

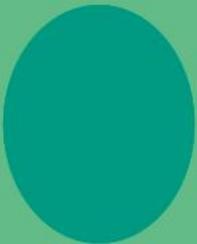
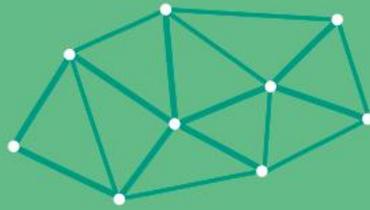
HOLLAND | BARNES

JACARANDA
MATHS QUEST

GENERAL 12
MATHEMATICS

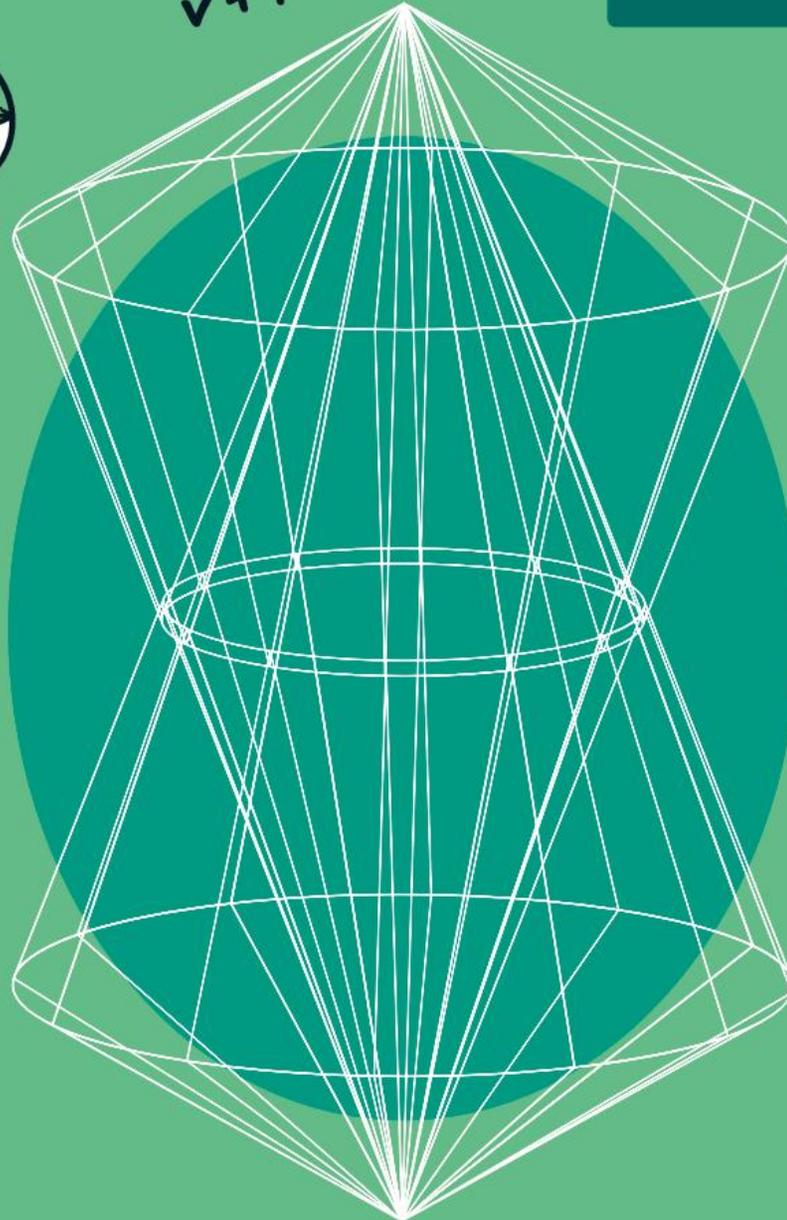
FOR QUEENSLAND

UNITS 3 & 4 | SECOND EDITION



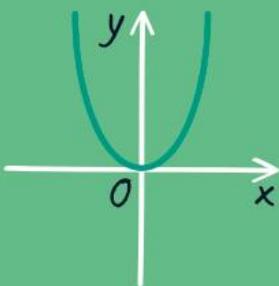
$$t_n = t_1 r^{(n-1)}$$

$$v + f - e = 2$$



$$A = \frac{d}{i}$$

$$\pi = 3.1415$$



JACARANDA
MATHS QUEST

GENERAL
MATHEMATICS **12**
FOR QUEENSLAND
UNITS 3 & 4 | SECOND EDITION

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The publisher acknowledges ongoing discussions related to gender-based population data. At the time of publishing, there was insufficient data available to allow for the meaningful analysis of trends and patterns to broaden our discussion of demographics beyond male and female gender identification.



A catalogue record for this
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National Library of Australia

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Unit 4 Examination

PRACTICE ASSESSMENT 4
Units 3 & 4 Examination

Glossary..... **online only**

The new jacTUTOR

An AI tutor to build confidence and get unstuck

With jacTUTOR, every student can have the help of a personal tutor when they get stuck, in class or at home. This purpose-built tool sits safely within your favourite Jacaranda resource, so guidance will always be specific to that lesson, question and curriculum.



A personal tutor for every student

Now every student can get the help they need, when they need it. jacTUTOR uses AI to create a fair and level playing field for all students.



Get guidance, not the answer

jacTUTOR doesn't just give the answer away. Students are given prompts to help understand what they should be asking to get to a correct answer.



Combat anxiety

Afraid to raise a hand or ask more questions? With jacTUTOR, students will find a safe space to ask questions, get clarification and try again.



Consistency over confusion

There are so many ways to learn. To avoid confusion, jacTUTOR is built on the same trusted method used in the Jacaranda text.



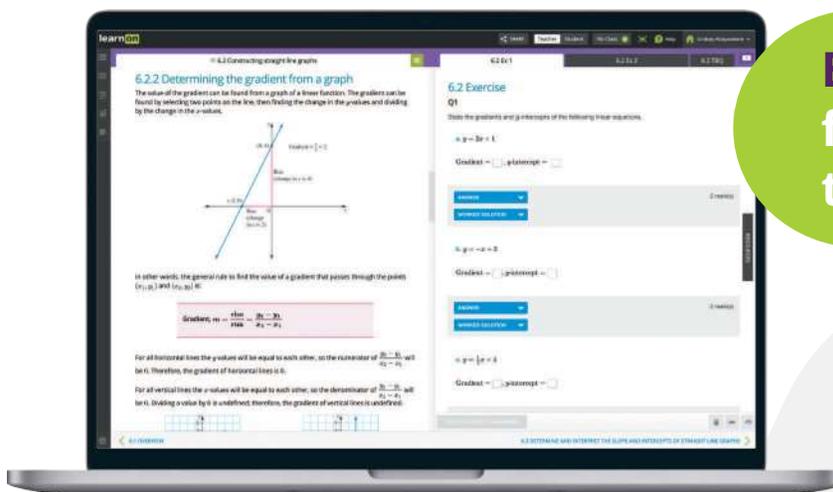
A safe space

To keep students safe, any concerning or inappropriate comments are automatically flagged and sent to their teacher.

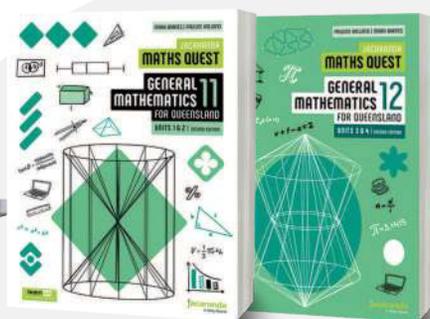
The screenshot shows the jacTUTOR interface overlaid on a laptop screen displaying a lesson page. The lesson page is titled "LESSON 5.4 The unitary method and best buys" and includes a "Fluency Q1" section with a table of chocolate weights and costs. The jacTUTOR interface has a header with "RESOURCES" and "jacTUTOR". A chat bubble contains the question "Can you help with subtopic 5.4 Exercise 2 Q1". A response bubble says "Sure, please select a help option below." Below this are three buttons: "WHAT IS THE QUESTION ASKING?", "CAN YOU SHOW ME HOW TO START?", and "HOW CAN I CHECK MY ANSWER?". At the bottom of the chat, it says "Assistant is in beta" and "Ask jacTutor ... to come".



Learning with learnON



Everything you need
for your students
to succeed



JACARANDA MATHS QUEST GENERAL MATHEMATICS 12 UNITS 3 AND 4 FOR QUEENSLAND | SECOND EDITION

Developed by expert teachers for students

Tried, tested and trusted. The completely revised and updated second edition of *Jacaranda Maths Quest General Mathematics 12 Units 3 & 4 for Queensland* continues to focus on helping teachers achieve learning success for every student – ensuring no student is left behind and no student is held back.

Because both what and how students learn matter



Learning is personal

Whether students need a challenge or a helping hand, you'll find what you need to create engaging lessons.

Whether in class or at home, students can get unstuck and progress! Scaffolded lessons with detailed worked examples and difficult concepts are supported by teacher-led video eLessons. Automatically marked, differentiated question sets are all supported by detailed worked solutions. And brand-new exam-style questions support in-depth skill acquisition in every lesson.



Learning is effortful

Learning happens when students push themselves. With learnON, Australia's most powerful online learning platform, students can challenge themselves, build confidence and ultimately achieve success.



Learning is rewarding

Through real-time results data, students can track and monitor their own progress and easily identify areas of strength and weakness.

And for teachers, Learning Analytics provide valuable insights to support student growth and drive informed intervention strategies.

Learn online with Australia's most

Everything you need for each of your lessons in one simple view

- Trusted, syllabus-aligned theory
- Engaging, rich multimedia
- All the teacher support resources you need
- Deep insights into progress
- Immediate feedback for students
- Create custom assignments in just a few clicks.

Practical teaching advice and ideas for each lesson provided in teachON

Each lesson linked to content points from the QCAA General Mathematics 2025 General senior syllabus

Reading content and rich media including embedded videos and interactivities

The screenshot shows the learnON interface for Lesson 1.3 Wages. The main content area features a 'SYLLABUS LINKS' section with a bullet point: 'Calculate weekly, fortnightly or monthly wages from an annual salary.' Below this is a source attribution: 'Source: QCE General Mathematics Syllabus extracts © QCAA; reproduced by permission.' A text block states: 'Employees may be paid for their work in a variety of ways. Most receive either a wage or a salary.' A flowchart titled 'Ways of paying employees' branches into 'Salary' and 'Wages'. The 'Salary' box defines it as a fixed amount paid per year, usually fortnightly or monthly, regardless of hours worked. It notes it's based on an annual amount divided into payments, with no extra pay for overtime and benefits like sick and annual leaves. Examples include architect, company director, data analyst, teacher, doctor, accountant, and government minister. The 'Wages' box defines it as a fixed amount per hour worked. It notes the weekly wage is the hourly rate multiplied by hours worked, with overtime paid at a higher rate and no additional payments like sick or annual leaves. Examples include waiter, kitchen hand, bar attendant, receptionist, technician, retail assistant, fruit picker, or packer. At the bottom, there are two images: one of a person in a red shirt looking at a document, and another of a person in a white shirt and blue apron in a kitchen setting. On the right side, there are exercise sections for '1.3 Exercises' with a 'SELECT YOUR' dropdown, and two questions: 'Q1 Nga is paid a sa' and 'Q2 Roger is paid a', each with a '\$' input field and 'ANSWER' and 'WORKED SOLUTION' buttons.

powerful learning tool, learnON

The image shows a screenshot of the learnON software interface on a laptop. The interface is divided into several sections. At the top, there is a navigation bar with options like 'SHARE', 'Teacher', 'Student', 'No Class', 'Help', and 'Teacher diacriTech'. Below this, there are tabs for '1.3 Ex 2' and '1.3 TBQ'. The main content area is split into two columns. The left column contains a 'PATHWAY' section with three buttons: 'SIMPLE FAMILIAR', 'COMPLEX FAMILIAR', and 'COMPLEX UNFAMILIAR'. Below these are question lists for each category. The right column is titled 'RESOURCES' and lists various items: 'Topic PDF', 'Solutions', 'Digital documents', 'Video eLessons', 'Interactivities', 'Exam question booklet', and 'TEACHER Digital documents'. Callout boxes with green lines point to specific features: 'Differentiated question sets' points to the '1.3 Ex 2' tab; 'Practice exam questions' points to the '1.3 TBQ' tab; 'Teacher and student views' points to the 'Teacher' button; 'Textbook questions' points to the 'Topic PDF' resource; 'Fully worked solutions' points to the 'Solutions' resource; 'Digital documents' points to the 'Digital documents' resource; 'Video eLessons' points to the 'Video eLessons' resource; 'Interactivities' points to the 'Interactivities' resource; 'Enhanced teaching support resources' points to the 'TEACHER Digital documents' resource; and 'Interactive questions with immediate feedback including answers and fully worked solutions' points to a question in the main content area. The question text is: 'Salary of \$67 860 per annum. Calculate Nga's fortnightly pay.' Below the question is a text input field and a '1 mark(s)' label. Another question is partially visible: 'Salary of \$86 840 per annum. Calculate Roger's weekly pay.'

Differentiated question sets

Practice exam questions

Teacher and student views

Textbook questions

Fully worked solutions

Digital documents

Video eLessons

Interactivities

Enhanced teaching support resources

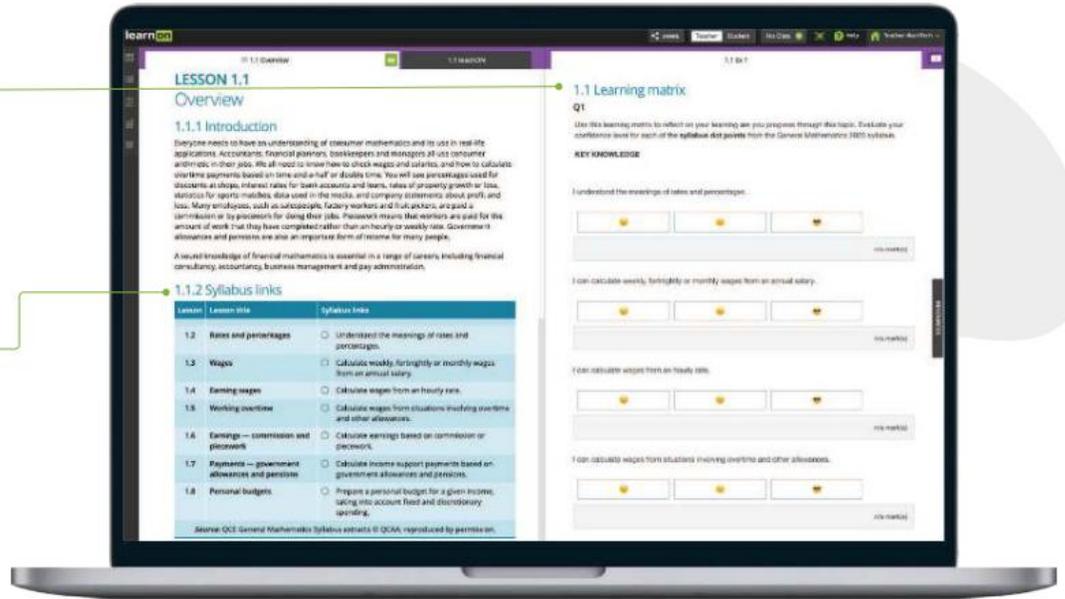
Interactive questions with immediate feedback including answers and fully worked solutions

Online, these new editions are the complete package

Trusted Jacaranda theory, plus tools to support teaching and make learning more engaging, personalised and visible.

Learning matrix to monitor student's confidence level throughout topics.

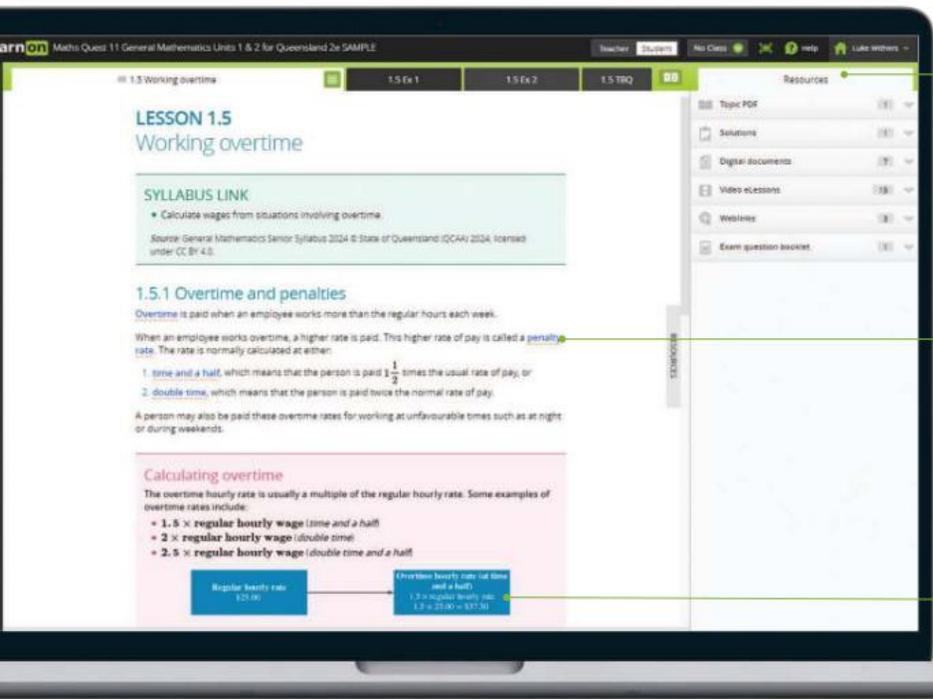
Each topic is linked to content points from the QCAA General Mathematics 2025 General senior syllabus.



onResources link to targeted digital resources including video eLessons and weblinks.

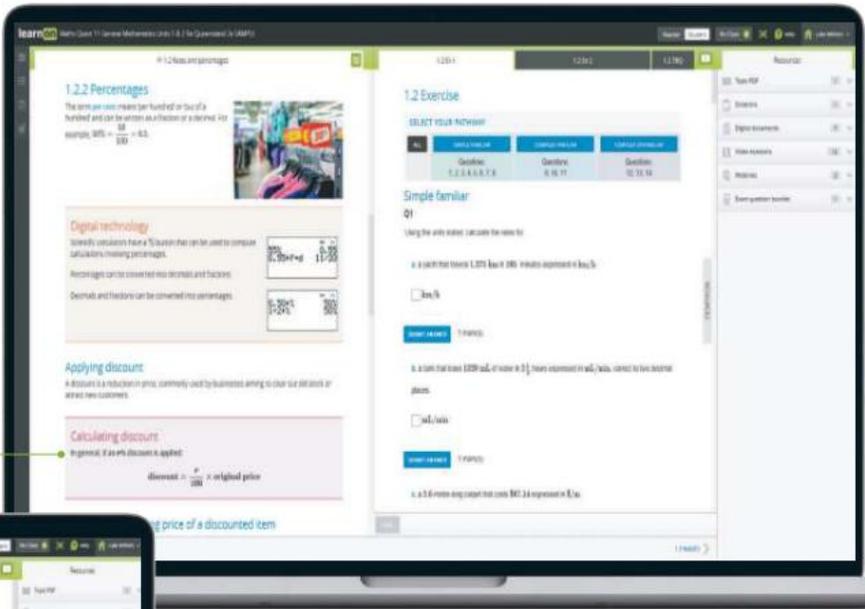
Interactive glossary terms help develop and support mathematical literacy.

Tables, flow charts and images break down content, allowing students to understand complex concepts.

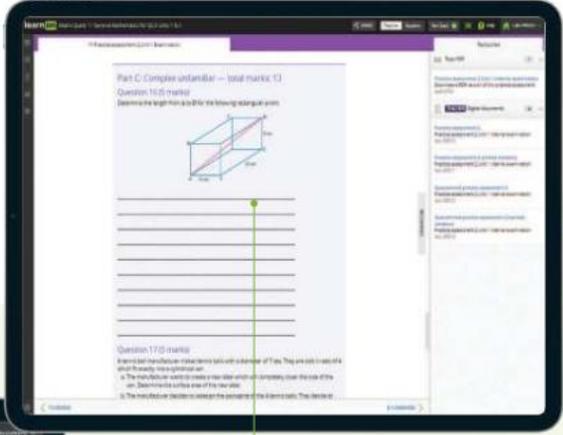
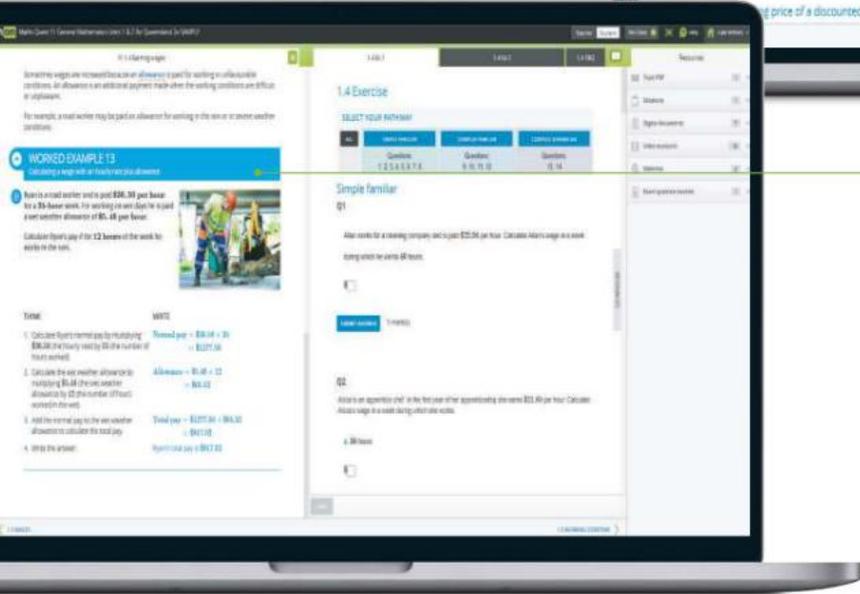




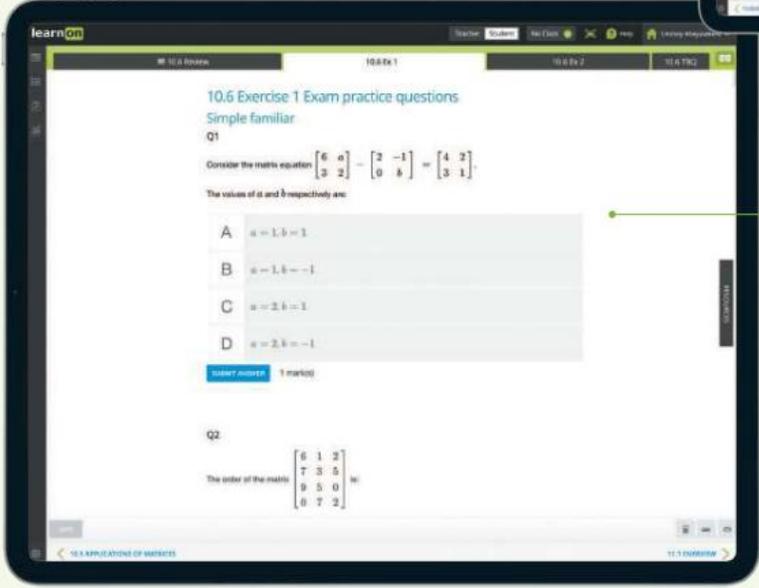
Pink highlight boxes summarise key information.



Worked examples, supported by teacher-led videos, break down the process of answering questions using a think/write format.



Each unit has practice assessments (Problem-solving and modelling task or examination) with sample responses to help build skills.



- Online and offline question sets contain practice questions and exam-style questions with exemplary responses and marking guides.
- Every question has immediate, corrective feedback to help students to overcome misconceptions as they occur and to study independently – in class and at home.

Teaching with learnON

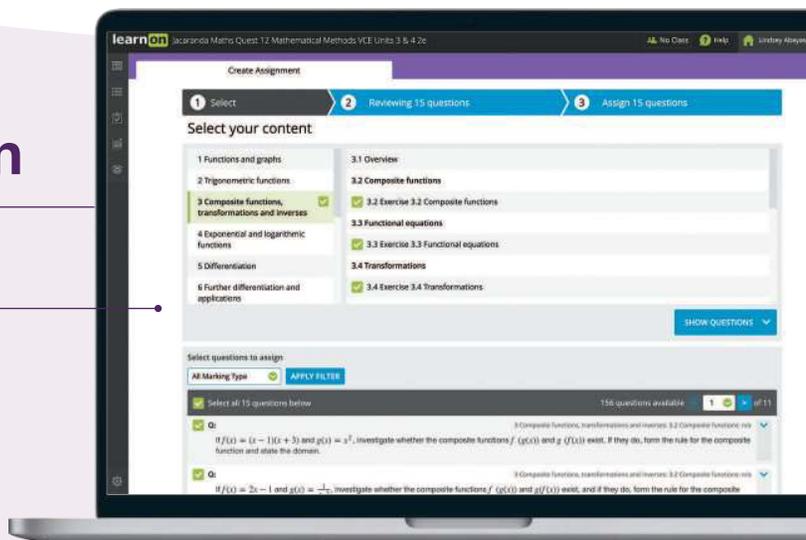


Enhanced teacher support resources, including:

- work programs and syllabus grids
 - teaching advice and additional activities
 - quarantined topic tests (with solutions)
 - Quarantined PSMTs and examinations
- Custom exam-builder with question differentiation (SF/CF/CU) question filters

Customise and assign

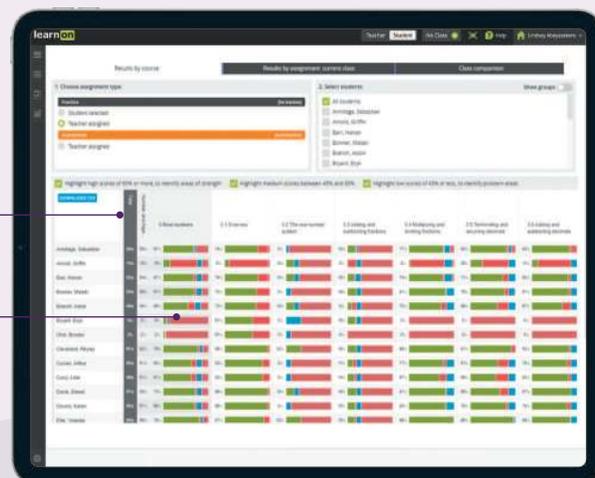
A testmaker enables you to create custom tests from the complete bank of thousands of questions (including past QCAA exam questions in year 12).



Reports and results

Data analytics and instant reports provide data-driven insights into performance across the entire course.

Show students (and their parents or carers) their own assessment data in fine detail. You can filter their results to identify areas of strength and weakness.



Acknowledgements

The authors and publisher would like to thank the following copyright holders, organisations and individuals for their assistance and for permission to reproduce copyright material in this book.

The full list of acknowledgements can be found here: www.jacaranda.com.au/acknowledgements/#2025

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UNIT

3

Bivariate data and time series analysis, sequences and Earth geometry

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1	Identifying and describing associations between two variables	3
TOPIC 2	Bivariate data analysis 2	45
2	Fitting a linear model to numerical data, and association and causation	45
TOPIC 3	Time series analysis	93
3	Time series analysis	93
TOPIC 4	Growth and decay in sequences	163
4	The arithmetic sequence	163
5	The geometric sequence	199
TOPIC 5	Earth geometry and time zones	231
6	Earth geometry and time zones	231
PRACTICE ASSESSMENT 1	Problem-solving and modelling task	online only
PRACTICE ASSESSMENT 2	Unit 3 Examination	online only

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1 Identifying and describing associations between two variables

LESSON SEQUENCE

1.1 Overview	4
1.2 Bivariate data	5
1.3 Two-way frequency tables	9
1.4 Scatterplots	19
1.5 Pearson's correlation coefficient and the coefficient of determination	28
1.6 Review	34
Answers	40

Fully worked solutions for this chapter are available online.

EXAM PREPARATION

Access exam-style questions in every lesson, available online and access past QCAA exam questions in every review.

Resources

-  **Solutions** Solutions — Chapter 1
-  **Exam questions** Exam question booklet — Chapter 1
-  **Digital documents** Learning matrix — Chapter 1
Chapter summary — Chapter 1

LESSON

1.1 Overview

Hey students! Bring these pages to life online



Watch videos



Engage with interactivities



Answer questions and check results

Find all this and MORE in jacPLUS



1.1.1 Introduction

Understanding bivariate data analysis is crucial for students, offering essential insights into comparing and contrasting data pairs. This chapter covers constructing and interpreting two-way frequency tables, allowing for efficient data organisation and pattern identification, which is vital in fields like market research and epidemiology.

Distinguishing between explanatory and response variables is key for valid statistical analysis. Scatterplots visually represent variable relationships, used in scientific research and financial modelling to detect trends.

Describing associations, calculating Pearson's correlation coefficient (r), and understanding the coefficient of determination (R^2) provide comprehensive analysis tools. These concepts are fundamental for tackling real-world data challenges and are applicable in any data-driven industry.



1.1.2 Syllabus links

Lesson	Lesson title	Syllabus links
1.2	Bivariate data	<ul style="list-style-type: none">○ Understand the meaning of bivariate data.
1.3	Two-way frequency tables	<ul style="list-style-type: none">○ Construct two-way frequency tables and determine the associated row and column sums and percentages.○ Use an appropriately percentaged two-way frequency table to identify patterns that suggest the presence of an association.○ Understand an association in terms of differences observed in percentages across categories in a systematic and concise manner, and interpret this in the context of the data.
1.4	Scatterplots	<ul style="list-style-type: none">○ Identify the explanatory variable and the response variable.○ Construct and use a scatterplot to identify the association between two numerical variables.○ Describe an association between two numerical variables in terms of direction (positive/negative), form (linear/non-linear) and strength (strong/moderate/weak).
1.5	Pearson's correlation coefficient and the coefficient of determination	<ul style="list-style-type: none">○ Calculate Pearson's correlation coefficient, r, from raw data using technology, and interpret it to quantify the strength of a linear association.○ Calculate the coefficient of determination, R^2, from raw data using technology, and interpret it to assess the strength of a linear association in terms of the explained variation.○ Use the correlation coefficient, r, to determine the coefficient of determination, R^2, and vice versa.

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LESSON

1.2 Bivariate data

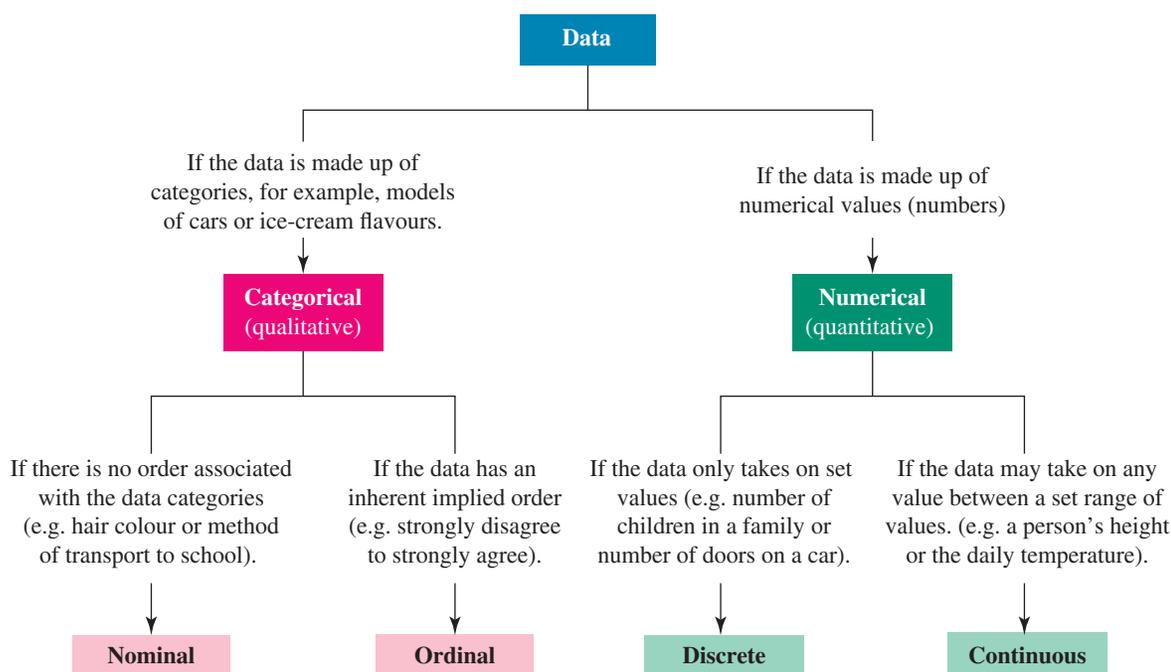
SYLLABUS LINKS

- Understand the meaning of bivariate data.

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1.2.1 Types of data

Recall from Unit 2 of General Mathematics that data are classified into two main types. The flowchart below distinguishes the types.



WORKED EXAMPLE 1 Categorising data

Categorise the following examples of data.

- The values of sales recorded at each branch of a fast-food outlet.
- The breeds of dog that appear at a dog show.
- The mass of a baby at birth.



THINK

- a. The values of sales at each branch can be measured, so the data must be numerical. As money the data can be only in a set amount (of cents).
- b. The breeds of dog at a show give non-numerical results, so the data must be categorical. There is no inherent implied order in dog breeds.
- c. A baby's mass is given as a set of numbers, so must be numerical. The mass can be measured to various degrees of accuracy.

WRITE

- a. The values of sales and numerical discrete data.
- b. The breeds of dog are categorical nominal data.
- c. The mass of baby's is numerical continuous data.

1.2.2 Univariate and bivariate data

In General Mathematics Unit 2, univariate data analysis was covered in detail. The study and analysis of univariate data involves the study of only one variable at a time, for example, the analysis of the heights of 100 basketballers or the favourite television series of 150 first year university students.



The study and analysis of **bivariate data** is the study of two variables at the same time to determine if a relationship exists between those variables and how that relationship can be used to make predictions. Examples of bivariate data are studying the relationship between hours studying and grades, average rainfall and number of plants, or blood pressure and age.

WORKED EXAMPLE 2 Determining univariate or bivariate data

State whether the variables below are examples of univariate or bivariate data, classifying them as categorical or numerical:

- a. The favourite spectator sport of 20 Year 12 students.
- b. Favourite ice-cream flavour and favourite colour of 30 people in a shopping centre.
- c. The weight and length of 40 newborn babies.

THINK

- a. The favourite spectator sport has only one variable involved, so univariate data. The data would be a category of types of sport.
- b. Two variables, favourite ice-cream and favourite colour, are involved so bivariate data. Both sets of data are in categories of flavours and colours respectively.
- c. Two variables, weight and length, are involved so bivariate data. Both sets of data would give numerical results.

WRITE

- a. The favourite spectator sport of 20 Year 12 students is univariate and categorical data.
- b. The favourite ice-cream flavour and favourite colour of 30 people in a shopping centre is bivariate and categorical data.
- c. The weight and length of 40 newborn babies is bivariate and numerical data.

1.2 Exercise

1.2 Exam questions **on**

Simple familiar
1, 2, 3, 4, 5, 6, 7, 8, 9

Complex familiar
10, 11

Complex unfamiliar
12

These questions are even better in jacPLUS!

- Receive immediate feedback
- Access sample responses
- Track results and progress



Find all this and MORE in jacPLUS

Simple familiar

1. A school principal is collecting data on students to analyse trends in the school population. For each of the following variables, classify whether the data is categorical or numerical.

Variable	Categorical or numerical
Year level of the student	
Student's weight (in kg)	
Favourite subject	
Number of books borrowed from the library per month	
Type of transport used to get to school	

2. **WE1** State whether the numerical data gathered in each of the following situations are discrete or continuous.

- The heights of 60 tomato plants at a plant nursery
- The number of jelly beans in each of 50 packets
- The time taken for each student in a class of six-year olds to tie his or her shoelaces
- The petrol consumption rate of a large sample of cars
- The IQ (intelligence quotient) of each student in a class



3. For each of the following, state if the data are categorical or numerical.

If numerical, state if the data are discrete or continuous.

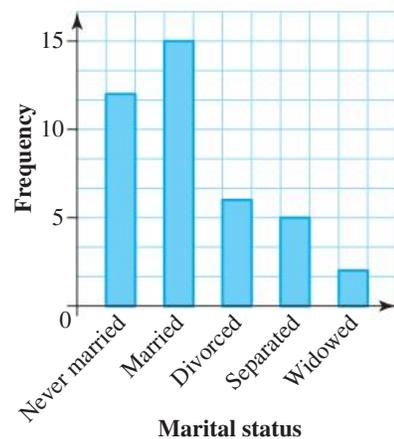
- The number of students in each class at your school
 - The teams people support at a football match
 - The brands of peanut butter sold at a supermarket
 - The heights of people in your class
 - The interest rate charged by each bank
 - A person's pulse rate
4. An opinion poll was conducted. A thousand people were given the statement 'Euthanasia should be legalised'. Each person was offered five responses: strongly agree, agree, unsure, disagree and strongly disagree. Describe the data type in this example.
5. A teacher marks her students' work with a grade A, B, C, D or E. Describe the data type used.
6. A teacher marks his students' work using a mark out of 100. Describe the data type used.

7. **WE2** State whether the following data are univariate or bivariate data, classifying them as categorical or numerical.

Variable	Categorical or numerical	Univariate or bivariate
The mark scored on a mathematics test and the hours of mathematics homework completed during the previous week.		
Water usage and time spent in the shower for 50 students in a hostel.		
The number of vehicles passing a point on a road during a given time period.		
A group of customers were asked to respond to the level of service they received in a store as poor, fair, excellent.		
The time each customer spends in a supermarket queue on a Saturday morning.		
Days of the week and choices of public transport.		

8. **MC** The chart shows the marital status of respondents to a survey. Choose the option below that best describes this data.

- A. Univariate and categorical data
- B. Univariate and numerical data
- C. Bivariate and categorical data
- D. Bivariate and numerical

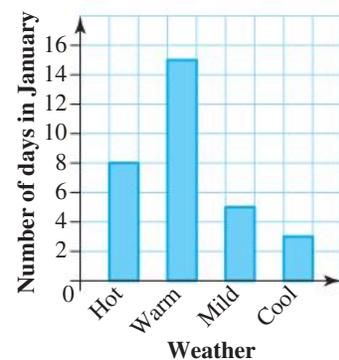


9. A scientist is investigating whether the amount of fertiliser used affects the height of tomato plants. Explain whether this is univariate or bivariate data, with reasoning.

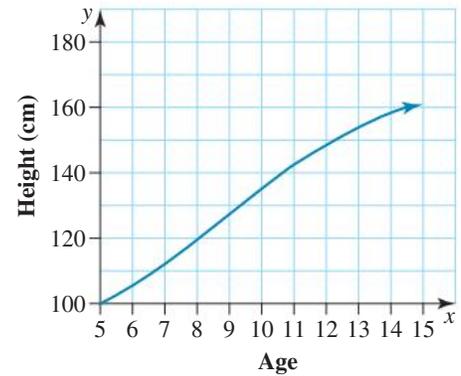


Complex familiar

10. The following graph shows the number of days of each weather type for the Gold Coast in January. Describe the data shown in this graph.



11. The following graph shows a girl's height each year for 10 years. Describe the data shown in this graph.



Complex unfamiliar

12. An online department store decides to send out a survey to collect some information about their customers. The survey asks their customers to provide the following information:

- Age (< 18, 18 – 29, 30 – 50, 50+)
- Gender
- Income
- Postcode

The customers' digital footprints provide records of their purchases, the amount of time the customer spends browsing and the amount of money the customer spends on a weekly basis.

Using this information identify five questions that could be answered by analysing this data and state whether this analysis would be univariate or bivariate.



Fully worked solutions for this chapter are available online.

LESSON

1.3 Two-way frequency tables

SYLLABUS LINKS

- Construct two-way frequency tables and determine the associated row and column sums and percentages.
- Use an appropriately percentaged two-way frequency table to identify patterns that suggest the presence of an association.
- Understand an association in terms of differences observed in percentages across categories in a systematic and concise manner, and interpret this in the context of the data.

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1.3.1 Categorical variables and two-way frequency tables

Two-way frequency tables are a useful means of organising, identifying and describing an association between two categorical variables for a group of individuals or objects.

Consider, for example, data collected on the eye colour of 200 male/female couples. It may be represented in a table such as the one on the following page.

		Female			Total
		Green	Brown	Blue	
Male	Blue	11	25	9	45
	Brown	19	51	28	98
	Green	17	27	13	57
Total		47	103	50	200

Splitting the data into categories based on the eye colour of the **males** and calculating percentages in each category would yield the following results. That is, 11 of the 45 blue-eyed males have female partners who have green eyes. So, $\frac{11}{45} \times 100 = 24.4\%$ of blue-eyed males have female partners with green eyes, and so on.

		Female			Total
		Green	Brown	Blue	
Male	Blue	$\frac{11}{45} \times 100 = 24.4\%$	$\frac{25}{45} \times 100 = 55.6\%$	$\frac{9}{45} \times 100 = 20\%$	100%
	Brown	$\frac{19}{98} \times 100 = 19.4\%$	$\frac{51}{98} \times 100 = 52.0\%$	$\frac{28}{98} \times 100 = 28.6\%$	100%
	Green	$\frac{17}{57} \times 100 = 29.8\%$	$\frac{27}{57} \times 100 = 47.4\%$	$\frac{13}{57} \times 100 = 22.8\%$	100%

The following table gives the percentages rounded to the nearest whole number.

		Female			Total
		Green	Brown	Blue	
Male	Blue	24%	56%	20%	100%
	Brown	19%	52%	29%	100%
	Green	30%	47%	23%	100%

Splitting the data into categories based on the eye colour of the **females** and calculating percentages in each category would yield the following results. That is, 11 of the 47 green-eyed females have male partners who have blue eyes. So, $\frac{11}{47} \times 100 = 23.4\%$ of green-eyed females have male partners with blue eyes.

		Female		
		Green	Brown	Blue
Male	Blue	$\frac{11}{47} \times 100 = 23.4\%$	$\frac{25}{103} \times 100 = 24.3\%$	$\frac{9}{50} \times 100 = 18\%$
	Brown	$\frac{19}{47} \times 100 = 40.4\%$	$\frac{51}{103} \times 100 = 49.5\%$	$\frac{28}{50} \times 100 = 56\%$
	Green	$\frac{17}{47} \times 100 = 36.2\%$	$\frac{27}{103} \times 100 = 26.2\%$	$\frac{13}{50} \times 100 = 26\%$
Total		100%	100%	100%

The following table gives the percentages rounded to the nearest whole number.

		Female		
		Green	Brown	Blue
Male	Blue	23%	24%	18%
	Brown	41%	50%	56%
	Green	36%	26%	26%
Total		100%	100%	100%

It is obvious that the interpretation of the data depends on the reference basis. We may wish to consider those couples where the male is blue-eyed and the female brown-eyed. Note that this represents 25 couples. What if we talk about percentages? Comparing the percentages in the two tables, it can be seen that:

1. 56% of blue-eyed males have female partners with brown eyes (see pink shading).
2. 24% of brown-eyed females have male partners with blue eyes (green shading).

These percentages have vastly different values, yet they both describe the same set of 25 couples of blue-eyed males and brown-eyed females. It is important, particularly when dealing with two-way frequency tables, to consider the reference basis for percentages.

WORKED EXAMPLE 3 Constructing two-way tables

A new test was designed to assess the reading ability of students entering high school.

The results were used to determine if the students' reading level was adequate to cope with high school.

The students' results were then checked against existing records.

Of the 150 adequate readers who sat for the test, 147 of them passed.

Of the 50 inadequate readers who sat for the test, 9 of them passed.

Present this information in a two-way frequency table.



THINK

Construct the table showing the existing records (adequate readers and inadequate readers) and the results of the new test (passed and did not pass).

WRITE

	New test results		
	Passed	Did not pass	Total
Adequate readers	147	3	150
Inadequate readers	9	41	50
Total	156	44	200

When information on a test is presented in a two-way frequency table, conclusions can be made about the accuracy of the test.

WORKED EXAMPLE 4 Interpreting two-way tables

A batch of sniffer dogs are trained by customs to smell drugs in suitcases. Before the dogs are used at airports they must pass a test by detecting the smell of drugs and selecting the bags. The results of that test are shown in the following two-way frequency table.

		Test results		
		Detected	Not detected	Total
Bags	With drugs	24	1	25
	Without drugs	11	164	175
	Total	35	165	200

- Determine how many bags did the sniffer dogs examine.
- Determine in how many bags did the dogs detect drugs.
- Determine in what percentage of bags without drugs the dogs incorrectly detected drugs. Write your answer correct to 1 decimal place.
- Based on the above results, determine the percentage of the time that the dogs will not detect a bag carrying drugs.

THINK

- The total number is located in the bottom right cell.
- The total of the detected column.
- There were 175 bags without drugs but dogs incorrectly detected drugs in 11 bags. Write this as a percentage, rounding to one decimal place.
- Of 25 bags with drugs, 1 went undetected. Write this as a percentage.

WRITE

- 200 bags were examined.
- The dogs detected drugs in 35 bags.
- Percentage incorrectly detected
$$= \frac{11}{175} \times 100\%$$
$$= 6.3\%$$
- Percentage not detected
$$= \frac{1}{25} \times 100\%$$
$$= 4\%$$

After studying a two-way frequency table, we should also be able to make judgements about the information given in the tables. Many two-way frequency tables will require you to make your own value judgements about the conclusions established. For example, the 94% overall accuracy recorded may be considered 'very acceptable'.

WORKED EXAMPLE 5 Constructing a percentage two-way table

The following data table shows the composition of the employees of a small law firm.

	Full-time	Part-time
Female	4	11
Male	30	5

- Extend the table to show totals in all categories and an overall total.
- Construct a table showing percentages with respect to type of employment (full or part-time).
- Construct a table showing percentages with respect to the gender of the employee.
- Determine the percentage of females who work full time.
- Determine the percentage of full-time workers who are female.

THINK

- a. Add the numbers in the cells for all the rows and columns and enter the totals. Check that the overall total is consistent for the rows and columns.
- b. Percentages with respect to type of employment are based on totals in columns. The totals in the columns are on the denominator when calculating percentages.

WRITE

a.

	Full-time	Part-time	Total
Female	4	11	15
Male	30	5	35
Total	34	16	50

b.

	Full-time	Part-time
Female	$\frac{4}{34} \times 100 = 12\%$	$\frac{11}{16} \times 100 = 69\%$
Male	$\frac{30}{34} \times 100 = 88\%$	$\frac{5}{16} \times 100 = 31\%$
Total	100%	100%

- c. Percentages with respect to gender are based on totals in rows. The totals in the rows are on the denominator when calculating percentages.

c.

	Full-time	Part-time	Total
Female	$\frac{4}{15} \times 100 = 27\%$	$\frac{11}{15} \times 100 = 73\%$	100%
Male	$\frac{30}{35} \times 100 = 86\%$	$\frac{5}{35} \times 100 = 14\%$	100%

- d. 1. This is based on female totals shown in the table in part c.
2. Write the answer.

d. $\frac{\text{full time}}{\text{female total}} \times 100 = \frac{4}{15} \times 100 = 27\%$

Percentage of females who work full time = 27%

- e. 1. This is based on full-time totals shown in the table in part b.
2. Write the answer.

e. $\frac{\text{female}}{\text{full-time total}} \times 100 = \frac{4}{34} \times 100 = 12\%$

Percentage of full-time workers who are female = 12%

We are constantly bombarded with statistics, some of which are a valid interpretation of data, and some of which are not. On occasions, the misuse of statistics may be unintentional or through ignorance, but there are occasions when misleading figures are quoted intentionally.

WORKED EXAMPLE 6 Filling in unknown information

When discussing the probability of female attendance of 15–19-year-olds at an educational institution in a large city, it was claimed that 51% of the females in this age group attended an educational institution. Some of the information is contained in the following two-way frequency table.

15–19-year-olds	Male	Female	Total
Attend an educational institution	40 549	42 704	
Do not attend an educational institution			
Total	63 978	61 948	

- a. Complete the two-way frequency table displaying the attendance/non-attendance of 15–19-year-old males and females at an educational institution and the totals.
- b. Use the two-way frequency table to discuss the validity of the claim that 51% of the females in the group attended an educational institution.

THINK

- a. Complete the two-way frequency table with totals for rows and columns. Subtract the attendance figure from its relevant total to determine the non-attendance figure.

- b. 1. Calculate the percentage of females who attend an educational institution; that is,

$$\frac{\text{no. females at education institute}}{\text{total number of females}} \times 100$$
2. Calculate the percentage of those in an educational institution who are female; that is,

$$\frac{\text{no. females at education institute}}{\text{total number in an educational institution}} \times 100$$
3. Compare these figures with the claim above to determine if the claim is correct.

WRITE

- a.

15–19-year-olds	Male	Female	Total
Attend an educational institution	40 549	42 704	83 253
Do not attend an educational institution	23 429	19 244	42 673
Total	63 978	61 948	125 926

- b. The percentage of females who attend an educational institution = $\frac{42\,704}{61\,948} \times 100$
= 69%

69% of females attend an educational institution.

The percentage of people in an educational institution who are female = $\frac{42\,704}{82\,253} \times 100$
= 51%

51% of those in an educational institution are female.

The claim that 51% of females in the group attended an educational institution is not correct. It should have said that 69% of 15–19-year-old females attend an educational institution.

1.3 Exercise

1.3 Exam questions **on**

Simple familiar

1, 2, 3, 4, 5, 6, 7, 8, 9

Complex familiar

10, 11, 12, 13

Complex unfamiliar

14, 15

These questions are even better in jacPLUS!

- Receive immediate feedback
- Access sample responses
- Track results and progress



Find all this and MORE in jacPLUS

Simple familiar

1. **WE3** A test is developed to test for infection with the flu virus. To test the accuracy, the following 500 people are tested.

- Of the 100 people who are known to have the flu who are tested, the test returns 98 positive results.
- Of the 400 people who are known not to be infected with the virus who are tested, 12 false positives are returned.

	Test results		
	Accurate	Not accurate	Total
With virus			
Without virus			
Total			

Display this information in the two-way frequency table provided.

2. One thousand people take a lie detector test. Of 800 people known to be telling the truth, the lie detector indicates that 23 are lying. Of 200 people known to be lying, the lie detector indicates that 156 are lying. Present this information in a two-way frequency table.

3. **WE4** The following table indicates the results of a radar surveillance system. If the system detects an intruder, an alarm is activated.

	Test results		
	Alarm activated	Not activated	Total
Intruders	40	8	48
No intruders	4	148	152
Total	44	156	200

- Determine over how many nights the system was tested.
- Determine on how many occasions the alarm was activated.
- If the alarm is activated, determine the percentage chance that there actually is an intruder.
- If the alarm was not activated, determine the percentage chance that there was an intruder.
- Determine the percentage of accurate results over the test period.
- Comment on the overall performance of the radar detection system.

4. **WE5** The following table displays the accuracy gained from a medical test screening for a virus.

Note: A positive test indicates that the patient has the virus.

	Test results		
	Accurate	Not accurate	Total
With virus	45	3	48
Without virus	922	30	952
Total	967	33	1000

- Determine how many patients were screened for the virus.
- Determine how many positive tests were recorded (that is, in how many tests was the virus detected).
- Determine what percentage of test results were accurate.

The information below is to be used in questions 5 to 7.

A test for a medical disease does not always produce the correct result. A positive test indicates that the patient has the condition. The following table indicates the results of a trial on a number of patients who were known to either have the disease or known not to have the disease.

	Test results		
	Accurate	Not accurate	Total
With disease	57	3	60
Without disease	486	54	540
Total	543	57	600

5. **MC** The overall accuracy of the test is:
A. 9.5% **B.** 90% **C.** 90.5% **D.** 92.5%
6. **MC** Based on the table, determine the probability that a patient who has the disease has it detected by the test.
A. 9.5% **B.** 90% **C.** 90.5% **D.** 95%
7. **MC** Determine which of the following statements is correct.
A. The test has a greater accuracy with positive results than with negative results.
B. The test has a greater accuracy with negative results than with positive results.
C. The test is equally accurate with positive and negative test results.
D. The test is equally inaccurate with positive and negative test results.
8. **WE6** Airport scanning equipment is tested by scanning 200 pieces of luggage. Prohibited items were placed in 50 bags and the scanning equipment detected 48 of them. The equipment detected prohibited items in five bags that did not have any forbidden items in them.
a. Use the above information to complete the following two-way frequency table.

	Test results		
	Accurate	Not accurate	Total
Bags with prohibited items			
Bags with no prohibited items			
Total			

- b.** Use the table to answer the following.
- Determine the percentage of bags with prohibited items that were detected as having prohibited items.
 - Determine the percentage of false positives among the bags that had no prohibited items.
 - Determine the percentage of the original 50 prohibited items that pass through the scanning equipment undetected.
 - Determine the overall percentage accuracy of the scanning equipment.
9. The following two-way frequency table shows the results of a test done on a speed camera. A positive result means that the camera detected the car as speeding.
- | | Positive | Negative | Total |
|--------------|----------|----------|-------|
| Speeding | 18 | 2 | 20 |
| Not speeding | 5 | 95 | 100 |
| Total | 23 | 97 | |
- Determine how many cars in total were involved in the test.
 - Determine how many cars were speeding.
 - Determine how many cars were recorded as speeding.
 - Based on the above results, determine the percentage chance (probability) that a car that is speeding escapes detection by the camera.

Complex familiar

10. The following two-way frequency table shows the results of the tests done on a car alarm system.

	Alarm activated	Alarm not activated	Total
Car break-in	43	7	50
No break-in	2	298	300
Total	45	305	

- Determine how many tests were conducted on the alarm.
 - Determine the number of times the alarm was activated.
 - Based on the above results, if the alarm is ringing, determine the percentage chance (probability) that a break-in is actually taking place.
11. A medical test is designed to detect the presence of a certain gene in unborn babies. The test is conducted on a sample of unborn babies with the following results.
- Of 100 babies with the gene, 95 were detected as having the gene during the test.
 - Of 900 babies without the gene, 37 were detected as having the gene during the test.
- Display the results in a two-way frequency table.
 - Determine the percentage probability that, if the gene is present, it is detected.
 - After birth, a baby that is thought to have the gene is chosen at random. Determine the percentage probability that the baby does actually have the gene being tested for.
 - Determine the percentage probability that a baby chosen at random from the sample was correctly diagnosed.
12. The following table shows information about a particular survey of a sample of men. It shows the percentage of men by age and their preferred type of movie.

