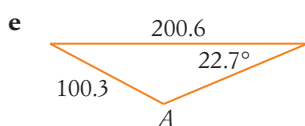
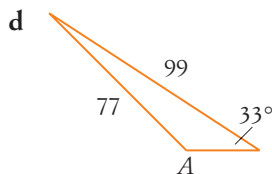
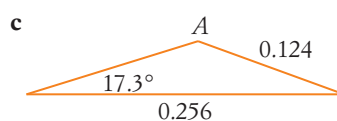
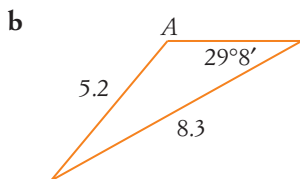
a $\sin X = \frac{5.3 \sin 123^\circ}{9.7}$ b $\sin X = \frac{39 \sin 85^\circ 29'}{64}$ c $\sin X = \frac{467 \sin 63.8^\circ}{518}$

2 Find α in each triangle if α is acute, correct to the nearest 0.1 degree.



See Example 18

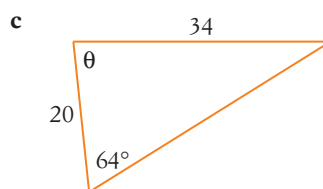
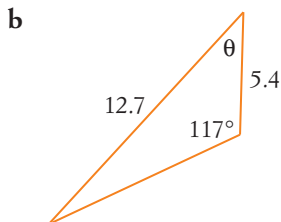
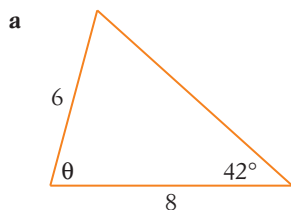
- 3 Find the size of $\angle A$ to the nearest minute if $\angle A$ is obtuse.



- 4 Find all possible angles for each triangle, correct to the nearest degree, after sketching a diagram.

- a In $\triangle PQR$, $\angle P = 35^\circ$, $p = 8$ cm, and $q = 10$ cm. Find $\angle Q$.
b In $\triangle UVW$, $\angle W = 95^\circ$, $w = 16$ m, and $v = 10$ cm. Find $\angle V$.
c In $\triangle XYZ$, $\angle Y = 24^\circ$, $y = 3.4$ km, and $z = 5.7$ km. Find $\angle Z$.
d In $\triangle DEF$, $\angle E = 37^\circ$, $e = 107$ mm, and $d = 121$ mm. Find $\angle D$.

- 5 Find θ in each triangle correct to the nearest degree, given that θ is acute.



9-09 The cosine rule

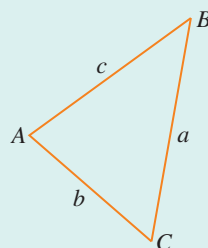
The **cosine rule** is a relationship between the three sides of a triangle and one of its angles. It is an extension of Pythagoras' triangle that can be applied to any triangle, not just right-angled ones.

Summary

For any triangle ABC :

$$a^2 = b^2 + c^2 - 2bc \cos A$$

where a is the unknown side, A is the angle opposite a , and b and c are the other two sides.



Stage 5.3

See Example 19

See Example 20

Stage 5.3

Proof:

In $\triangle ABC$, draw CX for the perpendicular height, h , of the triangle.
 CX divides $\triangle ABC$ into two right-angled triangles.

Let $AX = x$, $\therefore XB = c - x$

In $\triangle AXC$, $b^2 = h^2 + x^2$

$$\therefore h^2 = b^2 - x^2$$

In $\triangle BXC$, $a^2 = h^2 + (c - x)^2$

$$\therefore h^2 = a^2 - (c - x)^2$$

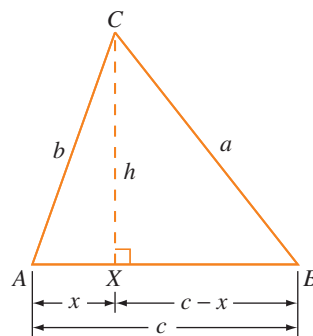
$$\therefore b^2 - x^2 = a^2 - (c - x)^2$$

$$b^2 - x^2 = a^2 - (c^2 - 2cx + x^2)$$

$$b^2 - x^2 = a^2 - c^2 + 2cx - x^2$$

$$b^2 = a^2 - c^2 + 2cx$$

$$\therefore a^2 = b^2 + c^2 - 2cx \quad (*) \quad \text{Making } a^2 \text{ the subject}$$



In $\triangle AXC$, $\cos A = \frac{x}{b}$

$\therefore x = b \cos A$

Substituting for x in (*):

$$a^2 = b^2 + c^2 - 2cb \cos A$$

The cosine rule can be used in problems involving three sides of a triangle and one of the angles.

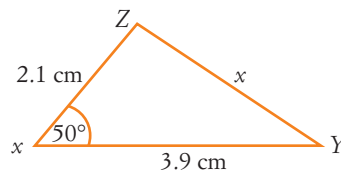
Video tutorial

The cosine rule

MAT10MGVT10025

Example 21

Find x correct to two decimal places.



Solution

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$x^2 = 2.1^2 + 3.9^2 - 2 \times 2.1 \times 3.9 \cos 50^\circ$$

$$= 9.091\,138\dots$$

$$x = \sqrt{9.091\,138\dots}$$

$$= 3.01515\dots$$

$$\approx 3.02 \text{ cm}$$

50° is the angle opposite x .

From the diagram, an answer of 3.02 cm looks reasonable.

Exercise 9-09 The cosine rule

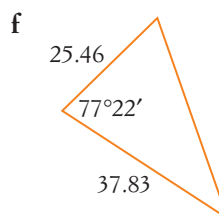
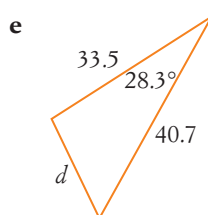
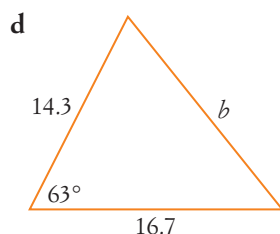
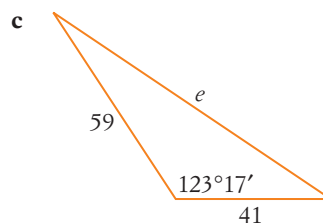
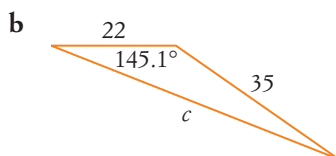
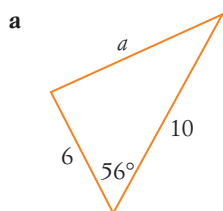
1 Solve each equation for x , correct to one decimal place.

a $x^2 = 8^2 + 9^2 - 2 \times 8 \times 9 \times \cos 38^\circ$

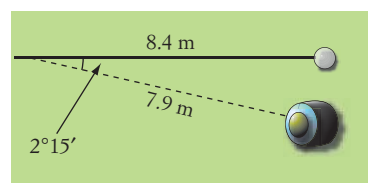
b $x^2 = 11.3^2 + 9.7^2 - 2 \times 11.3 \times 9.7 \times \cos 76.9^\circ$

c $x^2 = 17^2 + 20.1^2 - 2 \times 17 \times 20.1 \times \cos 149^\circ 45'$

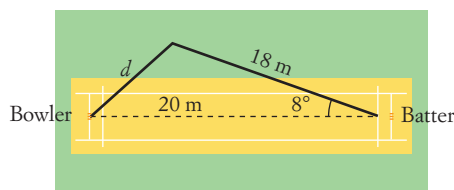
- 2 Find, correct to two decimal places, the value of each pronumeral.



- 3 In a game of lawn bowls, Jayden is aiming to hit the jack (target ball) 8.4 m away. If he bowls $2^\circ 15'$ off-centre and his bowl travels 7.9 m, how far is his bowl from the jack? Answer correct to one decimal place.



- 4 In a cricket match, the distance between the bowler and the batter was 20 m. During one bowl, the batter hit the ball at an angle of 8° to the line of the pitch and the bowler ran and caught the ball after it had travelled 18 m.

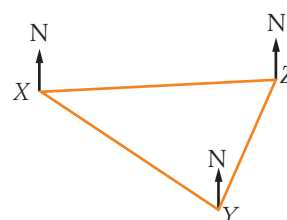


How far did the bowler run to catch the ball? Select the correct answer **A**, **B**, **C** or **D**.

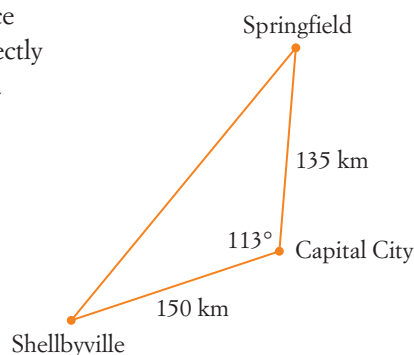
A 1.1 m **B** 2.0 m **C** 3.3 m **D** 4.0 m

- 5 A yacht sails from X to Y on a bearing of 130° for 4.2 km. It then turns and travels to Z on a bearing of 025° for 2.9 km.

- a** Copy the diagram and mark the given information on it.
b Explain why $\angle XYZ = 75^\circ$.
c Calculate the distance XZ , correct to one decimal place.



- 6 Three towns are joined by straight roads. What distance (correct to the nearest kilometre) is saved by going directly from Springfield to Shellbyville instead of travelling via Capital City?



Stage 5.3

- 7 a What is the value of $\cos 90^\circ$?
b What does $a^2 = b^2 + c^2 - 2bc \cos A$ simplify to if $A = 90^\circ$?
c Hence what happens to the cosine rule when it is applied to a right-angled triangle?

9-10 The cosine rule for angles

If we rewrite the cosine rule so that $\cos A$ is the subject, then we will have a formula for finding an unknown angle when the three sides of a triangle are known.

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$a^2 + 2bc \cos A = b^2 + c^2$$

Adding $2bc \cos A$ to both sides so that $\cos A$ appears on the LHS

$$2bc \cos A = b^2 + c^2 - a^2$$

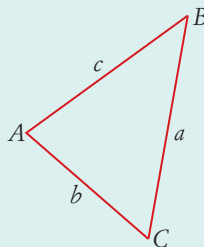
$$\therefore \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Summary

For any triangle ABC :

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

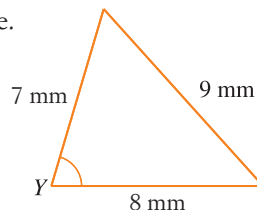
where A is the unknown angle, a is the side opposite A , and b and c are the other two sides.



The cosine rule can be used to find an unknown angle if the lengths of the three sides are known.

Example 22

Find the size of the marked angle Y , correct to the nearest degree.



Solution

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos Y = \frac{8^2 + 7^2 - 9^2}{2 \times 8 \times 7}$$

$$= \frac{32}{112}$$

$$Y = 73.398 \dots$$

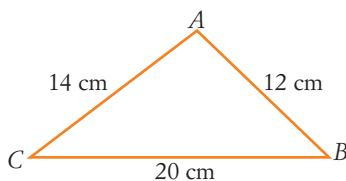
$$\approx 73^\circ$$

9 mm is opposite angle Y

From the diagram, an answer of 73° looks reasonable.

Example 23

Calculate, correct to the nearest minute, the size of the largest angle in this triangle.



Solution

The largest angle is opposite the longest side, so it is $\angle A$.

$$\begin{aligned}\cos A &= \frac{b^2 + c^2 - a^2}{2bc} \\ &= \frac{14^2 + 12^2 - 20^2}{2 \times 14 \times 12} \\ &= \frac{-60}{336}\end{aligned}$$

$$\begin{aligned}A &= 100.28656 \dots \\ &= 100^\circ 17' 11.6'' \\ &\approx 100^\circ 17'\end{aligned}$$

Cos is negative so the angle will be obtuse.

From the diagram, an answer of $100^\circ 17'$ looks reasonable.

Exercise 9-10 The cosine rule for angles

1 Solve each equation for X , correct to the nearest degree.

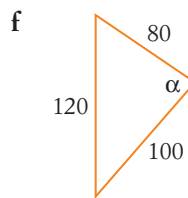
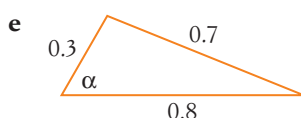
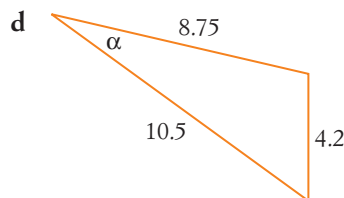
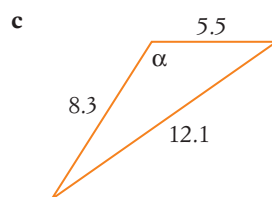
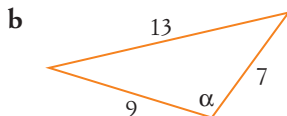
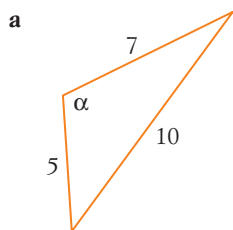
a $\cos X = \frac{12^2 + 14^2 - 15^2}{2 \times 12 \times 14}$

b $\cos X = \frac{5.7^2 + 6.8^2 - 3.7^2}{2 \times 5.7 \times 6.8}$

c $\cos X = \frac{5^2 + 6^2 - 9^2}{2 \times 5 \times 6}$

d $\cos X = \frac{9.2^2 + 4.7^2 - 12.8^2}{2 \times 9.2 \times 4.7}$

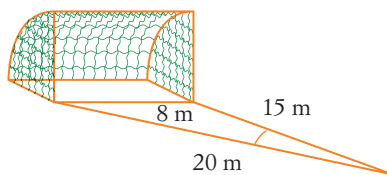
2 Find the size of α in each triangle, correct to the nearest degree.



See Example 22

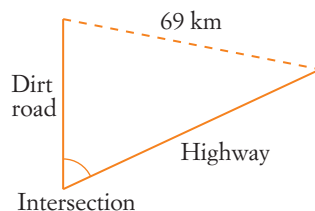
Stage 5.3

- 3 A soccer goal is 8 m wide. A player shoots for goal (along the ground) when 20 m from one post and 15 m from the other post. Within what angle (correct to the nearest 0.1 degree) must the shot be made for the player to have a chance of scoring a goal?



See Example 23

- 4 Two cars leave an intersection at the same time. Car *A* drives down the dirt road at 60 km/h and car *B* drives down the highway at 100 km/h. After 45 minutes they are 69 km apart. Find the angle between the two roads, correct to the nearest minute.



- 5 A triangle has sides of 21 m, 17 m and 10 m. Find the size of the largest angle, correct to the nearest degree.

9-11 The area of a triangle

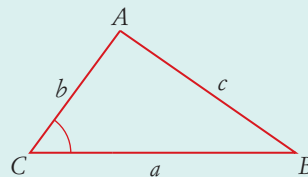
We already know that the formula for the area of a triangle is $A = \frac{1}{2}bh$ but there is also a trigonometric formula if we know the lengths of two sides of the triangle and the size of the **included angle** between them.

Summary

The area of a triangle

$$A = \frac{1}{2}ab \sin C$$

where C is the included angle between sides a and b .



Proof:

In $\triangle ABC$, draw AX for the perpendicular height, h , of the triangle. AX divides $\triangle ABC$ into two right-angled triangles.

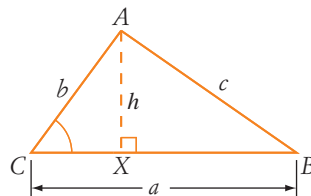
$$\text{Area} = \frac{1}{2} \times \text{base} \times \text{height}$$

$$A = \frac{1}{2}ah$$

$$\text{In } \triangle AXC, \sin C = \frac{h}{b}$$

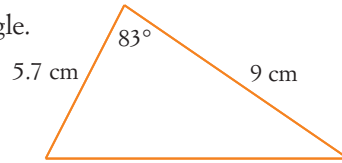
$$\therefore h = b \sin C$$

$$\therefore A = \frac{1}{2}ab \sin C$$



Example 24

Find, correct to one decimal place, the area of this triangle.



Solution

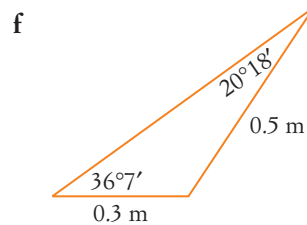
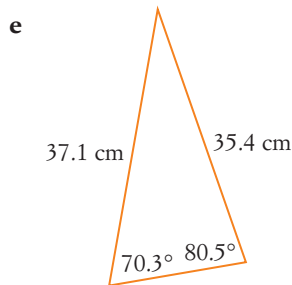
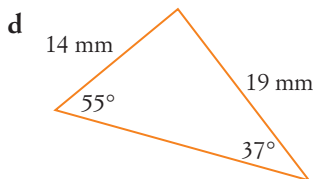
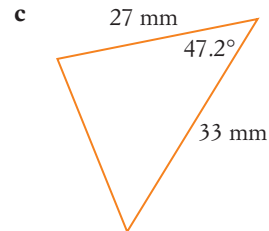
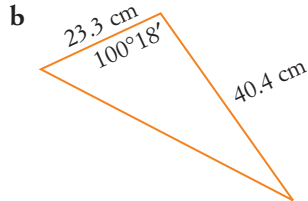
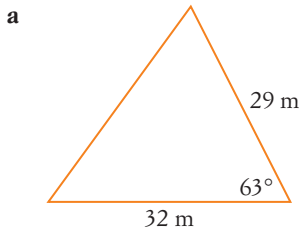
$$\begin{aligned} A &= \frac{1}{2}ab \sin C \\ &= \frac{1}{2} \times 5.7 \times 9 \times \sin 83^\circ \\ &= 25.458 \dots \\ &\approx 25.5 \text{ cm}^2 \end{aligned}$$

83° is the included angle between 5.7 cm and 9 cm.

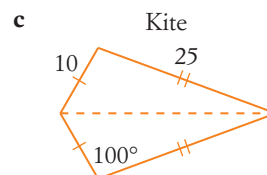
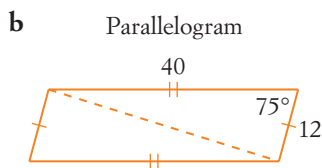
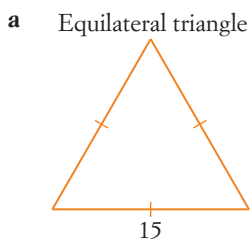
See Example 24

Exercise 9-11 The area of a triangle

1 Find, correct to one decimal place, the area of each triangle.

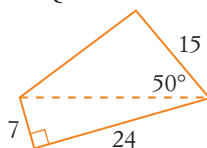


2 Calculate, correct to one decimal place, the area of each shape. All measurements are in metres.

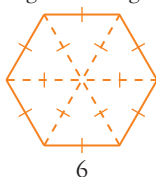


Stage 5.3

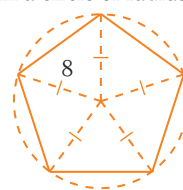
d Quadrilateral



e Regular hexagon

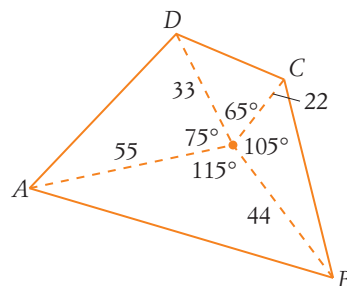


f Regular pentagon inscribed in a circle of radius 8



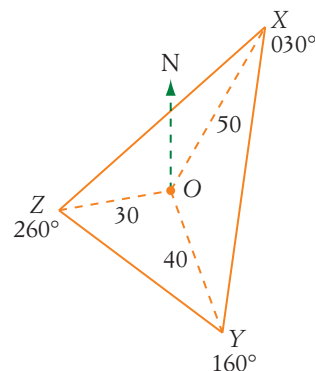
- 3 The diagram shows the results of a radial survey of a block of land. All distances are in metres.

- Use the cosine rule to find the lengths of AB , BC , CD and AD and, hence, find the perimeter of the block of land (to the nearest metre).
- Use the area formula to find the area of each triangle and, hence, calculate the area of the block of land (to the nearest m^2).



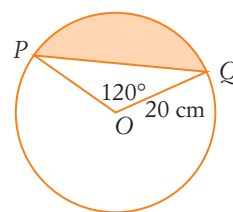
- 4 The results of a radial survey are shown in the diagram. All measurements are in metres.

- Find the size of $\angle XOY$.
- Calculate, correct to two decimal places, the area of $\triangle XOY$.



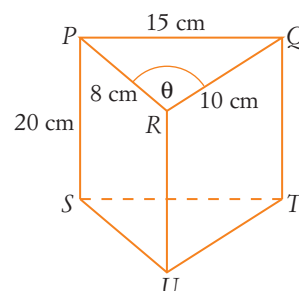
- 5 O is the centre of a circle of radius 20 cm. Calculate, correct to one decimal place, the area of:

- sector OPQ
- triangle OPQ
- the shaded segment.



- 6 A triangular prism has base edges of 8 cm, 10 cm and 15 cm, and a height of 20 cm.

- Calculate the size $\angle PRQ$, correct to nearest degree.
- Find the area of $\triangle PQR$, correct to the nearest cm^2 .
- Find the volume of the prism.



Just for the record

Triangle area formulas

Stage 5.3

How many formulas are there for calculating the area, A , of a triangle?
Here are some different ones.

- Given three sides a, b, c : $A = \sqrt{s(s-a)(s-b)(s-c)}$ where s is the semi-perimeter $\frac{a+b+c}{2}$
 - Given two sides, a, b and the included angle, C : $A = \frac{1}{2}ab \sin C$
 - Given one side, a and three angles, A, B, C : $A = \frac{1}{2}a^2 \frac{\sin B \sin C}{\sin A}$
- Use each formula to find the area of a triangle with sides 1, $\sqrt{3}$, 2 and angles 30° , 60° , 90° .
 - Show how the result $A = \frac{1}{2}a^2 \frac{\sin B \sin C}{\sin A}$ can be derived from $A = \frac{1}{2}ab \sin C$ using the sine rule.
 - Find other formulas for the area of a triangle.

9-12

Problems involving the sine and cosine rules

Worksheet

Finding an unknown side

MAT10MGWK10222

Worksheet

Finding an unknown angle

MAT10MGWK10223

Puzzle sheet

The sine and cosine rules

MAT10MGPS00058

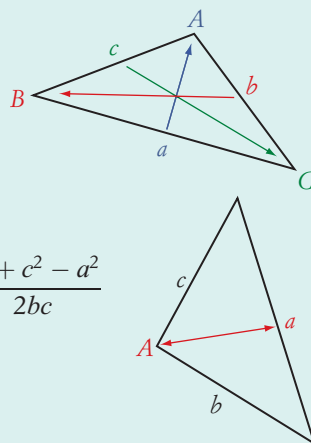
Summary

The **sine rule** is used for triangle problems involving two sides and two angles opposite them.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

The **cosine rule** is used for triangle problems involving three sides and one angle.

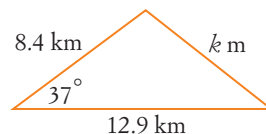
$$a^2 = b^2 + c^2 - 2bc \cos A \text{ and } \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$



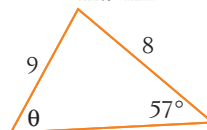
Stage 5.3

Example 25

- a Find the value of k , correct to one decimal place.



- b Find the value of θ , correct to the nearest minute.



Solution

- a The problem involves three sides and one angle so use the cosine rule.

$$\begin{aligned} k^2 &= 8.4^2 + 12.9^2 - 2 \times 8.4 \times 12.9 \times \cos 37^\circ \\ &= 63.889 \dots \\ k &= \sqrt{63.889 \dots} \\ &= 7.993 \dots \\ &\approx 8.0 \text{ m} \end{aligned}$$

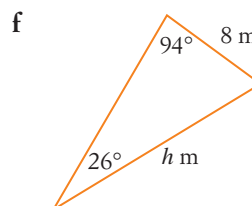
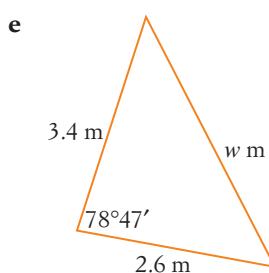
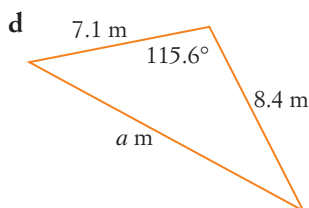
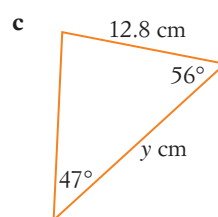
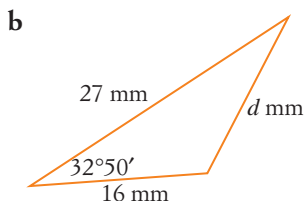
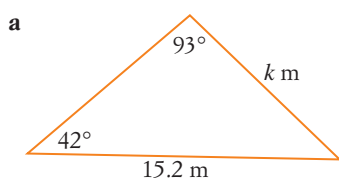
- b The problem involves two sides and the two angles opposite them, so use the sine rule.

$$\begin{aligned} \frac{\sin \theta}{8} &= \frac{\sin 57^\circ}{9} \\ \sin \theta &= \frac{8 \sin 57^\circ}{9} \\ &= 0.7454 \dots \\ \theta &= 48.2007 \dots \\ &= 48^\circ 12' 2.77'' \\ &\approx 48^\circ 12' \end{aligned}$$

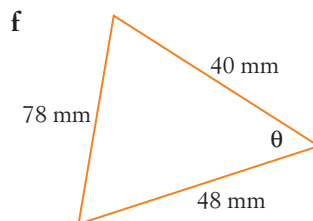
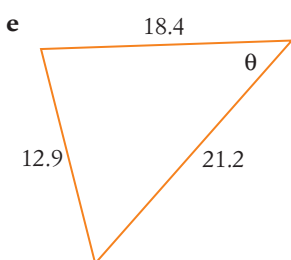
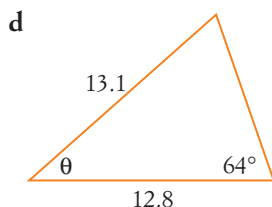
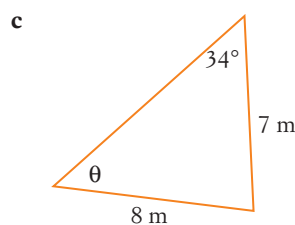
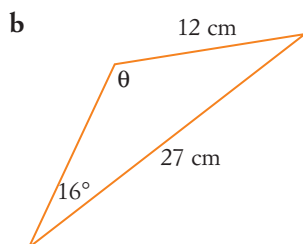
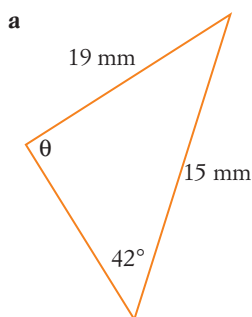
Exercise 9-12 Problems involving the sine and cosine rules

See Example 25

- 1 Find, correct to one decimal place, the value of each pronumeral.

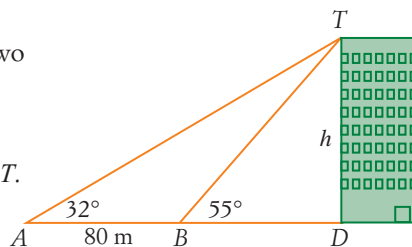


- 2 Find the value of θ to the nearest degree. Use diagrams to note whether θ is acute or obtuse.

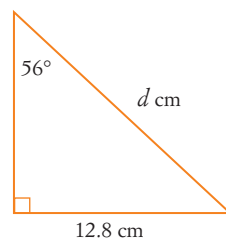


- 3 The angles of elevation of a building measured from two positions 80 m apart are 32° and 55° .

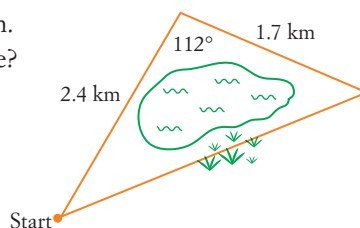
- a** Explain why $\angle ATB = 23^\circ$.
b Find, correct to two decimal places, the length of BT .
c Hence find the height, h , of the building, correct to the nearest metre.



- 4 **a** What is the value of $\sin 90^\circ$?
b Find, correct to one decimal place, the value of d using:
i the sine rule **ii** the sine ratio for right-angled triangles.
c What do you notice about your results? Give reasons.



- 5 Mikayla needs to run around a cross-country course as shown. What is the length of the course, correct to one decimal place?

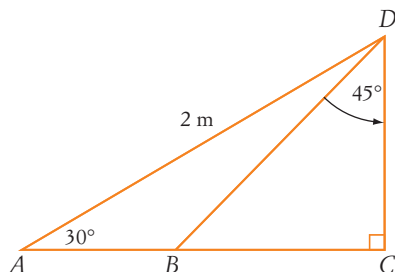


- 6 A plane flew on a bearing of 150° for 370 km. It then changed direction and flew another 285 km on a bearing of 235° . How far, correct to the nearest kilometre, is the plane from its starting point?

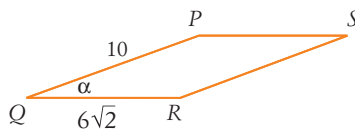
Stage 5.3

Power plus

- 1 For this diagram, find the exact value of:
a the length of AB **b** the area of $\triangle ADB$



- 2 Find θ if θ is between 0° and 180° , and:
a $\sin \theta = \cos \theta$ **b** $\sin \theta = \sqrt{3} \cos \theta$ **c** $\sqrt{3} \sin \theta = \cos \theta$
d $\cos \theta = -\sqrt{3} \sin \theta$ **e** $\sin \theta = -\sqrt{3} \cos \theta$ **f** $\cos \theta = -\sin \theta$
g $\sin^2 \theta = \frac{3}{4}$ **h** $\tan^2 \theta = \frac{1}{3}$ **i** $\cos^2 \theta = \frac{1}{2}$
- 3 Show that:
a $\sin(180^\circ - B) + \cos(90^\circ - B) = 2 \sin B$ **b** $\frac{\sin(180^\circ - B)}{\cos(180^\circ - B)} = -\tan B$
c $\sin(180^\circ - B) \cos(180^\circ - B) \tan(180^\circ - B) = \sin^2 B$ **d** $\frac{(\sin \theta)^2 + \sin \theta \cos \theta}{(\cos \theta)^2 + \sin \theta \cos \theta} = \tan \theta$
- 4 Express $\cos 45^\circ + \sin 45^\circ$ as a surd in simplest form.
- 5 The area of the parallelogram shown is 30 cm^2 . Find two possible values for angle α , correct to the nearest degree.



Language of maths

angle of depression	angle of elevation	bearing	compass bearing
complementary	cosine rule	degree ($^{\circ}$)	denominator
exact ratio	included angle	minute ($'$)	obtuse
Pythagoras' theorem	right-angled	second ($''$)	sine rule
slant height	supplementary	surd	tangent ($\tan$)
theta (θ)	three-figure bearing	trigonometric ratio	unit circle

- Copy and complete: A **bearing** is an _____ used to describe precisely the _____ of one location from a given reference point.
- The word **minute** has an alternative pronunciation and meaning. What is its alternative meaning?
- If two angles have a sum of 90° , what special name do we give them?
- What is an **exact ratio**? Give an example of one.
- Describe the shape of the **sine curve**.
- What is the name given to the angle that is between two known sides?

Topic overview

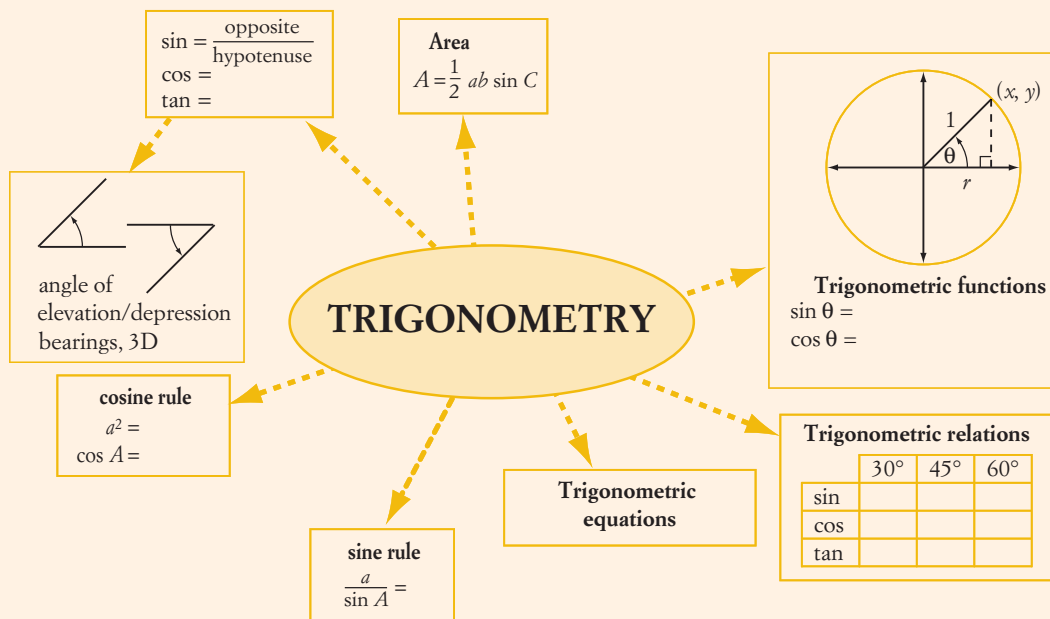
For each statement about the topic, give a rating from 0 to 5 using this scale.



- I can use Pythagoras' theorem and trigonometric ratios to solve problems involving right-angled triangles in 2D, including problems involving angles of elevation and depression, and bearings
- I can calculate and graph the trigonometric ratios for angles between 0° and 360°
- I can calculate the exact trigonometric ratios and solve trigonometric equations
- I can use the sine and cosine rules to find unknown sides and angles in triangles
- I can use the trigonometric formula for the area of a triangle

Chapter 9 review

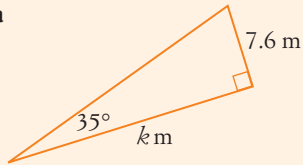
Copy and complete this mind map of the topic, adding detail to its branches and using pictures, symbols and colour where needed. Ask your teacher to check your work.



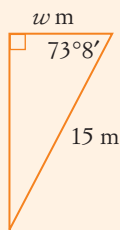
- 1 Find, correct to one decimal place, the value of each pronumeral.

See Exercise 9-01

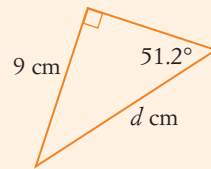
a



b



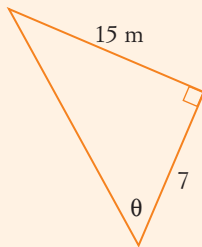
c



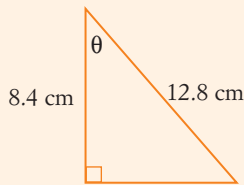
- 2 Find the value of θ , correct to the nearest minute.

See Exercise 9-01

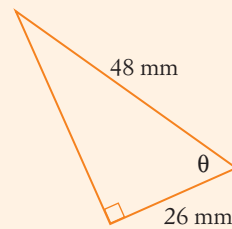
a



b



c



- 3 From the top of a 300 m cliff Renee observes a boat at a distance of 1.3 km from the base of the cliff. Find, correct to the nearest degree, the angle of depression of the boat.

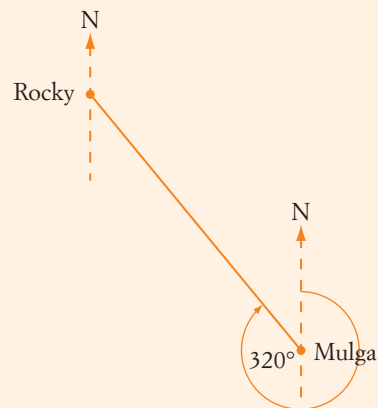
See Exercise 9-01

- 4 What is the bearing of:

See Exercise 9-02

a Rocky from Mulga?

b Mulga from Rocky?



- 5 Two planes leave an airport at the same time. The first travels on a bearing of 063° at 500 km/h. The second travels on a bearing of 153° at 400 km/h.

See Exercise 9-02

a How far apart are the planes after 2 hours, to the nearest km?

b Calculate, correct to the nearest degree, the bearing of the first plane from the second plane.

- 6 A box in the shape of a square prism has a base of 10 cm by 10 cm and is 30 cm tall. Find to the nearest whole number:

a the length of the longest diagonal of the box

b the angle that the longest diagonal makes with the base.

Stage 5.3

See Exercise 9-03

Chapter 9 revision

Stage 5.3

See Exercise 9-04

See Exercise 9-05

See Exercise 9-05

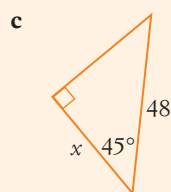
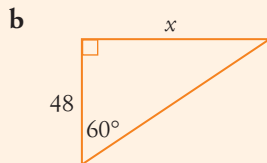
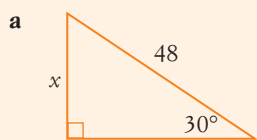
See Exercise 9-06

See Exercise 9-07

See Exercise 9-08

See Exercise 9-09

- 7 Find the exact value of x in each triangle.



- 8 Graph $y = \cos \theta$ for θ from 0° to 360° .

- 9 Find the acute angle α if:

a $\sin 123^\circ = \sin \alpha$

b $\tan 93^\circ = -\tan \alpha$

c $\cos 110^\circ = -\cos \alpha$

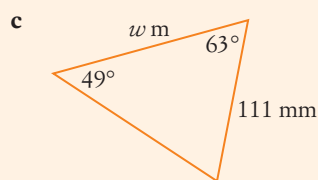
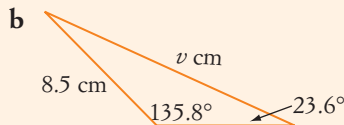
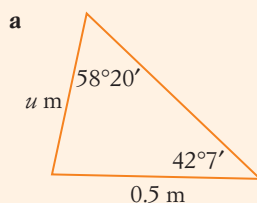
- 10 Solve each trigonometric equation, giving all possible acute and obtuse solutions correct to the nearest minute.

a $\cos x = -\frac{5}{11}$

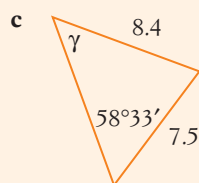
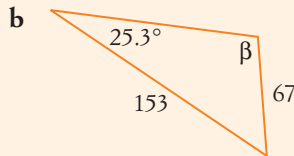
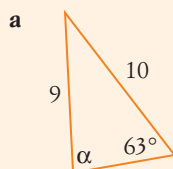
b $\tan x = \frac{11}{5}$

c $\sin x = \frac{5}{11}$

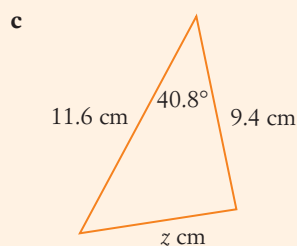
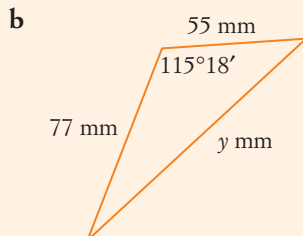
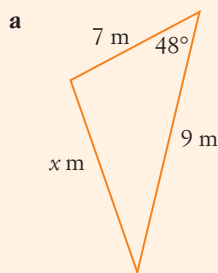
- 11 Find the value of each pronumeral, correct to one decimal place.



- 12 Find the value of each pronumeral, correct to the nearest minute.

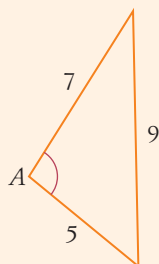


- 13 Find the value of each pronumeral, correct to one decimal place.

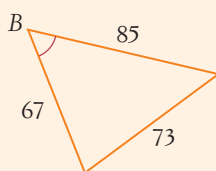


- 14 Find the value of each pronumeral, correct to the nearest degree.

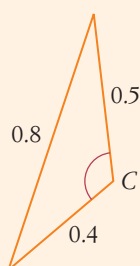
a



b

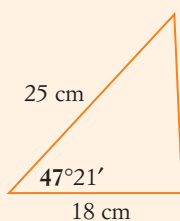


c

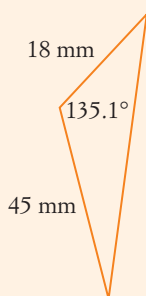


- 15 Find, correct to the nearest whole number, the area of each triangle.

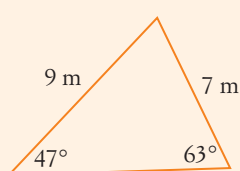
a



b



c



- 16 A parallelogram has sides of 12 cm and 6 cm and one interior angle of $65^{\circ}35'$. Find the length of the longer diagonal, correct to one decimal place.

Stage 5.3

See Exercise 9-10

See Exercise 9-11

See Exercise 9-12



10

Number and Algebra

Simultaneous equations

Many scientific, natural, economic and social phenomena can be modelled by equations. Often these models consist of more than one equation. For example, when manufacturing milk, equations can be written that describe relationships between quantity, cost and income. These equations can then be solved simultaneously to obtain information on pricing and the quantities that need to be produced and sold to make a profit.



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Chapter outline

	Proficiency strands			
10-01 Solving simultaneous equations graphically	U	F	R	C
10-02 The elimination method	U	F	R	C
10-03 The substitution method	U	F	R	C
10-04 Simultaneous equations problems	U	F	PS	C

Wordbank

coefficient The numerical part of an algebraic term. For example, in $3x^2 + 7x - 1$ the coefficient of x is 7.

elimination method A method of solving simultaneous equations that involves combining them to eliminate one of the variables

graphical method A method of solving simultaneous equations that involves graphing them on a number plane and identifying the point(s) of intersection

simultaneous equations Two (or more) equations that must be solved together so that the solution satisfies both equations. For example, $y = 2x + 1$ and $y = 3x$ are simultaneous equations that have a solution of $x = 1, y = 3$.

substitution method A method of solving simultaneous equations that involves substituting one equation into another equation

In this chapter you will:

- solve linear simultaneous equations, using algebraic and graphical techniques including using digital technology 
- solve linear simultaneous equations by graphing them on a number plane and finding the point of intersection of the lines
- solve linear simultaneous equations algebraically using the elimination and substitution methods
- solve problems using linear simultaneous equations

SkillCheck

Worksheet

StartUp assignment 9

MAT10NAWK10064

- Given the equation $y = 2x + 5$, find y when:
 - $x = 0$
 - $x = 4$
 - $x = \frac{1}{2}$
 - $x = -3$
- Given the equation $y = 4 - 3x$, find y when:
 - $x = 5$
 - $x = 1$
 - $x = -1$
 - $x = -\frac{1}{2}$
- By completing a table of values, graph each equation.
 - $y = x + 1$
 - $y = 3x$
 - $y = \frac{x}{2} - 1$
 - $y = 3 - x$
 - $x + y = 4$
 - $2x - y = 5$
- Test whether the point $(-2, 3)$ lies on the line represented by each equation.
 - $y = 1 - x$
 - $x + y = 3$
 - $2x - y = 7$
 - $\frac{1}{2}x + y = 2$
 - $y = 3x + 7$
 - $2y = 3x$
- Show that the point $(2, 5)$ lies on both the lines $y = 2x + 1$ and $x + y = 7$.
 - At what point do these two lines intersect?
- Use the y -intercept and the gradient to graph each equation.
 - $y = -2x + 3$
 - $y = \frac{5}{2}x - 2$
 - $y = -\frac{4}{3}x + 5$

Investigation: When two lines meet

- Copy and complete the table of values for each equation.

a $x + 2y = 0$

x	-2	-1	0	1	2
y					

b $y = x + 4$

x	-2	-1	0	1	2
y					

- Which coordinates satisfy both equations?
- On the same set of axes draw the graphs of $x + 2y = 0$ and $y = x + 4$.
- Do the lines you drew in question 3 intersect?
 - What are the coordinates of the point of intersection?

5 Repeat questions 1 to 4 for these pairs of equations.

a $x - y = 5$

b $3x + y = 8$

$2x + y = 1$

$x + 2y = 1$

6 Copy and complete.

a The coordinates of the p_____ of intersection between two lines satisfy both equations.

b The values of x and y that satisfy both equations are the coordinates of the _____.

10-01 Solving simultaneous equations graphically

A linear equation in one variable such as $3x + 5 = 11$ has only one solution for x ($x = 2$). However, a linear equation in **two** variables, such as $x + 3y = 5$, has more than one solution (for example, $x = 2, y = 1$, or $x = 5, y = 0$, and so on). The equation actually has an **infinite number** of solutions.

We will now look at solving two equations simultaneously to see if there is a solution that satisfies both equations.

Simultaneous equations can be solved **graphically** or **algebraically**.

Summary

- Linear simultaneous equations can be graphed as lines on the same number plane.
- If two (non-parallel) lines are drawn, the lines will intersect.
- At the point of intersection, the x -coordinate and y -coordinate represent the solution to the simultaneous equations.

Example 1

On the same set of axes, graph $3x + y = 4$ and $x + y = -2$, then solve the equations simultaneously.

Solution

Step 1

Construct tables of values.

$3x + y = 4$

x	0	1	2
y	4	1	-2

$x + y = -2$

x	0	1	2
y	-2	-3	-4

Worksheet

Testing simultaneous equations

MAT10NAWK10065

Worksheet

Intersection of lines

MAT10NAWK10066

Technology worksheet

Excel worksheet:
Solving simultaneous equations

MAT10NACT00017

Technology worksheet

Excel spreadsheet:
Simultaneous equations solver

MAT10NACT00047

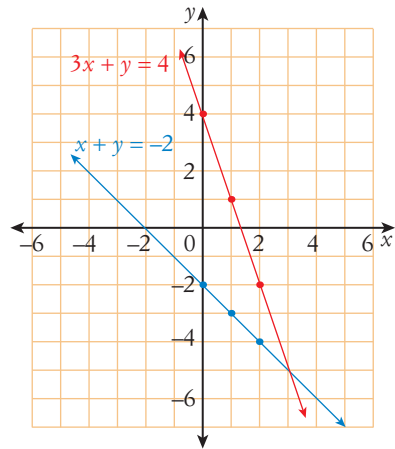
Step 2

Graph the equations.

The lines intersect at $(3, -5)$.

$\therefore$ The solution of the simultaneous equations $3x + y = 4$ and $x + y = -2$ is $x = 3, y = -5$.

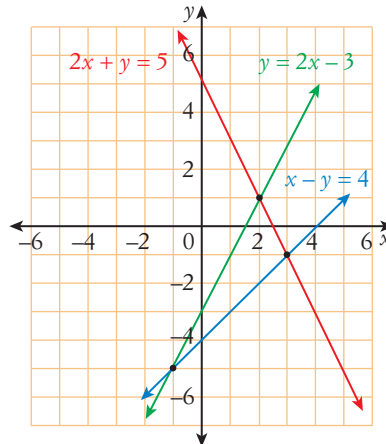
Check that $x = 3, y = -5$ satisfies both equations.



Exercise 10-01 Solving simultaneous equations graphically

- 1 Use the graph to write the solution to each pair of simultaneous equations.

- a $x - y = 4$ and $2x + y = 5$
- b $2x + y = 5$ and $y = 2x - 3$
- c $x - y = 4$ and $y = 2x - 3$



- See Example 1** 2 Graph each pair of equations on the same set of axes. Then find the solution to the pair of simultaneous equations.

- a $y = 2x$ and $y = 3 - x$
- b $y = 2x + 1$ and $y = x - 4$
- c $x + y = 3$ and $4x + y = 6$
- d $y = -x + 2$ and $y = 3x + 4$
- e $y = 2x - 5$ and $y = 5x + 1$
- f $2x + y = 6$ and $y = 1 - x$
- g $y = 7 - x$ and $y = 3x + 5$
- h $x + 2y = 7$ and $2x - y = 4$
- i $3x - 2y = 12$ and $x + 2y = 8$
- j $y = x + 3$ and $2x - y = 2$
- k $5x - y = 5$ and $x + y = 4$
- l $5x + 3y = 20$ and $y = x - 4$

- 3 a On the same set of axes, draw the graphs of $y = 1 - 2x$ and $2x + y = 4$.
b Why isn't there a solution to the simultaneous equations $y = 1 - 2x$ and $2x + y = 4$?

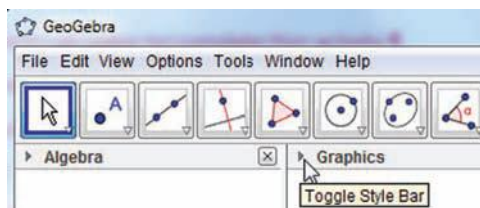
Technology Solving simultaneous equations graphically

You can use GeoGebra or other graphing software to solve simultaneous equations graphically. Write each answer as coordinates in the form (x, y) representing the **point of intersection**.

Using GeoGebra

Before you start, apply these settings.

- 1 Open up **GeoGebra** and click the little arrow in front of **Graphics**.

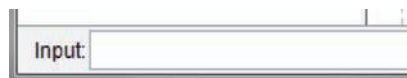


From the new panel that pops up, select the grid option at the top left-hand side.

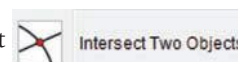
- 2 Enter these equations in the **Input bar**.

$$y = -x + 1$$

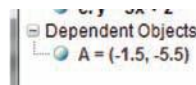
$$y = x + 3$$



- 3 If the points are difficult to read on screen, select **Intersect Two Objects** from the second icon drop-down menu.



- 4 In the **Algebra View** (left) you will see the exact coordinates of the point of intersection. They are listed as **Dependent Objects**.



- 5 Enter each pair of equations using step 2 above.

a $y = 2$

$$x = 3$$

c $y = 5x + 2$

$$y = 3x - 1$$

e $y = -x - 8$

$$y = -3x + 4$$

g $y = -x$

$$x = -4$$

i $y = -2x + 2$

$$y = -2x$$

HINT: Click  to locate the point of intersection

What do you notice about these equations? Do they intersect?

b $y = -2x + 4$

$$y = x - 5$$

d $y = -1$

$$x = 0$$

f $y = 2x + 6$

$$y = x + 9$$

h $y = x + 4$

$$y = x + 6$$

- 6 Enter each set of equations using step 2 above. Find the point of intersection.

a $y = 3x, y = -x + 2$ and $x = 0.5$

b $y = -4x + 1, y = -5x$ and $y = x + 6$

Technology

GeoGebra:
Simultaneous
equations

MAT10NATC00011

10-02 The elimination method

Using graphs to solve simultaneous equations can be time-consuming and inaccurate. **Algebraic methods** provide a better way of solving things. There are two algebraic methods: the **elimination method** and the **substitution method**.

In the elimination method, equations are added or subtracted to eliminate one of the variables.

Example 2

Video tutorial

Simultaneous
equations

MAT10NAV10027

Solve the simultaneous equations $x + 3y = 7$ and $4x - 3y = 13$.

Solution

Label each equation.

$$x + 3y = 7 \quad [1]$$

$$4x - 3y = 13 \quad [2]$$

Since there is the same number of y s in each equation, and since they are **opposite** in sign ($3y$ and $-3y$), **add** equations [1] and [2] to eliminate the variable y .

$$5x = 20 \quad [1] + [2]$$

$$\frac{5x}{5} = \frac{20}{5}$$

$$\therefore x = 4$$

Substitute $x = 4$ into equation [1] to find the y -value.

$$x + 3y = 7 \quad [1]$$

$$4 + 3y = 7$$

$$4 + 3y - 4 = 7 - 4$$

$$3y = 3$$

$$\therefore y = 1$$

$\therefore$ The solution is $x = 4, y = 1$.

Example 3

Solve $2k + 3m = 9$ and $2k - 5m = 1$.

Solution

Label each equation.

$$2k + 3m = 9 \quad [1]$$

$$2k - 5m = 1 \quad [2]$$

Since there is the same number of k s in each equation, and because they have the **same** sign ($2k$ and $2k$), **subtract** equation [2] from [1] to eliminate k .

$$\begin{aligned} 8m &= 8 & [1] - [2] \\ \frac{8m}{8} &= \frac{8}{8} \\ \therefore m &= 1 \end{aligned}$$

Substitute $m = 1$ into equation [1] to find the value of k .

$$\begin{aligned} 2k + 3m &= 9 \\ 2k + 3 \times 1 &= 9 \\ 2k + 3 &= 9 \\ 2k + 3 - 3 &= 9 - 3 \\ 2k &= 6 \\ \frac{2k}{2} &= \frac{6}{2} \\ \therefore k &= 3 \end{aligned}$$

$\therefore$ The solution is $m = 1, k = 3$.

Example 4

Solve $3a + 4c = 8$ and $2a - 3c = 11$.

Solution

Label each equation.

$$\begin{aligned} 3a + 4c &= 8 & [1] \\ 2a - 3c &= 11 & [2] \end{aligned}$$

In this case, neither adding nor subtracting equations [1] and [2] will eliminate a variable. Let's choose to eliminate c . We need to make the coefficient of c the same in both equations ($12c$).

$$\begin{aligned} 9a + 12c &= 24 & [3] & \text{Multiplying both sides of [1] by 3.} \\ 8a - 12c &= 44 & [4] & \text{Multiplying both sides of [2] by 4.} \\ 17a &= 68 & [3] + [4] \end{aligned}$$

$$\therefore a = 4$$

Substitute $a = 4$ in [1] to find c .

$$\begin{aligned} 3a + 4c &= 8 \\ 3 \times 4 + 4c &= 8 \\ 12 + 4c &= 8 \\ 4c &= -4 \\ c &= -1 \end{aligned}$$

$\therefore$ The solution is $a = 4, c = -1$.

The **coefficient** of c is the number in front of the c in the equation

Exercise 10-02 The elimination method

See Example 2 1 For each pair of simultaneous equations, eliminate one variable by adding the equations, then solve the equations.

a $4k + d = 5$
 $2k - d = 7$

b $2x - w = 6$
 $x + w = 9$

c $3g + 5h = 4$
 $2g - 5h = 6$

d $7p - 4n = -20$
 $3p + 4n = 10$

e $4q + 3r = 8$
 $-q - 3r = 7$

f $-5k - 3x = 8$
 $5k + 4x = -3$

g $-4c - 6e = -12$
 $4c - 10e = -4$

h $-3y + 5k = 21$
 $3y + k = -3$

i $a + 3f = 8$
 $-a + 4f = 6$

See Example 3 2 For each pair of simultaneous equations, eliminate one variable by subtracting the equations, then solve the equations.

a $5k + d = 16$
 $3k + d = 4$

b $4a + 3c = 7$
 $a + 3c = 4$

c $4h + 3y = 24$
 $4h - y = 8$

d $3x + 5e = 16$
 $3x - 2e = -5$

e $4q - 2w = -1$
 $7q - 2w = 8$

f $6p + c = 16$
 $4p + c = 10$

g $5y + 3m = 18$
 $2y + 3m = 6$

h $3a + 2r = 8$
 $a + 2r = 10$

i $-x + 5w = 8$
 $-x + 3w = 4$

See Example 4 3 Solve each pair of simultaneous equations.

a $3w + q = 6$
 $2w - 3q = 15$

b $2x + m = 5$
 $3x + 2m = 3$

c $2d + 3h = 25$
 $d + 4h = -5$

d $-3g + 2n = 9$
 $g + 5n = 14$

e $5m - h = 10$
 $m - 3h = 2$

f $2y + 3e = -6$
 $5y - 2e = 23$

g $3q - 2w = 11$
 $2q - 5w = 22$

h $5a + 3d = 4$
 $4a + 2d = 3$

i $-2p + 3k = 19$
 $7p + 4k = 6$

j $5a + 2f = -14$
 $2a - 3f = 2$

k $5r - 3c = 2$
 $-3r + 2c = -14$

l $5y - 4x = 1$
 $2y - 3x = 6$

Puzzle sheet

Simultaneous
equations order activity

MAT10NAPS10067

10-03 The substitution method

With the substitution method, substitute the x or y variables from one equation into the other equation.

Puzzle sheet

Simultaneous
equations by
substitution

MAT10NAPS00039

Example 5

Solve the simultaneous equations $y = x + 4$ and $y = 3x - 2$.

Solution

Label each equation.

$$y = x + 4 \quad [1]$$

$$y = 3x - 2 \quad [2]$$

Use equation [1] to substitute for y in equation [2] and solve for x .

$$\begin{aligned} x + 4 &= 3x - 2 \\ x + 4 - 3x &= 3x - 2 - 3x \\ -2x + 4 &= -2 \\ -2x + 4 - 4 &= -2 - 4 \\ -2x &= -6 \\ \frac{-2x}{-2} &= \frac{-6}{-2} \\ x &= 3 \end{aligned}$$

Now substitute $x = 3$ in equation [1] to find y .

$$\begin{aligned} y &= x + 4 \\ y &= 3 + 4 \\ &= 7 \end{aligned}$$

$\therefore$ The solution is $x = 3$ and $y = 7$.

Example 6

Solve the simultaneous equations $5x + 3y = 9$ and $y = 7 - 3x$.

Solution

Label each equation.

$$5x + 3y = 9 \quad [1]$$

$$y = 7 - 3x \quad [2]$$

Since y is the subject in [2], substitute equation [2] into equation [1] to give an equation using x only.

$$\begin{aligned} 5x + 3(7 - 3x) &= 9 \\ 5x + 21 - 9x &= 9 \\ -4x &= -12 \\ \frac{-4x}{-4} &= \frac{-12}{-4} \\ x &= 3 \end{aligned}$$

Video tutorial

Simultaneous
equations

MAT10NAVT10027

Now substitute $x = 3$ into equation [2] to find y .

$$y = 7 - 3x$$

$$\begin{aligned} y &= 7 - 3 \times 3 \\ &= -2 \end{aligned}$$

$\therefore$ The solution is $x = 3$ and $y = -2$.

Exercise 10-03 The substitution method

See Example 5

- 1 Use the substitution method to solve each pair of simultaneous equations.

a $y = 2x + 1$ and $y = x + 3$

b $y = 5 - 2x$ and $y = 3x + 2$

c $x = 3 + 2y$ and $x = 9 - y$

d $y = -x$ and $y = 3x - 8$

e $x = 1 - 4y$ and $x = 2y + 7$

f $x = 2y$ and $x = 6 - y$

See Example 6

- 2 Solve each pair of simultaneous equations.

a $y = 2x + 3$ and $3x - y = 6$

b $y = x - 2$ and $3x + y = 18$

c $y = 1 - 4x$ and $4x + 2y = 3$

d $x = 2y - 5$ and $4x - y = -13$

e $x = 3y - 4$ and $5x - 4y = 2$

f $x = 5 - 3y$ and $4y - x = 23$

g $2x - 5y = -1$ and $y = 10 - x$

h $6y - 2x = 9$ and $y = \frac{x+2}{2}$

i $x = \frac{9-y}{3}$ and $3x + 2y = 10$

j $y = 3x + 5$ and $4x - 3y = 1$

Investigation: Elimination or substitution method?

With two algebraic methods for solving simultaneous equations, often it is more efficient to use one method than another.

- 1 Consider these pairs of simultaneous equations.

a $x - 2y = 9$ [1]

b $4a + 3c = 18$ [1]

c $3a - 2y = -5$ [1]

$3x + 2y = 11$ [2]

$4a - 3c = -6$ [2]

$2a + 5y = 3$ [2]

i Why might the elimination method be the more appropriate method to use with these equations?

ii What feature in the pairs of equations do you look for to decide if the elimination method is the best one to use?

iii Solve the three pairs of simultaneous equations using the elimination method.

- 2 Consider these pairs of simultaneous equations.

a $m = 2p$ [1]

b $m = 4 - p$ [1]

c $p = 2m - 5$ [1]

$m + p = 15$ [2]

$4m - 3p = -6$ [2]

$5m - 3p = 11$ [2]

i Why might the substitution method be the more appropriate method to use with these equations?

ii What feature in the pairs of equations do you look for to decide if the substitution method is the best one to use?

iii Solve the three pairs of simultaneous equations using the substitution method.

3 Using whichever method is more efficient, solve each of these pairs of simultaneous equations.

a $7c + 2y = 13$ [1]

$3c + 2y = 1$ [2]

b $m = 5 - k$ [1]

$2m - k = 4$ [2]

c $3x + 8y = 10$ [1]

$x = 3 - 2y$ [2]

d $4h - 3w = 8$ [1]

$4h + 7w = 12$ [2]

e $3d = q$ [1]

$q + 4d = 14$ [2]

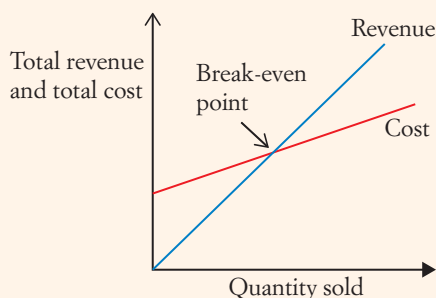
f $3h + 5r = 7$ [1]

$2h - 3r = -8$ [2]

Just for the record

Break-even point

Manufacturers use simultaneous equations to make decisions about how many products they should make and sell. Linear equations can be formed to determine total revenue (the amount made from selling products) and total costs (the cost of making the products). Total revenue = cost per item $\times$ number of items made, while total costs includes rent and production costs.



The equations can be graphed as shown. The point where the two lines intersect is called the **break-even point** and occurs when total revenue is equal to total cost.

A publisher receives \$35 per book sold. There are fixed costs of \$110 000 and production costs per book are \$8.50.

- Determine the equations for total revenue and total costs.
- Graph the equations to find the break-even point.
- How many books must be sold before the publisher makes a profit?

10-04

Problems involving simultaneous equations

Worksheet

Simultaneous
equations problems

MAT10NAWK10068

Animated example

Simultaneous
equations

MAT10NAEE00011

Sometimes, worded problems can be solved using simultaneous equations.

- Read the problem carefully
- Identify the variables to be used
- Use the variables to write simultaneous equations from the information given in the problem
- Solve the equations
- Solve the problem by answering in words

Example 7

At an art show there were 520 guests. If there are 46 more women than men, how many women attended the show?

Solution

Let the number of women attending be w .

Let the number of men attending be m .

$$\therefore w + m = 520 \quad [1]$$

$$\text{and } w = m + 46 \quad [2]$$

520 people altogether.
46 more women than men.

Use equation [2] to substitute for w in equation [1].

$$m + 46 + m = 520$$

$$2m + 46 = 520$$

$$2m + 46 - 46 = 520 - 46$$

$$2m = 474$$

$$m = 237$$

Substitute $m = 237$ into equation [2] to find w .

$$w = 237 + 46$$

$$= 283$$

$\therefore$ There were 283 women who attended the art show.

Example 8

Anita and Ben spent \$931 on shrubs and trees for their new home. Altogether they bought 70 plants. The shrubs cost \$11 each while the trees cost \$18 each. How many of each plant did they buy?



123rf/Shariff Che'Lah

Solution

Let x be the number of shrubs.

Let y be the number of trees.

$$\therefore x + y = 70 \quad [1]$$

$$\text{and } 11x + 18y = 931 \quad [2]$$

Neither adding nor subtracting equations [1] and [2] will eliminate a variable. Let's choose to eliminate x . We will need to make the coefficient of x the same in both equations ($11x$).

$$11x + 18y = 931 \quad [2]$$

$$11x + 11y = 770 \quad [3]$$

$$7y = 161$$

$$y = 23$$

Substitute $y = 23$ in [1] to find the value of x .

$$x + y = 70$$

$$x + 23 = 70$$

$$\therefore x = 47$$

So Anita and Ben bought 47 shrubs and 23 trees.

Multiplying both sides of equation [1] by 11.

$$[2] - [3]$$

Exercise 10-04 Problems involving simultaneous equations

- At a school concert there were 640 guests. There were 70 more women than men. How many of the audience were men?
- At a circus, there were twice as many children as there were adults in attendance. Altogether, 1020 attended the circus. How many were children?
- Tickets to a concert cost \$5 for children and \$14 for adults. Altogether, 650 people attended the concert and ticket sales totalled \$5824. Let a stand for the number of adults and c stand for the number of children.
 - Explain why the equations $a + c = 650$ and $14a + 5c = 5824$ correctly match the information.
 - Solve the equations simultaneously to find the number of children that attended the concert.
- Tracey bought a total of 17 DVDs and CDs. Each DVD cost her \$25 and each CD cost \$18. Altogether, Tracey spent \$390. How many DVDs did she buy?
- Aaron is three times as old as Sejuti. The sum of their ages is 48. How old are Aaron and Sejuti?
- The sum of the ages of Mrs Bui and her daughter Hayley is 70. The difference between their ages is 38 years. How old is Hayley?

See Example 7

See Example 8

- 7 A business bought a total of 60 ink cartridges. Some of them were black, costing \$42 each. The others were colour, each costing \$35. How many of each type did the business buy if the total cost of the ink cartridges was \$2352?
- 8 Five pies and two sausage rolls cost a total of \$23.40, while two pies and 3 sausage rolls cost \$15.30. Find the cost of a pie and the cost of a sausage roll.
- 9 Pete's Pizzas sells Supreme pizzas for \$15.90 each and Vegetarian pizzas for \$13.50 each. If 45 pizzas were sold at lunchtime, totalling \$684.30, how many of each pizza were sold?



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- 10 Nasser bought 3 punnets of strawberries and 5 punnets of blueberries for \$35.45 and Sarah bought 7 punnets of strawberries and 2 punnets of blueberries for \$34.48. What was the cost of each punnet of strawberries and blueberries?
- 11 A money box contains only 20-cent coins and 50-cent coins. Altogether, there are 245 coins in the money box and they amount to \$76.30. Let x be the number of 20-cent coins and y be the number of 50-cent coins.
- Explain why the equations $x + y = 245$ and $20x + 50y = 7630$ correctly match the information.
 - Solve the equations to determine the number of 20-cent and 50-cent coins in the money box.
- 12 The initial cost for producing bottles of fresh orange juice is \$135 plus \$1.20 for each bottle. The bottles of juice are sold for \$3 each. C is the cost in dollars, R is the total sales in dollars and n is the number of bottles produced and/or sold.
- Explain why the equations $C = 135 + 1.2n$ and $R = 3n$ correctly match the information.
 - Copy and complete the table of values below for both equations.

$$C = 135 + 1.2n$$

n	0	50	100
C			

$$R = 3n$$

n	0	50	100
R			

- Draw the graphs of both equations on the same axes for values of 0 to 100 for n on the horizontal axis and values of \$0 to \$300 on the vertical axis.
- For what value of n is total sales equal to total cost (the break-even point)?

Mental skills 10

Maths without calculators

Simplifying fractions and ratios

When simplifying a fraction or a ratio, look for a common factor to divide into both the numerator and the denominator, preferably the highest common factor (HCF).

1 Study each example.

a Simplify $\frac{27}{45}$

$$\frac{\cancel{27}^9}{\cancel{45}_{15}} = \frac{9}{15}$$

Dividing numerator and denominator by 3.

$$\frac{\cancel{9}^3}{\cancel{15}_5} = \frac{3}{5}$$

Dividing numerator and denominator by 3 again.

Note: This fraction could be simplified in one step if you divided by 9, the highest common factor (HCF) of 27 and 45.

b Simplify $\frac{160}{400}$.

$$\frac{\cancel{160}^{10}}{\cancel{400}^{10}} = \frac{16}{40}$$

Dividing numerator and denominator by 10.

$$\frac{\cancel{16}^2}{\cancel{40}_5} = \frac{2}{5}$$

Dividing numerator and denominator by 8.

Note: This fraction could be simplified in one step if you divided by 80, the HCF of 160 and 400.

c Simplify 24 : 36.

$$24 : 36 = \cancel{24}^4 : \cancel{36}^6 = 4 : 6$$

Dividing both terms by 6.

$$\cancel{4}^2 : \cancel{6}^3 = 2 : 3$$

Dividing both terms by 2.

Note: This fraction could be simplified in one step if you divided by 12, the HCF of 160 and 400.

d Simplify 135 : 90.

$$135 : 90 = 135^{\cancel{27}} : \cancel{90}^{18} = 27 : 18$$

Dividing both terms by 5.

$$\cancel{27}^3 : \cancel{18}^2 = 3 : 2$$

Dividing both terms by 9.

e Calculate $\frac{3}{8} \times \frac{2}{15}$ in simplest form.

$$\frac{3}{8} \times \frac{2}{15} = \frac{3}{\cancel{4}^2} \times \frac{\cancel{2}^1}{15}$$

Dividing 2 and 8 by 2.

$$= \frac{\cancel{3}^1}{\cancel{4}^2} \times \frac{\cancel{2}^1}{\cancel{15}_5}$$

Dividing 3 and 15 by 3.

$$= \frac{1}{20}$$

f What fraction is 36 minutes of 1 hour?

$$\frac{36}{1 \text{ h}} = \frac{36 \text{ min}}{60 \text{ min}} = \frac{3}{5}$$

2 Now simplify each fraction or ratio.

a $\frac{10}{15}$

b $\frac{16}{20}$

c $\frac{30}{42}$

d $\frac{8}{16}$

e $\frac{20}{80}$

f $\frac{6}{36}$

g $\frac{20}{24}$

h $\frac{12}{30}$

i 20 : 36

j 25 : 45

k 18 : 40

l 28 : 35

m 27 : 21

n 16 : 12

o $\frac{5}{6} \times \frac{18}{25}$

p $\frac{12}{50} \times \frac{10}{21}$

3 Express each as a simplified fraction.

a 425 g of 1 kg

b 8 months of 1 year

c 64 cm of 1 m

d 750 mL of 3 L

e 10 hours of 2 days

f 80c of \$10

Technology SMS plans

Use a spreadsheet to solve the problem below.

Madhu wants to change her mobile phone provider. She is looking for a better deal on text messages as this is her preferred method of contacting her friends and family. After Madhu has researched various company rates, she decides on a comparison of 2 providers.

Company A: Costs \$49 per month plus 25 cents per SMS, per recipient.

Company B: Monthly fee is \$35 but charges 42c per SMS, per recipient.

To calculate and graph the cost of each mobile phone text messaging plan, create a spreadsheet for both companies for $t = 0, 5, 10, 15, 20, \dots, 95, 100$ text messages.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W
1	Company A																						
2	No of Texts (t)	0	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100	
3	Cost (C)																						
4	Company B																						
5	No of Texts (t)	0	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100	
6	Cost (C)																						

1 In cell B3, enter the formula for *Company A*. Formula: $=49+0.25*B2$

Fill Right to continue the formula from cell B3 to V3.

2 Highlight B3 to V3 and B6 to V6. Click **Format Cells** to convert to currency (to 2 decimal places).

3 In cell B6, create a formula for *Company B*. [Hint: a formula must start with =]

Fill Right to continue the formula from cell B6 to V6.

4 Create an **XY Scatter** graph of the data for rows 2, 3 and 6 only.

Give your graph an appropriate title and label both axes.

5 Answer the following questions in the specified cells on your spreadsheet.

[Hint: some questions might need a formula]

a Cell A8: For small numbers of text messages, which company charges more?

b Cell A9: How much more did Company A charge if 30 text messages were sent per month?
[Write a formula]

c Cell A10: How much cheaper was Company B if 45 text messages were sent in the month?
[Write a formula]

d Cell A11: Between which 2 values for text messages sent per month did the graphs intersect (e.g., between 45 and 50)? Explain the significance of the point of intersection between the two graphs.

- e If Madhu's mother sets a limit of \$60 per month for text messaging:
- i A12: What is the most number of text messages that Madhu could possibly send? [Answer correct to nearest whole number]
 - ii A13: How many **more** texts will Madhu be able to send with one company than the other?
- f A14: Extend rows 2, 3, 5 and 6 to 120 text messages sent per month.
- g i A15: If Madhu's mother sets a limit of 60 text messages per month, which is the better plan? Give a reason for your answer.
- ii A16: If Madhu starts a part time job and contributes \$20 extra per month to the limit her mother has set, how many **more** texts will Madhu be able to send?
[Answer correct to nearest whole number]



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Power plus

- 1 With simultaneous equations in two variables, we have two equations to solve. With simultaneous equations in **three** variables, we have **three** equations to solve.

Step 1: Take two of the equations and eliminate one of the variables.
 Step 2: Take another two of the equations and eliminate the same variable.
 Step 3: Solve the two new simultaneous equations from Steps 1 and 2.
 Step 4: Use substitution to find the values of the other two variables.

Use the above steps to solve the following sets of simultaneous equations.

a $2x + y - 3w = -16$	b $3a - 2c + d = 5$	c $2m + 3n - p = 9$
$x - y + 4w = 25$	$5a + 2c + d = 25$	$3m - 2n + 5p = 27$
$3x - y + 2w = 19$	$4a + 3c - d = 10$	$4m + 3n + 2p = 13$

- 2 a Show that the solutions to the simultaneous equations $ax + by = c$ and $dx + ey = f$ are

$$x = \frac{ce - bf}{ae - bd} \text{ and } y = \frac{af - cd}{ae - bd}$$

- b The above solutions do not work when $ae = bd$. Explain why.
- c Solve the equations $3x - 2y = 11$ and $5x + y = 14$ by either the substitution or elimination method. Check that the results in part a give the same answer.
- d Set up a spreadsheet to solve simultaneous equations of the form $ax + by = c$ and $dx + ey = f$ using the solutions $x = \frac{ce - bf}{ae - bd}$ and $y = \frac{af - cd}{ae - bd}$.

Use your spreadsheet to solve the simultaneous equations below.

i $3x + y = 4$	ii $3x - 5y = 4$	iii $15x + 6y = 17$
$2x - y = 6$	$2x - 3y = 8$	$2x + 3y = 8$

Chapter 10 review

Language of maths

Puzzle sheet

Simultaneous
equations crossword

MAT10NAPS10069

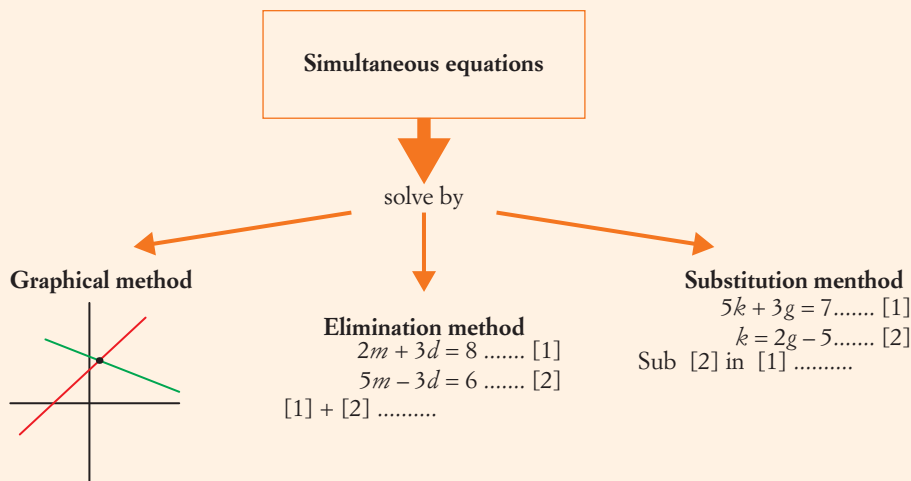
algebraic	axes	coefficient	elimination method
graphical	linear	point of intersection	satisfy
simultaneous equations	solution	substitution method	variable

- 1 How do you think **simultaneous equations** got their name?
- 2 What are the two algebraic methods for solving simultaneous equations?
- 3 Which algebraic method involves cancelling one of the variables?
- 4 What word means the answer to an equation or problem?
- 5 What does '**linear**' mean?
- 6 Which method of solving simultaneous equations involves the point of intersection of lines on a number plane?

Topic overview

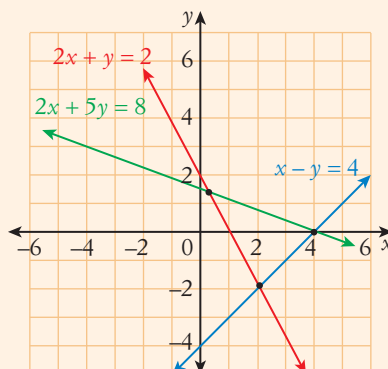
- In your own words, write down the new things you have learnt about simultaneous equations.
- What parts of this topic did you like?
- What parts of the topic did you find difficult or not understand?
- Copy and complete the following topic overview, and refer to the **Language of maths** word list for keywords you might like to include.

Copy and complete this mind map of the topic, adding detail to its branches and using pictures, symbols and colour where needed. Ask your teacher to check your work.



- 1 Use the graph to write the solution to each pair of simultaneous equations.

- a $x - y = 4$ and $2x + y = 2$
b $2x + 5y = 8$ and $x - y = 4$



See Exercise 10-01

- 2 Graph each pair of simultaneous equations on the same set of axes. By finding their point of intersection, write the solution to each pair of equations.

See Exercise 10-01

a $y = x + 2$	b $y = 3 - \frac{x}{2}$	c $y = 4 - 3x$
$y = 6 + 2x$	$y = 2x - 7$	$y = x$
d $y = 2x + 3$	e $x + y = 7$	f $y = 5 - 2x$
$y = 9 - x$	$y = 2x + 1$	$y = -1 - x$

- 3 Use the elimination method to solve each pair of simultaneous equations.

See Exercise 10-02

a $5m + 2c = 6$	b $2x + 3y = 5$	c $3a + 4d = 7$
$3m + 2c = -4$	$5x - 3y = 9$	$3a + d = 4$
d $4x - y = 9$	e $x - 4y = 3$	f $3d - 2w = 11$
$x - y = -9$	$x + 2y = -9$	$2d - 5w = 44$

- 4 Use the substitution method to solve each pair of simultaneous equations.

See Exercise 10-03

a $y = 7x - 3$	b $m = 4 - p$	c $h = 3t - 2$
$y = x + 9$	$m = -2 + p$	$h = t + 6$
d $a = 4 - 2c$	e $x + 2y = 3$	f $p = 4 - 2q$
$a = 6c$	$y = 2 - x$	$p = 3q + 24$

- 5 Solve each problem using simultaneous equations.

See Exercise 10-04

- In an audience of 2500, there were 700 more adults than children. Find the number of adults and the number of children that were in the audience.
- Robyn bought 30 CDs and DVDs for a total cost of \$696. Each CD cost \$25 and each DVD cost \$22. How many of each did she buy?
- It costs 2 adults and 5 children \$191 to go to a football game, while the cost of 3 adults and 2 children is \$160. Find the cost of an adult ticket.
- At the cake stall, the SRC sell two types of cakes – cheesecakes for \$4 each and mudcakes for \$3 each. In total, they sold 75 cakes for a total of \$253. How many of each cake did they sell?
- In Year 10, there are 213 students. There are 27 more boys than girls. Find the number of boys and girls in Year 10.

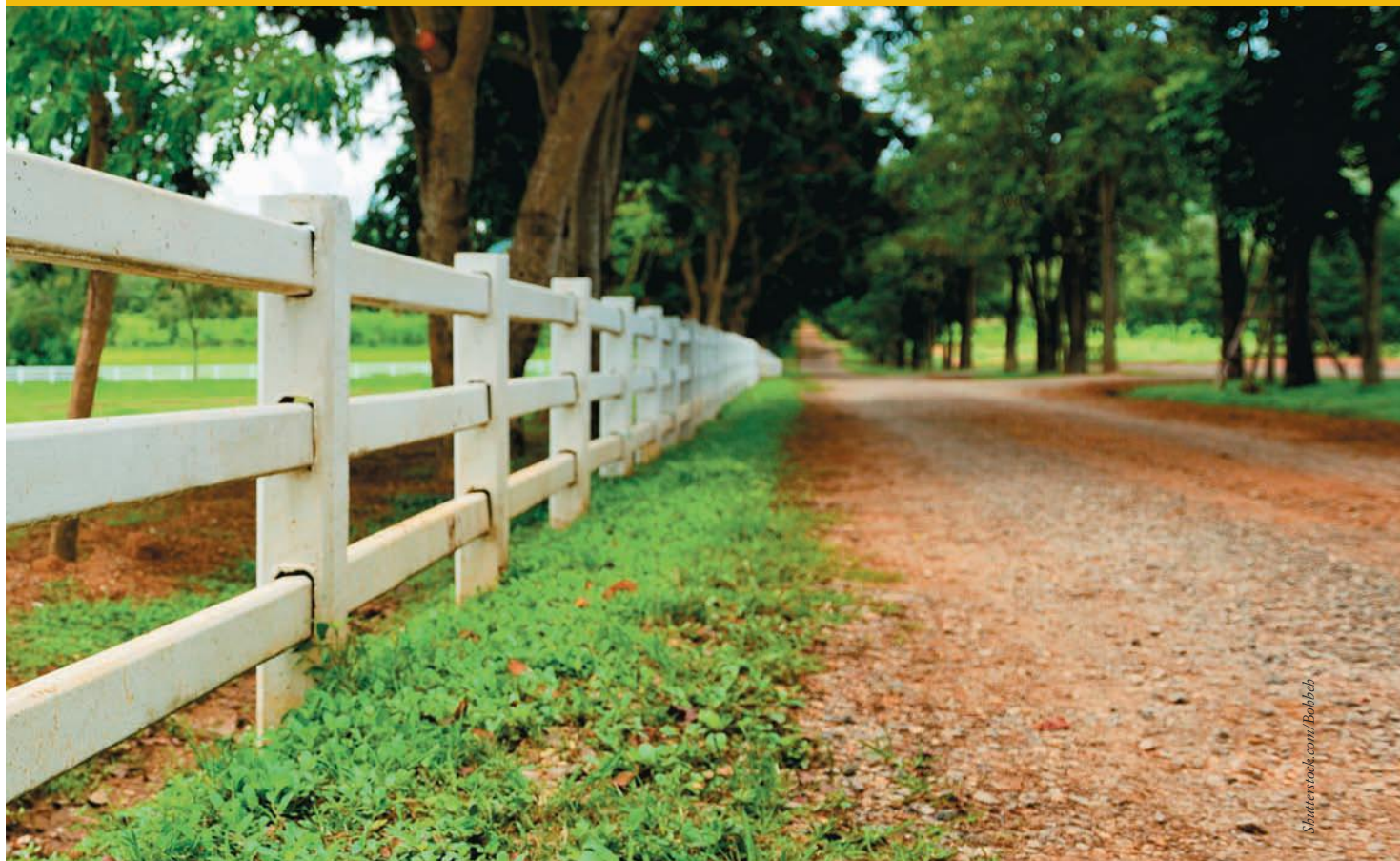


11

Number and Algebra

Quadratic equations and the parabola

A farmer wants to make two rectangular paddocks with 5000 metres of fencing. The length has to be twice the width and he wants the greatest area for his paddocks. What dimensions should the paddocks have? Problems like this can be solved using quadratic equations.



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Chapter outline

	Proficiency strands			
11-01 Quadratic equations $ax^2 + bx + c = 0$ *	U	F	R	C
11-02 Completing the square*	U	F	R	C
11-03 The quadratic formula*	U	F	R	C
11-04 Higher-order quadratic equations*	U	F	R	C
11-05 Quadratic equation problems*	U	F	PS	C
11-06 The parabola $y = ax^2 + bx + c$ *	U	F	R	C
11-07 The axis of symmetry and vertex of a parabola*	U	F	R	C
11-08 Point of intersection of a line and a curve*	U	F	R	C

*STAGE 5.3

Wordbank

coefficient The number in front of a variable; for example, in $y = 3x^2 + 4x - 6$, the coefficient of x^2 is 3

quadratic equation An equation in which the highest power of the variable is 2; for example, $2x^2 - 12x + 10 = 0$

quadratic formula A formula for solving quadratic equations of the form $ax^2 + bx + c = 0$

solution The answer to an equation, inequality or problem, the correct value(s) of the variable that makes an equation or inequality true

x-intercept The x value(s) at which a graph intersects the x-axis

In this chapter you will:

- (STAGE 5.3) factorise monic and non-monic quadratic expressions and solve a wide range of quadratic equations derived from a variety of contexts 
- (STAGE 5.3) solve quadratic equations of the form $ax^2 + bx + c = 0$ by factorising, completing the square and using the quadratic formula
- (STAGE 5.3) simplify and solve higher-order quadratic equations such as $x^4 - 2x^2 - 8 = 0$
- (STAGE 5.3) solve problems involving quadratic equations
- (STAGE 5.3) graph parabolas of the form $y = ax^2 + bx + c$ and identify their features
- (STAGE 5.3) find the axis of symmetry and the vertex of a parabola
- (STAGE 5.3) find the point of intersection of a line with a parabola, circle or hyperbola by solving linear and non-linear simultaneous equations

SkillCheck

Worksheet

StartUp assignment 14

MAT10NAWK10096

1 Solve each quadratic equation.

a $4x^2 = 100$

b $3m^2 - 12 = 0$

c $x^2 - x - 2 = 0$

d $u^2 + 3u - 28 = 0$

e $k^2 + 3k = 0$

f $w^2 - 10w + 25 = 0$

2 Factorise each expression.

a $16 - m^2$

b $d^2 - 121$

c $14y - 2y^2$

d $10p^2 + 25p$

e $5x^2 - 320$

f $18w^2 - 50$

g $k^2 + 5k + 4$

h $y^2 - 10y + 16$

i $m^2 - m - 56$

j $u^2 + 8u - 65$

k $w^2 - 10w + 21$

l $x^2 - 2x - 24$

3 If $y = x^2 + 4x - 7$, find the value of y if:

a $x = 1$

b $x = -1$

c $x = 2$

d $x = -3$

4 Factorise each expression.

a $3a^2 + 10a + 3$

b $5x^2 - 13x - 6$

c $6y^2 + y - 40$

d $15t^2 + 7t - 4$

e $5v^2 - 32v - 21$

f $8y^2 + 34y + 35$

g $15h^2 - 23h + 4$

h $12p^2 + 11p - 15$

i $16d^2 + 40d + 25$

Stage 5.3

11-01 Quadratic equations $ax^2 + bx + c = 0$

Puzzle sheet

Factorising quadratic equations

MAT10NAPS00037

In Chapter 7, Equations and logarithms, we learnt that in a **quadratic equation**, the highest power of the variable is 2; for example, $x^2 = 5$, $3m^2 + 7 = 10$, $d^2 - d - 6 = 0$ and $4y^2 - 3y = 8$. The general form of a quadratic equation is $ax^2 + bx + c = 0$, where a , b and c are constants (numbers).

Solving a quadratic equation means finding those values of the variable that satisfy the equation (or make the statement true). When checking solutions, substitute the possible solutions into the equation and show that the left-hand side (LHS) of the equation is equal to the right-hand side (RHS) of the equation.

Solving $ax^2 + bx + c = 0$ by factorising

To solve quadratic equations of the form $ax^2 + bx + c = 0$, we need to factorise the quadratic expression on the LHS, which we learnt in Chapter 5, Products and factors.

Summary

When **solving quadratic equations by factorising**, the following property is used.
If $pq = 0$, then $p = 0$ or $q = 0$.

Example 1

Solve each quadratic equation.

a $4w(3w + 2) = 0$

b $(5m + 2)(2m - 7) = 0$

Solution

a $4w(3w + 2) = 0$

$\therefore 4w = 0$ or $3w + 2 = 0$

$w = 0$ or $3w = -2$

$w = 0$ or $w = -\frac{2}{3}$

b $(5m + 2)(2m - 7) = 0$

$5m + 2 = 0$ or $2m - 7 = 0$

$5m = -2$ or $2m = 7$

$m = -\frac{2}{5}$ or $m = \frac{7}{2}$

$m = -\frac{2}{5}$ or $m = 3\frac{1}{2}$

Example 2

Solve each quadratic equation. Use substitution to check your solutions.

a $6p^2 - 9p = 0$

b $2x^2 - x - 15 = 0$

c $6y^2 = -7y + 5$

Solution

a $6p^2 - 9p = 0$

$3p(2p - 3) = 0$

$3p = 0$ or $2p - 3 = 0$

$p = 0$ or $2p = 3$

$p = 0$ or $p = \frac{3}{2}$

$p = 0$ or $p = 1\frac{1}{2}$

Check: When $p = 0$, LHS $= 6 \times 0^2 - 9 \times 0 = 0 = \text{RHS}$

When $p = 1\frac{1}{2}$, LHS $= 6 \times \left(1\frac{1}{2}\right)^2 - 9 \times 1\frac{1}{2} = 0 = \text{RHS}$

Factorising

Stage 5.3

Video tutorial

Quadratic equations by factorising

MAT10NAVT10029

Stage 5.3

b $2x^2 - x - 15 = 0$

$2x^2 - 6x + 5x - 15 = 0$

$2x(x - 3) + 5(x - 3) = 0$

$(x - 3)(2x + 5) = 0$

$x - 3 = 0 \quad \text{or} \quad 2x + 5 = 0$

$x = 3 \quad \text{or} \quad 2x = -5$

$x = 3 \quad \text{or} \quad x = -\frac{5}{2}$

$x = 3 \quad \text{or} \quad x = -2\frac{1}{2}$

Check: When $x = 3$, $\text{LHS} = 2 \times 3^2 - 3 - 15 = 0 = \text{RHS}$

When $x = -2\frac{1}{2}$, $\text{LHS} = 2 \times \left(-2\frac{1}{2}\right)^2 - \left(-2\frac{1}{2}\right) - 15 = 0 = \text{RHS}$

c $6y^2 = -7y + 5$

$6y^2 + 7y - 5 = 0$

$6y^2 - 3y + 10y - 5 = 0$

$3y(2y - 1) + 5(2y - 1) = 0$

$(2y - 1)(3y + 5) = 0$

$2y - 1 = 0 \quad \text{or} \quad 3y + 5 = 0$

$2y = 1 \quad \text{or} \quad 3y = -5$

$y = \frac{1}{2} \quad \text{or} \quad y = -\frac{5}{3}$

$y = \frac{1}{2} \quad \text{or} \quad y = -1\frac{2}{3}$

Check: When $y = \frac{1}{2}$:

$\text{LHS} = 6 \times \left(\frac{1}{2}\right)^2 = 1\frac{1}{2}$

$\text{RHS} = -7 \times \left(\frac{1}{2}\right) + 5 = 1\frac{1}{2}$

$\text{LHS} = \text{RHS}$

When $y = -1\frac{2}{3}$:

$\text{LHS} = 6 \times \left(-1\frac{2}{3}\right)^2 = 16\frac{2}{3}$

$\text{RHS} = -7 \times \left(-1\frac{2}{3}\right) + 5 = 16\frac{2}{3}$

$\text{LHS} = \text{RHS}$

Exercise 11-01 Quadratic equations

1 Solve each quadratic equation.

a $(m + 7)(m + 3) = 0$

b $(d - 3)(d - 7) = 0$

c $(y - 3)(y + 5) = 0$

d $k(k - 3) = 0$

e $t(t + 7) = 0$

f $2p(p - 3) = 0$

g $w(3w - 2) = 0$

h $(2n + 1)(n - 3) = 0$

i $(5a - 3)(2a - 1) = 0$

j $(3x + 1)(2x + 3) = 0$

k $(2c - 5)^2 = 0$

l $(2f - 1)^2 = 0$

m $(3c + 1)(4c + 1) = 0$

n $(1 - 2h)(h + 1) = 0$

o $(5 - 7e)(1 - e) = 0$

2 Solve each quadratic equation.

a $2y^2 + 7y + 6 = 0$

b $2g^2 + 5g + 3 = 0$

c $3d^2 + 5d + 2 = 0$

d $5t^2 + 16t + 11 = 0$

e $2p^2 - 11p + 12 = 0$

f $10x^2 - 19x + 6 = 0$

g $8y^2 - 2y - 3 = 0$

h $6a^2 - 5a - 4 = 0$

i $4w^2 - 7w - 15 = 0$

j $5c^2 + 2c - 7 = 0$

k $8e^2 + 10e - 3 = 0$

l $3q^2 + 4q - 15 = 0$

m $4g^2 - 20g + 25 = 0$

n $18m^2 - 3m - 10 = 0$

o $16 - 8w - 3w^2 = 0$

p $36 + 3y - 3y^2 = 0$

q $2f^2 - 24f + 72 = 0$

r $12h^2 + 3h - 9 = 0$

3 Express each quadratic equation in the form $ax^2 + bx + c = 0$ and solve.

a $2x^2 = x + 15$

b $4t(t + 2) = 5$

c $41u = -8u^2 - 5$

d $7m^2 = 8m - 1$

e $p(p - 3) = 28$

f $(e - 2)^2 = 9$

g $t(2t - 13) = -15$

h $7 = 6d^2 + 11d$

i $5h^2 = 125$

j $8f^2 = 4f$

k $6w^2 + 3 = 19w$

l $4a(3a + 5) - 8 = 0$

4 A certain positive number, plus its square, minus 72, equals zero. Find the number.

11-02 Completing the square

Quadratic equations can be solved by using a method called **completing the square**. We try to make the LHS of the equation a perfect square.

The method of **completing the square** is based on the following results for perfect squares.

$$(x + a)^2 = x^2 + 2ax + a^2$$

$$(x - a)^2 = x^2 - 2ax + a^2$$

We note that the last term, a^2 , is the square of 'half the coefficient of x '.

Example 3

Find the numbers that complete the square in each equation.

a $x^2 + 10x + \dots = (x + \dots)^2$

b $x^2 - 14x + \dots = (x - \dots)^2$

Solution

a The coefficient of x is 10.
Half of 10 is 5, and $5^2 = 25$.
The perfect square is
 $x^2 + 10x + 25 = (x + 5)^2$

b The coefficient of x is -14 .
Half of -14 is -7 , and $(-7)^2 = 49$.
The perfect square is
 $x^2 - 14x + 49 = (x - 7)^2$

Stage 5.3

See Example 1

See Example 2

Puzzle sheet

Complete the square
order activity

MAT10NAPS10097

Video tutorial

Solving equations

MAT10NAVT00011

Stage 5.3

Video tutorial

Simple quadratic
equations

MAT10NAVT10028

Example 4

Solve $(k + 3)^2 = 7$.

Solution

$$(k + 3)^2 = 7$$

$$k + 3 = \pm\sqrt{7}$$

$$k + 3 - 3 = \sqrt{7} - 3 \quad \text{or} \quad k + 3 - 3 = -\sqrt{7} - 3$$

This solution is usually written as:

$$k = -3 + \sqrt{7} \quad \text{or} \quad k = -3 - \sqrt{7}$$

Notice that the two solutions are surds (not rational),
of the form $a + \sqrt{b}$ and $a - \sqrt{b}$.

Taking the square root of both sides.

Example 5

Solve $x^2 + 6x + 7 = 0$ by completing the square.

Solution

Step 1

Move the constant term to the RHS.

$$x^2 + 6x = -7$$

Step 2

Halve the coefficient of x , square it and then
add the square to both sides.

$$x^2 + 6x + 3^2 = -7 + 3^2$$

$$x^2 + 6x + 9 = 2$$

Step 3

Express the LHS as a perfect square.

$$(x + 3)^2 = 2$$

Step 4

Solve the resulting equation.

$$x + 3 = \pm\sqrt{2}$$

$$x = -3 \pm \sqrt{2}$$

$$x = -3 + \sqrt{2} \quad \text{or} \quad x = -3 - \sqrt{2}$$

Taking the square root of both sides.

This answer in surd form is
called the 'exact answer'.

Example 6

Solve $2x^2 - 3x - 4 = 0$, writing the solution correct to two decimal places.

Solution

$$2x^2 - 3x - 4 = 0$$

$$2x^2 - 3x = 4$$

$$x^2 - \frac{3}{2}x = 2$$

$$x^2 - \frac{3}{2}x + \left(-\frac{3}{4}\right)^2 = 2 + \left(-\frac{3}{4}\right)^2$$

$$\left(x - \frac{3}{4}\right)^2 = \frac{41}{16}$$

$$x - \frac{3}{4} = \pm \sqrt{\frac{41}{16}}$$

$$= \pm \frac{\sqrt{41}}{4}$$

$$x = \frac{3}{4} \pm \frac{\sqrt{41}}{4}$$

$$= \frac{3 \pm \sqrt{41}}{4}$$

$$x = \frac{3 + \sqrt{41}}{4} \quad \text{or} \quad x = \frac{3 - \sqrt{41}}{4}$$

$$x = 2.35078 \dots \quad \text{or} \quad x = -0.85078 \dots$$

$$x \approx 2.35 \quad \text{or} \quad x \approx -0.85$$

Moving the constant term to the RHS.

Dividing both sides by 2, the coefficient of x^2 .

Completing the square.

Expressing the LHS as a perfect square and leaving the RHS as an improper fraction.

Taking the square root of both sides.

Simplifying the RHS.

This is the exact answer in surd form.

This is an approximate answer in decimal form.

Stage 5.3

See Example 3

Exercise 11-02 Completing the square

1 Find the numbers that 'complete the square' in each equation.

a $x^2 + 2x + \underline{\hspace{1cm}} = (x + \underline{\hspace{1cm}})^2$

c $m^2 - 8m + \underline{\hspace{1cm}} = (m - \underline{\hspace{1cm}})^2$

e $y^2 + 7y + \underline{\hspace{1cm}} = (y + \underline{\hspace{1cm}})^2$

g $x^2 + x + \underline{\hspace{1cm}} = (x + \underline{\hspace{1cm}})^2$

i $a^2 + \frac{7}{2}a + \underline{\hspace{1cm}} = (a + \underline{\hspace{1cm}})^2$

b $p^2 - 6p + \underline{\hspace{1cm}} = (p - \underline{\hspace{1cm}})^2$

d $k^2 + 4k + \underline{\hspace{1cm}} = (k + \underline{\hspace{1cm}})^2$

f $w^2 - 3w + \underline{\hspace{1cm}} = (w - \underline{\hspace{1cm}})^2$

h $h^2 - 5h + \underline{\hspace{1cm}} = (h - \underline{\hspace{1cm}})^2$

j $v^2 - \frac{5}{3}v + \underline{\hspace{1cm}} = (v - \underline{\hspace{1cm}})^2$

Stage 5.3

See Example 4

2 Solve each equation, writing the solution in surd form.

a $(d + 3)^2 = 7$

b $(x - 5)^2 = 5$

c $(p + 1)^2 = 10$

d $(y - 1)^2 = 2$

e $\left(m - \frac{1}{2}\right)^2 = 5$

f $\left(t + \frac{2}{3}\right)^2 = 3$

g $(c + 1)^2 = \frac{21}{2}$

h $(w - 3)^2 = \frac{41}{2}$

i $\left(n + \frac{2}{3}\right)^2 = \frac{7}{9}$

j $\left(e - \frac{3}{2}\right)^2 = \frac{71}{4}$

k $(d - 2)^2 = 5$

l $\left(x - \frac{3}{4}\right)^2 = 2$

See Example 5

3 Solve each equation by completing the square. Leave your answers in exact form.

a $h^2 + 2h - 5 = 0$

b $r^2 - 2r - 1 = 0$

c $m^2 + 6m + 2 = 0$

d $w^2 - 4w - 60 = 0$

e $a^2 - 10a - 5 = 0$

f $y^2 + 8y - 3 = 0$

g $p^2 + 12p - 5 = 0$

h $x^2 - 4x + 2 = 0$

i $u^2 + 9u + 14 = 0$

j $d^2 + d - 7 = 0$

k $c^2 - 9c + 2 = 0$

l $e^2 + 5e + 2 = 0$

m $y^2 - 3y - 8 = 0$

n $b^2 - b - 5 = 0$

o $q^2 - 3q + 1 = 0$

p $2g^2 + 7g - 3 = 0$

q $2x^2 + 5x - 7 = 0$

r $3f^2 + 4f - 6 = 0$

See Example 6

4 Solve each of these by completing the square. Give your answers correct to two decimal places.

a $x^2 + 12x + 9 = 0$

b $m^2 - 16m - 7 = 0$

c $g^2 + 4g - 3 = 0$

d $2h^2 + 3h - 7 = 0$

e $5w^2 - 4w - 3 = 0$

f $3y^2 + y - 5 = 0$

g $3p^2 + 2p - 8 = 0$

h $4e^2 - e - 4 = 0$

i $2n^2 + 3n - 5 = 0$

Worksheet

Investigating quadratic equations

MAT10NAWK10098

Puzzle sheet

Quadratic equations puzzle

MAT10NAPS10099

Video tutorial

The quadratic formula

MAT10NAVT10030

Video tutorial

Solving equations

MAT10NAVT00011

Technology worksheet

Excel worksheet: The quadratic formula

MAT10NACT00018

Technology worksheet

Excel spreadsheet: The quadratic formula

MAT10NACT00048

Puzzle sheet

The quadratic formula

MAT10NAPS00038

11-03 The quadratic formula

There is a formula for solving a quadratic equation of the form $ax^2 + bx + c = 0$ that involves the coefficients a , b and c .

Summary

The **quadratic formula** for the solution to the quadratic equation $ax^2 + bx + c = 0$ is:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 7

Solve each quadratic equation using the quadratic formula.

a $x^2 - 3x + 2 = 0$

b $6x^2 + x - 2 = 0$

c $3x^2 + 11x + 2 = 0$

Solution

a For $x^2 - 3x + 2 = 0$:

$$a = 1, b = -3, \text{ and } c = 2$$

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-3) \pm \sqrt{(-3)^2 - 4 \times 1 \times 2}}{2 \times 1} \\ &= \frac{3 \pm \sqrt{1}}{2} = \frac{3 \pm 1}{2} \end{aligned}$$

$$\begin{aligned} x &= \frac{3+1}{2} & \text{or} & & x &= \frac{3-1}{2} \\ x &= 2 & \text{or} & & x &= 1 \end{aligned}$$

c For $3x^2 + 11x + 2 = 0$:

$$a = 3, b = 11 \text{ and } c = 2$$

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-11 \pm \sqrt{11^2 - 4 \times 3 \times 2}}{2 \times 3} = \frac{-11 \pm \sqrt{97}}{6} \\ x &= \frac{-11 + \sqrt{97}}{6} & \text{or} & & x &= \frac{-11 - \sqrt{97}}{6} \end{aligned}$$

$$\begin{aligned} x &= -0.1918 \dots & \text{or} & & x &= -3.4748 \dots \\ x &\approx -0.19 & \text{or} & & x &\approx -3.47 \end{aligned}$$

b For $6x^2 + x - 2 = 0$:

$$a = 6, b = 1, \text{ and } c = -2$$

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-1 \pm \sqrt{1^2 - 4 \times 6 \times (-2)}}{2 \times 6} \\ &= \frac{-1 \pm \sqrt{49}}{12} = \frac{-1 \pm 7}{12} \end{aligned}$$

$$\begin{aligned} x &= \frac{-1+7}{12} & \text{or} & & x &= \frac{-1-7}{12} \\ x &= \frac{6}{12} & \text{or} & & x &= \frac{-8}{12} \\ x &= \frac{1}{2} & \text{or} & & x &= -\frac{2}{3} \end{aligned}$$

In exact surd form.

Rounded correct to two decimal places.

Example 8

Solve $2x^2 - 6 = 9x$ using the quadratic formula, expressing the answer correct to two decimal places.

Solution

$$\begin{aligned} 2x^2 - 6 &= 9x \\ 2x^2 - 9x - 6 &= 0 \end{aligned}$$

Substitute $a = 2, b = -9, c = -6$

$$\begin{aligned} x &= \frac{-(-9) \pm \sqrt{(-9)^2 - 4 \times 2 \times (-6)}}{2 \times 2} = \frac{9 \pm \sqrt{129}}{4} \\ x &= \frac{9 + \sqrt{129}}{4} & \text{or} & & x &= \frac{9 - \sqrt{129}}{4} \\ x &= 5.0894 \dots & \text{or} & & x &= -0.5894 \dots \\ x &\approx 5.09 & \text{or} & & x &\approx -0.59 \end{aligned}$$

Writing the quadratic equation in the form $ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Stage 5.3

Exercise 11-03 The quadratic formula

See Example 7

1 Solve each quadratic equation using the quadratic formula. Write each solution in exact form.

a $x^2 + 6x + 2 = 0$

b $m^2 - 5m - 3 = 0$

c $w^2 - 8w + 3 = 0$

d $k^2 + 3k - 5 = 0$

e $y^2 - 4y - 1 = 0$

f $p^2 + p - 5 = 0$

g $u^2 - 7u - 3 = 0$

h $2a^2 + 3a - 7 = 0$

i $5q^2 - 6q + 1 = 0$

j $3c^2 + 2c - 2 = 0$

j $4e^2 - 5e - 2 = 0$

l $3x^2 + 8x + 2 = 0$

m $2d^2 - 4d - 5 = 0$

n $3a^2 - 10a - 2 = 0$

o $2t^2 + 3t - 5 = 0$

p $3y^2 + 8y + 4 = 0$

q $6k^2 - 11k + 5 = 0$

r $2n^2 - 5n - 11 = 0$

See Example 8

2 Solve each quadratic equation using the quadratic formula, expressing the solution as a surd.

a $5y^2 - 9y = 3$

b $3m^2 = 7 - 2m$

c $4x^2 = 3x + 2$

d $1 - 4k - k^2 = 0$

e $3m^2 - 1 - 3m = 0$

f $1 - 2g - 5g^2 = 0$

g $8 = 9h - 2h^2$

h $2w + 2 = 3w^2$

i $4p - 3p^2 = -1$

j $2 - 4u - 5u^2 = 0$

k $6a^2 = 9 - 4a$

l $10 = 3y + 2y^2$

3 Solve each quadratic equation, writing the solutions correct to two decimal places.

a $k^2 - 9k + 1 = 0$

b $c^2 - 2 = 0$

c $m^2 - 5 = 2$

d $2n^2 + 2 = 7n$

e $2p^2 + 3p - 4 = 0$

f $6w^2 + 5w - 2 = 0$

g $3x^2 + 2 - 8x = 0$

h $h^2 = 7 + 2h$

i $1 + x - x^2 = 0$

j $36 = 13a - a^2$

k $5v^2 - 11 = 0$

l $5c^2 + 8 = 15c$

m $t^2 = 5(t + 5)$

n $(x - 6)^2 = 3$

o $12 = 2d^2 - 3d$

Investigation: Proving the quadratic formula

The 'completing the square' method can be used to prove the quadratic formula. Copy and complete the following proof that solves $ax^2 + bx + c = 0$ by completing the square.

$$ax^2 + bx + c = 0$$

$$ax^2 + bx = \text{_____}$$

$$x^2 + \frac{b}{a}x = \text{_____}$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \text{_____} + \left(\frac{b}{2a}\right)^2$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} + \frac{b^2}{4a^2}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 + 4ac}{4a^2}$$

$$\therefore x + \frac{b}{2a} = \pm \sqrt{\frac{b^2 + 4ac}{4a^2}}$$

$$x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 + 4ac}}{2a}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 + 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Moving c to the RHS

Dividing both sides by a

Completing the square

Expressing the LHS as a perfect square

Simplifying the RHS

Taking the square root of both sides

Simplifying the RHS

Moving $\frac{b}{2a}$ to the RHS

Technology The quadratic formula

Stage 5.3

Set up your spreadsheet as shown below.

	A	B	C	D	E
1	a	b	c	x_1	x_2
2					
⋮					
⋮					
⋮					
10					

- a , b and c are the coefficients from the quadratic equation $ax^2 + bx + c = 0$.
 - x_1 , x_2 are the possible solutions.
- Enter the formula $=(-B2+SQRT(B2^2-4*A2*C2))/2*A2$ in cell D2 and enter the formula $=(-B2-SQRT(B2^2-4*A2*C2))/2*A2$ in cell E2.
 - Copy the formulas down to row 10.
 - Use your spreadsheet to solve each of the following.

a $x^2 + 9x - 20 = 0$	b $2x^2 + 6x + 1 = 0$	c $3x^2 + 2x - 5 = 0$
d $x^2 + 2x - 15 = 0$	e $8x^2 - 10x + 3 = 0$	f $x^2 + 2x - 3 = 0$
g $4x^2 - 20x + 4 = 0$	h $x^2 - 6x - 8 = 0$	i $10x^2 - 3x - 1 = 0$
 - Use your spreadsheet to solve $x^2 - x + 3 = 0$. Explain your result.



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Stage 5.3

Investigation: How many solutions?

1 A quadratic equation can have 0, 1 or 2 solutions. When does a quadratic equation have:

a no solutions?

b only one solution?

2 In the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, the expression under the square root symbol, $b^2 - 4ac$, has a special name, the discriminant. The table below shows twelve quadratic equations and the number of solutions for each one. Copy and complete the table by calculating the value of $b^2 - 4ac$ for each equation.

	Equation	Number of solutions	$b^2 - 4ac$
a	$x^2 - 4x + 4 = 0$	1	
b	$x^2 - 3x + 6 = 0$	0	
c	$x^2 + 4x + 1 = 0$	2	
d	$9x^2 - 60x + 100 = 0$	1	
e	$x^2 + 6x + 10 = 0$	0	
f	$5x^2 - 2x + 3 = 0$	0	
g	$49x^2 - 28x + 4 = 0$	1	
h	$x^2 + 5x + 6 = 0$	2	
i	$x^2 - 4x + 3 = 0$	2	
j	$2x^2 + 4x + 5 = 0$	0	
k	$3x^2 + 7x + 4 = 0$	2	
l	$x^2 - 16x + 64 = 0$	1	

3 What do you notice about the value of the discriminant, $b^2 - 4ac$, when the quadratic equation has:

a no solutions?

b one solution?

c two solutions?

4 Explain how the discriminant, $b^2 - 4ac$, in the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, determines whether a quadratic equation has two solutions, one solution or no solutions.

5 Without solving the equations, determine the number of solutions for each quadratic equation.

a $3x^2 - 2x + 5 = 0$

b $2x^2 - 6x + 1 = 0$

c $4x^2 - 12x + 9 = 0$

d $25x^2 + 10x + 1 = 0$

e $x^2 - 6x + 5 = 0$

f $x^2 + x + 4 = 0$

g $x^2 = 3 - 8x$

h $20x = 2 - 5x^2$

i $x(x + 6) = 12$

j $y = (x - 2)(2x - 5)$

k $y = -(x + 1)(3x + 2)$

l $y = x^2 + 6x + 8$

m $y = -x^2 + 4x + 5$

n $y = 3x^2 + 11x - 20$

o $y = -2x^2 + 5x - 2$

11-04 Higher-order quadratic equations

Stage 5.3

NSW

Higher-order equations are equations where the highest power of the variable is greater than 2. Equations such as $x^4 - 6x^2 + 8 = 0$, $2y^6 + 15y^3 - 8 = 0$ are examples of higher-order 'quadratic' equations. Substitution can be used so that the equation is 'simplified' or reduced to a quadratic equation.

Example 9

Solve each quadratic equation.

a $x^4 - 11x^2 + 18 = 0$

b $y^4 + 3y^2 - 4 = 0$

Solution

a This is a quadratic equation in x^2 .

Let $u = x^2$.