Age group of men surveyed	Preferred type of movie		
	Action	Comedy	Drama
19 years and under	9.5%	4.7%	3.0%
20 to 35 years	41.1%	30.3%	20.5%
36 to 59 years	22.4%	30.7%	35.5%
60 years and over	27.0%	34.3%	41.0%

- Of the men who preferred drama series, determine what percentage were between 20 and 59 years inclusive.
 - State whether the information in the table supports the opinion that, for this sample of men, the preferred type of movie is associated with age. Justify your answer with the appropriate percentages.
13. A medical test screens 200 people for a virus. A positive test result indicates that the patient has the virus.
- Of 50 people known to have the virus, the test produced 48 positive results.
 - Of the remainder who were known not to have the virus, the test produced one positive result.
- Determine the percentage of people with the virus who returned a positive result.
 - Determine the probability of returning a positive test when not infected.

Complex unfamiliar

14. 67 primary and 47 secondary school students were asked about their attitude towards the number of school holidays which should be given.



They were asked whether there should be more, fewer or the same number. Five primary students and 2 secondary students wanted fewer holidays, 29 primary and 9 secondary students thought that they had enough holidays (that is, they chose the same number) and the rest thought that they needed to be given more holidays.

Form a two-way frequency table with reference to primary and secondary school percentages and determine if there is an association between the age of the student (primary or secondary) and their view about holidays. Use percentages to justify your answer.

15. A researcher investigates whether there is an association between regular exercise and preferred mode of transport to work. A survey of 500 people produces the following two-way frequency table:

Mode of transport	Exercises regularly	Does not exercise regularly	Total
Car	120	180	300
Public transport	90	50	140
Active (walk/bike)	40	20	60
Total	250	250	500



Using appropriate percentages, determine whether there is an association between exercising regularly and mode of transport. Justify your conclusion.

Fully worked solutions for this chapter are available online.

LESSON

1.4 Scatterplots

SYLLABUS LINKS

- Identify the explanatory variable and the response variable.
- Construct and use a scatterplot to identify the association between two numerical variables.
- Describe an association between two numerical variables in terms of direction (positive/negative), form (linear/non-linear) and strength (strong/moderate/weak).

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1.4.1 Numerical variables and scatterplots

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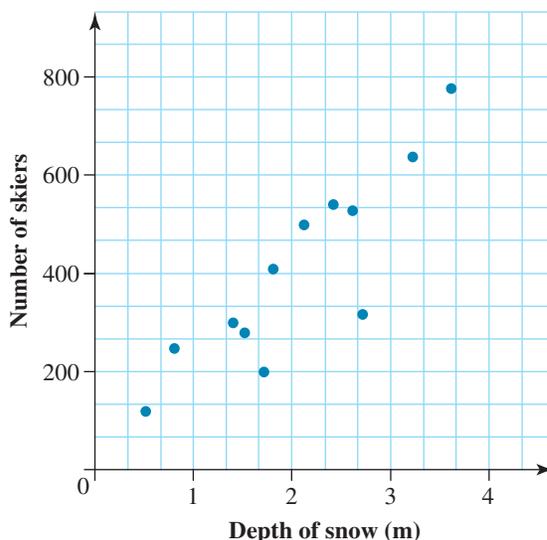
The manager of a ski resort collects the following numerical data over 12 consecutive weekends at his resort.

Depth of snow (m)	0.5	0.8	2.1	3.6	1.4	1.5	1.8	2.7	3.2	2.4	2.6	1.7
Number of skiers	120	250	500	780	300	280	410	320	640	540	530	200

As there are two types of data in this example, this is an example of bivariate data. For each item (weekend), two variables are considered (depth of snow and number of skiers). In the case of the ski resort data, the manager might be interested in answering the following questions.

1. Are visitor numbers related to depth of snow?
2. If there is a relationship between visitor numbers and depth of snow, is it always true or is it just a guide? In other words, how strong is the relationship?
3. How much confidence could be placed in the prediction?

Scatterplots are a useful means of identifying and describing an association between two numerical variables. The data from the ski resort is used to create the scatterplot shown. Each of the data points is represented by a single visible point on the graph.



1.4.2 Explanatory and response variables

When constructing a scatterplot, it is important to choose the correct variable to assign to each of the axes. The convention is to place the **explanatory** (independent) **variable** on the x -axis and the **response** (dependent) **variable** on the y -axis. The explanatory variable in an experiment or investigation is the variable that is deliberately controlled or adjusted by the investigator. The response variable is the variable that responds to changes in the explanatory variable.

Neither of the variables involved in the ski resort data were controlled directly by the investigator but ‘Number of skiers’ would be considered the response variable because it is likely to change depending on depth of snow. (The snow depth does not depend on numbers of skiers.) Notice how the scatterplot above, for the ski resort data shows a general upward trend. It is not a perfectly straight line, but it is still clear that a general trend or relationship has formed: as the depth of snow increases, so too does the number of skiers.

WORKED EXAMPLE 7 Constructing a scatterplot

The following table shows the height and mass of ten Year 11 students.

Height (cm)	120	124	130	135	142	148	160	164	170	175
Mass (kg)	45	50	54	59	60	65	70	78	75	80

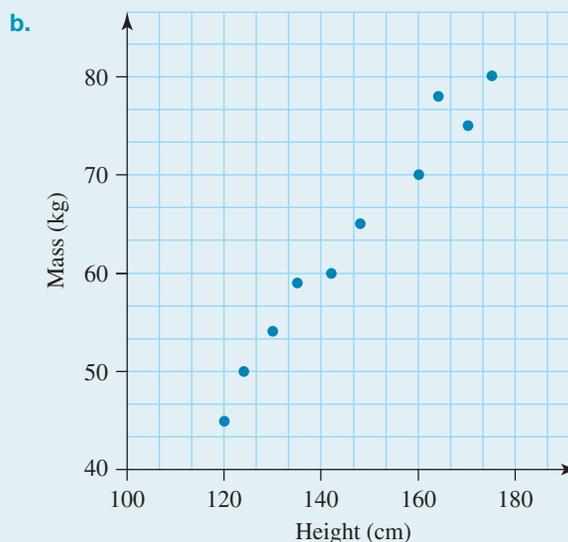
- Identify the explanatory and response variables
- Construct a scatterplot of the data.

THINK

- Consider which variable does not respond to the other. The height of a person will not be changed due to their mass; however, their height could explain their mass.
- Place the height on the x -axis and the mass on the y -axis since height is the explanatory variable and mass is the response variable.
 - Plot the point given by each pair.

WRITE

- Explanatory variable = Height
Response variable = Mass



1.4.3 Linear and non-linear relationships

When we are describing the relationships between two variables displayed on a scatterplot, we need to comment on:

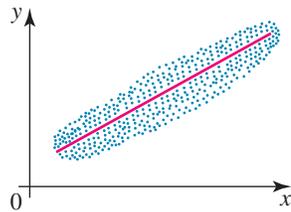
- the form — whether it is linear or non-linear
- the direction — whether it is positive or negative
- the strength — whether it is strong, moderate or weak.

The form and direction

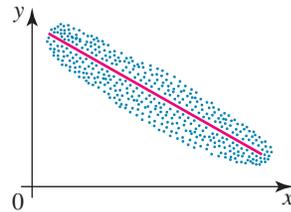
The form of a bivariate data set can be categorised as either linear or non-linear. The points on the scatterplot will produce a linear pattern or a non-linear pattern.

In a linear pattern, the points tend to form a straight line, whereas in a non-linear pattern, the points tend to form a curve.

Linear relationships

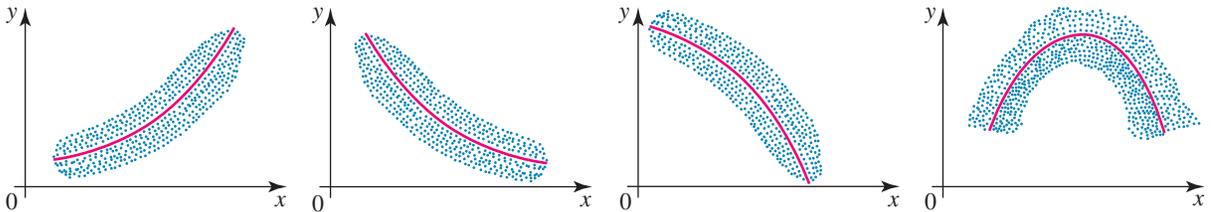


There is a positive relationship between variables; that is, as one variable increases, the other variable also increases.



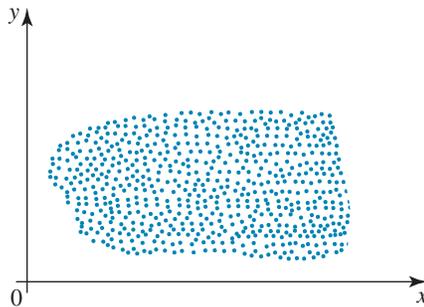
There is a negative relationship between variables; that is, as one variable increases, the other variable decreases.

Non-linear relationships



No relationship

In other cases it may be that there is no relationship at all between the two variables. Such a scatterplot would look like the one shown.



WORKED EXAMPLE 8 Determining a relationship

The following table shows the length and mass of a dozen eggs.

Length (cm)	6.2	3.9	4.5	5.8	7.2	7.6	6.1	6.7	7.3	5.1	6.0	7.3
Mass (g)	60	15	25	50	95	110	55	75	95	35	54	96

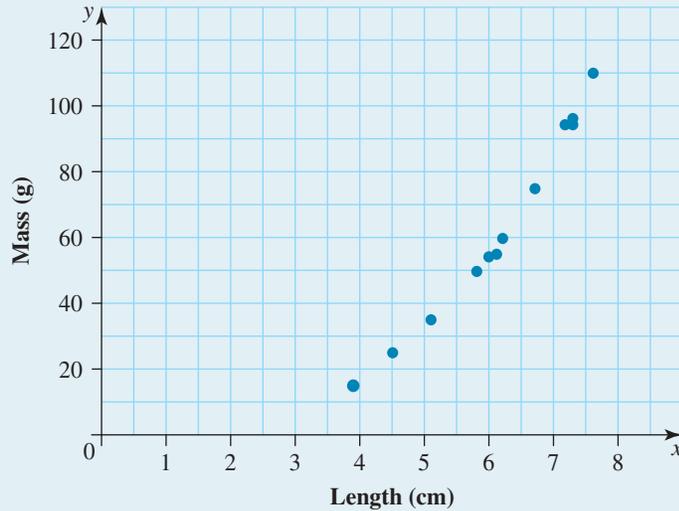
- Display this information in a scatterplot.
- Determine if there is any relationship between the length and mass of the eggs and state if the relationship is linear.

THINK

- Display length on the x -axis and mass on the y -axis since the length is the explanatory variable and the mass is the response variable.
 - Plot the point given by each pair.
- Study the scatterplot to see if mass increases as length increases.
 - Study the scatterplot to see if the points seem to approximate a straight line.

WRITE

a.



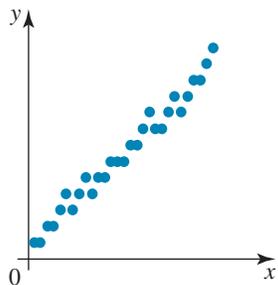
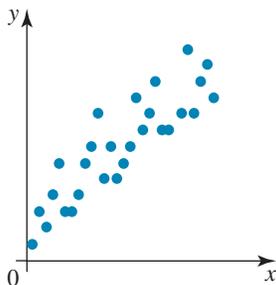
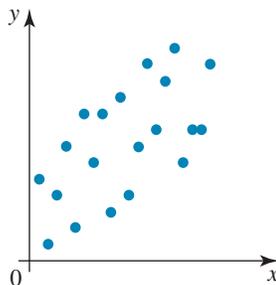
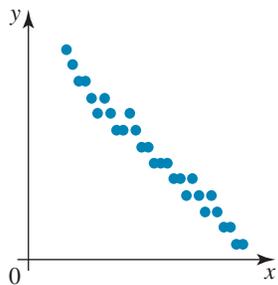
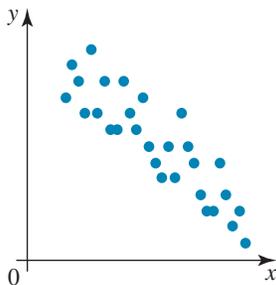
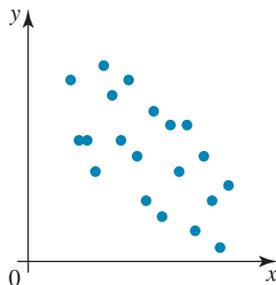
- As length increases, so does the mass of the egg.

The points do not approximate a straight line and so the relationship is non-linear.

The strength

When interpreting a scatterplot, the association or correlation provides an insight into the relationship between the two variables. The association is a measure of the strength of the linear relationship between the two variables.

To measure how strongly the scatterplot points tend to form a straight line, we begin by estimating the strength of the association as strong, moderate or weak based on inspection of the scatterplot.

Strong positive correlation**Moderate positive correlation****Weak positive correlation****Strong negative correlation****Moderate negative correlation****Weak negative correlation**

1.4.4 Graphing a scatterplot using a spreadsheet

Using a spreadsheet to graph a scatterplot involves inputting the data into the cells, selecting the appropriate chart type, and customising the plot to clearly display the relationship between the two variables.

WORKED EXAMPLE 9 Using digital tools

This table shows the hearing test scores of people of different ages.

Age (years)	55	40	35	30	42	48	50	48	32	45	30	56
Hearing test score	2.5	3.8	4.0	3.9	2.5	3.2	2.2	1.8	3.0	2.0	4.0	1.8

- Identify the explanatory variable (put on the x -axis).
- Using Excel, construct a scatterplot showing hearing test score against age.
- By examining the scatterplot, determine if there is an association between age and hearing loss.



THINK

- The explanatory variable in an experiment or investigation is the variable that is deliberately controlled or adjusted by the investigator. This would be the age of the participants.

WRITE

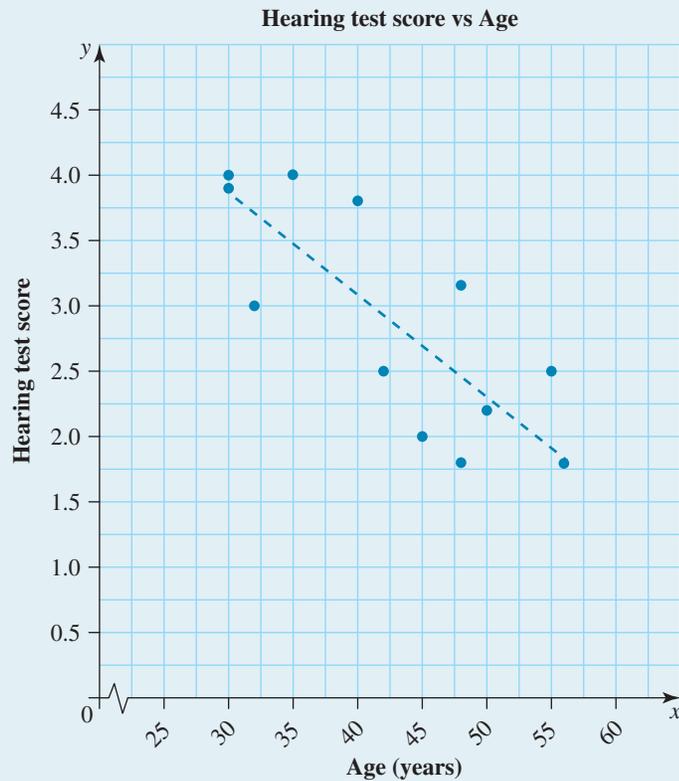
- The explanatory variable is 'age'.

- b. 1. Open a new spreadsheet and enter the data for 'Age' in Column A and 'Hearing test score' in Column B.

b.

	A	B
1	Age (Years)	Hearing test score
2	55	2.5
3	40	3.8
4	35	4.0
5	30	3.9
6	42	2.5
7	48	3.2
8	50	2.2
9	48	1.8
10	32	3.0
11	45	2.0
12	30	4.0
13	56	1.8

2. Construct a scatterplot in Excel with age on the horizontal axis and hearing test score on the vertical axis.



- c. The points appear to follow a weak negative linear pattern.
- c. The scatterplot shows a weak, negative linear pattern which would suggest there is a weak negative association between age and hearing loss, this suggests that as a person ages, their hearing decreases.

1.4 Exercise

1.4 Exam questions **on**

Simple familiar
1, 2, 3, 4, 5, 6, 7, 8, 9

Complex familiar
10, 11, 12

Complex unfamiliar
13, 14

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- Receive immediate feedback
- Access sample responses
- Track results and progress



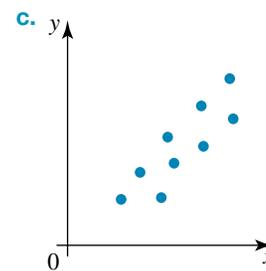
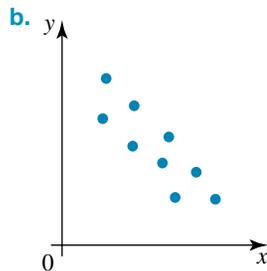
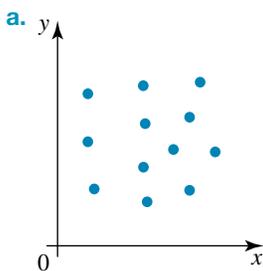
Find all this and MORE in jacPLUS

Simple familiar

- For each of the following:
 - Identify the explanatory and response variables in each of the following scenarios.
 - state what happens to the value of the second variable when there is an increase in the value of the first variable.
 - Age and height
 - Distance travelled and time taken (at a fixed speed)
 - Temperature and elevation
 - Blood alcohol level and reaction time
 - IQ and results on an academic test
 - Overtime pay and hours worked
 - Value of a car and its age
 - Time taken to travel a given distance and speed of travel.



- For the graphs below, indicate whether the two variables graphed display a positive, negative or no relationship between their values.



Where appropriate, use a spreadsheet to graph the scatterplots in the following questions.

- WE7** The following table shows the maximum temperature each day, together with the number of people who attend the cinema that day.

Temperature (°C)	25	33	30	22	15	18	27	22	28	20
Number at cinema	256	184	190	312	458	401	200	357	312	423

- Identify the explanatory and response variables.
 - Display this information on a scatterplot.
- The following table shows the marks obtained by a group of 10 students in history and geography.

History	36	65	82	72	58	39	58	74	82	66
Geography	45	78	66	72	50	51	61	70	60	88

Display the information on a scatterplot.

5. The following table shows the age of 20 people and the number of hours they play video games each week.

Age	12	28	32	44	52	63	27	15	19	27
Time spent playing video games (h)	1.5	2.5	3.5	1.5	3.5	5.5	4.5	6.5	2.5	3.0
Age	16	39	42	74	67	13	21	30	52	44
Time spent playing video games (h)	3.5	1.0	1.5	2.0	3.5	6.5	4.0	1.5	2.5	3.5



Display this information on a scatterplot.

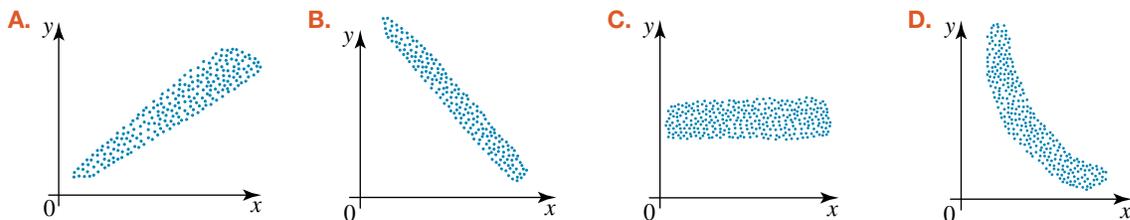
6. **WEB** The following table shows the marks obtained by 10 students in chemistry and physics

Chemistry	84	74	53	81	68	49	57	70	89	37
Physics	72	76	71	75	73	51	55	82	81	33

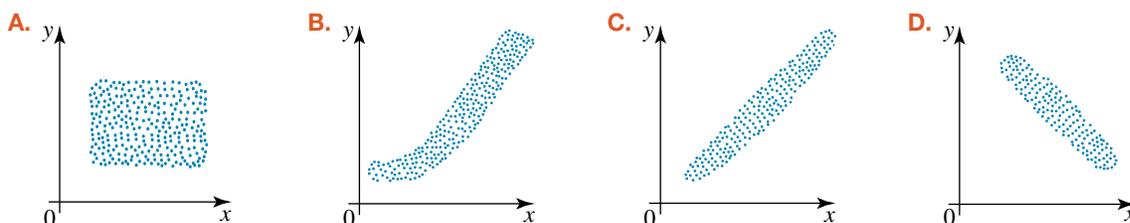
- Display the information on a scatterplot.
 - Determine whether there is any relationship between the marks obtained in Chemistry and Physics. If there does appear to be a relationship, state whether the relationship is linear.
7. **WEB** The following table shows the daily temperature and the number of hot pies sold at the school canteen.

Temperature (°C)	24	32	28	23	16	14	26	20	29	21
Number of pies sold	56	20	24	60	84	120	70	95	36	63

- Identify the explanatory variable.
 - Using Excel, construct a scatterplot showing the number of pies sold against temperature.
 - By examining the scatterplot, determine if there appears to be any relationship between the two variables and if the relationship appears to be linear.
8. **MC** Identify the scatterplot that does *not* display a linear relationship.



9. **MC** Identify the scatterplot in which there is no relationship evident between the variables.



Complex familiar

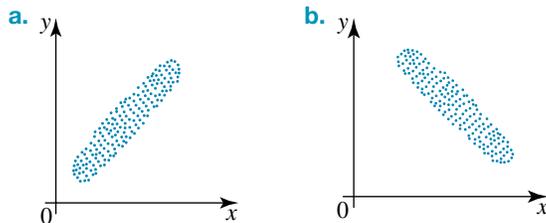
10. Container ships arriving on a wharf are unloaded by work teams.



The following table shows the number of people in the work team and the time taken to unload the container ship.

Number in work team	15	18	12	19	22	21	17	16	18	20
Hours taken	20	16	25	15	14	13	18	20	17	14

- a. Display the information on a scatterplot.
 b. Describe the relationship between the number in the work team and hours taken, referring to strength, form and direction.
11. Give an example of a situation where the scatterplot may look like the ones below.



12. The following table shows weight and height for a group of men and women.

Height (cm)	150	160	170	180	190	200
Weight (kg)	50	57	65	72	81	90

- a. Construct a scatterplot showing this data.
 b. Jared's height is 175 cm and his weight is 62 kg. Determine whether Jared's height and weight follow the trend of this data.

Complex unfamiliar

13. The table below shows the number of tickets sold for a new movie in the days following its release.

Number of days since first release	5	10	15	20	25
Total number of tickets sold in millions	8	12	18	26	40

Evaluate the reasonableness of the claim that there is a strong linear association between the two variables.

14. A sports scientist is studying the relationship between the number of weekly training sessions and the average running speed of athletes. She also records the athletes' age and gender. Identify if age would be considered an explanatory variable in this study. Explain why or why not.

Fully worked solutions for this chapter are available online.

LESSON

1.5 Pearson's correlation coefficient and the coefficient of determination

SYLLABUS LINKS

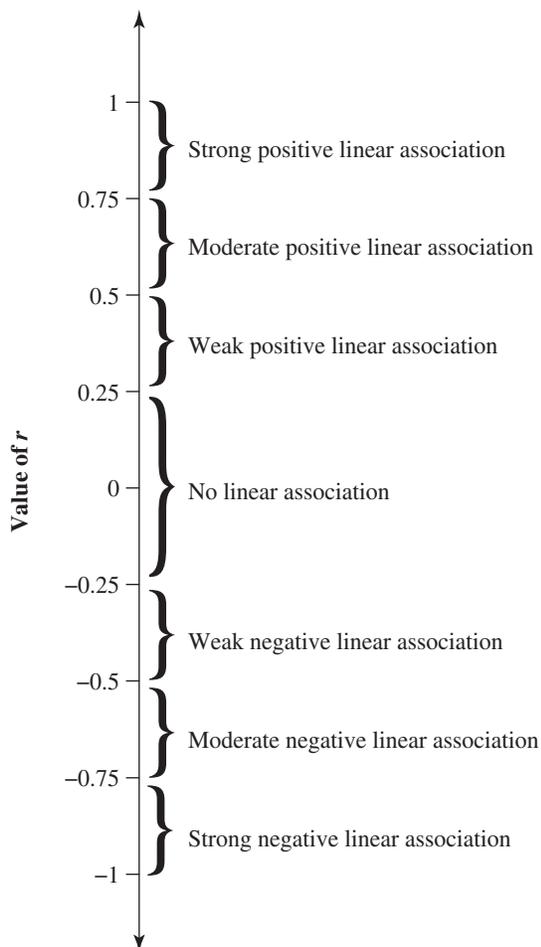
- Calculate Pearson's correlation coefficient, r , from raw data using technology and interpret it to quantify the strength of a linear association.
- Calculate the coefficient of determination, R^2 , from raw data using technology and interpret it to assess the strength of a linear association in terms of the explained variation.
- Use the correlation coefficient, r , to determine the coefficient of determination, R^2 and vice versa.

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1.5.1 Calculating Pearson's correlation coefficient (r)

To determine exactly how strong this relationship is we can use Pearson's correlation coefficient, r , which measures the strength of a linear trend and associates it with a numerical value between -1 and $+1$.

A value of either -1 or $+1$ indicates a perfect linear correlation, while a result closer to zero indicates no correlation between the variables. The diagram below describes the strength of the association between variables for a given value of the Pearson's correlation coefficient.



WORKED EXAMPLE 10 Calculating r

Consider the following set of data.

x	2	3	4	5	6	7	8
y	1.2	2.7	2.3	4.1	3.9	8.4	6.5

Calculate Pearson's correlation coefficient, r , using technology and interpret it to quantify the strength of the linear association.

THINK

- To calculate Pearson's correlation coefficient, r , input the data into your calculator using the STAT mode.
- Accessing the STAT menu, find the regression (reg) function.
- Choose the ' r ' option.
- Interpret the answer.

WRITE



$r = 0.89$ (correct to 2 decimal places)

This suggests a strong positive association between the variables x and y .

1.5.2 The coefficient of determination (R^2)

The **coefficient of determination**, R^2 , is Pearson's correlation coefficient squared.

$$r^2 = R^2$$

If a value for R^2 of 0.71 is found, for example, this would indicate that 71% of the variation in the y -variable is explained by the variation in the x -variable and 29% can be explained by other factors.

The coefficient of determination can be calculated using an Excel spreadsheet or other suitable technology.

WORKED EXAMPLE 11 Calculating the coefficient of determination

Data was collected on the time it takes students to get to school and their ATAR scores.

Time (mins)	12	35	19	42	33	31	25	46	45	40	14	44	39	31	22
ATAR score	53	75	97	59	87	70	71	66	37	48	94	68	33	59	42

Appropriate technology should be used to answer the following questions.

- Construct a scatterplot of this data.
- Determine Pearson's correlation coefficient, r and the coefficient of determination, R^2 .
- Interpret the values of r and R^2 .



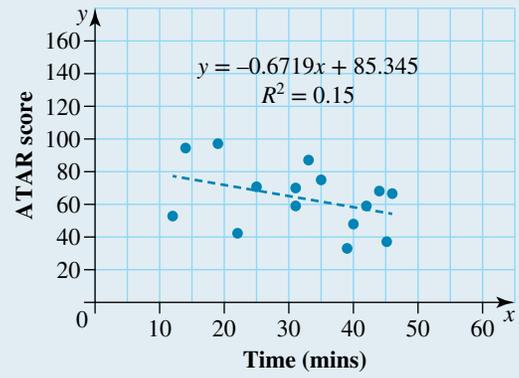
THINK

a. Using Excel, construct a scatterplot representing the data collected. Time is the explanatory variable and ATAR score is the response variable.

b. Write the value of r and R^2 . R^2 is given by Excel when a line of best fit is added.

c. Interpret the r value.
Interpret the R^2 value

WRITE

- a. 
- b. $R^2 = 0.15$
 $r = \sqrt{0.15}$
 $= -0.39$ (negative due to the trend of the data)
- c. An r value of -0.39 implies that there is a weak negative linear relationship, which is not a clear indication that an increased distance from school could have a negative effect on your ATAR score. An R^2 value of 0.15 implies that 15% of the variation in the ATAR score can be explained by the variation in the distance students live from their schools and 85% is explained by other factors.

Exercise 1.5 Pearson's correlation coefficient and the coefficient of determination

learnon

1.5 Exercise

1.5 Exam questions **on**

Simple familiar

1, 2, 3, 4, 5, 6, 7,
8, 9

Complex familiar

10, 11, 12

Complex unfamiliar

13, 14, 15

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1. **WE10** Consider the following set of data.

x	1	3	5	7	8	9	11
y	12	11	14	10	17	16	18

Calculate Pearson's correlation coefficient (r) and describe the strength of the association between the variables x and y .

2. Consider the following set of data.

x	1	2	3	4	5	6
y	10	7	8	6	5	4

Calculate Pearson's correlation coefficient (r) and describe the strength of the association between the variables x and y .

3. **WE11** A researcher collected data on the number of hours studied and the scores obtained by 10 students in a test. The data is as follows:

Hours studied	1	2	3	4	5	6	7	8	9	10
Scores obtained	53	55	60	69	70	75	78	85	90	95

- Construct a scatterplot of this data.
 - Determine Pearson's correlation coefficient, r , and the coefficient of determination, R^2 , both correct to 2 decimal places.
 - Interpret the values of r and R^2 .
4. Data on 15 people's shoe size and the length of their hair was collected.



Shoe size	6	8	7	8	9	6	7	12	8	9	10	12	7	9	11
Length of hair (cm)	9	14	12	1	7	8	5	22	15	8	18	4	5	9	3

Calculate the Pearson's correlation coefficient, r , for this data and comment on the strength of the relationship.

5. A class of Year 12 students was asked to record the amount of time in hours that they spent on a Practice Internal assessment and the mark out of 100 that they received for the assessment.

Time spent (hours)	2	0.5	1.5	2.5	0.25	2	2.5	2.5	2	0.5
Mark	72	52	76	82	36	73	84	80	74	48
Time spent (hours)	0.75	1.5	1	2	3	3.5	1	3	2.5	3
Mark	58	69	62	78	90	94	70	92	88	97

- Determine the explanatory variable and response variable for this data and construct a scatterplot.
- State whether the scatterplot indicates a relationship between the variables. If so, describe the relationship.
- Calculate r and comment on the strength of the relationship.

6. A researcher who is investigating the proposition that ‘tall mothers have tall sons’ measures the heights of twelve mothers and the heights of their adult sons. The results are shown below.

Height of mother (cm)	185	155	171	169	170	175	158	156	168	169	179	173
Height of son (cm)	188	157	172	173	174	180	159	150	172	175	180	190

- Construct a scatterplot of this data.
- Calculate r for this data and use this to comment on the statement ‘tall mothers have tall sons’.



7. The following table shows the number of Icy-poles sold and the maximum temperature of 10 days recorded by a shopkeeper.

Maximum temperature (°C)	14	10	22	26	30	20	18	12	8	6
Number of icy-poles sold	250	200	365	500	630	420	320	280	220	150

- Plot the data upon a scatterplot.
 - Calculate r and comment on the strength of the relationship.
8. The heights and ulna (elbow to the centre of wrist bone) lengths of twelve Year 12 students from Clever High are shown below.



Ulna (cm)	26	28	25	25	24	23	25	26	25	27	25	26
Height (cm)	170	174	178	167	166	164	176	177	170	172	183	175

- Construct a scatterplot of height against ulna lengths.
 - Calculate r and comment on the strength of the relationship.
9. A researcher finds that the coefficient of determination between hours of study and final exam scores is $R^2 = 0.7225$.
- Calculate the correlation coefficient.
 - Describe the strength of the relationship.

Complex familiar

10. The 10 am temperature ($^{\circ}\text{C}$) and rainfall (mm) over two weeks during the month of April in a city is shown below.

Temperature ($^{\circ}\text{C}$)	5	16	20	10	14	8	18	13	9	17	7	14	4	3
Rainfall (mm)	3	5	8	9	6	7	8	7	6	7	4	10	2	1

Comment on how a freak rainfall of 25 mm on an 18°C day would affect the overall strength of the association

11. A study examines the relationship between the number of weekly gym sessions (x) and the bench press weight lifted (y) in kilograms. The correlation coefficient is $r = 0.78$.
- Calculate the coefficient of determination, R^2 , as a percentage to 2 decimal places.
 - Interpret what this value means in the context of the data.
12. A survey finds that the correlation between the amount of time spent on social media per day and students' test scores is $r = -0.62$.
- Explain what the negative sign of r indicates.
 - Calculate R^2 and interpret its meaning.

Complex unfamiliar

13. The manager of a ski resort collects the following numerical data over 12 consecutive weekends at his resort.

Depth of snow (m)	0.5	0.8	2.1	3.6	1.4	1.5	1.8	2.7	3.2	2.4	2.6	1.7
Number of skiers	120	250	500	780	300	280	410	320	640	540	530	200

Evaluate the reasonableness of the statement that the depth of snow will always explain how many skiers to expect

14. A car manufacturer examines the relationship between the horsepower of a vehicle and its fuel efficiency (measured in kilometres per litre). The correlation coefficient is $r = -0.85$. Discuss the meaning of this value and identify potential limitations of using horsepower alone to predict fuel efficiency.
15. A study on employee productivity examines the relationship between daily coffee consumption and the number of tasks completed per day. The correlation coefficient is $r = 0.40$. Evaluate the strength and usefulness of this correlation, and discuss why it may not imply a causal relationship.



Fully worked solutions for this chapter are available online.

LESSON

1.6 Review

1.6.1 Summary

Hey students! Now that it's time to revise this chapter, go online to:

 Access the chapter summary

 Review your results

 Practise exam questions

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1.6 Exercise

learnon

1.6 Exercise

1.6 QCAA questions 

Simple familiar

1, 2, 3, 4, 5, 6, 7, 8,
9, 10, 11, 12

Complex familiar

13, 14, 15, 16

Complex unfamiliar

17, 18, 19, 20

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- Access sample responses
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Simple familiar

- State whether the data collected in each of the following situations would be categorical or numerical.
 - The number of matches in each box is counted for a large sample of boxes.
 - The sex of respondents to a questionnaire is recorded as either M or F.
 - A fisheries inspector records the lengths of 40 cod.
 - The occurrence of hot, warm, mild and cool weather for each day in January is recorded.
 - The actual temperature for each day in January is recorded.
 - Cinema critics are asked to judge a film by awarding it a rating from one to five stars.
- In some cases it is easier to count numbers in a particular category by considering a different population. In each of the following pairs of proportions, identify which one would be easier to determine.
 - Proportion of males who are left-handed
 - Proportion of left-handers who are males
 - Proportion of Mathematics A students in your school who are over 16
 - Proportion of over 16 year olds in your school who study Mathematics A
 - Proportion of state school students who live in Queensland
 - Proportion of Queensland school students who attend a state school.
- The following table shows information about a particular survey of a sample of women. It shows the percentage of women by age and their preferred type of music.
 - Of the women who preferred jazz, determine the percentage who were between 20 and 59 years inclusive.
 - State whether the information in the table supports the opinion that, for this sample of women, the preferred type of music is associated with age. Justify your answer with the appropriate percentages.

Age group of women surveyed	Preferred type of movie		
	Pop	Classical	Jazz
19 years and under	9.5%	4.7%	3.0%
20 to 35 years	41.1%	30.3%	20.5%
36 to 59 years	22.4%	30.7%	35.5%
60 years and over	27.0%	34.3%	41.0%

4. The following table shows the wages of 20 people and the amount of money they spend each week on entertainment.

Wages (\$)	370	380	500	510	395	430	535	490	495	550
Amount spent on entertainment (\$)	55	85	150	75	145	100	130	115	70	150
Wages (\$)	810	460	475	520	530	475	610	780	350	460
Amount spent on entertainment (\$)	220	50	100	150	140	160	90	130	40	50

Display this information on a scatterplot.

5. The following table shows the marks obtained by nine students in English and History.

English	55	20	27	33	73	18	37	51	79
History	72	37	53	74	73	44	59	55	84

- a. Display the information on a scatterplot.
 b. State whether there is any relationship between the marks obtained in English and in History. If there does appear to be a relationship, state whether the relationship is linear.
6. A group of 10 Year 12 students were asked to record the number of books they read during the year and the mark out of 100 that they received for their English assessment. The data collected is shown in the table.

Number of books read	1	3	6	8	11	13	18	21	23	27
Mark	55	63	72	73	79	81	84	86	92	98

- a. Determine the explanatory variable and response variable for this data.
 b. Display the information in a scatterplot.
 c. Calculate r and comment on the strength of the relationship.
7. The following two-way frequency table shows the composition of left-handed and right-handed students in Years 11 and 12 at a school.

	Left handers	Right handers
Year 11	13	106
Year 12	9	89

Determine the percentage of left-handers who are in Year 12. Give your answer to 2 decimal places.

8. A sample of the Year 11 and Year 12 students in a school were asked whether they prefer reading fantasy, science fiction or mystery books. The data collected is shown in the table.

	Year 11	Year 12
Fantasy	43	26
Science fiction	32	15
Mystery	10	21

Determine the percentage of students who prefer science fiction or fantasy. Give your answer to 2 decimal places.

9. A researcher took 100 blood samples from a group of volunteers. He wanted to determine if there is a relationship between the amount of iron present in the blood and the amount of vitamin C present in the blood. Identify which data display he should use to best show this type of data.

10. The following two-way frequency table shows the results of a trial on new metal detectors for aircraft. The metal detector scans a piece of hand luggage and lights up if metal is found:

	Test results		
	Accurate	Non accurate	Total
With metal	9	1	10
Without metal	87	3	90
Total	96	4	

Determine the percentage of metal going undetected in the hand luggage.

11. a. Describe the correlation between a set of bivariate data giving the correlation coefficient of $r = 0.64$.
 b. If the Pearson's correlation coefficient, r , is found to be -0.7564 , determine the coefficient of determination, R^2 .
12. Determine the coefficient of determination of the following data.

Wing length (cm)	10.6	10.4	10.8	11.2	10.8	10.5	11.1	10.2	10.3	10.7	10.2	11.4
Tail length (cm)	7.8	7.4	7.9	7.7	7.8	7.2	7.9	7.2	7.4	7.4	7.1	8.3

Complex familiar

13. A reading test for people with dyslexia is given and the results are shown in the following two-way frequency table.

	Test results		
	Accurate	Not accurate	Total
With dyslexia	39	1	40
Without dyslexia	85	5	90
Total	124	6	

- a. Determine how many people were tested.
 b. Determine the percentage of people who tested positive for dyslexia.
 c. Based on the above results, if a person with dyslexia takes the test, determine the percentage chance that they will be accurately diagnosed.
14. The following table shows the number of sick days taken by 10 employees and relates this to the number of children that they have.

Number of children	1	0	3	2	2	4	6	0	1	2
Number of sick days	5	3	10	8	4	12	12	0	1	5

- a. Show this information on a scatterplot.
 b. Determine whether a relationship appears to exist between the number of sick days taken and the number of children they have. If so, state whether the relationship is linear.

15. The following table shows the number of cars and number of televisions in each household.

Number of cars	1	1	2	2	2	3	1	0	1	2
Number of televisions	2	1	1	2	0	1	4	3	1	1

- Show this information on a scatterplot.
 - Determine if a relationship appears to exist between the number of televisions in each household and the number of cars they have. If so, state whether the relationship is linear.
16. A potato farmer records the yield in kilograms and the length in metres of 10 commercial potato plots as shown.

Length (m)	10	6	16	2	13	7	4	12	5	8
Yield (kg)	220	250	400	25	500	430	120	350	310	280

- Identify the explanatory variable and the response variable.
- Construct a scatterplot to illustrate this data.
- State the type of relationship shown by the scatterplot.
- If the scatterplot shows a suitable relationship, calculate r using technology and comment on the strength of the relationship in terms of strength, direction and form.



Complex unfamiliar

17. The results of a lie detector test are given below.
- Of 80 people who are known to be telling the truth, the lie detector indicates that three are lying.
 - Of 20 people known to be lying, the lie detector indicates that 17 are lying.
- For this particular lie detector test, the acceptable % error is 5%. Determine whether this test passes the acceptable level of error.
18. A teacher surveys 64 children on how they travelled to school. Twenty of the students were in Year 7. The teacher surveyed 30% more students in Year 9 than in Year 7. The rest of the students surveyed were in Year 11. Seventy-five per cent of the students in Year 7 walked to school. Eight more students in Year 9 walked to school than did not walk. Out of students surveyed, more Year 11 students walked to school than Year 9 students. Determine what percentage of students a new safe walking trail should affect.
19. The following is data from an umbrella store depending on average weekly rainfall:

Rainfall (mm)	5.1	5.8	6.6	5.4	12.8	12	7	11	9	9.4	9	11.4	10	8.1
Umbrellas sold	27	33	60	48	93	102	75	99	66	108	114	111	90	87

Determine why this graph wouldn't be useful to predict umbrella sales in large amounts of rainfall.

20. Given the following data, determine with mathematical justification the amount of effect the number of meals produced by a chef has on the total cost.

Number of meals	60	50	90	70	75	100	25	30
Cost (\$)	530	485	720	595	585	750	300	345

1.6 Past QCAA exam questions

Question 1 (1 mark)

Source: QCAA 2023 General Mathematics, Paper 1, Section 1, Q4; © State of Queensland (QCAA) 2024

Athletes were surveyed about their preferred shoe brand: X, Y or Z. The results are shown in the frequency table.

	X	Y	Z	Total
Field athletes	26	12	2	40
Track athletes	14	18	8	40
Total	40	30	10	80

The percentage of field athletes who prefer brand Y is

- A. 12%
- B. 15%
- C. 30%
- D. 40%

Question 2 (1 mark)

Source: QCAA 2023 General Mathematics, Paper 1, Section 1, Q5; © State of Queensland (QCAA) 2024

A scatterplot is created to identify the nature of the relationship between two variables: vehicle age and distance travelled.

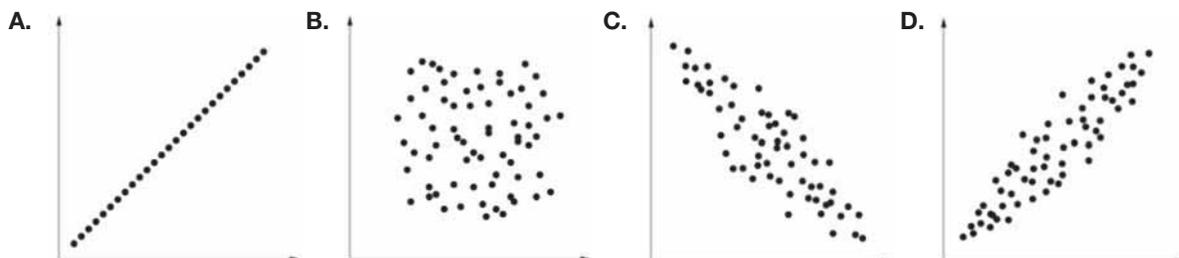
Which statement is correct?

- A. The vertical axis should show vehicle age as the response variable.
- B. The horizontal axis should show vehicle age as the explanatory variable.
- C. The horizontal axis should show distance travelled as the response variable.
- D. The vertical axis should show distance travelled as the explanatory variable.

Question 3 (1 mark)

Source: QCAA 2022 General Mathematics, Paper 1, Section 1, Q12; © State of Queensland (QCAA) 2024

Identify the scatterplot that best demonstrates a strong negative association.



Question 4 (1 mark)

Source: QCAA 2022 General Mathematics, Paper 1, Section 1, Q10; © State of Queensland (QCAA) 2024

Which example states an explanatory variable followed by a response variable?

- A. car manufacturers and car colours
- B. dog breeds and frequency of names
- C. plant growth and amount of fertiliser used
- D. daily temperatures and daily ice cream sales

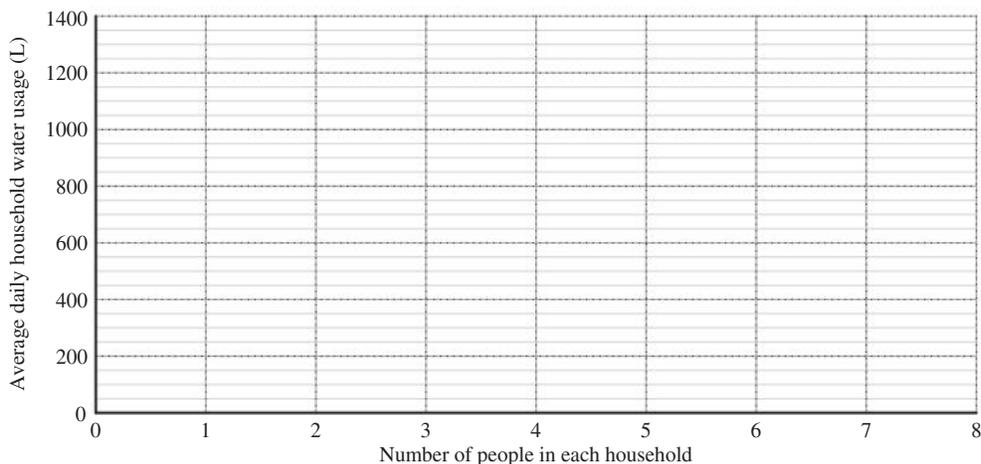
Question 5 (5 marks)

Source: QCAA 2020 General Mathematics, Paper 2, Section 1, Q2; © State of Queensland (QCAA) 2024

The number of people living in each household and the average daily household water usage, measured in litres (L), were recorded for 10 households.

Number of people in each household	6	2	4	5	5	4	3	1	6	7
Average daily household water usage (L)	990	160	320	480	410	280	240	130	940	1340

Calculate Pearson's correlation coefficient and then evaluate the appropriateness of using this coefficient for the association between daily water usage and the number of people living in a household.



Fully worked solutions for this chapter are available online.

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Answers

Chapter 1 Identifying and describing associations between two variables

1.2 Bivariate data

1.2 Exercise

Variable	Categorical or numerical
Year level of the student	Categorical
Student's weight (in kg)	Numerical
Favourite subject	Categorical
Number of books borrowed from the library per month	Numerical
Type of transport used to get to school	Categorical

2. a. Continuous b. Discrete c. Continuous
d. Continuous e. Discrete
3. a. Numerical and discrete b. Categorical
c. Categorical d. Numerical and continuous
e. Numerical and continuous
f. Numerical and discrete
4. Categorical, ordinal
5. Categorical, ordinal
6. Numerical and discrete
- 7.

Variable	Categorical or numerical	Univariate or bivariate
The mark scored on a mathematics test and the hours of mathematics homework completed during the previous week.	Numerical	Bivariate
Water usage and time spent in the shower for 50 students in a hostel.	Numerical	Bivariate
The number of vehicles passing a point on a road during a given time period.	Numerical	Univariate
A group of customers were asked to respond to the level of service they received in a store as poor, fair, excellent.	Categorical	Univariate
The time each customer spends in a supermarket queue on a Saturday morning.	Numerical	Univariate
Days of the week and choices of public transport.	Categorical	Bivariate

8. A
9. Bivariate data
10. Categorical and ordinal
11. Numerical and continuous
12. Possible questions could include:
What is the average age of our customers? (univariate)
Is there a relationship between amount of money spent and time spent on the website? (bivariate)
Is there a relationship between income and postcode? (bivariate)
Do more males or females visit the website? (univariate)
What is the average amount of time customers spend on the website? (univariate)
Is there a relationship between the customer's postcode and the amount of time spent on the website? (bivariate)

1.3 Two-way frequency tables

1.3 Exercise

1.

	Test results		
	Accurate	Not accurate	Total
With virus	98	2	100
Without virus	388	12	400
Total	486	14	500

2.

	Test results		
	Accurate	Not accurate	Total
Telling truth	777	23	800
Telling lies	156	44	200
Total	933	67	1000

- 3 a. 200 b. 44 c. 90.9% d. 5.1% e. 94%
f. Sample response: The alarm appears to be reliable in detecting intruders and activating the alarm (80 + %).
- 4 a. 1000 b. 75 c. 96.7%
5. C
6. D
7. A
8. a.

	Test results		
	Accurate	Not accurate	Total
Bags with prohibited items	48	2	50
Bags with no prohibited items	145	5	150
Total	193	7	200

- b. i. 96%
ii. 3.3%
iii. 4%
iv. 96.5%

9. a. 120 b. 20 c. 23

$$d. P(\text{avoids detection}) = \frac{2}{20} \times 100 = 10\%$$

10. a. 350
b. 45

$$c. P(\text{break in taking place}) = \frac{43}{45} \times 100 = 95.6\%$$

11. a.

	Detected	Not detected	Total
Gene present	95	5	100
Not present	37	863	900
Total	132	868	1000

$$b. P(\text{gene detected}) = \frac{95}{100} \times 100 = 95\%$$

$$c. P(\text{has gene}) = \frac{95}{132} \times 100 = 72.0\%$$

$$d. P(\text{correct diagnosis}) = \frac{958}{1000} \times 100 = 95.8\%$$

12. a. 56%
b. Yes, if you consider the 20- to 35-year-old age group, the percentages, 41.1%, 30.3% and 20.5% show a decrease in preferred type from action being the most popular to drama the least popular. This is reversed for the 60 years and over group.

13. a.

	Test results		
	Accurate	Not accurate	Total
With virus	48	2	50
Without virus	149	1	150
Total	197	3	200

The percentage of people with the virus who returned a positive result was 96%.

- b. The probability of returning a positive test when not infected was 0.67%.

14.

Attitude	Primary	Secondary
Fewer	7.5%	4.3%
Same	43.3%	19.1%
More	49.2%	76.6%
Total	100%	100%

Secondary students were much keener on having more holidays than primary students, as the percentage of secondary students who desired more holidays was 76.6%, compared to primary students who desired more holidays at 49.2%.

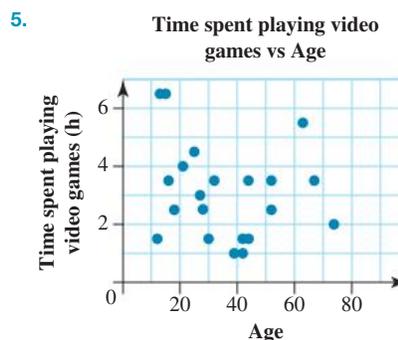
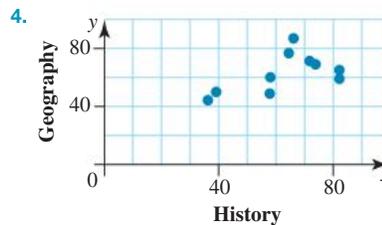
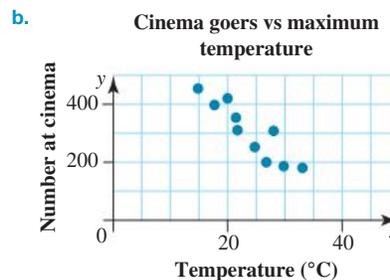
15. A higher proportion of people who use public or active transport exercise regularly compared to those who drive. Only 40% of car users exercise regularly, compared to 64.3% of public transport users and 66.7% of active transport users.

1.4 Scatterplots

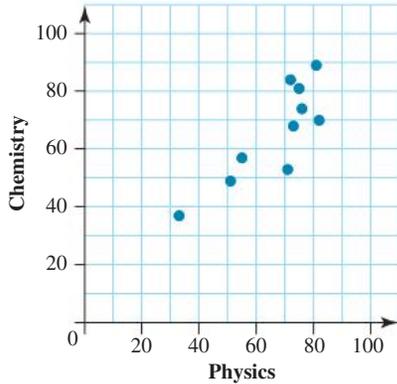
1.4 Exercise

1. a. i. Height
ii. increase
b. i. Distance travelled
ii. Increase
c. i. Temperature
ii. Decrease
d. i. Reaction time
ii. Decrease
e. i. Test results
ii. Increase
f. i. Overtime pay
ii. Increase
g. i. Value of car
ii. Decrease
h. i. Travelling time
ii. Decrease

2. a. No relationship
b. Negative relationship
c. Positive relationship
3. a. Explanatory variable = Temperature
Response variable = Number of cinema



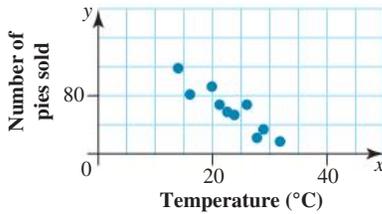
6. a. Marks in Chemistry vs marks in Physics



- b. The greater the physics mark, the greater the chemistry mark, generally. However, as the points on the scatterplot do not form a straight line, the relationship is non-linear.

7. a. Temperature

- b. Number of pie sold vs temperature

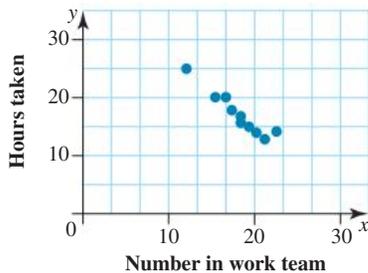


- c. The greater the temperature, the fewer pies are sold. The points on the scatterplot approximate a straight line and so the relationship can be said to be linear.

8. D

9. A

10. a. Time taken vs number of people in team

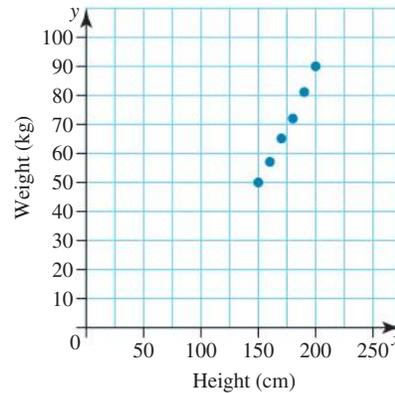


- b. More workers on the team reduces the amount of time taken to unload the ship and, as the points on the scatterplot approximate a straight line, the relationship is linear.

11. Sample response:

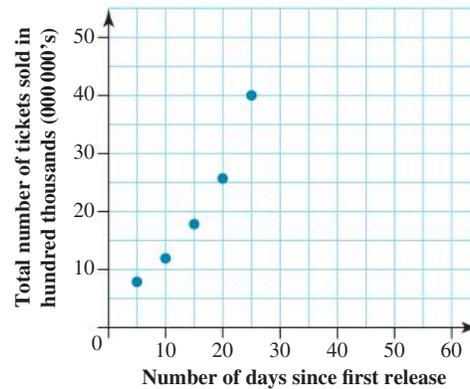
- a. Hours spent practicing basketball and points scored
b. The density of an object and its volume

12. a. Weight (kg) vs Height (cm)



- b. From the graph, someone with height 175 cm should be on average about 70 kg, so is around 8 kg heavier. Jared's height and weight do not really follow the trend of this data.

13. Tickets sold vs number of days since first release



The scatterplot clearly shows a non-linear relationship. Therefore, the claim is not reasonable.

14. Age would not be considered an explanatory variable, due to the focus being on weekly sessions and speed.

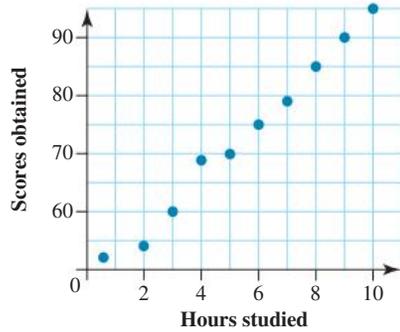
1.5 Pearson's correlation coefficient and the coefficient of determination

1.5 Exercise

- Using technology, $r = 0.72$
This suggests a moderate positive association between the variables x and y .
- Using technology, $r = -0.94$
This suggests a strong negative association between the variables x and y .

3. a.

Scatterplot of hours studies vs scores obtained



b. $r = 0.99$, $R^2 = 0.99$

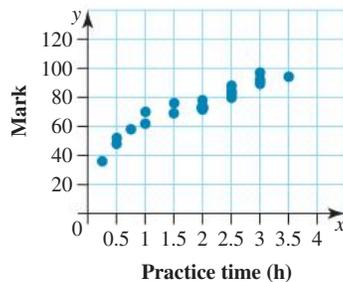
c. Both indicate a strong positive linear relationship.

4. $r = 0.2055$

You would not expect to find a relationship between shoe size and hair length and as $r = 0.2055$, this indicates that there is no relationship between the two variables.

5. a. Time spent – explanatory variable, mark – response variable

Mark vs Time spent

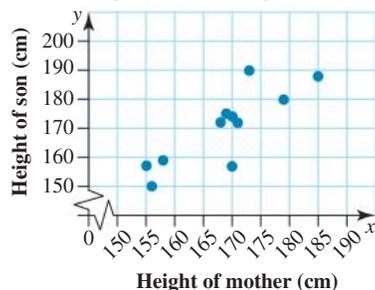


b. The scatterplot indicates a strong, positive linear relationship.

c. $r = 0.95$. This indicates a strong positive relationship between the time spent on the assignment and the mark achieved. The more time spent on the assessment, the higher the mark they achieved.

6. a.

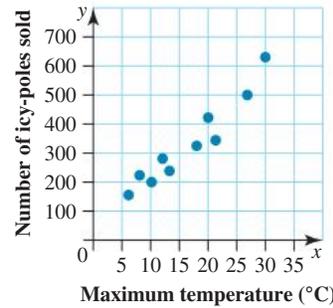
Height of son vs Height of mother



b. $r = 0.92$. This indicates a strong positive relationship between the height of the mother and the height of the son. It can be said that tall mothers tend to have tall sons based on this data.

7. a.

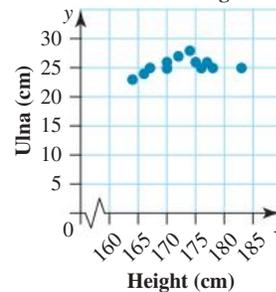
Number of icy-poles sold vs Maximum temperature



b. $r = 0.96$. This indicates a strong positive relationship between the maximum temperature and the number of icy-poles sold. The hotter the day more icy-poles sold.

8. a.

Ulna vs Height



b. $r = 0.38$. This indicates a weak positive relationship between the Year 12 students' height and length of their ulna.

9. a. $r = 0.85$

b. The relationship is strong.

10. $r = 0.72$. This indicates a moderate positive relationship between the temperature and the amount of rainfall.

11. a. $R^2 = 60.84\%$

b. 60.48% of the variation can be explained by the number of gym sessions.

12. a. As social media increases, test scores decrease.

b. 38.44% of the variation in test scores is due to social media use.

13. $R^2 = 0.78$; therefore, only 78% of the increase in the number of skiers can be attributed to the increase in the depth of snow. Thus, it is unreasonable to state that the depth of snow will always explain how many skiers to expect.

14. As horsepower increases, fuel efficiency decreases. A variety of limitations is possible.

15. There is a weak positive correlation, which is not likely to have a high level of impact on results.

1.6 Review

1.6 Exercise

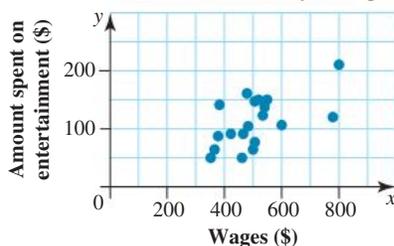
1. a. Numerical b. Categorical c. Numerical
d. Categorical e. Numerical f. Numerical

2. a. ii
b. ii
c. ii

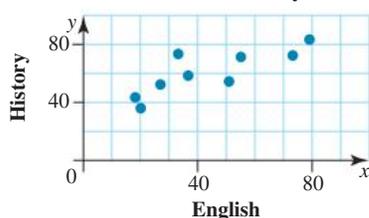
3. a. 56%

- b. Yes, if you consider the 20 to 35-year-old age group, the percentages, 41.1%, 30.3% and 20.5% show a decrease in preferred type from pop being the most popular to jazz the least popular. This is reversed for the 60 years and over group.

4. Entertainment money vs wages



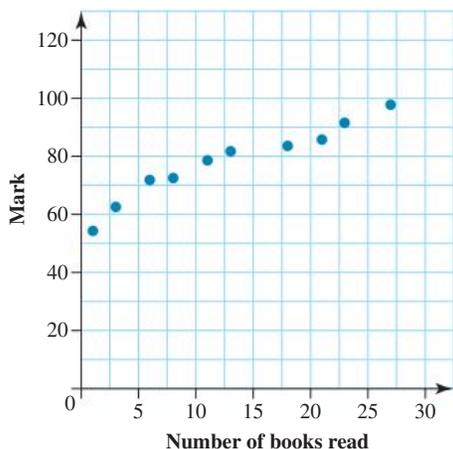
5. a. Marks in Geography vs marks in History



- b. The greater the English mark, the greater the history mark, generally. However, as the points on the scatterplot do not form a straight line, the relationship is non-linear.

6. a. Explanatory variable: number of books read
Response variable: mark

b. Mark vs number of books read



- c. $R^2 = 0.9394$ thus, $r = 96.92\%$.
An r value of 96.92% indicates a strong positive relationship between the number of books read and the mark achieved.

7. 9.18%
8. 78.91%
9. Scatterplot
10. 10%

11. a. Moderate, positive correlation

b. 0.5721

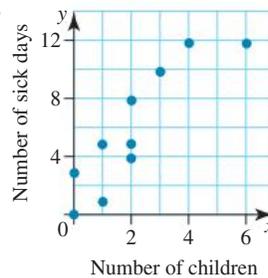
12. 0.7417

13. a. 130

b. 33.8%

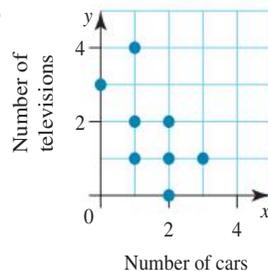
c. 97.5%

14. a.



- b. There appears to be a positive relationship which is linear.

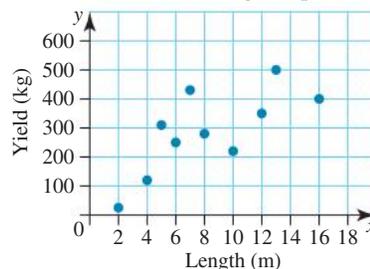
15. a.



- b. There is no apparent relationship.

16. a. Explanatory variable – length
Response variable – yield

b. Yield vs Length of plot



- c. The scatterplot indicates a positive linear relationship.
d. Using Excel, $r = 0.74$. There is a moderate positive relationship between the length of the plot and the yield of potatoes.

17. No, the test does not pass the acceptable level of error.

18. 78.125%

19. When plotted, the data diverges from a line of best fit more and more with larger amounts of rainfall.

20. 99% of the change in cost is from the change in number of meals due to $R^2 = 0.988$.

1.6 Past QCAA exam questions

1. C
2. B
3. C
4. D
5. The correlation coefficient should not be used as the scatterplot is not linear.

2 Fitting a linear model to numerical data, and association and causation

LESSON SEQUENCE

2.1 Overview	46
2.2 Review of the general equation of a straight line	47
2.3 Fitting a least-squares line to data	55
2.4 Residual plots	68
2.5 Association and causation	75
2.6 Review	80
Answers	87

Fully worked solutions for this chapter are available online.

EXAM PREPARATION

Access exam-style questions in every lesson, available online and access past QCAA exam questions in every review.

Resources

	Solutions	Solutions — Chapter 2
	Exam questions	Exam question booklet — Chapter 2
	Digital documents	Learning matrix — Chapter 2 Chapter summary — Chapter 2

LESSON

2.1 Overview

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2.1.1 Introduction

A mathematical model uses mathematical concepts to illustrate real-world phenomena. These models help researchers study aspects of nature or society without being physically present. For instance, the formula $V = l \times w \times h$ calculates the volume of a rectangular box without needing to construct and measure multiple different-sized boxes. More complex models, like those used for launching rockets, incorporate factors such as gravity, rocket size and fuel energy to ensure safety and efficiency.



2.1.2 Syllabus links

Lesson	Lesson title	Syllabus links
2.2	Review of the general equation of a straight line	<ul style="list-style-type: none"><input type="radio"/> [This lesson is a review of straight-lines from Unit 1, Topic 5.]
2.3	Fitting a least-squares line to data	<ul style="list-style-type: none"><input type="radio"/> Model a linear relationship by using technology to fit a least-squares line to the data, in the form of $y = mx + c$ where m is slope (slope) and c is y-intercept.<input type="radio"/> Understand and use $m = r \frac{s_y}{s_x}$ and $c = \bar{y} - m\bar{x}$ to determine the equation of a least-squares line, where m is slope (gradient), r is correlation coefficient, s_y is (sample) standard deviation of y values, s_x is (sample) standard deviation of x values, c is y-intercept, \bar{y} is mean of y values and \bar{x} is mean of x values.<input type="radio"/> Interpret the y-intercept and slope (gradient) of the fitted line.<input type="radio"/> Distinguish between interpolation and extrapolation.<input type="radio"/> Use the equation of the least-squares line to make predictions.<input type="radio"/> Recognise and explain the potential dangers of extrapolation.<input type="radio"/> Solve practical problems by identifying, analysing and describing associations between two variables (categorical and/or numerical).
2.4	Residual plots	<ul style="list-style-type: none"><input type="radio"/> Construct a residual plot and use it to assess the appropriateness of fitting a linear model to the data.<input type="radio"/> Solve practical problems by identifying, analysing and describing associations between two variables (categorical and/or numerical).
2.5	Association and causation	<ul style="list-style-type: none"><input type="radio"/> Recognise and explain that an observed association between two variables (categorical and/or numerical) does not necessarily mean that there is a causal relationship between them.<input type="radio"/> Identify and communicate possible non-causal explanations for an association, including coincidence or the influence of another variable.<input type="radio"/> Solve practical problems by identifying, analysing and describing associations between two variables (categorical and/or numerical).

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LESSON

2.2 Review of the general equation of a straight line

SYLLABUS LINKS

- [This lesson is a review of straight-lines from Unit 1, Topic 5.]

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2.2.1 Linear relationships

When a straight line is formed by a set of points on a **Cartesian plane**, there is a linear relationship between the x -coordinates and the y -coordinates of the points on the line.

Equation of a straight line.

The general form of a linear equation is:

$$y = mx + c$$

Linear relationships can be identified by using a table of values, plotting points or looking at an equation.

x	-2	-1	0	1	2
y	-2	1	4	7	10

Blue arrows above the table show a constant difference of +1 between consecutive x values. Red arrows below the table show a constant difference of +3 between consecutive y values.

Each variable has a constant difference.

$$y = mx + c$$

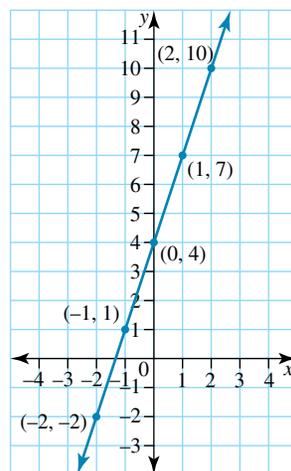
$$y = \frac{3}{1}x + 4$$

$$y = 3x + 4$$

The equation can be written in the form

$$y = mx + c.$$

The points form a straight line.

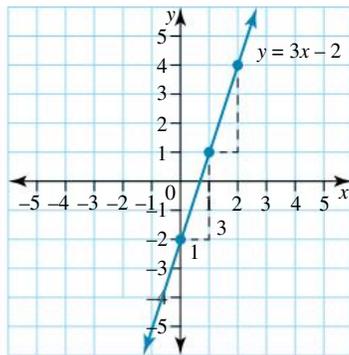


2.2.2 The slope

int-6484

The **slope (gradient)**, m , of the line is a measure of the steepness of the line. The slope is also a measure of the rate of change in y with respect to x .

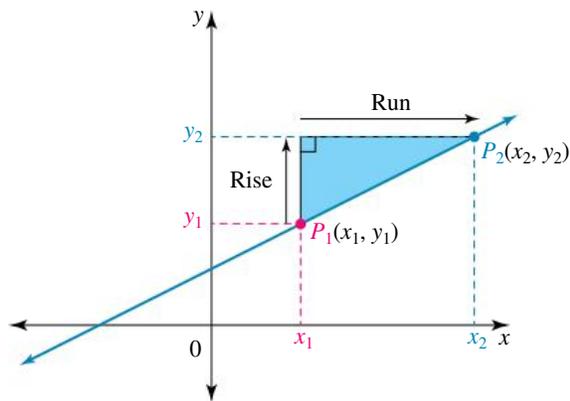
The straight line $y = 3x - 2$ crosses the y -axis at $y = -2$ and has a slope of 3, meaning for each 1 unit increase in x , the y -value increases by 3 units.



The slope of a straight line can be positive, negative, zero or undefined.

$m > 0$	$m < 0$	$m = 0$	m is undefined

The slope of a line can be determined from any two points on the line.



Slope (gradient)

$$\begin{aligned}
 m &= \frac{\text{rise}}{\text{run}} \\
 &= \frac{y_2 - y_1}{x_2 - x_1}
 \end{aligned}$$

Subscripts are used to show which points the coordinates belong to. For example, the x - and y -coordinates of point 1 are (x_1, y_1) ; the x - and y -coordinates of point 2 are (x_2, y_2) .

WORKED EXAMPLE 1 Determining the slope of two points

Calculate the slope of the line that passes through the points $(3, -2)$ and $(-4, 6)$.

THINK

1. Let one point be point 1 and the other point be point 2. Record their x - and y -coordinates.

2. Substitute the coordinates into the slope formula

$$m = \frac{y_2 - y_1}{x_2 - x_1} \text{ and simplify.}$$

WRITE

$$\text{Let: } (x_1, y_1) = (3, -2)$$

$$(x_2, y_2) = (-4, 6)$$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{6 - (-2)}{-4 - 3}$$

$$= \frac{8}{-7}$$

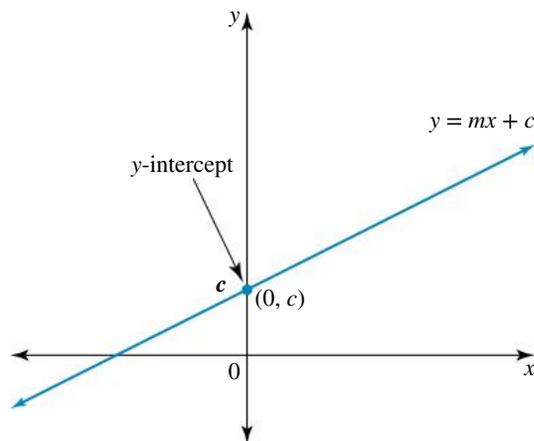
$$= -1\frac{1}{7}$$

2.2.3 The y -intercept

The y -intercept, c , of the line is the point where the line intersects the y -axis (the line $x = 0$).

The value of the y -intercept can be determined by any of these methods:

- looking at the graph of the line and determining the point at which the line crosses the y -axis
- writing the equation of the line in the form of $y = mx + c$ and identifying the constant value c
- substituting $x = 0$ into the equation of the line and solving for y .



2.2.4 Determining the equation of a line using two points

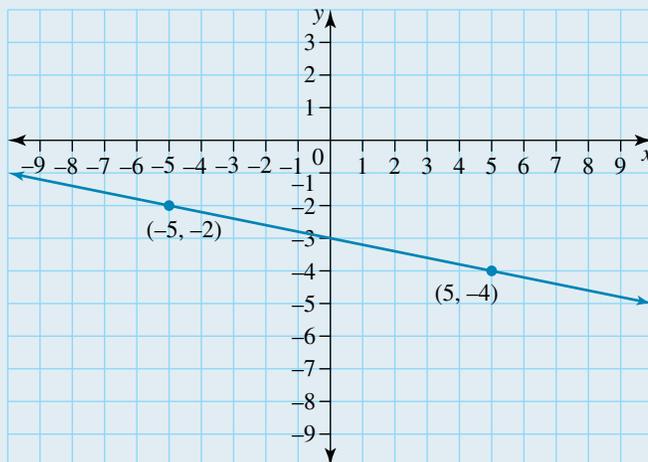
int-6485

The coordinates of two points on a line are all that is needed to determine the equation of a line. To determine the equation of the line, the values of both the slope and the y -intercept are required.

- Calculate the slope using the two points.
- Use the general form of the equation, the coordinates of one point and the slope to calculate the value of the y -intercept.
- Write the equation of the line in the form $y = mx + c$.

WORKED EXAMPLE 2 Determining the equation of a line using two points

Determine the equation of the line that passes through the points $(-5, -2)$ and $(5, -4)$.



THINK

- To determine the equation of the line, both the slope and the y -intercept are required. Calculate the slope first. Let one point be point 1 and the other point be point 2. Record their x - and y -coordinates.
- Substitute the coordinates into the slope formula

$$m = \frac{y_2 - y_1}{x_2 - x_1} \text{ and simplify.}$$

- Write the general equation of a straight line. Substitute the value of the slope m , into the formula.
To find the value of the y -intercept, c , substitute the coordinates of one of the points into the equation as the values for x and y .

- Write the equation of the line.

WRITE

$$\text{Let: } (x_1, y_1) = (-5, -2)$$

$$(x_2, y_2) = (5, -4)$$

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-4 - (-2)}{5 - (-5)} \\ &= \frac{-4 + 2}{5 + 5} \\ &= \frac{-2}{10} \\ &= -\frac{1}{5} \end{aligned}$$

$$y = mx + c$$

$$\text{Let } m = -\frac{1}{5}$$

$$y = -\frac{1}{5}x + c$$

$$\text{Let } (x, y) = (5, -4)$$

$$-4 = -\frac{1}{5}(5) + c$$

$$-4 = -1 + c$$

$$c = -3$$

$$y = -\frac{1}{5}x - 3 \text{ or } y = \frac{-x - 15}{5}$$

WORKED EXAMPLE 3 Sketching linear equations

For each of the linear equations below:

- state the slope and y-intercept
- sketch the graph of the equation.

a. $y = 4x - 11$

b. $y = -4x$

THINK

a. i. Compare the equation given with the general form of a linear equation: $y = mx + c$. The coefficient of x is m (the slope), and the constant c is the y-intercept.

ii. Construct a set of axes and mark the position of the y-intercept.

The y-intercept is -11 , as shown in blue.

The slope is 4, so $\frac{\text{rise}}{\text{run}} = \frac{4}{1}$.

From the y-intercept, rise 4 and run 1, then mark in a second point $(1, -7)$, as shown in pink.

The two points can now be connected with a straight line.

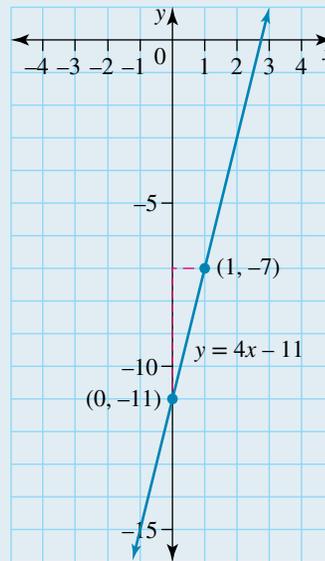
Write the equation next to the line.

WRITE

a. For $y = 4x - 11$

Slope (m) = 4

y-intercept (c) = -11 .



b. i. Compare the equation given with the general form of a linear equation: $y = mx + c$. Identify m as the slope and c as the y-intercept.

ii. Construct a set of axes.

Mark in the position of the y-intercept at 0, as shown in blue. The slope is -4 ,

so $\frac{\text{rise}}{\text{run}} = -\frac{4}{1}$.

From the y-intercept, rise -4 and run 1, then mark in a second point $(1, -4)$, as shown in pink.

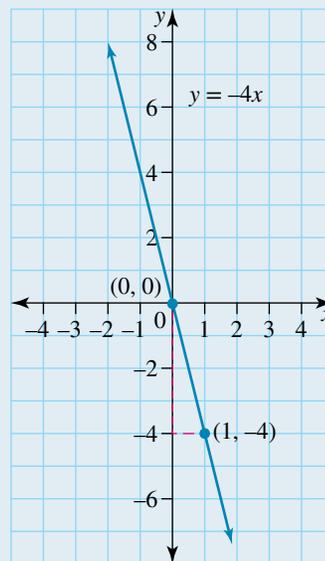
The two points can now be connected with a straight line to form the graph.

Write the equation next to the line.

b. For $y = -4x$ or $(y = -4x + 0)$

Slope (m) = -4 and

y-intercept (c) = 0.



2.2 Exercise

2.2 Exam questions

Simple familiar

1, 2, 3, 4, 5, 6, 7, 8,
9, 10, 11, 12

Complex familiar

13, 14

Complex unfamiliar

15

These questions are even better in jacPLUS!

- Receive immediate feedback
- Access sample responses
- Track results and progress



Find all this and MORE in jacPLUS

Simple familiar

1. In your own words, explain:

- the general form of a straight line
- what the letters represent in the general form of a straight line, $y = mx + c$.

2. State the value of m and c in each of the following linear relationships.

- | | |
|---------------------------|---------------------------|
| a. $y = -5x + 4$ | b. $y = 3x + 11$ |
| c. $y = -6x$ | d. $y = \frac{5}{2}x + 5$ |
| e. $y = \frac{2}{3}x - 6$ | f. $y = 2x - 1$ |

3. Complete the sentences below.

- The equation $y = 3x - 1$ has a slope of _____ units.
This means that for every increase of 1 unit in the horizontal direction there is an increase of _____ units in the vertical direction.
- The equation $y = -x + 1$ has a slope of _____ units.
This means that for every increase of _____ unit in the horizontal direction there is an increase of _____ units in the vertical direction.
- The equation $y = \frac{1}{3}x - 1$ has a slope of _____ units.
This means that for every increase of _____ unit in the horizontal direction there is an increase of _____ units in the vertical direction.

4. **WE1** Calculate the slope between each pair of points for each of the following.

- | | |
|----------------------|------------------------|
| a. (3, 4), (1, 0) | b. (1, 3), (4, 6) |
| c. (-2, 4), (-4, 2) | d. (6, 3), (2, 5) |
| e. (3, -1), (-1, 0) | f. (6, 8), (-3, -4) |
| g. (-2, -3), (-5, 1) | h. (-3, -6), (-10, -4) |
| i. (2, 5), (-1, 5) | |

5. Write the linear relationship for the line with the following properties:

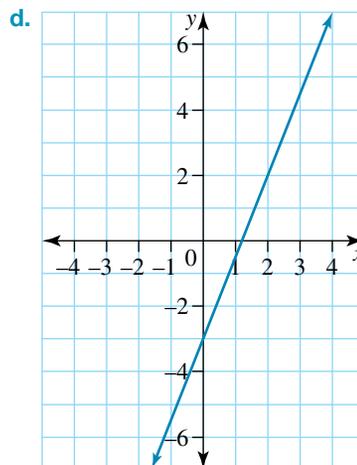
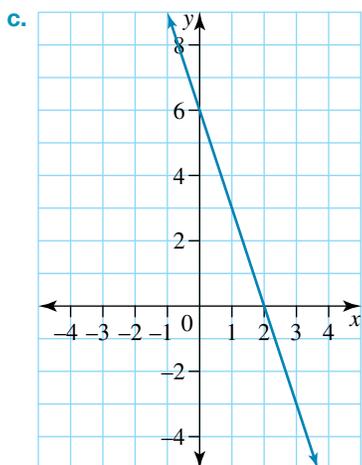
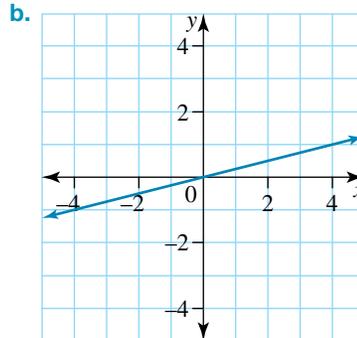
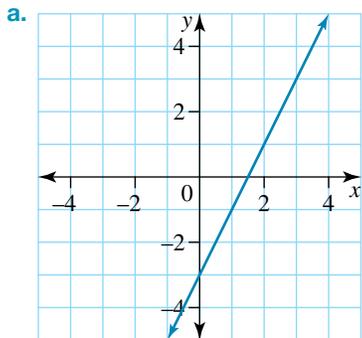
- | | |
|---|------------------------------------|
| a. slope = 2, y-intercept = -2 | b. slope = -2, y-intercept = -1 |
| c. slope = $-\frac{3}{8}$, y-intercept = 2 | d. slope = 0, y-intercept = 4 |
| e. slope = 1, y-intercept = 0 | f. slope = -0.25, y-intercept = -4 |

6. **WE2** Determine the equation of the straight lines that join each of the following pairs of points.

- | | |
|-----------------------|---------------------|
| a. (3, 4), (-2, -3) | b. (4, 6), (-5, 1) |
| c. (-2, 4), (-4, 2) | d. (6, 3), (1, 3) |
| e. (-1, 0), (-10, -4) | f. (-3, -4), (2, 5) |

7. For each of the linear graphs shown:

- i. state the y-intercept
- ii. calculate the slope
- iii. determine a linear relationship to describe the graph.



8. **WE3** For each of the linear relationships below:

- i. state the slope and the y-intercept
- ii. sketch the graph of the equation.

a. $y = 2x + 5$

b. $y = -4x - 3$

c. $y = -3x + 5$

d. $y = \frac{1}{2}x - 3$

e. $y = -\frac{2}{3}x + 3$

f. $y = \frac{4}{5}x$

9. Construct graphs for each of the following linear relationships.

a. $y = -5x + 4$

b. $y = 3x + 11$

c. $y = -6x$

d. $y = \frac{5}{2}x + 5$

e. $y = \frac{2}{3}x - 6$

f. $y = 2x - 1$

10. a. If t represents the time in hours and C represents cost (\$), construct a table of values for 0–3 hours for the cost of playing tenpin bowling at a new alley.

NEW Tenpin Bowling Alley

Shoe rental just \$2 (fixed fee)

Rent a lane for ONLY \$6/hour!

- b. Use your table of values to plot a graph of time versus cost. (*Hint:* Ensure your time axis (horizontal axis) extends to 6 hours and your cost axis (vertical axis) extends to \$40.)
- c. Calculate the slope.
- d. Write a linear equation to describe the relationship between cost and time.
- e. Use your linear equation from part d to calculate the cost of a 5-hour tournament.
- f. Use your graph to check your answer to part e.
11. A straight line has a slope of 4 and passes through the point (2, 9). Determine the y -intercept of the line.
12. A straight line has an equation of $2x = 3y - 1$. Rearrange the equation into the form $y = mx + c$ and hence state the y -intercept and the slope.

Complex familiar

13. The equation $C = 3.5g + 55$ represents the cost of hiring online computer games from a gaming club, where C is the cost of hiring g games.
If a teenager budgets on spending \$300 on games for the year, determine how many games he can hire.
14. A family dental fund charges an annual membership fee of \$200 and then \$20 per dental appointment.
If the average cost of a dental appointment is \$60 without a dental plan, determine which of the following families will have saved money by being in the fund if they each visit the dentist twice a year. Calculate how much they will have saved.
- a. A family of 2 b. A family of 4 c. A family of 10



Complex unfamiliar

15. A local coffee shop sells coffee at a price of \$4.50 per cup and charges a one-time membership fee of \$12.50, which includes a reusable coffee cup. If you don't become a member, coffee costs \$6.25 per cup.
Julia buys a coffee every weekday and decides to become a member. Her friend Tessa only buys coffee three times a week and decides not to join. Julia says that she will have spent less money than Tessa after 6 weeks. Use a mathematical model to determine if Julia is correct.



Fully worked solutions for this chapter are available online.

LESSON

2.3 Fitting a least-squares line to data

SYLLABUS LINKS

- Model a linear relationship by using technology to fit a least-squares line to the data, in the form of $y = mx + c$ where m is slope (gradient) and c is y -intercept.
- Understand and use $m = r \frac{s_y}{s_x}$ and $c = \bar{y} - m\bar{x}$ to determine the equation of a least-squares line, where m is slope (gradient), r is correlation coefficient, s_y is (sample) standard deviation of y values, s_x is (sample) standard deviation of x values, c is y -intercept, \bar{y} is mean of y values and \bar{x} is mean of x values.
- Interpret the y -intercept and slope (gradient) of the fitted line.
- Distinguish between interpolation and extrapolation.
- Use the equation of the least-squares line to make predictions.
- Recognise and explain the potential dangers of extrapolation.
- Solve practical problems by identifying, analysing and describing associations between two variables (categorical and/or numerical).

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2.3.1 Lines of best fit

In Chapter 1 scatterplots were constructed from raw data and studied to determine if a linear association existed between the two variables. If the points on a scatterplot appear to be distributed in a linear pattern, a straight line can be drawn through the data.

A **line of best fit** is the straight line that is positioned as close as possible to all the data points, that is, the average distance between the data points and the line is minimised. It is used to generalise the relationship between two variables.

There are a number of ways to draw a line of best fit. This General Mathematics course focuses on the least-squares regression line as a mathematical means of constructing a line of best fit.



2.3.2 The least-squares regression line

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Least-squares regression involves an exact mathematical approach to fitting a line of best fit to bivariate data that shows a strong linear relationship.

This line minimises the vertical distances between the data points and the line of best fit. It is called the least-squares regression line because if we took the squares of these vertical distances, this line would represent the smallest possible sum of all these squares.

The equation for the least-squares regression line takes the form: $y = mx + c$, where y is the response (dependent) variable, x is the explanatory (independent) variable, m is the slope or gradient of the line and c is the y -intercept.

Technology in the form of a spreadsheet program such as Excel or a scientific calculator can be used to calculate the equation of the least-squares regression line.

The equation of a least-squares regression line

To determine the equation of the least-squares regression line, the following summary data is required:

\bar{x} — the mean of the x values (explanatory variable)

\bar{y} — the mean of the y values (response variable)

s_x — the (sample) standard deviation of the x values

s_y — the (sample) standard deviation of the y values

r — correlation coefficient.

The general form of the least-squares regression line is:

$$y = mx + c$$

where:

the slope of the regression line is $m = r \frac{s_y}{s_x}$

the y -intercept of the regression line is $c = \bar{y} - m\bar{x}$.



tivd-12125

WORKED EXAMPLE 4 Calculating the least-squares regression line

A study to find a relationship between the height of men and the height of their female partner revealed the following details.

- Mean height of the men: 180 cm
- Mean height of the female partners: 169 cm
- Sample standard deviation of the heights of the men: 5.3 cm
- Sample standard deviation of the heights of women: 4.8 cm
- Correlation coefficient: $r = 0.85$

The form of the least-squares regression line is:

Height of female partner = $m \times$ height of man + c

- Determine which variable is the response variable.
- Calculate the value of m for the regression line (to 2 decimal places).
- Calculate the value of c for the regression line (to 2 decimal places).
- Use the equation of the regression line to predict the height of a female whose male partner is 195 cm tall (to nearest cm).



THINK

- Recall that the response variable is the subject of the equation in $y = mx + c$ form; that is, y .
- The value of m is the slope of the regression line. Write the formula and state the required values.
 - Substitute the values into the formula and evaluate m .

WRITE

- The response variable is the height of the female.

$$\begin{aligned} \text{b. } m &= r \frac{s_y}{s_x} \\ r &= 0.85, s_y = 4.8 \text{ and } s_x = 5.3 \end{aligned}$$

$$\begin{aligned} m &= 0.85 \times \frac{4.8}{5.3} \\ &= 0.7698 \\ m &= 0.77 \end{aligned}$$

- | | |
|---|--|
| <p>c. 1. The value of c is the y-intercept of the regression line. Write the formula and state the required values.</p> <p>2. Substitute the values into the formula and evaluate c.</p> | <p>c. $c = \bar{y} - m\bar{x}$
 $\bar{y} = 169$, $\bar{x} = 180$ and $m = 0.7698$ (from part b)</p> <p>$c = 169 - 0.7698 \times 180$
 $= 30.436$
 $= 30.44$</p> |
| <p>d. 1. State the equation of the regression line, using the values calculated from parts b and c. In this equation, y represents the height of the female and x represents the height of the male.</p> <p>2. The height of the male is 195 cm, so substitute $x = 195$ into the equation and evaluate.</p> <p>3. Write a statement, rounding your answer to the nearest centimetre.</p> | <p>d. $y = 0.77x + 30.44$</p> <p>$y = 0.77 \times 195 + 30.44$
 $= 180.59$</p> <p>Using the equation of the regression line, the female's height is predicted to be 181 cm.</p> |

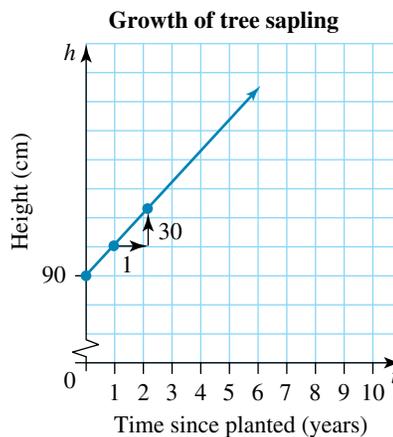
2.3.3 Interpreting the intercept and slope

Often data is collected in order to analyse a situation and make informed decisions or predictions about what might happen in other situations. The regression line equation from a scatterplot can be used for this purpose.

Recall from section 2.2.2 that the slope of a line represents the change in the y -value for every 1 unit increase in the x -value. The y -intercept is the y -value when the x -value is zero, that is where the line crosses the y -axis.

Consider the equation below, which represents the height of a tree, h (in cm), given the number of years since it was planted, t .

$$h = 30t + 90$$



From this equation we can see that the slope is 30 and the y -intercept is 90. To interpret the practical meaning of these values in this context, the sketch of the function is useful.

As can be seen from the sketch, the slope of 30 indicates that the tree is growing at a rate of 30 cm per year, and the y -intercept of 90 indicates the height of the tree when it was first planted (at time zero).

WORKED EXAMPLE 5 Using technology to determine the least-squares regression line

The following table shows data from Bilbo's Real Estate for house sales in the Shire in November 2025.

House	Number of bedrooms	Number of bathrooms	Size of garage (cars)	Size of land (m ²)	Price (\$)
1	2	1	1	117	730 000
2	4	2	1	630	1 875 000
3	3	1	2	688	1 300 000
4	2	1	1	228	790 000
5	3	1	2	858	1 610 000
6	2	1	1	637	670 000
7	3	1	1	588	1 400 000
8	6	4	1	700	2 060 000
9	2	1	1	93	520 000
10	2	1	1	73	639 000
11	3	1	1	242	720 000
12	1	1	1	112	460 000
13	2	1	1	167	737 000

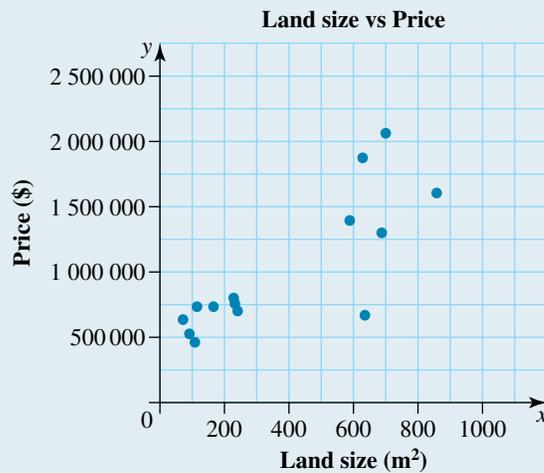
- Using Excel, draw a scatterplot of S (size of land) against P (price of house).
- Using Excel, determine the least-squares regression line in terms of S and P .
- Using a scientific calculator, determine the least-squares regression line in terms of S and P .
- Describe what the least-squares regression line tells you about property prices in the Shire.

THINK

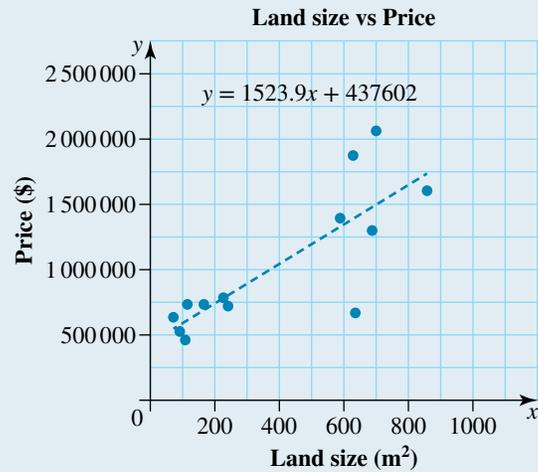
- Size of land (S) is the explanatory variable.
Price of house (P) is the response variable.
Plot the table of values using Excel.

WRITE

-



- b. Determine the regression line using Excel. **b.**



The regression line calculated is: $y = 1523.9x + 437\ 602$.
 In terms of price and size of land, the equation is:
 $P = 1523.9 \times S + 437\ 602$.

- c. **1.** Enter the Stat mode on your calculator and choose 2 variables. Enter the x -values in L1, move the cursor to L2 and enter the y -values.

c.

L1	L2
117	730 000
630	1 875 000
688	1 300 000
228	790 000
858	1 610 000
637	670 000
588	1 400 000
700	2 060 000
93	520 000
73	639 000
242	720 000
112	460 000
167	737 000

- 2.** Enter Stats mode and find the regression calculations for your calculator.
Note: Some calculators will display m and c as a and b .
- 3.** The answer appears on screen.
 Replace the variables with P and S .

$$a = 1523.9$$

$$b = 437\ 602.31$$

$$y = 1523.9x + 437\ 602$$

$$P = 1523.9 \times S + 437\ 602$$

- d.** Interpret the least-squares regression line by referring to the slope (\$1500) and the y -intercept (\$437 602).
- d.** Property prices begin at \$437 602 and increase by over \$1500/m² after that.

Digital technology

To determine a least-squares regression line on your calculator, you need to determine the coefficients a and b in the linear regression equation $y = a + bx$ using the stat function. The following instructions show how on a Casio fx-82AU Plus calculator.

Note: $y = a + bx$ is another form of $y = mx + c$ where $a = c$ and $b = m$.

1. Press MODE then select STAT mode. Choose the linear regression option (usually indicated by '2: A + BX' or similar).
2. Input your data into both columns then press equals to store the data. Press AC or similar to clear the screen.
3. Press SHIFT + 1 to access the statistical menu. Select '5: Reg' for regression calculations.
4. Press 1 to display the value of a . Press 2 to display the value of b .
5. Record these values for your linear regression equation $y = a + bx$.
6. Therefore, the equation is $y = 1523.9x + 437602$.

```

1: 1-VAR  2: A+BX
3: _+CX^2 4: 1n X
5: e^X    6: A·B^X
7: A·X^B  8: 1/X
    
```

```

          STAT      D
          X      Y
11 | 242 | 720000
12 | 112 | 460000
13 | 167 | 660000
          =
          737000
    
```

```

1: Type    2: Data
3: Sum     4: Var
5: Reg     6: MinMax
    
```

```

1: A      2: B
3: r      4: Δ
5: Δ
    
```

```

          STAT      D
          A
          =
          437602.3135
    
```

```

          STAT      D
          B
          =
          1523.89829
    
```

WORKED EXAMPLE 6 Interpreting the intercept and slope

The least-squares regression equation for a line is $y = -8x + 62$.

- Identify the y -intercept.
- Determine the change in the response variable for each unit of change in the explanatory variable.
- Explain what your answer to part **b** reveals about the direction of the line.

THINK

- Consider the equation in the form $y = mx + c$. Identify the value that represents c .
- The change in the response variable due to the explanatory variable is reflected in the slope. Identify the m value in the equation.
- A positive m value indicates a line pointing in a positive direction, while a negative m value indicates a line pointing in a negative direction.

WRITE

- y -intercept (c) = 62
- Slope (m) = -8
- As the m value is negative, the direction of the line is negative.

2.3.4 Interpolation and extrapolation

The regression line can be used to explore data points both inside and outside of the given scatterplot range. When investigating data inside the variable range, the data is being **interpolated**. Predictions outside the range of data is **extrapolation**.

The regression equation can be used to make predictions from the data by substituting in a value for either the explanatory variable (x) or the response variable (y) to calculate the value of the other variable.



tlvd-12126

WORKED EXAMPLE 7 Using interpolation and extrapolation to make predictions

Flowers with a diameter of 5 to 17 cm were measured and the number of petals for each flower was documented. A regression equation of $N = 1.88d + 0.41$ was established, where N is the number of petals and d is the diameter of the flower (in cm).

- Identify the explanatory variable.
- Determine the number of petals that would be expected on a flower with a diameter of 15 cm. Round to the nearest whole number.
- Identify whether the value found in part **b** is an example of interpolated or extrapolated data.
- Use the equation to predict the diameter of a flower with 35 petals, rounding to 1 decimal place.
- Determine whether part **d** represents interpolated or extrapolated data.

THINK

- Consider the format of the equation. The variable on the right-hand side of the equation will be the explanatory variable.
- Using the equation, substitute 15 for d and evaluate.
 - Round N to the nearest whole value.
Note: Round down as you can't have 0.61 of a petal.

WRITE

- Explanatory variable = flower diameter (d)
- $$N = 0.41 + 1.88d$$
$$= 0.41 + 1.88 \times 15$$
$$= 28.61$$

28 petals

c. Consider the data range given in the opening statement.

d. 1. Using the equation, substitute 35 in place of N .

2. Transpose the equation to solve for d .

3. Round to 1 decimal place.

e. Consider the data range given in the opening statement.

c. 15 cm is inside the data range, so this is interpolation, not extrapolation.

d. $35 = 0.41 + 1.88d$

$$d = \frac{35 - 0.41}{1.88}$$

$$= 18.40$$

$$= 18.4 \text{ (correct to 1 decimal place)}$$

e. 18.4 cm is outside the data range, so this is an example of extrapolated data.

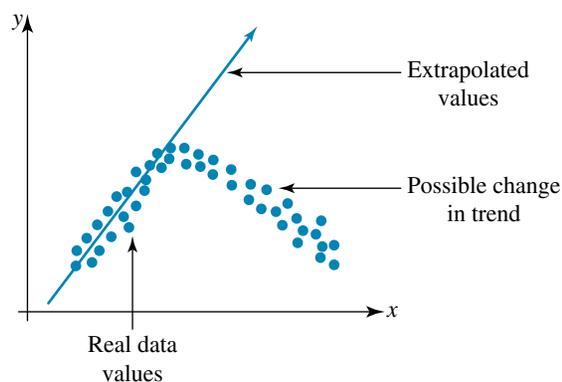
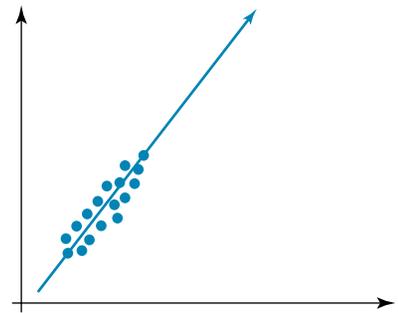
2.3.5 Limitations of regression line predictions

When reviewing predictions drawn from a scatterplot, it is necessary to question the reliability of the results. As with any conclusion or prediction, the results rely heavily on the initial data. If the data was collected from a small sample, then the limited information could contain biases or a lack of diversity that would not be present in a larger sample. The more data that can be provided at the start, the more accurate a result will be produced.

The strength of the association between the variables also provides an indication of the reliability of the data. Data that produces no association or a low association would suggest that any conclusions drawn from the data will be unreliable.

When extrapolating data, it is assumed that additional data will follow the same pattern as the data already in use. This assumption means extrapolated data is not as reliable as interpolated data.

Consider the following data. Given the trend of the data, the least-squares regression line would be a positive linear function.



If, however, once more data was collected the data trend changed, any earlier predictions made using extrapolation would not be reasonable. This highlights the dangers of extrapolation based on the assumption that the data will continue to follow the same trend.

2.3 Exercise

2.3 Exam questions **ON**

Simple familiar
1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13

Complex familiar
14, 15, 16, 17, 18

Complex unfamiliar
19, 20

These questions are even better in jacPLUS!

- Receive immediate feedback
- Access sample responses
- Track results and progress



Find all this and MORE in jacPLUS

Simple familiar

1. **WE4** The following summary details were calculated from a study to find a relationship between Mathematics exam marks and English exam marks from the results of 120 Year 10 students.

Mean Mathematics exam mark = 64%
 Mean English exam mark = 74%
 Sample standard deviation of Mathematics exam mark = 14.5%
 Sample standard deviation of English exam mark = 9.8%
 Correlation coefficient, $r = 0.64$.
 The form of the least-squares regression line is:
 Mathematics exam mark = $m \times$ English exam mark + c



- Determine which variable is the response variable (y-variable).
 - Calculate the value of m for the least-squares regression line.
 - Calculate the value of c for the least-squares regression line.
 - Use the regression line to predict the expected Mathematics exam mark if a student scores 85% in an English exam (to the nearest percentage).
2. Determine the least-squares regression equation, given the following summary data.
- $\bar{x} = 5.6, s_x = 1.2, \bar{y} = 110.4, s_y = 5.7, r = 0.7$
 - $\bar{x} = 110.4, s_x = 5.7, \bar{y} = 5.6, s_y = 1.2, r = -0.7$
 - $\bar{x} = 25, s_x = 4.2, \bar{y} = 10\,200, s_y = 250, r = 0.88$
 - $\bar{x} = 10, s_x = 1, \bar{y} = 20, s_y = 2, r = -0.5$
3. **WE5** A researcher investigating the proposition that ‘tall mothers have tall sons’ measures the height of 12 mothers and the height of their adult sons. The results are shown.
- Using Excel, draw a scatterplot.
 - Determine the least-squares regression line in terms of height of mother (M) and height of son (S). Give values correct to 4 significant figures.
 - Explain what the least-squares regression line tells you about the relationship between the height of the mother and the height of the son.

Height of mother (cm)	Height of son (cm)
185	188
155	157
171	172
169	173
170	174
175	180
158	159
156	150
168	172
169	175
179	180
173	190

4. **WE6** The least-squares regression equation for a line is $y = 1.701x - 1.837$.
- Identify the y -intercept.
 - For each unit of change in the explanatory variable, determine how much the response variable changes by.
 - Explain what your answer to part **b** reveals about the direction of the line.
5. The least-squares regression equation for a line is $y = 105.90 - 1.476x$.
- Identify the y -intercept.
 - For each unit of change in the explanatory variable, determine how much the response variable changes by.
 - Explain what your answer to part **b** reveals about the direction of the line.
6. **WE7** The ages of toddlers between 9 and 36 months were collected and the number of recognisable words in their vocabulary for each toddler was documented. A regression equation $W = 2.06a - 16.54$ was established, where W is the number of recognisable words and a is the age in months.
- Identify the explanatory variable.
 - Determine the number of recognisable words that would be expected from a toddler of age 16 months. Round to the nearest whole number.
 - Identify whether the value found in part **b** is an example of interpolated or extrapolated data.
 - Use the equation to predict the age of a toddler who uses 60 recognisable words, rounding to 1 month.
 - Determine whether part **d** represents interpolation or extrapolation.
7. A brand of medication for babies bases the dosage on the age (in months) of the child.



The regression equation for this situation is $M = 0.312A + 0.157$, where M is the amount of medication in mL and A is the age in months.

- Identify the explanatory variable.
 - Calculate the amount of medication required for a child aged 6 months.
 - Determine the age of a child who requires 2.5 mL of the medication. Give your answer correct to 1 decimal place.
8. A survey of the nightly room rate for Sydney hotels and their proximity to the Sydney Harbour Bridge produced the regression equation $C = -50.471d + 281.92$, where C is the cost of a room per night in dollars and d is the distance to the bridge in kilometres.
- Identify the response variable.
 - Based on this equation, calculate the cost of a hotel room 2.5 km from the bridge. Give your answer correct to the nearest cent.
 - Determine the distance of a hotel room from the bridge if the cost of the room was \$115. Give your answer correct to 2 decimal places.

9. An equation for a regression line is $y = -1.56x + 3.2$. Describe the conclusions that can be drawn from the equation about the direction of the regression line.
10. Answer the following questions for the equation $y = -5x + 60$.
- Identify the y -intercept.
 - For each unit of change in the explanatory variable, determine how much the response variable changes by.
 - State whether the direction of the data is positive or negative.
 - Calculate the value of y when $x = 40$.
11. Lucy was given the equation $y = 7.32x - 12.9$ and asked to find the value of x when $y = 15.68$. Her working steps are below.

$$\begin{aligned}
 y &= 7.32x - 12.9 \\
 15.68 &= 7.32x - 12.9 \\
 x &= \frac{15.68}{7.32} + 12.9 \\
 x &= 15.04
 \end{aligned}$$

Her teacher indicates her answer is wrong.

- Calculate the correct value of x . Give your answer correct to 2 decimal places.
 - Identify and explain Lucy's error.
12. Answer the following questions for the equation $y = 25x - 12$.
- Identify the y -intercept.
 - For each unit of change in the explanatory variable, determine how much the response variable changes by.
 - State whether the direction of the data is positive or negative.
 - Calculate the value of y when $x = 3.5$.
13. While camping a mathematician estimated that number of mosquitoes around the fire $= 10.2 + 0.5 \times$ temperature of the fire ($^{\circ}\text{C}$).



- Determine the number of mosquitoes that would be expected if the temperature of the fire was 240°C . Give your answer correct to the nearest whole number.
- Determine what the temperature of the fire would be if there were only 12 mosquitoes in the area.
- Identify some factors that could affect the reliability of this equation.

Complex familiar

14. After studying the linear relationship between the size of a spider's web and the number of insects caught, a scientist establishes that for every increase in area of 1 cm^2 , the number of insects caught will increase by 1.1. The size of the web starts at 0.43 cm^2 . Determine how many insects are likely to be caught if the area of the spider's web is 60 cm^2 . Give your answer correct to the nearest whole number.



15. A data set produced a positive direction and for each incremental increase in the explanatory variable, the response variable increased by 2.5. If $y = 4$ when $x = 0$, determine the equation for the regression line.
16. Assuming a linear model, use the data given below and appropriate technology to determine the predicted values for the entries missing in the table below.

x	10	11	12	13	14	15	16		18
y	22	18		15	17	11	11	7	9

17. Data on 15 people's average monthly income and the amount of money they spend at restaurants was collected.

Average monthly income (\$000s)	Money spent at restaurants per month (\$)
4.2	620
3.6	395
2.7	185
2.8	150
2.5	130
3.0	220
3.1	245
2.2	100
4.0	400
3.7	380
3.8	200
3.5	360
2.9	175
3.6	350
4.1	600

Hayden spent \$265 eating out last month. Estimate his monthly income and evaluate the reasonableness of this estimate.

18. Data on 15 students' marks in Geography and Music assessments were collected.

Geography marks	Music marks
65	91
80	57
72	77
61	89
99	51
54	76
39	62
66	87
78	88
89	64
84	90
73	45
68	60
57	79
60	69

Predict the mark a student would receive for Music if they received a mark of 85 for Geography, rounding to the nearest whole number. Evaluate the reasonableness of your prediction. You could use other statistical analysis values to justify your response.

Complex unfamiliar

19. For three months, Cameron has been wearing an exercise-tracking wristband that records the distance he walks and the number of calories he burns. A graph shows his weekly totals. The regression line equation for the data where y represents the number of calories burned and x represents the distance walked is $y = 14\,301 + 115.02x$. Pearson's product-moment correlation coefficient for this data is 0.9678 and the mean of the number of calories burned each week is 17.114. Calculate the mean number of kilometres Cameron has walked each week. Give your answer correct to 2 decimal places.



20. A phone carrier company used fingerprint biometric technology to collate some results about people using a particular social media application. A graph is drawn that shows their results for people's ages and the number of social media friends they had. The regression line equation for this data set is $y = -13.613x + 777.84$, where y represents the number of social media friends and x represents the ages of the people. Interpret the slope and the intercept for this equation, and suggest a suitable range for this data. Justify your answer.



Fully worked solutions for this chapter are available online.

LESSON

2.4 Residual plots

SYLLABUS LINKS

- Construct a residual plot and use it to assess the appropriateness of fitting a linear model to the data
- Solve practical problems by identifying, analysing and describing associations between two variables (categorical and/or numerical).

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2.4.1 Residual plots and analysis

In the world of statistical modelling, it is often not enough to use a scatterplot to determine if a linear model is appropriate. There may be other underlying patterns that are difficult to recognise simply by studying the scatterplot. A **residual plot** can be constructed using the least-squares regression line to highlight any underlying patterns which would indicate that the data does not fit a linear model.

Calculating residuals

Consider the following set of data.

x	3	4	5	6	7	8
y	4	8	12	2	10	15

The following graph shows the data plotted on a scatterplot and a regression line fitted.