$\therefore x^4 - 11x^2 + 18 = 0$ becomes

$$u^2 - 11u + 18 = 0$$

$$(u - 9)(u - 2) = 0$$

$$u - 9 = 0 \quad \text{or} \quad u - 2 = 0$$

$$u = 9 \quad \text{or} \quad u = 2$$

But $u = x^2$

$$\therefore x^2 = 9 \quad \text{or} \quad x^2 = 2$$

$$x = \pm 3 \quad \text{or} \quad x = \pm\sqrt{2}$$

The solutions of the equation are:

$$x = \pm\sqrt{2} \quad \text{or} \quad x = \pm 3$$

That is $x = \sqrt{2}, -\sqrt{2}, 3$ or -3 .

This quadratic equation has 4 solutions

b $y^4 + 3y^2 - 4 = 0$ is a quadratic equation in y^2 .

Let $u = y^2$

$\therefore y^4 + 3y^2 - 4 = 0$ becomes

$$u^2 + 3u - 4 = 0$$

$$(u + 4)(u - 1) = 0$$

$$\therefore u + 4 = 0 \quad \text{or} \quad u - 1 = 0$$

$$u = -4 \quad \text{or} \quad u = 1$$

But $u = y^2$

$$y^2 = -4 \quad \text{or} \quad y^2 = 1$$

which has no solution or $y = \pm 1$

$\therefore$ The solutions of the equation are $y = \pm 1$.

This quadratic equation has 2 solutions

Stage 5.3

Exercise 11-04 Higher-order quadratic equations

See Example 9

- 1 Use the substitution given in the brackets to solve each quadratic equation.
 - a $y^4 - 12y^2 + 32 = 0$ [let $u = y^2$]
 - b $m^4 - 13m^2 - 48 = 0$ [let $u = m^2$]
 - c $w^4 - 8w^2 + 12 = 0$ [let $u = w^2$]
 - d $m^6 - 9m^3 + 8 = 0$ [let $u = m^3$]
 - e $k^6 - 26k^3 - 27 = 0$ [let $u = k^3$]
 - f $w^6 - 4w^3 - 32 = 0$ [let $u = w^3$]
 - g $(3x + 5)^2 - 8(3x + 5) + 12 = 0$ [let $u = 3x + 5$]
 - h $(5y - 1)^2 - (5y - 1) - 30 = 0$ [let $u = 5y - 1$]
 - i $(2a + 7)^2 - 10(2a + 7) + 24 = 0$ [let $u = 2a + 7$]
 - j $p^6 + 6p^3 + 5 = 0$ [let $u = p^3$]
 - k $8g^6 - 65g^3 + 8 = 0$ [let $u = g^3$]
 - l $125c^6 + 999c^3 - 8 = 0$ [let $u = c^3$]
- 2 Solve each equation, and give your solutions correct to one decimal place.
 - a $2m^4 - 19m^2 + 24 = 0$ [let $u = m^2$]
 - b $5x^4 - 14x^2 + 8 = 0$ [let $u = x^2$]
 - c $7a^6 - 9a^3 - 10 = 0$ [let $u = a^3$]
 - d $2y^6 + y^3 - 10 = 0$ [let $u = y^3$]
 - e $3w^6 - 16w^3 - 35 = 0$ [let $u = w^3$]
 - f $12e^4 + 5e^2 - 2 = 0$ [let $u = e^2$]
- 3 Solve each equation, correct to one decimal place where necessary.

a $w^4 - 21w^2 - 100 = 0$	b $x^6 + 6x^3 - 16 = 0$	c $k^6 - 64 = 0$
d $m^4 - 4m^2 + 4 = 0$	e $2y^4 + 7y^2 - 15 = 0$	f $v^6 + 5v^3 - 24 = 0$

Mental skills 11 Maths without calculators

Dividing a quantity in a given ratio

- 1 Study this example.
 Divide \$5600 between Alice and Peter in the ratio 5 : 3.
 Total number of parts = $5 + 3 = 8$.
 1 part = $\$5600 \div 8 = \$700$
 Alice's share = $5 \times \$700 = \$3500$
 Peter's share = $3 \times \$700 = \$2100$
 Check: $\$3500 + \$2100 = \$5600$ (original amount)

- 2 Now divide each of these quantities in the given ratio.
- a Divide \$150 between Mark and Jenni in the ratio 2 : 1.
 - b Divide \$2100 between Simon and Sunil in the ratio 4 : 3.
 - c Divide \$720 between Lisa and Bree in the ratio 2 : 7.
 - d Divide \$2000 between William and Adriana in the ratio 1 : 3.
 - e Divide \$4500 between Ed and Pete in the ratio 3 : 2.
 - f Divide \$3000 between Sharanya and Asam in the ratio 3 : 7.
 - g Divide \$3600 between Cindy and Carmen in the ratio 5 : 1.
 - h Divide \$1600 between Nancy and John in the ratio 3 : 5.
 - i Divide \$990 between Carol and Louis in the ratio 5 : 4.
 - j Divide \$4000 between Yvette and Andre in the ratio 1 : 4.
 - k Divide \$4900 between Arden and Ivan in the ratio 3 : 4.
 - l Divide \$3200 between Tan and Mai in the ratio 5 : 3.

Stage 5.3

11-05 Quadratic equation problems

Worksheet

Problems involving
quadratic equations

MAT10NAWK00029

Example 10

A rectangular garden is 5 m longer than it is wide. The area of the rectangle is 84 m^2 . Find the dimensions of the garden.

Solution

Let the width of the rectangular garden be $x \text{ m}$.

$\therefore$ The length of the garden is $(x + 5) \text{ m}$.

Area = length $\times$ width

$$= x(x + 5) \text{ m}^2$$

$$\therefore x(x + 5) = 84$$

$$x^2 + 5x = 84$$

$$x^2 + 5x - 84 = 0$$

$$(x + 12)(x - 7) = 0$$

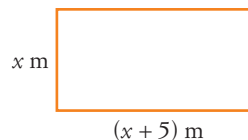
$$x + 12 = 0 \quad \text{or} \quad x - 7 = 0$$

$$\therefore x = -12 \quad \text{or} \quad x = 7$$

Since x represents a measurement of width, $x \neq -12$.

$\therefore$ The width is 7 m and the length is $7 + 5 = 12 \text{ m}$

Check: Area = $7 \text{ m} \times 12 \text{ m} = 84 \text{ m}^2$



Stage 5.3

Example 11

A ball is thrown upwards and its height, h metres, after t seconds is given by the formula $h = 30t - 5t^2$. At what times did the ball reach a height of 24 m? Express the answer correct to two decimal places.

Solution

The equation is $h = 30t - 5t^2$

When $h = 24$,

$$24 = 30t - 5t^2$$

Rearranging the equation,

$$5t^2 - 30t + 24 = 0$$

Using the quadratic formula,

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-30) \pm \sqrt{(-30)^2 - 4 \times 5 \times 24}}{2 \times 5}$$

$$= \frac{30 \pm \sqrt{420}}{10}$$

$$t = \frac{30 + \sqrt{420}}{10} \quad \text{or} \quad t = \frac{30 - \sqrt{420}}{10}$$

$$t = 5.049 \dots \quad \text{or} \quad t = 0.950 \dots$$

$$t \approx 5.05 \quad \text{or} \quad t \approx 0.95$$

The ball reaches a height of 24 m after 0.95 s and after 5.05 s.



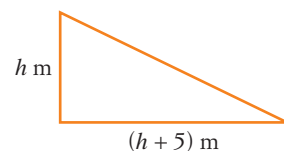
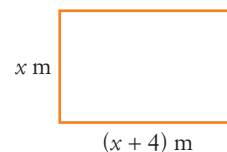
Science Photo Library/Dr Gary Settles

Why are there two answers to this problem? Why is the ball at a height of 24 m at two different times?

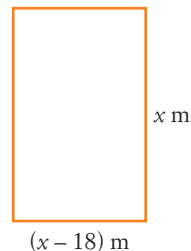
Exercise 11-05 Quadratic equation problems

See Example 10

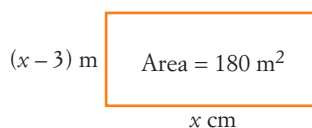
- 1 A garden is in the shape of a rectangle and its length is 4 m longer than its width. If the area of the garden is 96 m^2 , find the dimensions of the garden.
- 2 A park is in the shape of a right-angled triangle, with its base 5 m longer than its perpendicular height. If the area of the park is 700 m^2 , find the dimensions of the park.



- 3 A rectangular block of land has its width 18 m shorter than its length. If the area of the block is 1008 m^2 , find the dimensions of the block of land.



- 4 A block of land is in the shape of a rectangle with its width 20 m shorter than its length. If the area of the block is 2204 m^2 , find the length and width of the block of land.
- 5 A rectangular garden bed is twice as long as it is wide. Its area is 84.5 m^2 . Find the length of the garden bed.
- 6 Find the dimensions of this rectangle.



- 7 A golf ball is thrown upwards and its height, h metres, after t seconds is given by the formula $h = 18t - 5t^2$. At what times did the ball reach a height of 8 m? Answer correct to two decimal places.
- 8 After jumping from a plane, the height (in metres) of a skydiver above the ground is given by $h = 4000 - 5t^2$, where t is the time (in seconds) after jumping.
- How high was the plane at the moment the skydiver jumped?
 - What was the skydiver's height after 20 seconds?
 - The skydiver opened his parachute at 1000 m. How long did it take the skydiver to reach this height? (Give your answer correct to one decimal place.)
- 9 A ball is thrown from a balcony and its height (in metres) after t seconds is given by the formula $h = 30 + 12t - 5t^2$.
- What is the height of the ball after 2 seconds?
 - When does the ball hit the ground? Answer correct to one decimal place.
 - At what time (correct to one decimal place) is the ball at a height of:
 - 35 m?
 - 10 m?
- 10 The sum of a number and its square is 72. What is the number?
- 11 The product of two consecutive integers is 600. Find the integers.
- 12 When a number is subtracted from its square, the result is 1190. What is the number?



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Stage 5.3

See Example 11

Stage 5.3

Worksheet

Graphing parabolas 2

MAT10NAWK10100

Technology worksheet

Investigating
parabolas 1

MAT10NACT00010

Technology worksheet

Excel spreadsheet:
Investigating
parabolas 1

MAT10NACT00040

Technology worksheet

GeoGebra: Parabola

MAT10NATC00007

11-06 The parabola $y = ax^2 + bx + c$

The graph of a quadratic equation is a smooth U-shaped curve called a **parabola**. We have already graphed simple parabolas of the form $y = ax^2 + c$ in Chapter 8, Graphs. Now we will graph parabolas of the form $y = ax^2 + bx + c$.

Example 12

For each quadratic equation:

- complete a table of values and draw the graph
- find the x -intercepts
- find the y -intercept
- solve $y = 0$ and compare the solutions to the x -intercepts.

a $y = x^2 + 6x + 5$

b $y = 2x^2 - x - 10$

Solution

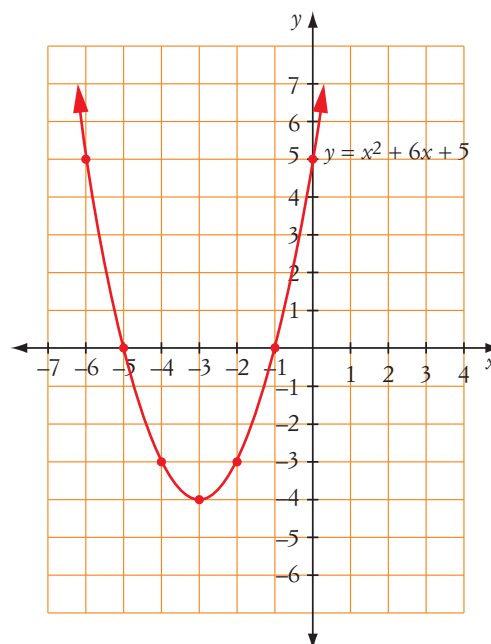
- a
- $y = x^2 + 6x + 5$
 - The x -intercepts are -5 and -1 .
 - The y -intercept is 5 .

Note that this is the constant term, $c = 5$, in $y = x^2 + 6x + 5$.

iv $x^2 + 6x + 5 = 0$
 $(x + 5)(x + 1) = 0$
 $x + 5 = 0$ or $x + 1 = 0$
 $x = -5$ or $x = -1$

The solutions to the quadratic equation are the same as the x -intercepts of the graph of $y = x^2 + 6x + 5$.

x	-6	-5	-4	-3	-2	-1	0	1
y	5	0	-3	-4	-3	0	5	12



- b
- $y = 2x^2 - x - 10$
 - The x -intercepts are -2 and $2\frac{1}{2}$.
 - The y -intercept is -10 .

x	-3	-2	-1	0	1	2	3
y	11	0	-7	-10	-7	0	5

Note that this is the constant term, $c = -10$, in $y = 2x^2 - x - 10$

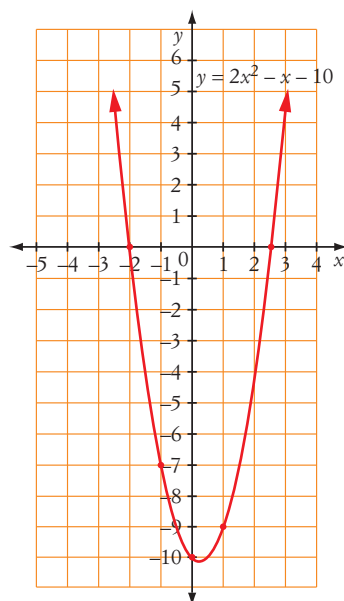
iv $2x^2 - x - 10 = 0$

$$(2x - 5)(x + 2) = 0$$

$$2x - 5 = 0 \quad \text{or} \quad x + 2 = 0$$

$$x = 2\frac{1}{2} \quad \text{or} \quad x = -2$$

The solutions to the quadratic equation are the same as the x -intercepts of the graph of $y = 2x^2 - x - 10$.



Stage 5.3

Summary

For the graph of the parabola $y = ax^2 + bx + c$:

- if $a > 0$ (positive), the parabola is **concave up**
- if $a < 0$ (negative), the parabola is **concave down**
- the **y-intercept** of the parabola is c
- the **x-intercepts** of the parabola are the solutions to the quadratic equation $ax^2 + bx + c = 0$

Example 13

Graph each quadratic equation, showing its x - and y -intercepts.

a $y = 2x(x - 5)$

b $y = 2x^2 + x - 6$

c $y = -2x^2 - 3x + 9$

Solution

a $y = x(2x - 5)$

$$= 2x^2 - 5x$$

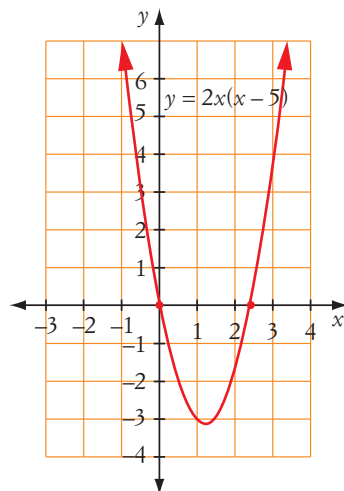
$a = 2 > 0$, so the parabola is concave up.

x -intercepts: $x(2x - 5) = 0$

$$x = 0 \quad \text{and} \quad 2x - 5 = 0$$

$$x = 0 \quad \text{and} \quad x = 2\frac{1}{2}$$

y -intercept: $c = 0$



Stage 5.3

b $y = 2x^2 + x - 6$
 $a = 2 > 0$, so the parabola is concave up.

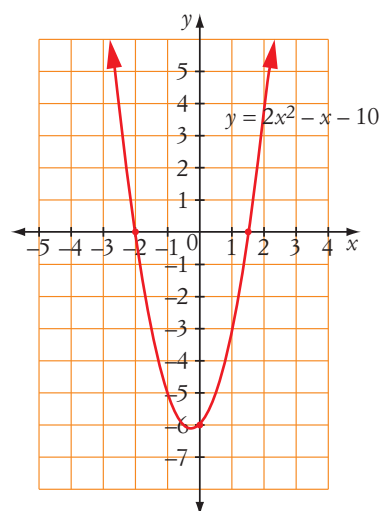
x -intercepts: $2x^2 + x - 6 = 0$

$(2x - 3)(x + 2) = 0$

$2x - 3 = 0$ and $x + 2 = 0$

$x = 1\frac{1}{2}$ and $x = -2$

y -intercept: $c = -6$



c $y = -2x^2 - 3x + 9$
 $a = -2 < 0$, so the parabola is concave down.

x -intercepts: $-2x^2 - 3x + 9 = 0$

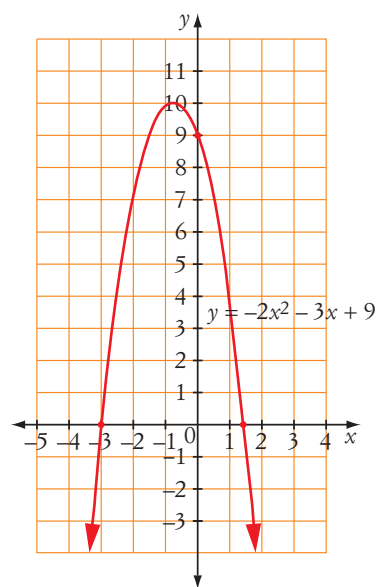
$2x^2 + 3x - 9 = 0$ Dividing by -1

$(2x - 3)(x + 3) = 0$

$2x - 3 = 0$ and $x + 3 = 0$

$x = 1\frac{1}{2}$ and $x = -3$

y -intercept: $c = 9$



Exercise 11-06 The parabola $y = ax^2 + bx + c$

See Example 12

- 1 For each quadratic equation:
 - i complete a table of values and draw the graph
 - ii find the x -intercepts
 - iii find the y -intercept
 - iv solve $y = 0$ and compare the solutions to the x -intercepts.

a $y = x^2 + 4x + 3$

b $y = 2x^2 + 7x + 3$

c $y = 2x^2 - 3x - 9$

d $y = x^2 - 2x$

2 Find the y -intercept of the parabolas with equation:

a $y = 3x^2 - 2x - 5$

b $y = 2x^2 + 6x + 3$

c $y = 5x^2 - 10x$

3 Graph each quadratic equation, showing its x - and y -intercepts.

a $y = x(x - 4)$

b $y = 3x(x + 2)$

c $y = (x + 3)(x - 5)$

d $y = (x - 2)(2x - 5)$

e $y = -(x + 1)(3x + 2)$

f $y = x^2 + 6x + 8$

g $y = -x^2 + 4x + 5$

h $y = 3x^2 + 11x - 20$

i $y = -2x^2 + 5x - 2$

Stage 5.3

See Example 13

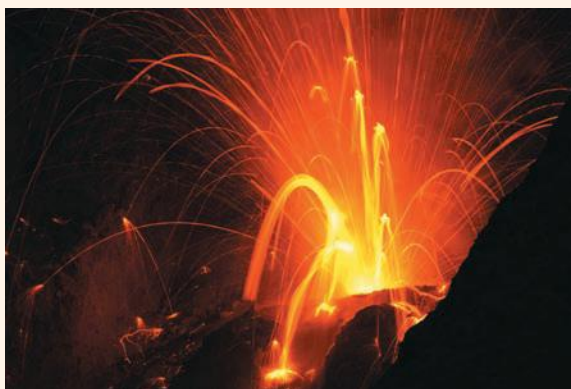
Just for the record

Parabolas in action

When an object is thrown or when a bullet is fired from a gun, its path is a parabola. The length of the path will depend on the initial speed. The photo shows parabolas of lava being expelled from a volcano.

Parabola-shaped mirrors or reflectors are used in car headlights, spotlights and satellite dishes.

Investigate the property of parabolas that makes them useful as parabolic reflectors.

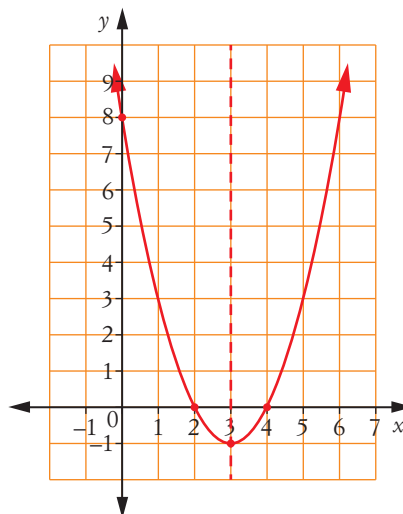


Shutterstock.com/Photovoltanica.com

Investigation: The axis of symmetry and vertex of a parabola

1 The graph of $y = x^2 - 6x + 8$ is shown.

- a Write the equation of the axis of symmetry of the parabola.
- b What are the x -intercepts of the parabola?
- c What is the midpoint of the interval joining the x -intercepts?
- d How can the x -intercepts be used to find the equation of the axis of symmetry?
- e Copy and complete: The equation of the axis of symmetry can be determined by finding the _____ of the x -intercepts.



Stage 5.3

- 2 According to the quadratic formula, the solutions to $ax^2 + bx + c = 0$ are given by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. This means that the x -intercepts of the parabola $y = ax^2 + bx + c$ are $x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$. Hence find the equation of the axis of symmetry of $y = ax^2 + bx + c$.
- 3 a For $y = x^2 - 6x + 8$, what are the values of a (the coefficient of x^2) and b (the coefficient of x)?
- b Find the value of $-\frac{b}{2a}$.
- c How is the value of $-\frac{b}{2a}$ related to the axis of symmetry for the parabola $y = x^2 - 6x + 8$?

Worksheet
Features of a parabola
MAT10NAWK10102
Worksheet
A page of parabolas
MAT10NAWK10103
Worksheet
Parabolas
MAT10NAWK00019

11-07 The axis of symmetry and vertex of a parabola

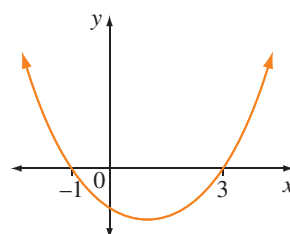
Summary

For the **graph of the parabola** $y = ax^2 + bx + c$:

- the equation of its **axis of symmetry** is $x = -\frac{b}{2a}$
- the **axis of symmetry** passes through the point halfway between the two x -intercepts of the parabola
- the **vertex** of the parabola lies on the axis of symmetry, so its **x -coordinate** is $x = -\frac{b}{2a}$ and its **y -coordinate** can be found by substituting the x -coordinate into the equation of the parabola

Example 14

Find the equation of the axis of symmetry for the parabola shown.



Solution

The x -intercepts are -1 and 3 .

The equation of the axis of symmetry is:

$$\begin{aligned} x &= \frac{-1 + 3}{2} \\ &= 1 \end{aligned}$$

Finding the average of the x -intercepts.

Example 15

For the parabola with equation $y = 2x^2 - 4x + 3$, find:

- a the equation of its axis of symmetry
- b the coordinates of the vertex of the parabola.

Solution

- a For $y = 2x^2 - 4x + 3$, $a = 2$, $b = -4$ and $c = 3$.

The axis of symmetry is:

$$\begin{aligned} x &= -\frac{b}{2a} \\ &= -\frac{-4}{2 \times 2} \\ &= 1 \end{aligned}$$

$x = 1$ is the equation of the axis of symmetry.

- b The vertex lies on the axis of symmetry.

Substitute $x = 1$ in $y = 2x^2 - 4x + 3$

$$\begin{aligned} y &= 2 \times 1^2 - 4 \times 1 + 3 \\ &= 1 \end{aligned}$$

The vertex is $(1, 1)$.

Finding the vertex by completing the square

Example 16

Complete the square to find the coordinates of the vertex for the parabola with the equation $y = x^2 + 8x + 5$.

Solution

$$\begin{aligned} y &= x^2 + 8x + 5 \\ &= (x^2 + 8x + 4^2) - 4^2 + 5 \\ &= (x + 4)^2 - 11 \end{aligned}$$

Completing the square on $x^2 + 8x$

This is the equation of a parabola of the form $y = a(x - r)^2 + c$, where the vertex is at (r, c) .

$\therefore$ The vertex is at $(-4, -11)$

$r = -4, c = -11$.

Stage 5.3

Example 17

Sketch the parabola $y = 2x^2 + 5x + 1$.

Solution

$a = 2 > 0$, so the parabola is concave up.

y -intercept $= 1$.

$y = 2x^2 + 5x + 1$ cannot be factorised, so we cannot find the x -intercepts precisely.

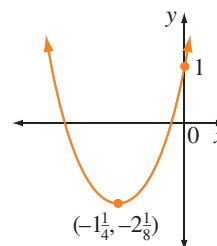
For the vertex:

$$\begin{aligned} x &= -\frac{b}{2a} \quad \text{where } a = 2, \text{ and } b = 5 \\ &= -\frac{5}{2 \times 2} \\ &= -1\frac{1}{4} \end{aligned}$$

Substitute $x = -1\frac{1}{4}$ in $y = 2x^2 + 5x + 1$

$$\begin{aligned} y &= 2 \times \left(-1\frac{1}{4}\right)^2 + 5 \times \left(-1\frac{1}{4}\right) + 1 \\ &= -2\frac{1}{8} \end{aligned}$$

$\therefore$ The vertex has coordinates $\left(-1\frac{1}{4}, -2\frac{1}{8}\right)$.



Summary

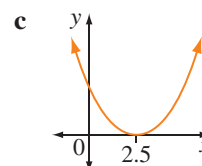
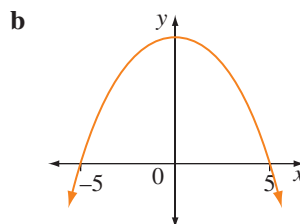
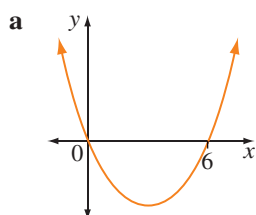
To graph $y = ax^2 + bx + c$:

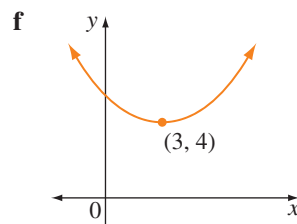
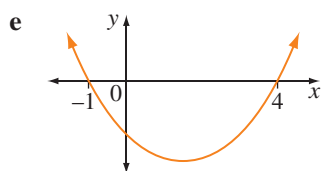
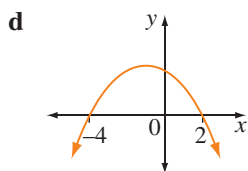
- use the sign of a (the coefficient of x^2) to determine whether the parabola is **concave up** or **down**
- find the **y -intercept**
- find the **x -intercepts** where possible by solving a quadratic equation
- use $x = -\frac{b}{2a}$ to find the **axis of symmetry** and the **vertex**

Exercise 11-07 The axis of symmetry and vertex of a parabola

See Example 14

- 1 Write the equation for the axis of symmetry in each parabola.





2 For each parabola whose equation is given below:

- i find the equation for its axis of symmetry ii find the coordinates of its vertex

a $y = x^2 - 6x + 8$

b $y = -x^2 + 10x - 9$

c $y = x^2 - 2x + 10$

d $y = -x^2 + 8x + 9$

e $y = -x^2 + x - 25$

f $y = 5x^2 + 40x$

g $y = 24x - 3x^2$

h $y = 4x^2 + 2x - 1$

i $y = 1 - 3x - 9x^2$

3 Use the method of completing the square to find the coordinates of the vertex of the parabola for each of the following equations.

a $y = x^2 + 4x + 1$

b $y = x^2 + 5x - 4$

c $y = -x^2 + 2x + 5$

d $y = x^2 - 6x$

e $y = 2 - 2x + x^2$

f $y = 2x^2 - 5x + 2$

4 Graph the parabola with each given equation, showing:

- i the x -intercepts (correct to one decimal place where necessary)

- ii the y -intercept

- iii the equation of the axis of symmetry

- iv the coordinates of the vertex

- v the concavity

a $y = x^2 - 6x - 40$

b $y = x^2 - 3x$

c $y = 2x^2 + 3x + 4$

d $y = -x^2 + 6x + 5$

e $y = -4x^2 - 12x + 21$

f $y = x^2 - 8x + 3$

g $y = 5x^2 + 7x + 4$

h $y = 8x - 2x^2$

i $y = -2x^2 + 7x - 6$

See Example 15

See Example 16

See Example 17

Technology Graphing parabolas

In this activity, we will use GeoGebra or other graphing technology to graph parabolas and identify their axes of symmetry and vertices.

1 Graph $y = x^2 + 5x + 6$ by entering $y=x^2+5x+6$

Use  to obtain a better view of the graph.

a What is the axis of symmetry?

b What is the vertex?

2 Graph $y = 2x^2 - 9x - 35$ and find its axis of symmetry and vertex.

3 Enter each equation and identify its axis of symmetry and vertex.

a $y = x^2 - 14x + 49$

b $y = 2x^2 - 11x - 21$

c $y = 8x^2 + 28x + 20$

d $y = 8 + 2x - x^2$

e $y = 20 - x - x^2$

f $y = 40 + 2x - 3x^2$

Stage 5.3

NSW

Puzzle sheet

Simultaneous
equations order
activity 2

MAT10NAPS10225

Technology worksheet

Excel spreadsheet:
Intersection of a
parabola and a straight
line

MAT10NACT00051

Technology worksheet

Excel worksheet:
Intersection of a
parabola and a straight
line

MAT10NACT00021

11-08 Point of intersection of a line and a curve

The points of intersection of a line with a parabola, circle or hyperbola may be found either graphically or algebraically. The algebraic method involves solving simultaneous equations using the substitution method.

Example 18

Find the points of intersection of:

- the line $y = 2x - 3$ and the parabola $y = x^2 - 3x + 1$
- the line $y = x + 2$ and the circle $x^2 + y^2 = 4$
- the line $y = x + 5$ and the hyperbola $y = \frac{6}{x}$

Solution

$$\begin{array}{ll} \text{a} & y = 2x - 3 \quad [1] \\ & y = x^2 - 3x + 1 \quad [2] \end{array}$$

Use [2] to substitute for y in [1].

$$\begin{aligned} x^2 - 3x + 1 &= 2x - 3 \\ x^2 - 5x + 4 &= 0 \\ (x - 4)(x - 1) &= 0 \\ x - 4 = 0 &\text{ or } x - 1 = 0 \\ x = 4 &\text{ or } x = 1 \end{aligned}$$

Substitute $x = 4$ and $x = 1$ into [1] to find y .

$$\begin{aligned} y &= 2 \times 4 - 3 & \text{ and } & y = 2 \times 1 - 3 \\ &= 5 & & = -1 \end{aligned}$$

The solutions are $x = 4, y = 5$ and $x = 1, y = -1$

The points of intersection are $(4, 5)$ and $(1, -1)$.

$$\begin{array}{ll} \text{b} & y = x + 2 \quad [1] \\ & x^2 + y^2 = 4 \quad [2] \end{array}$$

Use [1] to substitute for y in [2].

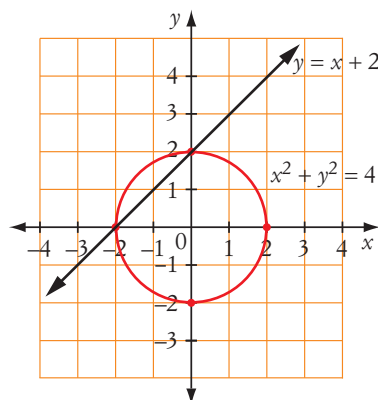
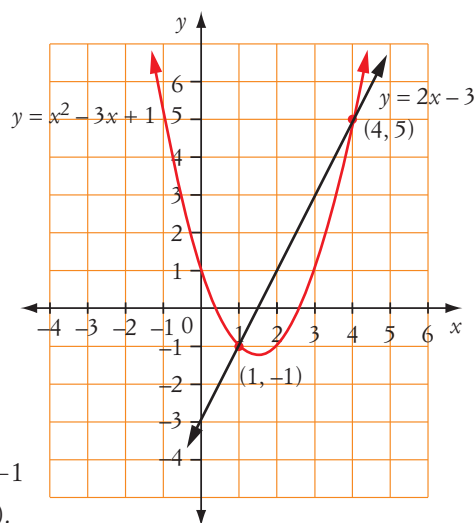
$$\begin{aligned} x^2 + (x + 2)^2 &= 4 \\ x^2 + x^2 + 4x + 4 &= 4 \\ 2x^2 + 4x &= 0 \\ 2x(x + 2) &= 0 \end{aligned}$$

$$\begin{aligned} 2x = 0 &\text{ or } x + 2 = 0 \\ x = 0 &\text{ or } x = -2 \end{aligned}$$

Substitute $x = 0$ and $x = -2$ into [1] to find y .

$$\begin{aligned} y &= 0 + 2 & \text{ and } & y = -2 + 2 \\ &= 2 & & = 0 \end{aligned}$$

The points of intersection are $(0, 2)$ and $(-2, 0)$.



c $y = x + 5$ [1]

$y = \frac{6}{x}$ [2]

Use [1] to substitute for y in [2].

$$x + 5 = \frac{6}{x}$$

$$x(x + 5) = x \times \frac{6}{x}$$

$$x^2 + 5x = 6$$

$$x^2 + 5x - 6 = 0$$

$$(x + 6)(x - 1) = 0$$

$$x + 6 = 0 \quad \text{or} \quad x - 1 = 0$$

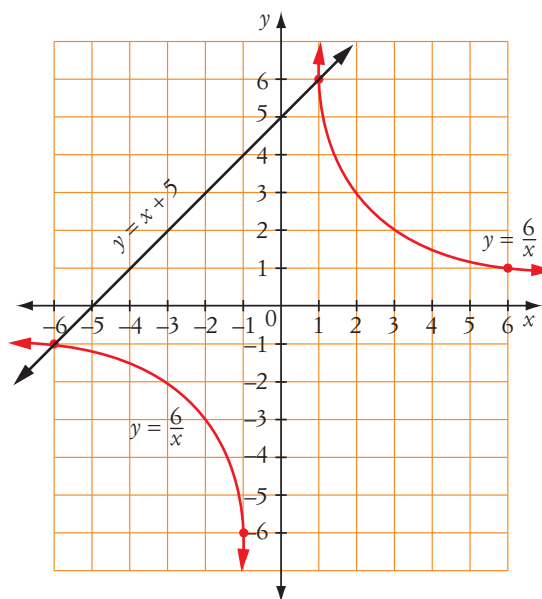
$$x = -6 \quad \text{or} \quad x = 1$$

Substitute $x = -6$ and $x = 1$ into [1] to find y .

$$y = -6 + 5 \quad \text{and} \quad y = 1 + 5$$

$$= -1 \quad \quad \quad = 6$$

The points of intersection are $(-6, -1)$ and $(1, 6)$.



Stage 5.3

Exercise 11-08 Point of intersection of a line and a curve

1 Find the points of intersection between the line and the parabola with equations:

a $y = x^2$ and $y = 5x - 6$

b $y = 2x^2$ and $y = 8x$

c $y = x^2 + 10$ and $y = 2x + 18$

d $y = 5x^2$ and $y = x + 6$

e $y = 3x^2 + 2x + 10$ and $y = 12 - 3x$

f $y = 9x - x^2$ and $y = 6x - 10$

g $y = 6x - 3$ and $y = x^2 - 8x + 46$

h $x + y = 2$ and $y = x^2 - 2x$

2 Find the points of intersection of:

a $y = x - 3$ and the hyperbola $y = -\frac{2}{x}$

b $y = 1 - x$ and the circle $x^2 + y^2 = 1$

c $x + y = 3$ and the circle $x^2 + y^2 = 9$

d $y = x + 5$ and the circle $x^2 + y^2 = 25$

e $y = 5x + 2$ and the hyperbola $y = \frac{7}{x}$

f $2x - y = 1$ and the hyperbola $y = \frac{15}{x}$

g $y = 2x + 5$ and the hyperbola $y = -\frac{3}{x}$

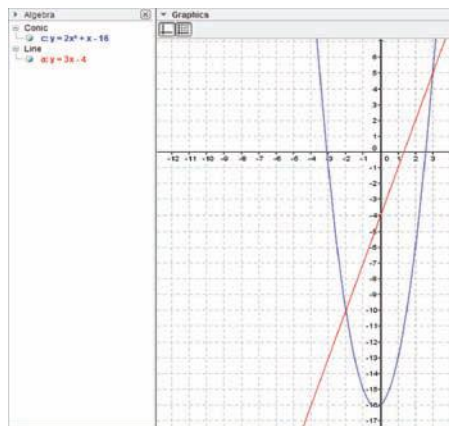
See Example 18

Stage 5.3

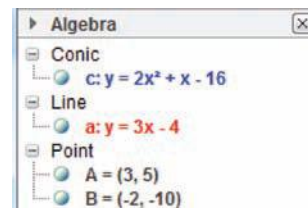
Technology Points of intersection

In this activity, we will use GeoGebra to find the point of intersection between a line and a curve.

- 1 Select **Axes** and **Grid**.
- 2 Select **View**, **Algebra**. Press **Ctrl** and hold your mouse down anywhere on the grid (except on the axes) to move the axes. Do this until they are centred in the middle of the screen.
- 3 Enter the following equations for a line and a parabola in the Input bar:
 $y = 3x - 4$ and $y = 2x^2 + x - 16$



- 4 Select **New Point**  and right-click on the intersection points between the line and curve.
- 5 The points of intersection will be displayed in the **Algebra View**.



- 6 Check your answers to **Exercise 14-08** by graphing the pairs of equations to find the points of intersection.

Power plus

- 1 a Show that $(3x^2 - 7x)^2 + 5(3x^2 - 7x) + 4 = 9x^4 - 42x^3 + 64x^2 - 35x + 4$.
 b Solve the equation $9x^4 - 42x^3 + 64x^2 - 35x + 4 = 0$.
- 2 Without finding the coordinates of the point of intersection (if it exists) determine whether the line $y = 8x + 5$ intersects twice, is a tangent to, or completely misses the parabola $y = 2x^2 + 7x - 6$.
- 3 A gift box is made from a rectangular piece of cardboard that is three times as long as it is wide. 5 cm squares are cut from each corner and the ends are then folded up to make the box. If the box's volume is 4340 cm^3 , find the length and width of the cardboard.

Chapter 11 review

Language of maths

axis	axis of symmetry	check	circle
coefficient	complete the square	concave down	concave up
exact	factorise	hyperbola	left-hand side (LHS)
parabola	quadratic equation	quadratic formula	right-hand side (RHS)
simultaneous equations	solution	square root	substitution
surd	vertex	x-intercept	y-intercept

Puzzle sheet

Quadratic equations
crossword (Advanced)

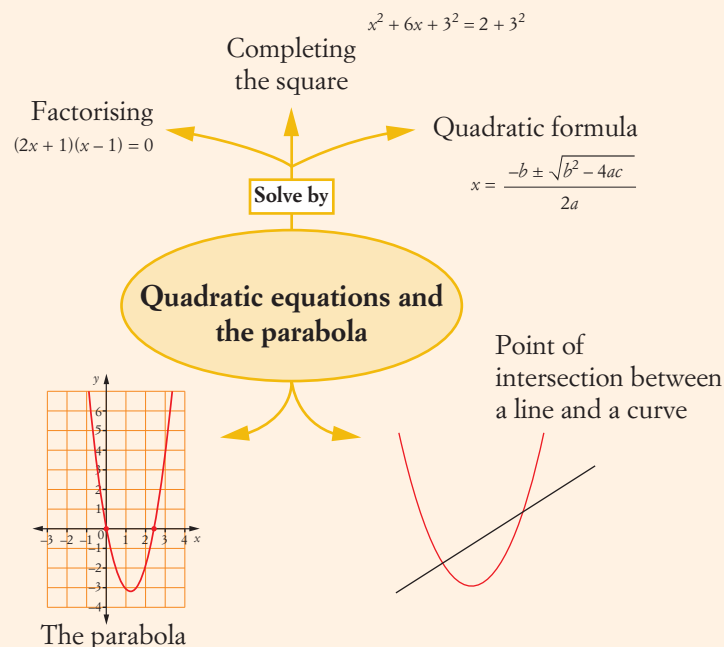
MAT10NAPS10226

- 1 What is a **quadratic equation**?
- 2 What is the **coefficient** of x in $ax^2 + bx + c = 0$?
- 3 What are the three methods for solving a quadratic equation?
- 4 How do you write the solution to a quadratic equation in **exact form**?
- 5 How many solutions can a quadratic equation have?
- 6 What are higher-order order quadratic equations?

Topic overview

- Write in your own words what you have learnt in this chapter.
- Write down the parts of this chapter that were new to you.
- What parts of this chapter did you find difficult?

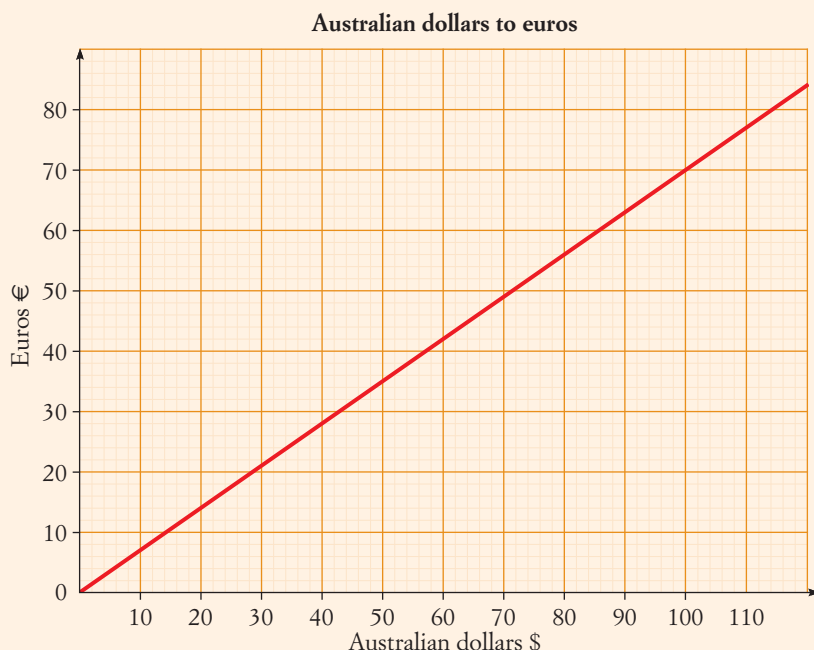
Copy and complete this mind map of the topic, adding detail to its branches and using pictures, symbols and colour where needed. Ask your teacher to check your work.



Chapter 11 revision

- See Exercise 11-01 **1** Solve each quadratic equation.
- a** $(m - 2)(2m + 3) = 0$ **b** $3x(x - 5) = 0$ **c** $p^2 - 2p - 48 = 0$
d $y^2 - 8y + 12 = 0$ **e** $3w^2 + 19w + 20 = 0$ **f** $8k^2 = 2k + 3$
g $n(2n - 4) = 30$ **h** $(2d + 3)^2 = 36$ **i** $4x^2 - 15 = 4x$
- See Exercise 11-02 **2** Solve each quadratic equation by completing the square. Leave your answers in exact form.
- a** $y^2 + 4y - 7 = 0$ **b** $2p^2 - 3p - 3 = 0$ **c** $5w^2 + 2w - 1 = 0$
- See Exercise 11-03 **3** Solve each quadratic equation using the quadratic formula. Write each solution in exact form.
- a** $m^2 + 5m - 3 = 0$ **b** $2d^2 - 7d - 11 = 0$ **c** $3k^2 - k = 21$
- See Exercise 11-03 **4** Solve each quadratic equation, writing the solution correct to two decimal places.
- a** $x^2 + 3x - 1 = 0$ **b** $4h^2 - 3h - 5 = 0$ **c** $w^2 = 5w + 2$
- See Exercise 11-04 **5** Solve each quadratic equation.
- a** $m^4 - 6m^2 + 8 = 0$ **b** $2y^4 - 5y^2 - 12 = 0$ **c** $x^6 - 7x^3 - 8 = 0$
- See Exercise 11-05 **6** Write a quadratic equation for each problem, then solve the problem.
- a** The product of two consecutive even numbers is 1088. Find the numbers.
b A rectangular garden bed is 8 m longer than it is wide. If the area of the garden bed is 308 m^2 , find the dimensions of the garden bed.
- See Exercise 11-06 **7** Graph $y = 2x^2 + 5x - 7$, showing its x - and y -intercepts.
- See Exercise 11-07 **8** Graph each quadratic equation, showing the x -intercepts (correct to one decimal place where necessary), the y -intercept, the equation of the axis of symmetry and the coordinates of the vertex.
- a** $y = x^2 - 2x - 4$ **b** $y = -8x - 3x^2$ **c** $y = 5x^2 + 3x - 8$
- See Exercise 11-08 **9** Find the point of intersection of the line:
- a** $y = x + 3$ and the parabola $y = 2x^2$
b $y = 3 - 2x$ and the parabola $y = x^2 + 2x - 2$
c $y = 4 - x$ and the circle $x^2 + y^2 = 16$
d $y = x - 6$ and the hyperbola $y = -\frac{8}{x}$

- 1 Graph the simultaneous equations $y = 2x - 3$ and $x + y = 6$ on a number plane to solve them. See Exercise 10-01
- 2 This conversion graph shows the exchange rate between the Australian dollar and the euro. See Exercise 8-03

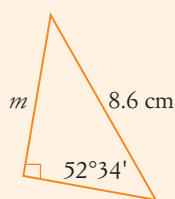


Use the graph to convert:

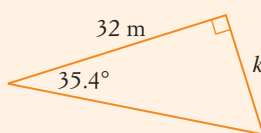
- a AU\$25 to euros
- b €60 to Australian dollars
- c AU\$140 to euros

- 3 Find the value of each pronumeral, correct to one decimal place. See Exercise 9-01

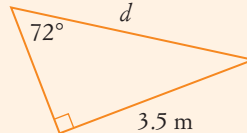
a



b



c



- 4 The time taken, t seconds, to complete a race varies inversely with the speed, S m/s, of the sprinter. See Exercise 8-02
 - a If it takes 9.80 seconds to complete a race at a speed of 10.2 m/s, find the variation equation for t .
 - b If the time taken is 10.5 seconds, what is the speed of the sprinter?
 - c If the sprinter's average speed for the race is 10.3 m/s, how long will it take to complete the race?
- 5 For the parabola with equation $y = -3x^2 + 6x - 1$, find: See Exercise 11-07
 - a the equation of the axis of symmetry
 - b the coordinates of the vertex

Mixed revision 3

See Exercise 10-02

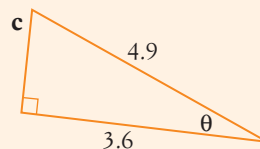
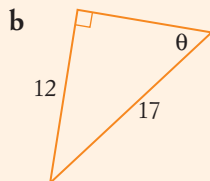
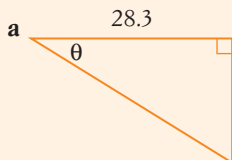
- 6 Solve this pair of simultaneous equations.

$$2y + 3f = 15$$

$$5y - 2f = 9$$

See Exercise 9-01

- 7 Find the value of θ , correct to the nearest degree.



Stage 5.3

See Exercise 11-03

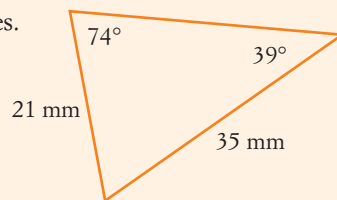
- 8 Solve $3x^2 - 4x - 1 = 0$, writing the solution in exact form.

See Exercise 8-08

- 9 Sketch the graph of $y = 2x^3 - 2$.

See Exercise 9-11

- 10 Find the area of this triangle, correct to two significant figures.



See Exercise 11-01

- 11 Solve each quadratic equation.

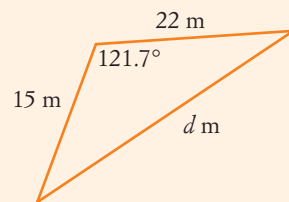
a $4m^2 + 5m + 1 = 0$

b $y(10y - 3) = 4$

c $2w^2 = 3w + 5$

See Exercise 9-09

- 12 Find the value of d , correct to one decimal place.



See Exercise 10-04

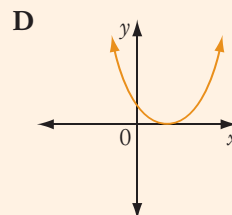
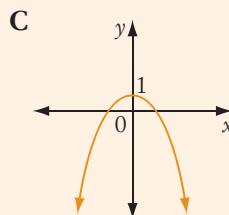
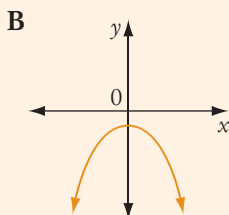
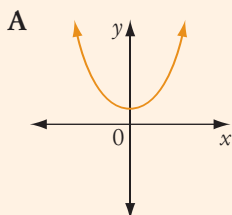
- 13 Tickets to the Dolphin swimming club's presentation day cost \$20 for adults and \$15 for children. Altogether, 395 people attended and ticket sales totalled \$6700. Let A stand for the number of adults and C for the number of children.

a Explain why $A + C = 395$ and $20A + 15C = 6700$.

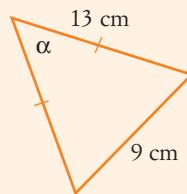
b Solve the equations simultaneously to find the number of children that attended presentation day.

See Exercise 8-06

- 14 Which of these could be a graph of $y = 1 - 2x^2$? Select the correct answer **A**, **B**, **C** or **D**.



- 15 Find α , correct to the nearest minute.



Stage 5.3

See Exercise 9-10

- 16 Solve this pair of simultaneous equations.

$$y = x + 3$$

$$y = 5x - 7$$

See Exercise 10-03

- 17 In $\triangle ABC$, $\angle A = 38^\circ$, $a = 7$ m and $c = 9$ cm. Find the possible values for the size of $\angle C$ (correct to the nearest degree).

Stage 5.3

See Exercise 9-08

- 18 Sketch the exponential curve of the equation $y = 4^{-x}$, showing the y -intercept.

See Exercise 8-11

- 19 Find the exact value of each expression.

a $\cos 45^\circ$

b $\sin 60^\circ$

c $\tan 30^\circ$

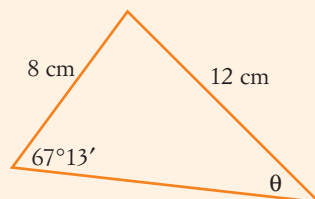
Stage 5.3

See Exercise 9-04

- 20 Substitute k for x^2 to solve $x^4 - 13x^2 + 36 = 0$.

See Exercise 11-04

- 21 Find the value of θ , correct to the nearest minute.



See Exercise 9-08

- 22 The length of a block of land is 5 metres less than 4 times its width. The area of the block is 1071 m^2 .

See Exercise 11-05

a Write a quadratic equation to show this information.

b Solve the equation to find the dimensions of the block of land.

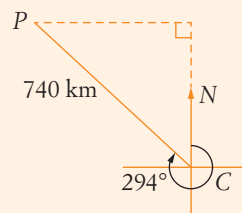
- 23 Find the point of intersection of the line $y = x - 3$ and the hyperbola $y = \frac{4}{x}$.

See Exercise 11-08

- 24 A plane flies on a bearing of 294° from Canberra for a distance of 740 km.

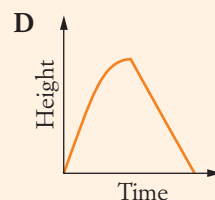
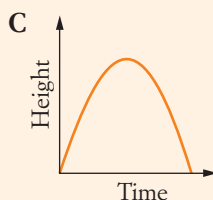
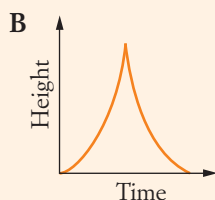
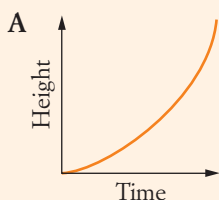
a How far north has the plane travelled (to the nearest kilometre)?

b What is the bearing of Canberra from the plane's position?



See Exercise 9-02

- 25 Select the graph that best describes the height of a ball thrown upwards.



Stage 5.3

See Exercise 8-05

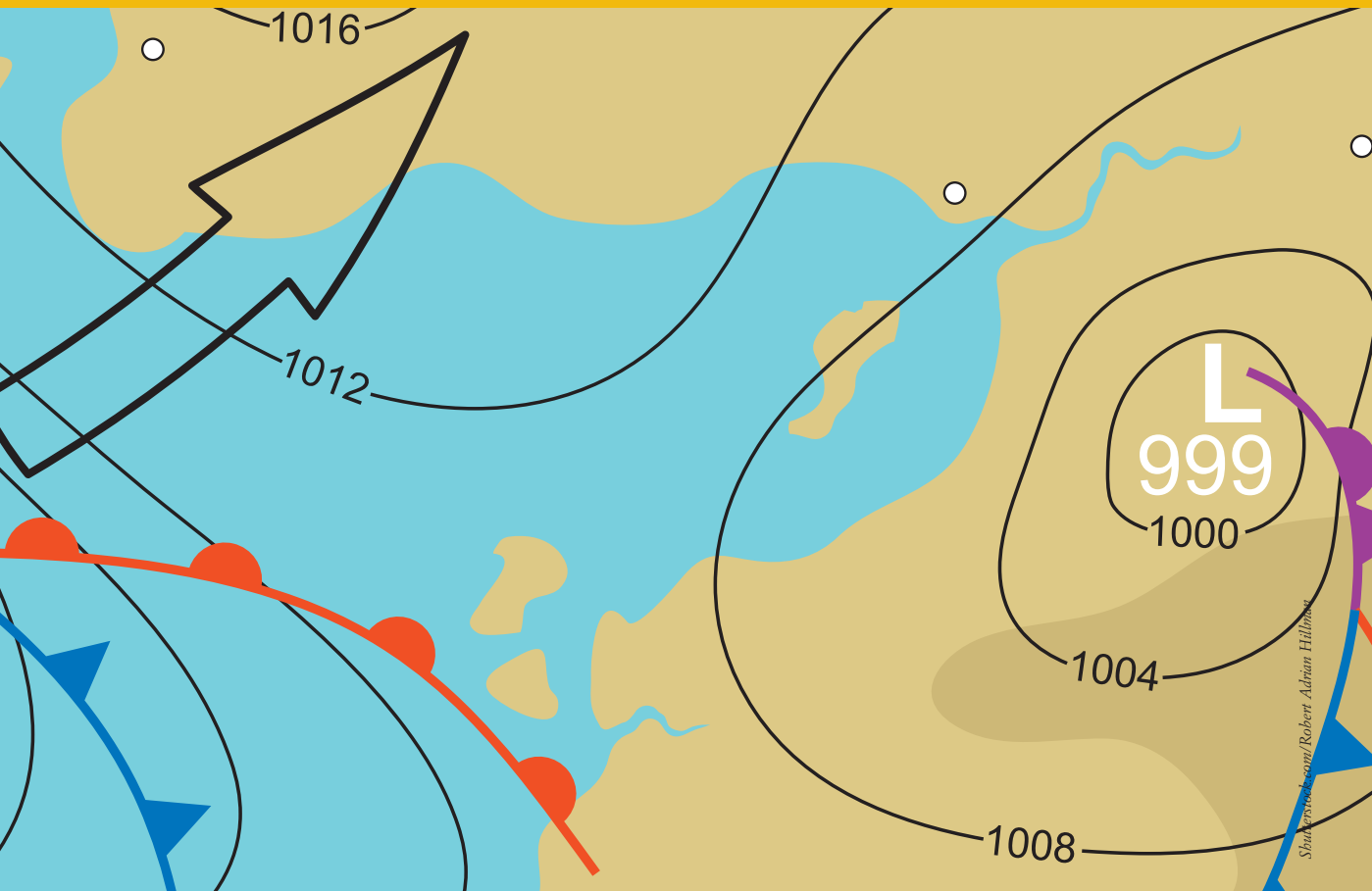


12

Statistics and Probability

Probability

Probability theory, the study of chance, began in the 17th century when two great mathematicians, Blaise Pascal and Pierre de Fermat, corresponded over problems arising from games of chance. Since then, probability has become an essential branch of mathematics that is used widely in fields such as weather forecasting, finance, insurance, politics and risk assessment of new medical treatments.



Chapter outline

	Proficiency strands				
12-01 Relative frequency	U	F	PS	R	C
12-02 Venn diagrams	U	F	PS	R	C
12-03 Two-way tables	U	F	PS	R	C
12-04 Two- and three-step experiments	U	F	PS	R	C
12-05 Selecting with and without replacement	U	F	PS	R	C
12-06 Dependent and independent events	U	F	PS	R	C
12-07 Conditional probability	U	F	PS	R	C

Wordbank

relative frequency The frequency of an event over repeated trials as a fraction of the total number of trials

tree diagram A diagram of branches for listing all the possible outcomes of a multi-step chance experiment

trial One go or run of a repeated probability experiment, for example, one roll of a die

two-step experiment A chance experiment with two steps or stages, such as rolling a pair of dice

two-way table A way of grouping items into two overlapping categories, such as gender and the ability to drive a car

Venn diagram A diagram of circles (usually overlapping) for grouping items into categories

■ In this chapter you will:

- calculate relative frequencies from given and collected data to estimate probabilities of events involving 'and' or 'or'
- describe events using language of probability of 'at least', 'A or B but not both', 'A or B or both' and 'and'
- list all outcomes for two- and three-step chance experiments, with and without replacement, using tree diagrams or arrays, and assign probabilities to outcomes and determine the probabilities for events 
- represent events in Venn diagrams and two-way tables and solve related problems
- understand the difference between dependent and independent events, and the product rule for independent events
- solve problems involving compound events and conditional probability

SkillCheck

MAT10SPWK10070

-

12-01 Relative frequency

Experimental probability is based on the results of a trial that has been repeated many times, such as testing the effectiveness of 100 light globes, or relies on past statistics, such as the number of babies born last year who were girls. It estimates **theoretical probability** using **relative frequency**, the number of times an event occurs compared to the total frequency of outcomes.

Technology worksheet

Excel worksheet: Long run proportion

MAT10SPCT00023

Technology worksheet

Excel spreadsheet: Long run proportion

MAT10SPCT00053

Puzzle sheet

Dice probability

MAT10SPPS00017

Summary

Experimental probability

$$P(E) = \frac{\text{number of times the event happened}}{\text{total number of trials}}$$

$$\text{or } P(E) = \frac{\text{frequency of } E}{\text{total frequency}}$$

Expected frequency is the expected number of times an event will occur over repeated trials.

$$\text{Expected frequency} = \text{theoretical probability} \times \text{number of trials.}$$

Example 1

James rolled a die 100 times and recorded the results in a table.

- Find the experimental probability of rolling:
 - an even number
 - an even number or a number greater than 4
 - an even number less than or equal to 4.
- Calculate the probability of rolling a 2 or 3
 - as an experimental probability
 - as a theoretical probability.
- If the die is rolled 100 times, what is the expected number of times of obtaining a 2 or a 3? How does this compare with James' observed number of times?

Outcome	Frequency
1	23
2	19
3	11
4	12
5	18
6	17

Solution

- Rolls of even numbers = $19 + 12 + 17$
= 48

Frequencies of 2, 4, 6

$$\begin{aligned} \text{Experimental probability } P(\text{even}) &= \frac{48}{100} \\ &= \frac{12}{25} \end{aligned}$$

- ii Rolls of even numbers or numbers greater than 4
 $= 19 + 12 + 17 + 18$
 $= 66$

Frequencies of 2, 4, 6, 5

$$\text{Experimental probability } P(\text{even or } > 4) = \frac{66}{100} \\ = \frac{33}{50}$$

- iii Rolls of even numbers less than or equal to 4 = 19 + 12
 $= 31$

Frequencies of 2 and 4

Experimental probability

$$P(\text{even and } \leq 4) = \frac{31}{100}$$

- b i Rolls of 2 or 3 = 19 + 11
 $= 30$

Frequencies of 2 and 3

$$\text{Experimental probability } P(2 \text{ or } 3) = \frac{30}{100} \\ = \frac{3}{10}$$

- ii Theoretical probability $P(2 \text{ or } 3) = \frac{2}{6}$
 $= \frac{1}{3}$

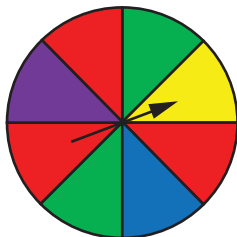
- c Expected number of rolls of 2 or 3 = $\frac{1}{3} \times 100$
 $= 33.333 \dots$
 ≈ 33

Probability $\times$ number of trials

From the table, the observed number of heads = 19 + 11 = 30, which is close to 33.

Exercise 12-01 Relative frequency

- See Example 1 1 Aashima spun this spinner 200 times and recorded the results in the table.

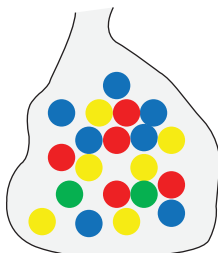


Event	Frequency
Red	85
Green	42
Blue	28
Yellow	15
Purple	30

- a Calculate, as a decimal, the experimental probability (relative frequency) that the arrow points to:
- i red ii blue iii green
- b Calculate, as a decimal, the theoretical probability that the arrow points to:
- i red ii blue iii green

- c Are the experimental and theoretical probabilities similar?
- d For 200 spins, what is the expected frequency of red or purple based on the theoretical probability? How does this compare with the observed frequency?

- 2 A bag contains 7 blue, 6 yellow, 5 red and 2 green marbles. Lamisa selects a marble at random, records its colour and then returns it to the bag. Lamisa repeats this process 100 times and the results are shown in the table.



Outcome	Frequency
Red	20
Blue	38
Yellow	33
Green	9

- a Find the relative frequency of selecting a marble that is:
- i red ii blue iii yellow iv green
- b What is the theoretical probability of selecting a marble that is:
- i red? ii blue? iii yellow? iv green?
- c Are the experimental and theoretical probabilities similar?
- d If the process is repeated 100 times, what is the expected frequency of a selecting a yellow or green marble? How does this compare with the observed frequency?

- 3 A coin is tossed.

- a What is the expected number of obtaining heads if the coin is tossed 100 times?
- b Toss a coin 100 times. Copy this table and record your results in it.
- c Calculate, as a decimal:
- i the experimental probability of tossing a head
- ii the theoretical probability of tossing a tail.
- d Are the experimental and theoretical probabilities similar?

Outcome	Frequency
Head	
Tail	

- 4 A die was repeatedly rolled and the results are shown in the table.

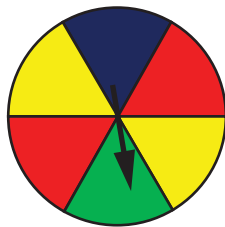
- a How many times was the die rolled?
- b Find the experimental probability (as a decimal) of rolling:
- i an odd number
- ii a number less than 4
- iii a 2 or a 3
- iv a number less than 4 or even.
- c Find the theoretical probability (as a decimal) of rolling:
- i an odd number
- ii a number less than 4
- iii a 2 or a 3
- iv a number less than 4 or even.
- d Compare the experimental probabilities to the theoretical probabilities.

Outcome	Frequency
1	95
2	119
3	108
4	87
5	78
6	113

- 5 Place 5 blue counters, 2 red counters and 3 yellow counters in a bag. Select a counter at random from the bag, note its colour and return the counter to the bag. Repeat this 60 times. Copy this table and record your results in it.

Event	Tally	Frequency
Blue		
Red		
Yellow		

- a What is the experimental probability of selecting:
- i a blue counter?
 - ii a red counter?
 - iii a yellow counter?
 - iv a red or blue counter?
- b What is the theoretical probability of selecting:
- i a blue counter?
 - ii a red counter?
 - iii a yellow counter?
 - iv a red or blue counter?
- c Are the experimental probabilities similar to the theoretical probabilities?
- 6 Denise spun this spinner 50 times and the results are shown in the table.



Event	Frequency
Red	15
Blue	6
Yellow	24
Green	5

- a What is the experimental probability (as a decimal) of the arrow stopping on
- i red?
 - ii blue?
 - iii yellow?
 - iv green?
- b What is the calculated probability (as a decimal) of the arrow stopping on
- i red?
 - ii blue?
 - iii yellow?
 - iv green?
- c Are the experimental and calculated probabilities similar?
- d What is the expected number of times of the arrow stopping on a colour that is not yellow? How does this compare with Denise's observed number of times?
- 7 A die is rolled 100 times.
- a What is the calculated probability of rolling a 6? (Express your answer as a fraction and as a decimal.)
 - b How many times would you expect a 6 to appear if the die was rolled 100 times?
 - c Roll a die 100 times and record your results in a table similar to the one shown.

Outcome	Tally	Frequency
1		
2		
3		
4		
5		
6		

- d What is the experimental probability or relative frequency of rolling a 6? (Express your answer as a fraction and as a decimal.)
- e How does the calculated probability of rolling a 6 compare with the experimental probability?

8 Two dice are rolled and the sum of the numbers appearing uppermost was recorded in the frequency histogram.

a How many times were the two dice rolled?

b Based on these results, what is the experimental probability (or relative frequency) of obtaining:

i a sum of 2?

ii a sum of 7?

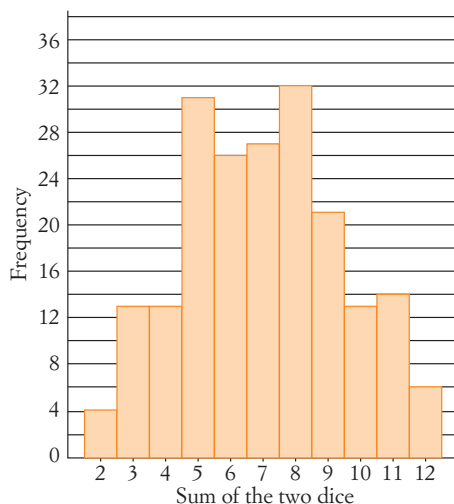
iii a sum of 10?

iv a sum greater than 7?

v a sum less than 7?

vi a sum of 7 or 8?

vii a sum that is even and greater than 6?



9 School-aged children at several shopping centres were asked how they travelled to school.

a How many students were surveyed?

b Based on these results, find the probability that a student chosen at random will:

i walk to school

ii be driven to school

iii catch a bus to school

iv catch a train to school

v ride a skateboard to school

c What mode of transport could 'Other' include?

d Survey 100 students at your school and make up a table showing the results. How do the results from your school compare with the results from the survey?

Mode of transport	Frequency
Walk	27
Bus	80
Car	62
Train	21
Bicycle	5
Skateboard	1
Other	4

12-02 Venn diagrams

A **Venn diagram** is a diagram of circles (usually overlapping) that is used to group items into categories. A rectangle represents the whole group while the circles represent categories. Items common to two or more categories are placed in the intersection (overlapping region) of the circles. The Venn diagram was invented in 1880 by English mathematician and priest, John Venn (1834–1923).



Pictures from History

Video tutorial

Venn diagrams

MAT10SPVT10014

Puzzle sheet

Venn diagrams
matching activity

MAT10SPPS10071

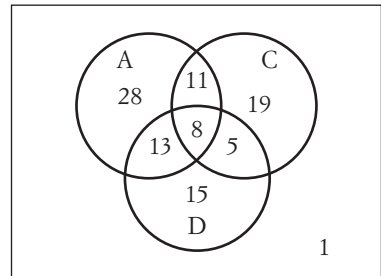
Puzzle sheet

And/or problems

MAT10SPPS00016

Example 2

The Venn diagram shows the results of a survey on what type of movies – action (A), comedy (C) or drama (D) – that students prefer to watch.



- a How many students were surveyed?
- b How many students preferred to watch two types of movies only?
- c Calculate, as a decimal, the probability of selecting a student who prefers to watch:
 - i action movies only
 - ii action or comedy movies but not dramas
 - iii action and drama movies
 - iv all types.
- d A student is chosen from those who like action and comedy movies. What is the probability that they also like to watch drama movies?
- e What is the probability of selecting a student who does not like watching any of the three types?

Solution

a Number of students = $28 + 11 + 8 + 13 + 5 + 19 + 15 + 1$
= 100

b 29 students only preferred two types of movies

$$11 + 13 + 5 = 29$$

c i Students preferring action movies only = 28

The region of A that doesn't overlap C or D.

$$P(\text{action only}) = \frac{28}{100}$$

$$= 0.28$$

ii Students preferring action or comedy only = $28 + 19 + 11$
= 58

The regions of A and C that don't overlap with D.

$$P(\text{action or comedy only}) = \frac{58}{100}$$

$$= 0.58$$

iii Students preferring action and drama = $13 + 8$
= 21

The regions where A and D intersect.

$$P(\text{action and drama}) = \frac{21}{100}$$

$$= 0.21$$

iv Students preferring all types = 8

The region where the three circles intersect.

$$P(\text{all types}) = \frac{8}{100}$$

$$= 0.08$$

d Students preferring action and comedy = $11 + 8$
= 19

Students preferring action and comedy and drama = 8

$$P(\text{drama if preferring action and comedy}) = \frac{8}{19}$$

$$= 0.42 \text{ (to 2 decimal places)}$$

- e There is one student who doesn't prefer action, comedy or drama.

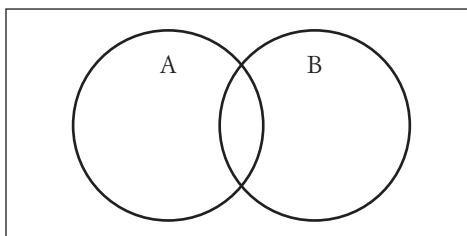
$$P(\text{not action, comedy or drama}) = \frac{1}{100} \\ = 0.01$$

'And' vs 'or'

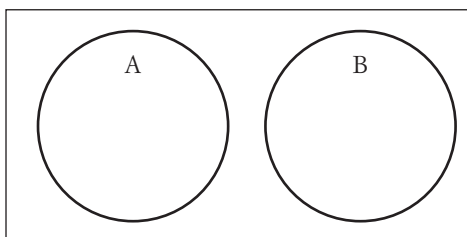
For two categories or events A and B, the phrase '**A and B**' means to have both of them occurring together. For example, 'to drive a car' **and** 'to ride a bus' means to do both things.

If A and B are **overlapping**, the phrase '**A or B**' means to have A or B or both. For example, 'to drive a car' **or** 'to ride a bus' means to drive a car only, or to ride a bus only, or to do both. In this case, 'A or B' actually **includes** 'A and B' so this is an example of an **inclusive** 'or'.

If A and B are **mutually exclusive**, this means that they are **not overlapping** and on a Venn diagram they appear as two separate circles. For mutually exclusive categories or events, the phrase '**A or B**' means to have A only or B only (but not both). For example, 'male' **or** 'female' means to be male, or female, but not both. In this case, 'A or B' **excludes** 'A and B' so this is an example of an **exclusive** 'or'.



Overlapping events: 'A or B'
means A or B or both



Mutually exclusive events: 'A or B'
means A or B but not both

Example 3

A survey of 110 students at Lambavare High showed that 34 students do Art, 65 students do Computer Studies, and 23 students do Computer Studies and Art.

- Represent this information on a Venn diagram.
- How many students do Art or Computer Studies but not both?
- What is the probability of randomly selecting a student from this group who:
 - does Computer Studies?
 - does Art and Computer Studies?
 - does Art or Computer Studies?
 - does neither Art nor Computer Studies?

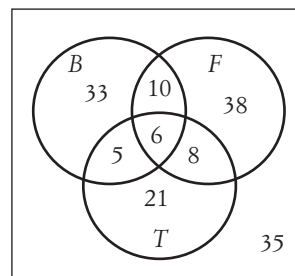


Corbis/Purestock

A Venn diagram showing two overlapping circles, A and C, within a rectangular frame labeled S. Circle A contains the number 11, circle C contains the number 42, and their intersection contains the number 23. The number 34 is written outside the frame at the bottom right, representing the total number of elements in S.

A Venn diagram with two overlapping circles labeled B and L. Circle B on the left contains the number 10. Circle L on the right contains the number 28. The overlapping region between the two circles contains the number 12.

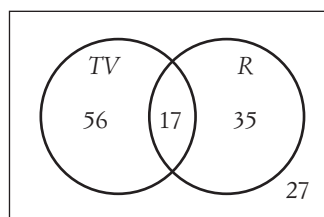
- 2 The Venn diagram shows the number of Year 10 students who play basketball (B), touch football (F) or tennis (T).



See Example 3

- How many students are in Year 10?
- Find the probability of selecting a student who plays:
 - basketball only
 - tennis only
 - touch football and tennis
 - touch football or tennis
 - basketball but not touch football
 - all three sports.
- Of the students who play touch football, find the probability of selecting a student who also plays tennis.

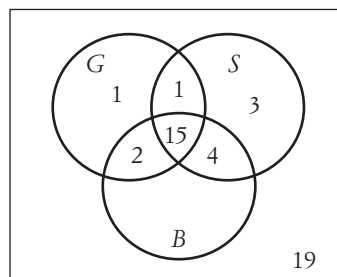
- 3 The Venn diagram shows the results of a survey asking people how they relax at home: watching television (TV) or reading (R).



- How many people were surveyed?
 - Find the probability of selecting a person who only watches TV to relax.
 - What is the probability of selecting a person who doesn't watch TV or read to relax?
 - Of the people who read, find the probability that they also watch TV to relax.
- 4 Of the 54 Year 10 Music students, 23 students sing (S), 43 students play a musical instrument (P) and 12 students sing and play a musical instrument.

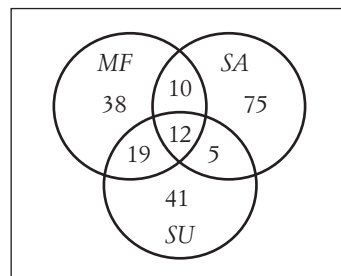
- Show this information on a Venn diagram.
- Find the probability of selecting a Music student who:
 - sings or plays an instrument
 - sings only
 - plays a musical instrument only
 - sings or plays an instrument, but not both

- 5 The Venn diagram shows the number of countries that won medals at the XXI Winter Olympic Games in Vancouver, 2010.



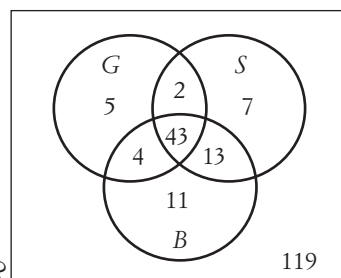
- How many countries competed at these winter Olympics?
 - What is the probability of selecting a country at random that:
 - won only gold medals?
 - won gold, silver and bronze medals?
 - won gold or silver medals, but not bronze?
 - did not win a gold or silver medal?
 - Of the countries that won medals, what is the probability of selecting a country that
 - won gold medals?
 - won bronze, but not gold or silver?
- 6 At Riverside College, Year 10 students are asked what language they are studying. 64 students take Indonesian (I), 47 students take Japanese (J), 15 students take both Indonesian and Japanese, and 27 do not study a language.
- How many students are in Year 10?
 - Show the information on a Venn diagram.
 - How many students studied only one language?
 - Find the probability of selecting a Year 10 student at random who studies:
 - Indonesian but not Japanese
 - Japanese but not Indonesian
 - no languages
 - only one language.

- 7 People at several shopping centres were asked on which day they preferred to shop – Monday to Friday (*MF*), Saturday (*SA*) or Sunday (*SU*). The results are shown in the Venn diagram.



- How many people were surveyed?
- What is the probability of selecting a person who prefers to shop on:
 - Monday to Friday?
 - Saturday?
 - Sunday?
 - on the weekend only?
 - on Saturday or Sunday?
 - any day (has no preference)?
- Find the probability of selecting a person who only prefers Saturday or Sunday but not both.
- Is it necessary to include the rectangle in this Venn diagram? Give reasons.

- 8 The Venn diagram shows the number of countries that won gold, silver, bronze or no medals at the 2012 Summer Olympic Games in London.



- Find the total number of countries that competed at the 2012 London games.
- What is the probability of randomly selecting a country that:
 - won a silver medal only?
 - won one medal only?
 - won at least two medals?
 - won at most one medal only?
- Out of the countries that won gold medals, find the probability of selecting a country that:
 - won gold and silver but not bronze
 - won gold, silver and bronze.

Worksheet

Two-way tables

MAT10SPWK10072

Puzzle sheet

Combined events:
Two-way tables

MAT10SPPS00014

12-03 Two-way tables

A **two-way table** is another way of grouping items into overlapping categories, especially when there are many overlaps that cannot easily be represented by a Venn diagram.

Example 4

Year 11 students at Southbank College were surveyed on whether they had part-time jobs.

	Part-time work	No part-time work
Male	43	27
Female	35	31

- How many students are in Year 11 at Southbank College?
- How many students had part-time work?
- How many male students were in Year 11?
- What is the probability of selecting a student at random who:
 - works part-time?
 - is female and works part-time?
 - is male and doesn't work?
 - doesn't work?
- What is the probability of selecting a student working part-time given that:
 - the student is male?
 - the student is female?

Solution

a Number of Year 11 students = $43 + 27 + 35 + 31$
= 136

b Students with part-time work = $43 + 35$
= 78

c Male students in Year 11 = $43 + 27$
= 70

d i $P(\text{student works part-time}) = \frac{78}{136}$
= $\frac{39}{68}$

ii There are 35 female students who work part-time.

$$P(\text{female and part-time work}) = \frac{35}{136}$$

iii There are 27 males who don't work.

$$P(\text{male and not working}) = \frac{27}{136}$$

iv Number of students not working = $27 + 31 = 58$

$$P(\text{not working}) = \frac{58}{136} = \frac{29}{68}$$

e i There are 70 male students and 43 of them work part-time.

$$P(\text{working part-time given that student is male}) = \frac{43}{70}$$

ii There are 66 female students and 35 of them work part-time.

$$P(\text{working part-time given that student is female}) = \frac{35}{66}$$

Exercise 12-03 Two-way tables

- 1 People attending the Staying Alive Fitness Centre early on a Saturday morning either went swimming or did a workout in the gym. The numbers are shown in the table on the right.

	Swimming	Gym
Male	32	53
Female	24	41

See Example 4

- a How many people went to the fitness centre?
- b Find the probability that a person selected at random:
- was female and went swimming
 - was male and did a workout in the gym
 - went swimming.
- c Find the percentage (to the nearest whole number) of females who did a workout in the gym.

- 2 Year 10 students at Baramvale High were given a choice of two activities, bowling or indoor soccer on a wet sports afternoon. Their choices are shown in the table.

	Bowling	Indoor soccer
Boys	25	48
Girls	43	12

- a How many students were in Year 10?
b How many students:
i went bowling? ii played indoor soccer?
c What is the probability of randomly selecting a student who went bowling?
d What is the probability of randomly selecting a girl who played indoor soccer?
- 3 The composition of the Legislative Assembly (the lower house) in the NSW State Parliament (in 2012) is shown in the table.

	Liberal/Nationals	Labor	Independents/Greens
Male	59	11	3
Female	11	9	0

- a How many members of parliament (MPs) are there in the Legislative Assembly?
b Find the percentage probability of randomly selecting an MP who is:
i female ii a male and in the Opposition iii an independent or Green
c What percentage probability of:
i Government MPs are female? ii Opposition MPs are female?
d Compare your answers to part c and comment on the difference between the two results.

- 4 People were asked to name their favourite takeaway food. The results are shown in the table below.

	Pizza	Hamburger	Fish and Chips
Men	22	35	18
Women	43	26	6

- a How many people were surveyed?
b Find the probability (as a decimal) that a person selected at random:
i is male ii is female and likes fish and chips
iii likes pizza iv is male and likes hamburgers.
c If a male is selected at random, what is the probability that his favourite takeaway food is pizza?

- 5 Year 7 students were asked about their favourite drink. The results are in the table below.

	Boys	Girls
Water	21	35
Milk	11	12
Juice	15	17
Soft drink	31	18

- a How many students were in Year 7?
b What is the probability of randomly selecting a student that:
i prefers water?
ii is a boy and likes milk?
iii is a girl and likes soft drinks?
c What is the probability that if a girl is randomly selected, she prefers water?

- 6 A survey looked at whether people ate breakfast and whether they exercised regularly. The results are shown in the two-way table.
- How many people were surveyed?
 - What percentage of people exercised?
 - Find the percentage probability of picking a person at random who:
 - eats breakfast
 - does not exercise regularly
 - eats breakfast and exercises regularly
 - does not eat breakfast and doesn't exercise.
 - Of the people who exercise regularly, what is the probability of picking someone who eats breakfast?

	Exercise	No exercise
Ate breakfast	72	27
Did not eat breakfast	38	63



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- 7 Students at Granthorp High School were asked to indicate their preference for dark or milk chocolate in a survey.

		Milk chocolate	
		Like	Dislike
Dark chocolate	Like	545	134
	Dislike	157	42

- How many students attended Granthorp High?
- What is the probability of selecting a student at random who:
 - likes dark chocolate?
 - likes both milk chocolate and dark chocolate?
 - likes dark chocolate but dislikes milk chocolate?
 - dislikes both dark and milk chocolate?

12-04 Two- and three-step experiments

A **two-** or **three-step experiment** is a chance experiment that has two or three parts or stages, for example:

- rolling two or three dice
- drawing two or three prizes in a raffle
- observing the weather each day over a weekend or a long weekend
- throwing two or three coins together

The sample space for two-step experiments can be displayed using lists, tables or tree diagrams but the sample space for three-step experiments is best displayed using a tree diagram.

Example 5

A coin is tossed and a die is rolled.

- a Use a table to display the sample space.
- b Find the probability of obtaining:
 - i a tail and a 3
 - ii a head and an even number.

Solution

- a The sample space of a coin is a head (H) and a tail (T).

The sample space for a die is 1, 2, 3, 4, 5 and 6.

The sample space of tossing a coin and rolling a die is shown in the table below.

		Die					
		1	2	3	4	5	6
Coin	H	H1	H2	H3	H4	H5	H6
	T	T1	T2	T3	T4	T5	T6

Using a table ensures that all outcomes are counted

- b i There are 12 outcomes in the sample space.

$$\therefore P(\text{a tail and a 3}) = P(T3) = \frac{1}{12}$$

- ii There are three outcomes that make up the event a head and an even number: H2, H4, H6

$$\begin{aligned} \therefore P(\text{a head and an even number}) &= \frac{3}{12} \\ &= \frac{1}{4} \end{aligned}$$

Video tutorial

Tree diagrams

MAT10SPVT10013

Worksheet

Tree diagrams

MAT10SPWK10073

Puzzle sheet

Combined events: Tree diagrams

MAT10SPPS00015

Tree diagrams

A **tree diagram** lists all the possible outcomes of each stage. Branches stretch out to show the possible pathways of outcomes at each step or stage. An outcomes column at the end of the diagram lists the sample space.

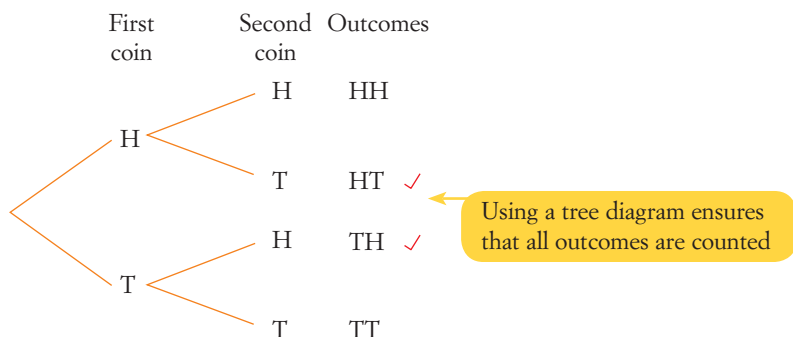
Example 6

Two coins are tossed.

- a Use a tree diagram to list the sample space.
- b Find the probability of tossing:
 - i two heads
 - ii a head and a tail (in any order).

Solution

- a There are 2 outcomes for the first coin, followed by 2 outcomes for the second coin. There are $2 \times 2 = 4$ possible outcomes.



- b i There is one outcome out of a possible 4 for two heads.

$$\therefore P(2 \text{ heads}) = \frac{1}{4}$$
- ii There are 2 outcomes for a head and a tail (ticked on the tree diagram).

$$\therefore P(\text{a head and a tail}) = \frac{2}{4}$$

$$= \frac{1}{2}$$

Exercise 12-04 Two- and three-step experiments

- 1 The positions of captain and vice-captain of a netball team are to be selected from Cassandra, Ashleigh, Gemma, Jodie, Erin and Robyn.
 - a List the possible pairings of captain and vice-captain.
 - b What is the probability of Ashleigh being captain or vice-captain?
 - c What is the probability of Erin becoming vice-captain?

See Example 5

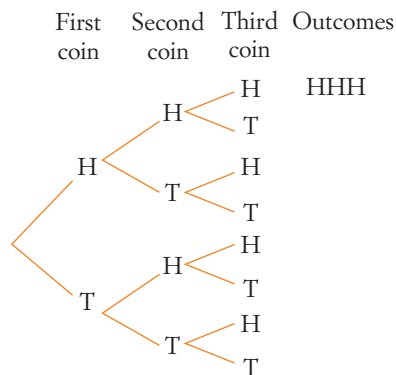
- 2 Two dice are rolled.
a Copy and complete the table below to list the sample space.

		Second die					
		1	2	3	4	5	6
First die	1	1, 1	1, 2				
	2						
	3						
	4			4, 3			
	5						
	6						

- b How many possible outcomes are there?
c Find the probability of:
i rolling doubles ii rolling two even numbers
iii rolling at least one 2 iv rolling two numbers greater than 3
v rolling one odd and one even number
vi rolling two numbers so that the first number is greater than the second number.

See Example 6 3 Three coins are tossed.

- a Copy and complete the tree diagram to list the sample space.
b How many outcomes are there in the sample space?
c Use the tree diagram to find these probabilities.
i $P(3 \text{ heads})$
ii $P(2 \text{ heads})$
iii $P(3 \text{ tails})$
iv $P(\text{head, then tail and then head})$
v $P(2 \text{ heads or } 3 \text{ heads})$



- d Find the probability of:
i tossing at least 1 tail ii tossing at most 2 tails.
e If three coins are tossed 200 times, find the expected number of:
i tossing two heads ii tossing no tails.
4 Use a tree diagram to display all possible outcomes when a coin is tossed and a die is rolled.
5 A tetrahedral die (numbered 1, 2, 3 and 4) and a normal six-sided die are rolled.
a Construct a table to list the outcomes in the sample space.
b How many outcomes are in the sample space?
c Find the probability of rolling:
i doubles
ii two even numbers
iii one even and one odd number
iv a pair of numbers that are both less than 4
v a pair of numbers that are both greater than 4.

6 Two normal dice are rolled and the sum of the two numbers is calculated.

a Copy and complete this table to show all possible sums.

		Second die					
		1	2	3	4	5	6
First die	1	2					
	2						
	3				7		
	4						
	5						
	6					11	

b Find the of probability of rolling a sum:

i of 5

ii of 12

iii of 7

iv that is even

v less than 2

vi more than 7

vii at least 7

viii between 4 and 8.

7 Four coins are tossed.

a Use a tree diagram to list the sample space.

b Find the probability of tossing:

i 4 heads

ii 1 head

iii 2 tails

iv at least 1 tail

v 2 heads and then 2 tails

vi not more than 1 tail

c If four coins are tossed 1000 times, find the expected number of:

i 4 heads

ii 2 heads and 2 tails

iii at least one tail

Mental skills 12

Maths without calculators

The unitary method with percentages

The unitary method is used when you are only given a *percentage* of an amount and you need to find the amount. It is called the unitary method because we find 1% of the amount first, then multiply that by 100 to find the whole (100%).

1 Study each example.

a If 8% of a number is 24, what is the number?

8% of the number = 24

$\therefore$ 1% of the number = $24 \div 8 = 3$

$\therefore$ 100% of the number = $3 \times 100 = 300$.

The number is 300. Check: $8\% \times 300 = 24$

b If 15% of an amount is \$90, what is the whole amount?

15% of the amount = \$90

$\therefore$ 1% of the amount = $\$90 \div 15 = \$6$

$\therefore$ 100% of the amount = $\$6 \times 100 = \$600$.

The amount is \$600. Check: $15\% \times \$600 = \$90$

- 2 Find the whole amount if:
- a 5% of the amount is \$35
 - b 11% of the amount is \$88
 - c 20% of the amount is 80
 - d 6% of the amount is 42
 - e 90% of the amount is \$270
 - f 15% of the amount is \$60
 - g 40% of the amount is 100
 - h 120% of the amount of \$360
 - i 25% of the amount is \$75
 - j 8% of the amount is 40

Investigation: The birth month paradox

A **paradox** is a statement or proposition that seems impossible but is actually true.

- 1 Copy this table.

Group	Outcome (Y or N)
1	
2	
3	
4	
5	

- 2 Randomly select a group of 5 people and ask them what month they were born in. If two or more people have the same birth month, record a Y in the table for Group 1, otherwise write N.
- 3 Repeat this process 4 more times, recording your results in the table.
- 4 Combine your results with those of six other students so that you have the outcomes for 30 groups.
- 5 What fraction of the groups had repeated birth months?
- 6 Obtain the results of another group of six students. What fraction of the groups had a repeated birth month?
- 7 The birth month paradox is that in any randomly selected group of 5 people, the probability that at least two people have the same birth month is greater than 0.5. Have your results shown this to be true?
- 8 Can you show the following?
- a For every 23 people selected at random, the probability that at least 2 people will share the same birthday is 50%.
 - b If 30 people are selected at random, this probability is 70%.
 - c If 50 people are selected at random, this probability is 97%.

12-05

Selecting with and without replacement

Worksheet

Multi-step experiments

MAT10SPWK10074

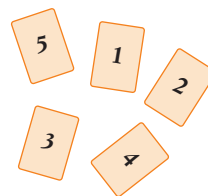
In two- and three-step experiments where an item is selected repeatedly, the outcome of the second or third step may or may not be affected by the outcome of the previous step. This depends upon whether each selected item is returned to the set of items before the next item is selected. If it is, then this is called **selecting 'with replacement'**. If it isn't, then it is called **selecting 'without replacement'**.

Example 7

Two cards are drawn from a set of cards numbered 1 to 5, to form a two-digit number.

a Make a list of all possible outcomes if the cards are drawn:

- i with replacement ii without replacement.



b If the first card is replaced before the second card is drawn, find the probability that the number formed is:

- i even ii greater than 30 iii divisible by 5.

c If the first card is not replaced, find the probability that the number formed is:

- i even ii odd iii less than 20.

Solution

a i The possible outcomes, **with replacement**, are:

11	21	31	41	51
12	22	32	42	52
13	23	33	43	53
14	24	34	44	54
15	25	35	45	55

There are $5 \times 5 = 25$ different outcomes possible.

ii The possible outcomes, **without replacement**, are:

12	21	31	41	51
13	23	32	42	52
14	24	34	43	53
15	25	35	45	54

There are $5 \times 4 = 20$ different outcomes possible.

b i There are 10 even numbers.

$$P(\text{even number}) = \frac{10}{25} = \frac{2}{5}$$

ii There are 15 numbers greater than 30.

$$P(\text{number} > 30) = \frac{15}{25} = \frac{3}{5}$$

- iii There are 5 numbers divisible by 5.

15, 25, 35, 45 and 55

$$P(\text{number divisible by } 5) = \frac{5}{25} = \frac{1}{5}$$

- c i There are 8 even numbers without replacement.

$$P(\text{even}) = \frac{8}{20} = \frac{2}{5}$$

- ii There are 12 odd numbers.

$$P(\text{odd}) = \frac{12}{20} = \frac{3}{5}$$

- iii There are 4 numbers less than 20.

$$P(\text{number} < 20) = \frac{4}{20} = \frac{1}{5}$$

Example 8

A bag contains 3 red counters and a blue counter.
Three counters are drawn at random without replacement.

- a Use a tree diagram to display all possible outcomes.
b Find the probability of drawing:
i two red counters
ii a red, blue, and red in that order
iii at least one red counter.



Solution

- a The tree diagram will have 4 branches for the first step or stage, followed by 3 branches for the second step, followed by 2 branches for the third step.

So there are $4 \times 3 \times 2 = 24$ outcomes in the sample space.

- b i There are 18 outcomes with two red counters (ticked ✓ on the tree diagram)

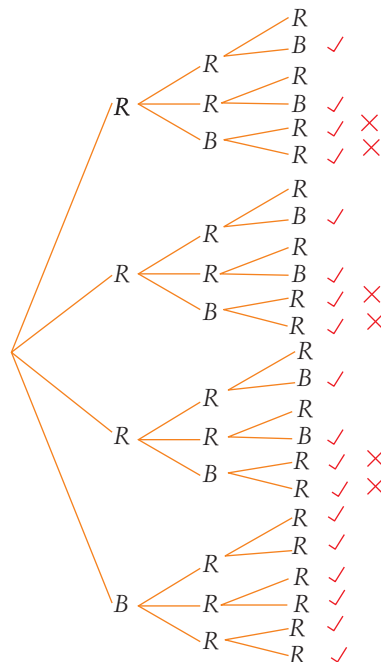
$$\therefore P(\text{two red counters}) = \frac{18}{24} = \frac{3}{4}$$

- ii Red, blue, red occurs 6 times (crossed × on the tree diagram).

$$\therefore P(\text{red, blue, red}) = \frac{6}{24} = \frac{1}{4}$$

- iii All outcomes contain at least one red counter.

$$\therefore P(\text{at least one red counter}) = \frac{24}{24} = 1$$



Exercise 12-05 Selecting with and without replacement

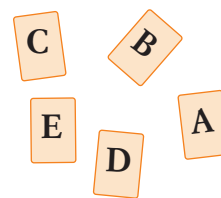
- 1 A boy and a girl are to be chosen from a group of 4 boys and 6 girls: Ben, Christian, Ewan, William, Becky, Cassandra, Emily, Marianne, Roz and Siana.

See Example 7

- List all the possible pairs of a boy and a girl.
- Find the probability of selecting:
 - Christian and Roz
 - a boy and a girl whose names begin with a B or a C
 - a pair that includes Emily.

- 2 Two cards are drawn from a set of cards labelled A, B, C, D and E.

- Make a list of all possible outcomes if the cards are drawn:
 - with replacement
 - without replacement.
- If the first card is replaced before the second card is drawn, find the probability that:
 - both letters are the same
 - both letters are vowels
 - one letter is a vowel and the other is a consonant.
- If the first card is not replaced, find the probability that:
 - both letters are vowels
 - one letter is a vowel and the other is a consonant
 - the first letter is a B or a D
 - the last letter is not A.



- 3 Two coins are tossed together.

- Copy and complete the table to find all the outcomes in the sample space.
- What is the probability of tossing:
 - two tails?
 - a head and a tail?
 - at least one head?

		Second coin	
		H	T
First coin	H		
	T		

- 4 When staying at a hotel, Sarah and David can select one item from each course of a breakfast menu.

1st course	2nd course
Cereal (C)	Bacon and eggs (B)
Fruit (F)	Ham and cheese croissants (H)
Yoghurt (Y)	Pancakes (P)
	Sausages and tomatoes (S)
	Toast and jam (T)

- Copy and complete the table to list all the different 2-course breakfasts available.

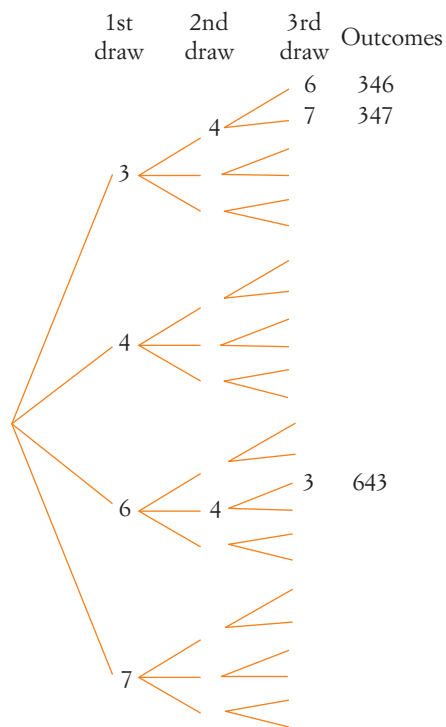
		2nd course				
		B	H	P	S	T
1st course	C					
	F					
	Y					

- If one of the combinations of breakfasts is chosen at random, what is the probability that it includes:
 - fruit?
 - cereal but not bacon and eggs?
 - fruit and croissants?

See Example 8

- 5 The numbers 3, 4, 6 and 7 are written on separate cards and placed in a bag. Three cards are drawn at random without replacement to form a three-digit number.

- a Copy and complete the tree diagram to list the sample space.
b Find the probability of forming:
i an even number
ii a number greater than 400
iii a number between 400 and 700
iv an even number greater than 400.

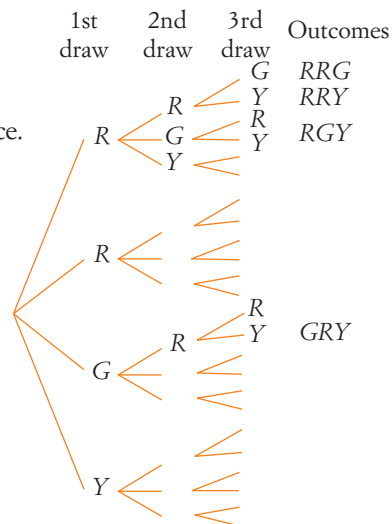


- 6 The cards in question 5 are to be now drawn *with replacement*, that is, the same card can be used twice.

- a Use a tree diagram to show all possible outcomes.
b Find the probability that the three-digit number formed is:
i a number with all digits the same
ii an odd number
iii greater than 500
iv begins and ends with a 4.

- 7 A bag contains 2 red marbles, 1 green marble and 1 yellow marble. Three marbles are drawn from the bag at random without replacement.

- a Copy and complete the tree diagram to list the sample space.
b Find the probability of drawing:
i two red marbles
ii a red, green, and red in that order
iii at least one red marble.



- 8 A family has three children.
- Use a tree diagram to list all possible outcomes in the sample space.
 - What is the probability that the family consists of:
 - 3 boys?
 - 3 girls?
 - 2 girls and a boy?
 - a girl and then two boys?
- 9 The weather on a long weekend will either be fine or rain each day, with each outcome being equally likely.
- Draw a tree diagram to show the possible outcomes for Saturday, Sunday and Monday.
 - What is the probability that:
 - it rains on all three days?
 - it is fine on two of the three days?
 - it is fine on Saturday and Sunday, but rains on Monday?
 - it rains on at least one day of the long weekend?
- 10 Two dice are rolled.
- How many outcomes are possible?
 - What is the probability of rolling two 6s?
 - What is the probability of rolling at least one 6?
- 11 Three counters are drawn from a bag containing 3 yellow counters, 1 blue counter and 1 white counter.
- Draw a tree diagram to show the sample space if the counters are drawn:
 - with replacement
 - without replacement.
 - If the counters are drawn with replacement, find the probability of obtaining:
 - three counters of the same colour
 - three counters of different colours
 - no blue counters
 - at least one yellow counter.
 - If the counters are drawn without replacement, find the probability of drawing:
 - three counters of the same colour
 - three of different colours
 - 2 blue counters and a white counter
 - no white counter.

Investigation: Dependent or independent?

Work in pairs.

You will need: a coin, 3 blue counters and 2 red counters.

- Toss a coin and record the outcome.
 - What is the probability of obtaining your outcome?
 - Toss the coin a second time and record the outcome.
 - What is the probability of obtaining the second outcome?
 - Is the outcome of the second toss affected by the outcome of the first toss? Is the probability of the second outcome independent or dependent on the first outcome? Justify your answer.
- Copy this table.
 - Put 3 blue counters and 2 red counters in a bag. Randomly draw a counter from the bag and record the outcome.
 - Put back the counter you drew in part **a** and shake the bag. Again draw a counter and record your outcome.
 - Repeat the procedure from parts **a** and **b** 40 times and record the totals of each outcome in the table.

With replacement	1st draw	2nd draw
Blue		
Red		
	40	40

- e Use your results to find:
- i $P(\text{blue counter drawn first})$ ii $P(\text{blue counter drawn second})$
- f i Are your two results for part e the same?
ii Would you expect the results to be the same? Give reasons.
- g Is the outcome of the second draw dependent on the outcome of the first draw?
- 3 a Copy this table.
- | Without replacement | 1st draw | 2nd draw |
|---------------------|----------|----------|
| Blue | | |
| Red | | |
| | 40 | 40 |
- b Again, place 3 blue counters and 2 red counters in a bag. Randomly draw a counter from the bag and record your outcome.
- c **Without replacing** the counter you drew in part a, shake the bag and draw a second counter, recording your outcome.
- d Repeat the procedure from parts a and b 40 times and record the totals of each outcome in the table.
- e Use your results to find:
- i $P(\text{blue counter drawn first})$ ii $P(\text{blue counter drawn second})$
- f i Are your two results for part e the same?
ii Would you expect your results to be the same? Give reasons.
- g Is the outcome of the second draw dependent on the outcome of the first draw? Compare your results with those of other students in your class.

12-06 Dependent and independent events

Two events are **independent** if the outcome of one event **does not affect** the outcome of the other event and so the probability of one event is not affected by the probability of the other event occurring. For example, if a coin is tossed and a die is rolled, the two events are independent as the outcome on the coin does not affect the outcome on the die.

Two events are **dependent** if the outcome of one event **affects** the outcome of the other event and so the probability of one event is affected by the probability of the other event occurring. For example, when drawing two marbles from a bag without replacement of the first marble, the two events are dependent because the outcome of the second draw is affected by the outcome of the first draw.

Example 9

A coin is tossed and a die is rolled.

- a List the outcomes in the sample space.
- b Find:
 - i $P(\text{tossing a head})$ ii $P(\text{rolling an even number})$
 - iii $P(\text{tossing a head and rolling an even number})$
- c Is $P(\text{tossing a head and rolling an even number}) = P(\text{tossing a head}) \times P(\text{rolling an even number})$?
- d Are the two events dependent or independent?

Solution

a The outcomes are H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5 and T6.

b i $P(H) = \frac{1}{2}$

ii $P(\text{even}) = \frac{3}{6} = \frac{1}{2}$

iii $P(H \text{ and even}) = \frac{3}{12} = \frac{1}{4}$

H2, H4 and H6

c Yes, since $P(H \text{ and even}) = \frac{1}{4}$ and

$$P(H) \times P(\text{even}) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

d The two events are independent since the outcome when tossing a coin does not affect the outcome when rolling a die.

Summary

Two events are **independent** if the outcome of one event does not affect the outcome of the other event.

If A and B are two independent events, then $P(A \text{ and } B) = P(A) \times P(B)$.

Example 10

A bag contains 3 blue marbles and 1 yellow marble. Two marbles are drawn from the bag, without replacing the marble from the first draw.

a Find the probability of:

i selecting a blue marble with the first draw

ii selecting a blue marble with the second draw if the first marble was blue.

b Are the two events dependent or independent?

Solution

a i $P(\text{blue on the first draw}) = \frac{3}{4}$

ii After drawing a blue marble, there are 3 marbles left, of which 2 are blue.

$$\therefore P(\text{blue on the second draw}) = \frac{2}{3}$$

b The bag contains 2 blue marbles and 1 yellow marble for the second draw, so

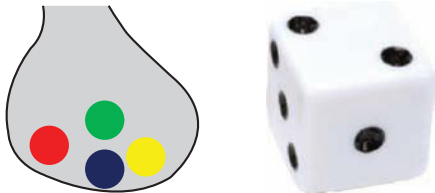
$P(\text{blue})$ changes from $\frac{3}{4}$ to $\frac{2}{3}$.

The second event is dependent on the first event.

Exercise 12-06 Dependent and independent events

- 1 State whether the following pairs of events are dependent or independent.
 - a rolling a die to obtain a 4 and rolling a second die to obtain an even number
 - b rolling a die and obtaining a 6 and tossing a coin to obtain a tail
 - c a person who trains hard and that person succeeding in sport
 - d drawing a red ball from a bag containing red and blue balls, replacing it and then drawing a blue ball from the bag
 - e electing a captain from a group of players and then electing a vice-captain from the same group
 - f tossing two coins and obtaining a head on the first coin and a head on the second coin
 - g drawing a King from a normal pack of cards, not replacing it and then drawing an Ace
- 2 In Lotto, 6 balls and 2 supplementary balls are drawn without replacement. Are the events of drawing each of the balls dependent or independent events? Give reasons.
- 3 A coin is tossed 3 times and the result is heads each time.
 - a Are each of the three coin tosses dependent or independent events?
 - b The coin is tossed a fourth time. What is the probability of obtaining a head on the fourth toss?

See Example 9

- 4 A normal die is rolled and a marble is drawn from a bag containing 1 yellow marble, 1 green marble, 1 blue marble and 1 red marble.
 
 - a Find the probability of:
 - i rolling a number less than 3 with the die
 - ii drawing a green marble from the bag.
 - b List the outcomes for rolling the die and drawing a marble from the bag.
 - c What is the probability of rolling a number less than 3 and drawing a green marble?
 - d Is $P(\text{rolling a number less than 3}) \times P(\text{drawing a green marble}) = P(\text{a number less than 3 and a green marble})$?
 - e Are the events of rolling a number less than 3 and drawing a green marble dependent or independent?
- 5 A red die and a blue die are rolled.
 - a Find the following probabilities.
 - i $P(5 \text{ on the red die})$
 - ii $P(\text{an even number on the blue die})$
 - b Are the events '5 on the red die' and 'an even number on the blue die' dependent or independent?

See Example 10

- 6 A bag contains 5 red balls and 4 yellow balls. Two balls are drawn at random without replacement.
 - a What is the probability of drawing a red ball first?
 - b What is the probability of obtaining a red ball on the second draw if the first ball was red?
 - c Are the two events dependent or independent? Give reasons.

- 7 A bag contains 5 yellow counters and 3 red counters. Two draws are made with no replacement. Find the probability of drawing:
- a
 - i a yellow counter on the first draw
 - ii a yellow counter on the second draw after a yellow counter was drawn with the first draw
 - b
 - i a red counter on the first draw
 - ii a yellow counter on the second draw after a red counter was drawn on the first draw
 - c
 - i a yellow counter on the first draw
 - ii a red counter on the second draw after a yellow counter was drawn on the first draw
 - d
 - i a red counter on the first draw
 - ii a red counter on the second draw after a red counter was drawn on the first draw.
- 8 Three children in a family are all girls. What is the probability that the next child in this family will be a girl?

Just for the record

Lotteries and Lotto

A lottery is a game of chance in which numbered tickets are drawn from tickets that have been sold. Lotteries were introduced by the State Government to raise money for hospitals. The first lottery was drawn on 20 August 1931 with a first prize of £5000.

Lotteries have been used to celebrate special events

and to help finance special projects. The Opera House Lottery, which commenced selling on 25 November 1957, was used to finance the construction of the Sydney Opera House.

Other games of chance have been introduced, including Lotto (1979), Instant Scratchies (1982), Oz Lotto (1994) and Powerball (1996).

Research the probability of winning Lotto, OZ Lotto and Powerball.



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12-07 Conditional probability

In many practical situations, events are not independent. For example, the probability of a student arriving to school on time if catching a bus may be dependent on the amount of traffic. Conditional probability is used to calculate probabilities for dependent events.

Summary

The **conditional probability** of an event B given event A, is the probability that event B occurs, given that event A has already occurred.

Worksheet

Conditional probability

MAT10SPWK10075

Puzzle sheet

Conditional probability:

Two-way tables

MAT10SPPS00018

Example 11

A bag contains 3 red marbles and 2 yellow marbles. Two marbles are drawn at random from the bag without replacement. What is the probability that the second marble is yellow, given that the first marble was also yellow?

Solution

If the first marble is yellow, there are 3 red marbles and 1 yellow marble left in the bag.

$$\therefore P(\text{second marble yellow, given the first marble is yellow}) = \frac{1}{4}$$

Example 12

Two dice are rolled and their total is calculated.

- Use a table to show all possible totals.
- Given that the total is 7, what is the probability that one of the dice shows a 3?
- Given that one of the dice shows a 4, what is the probability that the total is 10?
- Given that the total is 6, what is the probability of a double?
- Given that a double is rolled, what is the probability of:
 - a total of 12?
 - a total less than 10?

Solution

a

		2nd die					
		1	2	3	4	5	6
1st die	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

- b There are 6 outcomes that give a total of 7.

(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 2)

If one of the dice shows a three, the possible outcomes are (3, 4) and (4, 3).

$$P(\text{one die shows 3, given total} = 7) = \frac{2}{6} = \frac{1}{3}$$

- c There are 11 outcomes that have 4 showing on one of the dice.

(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6),
(1, 4), (2, 4), (3, 4), (5, 4), (6, 4)

Of these outcomes, only two have a total of 10.

(6, 4) and (4, 6)

$$P(\text{total} = 10, \text{ given one die shows 4}) = \frac{2}{11}$$

- d There are 5 outcomes that give a total of 6. (1, 5), (2, 4), (3, 3), (4, 2), (5, 1)
There is only 1 double. (3, 3)
 $P(\text{double, given total} = 6) = \frac{1}{5}$
- e There are 6 doubles that can be rolled. (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)
i (6, 6) is the only double with a total of 12.
 $P(\text{total} = 12, \text{ given double rolled}) = \frac{1}{6}$
ii (1, 1), (2, 2), (3, 3) and (4, 4) are the doubles with a total less than 10.
 $P(\text{total} < 10, \text{ given double rolled}) = \frac{4}{6} = \frac{2}{3}$

Exercise 12-07 Conditional probability

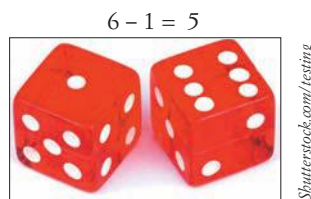
- A bag contains 4 yellow and 3 red marbles. Two marbles are drawn from the bag without replacement.
 - Given that the first marble was red, what is the probability that the second marble is also red?
 - Given that the first marble was yellow, what is the probability that the second marble is red?
- A shop takes delivery of 10 radio controlled cars, 3 of which are defective. Two cars are randomly selected and tested. What is the probability that the second car tested is defective, given that the first car tested was defective and was not replaced?
- In a chemistry class there are 5 boys and 7 girls. Two students are asked to come and help with an experiment. If the first student was a boy, what is the probability that:
 - both students chosen were boys?
 - a boy and a girl were chosen?
- In Aiden's pencil case, there are 3 red pens, 4 blue pens and 5 black pens. Aiden takes out two pens at random. What is the probability that:
 - if one pen is red, the other one is also red?
 - if one pen is red, the other pen is blue?
 - if one pen is black, the other pen is black?
 - if one pen is blue, the other pen is black?
- A die is rolled and a number less than 4 is the result. What is the probability that the number is even?
- A coin is tossed and a die is rolled at the same time. Knowing that an even number has been rolled, what is the probability of the result being a head and a 4?
- Two dice are rolled and the sum of the numbers is calculated.
 - Make up a table to show all possible sums.
 - Given that the sum is 9, find the probability that:
 - one of the dice shows a 4
 - one of the dice shows an even number.
 - Knowing that one of the dice shows a 6, find the probability that the sum is 11.
 - Given that one of the dice shows an even number, find the probability that:
 - the sum is even
 - the sum is 7.
 - If the dice show a double, what is the probability of a sum of 4?