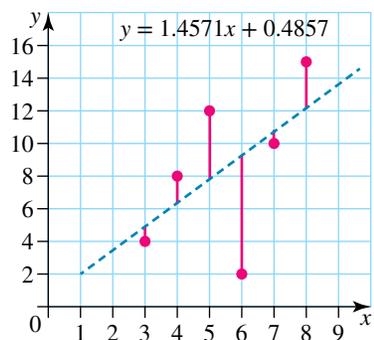
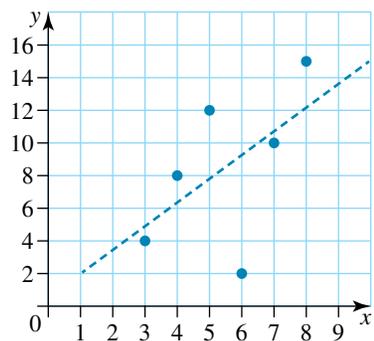
For this set of data, the linear regression equation is $y = 1.4571x + 0.4857$.

The lengths of the vertical lines joining the data points to the regression line are the **residuals**.

For each value of x (explanatory variable), there is the actual value y (response variable) from the data supplied and there is the predicted value, found from the linear regression equation.

Residuals

$$\text{Residual value} = \text{actual } y\text{-value} - \text{predicted } y\text{-value}$$



In the data in the previous table for the x -value of 3, the actual y -value is 4 and the predicted y -value using the linear regression equation is:

$$\begin{aligned} y &= 1.4571x + 0.4857 \\ &= 1.4571 \times 3 + 0.4857 \\ &= 4.857 \end{aligned}$$

$$\begin{aligned} \text{Residual value} &= \text{actual value} - \text{predicted value} \\ &= 4 - 4.857 \\ &= -0.86 \text{ (correct to 2 decimal places)} \end{aligned}$$

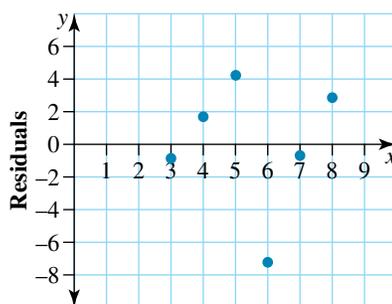
The following table is completed for all the predicted y -values and the residuals.

x	3	4	5	6	7	8
Actual y -value	4	8	12	2	10	15
Predicted y -value	4.86	6.31	7.77	9.23	10.69	12.14
Residual value	-0.86	1.69	4.23	-7.23	-0.69	2.86

Residual plots

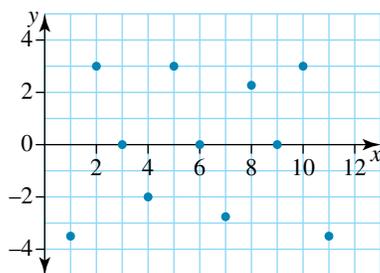
A **residual plot** is a graph of the points obtained by plotting the residuals on the vertical axis and the explanatory variable (x -value) on the horizontal axis. When the points in a residual plot are randomly spread around the horizontal axis, a linear regression model is appropriate for the data; otherwise a non-linear model should be considered.

For the data in the previous table a residual plot is completed below.

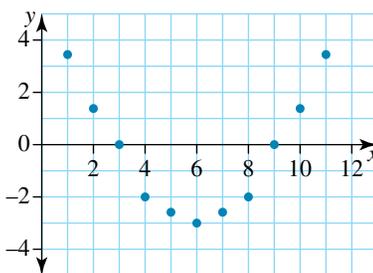


The previous residual plot shows a random pattern, therefore the linear model is seen as a good fit.

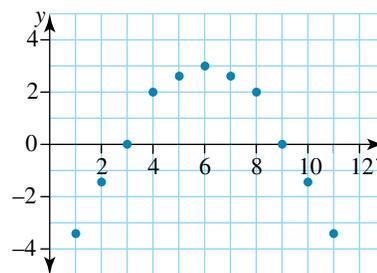
The following residual plots show some typical patterns for residuals.



Random pattern
Linear model appropriate



Non-random
Linear model *not* appropriate



Non-random
Linear model *not* appropriate

If a residual plot showed non-random patterns then a non-linear model for the original data should be investigated. Non-linear models are outside the scope of this course.

WORKED EXAMPLE 8 Calculating residual values and constructing residual plots

For the set of data below, the least-squares regression equation is $y = -0.56x + 45.6$.

a. Complete the table.

x	50	55	60	65	70
Actual y-value	20	10	15	8	7
Predicted y-value					
Residual value					

b. Construct a residual plot and interpret the graph.

THINK

a. 1. Calculate the predicted y-values:

$$y = -0.56x + 45.6$$

$$y = -0.56 \times 50 + 45.6 = 17.6$$

$$y = -0.56 \times 55 + 45.6 = 14.8$$

$$y = -0.56 \times 60 + 45.6 = 12$$

$$y = -0.56 \times 65 + 45.6 = 9.2$$

$$y = -0.56 \times 70 + 45.6 = 6.4$$

2. Calculate the residual values:

Residual value = actual y-value – predicted y-value

$$20 - 17.6 = 2.4$$

$$10 - 14.8 = -4.8$$

$$15 - 12 = 3$$

$$8 - 9.2 = -1.2$$

$$7 - 6.4 = 0.6$$

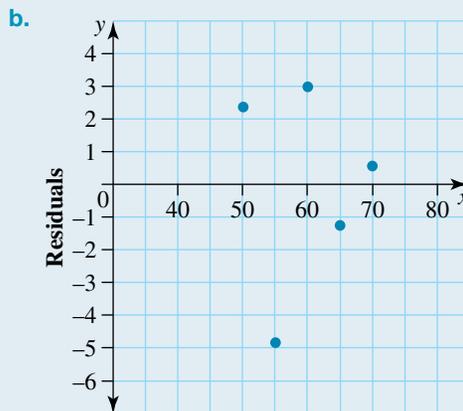
b. 1. Draw a set of axes with the x value on the horizontal axis and the residual value on the vertical axis. Plot the points on the axes.

WRITE

a.

x	50	55	60	65	70
Actual y-value	20	10	15	8	7
Predicted y-value	17.6	14.8	12	9.2	6.4
Residual value					

x	50	55	60	65	70
Actual y-value	20	10	15	8	7
Predicted y-value	17.6	14.8	12	9.2	6.4
Residual value	2.4	-4.8	3	-1.2	0.6



2. The points do not appear to form a pattern.

As the residuals appear to be randomly spread above and below the x-axis, then a linear model is an appropriate model for this set of data.

2.4 Exercise

2.4 Exam questions **ON**

Simple familiar

1, 2, 3, 4, 5, 6, 7, 8, 9

Complex familiar

10, 11

Complex unfamiliar

12, 13

These questions are even better in jacPLUS!

- Receive immediate feedback
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Simple familiar

1. **WEB** For the set of data below, the least-squares regression equation is $y = 1.27x + 7.65$

a. Complete the following table.

x	13	18	23	28	33	38
Actual y-value	25	31	40	36	48	60
Predicted y-value						
Residual value						

b. Construct a residual plot and interpret the graph.

2. Consider the following data set.

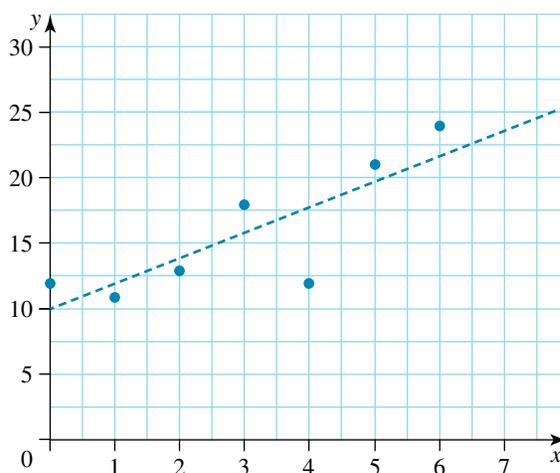
x	1	2	3	4	5	6	7	8	9	10
y	1	5	10	16	26	34	50	62	80	101

a. A linear regression equation of the form $y = mx + c$ was calculated for this data and the results were $c = -21.333$, $m = 10.879$. Use this information to calculate the predicted values of y and the residuals.

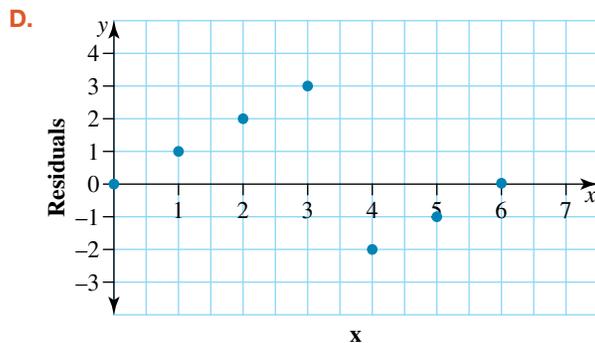
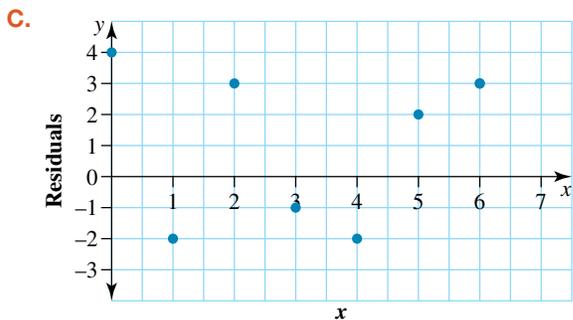
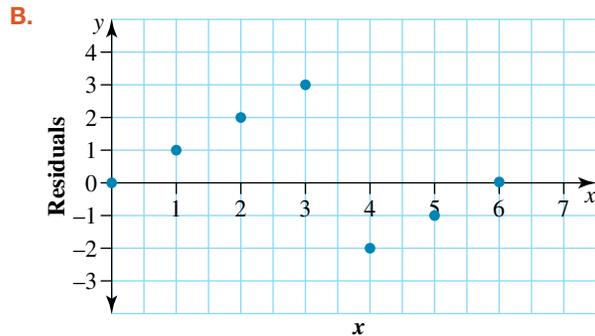
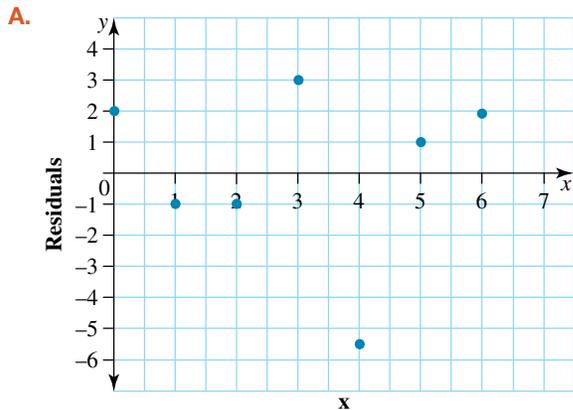
b. Plot the residuals and comment on the likely linearity of the data.

The following information relates to questions 3–5.

The following graph shows a least-squares regression line fitted to a set of data.



3. **MC** From the following options, choose the residual plot that best represents the residuals for this line.



4. **MC** The value of Pearson's correlation coefficient (r) for this data is closest to:

- A. -0.81 B. -0.33 C. 0.33 D. 0.81

5. **MC** The residual for the data value 4 is closest to:

- A. 6 B. 2 C. -6 D. -2

6. A linear model predicting the amount of a product sold based on advertising expenditure is given by:

$$y = 0.8x + 200$$

where y is the predicted sales in units and x is the advertising expenditure in thousands of dollars. If the actual sales are 1200 units and the advertising expenditure is \$1500, calculate the residual.

7. A regression line on a graph has the equation:

$$y = 2.817x - 5.65$$

A data point is located at $(5.7, 10.41)$. Determine whether this point lies on the regression line. If not, calculate its residual.

8. A company predicts employee salaries based on experience (years) using:

$$S = 5000E + 30\,000$$

where S is salary (in dollars) and E is experience (in years).

Complete the following sentences.

- For every increase of 1 year in experience, the employee's salary will increase by _____.
- The starting salary at this company is _____.
- An employee has 9 years' experience. The regression line predicts a salary of _____.
- The employee with 9 years' experience has an actual salary of \$71 500. The residual is _____.

9. **MC** A farmer uses a linear regression model to predict the crop yield (Y in tonnes) based on the amount of fertiliser applied (F in kilograms per hectare). The model is given as:

$$Y = 2.5F + 10$$

The observed data includes the following:

Fertiliser (kg/ha)	Observed yield (tonnes)
5	23
8	28
10	37

Determine the residual for the field where 8 kg/ha of fertiliser was applied and what it indicates.

- A. Residual = -2 , meaning the model overestimated the yield.
 - B. Residual = 2 , meaning the model underestimated the yield.
 - C. Residual = 3 , meaning the model underestimated the yield.
 - D. Residual = -3 , meaning the model overestimated the yield.
- Complex familiar**
10. The following table represents the costs for transporting a consignment of surfboards from Brisbane factories.



The cost is given in terms of distance from Brisbane. There are two factories which can be used. The data are summarised below.

Distance from Brisbane (km)	10	20	30	40	50	60	70	80
Factory 1 cost (\$)	70	70	90	100	110	120	150	180
Factory 2 cost (\$)	70	75	80	100	100	115	125	135

Determine if a linear model is suitable to describe the relationship between cost and distance from Brisbane for both of the factories.

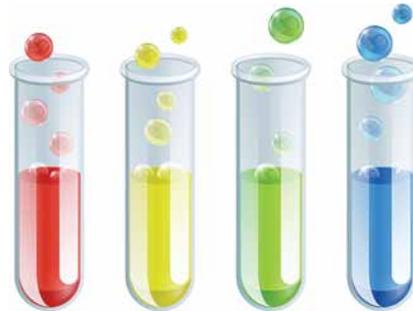
11. The following table contains the age and systolic blood pressure (SBP) for a group of volunteers at a university's Health Science department.

Age	37	38	40	42	45	48	50	52	53	55
Systolic blood pressure	130	140	132	149	144	157	161	145	165	162

Estimate the systolic blood pressure at age 75 and comment on the reliability of this estimation. Ensure you assess the likely linearity of the data as part of your response.

Complex unfamiliar

12. A woman diagnosed as anaemic has a level of 120 g/L of iron at her initial blood test. She agreed to join a research group to determine how quickly an iron supplement in capsule form, administered daily, would impact on her iron levels.
Her iron level was measured once a week. The following data were collected.



Week of experiment	1	2	3	4	5	6
Iron level	120	122	130	135	135	140

Using a residual analysis approach, discuss the reasonableness of fitting a linear model to this data and hence determine if it is possible to predict when the woman will reach a healthy iron count of 155 g/L of iron.

13. A farmer's market is held once a month and features local produce, handicrafts and trash and treasure stalls.
The number of local produce and handicraft stalls varies each weekend and is announced prior to the weekend on the website for the market. The number of local produce and handicraft stalls participating, together with the number of visitors to the market each month, is shown in the following table.



Number of stalls	53	34	61	32	61	25
Number of visitors	501	339	611	300	450	333

The least-squares regression line for this data is

$$\text{Number of visitors} = 6.64 \times \text{Number of stalls} + 128.11$$

Determine whether this is a suitable model to use to predict the number of visitors based on the number of stalls. Use mathematics to justify your answer.

Fully worked solutions for this chapter are available online.

LESSON

2.5 Association and causation

SYLLABUS LINKS

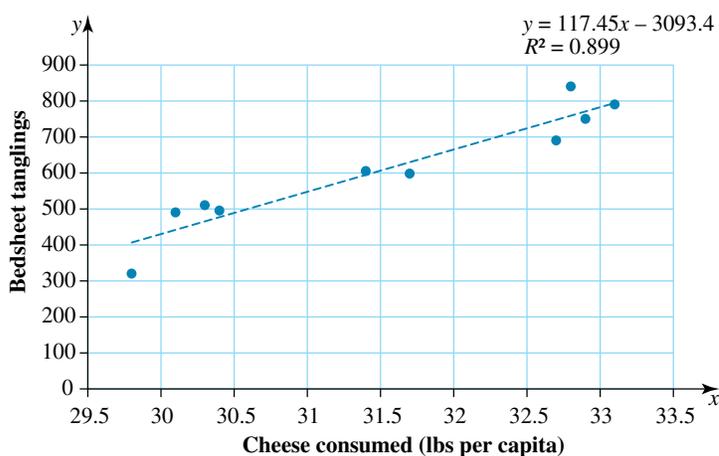
- Recognise and explain that an observed association between two variables (categorical and/or numerical) does not necessarily mean that there is a causal relationship between them.
- Identify and communicate possible non-causal explanations for an association, including coincidence or the influence of another variable.
- Solve practical problems by identifying, analysing and describing associations between two variables (categorical and/or numerical).

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2.5.1 Observation and association

The following scatterplot shows an association of $r = 0.95$ ($r = \sqrt{0.899} = 0.95$) for the number of people who died by becoming entangled in their bedsheets and per capita cheese consumption from 2000 to 2009.

The scatterplot shows a strong positive linear association between the variables, supported by a correlation coefficient of 0.948. This suggests that as cheese consumption increases, so does the number of people who die by becoming tangled in their bedsheets. However, this is an extreme example of why association does not imply causation. It is unrealistic to think increased cheese consumption causes such deaths.



Identifying a strong association can be a starting point for investigation, but proving causation requires further observation and experimentation. For instance, a high positive association between a basketball player's height and points scored does not mean recruiting tall players guarantees better scorers. Other factors like fitness, skill, and coordination are also essential.

Experiments using control groups can help establish causation. For example, agricultural scientists tested a new fertilizer by splitting tomato plants into two groups, one receiving the fertilizer and the other just water, keeping all other variables constant. This helps establish causation, with the fertilizer as the explanatory variable and crop size as the response variable.

2.5.2 The influence of another variable or coincidence explanations

Association between two variables can exist without causation. Other explanations include coincidence or the influence of another variable. For example, a study may show a link between a country's GDP and infant mortality rate. Although GDP does not directly reduce infant mortality, increased spending on vaccination programs, related to GDP, is the true cause.

Another example is the association between heart attack deaths and distance from a major hospital. Factors like diet, age, weight, and gender also influence heart attack deaths. Without considering these variables, the study cannot prove causation.

Sometimes associations are **coincidences**, like the link between cheese consumption and deaths from bed sheet entanglement. In research, it is essential not to assume causation from an observed association. Controlled experiments are required to establish a true causal relationship.

WORKED EXAMPLE 9 Identifying association and causation

Data showing the average weekly hours of exercise for a group of 8 people and their LDL-cholesterol reading are as given.

Average weekly hours of moderate exercise	7	1	3	8	3	10	6	2
LDL-C	100	125	130	120	128	90	115	135

- Construct a scatterplot for the data.
- Comment on the association between the number of hours of exercise and the LDL-C reading.
- Calculate r .
- Based on the value of r obtained in part c, justify whether it is appropriate to conclude that the decrease in the LDL-C reading is caused by the increasing hours of exercise.



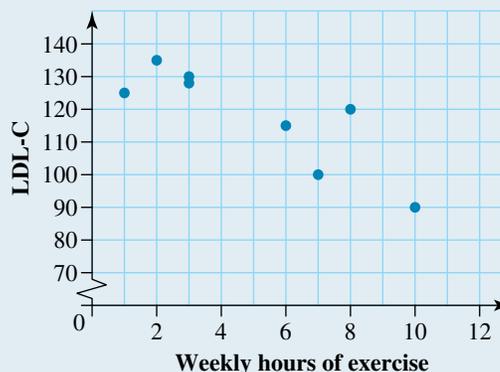
THINK

- Determine the explanatory and response variables
 - Sketch a scatterplot

- A negative association can be observed.
- Use technology to find the value of r .
- Association does not imply causation.

WRITE

- Weekly hours of exercise (explanatory) LDL-C reading (response)



- From the scatterplot there appears to be a strong negative association between the weekly hours of exercise and the LDL-C reading.
- $r = -0.85$
- Just because $r = -0.85$, it cannot be stated that increasing weekly hours of exercise will cause a decrease in the LDL-C reading. Other factors may need to be considered. Further investigation is needed before any conclusions can be drawn.

2.5 Exercise

2.5 Exam questions **on**

Simple familiar

1, 2, 3, 4, 5, 6, 7, 8, 9, 10

Complex familiar

11, 12

Complex unfamiliar

13, 14

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- Receive immediate feedback
- Access sample responses
- Track results and progress



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Simple familiar

1. **WE9** Data showing the number of rose bushes in the gardens of 10 houses and the annual income for each house is given.

Number of roses	4	2	10	5	3	11	6	7
Annual household income (\$)	80 000	32 000	120 000	65 000	21 000	122 000	75 000	82 000

- Using technology, construct a scatterplot for the data
- Comment on the association between the number of rose bushes and the household income.
- Calculate r using technology.
- Based on the value of r obtained in part **c**, justify whether it is appropriate to conclude that the increase in household income is caused by an increase in the number of rose bushes in the garden.



2. Data was collected on the level of aerobic fitness of 100 15-year olds and the amount of time they spent playing computer games. The correlation coefficient for this data was -0.86 . What can be said about the association between the level of aerobic fitness and the amount of time spent playing computer games for this group of 15-year olds?

3. **MC** During the months of August and September there was a strong positive association ($r = 0.93$) between the number of sightings of whales in the Whitsundays each week and the number of fines issued for boating infringements off Airlie Beach. A strong positive association was also found both between the number of whale sightings ($r = 0.96$), the number of fines issued for boating infringements ($r = 0.82$), and the number of visitors to Airlie Beach.



Using this information, determine which of the following statements is true.

- It is just a coincidence that there is a strong positive association between the number of sightings of whales in the Whitsundays and the number of fines issued for boating infringements.
- During the months of August and September, people are so excited about seeing whales that they forget to follow the rules for driving a boat.
- Tourists don't know that there are rules to be followed when at sea.
- August and September attract larger numbers of visitors and more visitors means more people taking out boats which leads to more boating infringements. The association between the number of whale sightings and the number of boating infringements can be explained by the common response variable, the number of tourists at Airlie Beach.

The following information applies to questions 4 and 5.

A set of data was collected from a large group of professional sportspeople. They were asked the number of hours they trained per week and the amount of money they earned. The results were recorded, and the value of Pearson's correlation coefficient was found to be 0.87.

4. **MC** Determine which of the following is NOT true.
- A. There is a positive association between the number of hours of training and the amount of money earned.
 - B. The association between the number of training hours and the amount of money earned can be classified as strong.
 - C. The linear relationship between the two variables suggests that as the number of training hours increase, so too does the amount of money earned.
 - D. The increase in the number of training hours causes the increase in the amount of money earned.
5. **MC** A study finds that people who exercise regularly tend to have lower stress levels. Determine which of the following is the best conclusion.
- A. Exercising reduces stress, proving a causal relationship.
 - B. People with low stress are more likely to have time for exercise, meaning stress causes exercise.
 - C. There is an association between exercise and stress, but causation is not proven.
 - D. Exercise and stress are completely unrelated.
6. **MC** A study finds a positive relationship between the number of firefighters at a fire and the amount of damage caused by the fire. Determine which of the following best explains this relationship.
- A. Firefighters cause more damage when they arrive at a fire.
 - B. Larger fires cause more damage and require more firefighters.
 - C. Having more firefighters causes the fire to spread.
 - D. The correlation is random and meaningless.
7. **MC** A researcher finds that the number of people who drown in swimming pools each year is strongly associated with the number of movies actor Nicolas Cage appears in. Identify the best explanation for this correlation
- A. Watching Nicolas Cage movies makes people more likely to go swimming.
 - B. More movies being made leads to more people going to the pool.
 - C. Drowning incidents cause more Nicolas Cage movies to be produced.
 - D. The correlation is a coincidence, with no causal link.
8. **MC** Determine the most effective method to evaluate whether consumption of a new energy drink enhances athletic performance.
- A. Conduct a survey asking athletes if they think the drink helps.
 - B. Perform a randomised controlled experiment where some athletes drink the energy drink and others do not.
 - C. Find a strong correlation between energy drink sales and athletic success.
 - D. Compare the performance of athletes who regularly drink energy drinks to those who do not.
9. **MC** A researcher finds that people who take daily vitamin supplements tend to be healthier than those who do not. Determine which of the following could exemplify the influence of another variable in this study.
- A. The colour of the vitamin pills
 - B. The price of the vitamins
 - C. The overall health-conscious lifestyle of vitamin users
 - D. The brand of vitamins consumed

10. **MC** A study finds a strong correlation between the number of televisions in a household and the household's income.
Determine which of the following is the best explanation.
- A. Households with higher incomes can afford more televisions, but owning televisions does not increase income.
 - B. Owning more televisions causes an increase in household income.
 - C. There is no association between income and the number of televisions.
 - D. The government mandates that wealthier households own more televisions.

Complex familiar

11. It has been found that there is a strong positive association between the number of firefighters that attend a fire and the amount of damage caused by the fire. To what extent can it be concluded that more firefighters cause more fire damage? Evaluate the potential common cause that could explain the relationship between these variables.
12. There is a strong positive association between the shoe size of a child and their academic knowledge. To what extent can it be inferred that a larger shoe size mean a smarter child? Explain the possible common cause(s) that may account for this association.



Complex unfamiliar

13. A group of university students volunteered to be part of a healthy food supplement trial. Their health was monitored every month for 10 months. A positive association was found between improvements in good health statistics such as cholesterol levels and blood pressure, and the consumption of the supplement. To what extent can it be concluded that the consumption of the supplement causes improvement in health? Assess other confounding variables that could contribute to this association.
14. A public health researcher analyses data from multiple cities and finds a strong association between the number of fast-food restaurants per capita and the obesity rate in those cities. The researcher concludes that increasing the number of fast-food restaurants in a city causes higher obesity rates. Identify at least two other variables that could influence this association and suggest an alternative explanation for the observed relations.

Fully worked solutions for this chapter are available online.

LESSON

2.6 Review

2.6.1 Summary

Hey students! Now that it's time to revise this chapter, go online to:

 Access the chapter summary

 Review your results

 Practise exam questions

Find all this and MORE in jacPLUS 



2.6 Exercise

learnon

2.6 Exercise

2.6 QCAA questions 

Simple familiar

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12

Complex familiar

13, 14, 15, 16

Complex unfamiliar

17, 18, 19, 20

These questions are even better in jacPLUS!

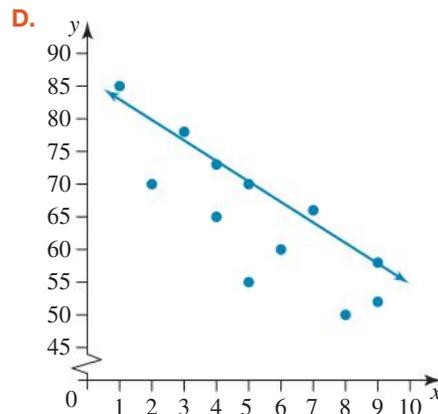
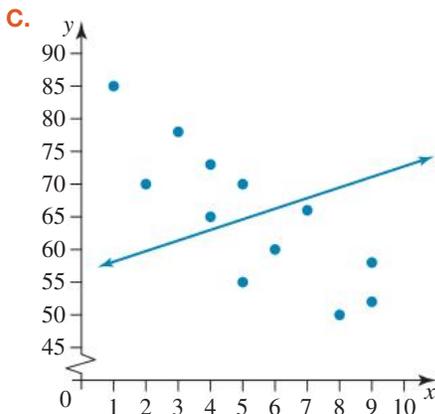
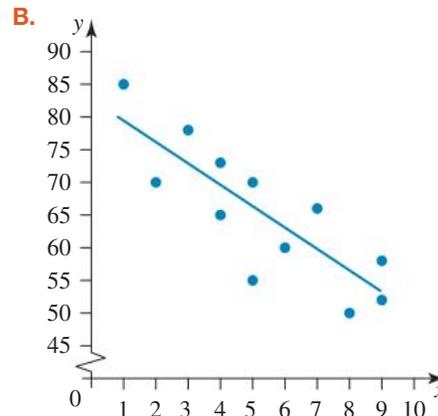
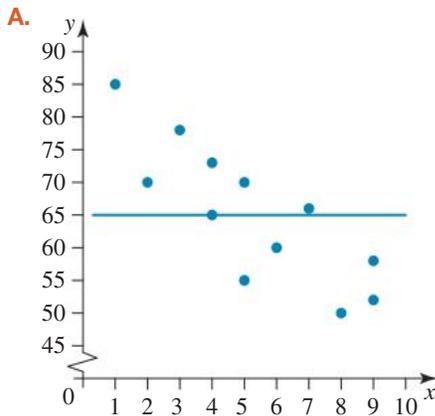
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- Track results and progress



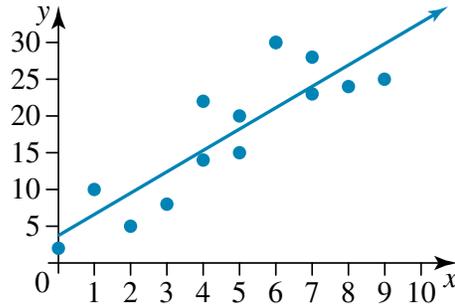
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Simple familiar

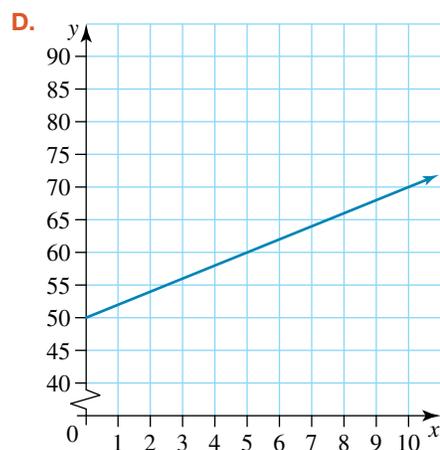
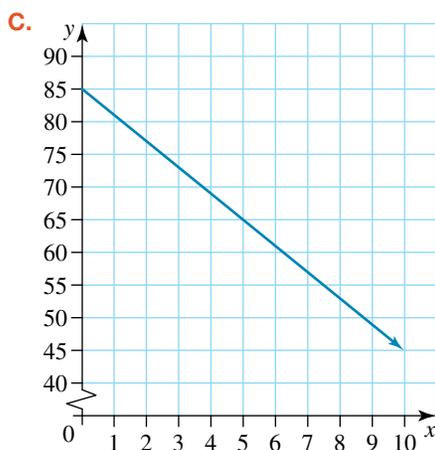
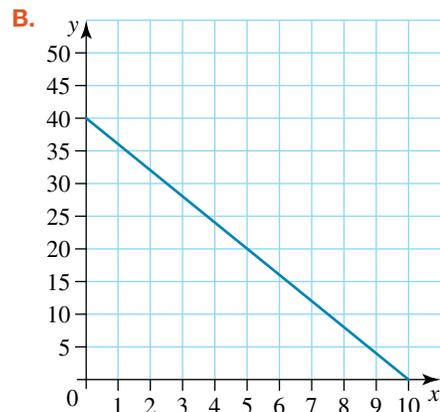
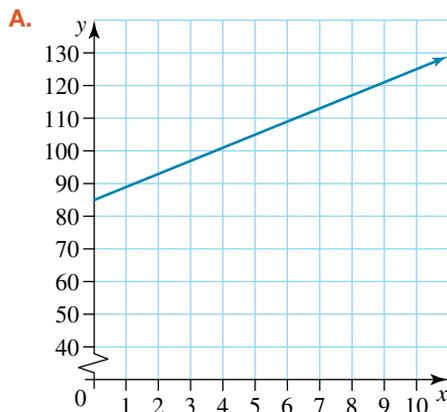
1. **MC** Determine which of the following scatterplots best demonstrates a line of best fit.



2. **MC** The regression line equation for the graph shown is closest to:



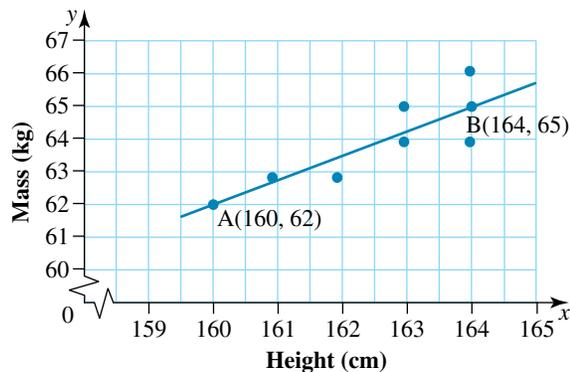
- A. $y = 3.8 + 2.9x$ B. $y = -3.8 - 2.9x$ C. $y = -3.8 + 2.9x$ D. $y = 3.8 - 2.9x$
3. **MC** If a linear regression line equation is of the form $y = mx + c$ and $m = 2.5$, $\bar{x} = 3$ and $\bar{y} = 6$ then the y -intercept, c will be:
- A. 1.5 B. -2.5 C. -1.5 D. 2.5
4. **MC** When $y = 15.87x + 0.54$, the value of y when $x = 2.5$ is:
- A. 18.91 B. 40.215 C. 39.135 D. 6.888
5. **MC** The graph for the regression line equation $y = -4x + 85$ is most likely to be:



6. **MC** For the linear equation $y = -3x + 12$, the slope of the equation can be interpreted as:
- for every one unit of increase in x , y increases by 3 units.
 - for every one unit of decrease in x , y increases by 12 units.
 - for every one unit of decrease in x , y decreases by 12 units.
 - for every one unit of increase in x , y decreases by 3 units.
7. **MC** For the following sample data set, determine which of the following is an example of interpolating data.

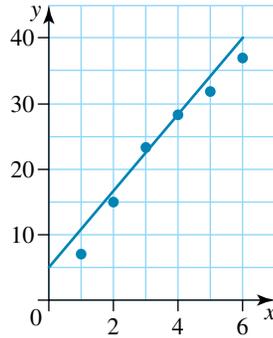
x	1	5	15	25
y	10	16	18	22

- Determining the value of x when $y = -7$
 - Determining the value of y when $x = 17$
 - Determining the value of x when $y = 27$
 - Determining the value of y when $x = 37$
8. For a data set where $\bar{x} = 11.5$, $\bar{y} = 16.5$, $s_x = 10.8$, $s_y = 5$, $r = 0.94$, determine the equation of the regression line. Round values to 3 decimal places.
9. A company analyses the relationship between the number of advertisements they run per month (n) and their total monthly sales revenue (R in thousands of dollars). They determine a least-squares regression line of $R = 2.5n + 50$.
- Interpret the slope.
 - Interpret the vertical intercept.
10. Consider the regression line drawn on the scatterplot shown below. Determine the equation of the regression line.

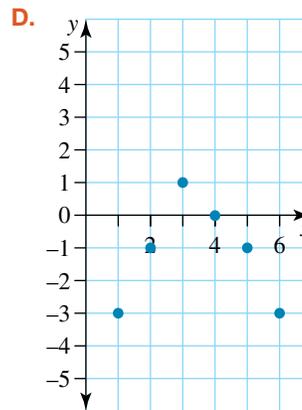
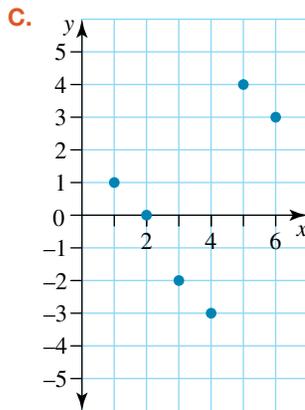
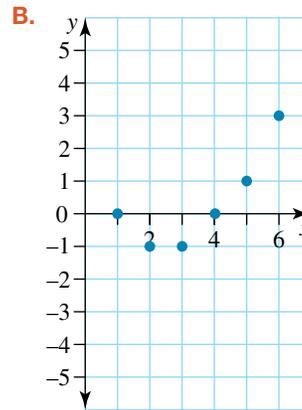
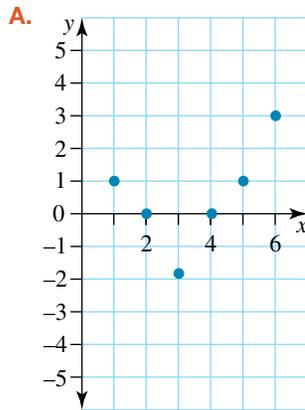


11. **MC** A study of a group of people found that there was a strong positive association ($r = 0.91$) between their blood pressure reading and the number of times they visited the podiatrist. A strong positive association was also found both between the blood pressure reading ($r = 0.95$), the number of times they visited the podiatrist ($r = 0.82$) and their age. Using this information, determine which of the following statements is true.
- High blood pressure is more common as one ages and problems with feet occur more often with ageing. The association between the blood pressure reading and the number of podiatry visits can be explained by the common response variable, the age of the person.
 - It is just a coincidence that there is a strong positive association between the blood pressure reading and the number of podiatrist visits.
 - No-one likes people touching their feet, so their blood pressure goes up when they visit the podiatrist.
 - Whenever a person has a blood pressure reading, they have to go to the podiatrist.

12. **MC** A least-squares regression line is fitted to the 6 points shown in the figure.



Determine which of the following looks most similar to the plot of residuals.



Complex familiar

13. Consider the following data set.

x	1	2	3	4	5	6	7	8	9	10
y	55	40	42	38	35	43	51	40	47	60

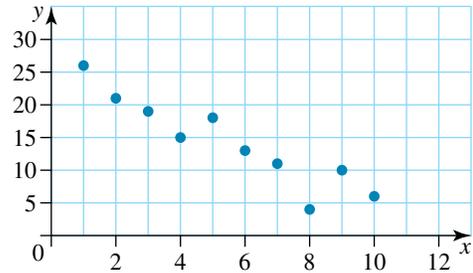
Determine the appropriateness of fitting a linear model to this data. Justify your answer with mathematical reasoning.

14. A scientist is studying the growth of a rare plant species under controlled conditions. She measures the plant's height (h in cm) at different weeks (n) and determines the least-squares regression line as $h = 3n + 5$. Determine the residual for the data point $(7, 30)$.

15. A scatterplot is drawn for a set of data.
The following summary statistics have been obtained for this data.

$$\bar{x} = 5.5, s_x = 3.03, \bar{y} = 14.3, s_y = 6.86, r = -0.93$$

Predict the value of x when y is 17, and comment on the accuracy of this prediction.



16. A linear regression models developed for a start-up business predict salary (y) based on years of experience (x). The equation is $y = 6000x + 62\,000$. Determine what the slope and y -intercept represent, and evaluate whether the company should place any limitations on this model.

Complex unfamiliar

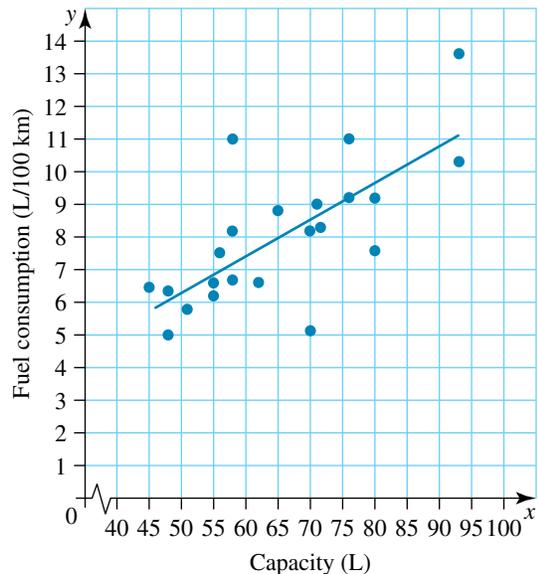
17. During an interview investigating the link between the sales of healthy snack foods (functional foods) and the increasing consumer demand for these products, an advertising expert made the following comment: 'There is an association but it's not causation ... our increasing need for healthy food and our laziness has resulted in mass innovation of functional foods.'

Explain why he might have stated there is no causative link between the sales of healthy foods and laziness.

18. An independent agency test-drove a random sample of current model vehicles and measured their fuel tank capacity against the average fuel consumption. Along with the following scatterplot, a regression equation of $y = 0.1119x + 0.6968$ with an r value of 0.516, was established.

Calculate, correct to the nearest whole number, the tank size of a vehicle that had a fuel consumption rate of 10.2 L per 100 km.

Determine the reasonableness of your answer and list at least two other factors that could influence the data.



19. A regression model predicts monthly sales (y) based on the number of salespeople (x).
The mean number of salespeople is 50.
The mean sales is \$100 000.

$$r = 0.85$$

The y -intercept for the regression model is (99, 900).

Determine the ratio of the standard deviations for number of salespeople to number of monthly sales.

20. The Bureau of Meteorology records data such as maximum temperatures and solar exposure on a daily and monthly basis. The following data table, for the Botanical Gardens in Melbourne, shows the monthly average amount of solar energy that falls on a horizontal surface and the monthly average maximum temperature. (Note: The data values have been rounded to the nearest whole number.)

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Average solar exposure (MJ)	25	21	17	11	8	6	7	10	13	18	21	24
Average max daily temp. (°C)	43	41	34	33	24	19	24	24	28	32	25	40

Conduct a linear regression analysis of this data, including a residual analysis and determine if both the predicted amount of solar exposure for a monthly average maximum temperature of 37°C and the predicted average maximum temperature for an average solar exposure of 3 MJ would be reasonable predictions.

2.6 Past QCAA exam questions

Question 1 (1 mark)

Source: 2023 QCAA General Mathematics, Paper 1, Section 1, Q7; © State of Queensland (QCAA) 2024

Which statement is always true for a causal relationship between an explanatory variable and a response variable?

- A. One of the variables is a confounding variable.
- B. The relationship is explained by a third variable.
- C. There is a positive association between the variables.
- D. The response variable is dependent on the explanatory variable.

Question 2 (1 mark)

Source: 2023 QCAA General Mathematics, Paper 1, Section 1, Q14; © State of Queensland (QCAA) 2024

A calculator is used to determine the equation of the least-squares line for the plant growth data in the table.

Number of days, d	6	15	20	24	35
Height of plant, h	12	14	16	18	30

What is the correct equation?

- A. $d = 0.6h + 5.7$
- B. $h = 0.6d + 5.7$
- C. $d = 5.7h + 0.6$
- D. $h = 5.7d + 0.6$

Question 3 (4 marks)

Source: 2023 QCAA General Mathematics, Paper 1, Section 2, Q22; © State of Queensland (QCAA) 2024

A person used a fitness tracker to monitor their hours of sleep and the distance they travelled each day.

Sleep (hours)	7	7.5	8	6.5	9.5	6	8.5
Distance (km)	5	7	5	7	4	8	6

- a. Using distance as the response variable, display the data in a scatterplot with labelled axes. **[3 marks]**
- b. Identify the direction of the association between hours of sleep and distance travelled. **[1 mark]**

Question 4 (5 marks)

Source: 2023 QCAA General Mathematics, Paper 2, Section 1, Q4; © State of Queensland (QCAA) 2024

Hiroki believes that more fish are caught on warmer days. Jiro believes that the number of fish caught in a day is more dependent on the number of people fishing.

Bivariate datasets for six days are shown.

Temperature, t ($^{\circ}\text{C}$)	32	26	20	27	23	29
Number of fish caught, f	530	400	320	220	180	120

Number of people fishing, p	46	58	38	34	30	28
Number of fish caught, f	530	400	320	220	180	120

Calculate the correlation coefficient for each dataset and use the results to identify the explanatory variable for the stronger linear association. Use the least-squares line equation for the stronger linear association to predict the number of fish caught on a 25°C day when 50 people are fishing.

Question 5 (7 marks)

Source: 2023 QCAA General Mathematics, Paper 2, Section 1, Q6; © State of Queensland (QCAA) 2024

The table shows the average superannuation account balance for workers of various ages in two different industries. The coefficient of determination, R^2 , for age versus account balance is 0.95 for industry A and 0.96 for industry B. 40-year-old Leigh works in the industry for which age explains a higher percentage of the account balance variation. Tony is 10 years older than Leigh and works in the other industry.

Age (years)	Account balance (\$)	
	Industry A	Industry B
22	7500	8100
32	42 000	60 000
42	98 000	120 000
52	160 000	210 000
62	290 000	360 000
72	400 000	480 000

Use linear models to predict the difference in current superannuation account balances for Leigh and Tony.

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Answers

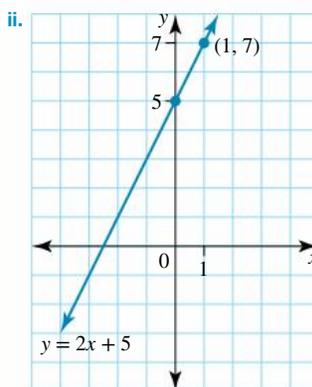
Chapter 2 Fitting a linear model to numerical data and association and causation

2.2 Review of the general equation of a straight line

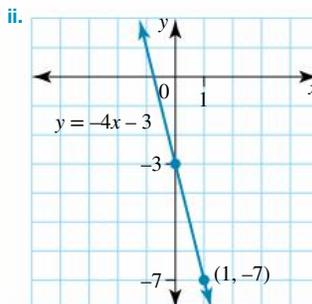
2.2 Exercise

1. a. $y = mx + c$
b. m is the slope of the graph and c is the y -intercept.
2. a. $m = -5, c = 4$ b. $m = 3, c = 11$
c. $m = -6, c = 0$ d. $m = \frac{5}{2}, c = 5$
e. $m = \frac{2}{3}, c = -6$ f. $m = 2, c = -1$
3. Complete the sentences below.
 - a. The equation $y = 3x - 1$ has a slope of 3 units. This means that for every increase of 1 unit in the horizontal direction there is an increase of 3 units in the vertical direction.
 - b. The equation $y = -x + 1$ has a slope of -1 unit. This means that for every increase of 1 unit in the horizontal direction there is an increase of -1 unit in the vertical direction.
 - c. The equation $y = \frac{1}{3}x - 1$ has a slope of $\frac{1}{3}$ units. This means that for every increase of 3 units in the horizontal direction there is an increase of 1 unit in the vertical direction.
4. a. 2 b. 1 c. 1
d. $-\frac{1}{2}$ e. $-\frac{1}{4}$ f. $\frac{4}{3}$
g. $-\frac{4}{3}$ h. $-\frac{2}{7}$ i. 0
5. a. $y = 2x - 2$ b. $y = -2x - 1$ c. $y = -\frac{3}{8}x + 2$
d. $y = 4$ e. $y = x$ f. $y = -0.25x - 4$
6. a. $y = \frac{7x}{5} - \frac{1}{5}$ or $5y = 7x - 1$
b. $y = \frac{5x}{9} + \frac{34}{9}$ or $9y = 5x + 34$
c. $y = x + 6$
d. $y = 3$
e. $y = \frac{4x}{9} + \frac{4}{9}$ or $9y = 4x + 4$
f. $y = \frac{9x}{5} + \frac{7}{5}$ or $5y = 9x + 7$
7. a. i. $(0, -3)$ ii. $m = 2$ iii. $y = 2x - 3$
b. i. $(0, 0)$ ii. $m = \frac{1}{4}$ iii. $y = \frac{1}{4}x$
c. i. $(0, 6)$ ii. $m = -3$ iii. $y = -3x + 6$
d. i. $(0, -3)$ ii. $m = \frac{5}{2}$ iii. $y = \frac{5}{2}x - 3$

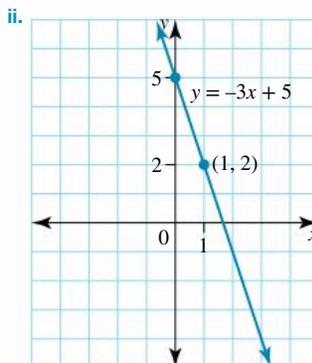
8. a. i. $m = 2, (0, 5)$



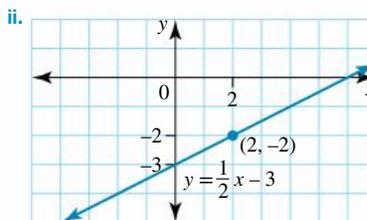
b. i. $m = -4, (0, -3)$



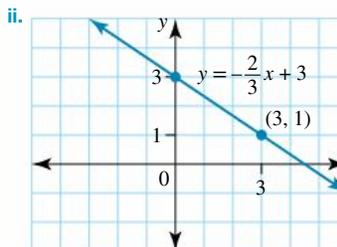
c. i. $m = -3, (0, 5)$



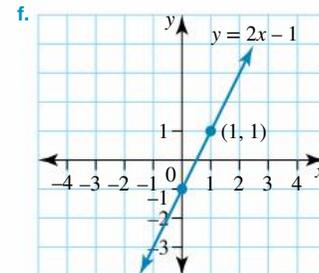
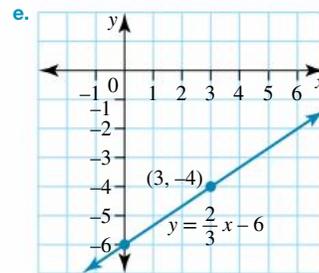
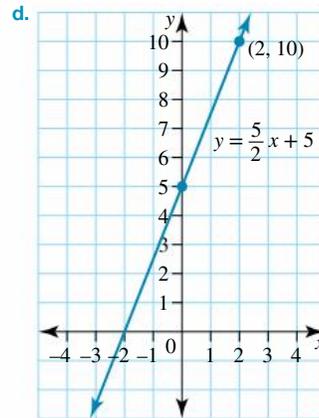
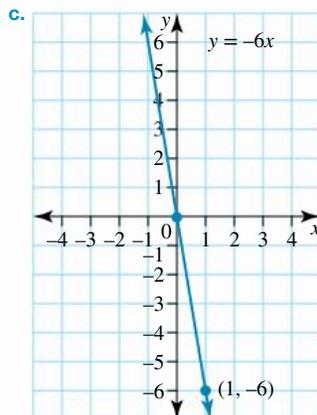
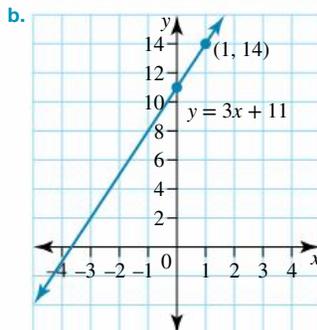
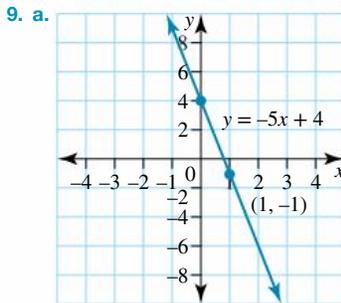
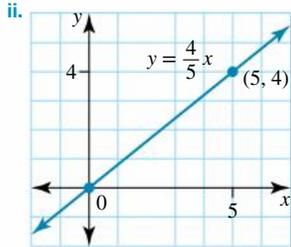
d. i. $m = \frac{1}{2}, (0, -3)$



e. i. $m = -\frac{2}{3}, (0, 3)$

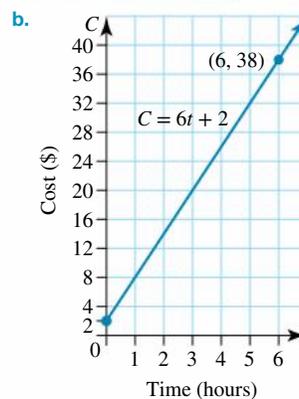


f. i. $m = \frac{4}{5}, (0, 0)$



10. a.

t	1	2	3
C	8	14	20



c. $m = 6$

d. $C = 6t + 2$

e. \$32

f. Answers will vary.

11. 1

12. $y = \frac{2}{3}x + \frac{1}{3}$

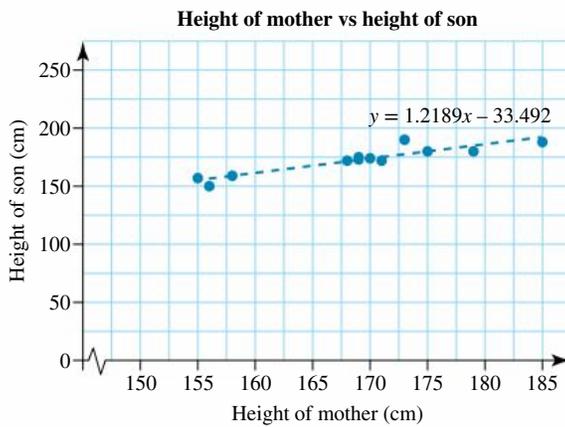
The gradient is $\frac{2}{3}$ and the y -intercept is $\frac{1}{3}$

13. 70
14. a. No saving, lose \$40.
b. Save \$120
c. Save \$600
15. No, Julia is not correct. Julia will have spent \$147.50 after 6 weeks and Tessa will have spent \$125.

2.3 Fitting a least-squares line to data

2.3 Exercise

1. a. The Mathematics exam mark
b. 0.95
c. -6.30
d. 74%
2. a. $y = 91.78 + 3.33x$
b. $y = -0.15x + 21.87$
c. $y = 52.38x + 8890.48$
d. $y = 30 - x$
3. a.



- b. $S = -33.49 + 1.219M$
- c. The regression line suggests that as the height of the mother increases, so does the height of the son.
4. a. -1.837 b. 1.701 c. Positive direction
5. a. 105.9 b. -1.476 c. Negative direction
6. a. a b. 16 c. Interpolation
d. 37 e. Extrapolation
7. a. Age in months b. 2.029 mL c. 7.5 months old
8. a. Cost per night b. \$155.74 c. 3.31 km
9. As the b value (slope) is negative, the direction is negative. The y -intercept is 3.2: therefore, when $x = 0$, $y = 3.2$. The trend line is negative.
10. a. 60 b. -5
c. Negative d. $y = -140$
11. a. 3.90
b. Lucy incorrectly transposed the 12.9. She should have moved this first before dividing by 7.32.
12. a. -12 b. 25
c. Positive d. 75.5

13. a. No. of mosquitoes = $10.2 + 0.5 \times \text{temp. of fire}$
 $= 10.2 + 0.5 \times 240$
 $= 130.2$

The number of mosquitoes is 130 (correct to the nearest whole number)

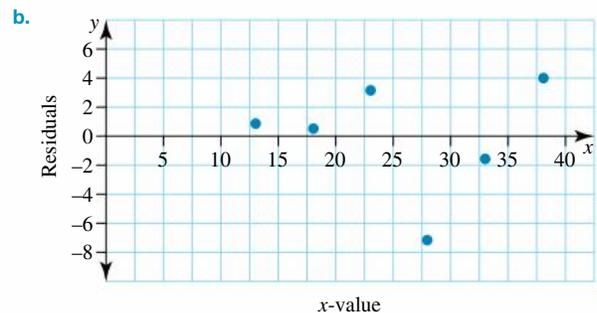
- b. $t = 3.6^\circ\text{C}$
- c. Mosquitoes are in hibernation in the cooler months of the year, so once the temperature drops below a certain level this model would not be appropriate. Also, the location of the fire, air temperature, wind conditions, proximity to water, etc. would impact on the mosquito population.
14. 66
15. $y = 4 + 2.5x$
16. 18.04, 18.18
17. \$3160
18. 69
 $r = -0.2172$. The value of r is low, showing that the line of best fit is poor and not a reliable predictor
19. 24.46 km
20. The slope of -13.613 says that for every increase of one year in age, the number of friends on social media decreases by 13.613. The intercept is when the age of a person is 0, the number of friends is 777.84. The x -intercept is 57.13, which indicates 0 friends after the age of 57. As young children cannot be on social media, the ages could start at 16. The upper limit for this model could be 50.

2.4 Residual plots

2.4 Exercise

1. a.

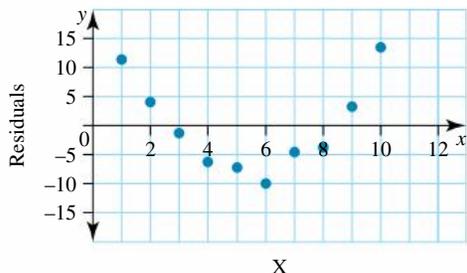
x	13	18	23	28	33	38
Actual y-value	25	31	40	36	48	60
Predicted y-value	24.16	30.51	36.86	43.21	49.56	55.91
Residual value	0.84	0.49	3.14	-7.21	-1.56	4.09



As the residuals appear to be randomly spread above and below the x -axis, this linear model is an appropriate model for this set of data.

2. a. See the table at the bottom of the page*

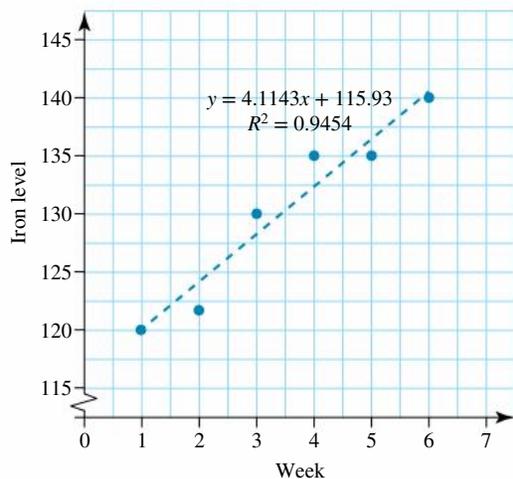
b.



The residuals form a pattern around the horizontal axis, therefore the linear model is not a good model for this set of data.

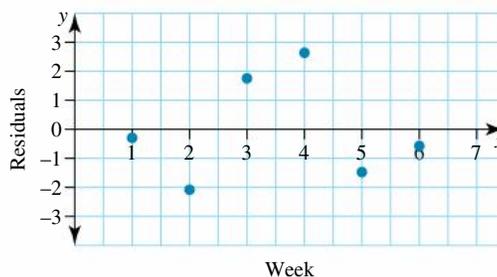
3. A
4. D
5. C
6. 998.8 units
7. The point is on the regression line.
8. a. \$5000 b. \$30 000 c. \$73 000
d. \$2500
9. A
10. The residual plots for both factories appear to be randomly distributed about the horizontal axes, so the linear model is suitable for predictions.
11. The dots are scattered randomly around the horizontal axis, so the linear model is suitable for this data. Extrapolating outside the range of data given gives a predicted systolic blood pressure for someone 75 aged, which could not be considered reliable.

12.



$$\text{Iron level} = 4.11 \times \text{week} + 115.93$$

Week of experiment	1	2	3	4	5	6
Iron level	120	122	130	135	135	140
Predicted iron level	120.04	124.15	128.26	132.37	136.48	140.59
Residual value	-0.04	-2.15	1.74	2.63	-1.48	-0.59



The residuals appear to be scattered randomly about the horizontal axis, so this suggests that the linear model is a suitable model for this data.

$$r^2 = 0.9454, \quad r = 0.9723$$

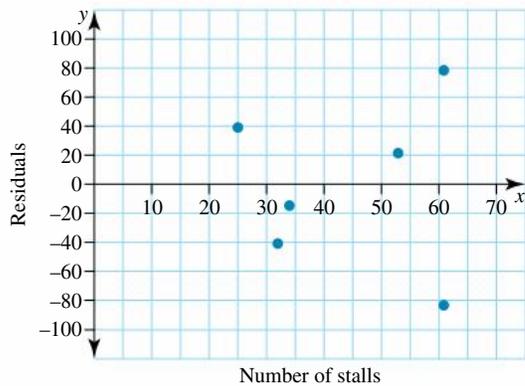
The iron level reaches 155 g/L by week 9.5, so according to this model the iron level will reach 155 g/L in week 10. A study of the initial scatterplot of the data suggests that the data shows a linear trend. The residual plot shows no pattern about the horizontal axis and Pearson's product-moment correlation coefficient of 0.9723 indicates a strong positive linear correlation, so the model is suitable to make predictions. However, the iron level of 155 g/L goes outside the range of the given data, so this needs to be considered when making the prediction.

13.

Number of stalls	53	34	61	32	61	25
Number of visitors	501	339	611	300	450	333
Predicted number of visitors	480	354	533	341	533	294
Residual value	21	-15	78	-41	-83	39

*2.

x	1	2	3	4	5	6	7	8	9	10
y	1	5	10	16	26	34	50	62	80	101
Predicted	-10.454	0.425	11.304	22.183	33.062	43.941	54.82	65.699	76.578	87.457
Residuals	11.454	4.575	-1.304	-6.183	-7.062	-9.941	-4.82	-3.699	3.422	13.543

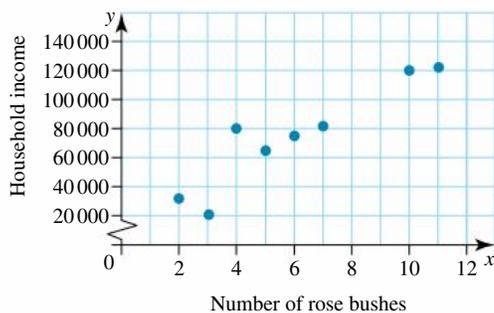


The residuals are scattered randomly above and below the horizontal axis, so this suggests that the linear model is suitable.

2.5 Association and causation

2.5 Exercise

1. a.



- b. From the scatterplot there appears to be a strong positive association between the number of rose bushes in a home garden and the annual household income.
- c. $r = 0.93$
- d. Just because $r = 0.93$, it cannot be stated that increasing the number of rose bushes in a home garden will increase the annual household income. Other factors may need to be considered. Further investigation is needed before any conclusions can be drawn.
2. There is a negative association of -0.86 between the amount of time spent playing computer games and the level of aerobic fitness.
3. D
4. D
5. C
6. B
7. D
8. B
9. C
10. A
11. No. More fire fighters are called to larger fires, so the damage is due to the size of the fire.

12. No. Larger shoe sizes are associated with age and so is greater academic knowledge. Age is the common cause.
13. Answers will vary. Sample responses can be found in the worked solutions in the online resources.
14. Answers will vary. Sample responses can be found in the worked solutions in the online resources.

2.6 Review

2.6 Exercise

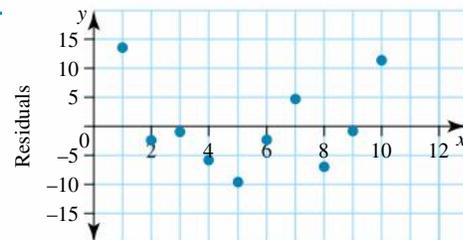
1. B
2. A
3. C
4. B
5. C
6. D
7. B
8. $y = 0.435x + 11.498$
9. a. The slope of 2.5 means that for each additional advertisement run, the company's sales revenue increases by \$2500 (since the unit is thousands of dollars).
- b. The intercept of 50 represents the predicted sales revenue when zero advertisements are run. This means that even if the company does not advertise at all, their baseline sales revenue is \$50 000.

10. $\text{Mass} = 0.75 \times \text{height} - 0.58$

11. A

12. D

13.



The residuals do not appear to be randomly scattered about the horizontal axis. The first point on the left is at about 13, then the points fluctuate above and below the axis, finishing at about 11 on the right. Given the value of r is 0.30, indicating a weak association, it is not appropriate to assume that the relationship between these variables is linear.

14. 4

15. $x = 4.22$

As the data is interpolated data, it is fairly accurate.

16. The slope represents an increase of \$6000 per year. The y-intercept represents the starting salary of \$62 000. The company may want to limit this, particularly for staff with extensive experience, to save the company money.

17. Although there appears to be a link between the laziness of people and the increase in sales of healthy foods, there are also many other possible factors besides laziness; for example, people are very time poor, unsure of what constitutes healthy food, and are lacking confidence and the skills to cook for themselves. Based on this observation alone, the cause of an increase in sales of healthy foods cannot be concluded to be due to laziness.

18. Outside the range of data given.

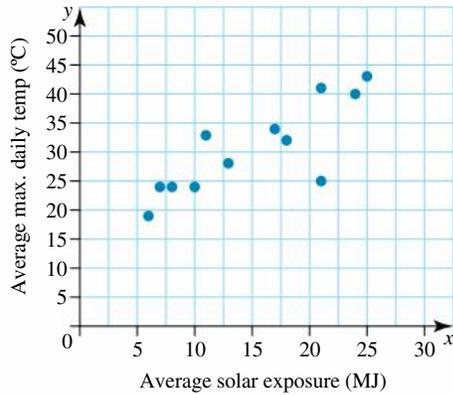
85 L

This value indicates a moderate relationship between the variables. Therefore, the data can be used, but other factors should also be considered.

Fuel consumption can be influenced by the size of the engine, the size of the vehicle, the type of driving (city, country) and the age of the vehicle.

19. 2.35

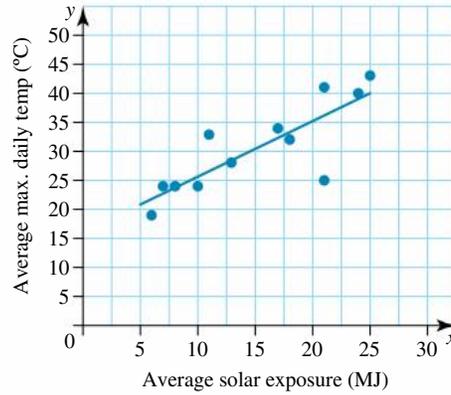
20.



Strong positive correlation

$$r = 0.8242; r^2 = 0.6793$$

These values indicate a strong relationship between the two variables. The coefficient of determination suggests that nearly 70% of the maximum daily temperature is due to the amount of solar exposure.



$$\text{Maximum daily temperature} = 16.232 + 0.9515 \times \text{average solar exposure}$$

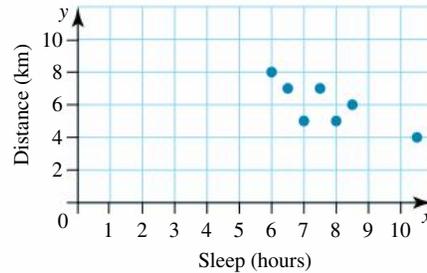
As the data demonstrates a strong positive correlation, these predictions are reasonable, however as the 3 MJ is outside the data set, caution should be used in regard to this prediction.

2.6 Past QCAA exam questions

1. D

2. B

3. a.



b. The association is negative.

4. It is predicted that 420 fish will be caught.

5. The difference in account balances for Leigh and Tony is predicted to be \$50 620.

3 Time series analysis

LESSON SEQUENCE

3.1 Overview	94
3.2 Constructing and describing time series plots	95
3.3 Fitting a least-squares line to time series data	105
3.4 Smoothing time series data using simple moving averages	112
3.5 Deseasonalising time series data	125
3.6 Review	140
Answers	150

Fully worked solutions for this chapter are available online.

EXAM PREPARATION

Access exam-style questions in every lesson, available online and access past QCAA exam questions in every review.

Resources

-  **Solutions** Solutions — Chapter 3
-  **Exam questions** Exam question booklet — Chapter 3
-  **Digital documents** Learning matrix — Chapter 3
Chapter summary — Chapter 3

LESSON

3.1 Overview

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3.1.1 Introduction

Modelling using time series data can be used for forecasting and monitoring predicted sales figures, population growth, mortality rates, long-term health outcomes, weather patterns, economic growth and future consumer trends. This information helps in decision making, resource allocation, strategy adjustments, and policy implementation to address challenges and opportunities effectively. Organisations can better prepare for future scenarios, optimise operations, and improve overall performance and sustainability.



Researchers study time series data to identify trends and patterns. For example, NASA scientists have found that the Earth's average surface temperature has increased by 1.1 °C since the late nineteenth century, with 16 of the 17 warmest years on record occurring since 2001.

Digital footprints from website interactions can be analysed using time series techniques to reveal trends in political ideology, friendships, consumer habits and interests. These insights can impact employment, property purchases, relationships and travel. Understanding time series analysis helps individuals make informed decisions about their digital interactions and data privacy.

3.1.2 Syllabus links

Lesson	Lesson title	Syllabus links
3.2	Constructing and describing time series plots	<ul style="list-style-type: none">○ Construct and use time series plots.○ Describe time series plots by identifying features, including trend (long-term direction, e.g. increasing/decreasing), seasonality (systematic, calendar-related movements) and irregular fluctuations (unsystematic, short-term fluctuations).
3.3	Fitting a least-squares line to time series data	<ul style="list-style-type: none">○ Fit a least-squares line to model long-term trends in time series data.○ Solve practical problems that involve the analysis of time series data.
3.4	Smoothing time series data using simple moving averages	<ul style="list-style-type: none">○ Smooth time series data by calculating a simple moving average using the mean or median for an odd number of data, including the use of spreadsheets.○ Solve practical problems that involve the analysis of time series data.
3.5	Deseasonalising time series data	<ul style="list-style-type: none">○ Deseasonalise a time series by calculating the seasonal indices using the average percentage method, including the use of spreadsheets.○ Solve practical problems that involve the analysis of time series data.

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LESSON

3.2 Constructing and describing time series plots

SYLLABUS LINKS

- Construct and use time series plots.
- Describe time series plots by identifying features, including trend (long-term direction, e.g. increasing/decreasing), seasonality (systematic, calendar-related movements) and irregular fluctuations (unsystematic, short-term fluctuations).

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3.2.1 Time series plots

As the name suggests, a **time series plot** is a line graph that represents a time series, with time plotted on the horizontal axis and the variable being measured on the vertical axis. Time series data is typically collected at regular intervals over a specific period to make observations, judgements and predictions about long-term trends.

3.2.2 Features for describing a time series plot

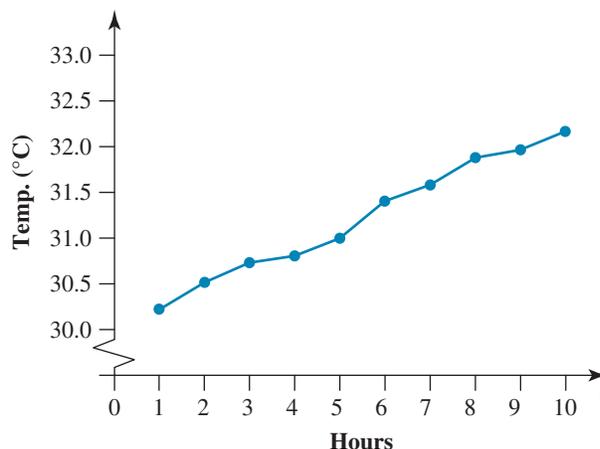
The behaviour of time series data can be described according to various patterns that can be observed. These qualities can help in analysing trends and making future predictions.

Trend

Trend in a time series is the general direction of the series over a long period of time. This could be increasing (upward/positive), decreasing (downward/negative) or no trend (flat).

An increasing trend indicates that the values are steadily rising over time, such as the growth in population, sales revenue or stock market performance. A decreasing trend suggests a steady decline, such as in the case of falling demand for a product, declining profits or a decreasing population. In some cases, the time series might exhibit no trend or a flat trend, meaning that the values remain relatively constant over time, which might be observed in mature industries or stable markets where little change is expected.

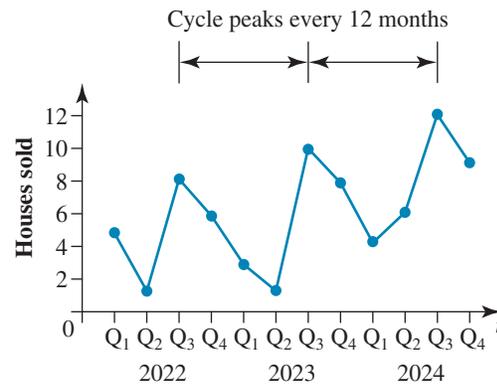
The following graph is an illustration of an overall upward trend indicating an increase in temperature over time.



Seasonality

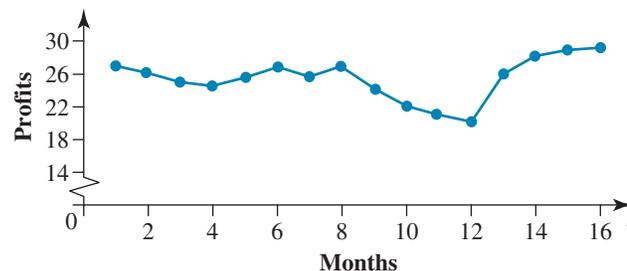
A **seasonal pattern** displays fluctuations that are systematic and repeat at the same time each week, month or quarter, usually lasting less than one year. These patterns are calendar-related and are often driven by external factors such as the weather, holidays or cultural events. For example, retail sales tend to peak during holiday seasons such as Christmas, while demand for air conditioning rises in summer months. Seasonal patterns are predictable and help businesses plan for fluctuations in demand, stock levels and staffing needs.

The following graph illustrates that the peak selling time for houses is spring (Q3). Historically, the real estate market tends to be stronger in spring and summer months, with slower activity in winter when fewer people are willing to move due to holidays and harsh weather. Other influential factors could also contribute to this seasonal pattern,



Irregular fluctuations

Irregular fluctuations are characterised by their unpredictability and lack of any discernible trend or pattern. Unlike seasonal fluctuations, which occur at regular intervals due to predictable factors, irregular fluctuations are typically the result of external shocks or unforeseen events. These events might include natural disasters like droughts, floods or earthquakes, as well as economic downturns, political instability or sudden technological disruptions, as illustrated in the following graph.

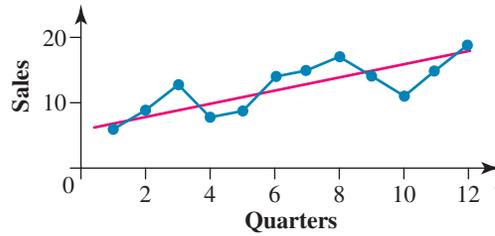


Such fluctuations are inherently random and cannot be forecasted with accuracy. As a result, they often create short-term volatility, disrupting markets, supply chains and production processes. Because they arise from unique, one-off occurrences, these fluctuations are difficult to model or predict, making it challenging for businesses, governments and individuals to prepare for or mitigate their effects.

Describing overall behaviour

Trends can indeed work in combination, often creating more complex patterns in data. For example, a seasonal pattern where certain fluctuations occur at predictable intervals, such as increased sales during holidays or higher demand for certain products in specific weather conditions, can coexist with an underlying upward trend, such as long-term economic growth, technological advancements or a company's expanding market share.

In this case, the data would show a regular seasonal variation (e.g. higher sales in December) while simultaneously reflecting an overall increase in sales or revenue over time due to the broader upward trend. An example of this is in the following graph.

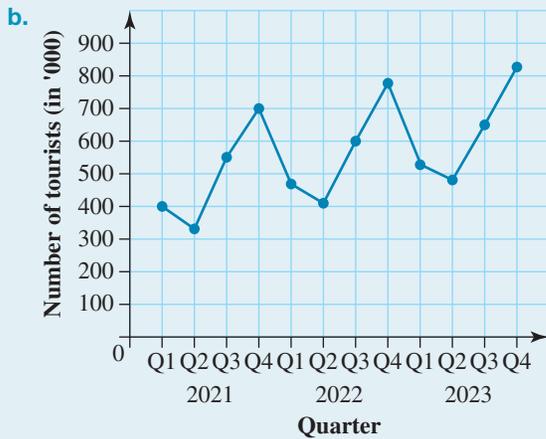
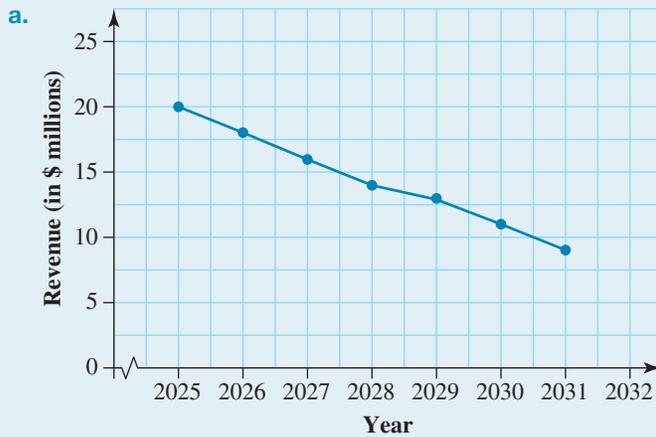


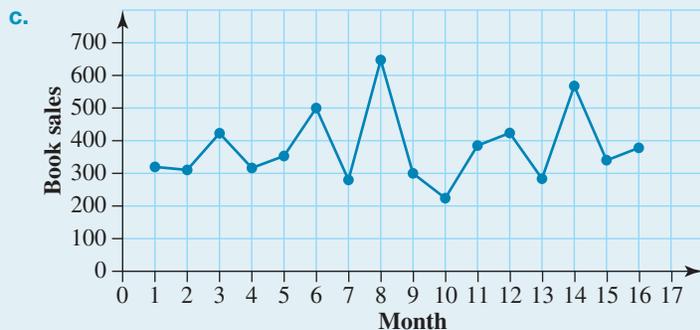
These combined patterns offer nuanced insights and refined forecasting. Understanding both the cyclical seasonal pattern and the consistent upward trend helps analysts predict future behavior, plan for demand fluctuations, and make strategic decisions. Such combinations are common in consumer behavior, agricultural production, and financial markets, where long-term growth is influenced by recurring seasonal cycles.

When describing a time series plot, identifying patterns is often straightforward, but considering contextual information can also help describe overall behavior. Note: Outliers may occur as one-off unanticipated events.

WORKED EXAMPLE 1 Describing a time series plot

Identify and describe the features of each time series plot.





THINK

- Observe and describe the time series plot by identifying its features. Consider contextual information as required.
- Observe and describe the time series plot by identifying its features. Consider contextual information as required.
- Observe and describe the time series plot by identifying its features. Consider contextual information as required.

WRITE

- This data indicates a decreasing trend as shown by the overall long-term decrease in measured values over time and downward trajectory of the graph as the revenue becomes less as each year progresses.
- This time series plot indicates a seasonal and long-term increasing pattern as shown by the periodic troughs at Q2 and periodic peaks at Q4 of each year, and the overall long-term increase in measured values over time as the number of tourists becomes more as each year progresses.
- The time series plot indicates irregular fluctuations as shown by unexplained spikes and drops in book sales, as shown by the fluctuations between months 6 and 9 or between months 13 and 15. There is no consistent pattern or explained cycle to the changes, indicating random and unpredictable variations in the data.

3.2.3 Constructing a time series plot

Time series data can be graphed using manual or technological methods. Both methods require a general understanding of the basic requirements for these graphical displays.

Conventional graphing techniques should be applied when constructing a time series plot. For accurate and appropriate graphical displays of time series data ensure that:

- axis labels are clearly presented with time represented along the x -axis (explanatory variable) and the variable being measured represented along the y -axis (response variable).
- appropriate interval measurements (equally spaced) are chosen for the data with units of measurement clearly indicated.
- the y -axis is scaled appropriately to the data range to avoid distorting the perception of the data by unnecessary stretching or compressing.
- all data points are plotted as indicated by a dot and a line segment is drawn between each data point as time progresses.

Once the data is graphed this can be used to analyse any behavioural patterns or lack thereof which can be described by the features as discussed earlier. Identifying these features can assist in making appropriate judgements about the data and possible predictions about the future.

WORKED EXAMPLE 2 Constructing a time series plot

Construct a time series plot of the following data that show the sales revenues (\$) for the first 8 years of a business selling homewares online.

Identify and describe any features of this time series plot.

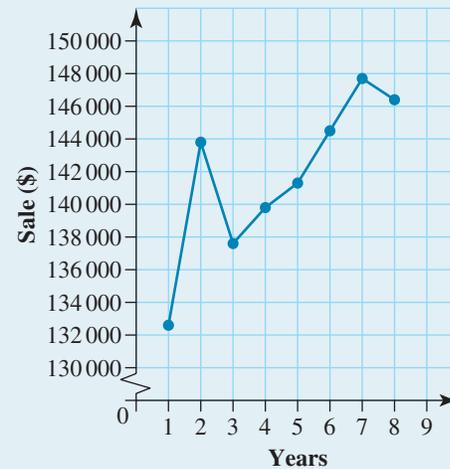
Year	1	2	3	4	5	6	7	8
Sales(\$)	132 600	143 800	137 600	139 800	141 300	144 500	147 700	146 400

THINK

1. Identify the explanatory and the response variables.
As this is times series data, the explanatory variable is the year.
2. Draw the scatterplot.
 - Scale the x -axis from 1 to 8 to represent each year. Label the x -axis as appropriate.
 - Insert an axis break in the y -axis since sales values start significantly above 0.
 - Scale the y -axis to accommodate the range of values, ensuring equal intervals between each. In this example begin with 130 000 and end with 150 000 in order to account for all the sales amounts in the data.
 - Plot each individual data point.
 - Rule line segments between data points.
3. Examine the graph to identify any trends present.

WRITE

Explanatory variable — Year
Response variable — Sales (\$)



The second data point is possibly an outlier, but overall the data appears to be following an increasing trend.

3.2 Exercise

3.2 Exam questions on

Simple familiar

1, 2, 3, 4, 5, 6, 7, 8,
9, 10, 11, 12

Complex familiar

13, 14

Complex unfamiliar

—

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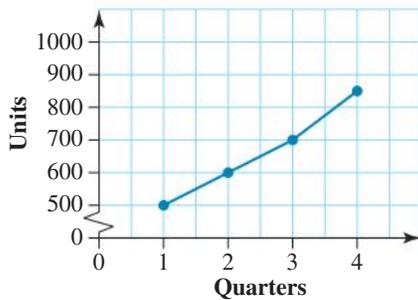


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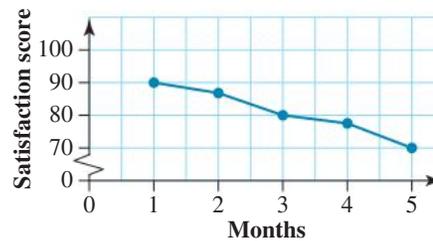
Simple familiar

- Describe the different time series plot features in your own words: increasing, decreasing and no trends, seasonal patterns and irregular fluctuations.
- WE1** Identify and describe the features of each time series plot.

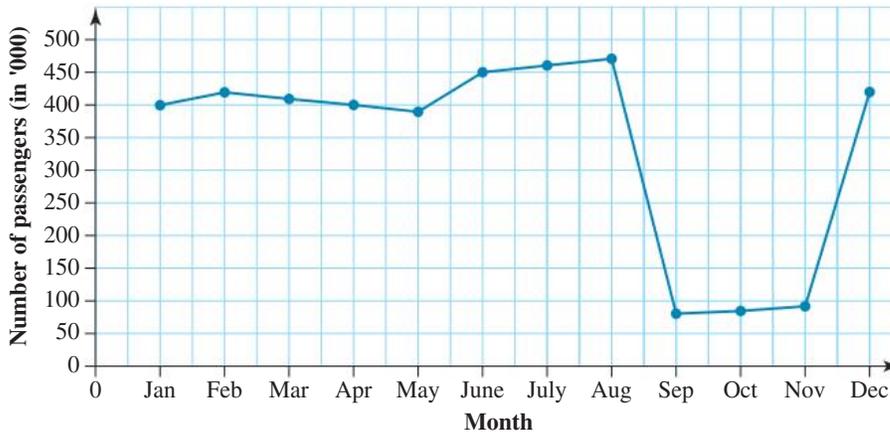
a.



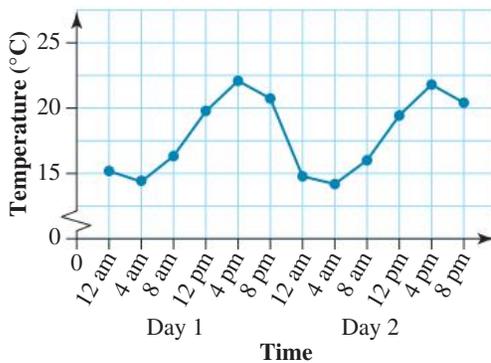
b.



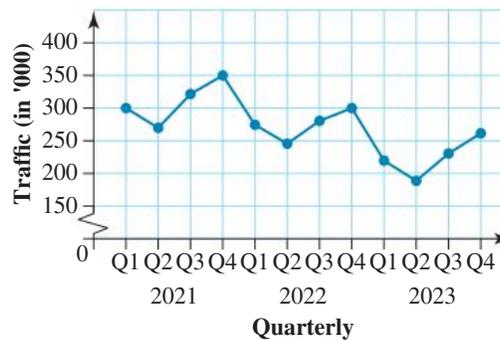
c.



d.



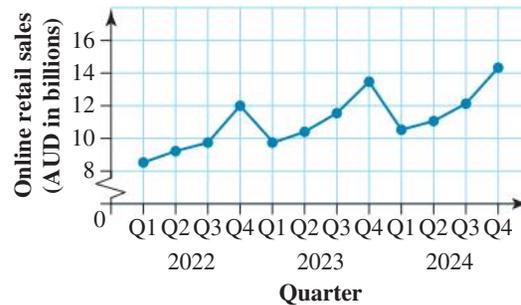
e.



3. **WE2** Construct a time series plot of the following data that shows the share price of a new technology company taken at the close of business on a Friday for eight weeks. Identify and describe any features of this time series plot.

Week	1	2	3	4	5	6	7	8
Price	5.50	10.80	16.20	21.40	26.20	30.80	34.40	38.00

4. Describe the long-term trend and seasonality of the time series data for the graph below.



5. Describe the patterns of the following time series situations.

- The amount of rainfall, per month, in north Queensland
- The number of soldiers in the Australian army, measured annually
- The number of people living in Australia, measured annually
- The share price of BHP, measured monthly
- Average UV index data, measured monthly

6. Construct a time series plot of the following data that show the number of people who watch a news broadcast on a given day of the week over a two week period. Identify and describe any features of this time series plot.

Day	M	T	W	Th	F	S	Su	M	T	W	Th	F	S	Su
Number of views (000 000s)	1.20	1.18	1.16	1.18	0.90	0.75	1.00	1.21	1.23	1.19	1.16	0.95	0.68	0.98

7. The following time series graph shows the standardised death rates for alcohol induced deaths over a 10-year period. A standardised death rate is a statistical measure of the death rate of a population. Describe the long-term trend of this data for male and female alcohol induced deaths.



8. The following table shows the number of hot chocolate drinks sold at a café each month.

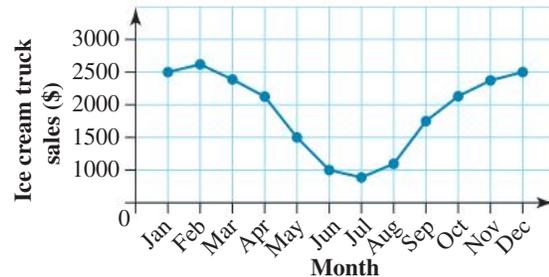
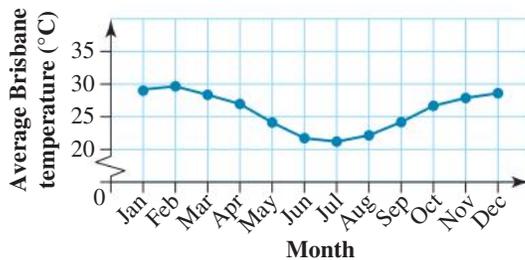
Month	Jan	Feb	Mar	Apr	May	June	July	Aug	Sep	Oct	Nov	Dec
Number of drinks sold	100	92	122	372	469	680	675	702	643	589	437	234

- a. Construct a time series graph for this data.
 b. Identify and describe any features of this time series plot.
9. The following table shows the number of swimsuits sold by a shop in four and a half years.

Season	Summer 14	Autumn 15	Winter 15	Spring 15	Summer 15	Autumn 16	Winter 16	Spring 16	Summer 16
Number of swimsuits	312	182	126	289	308	232	162	296	345

Season	Autumn 17	Winter 17	Spring 17	Summer 17	Autumn 18	Winter 18	Spring 18	Summer 18	Autumn 19
Number of swimsuits	233	180	266	455	321	233	388	516	409

- a. Construct a time series graph for this data.
 b. Identify and describe any features of this time series plot.
10. Examine and describe the seasonality in the time series graphs below, which depict the average monthly temperature and the monthly sales revenue of an ice cream truck in Brisbane. Evaluate the reasonableness of the following statement:
 ‘The seasonal pattern for both graphs indicates a correlation between temperature and ice cream sales.’



11. The following table includes weekly revenue data of Crystal’s local restaurant, which experienced a surge in popularity because of a customer’s viral social media post.

Weeks	1	2	3	4	5	6	7	8	9	10	11	12
Restaurant revenue (\$)	6 800	7 000	7 100	6 900	6 800	7 000	7 200	6 950	9 200	8 000	8 200	7 900

- a. Construct a time series plot for this data.
 b. Identify and describe any features of this time series plot.



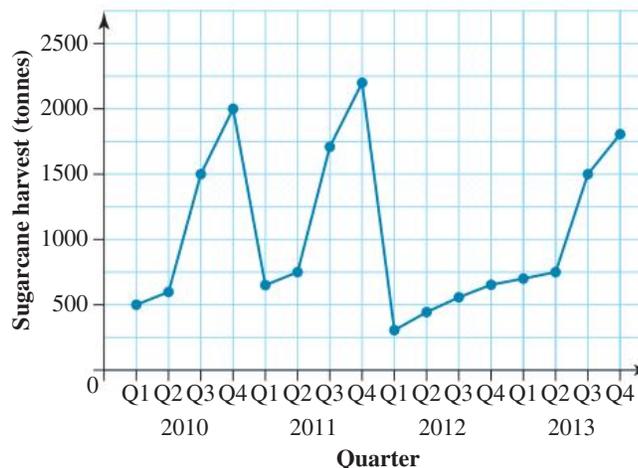
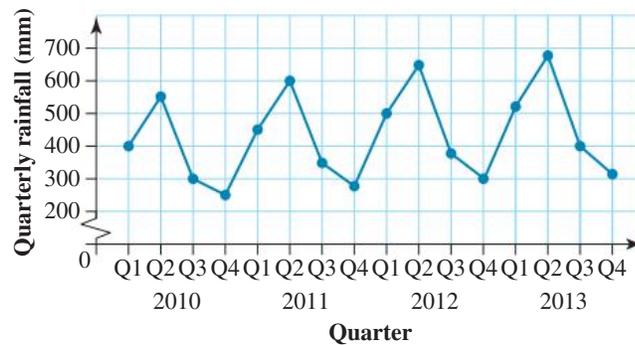
12. The following data shows the quarterly attendance figures (in hundreds) for a public park over the two-year period from 2023 to 2024.

Year	Quarter	Attendance
2023	Q1 (Jan–Mar)	145
2023	Q2 (Apr–Jun)	125
2023	Q3 (Jul–Sep)	105
2023	Q4 (Oct–Dec)	75
2024	Q1 (Jan–Mar)	140
2024	Q2 (Apr–Jun)	120
2024	Q3 (Jul–Sep)	100
2024	Q4 (Oct–Dec)	70

- Construct a time series plot for this data.
- Identify and describe any features of this time series plot.

Complex familiar

13. A township in a region of northeast Brazil is renowned for its sugarcane production. The local sugarcane producers claim that from 2010 to 2013, increasing rainfall has directly contributed to the growth of sugarcane crops. Analyse the following time series graphs, which display quarterly rainfall totals and the corresponding sugarcane harvest (in tonnes) for the region during this period.

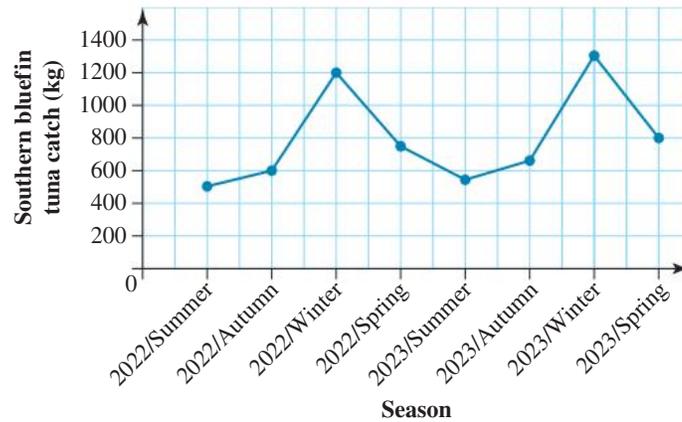
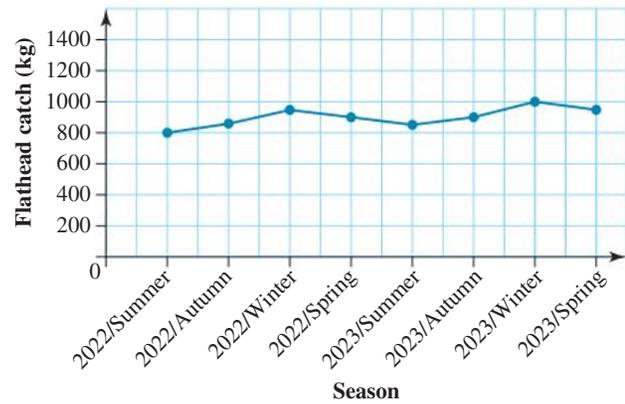
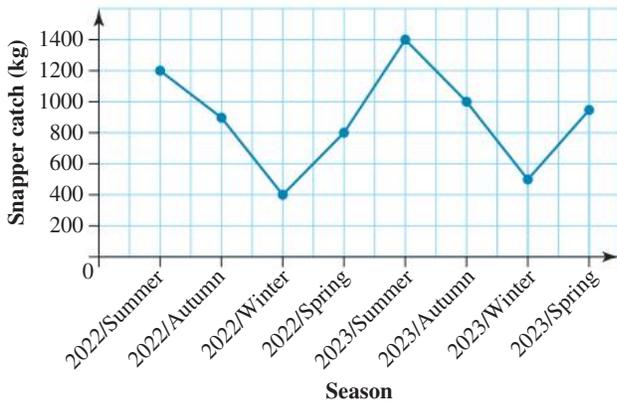


Describe the key features of both graphs and evaluate the reasonableness of the producers' claim regarding the relationship between rainfall and sugarcane growth.

14. A coastal town in southern Australia is known for its thriving fishing industry. A local fisherman, Tean Teong, claims that the seasonal water temperature fluctuations directly influence the abundance of certain fish species. The following data shows the average seasonal water temperatures and the catch data (in kg) of three different fish species (Snapper, Flathead and Southern Bluefin Tuna) over the years 2022 and 2023.

Evaluate the reasonableness of Tean Teong’s statement: “The catch of Snapper increases as water temperatures rise, while Southern Bluefin Tuna shows a preference for cooler waters, and Flathead is caught more evenly across the year, reflecting its tolerance for moderate temperatures.”

Season	2022 Summer	2022 Autumn	2022 Winter	2022 Spring	2023 Summer	2023 Autumn	2023 Winter	2023 Spring
Water temperature (°C)	23	21.5	19.25	20.75	23.5	22	19.25	20.75



Describe the key features of the time series temperature data for the coastal town and catch data for each fish. Evaluate the reasonableness of Tean Teong’s claim about the relationship between water temperature and the catch data.

Fully worked solutions for this chapter are available online.

LESSON

3.3 Fitting a least-squares line to time series data

SYLLABUS LINKS

- Fit a least-squares line to model long-term trends in time series data.
- Solve practical problems that involve the analysis of time series data.

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3.3.1 Modelling long-term trends with least-squares regression

In bivariate data analysis, the **least-squares regression line** models linear relationships between two variables. It also models **long-term trend** in time series data, showing whether data generally increases, decreases, or remains constant. The long-term trend represents the data's general direction over time, excluding short-term fluctuations. Fitting a least-squares line provides an accurate linear approximation of the data's behavior.

For example, in monthly sales data, a least-squares regression line helps determine long-term sales trends. This line can be calculated using software or a scientific calculator, making it easy to apply to large datasets. It can also forecast future values through extrapolation, though this should be done cautiously.

The slope of the regression line indicates the trend's rate and direction. A positive slope shows an increasing trend, while a negative slope shows a decreasing trend. The closer the data points are to the line, the stronger the relationship between the variables over time.

WORKED EXAMPLE 3 Using time series data to determine a least-squares regression line

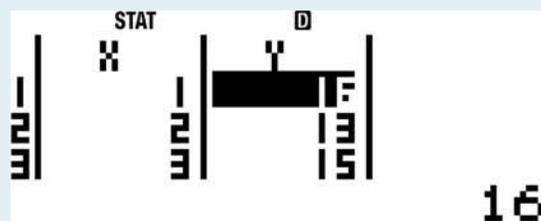
Determine the least-squares regression line for the following time series data. Describe the long-term trend of this data.

t	1	2	3	4	5	6	7	8	9	10
y	16	13	15	11	9	10	7	5	6	3

THINK

1. Depending on your scientific calculator brand, place the above data into a 2-variable table.

WRITE



2. Enter the values for t into the first column.
3. Enter the values for y into the second column.
4. Activate the Stat function and determine the equation for the least-squares regression line.

$$y = -1.38x + 17.1$$

This data indicates a decreasing trend as shown by the negative gradient in the least-squares regression line for the time series data. This will result in an overall downward trajectory long-term in measured values over time.

5. Describe the trend of the time series data.

Least-squares regression lines can also be created using other technology such as spreadsheets.

WORKED EXAMPLE 4 Using technology to determine and present a least-squares regression line

Data was recorded about the number of families who have moved from Brisbane to Maroochydore as represented by the time series data below.



- Construct a graph of the time series data using spreadsheet technology.
- Identify and describe any features of this time series plot.
- Determine the least-squares regression line and present its equation on the time series plot.
- Predict how many people will move in 2030. Evaluate the reasonableness of your solution.

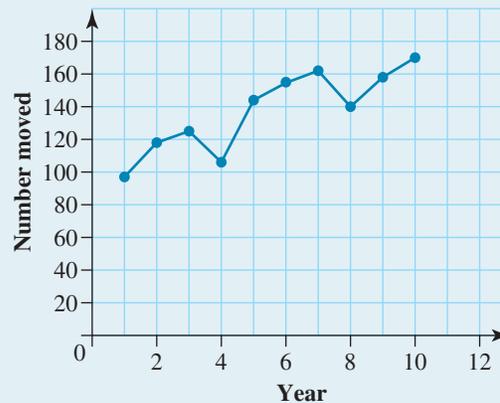
Year	2009	2010	2011	2012	2013	2014	2015	2016	2017	2018
Number moved	97	118	125	106	144	155	162	140	158	170

THINK

- Using a spreadsheet, enter the data then draw a scatterplot.
Note: The values for the years have been entered as 1, 2, 3 etc. We do this because it is easier to work with smaller numbers and convert back to the year when required.

WRITE

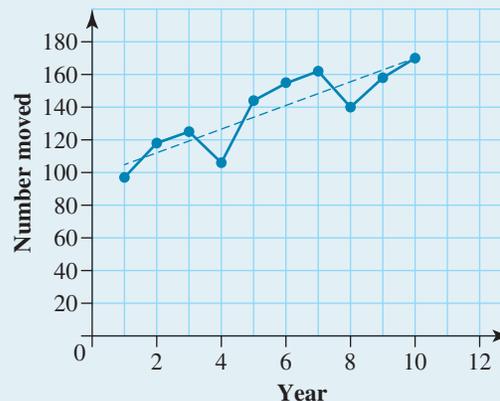
a.



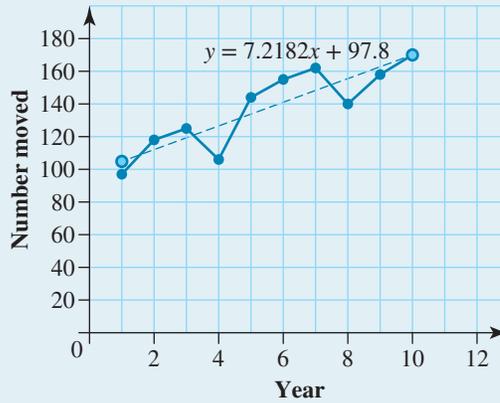
- Observe and describe the time series plot by identifying its features. Consider contextual information as required.

- This data indicates an increasing trend as shown by the overall long-term increase in measured values over time and upward trajectory of the graph as the number of people moving becomes more as each year progresses. There seems to be troughs in the data; however, there is no contextual information known that could lead to a seasonal pattern occurring.

1. To add the least-squares regression line, select the symbol for chart elements and select the trendline option.



2. To include the least-squares regression equation, click the cursor on the least-squares line then tick the box next to 'Display Equation on chart'.



3. Write the answer.

Let x be the number of years since 2008.

Let y be the number of people that have moved from Brisbane to Maroochydore.

$$y = 7.2182x + 97.8$$

- d. 1. Write the equation, substituting $x = 22$.

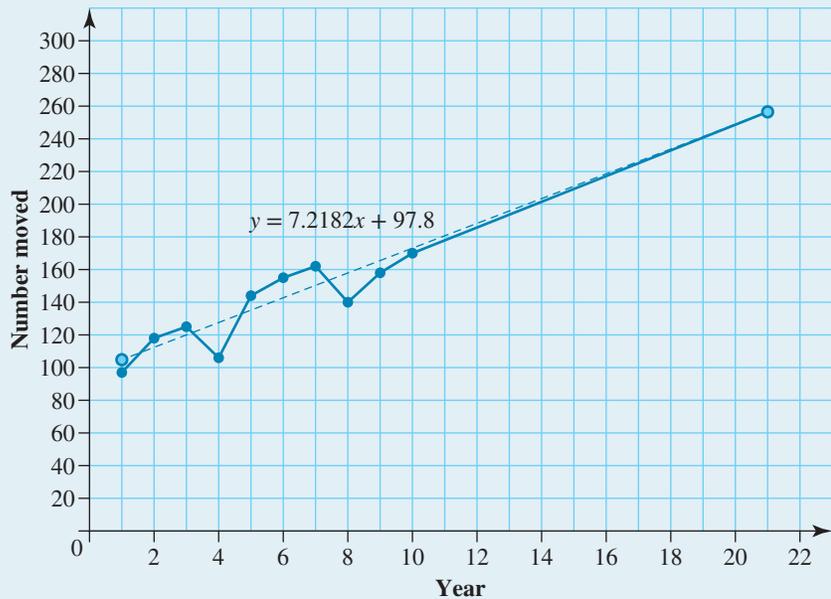
$$\begin{aligned} \text{d. Let } x &= 22, \\ y &= 7.2182x + 97.8 \\ &= 7.2182(22) + 97.8 \\ &= 256.6 \end{aligned}$$

2. Write the answer.

It is predicted that 256 people will move from Brisbane to Maroochydore in 2030.

3. Evaluate the reasonableness of the solution.

The answer is reasonable as it adheres to the overall linear model as calculated with the least-squares regression line and is also confirmed by the extrapolated graph below. The linear model does, however, assume a continuation of the increasing trend and does not allow for any fluctuations such as economic impacts or changes to house prices that could influence people moving.



3.3 Exercise

3.3 Exam questions **on**

Simple familiar

1, 2, 3, 4, 5, 6, 7, 8, 9, 10

Complex familiar

11, 12

Complex unfamiliar

13, 14

These questions are even better in jacPLUS!

- Receive immediate feedback
- Access sample responses
- Track results and progress



Find all this and MORE in jacPLUS

Simple familiar

1. Explain the relationship between a least-squares regression lines fitted to time series data and the underlying long-term trend of that data.
2. The following least-squares regression lines have been fitted to time series data. Describe their overall long-term trends.

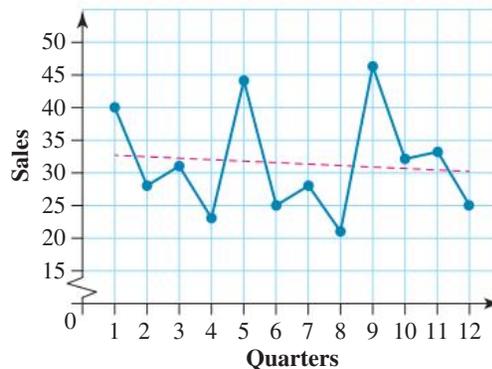
a. $y = 2x + 3x$	b. $y = -3x + 7$	c. $y = 0.75x - 2$
d. $y = -0.33x + 6.2$	e. $y = -1.2x - 5.76$	f. $y = 3.7$
3. **WE3** Use the data in the table to answer the following questions.

t	1	2	3	4	5	6	7	8	9	10	11
y	6	9	13	8	9	14	15	17	14	11	15

- a. Determine the least-squares regression line for the following time series data.
 - b. Describe the long-term trend of this data.
4. The number of employees at the Comnatpac Bank was recorded over a 10-month period.

Month	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Number of employees	6100	5700	5400	5200	4800	4400	4200	4000	3700	3300

- a. Determine the least-squares regression line for the following time series data.
 - b. Describe its long-term trend.
5. The time series plot shown represents the sales figures of a market stall over three quarters. A least-squares regression line is shown.



Describe the overall long-term trend of the data.

6. **WE4** The following data represents the body temperature of a patient with appendicitis, taken every hour.

Hour	1	2	3	4	5	6	7	8	9
Temp (°C)	37.2	37.5	37.8	37.9	38.0	38.5	38.6	39.0	39.2

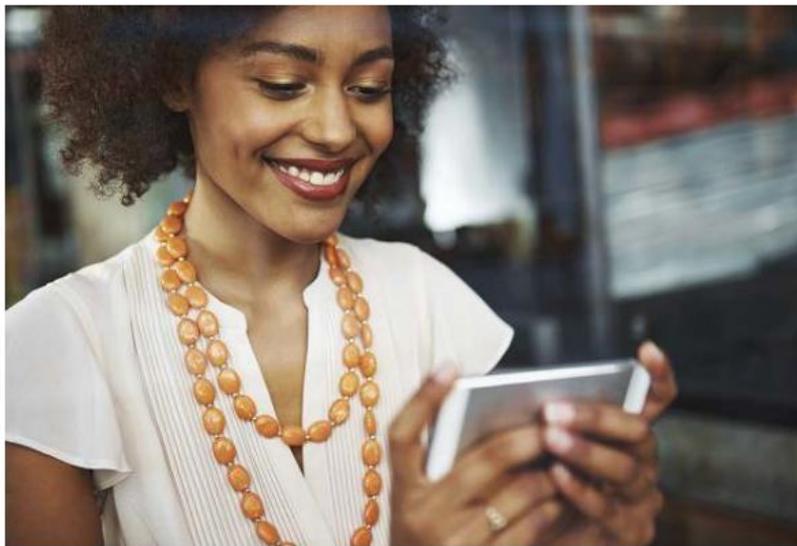


- Construct a graph of the time series data using spreadsheet technology.
- Identify and describe any features of this time series plot.
- Determine the least-squares regression line and present its equation on the time series plot.
- Predict the temperature at 12 hours. Evaluate the reasonableness of your solution.

7. The following table represents the quarterly sales figures (in thousands) of a popular software product.

Quarter	Q1-2022	Q2-2022	Q3-2022	Q4-2022	Q1-2023	Q2-2023	Q3-2023	Q4-2023	Q1-2024	Q2-2024	Q3-2024	Q4-2024
Sales	120	135	150	145	140	120	100	110	120	140	190	220

- Construct a graph of the time series data using spreadsheet technology.
- Identify and describe any features of this time series plot.
- Determine the least-squares regression line and present its equation on the time series plot.
- Determine the projected sales in quarter 1 of 2022. Evaluate the reasonableness of your solution.



8. The monthly share prices of a recently privatised telephone company were recorded as follows.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug
Price (\$)	2.50	2.70	3.00	3.20	3.60	3.70	3.90	4.20

- Determine the least-squares regression line.
- Predict the share price in June for the following year.

9. The following table represents the monthly sales figures of umbrellas.



Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Sales	100	90	60	35	20	10	5	10	15	40	70	95

- Determine the least-squares regression line.
- Determine the projected sales for January for the following year.

10. The following table shows weekly attendance at a community fitness centre over 10 weeks.

Week	1	2	3	4	5	6	7	8	9	10
Attendance	350	345	325	320	310	500	275	285	260	255

- Determine the least-squares regression line.
- Describe the overall long-term trend of the data.
- Determine the projected attendance for Week 16.

Complex familiar

11. Richard Real Estate claims that residential land doubles in value every 10 years. Below are the historical land valuations of a property in Coopersmith from 2015 to 2018:

- 2015: \$330 000
- 2016: \$375 000
- 2017: \$400 000
- 2018: \$430 000

Evaluate the reasonableness of this claim. Provide appropriate mathematical justification.



12. Milly is a competitive swimmer aiming to improve her 50-m freestyle times (as below).

Year	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct
Time (s)	44.5	44	43.6	43.2	42.8	42.4	55	41.7	41.4	41

After a knee injury in July set her back, she committed to a rigorous rehab program to resume training with successful outcomes. Milly wants to predict what her swimming time will be at the end of the year. She claims that accounting for her injury means that her end-of-year time will be 3 seconds lower than if she had not been injured. Evaluate the reasonableness of this claim.

Complex unfamiliar

13. Music streaming services Dexter Dynamics and Bettinson Beats hold a significant share of the market in the entertainment industry, as shown in the following tables, which show their historical average stock prices per year.