See Example 11

See Example 12

- 8 A drawer contains 5 different pairs of coloured socks – black, blue, red, brown and white.
- Haylee randomly takes two socks from the drawer. If one of the socks is blue, what is the probability that Haylee has:
 - a matching pair?
 - two different coloured socks?
 - Haylee selects two socks at random from the drawer. Given that she selected a red sock and a blue sock, what is the probability that the third sock she selects will form a matching pair?
 - What is the maximum number of socks that Haylee will need to take from the drawer before she has at least one matching pair of socks?
- 9 Lotto in NSW is a game of chance in which 6 balls are selected at random from a barrel containing balls numbered 1 to 45. What is the chance of Cassie winning Lotto with the sixth ball, given that she has the first five numbers?
- 10 Three cards are chosen at random from a normal deck of 52 cards. Given that the three cards are hearts, what is the probability that the fourth card is also a heart?
- 11 A card is drawn at random from a deck of 52 cards. What is the probability of drawing a queen, given that the card is a heart?
- 12 Two dice are rolled. What is the probability of rolling a 6 on the second die, given that the first die is a 6?
- 13 Two dice are rolled and the difference between the numbers is calculated.
- Copy and complete the table to show all possible outcomes.

		2nd die					
		1	2	3	4	5	6
1st die	1	0					
	2					3	
	3						
	4			1			
	5						
	6						



The difference between these numbers is 5

- What is the probability of obtaining a:
 - difference of 0?
 - difference of 5?
 - difference greater than 3?
- Knowing that the difference is 4, what is the probability that:
 - one of the dice shows a 1?
 - one of the dice shows a 3?
- If one of the dice shows a 4, find the probability that:
 - the difference is 0
 - the difference is 1.
- Given that the difference is odd, find the probability that:
 - one of the dice shows a 3
 - one of the dice shows an even number.

Power plus

- 1 Students at Arunel High School are surveyed about which sport they like to watch and what type of movies they like to see. The results are shown in the table below.

	Horror/Drama	Fantasy	Comedy	Action
Football	23	34	30	48
Cricket	25	12	45	34
Tennis	8	12	32	17

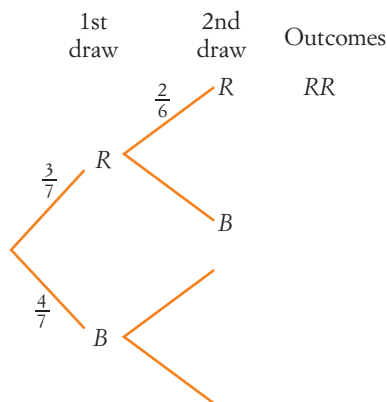
- a How many students were surveyed?
- b If a student is selected at random, what is the probability that the student likes to watch:
- i horror/drama movies? ii comedy and football?
- iii tennis, but not fantasy? iv action, but not cricket or tennis?
- c Given that a student likes to watch football, find the probability that the student also likes to watch action movies.
- d Of the students who like comedy, what is the probability that they also like to watch cricket?
- 2 A bag contains 3 red and 4 blue marbles. Two marbles are taken out of the bag without replacement.

- a A **probability tree diagram** is a tree diagram that has the probability of each step or stage listed on the branches.

Copy and complete the probability tree diagram shown on the right to show the possible outcomes.

- b Use the probability tree diagram to find the probability of drawing:

- i 2 red marbles
ii 2 blue marbles
iii a blue and a red marble
iv at least one blue marble.



- 3 The conditional probability of event A occurring, given that event B has occurred can be written as $P(A|B)$. Answer the following for the information shown in the Venn diagram.

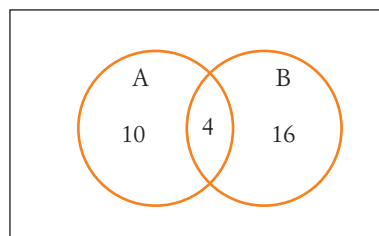
- a Find the probabilities:

- i $P(A)$ ii $P(B)$ iii $P(A \text{ and } B)$ iv $P(A|B)$ v $P(B|A)$

- b i Find the value of $\frac{P(A \text{ and } B)}{P(B)}$

- ii Is $P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$?

- c Show, by calculation, that $P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$.



Chapter 12 review

Language of maths

Puzzle sheet

Probability crossword

MAT10SPPS10076

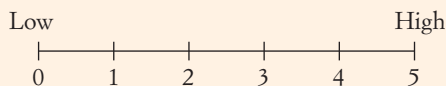
at least	compound event	conditional probability	dependent event
die/dice	event	expected frequency	experimental probability
independent event	mutually exclusive	overlapping	random
relative frequency	sample space	table	three-step experiment
theoretical probability	tree diagram	trial	two-step experiment
two-way table	Venn diagram	with replacement	without replacement

- 1 What is the meaning of **expected frequency**?
- 2 What term from the above list is another name for **experimental probability**?
- 3 On a **Venn diagram**, what does the rectangle represent?
- 4 Give an example of **dependent events**.
- 5 When are **tree diagrams** used in probability?
- 6 For two events A and B, what is the difference between '**A or B**' and '**A and B**'?

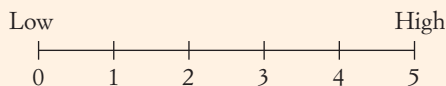
Topic overview

Rate your understanding of and your ability to do the work in this chapter by copying and completing the following scales. Circle your rating on each scale

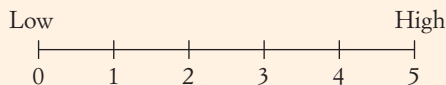
- a Calculate relative frequencies to estimate probabilities of events.



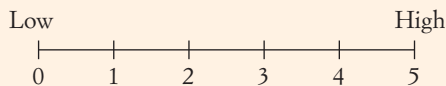
- b Represent events in Venn diagrams and two-way tables and solve related problems involving probabilities.



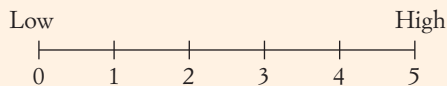
- c List all outcomes for two- and three-step chance experiments, with and without replacement, using tree diagrams or tables, and determine the probabilities of events.



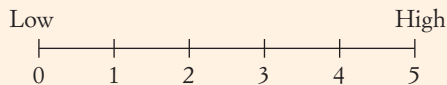
- d Calculate the probabilities of compound events.



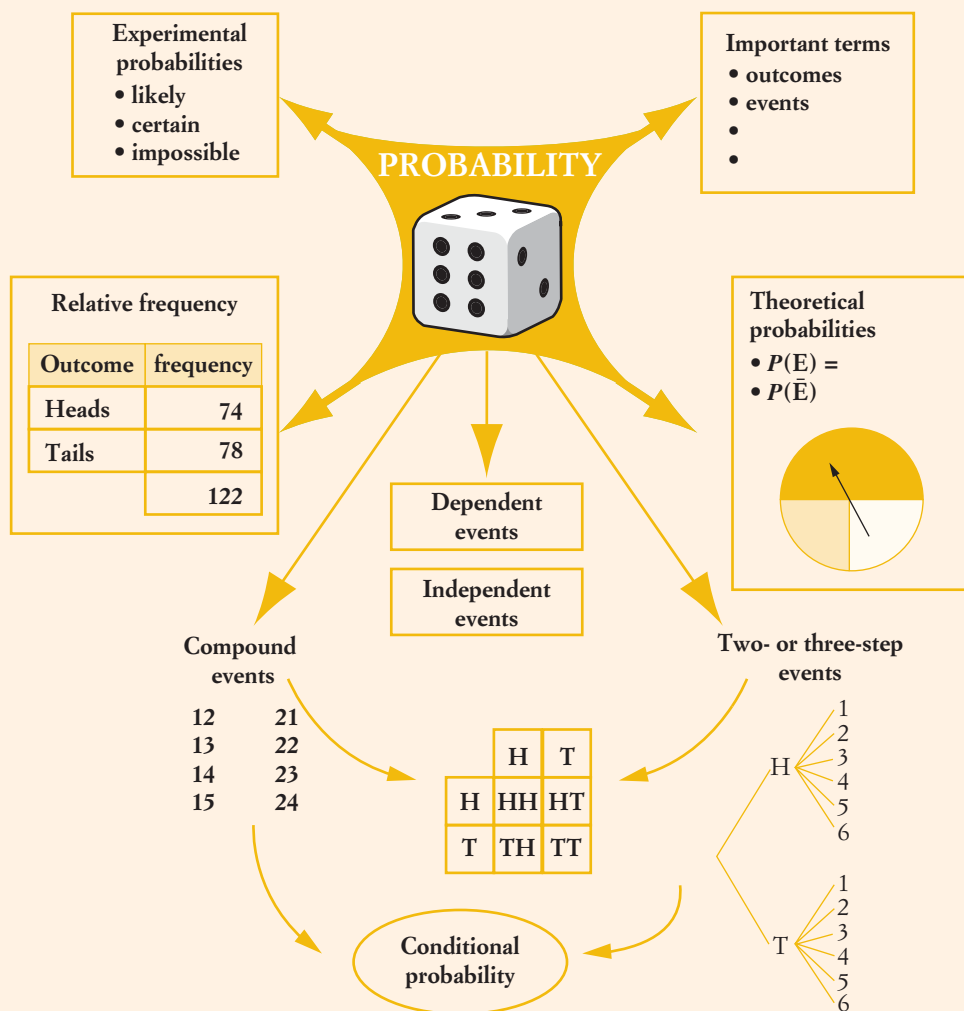
- e Understand the difference between dependent and independent events.



- f Calculate conditional probabilities.



Copy and complete this mind map of the topic, adding detail to its branches and using pictures, symbols and colour where needed. Ask your teacher to check your work.



Chapter 12 revision

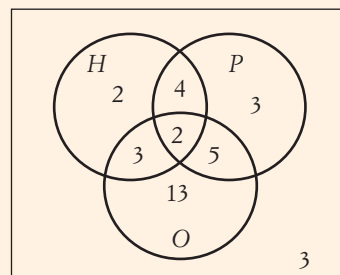
See Exercise 12-01

- 1 Three coins are tossed 150 times and the number of heads at each trial is recorded in the table.
 - a Find the relative frequency (experimental probability) of tossing:
 - i one head
 - ii two heads
 - iii three heads
 - iv at least two heads.
 - b Find the experimental probability of:
 - i at least one head
 - ii three tails.
 - c Are the answers in part b the same or different? Explain why.

Number of heads	Frequency
0	20
1	53
2	64
3	13

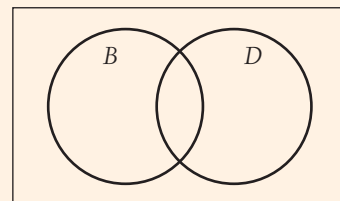
See Exercise 12-02

- 2 The Venn diagram shows the results of a survey on the types of music that school students listen to – Hip Hop/Rap (H), Pop (P) and Other (O), which includes R&B, Dance, Metal, Rock and Alternative.
 - a How many students were surveyed?
 - b Find the probability of selecting a student who likes to listen to all types of music.
 - c What is the probability of selecting a student who listens to
 - i Hip Hop/Rap and Pop?
 - ii Hip Hop/Rap or Pop?
 - iii Pop music only?
 - d Why are three students in the rectangle but not in the circles?



See Exercise 12-02

- 3 Of 20 people in a lift, 6 have blue eyes (B), 8 have dark hair (D) and 3 have blue eyes and dark hair.
 - a Copy and complete the Venn diagram to show the given information.
 - b What is the probability of selecting a person at random from the lift who has:
 - i blue eyes only?
 - ii dark hair?
 - iii blue eyes and dark hair?
 - iv hair that is not dark?
 - c What is the probability of selecting a person at random who has neither blue eyes nor dark hair?



See Exercise 12-03

- 4 Students were asked what type of activities they would like to do on a camp. The results are shown in the table.

	Hiking	Rock climbing	Kayaking
Boys	25	38	47
Girls	45	23	22

- a How many students were surveyed?
- b Find the probability (as a decimal) that a student selected at random:
 - i likes rock climbing
 - ii likes kayaking and is a girl
 - iii is a girl who likes hiking
 - iv is a boy who likes rock climbing or kayaking
- c If a boy is selected at random, what is the probability that his favourite activity is hiking?
- d If a student that likes kayaking is chosen, what is the probability that the student is:
 - i a boy?
 - ii a girl?

- 5 Two 4-sided dice (numbered 1, 2, 3 and 4) are rolled.
a Copy and complete the table below to list the sample space.

See Exercise 12-04

		Second die			
		1	2	3	4
First die	1				
	2		2, 2		
	3				
	4			4, 3	

- b How many possible outcomes are there?
c Find the probability of:
i rolling one odd and one even number
ii rolling two even numbers
iii rolling at least one 3
iv rolling two numbers less than 3
v rolling a double
vi rolling two numbers so that the first number is odd.

- 6 The numbers 2, 4, and 7 are written on separate cards and placed in a bag. Three cards are drawn at random to form a three-digit number.

See Exercise 12-05



- a Make up a tree diagram to list the sample space if the cards are drawn:
i with replacement
ii without replacement.
b If the cards are drawn with replacement, find the probability of forming:
i an even number
ii a number less than 400
iii the numbers 222, 444, or 777
iv an odd number greater than 400.
c If the cards are drawn without replacement, find the probability of forming:
i an odd number
ii a number greater than 400
iii a number beginning with 7
iv a number divisible by 4.

- 7 State whether the following pairs of events are dependent or independent.

See Exercise 12-06

- a obtaining a tail from tossing a coin and then tossing the coin again and obtaining a head
b drawing a ticket in a raffle and winning a first prize and then drawing a second ticket and winning a second prize.
c electing a president for a cricket club and then electing the vice-president of the cricket club
d a family's first three children are girls and then the fourth child is also a girl
e rolling a die and obtaining an even number and then rolling the die again and obtaining an even number again

- 8 Two 4-sided dice (with numbers 1, 2, 3 and 4) are rolled and the sum of the numbers is calculated.

See Exercise 12-07

- a Draw up a table to show all possible sums.
b Given that the sum is 5, find the probability that:
i one of the dice shows a 2
ii one of the dice shows an odd number.
c Knowing that one of the dice shows a 4, find the probability that the sum is 7.
d Given that one of the dice shows an even number, find the probability that:
i the sum is even
ii the sum is 5.
e If the dice show a double, what is the probability of a sum of 2?



13

Measurement and geometry

Geometry

The word 'geometry' comes from the Greek word *geometria* which means 'land measuring'. The principles and ideas of geometry are evident in many aspects of our lives. For example, geometry can be seen in the design of buildings, bridges, roads and transport networks.



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Chapter outline

	Proficiency strands			
13-01 Angle sum of a polygon	U	F	R	C
13-02 Congruent triangle proofs	U	F	PS	R C
13-03 Tests for quadrilaterals*	U	F	PS	R C
13-04 Proving properties of triangles and quadrilaterals	U	F	PS	R C
13-05 Formal geometrical proofs*	U	F	PS	R C
13-06 Similar figures	U	F		R C
13-07 Finding unknown sides in similar figures	U	F		R C
13-08 Tests for similar triangles	U	F	PS	R C
13-09 Similar triangle proofs*	U	F	PS	R C

*STAGE 5.3

Wordbank

congruence test One of four tests for proving that triangles are congruent: SSS, SAS, AAS and RHS

congruent Identical; exactly the same (symbol: $\equiv$)

included angle The angle between two given sides of a shape

quadrilateral test A property of a quadrilateral that proves that it is a particular type of quadrilateral, for example, if its opposite angles are equal, then it *must* be a parallelogram

rectangle A parallelogram with a right angle

regular polygon A polygon with all angles equal and all sides equal, such as an equilateral triangle or a square

similar To have the same shape but not necessarily the same size, an enlargement or reduction (symbol: $\parallel$)

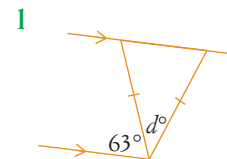
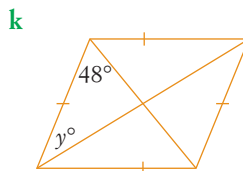
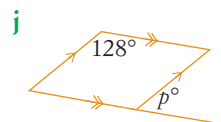
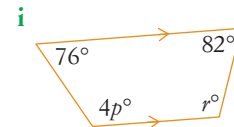
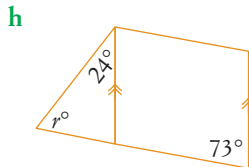
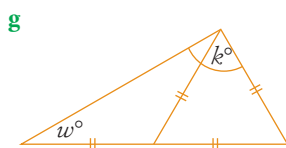
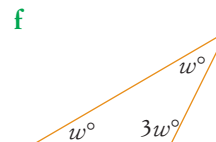
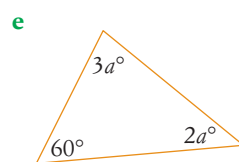
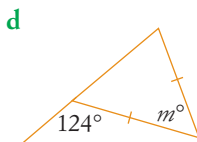
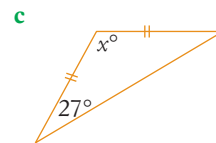
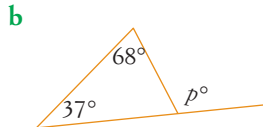
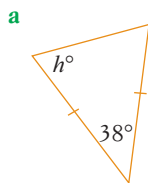
similarity test One of four tests for proving that triangles are similar: 'SSS', 'SAS', 'AA' and 'RHS'

In this chapter you will:

- formulate proofs involving congruent triangles and angle properties
- solve problems using ratio and scale factors in similar figures
- (STAGE 5.3) apply logical reasoning, including the use of congruence and similarity, to proofs and numerical exercises involving plane shapes 
- solve problems involving the angle sum of a polygon and the exterior angle sum of a convex polygon
- write formal proofs for congruent triangles
- (STAGE 5.3) understand and apply the definitions and tests for the special quadrilaterals
- prove properties of triangles and quadrilaterals using congruent triangles
- explain similarity and investigate the properties of similar figures
- identify and use the four tests for similar triangles
- (STAGE 5.3) write formal proofs of the similarity of triangles

SkillCheck

1 Find the value of each pronumeral.



Worksheet

StartUp assignment 11

MAT10MGWK10077

Puzzle sheet

Finding angles

MAT10MGPS00026

Video tutorial

Geometry

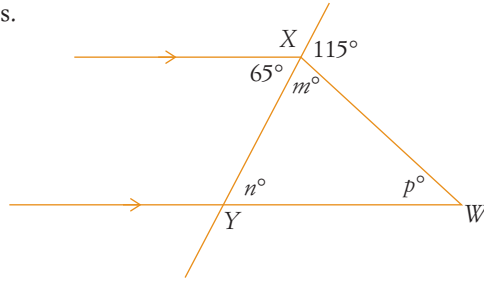
MAT10MGVT00008

Skillsheet

Starting Geometer's
SketchPad

MAT10MGSS10013

- 2 a Find the value of m , n and p , giving reasons.
b What type of triangle is $\triangle XYW$?



13-01 Angle sum of a polygon

NSW

Worksheet

Angles in polygons

MAT10MGWK10078

Technology

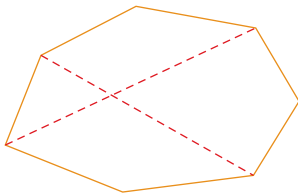
GeoGebra: Naming polygons

MAT10MGTC00008

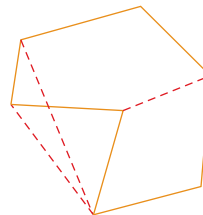


Alamy/Raymond Warren

A **polygon** is any shape with straight sides. A polygon may be either **convex** or **non-convex (concave)**.



Convex polygon



Non-convex polygon

In a convex polygon, all vertices point outwards, all diagonals lie within the shape and all angles are less than 180° . In a non-convex polygon, some vertices point inwards, some diagonals lie outside the shape and some angles are more than 180° (reflex angles).

Summary

The **angle sum of a polygon** with n sides is given by the formula $A = 180(n - 2)^\circ$. This formula applies to both convex and non-convex polygons.

Example 1

Find the angle sum of a 15-sided polygon.

Solution

$$\begin{aligned}\text{Angle sum} &= 180(15 - 2)^\circ & n &= 15 \\ &= (180 \times 13)^\circ \\ &= 2340^\circ\end{aligned}$$

Example 2

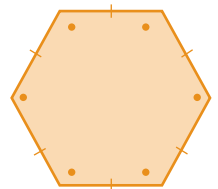
Find the number of sides in a polygon that has an angle sum of 1080° .

Solution

$$\begin{aligned}180(n - 2) &= 1080 \\ 180n - 360 &= 1080 \\ 180n &= 1440 \\ n &= \frac{1440}{180} \\ &= 8\end{aligned}$$

$\therefore$ The polygon has 8 sides (octagon).

A **regular polygon** has all its angles and sides equal. For example, a regular hexagon has 6 equal angles and 6 equal sides. A square is a regular polygon but a rhombus is not.



Summary

The size of each angle in a **regular polygon** with n sides $= \frac{\text{Angle sum}}{\text{Number of sides}} = \frac{180(n - 2)^\circ}{n}$

Example 3

Find the size of one angle in a regular pentagon.

Solution

A pentagon has 5 sides ($n = 5$).

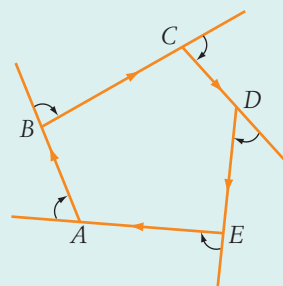
$$\begin{aligned}\text{Size of one angle} &= \frac{180(5 - 2)^\circ}{5} \\ &= \frac{(180 \times 3)^\circ}{5} \\ &= 108^\circ\end{aligned}$$

Each angle in a regular pentagon is 108° .

Exterior angle sum of a convex polygon

Summary

The **sum of the exterior angles** of a convex polygon is 360° .



Example 4

For a regular octagon, find the size of:

a each exterior angle

b each (interior) angle.

Solution

a Sum of exterior angles = 360°

$$\begin{aligned}\text{One exterior angle} &= 360^\circ \div 8 \\ &= 45^\circ\end{aligned}$$

b Each angle = $180^\circ - 45^\circ$ (angles on a straight line)
 $= 135^\circ$

$$\begin{aligned}\text{OR : Each angle} &= \frac{180(8 - 2)^\circ}{8} \\ &= 135^\circ\end{aligned}$$

Example 5

Find the number of sides in a regular polygon if:

- a** each exterior angle is 24° **b** each (interior) angle is 140° .

Solution

- a** Number of exterior angles = $360 \div 24$
= 15

$\therefore$ The regular polygon has 15 sides.

- b** Exterior angle = $180^\circ - 140^\circ$ (angles on a straight line)
 $= 40^\circ$

Sum of exterior angles = 360°

$$\begin{aligned}\text{Number of exterior angles} &= 360 \div 40 \\ &= 9\end{aligned}$$

$\therefore$ The regular polygon has 9 sides.

$$\text{OR: } \frac{180(n-2)}{n} = 140$$

$$180(n - 2) = 140n$$

$$180n - 360 = 140n$$

$$40n - 360 = 0$$

$$40n = 360$$

$$n = \frac{360}{40}$$

$$= 9$$

$\therefore$ The regular polygon has 9 sides.

Exercise 13-01 Angle sum of a polygon

- See Example 1** 1 Find the angle sum of a polygon with:
- a 12 sides b 10 sides c 9 sides d 20 sides e 15 sides
- See Example 2** 2 Find the number of sides in a polygon that has an angle sum of:
- a 720° b 3420° c 1980° d 5040° e 1260°
- 3 The angle sum of a regular polygon is 2520° .
- a How many sides does the polygon have?
- b Find the size of each angle.
- See Example 3** 4 Find the size of one angle in a regular:
- a decagon b octagon c hexagon d dodecagon
- 5 How many sides does a regular polygon have if each of its angles is:
- a 168° ? b 156° ? c 172° ? d 165.6° ?

- 6 Find the size of each exterior angle of a regular:

a pentagon
b dodecagon
c 18-sided polygon
d hexagon
- 7 Find the size of each angle in a regular:

a nonagon
b 20-sided polygon
c decagon
d 30-sided polygon
- 8 Find the number of sides in a regular polygon if each exterior angle is:

a 15°
b 72°
c 20°
d 40°
e 5°
f 12°
- 9 Find the number of sides in a regular polygon if each angle is:

a 135°
b 144°
c 156°
d 178°
e 165°
f 150°

See Example 4

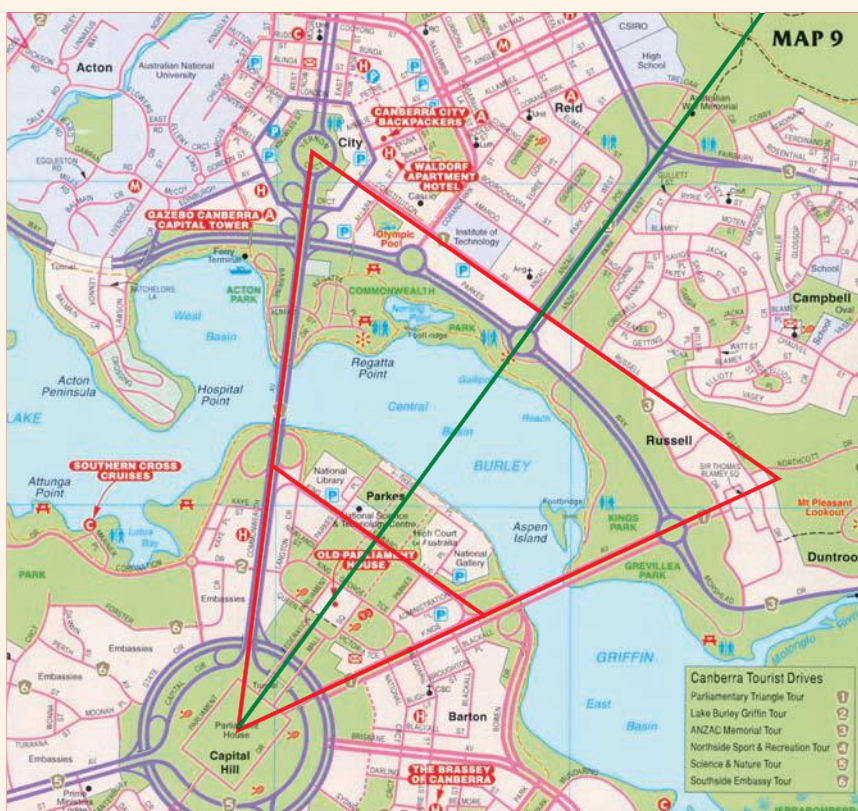
See Example 5

Just for the record

The geometry of Canberra

Canberra is located 300 km south-west of Sydney and was designed by the American architect Walter Burley Griffin. Construction of Australia's capital city began in 1913. The 'centre' of Canberra is based on an equilateral triangle, bounded by the 'sides' Commonwealth Avenue, Kings Avenue and Constitution Avenue. The smaller 'Parliamentary triangle' is bounded by Commonwealth Avenue, Kings Avenue and King Edward Terrace. The axis of symmetry of the triangle runs from Parliament House, across Lake Burley Griffin, and directly along Anzac Parade to the Australian War Memorial.

What other geometrical features can you see in Canberra's design?



Worksheet

Congruent triangles
proofs

MAT10MGWK10079

Congruent figures are identical in shape and size. **Matching sides** are equal, and **matching angles** are equal.

Worksheet

Congruent triangles

MAT10MGWK00022

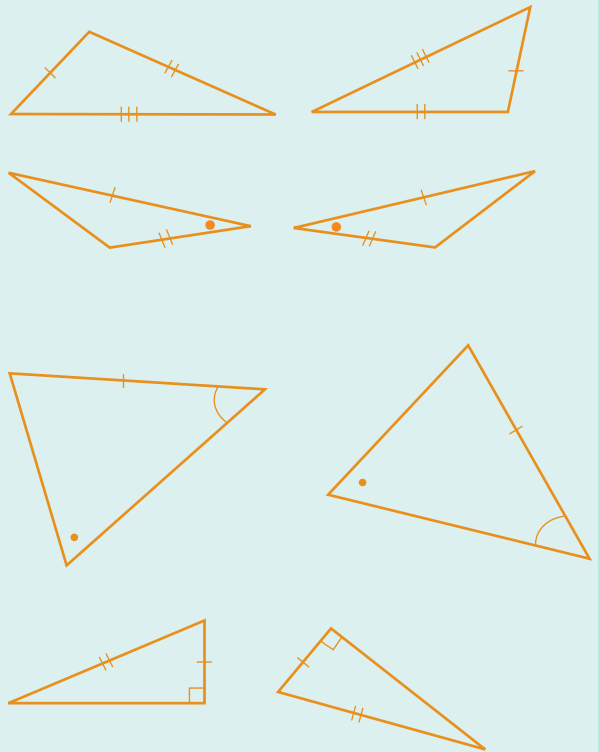
13-02 Congruent triangle proofs

Summary

There are four **congruence tests** for triangles: **SSS**, **SAS**, **AAS** or **RHS**.

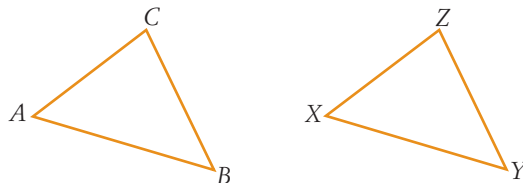
Two triangles are congruent if:

- the three sides of one triangle are respectively equal to the three sides of the other triangle (SSS rule)
- two sides and the **included angle** of one triangle are respectively equal to two sides and the **included angle** of the other triangle (SAS rule)
- two angles and one side of one triangle are respectively equal to two angles and the matching side of the other triangle (AAS rule)
- they are right-angled and the hypotenuse and another side of one triangle are respectively equal to the hypotenuse and another side of the other triangle (RHS rule).



The congruence symbol $\equiv$

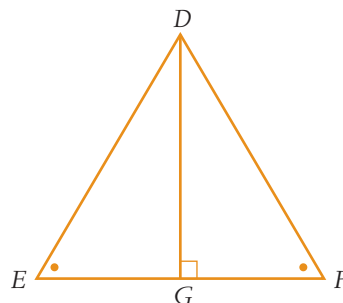
The symbol for 'is congruent to' is a special equals sign, written as ' $\equiv$ ' (which also means 'is identical to'). When using this notation, we must make sure that the vertices (angles) of the congruent figures are written in matching order. For example, ' $\triangle ABC \equiv \triangle XYZ$ ' means 'triangle ABC is congruent to triangle XYZ ' and means $\angle A = \angle X$, $\angle B = \angle Y$, $\angle C = \angle Z$.



Example 6

In this diagram, $\angle DEF = \angle DFE$ and $DG \perp EF$.

- Prove that $\triangle DEG \equiv \triangle DFG$.
- Hence show that G is the midpoint of EF .



Solution

- In $\triangle DEG$ and $\triangle DFG$:

$$\angle DEF = \angle DFE \quad (\text{given})$$

$$\angle DGE = \angle DGF = 90^\circ \quad (DG \perp EF)$$

DG is common.

$$\therefore \triangle DEG \equiv \triangle DFG \quad (\text{AAS})$$

- $\therefore EG = FG$ (matching sides of congruent triangles)
 $\therefore G$ is the midpoint of EF .

Identifying the triangles in matching order of vertices.

Stating each part of the congruence test, giving reasons.

Concluding the congruence proof, stating the test used.

Video tutorial

Congruent triangles proofs

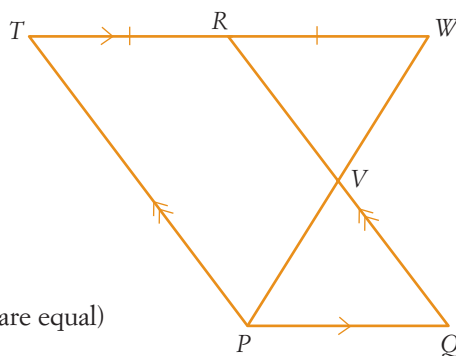
MAT10MGVT10019

Example 7

In the diagram, $PQRT$ is a parallelogram. TR is extended to W so that $TR = RW$.

Prove that:

- $\triangle PQV \equiv \triangle WRV$
- V is the midpoint of PW .



Solution

- In $\triangle PQV$ and $\triangle WRV$:

$$PQ = TR \quad (\text{opposite sides of a parallelogram are equal})$$

$$TR = RW \quad (\text{given})$$

$$\therefore PQ = RW$$

$$\angle Q = \angle WRV \quad (\text{alternate angles, } PQ \parallel TW)$$

$$\angle PVQ = \angle WVR \quad (\text{vertically opposite angles})$$

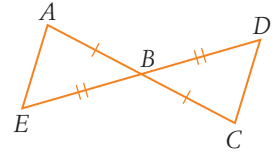
$$\therefore \triangle PQV \equiv \triangle WRV \quad (\text{AAS})$$

- $\therefore PV = WV$ (matching sides of congruent triangles)
 $\therefore V$ is the midpoint of PW .

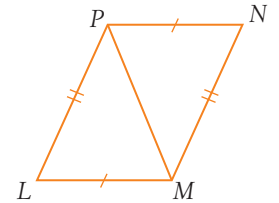
Exercise 13-02 Congruent triangle proofs

See Example 6

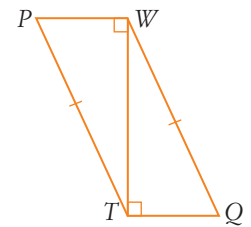
- 1 $AB = CB$ and $EB = DB$. Prove that $\triangle ABE \equiv \triangle CBD$.



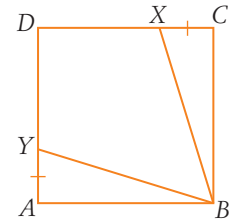
- 2 $LM = NP$ and $LP = NM$. Prove that $\triangle LMP \equiv \triangle NPM$.



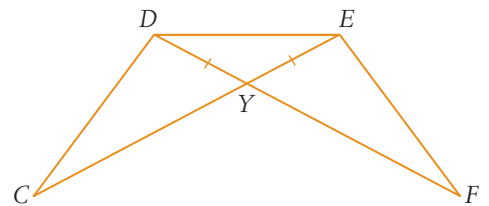
- 3 $QT \perp WT$, $PW \perp WT$ and $QW = PT$.
 Prove that $\triangle QTW \equiv \triangle PWT$.



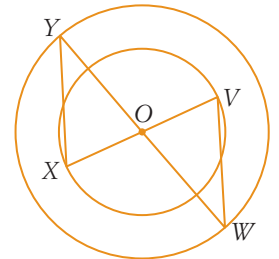
- 4 $ABCD$ is a square and $AY = CX$. Prove that $\triangle ABY \equiv \triangle CBX$.



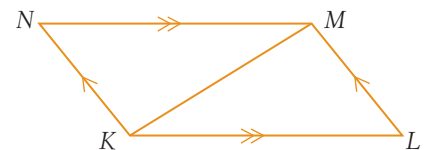
- 5 If $\angle CDE = \angle FED$ and $DY = EY$ in the diagram, prove that $\triangle CDE \equiv \triangle FED$.



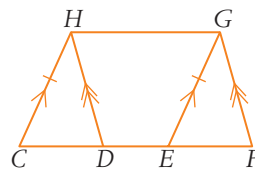
- 6 O is the centre of both circles. Prove that $\triangle XOY \equiv \triangle VOW$.



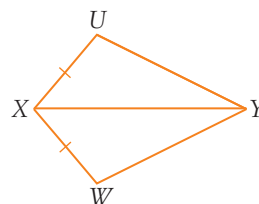
- 7 $KLMN$ is a quadrilateral with opposite sides parallel. Prove that $\triangle KLM \equiv \triangle MNK$.



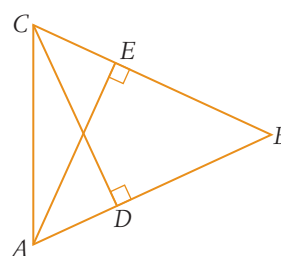
- 8 $CH \parallel EG$, $DH \parallel FG$ and $CH = EG$. Prove that $\triangle CDH \equiv \triangle EFG$.



- 9 YX bisects $\angle UXW$ and $UX = WX$. Prove that $\triangle UXY \equiv \triangle WXY$.

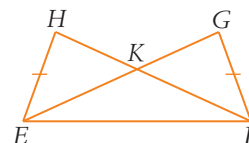


- 10 $\triangle ABC$ is isosceles with $BA = BC$. $AE \perp BC$ and $CD \perp BA$. Prove that $\triangle ABE \equiv \triangle CBD$.



- 11 $\angle HEF = \angle GFE$ and $EH = FG$. Prove that:

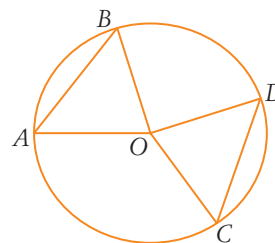
- a $\triangle HEF \equiv \triangle GFE$ b $\angle EHF = \angle FGE$.



See Example 7

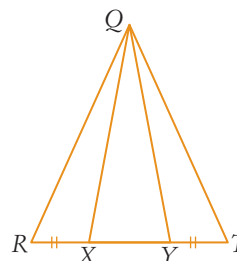
- 12 O is the centre of the circle and $AB = CD$. Prove that:

- a $\triangle AOB \equiv \triangle COD$ b $\angle AOB = \angle COD$.



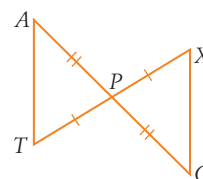
- 13 $\triangle QRT$ is isosceles with $QR = QT$. If $RX = TY$, prove that:

- a $\triangle QRX \equiv \triangle QTY$ b $\triangle QXY$ is isosceles.



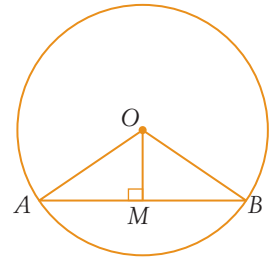
- 14 $TP = XP$ and $AP = CP$. Prove that:

- a $\triangle TAP \equiv \triangle XCP$
b $TA = XC$
c $TA \parallel XC$



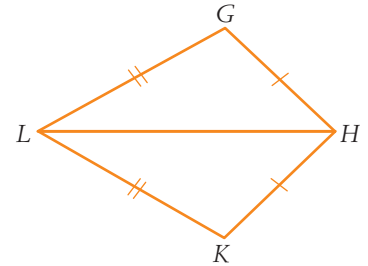
15 O is the centre of the circle and $OM \perp AB$. Prove that:

- $\triangle OAM \equiv \triangle OBM$
- OM bisects AB .



16 $GHKL$ is a kite with $GL = KL$ and $GH = KH$. Prove that:

- $\triangle GLH \equiv \triangle K LH$
- LH bisects $\angle GLK$ and $\angle GHK$.



Stage 5.3

Worksheet

Quadrilaterals: True or false?

MAT10MGWK00020

Technology

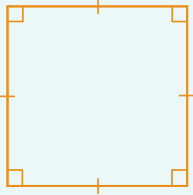
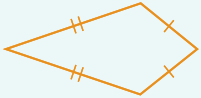
GeoGebra: Making quadrilaterals

MAT10MGTC00012

13-03 Tests for quadrilaterals

Summary

Quadrilateral	Formal definition	Other properties
Trapezium 	quadrilateral with at least one pair of opposite sides parallel	
Parallelogram 	quadrilateral with both pairs of opposite sides parallel	- opposite sides equal - opposite angles equal - diagonals bisect each other
Rhombus 	parallelogram with two adjacent sides equal in length	- all sides equal - diagonals bisect each other at right angles - diagonals bisect the angles of the rhombus
Rectangle 	parallelogram with a right angle	- opposite sides parallel/equal - all angles are right angles - diagonals are equal in length - diagonals bisect each other

Quadrilateral	Formal definition	Other properties
Square 	rectangle with two adjacent sides equal in length	- all sides equal - all angles are right angles - diagonals are equal in length - diagonals bisect each other at right angles - diagonals bisect the angles of a square
Kite 	convex quadrilateral with two pairs of equal adjacent sides	- one pair of opposite angles equal - diagonals intersect at right angles

Some properties of the special quadrilaterals can be used as minimum conditions to prove or test that a quadrilateral is a parallelogram, rectangle, square or rhombus, for example, if opposite angles are equal, then it *must* be a parallelogram.

Summary

Tests for quadrilaterals

A quadrilateral is a **parallelogram** if one of the following is true:

- both pairs of opposite angles are equal, or
- both pairs of opposite sides are equal, or
- both pairs of opposite sides are parallel, or
- one pair of opposite sides are equal and parallel, or
- the diagonals bisect each other.

A quadrilateral is a **rectangle** if one of the following is true:

- all angles are 90° , or
- diagonals are equal and bisect each other.

A quadrilateral is a **rhombus** if one of the following is true:

- all sides are equal, or
- diagonals bisect each other at right angles.

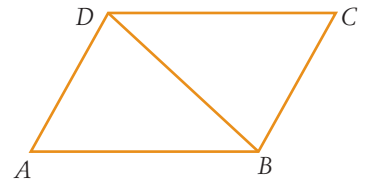
A quadrilateral is a **square** if one of the following is true:

- all sides are equal and one angle is 90° , or
- all angles are 90° and two adjacent sides are equal, or
- diagonals are equal and bisect each other at right angles.

Stage 5.3

Example 8

$ABCD$ is a quadrilateral with BD as a diagonal.
Prove that if the opposite sides of $ABCD$ are equal,
then it must be a parallelogram.



Solution

In $\triangle ABD$ and $\triangle CDB$:

$AD = CB$ (opposite sides of $ABCD$ are equal)

$AB = CD$ (opposite sides of $ABCD$ are equal)

BD is common.

$\therefore \triangle ABD \equiv \triangle CDB$ (SSS)

$\therefore \angle ABD = \angle CDB$ (matching angles of congruent triangles)

$\therefore AB \parallel CD$ (alternate angles are equal)

Also, $\angle ADB = \angle CBD$ (matching angles of congruent triangles)

$\therefore AD \parallel CB$ (alternate angles are equal)

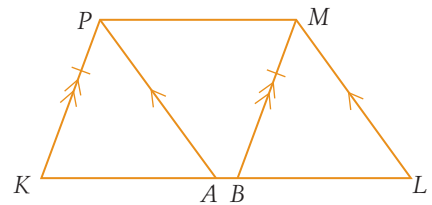
$\therefore ABCD$ is a parallelogram (opposite sides are parallel)

Example 9

In the diagram, $KP \parallel BM$, $AP \parallel LM$ and $KP = BM$.
Prove that:

a $\triangle KAP \equiv \triangle BLM$

b $ALMP$ is a parallelogram.



Solution

a In $\triangle KAP$ and $\triangle BLM$:

$\angle PKA = \angle MBL$ (corresponding angles, $KP \parallel BM$)

$\angle KAP = \angle BLM$ (corresponding angles, $AP \parallel LM$)

$KP = BM$ (given)

$\therefore \triangle KAP \equiv \triangle BLM$ (AAS)

b $\therefore AP = LM$ (matching sides of congruent triangles)

$AP \parallel LM$ (given)

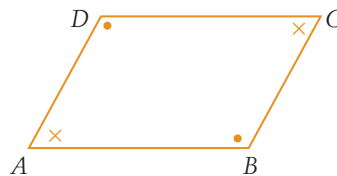
$\therefore ALMP$ is a parallelogram (one pair of opposite sides are parallel and equal)

Exercise 13-03 Tests for quadrilaterals

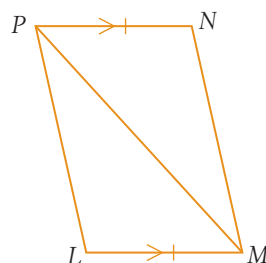
Stage 5.3

See Example 8

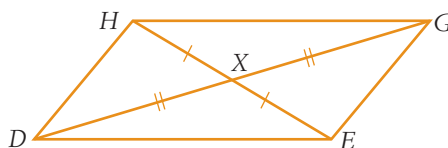
- 1 $ABCD$ is a quadrilateral in which opposite angles are equal. Prove that $ABCD$ must be a parallelogram.



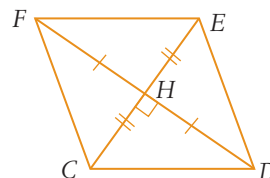
- 2 $LMNP$ is a quadrilateral in which $LM = NP$ and $LM \parallel NP$. Prove that, if a pair of opposite sides in a quadrilateral are equal and parallel, then the quadrilateral must be a parallelogram.



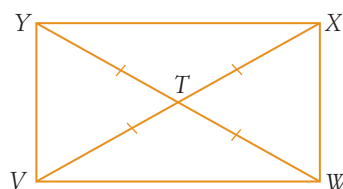
- 3 $DEGH$ is a quadrilateral whose diagonals DG and EH bisect each other. Prove that it must be a parallelogram.



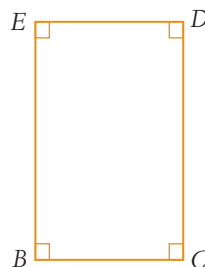
- 4 $CDEF$ is a quadrilateral whose diagonals CE and DF bisect each other at right angles. Prove that $CDEF$ must be a rhombus.



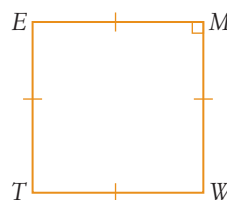
- 5 $VWXY$ is a quadrilateral whose diagonals VX and WY are equal and bisect each other. Prove that it must be a rectangle.



- 6 $BCDE$ is a quadrilateral with all its angles equal to 90° . Prove that its opposite sides are parallel and that hence it must be a rectangle.



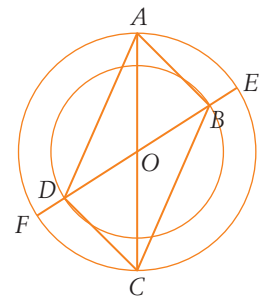
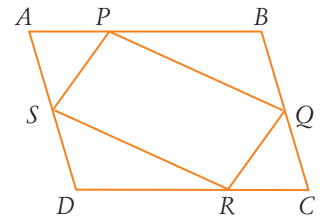
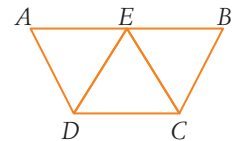
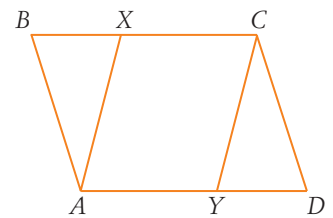
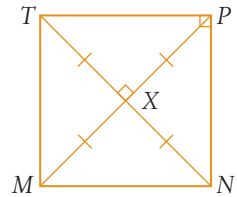
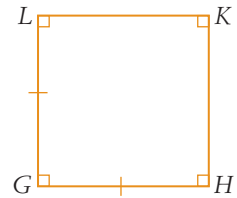
- 7 $TWME$ is a quadrilateral with all sides equal and $\angle M = 90^\circ$. Prove that the other angles are 90° as well and that hence it must be a square.



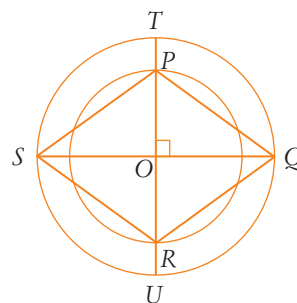
Stage 5.3

See Example 9

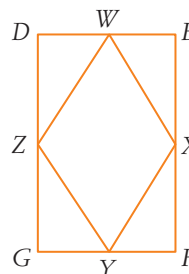
- 8 $GHKL$ is a quadrilateral with all angles equal to 90° and $GH = GL$. Prove that all sides are equal and hence that it must be a square.
- 9 $MNPT$ is a quadrilateral whose diagonals MP and NT are equal and bisect each other at right angles. Prove that $MNPT$ must be a square.
- 10 $ABCD$ is a parallelogram and $BX = DY$. Prove that:
 a $\triangle ABX \equiv \triangle CDY$
 b $AXCY$ is a parallelogram.
- 11 $AECD$ is a rhombus and $AE = EB$. Prove that:
 a $\triangle DAE \equiv \triangle CEB$
 b $BCDE$ is a parallelogram.
- 12 $ABCD$ is a parallelogram and $AP = AS = CQ = CR$. Prove that:
 a $RQ = PS$ and $PQ = RS$
 b $PQRS$ is a parallelogram.
- 13 AC and DB are diameters of concentric circles with centre O . Prove that $ABCD$ is a parallelogram.



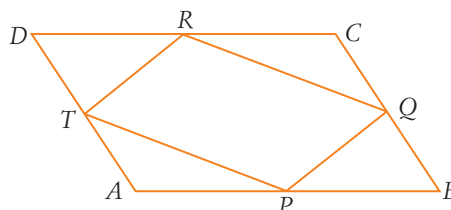
- 14 PR and SQ are diameters of concentric circles, centre O and $TU \perp SQ$. Prove that $PQRS$ is a rhombus.



- 15 $DEFG$ is a rectangle. W, X, Y and Z are the midpoints of the sides. Prove that $WXYZ$ is a rhombus.

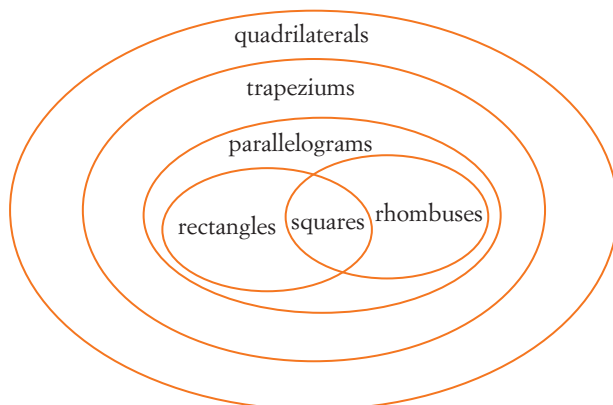


- 16 $ABCD$ is a parallelogram. P, Q, R and T are the midpoints of the sides. Prove that $PQRT$ is a parallelogram.



Investigation: Is a square a rhombus?

Using the definitions of the special quadrilaterals, we see that a parallelogram can also be classified as a trapezium since it has at least one pair of opposite sides parallel. This means that trapeziums are inclusive of parallelograms. Similarly, parallelograms are inclusive of rectangles and rectangles are inclusive of squares. This can be represented by a Venn diagram.



- Why is a rectangle a special type of parallelogram but a parallelogram is not always a rectangle? How can you use the Venn diagram to explain this?
- Where would you put kites on the Venn diagram?

Worksheet

Proving properties
of quadrilaterals

MAT10MGWK10080

Animated example

Geometric problems
and proofs

MAT10MGAE00008

13-04

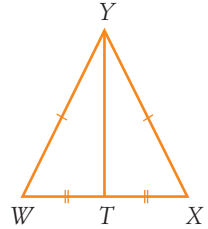
Proving properties of triangles
and quadrilaterals

The properties of triangles and quadrilaterals can be proved using the congruence tests.

Example 10

$\triangle WXY$ is an isosceles triangle with $YW = YX$. T is the midpoint of WX .

- Prove that $\triangle YTW \equiv \triangle YTX$.
- Explain why $\angle YWT = \angle YXT$.
- What geometrical result about isosceles triangles does this prove?



Solution

- For $\triangle YTW$ and $\triangle YTX$:

$$YW = YX \quad (\text{given})$$

YT is common.

$$WT = XT \quad (T \text{ is the midpoint of } WX)$$

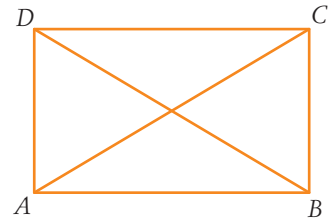
$$\therefore \triangle YTW \equiv \triangle YTX \text{ (SSS)}$$

- $\angle YWT = \angle YXT$ because they are matching angles of congruent triangles.
- The angles opposite the equal sides of an isosceles triangle are equal.

Example 11

$ABCD$ is a rectangle.

- Prove that $\triangle ABD \equiv \triangle BAC$
- Hence show that the diagonals of a rectangle are equal.



Solution

- In $\triangle ABD$ and $\triangle BAC$:

$$AD = BC \quad (\text{opposite sides of a rectangle})$$

AB is common.

$$\angle DAB = \angle CBA = 90^\circ \quad (\text{angles in a rectangle})$$

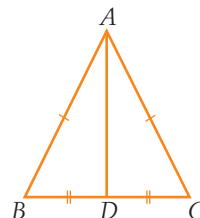
$$\therefore \triangle ABD \equiv \triangle BAC \text{ (SAS)}$$

- $BD = AC$ (matching sides of congruent triangles)
 $\therefore$ The diagonals of a rectangle are equal.

Exercise 13-04 Proving properties of triangles and quadrilaterals

- 1 $\triangle ABC$ is an isosceles triangle with $AB = AC$. D is the midpoint of BC .

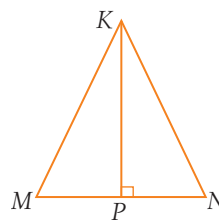
- Prove that $\triangle ABD \equiv \triangle ACD$.
- Explain why $\angle ADB = \angle ADC$.
- Hence prove that $AD \perp BC$.



See Example 10

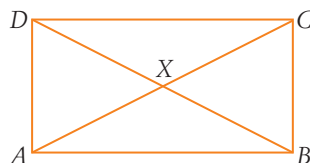
- 2 $\triangle KMN$ is an isosceles triangle in which $KM = KN$ and $KP \perp MN$.

- Prove that $\triangle KMP \equiv \triangle KNP$.
- Prove that the perpendicular from vertex K to the side MN bisects that side.



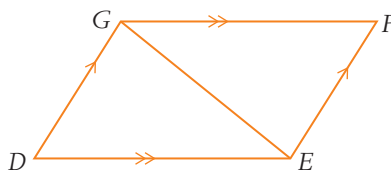
- 3 $ABCD$ is a rectangle.

- Prove that $\triangle AXB \equiv \triangle CXD$.
- Hence show that the diagonals of a rectangle bisect each other.



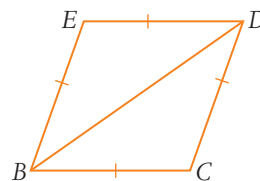
- 4 $DEFG$ is a parallelogram.

- Prove that $\triangle DEG \equiv \triangle FGE$.
- Draw the other diagonal DF and prove that $\triangle DGF \equiv \triangle FED$.
- Hence prove that the opposite angles of a parallelogram are equal.



- 5 $BCDE$ is a rhombus, so all of its sides are equal and opposite sides are parallel.

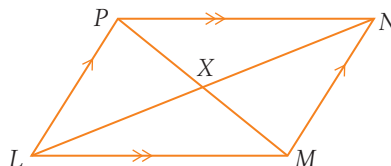
- Prove that $\triangle BED \equiv \triangle BCD$.
- Hence prove that the diagonal BD bisects $\angle EBC$ and $\angle EDC$.



See Example 11

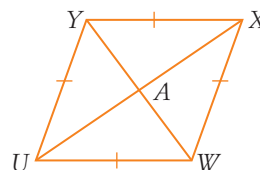
- 6 $LMNP$ is a parallelogram, so its opposite sides are parallel and equal.

- Prove that $\triangle LXM \equiv \triangle NXP$.
- Hence prove that the diagonals of a parallelogram bisect each other.

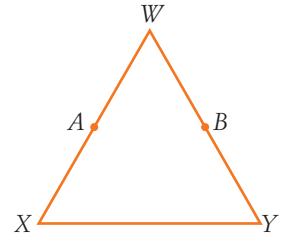
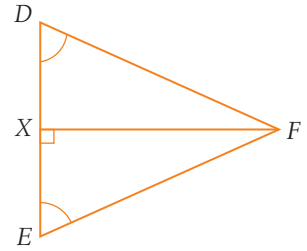


- 7 $UWXY$ is a rhombus.

- Prove that $\triangle UAW \equiv \triangle XAW$.
- Prove that $\triangle UAW \equiv \triangle UAY$.
- Hence prove that the diagonals of a rhombus bisect each other at right angles.



- 8 In $\triangle DFE$, $\angle D = \angle E$ and $FX \perp DE$.
- Prove that $\triangle DXF \equiv \triangle EXF$.
 - Hence prove that the sides opposite the equal angles in a triangle are equal.
- 9 $\triangle XYW$ is an equilateral triangle. A and B are the midpoints of the sides XW and YW respectively.
- Prove that $\triangle XBW \equiv \triangle XBY$.
 - Prove that $\triangle YAX \equiv \triangle YAW$.
 - Hence prove that $\angle X = \angle Y = \angle W = 60^\circ$.



Mental skills 13

Maths without calculators

Time before and time after

- 1 Study this example.

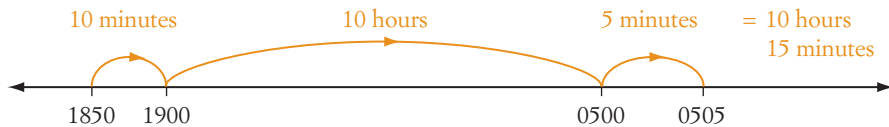
What is the time 10 hours and 15 minutes after 1850 hours?

$1850 + 10 \text{ hours} = 0450$ (next day).

Count: '1850, 1950, 2050, 2150, 2250, 2350, 0050, 0150, 0250, 0350, 0450'

$0450 + 15 \text{ min} = 0450 \text{ hours} + 10 \text{ min} + 5 \text{ min} = 0500 + 5 \text{ min} = 0505$ (next day)

OR:



- 2 Now find the time of day:

a 3 hours 20 minutes after 9:05 a.m.	b 5 hours 40 minutes after 7:30 p.m.
c 4 hours 35 minutes after 6:15 p.m.	d 11 hours 10 minutes after 11:45 a.m.
e 2 hours 45 minutes after 0325	f 7 hours 5 minutes after 1705
g 8 hours 30 minutes after 12:40 a.m.	h 4 hours 55 minutes after 10:20 p.m.
i 6 hours 25 minutes after 0435	j 2 hours 15 minutes after 2050
k 9 hours 50 minutes after 2:30 p.m.	l 3 hours 10 minutes after 8:25 a.m.

- 3 Study this example.

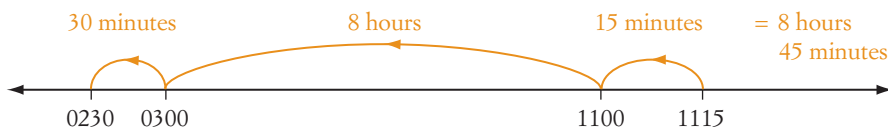
What is the time 8 hours and 45 minutes before 1115?

$1115 - 8 \text{ hours} = 0315$

Count back: '1115, 1015, 0915, 0815, 0715, 0615, 0515, 0415, 0315' (or $11 - 8 = 3$).

$0315 - 45 \text{ min} = 0315 - 15 \text{ min} - 30 \text{ min} = 0300 - 30 \text{ min} = 0230$

OR:



4 Now find the time of day:

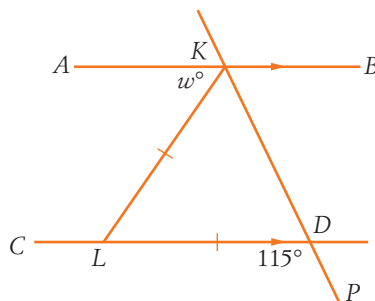
a 1 hour 15 minutes before 7:20 p.m.	b 4 hours 40 minutes before 11:20 a.m.
c 3 hours 20 minutes before 3:30 p.m.	d 5 hours 35 minutes before 8:25 a.m.
e 2 hours 10 minutes before 1455	f 3 hours 45 minutes before 0740
g 5 hours 25 minutes before 4:15 a.m.	h 9 hours 30 minutes before 9:45 p.m.
i 4 hours 20 minutes before 2005	j 2 hours 15 minutes before 0615

13-05 Formal geometrical proofs

General geometrical results can be proved by writing a geometrical argument, where reasons are given at each step of the argument. This is called **deductive geometry**.

Example 12

In the diagram, $AB \parallel CD$, $KL = DL$ and $\angle LDP = 115^\circ$.
Find the value of w , giving reasons.



Solution

$$\begin{aligned}\angle KDL &= 180^\circ - 115^\circ && \text{(angles on a straight line)} \\ &= 65^\circ\end{aligned}$$

$$\therefore \angle LKD = 65^\circ \quad \text{(equal angles of isosceles } \triangle KLD)$$

$$\angle AKD = 115^\circ \quad \text{(corresponding angles, } AB \parallel CD)$$

$$\begin{aligned}\therefore w &= 115^\circ - 65^\circ \\ &= 50^\circ\end{aligned}$$

Stage 5.3

Puzzle sheet

Geometrical proofs
order activity

MAT10MGPS10081

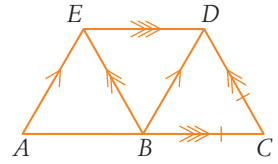
Stage 5.3

Example 13

In the diagram, $AC \parallel ED$, $AE \parallel BD$, $BE \parallel CD$ and $CB = CD$.
Prove that $\triangle ABE$ is an isosceles triangle.

Solution

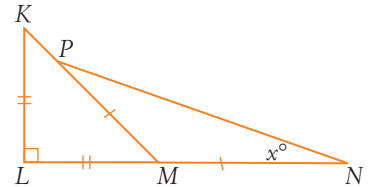
$\angle EBD = \angle CDB$ (alternate angles, $BE \parallel CD$)
and $\angle EBD = \angle AEB$ (alternate angles, $AE \parallel BD$)
 $\therefore \angle CDB = \angle AEB$
But $\angle CDB = \angle CBD$ (equal angles of isosceles $\triangle CBD$)
and $\angle BAE = \angle CBD$ (corresponding angles, $AE \parallel BD$)
 $\therefore \angle CDB = \angle BAE$
 $\therefore \angle AEB = \angle BAE$
 $\therefore \triangle ABE$ is an isosceles triangle. (two equal angles)



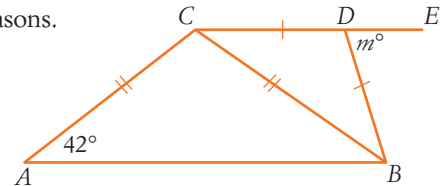
Exercise 13-05 Formal geometrical proofs

See Example 12

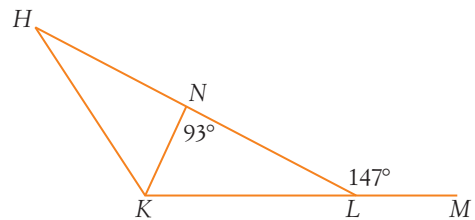
- 1 $KL = ML$ and $MN = MP$. Find x , giving reasons.



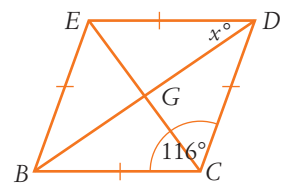
- 2 $CE \parallel AB$, $CD = BD$ and $AC = BC$. Find m , giving reasons.



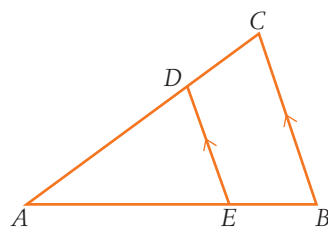
- 3 NK bisects $\angle HKL$. Find the size of $\angle NHK$, giving reasons.



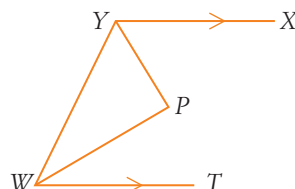
- 4 $BCDE$ is a rhombus with its diagonals intersecting at G . Find the value of x , giving reasons for each step.



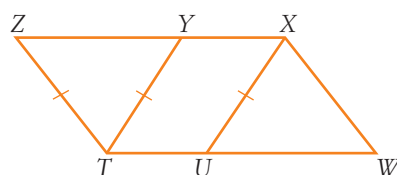
- 5 $\triangle ABC$ is an isosceles triangle, where $AB = AC$ and $BC \parallel ED$.
Prove that $\triangle ADE$ is an isosceles triangle.



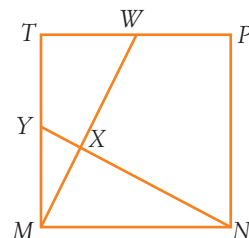
- 6 PY bisects $\angle XYW$, PW bisects $\angle TWY$ and $YX \parallel WT$.
Prove that $\angle YPW = 90^\circ$.



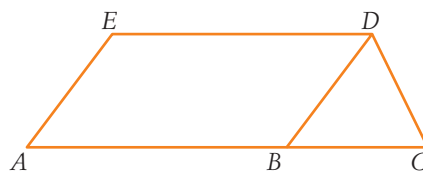
- 7 $TWXZ$ is a parallelogram and $TZ = TY = UX$.
a Prove that $\triangle TZY \equiv \triangle XWU$.
b Hence prove that $TUXY$ is a parallelogram.



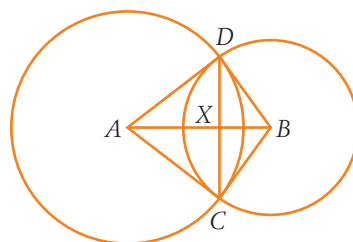
- 8 $MNPT$ is a square. W and Y are the midpoints of sides TP and MT .
a Prove that $\triangle MNY \equiv \triangle TMW$.
b Prove that $MW \perp NY$.



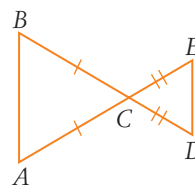
- 9 $ABDE$ is a parallelogram and $BC = BD$. Prove that $\angle AED = 2\angle BCD$.



- 10 A and B are the centres of two circles that intersect at C and D .
a Prove that $\triangle ADB \equiv \triangle ACB$.
b Hence prove that $\triangle DXB \equiv \triangle CXB$ and that $DX = CX$.

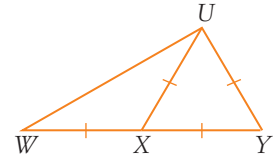


- 11 If $AC = BC$ and $DC = EC$, prove that $AB \parallel DE$.

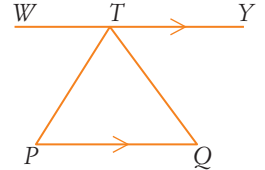


Stage 5.3

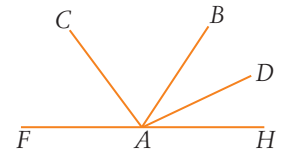
- 12 $\triangle UXY$ is an equilateral triangle and $WX = XU$. Prove that $\angle WUY$ is a right angle.



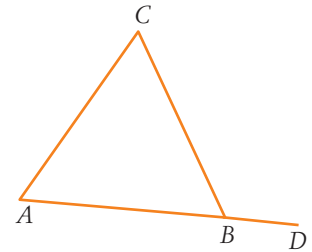
- 13 If $WY \parallel PQ$, prove that the angle sum of $\triangle PQT$ is 180° .



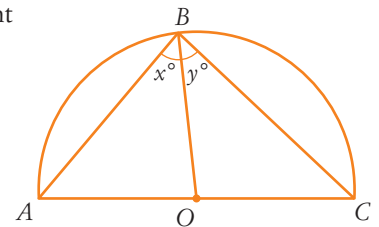
- 14 CA bisects $\angle FAB$ and DA bisects $\angle HAB$. Prove that $\angle CAD = 90^\circ$.



- 15 Prove that the exterior angle of a triangle is equal to the sum of the interior opposite angles (that is, prove that $\angle CBD = \angle CAB + \angle BCA$).



- 16 AC is the diameter of a semicircle with centre O . B is a point on the semicircle. Let $\angle ABO = x^\circ$ and $\angle CBO = y^\circ$. Prove that $\angle ABC$ is a right angle.



13-06 Similar figures

Similar figures have the same shape but are not necessarily the same size.

When a figure is enlarged or reduced, a **similar figure** is created. The original figure is called the **original**, while the enlarged or reduced figure is called the **image**.

The scale factor describes by how much a figure has been enlarged or reduced.

Summary

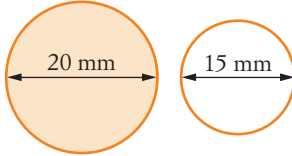
$$\text{Scale factor} = \frac{\text{image length}}{\text{original length}}$$

- If the scale factor is greater than 1, then the image is an enlargement.
- If the scale factor is between 0 and 1, then the image is a reduction.

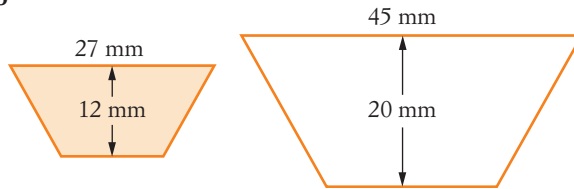
Example 14

Find the scale factor for each pair of similar figures.

a



b



Solution

$$\begin{aligned} \text{a Scale factor} &= \frac{15}{20} \\ &= \frac{3}{4} \end{aligned}$$

$$\frac{\text{Image length}}{\text{Original length}}$$

$$\begin{aligned} \text{b Scale factor} &= \frac{45}{27} \quad (\text{or } \frac{20}{12}) \\ &= \frac{5}{3} \end{aligned}$$

$$\frac{\text{Image length}}{\text{Original length}}$$

The similarity symbol |||

The symbol for 'is similar to' is '|||'. As with congruence notation, we must make sure that the vertices (angles) of the similar figures are written in matching order.

Summary

If two figures are similar, then:

- the matching angles are equal
- the matching sides are in the same ratio

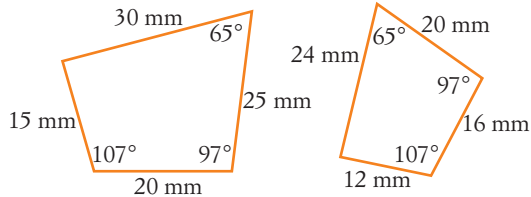


Shutterstock.com/Elizaveta Slagdy

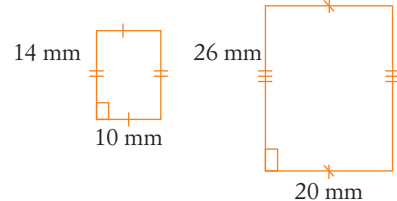
Example 15

Test whether each pair of figures are similar.

a



b



Solution

- a** For the two quadrilaterals, matching angles are equal and the ratios of matching sides are equal.

$$\frac{20}{16} = \frac{25}{20} = \frac{30}{24} = \frac{15}{12} = \frac{5}{4}$$

$\therefore$ The quadrilaterals are similar.

- b** For the two rectangles, matching angles are equal (90°) but the ratios of matching sides are not equal.

$$\frac{10}{20} = \frac{1}{2} \text{ but } \frac{14}{26} = \frac{7}{13}$$

$\therefore$ The rectangles are not similar.

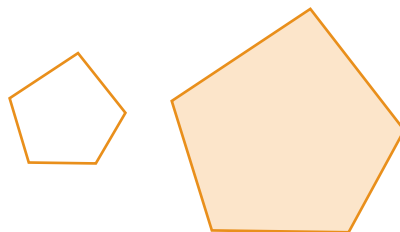
Exercise 13-06 Similar figures

- See Example 14** 1 By measurement, find the scale factor for each pair of similar figures.

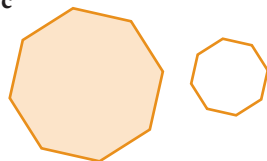
a



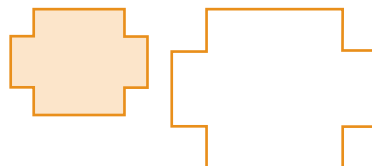
b



c

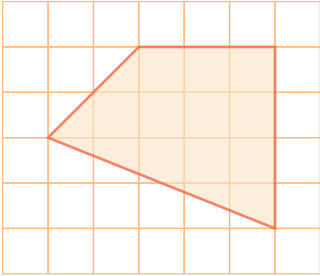


d

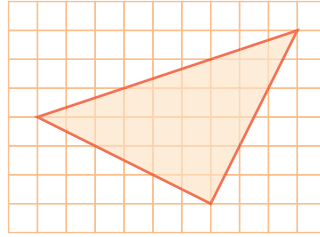


2 Copy each figure onto graph paper and draw its image using the given scale factor.

a Scale factor = 3



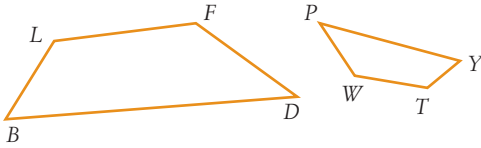
b Scale factor = $\frac{2}{3}$



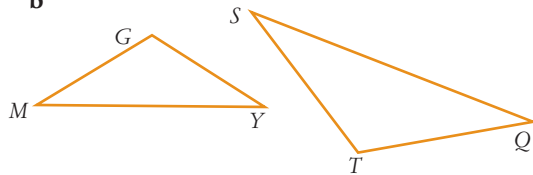
3 For each pair of similar figures:

- i list all pairs of matching angles
- ii list all pairs of matching sides
- iii use the correct notation to write a similarity statement relating them.

a



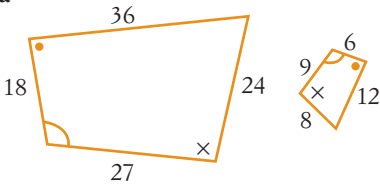
b



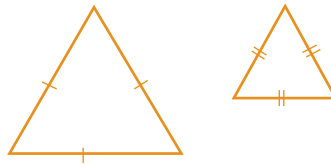
4 Test whether each pair of figures are similar.

See Example 15

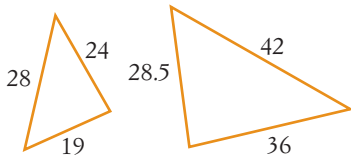
a



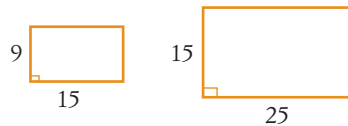
b



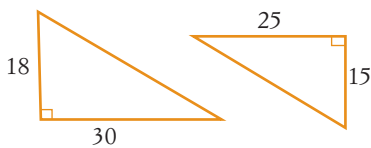
c



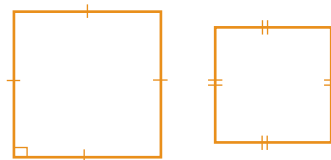
d



e



f



Skillsheet

Finding sides in similar triangles

MAT10MGSS10014

Worksheet

Finding sides in similar figures

MAT10MGWK10082

Puzzle sheet

Similar triangles

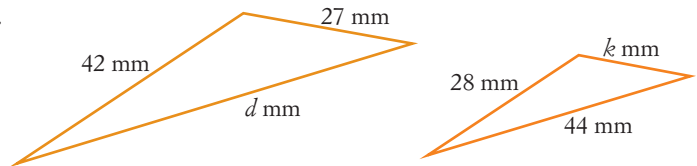
MAT10MGPS00025

13-07

Finding unknown lengths in similar figures

Example 16

The two triangles are similar.
Find the values of d and k .



Solution

Since the triangles are similar, the ratios of matching sides are equal.

$$\frac{d}{44} = \frac{28}{27}$$

$$d = \frac{28}{27} \times 44$$

$$= 66$$

$$\frac{k}{27} = \frac{28}{42}$$

$$k = \frac{28}{42} \times 27$$

$$= 18$$

OR

$$\text{Scale factor} = \frac{28}{42} = \frac{2}{3}$$

$$d = 44 \div \frac{2}{3}$$

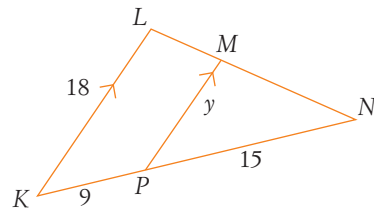
$$= 66$$

$$k = 27 \times \frac{2}{3}$$

$$= 18$$

Example 17

$\triangle KLN \parallel \triangle PMN$. Find the value of y .



Solution

$$\frac{MP}{LK} = \frac{PN}{KN}$$

$$\frac{y}{18} = \frac{15}{24}$$

$$y = \frac{15}{24} \times 18$$

$$= 11\frac{1}{4}$$

Ratios of matching sides are equal.

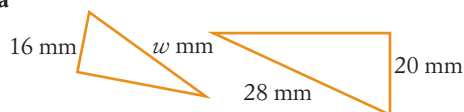
$$KN = 9 + 15 = 24$$

Exercise 13-07 Finding unknown lengths in similar figures

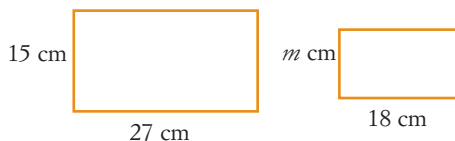
1 Find the value of every pronumeral in each pair of similar figures.

See Example 16

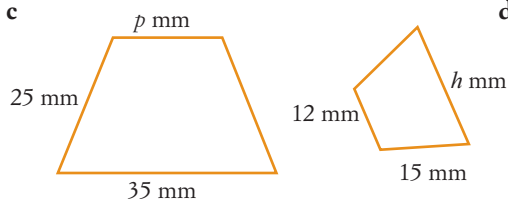
a



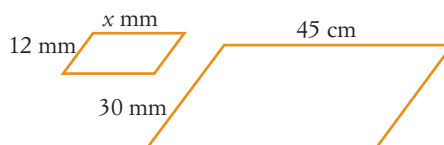
b



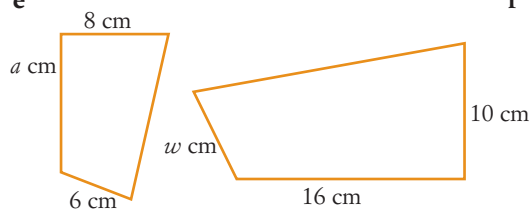
c



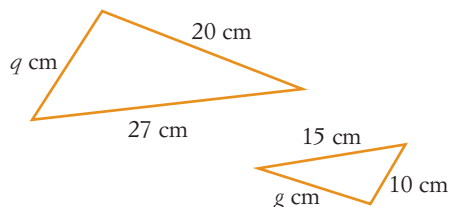
d



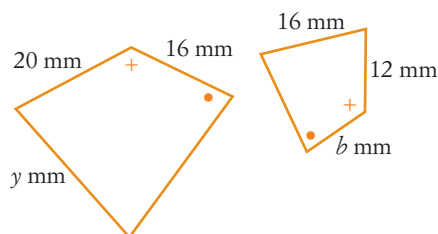
e



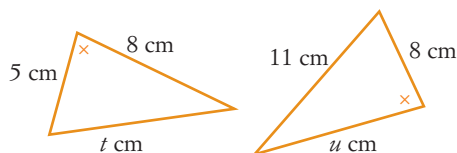
f



g

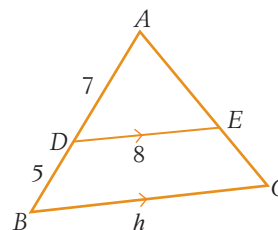


h

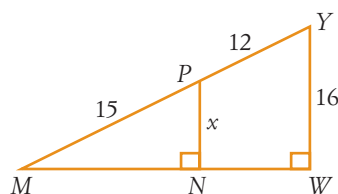


2 $\triangle ABC \sim \triangle ADE$. Find the value of h .

See Example 17



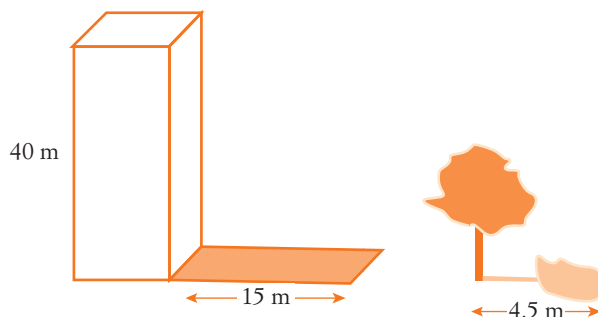
3 $\triangle MNP \sim \triangle MWY$. Find the value of x .



- 4 This photograph of the Sydney Harbour Bridge has been enlarged so that its length is 24 cm. If the dimensions of the original photo were $15\text{ cm} \times 10\text{ cm}$, what is the width of the enlargement?

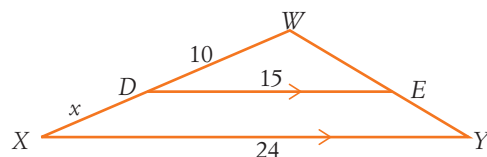


- 5 A building that is 40 m high casts a shadow 15 m long. At the same time, the shadow of a tree is 4.5 m long. What is the height of the tree?

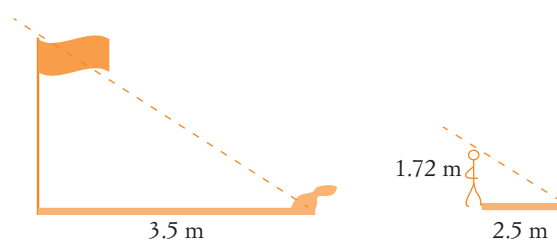


- 6 $\triangle WXY \sim \triangle WDE$. What is the value of x ?
Select the correct answer A, B, C or D.

A 4 B 6
C 8 D 10



- 7 Katrina is 1.72 m tall and casts a shadow that is 2.5 m long. At the same time, a flagpole casts a shadow that is 3.5 m long. How long is the flagpole?



- 8 Which two rectangles are similar? Select the correct answer A, B, C or D.



A M and N B K and P C M and P D K and N

- 9 A 2 m high fence casts a shadow of 1.4 m. How long is the shadow cast by a pole that is 3.2 m high at the same time?

13-08 Tests for similar triangles

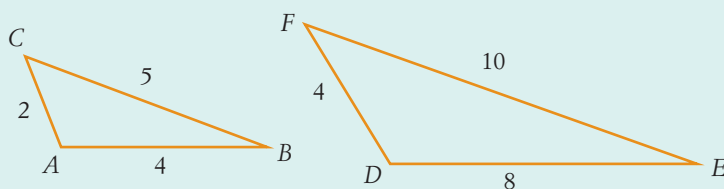
There are four sets of conditions that can be used to determine if two triangles are similar. These are called the **tests for similar triangles** or **similarity tests**.

Summary

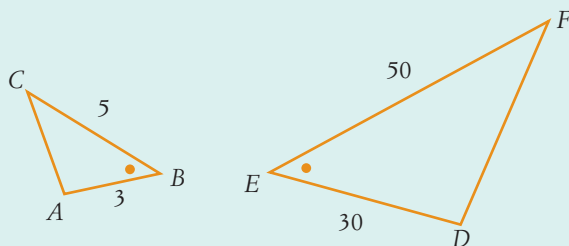
There are four tests for similar triangles.

Two triangles are similar if:

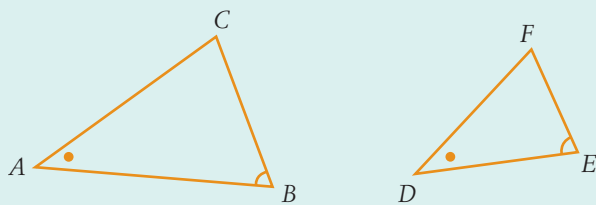
- the three sides of one triangle are proportional to the three sides of the other triangle ('SSS')



- two sides of one triangle are proportional to two sides of the other triangle, and the **included angles** are equal ('SAS')

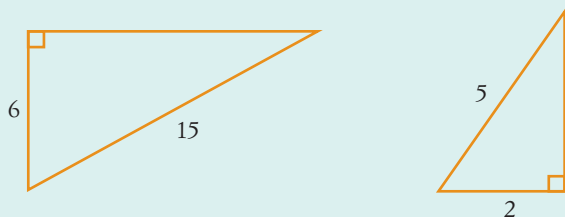


- two angles of one triangle are equal to two angles of the other triangle ('AA' or 'equiangular')



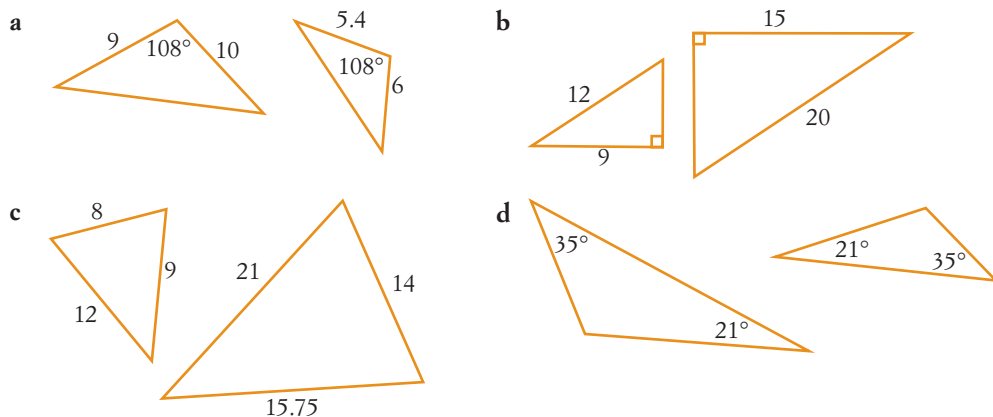
equiangular means 'equal angles'

- they are right-angled and the hypotenuse and a second side of one triangle are proportional to the hypotenuse and a second side of the other triangle ('RHS').



Example 18

Which test can be used to prove that each pair of triangles are similar?

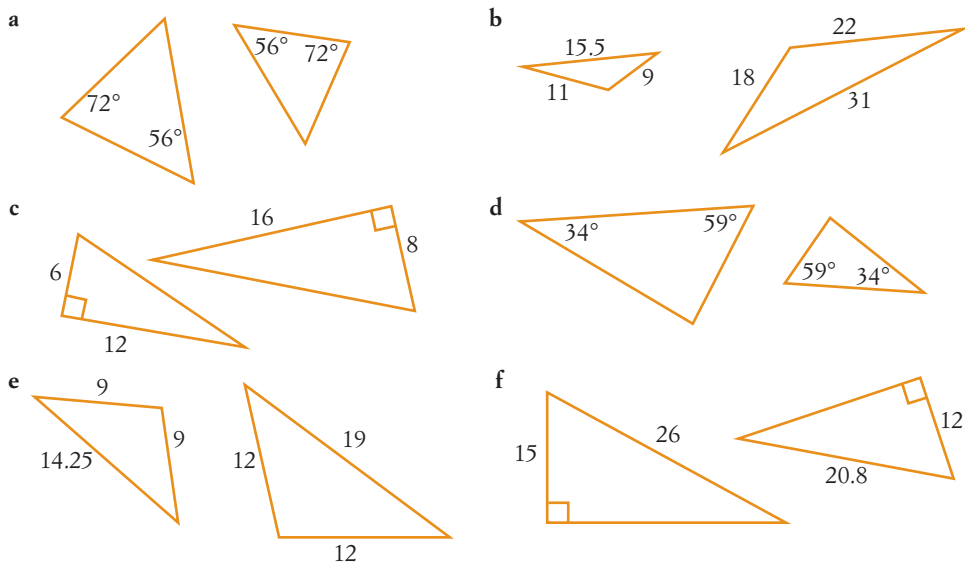


Solution

- a** Two pairs of matching sides are in the same ratio $\frac{10}{6} = \frac{9}{5.4} = \frac{5}{3}$ and the included angles in both are 108° . ('SAS')
- b** In both right-angled triangles, the pairs of hypotenuses and second sides are in the same ratio $\frac{12}{20} = \frac{9}{25} = \frac{3}{5}$. ('RHS')
- c** All three pairs of matching sides are in the same ratio $\frac{8}{21} = \frac{9}{14} = \frac{12}{15.75} = \frac{4}{7}$. ('SSS')
- d** Two pairs of angles are equal. ('AA')

Exercise 13-08 Tests for similar triangles

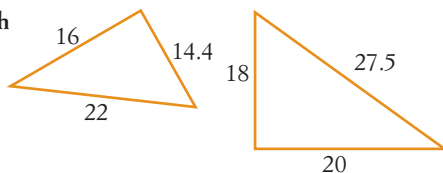
See Example 18 1 Which test can be used to prove that each pair of triangles are similar?



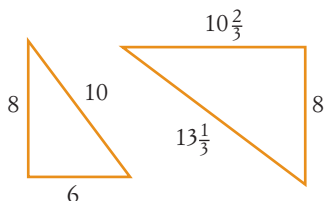
g



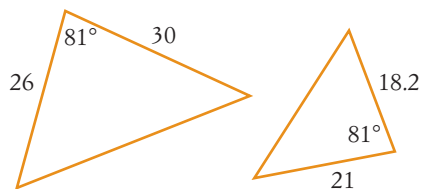
h



i

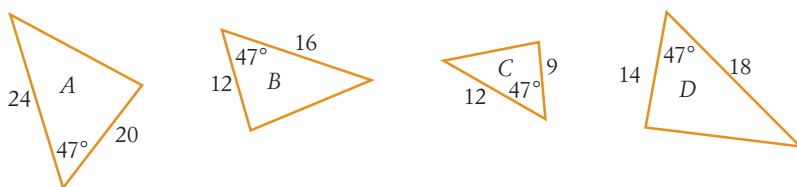


j

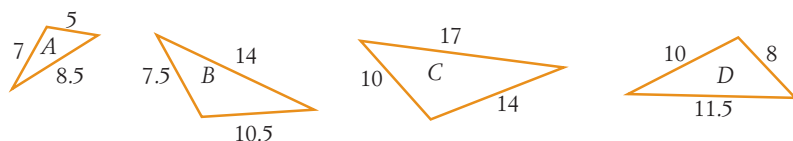


2 For each set of triangles, find the pair of similar triangles.

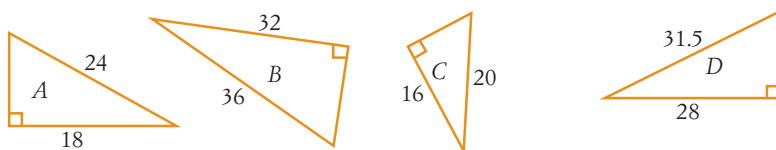
a



b

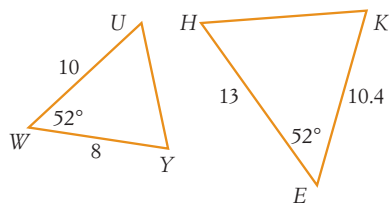


c

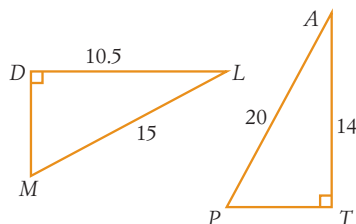


3 Use the correct notation to write a similarity statement relating each pair of similar triangles.

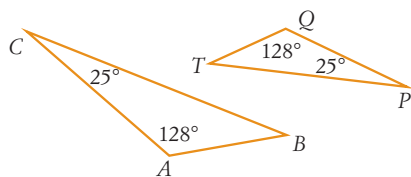
a



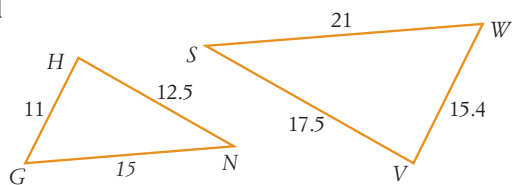
b



c



d



Stage 5.3

13-09 Similar triangle proofs

Worksheet

Congruent and similar triangle proofs

MAT10MGWK10083

Worksheet

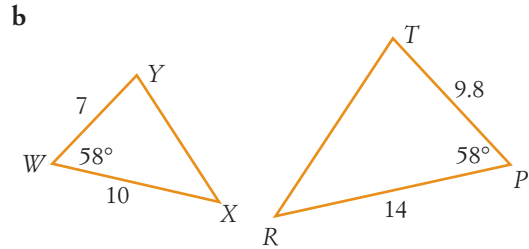
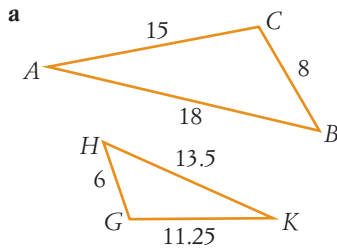
Complete the proofs

MAT10MGWK10227

To formally prove that two triangles are similar, we use a specific format that involves applying one of the four similarity tests: 'SSS', 'SAS', 'AA', 'RHS'.

Example 19

Prove that each pair of triangles are similar.



Solution

a In $\triangle ABC$ and $\triangle KHG$:

$$\frac{AB}{KH} = \frac{18}{13.5} = \frac{4}{3}$$

$$\frac{AC}{KG} = \frac{15}{11.25} = \frac{4}{3}$$

$$\frac{BC}{HG} = \frac{8}{6} = \frac{4}{3}$$

$$\frac{AB}{KH} = \frac{AC}{KG} = \frac{BC}{HG}$$

$\therefore \triangle ABC \parallel \triangle KHG$ (three pairs of matching sides are in proportion, or 'SSS')

b In $\triangle WXY$ and $\triangle RPT$:

$$\frac{WX}{PR} = \frac{10}{14} = \frac{5}{7}$$

$$\frac{WY}{PT} = \frac{7}{9.8} = \frac{5}{7}$$

$$\therefore \frac{WX}{PR} = \frac{WY}{PT}$$

$$\angle W = \angle P = 58^\circ$$

$\therefore \triangle WXY \parallel \triangle RPT$ (two pairs of matching sides are in proportion and the included angles are equal, or 'SAS')

Example 20

Prove that $\triangle ABC \parallel \triangle EDC$ and hence find the value of m .

Solution

In $\triangle ABC$ and $\triangle EDC$:

$$\angle A = \angle E \quad (\text{alternate angles, } AB \parallel DE)$$

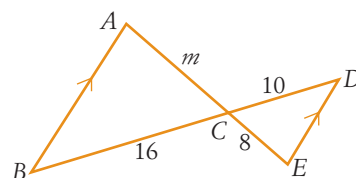
$$\angle B = \angle D \quad (\text{alternate angles, } AB \parallel DE)$$

$$\therefore \triangle ABC \parallel \triangle EDC \quad (\text{equiangular, or 'AA'})$$

$$\therefore \frac{m}{8} = \frac{16}{10} \quad (\text{matching sides in similar triangles})$$

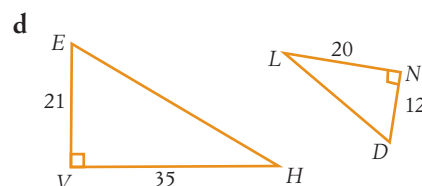
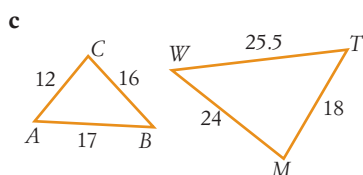
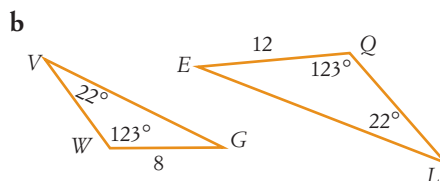
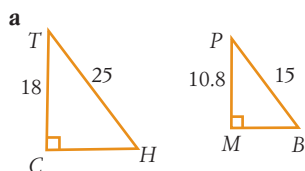
$$m = \frac{16}{10} \times 8$$

$$= 12.8$$

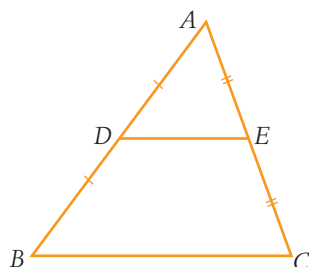


Exercise 13-09 Similar triangle proofs

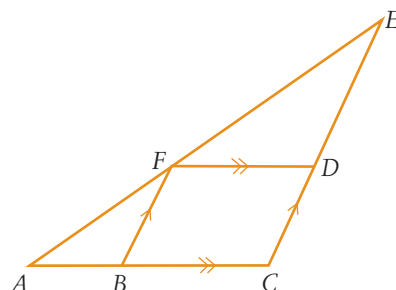
1 Prove that each pair of triangles are similar.



2 **a** D and E are the midpoints of AB and AC .
Prove that $\triangle ADE \parallel \triangle ABC$.



b $AC \parallel FD$ and $BF \parallel CE$. Prove that
 $\triangle ABF \parallel \triangle FDE$.

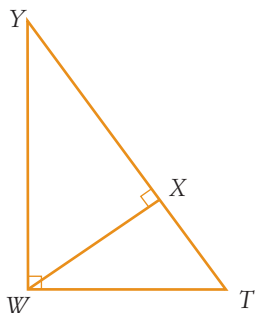


See Example 19

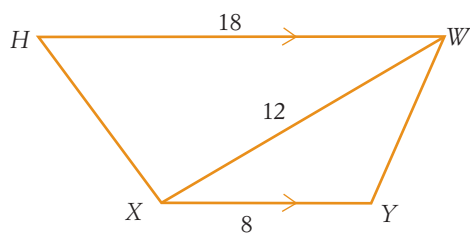
See Example 20

Stage 5.3

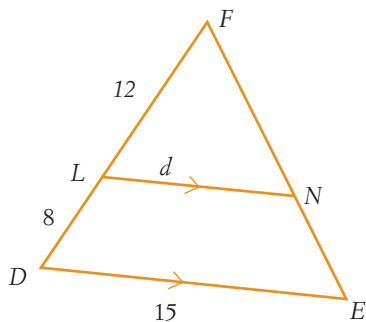
- c Prove that $\triangle WXY \parallel \triangle TXW$.



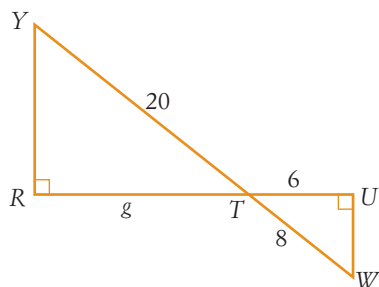
- e $HW \parallel XY$. Prove that $\triangle XWH \parallel \triangle YXW$.



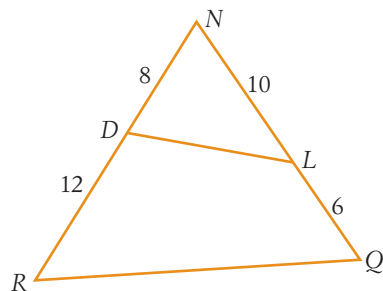
- 3 a i Prove that $\triangle FLN \parallel \triangle FDE$.
ii Find the value of d .



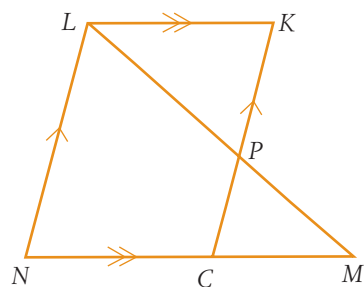
- c i Prove that $\triangle YRT \parallel \triangle WUT$.
ii Find the value of g .



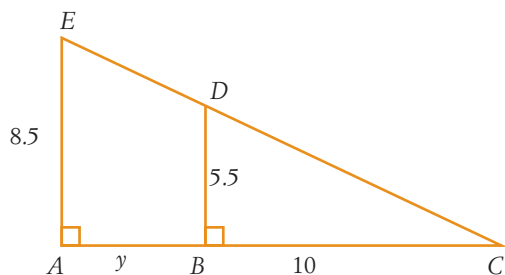
- d Prove that $\triangle NDL \parallel \triangle NQR$.



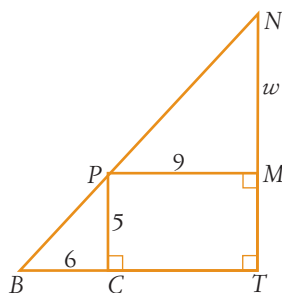
- f $NCKL$ is a parallelogram. Prove that $\triangle NML \parallel \triangle KLP$.



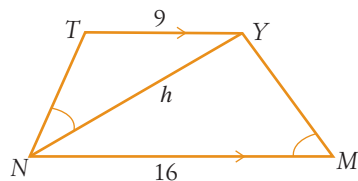
- b i Prove that $\triangle ACE \parallel \triangle BCD$.
ii Find the value of y .



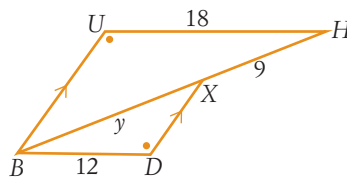
- d i Prove that $\triangle NMP \parallel \triangle PCB$.
ii Find the value of w .



- e i Prove that $\triangle TYN \parallel \triangle YNM$.
ii Find the value of h .

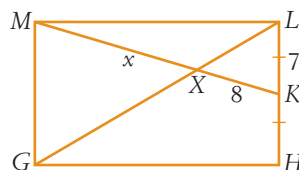


- f i Prove that $\triangle BHU \parallel \triangle XBD$.
ii Find the value of y .

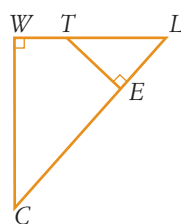


- 4 $GHLM$ is a rectangle and K is the midpoint of HL .

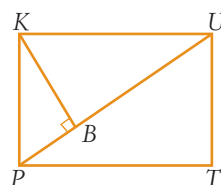
- a Prove that $\triangle MXG \parallel \triangle KXL$.
b Find the value of x .



- 5 a Prove that $\triangle CLW \parallel \triangle LTE$.
b If $WT = 5$ cm, $CE = 15$ cm and $EL = 6$ cm, find the length of TL .

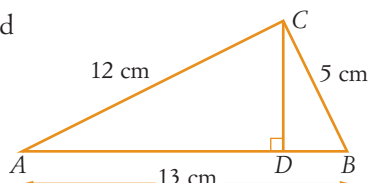


- 6 $PTUK$ is a rectangle and $KB \perp PU$.
a Prove that $\triangle PTU \parallel \triangle KBP$.
b If $BU = 21$ cm and $KP = 10$ cm, find the length of PB .



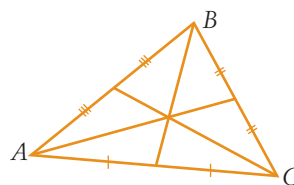
Power plus

- 1 a Prove that the three triangles ABC , ACD and CBD are similar.
b Find the length of CD .



- 2 $ABCD$ is a parallelogram. HG is any interval joining parallel sides AB and DC and passing through the midpoint of the diagonal BD . Prove that the interval through the midpoint of a diagonal in a parallelogram divides opposite sides equally (that is, prove $DG = BH$).

- 3 The median is a line joining a vertex of a triangle to the midpoint of the opposite side. Prove that the medians of a triangle are concurrent (that is, they meet at one intersection point).



Chapter 13 review

Language of maths

Puzzle sheet
Geometry crossword
MAT10MGPS10084

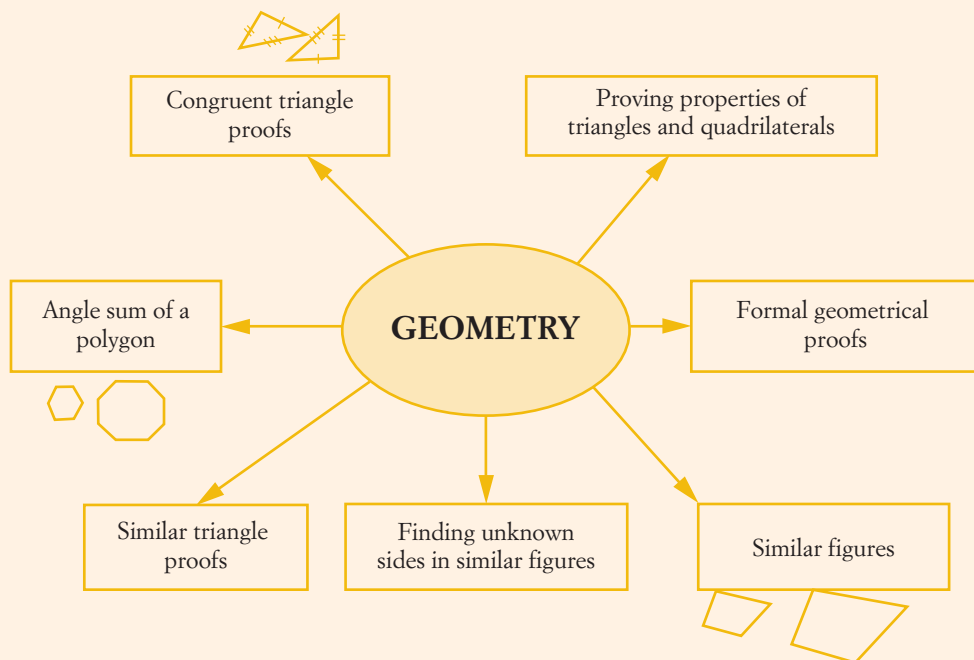
AAS	angle sum	congruence test	congruent ($\equiv$)
convex polygon	enlargement	equiangular	exterior angle
hypotenuse	image	included angle	matching
original	polygon	proof	proportional
quadrilateral test	regular polygon	RHS	SAS
scale factor	similar (III)	similarity test	SSS

- 1 What is a **convex polygon**?
- 2 Explain the difference between the **interior** and **exterior angles** of a polygon.
- 3 What is the symbol and meaning of 'is similar to'?
- 4 What happens to a figure that is changed by a scale factor of $\frac{1}{2}$?
- 5 What are the four tests for similar triangles?
- 6 What is the meaning of the 'A' in the SAS test for congruent triangles?
- 7 What does **equiangular** mean in the similarity tests?

Topic overview

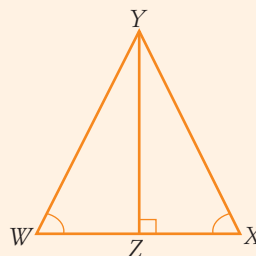
Quiz
Geometry
MAT10MGQZ00008

Copy and complete this mind map of the topic, adding detail to its branches and using pictures, symbols and colour where needed. Ask your teacher to check your work.



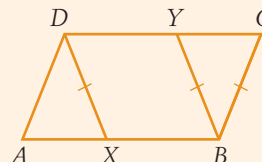
- 1 Find the size of one angle in a regular 15-sided polygon. See Exercise 13-01
- 2 The angle sum of a polygon is 6120° . How many sides does the polygon have? See Exercise 13-01
- 3 Find the number of sides in a regular polygon if each exterior angle is:
 a 10° b 24° c 45° d 15° See Exercise 13-01
- 4 Each angle in a regular polygon is 162° . How many sides does the polygon have? Select the correct answer **A**, **B**, **C** or **D**.
 A 18 B 20 C 22 D 24 See Exercise 13-01

- 5 In $\triangle WXY$, $\angle W = \angle X$ and $YZ \perp WX$. Prove that $\triangle WZY \equiv \triangle XZY$.



See Exercise 13-02

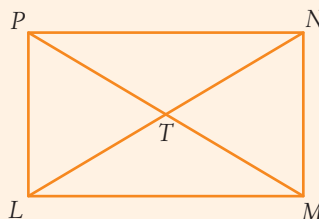
- 6 $ABCD$ is a parallelogram and $BC = BY = DX$.
 a Prove that $\triangle DAX \equiv \triangle BCY$.
 b Hence, prove that $BXDY$ is a parallelogram.



Stage 5.3

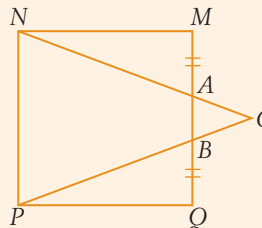
See Exercise 13-03

- 7 $PNML$ is a rectangle.
 a Prove that $\triangle PML \equiv \triangle NLM$.
 b Hence explain why $PM = NL$.
 c What geometrical result about rectangles does this prove?



See Exercise 13-04

- 8 $PNMQ$ is a square and $AM = BQ$. Prove that $\triangle NPC$ is isosceles.

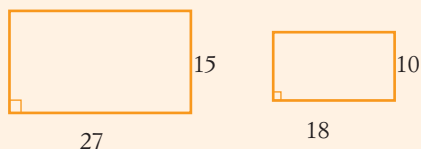


Stage 5.3

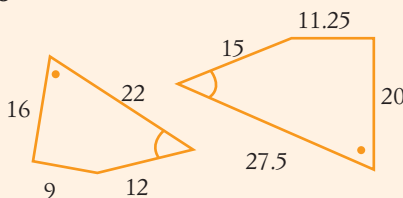
See Exercise 13-05

- 9 Test whether each pair of figures are similar.

a



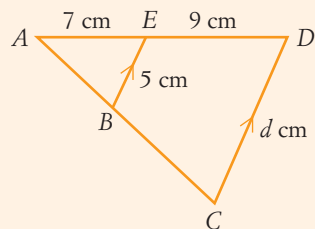
b



See Exercise 13-06

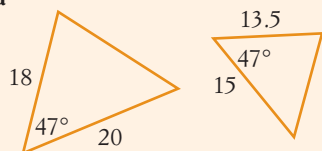
Chapter 13 revision

See Exercise 13-07 **10** If $\triangle ABE \parallel \triangle ACD$, find the value of d .

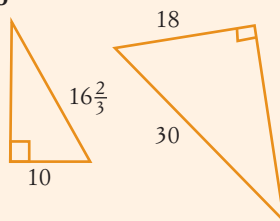


See Exercise 13-08 **11** Which test can be used to prove that each pair of triangles are similar?

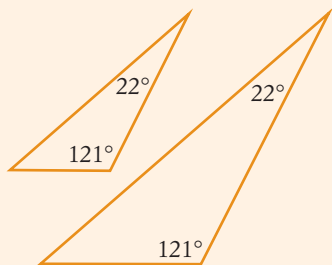
a



b



c



Stage 5.3

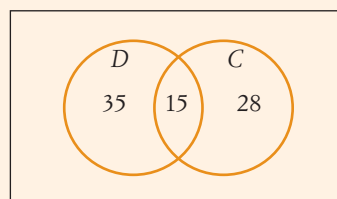
See Exercise 13-09

12 Use similar triangles to prove that the interval joining the midpoint of two sides in a triangle is parallel to the third side and is half its length.

- 1 The interior angle of regular polygon is 171° . How many sides does the polygon have?

See Exercise 13-01

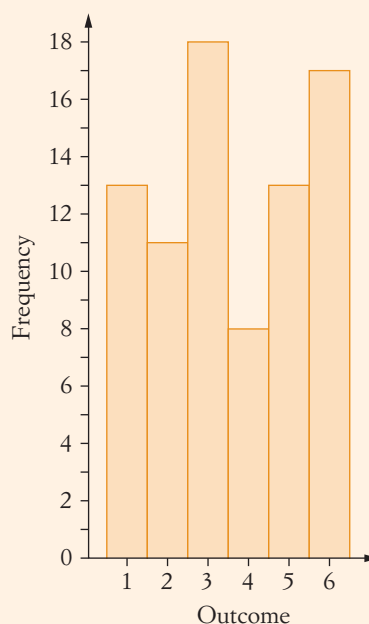
- 2 The Venn diagram shows the results of a survey asking people whether they prefer to watch movies on DVD (D) or at the cinema (C).