Year	2016	2017	2018	2019	2020	2021	2022	2023	2024	2025
DD share price (\$)	100	125.5	110	95	130	105.5	150	180	140	110

Year	2016	2017	2018	2019	2020	2021	2022	2023	2024	2025
BB share price (\$)	60	64	62	70	75	85	80	95	110	105

Analysts predict that, in the long term, Dexter Dynamics' stocks will outperform Bettinson Beats due to its higher historical stock value. Evaluate the reasonableness of this claim and determine the earliest year, if ever, one service will surpass the other in stock price.

14. Mark and Micah operate competing e-scooter businesses.



Below is the historical monthly sales data for each business in 2024.

Month	July	August	September	October	November	December
Mark units sold	323	328	375	383	421	431
Month	July	August	September	October	November	December
Micah units sold	180	230	290	340	420	500

Determine the first projected future month that one business will surpass the other by selling more than twice the number of units. Evaluate the reasonableness of your solution.

Fully worked solutions for this chapter are available online.

LESSON

3.4 Smoothing time series data using simple moving averages

SYLLABUS LINKS

- Smooth time series data by calculating a simple moving average using the mean or median for an odd number of data, including the use of spreadsheets.
- Solve practical problems that involve the analysis of time series data.

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3.4.1 Smoothing time series data

The previous lesson covered using a least-squares regression line to quantify long-term trends in time series data. This method fits a linear regression to raw data, capturing trends but also incorporating noise and fluctuations. High variability can distort the trend, so assessing a model's data spread and context helps us to evaluate the model's accuracy.

This lesson introduces a mathematical technique called **data smoothing**, which helps reduce noise and reveal underlying long-term trends more clearly. By filtering out short-term fluctuations — whether seasonal or irregular — data smoothing improves the interpretability of a time series, allowing for a clearer view of the data's long-term trend or behaviour. Unlike fitting a least-squares regression line directly to raw data, which can be influenced by short-term variability and outliers, data smoothing removes these fluctuations, allowing for a more stable and accurate representation of the overall trend.

3.4.2 Moving averages (means and medians)

There are several mathematical methods for smoothing time series, some of which are referred to as 'moving averages'. These include moving means and moving medians. Although a median is not technically an average, in this lesson 'moving average' is used to include both moving means and moving medians. (Individual questions will specify whether the mean or median is required.)

A moving mean smooths the data by averaging a set number of data points, known as the 'window size', around each point in the series. This produces a smoother series by reducing short-term fluctuations.

In a **simple moving mean**, the average (mean) of the values within the window is calculated, whereas in a **simple moving median**, the median value of the window is used instead. The median resists outliers, making it more robust. Both methods highlight trends by filtering noise.

3.4.3 Three-point and five-point moving averages

The three-point moving average smooths data by averaging each data point with its immediate neighbours: one before and one after. The new averaged value then takes the position of the middle point in the original data set, effectively replacing it and reducing the impact of short-term fluctuations.

The five-point moving average is similar but uses a window of five consecutive data points: the data value itself, two before and two after. It smooths the series further than the three-point moving average because it averages over a larger window of data.

Both 3- and 5-point moving averages can be calculated using either the mean or median, as seen in the worked examples below.

WORKED EXAMPLE 5 Calculating a simple moving mean

The following set of data gives the number of cups of coffee sold by a coffee cart over a 15-day period.

Day	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Number of sales	430	550	490	710	730	790	880	800	950	820	990	880	1000	990	1050

- Calculate the 3- and 5-point moving mean for this data.
- Graph the original data, the 3-point moving mean and the 5-point moving mean on the same set of axes.
- Comment on the smoothing effect.

THINK

- a. 1. Create a table with headings as shown. a.

To calculate the 3-point moving average, the average of the sales figures for the first three days is calculated and the value placed next to the second day. This process is then continued.

Note: All values are rounded to the nearest whole number.

$$\frac{430 + 550 + 490}{3} = 490$$

$$\frac{550 + 490 + 710}{3} = 583$$

$$\frac{490 + 710 + 730}{3} = 643$$

$$\frac{710 + 730 + 790}{3} = 743$$

$$\frac{730 + 790 + 880}{3} = 800$$

$$\frac{790 + 880 + 800}{3} = 823$$

$$\frac{880 + 800 + 950}{3} = 877$$

$$\frac{800 + 950 + 820}{3} = 857$$

$$\frac{950 + 820 + 990}{3} = 920$$

$$\frac{820 + 990 + 880}{3} = 897$$

$$\frac{990 + 880 + 1000}{3} = 957$$

$$\frac{880 + 1000 + 990}{3} = 957$$

$$\frac{1000 + 990 + 1050}{3} = 1013$$

WRITE

Day	Number of sales	3-point moving average	5-point moving average
1	430		
2	550	490	
3	490	583	
4	710	643	
5	730	743	
6	790	800	
7	880	823	
8	800	877	
9	950	857	
10	820	920	
11	990	897	
12	880	957	
13	1000	957	
14	990	1013	
15	1050		

2. To calculate the 5-point moving average, the average of the sales figures for the first 5 days is calculated and the value placed next to the third day. This process is then continued.

Note: All values are rounded to the nearest whole number.

$$\frac{430 + 550 + 490 + 710 + 730}{5} = 582$$

$$\frac{550 + 490 + 710 + 730 + 790}{5} = 654$$

$$\frac{490 + 710 + 730 + 790 + 880}{5} = 720$$

$$\frac{710 + 730 + 790 + 880 + 800}{5} = 782$$

$$\frac{730 + 790 + 880 + 800 + 950}{5} = 830$$

$$\frac{790 + 880 + 800 + 950 + 820}{5} = 848$$

$$\frac{880 + 800 + 950 + 820 + 990}{5} = 888$$

$$\frac{800 + 950 + 820 + 990 + 880}{5} = 888$$

$$\frac{950 + 820 + 990 + 880 + 1000}{5} = 928$$

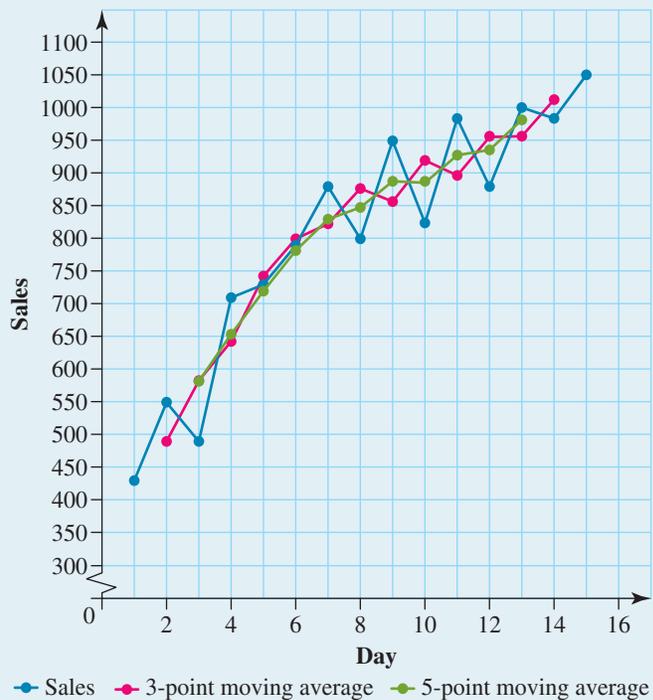
$$\frac{820 + 990 + 880 + 1000 + 990}{5} = 936$$

$$\frac{990 + 880 + 1000 + 990 + 1050}{5} = 982$$

- b. 1. Graph the original data, the 3-point moving average data and the 5-point moving average data on the same set of axes.

Day	Sales	3-point moving average	5-point moving average
1	430		
2	550	490	
3	490	583	582
4	710	643	654
5	730	743	720
6	790	800	782
7	880	823	830
8	800	877	848
9	950	857	888
10	820	920	888
11	990	897	928
12	880	957	936
13	1000	957	982
14	990	1013	
15	1050		

b.



c. Comment on the smoothing effect.

The 3- and 5-point moving average smooths out the data and hence the upward trend in sales can be seen more clearly. The 3-point moving average is more sensitive to the short-term fluctuations in the data as it includes less values in its average calculations which produce a plot that more closely follows the original data and its volatility. The 5-point moving average less sensitive to short-term fluctuations in the data as it includes more values in its average calculations which produce a plot that more effectively filters out noise making it better for identifying the long-term trend and overall pattern.

WORKED EXAMPLE 6 Calculating a simple moving median

Use the same set of data as in Worked example 5.

Day	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Number of sales	430	550	490	710	730	790	880	800	950	820	990	880	1000	990	1050

- Calculate the 3- and 5-point moving medians for this data.
- Graph the original data, the 3-point moving median and the 5-point moving median on the same set of axes.
- Comment on the smoothing effect.

THINK

- Create a table with headings as shown. To calculate the 3-point moving median, the median of the sales figures for the first three days is calculated and the value placed next to the second day. This process is then continued.
 430, 550, 490: Median = 490
 550, 490, 710: Median = 550
 490, 710, 730: Median = 710
 710, 730, 790: Median = 730
 730, 790, 880: Median = 790
 790, 880, 800: Median = 800
 880, 800, 950: Median = 880
 800, 950, 820: Median = 820
 950, 820, 990: Median = 950
 820, 990, 880: Median = 880
 990, 880, 1000: Median = 990
 880, 1000, 990: Median = 990
 1000, 990, 1050: Median = 1000

WRITE

- | Day | Number of sales | 3-point moving median | 5-point moving median |
|-----|-----------------|-----------------------|-----------------------|
| 1 | 430 | | |
| 2 | 550 | 490 | |
| 3 | 490 | 550 | |
| 4 | 710 | 710 | |
| 5 | 730 | 730 | |
| 6 | 790 | 790 | |
| 7 | 880 | 800 | |
| 8 | 800 | 880 | |
| 9 | 950 | 820 | |
| 10 | 820 | 950 | |
| 11 | 990 | 880 | |
| 12 | 880 | 990 | |
| 13 | 1000 | 990 | |
| 14 | 990 | 1000 | |
| 15 | 1050 | | |

2. Create a table with headings as shown.

To calculate the 5-point moving median, the median of the sales figures for the first five days is calculated and the value placed next to the third day.

This process is then continued.

430, 550, 490, 710, 730: Median = 550

550, 490, 710, 730, 790: Median = 710

490, 710, 730, 790, 880: Median = 730

710, 730, 790, 880, 800: Median = 790

730, 790, 880, 800, 950: Median = 800

790, 880, 800, 950, 820: Median = 820

880, 800, 950, 820, 990: Median = 880

800, 950, 820, 990, 880: Median = 880

950, 820, 990, 880, 1000: Median = 950

950, 820, 990, 880, 1000, 990:

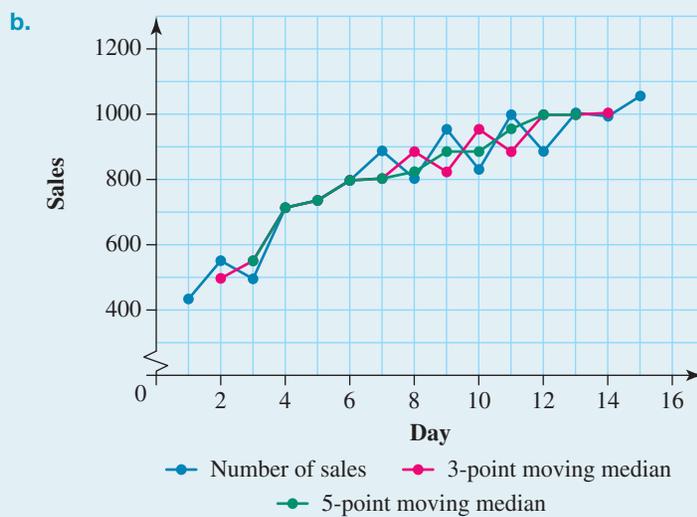
Median = 990

820, 990, 880, 1000, 990, 1050:

Median = 990

Day	Number of sales	3-point moving median	5-point moving median
1	430		
2	550	490	
3	490	550	550
4	710	710	710
5	730	730	730
6	790	790	790
7	880	800	800
8	800	880	820
9	950	820	880
10	820	950	880
11	990	880	950
12	880	990	990
13	1000	990	990
14	990	1000	
15	1050		

b. 1. Graph the original data, the 3-point moving average data and the 5-point moving average data on the same set of axes.



c. Comment on the smoothing effect.

The 3- and 5-point moving median smooths out the data and hence the upward trend in sales can be seen more clearly. Both are useful for filtering out sudden fluctuations in the data that may be deemed as anomaly situations or outliers. The 3-point moving median is a more sensitive filter as it uses less data points and preserves more of the data's original shape and its volatility. The 5-point moving average a stronger smoothing filter which is less sensitive to short-term fluctuations and sudden changes in the data. This means it does a better job filtering out noisy data which can be seen from day 7 to 12 identifying the long-term trend and overall pattern more effectively.

Other odd-numbered moving averages, such as seven or nine-point averages, can be used for even greater robustness against outliers, providing a more stable representation of the long-term trend with a slight sacrifice in data responsiveness.

3.4.4 Using a spreadsheet to smooth time series data

Moving averages can also be calculated using a spreadsheet to smooth time series data, helping to reduce fluctuations and reveal underlying trends. By organising the data in rows and columns, users can easily apply formulas and create charts to visualise the effects of smoothing, making it easier to analyse and interpret the data.

WORKED EXAMPLE 7 Calculating a simple moving average using a spreadsheet

Consider the data from previous worked example.

Day	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Sales	430	550	490	710	730	790	880	800	950	820	990	880	1000	990	1050

- Use a spreadsheet to calculate the 3- and 5-point moving means for this data.
- Determine and graph the least-squares regression line to the smoothed 3-point moving mean data and then determine the sales figure on day 20. Evaluate the reasonableness of your solution.

THINK

- Create a spreadsheet with the headings as shown and enter the Day and Sales data in the first and second columns. Place the cursor in cell C3 and enter the formula $= \text{AVERAGE}(B2:B4)$. Then press Enter.

Note: If using a spreadsheet to calculate a simple moving median use the formula $= \text{MEDIAN}(B2:B4)$. Then press Enter.

WRITE

a.

C3				
fx = AVERAGE(B2:B4)				
	A	B	C	D
1	Day	Sales	3-point moving average	5-point moving average
2	1	430		
3	2	550	= AVERAGE(B2:B4)	
4	3	490		
5	4	710		
6	5	730		
7	6	790		
8	7	880		
9	8	800		
10	9	950		
11	10	820		
12	11	990		
13	12	880		
14	13	1000		
15	14	990		
16	15	1050		

2. Place the cursor on the small box on the bottom right corner of cell C3 and fill down the column.

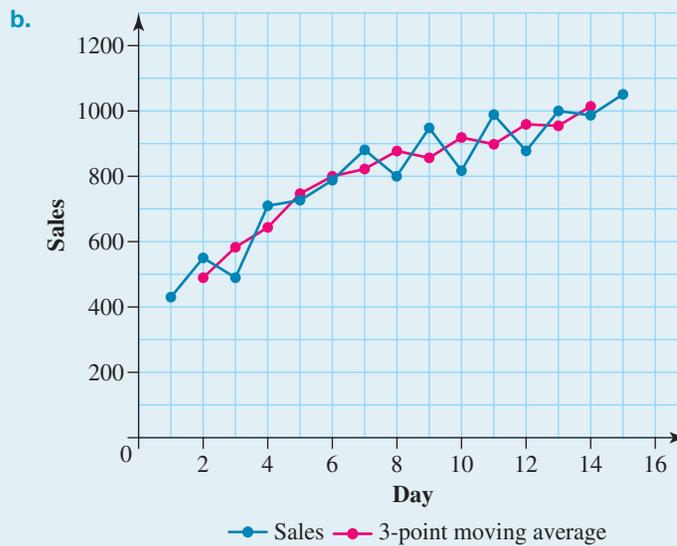
Note: To round the moving averages to whole numbers select: Format cells Number and change the number of decimal places to 0.

	A	B	C	D
1	Day	Sales	3-point moving average	5-point moving average
2	1	430		
3	2	550	490	
4	3	490	583	
5	4	710	643	
6	5	730	743	
7	6	790	800	
8	7	880	823	
9	8	800	877	
10	9	950	857	
11	10	820	920	
12	11	990	897	
13	12	880	957	
14	13	1000	957	
15	14	990	1013	
16	15	1050		

3. Repeat the process for calculating the 5-point moving average. The formula entered into cell D4 is:
= AVERAGE(B2:B6)

	A	B	C	D
1	Day	Sales	3-point moving average	5-point moving average
2	1	430		
3	2	550	490	
4	3	490	583	582
5	4	710	643	654
6	5	730	743	720
7	6	790	800	782
8	7	880	823	830
9	8	800	877	848
10	9	950	857	888
11	10	820	920	888
12	11	990	897	928
13	12	880	957	936
14	13	1000	957	982
15	14	990	1013	
16	15	1050		

- b. 1. Using the spreadsheet created in part a, graph the original data and the 3-point moving average data.



2. Add the trendline to the graph for the 3-point moving average data



3. Determine the sales figure on day 20.

$$\begin{aligned}
 \text{Let } x &= 20 \\
 y &= 38.425x + 504.91 \\
 &= 38.425(20) + 504.91 \\
 &= 1273.41
 \end{aligned}$$

4. Evaluate the reasonableness of the solution.

1273 cups of coffee will be sold on day 20 according to the linear model from the 3-point moving averages data. The answer is reasonable as it adheres to the overall linear model as calculated with the least-squares regression line. The linear model does, however, assume a continuation of the increasing trend and does not allow for any fluctuations such as economic impacts, seasonal impact or any limitations to the business that could prevent such high sales.

Exercise 3.4 Smoothing time series data using simple moving averages

3.4 Exercise

3.4 Exam questions **on**

Simple familiar

1, 2, 3, 4, 5, 6, 7, 8, 9, 10

Complex familiar

11, 12

Complex unfamiliar

13, 14

These questions are even better in jacPLUS!

- Receive immediate feedback
- Access sample responses
- Track results and progress



Find all this and MORE in jacPLUS

Simple familiar

1. Explain the purpose of smoothing time series data and the methods used to achieve this, specifically by calculating a simple moving average. Discuss how the mean and median methods can be applied for an odd number of data points.

The following relates to questions 2–5.

This set of data gives the number of houses sold by auction over a 12-month period for a real estate business.



Month	1	2	3	4	5	6	7	8	9	10	11	12
Number of houses sold	25	20	42	37	21	50	43	85	72	55	77	102

2. **WE5** Use the data to answer the following.
 - a. Calculate the 3- and 5-point moving means for this data. Round all values to the nearest whole number.
 - b. Graph the original data, the 3-point moving mean and the 5-point moving mean on the same set of axes. Comment on the smoothing effect.
3. **WE6** Answer the following.
 - a. Calculate the 3- and 5-point moving medians for this data.
 - b. Graph the original data, the 3-point moving median and the 5-point moving median on the same set of axes. Comment on the smoothing effect.
4. **WE7a** Answer the following.
 - a. Use a spreadsheet to calculate the 3- and 5-point moving means for this data.
 - b. Use a spreadsheet to calculate the 3- and 5-point moving median for this data.
5. Evaluate the effectiveness of using a moving average with a window size greater than 5 for smoothing this data.

6. Complete the following table.

Day	Sales (\$)	3-point moving mean	5-point moving mean
1	16		
2	24	19	
3	18		31
4	31	38	
5	66	63	
6	92		81
7	100		92
8	115	101	
9	88	92	
10	72		

7. Complete the following table.

Day	Sales (\$)	3-point moving median	5-point moving median
1	16		
2	24	18	
3	18	24	
4	31		31
5	66		
6	92	92	
7	100		92
8	115	100	
9	88		
10	72		

8. Historical unemployment rates for a certain country are shown below.

Year	1980	1981	1982	1983	1984	1985	1986	1987	1988
Rate	5.50%	5.70%	10.10%	10.00%	9.50%	8.10%	7.30%	6.70%	7.50%

Year	1989	1990	1991	1992	1993	1994	1995	1996	1997
Rate	7.40%	6.80%	7.50%	11.20%	10.90%	8.00%	8.00%	8.30%	8.10%

Use a spreadsheet to calculate and plot the 3- and 5-point moving means for this data against the unemployment rates. Compare the smoothing of both techniques.

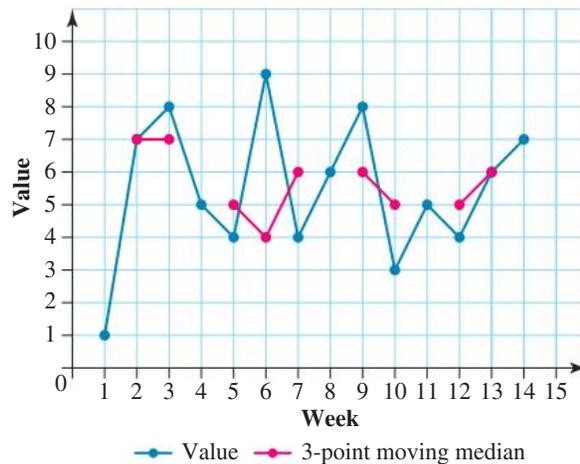
9. During times of economic instability, investors often turn to gold as a safe-haven asset. In more stable periods, they are more likely to take on risk by investing in other assets. The following table presents gold prices in US dollars per ounce from 2010 to 2023. The COVID-19 pandemic in 2020 led to significant economic disruption around the world.

Year	2010	2011	2012	2013	2014	2015	2016
Gold prices (US\$/ounce)	1300	1550	1700	1250	1300	1150	1250

Year	2017	2018	2019	2020	2021	2022	2023
Gold prices (US\$/ounce)	1300	1200	1400	1800	1750	1900	1800

Use a spreadsheet to calculate and plot the 3- and 5-point moving median for this data against the original data. Compare the smoothing of both techniques.

10. A time series plot, along with its incomplete 3-point moving median smoothing, is shown below. Determine the missing values for 3-point moving median.



Complex familiar

11. **WE7b** The total number of people using a new city car park every day was recorded each day for 10 days and the results are in the following table.



Day	1	2	3	4	5	6	7	8	9	10
Number of people	360	320	309	339	371	330	335	354	398	420

Determine and graph the least-squares regression line to the smoothed 5-point moving mean data and then predict the number of people using the new city car park on day 15. Evaluate the reasonableness of your solution.

12. A record of the maximum temperature recorded at the Richmond post office on 31 December for 12 years is recorded in the following table.

Year	2006	2007	2008	2009	2010	2011	2012	2013	2014	2015	2016	2017
Maximum temperature (°C)	37.7	36.0	36.8	33.1	34.8	37.0	37.2	43.6	40.4	36.2	38.4	41.9

Source: <http://www.bom.gov.au/climate/change/acorn/sat/data/acorn.sat.maxT.030045.daily.txt>.



Determine and graph the least-squares regression line to the smoothed 3-point moving median data and then predict the maximum temperature on 31 December 2025. Evaluate the reasonableness of your solution.

Complex unfamiliar

13. Irene owns a cosmetics company whose monthly sales data is shown below. Her sales have been steadily increasing, but an influx of orders for the month of July, caused by a sudden and unplanned stock shortage at a major competitor, resulted in an unusual spike in sales. Although this surge in orders was not something Irene had anticipated or could control, it led to higher-than-usual sales in that particular month.

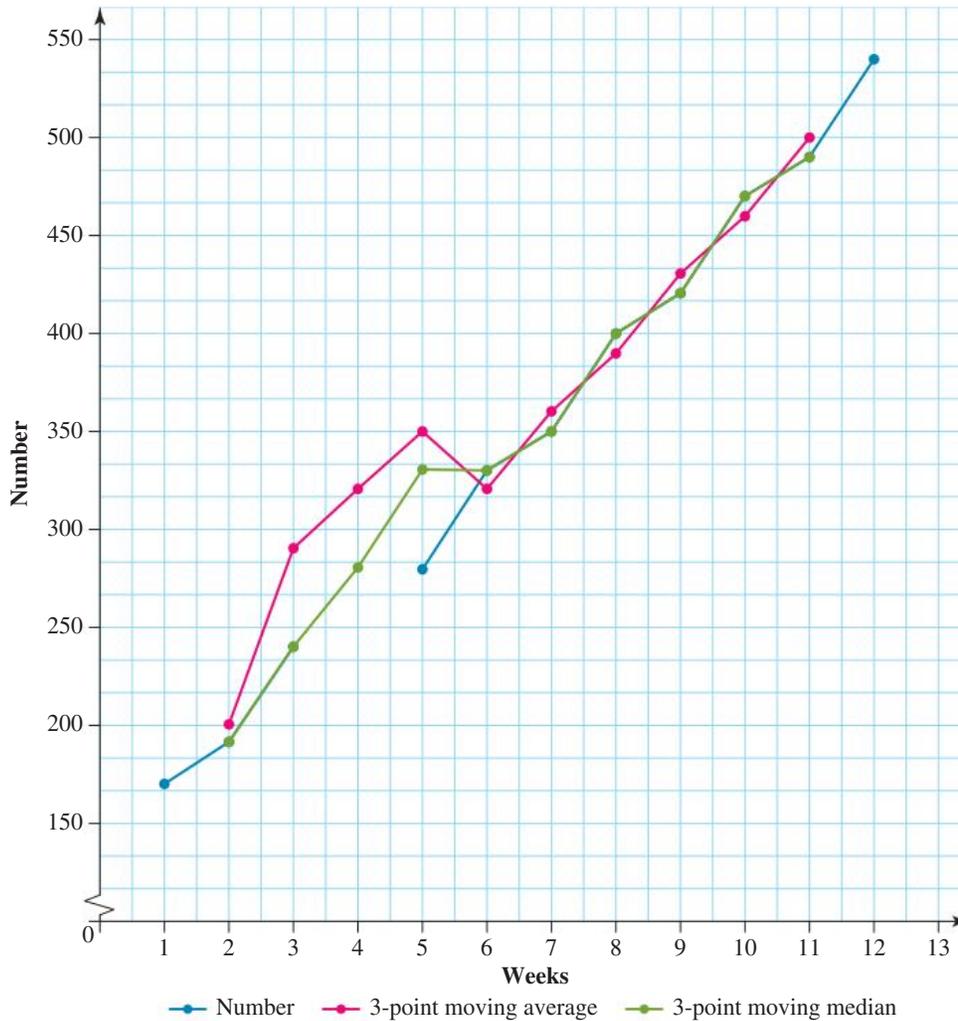
Irene now wishes to analyse the long-term trend of her sales in order to make a future prediction. Evaluate the most appropriate smoothing techniques to assist with this projection. Based on this analysis, determine her sales figure for the month at the end of the following financial year (June). Evaluate the reasonableness of your solution. Use spreadsheets to solve.



Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct
Sales (in thousands)	135	160	173	178	205	225	420	230	255	260

14. The table below presents time series data which has been plotted on the graph below alongside the corresponding 3-point moving mean and 3-point moving median values. All values graphed are non-rounded integers.

Week	1	2	3	4	5	6	7	8	9	10	11	12
Number	170	190	240	x	280	330	350	400	420	470	490	540



Determine the missing data value at Week 4. Evaluate the reasonableness of your solution both algebraically and graphically.

Fully worked solutions for this chapter are available online.

LESSON

3.5 Deseasonalising time series data

SYLLABUS LINKS

- Deseasonalise a time series by calculating the seasonal indices using the average percentage method, including the use of spreadsheets.
- Solve practical problems that involve the analysis of time series data.

Source: General Mathematics Senior Syllabus 2024 © State of Queensland (QCAA) 2024; licensed under CC BY 4.0.

3.5.1 Calculating seasonal indices using the average percentage method

The previous lesson covered data smoothing to reduce noise and reveal long-term trends. However, seasonal patterns can still obscure the true trend.

This lesson introduces **deseasonalising data**, which removes seasonal effects to highlight the long-term trend. A **seasonal index** expresses a period's deviation from the overall average, usually as a percentage. Deseasonalisation is achieved by calculating seasonal indices using the **average percentage method**, adjusting data to eliminate seasonal impact for more accurate trend analysis and forecasting.

The seasonal index

$$\text{Seasonal index} = \frac{\text{data value}}{\text{seasonal average}}$$

Seasonal indices show how each season compares to the overall average. An average index is 1 (or 100%), indicating the season follows the overall trend. Indices above 1 suggest better performance, whereas values below 1 indicate worse performance.

For example, an index of 1.3 for a set of sales data means sales are 30% higher than average, whereas 0.7 means sales are 30% lower.

For 4 periods (e.g. 4 quarters), the sum of all indices should be 4. Similarly, for 12 months, the sum should be 12.

WORKED EXAMPLE 8 Calculating seasonal indices using the average percentage method

Annual sales data is shown below.

Quarter	Q1	Q2	Q3	Q4
Sales	162	180	135	243

- Calculate the yearly sales average.
- Calculate the seasonal indices using the average percentage method.
- Interpret the meaning of each of the seasonal indices.

THINK

- Calculate yearly sales average.

WRITE

$$\begin{aligned} \text{a. Yearly sales average} &= \frac{162 + 180 + 135 + 243}{4} \\ &= 180 \end{aligned}$$

- b. 1. Use the following formula to calculate each of the seasonal indices:

$$\text{Seasonal index} = \frac{\text{data value}}{\text{seasonal average}}$$

2. Represent the seasonal indices in table form.

- c. Interpret each seasonal index.

b. Quarter 1: $\frac{162}{180} = 0.9$
 Quarter 2: $\frac{180}{180} = 1$
 Quarter 3: $\frac{135}{180} = 0.75$
 Quarter 4: $\frac{243}{180} = 1.35$

Quarter	1	2	3	4
Seasonal index	0.9	1	0.75	1.35

- c. Q1's seasonal index of 0.9 means that sales are 10% below the average (because 0.9 is 10% lower than 100, which is the baseline).
 Q2's seasonal index of 1 means that sales are exactly at the average level for the year.
 Q3's seasonal index of 0.75 means that sales are 25% below the average.
 Q4's seasonal index of 1.35 means that sales are 35% above the average.

3.5.2 Deseasonalising data

The following steps are used to deseasonalise a data set (also known as seasonally adjusting a data set).

Step 1: Observe the order of the data.

Understand how the data is organised. Identify whether the years are represented by columns or rows. This will guide the calculations in subsequent steps. Below are examples of both display formats.

Format 1: Years as columns

Quarter	2023	2024	2025
1	13	14	13
2	16	16	15
3	18	19	17
4	13	16	14

Format 2: Years as rows

Year	Quarter 1	Quarter 2	Quarter 3	Quarter 4
2023	13	16	18	13
2024	14	16	19	16
2025	13	15	17	14

Step 2: Calculate the average (mean) for each year.

For each year, calculate the average (mean) of the values by summing all the values in that year and dividing by the number of periods (quarters, months etc.).

$$\text{Yearly average} = \frac{\text{sum of all values for the year}}{\text{number of periods}}$$

For example, for 2023 (quarters 1–4):

$$\text{Yearly average} = \frac{13 + 16 + 18 + 13}{4} = 15$$

Step 3: Use the seasonal index formula.

After calculating the yearly average for each year, use the seasonal index formula by dividing each data point (for that year) by the corresponding yearly average. This step adjusts the values to account for each year's overall level of activity.

Construct a table with these newly adjusted values.

For example, if the yearly average for 2023 is 15, then for quarter 1 (13), calculate:

$$\begin{aligned} \text{Seasonal index} &= \frac{\text{data value}}{\text{seasonal average}} \\ \frac{13}{15} &= 0.8667 \end{aligned}$$

Step 4: Calculate the overall average seasonal index for each season (data sets consisting of more than 1 year).

Calculate the seasonal indices by averaging the adjusted values for each season (quarter, month etc.) from the newly adjusted table.

$$\text{Seasonal index} = \frac{\text{sum of all values for the season}}{\text{number of years}}$$

For instance, sum all values for quarter 1 across all years and divide by the number of years to calculate the seasonal index for quarter 1.

$$\text{Seasonal index} = \frac{0.8667 + 0.8615 + 0.8814}{3} = 0.87$$

Step 5: Deseasonalise the data.

Now that you have the seasonal indices, deseasonalise the original data by dividing each value by its corresponding seasonal index.

Deseasonalised value

$$\text{Deseasonalised value} = \frac{\text{data value}}{\text{seasonal index}}$$

For example, if the original value for 2023 quarter 1 is 13, then to deseasonalise this value, calculate:

$$\frac{13}{0.87} = 15$$

Construct a table with these all newly adjusted values. The deseasonalised values now represent the underlying trend without seasonal fluctuations.

Step 6: Plot the deseasonalised data.

Finally, plot the deseasonalised data alongside the raw data to visually compare the impact of removing the seasonal effects. This can help to assess how much seasonal variation was present in the data and can make it easier to identify long-term trends or cycles.



WORKED EXAMPLE 9 Deseasonalising data

The following set of data gives the number of new homes built by a company each quarter for 3 years.

Quarter	2022	2023	2024
1	14	14	14
2	15	15	16
3	17	17	16
4	14	15	14

Deseasonalise the data and plot on the same axis as the original data. Comment on the graph.

THINK

1. Refer to Step 1 in deseasonalising above about observing the order of the data. This data has years as columns, so calculate the averages for the year by working down each column.

WRITE

Quarter	2022	2023	2024
1	14	14	14
2	15	15	16
3	17	17	16
4	14	15	14
Yearly average	$= \frac{14 + 15 + 17 + 14}{4} = 15$	$= \frac{14 + 15 + 17 + 15}{4} = 15.25$	$= \frac{14 + 16 + 16 + 14}{4} = 15$

2. Calculate the seasonal index for each quarter using the formula:

$$\text{Seasonal index} = \frac{\text{data value}}{\text{seasonal average}}$$

Quarter	2022	2023	2024
1	$\frac{14}{15} = 0.9333$	$\frac{14}{15.25} = 0.9180$	$\frac{14}{15} = 0.9333$
2	$\frac{15}{15} = 1$	$\frac{15}{15.25} = 0.9836$	$\frac{16}{15} = 1.0667$
3	$\frac{17}{15} = 1.1333$	$\frac{17}{15.25} = 1.1148$	$\frac{16}{15} = 1.0667$
4	$\frac{14}{15} = 0.9333$	$\frac{15}{15.25} = 0.9836$	$\frac{14}{15} = 0.9333$

3. Since there are multiple years of data and therefore more than one seasonal index for each season, calculate the average seasonal index to arrive at a single value for each season.

Quarter 1: $\frac{0.9333 + 0.9180 + 0.9333}{3} = 0.9282$

Quarter 2: $\frac{1 + 0.9836 + 1.0667}{3} = 1.0168$

Quarter 3: $\frac{1.1333 + 1.1148 + 1.0667}{3} = 1.1049$

Quarter 4: $\frac{0.9333 + 0.9836 + 0.9333}{3} = 0.9501$

4. Write the answer.

Quarter	1	2	3	4
Seasonal index	0.9282	1.0168	1.1049	0.9501

5. Deseasonalise the data using the formula:

$$\text{Deseasonalised value} = \frac{\text{data value}}{\text{seasonal index}}$$

Quarter	2022	2023	2024
1	$\frac{14}{0.9282} = 15.083$	$\frac{14}{0.9282} = 15.083$	$\frac{14}{0.9282} = 15.083$
2	$\frac{15}{1.0168} = 14.752$	$\frac{15}{1.0168} = 14.752$	$\frac{16}{1.0168} = 15.736$
3	$\frac{17}{1.1049} = 15.386$	$\frac{17}{1.1049} = 15.386$	$\frac{16}{1.1049} = 14.481$
4	$\frac{14}{0.9501} = 14.735$	$\frac{15}{0.9501} = 15.788$	$\frac{14}{0.9501} = 14.735$

6. Write the answer.

The seasonally adjusted (deseasonalised) data are as follows.

Quarter	2022	2023	2024
1	15.083	15.083	15.083
2	14.752	14.752	15.736
3	15.386	15.386	14.481
4	14.735	15.788	14.735

7. Graph the original time series data and the seasonally adjusted data on the same axes.



8. Describe the trend.

The seasonally adjusted data has removed the obvious seasonal fluctuations. There appears to be a very slight downward trend within the seasonally adjusted graph.

3.5.3 Using a spreadsheet to deseasonalise data

Deseasonalising a time series can also be performed using a spreadsheet. While deseasonalising data by hand can be a very time-consuming process, using a spreadsheet is a far more efficient means of deseasonalising large amounts of time series data. By organising the data in rows and columns, users can calculate seasonal indices and apply them to adjust the data, making it quicker to visualise and analyse the true long-term patterns. The use of formulas and charts allows for an effective presentation of the deseasonalised data, improving forecasting and interpretation.

WORKED EXAMPLE 10 Deseasonalising data using a spreadsheet

The following set of data gives the population of Queensland (in millions) from the first quarter of 2010 to the last quarter of 2017.

Quarter	2010	2011	2012	2013	2014	2015	2016	2017
1	4.388	4.458	4.546	4.634	4.705	4.764	4.827	4.907
2	4.405	4.477	4.569	4.653	4.720	4.778	4.845	4.928
3	4.421	4.498	4.592	4.671	4.735	4.792	4.866	4.949
4	4.437	4.519	4.611	4.685	4.747	4.805	4.883	4.964

Source: <http://www.qgso.qld.gov.au/products/tables/erp-components-change-no/index.php>

Use a spreadsheet to deseasonalise the data and plot on the same axis as the original data. Comment on the graph.

THINK

- Enter the data into a new spreadsheet. To calculate the average for each year: Place the cursor in cell B6, type = AVERAGE (B2:B5), then press Enter. Repeat process to calculate the average for each year.

WRITE

	A	B	C	D	E	F
1	Quarter	2010	2011	2012	2013	2014
2	1	4.388	4.458	4.546	4.634	4.705
3	2	4.405	4.477	4.569	4.653	4.72
4	3	4.421	4.498	4.592	4.671	4.735
5	4	4.437	4.519	4.611	4.685	4.747
6		4.41275	4.488	4.5795	4.66075	4.72675

- Copy the completed table with the averages and paste below. To divide each value for that year by the average for that year, place the cursor in cell B9, type = B2/\$B\$6, then press Enter. Fill down the column for 2010 and repeat the process for the other years.

Note: Typing \$B\$6 ensures that the numerical value of 4.41275 from cell B6 is used for every calculation in the column.

	A	B	C	D	E
1	Quarter	2010	2011	2012	2013
2	1	4.388	4.458	4.546	4.634
3	2	4.405	4.477	4.569	4.653
4	3	4.421	4.498	4.592	4.671
5	4	4.437	4.519	4.611	4.685
6		4.41275	4.488	4.5795	4.66075
7					
8	Quarter	2010	2011	2012	2013
9	1	0.994391253	0.9933155	0.99268479	0.99426058
10	2	0.998243726	0.997549	0.99770717	0.99833718
11	3	1.001869582	1.0022282	1.00272956	1.00219922
12	4	1.005495439	1.0069073	1.00687848	1.00520303

3. To calculate the seasonal indices or average for the season (quarter), place the cursor in cell J9, type = AVERAGE(B9:B9), then press Enter. Repeat this process for each quarter.

fx =AVERAGE							
C	D	E	F	G	H	I	J
2011	2012	2013	2014	2015	2016	2017	
4.458	4.546	4.634	4.705	4.764	4.827	4.907	
4.477	4.569	4.653	4.72	4.778	4.845	4.928	
4.498	4.592	4.671	4.735	4.792	4.866	4.949	
4.519	4.611	4.685	4.747	4.805	4.883	4.964	
4.488	4.5795	4.66075	4.72675	4.78745	4.85525	4.937	
2011	2012	2013	2014	2015	2016	2017	
0.9933155	0.99268479	0.99426058	0.99539853	0.99566331	0.99418156	0.99392344	0.994227
0.997549	0.99770717	0.99833718	0.99857196	0.99858927	0.9978888	0.99817703	0.998133
1.0022282	1.00272956	1.00219922	1.00174539	1.00151523	1.0022141	1.00243063	1.002116
1.0069073	1.00687848	1.00520303	1.00428413	1.0042322	1.00571546	1.00546891	1.005523

4. Write the answer.

The seasonal indices as follow.

Quarter	1	2	3	4
Seasonal index	0.9943	0.9981	1.0021	1.0055

5. To divide each figure in the original table by its seasonal index, copy the original table of data, then place the cursor in cell B15 and type, = B2/\$J\$9, then press Enter. Repeat this process for the other quarters.

B5 : fx =B2/\$B\$6				
	A	B	C	D
1	Quarter	2010	2011	2012
2	1	4.388	4.458	4.546
3	2	4.405	4.477	4.569
4	3	4.421	4.498	4.592
5	4	4.437	4.519	4.611
6		4.41275	4.488	4.5795
7				
8	Quarter	2010	2011	2012
9	1	0.994391253	0.9933155	0.99268479
10	2	0.998243726	0.997549	0.99770717
11	3	1.001869582	1.0022282	1.00272956
12	4	1.005495439	1.0069073	1.00687848
13				
14	Quarter	2010	2011	2012
15	1	4.413	4.484	4.572
16	2	4.413	4.485	4.578
17	3	4.412	4.489	4.582
18	4	4.413	4.494	4.586

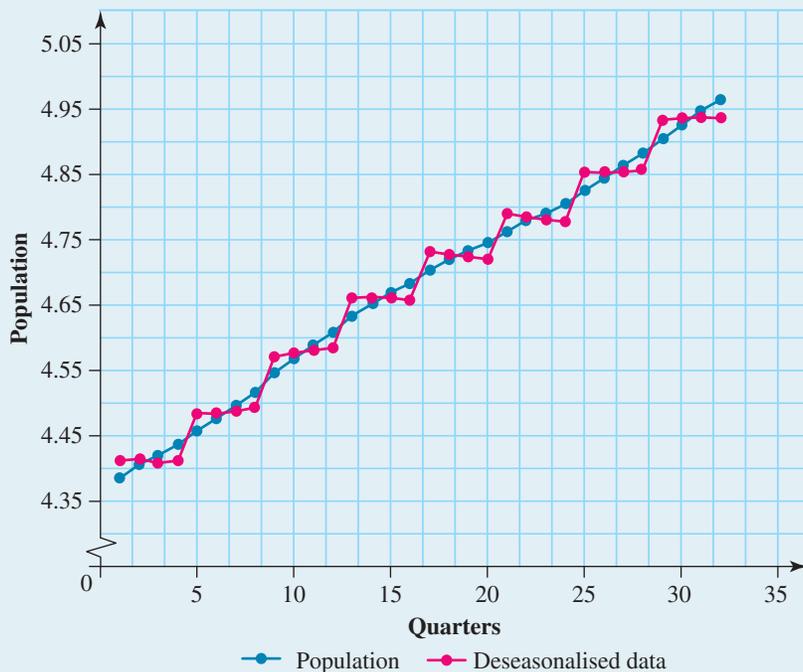
6. Write the answer.

Quarter	2010	2011	2012	2013	2014	2015	2016	2017
1	4.413	4.484	4.572	4.661	4.732	4.792	4.855	4.935
2	4.413	4.485	4.578	4.662	4.729	4.787	4.854	4.937
3	4.412	4.489	4.582	4.661	4.725	4.782	4.856	4.939
4	4.413	4.494	4.586	4.659	4.721	4.779	4.856	4.937

7. Create a table with headings as shown. Number the quarters as 1, 2, 3, 4, 5, 6, ... 32. Copy the original data into Column 2, with each set of quarters pasted below the previous four quarters and the deseasonalised data into Column 3.
Note: Copy the deseasonalised data and paste as 'values' into the table.

20	Quarter	Population	Deseasonalised data
21	1	4.388	4.413
22	2	4.405	4.413
23	3	4.421	4.412
24	4	4.437	4.413
25	5	4.458	4.484
26	6	4.477	4.485
27	7	4.498	4.489
28	8	4.519	4.494
29	9	4.546	4.572
30	10	4.569	4.578
31	11	4.592	4.582
32	12	4.611	4.586
33	13	4.634	4.661
34	14	4.653	4.662

Use Excel to create a graph with both the original data and the deseasonalised data on the same axes.



8. Describe the trend.

There is a very clear upward trend to both the original data and the seasonally adjusted data. The seasonally adjusted data has removed the obvious seasonal fluctuations.

3.5.4 Deseasonalising data for accurate forecasting

When working with seasonal data, it's crucial to deseasonalise it before fitting a trendline. Seasonal fluctuations can distort the true trend. Removing these effects helps focus on the overall pattern, leading to more accurate analysis and forecasting.

After fitting a trendline to the deseasonalised data, readjust for seasonality by multiplying the prediction by the seasonal index. This ensures the forecast includes expected seasonal peaks and troughs, resulting in a more realistic prediction.

Comparing predictions with and without deseasonalising data shows its importance. Predictions based on original data are influenced by short-term fluctuations, whereas those from deseasonalised data offer a reliable estimate of long-term trends. Combining the trend with seasonal adjustments achieves accurate seasonal forecasts.

WORKED EXAMPLE 11 Deseasonalising data for accurate forecasting

The following table shows the quarterly sales figures (in '000s) for a software company in 2023 and 2024:

Quarter	Q1	Q2	Q3	Q4
2023	120	150	180	250
2024	130	160	200	270

Seasonal indices and the deseasonalised data for these sales figures are as follows.

Quarter	Q1	Q2	Q3	Q4
Seasonal index	0.68	0.85	1.04	1.42

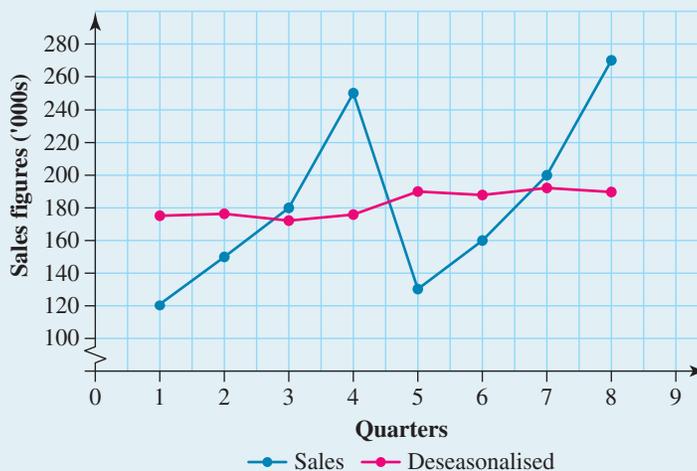
Quarter	Q1	Q2	Q3	Q4
2023	175.19	176.55	172.98	175.46
2024	189.79	188.32	192.20	189.50

- Using a spreadsheet, plot both the time series sales figures and the deseasonalised data on the same graph.
- Determine the least-squares regression line for both the sales data and the deseasonalised data by displaying their equations.
- Using the regression lines for both the original sales data and the deseasonalised data, predict the sales figures for Q1 2025. Evaluate the reasonableness of your solutions.

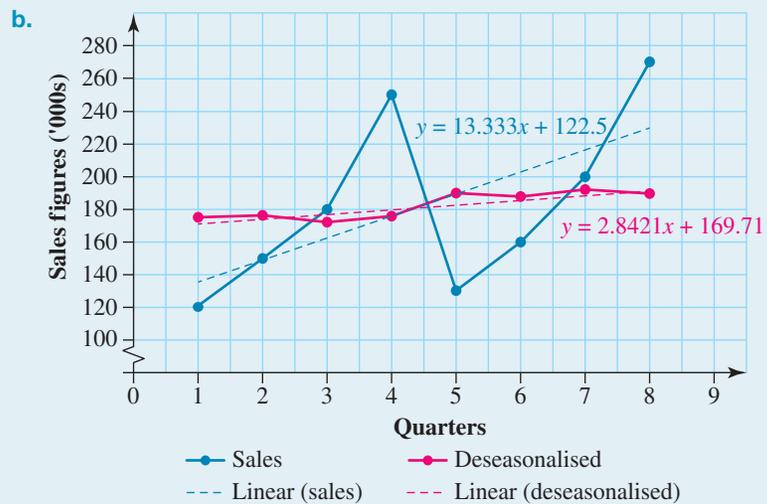
THINK

- Using a spreadsheet, enter the data and draw a scatterplot for both sales figures and deseasonalised data.

WRITE



b. To include the least-squares regression lines, click the cursor on the least-squares lines then tick the box next to 'Display Equation on chart'.



c. 1. Determine the x -value for Q1 2025 and write the equation for the sales figures.

2. Substitute and simplify.

3. Write the equation for the deseasonalised data.

4. Substitute and simplify

5. Seasonalise the amount by multiplying it by 0.68 since that is the relevant seasonal index for Q1.

6. Evaluate the reasonableness of the solutions.

c. Let $x = 9$.

$$y = 13.333x + 122.5$$

$$= 13.333(9) + 122.5$$

$$= 242.5$$

$$y = 2.8421x + 169.71$$

$$= 2.8421(9) + 169.71$$

$$= 195.2889$$

$$\text{Seasonalised amount} = 195.2889 \times 0.68$$

$$= 132.8$$

The predicted sales figure for Q1 2025, based on the regression line for the original data, is 242.5, whereas the prediction using the regression line for the deseasonalised data then seasonalised is 132.8.

According to the seasonality of the sales data, the latter prediction reflects the underlying trend adjusted for typical seasonal fluctuations, suggesting that the original data's prediction is likely influenced by seasonal peaks and might overestimate the actual sales for Q1 2025. The more reasonable result is 132.8.

3.5 Exercise

3.5 Exam questions on

Simple familiar

1, 2, 3, 4, 5, 6, 7, 8, 9, 10

Complex familiar

11, 12, 13

Complex unfamiliar

14, 15

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Simple familiar

1. WEB Annual sales data is shown below.

Quarter	Q1	Q2	Q3	Q4
Sales	204	300	240	216

- Calculate the yearly sales average.
 - Calculate the seasonal indices using the average percentage method.
 - Interpret the meaning of each of the seasonal indices.
2. The sales figures for Tiffany's Ice Cream Shop for the year are shown below.

Quarter	Sales
Summer	10 500
Autumn	6000
Winter	4500
Spring	9000



Calculate the seasonal indices for this data. Comment on your solution with respect to the sales figures.

3. The following table shows the monthly average rainfall (mm) for Cairns, Queensland, over the course of a year. Cairns experiences a tropical climate with a clear seasonal pattern: higher rainfall in the wet season (summer) and lower rainfall in the dry season (winter).

Month	Average rainfall (mm)
January	330
February	310
March	270
April	160
May	90
June	40
July	30
August	40
September	60
October	120
November	220
December	300

Calculate the seasonal indices for this data. Comment on your solution with respect to the average rainfall data.

4. Determine the seasonal index for the third quarter for the following table.

Quarter	1	2	3	4
Seasonal indices	1.4	0.70		0.88

5. Calculate the missing seasonal indices for June, August and November, assuming that the indices for these months are identical.

Month	Seasonal index
January	1.4
February	0.7
March	1.15
April	1
May	0.95
June	
July	0.65
August	
September	1.05
October	1.1
November	
December	1.3

6. Determine the seasonal indices for the following average temperature data.

Year	Summer	Autumn	Winter	Spring
2023	30	20	15	22
2024	31	21	16	23

7. Russell owns a gardening store, and the sales data, along with the corresponding seasonal indices, are provided below. Use this information to deseasonalise the data.



Year	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
2023	120	130	150	170	190	210	250	240	230	220	180	160
2024	125	135	155	175	195	215	255	245	235	225	185	165

Year	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Seasonal indices	0.64	0.70	0.80	0.91	1.01	1.12	1.33	1.28	1.22	1.17	0.96	0.86

8. The following table shows the quarterly deseasonalised water consumption data (in thousands of litres) for a household over three years, 2022, 2023 and 2024:

Quarter	Q1	Q2	Q3	Q4
2022	11.27	10.86	11.17	10.76
2023	11.27	11.94	11.17	11.59
2024	12.20	11.94	12.41	12.42

Quarter	Q1	Q2	Q3	Q4
Seasonal index	1.06	0.92	0.81	1.21

Determine the original water consumption data by seasonalising the data. Round answers to the nearest thousand.

9. Monthly sales figures for a company between January 2022 and December 2024 were used to calculate the following seasonal indices.

Month	Jan	Feb	Mar	Apr	May	June	July	Aug	Sept	Oct	Nov	Dec
Seasonal indices	1.22	0.79	1.01	1.23	0.89	0.80	1.02	1.10	0.94	1.12		1.11

- Determine the seasonal index for November.
- Determine the deseasonalised value for June 2024, given that the original sales figure for June 2024 was \$187 600.
- Determine the original sales figure for March 2023, given that the deseasonalised value for March 2023 was \$166 280.



10. **WE11** The following table shows the quarterly attendance figures (in thousands) for Nick's gym over two years, 2023 and 2024.

Quarter	Q1	Q2	Q3	Q4
2023	80	50	60	110
2024	90	55	65	120

Seasonal indices and the deseasonalised data for these sales figures are as follows.

Quarter	Q1	Q2	Q3	Q4
Seasonal index	1.08	0.67	0.79	1.46

Quarter	Q1	Q2	Q3	Q4
2023	74.16	75	75.57	75.31
2024	83.43	82.5	81.87	82.16

- Using a spreadsheet, plot both the time series attendance figures and the deseasonalised data on the same graph.
- Determine the least-squares regression line for both the attendance data and the deseasonalised data.
- Using the regression lines for both the original attendance data and the deseasonalised data, predict the attendance figures for Q2 2025. Evaluate the reasonableness of your solutions.

Complex familiar

11. **WE9** The following set of data gives the number of new cars sold by a car salesman each quarter for 3 years.

Quarter	2021	2022	2023
1	40	44	46
2	28	25	32
3	31	28	33
4	23	21	25

Deseasonalise the data and plot on the same axis as the original data. Comment on the graph.

12. **WE10** The following table below gives the number of sick days taken per month by the staff of a large company.

Month	2021	2022	2023	2024
Jan	26	29	33	31
Feb	25	28	30	35
Mar	29	33	38	29
Apr	31	35	37	33
May	32	36	41	37
June	35	40	42	39
July	36	43	44	42
Aug	40	45	47	44
Sept	34	38	43	46
Oct	31	25	38	39
Nov	29	33	36	45
Dec	25	30	32	48



Use a spreadsheet to deseasonalise the data and plot on the same axis as the original data. Comment on the graph.

13. The following table shows the amount of sparkling water (in millions of litres) produced by a new factory over a two-year period.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
2023	14.6	16.0	15.7	12.6	9.5	6.3	5.1	6.8	9.2	10.9	12.5	17
2024	14.1	15.8	16.1	13.1	8.7	5.6	4.5	4.5	7.9	9.9	9.9	12.4

Use a spreadsheet to deseasonalise the data and analyse it to predict the amount of sparkling water that will be produced in February 2025 to the nearest litre.

Complex unfamiliar

14. The staff of a Gold Coast hotel analyse their guest data over two years (shown below in thousands) and determine that, in response to forecasted economic demands, they must increase the number of guests by 20% in summer and autumn, 25% in winter and 15% in spring for the coming year.

Year	Summer	Autumn	Winter	Spring
2023	30	20	10	18
2024	35	25	12	20

Deseasonalise the guest data, incorporating these projected increases, and predict the total number of guests for 2026.

15. The following table shows the deseasonalised flu medication sales data for Tina and Chris's pharmacy.



Year	Summer	Autumn	Winter	Spring
2022	1145.83	1180.00	1306.91	1162.50
2023	1333.33	1380.00	1355.85	1439.77

The seasonal indices are as follows.

Season	Summer	Autumn	Winter	Spring
Seasonal index	0.24	1.00	1.88	0.88

In 2024, the actual flu medication sales were 410 units in summer, 1512 units in autumn, 2624 units in winter and 1312 units in spring.

Analyse the data to predict the number of units that would be sold in winter of 2027.

Fully worked solutions for this chapter are available online.

LESSON

3.6 Review

3.6.1 Summary

Hey students! Now that it's time to revise this chapter, go online to:

 Access the chapter summary

 Review your results

 Practise exam questions

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3.6 Exercise

learnon

3.6 Exercise

3.6 QCAA questions 

Simple familiar

1, 2, 3, 4, 5, 6, 7, 8,
9, 10, 11, 12

Complex familiar

13, 14, 15, 16

Complex unfamiliar

17, 18, 19, 20

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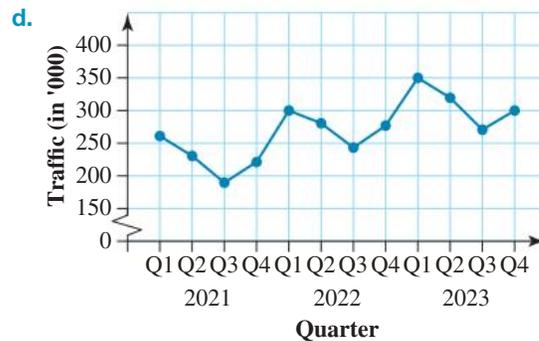
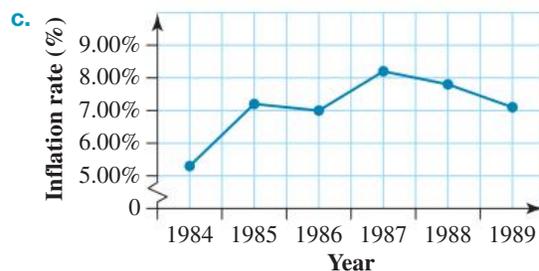
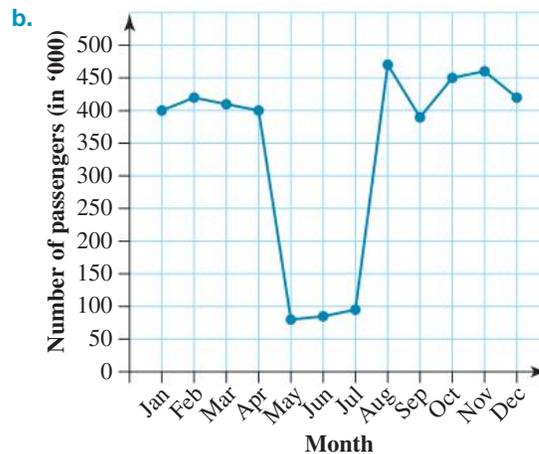
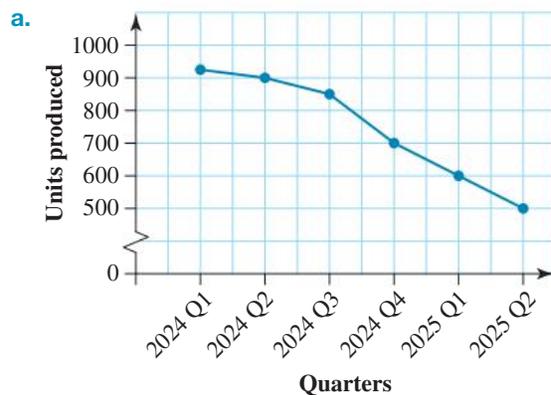
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Simple familiar

1. Identify and describe the features of each time series plot.



2. Construct a time series plot of the following data that shows the temperature across two days. Identify and describe any features of this time series plot.

Time	Day 1	Day 2										
	12 am	4 am	8 am	12 pm	4 pm	8 pm	12 am	4 am	8 am	12 pm	4 pm	8 pm
Temperature (°C)	24.1	23.4	26.5	30.2	32.7	28.9	24.3	23.6	26.2	30.5	32.5	29

3. The following data shows the satisfaction score for a manager over 5 months.

Months	Satisfaction score
1	90
2	87
3	80
4	78
5	70

- a. Determine the least-squares regression line for the following time series data.
 b. Describe the long-term trend of this data.
4. The following data represent the inflation rate from 1980 to 1995.

Year	1980	1981	1982	1983	1984	1985	1986	1987
Inflation rate (%)	12.40%	10.00%	10.10%	4.30%	5.30%	7.20%	7.00%	8.20%

Year	1988	1989	1990	1991	1992	1993	1994	1995
Inflation rate (%)	7.80%	7.10%	7.00%	5.60%	1.90%	1.80%	2.80%	6.10%

- a. Construct a graph of the time series data using spreadsheet technology.
 b. Identify and describe any features of this time series plot.
 c. Determine the least-squares regression line and present its equation on the time series plot.
 d. Determine the inflation rate in 2005. Evaluate the reasonableness of your solution.
5. The following table gives the number of new bicycles purchased in a large city.

Year	Number of bicycles sold	3-point moving mean	5-point moving mean	3-point moving median	5-point moving median
2017	7434				
2018	5680			7123	
2019	7123	7123	7608		7434
2020	8567	8308			8567
2021	9234		8636	8690	
2022	8690	9164	9329	9234	
2023	9567	9615			9567
2024	10 588			10 588	
2025	12 300				

Complete the table by filling in the missing values to the nearest whole number.

The following relates to questions 6–8.

Matthew is tracking the long-term trend of the Australian dollar (AUD) against the US dollar (USD) using the data below, covering the period from 2014 to 2023.



Year	2014	2015	2016	2017	2018	2019	2020	2021	2022	2023
AUD to USD	0.9	0.77	0.75	0.77	0.75	0.69	0.66	0.75	0.7	0.65

6. Answer the following.
- Calculate the 3- and 5-point moving mean for this data. Round all values to 2 decimal places.
 - Graph the original data, the 3-point moving mean and the 5-point moving mean on the same set of axes. Comment on the smoothing effect.
7. Answer the following.
- Calculate the 3- and 5-point moving medians for this data.
 - Graph the original data, the 3-point moving median and the 5-point moving median on the same set of axes. Comment on the smoothing effect.
8. Answer the following.
- Use a spreadsheet to calculate the 3- and 5-point moving means for this data.
 - Use a spreadsheet to calculate the 3- and 5-point moving medians for this data.
9. Annual sales data is shown below.

Quarter	Q1	Q2	Q3	Q4
Sales	3080	1760	1320	2640

- Calculate the yearly sales average.
 - Calculate the seasonal indices using the average percentage method.
10. The following table shows the seasonal indices from a time series.

Quarter	1	2	3	4
Index	0.98	0.81		1.02

Determine the a seasonally adjusted value for quarter 3 of 2025, when the original value was 445.

The following information relates to questions 11 and 12.

The following table gives the seasonal sales of gumboots in a garden supplies store for the years 2018 and 2019.

Year	Winter	Spring	Summer	Autumn
2018	710	530	450	780
2019	820	678	546	420



11. To calculate the seasonal indices for the data in the previous table, complete the following tables, filling in the missing values.

Year	Winter	Spring	Summer	Autumn	Averages
2018	710	530	450	780	
2019	820	678	546	420	616

Year	Winter	Spring	Summer	Autumn
2018	1.15	0.86	0.73	1.26
2019		1.10		0.68
Seasonal indices	1.24		0.81	0.97

12. Complete the following table with the deseasonalised data Round the deseasonalised values to the nearest whole number.

Year	Winter	Spring	Summer	Autumn
2018				
2019				

Complex familiar

13. Justin has been creating gaming content on his social media channel for several years. He claims that the increase in his video views from 2019 to 2022 is directly linked to the increase in the frequency of his uploads. Analyse the following time series data, which displays quarterly data on Justin's upload frequency and the corresponding total views (in thousands) for his channel during this period.

Quarter	Upload frequency	Views (thousands)
Q1 2019	12	5
Q2 2019	10	5.5
Q3 2019	13	6
Q4 2019	14	6.5
Q1 2020	14	6
Q2 2020	16	7
Q3 2020	15	7.5
Q4 2020	17	8
Q1 2021	18	8
Q2 2021	22	10
Q3 2021	21	11
Q4 2021	24	12
Q1 2022	20	9
Q2 2022	25	12
Q3 2022	28	13.5
Q4 2022	30	15

Graph both sets of data, describe their key features and evaluate the reasonableness of Justin's claim regarding the relationship between his upload frequency and video views.

14. Mia's Shop claims that the sales of her stock doubles in revenue every 5 years. Below are the historical sales figures for the shop from 2022 to 2025:

- 2022: \$50 000
- 2023: \$55 000
- 2024: \$58 500
- 2025: \$60 500

Evaluate the reasonableness of this claim. Provide appropriate mathematical justification.



15. The following table gives the sales of takeaway lunches sold each month in Tiffany's café in a shopping centre. Use spreadsheet to answer the following questions.

Month	Jan	Feb	Mar	Apr	May	June	July	Aug	Sept	Oct	Nov	Dec
Sales (\$)	4566	3125	4100	2890	3244	2533	3499	2105	2567	1345	2889	1400

Determine and graph the least-squares regression line to the smoothed 3-point moving mean data, and determine the sales of takeaway lunches sold at the end of the following financial year (June). Evaluate the reasonableness of your solution.

16. The following data provides the monthly rainfall data for the Alderley weather station in Brisbane for 2023 and 2024.



Year	Jan	Feb	Mar	Apr	May	Jun	July	Aug	Sept	Oct	Nov	Dec
2023	230.2	276.8	75.2	182.2	263.2	32.8	7.8	32.4	68.2	52.2	114.2	43.8
2024	107.8	120.2	121.0	10.8	27.0	311.6	24.8	33.2	99.8	24.0	53.0	121.4

Use a spreadsheet to deseasonalise the data and plot on the same axis as the original data. Comment on the graph.

Complex unfamiliar

17. Dennis and Maudrene operate competing ice kachang stores. Below is the monthly sales data for each store in the second half of 2025.



Owner	July	August	September	October	November	December
Dennis	120	140	145	150	165	170
Maudrene	80	100	130	160	200	220

Determine the first projected future month that one store will surpass the other by selling more than twice the number of ice kachang bowls.

Evaluate the reasonableness of your solution.

18. Beatrice is a competitive cyclist who has been diligently recording her time trial results each quarter for the past three years (as below). Her times have steadily improved over time, reflecting her hard work and consistent training. Beatrice's times improved steadily, but in the second and ninth quarters, she recorded unusually fast times. These exceptional performances were likely influenced by external factors, such as ideal weather or a particularly focused training period, and were not under her direct control, making them notable outliers in her overall progress.



Quarter	1	2	3	4	5	6	7	8	9	10	11	12
Time (mins)	52.10	46.01	51.44	50.43	50.42	50.42	49.80	49.40	46.35	48.45	49.43	48.13

Beatrice wishes to analyse the long-term trend of her cycling performance in order to predict her future race times. To assist with this projection, evaluate the most appropriate smoothing techniques to highlight her steady progress and account for any outliers or fluctuations. Based on this analysis, determine her expected time for the next race in the sixteenth quarter. Evaluate the reasonableness of your solution.

19. Margaret owns a successful puzzle business. Deseasonalised revenue data (in millions) for two years of this business is shown below, along with the relevant seasonal indices.

Quarter	Q1	Q2	Q3	Q4
Seasonal indices	0.34	0.49	0.79	2.39

Quarter	Q1	Q2	Q3	Q4
2023	59.62	61.78	63.62	62.68
2024	74.53	72.07	69.98	71.04

Margaret wishes to retire when the business's projected revenue hits 210 million. Determine in which year this will occur.

20. The following table shows the deseasonalised number of children (in thousands) visiting Coby's newly opened indoor play centre.

Year	Summer	Autumn	Winter	Spring
2022	24.56	25.43	25.38	22.33
2023	26.41	26.91	24.48	26.30



The seasonal indices are as follows.

Season	Summer	Autumn	Winter	Spring
Seasonal index	1.03	0.81	1.43	0.73

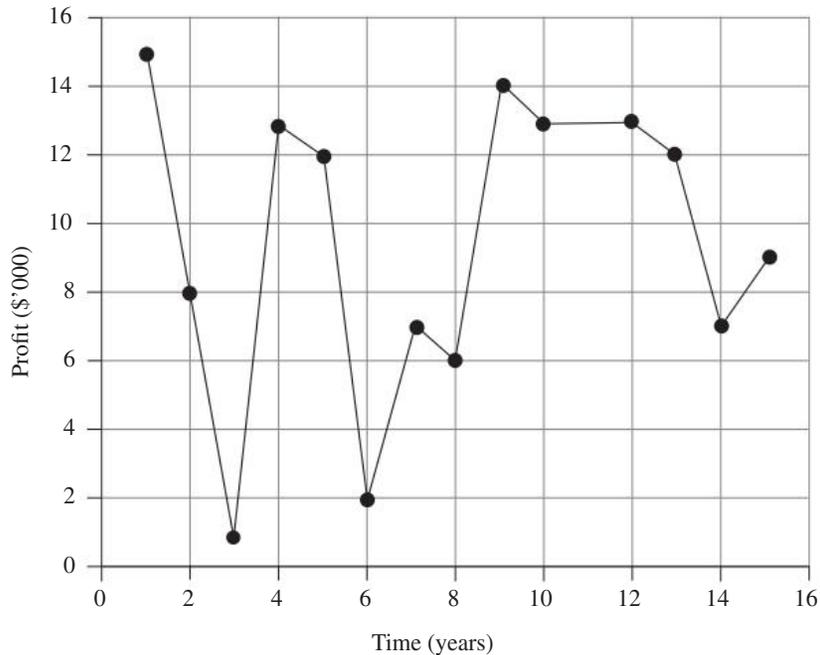
In 2024, the actual number of children (in units of a thousand) visiting in each season were 25.7 units in summer, 19.2 units in autumn, 37.1 units in winter and 20.3 units in spring. Analyse the data to predict the number of children that would visit in summer of 2025.

3.6 Past QCAA exam questions

Question 1 (1 mark)

Source: 2023 QCAA General Mathematics, Paper 1, Section 1, Q2; © State of Queensland (QCAA) 2024

A time series plot is shown.



It could best be described as

- A. cyclical.
- B. seasonal.
- C. irregular.
- D. increasing.

Question 2 (1 mark)

Source: 2023 QCAA General Mathematics, Paper 1, Section 1, Q10; © State of Queensland (QCAA) 2024

Annual sales data and related quarterly indices are shown. The quarterly indices were calculated by applying the average percentage method using the mean.

Quarter	Q1	Q2	Q3	Q4
Sales	160	x	128	200
Index	1.0	0.95	y	1.25

Determine the values for x and y .

	x	y
A.	122	0.8
B.	122	3.2
C.	152	0.8
D.	152	3.2

Question 3 (4 marks)

Source: 2023 QCAA General Mathematics, Paper 2, Section 1, Q2; © State of Queensland (QCAA) 2024

Buffalo fly bites cause skin wounds on cattle. The table shows the average number of skin wounds per animal in a herd for two years.

	Autumn	Winter	Spring	Summer
2021	285	28	195	460
2022	276	22	170	392

Deseasonalise the data.

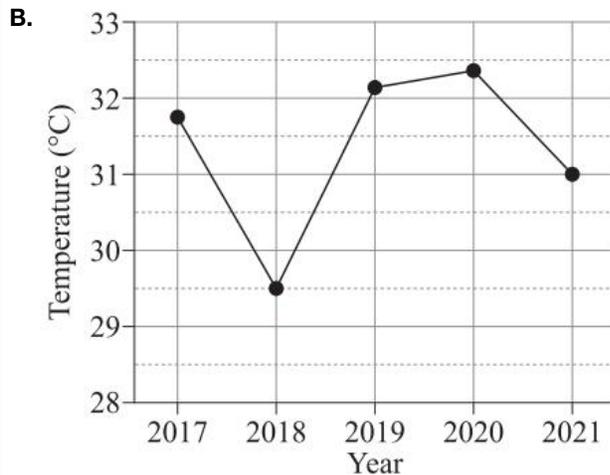
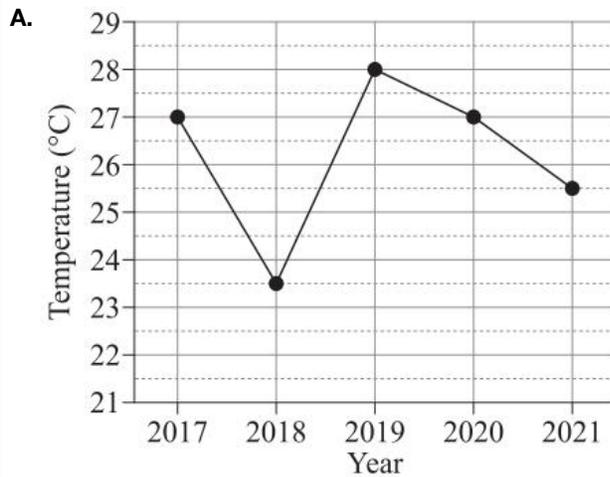
Question 4 (1 mark)

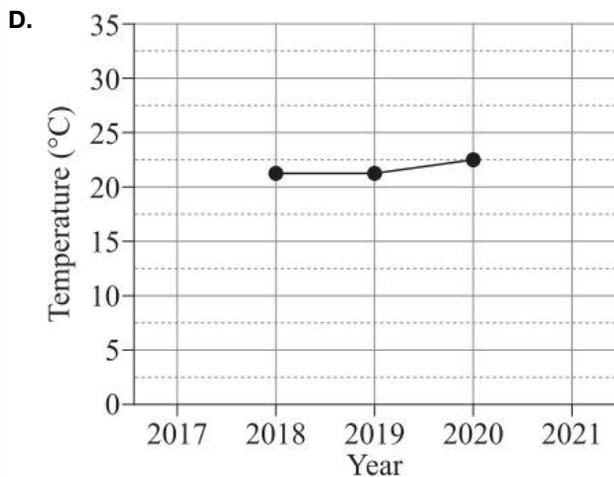
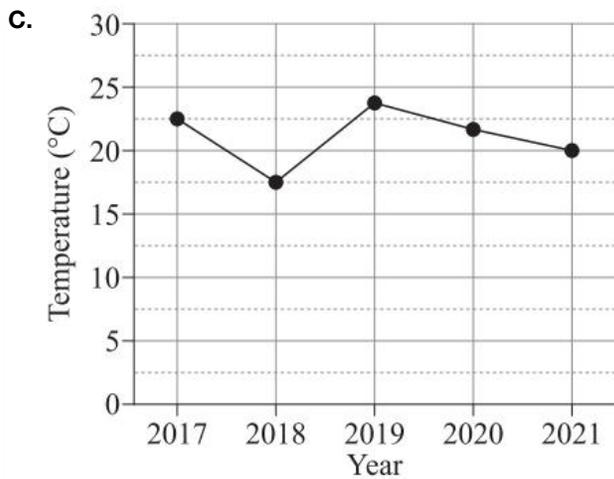
Source: 2022 QCAA General Mathematics, Paper 1, Section 1, Q3; © State of Queensland (QCAA) 2024

The table shows the minimum and maximum temperatures on January 1 each year in Bundaberg.

	Min (°C)	Max (°C)
2017	22.1	31.8
2018	17.8	29.6
2019	24.1	32.1
2020	22.1	32.3
2021	19.9	30.9

Which time series plot best represents the mean temperatures?





Question 5 (4 marks)

Source: 2022 QCAA General Mathematics, Paper 2, Section 1, Q1; © State of Queensland (QCAA) 2024

The table shows a swimwear company's seasonally adjusted swimsuit sales (in thousands).

	Season			
	Spring	Summer	Autumn	Winter
Seasonally adjusted swimsuit sales (in thousands)	33.3	34.8	36.4	35.8

The long-term seasonal indices for spring, summer and winter are 1.11, 1.42 and 0.62 respectively.

Determine the actual swimsuit sales for autumn.

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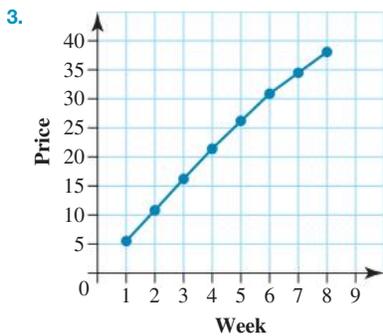
Answers

Chapter 3 Time series analysis

3.2 Constructing and describing time series plots

3.2 Exercise

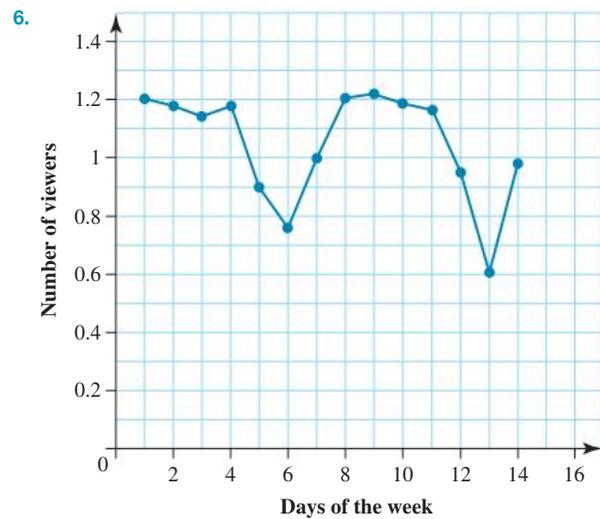
- Increasing trend: values gradually rise over time
Decreasing trend: values gradually decline over time
No trend: fluctuating data with no clear pattern
Seasonal patterns: regular repeating cycles
Irregular fluctuations: unpredictable, unexpected changes
- The plot shows an increasing trend with rising values each quarter.
 - The plot shows a decreasing trend with declining values each month.
 - The plot shows irregular fluctuations with a sudden drop in passengers from August.
 - The plot shows a seasonal pattern with temperature lows at night and peaks at 4 pm daily.
 - The plot shows a seasonal pattern with Q2 troughs and Q4 peaks, along with a long-term decreasing trend in traffic each year.



The plot shows an increasing trend with a long-term rise in share price each week.

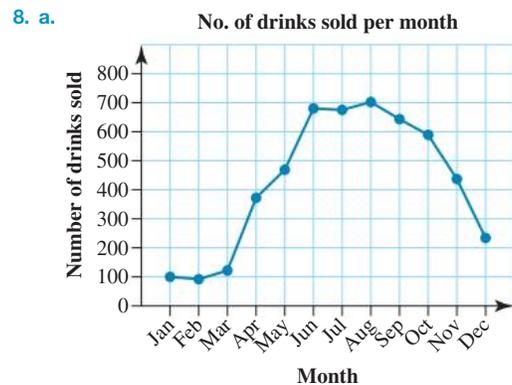
- The plot shows a seasonal pattern with Q1 troughs and Q4 peaks, along with a long-term increase in online sales each year.
- Seasonal
 - Irregular fluctuations
 - Increasing trend

- Irregular fluctuations
- Seasonal

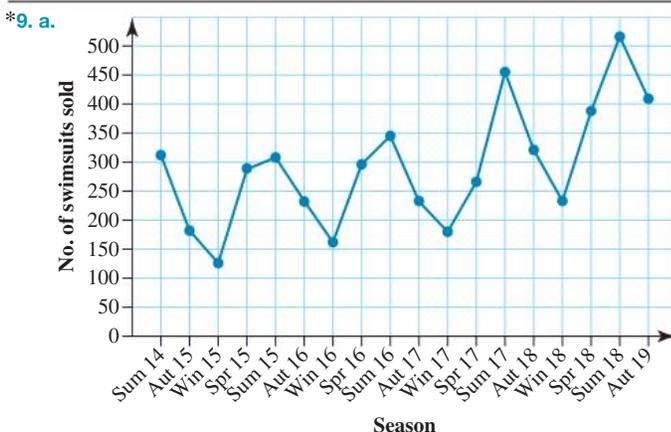


The plot shows a seasonal pattern with troughs on Saturdays and peaks from Monday to Thursday each week.

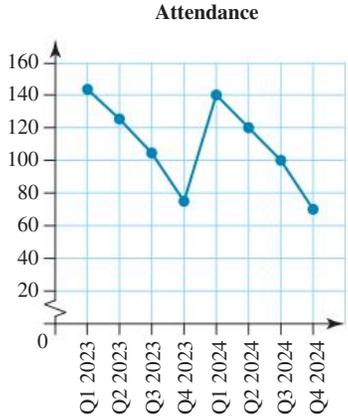
- The plot shows a decreasing trend in alcohol-induced deaths for males from 1997 to 2017, while the female death rate remains stable with minimal fluctuation.



- The plot suggests a seasonal pattern, with higher hot drink consumption in winter, but more data is needed to confirm the trend.
- See the figure at the bottom of the page*



- b. The plot shows a seasonal pattern with winter troughs and summer peaks, along with a long-term increase in swimsuit sales each year.
10. Both plots show seasonal patterns with peaks in summer and troughs in winter, suggesting a correlation between temperature and ice cream sales.
11. a. See the figure at the bottom of the page*
 b. The plot shows irregular fluctuations, with a sudden peak in revenue starting from week 8, leading to ongoing higher revenue, but no consistent pattern.
12. a.



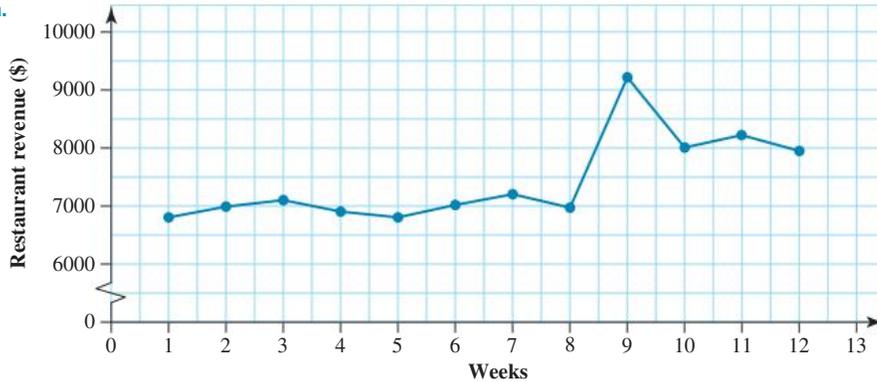
- b. There is a decreasing seasonal trend.
13. The statement is not completely reasonable. A sample response is available in the worked solutions online.
14. See the figure at the bottom of the page*
 The statement is reasonable. A sample response is available in the worked solutions online.

3.3 Fitting a least-squares line to time series data

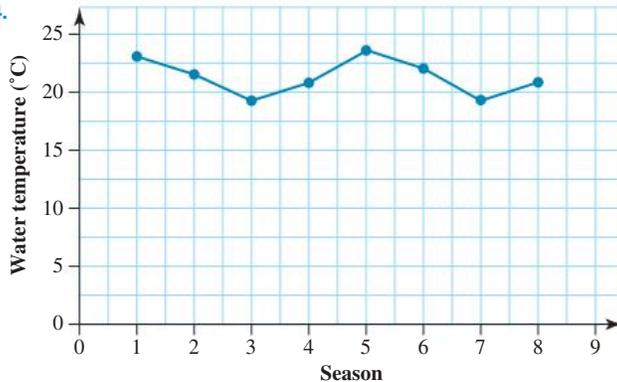
3.3 Exercise

- The least-squares regression line shows the long-term trend by minimising differences between observed and predicted values, smoothing short-term fluctuations for clearer direction and predictions.
- Positive slope → increasing trend
 - Negative slope → decreasing trend
 - Positive slope → increasing trend
 - Negative slope → decreasing trend
 - Negative slope → decreasing trend
 - Stable trend → y remains constant at 3.7.
- $y = 7.55x + 0.73$.
 - Positive, increasing trend.
- $y = -300.61x + 6333.33$.
 - Negative, decreasing trend.
- Negative slope → decreasing sales trend.

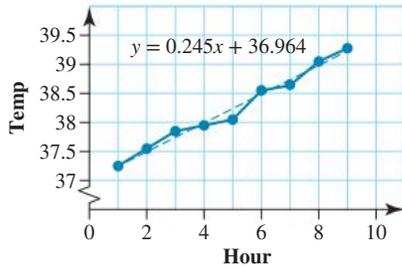
*11. a.



*14.

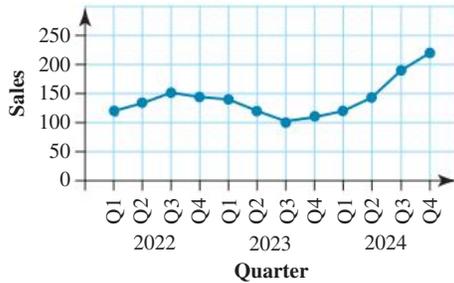


6. a. See the figure at the bottom of the page*
 b. Increasing trend → values rise over time.
 c. $y = 0.245x + 36.964$ where y is temperature of the patient and x is number of hours lapsed.

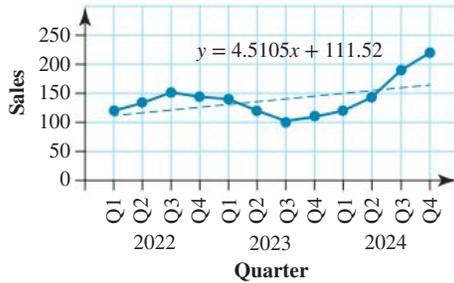


- d. 39.9 °C → Requires monitoring; the model is useful in the short term.

7. a.



- b. Irregular fluctuations, no seasonal pattern; overall increasing trend, especially in late 2019.
 c. $y = 4.5105x + 111.52$ where y is the sales for the quarter in thousands and x is the quarter and year.



- d. \$206 000 → Fits the linear model, reasonable within historical data.

8. a. $y = 0.24x + 2.26$ where y is the share price and x is the number of months lapsed.
 b. \$6.58.
 9. a. $y = -1.78x + 57.42$ where y is monthly sales figures of umbrellas and x is the number of months lapsed.
 b. 34
 10. a. $y = -9.67x + 375.67$
 b. The slope is negative, indicating the attendance is decreasing over time. Therefore, the overall long-term trend is decreasing.
 c. 220 people
 11. \$622 500 → This supports the claim that land value doubles every 10 years within a realistic range.
 12. Milly's claim is reasonable; 2.76 s rounds to 3 s.
 13. The claim is unreasonable. Bettinson Beats surpasses Dexter Dynamics in 2051.
 14. Dec 2026. The solution is reasonable.

3.4 Smoothing time series data using simple moving averages

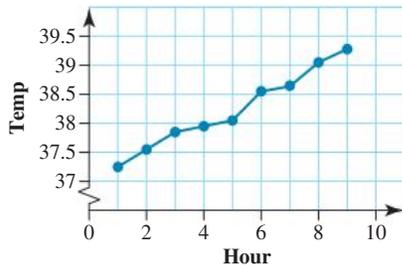
3.4 Exercise

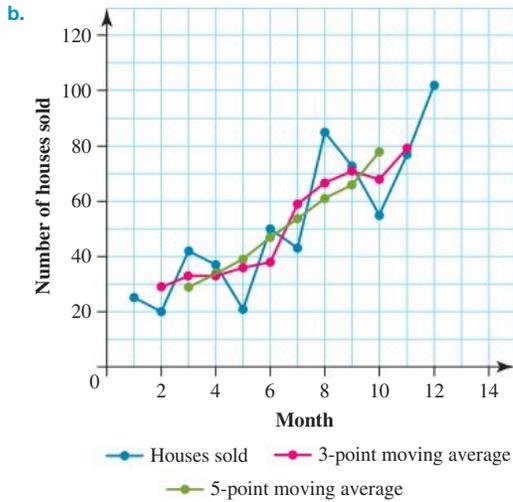
1. Smoothing data reveals long-term trends for better forecasts; a simple moving average uses the mean or median of data points.

2. a.

Month	Houses sold	3-point moving average	5-point moving average
1	25		
2	20	29	
3	42	33	29
4	37	33	34
5	21	36	39
6	50	38	47
7	43	59	54
8	85	67	61
9	72	71	66
10	55	68	78
11	77	78	
12	102		

*6. a.

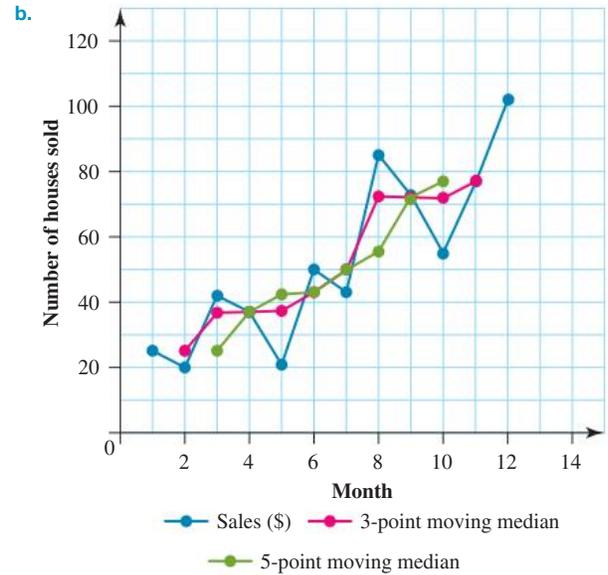




The 3-point moving mean is more sensitive, whereas the 5-point moving mean smooths data better, revealing long-term trends.

3. a.

Month	Number of houses sold	3-point moving median	5-point moving median
1	25		
2	20	25	
3	42	37	25
4	37	37	37
5	21	37	42
6	50	43	43
7	43	50	50
8	85	72	55
9	72	72	72
10	55	72	77
11	77	77	
12	102		



The 3-point median is sensitive to fluctuations, whereas the 5-point median better smooths data, filtering noise and revealing the long-term trend.

4. a. See the figure at the bottom of the page*

*4. a.

	A	B	C	D
1	Month	Houses sold	3-point moving mean	5-point moving mean
2	1	25		
3	2	20	29	
4	3	42	33	29
5	4	37	33	34
6	5	21	36	39
7	6	50	38	47
8	7	43	59	54
9	8	85	67	61
10	9	72	71	66
11	10	55	68	78
12	11	77	78	
13	12	102		

b. See the figure at the bottom of the page*

5. Larger moving averages reduce noise and highlight trends, but may cause lag and lose short-term details.

6.

Day	Sales (\$)	3-point moving average	5-point moving average
1	16		
2	24	19	
3	18	24	31
4	31	38	46
5	66	63	61
6	92	86	81
7	100	102	92
8	115	101	93
9	88	92	
10	72		

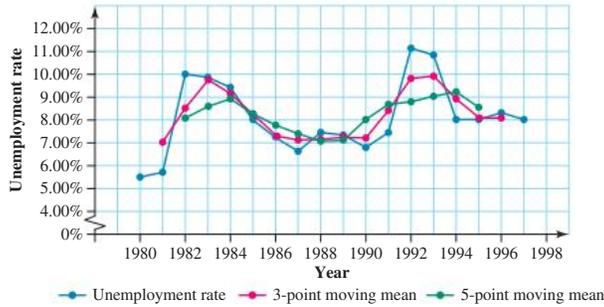
7.

Day	Sales (\$)	3-point moving median	5-point moving median
1	16		
2	24	18	
3	18	24	24
4	31	31	31
5	66	66	66
6	92	92	92
7	100	100	92
8	115	100	92
9	88	88	88
10	72	77	88
11	77	77	
12	102		

*4. b.

C3 : fx = MEDIAN(B2:B4)				
	A	B	C	D
1	Month	Houses sold	3-point moving median	5-point moving median
2	1	25		
3	2	20	25	
4	3	42	37	25
5	4	37	37	37
6	5	21	37	42
7	6	50	43	43
8	7	43	50	50
9	8	85	72	55
10	9	72	72	72
11	10	55	72	77
12	11	77	77	
13	12	102		

8. See the figure at the bottom of the page*



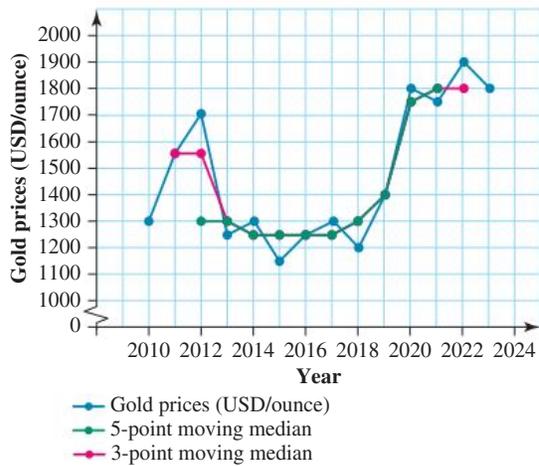
The 3-point moving mean captures the general trend with some variability, whereas the 5-point moving mean smooths the data more, filtering noise but possibly lagging behind changes.

9.

C3 fx = MEDIAN(B2:B4)				
	A	B	C	D
1	Year	Gold prices (USD/ounce)	3-point moving median	5-point moving median
2	2010	1,300		
3	2011	1,550		
4	2012	1,700	1,550	1,300
5	2013	1,250	1,550	1,300
6	2014	1,300	1,300	1,250
7	2015	1,150	1,250	1,250
8	2016	1,250	1,250	1,250
9	2017	1,300	1,250	1,250
10	2018	1,200	1,250	1,300
11	2019	1,400	1,300	1,400
12	2020	1,800	1,400	1,750
13	2021	1,750	1,750	1,800
14	2022	1,900	1,800	
15	2023	1,800	1,800	

*8.

C3 fx = AVERAGE(B2:B4)				
	A	B	C	D
1	Year	Unemployment rate	3-point moving mean	5-point moving mean
2	1980	5.50%		
3	1981	5.70%	7.10%	
4	1982	10.10%	8.60%	8.16%
5	1983	10.00%	9.87%	8.68%
6	1984	9.50%	9.20%	9.00%
7	1985	8.10%	8.30%	8.32%
8	1986	7.30%	7.37%	7.82%
9	1987	6.70%	7.17%	7.40%
10	1988	7.50%	7.20%	7.14%
11	1989	7.40%	7.23%	7.18%
12	1990	6.80%	7.23%	8.08%
13	1991	7.50%	8.50%	8.76%
14	1992	11.20%	9.87%	8.88%
15	1993	10.90%	10.03%	9.12%
16	1994	8.00%	8.97%	9.28%
17	1995	8.00%	8.10%	8.66%
18	1996	8.30%	8.13%	
19	1997	8.10%		

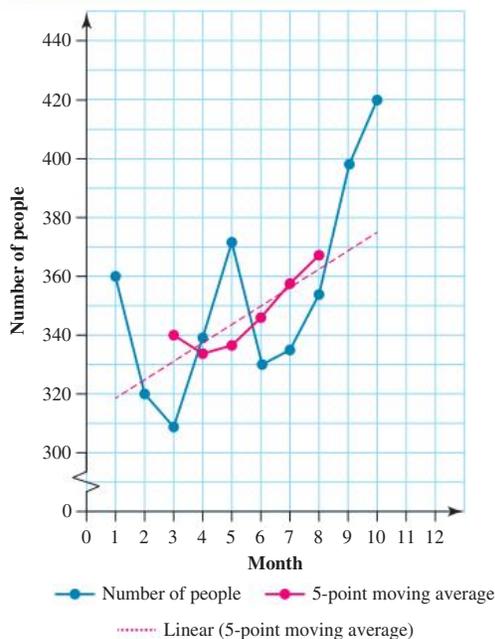


The 3-point moving median reacts quicker to fluctuations, capturing short-term trends but with more volatility. The 5-point moving median smooths data, highlighting long-term trends with less reaction to short-term peaks, such as in 2012.

10. Week 4: 5, Week 8: 6, Week 11: 4.

11.

Day	Number of people	5-point moving mean
1	360	
2	320	
3	309	340
4	339	334
5	371	337
6	330	346
7	335	358
8	354	367
9	398	
10	420	

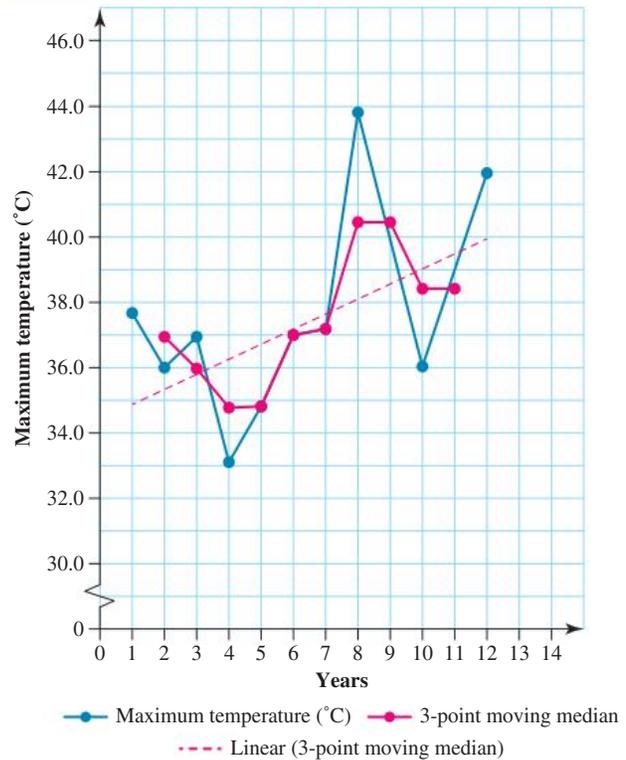


$$y = 6.17x + 313.06.$$

A total of 405 people on day 15 fits the linear model, but the increasing trend may be unrealistic due to capacity limits.

12.

Year	Maximum temperature (°C)	3-point moving median
2006	37.7	
2007	36.0	36.8
2008	36.8	36.0
2009	33.1	34.8
2010	34.8	34.8
2011	37.0	37.0
2012	37.2	37.2
2013	43.6	40.4
2014	40.4	40.4
2015	36.2	38.4
2016	38.4	38.4
2017	41.9	

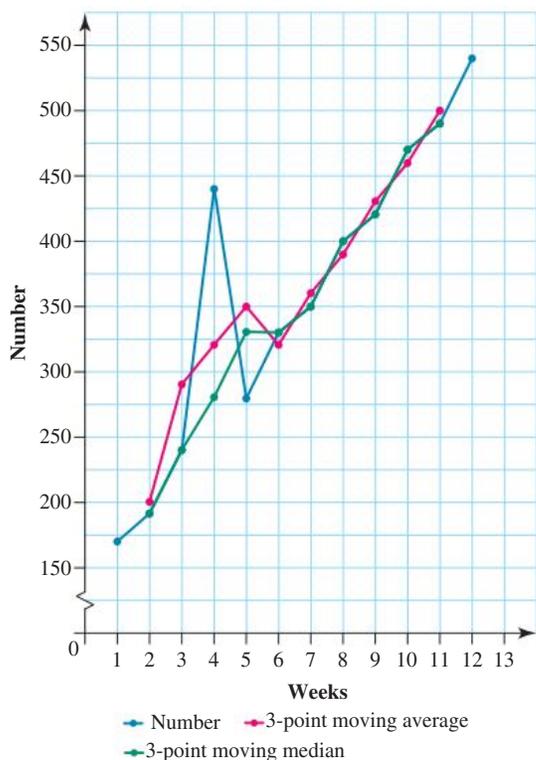


$$y = 0.46x + 34.42.$$

43.6 °C on 31 December 2025 fits the linear model, but a continuous increase may be unrealistic due to seasonal fluctuations and temperature limits.

13. \$396 000 for June aligns with the 3-point median linear model, but the assumption of constant growth may be unrealistic due to economic fluctuations or external factors.

14. The week 4 value of 440 is reasonable, verified algebraically and graphically, though it causes fluctuation in the 3-point moving average and median.



3.5 Deseasonalising time series data

3.5 Exercise

1. a. 240

Quarter	1	2	3	4
Seasonal index	0.85	1.25	1	0.9

- c. Q1: Sales 15% below average
 Q2: Sales 25% above average
 Q3: Sales at average
 Q4: Sales 10% below average

Quarter	Summer	Autumn	Winter	Spring
Seasonal index	1.4	0.8	0.6	1.2

The seasonal indices demonstrate that Tiffany's sales are highest in Summer where the business is operating 40% (1.4) above average and lowest in Winter at 40% (0.6) below average. This is to be expected as ice cream is more popular in warmer seasons.

3. See the table at the bottom of the page*
 4. 1.02
 5. June, August and November have a seasonal index of 0.9.

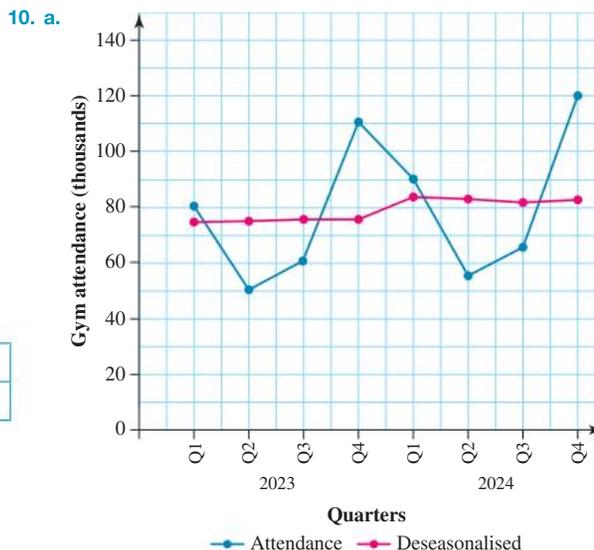
Season	Summer	Autumn	Winter	Spring
Seasonal indices	1.37	0.92	0.70	1.01

Year	Jan	Feb	Mar	Apr	May	Jun
2024	188	186	188	187	188	188
2025	195	193	194	192	193	192

Year	Jul	Aug	Sep	Oct	Nov	Dec
2024	188	188	189	188	188	186
2025	192	191	193	192	193	192

Quarter	Q1	Q2	Q3	Q4
2022	11.946	9.9912	9.048	13.020
2023	11.946	10.985	9.048	14.024
2024	12.932	10.985	10.052	15.028

9. a. 0.77
 b. \$234 500
 c. \$167 942.80



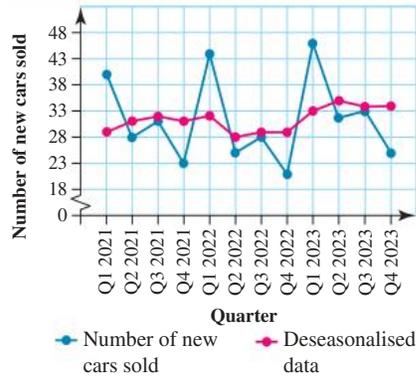
*3.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Seasonal indices	2.01	1.89	1.64	0.97	0.55	0.24	0.18	0.24	0.37	0.73	1.34	1.83

- b. See the figure at the bottom of the page*
 Attendance regression line: $y = 3.8096x + 61.607$
 Deseasonalised regression line: $y = 1.4197x + 72.361$
- c. The gym attendance prediction for Q2 2025 using the original data is 99 702. The gym attendance prediction for Q2 2025 using the deseasonalised data is 57 994. There is a significant difference in both results. Since the prediction using the original data will continue to follow an increasing trend, it does not reflect the seasonality of the data. The prediction calculated for the deseasonalised data that has been re-seasonalised is more reliable because it is in trend with the Q2 values, which are typically between around 50 000 and 60 000 attendees. The prediction of 57 994 is more reasonable.

11.

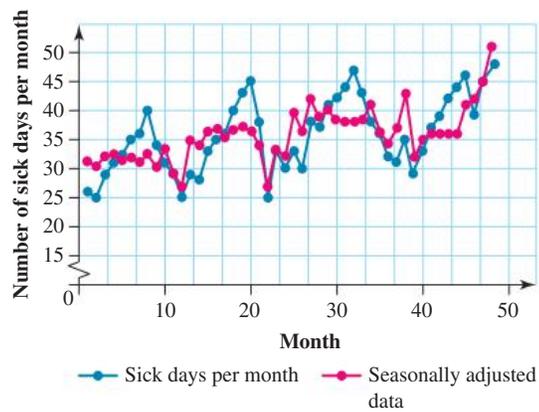
Quarter	2021	2022	2023
1	29	32	33
2	31	28	35
3	32	29	34
4	31	29	34



Deseasonalising the data has smoothed out the fluctuations caused by the seasonal nature of the data.

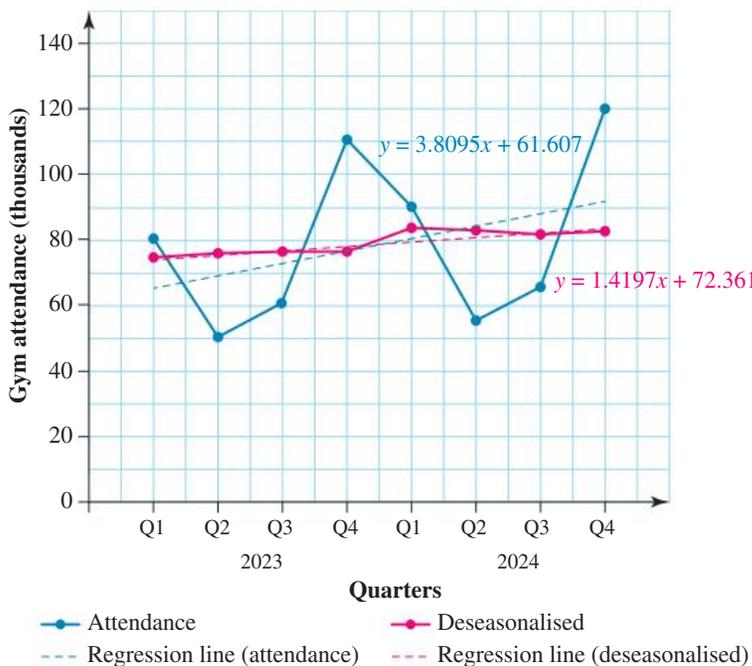
12.

Month	2021	2022	2023	2024
Jan	31	35	40	37
Feb	30	34	36	43
Mar	32	36	42	32
Apr	32	37	39	35
May	31	35	40	36
June	32	37	38	36
July	31	37	38	36
August	32	36	38	36
September	30	34	38	41
October	33	27	41	42
November	29	33	36	45
December	27	32	34	51



Although the deseasonalising has removed much of the seasonal fluctuations, there does seem to be some underlying pattern. There is a very slight upward trend.

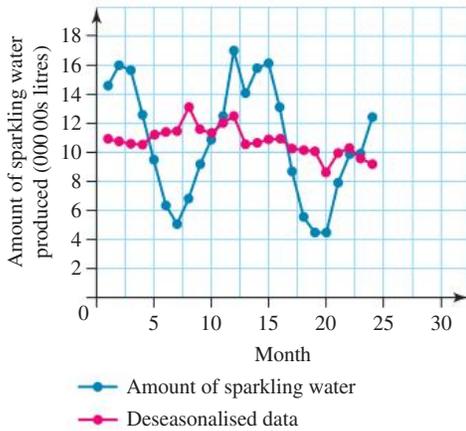
*10. b.



13.

Month	Jan	Feb	Mar	Apr	May	Jun
Seasonal indices	1.3338	1.4787	1.4802	1.1967	0.8446	0.5518
Month	Jul	Aug	Sep	Oct	Nov	Dec
Seasonal indices	0.4451	0.5200	0.7922	0.9651	1.0356	1.3562

Month	Jan	Feb	Mar	Apr	May	Jun
2023	10.9	10.8	10.6	10.5	11.2	11.4
2024	10.6	10.7	10.9	10.9	10.3	10.1
Month	Jul	Aug	Sep	Oct	Nov	Dec
2023	11.5	13.1	11.6	11.3	12.1	12.5
2024	10.1	8.7	10.0	10.3	9.6	9.1



The least-squares regression line is $y = -0.0836x + 11.830$.
 Due to the negative gradient, the long-term trend is that the production of sparkling water is decreasing over time.
 Deseasonalised amount = 9 656 400 L, Actual amount = 14 278 919 L
 This quantity assumes that the production rates decrease linearly as per the model; however, this amount does not account for additional factors that may occur contrary to any seasonal patterns.

14.

Year	Summer	Autumn	Winter	Spring
2023	19.619	18.729	18.987	20.540
2024	22.888	23.412	22.785	22.822
2025	27.466	28.094	28.481	26.246

The predicted amount of guests for 2026 is 121 428.

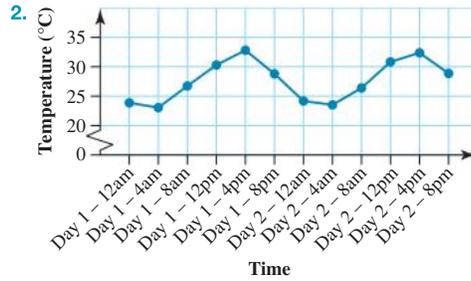
15. The predicted amount of flu medication units sold in winter of 2027 is 3677 units.