See Exercise 12-02

- How many people were surveyed?
- Find the probability of selecting a person who prefers to watch movies:
 - on DVD
 - only at the cinema
- What is the probability of selecting a person who only prefers to watch DVD?

- 3 A die was rolled and the frequency of each number is shown in the histogram.



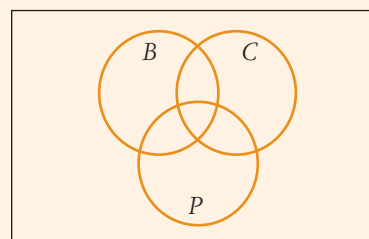
See Exercise 12-01

- How many times was the die rolled?
- Use these results to find the experimental probability of rolling:
 - a 1
 - an even number
 - a number less than 4
 - at least a 3
- What is the theoretical probability of rolling a 6? How does this compare with the experimental probability of rolling a 6?

- 4 Of 160 Year 11 students at Westvale High, 54 do Biology (B), 75 do Chemistry (C) and 68 study Physics (P). 55 students take both Chemistry and Physics, 20 study Biology and Chemistry and 10 students do all three.

See Exercise 12-02

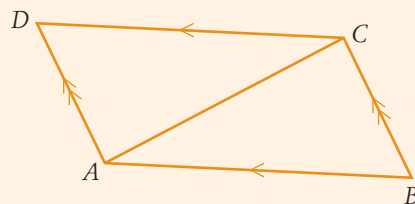
- Copy and complete the Venn diagram to show this information.
- Find the probability of selecting a student who:
 - only takes Physics
 - does not do any Science subjects
 - takes Chemistry and Physics but not Biology
 - does Chemistry or Biology
 - only takes one Science subject.
- From the students that do Biology, what is the probability that the student also studies Physics?



Mixed revision 4

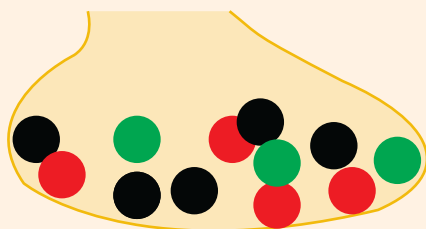
See Exercise 13-02

- 5 Prove that $\triangle ABC \equiv \triangle CDA$.



See Exercise 12-01

- 6 A bag contains 4 red, 5 black and 3 green marbles. Sadiya selects a marble at random, records its colour and then returns the marble to the bag. Sadiya repeats this process 150 times. The results are shown in the table.



Outcome	Frequency
red	58
black	65
green	27

- Find the relative frequency of selecting a marble that is:
 - red
 - black
 - green.
- What is the theoretical probability of selecting a marble that is
 - red?
 - black?
 - green?
- Are the experimental and theoretical probabilities similar?
- If the process is repeated 600 times, what is the expected frequency of selecting a red or green marble?

See Exercise 13-01

- 7 The angle sum of a regular polygon is 6120° .
- How many sides does the polygon have?
 - Find the size of each angle.

See Exercise 12-03

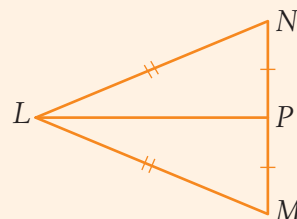
- 8 Shoppers at a mall were asked whether they had a pet dog or cat. The results are shown in the two-way table.

	Cat	No cat
Dog	28	35
No dog	32	40

- How many shoppers were surveyed?
- Find the probability of randomly selecting a shopper from the survey who does not have a cat or dog.
- What is the probability of randomly selecting a shopper who:
 - has only a dog or a cat (not both)?
 - has a cat as a pet?
 - does not have a dog?

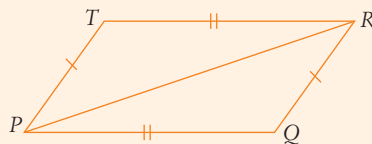
See Exercise 13-02

- 9 In the diagram, $LM = LN$ and P is the midpoint of MN . Prove that $\triangle LMP \equiv \triangle LNP$ and hence that $\angle LPM = \angle LPN = 90^\circ$.



- 10 $PQRT$ is a quadrilateral in which $PQ = RT$ and $PT = RQ$.
Prove that:

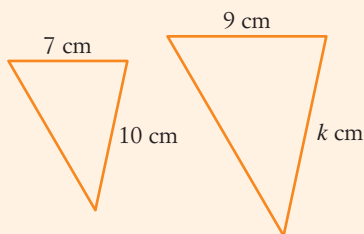
- a $\triangle PRT \cong \triangle RPQ$
b $PQRT$ is a parallelogram.



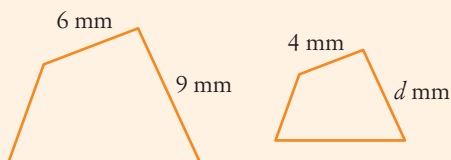
See Exercise 13-03

- 11 Find the value of the pronumeral in each pair of similar figures.

a

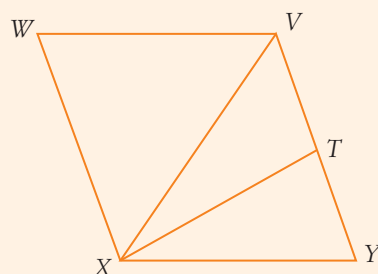


b



See Exercise 13-07

- 12 $XYVW$ is a rhombus and TX bisects $\angle VXY$.
Prove that $\angle XTY = 3 \times \angle TXY$.



Stage 5.3

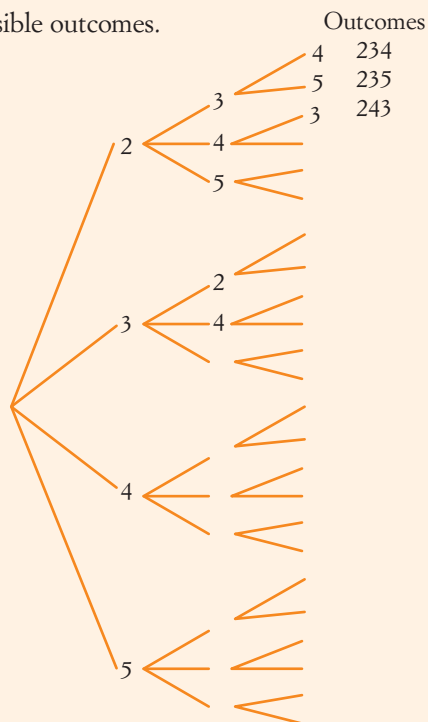
See Exercise 13-05

- 13 Three cards are drawn from a set of cards numbered 2, 3, 4 and 5, without replacement, to form a three digit number.

- a Copy and complete the tree diagram to list all possible outcomes.

- b Find the probability of forming:

- i an even number
ii a number ending in 3
iii a number greater than 400
iv a number between 200 and 500
v a number divisible by 5.



See Exercise 12-05

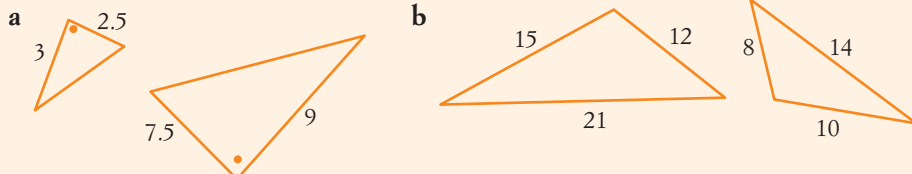
Mixed revision 4

See Exercise 13-01 **14** Find the angle sum of a polygon that has:

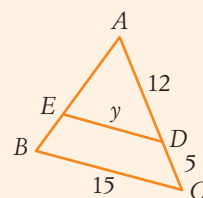
- a** 15 sides **b** 20 sides **c** 8 sides **d** 48 sides

See Exercise 13-01 **15** Find the number of sides in a regular polygon if each exterior angle is 10° .

See Exercise 13-08 **16** Which test can be used to prove that each pair of triangles is similar?



See Exercise 13-07 **17** If $\triangle ABC \parallel \triangle AED$, find the value of y (correct to one decimal place).



See Exercise 12-07 **18** Two dice are rolled and the product of the two numbers is calculated.

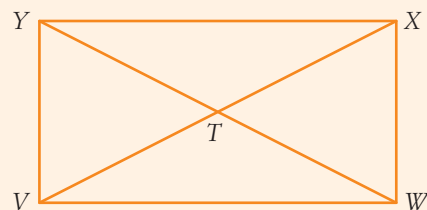
a Copy and complete the table to show all possible outcomes.

		1st die					
		1	2	3	4	5	6
2nd die	1	1					
	2				10		
	3						
	4		12				
	5						
	6						

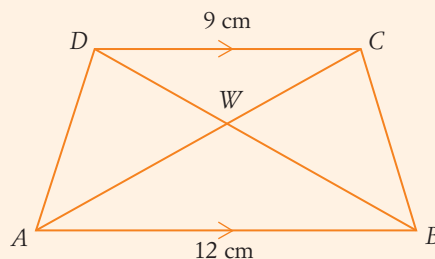
- b** Given that one of the numbers was a 3, what is the probability of obtaining a product that is odd?
- c** Given that one of the numbers is 4, what is the probability of obtaining a product:
- i** that is even? **ii** that is odd?
- d** Given that the first number is odd, find the probability of obtaining an even product.
- e** Given that an even number and an odd number are obtained, find the probability of obtaining an even product.

See Exercise 13-04 **19** $VWXY$ is a rectangle and the diagonals, VX and WY intersect at T .

- a** Prove that $\triangle YXT \equiv \triangle WVT$.
- b** Hence prove that the diagonals of a rectangle bisect each other.



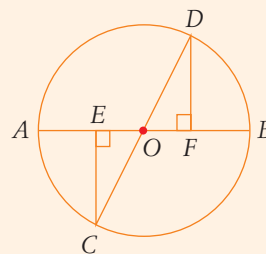
- 20** $ABCD$ is a trapezium.
- Prove that $\triangle CDW \parallel \triangle ABW$.
 - If $AC = 11$ cm, find the length of CW .



Stage 5.3

See Exercise 13-09

- 21** O is the centre of the circle shown. $CE \perp AB$ and $DF \perp AB$.
Prove that $\triangle CEO \equiv \triangle DFO$.



See Exercise 13-02

Stage 5.3

See Chapter 1
See Chapter 1

See Chapter 1

See Chapter 2

See Chapter 3

Stage 5.3

See Chapter 5

See Chapter 11

See Chapter 6

See Chapter 4

See Chapter 12

- 1 Simplify $3\sqrt{8} - \sqrt{32} + \sqrt{50}$.
- 2 Rationalise the denominator of $\frac{3\sqrt{2}}{2\sqrt{5}}$.
- 3 Expand and simplify $(3\sqrt{2} + 4\sqrt{5})^2$.
- 4 Kim is a real estate agent and is paid a commission of 2.5% on the value of the apartments she sells. She also receives a weekly retainer of \$1250. How much will Kim earn if she sells an apartment for \$578 000?

- 5 Find the gradient and y-intercept of the line with equation $4x + 5y - 10 = 0$.

- 6 Factorise each expression.

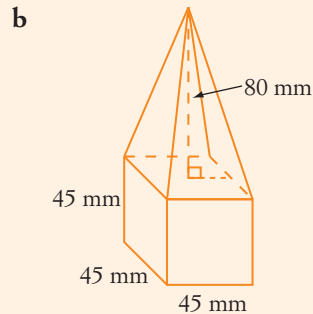
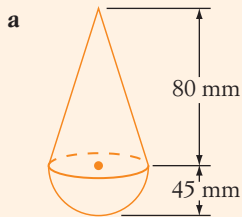
a $6x^2 + 12x$ **b** $25 - 100y^2$ **c** $a^2 - 6a - 40$ **d** $10p + 6 - 4p^2$

- 7 Solve the quadratic equation $x^2 + 7x + 2 = 0$, writing the solution in exact form.

- 8 Find, correct to one decimal place, the mean and standard deviation of this set of data.

Stem	Leaf				
3	2	5	8	9	
4	3	4	4	8	9
5	0	3	4		
6	5	8			
7	0	2			

- 9 Calculate the volume of each solid, correct to the nearest cubic millimetre.

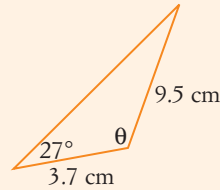


- 10 Students in Year 10 at a particular high school were asked if they had studied French at high school.

	Male	Female
French	35	87
No French	67	21

- a How many students are in Year 10 at the school?
- b What is the probability of selecting a Year 10 student at random who is:
 - i male and has studied French at high school?
 - ii female or studied French at high school?
- c Find the probability, expressed as a percentage, of randomly selecting a male in Year 10 who has not studied French at high school.
- d Given that a student is a Year 10 female, what is the probability that she has studied French at high school?

- 11 Find the value of θ , correct to the nearest minute.



Stage 5.3

See Chapter 9

- 12 Simplify each expression, writing the answer with a positive index where necessary.

a $7x^5 \times 8x^7$ b $4x^2 \div 16x^{-3}$ c $(3y)^{-2}$ d $m^{-6}n^3 \times mn^{-1} \div m^2n$

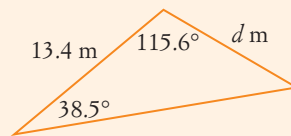
See Chapter 5

- 13 Solve each equation.

a $\frac{4y}{5} - 2 = 3$ b $\frac{5k}{3} - \frac{k}{2} = 4$ c $3x^2 = 27$

See Chapter 7

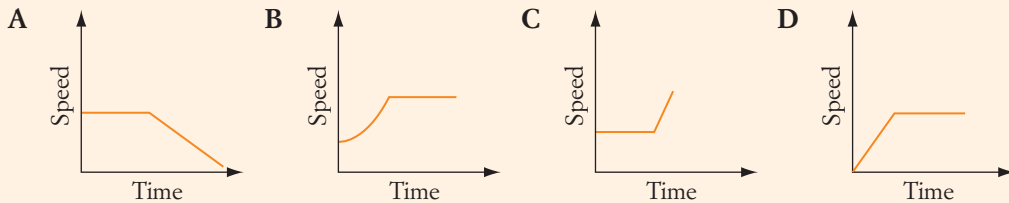
- 14 Find the value of d , correct to one decimal place.



Stage 5.3

See Chapter 9

- 15 A hiker climbs a mountain at a steady speed and then runs down the mountain. Which of the following graphs matches this situation?



- 16 Evaluate each logarithm.

a $\log_{10} 0.1$ b $\log_3 81$ c $\log_6 \sqrt{6}$

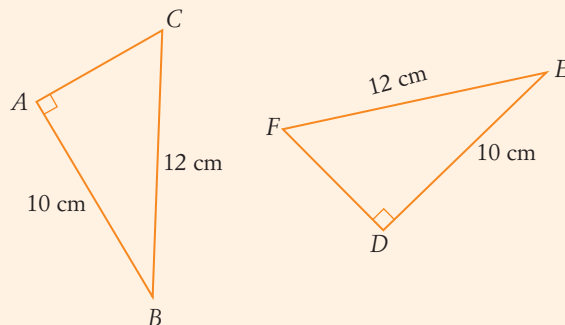
See Chapter 8

- 17 Simplify and evaluate each expression.

a $\log_3 9 + \log_3 27$ b $\log_5 125 - \log_5 25$ c $\log_2 128 - 5 \log_2 4$

See Chapter 8

- 18 Prove that $\triangle ABC \equiv \triangle DEF$.



See Chapter 13

- 19 Solve each inequality and graph the solution on a number line.

a $5y + 3 \geq -2$ b $\frac{2x-7}{2} < 4$ c $5 - 4x > 17$

See Chapter 7

Stage 5.3

See Chapter 3

See Chapter 10

See Chapter 8

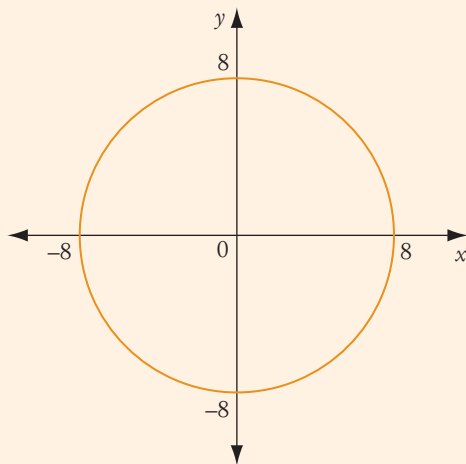
- 20** Find, in general form, the equation of a line that is parallel to $2x + 3y - 4 = 0$ and passes through $(2, -3)$.

- 21** Solve each pair of simultaneous equations.

$$3x - y = 4$$

$$2x + y = 6$$

- 22** Write the equation of this circle.



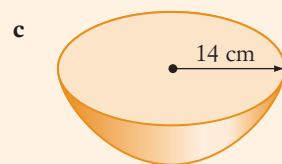
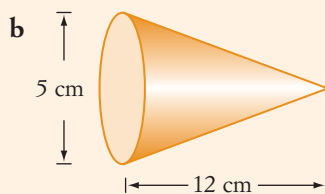
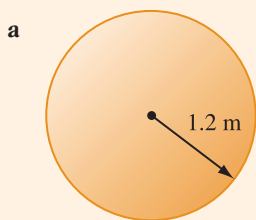
Stage 5.3

See Chapter 11

See Chapter 4

- 23** Graph the quadratic equation $y = 2x^2 - x - 15$, showing x - and y -intercepts, the axis of symmetry and the vertex.

- 24** Find, correct to one decimal place, the volume of each solid.



See Chapter 2

- 25** Calculate, correct to the nearest cent, the final amount if \$1260 is invested at 8.2% p.a. for 3 years, compounded quarterly.

See Chapter 3

- 26** $A(1, 3)$, $B(-2, 1)$, $C(4, -1)$ and $D(7, 1)$ are the vertices of a quadrilateral.

- a** Calculate the exact length and gradient of side:

i AB

ii BC

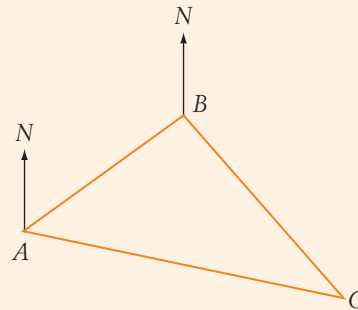
iii CD

iv AD

- b** What type of quadrilateral is $ABCD$?

- 27 A ship sails from port A for 100 km on a bearing of 040° to port B and then sails on a bearing of 162° for 324 km to port C .

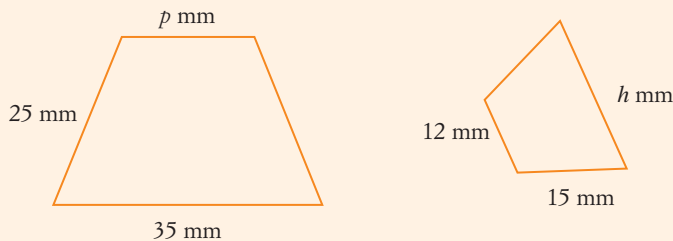
- Copy the diagram below and mark on it all the given information.
- Explain why $\angle ABC = 58^\circ$.
- What is the distance between port A and port C ? Answer correct to the nearest km.
- What is the bearing of port C from port A , correct to the nearest degree?



- 28 Graph each equation on a number plane.

a $y = \frac{3}{x}$ **b** $y = 3^x$

- 29 Find the value of h and p in this pair of similar figures.



- 30 The weather on a long weekend will either be fine or rain each day, with each outcome being equally likely.

- Draw a tree diagram to show the possible outcomes for Saturday, Sunday and Monday.
- What is the probability that it is fine:
 - on all three days?
 - on exactly two of the days?
 - on at least one of the days?

- 31 For an object that is cooling, the drop in temperature is directly proportional with time. If the temperature drops 5°C in 12 minutes, how long would it take to drop 8°C ? Answer correct to one decimal place.

Stage 5.3

See Chapter 9

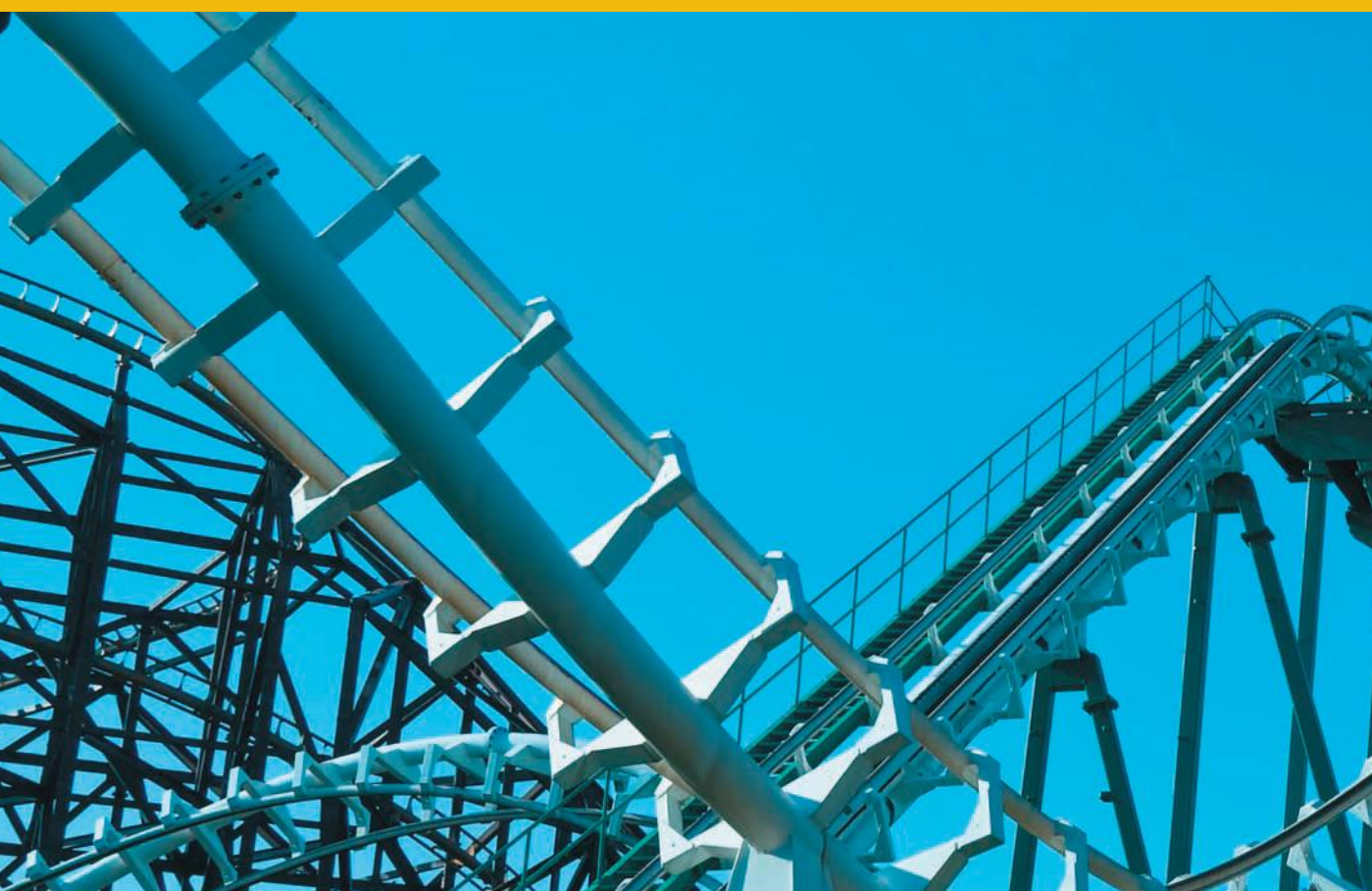
See Chapter 8

See Chapter 13

Stage 5.2

See Chapter 12

See Chapter 8



14

Number and algebra

Polynomials

Optional Stage 5.3 topic

We have analysed and graphed linear equations $y = mx + b$, quadratic equations $y = ax^2 + bx + c$ and simple cubic equations $y = ax^3 + c$. In this topic we will look at equations involving higher powers of x , their graphs and the methods for factorising them and sketching their graphs.



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Chapter outline

	Proficiency strands				
14-01 Polynomials*	U		R	C	
14-02 Adding and subtracting polynomials*	U	F	R	C	
14-03 Multiplying polynomials*	U	F	R	C	
14-04 Dividing polynomials*	U	F	R	C	
14-05 The remainder theorem*	U	F	R	C	
14-06 The factor theorem*	U	F	R	C	
14-07 The cubic curve $y = a(x - r)(x - s)(x - t)^*$	U	F	R	C	
14-08 Graphing polynomials*	U	F	PS	R	C
14-09 Transforming graphs of polynomials*	U	F	PS	R	C

*STAGE 5.3

Wordbank

degree of a polynomial The highest power in a polynomial. For example, the degree of $8x^3 + 4x - 7$ is 3.

polynomial An algebraic expression involving powers of x that are positive integers. For example, $P(x) = 8x^3 + 4x - 7$.

quotient The 'whole' part of the answer when a polynomial is divided by another polynomial

remainder The 'left-over' part of the answer when a polynomial is divided by another polynomial

root of an equation A value of x that is a solution to the equation $P(x) = 0$. A root of $P(x) = 0$ is the same as a *zero* of $P(x)$.

zero of a polynomial A value of x that makes the value of the polynomial $P(x)$ equal to 0

In this chapter you will:

- (STAGE 5.3) investigate the concept of a polynomial and apply the factor and remainder theorems to solve problems
- (STAGE 5.3) apply understanding of polynomials to sketch a range of curves and describe the features of these curves from their equation 
- (STAGE 5.3) recognise a polynomial and use polynomial notation
- (STAGE 5.3) add and subtract polynomials
- (STAGE 5.3) multiply and divide a polynomial by a linear polynomial
- (STAGE 5.3) understand and apply the remainder and factor theorems
- (STAGE 5.3) factorise polynomials and solve polynomial equations
- (STAGE 5.3) graph cubic equations of the form $y = a(x - r)(x - s)(x - t)$
- (STAGE 5.3) graph quadratic, cubic and quartic polynomials
- (STAGE 5.3) graph transformations of a basic polynomial $y = P(x)$

SkillCheck

1 If $x = 2$, evaluate each expression.

a $x^3 + 2x^2 - x - 1$

b $2x^2 + 3x - 11$

c $x^4 - x^3 + 2x$

2 Factorise each expression.

a $x^2 - 16$

b $x^3 - 16x$

c $3x^2 - 27$

d $3x^3 - 27x$

e $x^2 - 2x - 15$

f $x^2 + 5x - 24$

g $2x^2 + x - 10$

h $x^3 - 3x^2 - 70x$

3 Solve each equation.

a $(2x + 5)(x - 2) = 0$

b $x^2 - 10x = 0$

c $5x^2 - 3x = 0$

d $x^2 + 6x + 5 = 0$

e $x^2 - 2x - 120 = 0$

f $2x^2 + 7x + 6 = 0$

Stage 5.3

14-01 Polynomials

Worksheet

StartUp assignment 14
(Advanced)

MAT10NAWK10228

A **polynomial** is an algebraic expression involving powers of x that are positive integers, for example, $8x^3 + 4x - 7$. It is written using the notation $P(x)$, meaning 'a polynomial using the variable x '.

Summary

A **polynomial** has the general form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_2 x^2 + a_1 x + a_0$$

where the powers of x are positive integers, n

and $a_n, a_{n-1}, a_{n-2}, \dots, a_1, a_0$ are called **coefficients**.

The **leading term** of a polynomial is the term with the highest power. For example, the leading term of the polynomial $P(x) = 2x^5 + 8x^3 - 7$ is $2x^5$.

The **leading coefficient** of a polynomial is the coefficient of the leading term. For $P(x) = 2x^5 + 8x^3 - 7$, the leading coefficient is 2.

The highest power of the polynomial is the **degree** of the polynomial. For $P(x) = 2x^5 + 8x^3 - 7$, the degree is 5.

A **monic polynomial** is a polynomial that has a leading coefficient of 1. For example, $P(x) = x^4 - 7x^2 + x + 8$ is monic.

The **constant term** of a polynomial is the term 'at the end' of the polynomial that is independent of x . For $P(x) = x^4 - 7x^2 + x + 8$, the constant term is 8.

Names of polynomials

Linear polynomials have degree 1 and their graph is a straight line, for example, $P(x) = 2x - 7$.

Quadratic polynomials have degree 2 and their graph is a parabola, for example, $P(x) = 3x^2 - 2x + 5$.

Cubic polynomials have degree 3, for example $P(x) = 6x^3 - 7x^2 + 9x - 10$.

Quartic polynomials have degree 4, for example $P(x) = -2x^4 + x^3 - 8x^2 - 4x + 13$.

Example 1

Determine which of these expressions represents a polynomial.

$$A(x) = \sqrt{2}x^4 - 7x^5 + \frac{3}{x^2}$$

$$C(x) = 2\sqrt{x} + x^3 + 7$$

$$B(x) = 5x + 2$$

$$D(x) = (x^2 - 2)(x + 5)$$

Solution

- Since $A(x)$ can be written as $A(x) = \sqrt{2}x^4 - 7x^5 + 3x^{-2}$, it is *not* a polynomial because powers of x must be positive integers and $3x^{-2}$ has a negative power.
- $B(x) = 5x + 2$ is a polynomial.
- Since, $C(x) = 2x^{\frac{1}{2}} + x^3 + 7$, it is *not* a polynomial because $2x^{\frac{1}{2}}$ has a fractional power.
- $D(x) = (x^2 - 2)(x + 5) = x^3 + 5x^2 - 2x - 10$, which *is* a polynomial.

Example 2

For the polynomial $P(x) = 7x^6 + 5x^3 + 9x - 2$, state:

- a** the degree **b** the leading term **c** the leading coefficient
d the constant term **e** whether the polynomial is monic.

Solution

- a** The degree is 6.
b The leading term is $7x^6$.
c The leading coefficient is 7.
d The constant term is -2 .
e Since the leading coefficient is not 1, the polynomial is not monic.

The highest power of the polynomial is 6.
The term with the highest power

The term independent of x .

Stage 5.3

$P(x)$ notation

To find the value of a polynomial $P(x)$ for $x = c$, we write $P(c)$. This means that c is substituted for x in the polynomial.

Example 3

For the polynomial $P(x) = 3x^4 - 6x^2 + 4$, find:

- a** $P(3)$ **b** $P(-2)$ **c** $P(0) - P(-1)$

Solution

$$\begin{aligned} \mathbf{a} \quad P(3) &= 3(3^4) - 6(3^2) + 4 \\ &= 193 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad P(-2) &= 3(-2)^4 - 6(-2)^2 + 4 \\ &= 28 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad P(0) &= 3(0^4) - 6(0^2) + 4 \\ &= 4 \\ P(-1) &= 3(-1)^4 - 6(-1)^2 + 4 \\ &= 1 \\ \therefore P(0) - P(-1) &= 4 - 1 \\ &= 3 \end{aligned}$$

Exercise 14-01 Polynomials

See Example 1

- 1** Determine whether each expression is a polynomial. If it is a polynomial, state whether it is monic.

a $9x^2 - 5x$

b $\frac{5}{x^2} + \frac{2}{x^3} + x$

c $3x^5 + x^6 - \sqrt{7}$

d $10 - \frac{x}{2}$

e $4x + 2\sqrt{x}$

f $x^3 + x^2 - 11$

g $\sqrt{5}x^2 - \sqrt{10}x^4 + x^3$

h $3\sqrt{x} - 2x^4 + x^5$

i $-(20 - x)(20x + x^2)$

j $3^x - \frac{x}{3} + x^3$

k $(x^2 - 2)^2(x + 3)$

l $\frac{9x^3 + x^2 + x}{x}$

See Example 2

- 2** For each polynomial, state:

i the degree

ii the leading coefficient

iii the constant term.

a $9x^5 - 2x^4 + 5x^3 + 1$

b $7x^4 - 6x^5 + 3$

c $11x^2 - 10$

d $-6x$

e $8x - \frac{1}{2}x^2 + 7x^5 + 3$

f 9

g $4x^3 + \sqrt{2}x^4 - 11 + x^6$

h $22 - \frac{5x}{4}$

i $\frac{x^3}{3} + \frac{x^2}{2} + x$

See Example 3

- 3** If $P(x) = 3x^3 - 2x^2 + x - 1$, evaluate:

a $P(0)$

b $P(2)$

c $P(-1)$

d $P\left(\frac{1}{2}\right)$

e $P(\sqrt{2})$

- 4** If $P(x) = x^2 - x$ and $Q(x) = 1 - x^3$, find:

a $P(0) + Q(0)$

b $P(2) + Q(1)$

c $P(4) + Q(-1)$

d $P(-2) - Q(-2)$

e $P\left(\frac{1}{2}\right) + Q\left(\frac{1}{4}\right)$

f $P(2\sqrt{3}) + Q(\sqrt{3})$

g $P[Q(2)]$

h $Q[P(2)]$

14-02 Adding and subtracting polynomials

Stage 5.3

Example 4

If $P(x) = x^3 + 3x^2 - 2x - 5$ and $Q(x) = 3x^2 - 5x + 7$, simplify each expression.

a $P(x) + Q(x)$ **b** $P(x) - Q(x)$

Solution

$$\begin{aligned} \text{a } P(x) + Q(x) &= x^3 + 3x^2 - 2x - 5 + (3x^2 - 5x + 7) \\ &= x^3 + 6x^2 - 7x + 2 \end{aligned}$$

$$\begin{aligned} \text{b } P(x) - Q(x) &= x^3 + 3x^2 - 2x - 5 - (3x^2 - 5x + 7) \\ &= x^3 + 3x - 12 \end{aligned}$$

See Example 4

Exercise 14-02 Adding and subtracting polynomials

1 Simplify each expression.

a $(9x^3 + x^2 + x) + (7x^2 + 5x - 2)$

b $(3x + 2) - (x^3 - x)$

c $(8x - 2) - (11x + 4x^2)$

d $2(x^4 - x^2 + 2) - 3(x^3 + x^2 - x^4)$

e $x^4 - (x^3 - x^2 + 5x)$

f $2(5x^4 + x^3 + 2x) + 2(11 - 2x^2 + x^5)$

g $(7x^6 + x^5) + (x^2 + x^3) - (2 - 3x)$

h $8x^4 - x^3 + 6x^2 + 5x - (x + 6) + (x^3 + 9x^2 - 1)$

i $x^4 - 8x^3 + 2x^2 + 7x - 2 - (6x - 3x^2 + 5x^4)$

j $6x^3 - 2x^2 - x - (2x - 3) - (3x + 2)$

2 If $P(x) = 4x - 3$ and $Q(x) = x^2 + 7x + 2$, simplify each expression.

a $P(x) + Q(x)$ **b** $P(x) - Q(x)$ **c** $Q(x) - P(x)$ **d** $3P(x) + 2Q(x)$

3 If $A(x) = x^2 - 6$, $B(x) = 11x - 2x^2$ and $C(x) = 9 - 4x$, simplify each expression.

a $C(x) - B(x)$

b $A(x) - C(x)$

c $B(x) - A(x)$

d $A(x) + B(x) + C(x)$

e $A(x) - B(x) + C(x)$

f $A(x) - [B(x) + C(x)]$

4 If $R(x) = x^2 - 6x + 5$, $P(x) = x - 1$ and $Q(x) = R(x) - P(x)$, simplify each expression.

a $Q(x)$

b $Q(3\sqrt{2})$

c the values of x if $Q(x) = 0$

Stage 5.3

14-03 Multiplying polynomials

Example 5

If $P(x) = 3x + 1$ and $Q(x) = x^2 - 5x - 2$, find $P(x) \cdot Q(x)$.

Solution

$$\begin{aligned} P(x) \cdot Q(x) &= (3x + 1)(x^2 - 5x - 2) \\ &= 3x(x^2 - 5x - 2) + 1(x^2 - 5x - 2) \\ &= 3x^3 - 15x^2 - 6x + x^2 - 5x - 2 \\ &= 3x^3 - 14x^2 - 11x - 2 \end{aligned}$$

Exercise 14-03 Multiplying polynomials

See Example 5

- Expand each product.
 - $(3x + 2)(x^3 - x)$
 - $(11x + 4x^2)(8x - 2)$
 - $(9x^3 + x^2 + x)(5x - 2)$
 - $(6x^3 - 2x^2 - x - 3)(3x + 2)$
 - $(2 - 3x)(7x^6 + x^5 + x^3 + x^2)$
 - $(x + 6)(8x^4 - x^3 + 6x^2 + 5x - 2)$
- If $P(x) = 4x - 3$ and $Q(x) = x^2 + 7x + 2$, find $P(x) \cdot Q(x)$.
- If $A(x) = x^2 - 6$, $B(x) = 11x - 2x^2$ and $C(x) = 9 - 4x$, find:
 - $A(x) \cdot B(x)$
 - $A(x) \cdot C(x)$
 - $B(x) \cdot C(x)$

14-04 Dividing polynomials

Long division

The long division process for $9947 \div 65$ is shown on the right.

- 65 into 99 is 1, remainder 34
- Bring down the 4 from 9947
- 65 into 344 goes 5, remainder 19
- Bring down the 7 from 9947
- 65 into 197 goes 3, remainder 2

$$\therefore 9947 \div 65 = 153, \text{ remainder } 2$$

$$= 153 \frac{2}{65}$$

$$\therefore 9947 = 65 \times 153 + 2$$

$$\begin{array}{r} 153 \\ 65 \overline{)9947} \\ \underline{-65} \downarrow \\ 344 \downarrow \\ \underline{-325} \downarrow \\ 197 \downarrow \\ \underline{195} \\ 2 \end{array}$$

9947 is the **dividend**, 65 is the **divisor**, 153 is the **quotient**, 2 is the **remainder**.

Puzzle sheet

Finding the quotient

MAT10NAPS00052

The long division process can also be used to divide polynomials.

For example, $(x^3 + 5x^2 - 6x - 4) \div (x + 3)$ is

- $(x + 3)$ into $(x^3 + 5x^2)$ is x^2 , remainder $2x^2$ $x^3 \div x = x^2$
- Bring down the $(-6x)$ from the dividend
- $(x + 3)$ into $(2x^2 - 6x)$ goes $2x$, remainder $-12x$ $2x^2 \div x = 2x$
- Bring down the (-4) from the dividend
- $(x + 3)$ into $(-12x - 4)$ goes (-12) , remainder 32 $-12x \div x = -12$

$$\begin{array}{r} x^2 + 2x - 12 \\ x + 3 \overline{) x^3 + 5x^2 - 6x - 4} \\ \underline{x^3 + 3x^2} \\ 2x^2 - 6x \\ \underline{2x^2 + 6x} \\ -12x - 4 \\ \underline{-12x - 36} \\ 32 \end{array}$$

$\therefore (x^3 + 5x^2 - 6x - 4) \div (x + 3) = x^2 + 2x - 12$, remainder 32

$\therefore (x^3 + 5x^2 - 6x - 4) = (x + 3)(x^2 + 2x - 12) + 32$

$(x^3 + 5x^2 - 6x - 4)$ is the **dividend**, $(x + 3)$ is the **divisor**, $(x^2 + 2x - 12)$ is the **quotient**, 32 is the **remainder**.

Note that we can express the dividend $P(x)$ as the product of its factors plus the remainder:

$P(x) = \text{divisor} \times \text{quotient} + \text{remainder}$.

Example 6

Divide $P(x) = x^3 - x + 4$ by $A(x) = x - 3$. Then write $P(x)$ in the form $P(x) = A(x) \cdot Q(x) + R(x)$, where $Q(x)$ is the quotient and $R(x)$ is the remainder.

Solution

$$\begin{array}{r} x^2 - 3x + 8 \\ x - 3 \overline{) x^3 + 0x^2 - x + 4} \\ \underline{x^3 - 3x^2} \\ 3x^2 - x \\ \underline{3x^2 - 9x} \\ 8x + 4 \\ \underline{8x - 24} \\ 28 \end{array} \quad \begin{array}{l} \text{Write } 0x^2 \text{ as there is no } x^2 \text{ term} \\ \therefore x^3 - x + 4 = (x - 3)(x^2 + 3x + 8) + 28 \end{array}$$

Example 7

Show that $(2x + 5)$ is a factor of $2x^3 + 5x^2 - 12x - 30$.

Solution

If $2x^3 + 5x^2 - 12x - 30 = (2x + 5) \cdot Q(x)$ with no remainder, then $(2x + 5)$ is a factor.

$$\begin{array}{r} x^2 - 6 \\ 2x + 5 \overline{) 2x^3 + 5x^2 - 12x - 30} \\ \underline{2x^3 + 5x^2} \\ + 0x^2 - 12x - 30 \\ \underline{-12x - 30} \\ 0 \end{array}$$

Since the remainder is 0, this means that $2x^3 + 5x^2 - 12x - 30 = (2x + 5)(x^2 - 6)$

$\therefore (2x + 5)$ is a factor of $2x^3 + 5x^2 - 12x - 30$.

Stage 5.3

Exercise 14-04 Dividing polynomials

See Example 6

- 1 Perform the following divisions, then write the first polynomial in the form:
dividend = divisor $\times$ quotient + remainder.

a $(x^2 + 7x + 4) \div (x + 2)$ c $(4x^2 + 3x + 10) \div (x - 1)$ e $(x^3 + 6x^2 + 5x - 4) \div (x - 3)$ g $(2x^3 - x^2 + 5x + 3) \div (x + 6)$ i $(x^5 - x^4 + 8x^3 + 2x^2 - x - 1) \div (x + 1)$	b $(x^2 - 6x + 2) \div (x - 3)$ d $(8x^2 + 9x + 11) \div (2x + 1)$ f $(4x^3 + 2x^2 + x) \div (x + 4)$ h $(11 - x^2 + 3x^3) \div (x + 2)$ j $(x^4 - x^2 - 10) \div (x + 3)$
- 2 If $P(x) = 3x^2 - 7x + 5$, $W(x) = x^2 + 5x$ and $T(x) = x - 2$, then find the quotient and remainder in each expression.

a $P(x) \div T(x)$ c $[W(x) \times P(x)] \div T(x)$	b $W(x) \div T(x)$ d $[P(x) + W(x)] \div T(x)$

See Example 7

- 3 Show that $(2x - 1)$ is a factor of each polynomial. Express $P(x)$ as a product of the two factors.

a $P(x) = 6x^2 + x - 2$ c $P(x) = 8x^2 + 10x - 7$ e $P(x) = 2x^4 - 7x^3 - 5x^2 + 8x - 2$ g $P(x) = 6x^3 - 3x^2 + 2x - 1$	b $P(x) = 2x^3 + x^2 + x - 1$ d $P(x) = 6x^3 + x^2 - 1$ f $P(x) = 2x^4 - x^3 - 2x^2 + 7x - 3$ h $P(x) = 11x + x^5 - 6x^2 - 2x^6 - 4$

Puzzle sheet

The remainder theorem

MAT10NAPS00053

14-05 The remainder theorem

Summary

The remainder theorem

If a polynomial $P(x)$ is divided by the linear expression $(x - a)$, then the remainder is $P(a)$.

Proof:

Since $(x - a)$ is a polynomial of degree 1:

$P(x) = (x - a) \cdot Q(x) + R$, where the remainder R is a constant.

Substituting $x = a$, gives:

$$\begin{aligned} P(a) &= (a - a) \cdot Q(a) + R \\ &= 0 + R \\ &= R \end{aligned}$$

Example 8

Find the remainder when $P(x) = 6x^4 - 3x^3 + 2x + 5$ is divided by each linear expression.

a $(x - 2)$

b $(x + 3)$

Solution

a Dividing by $(x - 2)$ will give the remainder $P(2)$.

$$\begin{aligned} P(2) &= 6(2)^4 - 3(2)^3 + 2(2) + 5 \\ &= 81 \end{aligned}$$

b Dividing by $(x + 3)$ will give the remainder $P(-3)$.

$$\begin{aligned} P(-3) &= 6(-3)^4 - 3(-3)^3 + 2(-3) + 5 \\ &= 566 \end{aligned}$$

See Example 8

Puzzle sheet

Factorising cubic functions

MAT10NAPS00051

Exercise 14-05 The remainder theorem

1 Find the remainder when $P(x) = 2x^3 - 3x^2 + x - 1$ is divided by each linear expression.

a $(x - 2)$

b $(x + 4)$

c $(x - 1)$

d $(x - 5)$

e $(x + 1)$

f $(x - 10)$

g $(x + 3)$

h $(x - 3)$

2 Determine the remainder when the first polynomial is divided by the second polynomial.

a $x^2 + 11x - 6, x - 4$

b $x^2 - 2x + 3, x - 1$

c $3x^2 - x + 4, x - 2$

d $x^3 + x^2 - x, x + 2$

e $3x^3 + 2x^2 - 11, x + 1$

f $2x^2 - x - 16, x - 10$

g $5x^3 - x^2 - 6x, x + 1$

h $x^6 - 3x^4 + x^3 - 2, x + 2$

i $2x^4 - 3x^2 + 6x - 2, x + 3$

14-06 The factor theorem

If a polynomial $P(x)$ is divided by $(x - a)$ and the remainder is zero, then $(x - a)$ is a factor of $P(x)$. Furthermore, by the remainder theorem, $P(a) = 0$.

Summary

The factor theorem

If $(x - a)$ is a factor of $P(x)$, then $P(a) = 0$.

Stage 5.3

Factorising polynomials

Video tutorial

Factorising polynomials

MAT10NAVT10018

Example 9

- a** Show that $(x - 2)$ is a factor of $P(x) = 2x^3 - 7x^2 - 3x + 18$.
b Hence express $P(x)$ as a product of its factors.

Solution

$$\begin{aligned}\mathbf{a} \quad P(2) &= 2(2^3) - 7(2^2) - 3(2) + 18 \\ &= 0\end{aligned}$$

$\therefore (x - 2)$ is a factor of $P(x)$.

- b** Use long division to find the other factors.

$$\begin{array}{r} 2x^2 - 3x - 9 \\ x - 2 \overline{) 2x^3 - 7x^2 - 3x + 18} \\ \underline{2x^2 - 4x^2} \\ -3x^2 - 3x \\ \underline{-3x^2 + 6x} \\ -9x + 18 \\ \underline{-9x + 18} \\ 0 \end{array}$$

$$\begin{aligned}\therefore P(x) &= (x - 2)(2x^2 - 3x - 9) \\ &= (x - 2)(2x^2 - 6x + 3x - 9) \\ &= (x - 2)[2x(x - 3) + 3(x - 3)] \\ &= (x - 2)(x - 3)(2x + 3)\end{aligned}$$

$$\therefore 2x^3 - 7x^2 - 3x + 18 = (x - 2)(x - 3)(2x + 3)$$

Factorising $2x^2 - 3x - 9$

Zeroes of a polynomial

If $(x - a)$ is a factor of $P(x)$, $x = a$ is called a **zero** of the polynomial $P(x)$.

A **zero** of a polynomial $P(x)$ is a value of x that makes $P(x)$ equal to zero. It is a solution of $P(x) = 0$.

In Example 9 above, $x = 2$ is a **zero** of the polynomial $P(x) = 2x^3 - 7x^2 - 3x + 18$ because $P(2) = 0$.

Solving polynomial equations

Stage 5.3

Example 10

- a Factorise $2x^3 - 3x^2 - 29x - 30$.
b Hence solve the equation $2x^3 - 3x^2 - 29x - 30 = 0$.

Solution

- a Let $P(x) = 2x^3 - 3x^2 - 29x - 30$

Use 'guess and check' to find a zero of $P(x)$.

Any zero of $P(x)$ must be a factor of the constant term of $P(x)$, which is (-30) , because if $P(x)$ is expressed as a product of its factors, the constant terms of each factor must multiply together to make (-30) .

Factors of 30 are 1, -1 , 2, -2 , 3, -3 , 5, -5 , 6, -6 , 10, -10 , 15, -15 , 30 and -30 .

Guessing and checking:

$$P(1) = 2 - 3 - 29 - 30$$

$$= -60$$

$$\neq 0 \quad \therefore (x - 1) \text{ is not a factor}$$

$$P(-1) = -2 - 3 + 29 - 30$$

$$= -6$$

$$\neq 0 \quad \therefore (x + 1) \text{ is not a factor}$$

$$P(-2) = -16 - 12 + 58 - 30$$

$$= 0 \quad \therefore (x + 2) \text{ is a factor}$$

Now, by long division:

$$\begin{array}{r} 2x^2 - 7x - 15 \\ x + 2 \overline{) 2x^3 - 3x^2 - 29x - 30} \\ \underline{2x^3 + 4x^2} \\ -7x^2 - 29x \\ \underline{-7x^2 - 14x} \\ -15x - 30 \\ \underline{-15x - 30} \\ 0 \end{array}$$

$$\therefore P(x) = (x + 2)(2x^2 - 7x - 15)$$

$$= (x + 2)(2x^2 - 10x + 3x - 15)$$

$$= (x + 2)[2x(x - 5) + 3(x - 5)]$$

$$= (x + 2)(x - 5)(2x + 3)$$

$$\therefore 2x^3 - 3x^2 - 29x - 30 = (x + 2)(x - 5)(2x + 3)$$

Factorising $2x^2 - 7x - 15$

Stage 5.3

$$\begin{aligned} \text{b } 2x^3 - 3x^2 - 29x - 30 &= 0 \\ (x + 2)(x - 5)(2x + 3) &= 0 \end{aligned}$$

$$\begin{aligned} x + 2 = 0 \quad \text{or} \quad x - 5 = 0 \quad \text{or} \quad 2x + 3 = 0 \\ 2x = -3 \\ x = -\frac{3}{2} \\ x = -2 \quad \quad \quad x = 5 \quad \text{or} \quad x = -1\frac{1}{2} \end{aligned}$$

Exercise 14-06 The factor theorem

See Example 9

- 1 Determine which linear polynomial, A, B and/or C, is a factor of $P(x)$.

$P(x)$	A	B	C
a $x^2 - 8x + 7$	$(x - 2)$	$(x - 7)$	$(x - 1)$
b $x^3 + 2x^2 - x - 2$	$(x + 3)$	$(x + 1)$	$(x - 1)$
c $2x^3 + x + 3$	$(x + 1)$	$(x - 2)$	$(x - 3)$
d $x^3 - 5x^2 - 22x - 16$	$(x + 1)$	$(x + 2)$	$(x - 8)$
e $x^3 + 2x^2 - 13x + 10$	$(x + 5)$	$(x - 2)$	$(x - 3)$

- 2 Show that the second polynomial is a factor of the first polynomial.

a $x^2 + 10x - 24, x - 2$	b $x^3 + 5x^2 + 11x + 10, x + 2$
c $x^3 + 3x^2 - x - 3, x - 1$	d $x^4 - x^3 + 2x^2 - x - 1, x - 1$
e $x^3 - x^2 - 12x, x + 3$	f $x^5 - 4x^4 + 3x^3 + x^2 + 4, x - 2$
g $8x^4 + 2x^2 - 1, x - \frac{1}{2}$	h $2x^4 + 7x^3 - 56x^2 - 37x + 84, (x - 4)$

See Example 10

- 3 Factorise each polynomial.

a $x^3 + 6x^2 + 8x$	b $x^3 - x^2 - 2x$
c $x^3 + 2x^2 - x - 2$	d $2x^3 + 3x^2 - 18x + 8$
e $x^3 - 6x^2 + 11x - 6$	f $x^3 + x^2 - 46x + 80$
g $3x^3 - 16x^2 - 13x + 6$	h $6x^3 - 13x^2 + x + 2$
i $2x^4 - 5x^3 + 2x^2$	

- 4 Solve each equation.

a $(2x - 1)(x - 3)(x + 4) = 0$	b $(x + 4)(x^2 + 1) = 0$
c $(2x - 5)(x^2 - x - 6) = 0$	d $(x^2 - 25)(x^2 + 3) = 0$
e $(x - 2)(x + 3)(x^2 - 16) = 0$	f $x^3 + 5x^2 - 14x = 0$
g $x^3 + 2x^2 - 9x - 18 = 0$	h $x^3 - x^2 - 10x - 8 = 0$
i $x^3 - 21x - 20 = 0$	j $2x^3 - 7x^2 + 2x + 3 = 0$
k $12x^3 - 23x^2 + 13x - 2 = 0$	l $x^3 - x - 24 = 0$

14-07 The cubic curve $y = a(x - r)(x - s)(x - t)$

Stage 5.3

NSW

Worksheet

Graphing cubics 2

MAT10NAWK10230

In Chapter 11, when graphing parabolas of the form $y = ax^2 + bx + c$, we found its x -intercepts by solving the equation $ax^2 + bx + c = 0$.

Now we can graph cubic curves of the form $y = ax^3 + bx^2 + cx + d$ by factorising the RHS as $y = a(x - r)(x - s)(x - t)$ to find its x -intercepts at r , s and t .

Example 11

Sketch the graph of the cubic equation $y = x(x + 5)(x - 1)$.

Solution

Substitute $y = 0$ to find the x -intercepts.

$$0 = x(x + 5)(x - 1)$$

$$x = 0 \text{ or } x + 5 = 0 \text{ or } x - 1 = 0$$

$$\therefore x = 0, -5 \text{ and } 1$$

The x -intercepts are -5 , 0 and 1 .

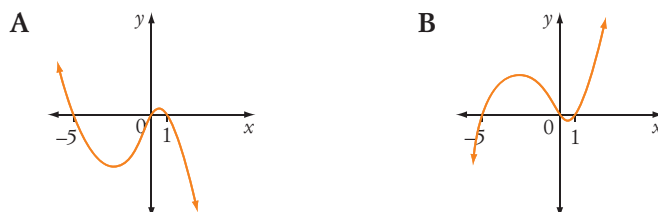
Substitute $x = 0$ to find the y -intercept.

$$y = 0(0 + 5)(0 - 1)$$

$$= 0$$

The y -intercept is 0 .

Possible graphs are:



To determine which graph is correct, we look at the leading coefficient of the cubic equation. In the expansion of $y = x(x + 5)(x - 1)$, the coefficient of x^3 is 1 , which is positive, so the correct graph is **B**, an increasing cubic curve.

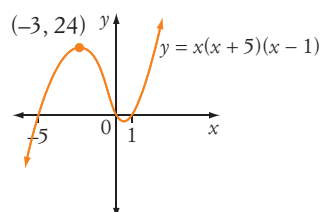
(If the coefficient of x^3 is negative, the correct graph is **A**, a decreasing cubic curve)

We can check this by substituting a value of x , say $x = -3$, into the equation to find a point on the curve:

$$y = -3(-3 + 5)(-3 - 1)$$

$$= 24$$

This means $(-3, 24)$ lies on the curve. So **B** must be the correct graph.



Use GeoGebra or other graphing technology to check the shape of this graph.

Stage 5.3

Exercise 14-07 The cubic curve $y = a(x - r)(x - s)(x - t)$

GeoGebra or other graphing technology can be used to check the shapes of the graphs in this exercise.

See Example 11

- For each cubic equation, find the x - and y -intercepts and sketch its graph.

a $y = (x - 2)(x + 2)(x + 3)$	b $y = x(x - 2)(x + 1)$	c $y = (x - 1)(x - 3)(x + 1)$
d $y = (4 - x)(x + 1)(x - 5)$	e $y = (1 - x)(2 + x)(x + 3)$	f $y = 2x(x - 6)(x - 3)$
g $y = -(x + 3)(x - 2)(x - 5)$	h $y = -2(x - 1)(x + 2)(x - 3)$	i $y = (x + 2)(x - 1)(x - 2)$
- What are the x -intercepts of the graph of $y = 2x(x + 1)(x - 3)$? Select the correct answer **A, B, C** or **D**.

A $x = 2, 1, -3$	B $x = 0, -1, 3$	C $x = 0, 1, -3$	D $x = 2, -1, 3$
- Which cubic equation has a graph with x -intercepts $-2, -5$ and 6 ? Select **A, B, C** or **D**.

A $y = (x + 2)(x + 5)(x - 6)$	B $y = (x - 2)(x - 5)(x + 6)$
C $y = (x + 2)(x - 5)(x + 6)$	D $y = (x - 2)(x + 5)(x + 6)$
- Sketch the graph of each cubic equation.

a $y = x(x - 3)(x + 1)$	b $y = -(x - 3)(x - 1)(x + 1)$	c $y = -2x(x - 1)(x + 6)$
d $y = (2x - 3)(x - 1)(x + 2)$	e $y = (x + 1)(x + 1)(x + 1)$	f $y = (x - 3)^2(x + 2)$

Worksheet

Polynomials review

MAT10NAWK10229

14-08 Graphing polynomials

Summary

To graph the polynomial $y = P(x)$:

- substitute $y = 0$ to find the x -intercepts
- substitute $x = 0$ to find the y -intercepts
- use the sign of the leading coefficient to sketch the shape of the curve

Example 12

Sketch $y = x^3 - x^2 - 10x - 8$.

Solution

Substitute $y = 0$ to find the x -intercepts.

$$0 = x^3 - x^2 - 10x - 8$$

Use the factor theorem to factorise the RHS.

$$\text{Let } P(x) = x^3 - x^2 - 10x - 8$$

Test factors of -8 .

$$\begin{aligned} P(1) &= 1 - 1 - 10 - 8 \\ &= -18 \\ &\neq 0 \end{aligned}$$

$$P(-1) = (-1)^3 - (-1)^2 - 10(-1) - 8 \\ = 0$$

$\therefore (x + 1)$ is a factor.

$$\begin{array}{r} x^2 - 2x - 8 \\ x + 1 \overline{) x^3 - x^2 - 10x - 8} \\ \underline{x^3 + x^2} \\ -2x^2 - 10x \\ \underline{-2x^2 - 2x} \\ -8x - 8 \\ \underline{-8x - 8} \\ 0 \end{array}$$

$$\therefore P(x) = (x + 1)(x^2 - 2x - 8) \\ = (x + 1)(x - 4)(x + 2)$$

$$\text{If } P(x) = 0: (x + 1)(x - 4)(x + 2) = 0$$

$\therefore$ The x -intercepts are -2 , -1 and 4 .

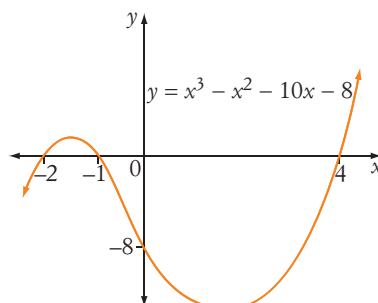
Substitute $x = 0$ to find the y -intercept.

$$y = 0^3 - 0^2 - 10(0) - 8 \\ = -8$$

$\therefore$ The y -intercept is -8 .

The leading coefficient is 1 , which is positive, so the cubic curve is increasing.

Factorising $x^2 - 2x - 8$



Use GeoGebra or other graphing technology to check the shape of this graph.

Single, double and triple roots of a polynomial equation

If $x = a$ is a solution of the polynomial equation $P(x) = 0$, then $x = a$ is called a **root** of the equation.

If $(x - a)^2$ is a factor of a polynomial $P(x)$, then $x = a$ is a **double root** of $P(x) = 0$.

For example, if $P(x) = (x - 3)^2(x^2 - x - 1)$, then $x = 3$ is a double root of $P(x) = 0$.

If $(x - a)^3$ is a factor of a polynomial $P(x)$, then $x = a$ is a **triple root** of $P(x) = 0$.

For example, if $P(x) = (x + 1)^3(x - 2)$, then $x = -1$ is a triple root of $P(x) = 0$.

If $(x - a)$ is a factor of a polynomial $P(x)$, then $x = a$ is a **single root** of $P(x) = 0$.

For example, if $P(x) = (x - 1)(x + 2)(x - 3)$, then $x = 1$, $x = -2$, $x = 3$ are all single roots of $P(x) = 0$.

Stage 5.3

Example 13

Sketch $P(x) = -(x - 3)^3(x + 1)$.

Solution

Solve $P(x) = 0$ to find the x -intercepts.

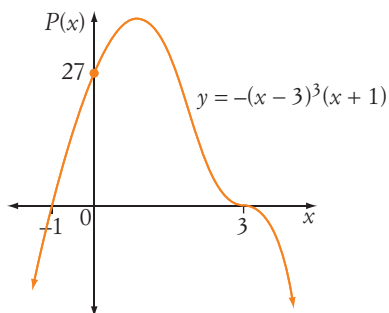
$$x = 3 \text{ and } x = -1$$

$x = 3$ is a triple root and $x = -1$ is a single root, so the graph will cross the x -axis at $x = 3$ (and take a shape similar to $y = x^3$ there) and at $x = -1$.

Substitute $x = 0$ to find the y -intercept.

$$\begin{aligned} y &= -(0 - 3)^3(0 + 1) \\ &= 27 \end{aligned}$$

The polynomial is a quartic (degree 4) and the leading term is $-x^3 \times x = -x^4$, which is negative, so the quartic curve will decrease as x increases.



Use GeoGebra or other graphing technology to check the shape of this graph.

Summary

If $P(x) = 0$ has a **single root** at $x = a$, then the graph of the polynomial **crosses** the x -axis at $x = a$.

If $P(x) = 0$ has a **double root** at $x = a$, then the graph of the polynomial **touches** the x -axis at $x = a$ with a flat gradient, taking the shape of a parabola there.

If $P(x) = 0$ has a **triple root** at $x = a$, then the graph of the polynomial **crosses** the x -axis at $x = a$ with a flat gradient, taking the shape of a cubic curve there.

Exercise 14-08 Graphing polynomials

GeoGebra or other graphing technology can be used to check the shapes of the graphs in this exercise.

1 Sketch the graph of each polynomial.

a $P(x) = x^3 + 6x^2 + 8x$

b $P(x) = x^3 + 2x^2 - x - 2$

c $P(x) = 2x^3 + 3x^2 - 18x + 8$

d $P(x) = 2x^4 - 5x^3 + 2x^2$

e $P(x) = 2x^3 + 17x^2 + 31x - 20$

f $P(x) = -x^3 + 5x^2 + 8x - 12$

g $P(x) = -x^4 + 3x^3 + 6x^2 - 8x$

h $P(x) = -x^4 + 13x^2 - 36$

See Example 12

2 Sketch the graph of each polynomial.

a $y = (x - 2)(x + 4)^2$

c $y = -(x + 1)^2(x - 2)^2$

e $y = -(x - 1)(x - 2)(x + 2)^2$

g $y = -(x - 2)^2(x + 1)^2$

i $y = (x^2 - 4)(x - 4)^2$

k $P(x) = x^4 - 5x^2 + 4$

b $y = -(x + 2)(x - 3)^2$

d $y = x(x + 1)(x - 3)^2$

f $y = -x(x - 4)^2$

h $y = (x - 4)^3(x + 1)$

j $y = -x^3 - x^2 + 5x - 3$

l $y = -x^4 - 3x^3 - 12x^2 - 44x - 48$

Stage 5.3

See Example 13

Technology Transforming graphs

1 Use GeoGebra or other graphing technology to graph $P(x) = x^2 - 3x + 2$.

2 Write the equations of each polynomial.

a $-P(x)$

b $P(-x)$

c $P(x) - 3$

d $2P(x)$

3 Graph the polynomials shown in question 2.

4 Describe how the graphs of the polynomials $y = -P(x)$, $y = P(-x)$, $y = P(x) - 3$, $y = 2P(x)$ can be drawn from the graph of $y = P(x)$.

14-09 Transforming graphs of polynomials

Worksheet

Advanced graphs

MAT10NAWK10231

Summary

If the graph of the polynomial $y = P(x)$ is drawn:

- the graph of $y = -P(x)$ is a reflection of $y = P(x)$ in the x -axis
- the graph of $y = P(-x)$ is a reflection of $y = P(x)$ in the y -axis
- the graph of $y = P(x) + c$ is a vertical translation of the graph $y = P(x)$
- the graph of $y = aP(x)$ is $y = P(x)$ either 'stretched' or 'compressed' vertically, with the same x -intercepts

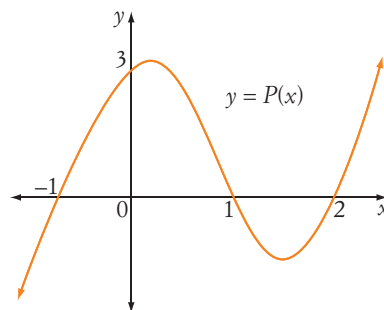
Stage 5.3

Example 14

The graph of $y = P(x)$ is shown.

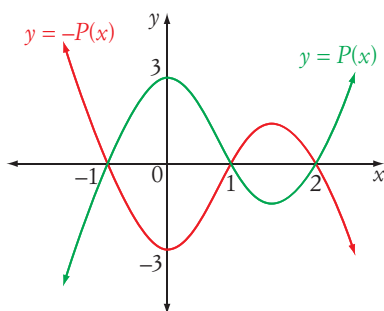
Draw the graphs of each polynomial.

- a** $y = -P(x)$ **b** $y = P(-x)$
c $y = P(x) + 2$ **d** $y = 3P(x)$

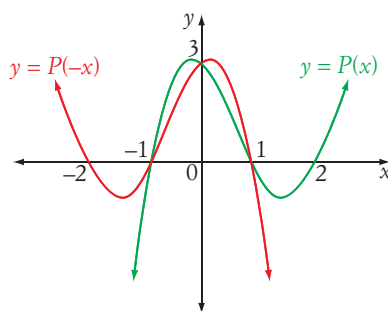


Solution

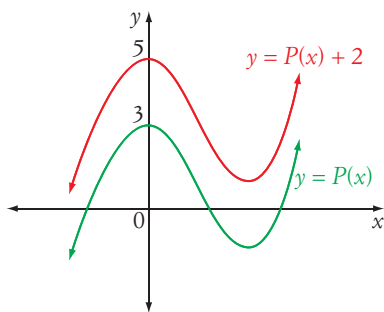
- a** $y = -P(x)$ is a reflection of $y = P(x)$ in the x-axis.



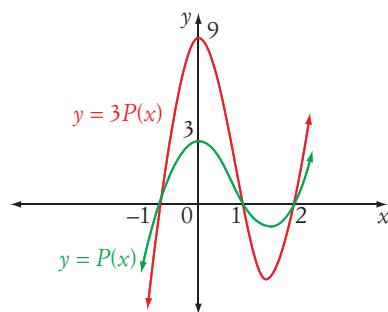
- b** $y = P(-x)$ is a reflection of $y = P(x)$ in the y-axis.



- c** $y = P(x) + 2$ is $y = P(x)$ vertically translated 2 units up.



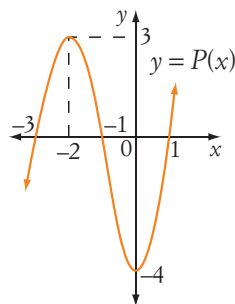
- d** $y = 3P(x)$ is $y = P(x)$ stretched vertically by a factor of 3.



Exercise 14-09 Transforming graphs of polynomials

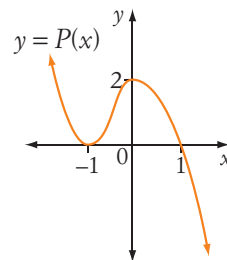
- 1 The graph of the polynomial $y = P(x)$ is shown. Sketch the graph of each polynomial below.

a $y = 2P(x)$ **b** $y = P(x) + 2$ **c** $y = \frac{1}{2}P(x)$
d $y = P(x) - 3$ **e** $y = -P(x)$ **f** $y = P(-x)$



- 2 Use the graph of $y = P(x)$ shown to sketch each polynomial.

a $y = P(x) + 1$ **b** $y = P(x) - 2$ **c** $y = 2P(x)$
d $y = -P(x)$ **e** $y = P(-x)$ **f** $y = -3P(x)$



- 3 Graph $P(x) = (x - 1)(x - 3)^2$ and use it to sketch the graph of each polynomial.

a $y = -P(x)$ **b** $y = P(-x)$ **c** $y = P(x) - 3$ **d** $y = 2P(x)$

- 4 Describe the transformation needed to graph each cubic equation using the graph of $y = x^3$.

a $y = x^3 - 2$ **b** $y = x^3 + 1$ **c** $y = 2x^3$
d $y = -x^3 + 3$ **e** $y = -3x^3$ **f** $y = 2 - x^3$
g $y = 2x^3 - 5$ **h** $y = 4 - 3x^3$ **i** $y = \frac{1}{2}x^3 + 4$



15

Measurement and geometry

Circle geometry

Optional Stage 5.3 topic

The ancient Greek philosopher, Thales of Miletus (624–546 BCE), is believed to have been the first mathematician and scientist to attempt to explain things by reasoning and deduction rather than by resorting to mythology and superstition. He lived before Pythagoras and proved the first theorem in mathematics: that any angle drawn on the circumference of a semicircle is a right angle. This rule is often called Thales' theorem.



Mathematics for the 21st Century

Chapter outline

	Proficiency strands				
15-01 Parts of a circle*	U				C
15-02 Chord properties of circles*	U	F	R		C
15-03 Angle properties of circles*	U	F	R		C
15-04 Tangent and secant properties of circles*	U	F	R		C
15-05 Proofs using circle theorems*	U	F	PS	R	C

*STAGE 5.3

Wordbank

collinear points Points that lie on a straight line.

converse A rule or statement turned back-to-front; the reverse statement

cyclic quadrilateral A quadrilateral bounded by a circle, whose vertices lie on the circle's circumference

intercept The distance between the points where a line intersects other lines or curves

perpendicular bisector An interval or line that bisects another interval at right angles

secant A line that intersects a curve at two points

subtend an angle To sit opposite an angle and 'hold' its arms

tangent A line that touches a curve at one point but does not cross it

In this chapter you will:

- (STAGE 5.3) Prove and apply angle and chord properties of circles 
- (STAGE 5.3) identify the parts of a circle
- (STAGE 5.3) prove chord and angle properties of circles and use them to find unknown lengths and angles in circles
- (STAGE 5.3) prove tangent and secant properties of circles and use them to find unknown lengths and angles in circles

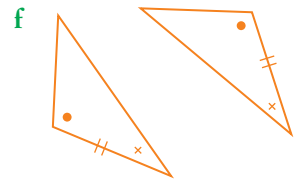
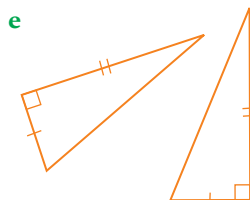
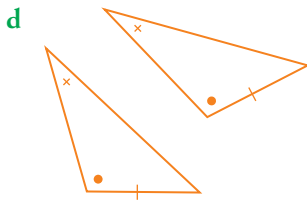
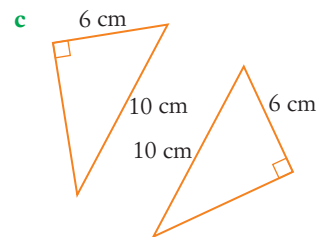
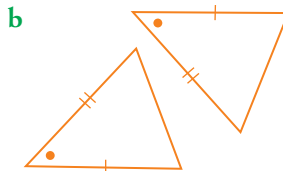
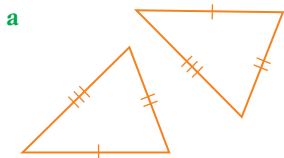
SkillCheck

Worksheet

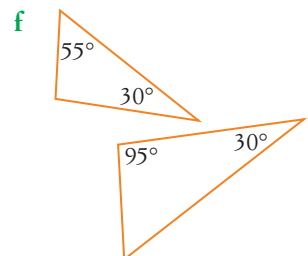
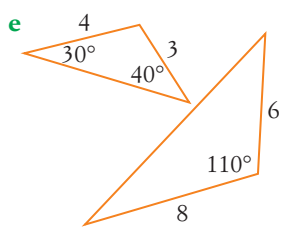
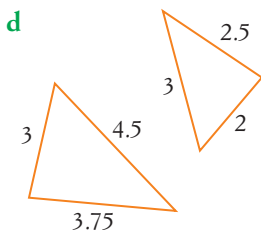
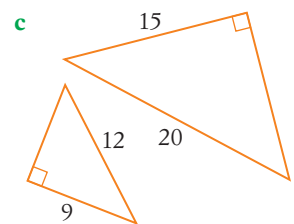
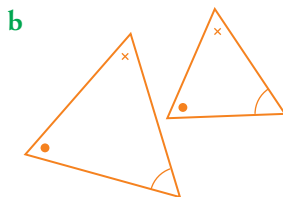
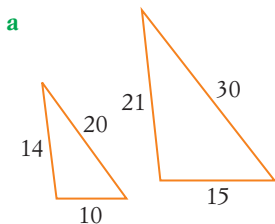
StartUp assignment 15

MAT10MGWK10232

- 1 Which congruence test (SSS, SAS, AAS or RHS) can be used to prove that each pair of triangles are congruent?



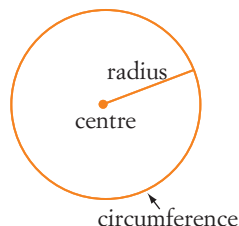
- 2 Which similarity test ('SSS', 'SAS', 'AA' or 'RHS') can be used to prove that each pair of triangles are similar?



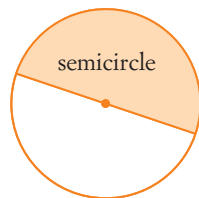
15-01 Parts of a circle

The **radius** is an interval joining the centre to any point on the circle.

The **circumference** is the perimeter of a circle.



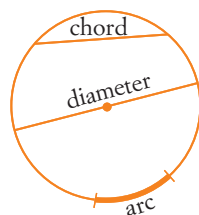
A **semicircle** is half of a circle.



A **chord** is an interval joining two points on the circle.

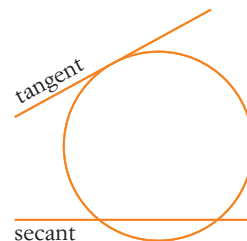
A **diameter** is a chord that passes through the centre.

An **arc** is a part of the circumference of the circle.



A **tangent** is a line touching the circle at one point.

A **secant** is a line that cuts the circle at two points.

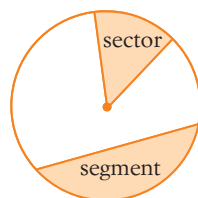


A **quadrant** is quarter of a circle.



A **sector** is part of a circle bounded by two radii and an arc.

A **segment** is part of a circle cut off by a chord.



Stage 5.3

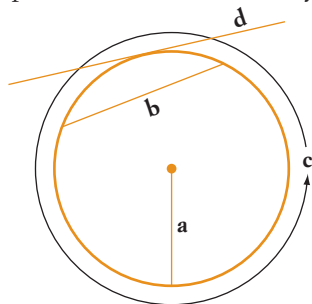
Worksheet

Parts of a circle

MAT10MGWK10233

Exercise 15-01 Parts of a circle

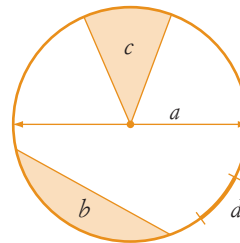
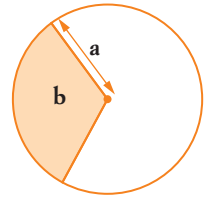
- 1 Name each part of the circle marked by a letter.



Stage 5.3

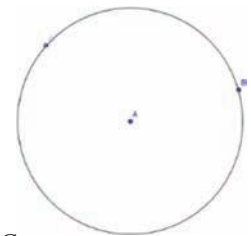
- 2 a Draw a sector and a segment in a circle.
b What is the difference between a sector and a segment?
- 3 State the mathematical rule connecting d (diameter) and r (radius) as an algebraic formula.
- 4 Which parts of a circle are labelled **a** and **b**? Select the correct answer **A**, **B**, **C** or **D**.

A diameter and segment C diameter and sector	B radius and segment D radius and sector
- 5 Write the correct word to match each description.
a the distance from the centre of a circle to its side
b quarter of a circle
c line that touches the outside of a circle once
d an interval from one side of a circle to the other side, through the circle's centre
e an interval from one side of the circle to the other side, not through the centre
f part of the circumference of a circle
g the area inside a circle formed by two radii and an arc
h the complete distance around a circle
i the area inside a circle formed by a chord and an arc
- 6 Name each part of the circle marked by a letter.



Technology Perpendicular to a chord

- 1 Click **View** and **Axes** in GeoGebra.
- 2 Draw a circle using **Circle with centre through point**.
- 3 Insert one **new point** on the circumference of the circle.
- 4 Use **Interval between two points** to join BC .
- 5 Click **Midpoint or Centre**  and chord BC .
- 6 Select **Perpendicular line**  and A and the midpoint of chord BC .
- 7 Select **Distance or Length** and measure C to the midpoint of BC and repeat for point B to the midpoint of BC . Use the **Move** tool to manipulate points B and C . What do you notice?



15-02 Chord properties of circles

Stage 5.3

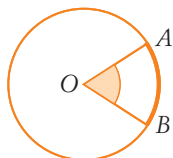
Worksheet

A page of circles

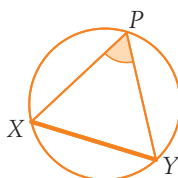
MAT10MGWK10234

Equal chords

When an arc or chord of a circle **subtends** an angle, it sits opposite or under the angle and 'holds' the angle's arms. The angle stands on the ends of the arc or chord. The angle is **subtended by** the arc or chord.



The arc AB subtends the angle at O .

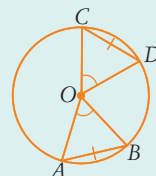


The chord XY subtends the angle at P .

A **theorem** is a rule or statement that can be proved from basic principles. We will now look at theorems relating to chords that can be proved by congruent triangles. These theorems can also be demonstrated using GeoGebra, other graphing technology, or pencil-and-paper constructions.

Summary

Theorem 1: Chords of equal length in a circle subtend equal angles at the centre of the circle.



If $AB = CD$,
 $\angle AOB = \angle COD$.

Proof:

In $\triangle AOB$ and $\triangle COD$:

$AB = CD$ (given)

$OA = OC$ (equal radii)

$OB = OD$ (equal radii)

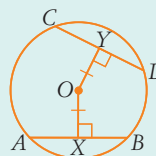
$\therefore \triangle AOB \equiv \triangle COD$ (SSS)

$\therefore \angle AOB = \angle COD$ (matching angles of congruent triangles)

So chords of equal length subtend equal angles at the centre of a circle.

Summary

Theorem 2: Chords of equal length in a circle are equidistant from the centre of the circle.



If $AB = CD$,
then $OX = OY$.

Stage 5.3

Proof:

Draw radii OA , OB , OC and OD .

In $\triangle AOX$ and $\triangle COY$:

$$OA = OC \quad (\text{equal radii})$$

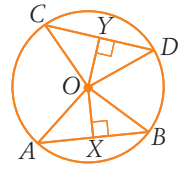
$$\angle A = \angle C \quad (\text{matching angles of congruent } \triangle AOB \text{ and } \triangle COD \text{ proved in Theorem 1})$$

$$\angle OXA = \angle OYC = 90^\circ \quad (\text{given})$$

$$\therefore \triangle AOX \equiv \triangle COY \quad (\text{AAS})$$

$$\therefore OX = OY \quad (\text{matching sides of congruent triangles})$$

So equal chords are equidistant from the centre of a circle.

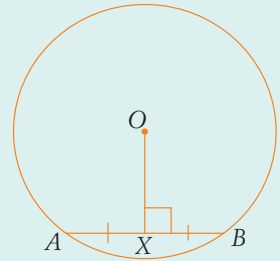


Perpendicular bisectors of chords

Summary

Theorem 3: The perpendicular from the centre of a circle to a chord bisects the chord.

The **converse** of this is also true. The line from the centre of a circle to the midpoint of a chord is perpendicular to the chord.



The word 'converse' means to 'turn around' or 'reverse'. So the converse of a theorem is the theorem written 'back-to-front'.

Proof:

Draw radii OA and OB .

In $\triangle AOX$ and $\triangle BOX$:

$$OA = OB \quad (\text{equal radii})$$

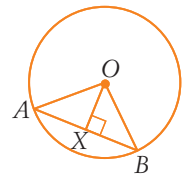
OX is common.

$$\angle OXA = \angle OXB = 90^\circ \quad (OX \perp AB)$$

$$\therefore \triangle AOX \equiv \triangle BOX \quad (\text{RHS})$$

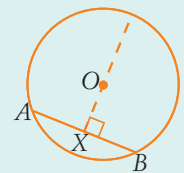
$$\therefore AX = BX \quad (\text{matching sides of congruent triangles})$$

So the perpendicular from the centre of a circle to a chord bisects the chord.



Summary

Theorem 4: The perpendicular bisector of a chord in a circle passes through the centre of the circle.



Proof:

The perpendicular from O to AB bisects AB . (proved in Theorem 3)

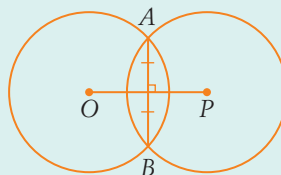
$\therefore OX$ is the perpendicular bisector of AB .

$\therefore$ The perpendicular bisector of AB passes through the centre O .

Common chords

Summary

Theorem 5: When two circles intersect, the line joining their centres bisects their common chord at right angles.



Proof:

Draw radii OA , OB , PA and PB and let OP and AB bisect at M .

In $\triangle OAP$ and $\triangle OBP$:

$OA = OB$ (equal radii)

$PA = PB$ (equal radii)

OP is common.

$\therefore \triangle OAP \equiv \triangle OBP$ (SSS)

$\therefore \angle AOM = \angle BOM$ (matching angles of congruent triangles)

In $\triangle OAM$ and $\triangle OBM$:

$OA = OB$ (equal radii)

OM is common.

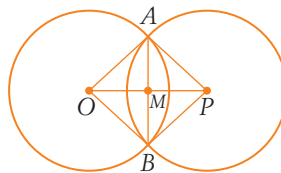
$\therefore \angle AOM = \angle BOM$ (proved above)

$\therefore \triangle OAM \equiv \triangle OBM$ (SAS)

$\therefore AM = MB$ (matching sides of congruent triangles)

$\therefore AB \perp OP$ (line from the centre of a circle to the midpoint of a chord)

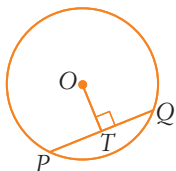
So the perpendicular from the centre of a circle to a chord bisects the chord.



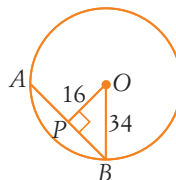
Example 1

In each diagram, O is the centre of a circle.

a If $PQ = 35$ cm, find PT .



b If $OP = 16$ cm, find AB .



Stage 5.3

Solution

$$\begin{aligned} \text{a } PT &= \frac{1}{2}(PQ) \quad (\text{the perpendicular from the centre bisects the chord}) \\ &= \frac{1}{2} \times 35 \\ &= 17.5 \text{ cm} \end{aligned}$$

$$\begin{aligned} \text{b } PB^2 &= 34^2 - 16^2 \\ &= 900 \end{aligned}$$

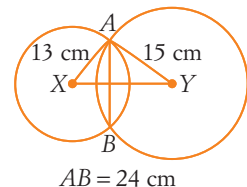
by Pythagoras' theorem

$$\begin{aligned} PB &= \sqrt{900} \\ &= 30 \text{ cm} \end{aligned}$$

$$\begin{aligned} \therefore AB &= 2 \times PB \quad (\text{the perpendicular from the centre bisects the chord}) \\ &= 60 \text{ cm} \end{aligned}$$

Example 2

The length AB of the common chord of two circles is 24 cm. If the radii of the circles are 13 cm and 15 cm, find the distance between their centres.



Solution

We need to find XY .

Let XY and AB cross at M .

$XY \perp AB$ (common chord of circles)

$$\begin{aligned} AM &= \frac{1}{2} \times AB \\ &= \frac{1}{2} \times 24 \\ &= 12 \text{ cm} \end{aligned}$$

$$\begin{aligned} XM^2 &= 13^2 - 12^2 \\ &= 25 \end{aligned}$$

by Pythagoras' theorem

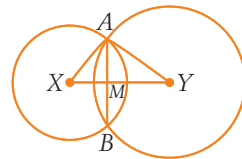
$$\begin{aligned} XM &= \sqrt{25} \\ &= 5 \text{ cm} \end{aligned}$$

$$\begin{aligned} YM^2 &= 15^2 - 12^2 \\ &= 81 \end{aligned}$$

$$\begin{aligned} YM &= \sqrt{81} \\ &= 9 \text{ cm} \end{aligned}$$

$$\begin{aligned} \therefore XY &= 5 + 9 \\ &= 14 \text{ cm} \end{aligned}$$

The distance between the centres is 14 cm.



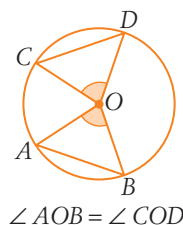
Exercise 15-02 Chord properties of circles

Stage 5.3

See Example 1

- 1 The converse of Theorem 1 is: 'Chords subtending equal angles at the centre of a circle have equal lengths'.

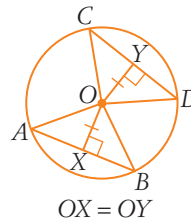
- Prove that $\triangle AOB \equiv \triangle COD$, given that $\angle AOB = \angle COD$.
- Hence, prove that $AB = CD$.



$\angle AOB = \angle COD$

- 2 The converse of Theorem 2 is: 'Chords that are equidistant from the centre of a circle are equal'.

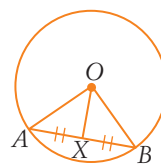
- Prove that $\triangle OXA \equiv \triangle OYC$ and therefore $AX = CY$.
- Prove that $\triangle OXB \equiv \triangle OYD$ and therefore $BX = DY$.
- Hence show that $AB = CD$.



$OX = OY$

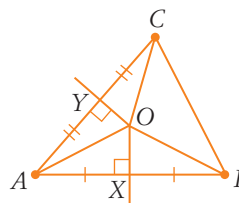
- 3 The converse of Theorem 3 is: 'The line from the centre of a circle to the midpoint of a chord is perpendicular to the chord.'

- Prove that $\triangle OXA \equiv \triangle OXB$ and therefore $\angle OXA = \angle OXB$.
- Prove that $OX \perp AB$.



$AX = BX$

- Draw a circle and construct any two non-parallel chords.
 - Bisect these chords.
 - Through what point must each perpendicular bisector pass? (Use Theorem 4.)
 - Explain why the point of intersection between the two perpendicular bisectors is the centre of the circle.
- 5 The diagram shows the perpendicular bisectors of two sides AB and CA of a triangle, intersecting at O . It can be proved that O is the centre of a circle passing through all three vertices of the triangle.

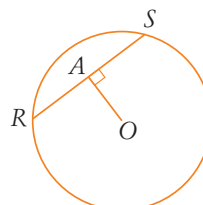


The full statement on this theorem is: 'Given any three non-collinear* points, the point of intersection of the perpendicular bisectors of any two sides of the triangle formed by the three points is the centre of the circle through all three points.'

(*Non-collinear means *not* on the same straight line.)

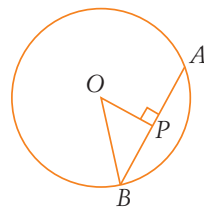
Prove this theorem as follows:

- prove that $\triangle AOY \equiv \triangle COY$ and therefore $OA = OC$
 - prove that $\triangle AOX \equiv \triangle BOX$ and therefore $OA = OB$
 - hence prove that the circle with centre O and radius OA also passes through B and C .
- 6 This circle has a radius of 7 cm and the chord RS is 8 cm. Find, correct to 2 decimal places, the length of OA .



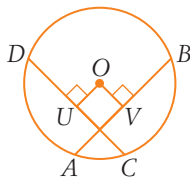
Stage 5.3

- 7 In the diagram, $AB \perp OP$. If $OB = 32.5$ cm and $OP = 12.5$ cm, find AB .

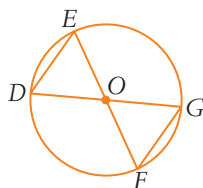


- 8 In each problem, state which chord theorem you use.

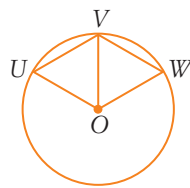
- a $OU = OV$
 $AB = 9$ m
 Find the length of UC .



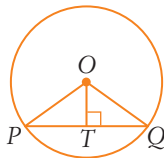
- b $GF = 12$ m
 Find the length of DE .



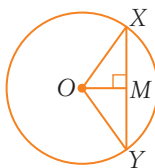
- c $UV = VW$
 $\angle VOW = 64^\circ$
 Find the size of $\angle UVO$.



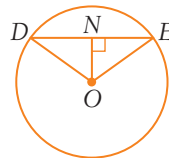
- d $OT \perp PQ$
 $OQ = 17$ mm
 $OT = 8$ mm
 Find PQ .



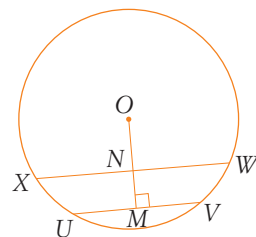
- e $OM \perp XY$
 $XY = 40$ cm
 $OX = 29$ cm
 Find OM .



- f $DE \perp ON$
 $ON = 18$ m
 $DE = 36$ m
 Find the radius.

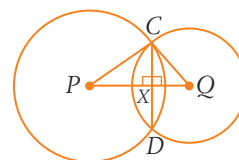
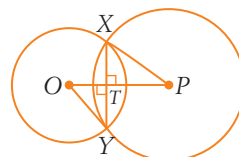


- 9 Find the radius of a circle in which a chord of length 96 cm is 20 cm from the centre.
- 10 On a clear day, you can see 11.5 km in any direction from a particular lookout. If a straight road is 6.9 km from the lookout, what length of the road can be seen from the lookout?
- 11 In the diagram, $OM \perp UV$, $XW \parallel UV$, $UV = 24$ cm and $XW = 32$ cm. If the radius of the circle is 20 cm, find the length of MN .



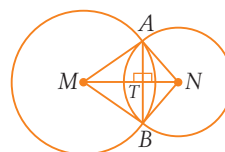
See Example 2

- 12 O and P are the centres of circles. If $OY = 40$ cm, $XP = 51$ cm and $XY = 48$ cm, find the distance between the centres.
- 13 P and Q are the centres of circles. If $DC = 32$ cm, $PX = 30$ cm and $XQ = 12$ cm, find the radius of each circle.



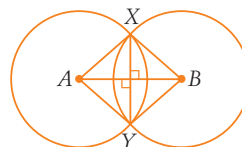
- 14 M and N are the centres of circles. $MT = 36$ cm, $NT = 20$ cm and $AM = 39$ cm. Find:

a AB b the area of kite $ANBM$



- 15 A and B are the centres of circles with the same radius. $XY = 40$ cm and $AB = 96$ cm. Find:

a the radius of each circle
b the area of rhombus $AXBY$



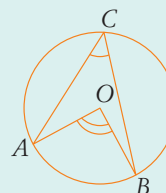
15-03 Angle properties of circles

We will now look at proofs of theorems relating to angles in a circle. These theorems can also be demonstrated using GeoGebra, other graphing technology, or pencil-and-paper constructions.

Summary

Theorem 6: The angle at the centre of a circle is twice the angle at the circumference, standing on the same arc.

$$\angle AOB = 2 \times \angle ACB$$



The arms of $\angle ACB$ touch the endpoints of the arc AB , so we say that $\angle ACB$ **stands on** the arc AB . $\angle AOB$ is the **angle at the centre** and $\angle ACB$ is the **angle at the circumference**. Both angles stand on the same arc AB .

Proof:

Draw radius CO and produce (extend) to point D .

Let $\angle ACD = x$ and $\angle BCD = y$.

$$\therefore \angle ACB = x + y$$

$$OA = OC = OB \quad (\text{equal radii})$$

$$\therefore \angle OAC = x \quad (\text{equal angles in isosceles } \triangle OAC)$$

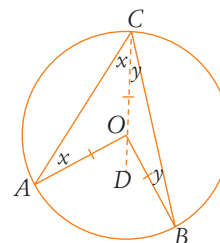
$$\text{Similarly, } \angle OBC = y \quad (\text{equal angles in isosceles } \triangle OBC)$$

$$\begin{aligned} \angle AOD &= x + x \quad (\text{exterior angle of } \triangle OAC) \\ &= 2x \end{aligned}$$

$$\begin{aligned} \text{Similarly, } \angle BOD &= y + y \quad (\text{exterior angle of } \triangle OBC) \\ &= 2y \end{aligned}$$

$$\begin{aligned} \therefore \angle AOB &= 2x + 2y \\ &= 2(x + y) \\ &= 2 \times \angle ACB \end{aligned}$$

So the angle at the centre of the circle is twice the angle at the circumference, standing on the same arc.



Stage 5.3

Worksheet

A page of circles

MAT10MGWK10234

Puzzle sheet

Finding angles in circles

MAT10MGPS00054

Worksheet

Angles problems with algebra

MAT10MGWK00040

Stage 5.3

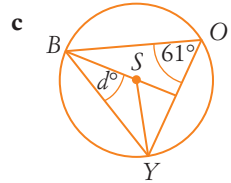
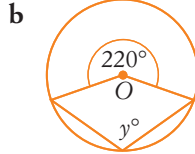
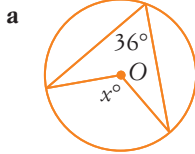
Video tutorial

Circle geometry

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Example 3

Find the value of each pronumeral.



Solution

a $x = 2 \times 36$ (angle at centre is twice angle at circumference)
 $= 72$

b $y = \frac{1}{2} \times 220$ (angle at centre is twice angle at circumference)
 $= 110$

c $\angle BSY = 2 \times 61^\circ$ (angle at centre is twice angle at circumference)
 $= 122^\circ$

$SB = SY$ (equal radii)

$\therefore \angle SBY = \angle SYB = d^\circ$ (equal angles in isosceles $\triangle SYB$)

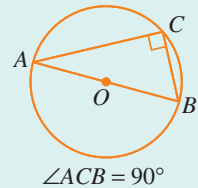
$2d + 122 = 180$ (angle sum of $\triangle SYB$)

$2d = 58$

$d = 29$

Summary

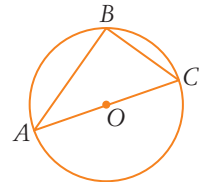
Theorem 7: The angle in a semicircle is a right angle.



Proof:

$\angle AOC = 180^\circ$ (a straight angle)

$\angle ABC = \frac{1}{2}$ of $\angle AOC$ (angle at centre is twice angle at circumference)
 $= \frac{1}{2} \times 180^\circ$
 $= 90^\circ$



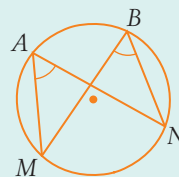
The angle in a semicircle is a right angle.

Summary

Theorem 8: Angles at the circumference of a circle, standing on the same arc, are equal.

Or put another way, angles in the same segment are equal.

$$\angle A = \angle B$$



Angles $\angle A$ and $\angle B$ both stand on the same arc MN .

Proof:

Join radii OM and ON .

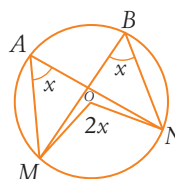
Let $\angle MON = 2x$

$$\begin{aligned}\therefore \angle A &= \frac{1}{2} \times \angle MON \quad (\text{angle at the centre is twice the angle at the circumference}) \\ &= \frac{1}{2} \times 2x \\ &= x\end{aligned}$$

$$\begin{aligned}\text{Similarly, } \angle B &= \frac{1}{2} \times \angle MON \\ &= \frac{1}{2} \times 2x \\ &= x\end{aligned}$$

$$\therefore \angle A = \angle B$$

So angles standing on the same arc are equal.



Example 4

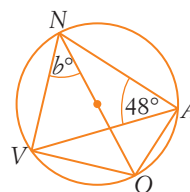
Find b , giving reasons.

Solution

$$\angle NAO = 90^\circ \quad (\text{angle in a semicircle})$$

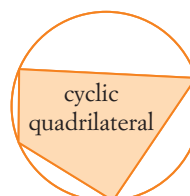
$$\begin{aligned}\therefore \angle VAO &= 90^\circ - 48^\circ \\ &= 42^\circ\end{aligned}$$

$$\therefore b = 42 \quad (\text{angles standing on the same arc})$$



Cyclic quadrilaterals

A **cyclic quadrilateral** is a quadrilateral whose four vertices lie on the circumference of a circle.



Video tutorial

Circle geometry

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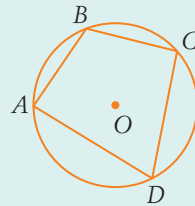
Stage 5.3

Summary

Theorem 9: The opposite angles in a cyclic quadrilateral are supplementary.

$$\angle A + \angle C = 180^\circ$$

$$\angle B + \angle D = 180^\circ$$

**Proof:**

Draw radii OA and OC .

Let $\angle B = x^\circ$ and $\angle D = y^\circ$.

$\therefore$ Reflex $\angle AOC = 2x^\circ$ (angle at centre is twice angle at the circumference)

$\therefore$ Obtuse $\angle AOC = 2y^\circ$ (angle at centre is twice angle at the circumference)

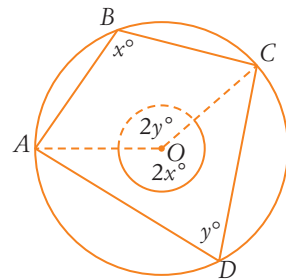
$$2x + 2y = 360 \quad (\text{angles at a point})$$

$$x + y = 180$$

$$\therefore \angle B + \angle D = 180^\circ$$

Hence, $\angle A + \angle C = 180^\circ$ (angle sum of a quadrilateral)

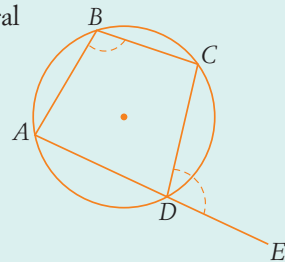
So the opposite angles of a cyclic quadrilateral are supplementary.



Summary

Theorem 10: The exterior angle at a vertex of a cyclic quadrilateral is equal to the interior opposite angle.

$$\angle CDE = \angle B$$

**Proof:**

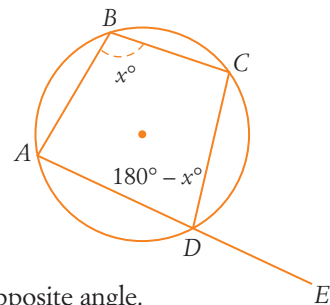
Let $\angle B = x^\circ$.

$$\angle ADC = 180^\circ - x^\circ \quad (\text{opposite angles of a cyclic quadrilateral})$$

$$\begin{aligned} \angle CDE &= 180^\circ - (180^\circ - x^\circ) \quad (\text{angles on a straight line}) \\ &= 180 - 180 + x^\circ \\ &= x^\circ \end{aligned}$$

$$\therefore \angle ABC = \angle CDE = x^\circ$$

The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.



Example 5

Find p and q , giving reasons.

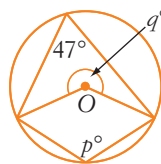
Solution

$$p = 180 - 47 \quad (\text{opposite angles of a cyclic quadrilateral})$$

$$= 133$$

$$q = 2 \times 133 \quad (\text{angle at centre is twice angle at circumference})$$

$$= 266$$



Example 6

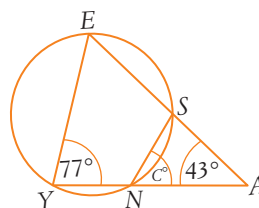
Find c , giving reasons.

Solution

$$\angle ASN = 77^\circ \quad (\text{exterior angle of cyclic quadrilateral YESN})$$

$$c = 180 - 43 - 77 \quad (\text{angle sum of } \triangle SAN)$$

$$= 60$$



Stage 5.3

Video tutorial

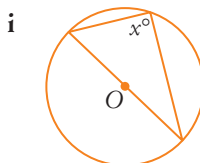
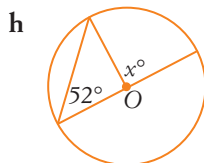
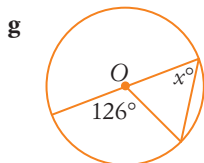
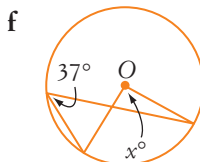
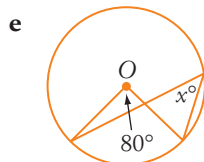
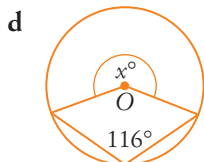
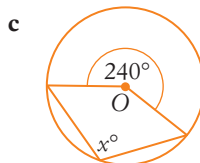
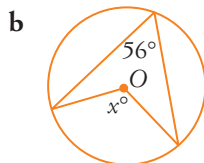
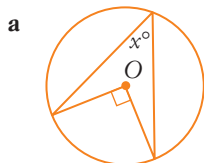
Circle geometry

MAT10MGVT10020

See Example 3

Exercise 15-03 Angle properties of circles

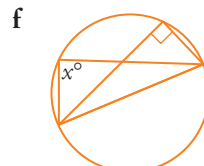
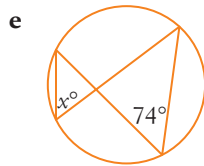
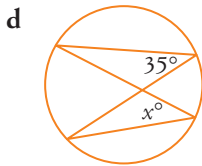
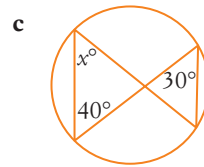
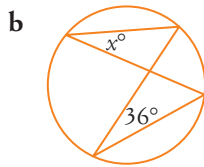
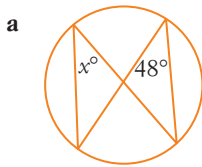
1 Find x , giving reasons, if O is the centre of the circle.



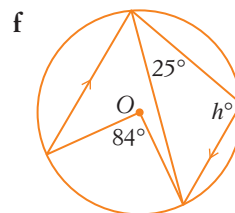
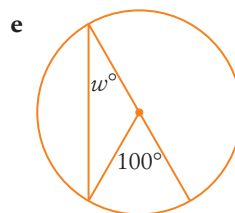
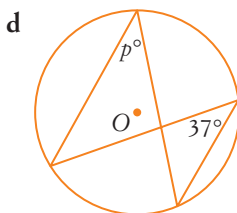
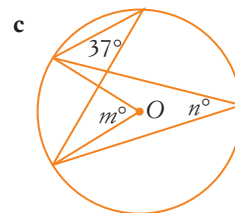
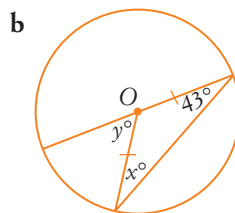
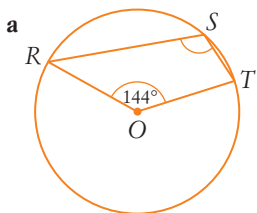
Stage 5.3

See Example 4

2 Find x , giving reasons.

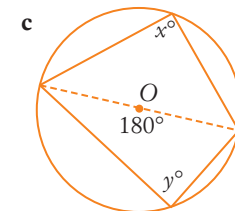
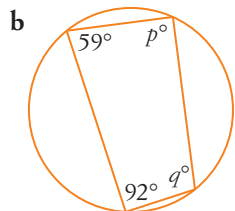
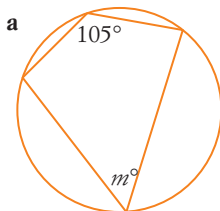


3 Find the value of each pronumeral, giving reasons.



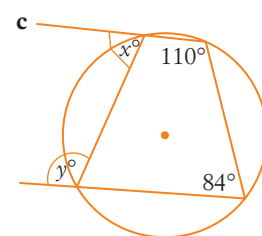
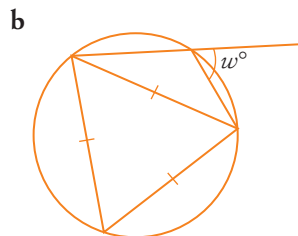
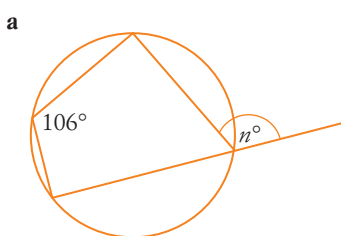
See Example 5

4 Find the value of each pronumeral, giving reasons.

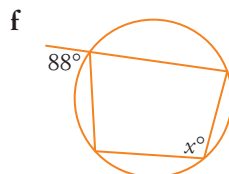
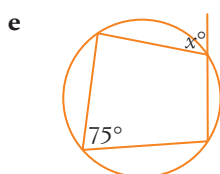
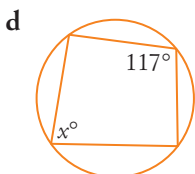
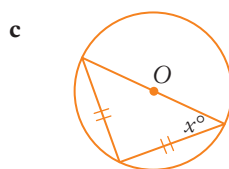
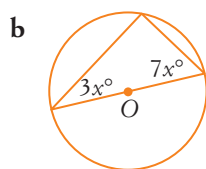
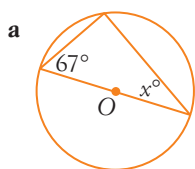


See Example 6

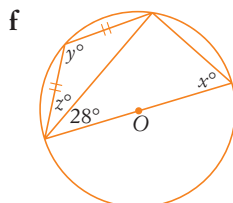
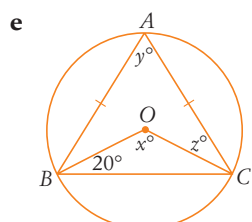
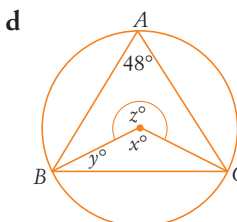
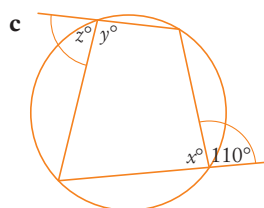
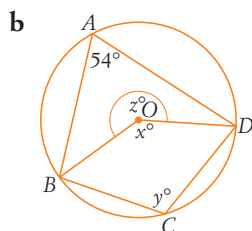
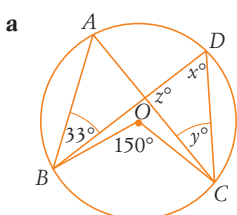
5 Find the value of each pronumeral, giving reasons.



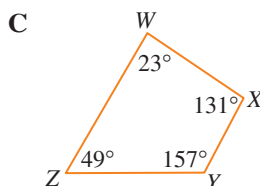
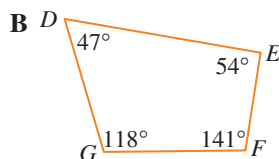
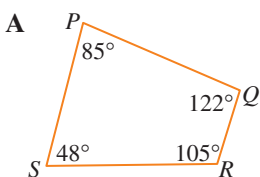
6 Find x , giving reasons.



7 Find the value of each pronumeral, giving reasons.



8 Which of these quadrilaterals can also be a cyclic quadrilateral? Why?



15-04 Tangent and secant properties of circles

We will now look at proofs of theorems relating to tangents and secants in a circle. These theorems can also be demonstrated using GeoGebra, other graphing technology, or pencil-and-paper constructions.

NSW

Worksheet

Circle geometry cards

MAT10MGWK10235

Worksheet

Circle geometry review

MAT10MGWK10236

Stage 5.3

Worksheet

A page of circles

MAT10MGWK10234

Tangents to a circle

A **tangent** to a circle is a line that touches the circle at only one point. The point of intersection is called the **point of contact**.

At any point on a circle, there is only one tangent to the circle at that point.

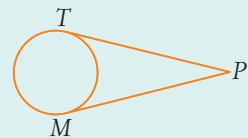
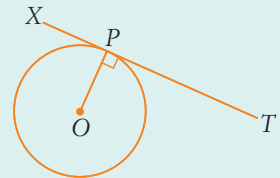
Summary

Theorem 11: A tangent to a circle is perpendicular to the radius drawn to the point of contact.

$$\angle OPT = 90^\circ$$

Theorem 12: The two tangents drawn to a circle from an external point are equal in length.

$$PT = PM.$$



Proof:

Draw radii OT and OM , and join OP .

In $\triangle PTO$ and $\triangle PMO$:

OP is common.

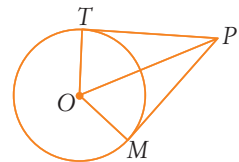
$OT = OM$ (equal radii)

$\angle OTP = \angle OMP = 90^\circ$ (angle between tangent and radius)

$\therefore \triangle PTO \equiv \triangle PMO$ (RHS)

$\therefore PT = PM$ (matching sides of congruent triangles)

So tangents drawn from an external point are equal in length.

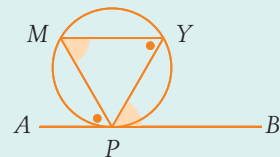


Alternate segment theorem

Summary

Theorem 13: The angle between a tangent and a chord drawn to the point of contact is equal to the angle in the alternate segment.

$$\angle YPB = \angle M$$



(To be proved in question 1 of Exercise 15-04)

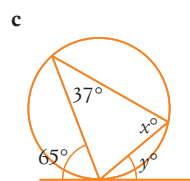
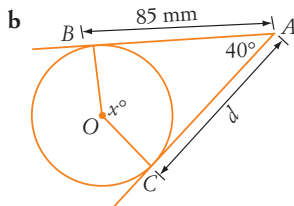
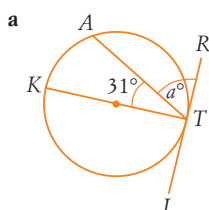
AB is a tangent to the circle. The chord PY divides the circle into two segments.

$\angle M$ is in the alternate (opposite) segment to $\angle YPB$.

Similarly, $\angle Y$ is in the alternate segment to $\angle APM$, so $\angle APM = \angle Y$.

Example 7

Find the value of each pronumeral, giving reasons.



Solution

a $\angle KTR = 90^\circ$ (angle between a tangent and a radius)

$$a + 31 = 90$$

$$a = 59$$

b $d = 85$ (tangents from an external point)

$\angle ABO = \angle ACO = 90^\circ$ (angle between a tangent and a radius)

$\therefore x + 90 + 40 + 90 = 360$ (angle sum of a quadrilateral)

$$x + 220 = 360$$

$$x = 140$$

c $x = 65$ (angle in alternate segment)

$y = 37$ (angle in alternate segment)

Stage 5.3

Video tutorial

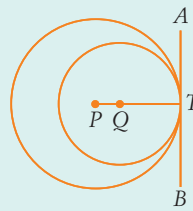
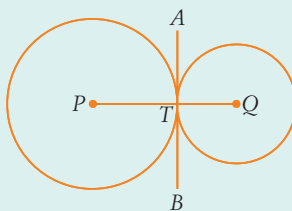
Circle geometry

MAT10MGVT10020

Summary

Theorem 14: When two circles touch, their centres and the point of contact are **collinear**.

The circles have a common tangent AB . P , T and Q lie on a straight line (to be proved in question 2 of Exercise 15-04).



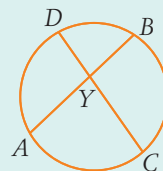
Products of intercepts

When a line crosses another line or curve at two points, the distance between the points is called an **intercept**.

Summary

Theorem 15: The products of the intercepts of two intersecting chords are equal.

$$AY \times YB = CY \times YD$$



AY , YB , CY and YD are called intercepts.

Stage 5.3

Proof:

Draw AD and CB .

In $\triangle ADY$ and $\triangle CBY$:

$$\angle D = \angle B \quad (\text{angles in the same segment})$$

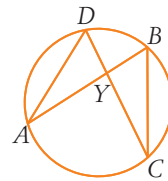
$$\angle A = \angle C \quad (\text{angles in the same segment})$$

$$\therefore \triangle ADY \parallel \triangle CBY \quad (\text{equiangular})$$

$$\therefore \frac{AY}{CY} = \frac{YD}{YB} \quad (\text{matching sides of similar triangles})$$

$$\therefore AY \times YB = CY \times YD$$

Therefore, the products of intercepts of two intersecting chords are equal.

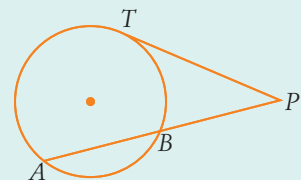
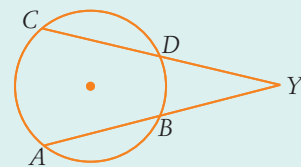
**Summary**

Theorem 16: The product of the intercepts of two intersecting secants to a circle from an external point are equal.

$$YA \times YB = YC \times YD$$

Theorem 17: The square of a tangent to a circle from an external point equals the product of the intercepts of any secant from the point.

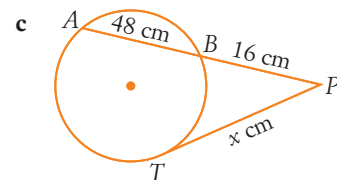
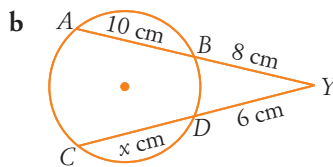
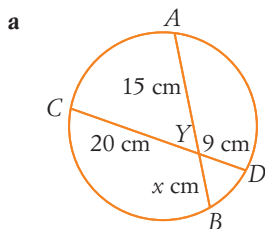
$$PT^2 = PB \times PA$$



(To be proved in question 3 of Exercise 15-04.)

Example 8

Find x , giving reasons.

**Solution**

a $x \times 15 = 9 \times 20$ (intercepts of intersecting chords)

$$15x = 180$$

$$x = 12$$

b $6 \times (6 + x) = 8 \times 18$ (intercepts of intersecting secants)

$$36 + 6x = 144$$

$$6x = 108$$

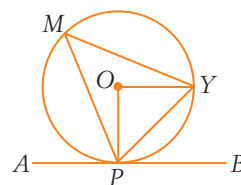
$$x = 18$$

$$\begin{aligned} \text{c } x^2 &= 16 \times (16 + 48) && (\text{tangent - secant theorem}) \\ &= 1024 \\ x &= \sqrt{1024} \\ &= 32 \end{aligned}$$

Exercise 15-04 Tangent and secant properties of circles

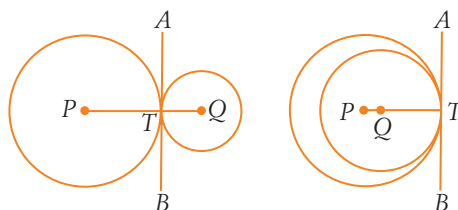
1 AB is a tangent to a circle at P , with centre O .

- If $\angle YPB$ is an angle between a tangent and chord, what is the angle in the alternate segment?
- If $\angle YPB = x^\circ$, why is $\angle OPY = 90^\circ - x^\circ$?
- Show that $\angle POY = 2x^\circ$.
- Hence show that $\angle PMY = \angle YPB$.



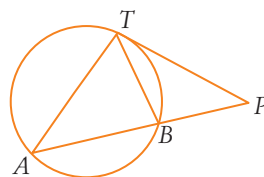
2 Two circles with centres P and Q touch at T with a common tangent AB .

- Explain why $\angle ATP$ and $\angle ATQ$ are right angles.
- Hence, show that P , T and Q are on the same line.

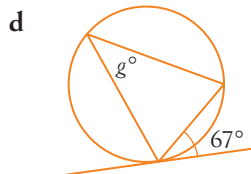
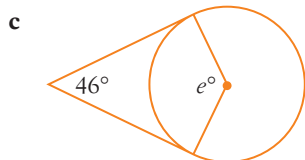
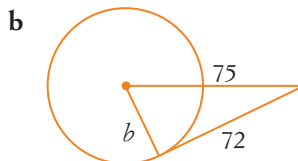
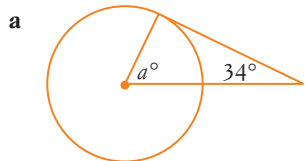


3 PT is a tangent and PA is a secant.

- Prove that $\triangle PTA \sim \triangle PBT$.
- Hence, show that $\frac{PT}{PB} = \frac{PA}{PT}$ and $PT^2 = PB \times PA$.



4 Find the value of each pronumeral, giving reasons.

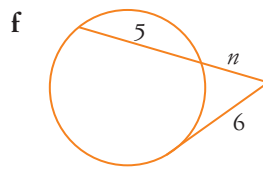
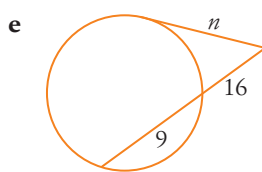
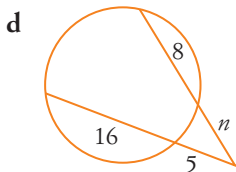
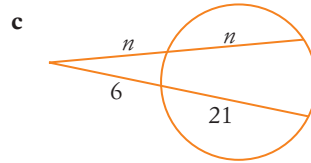
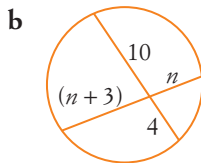
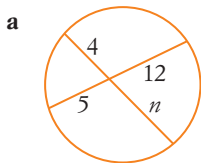


See Example 7

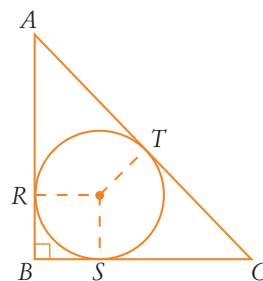
Stage 5.3

See Example 8

- 5 Find the value of n .

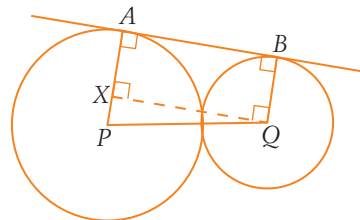


- 6 ABC is a right-angled triangle with an internal circle such that each side is a tangent to the circle. R , S and T are the respective points of contact of each of the tangents. If $SC = 30$ cm, $RB = 5$ cm and $AT = x$, find the value of x .



- 7 Two circles with centres P and Q and radii 18 cm and 8 cm respectively, touch each other externally as shown. Find the length of:

- a XP b AB



NSW

15-05 Proofs using circle theorems

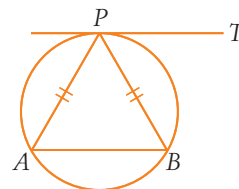
Example 9

PT is a tangent to a circle, and $PA = PB$.

Prove that $PT \parallel AB$.

Solution

$\angle TPB = \angle A$ (angle in alternate segment)
 $\angle A = \angle B$ (equal angles of isosceles $\triangle PAB$)
 $\therefore \angle TPB = \angle B$
 $\therefore PT \parallel AB$ (alternate angles are equal)

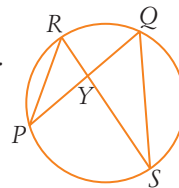


Exercise 15-05 Proofs using circle theorems

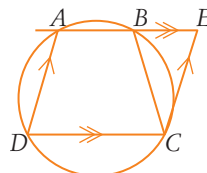
Stage 5.3

See Example 9

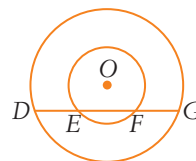
- 1 PQ and RS are two chords of a circle intersecting at the point Y . Prove that the triangles PYR and SYQ are similar and hence $PY \times YQ = RY \times YS$.



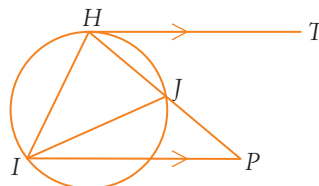
- 2 $ABCD$ is a cyclic quadrilateral. Side AB is produced to E so that $AECD$ is a parallelogram. Prove that $\triangle CBE$ is an isosceles triangle.



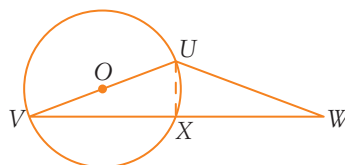
- 3 O is the centre of two concentric circles. $DEFG$ is a straight line. Prove that $DE = FG$.



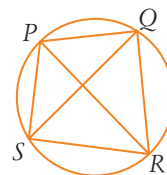
- 4 HT is a tangent and parallel to IP . HJP is a straight line. Prove that $\angle HIP = \angle HJI$.



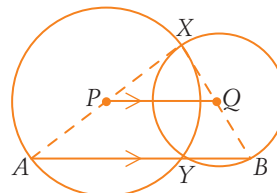
- 5 UVW is an isosceles triangle, where $UV = UW$. A circle with centre O is drawn on one of the equal sides. Prove that the circle bisects the base of the triangle.



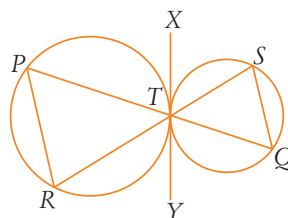
- 6 $PQRS$ is a cyclic quadrilateral. PR bisects $\angle QRS$. Prove that $\triangle SPQ$ is isosceles.

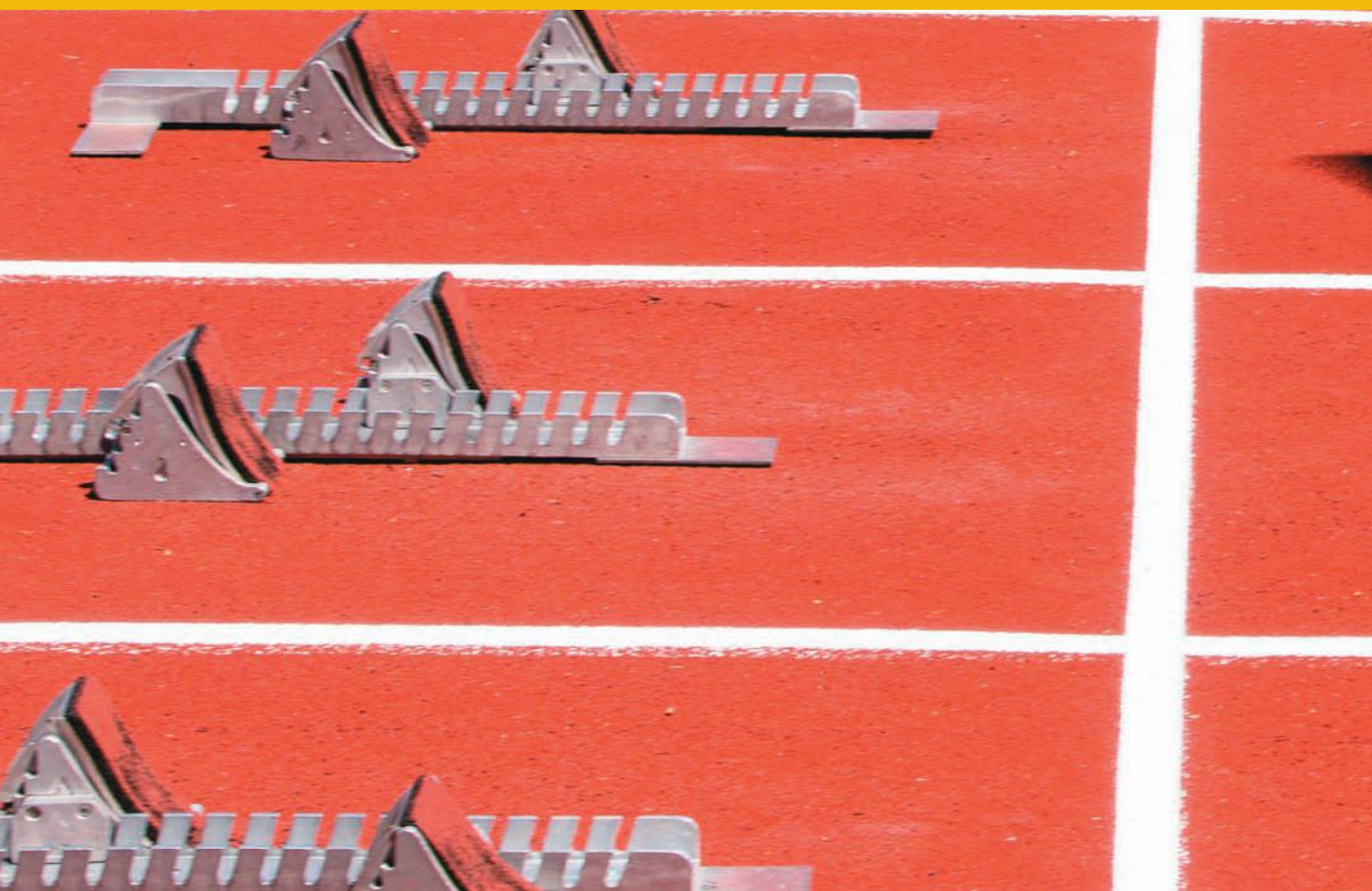


- 7 Two circles with centres P and Q intersect at X and Y . AB is parallel to PQ , the interval joining the centres of the circles. Prove that $PQ = \frac{1}{2}AB$.



- 8 Two circles touch externally at T . XY is the common tangent. PTQ and RTS are straight lines. Prove that $PR \parallel SQ$.





16

Number and algebra

Functions

Optional Stage 5.3 topic

Functions are used in many areas of mathematics and science and can be used to describe the relationship between two quantities such as the distance run by an athlete and the time taken. It is thought that the German mathematician, Gottfried Leibniz (1646–1716), was the first person to use the word *function* in 1694.



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Chapter outline

16-01 Functions*

16-02 Function notation*

16-03 Inverse functions*

16-04 Graphing $y = a^x$ and $y = \log_a x^*$

16-05 Graphing translations of functions*

Proficiency strands

U	F	R	C
U	F	R	C
U	F	R	C
U	F	R	C
U	F	R	C

*STAGE 5.3

Wordbank

domain The set of permissible x -values for the function $y = f(x)$

function A rule or relationship between two variables where, for each value of the independent variable (input value), x , there is only one value of the dependent variable (output value), y

function notation A way of writing a function, using the form $y = f(x)$

inverse function The reverse of a function, which 'undoes' the original function, found by interchanging the dependent and independent variables (x and y)

range The set of permissible y -values for the function $y = f(x)$

vertical line test A test applied to a graph to determine whether it is the graph of a function

In this chapter you will:

- (STAGE 5.3) understand the definition of a function and use function notation
- (STAGE 5.3) use the vertical line test on a graph to test whether it represents a function
- (STAGE 5.3) find the domain and range of a function
- (STAGE 5.3) understand and graph inverse functions
- (STAGE 5.3) graph exponential and logarithmic functions
- (STAGE 5.3) sketch $y = f(x) + k$ and $y = f(x - a)$ given the graph of $y = f(x)$

SkillCheck

Worksheet

StartUp assignment 16

MAT10NAWK10237

- Sketch the graph of each parabola.
a $y = x^2$ **b** $y = x^2 - 3$ **c** $y = x^2 + 3x$
- Sketch the graph of each cubic curve.
a $y = x^3$ **b** $y = x^3 + 3$ **c** $y = x^3 - 1$
- Sketch the graphs of each hyperbola.
a $y = \frac{1}{x}$ **b** $y = \frac{1}{x-1}$ **c** $y = \frac{1}{x+2}$
- Sketch the graphs of each exponential curve.
a $y = 2^x$ **b** $y = 3^{-x}$ **c** $y = -4^x$
- Make x the subject of each equation.
a $y = 2x - 1$ **b** $y = \frac{3}{x+1}$ **c** $y = x^2 + 4$

Stage 5.3

NSW

16-01 Functions

The concept of a function may be used when describing the relationship between two variables.

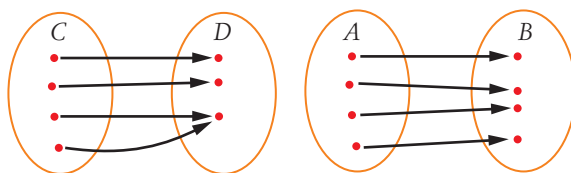
Summary

A **function** is a relationship between two variables where, for each value of the independent variable, there exists only one value of the dependent variable.

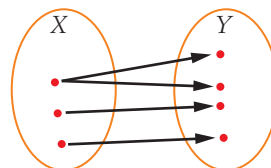
If x is the independent variable and y is the dependent variable, then a function is a relationship where for each x -value, there is only one y -value.

A function can also be visualised as a relationship between two sets of values where every value of one set associates with (or maps to) exactly one value of the second set.

Two different functions are shown in the diagrams below.



The diagram below does *not* represent a function because one value of set X associates with (or maps to) *two* members of set Y .



Example 1

Do the points $(3, 5)$, $(4, 3)$, $(4, -2)$ represent a function?

Solution

When $x = 4$, $y = 3$ and $y = -2$.

So, one x -value gives two y -values.

$\therefore$ The points do not represent a function.

Vertical line test

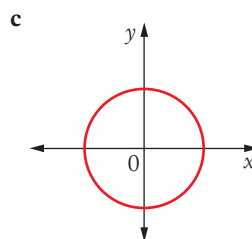
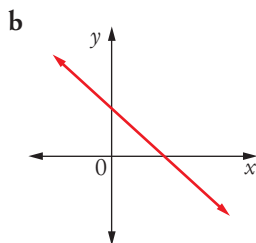
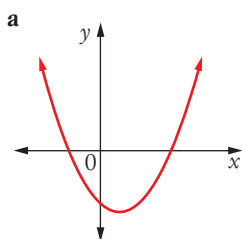
Summary

Any vertical line drawn on the graph of a function will cut the graph only once.

If any vertical line can cut a graph in more than one place, then the graph is not a function.

Example 2

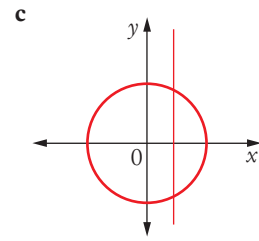
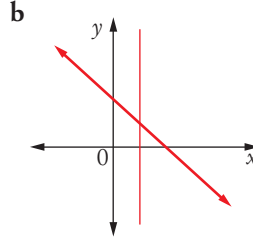
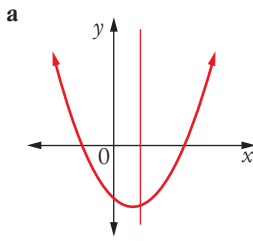
Use the vertical line test to determine whether each graph represents a function.



Stage 5.3

Solution

Draw a vertical line on each graph.



- a** Any vertical line will only cut the graph once.
 $\therefore$ The graph represents a function.
- b** Any vertical line will only cut the graph once.
 $\therefore$ The graph represents a function.
- c** A vertical line can cut the graph twice.
 $\therefore$ The graph does not represent a function.

For every x -value, there is only one y -value.

For this x -value, there are two y -values.

Exercise 16-01 Functions

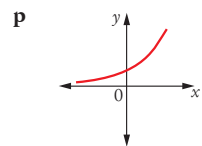
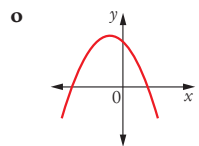
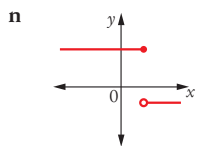
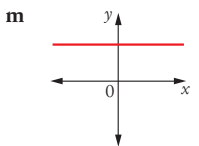
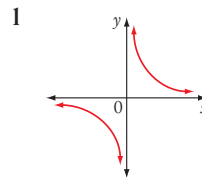
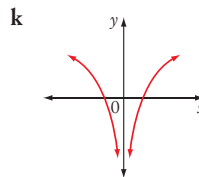
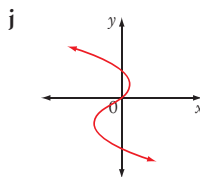
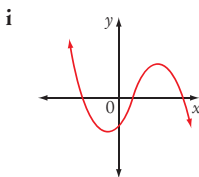
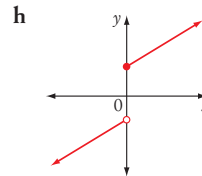
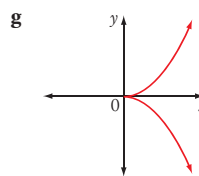
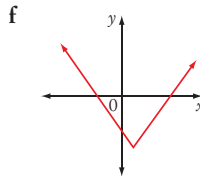
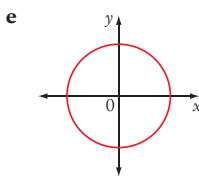
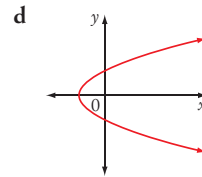
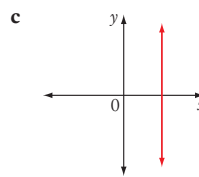
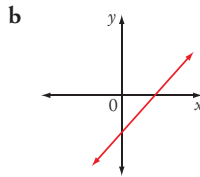
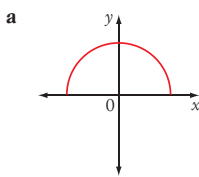
See Example 1

- 1 Determine whether each set of points represents a function.

a $(2, 4), (-2, 7), (5, 2)$	b $(-1, 7), (-1, 3), (4, 6)$
c $(-6, 3), (5, 1), (4, -5), (-3, -1)$	d $(2, 3), (-5, -3), (4, 4), (8, -3)$
e $(5, 8), (-1, -1), (3, 8), (4, 0)$	f $(7, 2), (-6, 0), (0, 3), (-3, 2)$
g $(3, -1), (3, 5), (3, -6), (3, 0)$	h $(0, 2), (7, 2), (-4, 2), (9, 2), (5, 2)$

See Example 2

- 2 Test whether each graph represents a function.



- 3 a Graph each linear equation and find its gradient.
 i $y = 3x - 2$ ii $x + 2y = 3$ iii $y = 4$ iv $x = -2$
 b Which of the lines drawn in part a represent a function?
 c For what values of the gradient is a linear equation:
 i always a function? ii never a function?

16-02 Function notation

Consider the equation $y = 3x - 1$. The variable y is the dependent variable because the value for y depends on the value for x .

We say that ' y is a function of x '. This is written as $y = f(x)$, which is read as ' y is f of x '.

So, for $y = 3x - 1$, we may write $y = 3x - 1$ or $f(x) = 3x - 1$.

Note that any variables can be used. For example, if the rule $h = 12t - 5t^2$ represents the height, h metres, of a ball after time t seconds, then h is a function of t and we write $f(t) = 12t - 5t^2$ or $h(t) = 12t - 5t^2$ and so on.

The value of a function

If $y = f(x)$, then the expression $f(2)$ means the value of the function when $x = 2$ (or the y -value when $x = 2$)

Summary

If $y = f(x)$, then $f(c)$ is the value of the function when $x = c$ (or the y -value when $x = c$).

Example 3

If $y = f(x) = x^2 - 2x$, find:

- a $f(3)$ b $f(-4)$ c $f(2a)$

Solution

a $f(x) = x^2 - 2x$

$\therefore f(3) = 3^2 - 2 \times 3$

$= 9 - 6$

$= 3$

Replacing x with 3 in $x^2 - 2x$

b $f(-4) = (-4)^2 - 2 \times (-4)$

$= 16 + 8$

$= 24$

Replacing x with (-4) in $x^2 - 2x$

c $f(2a) = (2a)^2 - 2 \times 2a$

$= 4a^2 - 4a$

Replacing x with $2a$ in $x^2 - 2x$

Stage 5.3

Example 4

For the function $g(h) = 3h + 5$, find:

- a $g(-2)$ b $g(3) + g(-1)$ c $g(k + 4)$
d the value of h for which $g(h) = 20$

Solution

$$\begin{aligned}\text{a } g(-2) &= 3(-2) + 5 \\ &= -6 + 5 \\ &= -1\end{aligned}$$

Replacing h with (-2) in $3h + 5$

$$\begin{aligned}\text{b } g(3) &= 3(3) + 5 & g(-1) &= 3(-1) + 5 \\ &= 14 & &= 2\end{aligned}$$

$$\begin{aligned}\therefore g(3) + g(-1) &= 14 + 2 \\ &= 16\end{aligned}$$

$$\begin{aligned}\text{c } \text{To find } g(k + 4), \text{ replace } h \text{ with } (k + 4) \text{ in } 3h + 5 \\ \therefore g(k + 4) &= 3(k + 4) + 5 \\ &= 3k + 12 + 5 \\ &= 3k + 17\end{aligned}$$

$$\begin{aligned}\text{d } g(h) &= 3h + 5 = 20 \\ 3h &= 20 - 5 \\ &= 15 \\ h &= \frac{15}{3} \\ h &= 5\end{aligned}$$

The domain and range of a function

For some functions $y = f(x)$, there may be restrictions on the x -values and y -values. The set of permissible x -values is called the **domain**, while the set of permissible y -values is called the **range**.

Example 5

Draw the graph of each function and find its domain and range.

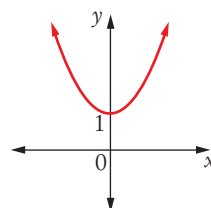
a $f(x) = x^2 + 1$ b $y = \frac{3}{x}$

Solution

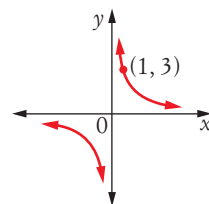
- a The graph of $y = f(x) = x^2 + 1$ is a parabola with a vertex at $(0, 1)$.

Note from the equation and graph that there is no restriction on the x values, so the domain of the function is 'all x '.

Note from the equation and graph that the y values are all greater than or equal to 1. So the range of the function is $y \geq 1$.



- b** The graph of $y = \frac{3}{x}$ is the hyperbola as shown.
- i** The permissible x values are all values of x except 0, which is written as $x \neq 0$. The domain of the function is $x \neq 0$.
- ii** The permissible y values are all y values except 0. The range of the function is $y \neq 0$.



Stage 5.3

See Example 3

Worksheet

Algebra 5

MAT10NAWK00034

See Example 4

Exercise 16-02 Function notation

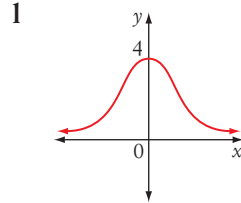
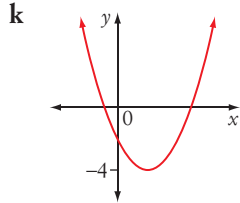
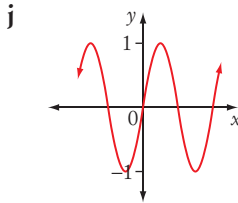
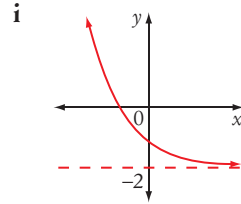
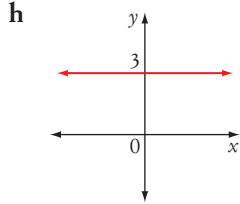
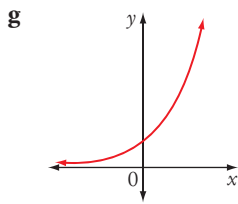
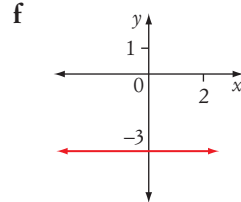
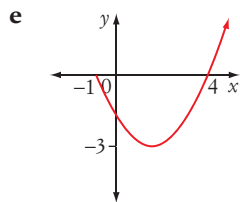
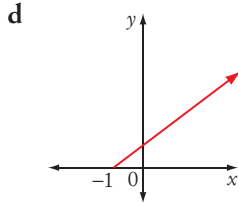
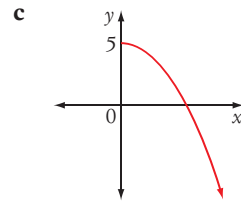
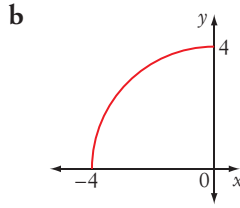
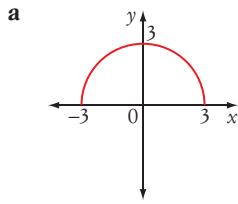
- For each function given, evaluate:
 - $f(3)$
 - $f(-1)$
 - $f(0)$
 - $f(x) = 2x$
 - $f(x) = 1 - x$
 - $f(x) = x^2 + 5x$
 - $f(x) = \frac{1}{2x+1}$
 - $f(x) = \sqrt{2x+3}$
 - $f(x) = \frac{2x+1}{2x-1}$
 - $f(x) = 1 - \frac{2}{1-2x}$
 - $f(x) = x^2 - 4x + 1$
 - $f(x) = (x+5)^2 - 4$
- If $f(x) = 3x - x^2$, find:
 - $f(3)$
 - $f(-4)$
 - $f(3k)$
- If $f(x) = 6x - 2$, find:
 - $f(3)$
 - $f(-2) + f(1)$
 - $f(d-3)$
 - $f(d+1) - f(d-1)$
 - the value of x for which $f(x) = 15$
- If $h(x) = \frac{x+2}{x-1}$, find:
 - $h(2)$
 - $h(-1)$
 - $h(0)$
 - x , if $h(x) = 3$
- If $g(t) = \frac{t}{2} + t^2$, find:
 - $g(2)$
 - $g(0)$
 - $g(-4)$
 - the values of t for which $g(t) = 3$
- Given $p(y) = 11 - 2y$,
 - find:
 - $p(0)$
 - $p(-4)$
 - $p(0.5)$
 - find:
 - $p(y+1)$
 - $p(y+1) - p(y)$
 - find y given that $p(y) = -21$.
- If $g(m) = m^2 + 2$ and $h(m) = 2m + 5$,
 - find:
 - $g(3)$
 - $h(-5)$
 - $g(-1) + h(-1)$
 - show $h(w^2) - 2g(w) = 1$
 - solve $g(m) = h(m)$
- If $k(x) = \sqrt{2x+1}$,
 - find $k(4)$
 - find x if $k(x) = 5$
 - explain why there is no value for $k(-1)$
 - what is the smallest value for x , such that $k(x)$ exists?
- If $h(t) = 3t^2 - 2t + 3$, find:
 - $h(-1)$
 - $h(2)$
 - $h(4) - h(0)$
 - $h(k^2)$
 - $h(-k)$
 - $h(k) - h(-k)$

Stage 5.3

See Example 5

10 For each graph of a function, find:

- i the domain ii the range



11 Graph each function and write its domain and range.

a $f(x) = 3x + 1$

b $y = 1 - x^2$

c $f(x) = 2^x$

d $y = \frac{2}{x}$

e $y = 1 - 2x$

f $f(x) = x^2 - 2x - 15$

g $f(x) = x^3$

h $f(x) = 4^x + 3$

i $y = (x + 3)^2 + 2$

NSW

16-03 Inverse functions

Puzzle sheet

Inverse functions
code puzzle

MAT10NAPS10239

The inverse function can be considered as the 'reverse operation' of the original function. The inverse function *undoes* the original function. For example, the inverse function to $y = x + 3$ is $y = x - 3$. The inverse function $y = x - 3$ will 'undo' the original function $y = x + 3$. Substituting $x = 10$ into $y = x + 3$ gives 13, and substituting $x = 13$ into $y = x - 3$ brings the answer back to 10. Other examples of inverse functions are:

$y = 2x$

and

$y = \frac{x}{2}$

$y = x^3$

and

$y = \sqrt[3]{x}$

$y = \sin x$

and

$y = \sin^{-1}x$

(SHIFT sin)

$y = a^x$

and

$y = \log_a x$

Inverse function notation and graph

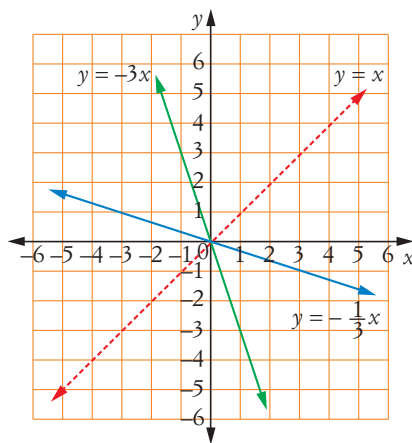
The inverse function for $y = f(x)$ is written as $y = f^{-1}(x)$.

Note that f^{-1} in this case means 'inverse' and not 'to the power of -1 '.

The graph of $y = f^{-1}(x)$ is always a reflection of the graph $y = f(x)$ in the line $y = x$.

The graph of $f(x) = -3x$ is shown as a green line on the right.

The graph of its inverse function $f^{-1}(x) = -\frac{1}{3}x$, shown as a light blue line, is a reflection of $f(x) = -3x$ across the dotted line $y = x$.



Summary

To find the inverse function of a function $y = f(x)$, rewrite the function with x and y interchanged, then make y the subject.

When the coordinates (x, y) are swapped to (y, x) , the point is reflected in the line $y = x$. This explains why the graph of $y = f^{-1}(x)$ is $y = f(x)$ reflected in this line.

Example 6

- If $f(x) = 3x - 4$, find the inverse function $f^{-1}(x)$.
- Graph $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes.

Solution

a $y = 3x - 4$

To find the inverse function, interchange the x and y variables.

$$\therefore x = 3y - 4$$

Make y the subject.

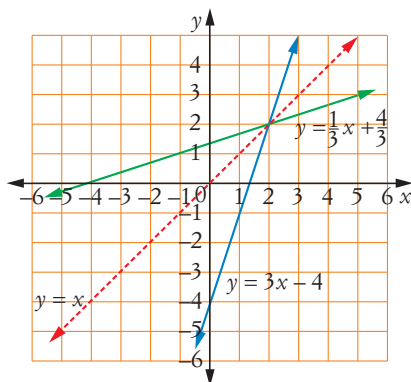
$$3y - 4 = x$$

$$\therefore 3y = x + 4$$

$$\therefore y = \frac{1}{3}x + \frac{4}{3}$$

$$\therefore f^{-1}(x) = \frac{1}{3}x + \frac{4}{3}$$

b



Stage 5.3

Condition for a function to have an inverse function

Not all functions have an inverse function.

Consider $f(x) = x^2 + 1$.

$$y = x^2 + 1$$

Swapping x and y to find the inverse function:

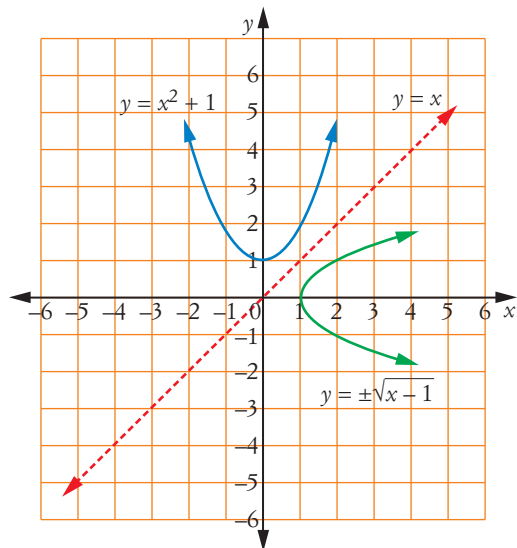
$$x = y^2 + 1$$

$$y^2 + 1 = x$$

$$y^2 = x - 1$$

$$y = \pm\sqrt{x-1}$$

$y = \pm\sqrt{x-1}$ is not a function because y takes on a positive and negative value (two values) for every value of x . Furthermore, when graphed on the number plane, it is a 'sideways parabola' (shown in green) that fails the vertical line test.



Note that when a graph is reflected in the line $y = x$, it is flipped sideways, so for an inverse function to exist, it must also pass a horizontal line test so that for every value of y there is only one value of x .

Summary

For a function to have an inverse function:

- for each **y value**, there is only one x value
- any **horizontal line** drawn on the graph of the function will cut the graph only once.

Example 7

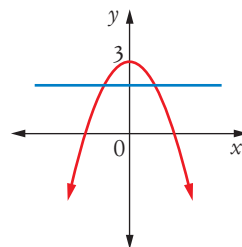
Does the function $f(x) = 3 - x^2$ have an inverse function?

Solution

The graph of $f(x) = 3 - x^2$ is shown.

A horizontal line cuts the graph of the function twice.

$\therefore f(x) = 3 - x^2$ does not have an inverse function.



Exercise 16-03 Inverse functions

Stage 5.3

See Example 6

1 For each function:

i find the inverse function $f^{-1}(x)$

ii sketch $y = x$, $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes.

a $y = 2x - 5$ b $f(x) = \frac{x}{3}$ c $f(x) = 3 - \frac{x}{2}$

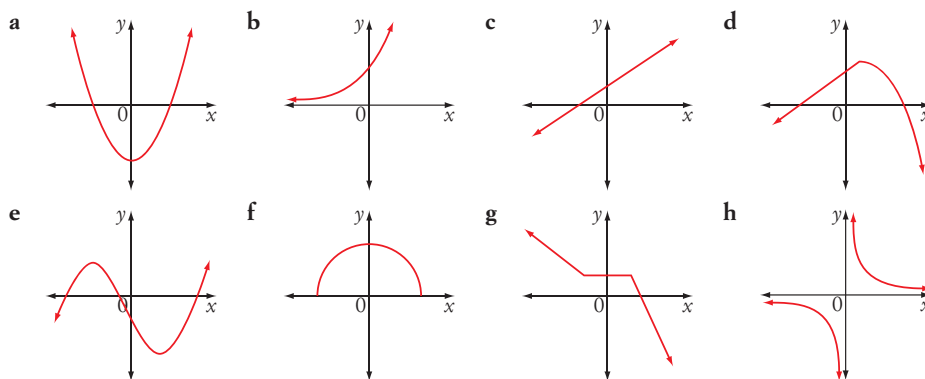
d $f(x) = \frac{4}{x}$ e $y = \frac{2x-3}{2}$ f $y = 1 - \frac{3x}{2}$

2 a If $f(x) = 2 - x$, show that $f^{-1}(x) = 2 - x$.

b By sketching $y = f(x)$, $y = f^{-1}(x)$ and $y = x$ on the same set of axes, explain why $f(x) = f^{-1}(x)$.

3 Which of the following represent functions whose inverse is a function?

See Example 6



4 a Sketch $y = x^2 - 2$.

b Does $y = x^2 - 2$ have an inverse function?

c Sketch $y = x^2 - 2$ for $x \geq 0$. Hence find and graph its inverse function.

d Sketch $y = x^2 - 2$ for $x \leq 0$. Hence find and graph its inverse function.

5 a Sketch $y = x^2 + 3$.

b Does $y = x^2 + 3$ have an inverse function?

c What restriction needs to be placed on $y = x^2 + 3$ so that it does have an inverse?

6 a Sketch $y = x^2 - 4x$ and find the coordinates of the vertex.

b What restriction needs to be placed on $y = x^2 - 4x$ so that it does have an inverse?

7 a Sketch the parabola $y = (x - 2)(x + 3)$ and find the coordinates of the vertex.

b Does $y = (x - 2)(x + 3)$ have an inverse function?

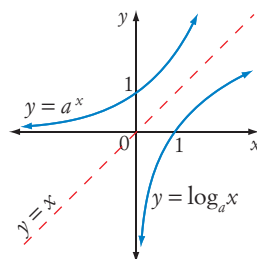
c What restriction needs to be placed on $y = (x - 2)(x + 3)$ so that it does have an inverse?

Stage 5.3

16-04 Graphing $y = a^x$ and $y = \log_a x$

The **exponential function** $y = a^x$ and the **logarithmic function** $y = \log_a x$ are inverse functions of each other. Remember from Chapter 9, Equations and logarithms, that the **logarithm** of a number is the power of the number, to a given positive base. For example, the logarithm of 32 to the base 2 is 5, written $\log_2 32 = 5$, because $2^5 = 32$.

Since $y = \log_a x$ is the inverse function of $y = a^x$, the graph of $y = \log_a x$ can be drawn by reflecting $y = a^x$ in the line $y = x$. Note that as x increases, the graph of $y = \log_a x$ increases slowly because y is a *power*. For example, since $\log_2 32 = 5$, this means that (32, 5) is a point on the graph of $y = \log_2 x$.



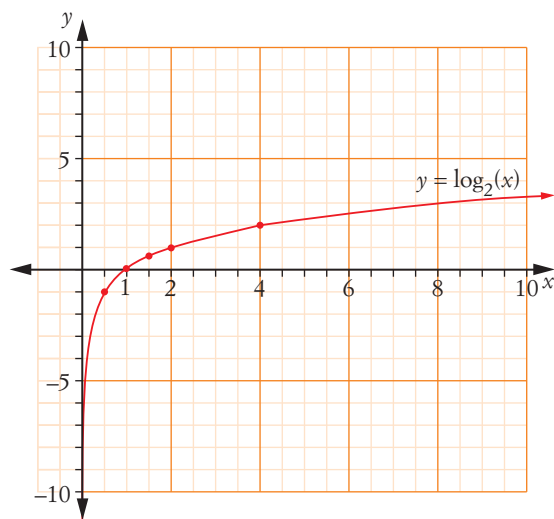
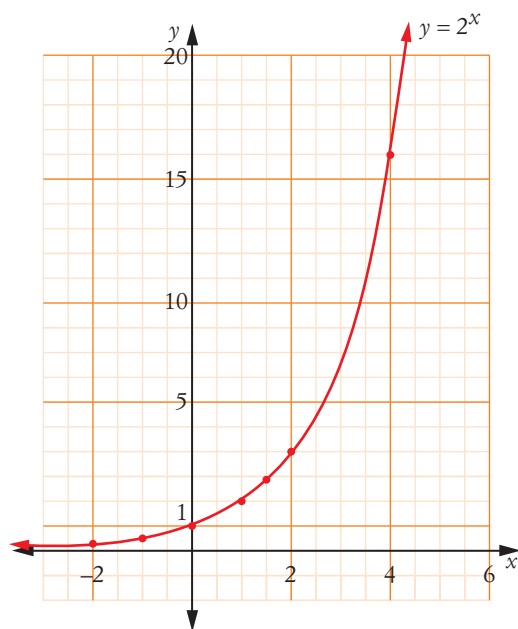
The table of values and graph for $y = 2^x$ and $y = \log_2 x$ are shown below for comparison.

$$y = 2^x$$

x	-2	-1	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	4
y	$\frac{1}{4}$	$\frac{1}{2}$	1	1.4	2	2.8	4	16

$$y = \log_2 x$$

x	-2	-1	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	4
y	-	-	-	-1	0	0.6	1	2



Exercise 16-04 Graphing $y = a^x$ and $y = \log_a x$

GeoGebra or other graphing technology can be used in this exercise.

- Show that the inverse function of $y = 2^x$ is $y = \log_2 x$, by making x the subject and then interchanging the x and y variables.
- Write the domain and range of $y = 2^x$.
 - Write the domain and range of $y = \log_2 x$.
 - What do you notice about the domain and range of $y = 2^x$ and $y = \log_2 x$?
- Describe the shape of the graph of $y = 2^x$ and $y = \log_2 x$. What are their x - and y -intercepts and asymptotes?
- On the same set of axes, graph the functions $y = 3^x$ and $y = 4^x$. Draw the reflections of each graph in the line $y = x$ to illustrate $y = \log_3 x$ and $y = \log_4 x$. Describe the differences and similarities in the graphs:
 - $y = 2^x$, $y = 3^x$ and $y = 4^x$
 - $y = \log_2 x$, $y = \log_3 x$ and $y = \log_4 x$
- Which is larger, 2^x , 3^x or 4^x , when:
 - $x < 0$?
 - $x > 0$?
- Which is larger, $\log_2 x$, $\log_3 x$ or $\log_4 x$, when:
 - $0 < x < 1$?
 - $x > 1$?
- The **log** key on a scientific calculator is used to find $\log_{10} x$, where $x > 0$. Evaluate each logarithm correct to four decimal places.
 - $\log_{10} 20$
 - $\log_{10} 627$
 - $\log_{10} 5426$
 - $\log_{10} 8.2$
 - $\log_{10} 2.5$
 - $\log_{10} 0.76$
- Graph of $y = \log_{10} x$ for values of x from 0 to 10.
- Use the graph of $y = \log_{10} x$ to estimate:
 - $\log_{10} 3$
 - $\log_{10} 6.5$
 - $\log_{10} 8.4$
 - $\log_{10} 0.5$

Compare your answers with those obtained using your calculator.

16-05 Graphing translations of functions

Summary

The graph of $y = f(x) + k$ is the graph of $y = f(x)$ shifted up k units (or down if k is negative).

The graph of $y = f(x - a)$ is the graph of $y = f(x)$ shifted right a units (or left if a is negative).

NSW

Worksheet

Graphing translations
of functions

MAT10NAWK10240

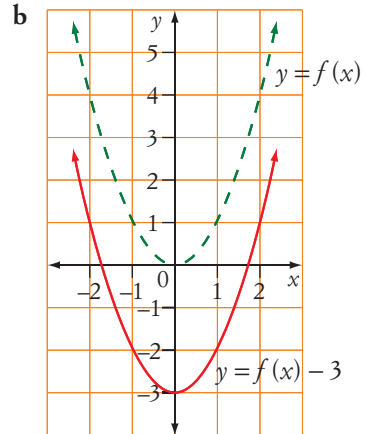
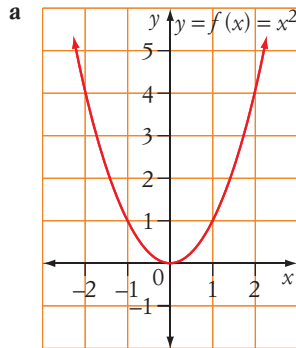
Stage 5.3

Example 8

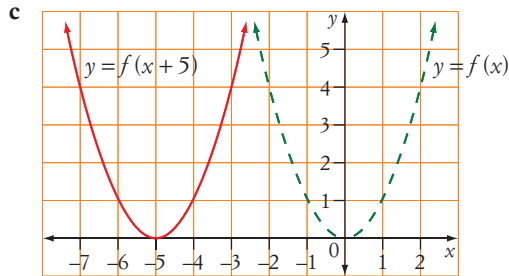
If $y = f(x) = x^2$, then graph:

- a $y = f(x)$ b $y = f(x) - 3$ c $y = f(x + 5)$

Solution



For $y = f(x) - 3$, the graph of $y = f(x)$ is shifted down 3 units.



For $y = f(x + 5)$, the graph of $y = f(x)$ is shifted 5 units to the left.

Exercise 16-05 Graphing translations of functions

See Example 7

- 1 If $y = f(x) = 2x$, then graph:

- a $y = f(x)$ b $y = f(x) + 1$ c $y = f(x) - 3$
 d $y = f(x - 2)$ e $y = f(x + 4)$

- 2 If $y = f(x) = x^2$, then graph:

- a $y = f(x) - 3$ b $y = f(x) + 2$ c $y = f(x) - 1$
 d $y = f(x - 3)$ e $y = f(x + 1)$ f $y = f(x + 2) - 1$

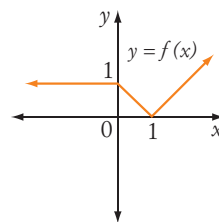
3 If $y = f(x) = x^3$, then graph:

a $y = f(x - 1)$ **b** $y = f(x) + 2$ **c** $y = f(x + 2) + 1$

4 The graph of $y = f(x)$ is shown. Draw the graphs of:

a $y = f(x) - 2$ **b** $y = f(x - 3)$ **c** $y = f(x + 2)$

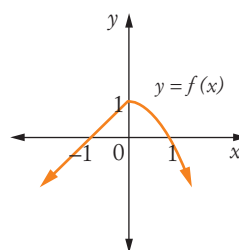
d $y = f(x) + 3$ **e** $y = f(x + 1) - 3$



5 The graph of $y = f(x)$ is shown. Draw the graphs of:

a $y = f(x) + 2$ **b** $y = f(x) - 3$

c $y = f(x - 1)$ **d** $y = f(x - 2) + 1$



Instructional glossary

Mathematical 'doing' words

Analyse To study something in great detail by breaking it into its parts.

Bisect To cut in half.

Classify To sort into categories or types.

Complete To fill in detail that makes a statement or diagram correct or finished.

Construct To draw a geometrical figure accurately.

Convert To change from one form to another. For example, convert a fraction to a decimal, or convert dollars to cents.

Decrease To make smaller by subtracting or dividing.

Estimate To make an educated guess for a number or answer; to find roughly or approximately.

Evaluate To find the value of an expression.

Expand To remove the brackets in an algebraic expression; the opposite of **factorise**. Expanding $3(2y + 1)$ gives $6y + 3$.

Factorise To take out the highest common factor (HCF) of an expression and insert brackets; the opposite of **expand**. Factorising $5x - 20$ gives $5(x - 4)$.

Give reasons When solving a problem, to show the mathematical rules or thinking used.

Graph To display on a number line, number plane or statistical graph.

Hence find/prove To find an answer or prove a result using previous answers or information supplied.

Increase To make larger by adding or multiplying.

Measure To find the size of something using an instrument. For example, to find a length using a ruler.

Prove/show that In questions where the answer is given, to use mathematical reasoning to prove that the answer is true.

Rationalise the denominator To simplify a fraction involving a surd by making its denominator rational (that is, not a surd)

Show working To show the steps you used to find an answer.

Simplify To give the answer in its simplest, shortest, neatest form.

Simplify (a fraction) To reduce the numerator and denominator of a fraction by dividing by their highest common factor (HCF).

Simplify (a ratio or rate) To reduce the terms or units of a ratio or rate by dividing by a common factor, in a similar way to simplifying a fraction.

Simplify (a surd) To write a surd $\sqrt{x}$ in its simplest form so that x has no factors that are perfect squares.

Sketch To draw a rough diagram. Less accurate than to **construct**.

Solve To find the value of the unknown variable in an equation or inequality.

Substitute To replace a variable with a number, to evaluate an expression. For example, substituting $t = 5$ in the expression $t^2 + 6$ gives 31.

Write/state To write an answer, formula or result without showing any working or explanation. (This usually means that the answer can be found mentally, or in one step.)

Glossaries

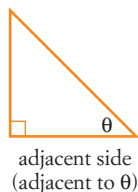
Mathematical glossary

Some common symbols

%	per cent	π	pi, the decimal 3.14159 . . .
()	parentheses, round brackets	$3x$	3 multiplied by x , $3 \times x$
[]	(square) brackets	$\frac{x}{2}$	x divided by 2, $x \div 2$
{ }	braces	x^2	x squared, $x \times x$
=	is equal to	x^3	x cubed, $x \times x \times x$
$\approx$	is approximately equal to	$\bar{x}$	the mean (average)
$\neq$	is not equal to	Σ	the sum of
$<$	is less than	σ_n	standard deviation
$>$	is greater than	$42^\circ 17' 54''$	42 degrees, 17 minutes, 54 seconds
$\leq$	is less than or equal to	$\angle ABC$	angle ABC
$\geq$	is greater than or equal to	$\triangle ABC$	triangle ABC
$\therefore$	therefore	$\parallel$	is parallel to
-3	negative 3	$\perp$	is perpendicular to
$\sqrt{\quad}$	the square root of	$\equiv$	is congruent (identical) to
$\sqrt[3]{\quad}$	the cube root of	$\sim$	is similar to
± 3	positive or negative 3	$P(E)$	The probability of an event, E
$0.\dot{6}$	the recurring decimal 0.666 666 . . .	$P(\bar{E})$	The probability of the complementary event, $\bar{E}$

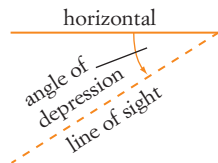
A

Adjacent side In a right-angled triangle, the side 'next to' the given angle, leading to the right angle.

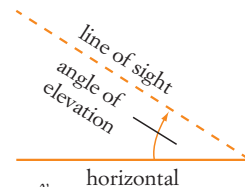


Allowable (tax) deduction A part of a person's yearly income that is not taxed, such as work-related expenses or donations to charities. All deductions are subtracted from yearly income to determine **taxable income**.

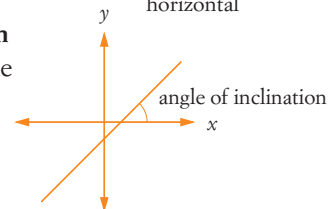
Angle of depression The angle of looking **down**, measured from the horizontal

**Angle of elevation**

The angle of looking **up**, measured from the horizontal

**Angle of inclination**

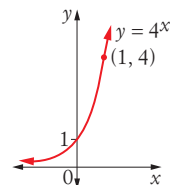
The angle that a line makes with the x -axis in the positive direction.

**Annual leave loading (or holiday loading)**

Extra payment to a worker during annual leave based on 17.5% of four weeks' pay.

Apex The pointed top of a **pyramid** or **cone**.

Asymptote A line that a curve gets very close to but never touches, for example, the x -axis is an asymptote of the exponential curve.



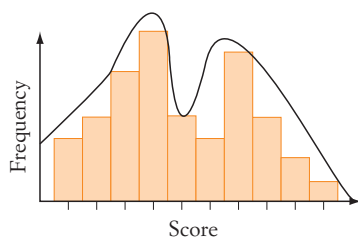
Average See **measure of location**.

B

Base (in index notation) When a number is raised to a power, the number raised is the base. In the expression 3^5 , the 3 is called the base.

Bisect To cut in half.

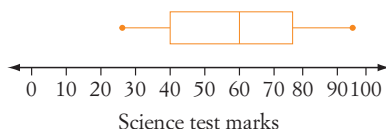
Bimodal distribution A statistical distribution that has two peaks.



Binomial expression An algebraic expression with two terms, for example, $x + 9$, $2y - 12$. See also **trinomial**.

Bivariate data Data that measures two variables, such as a person's height and arm span, represented by an ordered pair of values that can be graphed on a **scatter plot** for analysis.

Boxplot (or box-and-whisker plot). A graph that shows the quartiles of a set of data and the highest and lowest scores, the 'box' contains the middle 50% of scores while the 'whiskers' extend to the two extremes.



C

Capacity The amount of material (usually liquid) that a container can hold, measured in millilitres (mL), litres (L), kilolitres (kL) and megalitres (ML). See also **volume**.

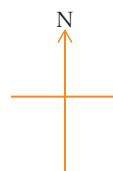
Cluster A group of data scores that are bunched or close together.

Coefficient The number in front of a variable in an algebraic term. For example, the coefficient of x in $2x - 5$ is 2.

Collinear points Points that lie on a straight line.

Compass bearing A bearing that refers to one of the sixteen points of a mariner's compass, for example, north-northwest (NNW). See also **bearing** and **three-figure bearing**.

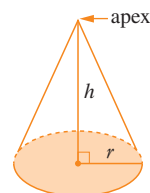
Compass rose A cross-shaped diagram that shows the direction of north. See also **compass bearing**.



Complementary event All the outcomes that are *not* the event; the 'opposite' event. For example, the complementary event to rolling 1 on a die is rolling a number that is not 1.

Compound interest Interest that is calculated as a percentage of the original principal and the accumulated interest. See also **simple interest**.

Cone A solid shape with a circular base, a curved surface and an **apex**.



Congruent Identical, exactly the same. The symbol ' $\equiv$ ' means 'is congruent to' or 'is identical to'.

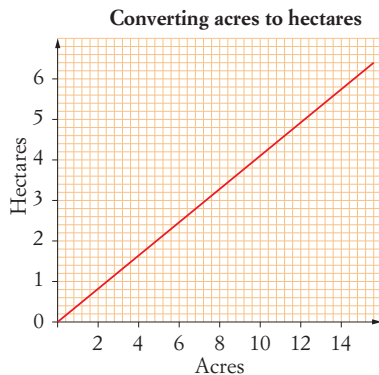
Congruence test One of four tests for proving that two triangles are congruent: SSS, SAS, AAS and RHS.

Constant term The term in an algebraic expression that is a number only, with no variable. For example, the constant term in $x^2 - 4x + 6$ is 6.

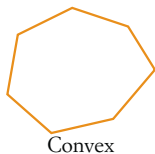
Converse A rule or statement turned back-to-front; the reverse statement.

Glossaries

Conversion graph A line graph for converting between different units or currencies, for example, miles to kilometres, or Australian dollars to US dollars. It usually contains one straight line that begins at the origin (0, 0).



Convex polygon A polygon whose vertices all point outwards. All diagonals lie within the shape, and all angles are less than 180° .



Cosine A ratio in a right-angled triangle:

$$\cos \theta = \frac{\text{side adjacent to } \theta}{\text{hypotenuse}}$$

where θ is an angle. See also **sine** and **tangent**.

Cosine rule A rule that relates the three sides and one of the angles of any triangle:

$$a^2 = b^2 + c^2 - 2bc \cos A$$

Cross-section A 'slice' of a solid cut across it rather than along it.



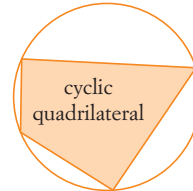
Cubic curve The graph of a cubic equation such as $y = 2x^3 + 1$.

Cubic equation An equation in which the highest power of the variable is 3, that is, a variable cubed, for example, $x^3 = 12$.

Curved surface area The area of the curved surface of a solid such as a cylinder, cone or sphere.

Cyclic quadrilateral

A quadrilateral bounded by a circle, whose vertices lie on the circle's circumference.



D

Data Information, a collection of facts.

Degree of a polynomial The highest power in a polynomial. For example the degree of $8x^3 + 4x - 7$ is 3.

Dependent event An event whose outcome (and probability) depends upon the outcome of another event, for example, the colour of the second marble drawn from a bag depends on the colour of the first marble drawn.

Dependent variable A variable that depends on another variable for its value. For example, if y depends on x , then the dependent variable is y and the **independent variable** is x .

Depreciation The decrease in value of items over time due to aging or use.

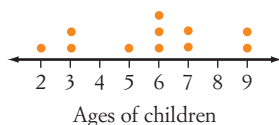
Difference of two squares An algebraic expression of the form $a^2 - b^2$, that can be factorised into $(a + b)(a - b)$. For example, $x^2 - 25 = (x + 5)(x - 5)$.

Distance–time graph A line graph illustrating a journey, showing distance travelled against time. Also called a **travel graph**.

Direct proportion (or direct variation) A relationship between two variables of the form $y = kx$, where k is a constant, for example, if $y = 8.5x$, then y is directly proportional to x .

Domain The set of permissible x values for the function $y = f(x)$.

Dot plot A graph that uses dots above a number line to show frequencies of data scores.



Double time Overtime pay that is calculated at 2 times the normal pay rate.

E

Equation A mathematical statement that two quantities are equal. For example, $8 + 2 = 10$ or $3b - 7 = 5$.

Equiangular All angles equal.

Equilateral triangle A triangle with all three sides equal (and all angles 60°).

Elimination method A method of solving simultaneous equations that involves combining them to eliminate one of the variables.

Exact ratio The sine, cosine and tangent of the special angles 30° , 45° and 60° , which can be expressed as exact fractions or surds rather than decimal approximations.

Expected frequency The expected number of times an event will occur over repeated trials, calculated by multiplying the probability of the event by the number of trials.

Experimental probability An estimate of theoretical probability; the **relative frequency** of an event in repeated trials of an experiment, found using the formula

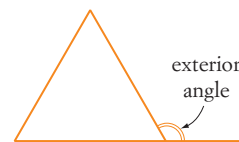
$$P(E) = \frac{\text{frequency of } E}{\text{total frequency}}$$

Exponential curve The graph of an exponential equation $y = a^x$. See **asymptote** for diagram.

Exponential equation or exponential function An equation of the form $y = a^x$, where a is a positive constant and the variable x is a power, for example, $y = 4^x$.

Exterior angle

An 'outside' angle of a shape created by extending one of the sides of the shape.



Event In probability, a result involving one or more outcomes. For example, when rolling a die, the event 'rolling an even number' contains the three outcomes {2, 4, 6}.

F

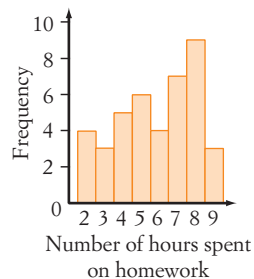
Factor theorem The rule that if $(x - a)$ is a factor of the polynomial $P(x)$, then $P(a) = 0$.

Five-number summary For a set of numerical data, the lowest score, lower quartile, median, upper quartile, highest score, used to draw a **boxplot**.

Formula (Plural: **formulas** or **formulae**) A rule written as an algebraic equation, using variables. The formula for the area of a triangle is $A = \frac{1}{2}bh$.

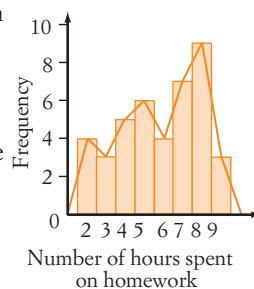
Frequency

histogram A column graph that shows the frequencies of numerical data. There are no spaces between the columns, and the graph looks like a row of office buildings.



Frequency

polygon A line graph that shows the frequencies of numerical data. It can be made by joining the midpoints of the tops of the columns of a histogram. The graph looks like a mountain.



Glossaries

Frequency (distribution) table A table listing the frequency of each value in a set of data, with columns for Score (x), Frequency (f) and sometimes Tally and fx .

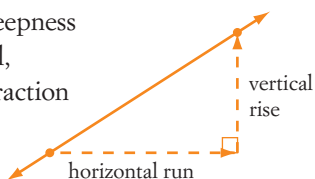
Function A rule or relationship between two variables where, for each value of the independent variable (input value), x , there is only one value of the dependent variable (output value), y .

Function notation A way of writing a function using the form $y = f(x)$.

G

General form of a linear equation The equation of a straight line $ax + by + c = 0$, where a , b and c are integers and a is positive.

Gradient The steepness of a line or interval, measured by the fraction $\frac{\text{rise}}{\text{run}}$

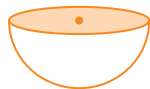


Gradient–intercept form of a linear equation The equation of a straight line $y = mx + b$, where m is the **gradient** and b is the y -intercept.

Gross pay Pay received before tax and other deductions are taken out.

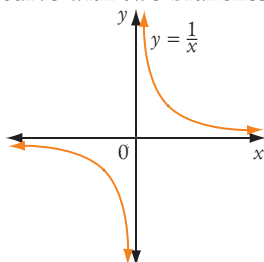
H

Hemisphere Half a **sphere**.

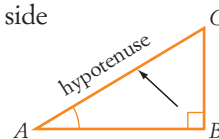


Hire-purchase See **term payments**.

Hyperbola The graph of $y = \frac{k}{x}$, where k is a constant, a curve with two branches.



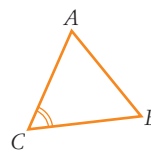
Hypotenuse The longest side of a right-angled triangle, opposite the right angle.



I

Image A transformed shape after it has been enlarged or reduced.

Included angle The angle between two given sides of a shape. For example, the included angle for sides AC and CB in this triangle is $\angle C$.



Income tax A tax paid to the government based on the size of a person's income.

Independent event An event whose outcome (and probability) does not depend upon the outcome of another event, for example, the number rolled on the second die does not depend on the number rolled on the first die.

Independent variable A variable whose value does not depend on another variable. For example, if y depends on x , then the **dependent variable** is y and the independent variable is x .

Inequality A mathematical statement that two quantities are not equal, involving algebraic expressions and an inequality sign ($>$, $\geq$, $<$, or $\leq$), for example, $-3 > -10$ or $2x - 7 \leq 15$.

Inverse function The reverse of a function $f(x)$, written $f^{-1}(x)$, which 'undoes' the original function, found by interchanging the dependent and independent variables (x and y).

Index (Plural: **indices**, pronounced 'in-da-sees'). See **Power**.

Index law An algebraic rule for simplifying expressions involving powers of the same base, for example, $a^m \times a^n = a^{m+n}$.

Instalment (or repayment) The amount of money paid at regular time periods (weekly, fortnightly, monthly) to pay off a loan.

Interquartile range (IQR) The difference between the upper quartile and lower quartiles, $IQR = Q_3 - Q_1$, representing the middle 50% of scores.

Intercept The distance between the points when a line crosses another line or curve at two points.

Interval A section of a line with a definite length, such as AB below.



Inverse proportion (or inverse variation) A relationship between two variables of the form

$y = \frac{k}{x}$, where k is a constant, for example, if $y = \frac{50}{x}$, then y is inversely proportional to x .

Irrational number A number such as π or $\sqrt{2}$ that cannot be expressed as a fraction (rational number). In decimal form, its digits run endlessly without repeating. See also **rational number** and **real number**.

Isosceles triangle A triangle with two equal sides (and two equal angles opposite those sides).

K

Kite A quadrilateral with two pairs of equal adjacent sides.

L

Leading coefficient of a polynomial The coefficient of the **leading term**. For example, the leading coefficient of $8x^3 + 4x - 7$ is 8.

Leading term of a polynomial The term with the highest power. For example, the leading term of $8x^3 + 4x - 7$ is $8x^3$.

LHS The left-hand side (of an equation).

Line of best fit A straight line drawn through the points on a scatter plot that best describes the bivariate data.

Linear equation A formula whose graph is a straight line, or an equation involving a variable that is not raised to a power, such as $2x + 9 = 17$.

Logarithm The power of a number, to a given base. For example, $\log_{10} 1000 = 3$, meaning that the logarithm of 1000 to base 10 is 3, because $1000 = 10^3$.

Logarithmic equation An equation involving a logarithm such as $\log_5 x = -3$, which can be solved by rewriting the equation in index form.

Logarithmic function A function of the form $y = \log_a x$, whose inverse is the exponential function.

M

Mean The average of a set of data, represented by $\bar{x}$, calculated by dividing the sum of the scores by the number of scores.

Measure of location An average, middle or typical value of a set of data. The three measures of location are the **mean**, **median** and **mode**.

Measure of spread A statistical value that describes how the scores in a data set are spread, for example, **range**, **interquartile range** or **standard deviation**.

Median The middle score when the scores are arranged in order. If the number of values is even, then the median is the average of the two middle values.

Midpoint The point in the middle of an interval or halfway between two given points.

Minute (symbol ') A measure of angle size. $\frac{1}{60}$ of a degree. $1^\circ = 60'$.

Glossaries

Mode The most common or frequent score(s) in a set of data.

Monic polynomial A polynomial that has a **leading coefficient** of 1, for example, $P(x) = x^4 - 7x^2 + x + 8$.

Mutually exclusive events Events or categories that have no items in common.

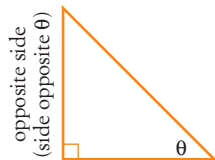
N

Negatively skewed See **skewed distribution**.

Net pay Pay received after deductions from gross pay; 'take-home' pay.

O

Opposite side In a right-angled triangle, the side directly facing the given angle.



Outcome In probability, a result of a situation or experiment. For example, when rolling a die, one possible outcome is rolling a 4.

Outlier An extreme data value that is much different from the other values in a set.

P

Parabola A U-shaped curve that is the graph of a quadratic equation such as $y = x^2$.

Parallelogram A quadrilateral in which the opposite sides are parallel.

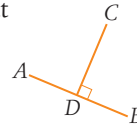
PAYG (Pay As You Go) tax Income tax deducted from your pay each payday by your employer.

Perfect square A square number or an algebraic expression that represents one, for example, 64 , $(x + 9)^2$, $(a - b)^2$.

Per annum (p.a) Per year.

Perpendicular bisector An interval or line that bisects another interval at right angles.

Perpendicular lines Lines that intersect to form a right angle. $AB \perp CD$ means 'AB is perpendicular to CD'.



Point-gradient form of a linear equation

The equation of a line with gradient m and that passes through the point (x_1, y_1) , which is $y - y_1 = m(x - x_1)$.

Polygon Any flat shape made up of straight sides.



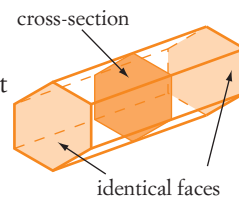
Polynomial An algebraic expression involving powers of x that are positive integers. For example $P(x) = 8x^3 + 4x - 7$.

Positively skewed See **skewed distribution**.

Power (or index) The number of times a base is multiplied by itself. In 2^5 , the power is 5. Also called the *exponent*.

Principal An amount of money invested or borrowed, on which interest is calculated.

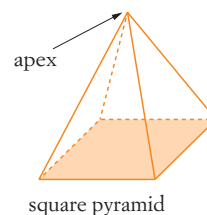
Prism A solid shape with identical cross-sections that has straight sides.



Probability The chance of an event occurring, measured as a fraction, decimal or percentage between 0 and 1.

Pronumeral Another name for **variable**.

Pyramid A solid with a **polygon** for a base and triangular faces that meet at a point called the **apex**.



Q

Quadrant (of a circle) A sector that is a quarter of a circle, containing a right angle.



Quadratic expression An algebraic expression in which the highest power of the variable is 2, for example, $x^2 - 5x + 7$, $x^2 - 15$, $2x^2 - 3x + 9$ and $-4x^2 + 7x$.

Quadratic equation An equation in which the highest power of the variable is 2, that is, a variable squared, for example, $3x^2 - 6 = 69$ or an equation such as $y = 3x^2 - 6$ whose graph is a **parabola**.

Quadratic formula The formula for solving quadratic equations of the form $ax^2 + bx + c = 0$, which is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Quadratic trinomial See **trinomial**.

Quadrilateral test A property of a quadrilateral that proves that it is a particular type of quadrilateral, for example, if opposite angles are equal, then it *must* be a parallelogram.

Quarterly Occurring regularly four times a year, that is, every three months.

Quartic polynomial A polynomial with degree 4, for example, $P(x) = -6x^4 + x^3 - 2x^2 + 9x$.

Quartile The values Q_1 , Q_2 , Q_3 that divide a set of data into four equal parts. The 1st quartile Q_1 is the lower quartile, the 2nd quartile Q_2 is the **median**, the 3rd quartile Q_3 is the upper quartile.

Quotient The 'whole' part of the answer when a polynomial is divided by another polynomial.

R

Random In probability, describing a situation where every possible outcome has an equal chance, or is equally likely.

Range In a set of data, the difference between the highest and lowest scores.

Range (of a function) The set of permissible y values for the function $y = f(x)$.

Rational number A number that can be written as a fraction in the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$. See also **irrational number** and **real number**.

Real number A **rational** or **irrational number**, that can be ordered on a number line.

Reciprocal The product of any number and its reciprocal is 1. The reciprocal of any number is found by first writing the number as a fraction and then swapping the numerator with the denominator. The reciprocal of 5 is $\frac{1}{5}$ and the reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$.

Rectangle A quadrilateral with four right angles.

Regular polygon A polygon that has all sides equal and all angles equal.

Relative frequency The number of times an event or score occurred, written as a fraction of the total number of events or scores. See also **experimental probability**.

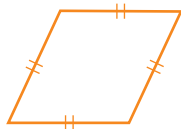
Remainder The 'left-over' part of the answer when a polynomial is divided by another polynomial.

Remainder theorem The rule that if a polynomial $P(x)$ is divided by the linear expression $(x - a)$, then the remainder is $P(a)$.

Repayment See **instalment**.

Glossaries

Rhombus A quadrilateral with four equal sides.



RHS The right-hand side (of an equation).

Rise Short for 'vertical rise', this is the change in vertical position between two points on a line or interval, the number of units 'going up', used with the **run** to calculate the **gradient** of a line or interval. See **gradient**.

Root of an equation A value of x that is a solution to the polynomial equation $P(x) = 0$. A root of $P(x) = 0$ is the same as a **zero** of $P(x)$.

Run Short for 'horizontal run', this is the change in horizontal position between two points on a line or interval, the number of units 'going right', used with the **rise** to calculate the **gradient** of a line or interval. See **gradient**.

S

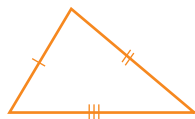
Sample In statistics, a group of people or items selected from a population for study.

Sample space In a probability situation, the set of all possible outcomes.

Scale factor The amount by which a shape has been enlarged or reduced, equal to

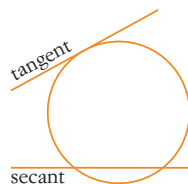
$$\frac{\text{image length}}{\text{original length}}$$

Scalene triangle A triangle with no equal sides.



Scatter plot A graph of points on a number plane. Each point represents the values of the two different variables and the resulting graph may show a pattern.

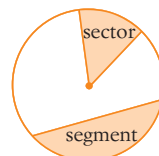
Secant A line that intersects a curve at two points. See also **tangent**.



Second (") A measure of angle size. $\frac{1}{60}$ of a minute. $1' = 60''$.

Sector A region of a circle cut off by two radii, shaped like a slice of pizza.

Segment A region of a circle cut off by a chord.



Shape of a distribution The way the data in a frequency distribution is spread, can be **symmetrical**, positively **skewed** or negatively **skewed**.

Significant figures The meaningful digits in a number that show its level of accuracy, the first non-zero digits, for example, 31 487 000 has 5 significant figures.

Similar To have the same shape but not necessarily the same size, an enlargement or reduction (symbol: |||).

Similarity test One of four tests for proving that two triangles are similar.

Simple interest Interest that is calculated as a percentage of the original principal. See also **compound interest**.

Simultaneous equations Two (or more) equations that must be solved together so that the solution satisfies both equations. For example, $y = 2x + 1$ and $y = 3x$ are simultaneous equations that have a solution of $x = 1, y = 3$.

Sine A ratio in a right-angled triangle:

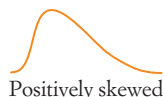
$$\sin \theta = \frac{\text{side opposite to } \theta}{\text{hypotenuse}}$$

where θ is an angle. See also **cosine** and **tangent**.

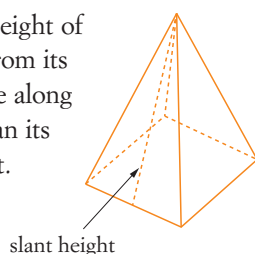
Sine rule A rule that relates the sides of any triangle to the sine of their opposite angles:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Skewed distribution A distribution in which most of the scores are clustered at one end, creating a 'tail' at the other end. The tail determines whether the skew is positive or negative. See also **symmetrical distribution**.



Slant height The height of a pyramid or cone from its **apex** (top) to its base along a side face rather than its perpendicular height.

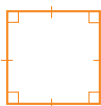


Solution The answer to an equation, inequality or problem, the correct value(s) of the variable that makes an equation or inequality true.

Sphere A ball shape, a solid that is completely round.



Square A quadrilateral with four equal sides and four right angles.



Standard deviation (symbol σ_n) A measure of spread that depends on every score in the data set and their mean.

Stem-and-leaf plot A 'number graph' that lists all the data scores, in groups. Each score is split into a 'stem' and a 'leaf'. This stem-and-leaf plot shows 12 test scores, from 42 to 82.

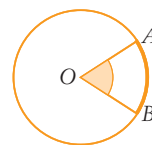
Stem	Leaf
4	2 5
5	0 2 8
6	6 7
7	3 5 7 7
8	2

Subject of a formula The variable for which a formula is written, the variable on the left-hand side of a formula. The subject of the formula

$$A = \frac{1}{2}bh \text{ is } A.$$

Substitution method A method of solving **simultaneous equations** that involves substituting one equation into another equation.

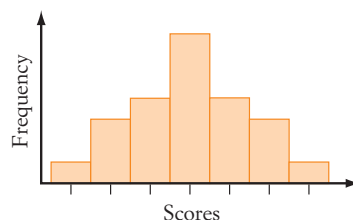
Subtend an angle To sit opposite an angle and 'hold' its arms. In the diagram, the arc AB subtends the angle at O , or angle O is subtended by the arc AB .



Surd A square root (or other root) whose exact value cannot be found because it is **irrational**, such as $\sqrt{10}$ or $\sqrt[3]{7}$.

Surface area The total area of all the faces of a solid shape.

Symmetrical distribution A distribution in which all scores are distributed equally on both sides of the centre, its shape having line symmetry. See also **skewed distribution**.



T

Tangent A ratio in a right-angled triangle:

$$\tan \theta = \frac{\text{side opposite to } \theta}{\text{side adjacent to } \theta}$$
 where θ is an angle. See also **sine** and **cosine**.

Tangent A line that touches a curve at one point but does not cross it. See also **secant**.

Tax deduction See **allowable deduction**.

Taxable income The part of a person's income that is taxed, equal to yearly income minus allowable deductions.

Term payments Paying for an expensive item through a loan in which regular instalments are made over time. Also called **hire-purchase**.

Test for a quadrilateral See **quadrilateral test**.

Glossaries

Theoretical probability (or calculated probability)

Probability calculated using the formula:

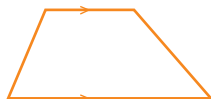
$$P(E) = \frac{\text{number of favourable outcomes}}{\text{total number of outcomes}}$$

Three-figure bearing (or true bearing)

A bearing that uses three-digit angles from 000° to 360° to show the amount of turning measured clockwise from north. See also **bearing** or **compass bearing**.

Time-and-a-half Overtime pay that is calculated at 1.5 times the normal pay rate.

Trapezium A quadrilateral with one pair of opposite sides parallel.



Travel graph See **distance–time graph**.

Trial One go or run of a repeated probability experiment, for example, one roll of a die.

Trinomial An algebraic expression with three terms, for example, $3x + 2y - 5$. In a **quadratic trinomial** such as $x^2 + 4x + 6$, the highest power of the pronumeral is 2. See also **binomial** and **quadratic expression**.

Tree diagram A diagram of branches for listing all of the possible outcomes of a multi-step chance experiment.

Two-step experiment (or two-stage experiment) A chance experiment with two steps or stages, such as rolling a pair of dice.

Two-way table A table that shows the number of items belonging to overlapping categories.

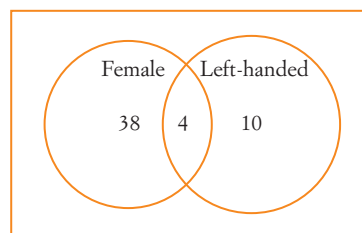
	Can swim	Cannot swim
Boys	13	2
Girls	9	3

V

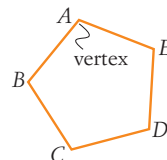
Variable A symbol, usually a letter of the alphabet, that stands for a number. Also called a **pronumeral** or **unknown**.

Variation See **direct proportion**.

Venn diagram A diagram of circles (usually overlapping) for grouping items into categories.



Vertex (Plural: **vertices**) A corner of a shape, angle or curve.



Vertical line test A test applied to a graph to determine whether it is the graph of a function.

Volume The amount of space occupied by a solid object, measured in cubic units.

X

x-intercept The x value at which a graph cuts the x -axis

Y

y-intercept The y value at which a line cuts the y -axis.

Z

Zero of a polynomial A value of x that makes the value of the polynomial $P(x)$ equal to 0.

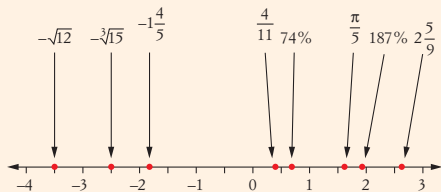
Chapter 1

SkillCheck

1 a $25y^2$	b $64m^3$	c $9x^2$
2 a $5x + 10$	b $4y - 12$	c $3 + 6w$
d $10 - 2y$	e $-10a - 15$	f $k + 2k^2$
- 3 81, 25, 100, 16, 64
- 4 a $m^2 + 10m + 21$
- c $n^2 - 5n + 6$
- e $4 - 17p - 15p^2$
- g $x^2 + 8m + 16$
- i $4k^2 + 4k + 1$
- k $t^2 - 49$
- b $y^2 - 3y - 4$
- d $6d^2 + 11d + 3$
- f $3a^2 + 17af + 10f^2$
- h $y^2 - 6y + 9$
- j $a^2 - 25$
- l $9m^2 - 16$

Exercise 1-01

- 1 C 2 B 3 $\sqrt{32}, \sqrt{125}, \sqrt{4.9}, \sqrt{52}, \sqrt{288}$
- 4 a R b I c R d R e R f R
- g R h R i I j R k I l I
- 5 a $1\frac{4}{7}, \frac{\pi}{2}, \sqrt{2}$
- b $2\frac{7}{9}, \sqrt[3]{20}, 2.\dot{6}$
- 6 a -1.8
- b 0.7
- c 0.4
- d -3.5
- e -2.5
- f 2.6
- g 1.6
- h 1.9



- 7 1.41
- 8 Teacher to check, $\sqrt{5} \approx 2.24$, $\sqrt{10} \approx 3.16$, $\sqrt{17} \approx 4.12$

Exercise 1-02

1 a 2	b 5	c 27	d 250
e 0.09	f 28	g 45	h 50
- 2 a $5\sqrt{2}$
- b $2\sqrt{3}$
- c $2\sqrt{7}$
- d $5\sqrt{6}$
- e $10\sqrt{7}$
- f $3\sqrt{5}$
- g $4\sqrt{3}$
- h $10\sqrt{2}$
- i $4\sqrt{6}$
- j $3\sqrt{7}$
- k $12\sqrt{2}$
- l $6\sqrt{3}$
- m $5\sqrt{3}$
- n $7\sqrt{3}$
- o $4\sqrt{2}$
- p $11\sqrt{2}$
- q $9\sqrt{2}$
- r $7\sqrt{5}$
- s $5\sqrt{5}$
- t $16\sqrt{2}$
- 3 a $6\sqrt{5}$
- b $16\sqrt{2}$
- c $48\sqrt{2}$
- d $\sqrt{10}$
- e $\sqrt{3}$
- f $\frac{\sqrt{7}}{3}$
- g $6\sqrt{6}$
- h $18\sqrt{17}$
- i $\frac{5\sqrt{5}}{2}$
- j $3\sqrt{2}$
- k $3\sqrt{3}$
- l $40\sqrt{10}$
- m $15\sqrt{3}$
- n $14\sqrt{17}$
- o $\frac{\sqrt{13}}{3}$
- 4 B
- 5 B
- 6 a false b false c true d true e true f false

Exercise 1-03

1 a $11\sqrt{3}$	b $3\sqrt{2}$	c $4\sqrt{6}$	d $4\sqrt{5}$
e 0	f $\sqrt{10}$	g $8\sqrt{15}$	h $-\sqrt{6}$
i $2\sqrt{3}$	j $10\sqrt{5}$	k $6\sqrt{10}$	l $-5\sqrt{3}$
- 2 a $7\sqrt{3} - 9$
- b $7\sqrt{10} - 7\sqrt{2}$
- c $5\sqrt{2} - 9\sqrt{3}$
- d $7\sqrt{15} + 8\sqrt{2}$
- e $-2\sqrt{5} - 3\sqrt{7}$
- f $2\sqrt{6} - 8\sqrt{3}$
- g $13\sqrt{11} - \sqrt{3}$
- h $11\sqrt{7} - 6\sqrt{13}$
- i $-6\sqrt{7}$
- j $-3\sqrt{5}$

3 a D	b A	c $-2\sqrt{5}$	d $-\sqrt{7}$
4 a $6\sqrt{2}$	b $3\sqrt{3}$	c $-\sqrt{10}$	d $8\sqrt{11}$
e $5\sqrt{6}$	f $7\sqrt{5}$	g $3\sqrt{2}$	h $9\sqrt{2}$
i $6\sqrt{2}$	j $8\sqrt{3}$	k $3\sqrt{2}$	l $30\sqrt{3}$
m $11\sqrt{3}$	n $-\sqrt{5}$	o $-6\sqrt{3}$	p $29\sqrt{2}$
q $5\sqrt{7}$	r $41\sqrt{2}$	s $5\sqrt{6}$	t $6\sqrt{2} + 2\sqrt{3}$
u $-15\sqrt{3}$	v 0	w $6\sqrt{2} + 2\sqrt{3}$	x $12\sqrt{3} + 3\sqrt{6}$
y $3\sqrt{2} - 6\sqrt{5}$	z $4\sqrt{6}$		

Exercise 1-04

1 a $\sqrt{30}$	b $-\sqrt{66}$	c $2\sqrt{10}$	d 6
e -5	f 144	g $15\sqrt{30}$	h $-10\sqrt{21}$
i 140	j $-30\sqrt{2}$	k 36	l $-60\sqrt{2}$
m -112	n $24\sqrt{6}$	o 80	p $90\sqrt{6}$
q -396	r $160\sqrt{5}$	s $216\sqrt{2}$	t $-96\sqrt{6}$
u $36\sqrt{5}$	v $-60\sqrt{10}$	w $252\sqrt{3}$	x 144
- 2 a $\sqrt{3}$
- b $-\sqrt{6}$
- c $8\sqrt{7}$
- d $2\sqrt{2}$
- e $-\frac{\sqrt{7}}{2}$
- f 21
- g 1
- h 8
- i $5\sqrt{3}$
- j $5\sqrt{2}$
- k $4\sqrt{3}$
- l $2\sqrt{6}$
- m $2\sqrt{14}$
- n $-\frac{\sqrt{2}}{4}$
- o 1
- p 10
- q 4
- r $-21\sqrt{2}$
- s 12
- t 2
- u $\frac{2}{3}$
- 3 a 6
- b 7
- c 6
- d $15y$
- e x
- f $a\sqrt{a}$
- 4 C
- 5 A
- 6 a 2
- b $4\sqrt{6}$
- c $\sqrt{30}$
- d $\frac{2}{45}$
- e $14\sqrt{3}$
- f 2

Mental skills 1

2 a 11	b 40	c 7	d 24	e 23
f 6	g 43	h 80	i 18	j 15
k 40	l 65	m 11	n 14	o 12
p 135				

Exercise 1-05

1 a $\sqrt{15} + \sqrt{10}$	b $2\sqrt{3} - \sqrt{6}$	c $\sqrt{6} + \sqrt{14}$
d $3\sqrt{10} - 5$	e $6 + 6\sqrt{6}$	f $\sqrt{55} - 4\sqrt{11}$
g $42 - 8\sqrt{7}$	h $5\sqrt{5} + 75$	i $24 + 3\sqrt{6}$
- 2 C
- 3 a $10 + \sqrt{10} - 6\sqrt{5} - 3\sqrt{2}$
- b $7 + 2\sqrt{7} - \sqrt{21} - 2\sqrt{3}$
- c $28\sqrt{6} + 21 + 8\sqrt{2} + 2\sqrt{3}$
- d $20 + \sqrt{10}$
- e $109 + 10\sqrt{77}$
- f $72 - 23\sqrt{6}$
- g $16\sqrt{10} + 54$
- h $-16 - \sqrt{35}$
- 4 C
- 5 a $8 - 2\sqrt{15}$
- b $9 + 2\sqrt{14}$
- c $9 - 4\sqrt{5}$
- d $19 + 6\sqrt{10}$
- e $77 + 30\sqrt{6}$
- f $179 - 20\sqrt{7}$
- g $38 + 12\sqrt{10}$
- h $23 + 4\sqrt{15}$
- 6 a 1
- b 22
- c 8
- d 2
- e 1
- f 166
- g 13
- h -43
- 7 C
- 8 a $88 - 30\sqrt{7}$
- b $21\sqrt{2} - 10$
- c $5\sqrt{35} + 29$
- d $73 + 40\sqrt{3}$
- e 29
- f $92 - 12\sqrt{5}$

Exercise 1-06

- 1 B
 2 a $\frac{\sqrt{2}}{2}$ b $\frac{\sqrt{7}}{7}$ c $\frac{\sqrt{3}}{3}$ d $\frac{3\sqrt{2}}{2}$ e $\frac{2\sqrt{7}}{7}$ f $\frac{\sqrt{2}}{6}$
 g $\frac{\sqrt{3}}{6}$ h $\frac{\sqrt{7}}{28}$ i $\frac{7\sqrt{5}}{15}$ j $\frac{\sqrt{10}}{15}$ k $\frac{\sqrt{3}}{2}$ l $\frac{\sqrt{15}}{4}$
 3 A 4 D
 5 a $\frac{2-\sqrt{2}}{2}$ b $\frac{\sqrt{3}-5}{5}$ c $\frac{5\sqrt{2}+\sqrt{6}}{4}$ d $\frac{2\sqrt{3}-3\sqrt{2}}{18}$
 6 a $\frac{2\sqrt{7}+7\sqrt{2}}{14}$ b $\frac{\sqrt{10}+5\sqrt{3}}{5}$ c $\frac{\sqrt{3}-\sqrt{2}}{2}$

Power plus

- 1 a Yes, because you are multiplying by 1. b $\frac{3-\sqrt{2}}{7}$
 2 a $5-2\sqrt{3}$ b $2+\sqrt{3}$ c $\sqrt{5}-1$ d $\sqrt{7}+\sqrt{3}$
 3 a $2+\sqrt{3}$ b $\frac{\sqrt{10}-\sqrt{2}}{4}$ c $\frac{\sqrt{7}+\sqrt{3}}{4}$ d $\frac{10-4\sqrt{3}}{13}$
 4 s = $\frac{\sqrt{3}D}{3}$ 5 $\frac{\sqrt{2}}{2}$ 6 $(2+2\sqrt{3})$ mm

Chapter 1 revision

- 1 A
 2 a I b R c R d I
 e R f R g R h I
 3 a $6\sqrt{2}$ b $7\sqrt{2}$ c $5\sqrt{11}$ d $8\sqrt{2}$ e $15\sqrt{6}$
 f $14\sqrt{7}$ g $48\sqrt{2}$ h $15\sqrt{5}$ i $28\sqrt{3}$ j $\sqrt{6}$
 k $\frac{20\sqrt{5}}{3}$ l $2\sqrt{2}$ m $4\sqrt{11}$ n 6 o $6\sqrt{2}$
 4 a $13\sqrt{2}$ b $5\sqrt{2}-7\sqrt{5}$ c $14\sqrt{2}+17\sqrt{3}$
 d $32\sqrt{5}-9\sqrt{7}$ e $38\sqrt{2}-24\sqrt{3}$ f $8\sqrt{11}$
 5 a $\sqrt{21}$ b $2\sqrt{10}$ c $4\sqrt{3}$ d $\sqrt{55}$
 e $\sqrt{6}$ f $7\sqrt{14}$ g $4\sqrt{6}$ h 125
 i $\frac{5}{3}$ j $\frac{3\sqrt{2}}{2}$ k $\frac{2}{3}$ l $\frac{7}{5}$
 6 a $9\sqrt{2}-12$ b $\sqrt{10}-10\sqrt{5}$ c $7\sqrt{35}-27$
 d $23-8\sqrt{7}$ e $77+10\sqrt{6}$ f 43
 g 103 h $70+38\sqrt{5}$
 7 a $\frac{\sqrt{10}}{10}$ b $\frac{3\sqrt{2}}{2}$ c $\frac{\sqrt{7}}{35}$ d $\frac{\sqrt{3}}{4}$ e $\frac{5\sqrt{6}}{6}$ f $\frac{2\sqrt{2}+1}{3}$

Chapter 2

SkillCheck

- 1 a 0.04 b 0.22 c 0.183 d 0.047
 e 0.095 f 0.0675 g 0.1525 h 0.2
 2 a \$72 b \$116.25 c \$4494
 3 a \$7350 b \$4034.10 c \$8737.60
 4 a 36 b 24 c 60
 5 a 52 b 26 c 365
 d 4 e 12 f 8 years 4 months
 6 a 1152 b 50 c 0.06
 7 a \$5962.59 b \$33 433.46 c \$18 481.63 d \$64 937.10

Exercise 2-01

- 1 a \$874 b \$938.80 c \$367.20
 2 Greta earns more per week by \$27.48.

- 3 a \$3461.86 b \$6923.73 c \$15 053.33
 4 Job 1: \$1104.64; Job 2: \$1160; Job 2 by \$55.36
 5 \$1096.10 6 \$735.23 7 \$761.24 8 A 9 \$13 312.50
 10 \$1394.40 11 \$2115 12 54 13 \$63.95
 14 a \$427 b \$700 c \$956.87 d \$625.55
 15 a \$972.12 b \$680.48 c \$4568.96

Exercise 2-02

- 1 a \$45 697 b \$6398.53
 2 a \$114 719 b \$30 393.03
 3 a \$90 904 b \$21 581.48
 4 C 5 \$19 924.99 6 \$45 456.10
 7 \$696.42 8 \$623.52
 9 a \$452 b \$1711.10 c 25.0%
 10 a \$458 b \$1747.65 c 24.0%
 11 a \$2296 b \$456 c \$1646.73
 12 a \$2297.59 b \$456 c \$1550.39
 13 Gross weekly income = \$816.90; Total deductions = \$369.10;
 Net income = \$447.80

Exercise 2-03

- 1 a \$5040 b \$2953.50 c \$102.50
 d \$96.95 e \$71.92 f \$451.20
 2 a \$87.50 b \$5925.15 c \$391 000
 d \$820 e \$279 f \$723.04
 3 A
 4 a \$11 200 b \$1569 c \$9392.50 d \$11 331.25
 5 a \$1440 b \$7440 6 4.5%
 7 a \$6750 b 18.75% 8 9.75% p.a.
 9 2 years 10 26 weeks 11 137 days
 12 C 13 2.5 years 14 2.6% p.a.
 15 a \$18.90 b \$1063.90

Exercise 2-04

- 1 a Check with your teacher. Investment after 1st yr = \$24 150;
 Investment after 2nd yr = \$25 357.50
 b Compound interest = \$2357.50
 2 a \$16 153.36 b \$1153.36
 3 a \$38 459.48 b \$4359.48
 4 a \$5408, \$408 b \$30 245.29, \$2445.29
 c \$11 113.20, \$1513.20 d \$41 905.55, \$2405.55
 e \$19 337.39, \$937.39
 5 a \$4791.80 b \$1642.38 c \$308.93
 d \$3913.84 e \$6834.42

Mental skills 2

- 2 a 18 b \$126 c 39 d \$30.30
 e \$7.50 f 10.8 g \$27 h 60
 i \$240 j \$3.30 k 900 l \$52.50
 4 a 10 b 166 c \$50 d \$22
 e 37.5 f \$5.80 g 135 h \$22.60

Answers

- 6 a 500 b \$20 c 4.5 d \$6.25
e 81 f \$35 g 16.5 h 74.5
i \$195 j \$425 k \$31.50 l 290
8 a 160 b \$1.50 c 7.5 d \$32.50
e \$67.50 f \$31.25 g 38 h 170

- 8 Yes, it will lose approximately 52% after 7 years.
9 a \$1800 b 5 years c \$798.67
d Yes, in the 30th year. e No

Power plus

- 1 4 years and 61 days 2 \$4444.44 3 \$12 838.71
4 \$63 367.49 5 \$2276.87 6 790 000
7 a 18 years. b 18 years.
c No. The size of the interest rate and the number of compounding periods determine how quickly the principal takes to double in value.

Chapter 2 revision

Exercise 2-05

- 1 \$14 332.50
2 a i \$9754.75 ii \$3254.75
b i \$13 858.59 ii \$3858.59
c i \$12 634.81 ii \$394.81
d i \$43 949.46 ii \$9349.46
e i \$8427.39 ii \$427.39
3 D 4 \$1 301 018.83 5 B
6 a i \$13 488.50 ii \$3488.50
b i \$52 751.13 ii \$17 251.13
c i \$9448.23 ii \$548.23
d i 53 366.91 ii \$11 366.91
e i \$19 473.44 ii \$2973.44
f i \$5177.03 ii \$277.03
7 C 8 a \$600 b \$615 c Tegan by \$15.
9 a \$10 510.31 b \$1969.48 less
10 a i \$7554.45 ii \$7688.85
iii \$7758.33 iv \$7805.54
b Monthly, because it earns the most interest.

- 1 \$13 045.75 2 \$1349.18
3 a \$797.45 b \$1011.40
4 a \$1052.51 b \$736.76 c \$4946.80
5 a \$67 725 b \$13 557.63
6 a \$2400 b \$392.50 c \$78.75 d \$621.37
7 a \$5955.08 b \$955.08 8 \$36 282.78
9 \$15 374.72 10 \$852.91 11 \$45 815.75
12 a \$487.50 b \$4387.50 c \$1908.56
d \$6296.06 e \$174.89 f \$6783.56
13 a \$8851.45 b \$6138.55 c 59%

Exercise 2-06

- 1 a \$175.50 b \$1579.50 c \$328.14
d \$1907.64 e \$105.98 f \$2083.14
2 a \$1275 b \$24 225 c \$10 416.75
d \$34 641.75 e \$577.36 f \$35 916.75
3 a \$1379 b \$2316.72 c \$217.58
4 a \$3420 b \$720 c \$1500 d 48%
5 a \$2080 b \$8320 c \$13 200
d \$4880 e 58.7%
6 a \$32.90 b \$437.42 c \$108.42 d 36.62%
7 a \$1073.40 b \$273.40 c 34.2%
8 a \$2599 b \$3576 c \$677 d 10.4%
9 a \$262.50 b 7.4%

Exercise 2-07

- 1 \$933.89
2 a \$20 429.69 b \$29 560.31
3 a i \$659.66 ii 60% b i \$2459.54 ii 45.2%
c i \$5073.42 ii 60% d i \$778.24 ii 41%
e i \$14 020.37 ii 51% f i \$851.35 ii 37%
g i \$403.03 ii 46.3% h i \$1097.20 ii 68.6%
4 a i 90% ii 73% iii 53% iv 48%
b By trial and error, in approx 6.6 years.
5 a i \$10 000 ii \$8000 iii \$4096 b 32.8%
6 a \$11 138.51 b \$4661.49
7 a \$6472.88 b \$3441
c 8 years and 9 months. d 23.2%

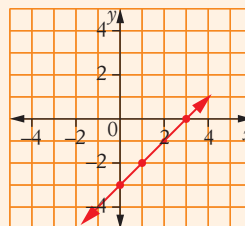
Chapter 3

SkillCheck

- 1 a (6, 1) b (-5, -4) c 6 d 6
e $AC = BC = 4.5$ f isosceles g $\frac{1}{3}$ h $-\frac{2}{3}$

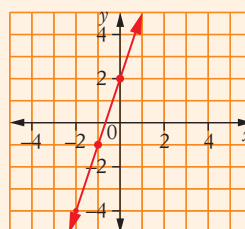
2 a

x	0	1	2	3
y	-3	-2	-1	0



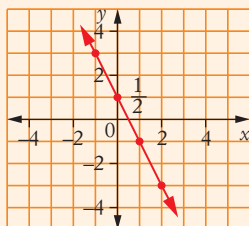
b

x	-2	-1	0	1
y	-4	-1	2	5



c

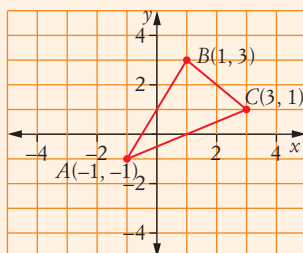
x	-1	0	1	2
y	3	1	-1	-3



3 a -2 b -8 c 5 d 4 e $-\frac{1}{4}$ f $2\sqrt{17}$

Exercise 3-01

- 1 B
4 a $\frac{1}{3}$
5 a i 2.2 b i 10.8 c i 7.1 d i 7.6 e i 10.2 f i 5.7
6 a $\sqrt{89}$
7 k: $m = \frac{1}{5}$; l: $m = -\frac{1}{2}$
8 B
- 2 C
b -2
ii (6, 2.5)
ii (3.5, 3)
ii (-2.5, -0.5)
ii (0.5, -7.5)
ii (-6, -3)
ii (5, 0)
b $\sqrt{194}$
- 3 A
c $\frac{7}{3}$
iii $-\frac{1}{2}$
iii $\frac{2}{3}$
iii -1
iii $-\frac{3}{7}$
iii -5
iii 1
c $\sqrt{82}$



- b $AB = AC = 2\sqrt{5}$, $BC = 2\sqrt{2}$ c $AB = AC = 2\sqrt{5}$
d isosceles e 11.8
- 10 a
b square
c $m_{KL} = \frac{2}{3}$, $m_{LM} = \frac{2}{3}$, $m_{PM} = -\frac{2}{3}$
d $m_{KL} = \frac{2}{3}$, $m_{LM} = \frac{2}{3}$, $m_{PM} = -\frac{2}{3}$
e the gradients are equal, they are parallel
f $KL = LM = PM = KP = 2\sqrt{13}$
g 28.8 h 52 sq. units
- 11 a $P(-2, 1)$, $Q(1, 3)$
b $PQ = 3.6$, $AC = 7.2$, $AC = 2 \times PQ$
c $m_{PQ} = \frac{2}{3}$, $m_{AC} = \frac{2}{3}$; the gradients are equal.

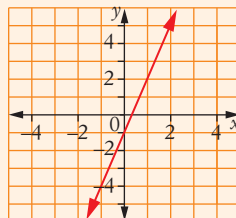
- 12 a 72° b 27° c 45° d 68°
e 117° f 37° g 174° h 146°
13 a 1.73 b -0.40 c 0.90 d -14.30
e 0.14 f -1 g -0.05 h 0

Exercise 3-02

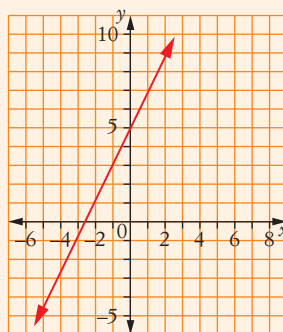
- 1 a neither b perpendicular c parallel
d neither e parallel f neither
2 a 4 b -2 c $\frac{1}{3}$ d -0.2
3 a -1 b $\frac{1}{6}$ c $\frac{2}{3}$ d $-\frac{2}{5}$
4 D 5 B 6 A
7 a $m_{AB} = -\frac{4}{3}$, $m_{CD} = -\frac{4}{3}$, $\therefore AB \parallel CD$
b $m_{PQ} = \frac{3}{4}$, $m_{CD} = -\frac{4}{3}$, $\therefore PQ \perp CD$
8 a $\frac{1}{3}$ b -3

Exercise 3-03

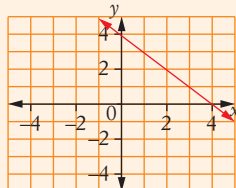
- 1 a i $\frac{1}{3}$ ii -1



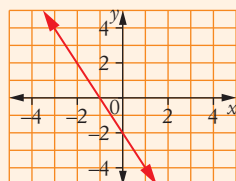
- b i -2.5 ii 5



- c i 4 ii 4

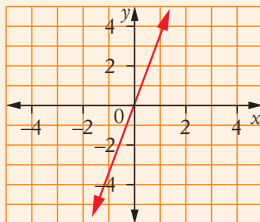


- d i -1 ii -2



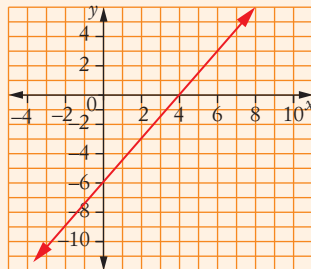
Answers

e i 0

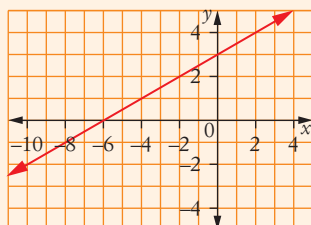


ii 0

d

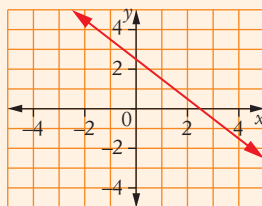


f i -6

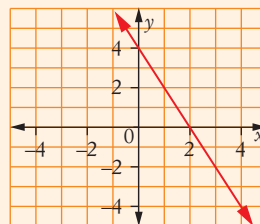


ii 3

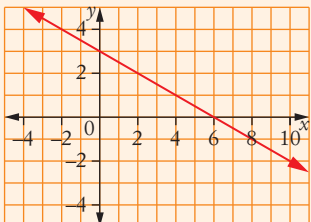
e



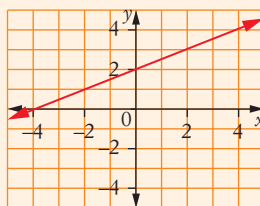
2 a



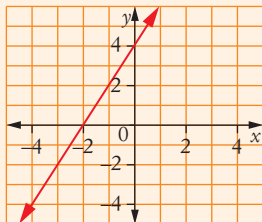
f



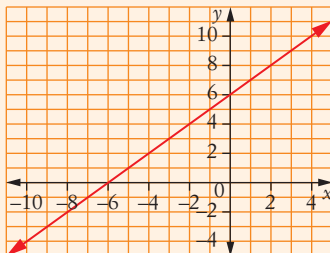
b



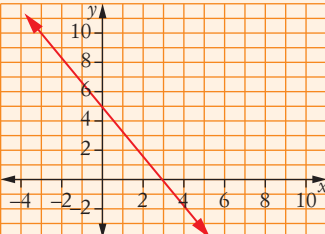
g

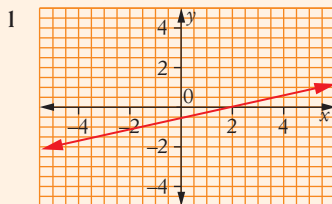
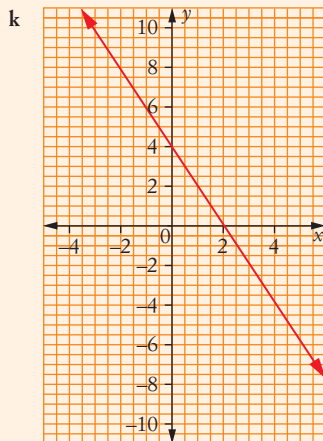
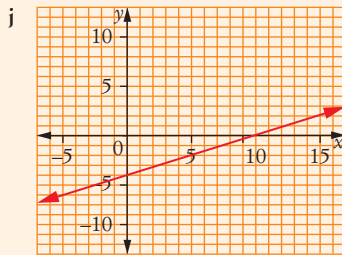
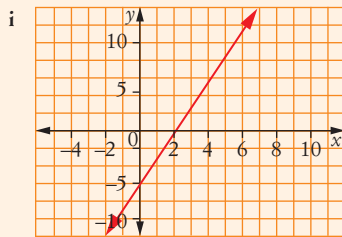


c



h



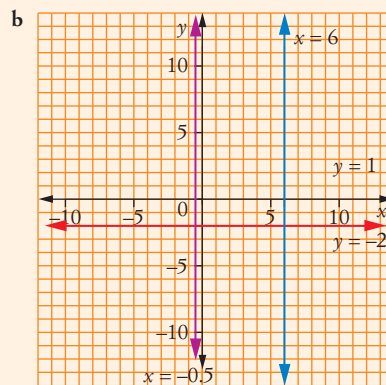
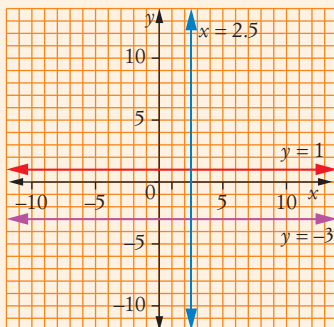


3 a no b yes c yes d yes e no f no

4 C

5 a $x = -4$ b $x = 1$ c $y = 5$ d $y = -3$

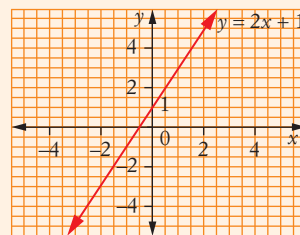
6 a



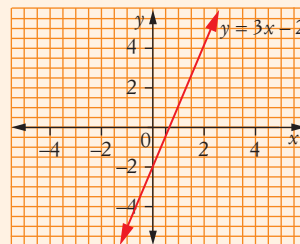
7 a $y = 2$ b $x = 4$ c $x = -1$ d $y = -2$
 e $y = 3$ f $x = -1$ g $y = 6$ h $x = -1$
 8 A 9 C 10 a x-axis b y-axis

Exercise 3-04

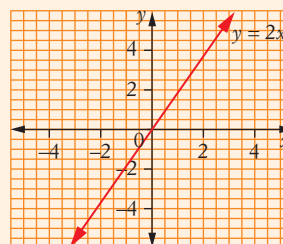
1 a $m = 3, b = -2$ b $m = -2, b = 7$ c $m = 1, b = 4$
 d $m = -1, b = 9$ e $m = \frac{3}{4}, b = 6$ f $m = 1, b = 0$
 g $m = \frac{1}{2}, b = -11$ h $m = \frac{2}{3}, b = 6$ i $m = -\frac{1}{3}, b = -8$
 j $m = 2, b = -6$ k $m = -3, b = 11$ l $m = 1, b = -\frac{7}{2}$
 2 a $y = 2x + 1$ b $y = \frac{3}{4}x + 2$ c $y = -7x + 5$
 d $y = -\frac{2}{5}x + 3$ e $y = -2x - 3$ f $y = -3x + \frac{1}{2}$
 3 a $m = 2, b = 1$



b $m = 3, b = -2$

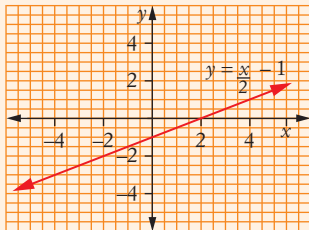


c $m = 2, b = 0$

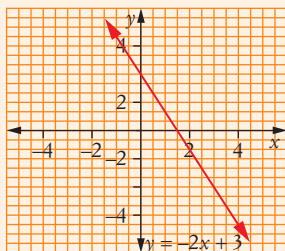


Answers

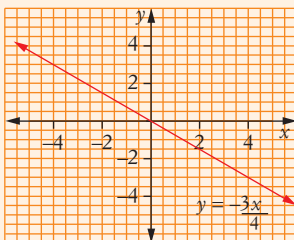
d $m = \frac{1}{2}, b = -1$



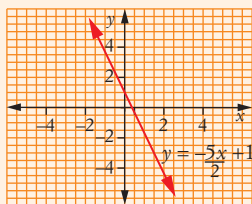
e $m = -2, b = 3$



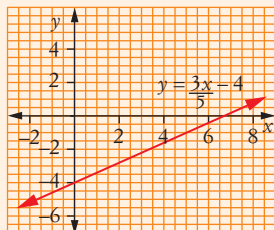
f $m = -\frac{3}{4}, b = 0$



g $m = -\frac{5}{2}, b = 1$



h $m = \frac{3}{5}, b = -4$



4 $y = 2x$

5 a C b B, D c B d C, D e A, B f D

6 a $y = 4x + 3, y = 4x - 6$ b $3x - y + 7 = 0, y = 3x - 2$

Mental skills 3

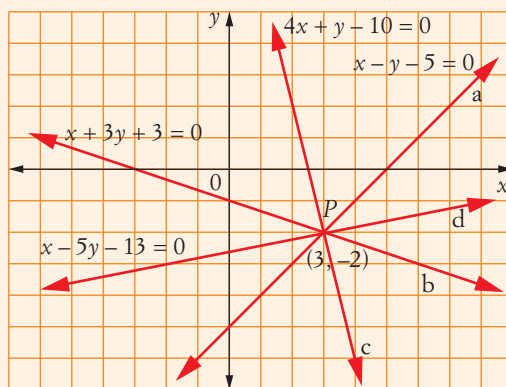
- 2 a 8 h 30 mins b 5 h 40 mins c 3 h 25 mins
 d 8 h 15 mins e 11 h 25 mins f 1 h 40 mins
 g 5 h 10 mins h 5 h 45 mins i 7 h 55 mins
 j 7 h 40 mins

Exercise 3-05

- 1 a $x - y + 2 = 0$ b $3x - y - 1 = 0$
 c $5x - y + 8 = 0$ d $x + 2y - 3 = 0$
 e $x - 2y - 6 = 0$ f $8x - y + 2 = 0$
 g $6x - y - 3 = 0$ h $x - 2y - 6 = 0$
 i $3x - 5y + 10 = 0$
 2 a $m = -2, b = 6$ b $m = 4, b = -5$
 c $m = \frac{3}{2}, b = 2$ d $m = -2, b = 1$
 e $m = -2, b = -5$ f $m = \frac{4}{3}, b = 4$
 3 B 4 B

Exercise 3-06

- 1 a $2x - y + 1 = 0$ b $x + y + 2 = 0$
 c $4x - y - 20 = 0$ d $2x - 3y - 4 = 0$
 e $x + 5y + 38 = 0$ f $3x + y - 4 = 0$
 g $4x + y + 1 = 0$ h $3x - 4y + 10 = 0$
 i $2x + y + 10 = 0$
 2 a and b



- 3 a $x - y - 4 = 0$ b $4x - 5y + 18 = 0$
 c $5x - 6y + 23 = 0$ d $8x + 3y - 10 = 0$
 e $3x + 2y - 6 = 0$ f $5x - 3y - 1 = 0$
 g $6x + 11y + 38 = 0$ h $x + y - 3 = 0$
 i $4x - 3y - 11 = 0$
 4 k: $x + 2y - 7 = 0, l: 3x - y + 7 = 0$
 5 $4x + y - 20 = 0$ 6 $5x - 7y + 42 = 0$
 7 $2x - 3y + 18 = 0$ 8 $3x + 5y - 30 = 0$
 9 a $2x - y + 1 = 0$ b same

10 a $m = \frac{y_2 - y_1}{x_2 - x_1}$
 $y - y_1 = m(x - x_1)$
 $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$
 $\therefore \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$

b $x - y - 4 = 0$ c same

Exercise 3-07

1 a $y = 2x + 5$ b $y = -\frac{3}{4}x + 3$ c $y = -3x + 6$
d $y = -x + 3$ e $y = \frac{1}{2}x + 3$ f $y = -3x - 3$
2 a $y = x + 2$ b $y = \frac{3}{4}x$ c $y = -\frac{1}{3}x + 6$
d $y = -\frac{1}{2}x + 4$ e $y = -3x - 3$ f $y = -x - 2$
g $y = 3x - 10$ h $y = \frac{2}{5}x + 2$ i $y = 2x - 3$

Exercise 3-08

1 a $y = 2x + 4$ b $y = 3x + 6$ c $y = -\frac{1}{2}x + \frac{11}{2}$
d $y = 2x - 12$ e $y = -5x - 13$ f $y = \frac{1}{2}x - 10$
2 a $y = -2x - 2$ b $y = \frac{1}{5}x - \frac{1}{5}$ c $y = -\frac{1}{3}x + \frac{4}{3}$
d $y = -3x - 3$ e $y = x + 6$ f $y = -\frac{1}{3}x - \frac{31}{3}$
3 a $m = 2$ b $M(0, 2)$ c $-\frac{1}{2}$ d $y = -\frac{1}{2}x + 2$
4 a $y = \frac{1}{3}x + 1$ b -3 c $y = -3x + 11$
5 a $y = -\frac{4}{5}x + 8$ b $A(10, 0)$ c $\frac{5}{4}$
d $y = \frac{5}{4}x - \frac{25}{2}$ e $(0, -12.5)$

Exercise 3-09

1 a i $5x + 2y - 18 = 0$ ii $3x - 4y - 16 = 0$
iii $(0, 9)$ iv $(0, -4)$ v 26 units^2
b i $x - 5y + 20 = 0$ ii $x + 2y + 6 = 0$
iii $(0, 4)$ iv $(0, -3)$ v 35 units^2
c i $3x - y - 46 = 0$ ii $7x + 15y + 66 = 0$
iii $(15\frac{1}{3}, 0)$ iv $(-9\frac{2}{3}, 0)$ v $123\frac{17}{21} \text{ units}^2$
2 a $5x - 2y - 25 = 0$ b $5x + 7y - 25 = 0$
c $w = 5$ d $t = -10\frac{2}{5}$
3 a $DE = EF = FG = DG = 5 \text{ units}$
b For DE and GF , $m = 0$
For DG and EF , $m = -\frac{4}{3}$
c Diagonal DF , $m = -\frac{1}{2}$
Diagonal EG , $m = 2$
Since $-\frac{1}{2} \times 2 = -1$ it is true that $DF \perp EG$.
d Midpoint of $DF = (0, 0)$
Midpoint of $EG = (0, 0)$
The diagonals bisect each other because their midpoints are the same.
e Opposite sides are equal and parallel, adjacent sides are equal, diagonals bisect each other at right angles.
4 b $6\sqrt{5} \text{ units}$ c $(-1, -1)$
d No, since $m_{PR} \times m_{QS} \neq -1$
e Rectangle, diagonals are equal and bisect each other but not at right angles.
5 b $CE = \sqrt{130} \text{ units}$, $DF = \sqrt{130} \text{ units}$
c $(\frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{2}, \frac{1}{2})$

d $m_{CE} = \frac{11}{3}$, $m_{PR} = -\frac{3}{11}$
 $\therefore CE \perp PR$ because $\frac{11}{3} \times (-\frac{3}{11}) = -1$
e Square, diagonals are equal and bisect each other at right angles.
6 a $BC = DE = \sqrt{61} \text{ units}$, $CD = BE = \sqrt{65} \text{ units}$
b $m_{BC} = \frac{5}{6}$, $m_{CD} = -\frac{4}{7}$, $m_{DE} = \frac{5}{6}$, $m_{BE} = -\frac{4}{7}$
c Midpoint of $BD = (1\frac{1}{2}, 2\frac{1}{2})$; Midpoint of $CE = (1\frac{1}{2}, 2\frac{1}{2})$
d Parallelogram, opposite sides are parallel and equal.
7 a $AC = BD = \sqrt{104} \text{ units}$
b Midpoint of $AC = (1, 2)$, midpoint of $BD = (1, 2)$
 $m_{AC} = -5$, $m_{BD} = \frac{1}{5}$, $\therefore AC \perp BD$
c The diagonals are equal and bisect each other at right angles.
8 Midpoint of $KM = \text{Midpoint of } LN = (2\frac{1}{2}, \frac{1}{2})$
 $m_{KM} \times m_{LN} = 1 \times (-1) = -1$
9 Teacher to check.
10 a $m_{JK} = -\frac{1}{3}$, $m_{LM} = -\frac{1}{3}$, $m_{KL} = -\frac{5}{2}$, $m_{JM} = -\frac{5}{2}$
b Parallelogram because opposite sides are parallel.
11 a $X(3, \frac{1}{2})$; $Y(-1\frac{1}{2}, 3\frac{1}{2})$
b $m_{XY} = -\frac{2}{3}$, $m_{CB} = -\frac{2}{3}$
 $\therefore XY \parallel CB$
c $XY = \frac{1}{2}\sqrt{117}$, $CB = \sqrt{117}$
 $\therefore CB \parallel 2XY$
12 a $m_{WN} = -1$ and $m_{CT} = 1$
 $m_{WN} \times m_{CT} = -1$
 $\therefore WN \perp CT$
 $M_{WN}(-1.5, -0.5)$ and $M_{CT}(-1.5, -0.5)$
 $\therefore$ Diagonals bisect at right angles.
b rhombus
13 Trapezium
14 $ST = WX = \sqrt{37} \text{ units}$
 $TW = SX = 2\sqrt{37} \text{ units}$
 $XS \perp ST$ because $m_{XS} = -\frac{1}{6}$, $m_{ST} = 6$
 $\therefore STWX$ is a rectangle because opposite sides are equal and angles are right angles.
15 a Midpoint of $TU = A(4, -1)$
Midpoint of $UV = B(0, 3)$
Midpoint of $SV = C(-5, -1)$
Midpoint of $ST = D(-1, -5)$
b Gradient of $AB = -1 = \text{gradient of } CD$
Gradient of $AD = \frac{4}{5} = \text{gradient of } BC$
 $AC = 9 \text{ units}$, $BD = \sqrt{65} \text{ units}$
 $\therefore ABCD$ is a parallelogram.
16 a $m_{LM} = \frac{1}{3}$, $m_{LN} = \frac{1}{3}$, $m_{MN} = \frac{1}{3}$
b L, M and N are collinear points.

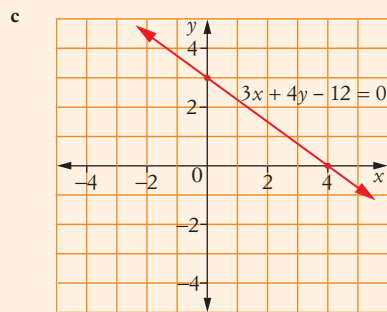
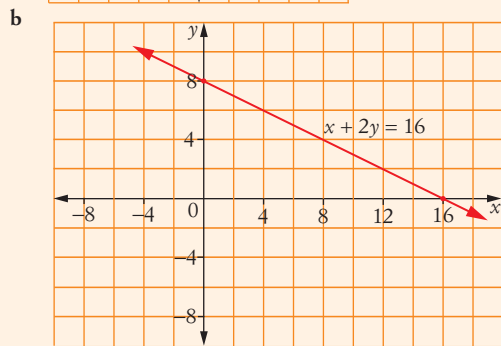
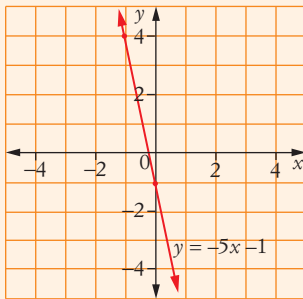
Power plus

1 a $\frac{2}{3}$ b $y = \frac{2}{3}x - 2$ c $y = 4$
2 $k = 5$ 3 $B(2, -1)$ 4 $X(-2, 3)$
5 a $-\frac{3}{2}$ b $3x + 2y + 2 = 0$ or $y = -\frac{3x}{2} - 1$
6 $D(-3, 5)$ or $(7, 9)$ or $(1, 3)$

Answers

Chapter 3 revision

- 1 a 12.6 b $M(-1, 4)$ c $\frac{1}{5}$
 2 a $HJ = JK = KL = HL = \sqrt{58}$
 b $m_{HJ} = \frac{3}{7}, m_{JK} = \frac{7}{3}, m_{KL} = \frac{3}{7}, m_{HL} = \frac{7}{3}$
 c $HK = 10\sqrt{2}, JL = 4\sqrt{2}$
 d rhombus
 3 a 72° b 51° c 135° d 146°
 4 a $-\frac{1}{2}$ b 2
 5 a



- 6 C 7 D
 8 a $m = 2, b = -10$ b $m = 4, b = 3$ c $m = -\frac{3}{8}, b = \frac{1}{2}$
 9 a $3x - y + 5 = 0$ b $2x - 5y - 50 = 0$
 c $x - 3y - 6 = 0$
 10 a $m = 1, b = 2$ b $m = \frac{1}{4}, b = 1$ c $m = -3, b = 9$
 11 a $3x + y - 20 = 0$ b $2x - 3y + 26 = 0$
 12 a $3x - 5y - 20 = 0$ b $x + y + 3 = 0$
 13 a $2x - y + 3 = 0$ b $x + 2y + 8 = 0$
 14 a $3x - y - 6 = 0$ b $2x + y = 0$
 15 $8x + 3y - 95 = 0$
 16 $PN = LM = PM = PL = \sqrt{34}$ units
 $M_{PN} = \frac{3}{5}, M_{PL} = -\frac{5}{3} \therefore PN \perp PL$
 $\therefore LMNP$ is a square because all sides are equal and it has a right angle.

Chapter 4

SkillCheck

- 1 a 182 cm^2 b 770 cm^2 c 680 cm^2
 2 a 1.51 m^2 b 5.59 m^2 c 8.73 m^2

Exercise 4-01

- 1 a 282 m^2 b 298 cm^2 c 2720 mm^2
 d 204 m^2 e 1288 mm^2 f 165 m^2
 2 a cube, 1944 m^3 b rectangular prism, 538 m^3
 c triangular prism, 9720 m^3 d open trapezoidal prism, 6378 m^3
 3 a $80 \text{ m}^3, \$8400$ b 171.4 m^2
 4 a 1036 cm^2 b 1020 mm^2 c 204 m^2
 d 390 cm^2 e 672 cm^2 f 5672 mm^2
 5 a 32.91 m^2 b 74.56 m^2

Exercise 4-02

- 1 a 200.81 m^2 b 3573.56 mm^2 c 206.47 cm^2 d 15.83 m^2
 2 a 35 m^2 b 3478 cm^2
 3 a 1009 m^2 b 2160 m^2 c 4 m^2 d 1895 m^2
 e 7 m^2 f 14 m^2 g 1131 m^2 h 95 m^2
 4 a 41.6 m^2 b 5 L
 5 The triangular prism tent by 1.2 m^2 .

Exercise 4-03

- 1 a 275 m^2 b 564 mm^2 c 87.4 cm^2
 2 a 166.4 m^2 b 3456 mm^2 c 743.1 cm^2
 3 a 843 cm^2 b 1592 cm^2 c 3116 cm^2
 4 a 432 cm^2 b 2150 cm^2 c 173 cm^2
 5 85 m^2
 6 a 1344 mm^2 b 180 cm^2 c 343.4 m^2
 7 a 42 m b $A = 735 \text{ m}^2$ c 35 m d 28 m

Exercise 4-04

- 1 a 101 cm^2 b 628 cm^2 c 2419 cm^2
 2 a 392.7 mm^2 b 62.8 m^2 c 192.4 cm^2
 3 a $90\pi \text{ m}^2$ b $224\pi \text{ mm}^2$ c $450\pi \text{ cm}^2$
 4 a 2827.43 mm^2 b 380.13 m^2 c 366.44 cm^2
 5 a $432\pi \text{ m}^2$ b $192\pi \text{ m}^2$ c $768\pi \text{ m}^2$
 6 a 314 m^2 b 628 m^2 c 628 m^2 d 402 m^2
 7 $5.1 \times 10^8 \text{ km}^2$ 8 5525 cm^2
 9 a 30.16 cm b 4.80 cm c 8 cm d 193.02 cm^2
 10 a 21.9 mm b 25.2 mm c 30.9 mm
 11 a 6.9 cm b 85.3 cm c 85 cm

Mental skills 4

Exact answers shown

- 2 a 331 b 157 c 1587 d 255 e 421
 f 203 g 413 h 734 i 6723 j 15 744
 k 276 l $72\frac{2}{7}$
 4 a 28.231 b 14.187 c 177.4967 d 416.752 e 2.4156
 f 5.0237 g 21.75 h 3.69 i 5.81 j 23.9121

Exercise 4-05

1 a 446.96 cm ²	b 49 270 cm ²	c 864 cm ²
2 a 352 cm ²	b 76 cm ²	
3 a 9721.7 cm ²	b 14 031.4 cm ²	c 14 778.1 cm ²
d 2858.8 cm ²	e 2793.5 cm ²	f 394.7 cm ²
4 a 26.14 m ²	b 19 m ²	5 2953 cm ²
6 1028.32 cm ²		
7 a 857.7 cm ²	b 412.3 cm ²	c 1042.0 cm ²
d 5969.0 cm ²	e 250.6 cm ²	f 628.3 cm ²
g 282.7 cm ²	h 652.9 cm ²	i 501.6 cm ²
j 3769.9 cm ²	k 1148.8 cm ²	l 3017.7 cm ²
m 6615.4 cm ²	n 3908.4 cm ²	o 328 cm ²

Exercise 4-06

1 a 9.7 m ³	b 94 247.8 m ³	c 7.4 m ³
d 135.7 m ³	e 5026.5 m ³	f 216.9 m ³
g 42.3 m ³	h 107.5 m ³	i 146.3 m ³
2 a 251.3 cm ³	b 320 cm ³	c 21.5%
3 500 kL	4 13 666 cm ³	5 63 L
6 a 4825.49 cm ³	b 5026.55 cm ³	c 1989.38 cm ³
d 6375.00 cm ³	e 5301.44 cm ³	f 3084.96 cm ³
g 536.19 cm ³	h 1884.96 cm ³	i 167.33 cm ³
j 12 900 cm ³	k 167.55 cm ³	l 794.12 cm ³
7 a 182.83 m ³	b \$21.94 per day	

Exercise 4-07

1 a 192 cm ³	b 200 cm ³	c 80 cm ³
d 336 m ³	e 1200 cm ³	f 80 m ³
2 a i 12 cm	ii 1296 cm ³	b i 40 m
c i 24 mm	ii 1568 mm ³	d i 60 mm
e i 7.7 m	ii 133.1 m ³	f i 84 cm
3 a 151 m ³	b 314 cm ³	c 1780 mm ³
d 616 cm ³	e 393 cm ³	f 2545 mm ³
4 a i 6.3 cm	ii 59.6 cm ³	
b i 3.9 m	ii 19.9 m ³	
c i 9.2 cm	ii 153.6 cm ³	
d i 3.5 m	ii 2.3 m ³	
e i 244.6 m	ii 296 103.1 m ³	
f i 71.9 cm	ii 129 674.2 cm ³	
5 a i 14 137 cm ³	ii 14 137 mL	
b i 697 m ³	ii 697 kL	
c i 660 cm ³	ii 660 mL	
d i 3619 m ³	ii 3619 kL	
e i 1072 cm ³	ii 1072 mL	
f i 8579 mm ³	ii 9 mL	
6 1.1×10^{12} km ³	7 a 33.75 m ³	b 19 t
8 22.5 m	9 14.1 cm	10 5.7 m
11 2.8 cm	12 26.9 mm	

Exercise 4-08

1 a 59 m ³	b 59 kL	
2 a 343 cm ³	b 240 cm ³	c 1152 cm ³
d 6100 cm ³	e 6048 cm ³	f 2500 cm ³

3 a i 31 416 cm ³	ii 31.416 L	b i 616 cm ³	ii 0.616 L
c i 264 cm ³	ii 0.264 L		
4 28.27 kL			
5 a 12 balls	b 60 balls	c 31 416 cm ³	d 48%
6 a 1963 cm ³	b 0.55 cm ³ /s		
7 a 250 m ³	b 210 kL	c \$415.80	

Exercise 4-09

1 a 9 : 1	b 9 : 25	c 81 : 25	d 4 : 9
2 a 3 : 5	b 1 : 10	c 8 : 5	d 4 : 9
3 a 3.5 cm	b 18	c 48	d 36.75
4 54 cm ²	5 44.1 cm ²	6 7.5 cm	7 154 cm
- 8 The area is quadrupled ($\times 4$).
 9 The sides are decreased by a factor of 3.
 10 The area has increased by a factor of 6.25.
 11 The sides have decreased by a factor of $\frac{1}{10}$.

Exercise 4-10

1 a i 9 : 25	ii 27 : 125	b i 4 : 9	ii 8 : 27
c i 16 : 25	ii 64 : 125	d i 4 : 25	ii 8 : 125
2 a 9 : 10	b 729 : 1000	3 a 5 : 7	b 25 : 49
4 76 800 mm ³	5 75.6 mL	6 2531.25 g or 2.531 kg	
7 78 L	8 a 2.25	b 3.375	
- 9 There has been a $\frac{27}{64}$ decrease in the volume.

Power plus

Teacher to check.

Chapter 4 revision

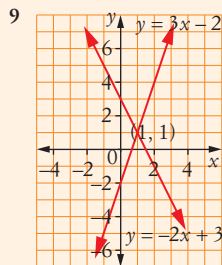
1 a 1.08 m ²	b 3150 mm ²	c 5236 cm ²
d 277.6 m ²	e 216 cm ²	f 482 mm ²
2 a 7389.0 m ²	b 1437.3 m ²	c 104.3 m ²
d 14 294.2 cm ²	e 5871.2 cm ²	f 4427.8 cm ²
3 a 960 cm ²	b 7776 cm ²	c 1370 cm ²
4 a 704 m ²	b 4524 m ²	c 2488 m ²
d 452 m ²	e 681 m ²	f 5890 m ²
5 a 3180 cm ²	b 1268 cm ²	c 395 cm ²
d 3318 cm ²	e 1728 cm ²	f 3436 cm ²
6 a 36 816 m ³	b 20 160 m ³	c 10 016 m ³
7 a 322.67 m ³	b 540 cm ³	c 1568 mm ³
d 1340.41 cm ³	e 10 262.54 mm ³	f 904.78 m ³
8 a 360 498 mm ³	b 145 125 mm ³	c 455 cm ³
d 60.75 m ³	e 3054 cm ³	f 18 096 m ³
9 a 234.375 cm ²	b $6\frac{1}{4}$	10 a 250 cm ³
		b 36 : 49

Mixed revision 1

1 a 12 124 cm ²	b 290 m ²	c 1568 mm ²
2 143°		
3 a \$1 001.72	b \$701.20	c \$4 708.08
4 a \$425	b \$860.63	c \$1 105
5 a $-\frac{3}{2}$	b $\frac{2}{3}$	

Answers

- 6 a 5629.7 m^2 b 135.4 m^2 c $21\,205.8 \text{ cm}^2$
 7 a $13\sqrt{2}$ b $-4\sqrt{11}$ c $6\sqrt{6} - 6\sqrt{3}$
 8 a \$47 210 b \$6890.25



- 10 13 824 cm^2 11 B
 12 a 565 m^2 b 817 m^2 c 804 m^2
 13 a \$4764.06 b \$4782.47 c \$4786.73
 14 $3x - 4y + 24 = 0$ 15 gradient 5, y-intercept 3
 16 a \$768 b \$6912 c \$3317.76
 d \$10 229.76 e \$213.12 f \$10 997.76
 17 a \$19 676.44 b \$10 313.56 c 65.6%
 18 $x + y - 2 = 0$ 19 43 m^3 20 $5x - 2y - 30 = 0$
 21 a $\frac{\sqrt{10}}{10}$ b $\frac{5\sqrt{6}}{6}$ c $\frac{4\sqrt{5+3}}{15}$
 22 Show that all sides have length $\sqrt{34}$ and two sides are perpendicular with gradients $\frac{3}{5}$ and $-\frac{5}{3}$. Teacher to check.
 23 a 1408.33 mm^3 b $39\,810.26 \text{ cm}^3$ c 11.49 m^3
 24 1000 cm^2
 25 a 218 b $55 - 14\sqrt{6}$

Chapter 5

SkillCheck

- 1 a g^9 b r^6 c d^{15} d k^2
 e h^{10} f m^4 g a h 1
 i $6e^7$ j $3n^4$ k $1000w^9$ l 25
 m v^5w^5 n $\frac{v^3}{w^3}$ o $\frac{1}{y}$ p $\frac{1}{y^2}$
 2 a $\frac{19a}{20}$ b $\frac{p}{6}$ c $\frac{5}{3}$ d $\frac{x}{7y}$
 3 a $18m^2 + 66m$ b $-15g + 40$
 4 a $4(x + 6)$ b $5(4 - 3a)$ c $q(q + 1)$
 d $6a(3a - 2)$ e $-2(y + 15)$ f $-6(3w - 4)$
 5 a 3 and 6 b -2 and -4 c 4 and -5 d 8 and -2

Exercise 5-01

- 1 a $6p^7$ b $5w^6$ c m^{30} d $9q^6$
 e $30n^{16}t^5$ f $4x^5y^4$ g $-5e^{10}g^4$ h $-9a^{10}b^5$
 i $100y^{20}$ j $-64p^3$ k $\frac{3pq^2r^4}{2}$ l $54u^4v^3w^8$
 2 a $l^{18}m^{50}$ b $\frac{n^2}{8}$ c 7 d $\frac{w^{10}}{r^{15}}$
 e 1 f $64k^2y^{10}$ g -15 h $\frac{16b^4}{81d^4}$
 i $-125d^9y^{15}$ j $\frac{-27k^{12}}{1000}$ k -9 l $81p^8q^{12}r^{16}$
 3 a 1 b 1 c 7 d 1 e -8 f 9
 g 625 h 128 i 1 j 64 k $\frac{1}{64}$ l 1
 m 25 n $\frac{1}{10000}$ o 1 p 1
 4 a $\frac{1}{25}$ b $\frac{1}{32}$ c $\frac{1}{20}$ d $\frac{1}{1000}$

- 5 a $\frac{1}{8^7}$ b $\frac{1}{3^3}$ c $\frac{1}{y}$ d $\frac{1}{x^3}$ e $\frac{1}{25b^2}$ f $\frac{5}{b^2}$
 g $\frac{1}{ab}$ h $\frac{a}{b}$ i $\frac{11}{t^3}$ j $\frac{1}{1331t^3}$ k $\frac{p^3}{q^5}$ l $\frac{m}{w^3}$
 m $\frac{8}{u^3v^4}$ n $\frac{-2r^6}{y^5}$ o $\frac{10r^3}{e}$ p $\frac{n^7}{2k^4}$
 6 a $\frac{4}{h^2}$ b $\frac{3}{k}$ c $\frac{1}{16h^2}$ d $-\frac{2}{m}$ e $\frac{3g^3}{5}$ f $\frac{3t}{2r}$
 g $\frac{m^3n}{p^2}$ h $\frac{4a^2}{5b}$
 7 a $\frac{243}{1024}$ b $\frac{256}{625}$ c $\frac{16}{81}$ d $-\frac{125}{343}$ e $\frac{9}{k^2}$
 f $\frac{x^3}{27}$ g $\frac{256}{a^8}$ h $\frac{9e^6}{16}$ i $\frac{25t^2}{4d^2}$ j $-\frac{m^{15}}{h^{10}}$
 k $\frac{9p^8}{25d^6}$ l $\frac{64d^6}{27e^3}$
 8 a $5000x^{28}y^6$ b $\frac{8x^{36}}{y^6}$ c $\frac{r^2}{4q^{12}}$ d $\frac{1}{12q^7r}$ e $\frac{256x^{16}}{a^4}$
 f $\frac{a^4}{256x^{16}}$ g $-32h^{11}$ h $\frac{-8}{p^{12}h^7}$ i $64p^{10}h^{20}$

Exercise 5-02

- 1 a 8 b -3 c 25 d 10 e 2 f -0.2
 g 0.1 h -2 i -2 j -9 k 2 l 5
 2 a $\sqrt{10}$ b $\sqrt[3]{12}$ c $\sqrt{g}$ d $\sqrt[4]{m}$
 e $\sqrt{8r}$ f $\sqrt[6]{6h}$ g $\sqrt[3]{5j^8}$ h $\sqrt[3]{90ab}$
 3 a $5^{\frac{1}{2}}$ b $100^{\frac{1}{3}}$ c $20^{\frac{1}{4}}$ d $400^{\frac{1}{5}}$
 e $a^{\frac{1}{2}}$ f $q^{\frac{1}{3}}$ g $(xy)^{\frac{1}{4}}$ h $(36w)^{\frac{1}{5}}$
 4 a 32 b 27 c 64 d 25 e $\frac{1}{2}$ f $\frac{1}{4}$
 g $\frac{1}{2}$ h $\frac{1}{3}$ i $\frac{1}{100}$ j $\frac{1}{32}$ k $\frac{1}{8000}$ l $\frac{1}{25}$
 5 a 6.69 b 14.66 c -5.24 d 3.98
 e 7.62 f 0.05 g 132.96 h 0.30
 6 a $n^{\frac{2}{3}}$ b $d^{\frac{2}{3}}$ c p^3 d m^{-4}
 e $a^{\frac{2}{3}}$ f $a^{\frac{2}{3}}$ g $x^{-\frac{3}{4}}$ h $x^{-\frac{3}{4}}$
 7 a $8x^3$ b $4y^{\frac{2}{3}}$ c $27d^6$ d $512m^{12}$
 e $243r^5$ f $64h^6$ g $\frac{1}{16s^8}$ h $\frac{1}{16p^6}$
 i $\frac{1}{25n^4}$ j $\frac{1}{125t^2}$ k $343a^6b^{15}$ l $\frac{1}{100x^2y^4}$

Exercise 5-03

- 1 a $\frac{5n}{14}$ b $\frac{17c}{10}$ c $\frac{15r}{14}$ d $\frac{-19v}{24}$
 e $\frac{13t}{9}$ f $\frac{-5v}{16}$ g $\frac{13t}{36}$ h $\frac{17a}{30}$
 i $\frac{p+3}{z}$ j $\frac{u}{4g}$ k $\frac{1}{3f}$ l $\frac{2e}{5}$
 m $\frac{5r+3t}{rt}$ n $\frac{3}{4h}$ o $\frac{29}{12b}$ p $\frac{9p-4n}{3np}$
 2 a $\frac{5m-8n}{40}$ b $\frac{17t}{20}$ c $\frac{4a+9h}{30}$ d $\frac{3d-2r}{48}$
 e $\frac{13c}{10}$ f $\frac{6d-11r}{33}$ g $\frac{9h+10a}{15}$ h $\frac{25+24w}{30}$
 i $\frac{21+8a}{28}$ j $\frac{5e}{24}$ k $\frac{7m-2n}{14}$ l $\frac{17k}{30}$
 3 a $\frac{9m+2}{20}$ b $\frac{8x+10}{15}$ c $\frac{7y-5}{12}$ d $\frac{9x-13}{20}$
 e $\frac{-2m+8}{15}$ f $\frac{-3k+31}{70}$ g $\frac{k-3}{6}$ h $\frac{23x-41}{20}$
 i $\frac{14m+49}{36}$ j $\frac{17-12m}{35}$ k $\frac{33-7k}{6}$ l $\frac{14-13x}{6}$

Exercise 5-04

- 1 a $\frac{3m}{20}$ b $\frac{kw}{12}$ c $\frac{28}{pt}$ d $\frac{6}{5qy}$ e $\frac{3d}{e}$ f $\frac{8}{v}$
 g $\frac{3x}{4y}$ h $\frac{2}{3}$ i $\frac{d}{g}$ j $\frac{d^2}{12}$ k $\frac{2}{3}$ l $\frac{12a^2}{k^2}$

2 a $\frac{5x}{2y}$ b $\frac{t}{6r}$ c 4 d $\frac{h^2}{k^2}$ e $\frac{3d}{10}$ f $\frac{1}{3}$
g $\frac{16}{27}$ h $\frac{15}{2}$ i 12 j $\frac{9}{b^2}$ k $6p$ l $\frac{3g}{20y}$
3 a $\frac{5x}{z}$ b $\frac{c}{3b}$ c $\frac{5}{4}$ d $\frac{25b}{3}$ e $\frac{4t}{27}$ f $25y^2$
g $\frac{50p^2t}{7}$ h $\frac{1}{6h^2}$ i $\frac{2s}{35}$ j $\frac{2}{3n}$ k $\frac{27ac}{w}$ l uy

Mental skills 5

2 a 176 b 363 c 261 d 405
e 682 f 707 g 1818 h 3564
i 152 j 540 k 2142 l 588
m 288 n 693 o 3939 p 852

Exercise 5-05

1 a $4h + 24$ b $-3r - 30$ c $7x - 63y$
d $-4a + 20z$ e $-2 + t^2$ f $-20e^2 - 30e$
g $6y + 42y^2$ h $12x^2y^2 - 4xy$ i $16rt^2 - 8r^2t$
j $12ab^2 - 21a^2b$ k $-6h^2 + 18h^3$ l $-25x^3 - 20xy$
2 C
3 a Yes b No c Yes
4 a $15m^2 + 21m$ b $15e^2 - 9e$ c $3w^3 - 15w$
d $49x^3 - 10x^4$ e $t^2 + 7t + 12$ f $12 - 11h - 2h^2$
g $6x^3 + 35x^2 + 8$ h $6 + 3v - 2v^2$ i $w^2 - 8w + 3$
j $9y^2 - 36y + 35$ k $16m^3 + 2m^2$ l $20xy + 20x - 60y$
5 a $6(4x + 5)$ b $9(4 - 3a)$ c $x(x + 1)$
d $10y(3 - 2y)$ e $12d(3d + 2)$ f $4r(4r - 3)$
g $(a - 3)(a + 6)$ h $(8 + t)(t - 3)$ i $(3b + 5)(b - 2)$
j $-q(q + 36)$ k $-2t(3 - 5t)$ l $-3y(y + 2x)$
m $-hn(n - h)$ n $-2e(10e + 11)$ o $9m(5m - 6)$
6 B
7 a $4xy(3x - 4)$ b $2pr(9p + 8)$
c $36mn(m - 3n)$ d $36bc(ab - 4)$
e $16vw(3v + 4w)$ f $25gh(3g^2h - 5)$
g $p(1 - 8p - 4p^2)$ h $3mn(2n + 1 + 16m)$
i $8pg(4p^2 + g - 1)$ j $3a^2(6a^3 - 4 + 5a^2)$

Exercise 5-06

1 a $m^2 + 7m + 12$ b $w^2 + 10w + 25$ c $y^2 - 144$
d $a^2 - 5a - 24$ e $b^2 + 7b - 18$ f $u^2 - 15u + 56$
g $15 + 14k - k^2$ h $r^2 - 18r + 77$ i $c^2 - 9c + 18$
j $t^2 + t - 2$ k $x^2 + 6x - 40$ l $99 - 2n - n^2$
2 D
3 a $2x^2 + 11x + 15$ b $9e^2 + 42e + 49$
c $3p^2 + 7p - 10$ d $49d^2 - 28d + 4$
e $6f^2 + 4f - 10$ f $12m^2 + 5m - 25$
g $6 + 13h - 5h^2$ h $16p^2 - 40p + 25$
i $-10m^2 + 23m - 12$ j $12t^2 - 4t - 1$
k $25y^2 - 25$ l $-49a^2 + 84a - 36$
4 a $8y$ b w^2 c m^2 d $-14u$
e $-18k$ f $+80f$ g $4d^2 + 12d$ h $36a^2, 1$
5 a $h^2 + 14h + 49$ b $k^2 - 10k + 25$
c $x^2 - 2x + 1$ d $q^2 + 20q + 100$

e $25 - 10h + h^2$ f $49 + 14k + k^2$
g $x^2 - 2xw + w^2$ h $a^2 + 2ag + g^2$
i $4m^2 - 12m + 9$ j $25x^2 - 60x + 36$
k $81a^2 + 36a + 4$ l $25 + 70b + 49b^2$
m $16 - 40p + 25p^2$ n $121d^2 - 44cd + 4c^2$
o $64a^2 - 48ay + 9y^2$ p $1 + \frac{2}{y} + \frac{1}{y^2}$
q $t^2 - 2 + \frac{1}{t^2}$ r $\frac{9}{w^2} + 6 + w^2$
6 a $k^2 - 9$ b $y^2 - 64$ c $w^2 - 121$
d $49 - m^2$ e $81 - k^2$ f $9d^2 - 25$
g $25r^2 - 16$ h $16p^2 - 49$ i $9 - 64k^2$
j $4 - 81m^2$ k $81k^2 - 16l^2$ l $49n^2 - 64m^2$
m $t^2 - \frac{1}{t^2}$ n $\frac{w^2}{9} - 4$ o $1 - \frac{1}{r^2}$
7 a $9t^2 - 6td + d^2$ b $4e^2 - 1$ c $25a^2 - 16$
d $-p^2 + 4p - 4$ e $100 - 36y^2$ f $h^2 - 6hg + 9g^2$
g $6x^2 - 3y^2 - 7xy$ h $49a^2 - 16b^2$ i $u^2 - \frac{1}{u^2}$
8 a $-3x^2 + 7x + 2$ b $16k^2 - 48$ c $5xy - 6x + 3y + 9$
d $y^2 + 18$ e $8m^2 + 2n^2$ f $12h + 18$
g $x - 2$ h $1 - 2b^2$

Exercise 5-07

1 a $(x + y)(3p + 2q)$ b $(h + k)(2w - 3u)$
c $(3k + 4g)(5m + 2n)$ d $(x - 2a)(4y + 7a)$
e $(2k - 5f)(a + 4)$ f $(d + y)(c - h)$
g $4(m + t)(a + e)$ h $3(k - 2b)(y + 4)$
i $(3m + p)(n - 2)$ j $(9 + q)(p^2 - 3)$
k $(f - 10)(g - h)$ l $3(l + n)(3k - 4m)$
m $(2 - p)(p - c)$ n $(l - 3)(l^2 + m^2)$
o $(a + y)(x + 1 - k)$ p $(a - b + 3q)(p - 2q)$
2 a $(d + 4)(d - 4)$ b $(x + 5)(x - 5)$
c $(p + 11)(p - 11)$ d $(y + 9)(y - 9)$
e $(5 + t)(5 - t)$ f $(10 + k)(10 - k)$
g $(2r + 3d)(2r - 3d)$ h $(5g + 2e)(5g - 2e)$
i $(12 + 7m)(12 - 7m)$ j $(9y + 4k)(9y - 4k)$
k $(1 + 9d)(1 - 9d)$ l $(m + 2n)(m - 2n)$
m $(y + z)(y - z)$ n $(7 + 4m)(7 - 4m)$
o $(b + 11d)(b - 11d)$ p $(6c + 5k)(6c - 5k)$
q $(4 + 9h)(4 - 9h)$ r $(5a + 8m)(5a - 8m)$
s $(10 + 7n)(10 - 7n)$ t $(11p + 12q)(11p - 12q)$
u $(\frac{1}{2} + 5c)(\frac{1}{2} - 5c)$ v $(2t + \frac{1}{3})(2t - \frac{1}{3})$
w $(5h + \frac{3}{2})(5h - \frac{3}{2})$ x $(1 + mn)(1 - mn)$
3 a $4(m + 2p)(m - 2p)$ b $3(d + 3)(d - 3)$
c $y(y + 5)(y - 5)$ d $2(3 + 5g)(3 - 5g)$
e $k(1 + 4k)(1 - 4k)$ f $2(5q + 1)(5q - 1)$
g $3(d + 2v)(d - 2v)$ h $5t^3(t + 5)(t - 5)$
i $2(ab + 1)(ab - 1)$ j $x^2(y + w)(y - w)$
k $12(4f + 3g)(4f - 3g)$
l $5(3d + \frac{1}{2})(3d - \frac{1}{2})$ or $\frac{5}{4}(6d + 1)(6d - 1)$
m $2(x + 2a)(x - 2a)$ n $25(2 + w)(2 - w)$
o $5(\frac{1}{2} + 4e)(\frac{1}{2} - 4e)$ or $\frac{5}{4}(1 + 8e)(1 - 8e)$
p $(3c + 2\frac{1}{2})(3c - 2\frac{1}{2})$ or $\frac{1}{4}(6c + 5)(6c - 5)$

Answers

- 4 a $(c + \frac{1}{4})(c - \frac{1}{4})$ b $(\frac{m}{4} + \frac{n}{3})(\frac{m}{4} - \frac{n}{3})$
 c $(\frac{w}{5} + \frac{3u}{4})(\frac{w}{5} - \frac{3u}{4})$ d $(k^2 + 5)(k^2 - 5)$
 e $(\frac{4a}{7} + \frac{5b}{2})(\frac{4a}{7} - \frac{5b}{2})$ f $(t + 3)(t - 3)(t^2 + 9)$
 g $(10 + n)(10 - n)(100 + n^2)$ h $y(2x + y)$
 i $-(p + 3q)(3p - q)$ j $(\frac{x}{2} + \frac{y}{6})(\frac{x}{2} - \frac{y}{6})$
 k $4ab$

Exercise 5-08

- 1 a $-3, -8$ b $-5, 2$ c $3, -5$ d $3, 4$
 2 a $(x + 3)(x + 5)$ b $(d + 7)(d + 2)$ c $(m + 9)(m + 3)$
 d $(e + 3)(e + 2)$ e $(h + 2)(h + 2)$ f $(n + 1)(n + 10)$
 g $(n - 3)(n + 1)$ h $(r - 7)(r + 2)$ i $(h - 4)(h + 1)$
 j $(w - 9)(w + 2)$ k $(f - 9)(f + 3)$ l $(a - 6)(a + 2)$
 m $(x + 4)(x - 1)$ n $(t + 8)(t - 3)$ o $(m + 5)(m - 2)$
 p $(a + 2)(a - 1)$ q $(k + 7)(k - 2)$ r $(w + 6)(w - 2)$
 s $(p - 6)(p - 4)$ t $(n - 2)(n - 1)$ u $(r - 3)(r - 3)$
 v $(m + 2)(m + 2)$ w $(p - 10)(p - 10)$ x $(c - 5)(c - 5)$

Exercise 5-09

- 1 a $3(m + 1)(m + 2)$ b $2(y + 2)(y - 1)$
 c $5(t - 10)(t + 8)$ d $5e^2(e + 8)(e - 3)$
 e $x(x - 11)(x + 10)$ f $4(b - 7)(b + 6)$
 g $4(w + 4)(w - 3)$ h $3a(a - 4)(a + 1)$
 i $2(e + 5)(e + 4)$ j $-(t + 8)(t - 3)$
 k $-(u - 7)(u + 6)$ l $-(x - 7)(x + 4)$
 m $-(b + 4)(b - 3)$ n $-(k - 3)(k - 4)$
 o $-(x - 5)(x - 7)$
 2 a $(2d + 3)(3d + 5)$ b $(4m + 3)(2m + 1)$
 c $(y + 5)(2y + 7)$ d $(d + 10)(2d + 7)$
 e $(w + 15)(2w + 1)$ f $(e + 3)(4e + 3)$
 g $(2f + 3)(4f + 1)$ h $(d + 1)(3d + 2)$
 i $(b + 1)(2b + 7)$ j $(y + 1)(5y + 11)$
 k $(4g + 3)(2g + 5)$ l $(3a + 7)(2a + 3)$
 3 a $(4k - 3)(k - 2)$ b $(2w - 5)(3w - 1)$
 c $(p - 3)(5p - 4)$ d $(2g - 7)^2$
 e $(3f - 4)(4f - 3)$ f $(2h - 9)^2$
 g $(y + 1)(5y - 11)$ h $(4d - 5)(d + 1)$
 i $(2m + 3)(m - 3)$ j $(2a + 1)(4a - 3)$
 k $(5u - 4)(3u + 1)$ l $(3c + 1)(3c - 5)$
 m $(5m + 7)(m - 1)$ n $(3g - 4)(2g + 3)$
 o $(3p - 2)(p + 2)$ p $(7w - 1)(w + 1)$
 q $(5y - 1)(y + 3)$ r $(3n - 2)(n + 4)$
 4 a $(9w - 10)^2$ b $4(y + 1)^2$ c $(5h - 4)^2$
 5 a $3(m + 4)(3m - 2)$ b $2(2y - 5)(y + 1)$
 c $5(3k - 2)(2k + 3)$ d $4(w - 4)(3w + 1)$
 e $4(t + 2)(3t - 1)$ f $-(5q + 3)(5q - 2)$
 g $-2(2m - 1)(3m - 2)$ h $-(3h + 4)(4h - 5)$
 i $6(2c + 3)(2c + 1)$ j $-3(z + 1)(2z - 5)$
 k $2(2d - 3)(3d + 5)$ l $-2(x - 3)(3x - 2)$
 6 a $(w - 1)(7w - 1)$ b $(h - 3)(4h + 5)$
 c $(4x - 3)(2x + 1)$ d $(r + 5)(5r + 1)$
 e $(d - 7)(2d - 1)$ f $(3n + 1)(2n - 3)$

- g $-(3m - 2)(3m + 4)$ h $-(5c - 3)(c + 1)$
 i $(3g + 2)(5g + 3)$ j $-(4q + 3)(2q - 5)$
 k $(x - 2)(3x - 7)$ l $-(3d - 4)(d + 4)$

Exercise 5-10

- 1 a $(m + 8)^2$ b $3(d + 1)(d - 1)$
 c $(d + 3)(3d - 5)$ d $(3 + h)(k - 5)$
 e $(5y + 8)(5y - 8)$ f $4(5f + 4)(5f - 4)$
 g $q(q + 3 - 3p)$ h $-(g - 3)(g + 1)$
 i $4(2b + 5)(3b - 2)$ j $(5r + 1)(5r - 1)$
 k $(b^2 + 1)(b + 1)$ l $(2x - 5)^2$
 m $-(5d - 4)(d + 1)$ n $(b - 1)^2(b + 1)$
 o $2(2 + v)(2 - v)$ p $m(n + 3)(n + p)$
 q $2(w - 6)^2$ r $(6h + 1)^2$
 s $(3r - 8t)(5r + 3t)$ t $(2d + 1)^2$
 u $9(g + 2k)(g - 2k)$
 2 a $e(e - 5)(e + 2)$ b $-20(2p - 3q)(3p - 2q)$
 c $7(2x + 1)(2x - 1)$ d $(a - b)(a + b + 4)$
 e $(c - 2)^2(c + 2)$ f $(3a - 1)(2a + 5)$
 g $(t + 7)(t - 5)$ h $2(3p + 2)^2$
 i $-(6a - 1)(4a + 1)$ j $9(x + 2)(x - 3)$
 k $(a - 3)(2ab - 3)$ l $2(a + 3)^2$
 m $(5u - 1)^2$ n $(k - 3)(4k + 7)$
 o $3(4 + w)(4 - w)$ p $3(1 + 3s)(1 - 3s)$
 q $(k + 4)^2(k - 4)$ r $5y(y^2 - 2y + 3)$
 s $mn(m + 2)(m - 2)$ t $-2(a + 2)(a - 2)$
 u $4(2c - 3)(4c + 1)$

Exercise 5-11

- 1 a $x + y$ b $\frac{1}{2(r-r)}$ c $\frac{b-c}{a}$
 d -1 e $w - 4$ f $\frac{5}{d+i}$
 g $\frac{k+5}{k-5}$ h $3(c - 1)$ i $\frac{a+1}{m+n}$
 j $\frac{y+4}{2}$ k $\frac{k+1}{k+4}$ l $\frac{4a+5c}{a-c}$
 m $\frac{s+2}{s-3}$ n $\frac{1-2c}{3c-1}$ o $\frac{a+4}{2(p+2)}$
 2 a $\frac{7m+10}{m(m+1)(m+2)}$ b $\frac{2w-20}{w(w+3)(w+5)}$ c $\frac{4b-7}{(b-1)(b+2)(b-3)}$
 d $\frac{-k-2}{k(k+1)(k-1)}$ e $\frac{5h+12}{4h(h+1)}$ f $\frac{-4d-1}{(d+2)(d+1)}$
 g $\frac{42-5r}{4(r+6)(r-6)}$ h $\frac{d^2+3d-6}{d(d+2)(d-2)}$ i $\frac{-k^2+9k-5}{(k+1)(k-1)(k-4)}$
 j $\frac{3q-1}{(q+1)(q-1)}$
 3 a $6m$ b $1\frac{1}{24}$ c $\frac{1}{2}$ d $6k$ e $\frac{10}{h+1}$
 f $\frac{3}{2(a-b)}$ g $\frac{-r}{5(r+i)}$ h $\frac{4m}{m-1}$ i $\frac{4}{p}$ j $\frac{3}{7}$
 k $\frac{2}{3(x-2)}$ l $3\frac{1}{3}$ m $\frac{(d+1)(d-3)}{6}$ n $\frac{f+3}{4(f-3)}$ o $\frac{3}{f-2}$

Power plus

- 1 a $x^3 + 15x^2 + 55x + 25$ b $y^3 - 6y^2 + 12y - 8$
 c $a^3 + 3a^2b + 3ab^2 + b^3$ d $27d^3 + 270d^2 + 900d + 1000$
 2 a 441 b 2025 c 841 d 3481 e 10 404 f 9604
 3 899
 4 a 399 b 2499 c 8099 d 6396
 5 a $3y^2 + 12y + 14$ b $3x^2 + 9$ c $34n^2 - 34$ d $-4b^2$

- 6 a $(n + 2m)^2$
 c $(5x - 4y)^2$
 e $(c^2 + 1)^2$
 g $(\frac{x}{4} + \frac{y}{5})(\frac{x}{4} - \frac{y}{5})$
 i $(5c^2 - 10)(5c^2 + 10)$
 k $(\frac{4a}{7} + \frac{5b}{2})(\frac{4a}{7} - \frac{5b}{2})$
- b $(x - y)^2$
 d $5(a - 3b)^2$
 f $-(t - 1)(t + 1)(t^2 + 8)$
 h $(x + 1)(x - 1)(x^2 + 1)$
 j $(a + b + c)(a + b - c)$
 l $4pq$

Chapter 5 revision

- 1 a $6v^5w^7$ b $8t^7h^6$ c $25x^2y^4$ d 1
 e $\frac{1}{4k}$ f $\frac{8p^3}{27}$ g $\frac{1}{16m^2}$ h $\frac{4}{m^2}$
 i $625b^{24}y^{12}$ j $512t^{14}u^{16}$ k $5c^4d^4$ l $\frac{6a}{5b}$
 2 a 100 000 b $\frac{64}{125t^3}$ c $\frac{4d^2}{81}$ d $\frac{1}{m^2}$
 3 a 20 b -3 c $\frac{1}{8}$ d 8
 4 a $64a^6$ b $243t^{15}$ c $\frac{1}{16p^8}$ d $\frac{1}{25x^2y^4}$
 5 a $\frac{-3t}{20}$ b $\frac{19g}{6}$ c $\frac{x}{16}$ d $\frac{1}{25x^2y^4}$
 6 a $\frac{7r-20}{10}$ b $\frac{4w+9}{10}$ c $\frac{2p+20}{12}$ d $\frac{4w}{3}$
 7 a $\frac{3m^2}{10}$ b $\frac{3}{4}$ c $\frac{b}{2a}$ d $\frac{4w}{3}$
 8 a $56tp^2 - 40t^2p$ b $-15h^3 - 35h^2$ d $93 - 22n$
 c $4fg^2 - 30f^2g$ f $10y^2 - 41y + 21$
 e $8x^3 + 7x^4$ b $-6(4p - 3q)$
 9 a $8ar(2r + 3)$ d $15xy^2(1 - 2x^2y)$
 c $(5x - 1)(2 - 3x)$ f $4r^2s^3(8s + 3r^2)$
 e $6p(t^2 + 2pt - 8p^2)$ h $-8p^3q^3(1 - 6q^3)$
 g $25x^3y^3(2x - 3y)$
 i $(n^2 + 6)(n - 1)$
 10 a $b^2 + 13b + 30$ b $d^2 + d - 56$
 c $15t - 54 - t^2$ d $20x^2 + 13x - 21$
 e $49y^2 - 9$ f $25p^2 - 80p + 64$
 11 a $n^2 + 18n + 81$ b $n^2 - 81$
 c $9y^2 - 12y + 4$ d $16n^2 - 121$
 12 a $(r + t)(3p + 2q)$ b $(2a + 3c)(2b - 3d)$
 c $(b + 10)(b - 10)$ d $(5 - 4y)(5 + 4y)$
 e $5(2x + 1)(2x - 1)$ f $3t(t + 3)(t - 3)$
 13 a $(y + 5)(y + 5)$ b $(x - 20)(x - 1)$
 c $(n + 11)(n - 3)$ d $(a - 7)(a - 4)$
 e $(m - 12)(m + 7)$ f $(p + 9)(p - 6)$
 14 a $(3w + 2)(w + 1)$ b $(2y + 3)(y - 3)$
 c $-5(b + 4)(b - 3)$ d $(3p - 2)(p + 4)$
 e $2(3x - 1)(2x - 7)$ f $(3n - 2)(2n - 3)$
 15 a $5(q - 3)(q + 3)$ b $4(5x - 3)(x - 2)$
 c $(t - 1)^2(t + 1)$
 16 a $2n + 3t$ b $\frac{6(m+2)}{5}$ c $\frac{a+2}{a+1}$
 d $\frac{2y+12}{y(y+3)(y-3)}$ e 9 f $\frac{3(b+2)}{b(b-3)}$

Chapter 6

SkillCheck

- 1 a i 10 ii 16.5 iii 15 iv 15
 b i 13 ii 1.8 iii 2.5 iv 3
 c i 7 ii 11.5 iii 11 iv 11
 d i 6 ii 43.6 iii 43 iv 43
 e i 48 ii 34.3 iii 34.5 iv 24, 35
 f i 5 ii 2.2 iii 2 iv 2

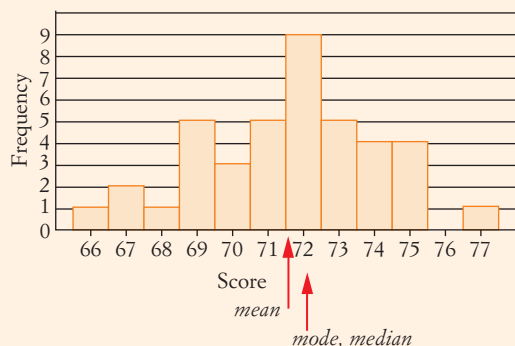
- 2 a i 31 ii 33.3 iii 62 b 78
 c i Median = 30, mean = 28.3, range = 25.
 ii The outlier has increased the median (by 1), the mean (by 5), and the range (by 37).

Exercise 6-01

- 1 a i symmetrical
 ii clustering at 9, no outliers
 b i not symmetrical, not skewed
 ii clustering in the 30s and 60s, no outliers
 c i positively skewed
 ii clustering at 1, no outliers
 d i negatively skewed
 ii clustering at 23–24, no outliers
 e i positively skewed
 ii clustering at 130–150, no outliers
 f i symmetrical
 ii clustering at 5, no outliers
 g i positively skewed
 ii clustering at 13, 23 is an outlier
 h i symmetrical
 ii clustering at 50s and 100s, 136 is an outlier

2 a

Score	Frequency
66	1
67	2
68	1
69	5
70	3
71	5
72	9
73	5
74	4
75	4
76	0
77	1



- b no outliers
 c negatively skewed
 d The lower the score below par, the fewer the golfers that achieve that score.
 e clustering at 72
 f mode = 72, $\bar{x}$ = 71.6, median = 72

Answers

- 3 a 45 b 1–9 hours (stem of 0)
 c no outliers d positively skewed
 e Most students spend limited time on their computers, and have other commitments and do activities such as sport. Only a few students spend many hours on the computer during the week.
 f Mode = 1, $\bar{x}$ = 14, median = 11

Stem	Leaf
12	3
13	
14	2 3
15	0 1 3 3 3 5 5
16	0 0 1 2 2 2 2 3 4 5 5 7 8 9
17	0 0 1 2 3
18	2

- b Symmetrical
 c 123 is an outlier.
 d Clustering occurs in the 160s.
 e mode = 162, median = 162, $\bar{x}$ = 160.2
 5 a slight positive skew b 13.8 is an outlier c 18.4
 d 19.5 e 19.5 f 10.5
 g No, the range has been affected by the outlier 13.8.

Exercise 6-02

- 1 a 5, 6.5, 8 b 18, 20, 26.5 c 32, 34.5, 38
 2 a range = 7, IQR = 3 b range = 22, IQR = 8.5
 c range = 16, IQR = 6
 3 a 7.5 b 3
 4 a 283 mm b 128 mm
 5 a 3 b 2.5 c 17.5 d 19 e 21.5 f 1.5
 6 a 34 b 13
 c i 68, 72, 72, 75, 77, 78, 79, 80 ii 50%
 d 75%
 7 a i 28 ii 9.5
 b The interquartile range, as it is not affected by the score of 35.
 c 48, 48, 48, 49, 51, 53, 55; 54%

Exercise 6-03

- 1 a 2.66 b 2.63 c 1.19 d 1.33 e 2.01
 2 a 7 b i 2.64 ii 2.28
 c Decreases the standard deviation.
 3 a C b A
 4 a 1.99 b 13.43
 5 a $\bar{x}$ = 165.89, σ = 8.37
 b i less than 157 or greater than 175 (to the nearest cm)
 ii between 157 and 174 (to the nearest cm)
 6 a $\bar{x}$ = 11.37, σ = 0.43
 b i less than 10.9, greater than 11.8 (correct to one decimal place)
 ii between 10.9 and 11.8 (correct to one decimal place)
 7 C

Exercise 6-04

- 1 a Men: $\bar{x}$ = 71.40, σ = 6.77; Women: $\bar{x}$ = 77.53, σ = 6.96
 b Yes, the mean of women's pulse rates is much higher, which may be due to stresses involved in shopping (and looking after children at the same time). The standard deviation for women is slightly higher.
 2 a Dominant hand: $\bar{x}$ = 0.40, σ = 0.11; Non-dominant hand: $\bar{x}$ = 0.52, σ = 0.51
 b Yes, the mean reaction time and standard deviation of the dominant hand are much lower than the mean and standard deviation of the non-dominant hand.
 c i 0.61 and 0.75 ii $\bar{x}$ = 0.37, σ = 0.05
 iii Removing the outliers has reduced the mean 0.40 to 0.37 and more than halved the standard deviation.
 d $\bar{x}$ = 0.40, σ = 0.06
 e The removal of the outlier from the non-dominant hand had the greater effect on the mean and standard deviation as the outlier of 2.60 was a more extreme score than the outliers for the dominant hand.
 3 a Western Tigers: $\bar{x}$ = 122.92, σ = 26.98; Barrington City: $\bar{x}$ = 120.92, σ = 23.62.
 b The Barrington City team is slightly more consistent as the standard deviation is 23.62 compared with 26.98 for Western Tigers.
 4 a Vatha: $\bar{x}$ = 13.76, σ = 0.55; Ana: $\bar{x}$ = 14.14, σ = 0.66
 b Vatha is more consistent as the standard deviation for her times is significantly lower than the standard deviation for Ana's times.
 5 B
 6 a Maths: i range = 47 ii IQR = 14 iii σ = 10.97
 Science: i range = 45 ii IQR = 19 iii σ = 13.16
 b Maths: $\bar{x}$ = 67.82; Science: $\bar{x}$ = 61.25
 c The students performed better in Maths as the mean was 67.82 compared to 61.25 for Science. The marks for maths were also more consistent as the IQR and standard deviation are both lower than those of Science.
 7 a Roosters: i range = 48 ii IQR = 20
 iii $\bar{x}$ = 26.67 iv σ = 12.55
 Dragons: i range = 32 ii IQR = 8
 iii $\bar{x}$ = 15.88 iv σ = 7.36
 b The range, IQR and the standard deviation for the Dragons are significantly lower than those of the Roosters, which show that the Dragons are more consistent in the number of points they scored per game. However the mean of the Roosters is significantly greater than the mean of the Dragons, which would indicate that they are a better team as they were able to score many more points per game.

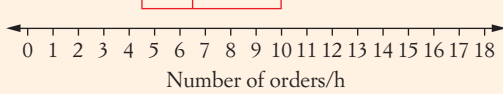
Mental skills 6

- 2 a 160 b 70 c 240 d 900 e 2600
 f 900 g 140 h 300 i 180 j 770
 k 18 l 34 m 46 n 26 o 18
 p 12 q 40 r 8 s 14 t 24

Exercise 6-05

- 1 a 1, 4.5, 6.5, 10, 18

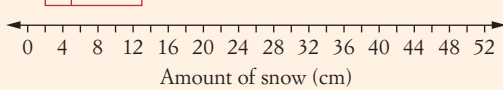
b



- 2 a 26

- b 1, 2, 5, 13, 50

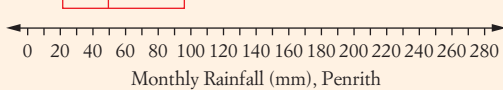
c



- 3 a 261

- b 5, 21.5, 49.5, 96, 266

c



- 4 a 27.5 h b 26 h c 30 h d 4 h e 50%

- 5 a 26 b 21 c 14 d i 25% ii 75%

- 6 a 6, 10, 19, 23, 29 b 13

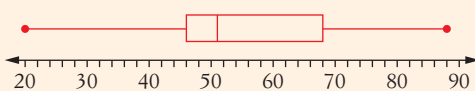
- c i 14

- ii 7

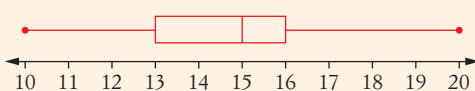
- iii 7

- iv 21

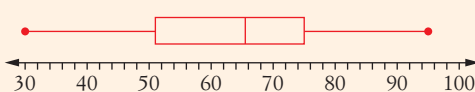
- 7 a 20, 46, 51, 68, 88



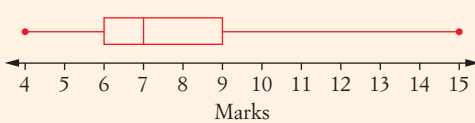
- b 10, 13, 15, 16, 20



- c 30, 51, 65.5, 75, 95



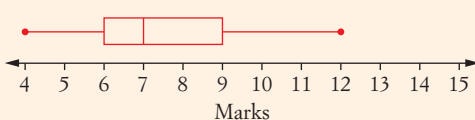
- 8 a 4, 6, 7, 9, 15



- b Dot plot is positively skewed. The length of the boxplot from the median to the highest score is greater than the length from the median to the lowest score.

- c 15

- d 4, 6, 7, 9, 12



- e i The boxplots are the same up to Q_3 .

- ii The whisker from Q_3 is reduced without the outlier.

Exercise 6-06

- 1 a i Year 10: 3.5; Year 8: 8 ii Year 10: 7.5; Year 8: 8.5

- iii Year 10: 1; Year 8: 2

- b i 25%

- ii 75%

- c i 10

- ii 0

- 2 a i 33 ii 28

- b Thunderbirds: 50; Swifts: 49

- c Thunderbirds: 15.5; Swifts: 9.5

- d The range for both teams is similar but the IQR of the Swifts is less than the IQR of the Thunderbirds, indicating that the Swifts are more consistent in their performance.

- e The position of the Thunderbirds' boxplot shows that the Thunderbirds scored more points in games than the Swifts and so performed better in the season.

- 3 a 10K: 9; 10N: 10

- b 10K: 6.5; 10N: 5.5

- c 10K: 3; 10N: 4

- d 10K – lower range and IQR.

- e 75%

- 4 C

- 5 a Brisbane: 26.9, 9.3, 4.7; Sydney: 23.5, 8.5, 4.9;

- Melbourne: 21.4, 13, 8.6; Hobart: 18.6, 11.2, 7

- b Melbourne – it has the highest range and IQR.

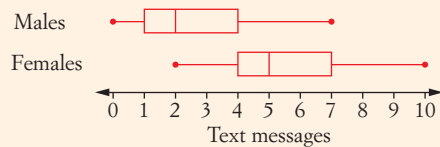
- c Brisbane, more than half of the mean monthly temperatures are higher than most of the mean monthly temperatures of the other cities.

- d Sydney's median temperature is significantly higher than Melbourne's, so Sydney is the warmer city.

- e Sydney has the smaller range and IQR of mean monthly temperatures, so it is more consistent.

- 6 a Male: 0, 1, 2, 4, 7; Female: 2, 4, 5, 7, 10

- b

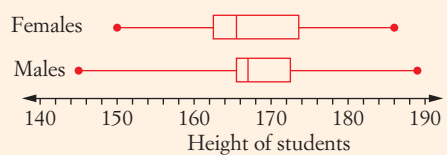


- c Male: 3; Females: 3

- d Males: 7; Females: 8

- e Both are positively skewed, the interquartile range is the same, and the range of females is one more than that of the males. Females do receive more text messages, as the boxplot shows that 75% of females receive more messages than 75% of males.

- 7 a Male: 145, 165, 167, 172.5, 189; Female: 150, 162.5, 165.5, 173.5, 186

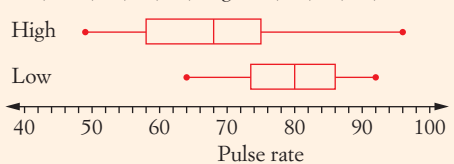


- b Male: Range 44 IQR 7.5

- Female: Range 36 IQR 11

- c Male students have a greater range (44 compared to 36), but a smaller interquartile range (7.5 compared to 12).

- 8 a Low: 64, 73.5, 80, 86, 92; High: 49, 58, 68, 75, 96



- b Low: 28, 12.5

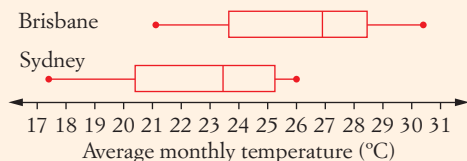
- High: 47, 17

- c The range and interquartile range of the High Frequency group are both greater than that of the Low Frequency group.

- d The high frequency group.

Answers

- 9 a Sydney: 17.6, 20.4, 23.45, 25.25, 26.1; Brisbane: 21.1, 23.65, 26.9, 28.45, 30.4



- b Sydney: 8.5° , 4.85° Brisbane: 9.3° , 4.8°
 c Sydney average monthly temperatures were slightly more consistent than Brisbane, since its range was 0.9° less than Brisbane's, while Brisbane's IQR was only 0.05° less than Sydney's.
 10 a Simone b Simone: 12; Amal: 10 c Amal, smaller range.
 d Simone: 10; Amal: 9 e Simone: 4; Amal: 5
 f Not enough information given to make a valid decision. The interquartile range and range only differ by 1.
 g 25%

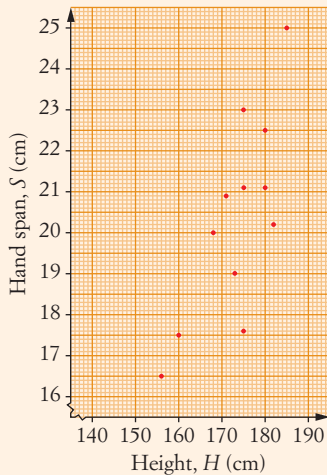
Exercise 6-07

- 1 a Boys: \$34.58, Girls: \$31.78 b Boys: \$33.50, Girls: \$28
 c Boys: Range = 72, IQR = 25, Girls: Range = 69, IQR = 30.5
 d i Boys are positively skewed slightly, girls are positively skewed.
 ii There are no outliers, clustering occurs for the boys in the 20–30s and for the girls in the 10–20s.
 e Boys generally carry more cash – they have a higher mean than the girls and the shape of the data for girls is more positively skewed.
 2 a 21 games b i 34 ii 51
 c Scorpions: $\bar{x} = 1.6$ goals; Vale United: $\bar{x} = 2.4$
 d Scorpions 5, Vale United 6
 e The shape of both teams' results is positively skewed. Clustering for Scorpions occurs at 1 and 2 and for Vale United it occurs at 2.
 f Vale United performed better as its mean was 2.4 goals/game compared to Scorpions 1.6 goals/game.
 3 a Sydney: $\bar{x} = 26.2$, median = 26.5, mode = 28
 Perth: $\bar{x} = 34.3$, median = 35, mode = 38
 b Sydney: Range = 9° , IQR = 3
 Perth: Range = 16° , IQR = 8
 c The temperatures for Sydney and Perth are both negatively skewed, there are no outliers. Sydney's temperatures are clustered in the high 20s, while Perth's are clustered at 34–38.
 d Sydney's temperatures are lower than Perth's, as evidenced by the significantly lower mean, median and mode. The range and interquartile range for Perth are greater than the range and interquartile range for Sydney, indicating greater spread.
 4 a 30
 b Quiz 1: $\bar{x} = 5.6$, mode = 6; Quiz 2: $\bar{x} = 6.3$, mode = 7
 c Quiz 1: 6; Quiz 2: 7
 d Quiz 1: i Range = 7 Quiz 2: i Range = 8
 ii IQR = 2 ii IQR = 2
 e Quiz 1: Results are symmetrical with clustering at 5–6, no outliers.
 Quiz 2: Results show negative skewness with clustering at 5 and 7–8, no outliers.
 f Scores for Quiz 2 are just better than Quiz 1, as the mean of Quiz 2 is higher than the mean of Quiz 1. The spread for both quizzes are similar as there is only a difference of 1 between the ranges and the IQRs are equal.
 5 a 39
 b i mode = 2 ii median = 2 iii range = 6 iv IQR = 1.5
 c positively skewed, no outliers d 50%
 e i By the highest columns.
 ii By the short length of the box when compared to the whole length of the boxplot.
 f i The shape of the distribution, the frequency for each household size and the mode. The mean can also be calculated from the histogram.
 ii The shape of the distribution, the median and the quartiles Q_1 and Q_2 .
 6 a i 5 ii 16
 b i mode = 22 ii range = 18 iii IQR = $24 - 16 = 8$
 c Negatively skewed.
 i The tail of the dot plot goes to the left.
 ii The length of the boxplot from the lowest score to the median is longer than from the median to the highest score.
 d i dot plot ii boxplot iii dot plot iv boxplot
 7 a Sunbeam Valley: range = 24, median = 71, IQR = $75 - 67 = 8$
 Bentley's Beach: range = 30, median = 73, IQR = $82 - 67 = 15$
 b Sunbeam Valley: negatively skewed (slight)
 Bentley's Beach: positively skewed
 c Sunbeam Valley's speeds are clustered in the 70s.
 d 25%
 e Bentley's Beach – higher median, positively skewed. 25% of drivers drive faster than all drivers in Sunbeam Valley. This may be due to more main roads with higher speed limits.
 8 a 36
 b Lamissa: mode 7, median = 7
 Anneka: mode = 7, median = 6
 c Lamissa: range = 8, IQR = $8 - 6 = 2$
 Anneka: range = 9, IQR = $7 - 4 = 3$
 d Lamissa's distribution of scores is negatively skewed with clustering at 7. Anneka's distribution is negatively skewed with clustering at 6 and 7.
 e i 30.5% ii 55.6%
 f Lamissa is the better archer. Her median score is higher than Anneka's, 30.5% of scores are less than 6 compared to Anneka's 55.6%. Also, from the boxplot, 50% of Lamissa's scores are equal to or better than 75% of Anneka's.
 9 a The range (47) is too large.
 b Women: 31 Men: 37
 c Women: Range = 38, IQR = $40 - 24 = 16$
 Men: Range = 47, IQR = $46 - 25 = 21$
 d Distribution for women is positively skewed with clustering in the 20s. Distribution for men is symmetrical with clustering in the 30s.

- e Men have the greater spread in the number of sit-ups completed, as the range and IQR are both greater than those for women.
- 10 a i 56 ii 38
 b i 10 Blue, 10 Yellow ii 10 Green iii 10 Red
 c i 10 Green ii 10 Yellow iii 10 Red, 10 Blue
 d 10 Blue. It shares the highest median with 10 Red but its lowest score is still higher than 25% of 10 Red's scores.

Exercise 6-08

1 a



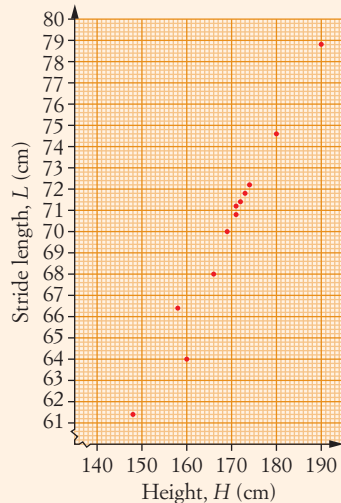
- b linear c As the heights of students increase, their handspans tend to increase.

- 2 a weak negative relationship b no relationship
 c strong positive relationship

3 Weak positive.

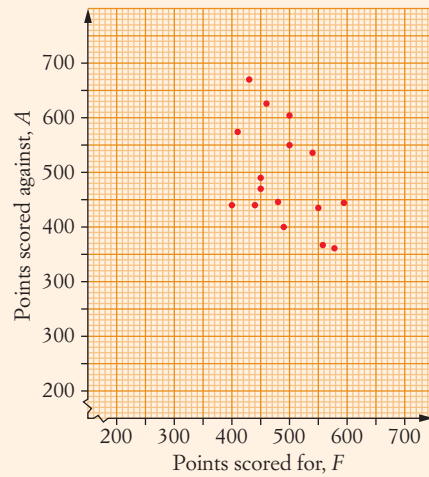
- 4 a Stride length depends on a person's height; the taller the person, is, the longer their legs are.

b



- c linear d Students' stride length increases with height.
 e strong positive relationship f Near 72.5 – 73 cm

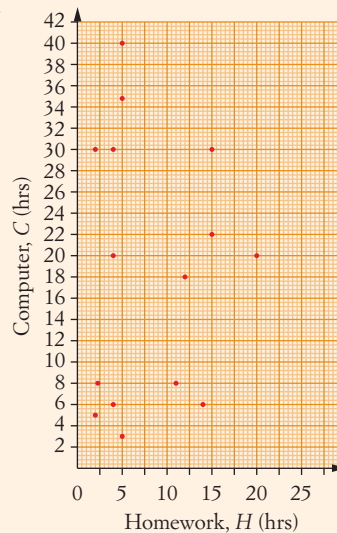
5 a



b Yes

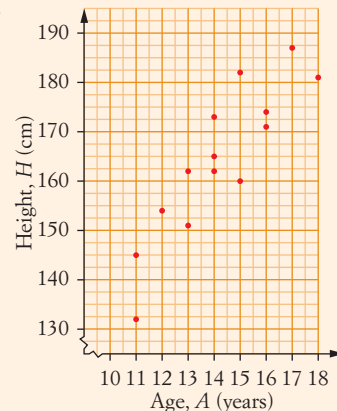
c Weak negative relationship

6 a



b no relationship

7 a



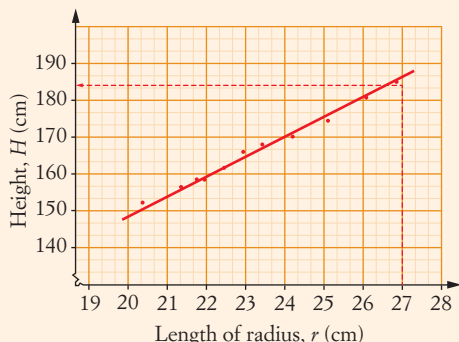
b Age, because as a young person ages, he usually grows in height.

c weak positive relationship

Answers

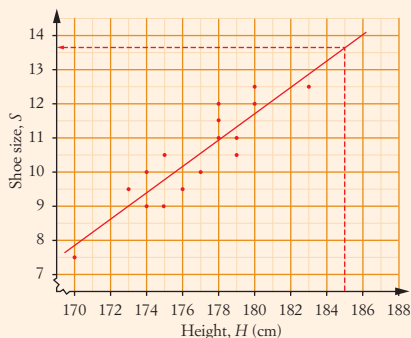
Exercise 6-09

1 a



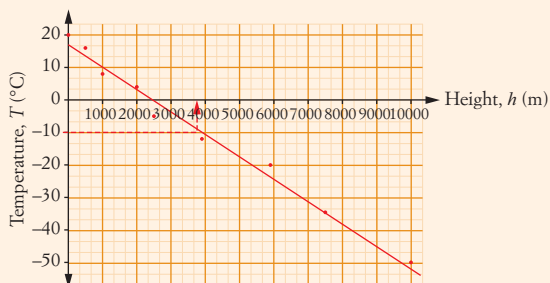
b $H = 5r + 48.5$ c 173.5 cm d 184 cm

2 a



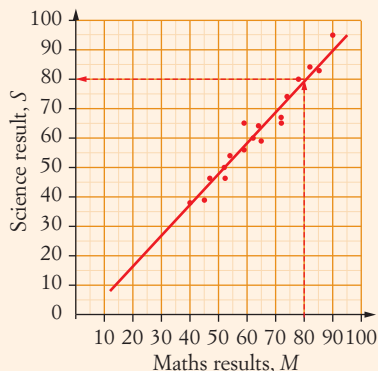
b $S = 0.393H - 59$ c 8.5
d 12 e 13.5

3 a



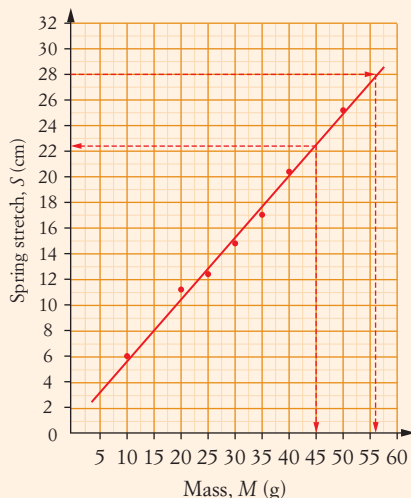
b $T = -0.0068h + 16$ c 5.8°C d 3800 m

4 a



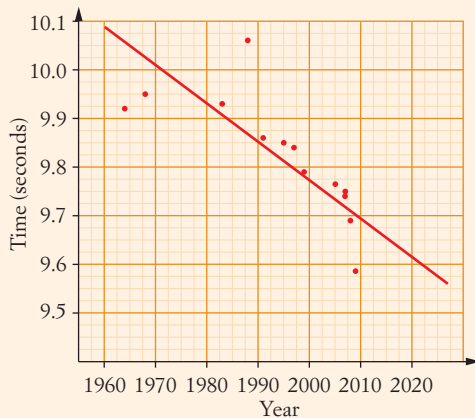
b 80 c 97

5 a



b 22.4 cm c 56 g
d Yes, because a spring has an elastic limit, which is the point at which a spring will not return to its original length as a result of the mass attached to the spring being too heavy.

6 a

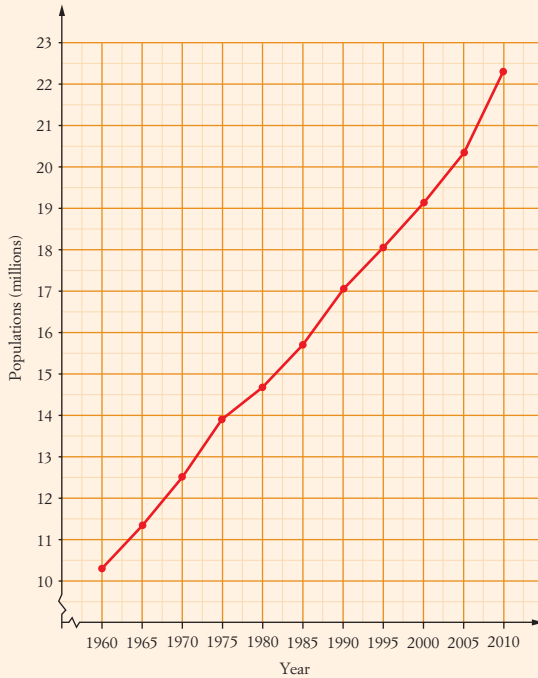


b 9.60
c There is a limit to how fast a person can run.

Exercise 6-10

- 1 a i 25 ii 42 iii 15
b December, more customers due to summer and Christmas holiday season.
c June, fewer customers due to winter, busy end-of-financial year season.
d Number of people employed peaks in December, then falls, only to increase in March, April (the Easter holiday period). It then falls again to a low in June, July and then slowly the number of people employed rises to a peak in December. From 2010 to 2012, the number of people employed is showing a slow increase.

2 a



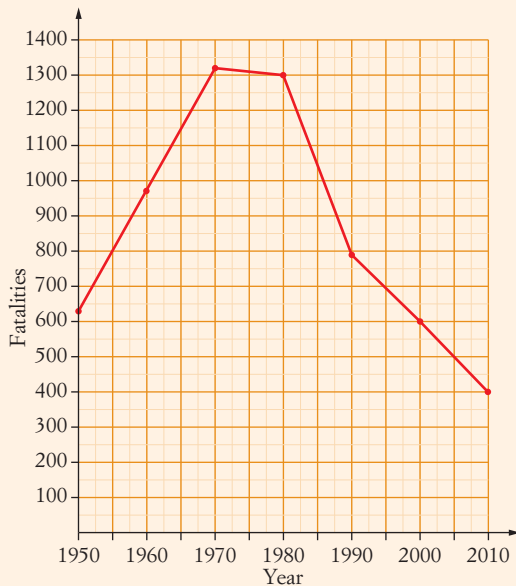
b 2005 – 2010

c Australia's population increased at 1.1–1.2 million every 5 years up to 1975. The population growth then slowed down for 5 years. From 1980, the population grew at a steady rate of just over a million people every 5 years but from in 2005–2010, the rate increased to 1.9 million for the 5-year period.

d i 26–27 million

ii 32–34 million, teacher to check

3 a

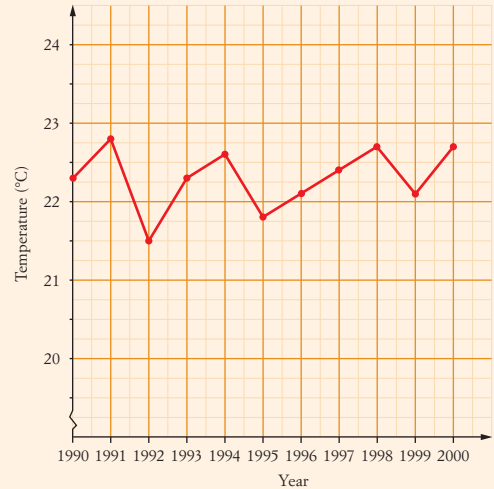


b From 1960, road fatalities rose at a steady rate, reaching a peak of approximately 1300 in 1970–1980. Road fatalities

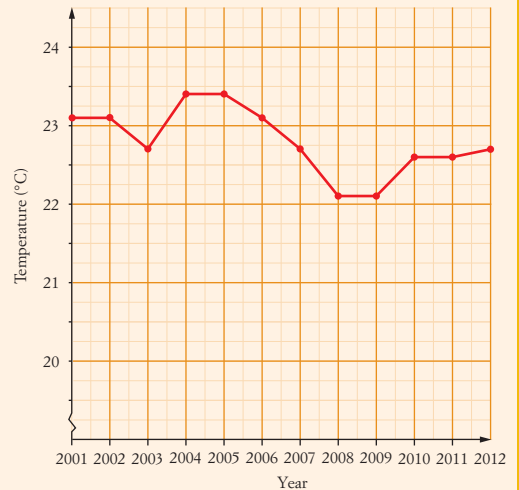
then fell at a rapid rate between 1980–1990 and continued to fall by about 200/5 years until 2010.

c Improved safety in cars with seat belts being compulsory, then drink driving laws introduced.

4 a i



ii

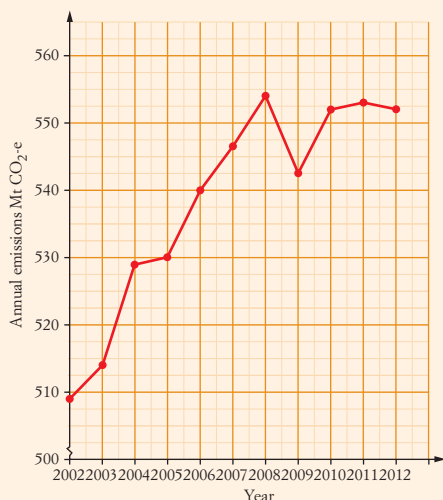


b i Starting at 22.3° in 1990, the temperature has seen a series of increases of less than 1°, followed by falls of less than 1°. The increase from 1990 to 2000 is 0.4°.

ii Starting at 23.1° in 2001, the temperature falls to 22.7° and then increases to a high of 23.4° for two years before falling to 22.1°. From 2009, the temperature rose, then remained steady before a slight increase to 22.7° in 2012.

c The temperature from 1990–2000 continually increased and decreased by less than 1°. The temperature in 2001–2012 started at 23.1°, rose to a high of 23.4° in 2004, before falling for three years. This was followed by a slight increase. The range of annual temperatures for both periods is 1.3°, but the minimum and maximum temperatures for 2001–2012 are 0.6° higher than for 1990–2000.

5 a



- b Carbon emissions increased by 55 Mt.
 c Carbon emissions stabilised.
 d More environmentally-friendly policies and practices in Australia.
 e i, ii Teacher to check and discuss.
- 6 a Approximately 4 million. b 18 million
 c 300 000 persons per year d 26.5 million
- 7 a Gradual increase in passenger movements with peaks in October and troughs in February.
 b i 3.9–4.0 million ii 4.25 million
 iii 4.2 million iv 4.5 million
 c 15%

Exercise 6-11

- 1 a Just surveying 300 people between 9 a.m. and 11 p.m. in shopping centres only targets a narrow group of people in certain areas.
 b The sample needs to be more random and over a large area, not just in shopping centres. A telephone survey should produce more accurate feedback.
- 2 The report does not say what conditions are needed for the hot water system to work effectively. The temperature in Queensland is much warmer than in NSW and Victoria. Consequently, with the cooler climate in NSW and Victoria, especially in winter, the heat pump system may not provide the savings that people in Queensland obtain.
- 3 a i The price of petrol has shown little increase from December to February.
 ii The price of petrol has shown marked rises and falls over the period from December to February.
 b Both graphs could be improved by starting the vertical scale at 0 cents/litre.
- 4 a That there is a marked difference between the fuel consumption of the different cars.
 b i 0.2 L/100 km ii 1 L/100 km iii 0.2 L/100 km
 c Begin the scale on the vertical axis with 0 and use a scale of 1 cm = 0.5 L/100 km instead of 1 cm = 0.2 L/100 km.

- 5 Yes, as there is no option for a customer to rate the product as unsatisfactory or poor.
- 6 a An example of a biased question could be: Which of these colours do you prefer – red, black, silver, blue?
 b Apart from surveying people, they need to look at sales figures of all cars. This will give information about the most popular car colour.
- 7–8 Teacher to check.

Exercise 6-12

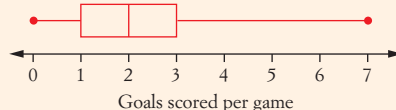
Teacher to check the investigations.

Power plus

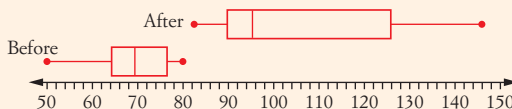
- 1 a -1 and 1
 b There is no relationship between the variables.
 c i 1 ii -0.2 iii -0.8
- 2 b, d, f
- 3 a $\bar{x} = 13.35$, median = 14, mode = 14
 b Range = 10, IQR = $15 - 12.5 = 2.5$
 c The mean, median, and mode will increase by 4, the range and the interquartile range remain unchanged.

Chapter 6 revision

- 1 a i negatively skewed ii clustering at 16 and 17, 10 is an outlier
 b i positively skewed ii clustering at 40s and 50s, no outliers
 c i symmetrical ii clustering at 4, no outliers
- 2 a 6.5 b 6 c 2.5 d 12.5 e 2
- 3 a $\bar{x} = 0.40$, $\sigma = 0.08$
 b Range = 0.33, IQR = 0.08
 c The interquartile range is the better measure as the standard deviation is affected by the outlier 0.62.
- 4 a Girls: $\bar{x} = 67.73$, $\sigma = 16.08$
 Boys: $\bar{x} = 61.67$, $\sigma = 12.35$
 b The girls performed better than the boys as their mean mark was about 6 more than the mean mark of the boys. However the boys' marks were less spread out than the girls.
- 5 a Range = 7, IQR = $3 - 1 = 2$
 b 0, 1, 2, 3, 7
 c

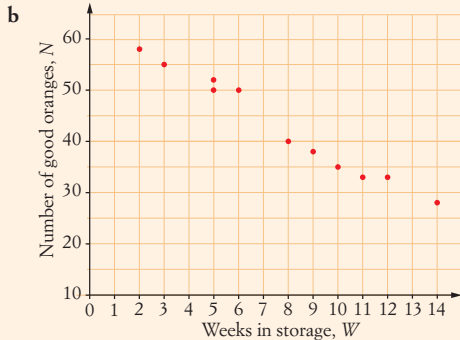


- 6 a Before: 50, 64, 69, 76, 80; After: 82, 89, 95.5, 126, 146

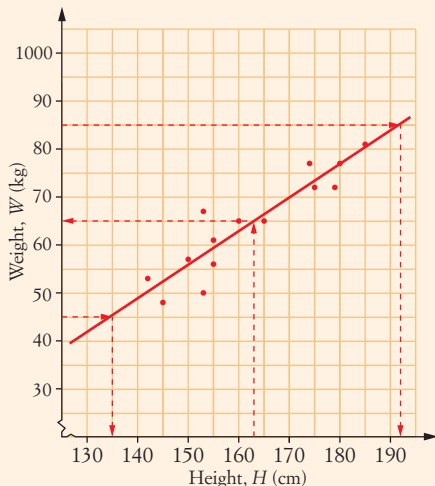


- b i Range = 30, IQR = 12 ii Range = 64, IQR = 37
 c The pulse rates for after exercise are significantly higher. In fact, all the rates for after exercise are above all the rates for before exercise. The median pulse for after exercise is 95.5 compared to the median pulse of 69 for before exercise. The range and interquartile range are also greater for the after exercise pulse rate.

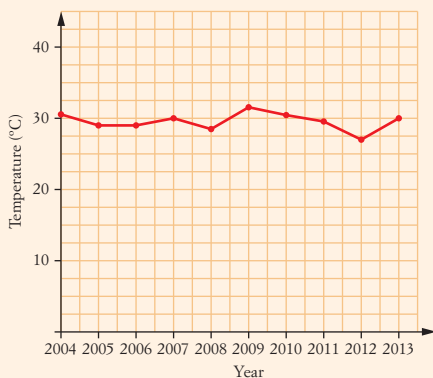
- 7 a i Both ii Stem-and-leaf plot
 b The range ($126 - 70 = 56$) is too large.
 c i median = 92 ii IQR = $99.5 - 84 = 15.5$
 d 50%
 8 a Weeks in storage – this determines how many oranges stay good.



- c linear
 d The longer the oranges remain in storage, the fewer good oranges there are in the box.
 e strong negative correlation
 9 a



- b $W = 0.714H - 51.4$ c 70 kg
 d 65 kg e i 192 cm ii 135 cm
 10 a Year
 b

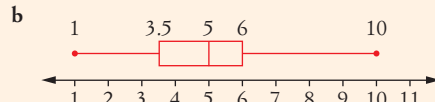


- c For the last 10 years, the mean maximum temperatures at Blacktown, after staying at 30.6° , have ranged from a low of 27.4° to a high of 31.7° , finishing at a temperature of 30.0° in 2013. This shows there has been little change in temperature for the month of January over the last 10 years.

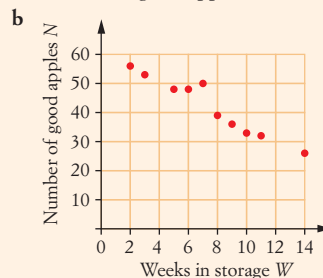
- 11 a That the product is healthy.
 b There is no data given on the actual fat content in the product. This should also be stated in terms of daily percentage requirement of fat or in mg of fat.

Mixed revision 2

- 1 a i negatively skewed ii no outliers, clusters at 7 and 8
 b i positively skewed ii outlier 98, clustering in 40s and 50s
 2 a $b = -7$ b $x = 10$
 3 a $81n^4m^8$ b $4a^4b^3$ c $\frac{1}{9}$
 4 a $\frac{6a-20}{15}$ b $\frac{5x+36}{24}$ c $\frac{7r-6}{10}$
 5 a Range = 33; Interquartile range = 7
 b Interquartile range, because it is not affected by the outlier of 112.
 6 a $x = \pm 6\sqrt{3}$ b $n = 0$ or 64 c $u = -15$ or 4
 7 a $\frac{9d}{2}$ b 2 c $8pv^2$
 8 a 1, 3.5, 5, 6, 10



- 9 a weak negative b strong negative c weak positive
 10 a -4 b 16 c $\frac{1}{12}$ d $\frac{1}{1000}$
 11 $y = -5.8$
 12 a $53gh - 45gh^2$ b $7y^4 + 2y^3$
 13 a Girls: mean = 62.7, standard deviation = 16.1
 Boys: mean = 66.9, standard deviation = 12.2
 b The boys performed better as their mean is higher.
 14 a $x^2 + 14x + 49$ b $25m^2 - 20m + 4$ c $9n^2 - 100$
 15 20, 21, 22
 16 a i stem-and-leaf plot ii stem-and-leaf plot
 b i median = 42 ii interquartile range = 16
 17 a $(y - 16)(y - 2)$ b $(n + 8)(n - 6)$ c $(a - 9)(a + 8)$
 18 a The independent variable is W , the weeks in storage. The number of weeks in storage is set first after which time the number of good apples is counted.

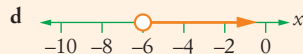
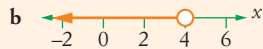


- c the number of good apples decreases the longer the apples are in storage.
 d there is a strong negative relationship between the variables W and N

Answers

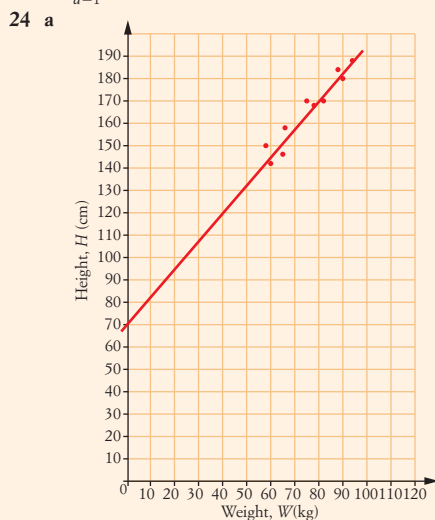
19 a $(5n+2)(n+1)$ b $(2a+3)(a-5)$ c $(3x-5)(x+2)$

20 $n = 3t^2$



22 a $n \geq 5$ b $k < -7$ c $x \geq 1\frac{1}{2}$

23 a $\frac{d-4}{d-1}$ b 16 c $\frac{3(y+1)}{y(y-3)}$



b $H = 1.2W + 70$; other answers possible.

c 48 kg d 170 cm

25 a 1 b 0 c 3 26 $x \approx 3.453$

Chapter 7

SkillCheck

1 a $a = -12$ b $x = 6$ c $x = 8$

2 a $(k+4)(k+1)$ b $(y-8)(y-2)$ c $(m-8)(m+7)$

d $(u+13)(u-5)$ e $(w-7)(w-3)$ f $(x-6)(x+4)$

Exercise 7-01

1 a $y = 15$ b $a = 9$ c $m = 7$ d $k = 57$

e $n = -35$ f $y = -7$ g $x = 31$ h $y = 46$

i $m = 18$ j $x = -29$ k $x = 24$ l $m = 10$

m $n = -\frac{1}{4}$ n $n = -\frac{3}{5}$ o $d = 3\frac{3}{4}$

2 a $k = 1\frac{7}{8}$ b $w = 1\frac{1}{3}$ c $x = -1\frac{1}{3}$ d $x = 3$

e $y = 3$ f $a = -8\frac{3}{5}$ g $p = 9\frac{2}{3}$ h $y = 3$

i $y = -\frac{5}{6}$ j $w = 10$ k $w = 50$ l $w = 9\frac{3}{5}$

m $a = \frac{6}{11}$ n $y = 60$ o $a = 1\frac{11}{13}$

3 a C b A

4 a $x = \frac{7}{15}$ b $p = \frac{109}{7}$ c $x = \frac{133}{7}$ d $x = \frac{76}{7}$

e $x = \frac{92}{9}$ f $y = \frac{23}{5}$ g $a = -41$ h $a = \frac{107}{14}$

i $w = 11$ j $a = \frac{75}{14}$

Exercise 7-02

1 a $m = \pm 12$ b $x = \pm 20$ c $y = \pm 15$ d $k = \pm 13$

e $y = \pm 1$ f $w = \pm 4$ g $x = \pm 2$ h $t = \pm 4$

i $a = \pm 4$ j $k = \pm 6$ k $w = \pm 10$ l $d = \pm 12$

m $k = \pm 1$ n $w = \pm 5$ o $y = \pm 1$ p $m = \pm 6$

q $x = \pm \frac{1}{2}$ r $p = \pm 3$ s $k = \pm 2$ t $y = \pm 10$

2 a $m = \pm 2$ b $a = \pm 9$ c $m = \pm 2\sqrt{7}$

d $k = \pm \frac{\sqrt{5}}{3}$ e $k = \pm 4\sqrt{6}$ f $k = \pm 3\sqrt{10}$

3 a $m \approx \pm 1.94$ b $x \approx \pm 7.58$ c $y \approx \pm 0.35$

d $w \approx \pm 7.07$ e $a \approx \pm 9.24$ f $y \approx \pm 6.20$

4 a $x = -2, -1$ b $y = -4, -1$ c $y = -4, -12$

d $x = -4, 3$ e $x = -3, 1$ f $x = -8, 5$

5 a $x = 6, -5$ b $x = 4$ c $x = 11, -6$

d $d = 0, 2$ e $q = 5, -2$ f $n = 0, -4$

g $k = 0, 7$ h $y = 0, 5$ i $v = 0, 12$

j $m = 0, 3$ k $a = -20, -4$ l $n = 0, -10$

m $u = -4, 2$ n $x = 7, -6$ o $p = -4, -5$

6 You cannot take the square root of a negative number.

7 a, c, f: cannot find square root of negative number.

Exercise 7-03

1 a $x = 1$ b $m = 5$ c $a = 11$ d $u = -2$

e $y = -9$ f $n = \sqrt[3]{20}$ g $h = \sqrt[3]{11}$ h $k = \sqrt[3]{-48}$

i $m = \sqrt[3]{-15}$ j $m = 4$ k $x = \sqrt[3]{\frac{-81}{4}}$ l $x = \sqrt[3]{-40}$

2 a $w = 2.5$ b $m = 2.5$ c $m = 6$ d $t = 3.2$

e $x = 3$ f $x = -5.5$ g $x = -2.4$ h $x = 0.8$

i $a = 5.5$ j $a = 0.4$ k $x = -2.3$ l $t = -4.4$

3 a yes b when c is positive c when c is negative d no

Exercise 7-04

1 100 cm $\times$ 25 cm 2 18 mm, 36 mm, 36 mm

3 child: \$21, adult: \$48 4 61, 62, 63 5 94, 96, 98

6 Anand: 3, Sunjay: 27 7 26 8 4

9 Vatha: 22, Chris: 14 10 213, 214, 215, 216

11 $x = 35$ 12 117

13 6 14 $x = 15.5$ 15 Scott: 11, Mother: 34

16 $25^\circ, 50^\circ, 105^\circ$ 17 72 L when full

18 8 teachers, 120 students

Mental skills 7

2 a 3.5 b 2.4 c 0.12 d 0.36

e 0.8 f 0.027 g 0.2 h 8.8

i 0.24 j 0.012 k 1.8 l 0.028

4 a 66.3 b 6630 c 6.63 d 0.663

e 6.63 f 663 g 0.663 h 663

i 6630 j 66.3 k 0.663 l 0.0663

Exercise 7-05

1 a $C = 15.1$ m b $r = 31.8$ cm 2 $w = 17$ m 3 30.6 m/s

4 a 36 km/h b 86.4 km/h c 180 km/h 5 43

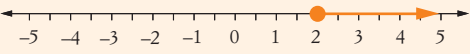
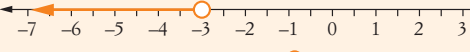
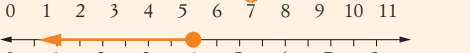
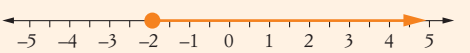
6 a 27°C b 0°C c 100°C d 39°C

- 7 a 21.0 b 105.8 kg 8 4.9 m
9 a 436 km b 7 h
10 a \$97.50 b 620 km 11 $h = 13.2$ cm

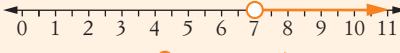
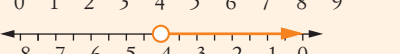
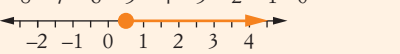
Exercise 7-06

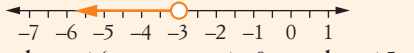
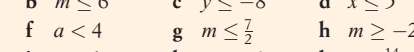
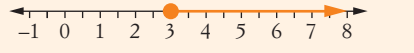
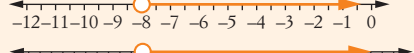
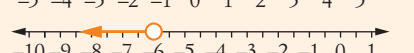
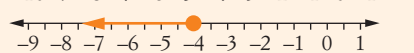
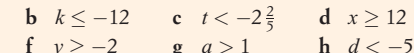
- 1 a $y = \frac{5-x}{2}$ b $y = \frac{k-m}{p}$
c $y = \frac{P-8}{k}$ d $y = \frac{5m}{3}$
e $y = \frac{K-D}{M}$ f $y = \frac{4d-5}{8}$
g $y = \frac{2e+k}{a}$ h $y = \frac{20m}{3} - 3$ or $y = \frac{20m-9}{3}$
i $y = \pm\sqrt{\frac{w-5}{x}}$ j $y = kx^2$
k $y = \frac{5n-d}{n}$ or $y = 5 - \frac{d}{n}$ l $y = cT^2 - k$
2 a $b = \pm\sqrt{c^2 - a^2}$ b $a = \frac{2(s-ut)}{t^2}$ c $a = \frac{v-u}{t}$
d $r = \sqrt{\frac{3V}{4\pi}}$ e $R = \pm\sqrt{\frac{A+\pi r^2}{\pi}}$ f $l = \frac{A-\pi r^2}{\pi r}$
g $n = \frac{S+360}{180}$ or $\frac{S}{180} + 2$ h $r = \frac{sx}{s-2m}$ i $b = \pm\sqrt{x^2 + 4ac}$
j $x = \frac{5-y}{4}$ k $A = \frac{mn}{5-2m}$ l $p = \frac{a}{a-s}$
m $a = \frac{b(X+Y)}{Y-X}$ n $x = \frac{5-2a}{5}$ or $1 - \frac{2a}{5}$ o $b = \frac{u(y-1)}{x-ay}$

Exercise 7-07

- 1 a  b  c  d  e  f  g  h 
2 a $x < 4$ b $x \geq 2$ c $x > -6$ d $x \leq 1$
3 B
4 a $x \geq 1$ b $x < 4$ c $x > 6$
d $x \leq -1$ e $x > -6$ f $x < 2$
g $x \geq -4$ h $x \geq 25$ i $x < 0$

Exercise 7-08

- 1 a $x > 7$  b $y \geq 4$  c $m \leq -2$  d $x \geq -100$  e $x < 5$  f $y > -4$  g $a \geq \frac{1}{2}$ 

- h $w \leq -10$  i $a \geq 5$  j $a \leq 3$  k $a \geq -1$  l $w < -3$ 
2 a $x \geq 1$ b $m \leq 6$ c $y \leq -8$ d $x \leq 5$
e $w > -3$ f $a < 4$ g $m \leq \frac{7}{2}$ h $m \geq -2\frac{1}{2}$
i $x < 35$ j $x < 6$ k $y \geq -2$ l $a \leq \frac{14}{5}$
3 C
4 a $x \geq 3$  b $y > -8$  c $k > -11$  d $m \leq 0$  e $p < -6$  f $t \leq -4$ 
5 a $x > -3$ b $k \leq -12$ c $t < -2\frac{2}{5}$ d $x \geq 12$
e $w < -1$ f $y \geq -2$ g $a > 1$ h $d < -5\frac{1}{2}$
i $w < -11$ j $p < -4$ k $m < 4$ l $x \geq -\frac{2}{7}$

Exercise 7-09

- 1 a 2 b 3 c 2 d 4 e 5 f 3
g 3 h 2 i 6 j 8 k 6 l 3
2 a $\log_5 25 = 2$ b $\log_4 64 = 3$ c $\log_{10} 10\,000 = 4$
d $\log_{25} 5 = \frac{1}{2}$ e $\log_2 \frac{1}{16} = -4$ f $\log_3 \frac{1}{9} = -2$
g $\log_8 4 = \frac{2}{3}$ h $\log_{10} 0.01 = -2$ i $\log_4 \sqrt{2} = \frac{1}{4}$
j $\log_{16} 4 = \frac{1}{2}$ k $\log_9 27 = \frac{3}{2}$ l $\log_6 \frac{1}{\sqrt{6}} = -\frac{1}{2}$
3 a $125 = 5^3$ b $10 = 10^1$ c $27 = \sqrt{3}^6$
d $8\sqrt{2} = 2^{3.5}$ e $64 = 2^6$ f $\frac{1}{81} = 3^{-4}$
g $\frac{1}{125} = 5^{-3}$ h $\sqrt{2} = 8^{\frac{1}{6}}$ i $10 = 100^{\frac{1}{2}}$
j $5\sqrt{5} = 5^{\frac{3}{2}}$ k $2 = 8^{\frac{1}{3}}$ l $\frac{1}{100} = 100^{-1}$
4 Because a base raised to any power always gives a positive number.

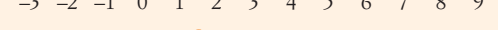
Exercise 7-10

- 1 a 7 b 3 c 2 d -1 e $\frac{1}{2}$
f -2 g -4 h -4 i 1 j 2
k 2 l 2 m 1 n 2
2 a $\log_x 30$ b $\log_x 5$ c $\log_x 8$
d $\log_x 2$ e $\log_x 40$ f $\log_x 10$
g $\log_x \frac{1}{4}$ h $\log_x \frac{1}{5}$ i $\log_x 12$
3 a 1.2042 b 2.6021 c 3.6021 d 0.301 05
e -0.3979 f 2.2042 g 0.3979 h 0.801 05
4 a 1 b 3 c 3 d 2 e -1 f 0.5
g 3 h 2 i -1 j -4
5 a 5 b 3 c 1
d 7 e 0 f $\frac{3}{2}\log_a x$ or $\log_a(\sqrt{x})^3$

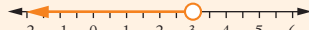
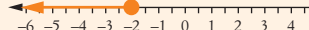
Exercise 7-11

- 1 a $k = 9$ b $m = 7$ c $d = 10$
 d $x = 2.5$ e $y = -4.5$ f $a = 3.5$
 g $k = 1.5$ h $n = -1.5$ i $d = 2.75$
 2 a $x = 1.425$ b $x = 2.227$ c $x = 2.519$
 d $x = -0.943$ e $x = 0.428$ f $x = -0.661$
 g $x = 7.555$ h $x = -0.107$ i $x = -1.121$
 j $x = 1.011$ k $y = 0.975$ l $k = -2.069$
 3 a $x = 2$ b $x = \frac{2}{3}$ c $x = \frac{5}{4}$ d $x = -\frac{1}{2}$
 e $x = \frac{7}{2}$ f $x = -\frac{13}{6}$ g $x = \frac{3}{4}$ h $x = -2$
 4 a $x = 8$ b $x = 1000$ c $x = \frac{1}{25}$ d $x = \frac{1}{64}$
 e $x = 3$ f $x = 2$ g $x = \frac{1}{1000}$ h $x = 16\sqrt{2}$
 i $x = \frac{1}{\sqrt{10}}$ j $x = \frac{1}{8}$ k $x = 128$ l $x = \frac{1}{25}$
 m $x = 2$ n $x = \frac{1}{5}$ o $x = \frac{1}{2}$ p $x = 0.1$
 q $x = 16$ r $x = 2$ s $x = 3.915$ t $x = 23.04$
 5 11.89 $\approx$ 12 years 6 22.43 $\approx$ 23 months
 7 a $A = 106$ g b $t = 20$ days c $t = 58$ days

Power plus

- 1 a $x = 1\frac{2}{25}$ b $x = \frac{1}{3}$ c $x = -8$
 2 a 
 b 
 c 
 3 The solution to $D = \frac{1}{2}n(n-3) = 100$ is not a positive integer.
 4 $p = 4, q = 3$
 5 a $a = 7$ b $x = 20$ c $m = 2$ d $h = 5$

Chapter 7 revision

- 1 a $w = 6$ b $y = \frac{35}{4}$ c $a = -3$
 d $m = 3$ e $s = 4$ f $x = 1\frac{2}{3}$
 2 a $m = \frac{23}{3}$ b $y = \frac{2}{7}$ c $m = \frac{3}{2}$
 3 a $y = \pm 2$ b $p = \pm 10$ c $x = \pm 3$
 d $m = \pm 1$ e $w = \pm 5$ f $x = -7, -1$
 g $h = 9, -1$ h $u = 7, -11$ i $k = 0, -5$
 4 a $u = 1.9$ b $m = 2.9$ c $x = 1.4$
 5 Jane: 16, Grace: 13 6 120 m
 7 a $a = \frac{y-b}{x}$ b $mP^2 = a$ c $a = \frac{1-M}{M+1}$
 8 a $x \geq 0$ 
 b $x < 3$ 
 c $x \leq -2$ 
 d $x > -5$ 
 9 a $y \geq 16$ b $y \leq -7\frac{1}{2}$ c $a > -5$
 d $x > -3$ e $a < -16$ f $x \leq -3$
 10 a $6^3 = 216$ b $2^{-4} = \frac{1}{16}$ c $7^{\frac{1}{2}} = 7\sqrt{7}$
 11 a 1 b 0 c 3
 12 a 0.9542 b 2.4771 c 0.5229 d 0.9771
 13 a $x = 1.490$ b $x = -0.943$ c $x = 0.236$ d $x = 1.420$

Chapter 8

SkillCheck

- 1 a -1 b 29 c -3 d 69
 2 a 4 b $-\frac{1}{2}$ c $\frac{8}{5}$ d -3.2
 3 a 625 b 3125 c 1 d $\frac{1}{25}$

Exercise 8-01

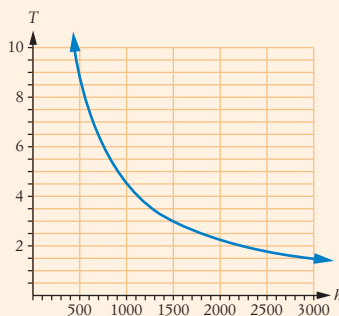
- 1 a $D = 190T$ b i 3.8 km ii 8.55 km c 1 h 5 min
 2 a $E = 26.2$ h b \$183.40 c 5.5 h
 3 a $I = \frac{16D}{425}$ b \$33.88 c \$67.76
 4 A 5 $b = 2.5a$

h	c
1	\$ 7.50
2	\$15
3	\$22.50

- b $c = 7.5$ h c \$45 d 11 e 7.5. It is the same.
 7 C 8 a $F = 0.006m$ b 15 L/100 km
 9 A 10 a 22.8 kg b 84.1 kg

Exercise 8-02

- 1 a $T = \frac{920}{s}$ b 10 h 13 min c 92 km/h 2 C
 3 a i 15°C ii 1.8°C b i 562.5 m ii 200 m
 c



- 4 4533 people 5 B 6 $\frac{14}{15}$
 7 a 8 min b 4 people 8 a $b = \frac{8}{a}$ b $b = \frac{100}{a}$
 9 a $F = \frac{112}{L}$ b 6 beats/sec c 25 cm
 10 a $y = \frac{1}{16}$ b $x = 1\frac{1}{4}$ 11 a 2.5 h b 5 friends

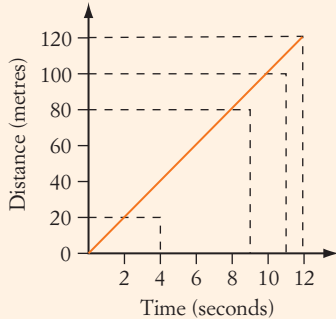
Exercise 8-03

- 1 a i £26 ii £57 b i \$A28 ii \$A92
 c i approx. £7 ii approx. \$A30
 2 a i 2 km ii 20 km iii 34 km
 b i 50 furlongs ii 122 furlongs iii 180 furlongs
 c 60 km d 500 furlongs
 3 a i ¥16 000 ii ¥63 000 iii ¥78 000
 b i \$A250 ii \$A760 iii \$A920
 4 a -18°C b 10°C c 26°C
 d 32°F e 14°F f 86°F
 5 a 4.9 ha b 2 ha c 10.8 acres d 12.2 acres
 e i 32 000 m² ii 3.2 ha iii 7.8 acres
 6 a i P620 ii P2100 iii P3600
 b \$A12 c \$A5 d P4900

Exercise 8-04

- 1 a The cyclist leaves the starting point, travels at a speed of 20 km/h for 1 hour; stops for 1 hour; then continues for another hour at a speed of 10 km/h. At D, the cyclist stops for half an hour, then cycles back towards the starting point at a speed of 30 km/h for 1 hour.
- b No, as the gradients of the intervals are all different.
- c The cyclist is moving back towards the starting point.
- 2 a Kate 133 m/min, Colleen 114 m/min
- b Kate, in 9 minutes c 1.5 min
- d 680 m e 110 m
- f The graph shows the distance they move down the slope and this increases as more time passes.
- 3 a The person starts the journey fast (the graph is steep), then slows down (the graph becomes less steep) before increasing speed again (the graph becomes steeper).
- b The person is travelling or returning towards a specific place. Initially the person's speed is fast, then he slows down and stops (the graph is horizontal).
- c The person starts the journey at a high speed and then gradually slows down to a stop.

- 4 a H b D c A–B d F e E f C
- 5



- 6 a b c d
- 7 a i C ii B iii A
- b i Is the steepest (has the greatest gradient) and must be the fastest (Jade).
- ii Is the least steep (the smallest gradient) and must be the slowest (Cameron).
- iii The slope of this graph is between the other two (Kiet).
- c Jade stopped to talk to a friend. (Other answers possible.)
- d This person speeds up slightly and maintains speed for a while, slowing down gradually to a stop.
- 8 a C b D c E d F e B f A

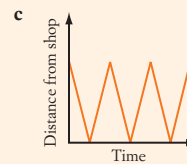
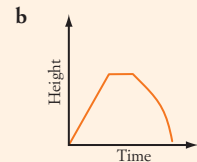
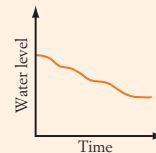
Mental skills 8

- 2 a 2, 5 b 3 c 2, 3, 6 d 3, 5
- e 2 f 2, 3, 5, 6 g 5 h 3
- 3 a 4, 9 b 4, 10 c 9 d 10
- e 4, 9 f none of these g 9 h 4, 9, 10

Exercise 8-05

- 1 a Temperature increases to maximum on first day then cools down to the initial temperature. Then second day temperature increases to slightly higher maximum than first day and then returns to similar minimum temperature as for previous day.
- b This graph could indicate four high tides and four low tides in a reasonably regular pattern.
- c Amount of petrol decreases at constant rate. Petrol tank is filled, then petrol is used at a constant rate.
- d Height of person increases slowly at first then more quickly (possibly teenage years/ growth spurt indicated by steeper graph) then slows down as final height reached.

- 2 a B b A c C d B e A
- 3 a



- 4 a viii b i c iii d vi e ix
- f iv g ii h v i vii
- 5 a A b B c B d A
- 6 a B b H c A d F e C
- 7 a C b E c F d D e A f B
- 8 a E b F c B d C e A f D

Exercise 8-06

- 1 a
- b i $y = x^2, y = x^2 + 2, y = x^2 - 1$
- ii $y = -x^2, y = -2x^2$
- iii $y = x^2, y = -x^2, y = -2x^2$

2 A

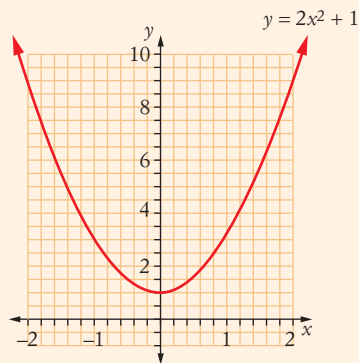
3 a F b I c A d K e J f C

g B h L i H j E k G l D

4 a $y = -x^2$ b $y = x^2$ c $y = -x^2 - \frac{1}{4}$

d $y = -x^2 - 9$ e $y = \frac{1}{2} - x^2$ f $y = x^2 + 9$

5 a

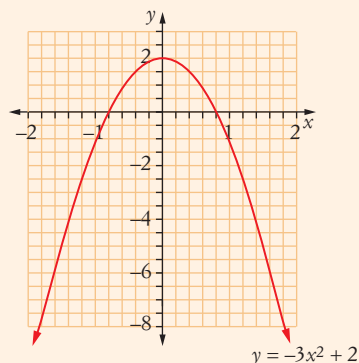


b (0, 1)

c concave up

d $y = 1$

6 a



b (0, 2)

c $x = 0$

d $y = 2$

7 A

8 a B b G c D d J e E f I

g A h K i H j L k F l C

9 a i narrower

ii up

iii 3

b i wider

ii up

iii 1

c i narrower

ii down

iii -5

d i wider

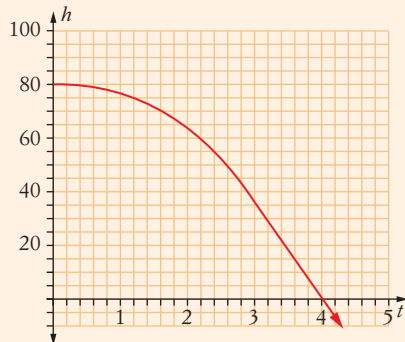
ii down

iii -12

10 a $x = \pm 4$

b $x = \pm 11$

11 a



b 80 m

c 35.9 m

d 4.0 s

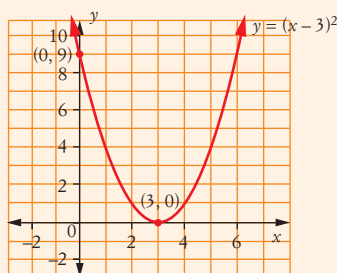
e 3.91 s

12 a $x = \pm 9$

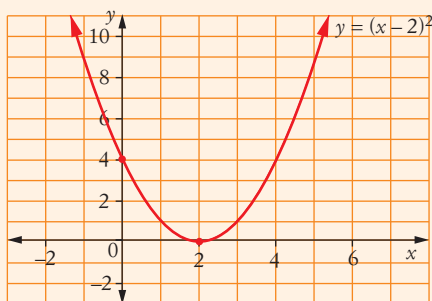
b $x = \pm 14.02$

Exercise 8-07

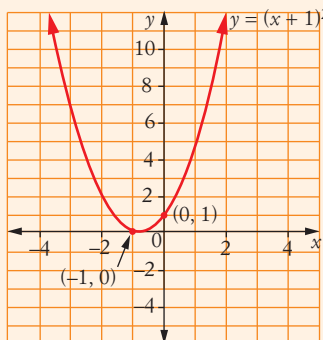
1 a



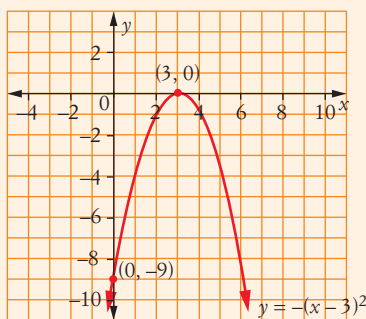
b



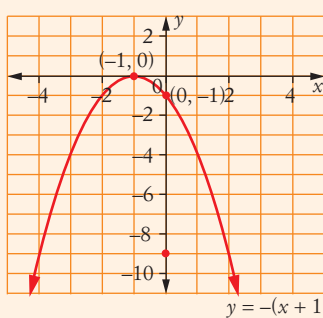
c

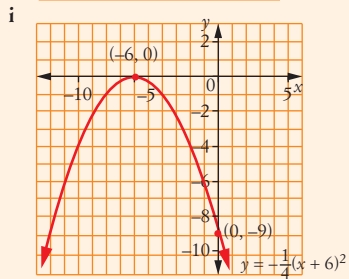
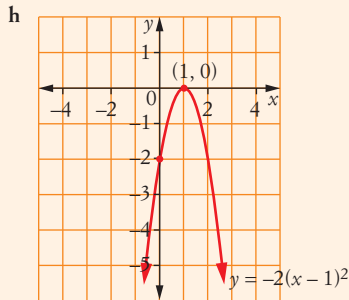
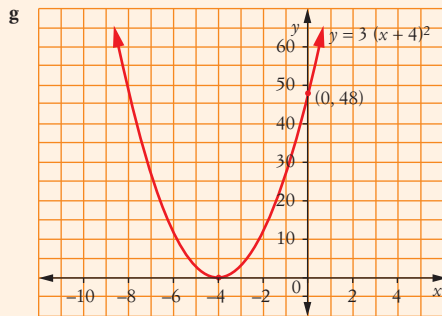
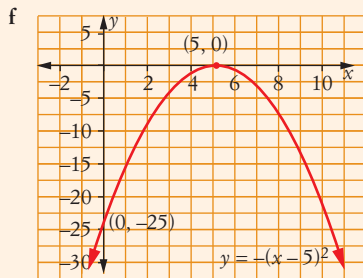


d



e





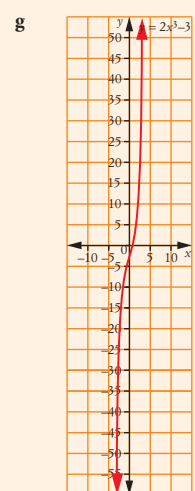
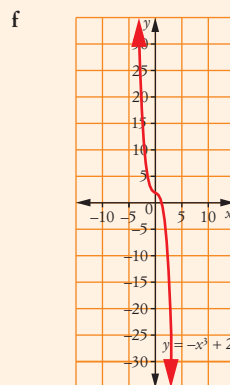
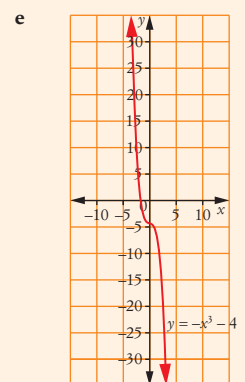
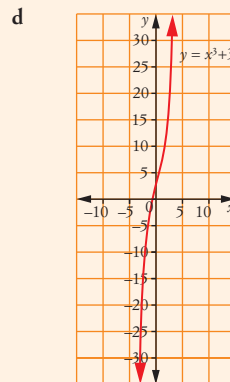
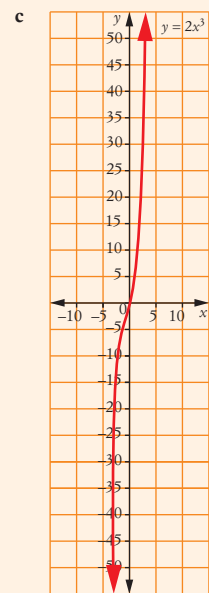
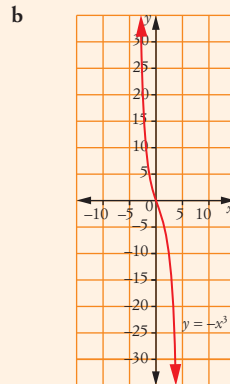
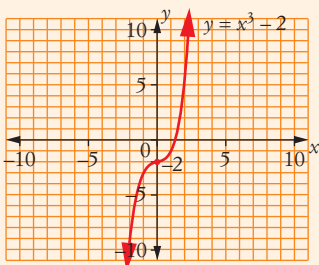
2 B

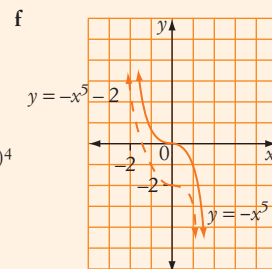
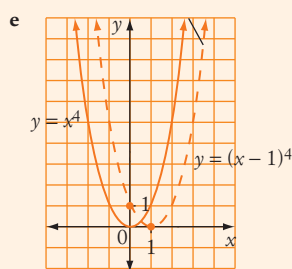
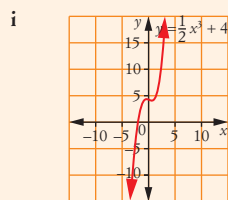
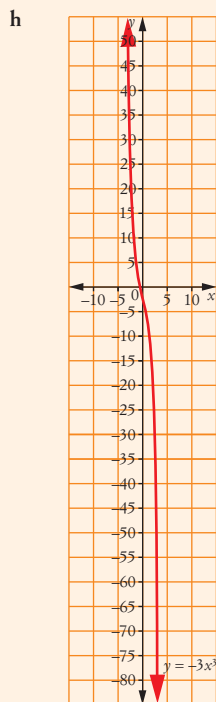
3 a D b G c A d E

e B f H g F h C

Exercise 8-08

1 a



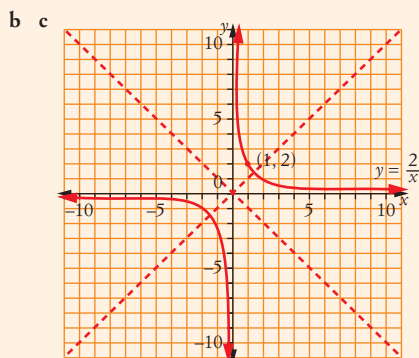


- 3 a Move up 4 units. b Move right 5 units.
 c Move left 3 units. d Move up 4 units.
 e Move right 3 units. f Move left 2 units.

Exercise 8-10

1 a

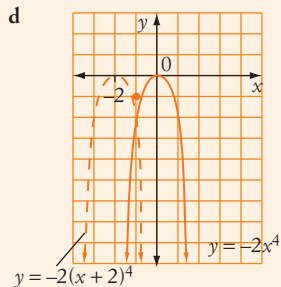
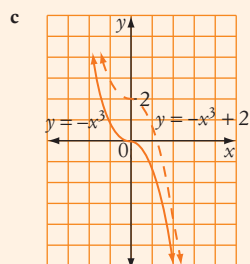
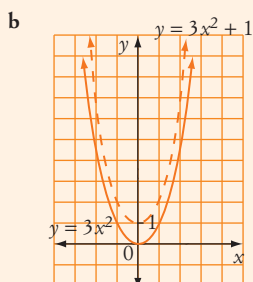
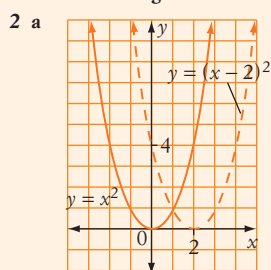
x	-3	-2	-1	0	1	2	3
y	$-\frac{2}{3}$	-1	-2	-	2	1	$\frac{2}{3}$



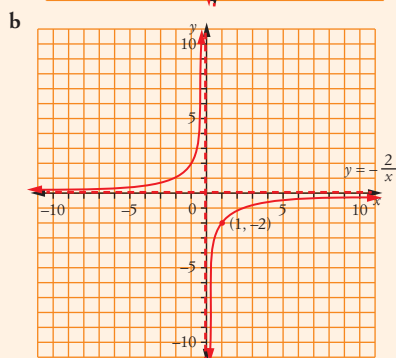
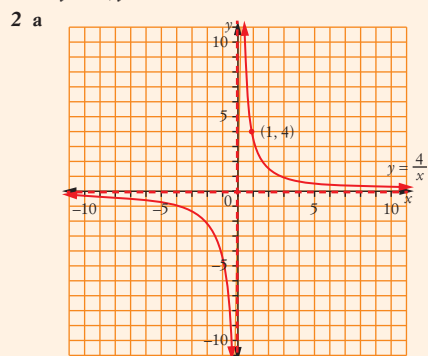
- 2 a A b E c G d C e F
 f D g H h I i B
 3 a $y = 2x^3 - 3$ b $y = -4x^3 + 1$

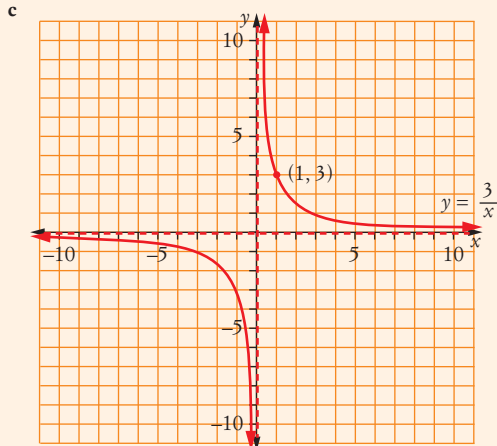
Exercise 8-09

- 1 a E b H c I d C e G
 f A g F h D i B



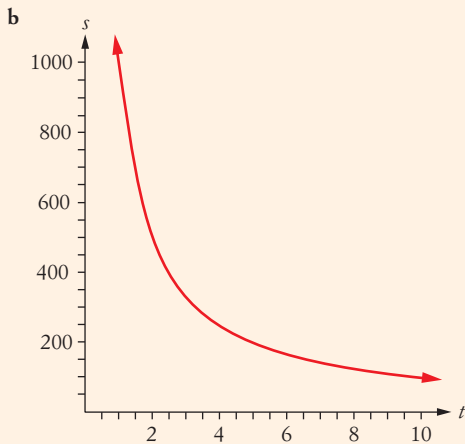
- d $y = x, y = -x$





3 a

t	1	2	3	4	5	6	7	8	9	10
S	1000	500	333	250	200	167	143	125	111	100

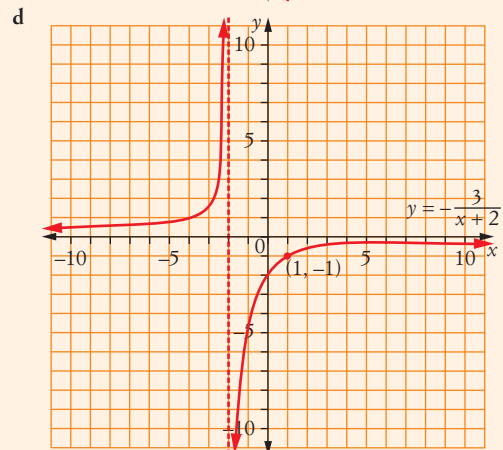
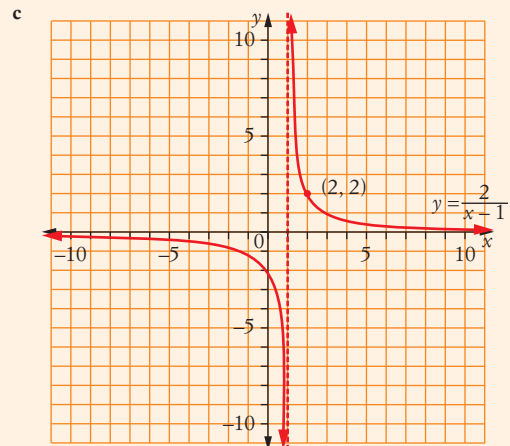
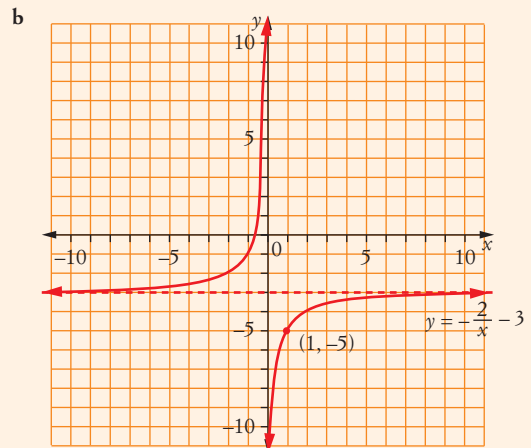
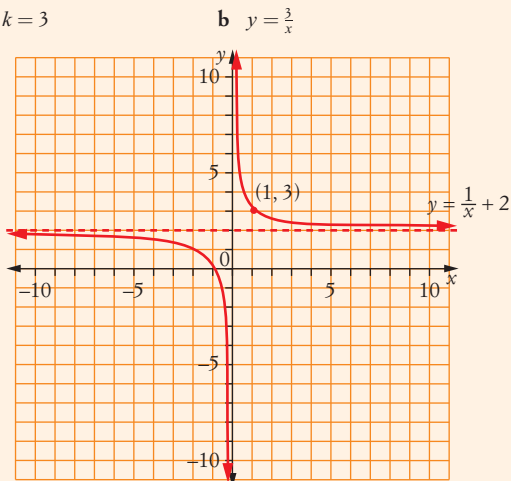


c The time taken is always positive and it is impossible to travel with zero time. Also, you cannot divide by zero.

d Yes, when $t = 2$ h, $s = 500$ km/h and when $t = 4$ h, $s = 250$ km/h.

4 a $k = 3$

5 a



6 a $c = 1, k = -6$

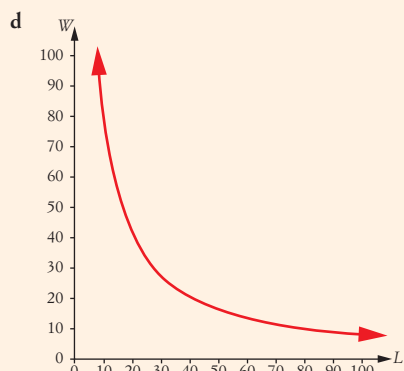
b $y = -\frac{6}{x} + 1$

7 a

L	10	20	30	40	50	60	70	80	90	100
W	80	40	27	20	16	13	11	10	9	8

b $WL = 800$ or $W = \frac{800}{L}$

c If the length or width equals zero, the block of land doesn't exist.

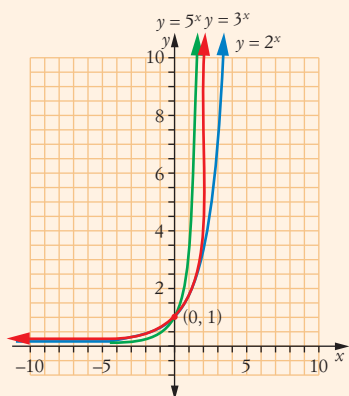


- e** As the length increases, the width decreases. The graph flattens out and gets closer to the horizontal axis, but never touches it (an asymptote).
f As the length decreases, the width increases. The graph is steeper and gets closer to the vertical axis, but never touches it (an asymptote).

8 A

Exercise 8-11

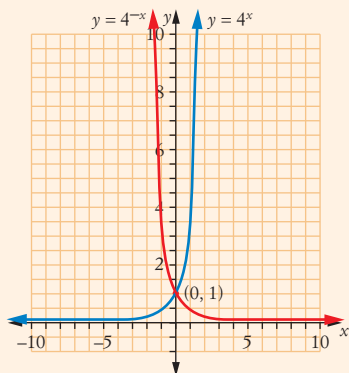
1 a



b 1

- c** For $y = a^x$ where $a = 2, 3$ or 5 , as a increases the graph increases more rapidly as x becomes larger in the first quadrant.

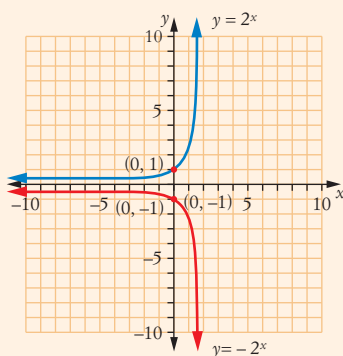
2 a



- b i** $y = 4^{-x}$ **ii** $y = a^{-x}$

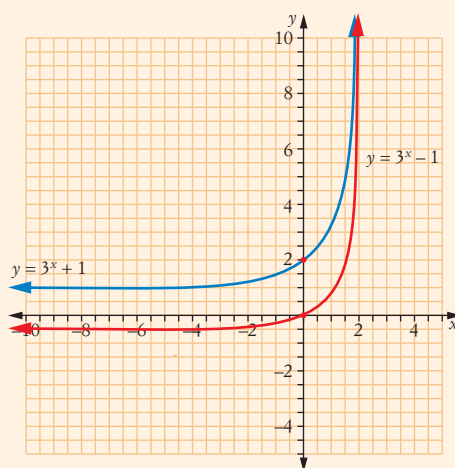
3 B

4 a



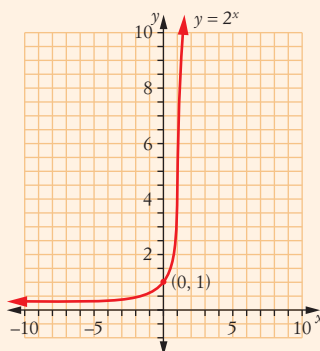
- b** They are the same graph reflected in the x -axis **c** $y = -a^x$

5

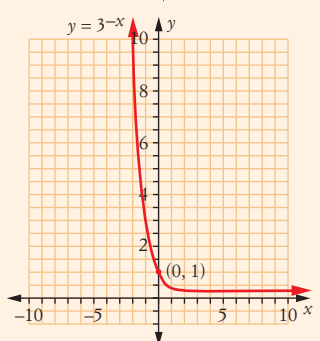


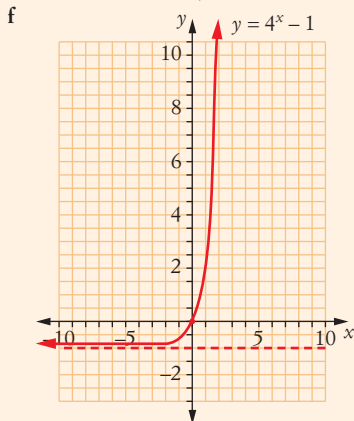
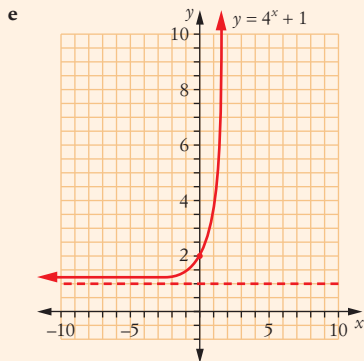
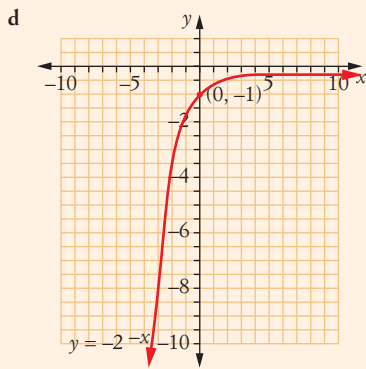
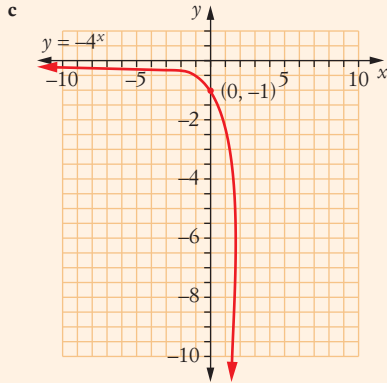
Same shape, shifted down 2 units.

6 a



b

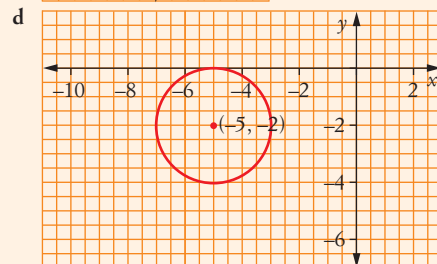
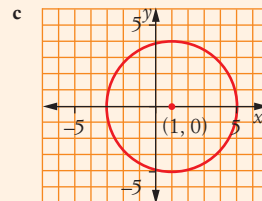
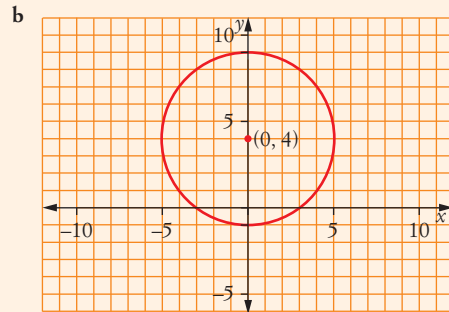
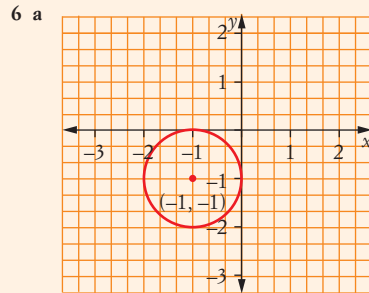




7 $y = 4^{-x}$

Exercise 8-12

- 1 a centre $(0, 0)$, $r = 2$ b centre $(0, 0)$, $r = 6$
 c centre $(0, 0)$, $r = 8$ d centre $(0, 0)$, $r = 10$
 e centre $(0, 0)$, $r = 9$ f centre $(0, 0)$, $r = 5$
- 2 D
- 3 a centre $(-2, 4)$, $r = 7$ b centre $(3, 1)$, $r = 1$
 c centre $(9, 12)$, $r = 15$ d centre $(0, -3)$, $r = 2$
 e centre $(-6, -1)$, $r = \sqrt{10}$ f centre $(-5, -8)$, $r = 4$
 g centre $(0, 0)$, $r = 6\sqrt{2}$ h centre $(-2, 1)$, $r = 5\sqrt{2}$
 i centre $(2, 0)$, $r = 8$ j centre $(4, -3)$, $r = 2.5$
 k centre $(3, 4)$, $r = 9$ l centre $(0, -1)$, $r = \frac{1}{3}$
- 4 B
- 5 a $(x - 1)^2 + (y + 2)^2 = 9$ b $(x - 10)^2 + (y + 11)^2 = 4$
 c $(x + 3)^2 + (y - 2)^2 = 100$ d $x^2 + (y + 1)^2 = 1$
 e $(x + 6)^2 + (y - 2)^2 = 5$ f $(x + 1)^2 + (y - 5)^2 = 8$



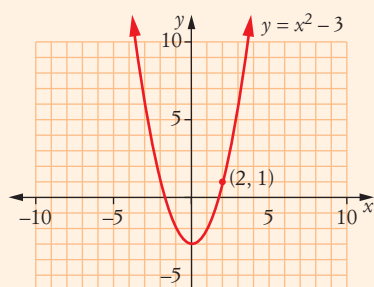
Answers

- 7 a 1, 1 b 9, 3 c 16, 4 d 4, 2
 e $\frac{49}{4}, \frac{7}{2}$ f $\frac{9}{4}, \frac{3}{2}$ g $\frac{1}{4}, \frac{1}{2}$ h $\frac{25}{4}, \frac{5}{2}$
 8 a $(-3, 1), r = 5$ b $(4, 2), r = 7$ c $(-2, 5), r = 6$
 d $(-10, 6), r = 1$ e $(2, -4), r = 5$ f $(6, -3), r = 4$
 g $(-3, 10), r = 9$ h $(4, -1), r = 2\sqrt{3}$

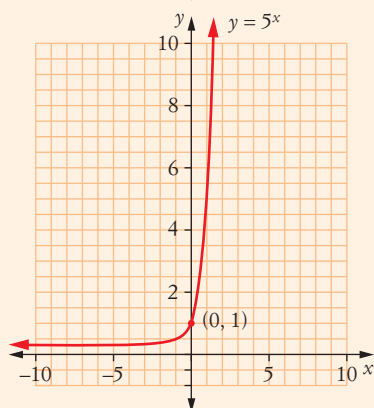
Exercise 8-13

- 1 a P b L c E d L e C f L
 g P h L i P j E k P l C
 2 a G b J c H d D e A
 f F g C h E i B j I

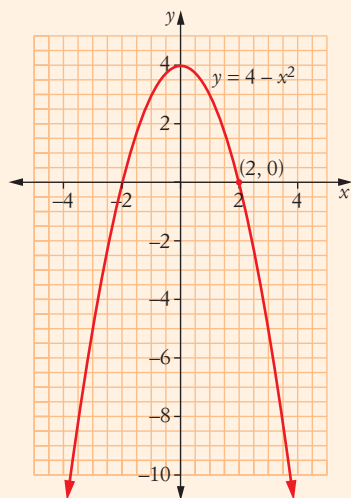
3 a



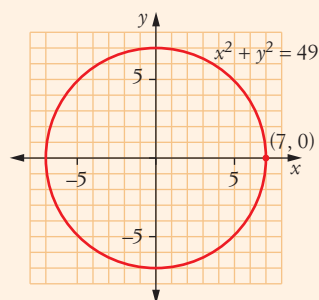
b



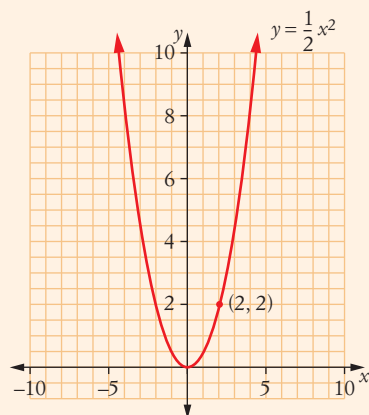
c



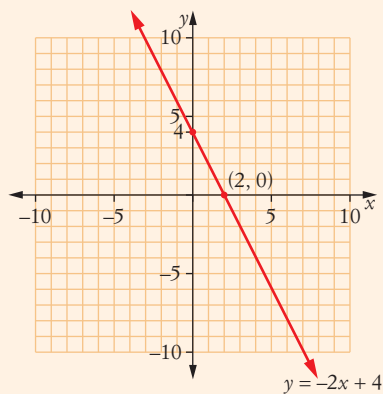
d



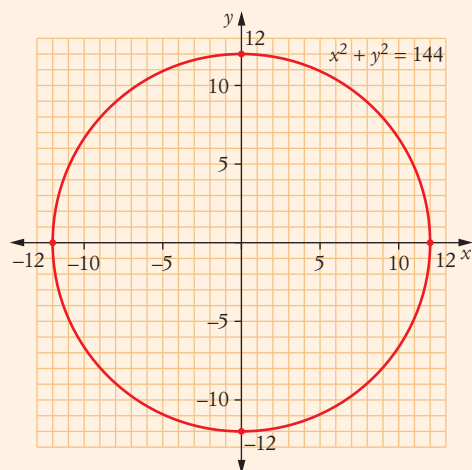
e



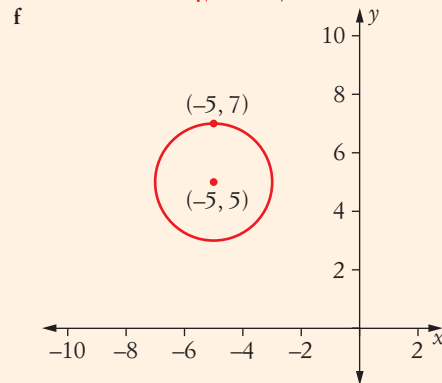
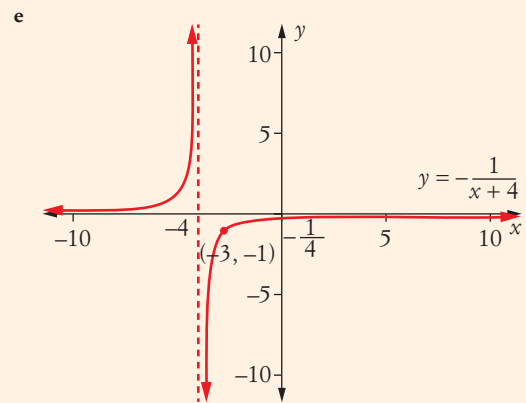
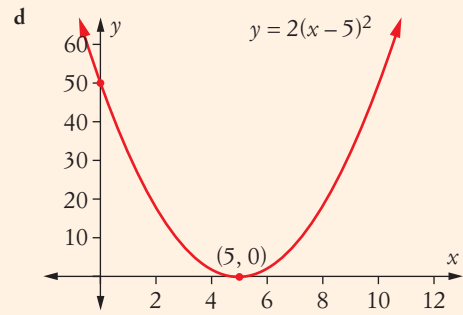
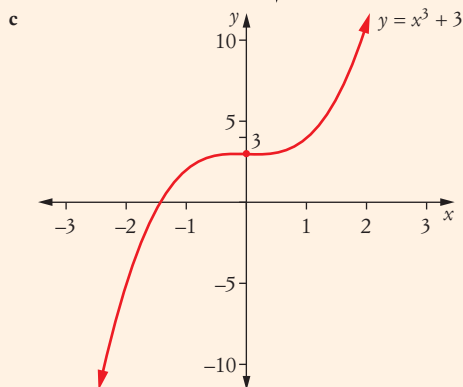
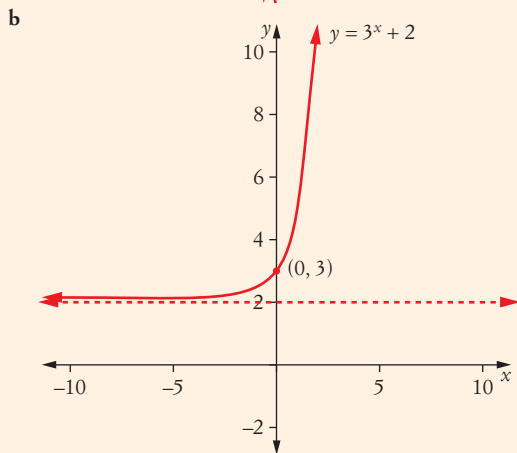
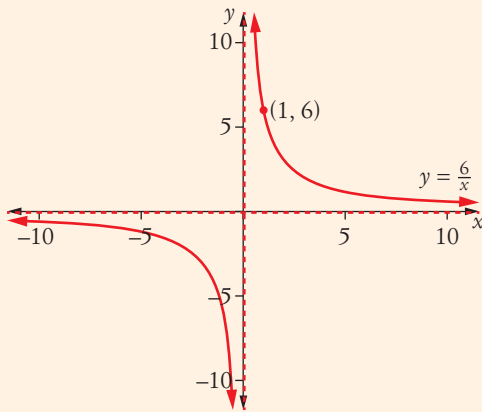
f



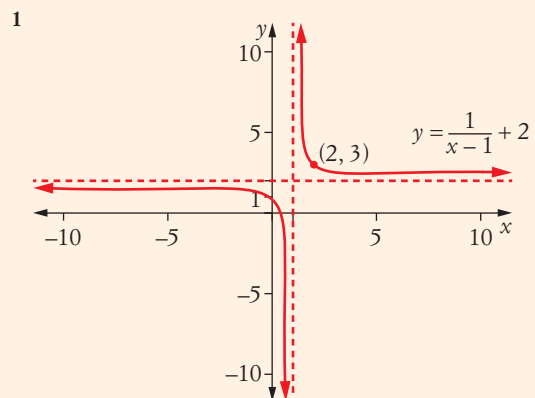
g



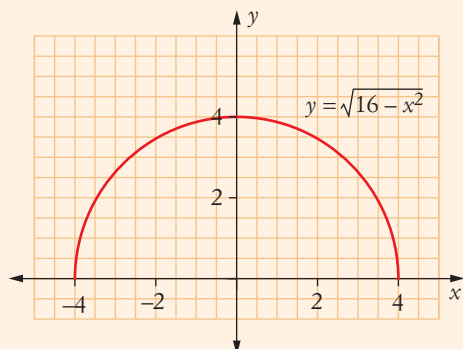
- 4 a 1 b 3 c -6 d 1
 5 a H b C c P d H
 e Q f C g E h H
 6 a D b E c C d A
 e H f B g F h G
- 7 a i exponential ii 1 iii $y = 0$
 b i exponential ii 2 iii $y = 1$
 c i exponential ii -2 iii $y = -3$
 d i hyperbola ii none iii $x = 0, y = 0$
 e i hyperbola ii $-\frac{2}{3}$ iii $x = 3, y = 0$
 f i hyperbola ii none iii $x = 0, y = 2$
- 8 a



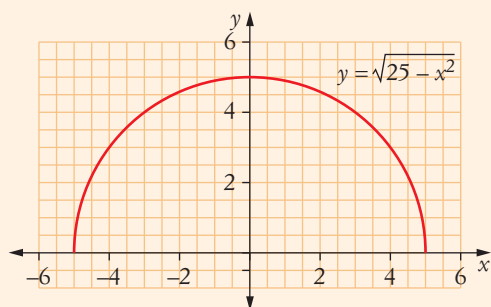
Power plus



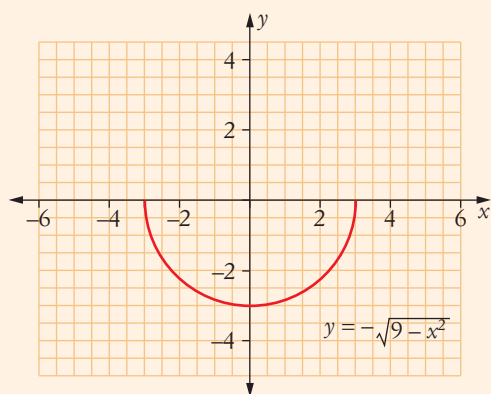
2 a centre (0, 0) and $r = 4$



b centre (0, 0) and $r = 5$

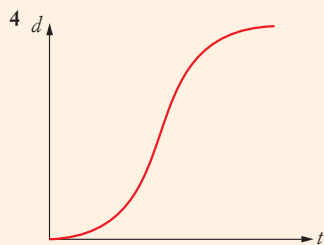


c centre (0, 0) and $r = 3$



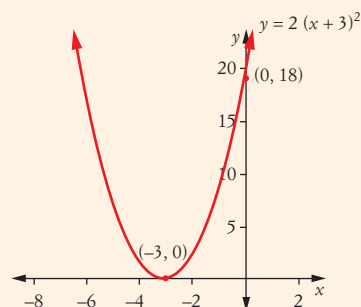
Chapter 8 revision

1 $H = 310.5$ 2 10°C 3 a £46 b \$A85

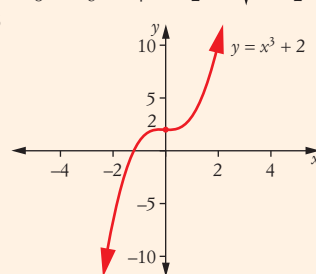


5 a B b C c A
6 a C b F c A d E e D f B

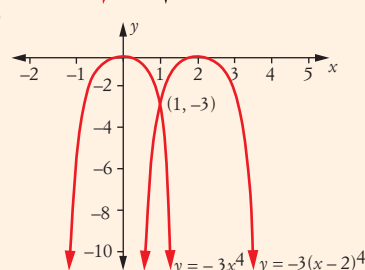
7



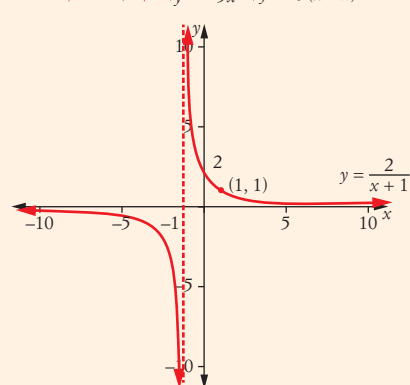
8



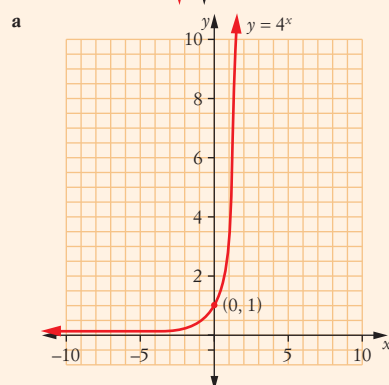
9



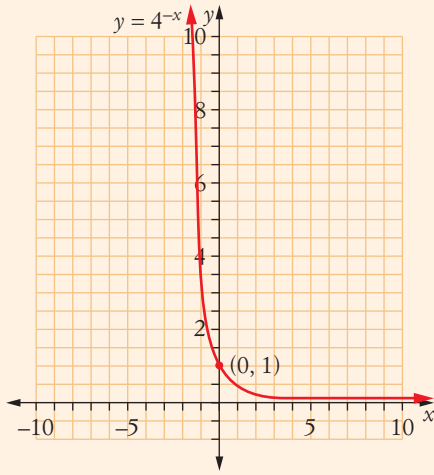
10



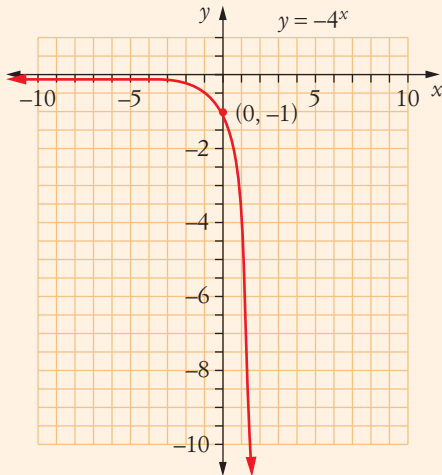
11 a



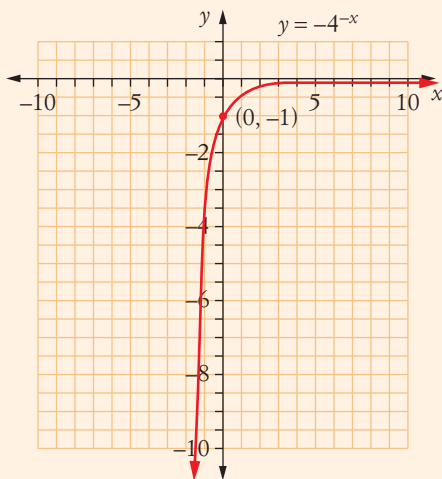
b



c



d



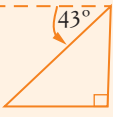
- 12 a centre (0, 0), $r = 10$ b centre (0, 0), $r = 6$
 c centre (0, 0), $r = 7$ d centre (-5, 6), $r = 9$
 e centre (10, 0), $r = 15$ f centre (-7, -10), $r = 4\sqrt{5}$
 13 a D b C c B d J e E f H
 g L h G i I j A k K l F

Chapter 9

SkillCheck

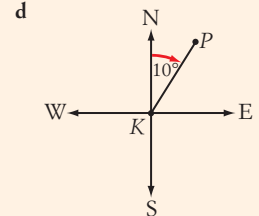
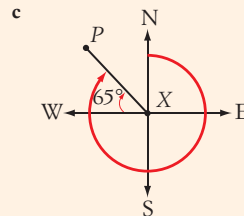
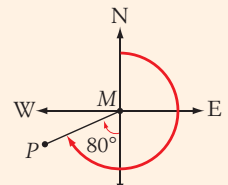
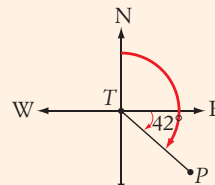
1 a 64°	b 26°	c 12°
2 a 0.8480	b 0.7760	c 0.1539
d 64.9839	e 13.9884	f 13.7044
3 a $45^\circ 48'$	b $33^\circ 11'$	c $5^\circ 21'$
4 a $64^\circ 37'$	b $69^\circ 41'$	c $28^\circ 8'$

Exercise 9-01

1 a 64.7 cm	b 14.2 cm	c 54.5 cm
d 18.5 cm	e 5.1 cm	f 17.4 cm
g 48.8 cm	h 59.0 cm	i 17.5 cm
2 a $38^\circ 41'$	b $56^\circ 7'$	c $42^\circ 56'$
d $52^\circ 57'$	e $64^\circ 37'$	f $45^\circ 1'$
3 a 73°	b 5.7 m	
4 4°	5 114 m	6 6.49 m
7 a $51^\circ 20'$	b $\frac{5}{4}$	8 2.6 km
9 a 11.6 m	b 11.2 m	
10 a 	b 	c 
11 127 m	12 1132 m	13 177 m
14 180 m	15 $14^\circ 29'$	16 $79^\circ 44'$
18 47.7 m	19 14.5 m	20 970 m

Exercise 9-02

1 a 237°	b 295°	c 046°	d 115°	e 210°
f 140°	g 312°	h 253°	i 065°	
2 a 000°	b 090°	c 180°	d 270°	e 038°
f 125°	g 330°	h 225°	i 072°	j 187°
3 SW	4 a NNW	b 337.5°		
5 a		b		



6 a 22 km	b 257°	
7 a 37°	b 163 km	c 323°
8 a 12.2 km	b 305°	9 a 18.5 km b NNW
10 45.7 km		11 a 13.509 km b 321°
12 a 2122 km	b 330°	13 a 15 km b 26 km
14 a 261.08 km	b 167.82 km	

Answers

Exercise 9-03

- 1 a $4\sqrt{13}$ cm b 15.6 cm c 23°
 2 a $\sqrt{800}$ cm or $20\sqrt{2}$ cm b 34.64 cm c $35^\circ 16'$
 3 a 20.40 cm b 79° 4 34°
 5 a 9.1 cm b 28° 6 a 53 m b 114 m
 7 37.5 m 8 a 50° b 868 m
 9 a 85 m, 50 m b 99 m apart
 10 a $\angle WHF = 52^\circ$, $\angle WFH = 38^\circ$ and $\angle HWF = 90^\circ$
 b Tower is 49.1 m.
 11 a 285° b 7°

Exercise 9-04

- 1 a 43° b 16° c 87.45° d 34.8° e $51^\circ 43'$ f $72^\circ 22'$
 2 a 0 b 1 c 1
 3 $\cos \beta = \frac{5}{13}$, $\cos \alpha = \frac{12}{13}$, $\sin \beta = \frac{12}{13}$
 4 $\sin F = \frac{40}{41}$, $\sin E = \frac{9}{41}$, $\cos F = \frac{9}{41}$
 5 $\cos Y = \frac{\sqrt{5}}{3}$, $\sin Y = \frac{2}{3}$, $\sin X = \frac{\sqrt{5}}{3}$
 6 $\cos \phi = \frac{\sqrt{5}}{4}$, $\sin \phi = \frac{\sqrt{11}}{4}$, $\cos \theta = \frac{\sqrt{11}}{4}$
 7 a 8 b 12 c $3\sqrt{2}$ d 45° e 30° f 30°
 8 $35\sqrt{3}$ m

Mental skills 9

- 2 a $\frac{2}{3}$ b $\frac{3}{4}$ c $\frac{5}{7}$ d $\frac{1}{2}$ e $\frac{1}{4}$ f $\frac{1}{6}$
 g $\frac{2}{6}$ h $\frac{2}{5}$ i 5 : 9 j 5 : 9 k 9 : 20 l 4 : 5
 m 9 : 7 n 4 : 3 o $\frac{3}{5}$ p $\frac{4}{35}$
 3 a $\frac{17}{40}$ b $\frac{2}{3}$ c $\frac{16}{25}$ d $\frac{1}{4}$ e $\frac{5}{24}$ f $\frac{2}{25}$

Exercise 9-05

- 1 a $\tan A = \frac{60}{91}$ b $\tan Y = 1.\dot{3}$ c $\tan X = \frac{2}{3}$
 d $\tan P = \frac{9}{40}$ e $\tan Q = \frac{\sqrt{40}}{3}$ f $\sin X = \frac{11}{61}$
 g $\cos X = \frac{7}{25}$ h $\sin X = \frac{2}{3}$
 2 a P b P c N d N
 e P f N g N h P
 3 a -0.89 b 0.19 c -0.77 d -0.11
 e 0.51 f -0.58 g 0.05 h -0.42
 i -0.78 j -0.87 k 0.18 l 0.28
 4 a, b Teacher to check.
 c The graph has a wave shape that repeats itself after 360° .
 Maximum $y = 1$ at $\theta = 90^\circ$; Minimum $y = -1$ and $\theta = 270^\circ$
 d No e Yes, centre of symmetry at $(180^\circ, 0)$.
 f i $0^\circ < \theta < 180^\circ$ (1st and 2nd quadrants)
 ii $180^\circ < \theta < 360^\circ$ (3rd and 4th quadrants)
 5 a, b Teacher to check.
 c The graph has a wave shape that repeats itself after 360° .
 Maximum $y = 1$ at $\theta = 0^\circ$ and $\theta = 360^\circ$; Minimum $y = -1$
 and $\theta = 180^\circ$
 d Yes, axis of symmetry $\theta = 180^\circ$ e No
 f i $0^\circ < \theta < 90^\circ$ and $270^\circ < \theta < 360^\circ$ (1st and 4th quadrants)
 ii $90^\circ < \theta < 270^\circ$ (3rd and 4th quadrants)
 g Similarities: Both graphs have the same wave shapes that run
 between $y = -1$ and $y = 1$ and repeat themselves after 360° .
 Differences: The graphs have different x - and θ -intercepts
 6 a 10° b 70° c 50° d 83° e 65° f 12°

- 7 a $-\cos 38^\circ$ b $\sin 75^\circ$ c $-\cos 25^\circ$
 d $-\tan 78^\circ$ e $-\cos 7.3^\circ$ f $-\sin 64.5^\circ$
 g $-\cos 40^\circ 25'$ h $-\tan 9.2^\circ$ i $\sin 59^\circ 25'$
 j $-\tan 19^\circ 50'$ k $\sin 84.5^\circ$ l $-\tan 40.5^\circ$
 8 a $\frac{1}{2\sqrt{3}}$ b -1 c $\frac{1}{\sqrt{2}}$ d $-\frac{1}{2}$ e $-\frac{1}{\sqrt{3}}$
 f $\frac{\sqrt{3}}{2}$ g $-\frac{\sqrt{3}}{2}$ h $-\sqrt{3}$ i 1 j $-\frac{1}{\sqrt{2}}$
 9 a, b Teacher to check.
 c The tan graph is broken into three sections and repeats
 itself after 180° . It has asymptotes at 90° and 270° .
 d No e centre of symmetry at $(180^\circ, 0)$.
 f i $0^\circ < \theta < 90^\circ$ and $180^\circ < \theta < 270^\circ$ (1st and 3rd quadrants).
 ii $90^\circ < \theta < 180^\circ$ and $270^\circ < \theta < 360^\circ$ (2nd and
 4th quadrants).

Exercise 9-06

- 1 a $57^\circ, 123^\circ$ b 143° c 110° d 130°
 e $7^\circ, 173^\circ$ f 135° g 100° h $25^\circ, 155^\circ$
 i 114° j $33^\circ, 147^\circ$ k 105° l 118°
 2 a $145^\circ 9'$ b $131^\circ 57'$ c $159^\circ 26'$ d $173^\circ 48'$
 e $152^\circ 58'$ f $115^\circ 51'$ g no solution h $126^\circ 52'$
 i no solution j $163^\circ 18'$ k $126^\circ 52'$ l $154^\circ 37'$
 3 a 137° b 136° c $61^\circ, 119^\circ$
 d 69° e $42^\circ, 138^\circ$ f 143°
 g 45° h 60° i $45^\circ, 135^\circ$

Exercise 9-07

- 1 a 18.4 b 21.1 c 105.0
 2 a $a = 20.51$ b $b = 11.91$ c $c = 12.58$
 d $d = 4.10$ e $e = 30.85$ f $f = 3.55$
 g $k = 5.99$ cm h $w = 29.17$ m i $p = 8.29$ m
 3 79 m 4 25 m 5 b 1042 cm 6 a 110° b 131.6 m
 7 561 km 8 d 124.7 m 9 b 595 m

Exercise 9-08

- 1 a 27° b 37° c 54°
 2 a 44.5° b 46.6° c 32.0°
 d 67.3° e 18.8° f 31.8°
 3 a $149^\circ 7'$ b $129^\circ 0'$ c $142^\circ 8'$
 d $135^\circ 33'$ e $129^\circ 29'$ f $162^\circ 13'$
 4 a 46° or 134° b 39° c 55° or 125° d 44° or 136°
 5 a 75° or 117° b 41° c 84°

Exercise 9-09

- 1 a 5.6 b 13.1 c 35.8
 2 a $a = 8.30$ b $c = 54.52$ c $e = 88.41$
 d $b = 16.33$ e $d = 19.44$ f $f = 40.72$
 3 0.6 m 4 C
 5 a Teacher to check.
 b $\angle XYN = 180^\circ - 130^\circ = 50^\circ$ (cointerior angles on
 parallel lines) $\angle XYZ = 50^\circ + 25^\circ = 75^\circ$
 c 4.4 km
 6 47 km
 7 a 0 b $a^2 = b^2 + c^2$
 c With $\cos 90^\circ = 0$, the cosine rule reverts to
 Pythagoras' theorem.

Exercise 9-10

- 1 a 70° b 33° c 109° d 131°
 2 a 112° b 108° c 121° d 23° e 60° f 83°
 3 20.8° 4 $64^\circ 40'$ 5 99°

Exercise 9-11

- 1 a 413.4 m^2 b 463.1 cm^2 c 326.9 mm^2
 d 132.9 mm^2 e 320.4 cm^2 f 0.1 m^2
 2 a 97.4 m^2 b 463.6 m^2 c 246.2 m^2
 d 227.6 m^2 e 93.5 m^2 f 152.2 m^2
 3 a 225 m b 2770 m^2 4 a 130° b 766.07 m^2
 5 a 418.9 cm^2 b 173.2 cm^2 c 245.7 cm^2
 6 a 112° b 37 cm^2 c 740 cm^3

Exercise 9-12

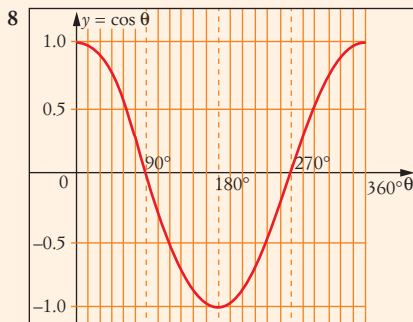
- 1 a 10.2 m b 16.1 mm c 17.1 cm
 d 13.1 m e 3.9 m f 18.2 m
 2 a 32° b 142° (or 38°) c 29°
 d 55° e 37° f 125°
 3 a $32^\circ + 23^\circ = 55^\circ$ (exterior angle of a triangle)
 b 108.50 m c 89 m
 4 a 1 b i 15.4 ii 15.4
 c The results are the same. The sine rule $\frac{d}{\sin 90^\circ} = \frac{12.8}{\sin 56^\circ}$
 becomes $d = \frac{12.8}{\sin 56^\circ}$ (since $\sin 90^\circ = 1$), which is the same
 result when using the sine ratio.
 5 7.5 km 6 486 km

Power plus

- 1 a $\sqrt{3} - 1$ b $\frac{1}{2}(\sqrt{3} - 1)$
 2 a 45° b 60° c 30°
 d 150° e 120° f 135°
 g $60^\circ, 120^\circ$ h $30^\circ, 150^\circ$ i $45^\circ, 135^\circ$
 3 Teacher to check. 4 $\sqrt{2}$ 5 $45^\circ, 135^\circ$

Chapter 9 revision

- 1 a 10.9 m b 4.4 m c 11.5 cm
 2 a $64^\circ 59'$ b $48^\circ 59'$ c $57^\circ 12'$
 3 13° 4 a 320° b 140°
 5 a 1281 km b 024° 6 a 33 cm b 65°
 7 a 48 b $48\sqrt{3}$ c $\frac{48}{\sqrt{2}} = 24\sqrt{2}$

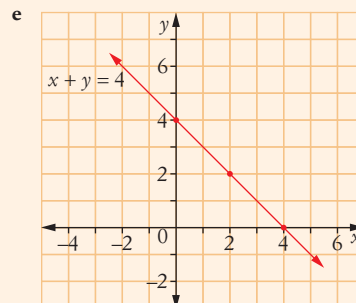
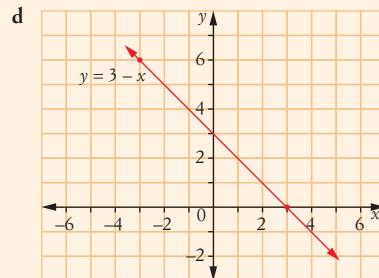
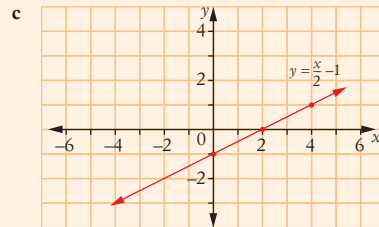
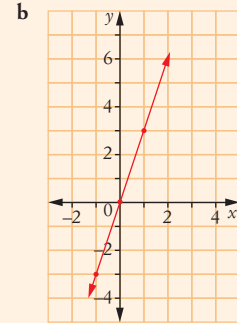
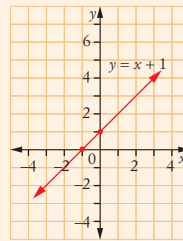


- 9 a 57° b 87° c 70°
 10 a $117^\circ 2'$ b $65^\circ 33'$ c $27^\circ 2', 152^\circ 58'$
 11 a 0.4 m b 14.8 cm c 136.4 mm
 12 a $81^\circ 54'$ or $98^\circ 6'$ b $77^\circ 24'$ or $102^\circ 36'$ c $49^\circ 37'$
 13 a 6.8 m b 112.1 mm c 7.6 cm
 14 a 96° b 56° c 125°
 15 a 165 cm^2 b 286 m^2 c 30 m^2
 16 15.5 cm

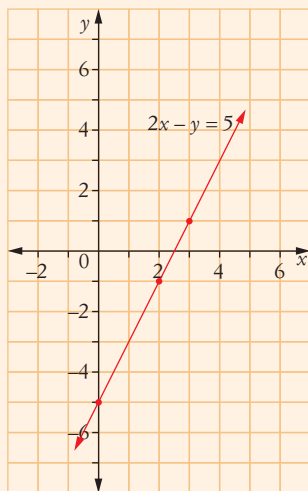
Chapter 10

SkillCheck

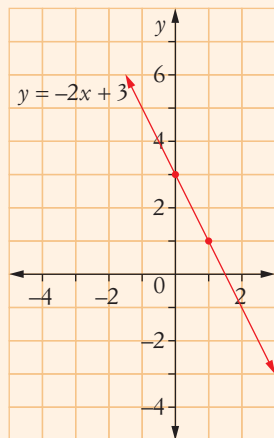
- 1 a 5 b 13 c 6 d -1
 2 a -11 b 1 c 7 d $5\frac{1}{2}$
 3 a



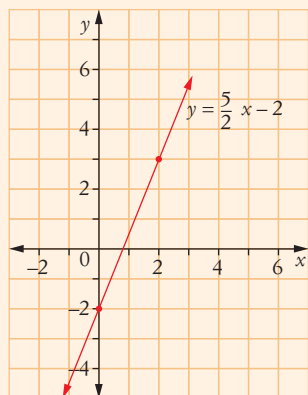
f



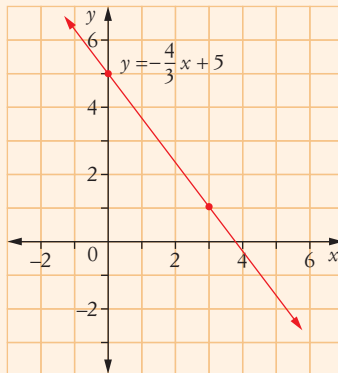
- 4 a yes b no c no d yes e no f no
 5 a For $y = 2x + 1$, when $x = 2$, $y = 2 \times 2 + 1 = 5$
 $\therefore (2, 5)$ lies on $y = 2x + 1$
 For $x + y = 7$, when $x = 2$, $y = 5$, $2 + 5 = 7$
 $\therefore (2, 5)$ lies on $x + y = 7$
 b $(2, 5)$
 6 a $m = -2$, $b = 3$



- b $m = \frac{5}{2}$, $b = -2$

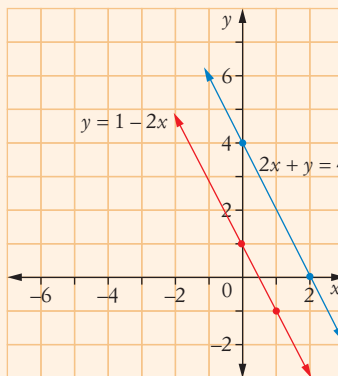


- c $m = -\frac{4}{3}$, $b = 5$



Exercise 10-01

- 1 a $x = 3$, $y = -1$ b $x = 2$, $y = 1$ c $x = -1$, $y = -5$
 2 a $x = 1$, $y = 2$ b $x = -5$, $y = -9$
 c $x = 1$, $y = 2$ d $x = -\frac{1}{2}$, $y = 2\frac{1}{2}$
 e $x = -2$, $y = -9$ f $x = 5$, $y = -4$
 g $x = \frac{1}{2}$, $y = 6\frac{1}{2}$ h $x = 3$, $y = 2$
 i $x = 5$, $y = 1\frac{1}{2}$ j $x = 5$, $y = 8$
 k $x = 1\frac{1}{2}$, $y = 2\frac{1}{2}$ l $x = 4$, $y = 0$
 3 a



- b The lines are parallel.

Exercise 10-02

- 1 a $d = -3$, $k = 2$ b $x = 5$, $w = 4$ c $g = 2$, $h = -\frac{2}{5}$
 d $n = 3.25$, $p = -1$ e $q = 5$, $r = -4$ f $k = -4\frac{3}{5}$, $x = 5$
 g $c = 1\frac{1}{2}$, $e = 1$ h $k = 3$, $y = -2$ i $a = 2$, $f = 2$
 2 a $d = -14$, $k = 6$ b $a = 1$, $c = 1$ c $h = 3$, $y = 4$
 d $e = 3$, $x = \frac{1}{3}$ e $q = 3$, $w = 6\frac{1}{2}$ f $c = -2$, $p = 3$
 g $m = -\frac{2}{3}$, $y = 4$ h $a = -1$, $r = 5\frac{1}{2}$ i $x = 2$, $w = 2$
 3 a $q = -3$, $w = 3$ b $m = -9$, $x = 7$ c $d = 23$, $h = -7$
 d $g = -1$, $n = 3$ e $h = 0$, $m = 2$ f $e = -4$, $y = 3$
 g $q = 1$, $w = -4$ h $a = \frac{1}{2}$, $d = \frac{1}{2}$ i $k = 5$, $p = -2$
 j $a = -2$, $f = -2$ k $c = -64$, $r = -38$ l $x = -4$, $y = -3$

Exercise 10-03

- 1 a $x = 2, y = 5$ b $x = \frac{3}{5}, y = 3\frac{4}{5}$ c $x = 7, y = 2$
 d $x = 2, y = -2$ e $x = 5, y = -1$ f $x = 4, y = 2$
 2 a $x = 9, y = 21$ b $x = 5, y = 3$
 c $x = -\frac{1}{4}, y = 2$ d $x = -3, y = 1$
 e $x = 2, y = 2$ f $x = -7, y = 4$
 g $x = 7, y = 3$ h $x = 3, y = 2\frac{1}{2}$
 i $x = 2\frac{2}{3}, y = 1$ j $x = -3\frac{1}{5}, y = -4\frac{3}{5}$

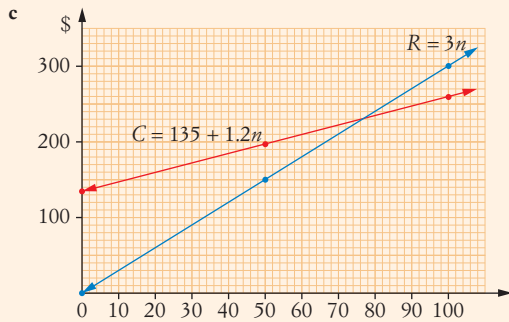
Exercise 10-04

- 1 285 2 680 3 b 364 4 12
 5 Aaron 36, Sejuti 12 6 16 7 black 36, colour 24
 8 Pie: \$3.60, Sausage roll: \$2.70
 9 Supreme 32, Vegetarian 13
 10 Strawberries \$3.50; Blueberries \$4.99
 11 b 20-cent coins: 154, 50-cent coins: 91
 12 a Teacher to check.
 b $C = 135 + 1.2n$

n	0	50	100
C	135	195	255

$$R = 3n$$

n	0	50	100
R	0	150	300



d $n = 75$

Mental skills 10

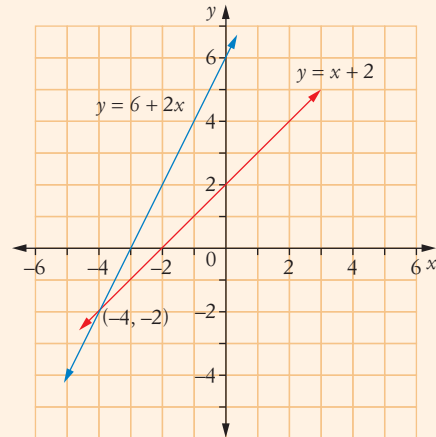
- 2 a $\frac{2}{3}$ b $\frac{4}{5}$ c $\frac{5}{7}$ d $\frac{1}{2}$
 e $\frac{1}{4}$ f $\frac{1}{6}$ g $\frac{2}{6}$ h $\frac{2}{5}$
 i 5 : 9 j 5 : 9 k 9 : 20 l 4 : 5
 m 9 : 7 n 4 : 3 o $\frac{3}{5}$ p $\frac{4}{35}$
 3 a $\frac{17}{40}$ b $\frac{2}{5}$ c $\frac{16}{25}$ d $\frac{1}{4}$ e $\frac{5}{24}$ f $\frac{2}{25}$

Power plus

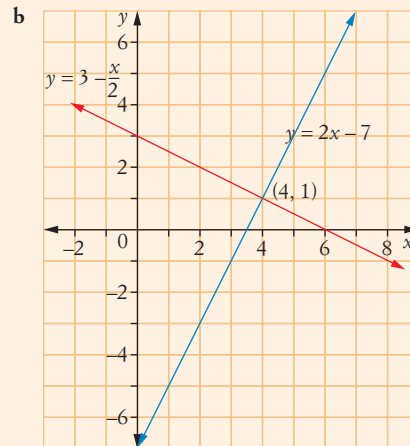
- 1 a $x = 1\frac{1}{2}, w = 4\frac{1}{2}, y = -5\frac{1}{2}$
 b $a = 1\frac{7}{13}, c = 4\frac{3}{13}, d = 8\frac{11}{13}$
 c $p = -11\frac{3}{13}, m = 18\frac{11}{13}, n = -13\frac{4}{13}$
 2 a Teacher to check.
 b $ae - bd = 0$ and a fraction cannot have denominator 0.
 c $x = 3, y = -1$
 d Teacher to check.
 i $x = 2, y = -2$ ii $x = 28, y = 16$ iii $x = \frac{1}{11}, y = 2\frac{20}{33}$

Chapter 10 revision

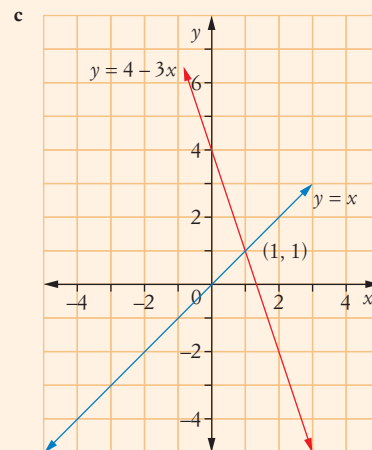
- 1 a $x = 2, y = -2$ b $x = 4, y = 0$
 2 a



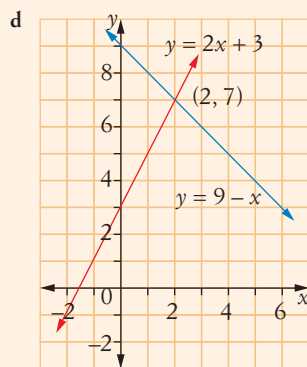
$x = -4, y = -2$



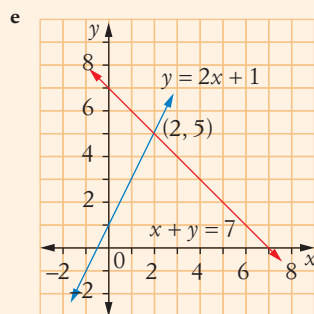
$x = 4, y = 1$



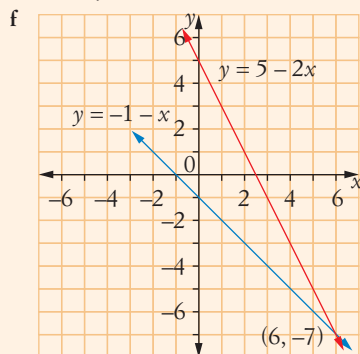
$x = 1, y = 1$



$$x = 2, y = 7$$



$$x = 2, y = 5$$



$$x = 6, y = -7$$

- 3 a $m = 5, y = -9\frac{1}{2}$ b $x = 2, y = \frac{1}{3}$
 c $a = 1, d = 1$ d $x = 6, y = 15$
 e $x = -5, y = -2$ f $d = -3, w = -10$
 4 a $x = 2, y = 11$ b $m = 1, p = 3$ c $h = 10, t = 4$
 d $a = 3, c = \frac{1}{2}$ e $x = 1, y = 1$ f $p = 12, q = -4$
 5 a 1600 adults, 900 children b 18 DVDs, 12 CDs
 c \$38 d 28 cheesecakes, 47 mudcakes
 e 120 boys, 93 girls

- 2 a $(4 - m)(4 + m)$ b $(d - 11)(d + 11)$ c $2y(7 - y)$
 d $5p(2p + 5)$ e $5(x - 8)(x + 8)$ f $2(3w - 5)(3w + 5)$
 g $(k + 4)(k + 1)$ h $(y - 8)(y - 2)$ i $(m - 8)(m + 7)$
 j $(u + 13)(u - 5)$ k $(w - 7)(w - 3)$ l $(x - 6)(x + 4)$
 3 a $y = -2$ b $y = -10$ c $y = 5$ d $y = -10$
 4 a $(3a + 1)(a + 3)$ b $(5x + 2)(x - 3)$ c $(2y - 5)(3y + 8)$
 d $(3t - 1)(5t + 4)$ e $(5v + 3)(v - 7)$ f $(2y + 5)(4y + 7)$
 g $(3h - 4)(5h - 1)$ h $(4p - 3)(3p + 5)$ i $(4d + 5)^2$

Exercise 11-01

- 1 a $m = -7$ or -3 b $d = 3$ or 7 c $y = -5$ or 3
 d $k = 0$ or 3 e $t = -7$ or 0 f $p = 0$ or 3
 g $w = 0$ or $\frac{2}{3}$ h $n = -\frac{1}{2}$ or 3 i $a = \frac{1}{2}$ or $\frac{3}{5}$
 j $x = -\frac{1}{3}$ or $-1\frac{1}{2}$ k $c = \frac{5}{2}$ l $f = \frac{1}{2}$
 m $c = -\frac{1}{3}$ or $-\frac{1}{4}$ n $h = -1$ or $\frac{1}{2}$ o $e = \frac{5}{7}$ or 1
 2 a $y = -2$ or $-1\frac{1}{2}$ b $g = -1$ or $-1\frac{1}{2}$ c $d = -1$ or $-\frac{2}{3}$
 d $t = -2\frac{1}{3}$ or -1 e $p = 1\frac{1}{2}$ or 4 f $x = \frac{2}{3}$ or $1\frac{1}{2}$
 g $y = \frac{3}{4}$ or $-\frac{1}{2}$ h $a = -\frac{1}{2}$ or $\frac{1}{3}$ i $w = -1\frac{1}{4}$ or 3
 j $c = 1$ or $-1\frac{1}{2}$ k $e = \frac{1}{4}$ or $-1\frac{1}{2}$ l $q = -3$ or $1\frac{2}{3}$
 m $g = 2\frac{1}{2}$ n $m = -\frac{2}{3}$ or $\frac{2}{6}$ o $w = -4$ or $1\frac{1}{3}$
 p $y = -3$ or 4 q $f = 6$ r $h = \frac{3}{4}$ or -1
 3 a $x = -2\frac{1}{2}$ or 3 b $t = -2\frac{1}{2}$ or $\frac{1}{2}$ c $u = -\frac{1}{8}$ or -5
 d $m = \frac{1}{7}$ or 1 e $p = -4$ or 7 f $e = -1$ or 5
 g $t = \frac{3}{2}$ or 5 h $d = -\frac{7}{3}$ or $\frac{1}{2}$ i $h = \pm 5$
 j $f = 0$ or $\frac{1}{2}$ k $w = \frac{1}{6}$ or 3 l $a = -2$ or $\frac{1}{3}$
 4 8

Exercise 11-02

- 1 a $x^2 + 2x + 1 = (x + 1)^2$ b $p^2 - 6p + 9 = (p - 3)^2$
 c $m^2 - 8m + 16 = (m - 4)^2$ d $k^2 + 4k + 4 = (k + 2)^2$
 e $y^2 + 7y + \frac{49}{4} = (y + \frac{7}{2})^2$ f $w^2 - 3w + \frac{9}{4} = (w - \frac{3}{2})^2$
 g $x^2 + x + \frac{1}{4} = (x + \frac{1}{2})^2$ h $h^2 - 5h + \frac{25}{4} = (h - \frac{5}{2})^2$
 i $a^2 + \frac{7}{2}a + \frac{49}{16} = (a + \frac{7}{4})^2$ j $v^2 - \frac{5}{3}v + \frac{25}{36} = (v - \frac{5}{6})^2$
 2 a $d = -3 + \sqrt{7}, -3 - \sqrt{7}$ b $x = 5 + \sqrt{5}, 5 - \sqrt{5}$
 c $p = -1 + \sqrt{10}, -1 - \sqrt{10}$ d $y = 1 + \sqrt{2}, 1 - \sqrt{2}$
 e $m = \frac{1+2\sqrt{5}}{2}, \frac{1-2\sqrt{5}}{2}$ f $t = \frac{-2+3\sqrt{3}}{3}, \frac{-2-3\sqrt{3}}{3}$
 g $c = \frac{-2+\sqrt{42}}{2}, \frac{-2-\sqrt{42}}{2}$ h $w = \frac{6+\sqrt{82}}{2}, \frac{6-\sqrt{82}}{2}$
 i $n = \frac{-2+\sqrt{7}}{3}, \frac{-2-\sqrt{7}}{3}$ j $e = \frac{3+\sqrt{71}}{2}, \frac{3-\sqrt{71}}{2}$
 k $d = 2 + \sqrt{5}, 2 - \sqrt{5}$ l $x = \frac{3+4\sqrt{2}}{4}, \frac{3-4\sqrt{2}}{4}$
 3 a $h = -1 \pm \sqrt{6}$ b $r = 1 \pm \sqrt{2}$ c $m = -3 \pm \sqrt{7}$
 d $w = -6, 10$ e $a = 5 \pm \sqrt{30}$ f $y = -4 \pm \sqrt{19}$
 g $p = -6 \pm \sqrt{41}$ h $h = 2 \pm \sqrt{2}$ i $u = -7, -2$
 j $d = \frac{-1 \pm \sqrt{29}}{2}$ k $c = \frac{9 \pm \sqrt{73}}{2}$ l $e = \frac{-5 \pm \sqrt{17}}{2}$
 m $y = \frac{3 \pm \sqrt{41}}{2}$ n $b = \frac{1 \pm \sqrt{21}}{2}$ o $q = \frac{3 \pm \sqrt{5}}{2}$
 p $g = \frac{-7 \pm \sqrt{73}}{4}$ q $x = -3\frac{1}{2}, 1$ r $u = \frac{-2 \pm \sqrt{22}}{3}$
 4 a $x = -11.20$ or -0.80 b $m = -0.43$ or 16.43
 c $g = -4.65$ or 0.65 d $h = 1.27$ or -2.77
 e $w = 1.27$ or -0.47 f $y = 1.14$ or -1.47
 g $p = -2$ or 1.33 h $e = 1.13$ or -0.88
 i $n = 1$ or -2.5

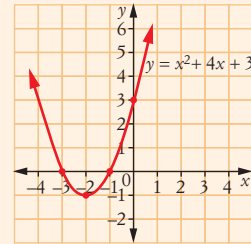
Chapter 11

SkillCheck

- 1 a $x = \pm 5$ b $m = \pm 2$ c $x = 2$ or -1
 d $u = -7$ or 4 e $k = 0$ or -3 f $w = 5$

Exercise 11-03

- 1 a $x = -3 \pm \sqrt{7}$ b $m = \frac{5 \pm \sqrt{57}}{2}$ c $w = 4 \pm \sqrt{13}$
d $k = \frac{-3 \pm \sqrt{29}}{2}$ e $y = 2 \pm \sqrt{5}$ f $p = \frac{-1 \pm \sqrt{21}}{2}$
g $u = \frac{7 \pm \sqrt{61}}{2}$ h $a = \frac{-3 \pm \sqrt{65}}{4}$ i $q = \frac{1}{5}, 1$
j $c = \frac{-1 \pm \sqrt{7}}{3}$ k $e = \frac{5 \pm \sqrt{57}}{8}$ l $x = \frac{-4 \pm \sqrt{10}}{3}$
m $d = \frac{2 \pm \sqrt{14}}{2}$ n $a = \frac{5 \pm \sqrt{31}}{3}$ o $t = -2\frac{1}{2}, 1$
p $y = -\frac{2}{3}, -2$ q $k = \frac{5}{6}, 1$ r $n = \frac{5 \pm \sqrt{113}}{4}$
- 2 a $y = \frac{9 \pm \sqrt{141}}{10}$ b $m = \frac{-1 \pm \sqrt{22}}{3}$ c $x = \frac{3 \pm \sqrt{41}}{8}$
d $k = -2 \pm \sqrt{5}$ e $m = \frac{3 \pm \sqrt{21}}{6}$ f $g = \frac{-1 \pm \sqrt{6}}{5}$
g $h = \frac{9 \pm \sqrt{17}}{4}$ h $w = \frac{1 \pm \sqrt{7}}{3}$ i $p = \frac{2 \pm \sqrt{7}}{3}$
j $u = \frac{-2 \pm \sqrt{14}}{5}$ k $a = \frac{-2 \pm \sqrt{58}}{6}$ l $y = \frac{-3 \pm \sqrt{89}}{4}$
- 3 a $k = 8.89, 0.11$ b $c = 1.41, -1.41$ c $m = 2.65, -2.65$
d $n = 3.19, 0.31$ e $p = 0.85, -2.35$ f $w = 0.30, -1.13$
g $x = 2.39, 0.28$ h $h = 3.83, -1.83$ i $x = 1.62, -0.62$
j $a = 4, 9$ k $v = 1.48, -1.48$ l $c = 2.31, 0.69$
m $t = 8.09, -3.09$ n $x = 4.27, 7.73$ o $d = 3.31, -1.81$



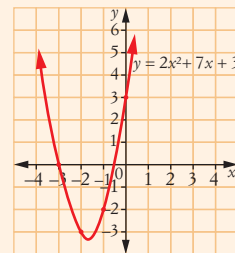
ii -3 and -1

iii 3

iv $x = -3$ and -1 , same as x -intercepts

b i

x	-3	-2	-1	0	1	2	3
y	0	-3	-2	3	12	25	42



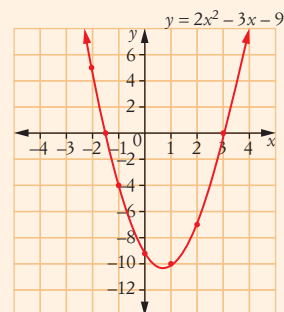
ii -3 and $-\frac{1}{2}$

iii 3

iv $x = -3$ and $-\frac{1}{2}$, same as x -intercepts

c i

x	-3	-2	-1	0	1	2	3
y	18	5	-4	-9	-10	-7	0



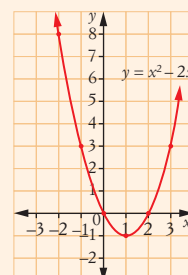
ii $-1\frac{1}{2}$ and 3

iii -9

iv $x = -1\frac{1}{2}$ and 3 , same as x -intercepts

d i

x	-3	-2	-1	0	1	2	3
y	15	8	3	0	-1	0	3



ii 0 and 2

iii 0

iv $x = 0$ and 2 , same as x -intercepts

2 a -5

b 3

c 0

Exercise 11-04

- 1 a $y = \pm 2, \pm 2\sqrt{2}$ b $m = \pm 4$ c $w = \pm\sqrt{2}, \pm\sqrt{6}$
d $m = 1, 2$ e $k = -1, 3$ f $w = 2, -\sqrt[3]{4}$
g $x = -1, \frac{1}{3}$ h $y = -\frac{4}{5}, 1\frac{2}{5}$ i $a = -1\frac{1}{2}, -\frac{1}{2}$
j $p = -\sqrt[3]{5}, -1$ k $g = \frac{1}{2}, 2$ l $c = -2, \frac{1}{5}$
- 2 a $m \approx \pm 1.2, \pm 2.8$ b $x \approx \pm 0.9, \pm 1.4$ c $a \approx -0.9, 1.3$
d $y \approx -1.4, 1.3$ e $w \approx -1.2, 1.9$ f $e \approx \pm 0.5$
- 3 a $w = \pm 5$ b $x = -2, 1.3$ c $k = \pm 2$
d $m = \pm 1.4$ e $y = \pm 1.2$ f $v = -2, 1.4$

Mental skills 11

- 2 a \$100, \$50 b \$1200, \$900 c \$160, \$560
d \$500, \$1500 e \$2700, \$1800 f \$900, \$2100
g \$3000, \$600 h \$600, \$1000 i \$550, \$440
j \$800, \$3200 k \$2100, \$2800 l \$2000, \$1200

Exercise 11-05

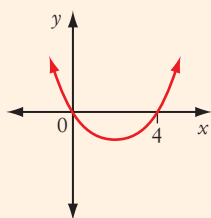
- 1 12 m by 8 m 2 40 m by 35 m
3 42 m by 24 m 4 Length 58 m, width 38 m
5 Length 13 m 6 15 m by 12 m
7 0.52 s, 3.08 s 8 a 4000 m b 2000 m c 24.5 s
9 a 34 m b 3.9 s c i 0.5 s and 1.9 s ii 3.5 s
10 8 or -9 11 24, 25 or $-24, -25$ 12 35 or -34

Exercise 11-06

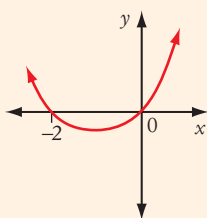
- 1 a i
- | | | | | | | | |
|-----|------|------|------|-----|-----|------|------|
| x | -3 | -2 | -1 | 0 | 1 | 2 | 3 |
| y | 0 | -1 | 0 | 3 | 8 | 15 | 24 |

Answers

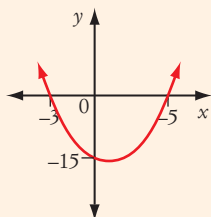
3 a



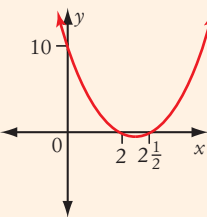
b



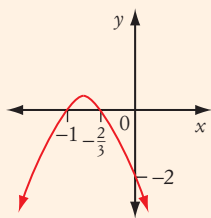
c



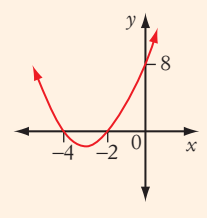
d



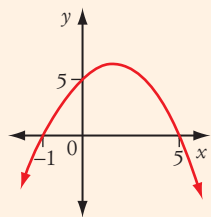
e



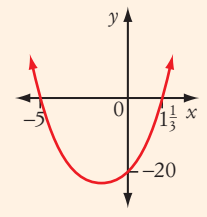
f



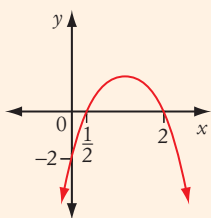
g



h



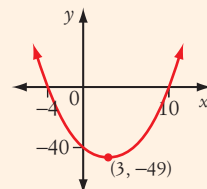
i



4 a i -4, 10
iv (3, -49)

ii -40
v concave up

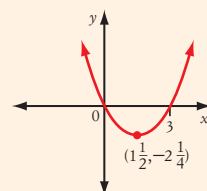
iii $x = 3$



b i 0, 3
iv $(1\frac{1}{2}, -2\frac{1}{4})$

ii 0
v concave up

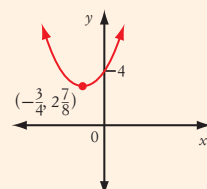
iii $x = 1\frac{1}{2}$



c i No x-intercepts
iv $(-\frac{3}{4}, 2\frac{7}{8})$

ii 4
v concave up

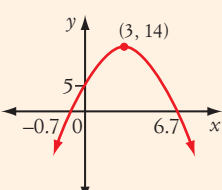
iii $x = -\frac{3}{4}$



d i -0.7, 6.7
iv (3, 14)

ii 5
v concave down

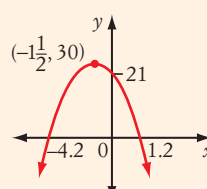
iii $x = 3$



e i -4.2, 1.2
iv $(-1\frac{1}{2}, 30)$

ii 21
v concave down

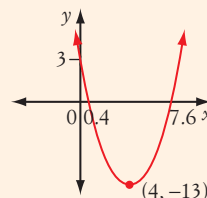
iii $x = -1\frac{1}{2}$



f i 0.4, 7.6
iv (4, -13)

ii 3
v concave up

iii $x = 4$

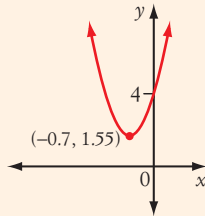


Exercise 11-07

- 1 a $x = 3$ b $x = 0$ c $x = 2\frac{1}{2}$
d $x = -1$ e $x = 1\frac{1}{2}$ f $x = 3$
- 2 a i $x = 3$ ii (3, -1) b i $x = 5$ ii (5, 16)
c i $x = 1$ ii (1, 9) d i $x = 4$ ii (4, 25)
e i $x = \frac{1}{2}$ ii $(\frac{1}{2}, -24\frac{3}{4})$ f i $x = -4$ ii (-4, -80)
g i $x = 4$ ii (4, 48) h i $x = -\frac{1}{4}$ ii $(-\frac{1}{4}, -1\frac{1}{4})$
i i $x = -\frac{1}{6}$ ii $(-\frac{1}{6}, 1\frac{1}{4})$
- 3 a $y = (x + 2)^2 - 3, (-2, -3)$
b $y = (x + \frac{5}{2})^2 - 10\frac{1}{4}, (-\frac{5}{2}, -10\frac{1}{4})$
c $y = -(x - 1)^2 + 6, (1, 6)$
d $y = (x - 3)^2 - 9, (3, -9)$
e $y = (x - 1)^2 + 1, (1, 1)$
f $y = 2(x - \frac{5}{4})^2 - \frac{9}{8}, (1\frac{1}{4}, -1\frac{1}{8})$

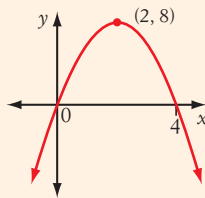
g i No x -intercepts
iv $(-0.7, 1.55)$

ii 4
v concave up



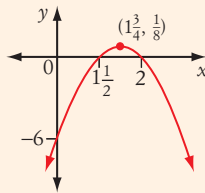
h i 0, 4
iv $(2, 8)$

ii 0
v concave down



i i $1\frac{1}{2}, 2$
iv $(1\frac{3}{4}, \frac{1}{8})$

ii -6
v concave down



iii $x = -0.7$

iii $x = 2$

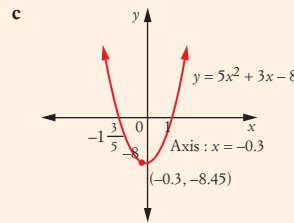
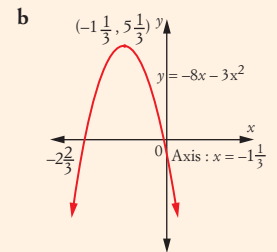
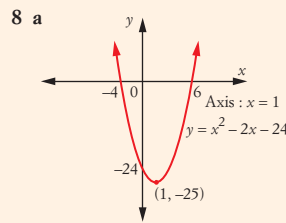
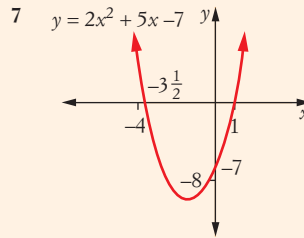
iii $x = 1\frac{3}{4}$

3 a $m = \frac{-5 \pm \sqrt{37}}{2}$ b $d = \frac{7 \pm \sqrt{137}}{4}$ c $k = \frac{1 \pm \sqrt{253}}{6}$

4 a $x \approx 0.30, -3.30$ b $h \approx 1.55, -0.80$ c $w \approx 5.37, -0.37$

5 a $m = \pm 2, \pm \sqrt{2}$ b $y = \pm 2$ c $x = -1, 2$

6 a 32 and 34 b $14 \text{ m} \times 22 \text{ m}$



9 a $(1\frac{1}{2}, 4\frac{1}{2}), (-1, 2)$

b $(-5, 13), (1, 1)$

c $(0, 4), (4, 0)$

d $(4, -2), (2, -4)$

Exercise 11-08

1 a $(3, 9), (2, 4)$	b $(0, 0), (4, 32)$
c $(4, 26), (-2, 14)$	d $(-1, 5), (1\frac{1}{5}, 7\frac{1}{5})$
e $(-2, 18), (\frac{1}{5}, 11)$	f $(-2, -22), (5, 20)$
g $(7, 39)$	h $(2, 0), (-1, 3)$
2 a $(1, -2), (2, -1)$	b $(0, 1), (1, 0)$
c $(0, 3), (3, 0)$	d $(0, 5), (-5, 0)$
e $(1, 7), (-1\frac{2}{5}, -5)$	f $(3, 5), (-2\frac{1}{2}, -6)$
g $(-1, 3), (-1\frac{1}{2}, 2)$	

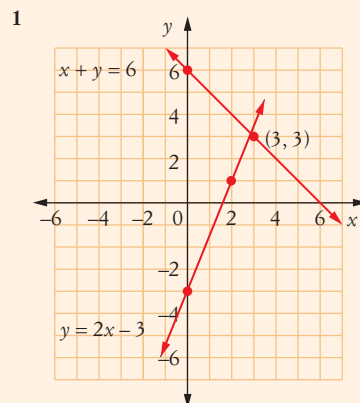
Power plus

1 a Teacher to check.	b $x = \frac{4}{3}, 1, \frac{7 \pm \sqrt{37}}{6}$
2 Intersects twice	3 $72 \text{ cm} \times 24 \text{ cm}$

Chapter 11 revision

1 a $m = 2, -\frac{3}{2}$	b $x = 0, 5$	c $p = -6, 8$
d $y = 2, 6$	e $w = -\frac{4}{3}, -5$	f $k = \frac{3}{4}, -\frac{1}{2}$
g $n = -3, 5$	h $d = -4\frac{1}{2}, 1\frac{1}{2}$	i $x = -1\frac{1}{2}, 2\frac{1}{2}$
2 a $y = -2 \pm \sqrt{11}$	b $p = \frac{3 \pm \sqrt{33}}{4}$	c $w = \frac{-1 \pm \sqrt{6}}{5}$

Mixed revision 3

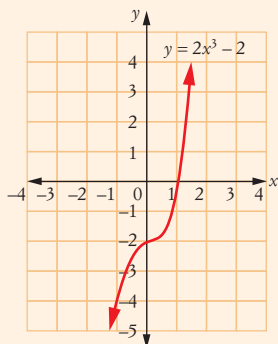


$x = 3, y = 3$

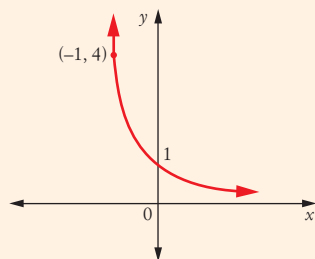
2 a €17.5	b AU\$86	c €98
3 a $m = 6.8 \text{ cm}$	b $k = 22.7 \text{ m}$	c $d = 3.7 \text{ m}$
4 a $t = \frac{99.96}{5}$	b 9.52 m/s	c 9.70 s

Answers

- 5 a $x = 1$ b $(1, 2)$ 6 $f = 3, y = 3$
 7 a 29° b 45° c 43° 8 $x = \frac{2 \pm \sqrt{7}}{3}$
 9



- 10 340 cm^2
 11 a $m = -\frac{1}{4}$ or -1 b $y = \frac{4}{5}$ or $-\frac{1}{2}$ c $w = \frac{5}{2}$ or -1
 12 $d = 32.5 \text{ m}$ 13 b 240 children 14 C
 15 $\alpha = 40^\circ 30'$ 16 $x = 2\frac{1}{2}, y = 5\frac{1}{2}$ 17 $52^\circ, 128^\circ$
 18



- 19 a $\frac{1}{\sqrt{2}}$ b $\frac{\sqrt{3}}{2}$ c $\frac{1}{\sqrt{3}}$
 20 $x = \pm 3, \pm 2$ 21 $\theta = 37^\circ 56'$
 22 a $4w^2 - 5w - 1071 = 0$ b $17 \text{ m} \times 63 \text{ m}$
 23 $(4, 1)$ and $(-1, -4)$ 24 a 301 km b 114° 25 C

Chapter 12

SkillCheck

- 1 C
 2 a 3
 b No, $P(10\text{c coin}) = \frac{5}{12}$, $P(20\text{c coin}) = \frac{1}{3}$, $P(50\text{c coin}) = \frac{1}{4}$
 3 a $\frac{1}{3}$ b $\frac{1}{3}$ c $\frac{5}{6}$
 4 a 0 b 1
 5 0.4 6 B 7 0.15

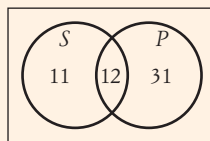
Exercise 12-01

- 1 a i 0.425 ii 0.14 iii 0.21
 b i 0.375 ii 0.125 iii 0.25
 c Yes
 d Expected frequency = 100. The observed frequency of red or purple is 115, which is more than the expected frequency.
 2 a i $\frac{1}{5} = 0.2$ ii $\frac{19}{50} = 0.38$ iii $\frac{33}{100} = 0.33$ iv $\frac{9}{100} = 0.09$
 b i $\frac{1}{4} = 0.25$ ii $\frac{7}{20} = 0.35$ iii $\frac{3}{10} = 0.3$ iv $\frac{1}{10} = 0.1$

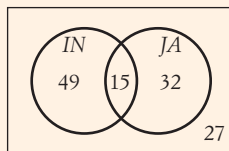
- c Yes
 d Expected frequency = 40. The expected frequency compares very favourably with the observed frequency of 42.
 3 a 50 b Teacher to check.
 c i Teacher to check. ii $\frac{1}{2}$
 d Teacher to check.
 4 a 600
 b i $\frac{281}{600} = 0.468$ ii $\frac{322}{600} = 0.537$
 iii $\frac{227}{600} = 0.378$ iv $\frac{322}{600} = 0.87$
 c i 0.5 ii 0.5 iii 0.33 iv 0.83
 d The probabilities are similar.
 5 a Teacher to check.
 b i $\frac{1}{2} = 0.5$ ii $\frac{1}{5} = 0.2$ iii $\frac{3}{10} = 0.3$ iv $\frac{7}{10} = 0.7$
 c Teacher to check.
 6 a i $\frac{3}{10} = 0.3$ ii $\frac{3}{25} = 0.12$ iii $\frac{12}{25} = 0.48$ iv $\frac{1}{10} = 0.1$
 b i 0.33 ii 0.17 iii 0.33 iv 0.17
 c Yes
 d Expected frequency of not yellow is 33. This is more than the observed frequency of 26.
 7 a $\frac{1}{6} \approx 0.17$ b 16 or 17 times c, d, e Teacher to check.
 8 a 200
 b i $\frac{4}{200} = 0.02$ ii $\frac{27}{200} = 0.135$ iii $\frac{13}{200} = 0.065$
 iv $\frac{86}{200} = 0.43$ v $\frac{87}{200} = 0.435$ vi $\frac{59}{200} = 0.295$
 vii $\frac{51}{200} = 0.255$
 9 a 200
 b i $\frac{27}{200} = 0.135$ ii $\frac{62}{200} = 0.31$ iii $\frac{80}{200} = 0.4$
 iv $\frac{21}{200} = 0.105$ v $\frac{1}{200} = 0.005$
 c Ferry, light rail (tram) d Teacher to check.

Exercise 12-02

- 1 a i $\frac{11}{25}$ ii $\frac{4}{5}$ iii $\frac{1}{5}$ iv $\frac{6}{25}$ v $\frac{19}{25}$ b $\frac{3}{10}$
 2 a 156
 b i $\frac{11}{52}$ ii $\frac{7}{52}$ iii $\frac{7}{78}$ iv $\frac{22}{39}$ v $\frac{19}{78}$ vi $\frac{1}{26}$
 c $\frac{7}{51}$
 3 a 135 b $\frac{56}{135}$ c $\frac{1}{5}$ d $\frac{17}{52}$
 4 a



- b i 1 ii $\frac{11}{54}$ iii $\frac{31}{54}$ iv $\frac{7}{9}$
 5 a 45 b i $\frac{1}{45}$ ii $\frac{1}{3}$ iii $\frac{1}{9}$ iv $\frac{19}{45}$
 c i $\frac{19}{26}$ ii 0
 6 a 123
 b



- c 81 d i $\frac{49}{123}$ ii $\frac{32}{123}$ iii $\frac{27}{123}$ iv $\frac{81}{123}$

- 7 a 200
 b i $\frac{79}{200}$ ii $\frac{51}{100}$ iii $\frac{77}{200}$ iv $\frac{121}{200}$ v $\frac{81}{100}$ vi $\frac{3}{50}$
 c $\frac{29}{40}$
 d No, because all the people surveyed indicated a day on which they preferred to shop.
- 8 a 204
 b i $\frac{7}{204}$ ii $\frac{23}{204}$ iii $\frac{31}{102}$ iv $\frac{71}{102}$
 c i $\frac{2}{54}$ ii $\frac{43}{54}$

Exercise 12-03

- 1 a 150 b i $\frac{4}{25}$ ii $\frac{53}{150}$ iii $\frac{28}{75}$ c 63%
 2 a 128 b i 68 ii 60 c $\frac{17}{32}$ d $\frac{12}{55}$
 3 a 93
 b i 21.5% ii 11.8% iii 3.2%
 c i 15.7% ii 45%
 d The percentage composition of women in the opposition is three times that of the percentage composition of women in the government.
- 4 a 150
 b i 0.5 ii 0.04 iii 0.43 iv 0.23
 c $\frac{22}{75} = 0.293$
 5 a 160
 b i $\frac{7}{20} = 0.35$ ii $\frac{11}{160} = 0.069$ iii $\frac{9}{80} = 0.113$
 c $\frac{35}{82} = 0.43$
 6 a 200 b 55%
 c i 49.5% ii 45% iii 36% iv 31.5%
 d 65.5%
 7 a 878
 b i $\frac{679}{878} = 0.773$ ii $\frac{545}{878} = 0.621$
 iii $\frac{67}{439} = 0.153$ iv $\frac{21}{439} = 0.048$

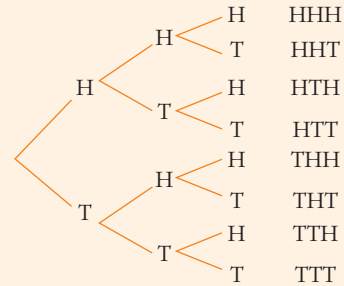
Exercise 12-04

- 1 a CA CG CJ CE CR AC
 AG AJ AE AR GC GA
 GJ GE GR JC JA JG
 JE JR EC EA EG EJ
 ER RC RA RG RJ RE
- b $\frac{1}{3}$ c $\frac{1}{6}$

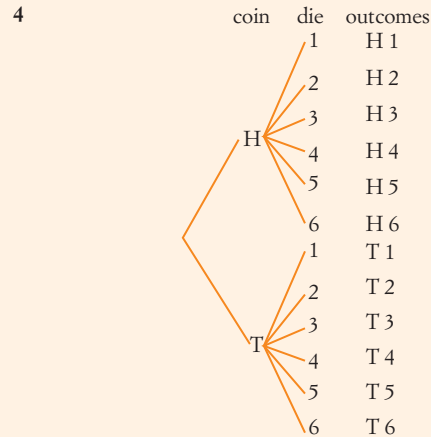
		Second die					
		1	2	3	4	5	6
First die	1	1, 1	1, 2	1, 3	1, 4	1, 5	1, 6
	2	2, 1	2, 2	2, 3	2, 4	2, 5	2, 6
	3	3, 1	3, 2	3, 3	3, 4	3, 5	3, 6
	4	4, 1	4, 2	4, 3	4, 4	4, 5	4, 6
	5	5, 1	5, 2	5, 3	5, 4	5, 5	5, 6
	6	6, 1	6, 2	6, 3	6, 4	6, 5	6, 6

- b 36
 c i $\frac{1}{6}$ ii $\frac{1}{4}$ iii $\frac{11}{36}$ iv $\frac{1}{4}$ v $\frac{1}{2}$ vi $\frac{5}{12}$

- 3 a First Second Third Outcomes



- b 8 c i $\frac{1}{8}$ ii $\frac{3}{8}$ iii $\frac{1}{8}$ iv $\frac{1}{8}$ v $\frac{1}{2}$
 d i $\frac{7}{8}$ ii $\frac{7}{8}$ e i 75 ii 25



- 5 a

		2nd die					
		1	2	3	4	5	6
1st die	1	1, 1	1, 2	1, 3	1, 4	1, 5	1, 6
	2	2, 1	2, 2	2, 3	2, 4	2, 5	2, 6
	3	3, 1	3, 2	3, 3	3, 4	3, 5	3, 6
	4	4, 1	4, 2	4, 3	4, 4	4, 5	4, 6

- b 24 c i $\frac{1}{6}$ ii $\frac{1}{4}$ iii $\frac{1}{2}$ iv $\frac{3}{8}$ v 0

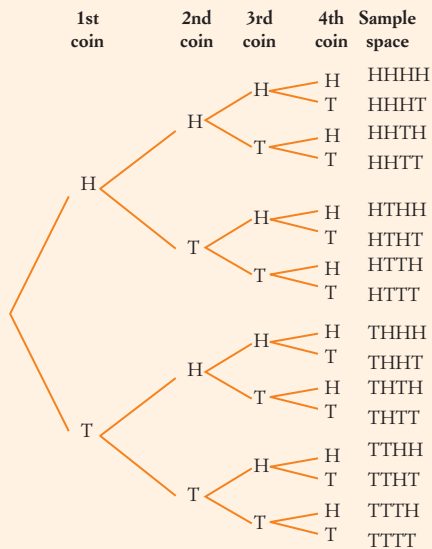
- 6 a

		2nd die					
		1	2	3	4	5	6
1st die	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

- b i $\frac{1}{9}$ ii $\frac{1}{36}$ iii $\frac{1}{6}$ iv $\frac{1}{2}$
 v 0 vi $\frac{5}{12}$ vii $\frac{7}{12}$ viii $\frac{5}{12}$

Answers

7 a



- b i $\frac{1}{16}$ ii $\frac{1}{4}$ iii $\frac{3}{8}$
 iv $\frac{15}{16}$ v $\frac{1}{16}$ vi $\frac{5}{16}$
 c i 63 ii 375 iii 937

Mental skills 12

- 2 a \$700 b \$800 c 400 d 700 e \$300
 f \$400 g 250 h \$300 i \$300 j 500

Exercise 12-05

1 a

		Girls					
		Be	Ca	Em	M	R	S
Boys	B	B, Be	B, Ca	B, Em	B, M	B, R	B, S
	C	C, Be	C, Ca	C, Em	C, M	C, R	C, S
	Ew	Ew, Be	Ew, Ca	Ew, Em	Ew, M	Ew, R	Ew, S
	W	W, Be	W, Ca	W, Em	W, M	W, R	W, S

- b i $\frac{1}{24}$ ii $\frac{1}{6}$ iii $\frac{1}{6}$

2 a i

	A	B	C	D	E
A	AA	AB	AC	AD	AE
B	BA	BB	BC	BD	BE
C	CA	CB	CC	CD	CE
D	DA	DB	DC	DD	DE
E	EA	EB	EC	ED	EE

ii

	A	B	C	D	E
A		AB	AC	AD	AE
B	BA		BC	BD	BE
C	CA	CB		CD	CE
D	DA	DB	DC		DE
E	EA	EB	EC	ED	

- b i $\frac{1}{5}$ ii $\frac{4}{25}$ iii $\frac{12}{25}$
 c i $\frac{1}{10}$ ii $\frac{3}{5}$ iii $\frac{2}{5}$ iv $\frac{4}{5}$

3 a

	H	T
H	HH	HT
T	TH	TT

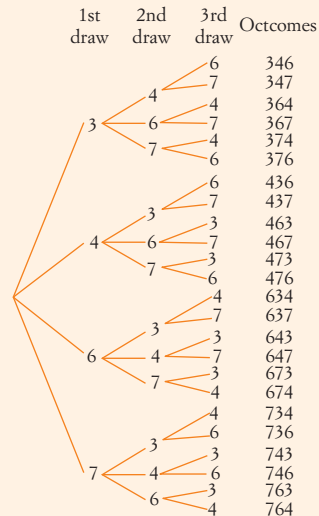
- b i $\frac{1}{4}$ ii $\frac{1}{2}$ iii $\frac{3}{4}$

4 a

		2nd course				
		B	H	P	S	T
1st course	C	CB	CH	CP	CS	CT
	F	FB	FH	FP	FS	FT
	Y	YB	YH	YP	YS	YT

- b i $\frac{1}{3}$ ii $\frac{4}{15}$ iii $\frac{1}{15}$

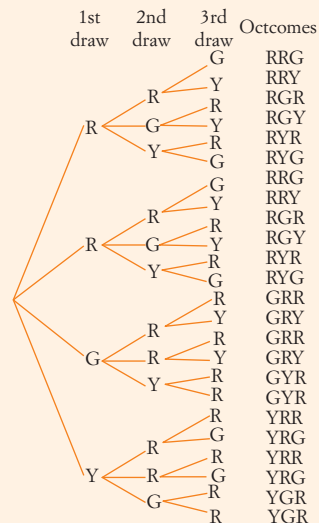
5 a



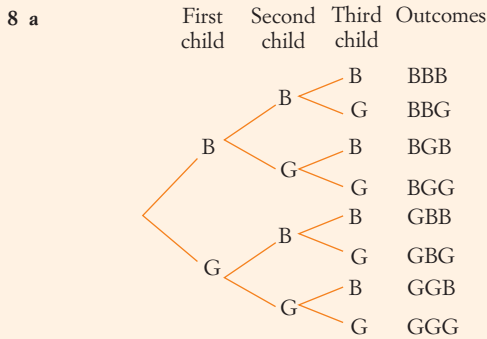
- b $\frac{1}{2}$ ii $\frac{3}{4}$ iii $\frac{1}{2}$ iv $\frac{1}{3}$
 6 a Teacher to check. 64 outcomes, beginning with 333, 334, 336, 337, 343, 344, 346, 347, ..., 776, 777.

- b i $\frac{1}{16}$ ii $\frac{1}{2}$ iii $\frac{1}{2}$ iv $\frac{1}{16}$

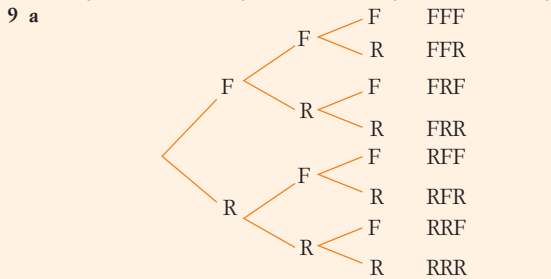
7 a



- b i $\frac{1}{2}$ ii $\frac{1}{12}$ iii 1



b i $\frac{1}{8}$ ii $\frac{1}{8}$ iii $\frac{3}{8}$ iv $\frac{1}{8}$



b i $\frac{1}{8}$ ii $\frac{3}{8}$ iii $\frac{1}{8}$ iv $\frac{7}{8}$

10 a 36

11 a i 125 outcomes. Teacher to check.

ii 60 outcomes. Teacher to check.

b i $\frac{29}{125}$ ii $\frac{18}{125}$ iii $\frac{64}{125}$ iv $\frac{117}{125}$

c i $\frac{1}{10}$ ii $\frac{3}{10}$ iii 0 iv $\frac{2}{5}$

Exercise 12-06

1 a independent b independent c dependent
d independent e dependent f independent
g dependent

2 Dependent, as the balls are not replaced when drawn.

3 a independent b $\frac{1}{2}$
4 a i $\frac{1}{3}$ ii $\frac{1}{4}$
b 1Y, 2Y, 3Y, 4Y, 5Y, 6Y, 1G, 2G, 3G, 4G, 5G, 6G, 1B, 2B, 3B, 4B, 5B, 6B, 1R, 2R, 3R, 4R, 5R, 6R,
c $\frac{1}{12}$ d Yes, $\frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$ e independent

5 a i $\frac{1}{6}$ ii $\frac{1}{2}$ b independent
6 a $\frac{2}{9}$ b $\frac{4}{8} = \frac{1}{2}$
c Dependent, as the first draw changes the contents of the bag.

7 a i $\frac{5}{8}$ ii $\frac{4}{7}$ b i $\frac{3}{8}$ ii $\frac{5}{7}$
c i $\frac{3}{8}$ ii $\frac{3}{7}$ d i $\frac{3}{8}$ ii $\frac{2}{7}$

8 $\frac{1}{2}$

Exercise 12-07

1 a $\frac{1}{3}$ b $\frac{3}{6} = \frac{1}{2}$ 2 $\frac{2}{9}$
3 a $\frac{4}{11}$ b $\frac{7}{11}$ c $\frac{4}{11}$ d $\frac{5}{11}$
4 a $\frac{2}{11}$ b $\frac{4}{11}$ c $\frac{4}{11}$ d $\frac{5}{11}$
5 $\frac{1}{3}$ 6 $\frac{1}{6}$

7 a

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

b i $\frac{1}{2}$ ii 1 c $\frac{2}{11}$
d i $\frac{9}{27} = \frac{1}{3}$ ii $\frac{6}{27} = \frac{2}{9}$ e $\frac{1}{6}$

8 a i $\frac{1}{9}$ ii $\frac{8}{9}$ b $\frac{2}{8} = \frac{1}{4}$ c 6

9 $\frac{1}{40}$ 10 $\frac{10}{49}$ 11 $\frac{1}{13}$ 12 $\frac{1}{6}$

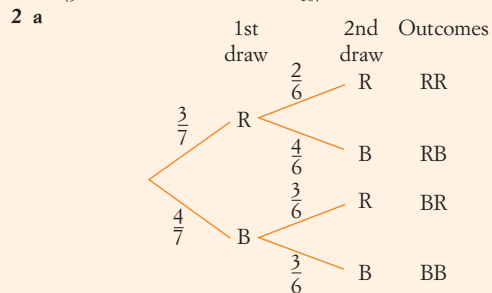
13 a

	1	2	3	4	5	6
1	0	1	2	3	4	5
2	1	0	1	2	3	4
3	2	1	0	1	2	3
4	3	2	1	0	1	2
5	4	3	2	1	0	1
6	5	4	3	2	1	0

b i $\frac{1}{6}$ ii $\frac{1}{18}$ iii $\frac{1}{6}$ c i $\frac{1}{2}$ ii 0
d i $\frac{1}{11}$ ii $\frac{4}{11}$ e i $\frac{6}{18} = \frac{1}{3}$ ii 1

Power plus

1 a 320
b i $\frac{7}{40} = 0.175$ ii $\frac{3}{32} = 0.094$
iii $\frac{27}{320} = 0.178$ iv $\frac{3}{20} = 0.15$
c $\frac{16}{45}$ d $\frac{45}{107}$



b i $\frac{1}{7}$ ii $\frac{2}{7}$ iii $\frac{4}{7}$ iv $\frac{6}{7}$
3 a i $\frac{14}{30} = \frac{7}{15}$ ii $\frac{20}{30} = \frac{2}{3}$ iii $\frac{4}{30} = \frac{2}{15}$
iv $\frac{4}{20} = \frac{1}{5}$ v $\frac{4}{14} = \frac{2}{7}$
b i $\frac{2}{3} = \frac{1}{5}$ ii Yes
c $P(B|A) = \frac{2}{7}, \frac{P(A \text{ and } B)}{P(A)} = \frac{\frac{2}{15}}{\frac{7}{15}} = \frac{2}{7}$

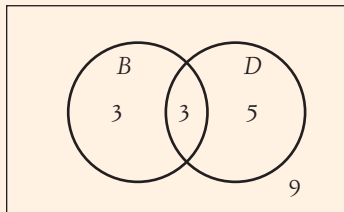
Chapter 12 revision

1 a i 0.353 ii 0.427 iii 0.087 iv 0.513
b i $\frac{13}{15} = 0.867$ ii $\frac{2}{15} = 0.133$
c Different – at least one head occurring excludes zero heads occurring, which is the same as three tails occurring. The events are complementary.

Answers

- 2 a 35 b $\frac{2}{35}$
 c i $\frac{6}{35}$ ii $\frac{19}{35}$ iii $\frac{3}{35}$
 d They don't like the types of music mentioned in the survey.

3 a



- b i $\frac{3}{20}$ ii $\frac{2}{5}$ iii $\frac{3}{20}$ iv $\frac{3}{5}$
 c $\frac{9}{20}$
 4 a 200
 b i 0.305 ii 0.11 iii 0.225 iv 0.425
 c $\frac{25}{110} = 0.227$
 d i $\frac{47}{69} = 0.681$ ii $\frac{22}{69} = 0.319$

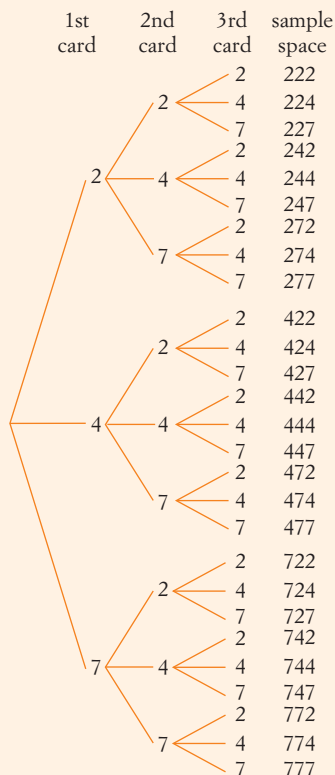
5 a

	1	2	3	4
1	1, 1	1, 2	1, 3	1, 4
2	2, 1	2, 2	2, 3	2, 4
3	3, 1	3, 2	3, 3	3, 4
4	4, 1	4, 2	4, 3	4, 4

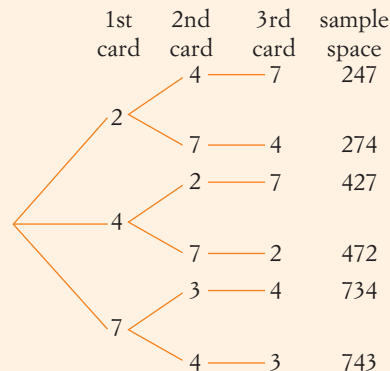
b 16

- c i $\frac{1}{2}$ ii $\frac{1}{4}$ iii $\frac{7}{16}$ iv $\frac{1}{4}$ v $\frac{1}{4}$ vi $\frac{1}{2}$

6 a i



ii



- b i $\frac{2}{3}$ ii $\frac{1}{3}$ iii $\frac{1}{9}$ iv $\frac{2}{9}$
 c i $\frac{1}{3}$ ii $\frac{2}{3}$ iii $\frac{1}{3}$ iv $\frac{1}{6}$
 7 a independent b dependent c dependent
 d independent e independent

8 a

	1	2	3	4
1	2	3	4	5
2	3	4	5	6
3	4	5	6	7
4	5	6	7	8

- b i $\frac{1}{2}$ ii 1 c $\frac{2}{7}$ d i $\frac{1}{3}$ ii $\frac{1}{3}$ e $\frac{1}{4}$

Chapter 13

SkillCheck

- 1 a $h = 71$ b $p = 105$ c $x = 126$
 d $m = 68$ e $a = 24$ f $w = 36$
 g $w = 30, k = 90$ h $r = 83$ i $p = 26, r = 98$
 j $p = 52$ k $y = 42$ l $d = 54$
 2 a $m = 65$ (angles on a straight line), $n = 65$ (alternate angles), $p = 50$ (angle sum of $\triangle XWY$)
 b Isosceles triangle ($m = n = 65$)

Exercise 13-01

- 1 a 1800° b 1440° c 1260° d 3240° e 2340°
 2 a 6 b 21 c 13 d 30 e 9
 3 a 16 b 157.5°
 4 a 144° b 135° c 120° d 150°
 5 a 30 b 15 c 45 d 25
 6 a 72° b 30° c 20° d 60°
 7 a 140° b 162° c 144° d 168°
 8 a 24 b 5 c 18 d 9 e 72 f 30
 9 a 8 b 10 c 15 d 180 e 24 f 12

Exercise 13-02

- 1 In $\triangle ABE$ and $\triangle CBD$:
 $\angle ABE = \angle CBD$ (vertically opposite angles)
 $AB = CB$ (given)
 $EB = DB$ (given)
 $\therefore \triangle ABE \equiv \triangle CBD$ (SAS)

- 2 In $\triangle LMP$ and $\triangle NPM$:
 $LP = NM$ (given)
 $LM = NP$ (given)
 MP is common.
 $\therefore \triangle LMP \equiv \triangle NPM$ (SSS)
- 3 In $\triangle QTW$ and $\triangle PWT$:
 $\angle QTW = \angle PWT = 90^\circ$ ($QT \perp WT$ and $PW \perp WT$)
 $QW = PT$ (given)
 WT is common.
 $\therefore \triangle QTW \equiv \triangle PWT$ (RHS)
- 4 In $\triangle ABY$ and $\triangle CBX$:
 $\angle BAY = \angle BCX = 90^\circ$ (equal angles of the square $ABCD$)
 $BA = BC$ (equal sides of the square $ABCD$)
 $AY = CX$ (given)
 $\therefore \triangle ABY \equiv \triangle CBX$ (SAS)
- 5 In $\triangle CDE$ and $\triangle FED$:
 $\angle CDE = \angle FED$ (given)
 $\angle DEC = \angle EDF$ (equal angles opposite equal sides of isosceles $\triangle DEY$)
 DE is common.
 $\therefore \triangle CDE \equiv \triangle FED$ (AAS)
- 6 In $\triangle XOY$ and $\triangle VOW$:
 $\angle XOY = \angle VOW$ (vertically opposite angles)
 $OX = OV$ (equal radii of small circle)
 $OY = OW$ (equal radii of large circle)
 $\therefore \triangle XOY \equiv \triangle VOW$ (SAS)
- 7 In $\triangle KLM$ and $\triangle MNK$:
 $\angle MKL = \angle KMN$ (alternate angles, $NM \parallel KL$)
 $\angle KML = \angle MKN$ (alternate angles, $KN \parallel LM$)
 KM is common.
 $\therefore \triangle KLM \equiv \triangle MNK$ (AAS)
- 8 In $\triangle CDH$ and $\triangle EFG$:
 $\angle HCD = \angle GEF$ (corresponding angles, $CH \parallel EG$)
 $\angle HDC = \angle GFE$ (corresponding angles, $DH \parallel FG$)
 $CH = EG$ (given)
 $\therefore \triangle CDH \equiv \triangle EFG$ (AAS)
- 9 In $\triangle UXY$ and $\triangle WXY$:
 $\angle UXY = \angle WXY$ (YX bisects $\angle UXW$)
 $UX = WX$ (given)
 XY is common.
 $\therefore \triangle UXY \equiv \triangle WXY$ (SAS)
- 10 In $\triangle ABE$ and $\triangle CBD$:
 $\angle AEB = \angle CDB = 90^\circ$ ($AE \perp BC$, $CD \perp AB$)
 $\angle B$ is common.
 $BA = BC$ (given)
 $\therefore \triangle ABE \equiv \triangle CBD$ (AAS)
- 11 a In $\triangle HEF$ and $\triangle GFE$:
 $\angle HEF = \angle GFE$ (given)
 $EH = FG$ (given)
 EF is common.
 $\therefore \triangle HEF \equiv \triangle GFE$ (SAS)
- b $\therefore \angle EHF = \angle FGE$ (matching angles of congruent triangles)
- 12 a In $\triangle AOB$ and $\triangle COD$:
 $AB = CD$ (given)
 $OA = OC$ (equal radii)
 $OB = OD$ (equal radii)
 $\therefore \triangle AOB \equiv \triangle COD$ (SSS)
- b $\therefore \angle AOB = \angle COD$ (matching angles of congruent triangles)
- 13 a In $\triangle QRX$ and $\triangle QTY$:
 $QR = QT$ (equal sides of isosceles $\triangle QRT$)
 $RX = TY$ (given)
 $\angle R = \angle T$ (equal angles of isosceles $\triangle QRT$)
 $\therefore \triangle QRX \equiv \triangle QTY$ (SAS)
- b $\therefore QX = QY$ (matching sides of congruent triangles)
 $\therefore \triangle QXY$ is isosceles (two sides of the triangle are equal)
- 14 a In $\triangle TAP$ and $\triangle XCP$:
 $\angle TPA = \angle XPC$ (vertically opposite angles)
 $TP = XP$ (given)
 $AP = CP$ (given)
 $\therefore \triangle TAP \equiv \triangle XCP$ (SAS)
- b $\therefore TA = XC$ (matching sides of congruent triangles)
- c $\therefore \angle A = \angle C$ (matching angles of congruent triangles)
 $\therefore TA \parallel XC$ (alternate angles are equal)
- 15 a In $\triangle OAM$ and $\triangle OBM$:
 $OA = OB$ (equal radii)
 $\angle OMA = \angle OMB = 90^\circ$ ($OM \perp AB$)
 OM is common.
 $\therefore \triangle OAM \equiv \triangle OBM$ (RHS)
- b $AM = BM$ (matching sides of congruent triangles)
 $\therefore OM$ bisects AB .
- 16 a In $\triangle GLH$ and $\triangle K LH$:
 $GL = KL$ (given)
 $GH = KH$ (given)
 LH is common.
 $\therefore \triangle GLH \equiv \triangle K LH$ (SSS)
- b $\angle GLH = \angle K LH$ (matching angles of congruent triangles)
 $\therefore LH$ bisects $\angle GLK$.
 $\angle GHL = \angle KHL$ (matching angles of congruent triangles)
 $\therefore LH$ bisects $\angle GHK$.

Exercise 13-03

- 1 $\angle A = \angle C$ and $\angle B = \angle D$
 Now $\angle A + \angle C + \angle B + \angle D = 360^\circ$ (angle sum of a quadrilateral)
 $\therefore 2\angle A + 2\angle B = 360^\circ$ ($\angle C = \angle A$, $\angle D = \angle B$)
 $\therefore \angle A + \angle B = 180^\circ$
 $\therefore AD \parallel BC$ (the pair of co-interior angles have a sum of 180°)
 Also, from $\angle A + \angle C + \angle B + \angle D = 360^\circ$ (angle sum of a quadrilateral)
 $\therefore 2\angle A + 2\angle D = 360^\circ$ ($\angle C = \angle A$, $\angle B = \angle D$)
 $\therefore \angle A + \angle D = 180^\circ$
 $\therefore AB \parallel DC$ (the pair of co-interior angles have a sum of 180°)
 $\therefore ABCD$ is a parallelogram (opposite sides are parallel).
- 2 In $\triangle LMP$ and $\triangle NPM$:
 $LM = NP$ (given)
 PM is common.
 $\angle LMP = \angle NPM$ (alternate angles, $LM \parallel NP$)
 $\therefore \triangle LMP \equiv \triangle NPM$ (SAS)
 $\angle LPM = \angle NMP$ (matching angles of congruent triangles)

Answers

- $\therefore LP \parallel NM$ (alternate angles proved equal)
 $\therefore LMNP$ is a parallelogram (opposite sides are parallel).
- 3 In $\triangle DXH$ and $\triangle GXE$:
 $\angle DXH = \angle GXE$ (vertically opposite angles)
 $HX = EX$ (given)
 $DX = GX$ (given)
 $\therefore \triangle DXH \equiv \triangle GXE$ (SAS)
 $\angle HDX = \angle EGX$ (matching angles of congruent triangles)
 $\therefore HD \parallel EG$ (alternate angles proved equal)
 Similarly, $\angle GHX = \angle DEX$ (matching angles of congruent triangles HXG and EXD)
 $\therefore HG \parallel ED$ (alternate angles proved equal)
 $\therefore DEGH$ is a parallelogram (opposite sides are parallel).
- 4 (Outline of proof only) $\triangle FHC \equiv \triangle FHE \equiv \triangle DHE \equiv \triangle DHC$ (SAS)
 $\therefore FC = FE = DE = DC$ (matching sides of congruent triangles)
 Also, $\angle CFH = \angle EDH$ and $\angle CDH = \angle EFH$ (matching angles of congruent triangles)
 $\therefore CDEF$ is a rhombus (opposite sides are parallel and all sides are equal).
- 5 (Outline of proof only) Since $WY = XV$ and the diagonals bisect each other,
 $TW = TV = TY = TX$
 $\therefore \triangle TWV \equiv \triangle TXY$ (SAS) and $\triangle TVY \equiv \triangle TWX$ (SAS)
 $\therefore \angle VWT = \angle XYT$ and $\angle TVY = \angle TXW$ (matching angles of congruent triangles)
 $\therefore VW \parallel YX$ and $VY \parallel XW$ (alternate angles proved equal)
 $\therefore VWXY$ is a parallelogram.
 Also, $\triangle YVW \equiv \triangle XYV \equiv \triangle VWX \equiv \triangle YXW$ (AAS)
 $\therefore \angle V = \angle W = \angle X = \angle Y$ (matching angles of congruent triangles)
 Since the angle sum of $VWXY = 360^\circ$
 $\angle V = \angle W = \angle X = \angle Y = 90^\circ$
 $\therefore VWXY$ is a rectangle.
- 6 $\angle B + \angle C = 180^\circ$ and $\angle B + \angle E = 180^\circ$
 $\therefore BE \parallel CD$ and $BC \parallel ED$ (pairs of co-interior angles have a sum of 180°)
 $\therefore BCDE$ is a parallelogram with right angles.
 $\therefore BCDE$ is a rectangle.
- 7 Since the sides are equal, $TWME$ is a rhombus (proved in question 4).
 Since $\angle M = 90^\circ$, $TWME$ is a square (a square is a rhombus with a right angle).
- 8 Since the angles of the quadrilateral are right angles, $GHLK$ is a rectangle (proved in question 6).
 If $GL = GH$,
 $GL = GH = LK = KH$ (opposite sides of a rectangle are equal)
 $\therefore GHLK$ is a square (all sides are equal, all angles are 90°).
- 9 The diagonals bisect each other at right angles, so $MNPT$ is a rhombus (proved in question 4).
 $\therefore MN = NP = PT = MT$
 Also, $\triangle MNT \equiv \triangle NPT \equiv \triangle PTM \equiv \triangle MNP$ (SSS, since $TN = PM$)
 $\therefore \angle M = \angle N = \angle P = \angle T = 90^\circ$ (angle sum of a quadrilateral and matching angles of congruent triangles)
 $\therefore MNPT$ is a square.
- 10 a In $\triangle ABX$ and $\triangle CDY$:
 $BX = DY$ (given)
 $\angle B = \angle D$ (opposite angles of a parallelogram)
 $AB = CD$ (opposite sides of a parallelogram)
 $\therefore \triangle ABX \equiv \triangle CDY$ (SAS)
- b $AX = CY$ (matching sides of congruent triangles)
 $XC = BC - BX$
 $= AD - DY$ ($BC = AD$, opposite sides of a parallelogram and $BX = DY$, given)
 $= AY$
 $\therefore AXCY$ is a parallelogram as pairs of opposite sides are equal.
- 11 a In $\triangle DAE$ and $\triangle CEB$:
 $AE = EB$ (given)
 $\angle DAE = \angle CEB$ (corresponding angles, $AD \parallel EC$)
 $AD = EC$ (equal sides of a rhombus)
 $\therefore \triangle DAE \equiv \triangle CEB$ (SAS)
- b $ED = BC$ (matching sides of congruent triangles from a)
 $AE = DC$ (equal sides of a rhombus) and $AE = EB$ (given)
 $\therefore DC = EB$
 $\therefore BCDE$ is a parallelogram because its opposite sides are equal.
- 12 a In $\triangle APS$ and $\triangle CRQ$:
 $AP = CR$ (given)
 $AS = CQ$ (given)
 $\angle A = \angle C$ (opposite angles of a parallelogram)
 $\therefore \triangle APS \equiv \triangle CRQ$ (SAS)
 $\therefore PS = QR$ (matching sides of congruent triangles)
 In $\triangle PBQ$ and $\triangle RDS$:
 $PB = RD$ (given $AP = CR$ and opposite sides of a parallelogram)
 $\angle B = \angle D$ (opposite angles of a parallelogram)
 $BQ = DS$ (given $CQ = AS$ and opposite sides of a parallelogram)
 $\therefore \triangle PBQ \equiv \triangle RDS$ (SAS)
 $\therefore PQ = RS$ (matching sides of congruent triangles)
- b $PQRS$ is a parallelogram because pairs of opposite sides are equal.
- 13 AC and DB are diagonals.
 $OD = OB$ (equal radii of small circle)
 $OA = OC$ (equal radii of large circle)
 $\therefore ABCD$ is a parallelogram because the diagonals bisect each other.
- 14 SQ and PR are diagonals.
 $OP = OR$ (equal radii of small circle)
 $OS = OQ$ (equal radii of large circle)
 $PR \perp SQ$ (given)
 $\therefore PQRS$ is a rhombus because its diagonals bisect each other at right angles.
- 15 Since $WD = WE = GY = YF$ (W and G are the midpoints of equal sides DE and GF)
 and $DZ = ZG = EX = XF$ (Z and X are the midpoints of equal sides DG and EF)
 $\therefore WZ = WX = YX = ZY$ (by Pythagoras' theorem)
 $\therefore WZYX$ is a rhombus (a quadrilateral with all sides equal)
- 16 In $\triangle APT$ and $\triangle CRQ$:
 $AT = CQ$ (T and Q are the midpoints of equal sides of a parallelogram)

$AP = CR$ (P and R are the midpoints of equal sides of a parallelogram)
 $\angle A = \angle C$ (opposite angles of a parallelogram)
 $\therefore \triangle APT \equiv \triangle CRQ$ (SAS)
 $\therefore PT = RQ$ (matching sides of congruent triangles)
 Similarly, proving $\triangle DRT$ and $\triangle BPQ$ congruent (SAS),
 $TR = QP$ (matching sides of congruent triangles)
 $\therefore PQRT$ is a parallelogram because its opposite sides are equal.

Exercise 13-04

- 1 a In $\triangle ABD$ and $\triangle ACD$:
 $AB = AC$ (given)
 $BD = CD$ (given)
 AD is common.
 $\therefore \triangle ABD \equiv \triangle ACD$ (SSS)
- b $\angle ADB = \angle ADC$ (matching angles of congruent triangles)
- c $\angle ADB + \angle ADC = 180^\circ$ (angles on a straight line)
 $\therefore \angle ADB = \angle ADC = 90^\circ$
 $\therefore AD \perp BC$
- 2 a In $\triangle KMP$ and $\triangle KNP$:
 $KM = KN$ (given)
 $\angle KPM = \angle KPN = 90^\circ$ ($KP \perp MN$)
 KP is common.
 $\therefore \triangle KMP \equiv \triangle KNP$ (RHS)
- b $MP = NP$ (matching sides of congruent triangles)
 $KP \perp MN$ (given)
 $\therefore$ Perpendicular from vertex K to MN bisects MN .
- 3 a In $\triangle ABX$ and $\triangle CDX$:
 $\angle ABX = \angle CDX$ (alternate angles, $AB \parallel CD$)
 $\angle BAX = \angle DCX$ (alternate angles, $AB \parallel CD$)
 $AB = CD$ (opposite sides of a rectangle)
 $\therefore \triangle ABX \equiv \triangle CDX$ (AAS)
- b $\therefore AX = CX$ and $BX = DX$ (matching sides of congruent triangles)
 $\therefore X$ is the midpoint of diagonals AB and CD .
 $\therefore$ The diagonals of a rectangle bisect each other.
- 4 a In $\triangle DEG$ and $\triangle FGE$:
 $\angle DEG = \angle FGE$ (alternate angles, $DE \parallel FG$)
 $\angle DGE = \angle FEG$ (alternate angles, $DG \parallel FE$)
 GE is common.
 $\therefore \triangle DEG \equiv \triangle FGE$ (AAS)
- b In $\triangle DGF$ and $\triangle FED$:
 $\angle DFG = \angle FDE$ (alternate angles, $FG \parallel DE$)
 $\angle FDG = \angle DFE$ (alternate angles, $DG \parallel FE$)
 DF is common.
 $\therefore \triangle DGF \equiv \triangle FED$ (AAS)
- c $\angle GDE = \angle EFG$ (matching angles of congruent triangles DEG and FGE)
 $\angle DGF = \angle FED$ (matching angles of congruent triangles DFG and FED)
 $\therefore$ Opposite angles of a parallelogram are equal.
- 5 a In $\triangle BED$ and $\triangle BCD$:
 $BE = BC$ (given)
 $DE = DC$ (given)
 BD is common.
 $\therefore \triangle BED \equiv \triangle BCD$ (SSS)
- b $\angle EBD = \angle CBD$ (matching angles of congruent triangles)
 $\angle EDB = \angle CDB$ (matching angles of congruent triangles)
 $\therefore$ Diagonal BD bisects $\angle EBC$ and $\angle EDC$.
- 6 a In $\triangle LXM$ and $\triangle NXP$:
 $\angle MLX = \angle PNX$ (alternate angles, $LM \parallel NP$)
 $\angle LMX = \angle NPX$ (alternate angles, $LM \parallel NP$)
 $LM = NP$ (given)
 $\therefore \triangle LXM \equiv \triangle NXP$ (AAS)
- b $LX = NX$ and $MX = PX$ (matching sides of congruent triangles)
 $\therefore$ Diagonals of a parallelogram bisect each other.
- 7 a In $\triangle UAW$ and $\triangle XAW$:
 $UW = XW$ (equal sides of a rhombus)
 AW is common.
 $\angle UWA = \angle XWA$ (diagonals of a rhombus bisect the angles)
 $\therefore \triangle UAW \equiv \triangle XAW$ (SAS)
- b In $\triangle UAW$ and $\triangle UAY$:
 $UW = UY$ (given)
 AU is common.
 $\angle WUA = \angle YUA$ (diagonals of a rhombus bisect the angles)
 $\therefore \triangle UAW \equiv \triangle UAY$ (SAS)
- c $UA = XA$ (matching sides of congruent triangles UAW and XAW)
 $WA = YA$ (matching sides of congruent triangles UAW and UAY)
 $\therefore$ Diagonals bisect each other.
 $\angle UAW = \angle XAW$ (matching angles of congruent triangles UAW and XAW).
 But $\angle UAW + \angle XAW = 180^\circ$
 $\therefore \angle UAW = \angle XAW = 90^\circ$
 $\therefore WA \perp UX$
 $\therefore$ Diagonals WY and UX are perpendicular.
 $\therefore$ Diagonals bisect each other at right angles.
- 8 a In $\triangle DXF$ and $\triangle EXF$:
 $\angle D = \angle E$ (given)
 $\angle DXF = \angle EXF = 90^\circ$ ($FX \perp DE$)
 FX is common.
 $\therefore \triangle DXF \equiv \triangle EXF$ (AAS)
- b $FD = FE$ (matching sides of congruent triangles)
 Also, FD is opposite $\angle E$ and FE is opposite $\angle D$.
 $\therefore$ Sides opposite the equal angles in a triangle are equal.
- 9 a Join X to B .
 In $\triangle XBW$ and $\triangle XBY$:
 $XW = XY$ (equal sides of equilateral $\triangle XYW$)
 $WB = YB$ (B is the midpoint of WY)
 XB is common.
 $\therefore \triangle XBW \equiv \triangle XBY$ (SSS)
- b Join Y to A .
 In $\triangle YAX$ and $\triangle YAW$:
 $YX = YW$ (equal sides of equilateral $\triangle XYW$)
 $XA = WA$ (A is the midpoint of XW)
 YA is common.
 $\therefore \triangle YAX \equiv \triangle YAW$ (SSS)

Answers

- c $\angle W = \angle Y$ (matching angles of congruent triangles XBW and XY)
 $\angle X = \angle W$ (matching angles of congruent triangles YAX and YAW)
 $\therefore \angle W = \angle Y = \angle X$
 But $\angle W + \angle Y + \angle X = 180^\circ$
 $\therefore \angle W = \angle Y = \angle X = 60^\circ$

Mental skills 13

2 a 12:25 p.m.	b 1:10 a.m.	c 10:50 p.m.	d 10:55 p.m.
e 0610	f 0010	g 9:10 a.m.	h 3:15 a.m.
i 1100	j 2305	k 12:20 a.m.	l 11:35 a.m.
4 a 6:05 p.m.	b 6:40 a.m.	c 12:10 p.m.	d 2:50 a.m.
e 1245	f 0355	g 10:50 p.m.	h 12:15 p.m.
i 1545	j 0400		

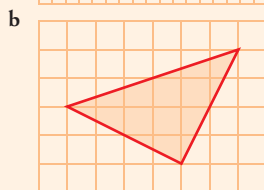
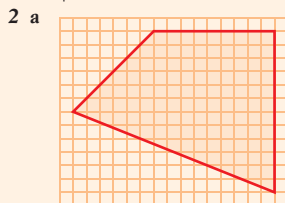
Exercise 13-05

- $\angle LMK = \angle LKM = 45^\circ$ (angle sum of a right-angled isosceles $\triangle KML$)
 $\therefore \angle PMN = 135^\circ$ (angles on a straight line)
 $\therefore 2x + 135 = 180$ (angle sum of isosceles $\triangle PMN$)
 $\therefore x = 22.5$
- $\angle ABC = 42^\circ$ (equal angles of isosceles $\triangle ABC$)
 $\therefore \angle BCD = \angle ABC = 42^\circ$ (alternate angles, $CE \parallel AB$)
 $\therefore \angle DBC = \angle BCD = 42^\circ$ (equal angles of isosceles $\triangle BCD$)
 $\angle EDB = \angle ABD$ (alternate angles, $CE \parallel AB$)
 $= \angle ABC + \angle DBC$
 $= 42^\circ + 42^\circ$
 $\therefore m = 84^\circ$
- $\angle NKL + 93^\circ = 147^\circ$ (exterior angle of $\triangle NKL$ equal to sum of interior opposite angles)
 $\therefore \angle NKL = 54^\circ$
 $\therefore \angle NKH = 54^\circ$ (NK bisects $\angle HKL$)
 $\therefore \angle HKL = 108^\circ$
 $\angle NHK + 108^\circ = 147^\circ$ (exterior angle of $\triangle HKL$ equal to sum of interior opposite angles)
 $\therefore \angle NHK = 39^\circ$
- $\angle EDC = 180^\circ - 116^\circ$ (co-interior angles, $BC \parallel ED$)
 $= 64^\circ$
 $\therefore x = 64 \div 2$ (diagonals bisect the angles of a rhombus)
 $= 32$
- $\angle AED = \angle ABC$ (corresponding angles, $ED \parallel BC$)
 $\angle ADE = \angle ACB$ (corresponding angles, $ED \parallel BC$)
 But $\angle ABC = \angle ACB$ (equal angles of isosceles $\triangle ABC$)
 $\therefore \angle AED = \angle ADE$
 $\therefore \triangle AED$ is isosceles (two equal angles)
- Let $\angle XYP = x^\circ$, $\angle TWP = y^\circ$
 $\therefore \angle PYW = x^\circ$ (YP bisects $\angle XYW$) and $\angle PWY = y^\circ$ (WP bisects $\angle TWY$)
 $\therefore 2x^\circ + 2y^\circ = 180^\circ$ (co-interior angles, $YX \parallel WT$)
 $\therefore x^\circ + y^\circ = 90^\circ$
 But $x^\circ + y^\circ + \angle YPW = 180^\circ$ (angle sum of $\triangle YPW$)
 $\therefore 90^\circ + \angle YPW = 180^\circ$
 $\therefore \angle YPW = 90^\circ$
- $TZ = TY$ (given)
 $\therefore \triangle TZY$ is isosceles.
 $\therefore \angle TZY = \angle TYZ$ (angles opposite equal sides)
 $TZ = UX$ (given)
 and $TZ = WX$ (equal opposite sides of a parallelogram)
 $\therefore UX = WX$
 $\therefore \triangle XUW$ is isosceles.
 $\therefore \angle XUW = \angle XWU$ (angles opposite equal sides)
 But $\angle TZY = \angle XWU$ (equal opposite angles of a parallelogram)
 $\therefore \angle TZY = \angle TYZ = \angle XUW = \angle XWU$
 In $\triangle TZY$ and $\triangle XWU$:
 $TZ = XU$ (given)
 $\angle TZY = \angle XWU$ (opposite angles of a parallelogram)
 $\angle TYZ = \angle XUW$ (proven)
 $\therefore \triangle TZY \equiv \triangle XWU$ (AAS)
 - $TY = UX$ (given)
 $ZY = UW$ (matching sides of congruent triangles)
 $TW = ZX$ (opposite sides of a parallelogram)
 $\therefore TU = TW - UW$
 $= ZX - ZY$
 $= XY$
 $\therefore TUXY$ is a parallelogram (opposite sides equal).
- In $\triangle MNY$ and $\triangle TMW$:
 $MN = MT$ (equal sides of a square)
 $MY = TW$ (Y and W are the midpoints of equal sides of a square)
 $\angle NMY = \angle MTW = 90^\circ$ (angles in a square)
 $\therefore \triangle MNY \equiv \triangle TMW$ (SAS).
 - $\angle MNY = \angle TMW = x^\circ$ (matching angles of congruent triangles)
 $\therefore \angle NMX = 90^\circ - x^\circ$
 $\therefore \angle MXN = 180^\circ - x^\circ - (90^\circ - x^\circ)$
 $= 90^\circ$
 $\therefore MW \perp NY$
- $\angle BCD = \angle BDC$ (equal angles of isosceles $\triangle BCD$)
 $\therefore \angle ABD = \angle BCD + \angle BDC$ (exterior angle of $\triangle BCD$ equal to sum of interior opposite angles)
 $= 2\angle BCD$
 But $\angle ABD = \angle AED$ (opposite angles of a parallelogram)
 $\therefore \angle AED = 2\angle BCD$
- In $\triangle ADB$ and $\triangle ACB$:
 $AD = AC$ (equal radii of large circle)
 $BC = BD$ (equal radii of small circle)
 AB is common.
 $\therefore \triangle ADB \equiv \triangle ACB$ (SSS)
 - In $\triangle DXB$ and $\triangle CXB$:
 BX is common.
 $BD = BC$ (equal radii)
 $\angle DBX = \angle CBX$ (matching angles of congruent triangles proved in a)
 $\therefore \triangle DXB \equiv \triangle CXB$ (SAS)
 $DX = CX$ (matching sides of congruent triangles proved in b)
- Let $\angle A = \angle B = x^\circ$ (equal angles of isosceles $\triangle ABC$)
 $\therefore \angle ACB = 180^\circ - 2x^\circ$ (angle sum of $\triangle ABC$)
 $\angle DCE = \angle ACB = 180^\circ - 2x^\circ$ (vertically opposite angles)

- $\therefore \angle D = \angle E = \frac{1}{2}[180^\circ - (180^\circ - 2x^\circ)]$ (angle sum of isosceles $\triangle DCE$)
 $\therefore \angle D = \angle E = x^\circ$
 $\therefore \angle A = \angle E$
 $\therefore AB \parallel DE$ (alternate angles are equal)
- 12** $\angle YUX = \angle UYX = \angle UXY = 60^\circ$ (angles in equilateral $\triangle UXY$)
 $\therefore \angle UXW = 120^\circ$ (angles on a straight line)
 $\angle XWU + \angle XUW + 120^\circ = 180^\circ$ (angle sum of $\triangle WXU$)
 $\therefore \angle XWU = \angle XUW = 30^\circ$ ($\triangle WXU$ is isosceles)
 $\angle WUY = \angle XUW + \angle YUX$
 $= 30^\circ + 60^\circ$
 $= 90^\circ$
- 13** $\angle WTP = \angle P$ and $\angle YTQ = \angle Q$ (alternate angles, $WY \parallel PQ$)
 $\angle WTP + \angle PTQ + \angle YTQ = 180^\circ$ (angles on a straight line)
 $\therefore$ angle sum of $\triangle PQT = \angle P + \angle PTQ + \angle Q$
 $= \angle WTP + \angle PTQ + \angle YTQ$ (from above)
 $= 180^\circ$
- 14** $\angle BAD + \angle DAH + \angle BAC + \angle CAF = 180^\circ$ (angles on a straight line)
 But $\angle BAD = \angle DAH$ (AD bisects $\angle HAB$)
 and $\angle BAC = \angle CAF$ (AC bisects $\angle FAB$)
 $\therefore 2\angle BAD + 2\angle BAC = 180^\circ$
 $\therefore \angle BAD + \angle BAC = 90^\circ$
 $\therefore \angle CAD = 90^\circ$
- 15** $\angle BAC + \angle BCA + \angle ABC = 180^\circ$ (angle sum of $\triangle ABC$)
 $\therefore \angle ABC = 180^\circ - (\angle BAC + \angle BCA)$
 $\angle CBD + \angle ABC = 180^\circ$ (angles on a straight line)
 $\therefore \angle CBD = 180^\circ - \angle ABC$
 $= 180^\circ - [180^\circ - (\angle BAC + \angle BCA)]$
 $= \angle BAC + \angle BCA$
- 16** $\angle ABO = x^\circ = \angle BAO$ and $\angle CBO = y^\circ = \angle BCO$ (equal angles of isosceles $\triangle ABO$ and $\triangle CBO$, equal radii)
 $\therefore 2x^\circ + 2y^\circ = 180^\circ$ (angle sum of $\triangle ABC$)
 $\therefore x^\circ + y^\circ = 90^\circ$
 $\therefore \angle ABO + \angle CBO = 90^\circ = \angle ABC$
 $\therefore \angle ABC$ is a right angle.

Exercise 13-06

- 1 a** $\frac{3}{4}$ **b** 2 **c** $\frac{1}{2}$ **d** 1.5



- 3 a i** $\angle L$ and $\angle T$, $\angle F$ and $\angle W$, $\angle D$ and $\angle P$, $\angle B$ and $\angle Y$.
ii LF and TW , FD and WP , BD and YP , LB and TY .
iii $LFDB \parallel TWPY$
b i $\angle G$ and $\angle T$, $\angle M$ and $\angle Q$, $\angle Y$ and $\angle S$,
ii MY and QS , GM and TQ , GY and TS .
iii $\triangle GYM \parallel \triangle TSQ$
- 4 a** Yes, ($\frac{36}{12} = \frac{27}{9} = \frac{24}{8} = \frac{18}{6} = 3$)
b Yes, all equilateral triangles are similar.
c Yes ($\frac{28}{42} = \frac{24}{36} = \frac{19}{28.5} = \frac{2}{3}$)
d Yes ($\frac{9}{15} = \frac{15}{25} = \frac{3}{5}$)
e Yes, ($\frac{18}{15} = \frac{30}{25} = \frac{6}{5}$, and the triangle is right-angled)
f Yes, all squares are similar.

Exercise 13-07

- 1 a** $w = 22.4$ **b** $m = 10$ **c** $p = 20, h = 21$
d $x = 18$ **e** $a = 12.8, w = 7.5$ **f** $g = 11\frac{1}{5}, q = 18$
g $y = 26\frac{2}{3}, b = 9\frac{1}{2}$ or 9.6 **h** $u = 12\frac{4}{5}$ or $12.8, t = 6\frac{7}{8}$ or 6.875
2 $h = 13\frac{5}{7}$ **3** $x = 8\frac{8}{9}$ **4** $w = 16$ cm **5** $h = 12$ m
6 B **7** $h = 2.408$ m **8** D **9** 2.24 m

Exercise 13-08

- 1 a** Two pairs of angles are equal (AA).
b All three pairs of matching sides are in the same ratio,
 $\frac{9}{18} = \frac{11}{22} = \frac{15.5}{31} = \frac{1}{2}$ (SSS)
c Two pairs of matching sides are in the same ratio
 $\frac{6}{8} = \frac{12}{16} = \frac{3}{4}$ and the included angles are equal (SAS).
d Two pairs of angles are equal (AA).
e All three pairs of matching sides are in the same ratio
 $\frac{9}{12} = \frac{9}{12} = \frac{14.25}{19} = \frac{3}{4}$ (SSS)
f In both right-angled triangles, the pairs of hypotenuses and second sides are in the same ratio $\frac{12}{15} = \frac{20.8}{26} = \frac{4}{5}$ (RHS).
g Two pairs of angles are equal (AA).
h All three pairs of matching sides are in the same ratio
 $\frac{18}{14.4} = \frac{27.5}{22} = \frac{20}{16} = \frac{5}{4}$ (SSS).
i All three pairs of matching sides are in the same ratio
 $\frac{6}{8} = \frac{8}{10\frac{2}{3}} = \frac{10}{13\frac{1}{3}} = \frac{3}{4}$ (SSS).
j Two pairs of matching sides are in the same ratio
 $\frac{26}{18.2} = \frac{30}{21} = \frac{10}{7}$ and the included angles are equal (SAS).
- 2 a** B and C (SAS) **b** A and C (SSS) **c** B and D (RHS)
- 3 a** $\triangle UWY \parallel \triangle HEK$ (SAS) **b** $\triangle DML \parallel \triangle TPA$ (RHS)
c $\triangle ABC \parallel \triangle QTP$ (AA) **d** $\triangle GHN \parallel \triangle WVS$ (SSS)

Exercise 13-09

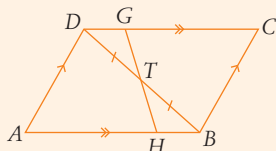
- 1 a** In $\triangle TCH$ and $\triangle PMB$:
 $\frac{TC}{PM} = \frac{18}{10.8} = \frac{5}{3}$
 $\frac{TH}{PB} = \frac{25}{15} = \frac{5}{3}$
 $\angle C = \angle M = 90^\circ$
 $\therefore \triangle TCH \parallel \triangle PMB$ (in a right-angled triangle, hypotenuses and two pairs of matching sides are in proportion or RHS)
- b** In $\triangle VWG$ and $\triangle LQE$:
 $\angle V = \angle L = 22^\circ$
 $\angle W = \angle Q = 123^\circ$
 $\therefore \triangle VWG \parallel \triangle LQE$ (equiangular or AA)

- c** In $\triangle ABC$ and $\triangle TWM$:
 $\frac{AB}{TW} = \frac{17}{25.5} = \frac{2}{3}$
 $\frac{AC}{TM} = \frac{12}{18} = \frac{2}{3}$
 $\frac{BC}{WM} = \frac{16}{24} = \frac{2}{3}$
 $\therefore \triangle ABC \parallel \triangle TWM$ (three pairs of matching sides in proportion or SSS)
- d** In $\triangle EVH$ and $\triangle DNL$:
 $\frac{EV}{DN} = \frac{21}{12} = \frac{7}{4}$
 $\frac{VH}{NL} = \frac{35}{20} = \frac{7}{4}$
 $\angle V = \angle N = 90^\circ$
 $\therefore \triangle EVH \parallel \triangle DNL$ (two pairs of matching sides in proportion and the included angles equal or SAS)
- 2 a** In $\triangle ADE$ and $\triangle ABC$:
 $\frac{AD}{AB} = \frac{1}{2}$ (D is the midpoint of AB)
 $\frac{AE}{AC} = \frac{1}{2}$ (E is the midpoint of AC)
 $\angle A$ is common.
 $\therefore \triangle ADE \parallel \triangle ABC$ (two pairs of matching sides in proportion and the included angles equal or SAS)
- b** In $\triangle ABF$ and $\triangle FDE$:
 $\angle AFB = \angle FED$ (corresponding angles, $BF \parallel CE$)
 $\angle FAB = \angle EFD$ (corresponding angles, $AC \parallel FD$)
 $\therefore \triangle ABF \parallel \triangle FDE$ (equiangular or AA)
- c** In $\triangle WXY$ and $\triangle TXW$:
 $\angle WXY = \angle TXW = 90^\circ$ (given)
 $\angle YWX = 90^\circ - \angle WYX$ (angle sum of $\triangle WXY$)
 $\angle XTW = 90^\circ - \angle WYX$ (angle sum of $\triangle WTY$)
 $= \angle YWX$
 $\therefore \triangle WXY \parallel \triangle TXW$ (equiangular or AA)
- d** In $\triangle NDL$ and $\triangle NQR$:
 $\frac{ND}{NQ} = \frac{8}{16} = \frac{1}{2}$
 $\frac{NL}{NR} = \frac{10}{20} = \frac{1}{2}$
 $\angle N$ is common.
 $\therefore \triangle NDL \parallel \triangle NQR$ (two pairs of matching sides in proportion and the included angles equal or SAS)
- e** In $\triangle XWH$ and $\triangle YXW$:
 $\frac{HW}{WX} = \frac{18}{12} = \frac{3}{2}$
 $\frac{XW}{YX} = \frac{12}{8} = \frac{3}{2}$
 $\angle HWX = \angle YXW$ (alternate angles, $HW \parallel YX$)
 $\therefore \triangle XWH \parallel \triangle YXW$ (two pairs of matching sides in proportion and the included angles equal or SAS)
- f** In $\triangle NML$ and $\triangle KLP$:
 $\angle NML = \angle KLP$ (alternate angles, $NM \parallel LK$)
 $\angle N = \angle K$ (opposite angles of a parallelogram)
 $\therefore \triangle NML \parallel \triangle KLP$ (equiangular or AA)
- 3 a i** In $\triangle FLN$ and $\triangle FDE$:
 $\angle FLN = \angle FDE$ (corresponding angles, $LN \parallel DE$)
 $\angle F$ is common.
 $\therefore \triangle FLN \parallel \triangle FDE$ (equiangular or AA)
ii $d = 9$
- b i** In $\triangle ACE$ and $\triangle BCD$:
 $\angle EAC = \angle DBC = 90^\circ$ (given)
 $\angle C$ is common.
 $\therefore \triangle ACE \parallel \triangle BCD$ (equiangular or AA)
ii $y = 5\frac{5}{11}$
- c i** In $\triangle YRT$ and $\triangle WUT$:
 $\angle YRT = \angle WUT = 90^\circ$ (given)
 $\angle YTR = \angle WTU$ (vertically opposite angles)
 $\therefore \triangle YRT \parallel \triangle WUT$ (equiangular or AA)
ii $g = 15$
- d i** $\angle T + \angle PCT = 90^\circ + 90^\circ = 180^\circ$
 $\therefore TN \parallel CP$ (co-interior angles are supplementary)
 In $\triangle NMP$ and $\triangle PCB$:
 $\angle NMP = \angle PCB = 90^\circ$ (given)
 $\angle N = \angle CPB$ (corresponding angles, $TN \parallel CP$)
 $\therefore \triangle NMP \parallel \triangle PCB$ (equiangular or AA)
ii $w = 7.5$
- e i** In $\triangle TYN$ and $\triangle YNM$:
 $\angle TYN = \angle MNY$ (alternate angles, $TY \parallel MN$)
 $\angle TNY = \angle YMN$ (given)
 $\therefore \triangle TYN \parallel \triangle YNM$ (equiangular or AA)
ii $h = 12$
- f i** In $\triangle BHU$ and $\triangle XBD$:
 $\angle BUH = \angle XDB$ (given)
 $\angle UBH = \angle DXB$ (alternate angles, $BU \parallel DX$)
 $\therefore \triangle BHU \parallel \triangle XBD$ (equiangular or AA)
ii $y = 18$
- 4 a** In $\triangle MXG$ and $\triangle KXL$:
 $MG \parallel LH$ (opposite sides of a rectangle)
 $\angle GMX = \angle LKX$ (alternate angles, $MG \parallel LH$)
 $\angle MGX = \angle KLX$ (alternate angles, $MG \parallel LH$)
 $\therefore \triangle MXG \parallel \triangle KXL$ (equiangular or AA)
- b** $x = 16$
- 5 a** In $\triangle CLW$ and $\triangle LTE$:
 $\angle CWL = \angle TEL = 90^\circ$ (given)
 $\angle L$ is common.
 $\therefore \triangle CLW \parallel \triangle LTE$ (equiangular or AA)
- b** $x = 2$
- 6 a** In $\triangle PTU$ and $\triangle KPB$:
 $\angle PUT = \angle KPB$ (alternate angles, $PK \parallel TU$)
 $\angle PTU = \angle KBP = 90^\circ$ (given)
 $\therefore \triangle BHU \parallel \triangle XBD$ (equiangular or AA)
- b** $PB = 4$

Power plus

- 1 a** In $\triangle ABC$ and $\triangle CBD$:
 $\angle ACB = 90^\circ$ (by Pythagoras' theorem in $\triangle ABC$)
 $\angle CDB = 90^\circ$ (given)
 $\therefore \angle ACB = \angle CDB$
 $\angle B$ is common.
 $\therefore \triangle ABC \parallel \triangle CBD$ (equiangular or AA)
 In $\triangle ABC$ and $\triangle ACD$:
 $\angle ACB = 90^\circ$ (by Pythagoras' theorem in $\triangle ABC$)
 $\angle CDA = 90^\circ$ (given)
 $\therefore \angle ACB = \angle CDA$
 $\angle A$ is common.
 $\therefore \triangle ABC \parallel \triangle ACD$ (equiangular or AA)
 $\therefore \triangle ABC \parallel \triangle CDB \parallel \triangle ACD$
- b** $CD = 4\frac{8}{13} \approx 4.62$

2



In $\triangle DGT$ and $\triangle BHT$:

$\angle DGT = \angle BHT$ (alternate angles, $DC \parallel AB$)

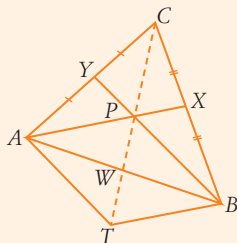
$\angle GDT = \angle HBT$ (alternate angles, $DC \parallel AB$)

$DT = BT$ (given)

$\therefore \triangle DGT \equiv \triangle BHT$ (AAS)

$\therefore DG = BH$ (matching sides of congruent triangles)

3



(Outline of proof only) X and Y are midpoints of BC and AD .

Medians AX and BY meet at P .

Draw CP to T , so that $CP = PT$.

Prove that $\triangle CYP \parallel \triangle CAT$ (SAS)

$\therefore YP \parallel AT$

$\therefore PB \parallel AT$

Similarly, prove $\triangle CXP \parallel \triangle CBT$ (SAS)

$\therefore PA \parallel BT$

$\therefore APBT$ is a parallelogram (opposite sides are parallel)

W is the midpoint of AB (the diagonals of a parallelogram bisect each other).

b $AX = CY$ (matching sides of congruent triangles)

$AB = CD$ (opposite sides of a parallelogram)

$\therefore XB = AB - AX$

$= CD - CY$

$= YD$

$DX = YB$ (given)

$\therefore BXDY$ is a parallelogram (opposite sides equal).

7 a In $\triangle PML$ and $\triangle NLM$:

$LP = MN$ (opposite sides of a rectangle)

LM is common.

$\angle PLM = \angle NML = 90^\circ$ (angles of a rectangle)

$\therefore \triangle PML \equiv \triangle NLM$ (SAS).

b $\therefore PM = NL$ (matching sides of congruent triangles)

c The diagonals of a rectangle are equal.

8 In $\triangle NMA$ and $\triangle PQB$:

$NM = PQ$ (sides of a square)

$AM = BQ$ (given)

$\angle NMA = \angle PQB = 90^\circ$ (angles of a square)

$\therefore \triangle NMA \equiv \triangle PQB$ (SAS).

$\angle NAM = \angle PBQ$ (matching angles of congruent triangles)

But $\angle NAM = \angle CAB$ and $\angle PBQ = \angle CBA$ (vertically opposite angles)

$\therefore \angle CAB = \angle CBA$

$\therefore \triangle CBA$ is isosceles (two equal angles).

$\therefore AC = BC$ (sides opposite the equal angles in isosceles $\triangle CBA$)

Also, $NA = PB$ (matching sides of congruent triangles)

$\therefore NC = NA + AC$

$= PB + BC$

$= PC$

$\therefore \triangle NPC$ is isosceles (two sides proved equal).

9 a Yes, $(\frac{27}{18} = \frac{15}{10} = \frac{3}{2})$

b Yes, $(\frac{22}{27.5} = \frac{16}{20} = \frac{12}{15} = \frac{9}{11.25} = \frac{4}{5} = 0.8)$

Chapter 13 revision

1 156° 2 36 3 a 36 b 15 c 8 d 24

4 B

5 In $\triangle WYZ$ and $\triangle XYZ$

$\angle W = \angle X$ (given)

$\angle WZY = \angle XZY = 90^\circ$ ($YZ \perp WX$)

YZ is common.

$\therefore \triangle WYZ \equiv \triangle XYZ$ (AAS)

6 a $\triangle BYC$ is isosceles ($BC = BY$).

$\therefore \angle BCY = \angle C$ (angles opposite equal sides)

$BC = AD$ (opposite sides of a parallelogram)

$\therefore \triangle ADX$ is isosceles ($AD = XD$).

$\therefore \angle A = \angle DXA$ (angles opposite equal sides)

But $\angle A = \angle BCY$ (opposite angles of a parallelogram)

$\therefore \angle A = \angle DXA = \angle C = \angle BCY$

In $\triangle ADX$ and $\triangle CBY$

$AD = CB$ (given)

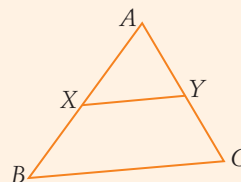
$\angle A = \angle C$ (opposite angles of a parallelogram)

$\angle DXA = \angle BCY$ (proven)

$\therefore \triangle ADX \equiv \triangle CBY$ (AAS)

10 $11\frac{3}{7}$ 11 a SAS b RHS c AA

12



In $\triangle AXY$ and $\triangle ABC$:

$\frac{AX}{AB} = \frac{1}{2}$ (X is the midpoint of AB)

$\frac{AY}{AC} = \frac{1}{2}$ (Y is the midpoint of AC)

$\angle A$ is common.

$\therefore \triangle AXY \parallel \triangle ABC$ (two pairs of matching sides in proportion and the included angles equal or SAS)

$\therefore \angle AXY = \angle B$ (matching angles in similar triangles)

$\therefore XY \parallel BC$ (corresponding angles are equal)

$\therefore \frac{XY}{BC} = \frac{AX}{AB}$ (matching pairs of sides in proportion)

$\therefore \frac{XY}{BC} = \frac{1}{2}$

$\therefore XY = \frac{1}{2} \times BC$

Mixed revision 4

- 1 40
- 2 a 78 b i $\frac{25}{39}$ ii $\frac{14}{39}$ c $\frac{35}{78}$
- 3 a 80
- b i $\frac{13}{80}$ ii $\frac{36}{80} = \frac{9}{20}$ iii $\frac{42}{80} = \frac{21}{40}$ iv $\frac{56}{80} = \frac{7}{10}$
- c $\frac{1}{6} = 0.1\bar{6}$, which is lower than the experimental probability of $\frac{17}{80} = 0.2125$
- 4 a
-
- b i $\frac{13}{160}$ ii $\frac{19}{80}$ iii $\frac{9}{32}$ iv $\frac{109}{160}$ v $\frac{57}{160}$
- c $\frac{5}{27}$
- 5 In $\triangle ABC$ and $\triangle CDA$:
 $\angle BAC = \angle DCA$ (alternate angles, $AB \parallel CD$)
 $\angle BCA = \angle DAC$ (alternate angles, $AD \parallel CB$)
 AC is common.
 $\therefore \triangle ABC \equiv \triangle CDA$ (AAS)
- 6 a i $\frac{29}{75} = 0.39$ ii $\frac{13}{30} = 0.43$ iii $\frac{9}{50} = 0.18$
- b i 0.33 ii 0.42 iii 0.25
- c The probabilities for drawing a black marble are similar.
- d 350
- 7 a 36 b 170°
- 8 a 135 b $\frac{40}{135} = \frac{8}{27}$
- c i $\frac{67}{135}$ ii $\frac{60}{135} = \frac{4}{9}$ iii $\frac{72}{135} = \frac{8}{15}$
- 9 In $\triangle LMP$ and $\triangle LNP$:
 $LM = LN$ (given)
 $MP = NP$ (P is the midpoint of MN)
 LP is common.
 $\therefore \triangle LMP \equiv \triangle LNP$ (SSS)
 $\therefore \angle LPM = \angle LPN$ (matching angle of congruent triangles)
 But $\angle LPM + \angle LPN = 180^\circ$ (angles on a line)
 $\therefore \angle LPM = \angle LPN = 90^\circ$
- 10 a In $\triangle PRT$ and $\triangle RPQ$:
 $PT = RQ$ (given)
 $RT = PQ$ (given)
 PR is common.
 $\therefore \triangle PRT \equiv \triangle RPQ$ (SSS)
- b $\angle PRT = \angle RPQ$ (matching angles of congruent triangles)
 $\therefore RT \parallel PQ$ (alternate angles are equal)
 $\angle RPT = \angle PRQ$ (matching angles of congruent triangles)
 $\therefore PT \parallel RQ$ (alternate angles are equal)
 $\therefore PQRT$ is a parallelogram (opposite sides are parallel)
- 11 a 12.86 b 6
- 12 Let $\angle VXY = 2x^\circ$
 $\therefore \angle VXT = \angle TXY = x^\circ$ (TX bisects $\angle VXY$)
 Since $XY = VY$ (equal sides of rhombus $XYVW$),
 $\triangle XYV$ is isosceles.

$$\begin{aligned}\therefore \angle VXY &= 2x^\circ = \angle XZY \\ \therefore \angle XTY &= \angle VXT + \angle XZY \text{ (exterior angle of } \triangle VXT \text{ equal to} \\ &\text{the interior opposite angles)} \\ &= x^\circ + 2x^\circ \\ &= 3x^\circ \\ &= 3 \times \angle TXY\end{aligned}$$

- 13 a Teacher to check.
- b i $\frac{1}{2}$ ii $\frac{1}{4}$ iii $\frac{1}{2}$ iv $\frac{3}{4}$ v $\frac{1}{4}$
- 14 a 2340° b 3240° c 1080° d 8280°
- 15 36 16 a SAS b SSS
- 17 10.6
- 18 a

	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	8	10	12
3	3	6	9	12	15	18
4	4	8	12	16	20	24
5	5	10	15	20	25	30
6	6	12	18	24	30	36

- b $\frac{5}{11}$ c i 1 ii 0 d $\frac{1}{2}$ e 1
- 19 a In $\triangle YXT$ and $\triangle WVT$:
 $YX = VW$ and $YX \parallel VW$ (opposite sides of a rectangle)
 $\angle YXT = \angle WVT$ (alternate angles, $YX \parallel VW$)
 $\angle YTX = \angle WTV$ (vertically opposite angles)
 $\therefore \triangle YXT \equiv \triangle WVT$ (AAS)
- b $\therefore YT = TW$ and $XT = TV$ (matching sides in congruent triangles)
 $\therefore$ The diagonals of a rectangle bisect each other.
- 20 a In $\triangle ABW$ and $\triangle CDW$:
 $\angle ABW = \angle CDW$ (alternate angles, $AB \parallel CD$)
 $\angle BAW = \angle DCW$ (alternate angles, $AB \parallel CD$)
 $\therefore \triangle ABW \equiv \triangle CDW$ (AA)
- b $CW = 4\frac{5}{7}$
- 21 In $\triangle CEO$ and $\triangle DFO$:
 $OC = OD$ (equal radii)
 $\angle CEO = \angle DFO = 90^\circ$ ($CE \perp AB$ and $DF \perp AB$)
 $\angle COE = \angle DOF$ (vertically opposite angles)
 $\therefore \triangle CEO \equiv \triangle DFO$ (AAS)

General revision

- 1 $7\sqrt{2}$ 2 $\frac{3\sqrt{10}}{10}$ 3 $98 + 24\sqrt{10}$ 4 \$15 700
- 5 gradient = $-\frac{4}{5}$, y-intercept = 2
- 6 a $6x(x+2)$ b $25(1+2y)(1-2y)$
- c $(a-10)(a+4)$ d $-2(2p+1)(p-3)$
- 7 $x = \frac{-7 \pm \sqrt{41}}{2}$ 8 $\bar{x} = 50.3$, $\sigma = 12.2$
- 9 a $360\,498\text{ mm}^3$ b $145\,125\text{ mm}^3$
- 10 a 210 b i $\frac{1}{6}$ ii $\frac{143}{210}$ c 31.9% d $\frac{29}{36}$
- 11 $\theta = 142^\circ 49'$
- 12 a $56x^{12}$ b $\frac{x^5}{4}$ c $\frac{1}{9y^2}$ d $\frac{n}{m^7}$
- 13 a $y = 6\frac{1}{4}$ b $k = 3\frac{3}{7}$ c $x = \pm 3$

14 $d = 19.1$ 15 C

16 a -1 b 4 c $\frac{1}{2}$
17 a 5 b 1 c -3

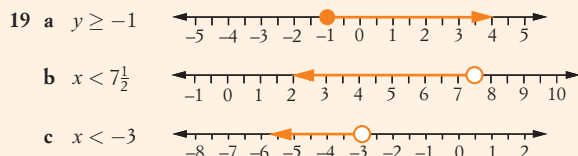
18 In $\triangle ABC \cong \triangle DEF$:

$AB = DE = 10$ cm (given)

$CB = FE = 12$ cm (given)

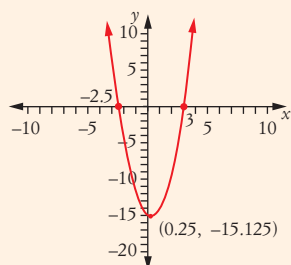
$\angle A = \angle D = 90^\circ$ (given)

$\therefore \triangle ABC \cong \triangle DEF$ (RHS)



20 $2x + 3y + 5 = 0$ 21 $x = 2, y = 2$ 22 $x^2 + y^2 = 64$

23 x -intercepts at $-2\frac{1}{2}$ and 3 , y -intercept at -15 , axis of symmetry $x = \frac{1}{4}$, vertex $(\frac{1}{4}, -15\frac{1}{8})$



24 a 7.2 m^3 b 78.5 cm^3 c 5747.0 cm^3

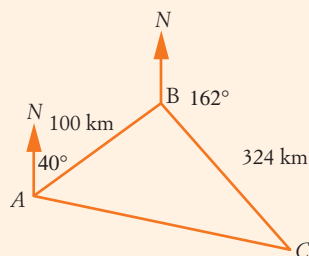
25 \$1607.41

26 a i $\sqrt{13}, m = \frac{2}{3}$ ii $2\sqrt{10}, m = \frac{-1}{3}$

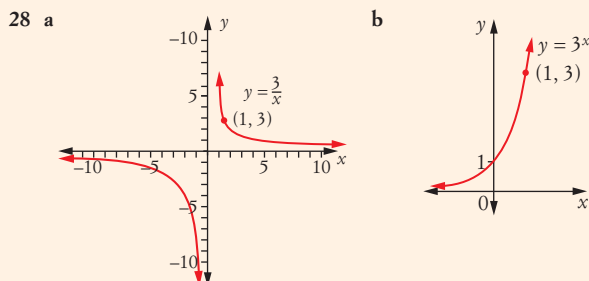
iii $\sqrt{13}, m = \frac{2}{3}$ iv $2\sqrt{10}, m = \frac{-1}{3}$

b parallelogram

27 a

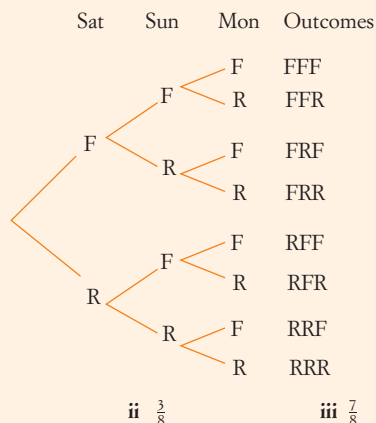


b $40^\circ + 18(180^\circ - 162^\circ) = 58^\circ$ c 284 km d 115°



29 $h = 21, p = 20$

30 a F = fine, R = rain



31 19.2 min

Chapter 14

SkillCheck

- 1 a 13 b 3 c 12
2 a $(x-4)(x+4)$ b $x(x-4)(x+4)$ c $3(x-3)(x+3)$
d $3x(x-3)(x+3)$ e $(x-5)(x+3)$ f $(x+8)(x-3)$
g $(x-2)(2x+5)$ h $x(x-10)(x+7)$
3 a $x = -\frac{2}{3}$ or $x = 2$ b $x = 0$ or $x = 10$
c $x = 0$ or $x = \frac{3}{5}$ d $x = -1$ or $x = -5$
e $x = -10$ or $x = 12$ f $x = -2$ or $x = -\frac{3}{2}$

Exercise 14-01

- 1 a Yes, not monic b No c Yes, monic
d Yes, not monic e No f Yes, monic
g Yes, not monic h No i Yes, monic
j No k Yes, monic l Yes, not monic
2 a i 5 ii 9 iii 1 b i 5 ii -6 iii 3
c i 2 ii 11 iii -10 d i 1 ii -6 iii 0
e i 5 ii 7 iii 3 f i 0 ii 9 iii 9
g i 6 ii 1 iii -11 h i 1 ii $-\frac{5}{4}$ iii 22
i i 3 ii $\frac{1}{3}$ iii 0
3 a -1 b 17 c -7 d $-\frac{5}{8}$ e $7\sqrt{2} - 5$
4 a 1 b 2 c 14 d -3
e $\frac{47}{64}$ f $13 - 5\sqrt{3}$ g 56 h -7

Exercise 14-02

- 1 a $9x^3 + 8x^2 + 6x - 2$ b $-x^3 + 4x + 2$ c $-4x^2 - 3x - 2$
d $5x^4 - 3x^3 - 5x^2 + 4$ e $x^4 - x^3 + x^2 - 5x$
f $2x^5 + 10x^4 + 2x^3 - 4x^2 + 4x + 22$
g $7x^6 + x^5 + x^3 + x^2 + 3x - 2$ h $8x^4 + 15x^2 + 4x - 7$
i $-4x^4 - 8x^3 + 5x^2 + x - 2$ j $6x^3 - 2x^2 - 6x + 1$
2 a $x^2 + 11x - 1$ b $-x^2 - 3x - 5$
c $x^2 + 3x + 5$ d $2x^2 + 26x - 5$
3 a $2x^2 - 15x + 9$ b $x^2 + 4x - 15$ c $-3x^2 + 11x + 6$
d $-x^2 + 7x + 3$ e $3x^2 - 15x + 3$ f $3x^2 - 7x - 15$
4 a $x^2 - 7x + 6$ b $24 - 21\sqrt{2}$ c 1, 6

Answers

Exercise 14-03

- 1 a $3x^4 + 2x^2 - 3x^2 - 2x$ b $32x^3 + 80x^2 - 22x$
 c $45x^4 - 13x^3 + 3x^2 - 2x$ d $18x^4 + 6x^3 - 7x^2 - 11x - 6$
 e $-21x^7 + 11x^6 + 2x^5 - 3x^4 - x^3 + 2x^2$
 f $8x^5 + 47x^4 + 41x^2 + 28x - 12$
 2 $4x^3 + 25x^2 - 13x - 6$
 3 a $-2x^4 + 11x^3 + 12x^2 - 66x$ b $-4x^3 + 9x^2 + 24x - 54$
 c $8x^3 - 62x^2 + 99x$

Exercise 14-04

- 1 a $x^2 + 7x + 4 = (x + 2)(x + 5) - 6$
 b $x^2 - 6x + 2 = (x - 3)(x - 3) - 7$
 c $4x^2 + 3x + 10 = (x - 1)(4x + 7) + 17$
 d $8x^2 + 9x + 11 = (2x + 1)(4x + 2\frac{1}{2}) + 8\frac{1}{2}$
 e $x^3 + 6x^2 + 5x - 4 = (x - 3)(x^2 + 9x + 32) + 92$
 f $4x^3 + 2x^2 + x = (x + 4)(4x^2 - 14x + 57) - 228$
 g $2x^3 - x^2 + 5x + 3 = (x + 6)(2x^2 - 13x + 83) - 495$
 h $3x^3 - x^2 + 11 = (x + 2)(3x^2 - 7x + 14) - 17$
 i $x^5 - x^4 + 8x^3 + 2x^2 - x - 1 = (x + 1)(x^4 - 2x^3 + 10x^2 - 8x + 7) - 8$
 j $x^4 - x^2 - 10 = (x + 3)(x^3 - 3x^2 + 8x - 24) + 62$
 2 a $(3x - 1), R = 3$ b $(x + 7), R = 14$
 c $(3x^3 + 14x^2 - 2x + 21), R = 42$ d $(4x + 6), R = 17$
 3 a $(2x - 1)(3x + 2)$ b $(2x - 1)(x^2 + x + 1)$
 c $(2x - 1)(4x + 7)$ d $(2x - 1)(3x^2 + 2x + 1)$
 e $(2x - 1)(x^3 - 3x^2 - 4x + 2)$ f $(2x - 1)(x^3 - x + 3)$
 g $(2x - 1)(3x^2 + 1)$ h $(2x - 1)(4 - 3x - x^5)$

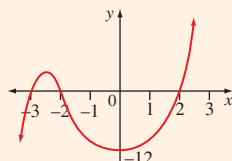
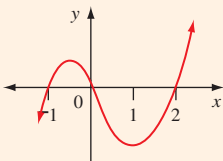
Exercise 14-05

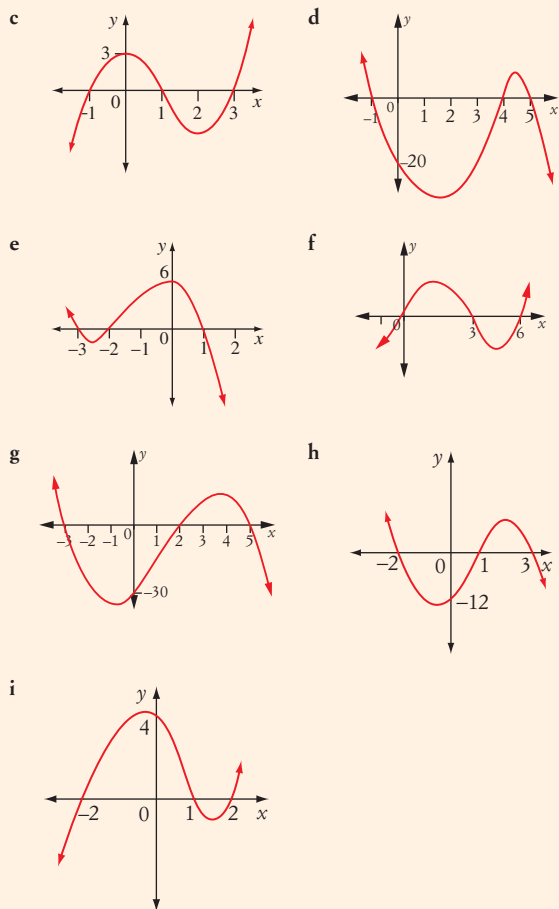
- 1 a 5 b -181 c -1 d 179
 e -7 f 1709 g -85 h 29
 2 a 54 b 2 c 14 d -2 e -12
 f 174 g 0 h 6 i 115

Exercise 14-06

- 1 a B, C b B, C c A d A, B, C e A, B
 2 Teacher to check.
 3 a $x(x + 2)(x + 4)$ b $x(x - 2)(x + 1)$
 c $(x - 1)(x + 1)(x + 2)$ d $(x - 2)(2x - 1)(x + 4)$
 e $(x - 1)(x - 2)(x - 3)$ f $(x - 2)(x + 8)(x - 5)$
 g $(x - 6)(x + 1)(3x - 1)$ h $(x - 2)(3x + 1)(2x - 1)$
 i $x^2(2x - 1)(x - 2)$
 4 a $x = -4, \frac{1}{2}, 3$ b $x = -4$ c $x = -2, \frac{5}{2}, 3$
 d $x = \pm 5$ e $x = \pm 4, -3, 2$ f $x = -7, 0, 2$
 g $x = -3, -2, 3$ h $x = -2, -1, 4$ i $x = -4, -1, 5$
 j $x = -\frac{1}{2}, 1, 3$ k $x = \frac{1}{4}, \frac{2}{3}, 1$ l $x = 3$

Exercise 14-07

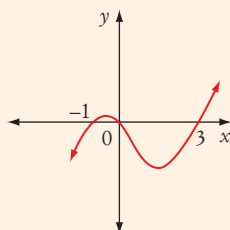
- 1 a  b 



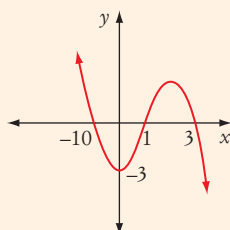
2 B

3 A

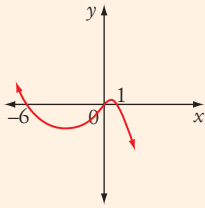
4 a x-intercepts are -1, 0, 3 and y-intercept is 0.



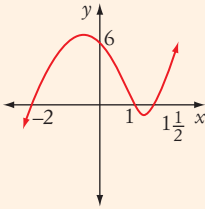
b x-intercepts are -1, 1, 3 and y-intercept is -3.



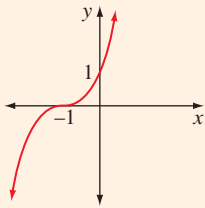
c x-intercepts are -6 , 0 , 1 and y-intercept is 0 .



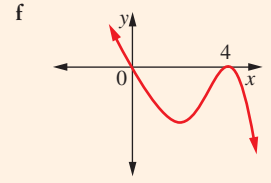
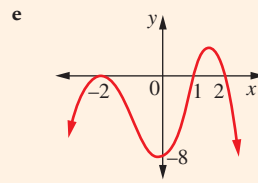
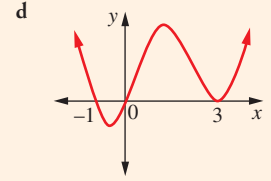
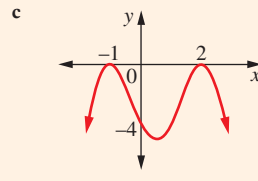
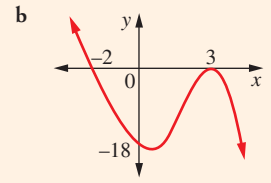
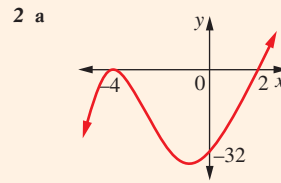
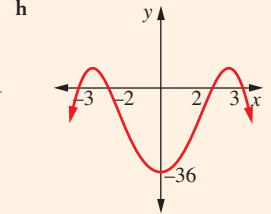
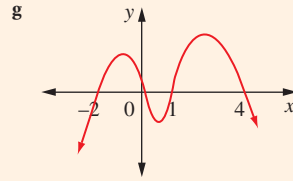
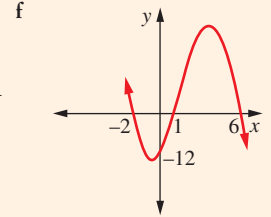
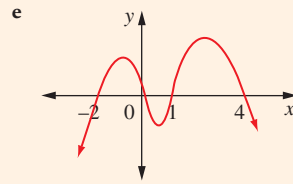
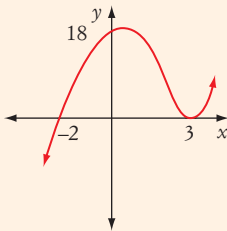
d x-intercepts are -2 , 1 and $1\frac{1}{2}$ and y-intercept is 6 .



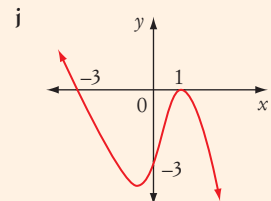
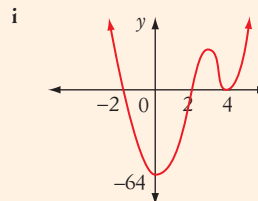
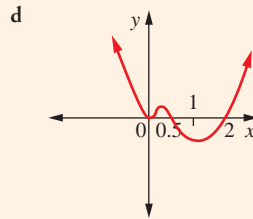
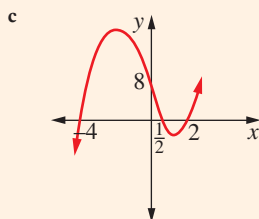
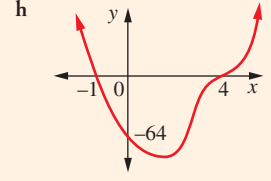
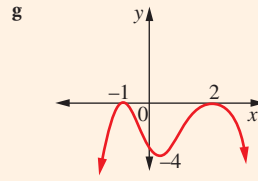
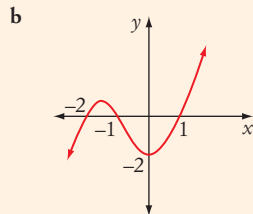
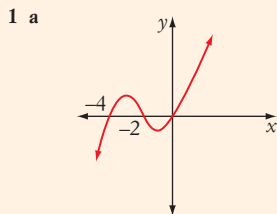
e x-intercept is -1 and y-intercept is 1 .

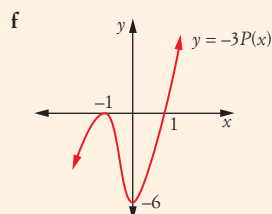
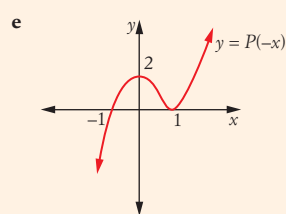
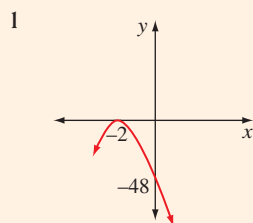
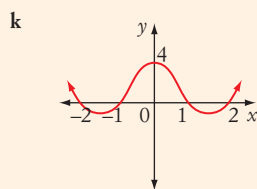


f x-intercepts are -2 and 3 and y-intercept is 18 .

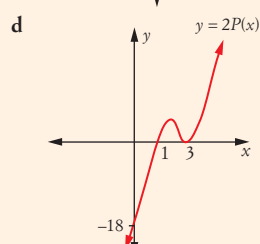
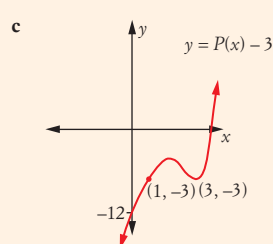
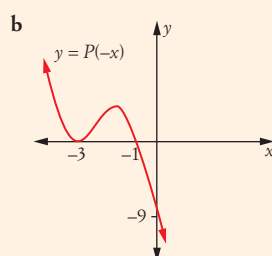
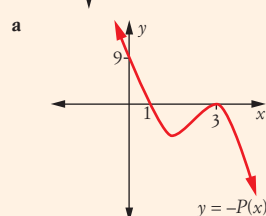
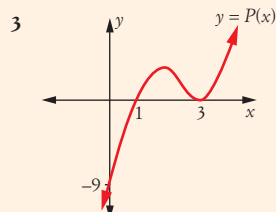
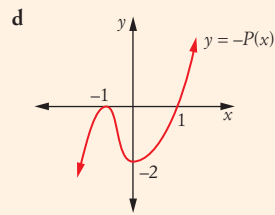
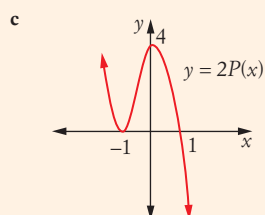
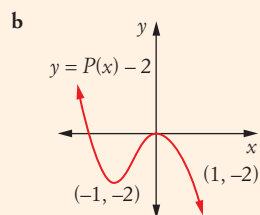
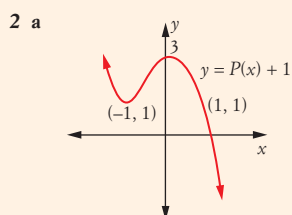
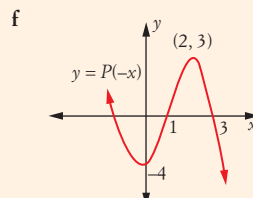
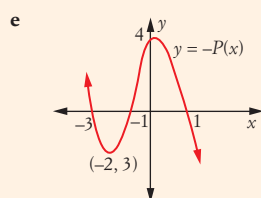
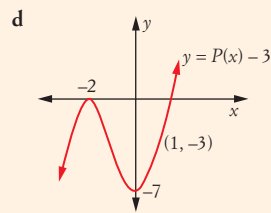
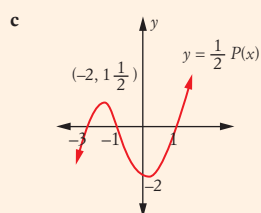
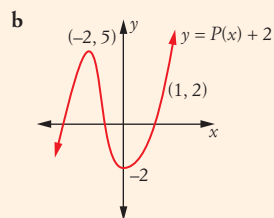
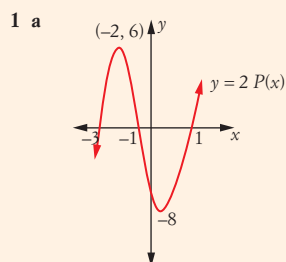


Exercise 14-08





Exercise 14-09



- 4 a** shift down 2 units
b shift up 1 unit
c stretch vertically by a factor of 2
d reflect in x -axis and shift up 3 units
e reflect in x -axis and stretch vertically by a factor of 3
f reflect in x -axis and shift up 2 units
g stretch vertically by a factor of 2 and shift down 5 units
h reflect in x -axis, stretch vertically by a factor of 3 and shift up 4 units
i compress vertically by a factor of $\frac{1}{2}$ and shift up 4 units

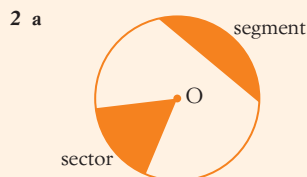
Chapter 15

SkillCheck

- 1 a** SSS **b** SAS **c** RHS **d** AAS **e** SAS **f** AAS
2 a SSS **b** AA **c** RHS **d** SSS **e** SAS **f** AA

Exercise 15-01

- 1 a radius b chord c circumference d tangent



- b sector drawn from centre of circle and bounded by 2 radii and arc. Segment is bounded by chord and arc

3 $d = 2r$

4 D

- 5 a radius b quadrant c tangent

- d diameter e chord f arc

- g sector h circumference i segment

- 6 a diameter b segment c sector d arc

Exercise 15-02

- 1 a Proof by SAS 2 a, b Proof by RHS
 3 a Proof by SSS 4 c The centre of the circle
 5 a, b Proof by SAS 6 5.74 cm 7 60 cm
 8 a $UC = 4.5$ m (the perpendicular from the centre bisects the chord and the chords are equal as they are the same distance from the centre)
 b $DE = 12$ m (chords of equal length subtend equal angles at the centre of a circle)
 c $\angle UVO = 58^\circ$ (chords of equal length subtend equal angles; angle sum of isosceles $\triangle$)
 d $PQ = 30$ mm (Pythagoras: the line from the centre is the perpendicular bisector of the chord)
 e $OM = 21$ cm (Pythagoras: the line from the centre is the perpendicular bisector of the chord)
 f $OD = 18\sqrt{2}$ (Pythagoras: the line from the centre is the perpendicular bisector of the chord)
 9 52 cm 10 18.4 km 11 $MN = 4$ cm
 12 77 cm 13 34 cm, 20 cm
 14 a $AB = 30$ cm b area = 840 cm^2
 15 a 52 cm b area = 1920 cm^2

Exercise 15-03

- 1 a 45
 b 112
 c 120
 d 232
 e 40
 f 74
 g 63
 h 104
 i 90 (angle in a semicircle)

- 2 a 48
 b 36
 c 30
 d 35
 e 74
 f 90

(angle at the centre is twice the angle on the circumference)

(angles at the circumference of a circle standing on the same arc)

- 3 a reflex $\angle ROT = 216$ (angles at a point), $\angle S = 108$ (angle at the centre is twice the angle at the circumference)
 b $x = 43$ (angles opposite equal sides in an isosceles triangle are equal), $y = 86$ (exterior angle of triangle equal to sum of two opposite interior angles, or angle at the centre is twice the angle at the circumference)
 c $n = 37$ (angles at the circumference standing on the same arc are equal), $m = 74$ (angle at the centre is twice the angle at the circumference)
 d $p = 37$ (angles at the circumference standing on the same arc equal)
 e $w = 50$ (angle at the centre is twice the angle at the circumference)
 f $h = 113$ (angle at the centre is twice the angle at the circumference, co-interior angles on parallel lines supplementary)
 4 a $m = 75$ (opposite angles of cyclic quadrilateral)
 b $p = 88$ (opposite angles of cyclic quadrilateral); $q = 121$ (opposite angles of cyclic quadrilateral)
 c $x = y = 90$ (angles in a semicircle)
 5 a $n = 106$ (exterior angle of cyclic quadrilateral)
 b $w = 60$ (angle in an equilateral triangle; exterior angle of cyclic quadrilateral)
 c $x = 84$ (exterior angle of cyclic quadrilateral), $y = 110$ (exterior angle of cyclic quadrilateral)
 6 a 23
 b 9
 c 45
 d 63 (opposite angles of a cyclic quadrilateral)
 e 75
 f 88
 (angle in semicircle, angle sum of a triangle) (exterior angle of cyclic quadrilateral)
 7 a $x = 75$ (angle at the centre is twice the angle at the circumference)
 $y = 33$ (angles at the circumference standing on the same arc)
 $z = 72$ (angle sum of a triangle)
 b $x = 108$ (angle at the centre is twice the angle at the circumference)
 $y = 126$ (opposite angles of a cyclic quadrilateral)
 $z = 252$ (angles at a point, or angle at the centre is twice the angle at the circumference)
 c $x = 70$ (straight line)
 $y = 110$ (exterior angle cyclic quadrilateral)
 $z = 70$ (straight line)
 d $x = 96$ (angle at the centre is twice the angle at the circumference)
 $y = 42$ (angles opposite equal sides of an isosceles $\triangle$, angle sum of a triangle)
 $z = 264$ (angles at a point)
 e $x = 140$ (angles opposite equal sides of an isosceles $\triangle$, angle sum of a triangle)
 $y = 70$ (angle at the centre is twice the angle at the circumference)
 $z = 35$ (angle sum of an isosceles $\triangle$, and by subtraction)
 f $x = 62$ (angle in a semicircle, angle sum of a triangle)
 $y = 118$ (opposite angles of a cyclic quadrilateral)
 $z = 31$ (angle sum of an isosceles $\triangle$)
 8 $WXYZ$ is a cyclic quadrilateral because
 $\angle W + \angle Y = 180^\circ$ and $\angle X = \angle Z = 180^\circ$
 i.e., opposite angles are supplementary

Exercise 15-04

- 1 a $\angle YMP$
b $OP \perp AB$ (angle between a tangent and radius)
c proof by angle sum of an isosceles triangle
d proof by angle at centre is twice angle at circumference
- 2 a angle between a tangent and radius
b proof by angles on a straight line
3 a proof by AA (equiangular triangles)
- 4 a $a = 56$ (the angle between the radius and the tangent is a right angle)
b $b = 21$ (radius is perpendicular to a tangent, and Pythagoras' theorem)
c $c = 134$ (a tangent is perpendicular to the radius; angle sum of a quadrilateral)
d $g = 67$ (alternate segment theorem)
- 5 a 15 b 5 c 9 d 7 e 20 f 4
- 6 $x = 7$ cm 7 a $XP = 10$ cm b $AB = 24$ cm

Exercise 15-05

- 1 $\left. \begin{array}{l} \angle R = \angle Q \\ \angle P = \angle S \end{array} \right\}$ (angles at the circumference standing on the same arc)
 $\therefore \triangle PYR \parallel \triangle SYQ$ (equiangular or AA)
 $\therefore \frac{PY}{SY} = \frac{RY}{QY}$ (matching sides in similar triangles)
 $\therefore PY \times YQ = RY \times YS$
- 2 $\angle ADC = \angle BEC$ (opposite angles of a parallelogram)
 $\angle ADC = \angle CBE$ (exterior angle of a cyclic quadrilateral)
 $\therefore \angle BEC = \angle CBE$
 $\therefore \triangle CBE$ is isosceles (two equal angles)
- 3 **Construction:** Draw a perpendicular from O to meet DG at P . Since the perpendicular from the centre to a chord bisects the chord:
 $DP = GP$ and $EP = FP$.
 $\therefore DE = DP - EP$
 $\quad = GP - FP$
 $\quad = FG$
- 4 $\angle THJ = \angle HIJ$ (alternate segment theorem)
 $\angle THJ = \angle HPI$ (alternate angles, $HT \parallel IP$)
 $\therefore \angle HIJ = \angle HPI$
In $\triangle HIP$ and $\triangle HJI$:
 $\angle HIJ = \angle HPI$ (proved above)
 $\angle IHJ = \angle IHP$ (common angles)
 $\therefore \triangle HIP \parallel \triangle HJI$ (equiangular)
 $\therefore \angle HIP = \angle HJI$ (third pair of equal angles in similar triangles)
- 5 In $\triangle UVX$ and $\triangle UWV$:
 $\angle UXV = 90^\circ = \angle UXW$ (angle in a semicircle, straight line)
 $UV = UW$ (given)
 UX is common.
 $\therefore \triangle UVX \cong \triangle UWV$ (RHS)
 $\therefore VX = VW$ (matching sides in congruent triangles)
 $\therefore$ circle bisects base of triangle.
- 6 Let $\angle QRP = x$
 $\therefore \angle SRP = x$ (PR bisects $\angle QRS$)
 $\therefore \angle PSQ = x = \angle QRP$ (angles at the circumference standing on the same arc)

$\angle PQS = x = \angle SRP$ (angles at the circumference standing on the same arc)

$$\therefore \angle PQS = \angle PSQ = x$$

$\therefore \triangle SPQ$ is isosceles (two angles are equal).

7 Join APX and BQX .

In $\triangle XPQ$ and $\triangle XAB$:

$\angle X$ is common

$\angle XPQ = \angle XAB$ (corresponding angles, $PQ \parallel AB$)

$\therefore \triangle XPQ \parallel \triangle XAB$ (equiangular or AA)

$$\therefore \frac{XP}{XA} = \frac{PQ}{AB} \text{ (matching sides in similar } \triangle\text{s)}$$

but $XP = \frac{1}{2}XA$ (radius is half the diameter)

$$\therefore \frac{\frac{1}{2}XA}{XA} = \frac{PQ}{AB}$$

$$\therefore \frac{1}{2} = \frac{PQ}{AB}$$

$$\therefore PQ = \frac{1}{2}AB$$

8 Let $\angle PTX = a$

$\therefore \angle R = a$ (alternate segment theorem)

Now $\angle QTY = a$ (vertically opposite angles)

$\therefore \angle S = a$ (alternate segment theorem)

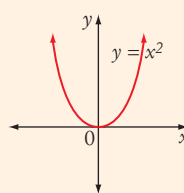
$$\therefore \angle R = \angle S = a$$

$\therefore PR \parallel SQ$ (alternate angles are equal).

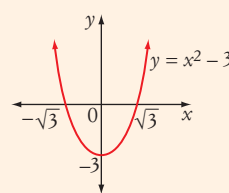
Chapter 16

SkillCheck

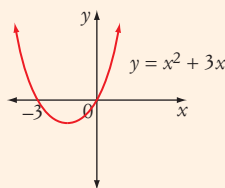
1 a



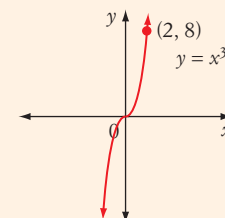
b



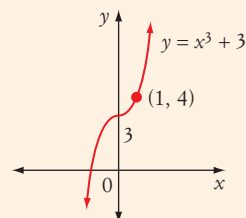
c



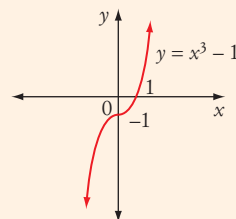
2 a



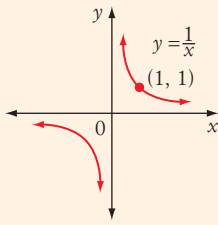
b



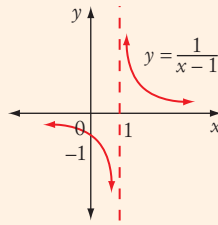
c



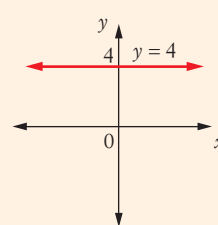
3 a



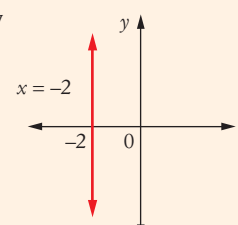
b



iii



iv



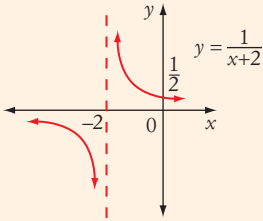
$m = 0$

No gradient or the gradient is undefined.

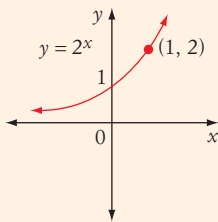
b i, ii and iii

c i For all values of m . ii If m is undefined (a vertical line).

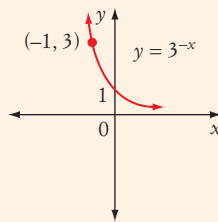
c



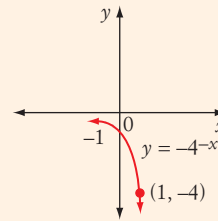
4 a



b



c



5 a $x = \frac{y+1}{2}$

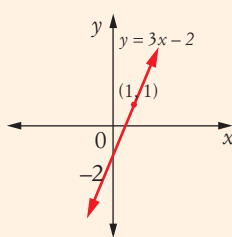
b $x = \frac{3}{y} - 1$

c $x = \pm\sqrt{y-4}$

Exercise 16-01

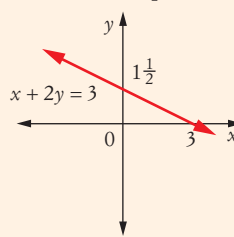
1 a Yes	b No	c Yes	d Yes
e Yes	f Yes	g No	h Yes
2 a Yes	b Yes	c No	d No
e No	f Yes	g No	h Yes
i Yes	j No	k Yes	l Yes
m Yes	n No	o Yes	p Yes

3 a i



$m = 3$

ii

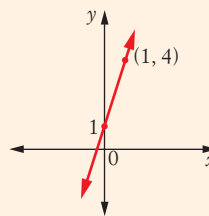


$m = -\frac{1}{2}$

Exercise 16-02

1 a i 6	ii -2	iii 0	b i -2	ii 2	iii 1
c i 24	ii -4	iii 0	d i $\frac{1}{7}$	ii -1	iii 1
e i ± 3	ii ± 1	iii $\pm\sqrt{3}$	f i $1\frac{2}{5}$	ii $\frac{1}{3}$	iii -1
g i $1\frac{2}{5}$	ii $\frac{1}{3}$	iii -1	h i -2	ii 6	iii 1
i i 60	ii 12	iii 21			
2 a 0	b -28	c $9k - 9k^2$			
3 a 16	b -10	c $6d - 20$	d 12	e $2\frac{5}{6}$	
4 a 4	b $-\frac{1}{2}$	c -2	d $x = 2\frac{1}{2}$		
5 a 5	b 0	c 14	d $t = -2, 1\frac{1}{2}$		
6 a i 11	ii 19	iii 10			
b $9 - 2y$	ii -2	c 16			
7 a i 11	ii -5	iii 6			
b Teacher to check	c $m = 3, -1$				
8 a ± 3	b 12	c $\sqrt{-1}$ has no value	d $x = -\frac{1}{2}$		
9 a 8	b 11	c 40			
d $3k^4 - 2k^2 + 3$	e $3k^2 + 2k + 3$	f $-4k$			
10 a $-3 \leq x \leq 3; 0 \leq y \leq 3$	b $-4 \leq x \leq 0; 0 \leq y \leq 4$				
c $x \geq 0; y \leq 5$	d $x \geq -1; y \geq 0$	e $x \geq -1; y \geq -3$			
f all $x; y = -3$	g all $x; y > 0$	h all $x; y = 3$			
i all $x; y > -2$	j all $x; -1 \leq y \leq 1$	k all $x; y \geq -4$			
l all $x; 0 < y \leq 4$					

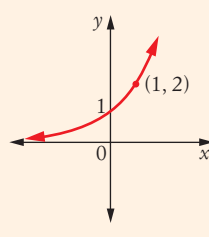
11 a



i All x

ii All y

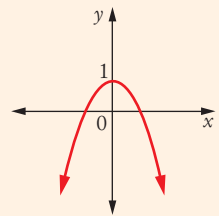
c



i All x

ii $y > 0$

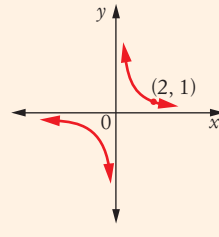
b



i All x

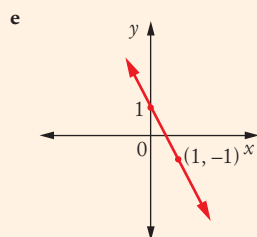
ii $y \leq 1$

d

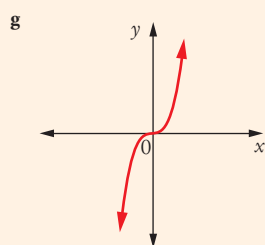


i $x \neq 0$

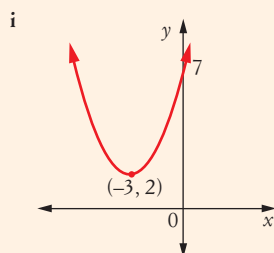
ii $y \neq 0$



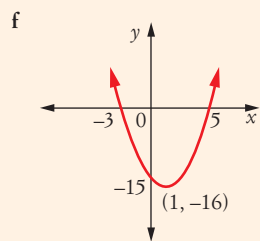
i All x **ii** All y



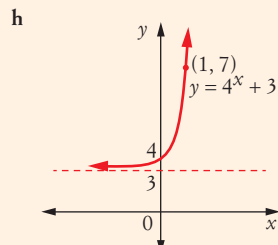
i All x **ii** All y



i All x **ii** $y \geq 2$

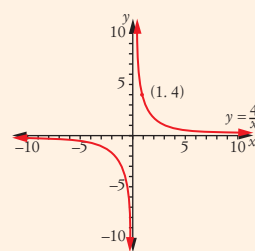


i All x **ii** $y \geq -16$

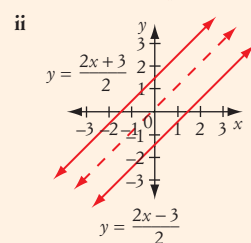


i All x **ii** $y > 3$

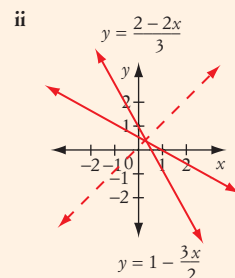
d i $f^{-1}(x) = f(x) = \frac{4}{x}$ **ii**



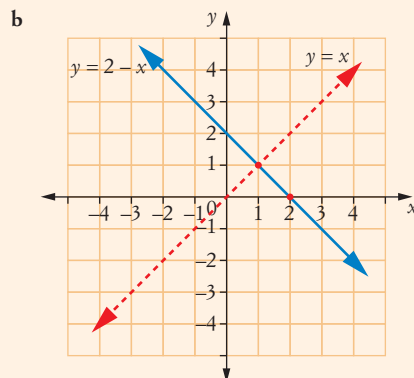
e i $f^{-1}(x) = \frac{2x+3}{2}$



f i $f^{-1}(x) = \frac{2-2x}{3}$



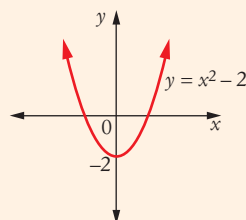
2 a Teacher to check.



The graph of $f(x) = 2 - x$ is itself symmetrical about the line $y = x$.

3 b, c and h

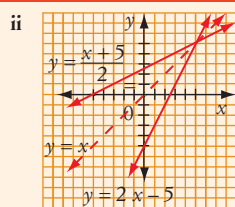
4 a



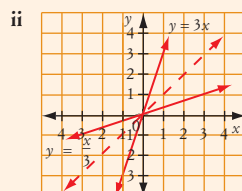
b No

Exercise 16-03

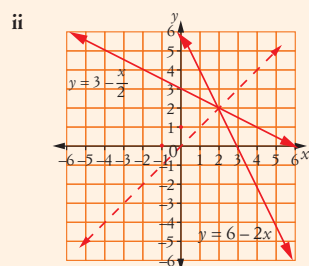
1 a i $f^{-1}(x) = \frac{x+5}{2}$



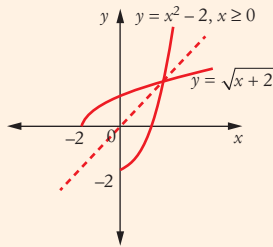
b i $f^{-1}(x) = 3x$



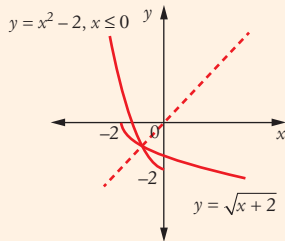
c i $f^{-1}(x) = 6 - 2x$



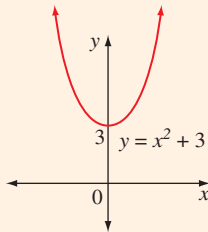
c $y = \sqrt{x+2}$



d $y = -\sqrt{x+2}$



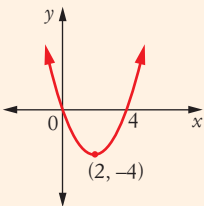
5 a



b No

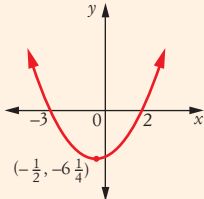
c $x \geq 0$ or $x \leq 0$

6 a



b $x \geq 2$ or $x \leq 2$

7 a



b No

c $x \geq -\frac{1}{2}$ or $x \leq -\frac{1}{2}$

Exercise 16-04

1 $y = 2^x$

Interchange x and y and make y the subject.

$\therefore x = 2^y$

$\therefore \log_2 x = \log_2 2^y$

$\therefore \log_2 x = y \log_2 2$ where $\log_2 2 = 1$

$\therefore y = \log_2 x$

2 a Domain: all x ; range: $y > 0$ b Domain: $x > 0$; range: all y

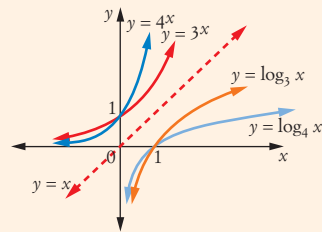
c They have interchanged.

3 Both graphs are increasing.

For $y = 2^x$, y -intercept = 1, no x -intercept.

For $y = \log_2 x$, x -intercept = 1, no y -intercept.

4



a They are all increasing graphs and have a y -intercept of 1. For $x > 0$, $y = 4^x$ is steeper than $y = 3^x$, which is steeper than $y = 2^x$.

For $x < 0$, $y = 4^x$ is closer to the x -axis than $y = 3^x$, which is closer to the x -axis than $y = 2^x$.

b They are all increasing graphs and have a x -intercept of 1. For $x > 1$, $y = \log_2 x$ is steeper than $y = \log_3 x$, which is steeper than $y = \log_4 x$.

For $x < 0$, $y = \log_4 x$ is closer to the y -axis than $y = \log_3 x$, which is closer to the y -axis than $y = \log_2 x$.

5 a

2^x

b 4^x

6 a $\log_4 x$

b $\log_2 x$

7 a 1.3010

b 2.7973

c 3.7345

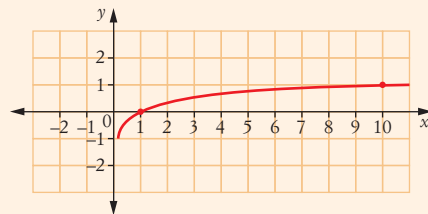
d 0.9138

e 0.3979

f -0.1192

8

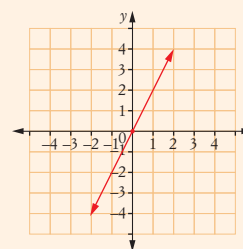
x	0.5	1	2	5	8	10
y	-0.3	0	0.3	0.7	0.9	1



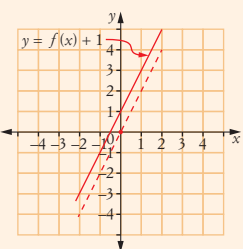
9 Teacher to check.

Exercise 16-05

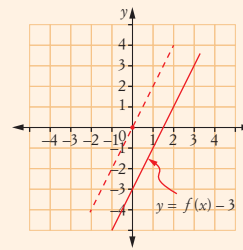
1 a



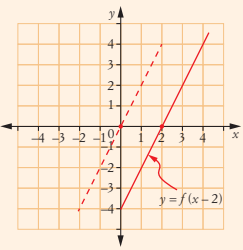
b

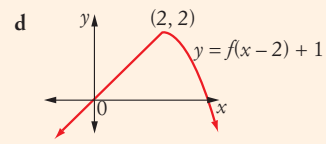
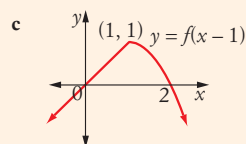
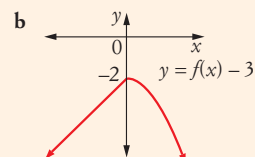
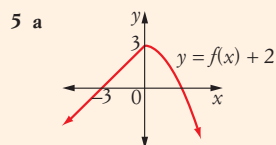
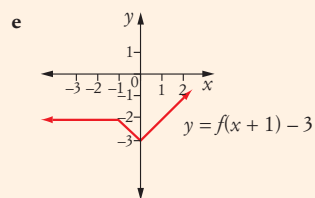
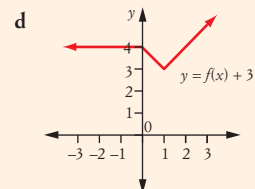
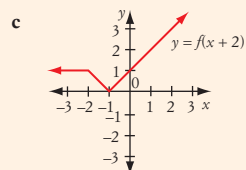
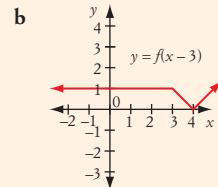
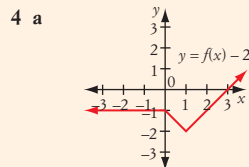
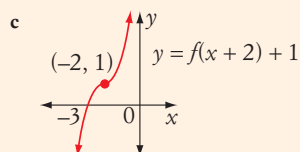
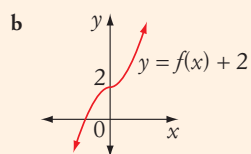
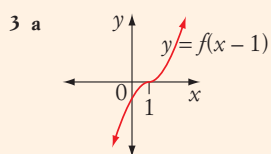
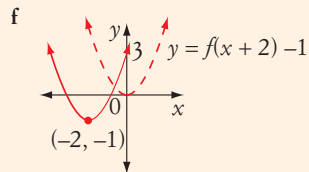
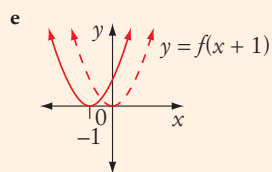
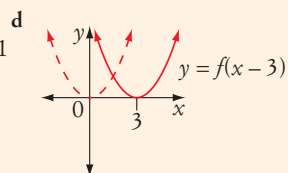
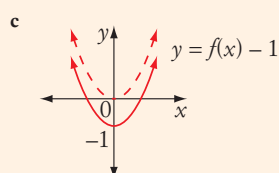
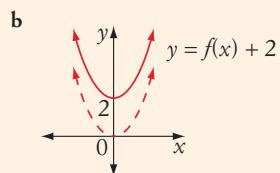
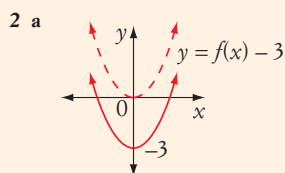
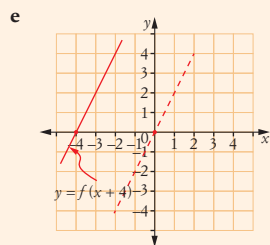


c



d





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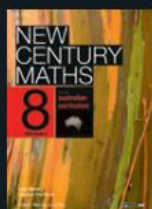
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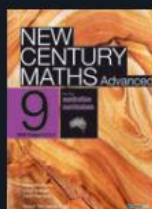
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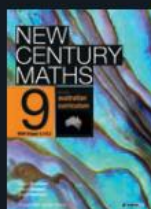
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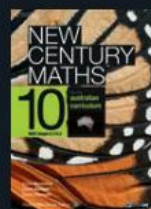
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