3.6 Review

3.6 Exercise

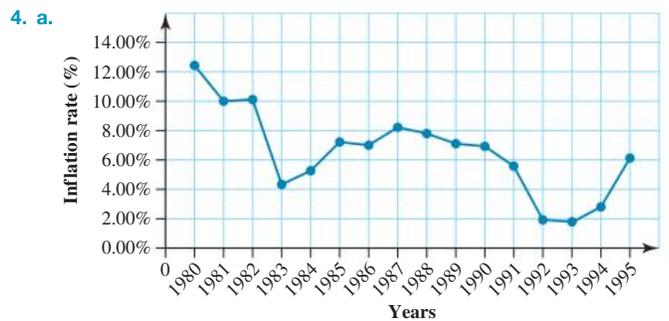
1. a. Decreasing trend, with fewer units produced each quarter.
- b. Irregular fluctuations, with a sudden drop in passengers and no clear pattern.
- c. This time series plot indicates an increasing trend, as shown by the overall long-term increase in measured values over time and the upward trajectory of the graph as the inflation rises as the years progress.

d. Seasonal and long-term increase, with peaks in Q1 and troughs in Q3, showing more traffic each year.

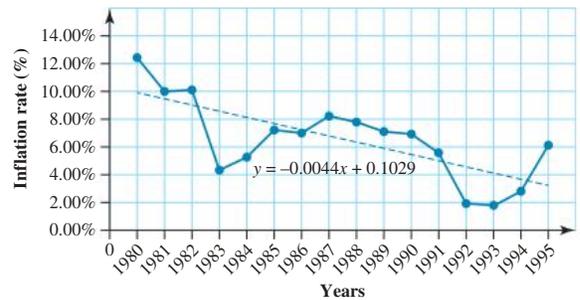


Seasonal pattern, with troughs at 4 am and peaks at 4 pm in daily temperatures.

3. a. $y = -4.9x + 95.7$.
- b. Decreasing trend, with a negative gradient in the regression line showing a downward long-term trajectory.



- b. This time series plot seems to indicate a decreasing trend shown by the downward trajectory of the graph.
- c. $y = 0.0044x + 0.1029$ where y is inflation rate and x is the year.



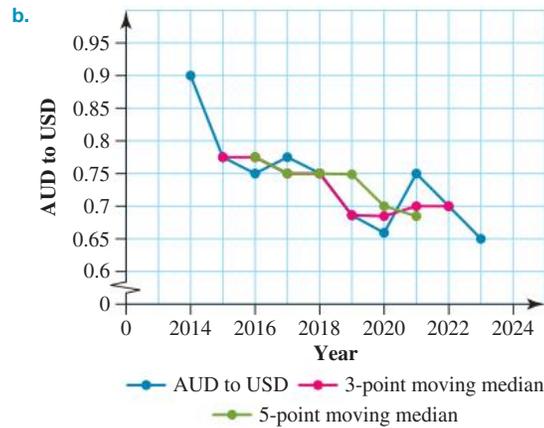
d. The linear model predicts a -1.15% inflation rate in 2005, but indefinite decline is unrealistic due to economic cycles and unpredictable fluctuations.

5.

Year	Number of bicycles sold	3-point moving mean	5-point moving mean	3-point moving median	5-point moving median
2017	7434				
2018	5680	6746		7123	
2019	7123	7123	7608	7123	7434
2020	8567	8308	7859	8567	8567
2021	9234	8830	8636	8690	8690
2022	8690	9164	9329	9234	9234
2023	9567	9615	10 076	9567	9567
2024	10 588	10 818		10 588	
2025	12 300				

6. a.

Year	AUD to USD	3-point moving mean	5-point moving mean
2014	0.9		
2015	0.77	0.81	
2016	0.75	0.76	0.79
2017	0.77	0.76	0.75
2018	0.75	0.74	0.72
2019	0.69	0.70	0.72
2020	0.66	0.70	0.71
2021	0.75	0.70	0.69
2022	0.7	0.70	
2023	0.65		



The 3-point moving median is more sensitive, preserving volatility, whereas the 5-point smooths better, filtering out noise and sudden fluctuations. Both reveal the upward sales trend.

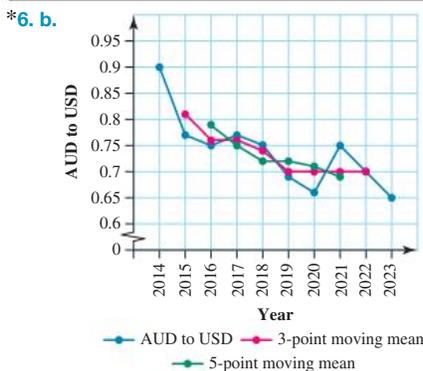
b. See image bottom of the page*
The 3-point moving average is more sensitive to short-term changes, whereas the 5-point smooths data for clearer long-term trends. Differences are subtle in exchange rate data.

7. a.

Year	AUD to USD	3-point moving median	5-point moving median
2014	0.9		
2015	0.77	0.77	
2016	0.75	0.77	0.77
2017	0.77	0.75	0.75
2018	0.75	0.75	0.75
2019	0.69	0.69	0.75
2020	0.66	0.69	0.70
2021	0.75	0.70	0.69
2022	0.7	0.70	
2023	0.65		

8. a.

C3				
fx = AVERAGE(B2:B4)				
	A	B	C	D
1	Year	AUD to USD	3-point moving average	5-point moving average
2	2014	0.9		
3	2015	0.77	0.81	
4	2016	0.75	0.76	0.79
5	2017	0.77	0.76	0.75
6	2018	0.75	0.74	0.72
7	2019	0.69	0.70	0.72
8	2020	0.66	0.70	0.71
9	2021	0.75	0.70	0.69
10	2022	0.7	0.70	
11	2023	0.65		



b.

	A	B	C	D
1	Year	AUD to USD	3-point moving average	5-point moving average
2	2014	0.9		
3	2015	0.77	0.77	
4	2016	0.75	0.77	0.77
5	2017	0.77	0.75	0.75
6	2018	0.75	0.75	0.75
7	2019	0.69	0.69	0.75
8	2020	0.66	0.69	0.70
9	2021	0.75	0.70	0.69
10	2022	0.7	0.70	
11	2023	0.65		

9. a. 2200

b.

Quarter	1	2	3	4
Seasonal index	1.4	0.8	0.6	1.2

10. 373.95

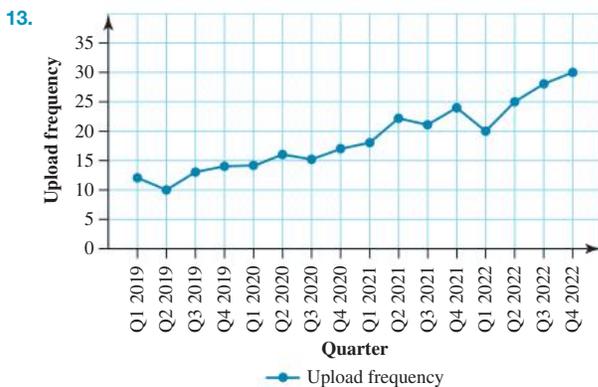
11.

Season	Winter	Spring	Summer	Autumn	Averages
2018	710	530	450	780	618
2019	820	678	546	420	616

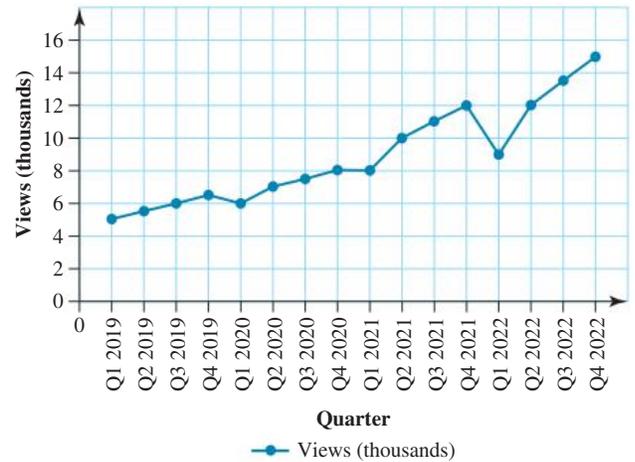
Season	Winter	Spring	Summer	Autumn
2018	1.15	0.86	0.73	1.26
2019	1.33	1.10	0.89	0.68
Seasonal indices	1.24	0.98	0.81	0.97

12.

Season	Winter	Spring	Summer	Autumn
2018	572	541	557	804
2019	661	692	676	433



This time series plot indicates a long-term increasing pattern as shown by the overall long-term increase in measured values over time as the amount of uploads becomes more as each quarter progresses.

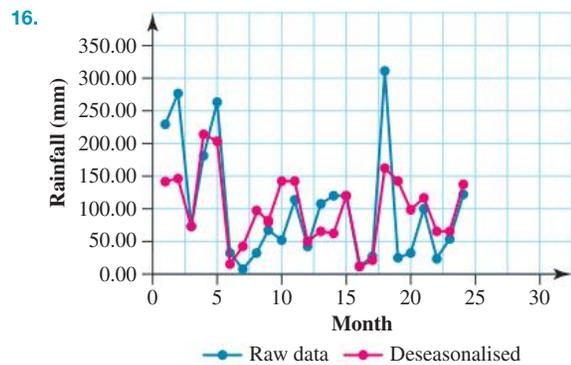


The time series shows a long-term increase in views, with a drop in Q1 2022 due to less uploading. Justin's claim is reasonable, as uploads correlate with views.

14. $y = 3500x + 47\,250$ where y is the revenue and x is the number of years.

The value revenue of the shop in 5 years' time is \$78 750 according to the linear model constructed for the time series data. The shop revenue should be around \$120 000 if this to be doubled by this time. This value is not close enough to for this claim to be reasonable.

15. $y = -206.71x + 4171.7$ where y is the sales and x is the number of months. There were \$451 in sales of takeaway lunches sold at the end of the following financial year (June) according to the linear model from the 3-point moving averages data. The answer is reasonable as it adheres to the overall linear model as calculated with the least-squares regression line. The linear model does, however, assume a continuation of the decreasing trend; the model will only be applicable within certain timeframes that it can account for.



Although the deseasonalising has removed much of the seasonal fluctuations, there does seem to be an existing underlying seasonal pattern. There is a slight downward trend.

17. Month 18 will be the first projected future month in which Maudrene's business will surpass Dennis's by selling more than twice the number of units.
This solution is reasonable according to the developed models.
18. The time for the sixteenth quarter is projected to be 46.64 minutes, according to the linear model derived from the 3-point moving median data. This estimate is reasonable, as it aligns with the overall linear trend calculated using the least-squares regression line and it is less than the 5-point moving median model's value having a steeper gradient hence faster projected time. However, the linear model for the 3-point median assumes a constant, uninterrupted decrease in times, which may not be entirely realistic due to factors such as changes in Beatrice's training routine, potential plateaus in performance, natural fluctuations in race conditions or increased competition in future races.
19. 2026
20. The predicted number of children that would visit in summer of 2025 is 27 080.

3.6 Past QCAA exam questions

1. C
2. C
3. See the table at the bottom of the page*
4. A
5. 30 940 swimsuits

*3.

	Autumn	Winter	Spring	Summer
2021	$\frac{285}{1.2307} = 231.6$	$\frac{28}{0.1090} = 256.9$	$\frac{195}{0.7982} = 244.3$	$\frac{460}{1.862} = 247.0$
2022	$\frac{276}{1.2307} = 224.3$	$\frac{22}{1.090} = 201.8$	$\frac{170}{0.7982} = 213.0$	$\frac{392}{1.862} = 210.5$

4 The arithmetic sequence

LESSON SEQUENCE

4.1 Overview	164
4.2 Using recursion to generate an arithmetic sequence	165
4.3 Using the rule for the n th term of an arithmetic sequence to make predictions	175
4.4 Using arithmetic sequences to model and analyse practical situations	182
4.5 Review	190
Answers	196

Fully worked solutions for this chapter are available online.

EXAM PREPARATION

Access exam-style questions in every lesson, available online and access past QCAA exam questions in every review.

Resources

 Solutions	Solutions — Chapter 4
 Exam questions	Exam question booklet — Chapter 4
 Digital documents	Learning matrix — Chapter 4 Chapter summary — Chapter 4

LESSON

4.1 Overview

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4.1.1 Introduction

Mathematical sequences, such as the famous Fibonacci sequence, play a significant role in various fields, including nature, science, finance and art. The Fibonacci sequence follows the pattern where each term is the sum of the two preceding ones as shown: 1, 1, 2, 3, 5, 8, 13, 21, ...

This recursive pattern is evident in nature, seen in the spirals of pinecones, flower petals and sunflower seed arrangements.

Similarly, the concept of recursion is fundamental in modelling growth and decay in other contexts. These sequences are used to analyse discrete situations, such as simple interest calculations, determining transportation costs based on initial charges and distance travelled, and calculating the value of an item using the straight line method of depreciation. Understanding these patterns is essential for navigating financial concepts related to loans and investments, which are explored in further detail. Together, these mathematical principles illustrate the profound connections between abstract sequences and real-world applications.



4.1.2 Syllabus links

Lesson	Lesson title	Syllabus links
4.2	Using recursion to generate an arithmetic sequence	<ul style="list-style-type: none">○ Use recursion to generate an arithmetic sequence.○ Display the terms of an arithmetic sequence in both tabular and graphical form and demonstrate that arithmetic sequences can be used to model linear growth and decay in discrete situations.
4.3	Using the rule for the n th term of an arithmetic sequence to make predictions	<ul style="list-style-type: none">○ Use the rule for the nth term of an arithmetic sequence.<ul style="list-style-type: none">• $t_n = t_1 + (n - 1)d$ where t_n is nth term, t_1 is first term, n is term number and d is common difference
4.4	Using arithmetic sequences to model and analyse practical situations	<ul style="list-style-type: none">○ Use arithmetic sequences to model and analyse practical situations involving linear growth or decay, e.g. analysing a simple interest loan or investment, calculating a taxi fare based on the flag fall and the charge per kilometre, calculating the value of an item using the straight-line method of depreciation.

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LESSON

4.2 Using recursion to generate an arithmetic sequence

SYLLABUS LINKS

- Use recursion to generate an arithmetic sequence.
- Display the terms of an arithmetic sequence in both tabular and graphical form and demonstrate that arithmetic sequences can be used to model linear growth and decay in discrete situations.

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4.2.1 Sequences

When a tradesperson charges for a job, there is typically a fixed call-out fee followed by an hourly rate. For example, if an electrician has a call-out fee of \$50 and charges \$30 for each hour of work, then beginning from the initial fee, the total cost after each hour would look like this:

\$50, \$80, \$110, \$140, \$170, \$200, . . .



This is called an arithmetic sequence, as each total is found by adding a certain amount to the previous total. In this case, \$30 is added to each successive total.

A **sequence** is an ordered list of numbers that follow a specific pattern. Each number in a sequence is defined as a **term**. It can be seen in the above example that the conventional way to express a sequence is by separating each term by a comma, and if such sequence is a continuous pattern, it will end with an ellipsis, ‘. . .’.

4.2.2 Arithmetic sequences

An arithmetic sequence is a sequence in which the difference between any two successive terms is the same. The first term is referred to as t_1 , and the n th term is t_n , where n is the term number. The successive terms are referred to as $t_1, t_2, t_3, t_4, \dots, t_n$.

The above sequence is an example of an arithmetic sequence because the difference between successive terms is 30. In general, this difference is known as the **common difference**, referred to as d .

The common difference is calculated by subtracting two consecutive terms as shown.

$$d = t_2 - t_1$$

$$d = t_3 - t_2$$

$$d = t_4 - t_3$$

If a sequence has a positive d value, the sequence is increasing; if a sequence has a negative d value, the sequence is decreasing.

Understanding this notation is important because it will allow rules to be constructed that will help calculate any term in a sequence.

Arithmetic sequences can be defined by providing the first term of the sequence and the common difference.

For example, the sequence in the example above can be defined as $t_1 = 50, d = 30$.

WORKED EXAMPLE 1 Identifying an arithmetic sequence

Determine if the following sequences are arithmetic. State t_1 and d .

a. 2, 5, 8, 11, 14, ...

b. 4, -1, -6, -11, -16, ...

c. 3, 5, 9, 17, 33, ...

THINK

a. 1. Calculate the difference between successive terms in the sequence.

2. Identify if the difference between successive terms is constant. If so, then the sequence can be called arithmetic.

b. 1. Calculate the difference between successive terms in the sequence.

2. Identify if the difference between successive terms is constant. If so, then the sequence can be called arithmetic.

c. 1. Calculate the difference between successive terms in the sequence.

2. Identify if the difference between successive terms is constant. If so, then the sequence can be called arithmetic.

WRITE

$$\begin{aligned} \text{a. } t_2 - t_1 &= 5 - 2 \\ &= 3 \end{aligned}$$

$$\begin{aligned} t_3 - t_2 &= 8 - 5 \\ &= 3 \end{aligned}$$

$$\begin{aligned} t_4 - t_3 &= 11 - 8 \\ &= 3 \end{aligned}$$

$$\begin{aligned} t_5 - t_4 &= 14 - 11 \\ &= 3 \end{aligned}$$

There is a common difference between successive terms. Therefore, this sequence is arithmetic; hence, $t_1 = 2$, $d = 3$.

$$\begin{aligned} \text{b. } t_2 - t_1 &= -1 - 4 \\ &= -5 \end{aligned}$$

$$\begin{aligned} t_3 - t_2 &= -6 - (-1) \\ &= -6 + 1 \\ &= -5 \end{aligned}$$

$$\begin{aligned} t_4 - t_3 &= -11 - (-6) \\ &= -11 + 6 \\ &= -5 \end{aligned}$$

$$\begin{aligned} t_5 - t_4 &= -16 - (-11) \\ &= -16 + 11 \\ &= -5 \end{aligned}$$

There is a common difference between successive terms. Therefore this sequence is arithmetic; hence, $t_1 = 4$, $d = -5$.

$$\begin{aligned} \text{c. } t_2 - t_1 &= 5 - 3 \\ &= 2 \end{aligned}$$

$$\begin{aligned} t_3 - t_2 &= 9 - 5 \\ &= 4 \end{aligned}$$

$$\begin{aligned} t_4 - t_3 &= 17 - 9 \\ &= 8 \end{aligned}$$

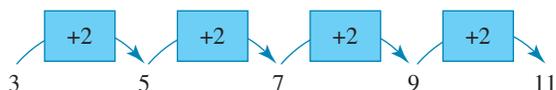
$$\begin{aligned} t_5 - t_4 &= 33 - 17 \\ &= 16 \end{aligned}$$

There is no common difference between successive terms, as the difference is not constant. Therefore, this sequence is not arithmetic; hence, $t_1 = 3$ but d does not exist.

4.2.3 Recursion and the arithmetic sequence

To recur means to occur again periodically or repeatedly. In the context of mathematical sequences, using recursion or a recursive rule allows us to use a term in a sequence to calculate the next term in that sequence. When this is done repeatedly, a sequence of terms is formed.

Consider the arithmetic sequence for which $t_1 = 3$ and $d = 2$.



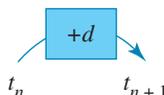
The generated sequence is 3, 5, 7, 9, 11,

Each successive term in the sequence can be calculated by adding the common difference to the previous term. The entire sequence can be generated if t_1 and d are defined.

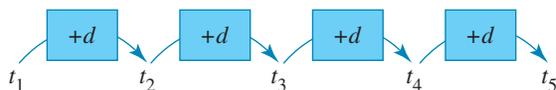
The **recurrence relation** for the above sequence can be defined as:

$$t_1 = 3, t_{n+1} = t_n + 2$$

Notice t_1 is clearly defined and d is represented in the recurrence relation. Remember that t_n denotes the n th term in a sequence. Therefore, t_{n+1} represents the term immediately following t_n , or the next term in the sequence. This can be shown visually as below.



The generated arithmetic sequence will then follow this general pattern.

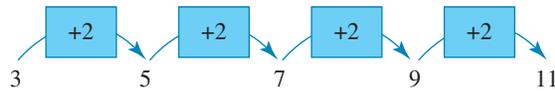


Recurrence relation for an arithmetic sequence

t_1 = the first term, $t_{n+1} = t_n + d$, where d is the common difference.

The diagram below demonstrates how each term is generated by the recurrence relation in the example provided. As can be seen, each successive term is generated by adding the common difference to the previous term.

$$t_1 = 3, t_{n+1} = t_n + 2$$



WORKED EXAMPLE 2 Generating terms in an arithmetic sequence using recursion

Use recursion to generate the first 5 terms of each of the following arithmetic sequences.

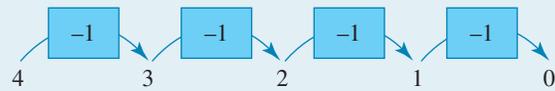
- $t_1 = 4, t_{n+1} = t_n - 1$
- $t_1 = -10, t_{n+1} = t_n + 6$

THINK

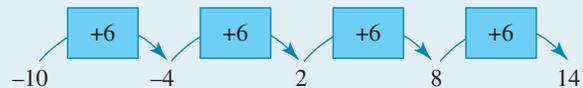
- To generate the terms of the arithmetic sequence recursively, start with $t_1 = 4$ and use $d = -1$ to determine each subsequent term.
- The recursive relation indicates that each term is derived from the previous term by adding 6 beginning with $t_1 = -10$.

WRITE

- The first 5 terms are: 4, 3, 2, 1, 0.



- The first 5 terms are: -10, -4, 2, 8, 14



WORKED EXAMPLE 3 Defining t_1 and d for arithmetic sequences

Define t_1 and d for the following arithmetic sequences.

- 2, 2, 6, 10, 14, ...
- 6, 3, 0, -3, -6, ...

THINK

- Since the sequence is defined as arithmetic, only one pair of terms is required to calculate d . Define t_1 and d .
- Since the sequence is defined as arithmetic, only one pair of terms is required to calculate d . Define t_1 and d .

WRITE

- $$t_2 - t_1 = 2 - (-2)$$

$$= 2 + 2$$

$$= 4$$

$$\therefore t_1 = -2, d = 4$$
- $$t_2 - t_1 = 3 - 6$$

$$= -3$$

$$\therefore t_1 = 6, d = -3$$

4.2.4 Using a scientific calculator to generate terms of an arithmetic sequence

A scientific calculator can be used to generate the terms of an arithmetic sequence.

WORKED EXAMPLE 4 Using a scientific calculator to generate an arithmetic sequence

Use a scientific calculator to generate the first 6 terms of the following arithmetic sequences.

a. $t_1 = -8, t_{n+1} = t_n + 1.5$

b. $t_1 = 150, d = -16$

THINK

a. 1. Type the first term of the sequence into the calculator screen, then press Enter or =.

2. Type +1.5, then press Enter or = 5 times.

3. Write the answer.

b. 1. Type the first term of the sequence into the calculator screen, then press Enter or =.

2. Type -16, then press Enter or = 5 times.

3. Write the answer.

WRITE

a.

-8
-6.5
-5
-3.5
-2
-0.5

The first 6 terms are -8, -6.5, -5, -3.5, -2, -0.5.

b.

150
134
118
102
86
70

The first 6 terms are 150, 134, 118, 102, 86, 70.

4.2.5 Tabular and graphical forms of arithmetic sequences

Table of values

When considering the graphical form of an arithmetic sequence, it is often helpful to first draw a table of values that contains the term number of the sequence, n , in the top row and the term value, t_n , in the bottom row.

Continuing with our generated sequence 3, 5, 7, 9, 11, ..., its tabulated form will be as shown.

Term number (n)	1	2	3	4	5
Term value (t_n)	3	5	7	9	11

The data from this table can be used to generate coordinates that can be plotted on a graph to represent this arithmetic sequence. The coordinates for this sequence are (1, 3), (2, 5), (3, 7), (4, 9) and (5, 11).

Graphs of arithmetic sequences

When graphs of an arithmetic sequence are drawn, the term number, n , is the explanatory variable, so it appears on the x -axis of the graph. The term value, t_n , is the response variable, so it appears on the y -axis of the graph.

Because there is a common difference between the terms of an arithmetic sequence, the relation between the terms is a linear relation. This means that when the terms of an arithmetic sequence are graphed, the points form a straight line.

When the graph has a positive slope, the common difference is a positive number and the relation can be described as a **model of linear growth**. When the graph has a negative slope, the common difference is a negative number and the relation can be described as a **model of linear decay**.

When the graph of an arithmetic sequence is drawn, the line can be extended to determine values of terms in the sequence that have yet to be determined.

WORKED EXAMPLE 5 Creating a table of values and graphing of an arithmetic sequence

An arithmetic sequence is described as follows.

$$t_1 = 7, d = 2$$

- Create a table of values showing the term number and term value for the first 5 terms of the sequence.
- Plot the graph of the sequence.
- Describe this arithmetic sequence as modelling linear growth or linear decay, and justify your response.

THINK

- Set up a table with the term number in the top row and the term value in the bottom row.
 - Use the t_1 and d values to determine the first 5 values of the sequence.
 - Complete the table with the calculated values.
- Use the table of values to identify the points to be plotted.
 - Plot the points on the graph.

WRITE

a.

Term number	1	2	3	4	5
Term value					

$$t_1 = 7$$

$$t_2 = 7 + 2 = 9$$

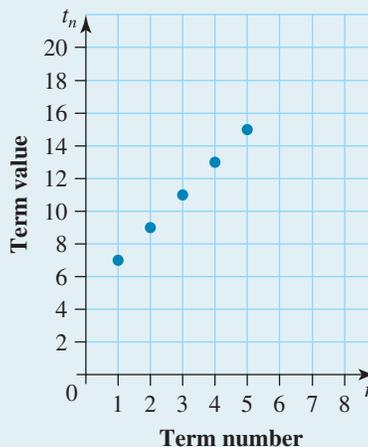
$$t_3 = 9 + 2 = 11$$

$$t_4 = 11 + 2 = 13$$

$$t_5 = 13 + 2 = 15$$

Term number	1	2	3	4	5
Term value	7	9	11	13	15

- b. The points to be plotted are (1, 7), (2, 9), (3, 11), (4, 13) and (5, 15).



- Observe the nature of the arithmetic sequence, in particular the value of its common difference. Positive values for d model linear growth, whereas negative values model linear decay.
- This arithmetic sequence models linear growth, as the common difference is a positive value. This is also indicated by the increasing term values as demonstrated by the graph.

4.2.6 Applying arithmetic sequence to contexts

Recursion can also be used in practical contexts that model linear growth or decay, which are often expressed by arithmetic sequences.

WORKED EXAMPLE 6 Applying arithmetic sequences to contexts

Summer is training for a marathon. The distances she runs (in kilometres) for each consecutive week form an arithmetic sequence with the following pattern:

$$x, 9, y, 15, \dots$$

Calculate the distance Summer will run in the sixth week of her training.

THINK

1. The sequence is arithmetic; hence, d can be determined.

WRITE

The sequence is arithmetic. Hence,

$$\begin{aligned}d &= t_3 - t_2 \\ &= y - 9\end{aligned}$$

$$\begin{aligned}d &= t_4 - t_3 \\ &= 15 - y\end{aligned}$$

Equate to solve for y .

$$y - 9 = 15 - y$$

$$2y = 24$$

$$y = 12$$

Calculate d .

$$\begin{aligned}d &= 15 - y \\ &= 15 - 12 \\ &= 3\end{aligned}$$

2. Since d is determined, calculate t_1 .

$$\begin{aligned}t_1 &= t_2 - d \\ &= 9 - 3 \\ &= 6\end{aligned}$$

$$= 6$$

$$\therefore t_1 = 6, d = 3$$

3. Use recursion to calculate t_6 .
4. Write the answer.

The sequence is 6, 9, 12, 15, 18, 21.

Summer will run 21 km in the sixth week of her training.

Exercise 4.2 Using recursion to generate an arithmetic sequence

4.2 Exercise

4.2 Exam questions **on**

Simple familiar

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12

Complex familiar

13, 14, 15

Complex unfamiliar

16, 17, 18

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Simple familiar

- Demonstrate that the following sequences are arithmetic.
 - 3, 6, 9, 12, 15, ...
 - 1, 6, 11, 16, 21, ...
 - 3, 9, 15, 21, 27, ...
 - 8, 7, 6, 5, 4, ...
 - 5, 2, -1, -4, -7, ...
 - 2, -3, -8, -13, -18, ...
- WE1** Determine if the following sequences are arithmetic. State t_1 and d .
 - 4, 8, 12, 16, 20, ...
 - 5, 1, -3, -7, -11, ...
 - 1, 3, 7, 15, 31, ...
 - 48, 24, 12, 6, 3, ...
 - 15, 10, 5, 0, -5, ...
 - 6, 1, -4, -9, -14, ...
- WE3** Define t_1 and d for the following arithmetic sequences.
 - 2, 5, 8, 11, 14, ...
 - 4, 9, 14, 19, 24, ...
 - 10, 4, -2, -8, -14, ...
 - 30, 20, 10, 0, -10, ...
 - 8, 12, 16, 20, 24, ...
 - 7, -10, -13, -16, -19, ...
- WE2,4** Use your calculator to generate the first 6 terms of the arithmetic sequences defined by the following recurrence relations.
 - $t_1 = -3.6, t_{n+1} = t_n + 5.2$
 - $t_1 = -0.4, d = -0.8$
- You are considering hiring two different handymen for a home repair project. Both handymen charge using the following recurrence rules where t_1 is the cost (in dollars) for a call out fee and d is the amount (in dollars) charged every hour:

Handyman A $t_1 = 20, t_{n+1} = t_n + 60$

Handyman B $t_1 = 80, t_{n+1} = t_n + 40$

Determine the number of hours at which both handymen will charge the same amount. Additionally, determine when each handyman is cheaper to hire. Discuss any limitations to the handyman models.



- Tanya receives \$50 from her grandparents for her birthday. The following week, she decides to save \$5 from her weekly pocket money to add to this gift. She continues to do this each week, following an arithmetic sequence.
 - Use recursion to generate the terms of this arithmetic sequence to show Tanya's savings at the end of each of the first 4 weeks.
 - Discuss if this situation demonstrates linear growth or linear decay.



7. Daniel is an ice sculptor and observes that his statue melts at a consistent rate of $5 \text{ cm}_3/\text{h}$. The volume of ice that remains, in cubic centimetres, can be modelled using the following information.

$$t_1 = 62, d = -5$$

- a. State the initial volume of the ice statue.
b. A shopping centre has hired Daniel to set up one of his ice statues for their full day of late-night trade. Determine the volume of ice that remains after 12 hours of trade using the situation above.
8. Determine the recursive rule for the following arithmetic sequences and any unknown variable(s).
- | | |
|-----------------------------|-----------------------------|
| a. $-5, -2, f, \dots$ | b. $3, f, 15, 21, \dots$ |
| c. $10, f, -10, -20, \dots$ | d. $3, f, g, 15, 19, \dots$ |
| e. $f, 20, 15, g, \dots$ | f. $f, 10, g, 2, -2, \dots$ |
9. Caster is organising a concert and is placing seats in rows. The number of seats in each row follows a pattern. The first row has 12 seats, the second row has 18 seats, the third row has 24 seats and the fourth row has 30 seats.
- a. Demonstrate that the number of seats per row forms an arithmetic sequence.
b. Define t_1 and d for this sequence.
c. Calculate the number of seats in the seventh row.
10. Clara the scientist is measuring the cooling rate of a chemical reaction. The initial temperature is 80° , and the temperature drops to 76° and 72° in the second and third minutes, respectively.
- a. Demonstrate how the temperatures form an arithmetic sequence.
b. Develop a recursive rule for this sequence.
c. Calculate the temperature of the reaction at 6 minutes using your recursive rule.
d. Evaluate the reasonableness of your solution.



11. **WE5** An arithmetic sequence is described as follows.

$$t_1 = 5, d = 10$$

- a. Create a table of values showing the term number and term value for the first 5 terms of the sequence.
b. Plot the graph of the sequence.
c. Describe this arithmetic sequence as modelling linear growth or linear decay, and justify your response.
12. An arithmetic sequence is described by the recurrence relation

$$t_1 = 12.8, t_{n+1} = t_n - 1.6$$

- a. Create a table of values showing the term number and term value for the first 5 terms of the sequence.
b. Plot the graph of the sequence.
c. Describe this arithmetic sequence as modelling linear growth or linear decay, and justify your response.

Complex familiar

13. **WE6** Annie has a savings account where her deposit amounts (in dollars) for each consecutive month follow the arithmetic sequence shown below.

$$x, 51, y, 39, \dots$$

Calculate the amount Annie deposits in her savings account in the seventh month.

14. In a park, the first bench is placed 10 m from the walking path, the second bench is 14 m from the path, and the third bench is 18 m from the path. The benches continue to be spaced out in this manner with a distance of 4 m between consecutive benches. If a decorative fence is installed 2 m between each bench, determine a recursive rule for the distance each decorative fence is from the walking path.
15. In a game of battleship, Brendon places his ships at the following coordinates:
- His aircraft carrier starts at (1, 9) and extends down for five positions.
 - His battleship starts at (2, 4) and extends horizontally right for four positions.
 - His submarine starts at (7, 4) and extends down for three positions.
- Determine the recurrence relation that, if graphed, will allow his opponent to hit all three of Brendon's ships.

Complex unfamiliar

16. Determine the value of the fifth term in the following arithmetic sequence.

$$2x + 1, x - 3, 1 - 4x, \dots$$

17. Farmer Sam is gradually decreasing the amount of water given to his plants each day to help them acclimatise to dryer conditions during a drought season. He decreases the amount each day by 4 L. Due to water restrictions, Sam only allocates 180 L of water for his plants for the next 6 days. Determine the amount of water he must use on the first day.



18. Below is a table of values representing an arithmetic sequence, where some values are known:

Term number (n)	Term value (t_n)
1	x
z	10
4	y
6	22

Using the information provided in the table, determine all possible values for x , y and z .

Fully worked solutions for this chapter are available online.

LESSON

4.3 Using the rule for the n th term of an arithmetic sequence to make predictions

SYLLABUS LINKS

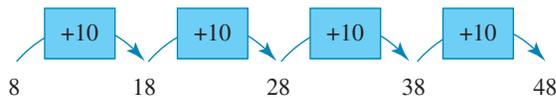
- Use the rule for the n th term of an arithmetic sequence.
 - $t_n = t_1 + (n - 1)d$ where t_n is n th term, t_1 is first term, n is term number and d is common difference

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4.3.1 Developing the rule for the n th term of an arithmetic sequence

The previous lesson explored arithmetic sequences and demonstrated how these quantities can be defined using recursion. However, using recursion can become cumbersome, particularly when calculating terms that are further along in the sequence.

Consider the arithmetic sequence for which $t_1 = 8$ and $d = 10$.



Remember that by recursion, the entire sequence can be defined and generated as shown.

$$t_1 = 8, t_{n+1} = t_n + 10$$

$$t_1 = 8$$

$$t_2 = t_1 + d \quad t_2 = 8 + 10 = 18$$

$$t_3 = t_2 + d \quad t_3 = 18 + 10 = 28$$

$$t_4 = t_3 + d \quad t_4 = 28 + 10 = 38$$

$$t_5 = t_4 + d \quad t_5 = 38 + 10 = 48$$

This system of equations can assist in producing a more efficient rule for calculating the n th term of an arithmetic sequence. By way of substitution, the below equations can be generated with respect to each term but with t_1 and d expressed on the right-hand side.

$$t_1 = 8$$

$$t_2 = 8 + 10$$

$$t_3 = 8 + 10 + 10$$

$$t_4 = 8 + 10 + 10 + 10$$

$$t_5 = 8 + 10 + 10 + 10 + 10$$

$$t_2 = t_1 + d$$

$$t_3 = t_1 + d + d$$

$$t_4 = t_1 + d + d + d$$

$$t_5 = t_1 + d + d + d + d$$

$$t_2 = t_1 + 1d$$

$$t_3 = t_1 + 2d$$

$$t_4 = t_1 + 3d$$

$$t_5 = t_1 + 4d$$

Notice that in the final system of equations, each term contains t_1 and multiples of d . Also notice that the coefficient for each d term in all equations is 1 less than the term number in its equation. This pattern gives rise to the following equation.

$$t_n = t_1 + (n - 1)d$$

Remember that n represents the term number. This rule can be applied to the example above, since it is known that $t_1 = 8$ and $d = 10$.

$$t_n = 8 + (n - 1) \times 10$$

For example, if $n = 4$, then the 4th term is:

$$t_4 = 8 + (4 - 1) \times 10$$

$$t_4 = 8 + 3 \times 10$$

$$t_4 = 38$$

Therefore, the 4th term is 38, as verified in the arithmetic sequence above.

The n th term rule for an arithmetic sequence

$$t_n = t_1 + (n - 1)d$$

where t_n is the n th term, t_1 is the first term and d is the common difference.

WORKED EXAMPLE 7 Determining the rule for an arithmetic sequence

Determine the rules that represent the following arithmetic sequences.

a. 3, 6, 9, 12, 15, ...

b. 40, 33, 26, 19, 12, ...

THINK

WRITE

a. 1. Determine the values of t_1 and d .

$$a. t_1 = 3$$

$$\begin{aligned} d &= t_2 - t_1 \\ &= 6 - 3 \\ &= 3 \end{aligned}$$

2. Substitute the values for t_1 and d into the formula for arithmetic sequences. Expand and simplify.

$$\begin{aligned} t_n &= t_1 + (n - 1)d \\ &= 3 + (n - 1) \times 3 \\ &= 3 + 3(n - 1) \\ &= 3 + 3n - 3 \\ &= 3n \end{aligned}$$

b. 1. Determine the values of t_1 and d .

$$b. t_1 = 40$$

$$\begin{aligned} d &= t_2 - t_1 \\ &= 33 - 40 \\ &= -7 \end{aligned}$$

2. Substitute the values for t_1 and d into the formula for arithmetic sequences. Expand and simplify.

$$\begin{aligned} t_n &= t_1 + (n - 1)d \\ &= 40 + (n - 1) \times -7 \\ &= 40 - 7(n - 1) \\ &= 40 - 7n + 7 \\ &= 47 - 7n \end{aligned}$$

4.3.2 Using the n th term rule for an arithmetic sequence

Using the n th term rule enables the prediction of a term value of an arithmetic sequence provided that the values of t_1 and d are known.

WORKED EXAMPLE 8 Determining a term value in an arithmetic sequence

Determine the 20th term of the following arithmetic sequence.

5, 40, 75, 110, 145, ...

THINK

1. Determine the value of t_1 .
2. Determine the value of d . Since the sequence is defined as arithmetic, only one pair of terms is required to calculate d .
3. Use the rule $t_n = t_1 + (n - 1)d$ where n is 20 for the 20th term.
4. Write the answer.

WRITE

$$\begin{aligned}t_1 &= 5 \\d &= t_2 - t_1 \\&= 40 - 5 \\&= 35 \\t_n &= t_1 + (n - 1)d \\t_{20} &= 5 + (20 - 1) \times 35 \\&= 5 + 19 \times 35 \\&= 670\end{aligned}$$

The 20th term is 670.

4.3.3 Using algebraic techniques with the n th term rule for an arithmetic sequence

The following example demonstrates the use of algebraic techniques with the n th term rule.

WORKED EXAMPLE 9 Solving for unknowns using the n th term rule

For the following arithmetic sequences:

- a. determine the 1st term, given $t_6 = 52$ and $d = 9$
- b. determine the common difference, given that the 1st term is 10 and the 6th term is -20
- c. determine which term number has a value of 205, given that the 1st term is 25 and the common difference is 15.

THINK

- a. 1. Substitute the given values into the n th term rule for an arithmetic sequence.
2. Solve for t_1 .
3. Write the answer.

WRITE

$$\begin{aligned}\text{Since } t_6 &= 52 \text{ and } d = 9, \\t_n &= t_1 + (n - 1)d \\t_6 &= t_1 + (6 - 1)9 \\52 &= t_1 + (5)9 \\52 &= t_1 + 45 \\t_1 &= 7\end{aligned}$$

b. 1. Substitute the given values into the n th term rule for an arithmetic sequence.

Since $t_1 = 10$ and $t_6 = -20$,

2. Solve for d .

$$\begin{aligned}t_n &= t_1 + (n - 1)d \\t_6 &= 10 + (6 - 1)d \\-20 &= 10 + 5d \\5d &= -30\end{aligned}$$

3. Write the answer.

$$d = -6$$

c. 1. Substitute the given values into the n th term rule for an arithmetic sequence.

Since $t_1 = 25$, $d = 15$ and $t_n = 205$,

2. Solve for n .

$$\begin{aligned}t_n &= t_1 + (n - 1)d \\t_n &= 25 + (n - 1)15 \\205 &= 25 + 15(n - 1) \\205 &= 25 + 15n - 15 \\205 &= 15n + 10 \\15n &= 195\end{aligned}$$

3. Write the answer.

$$d = 13$$

The n th term rule for an arithmetic sequence is particularly useful during circumstances when the first term and the common difference are not provided. In such cases, two other terms of the sequence will be provided. Simultaneous equations from Unit 2 will be useful in solving such problems.



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WORKED EXAMPLE 10 Using simultaneous equations to determine a term value

The 3rd term of an arithmetic sequence is -1 and the 5th term is 11 .

a. Determine the rule for the arithmetic sequence.

b. Determine the 50th term of the sequence.

THINK

a. 1. It is known that $t_3 = -1$ and therefore that $n = 3$. Substitute these values in the t_n rule.

2. It is known that $t_5 = 11$ and therefore that $n = 5$.

3. Solve the 2 equations simultaneously using the elimination technique.

Eliminate t_1 by subtracting equation [1] from equation [2]. Solve the resulting equation to determine the value of d .

WRITE

a. $t_3 = -1$; therefore, $n = 3$.

$$\begin{aligned}t_n &= t_1 + (n - 1)d \\t_3 &= t_1 + (3 - 1)d \\-1 &= t_1 + 2d\end{aligned}$$

$t_5 = 11$; therefore, $n = 5$.

$$\begin{aligned}t_n &= t_1 + (n - 1)d \\t_5 &= t_1 + (5 - 1)d \\11 &= t_1 + 4d\end{aligned}$$

$$\begin{aligned}t_1 + 2d &= -1 & [1] \\t_1 + 4d &= 11 & [2] \\2d &= 12 & [2] - [1] \\d &= 6\end{aligned}$$

4. Evaluate t_1 by substituting $d = 6$ into either of the two equations.

Substituting $d = 6$ into [1]:

$$t_1 + 2d = -1$$

$$t_1 + 2(6) = -1$$

$$t_1 = -1 - 12$$

$$t_1 = -13$$

5. To determine the rule, substitute values for t_1 and d into $t_n = t_1 + (n - 1)d$. Expand and simplify.

$$t_n = t_1 + (n - 1)d$$

$$t_n = -13 + (n - 1) \times 6$$

$$t_n = -13 + 6n - 6$$

$$t_n = -19 + 6n$$

- b. 1. To determine the 50th term or t_{50} , substitute $n = 50$ into the rule.

b. $t_n = -19 + 6n$

$$t_{50} = -19 + 6 \times 50$$

$$= -19 + 300$$

$$= 281$$

2. Write the answer.

The 50th term is 281.

Exercise 4.3 Using the rule for the n th term of an arithmetic sequence to make predictions

learn on

4.3 Exercise

4.3 Exam questions

Simple familiar

1, 2, 3, 4, 5, 6, 7,
8, 9

Complex familiar

10, 11, 12

Complex unfamiliar

13, 14, 15

These questions are even better in jacPLUS!

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Simple familiar

1. **WE7** Determine the rules that represent the following arithmetic sequences.

a. 2, 5, 8, 11, 14, ...

b. 3, 11, 19, 27, 35, ...

c. 8, 5, 2, -1, -4, ...

d. -4, -2, 0, 2, 4, ...

e. -10, -13, -16, -19, -22, ...

f. $\frac{7}{2}, \frac{11}{2}, \frac{15}{2}, \frac{19}{2}, \frac{23}{2}, \dots$

2. Determine the first 5 terms of the arithmetic sequences defined by the following rules.

a. $t_n = 2 + 3(n - 1)$

b. $t_n = 10 - 2(n - 1)$

c. $t_n = -5 + 4(n - 1)$

d. $t_n = 6 + 2.5(n - 1)$

e. $t_n = 3n + 2$

f. $t_n = \frac{2}{3}n - \frac{1}{3}$

3. **WE8** For each of the arithmetic sequences given, determine

a. the 12th term of the sequence 1, 3, 5, 7, 9 ...

b. the 27th term of the sequence 12, 11, 10, 9, 8, ...

c. the 23rd term of the sequence -2, 3, 8, 13, 18, ...

d. the 30th term of the sequence -10, -12, -14, -16, -18, ...

e. the 22nd term of the sequence -8, -7, -6, -5, -4, ...

4. **WE9a** Determine the first term in the arithmetic sequences for which:

a. $t_{10} = 20$ and $d = 2$

b. $t_5 = 15$ and $d = 3$

c. $t_{20} = -10$ and $d = -2$

d. $t_{12} = 6$ and $d = -1$

e. $t_{15} = -30$ and $d = 5$

f. $t_8 = -8$ and $d = -4$.

5. **WE9b** Determine the common difference for the arithmetic sequences where:

a. the 1st term is 3 and the 23rd term is 289

b. the 1st term is -15 and the 23rd term is 139

c. the 1st term is 7 and the 30th term is 529

d. the 1st term is -20 and the 12th term is -97

e. the 1st term is -30 and the 14th term is 61

f. the 1st term is 12 and the 27th term is -274 .

6. **WE9c** Answer the following.

a. The 1st term of an arithmetic sequence is 5 and the common difference is 40. Determine which term number has a value of 85.

b. The 1st term of an arithmetic sequence is 40 and the common difference is 12. Determine which term number has a value of 196.

c. For the arithmetic sequence 4, 13, 22, 31, ... determine which term number, t_n , will be equal to 58.

d. For the arithmetic sequence 9, 4.5, 0, ... determine which term number, t_n , will be equal to -18 .

e. For the arithmetic sequence $-60, -49, -38, \dots$ determine which term number, t_n , will be the first to be greater than 10.

f. For the arithmetic sequence 100, 87, 74, ... determine which term number, t_n , will be the first to be less than 58.

7. A batsman made 23 runs in his first innings, 33 in his second and 43 in his third.

If he continued to add

10 runs each innings, write down a rule for the number of runs he would have made in his n th innings.

8. The first fence post in a fence is 12 m from the road, the next is 15.5 m from the road and the next is 19 m from the road. The rest of the fence posts are spaced in this pattern.

a. Write down a rule for the distance of fence post n from the road.

b. If 100 posts are to be erected, determine how far the last post will be from the road.



9. An employee starts a new job with a \$60 000 salary in the first year and the promise of a pay rise of \$2500 at the end of each year.



a. Determine how much her salary will be in the sixth year.

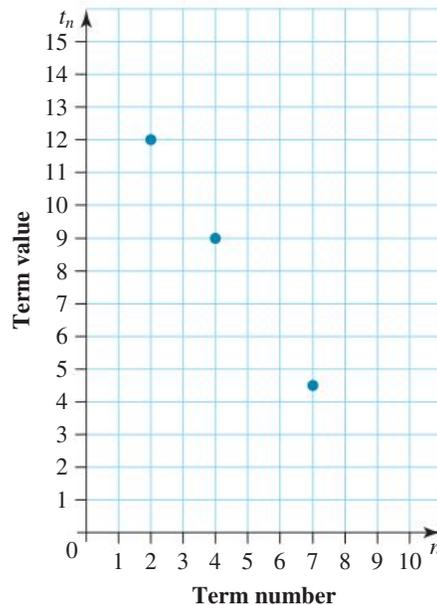
b. Determine how long it will take for her salary to reach \$85 000.

Complex familiar

10. WE10

- The 2nd term of an arithmetic sequence is 13 and the 5th term is 31. Determine the 17th term of this sequence.
- The 2nd term of an arithmetic sequence is -23 and the 5th term is 277. Determine the 20th term of this sequence.
- The 2nd term of an arithmetic sequence is 0 and the 6th term is -8 . Determine the 32nd term of this sequence.
- The 3rd term of an arithmetic sequence is 5 and the 7th term is -19 . Determine the 40th term of this sequence.
- The 4th term of an arithmetic sequence is 2 and the 9th term is -33 . Determine the 26th term of this sequence.
- The 5th term of an arithmetic sequence is 15, and the 10th term is -5 . Determine the 30th term of this sequence.

11. The graph shows some points of an arithmetic sequence.



Determine the value of the 12th term in this sequence.

12. Two arithmetic sequences are given as follows:

- The first sequence has a first term $t_1 = 5$ and a common difference $d = 3$.
- The second sequence has a first term $t_1 = 2$ and a common difference $d = 4$.

Using an algebraic method, determine the term position at which the values of the two sequences are equal. Evaluate the reasonableness of your solution.

Complex unfamiliar

13. Determine the rule for the arithmetic sequence if $t_5 = 2x$, $t_7 = 8x$ and $d = 9$.
14. If the following arithmetic sequence is plotted on a graph, determine the angle, to the nearest degree, that the line makes with the x -axis.

$$-9, -5, -1, 3, 7, 11, 15, 19, 23, 27, \dots$$

15. For an arithmetic sequence, if $t_3 = x - 2$, $t_5 = x + 4$ and $t_7 = 2x - 2$, determine t_{100} .

Fully worked solutions for this chapter are available online.

LESSON

4.4 Using arithmetic sequences to model and analyse practical situations

SYLLABUS LINKS

- Use arithmetic sequences to model and analyse practical situations involving linear growth or decay, e.g. analysing a simple interest loan or investment, calculating a taxi fare based on the flag fall and the charge per kilometre, calculating the value of an item using the straight-line method of depreciation.

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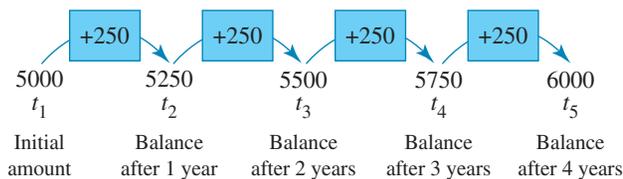
4.4.1 Modelling a simple interest loan or investment

When money is borrowed or invested, and interest is calculated as a percentage of the initial investment or loan, this is known as **simple interest**.

Consider an investment of \$5000 at a simple interest rate of 5% p.a., invested for 4 years.

As 5% of $\$5000 = \frac{5}{100} \times 5000 = \250 , the simple interest paid each year is \$250.

The arithmetic sequence for the first 4 years is illustrated by the following diagram.



The arithmetic sequence is then defined with $t_1 = 5000$ and $d = 250$.

The rule for the n th term in this arithmetic sequence would be as follows.

$$\begin{aligned}t_n &= t_1 + (n - 1)d \\ &= 5000 + 250(n - 1) \\ &= 5000 + 250n - 250 \\ &= 4750 + 250n \\ \therefore t_n &= 4750 + 250n\end{aligned}$$

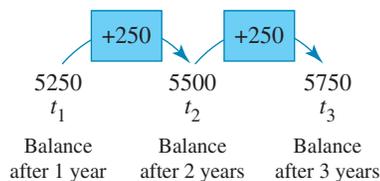
In this rule, n is the term number and t_n is the balance in the account.

Sometimes, however, it is more convenient to construct a different rule for a particular situation.

For example, using the investment figures given above, we might prefer a rule in which the n th term of the sequence is the value of the investment after n years; that is, we would start with t_1 being the balance of the account after 1 year.

In such a case, the sequence would then be determined as follows.

Balance of investment after 1 year = $5000 + 250 = 5250$ (t_1)



The arithmetic sequence is then defined with $t_1 = 5250$ and $d = 250$.

The rule for the n th term in this arithmetic sequence would be determined as follows.

$$\begin{aligned} t_n &= t_1 + (n - 1)d \\ &= 5250 + 250(n - 1) \\ &= 5250 + 250n - 250 \\ &= 5000 + 250n \\ \therefore t_n &= 5000 + 250n \end{aligned}$$

In this rule, n is the number of years lapsed and t_n is the balance in the account.

Note: Variables must be clearly defined whenever a mathematical model is being constructed.



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WORKED EXAMPLE 11 Using the n th term rule for simple interest loan

A small personal simple interest loan of \$1500 is taken out for a duration of 3 years at an interest rate of 6% p.a.

- Determine the rule for the n th term of the sequence defined, where n represents the number of years that have lapsed and t_n represents the amount owed in dollars.
- Calculate the amount owed at the end of 3 years.
- Use the developed rule to calculate the amount owed at the end of 5 years, if the loan continues to be taken out at the same rate.
- Determine how many full years it would take for the amount owed to double.

THINK

1. Calculate the amount of interest earned per year and the amount owing after 1 year.
2. Define t_1 and d .
3. Construct the n th term rule. Expand and simplify.
4. Write the answer.

WRITE

$$\begin{aligned} \text{Interest} &= \frac{6}{100} \times 1500 = \$90 \\ \text{Amount owing after 1 year} &= 1500 + 90 = 1590 \\ \text{Hence, } t_1 &= 1590 \text{ and } d = 90. \\ t_n &= t_1 + (n - 1)d \\ &= 1590 + 90(n - 1) \\ &= 1590 + 90n - 90 \\ &= 1500 + 90n \\ \therefore t_n &= 1500 + 90n \end{aligned}$$

- | | |
|---|--|
| <p>b. 1. Define the substituted value.</p> | Let $n = 3$. |
| 2. Write the n th term rule. | $t_n = 1500 + 90n$ |
| 3. Substitute the value $n = 3$ into the rule. | $t_3 = 1500 + 90(3)$ |
| 4. Simplify. | $= 1500 + 270$
$= 1770$ |
| 5. Write the answer. | The amount owed at the end of 3 years is \$1770. |
| c. 1. Define the substituted value. | Let $n = 5$. |
| 2. Write the n th term rule. | $t_n = 1500 + 90n$ |
| 3. Substitute the value $n = 5$ into the rule. | $t_5 = 1500 + 90(5)$ |
| 4. Simplify. | $= 1500 + 450$
$= 1950$ |
| 5. Write the answer. | The amount owed at the end of 5 years is \$1950. |
| d. 1. Calculate double the amount. | Let $t_n = 1500 \times 2 = 3000$ |
| 2. Write the n th term rule. | $t_n = 1500 + 90n$ |
| 3. Determine n . | $3000 = 1500 + 90n$
$90n = 1500$
$n = 16.67$ |
| 4. Write the answer. Round up due to rounding conventions required in the question. | It takes 17 years for the full amount to be doubled. |

4.4.2 Modelling straight-line depreciation

An asset is an item that has value to its owner. Many assets such as cars and computers lose value over time. This is called **depreciation**.

Straight-line depreciation is where the asset depreciates by a constant amount each year. When this type of depreciation is applied to an asset, the value of the asset over time is modelled by an arithmetic sequence.

After a certain period of time, or when an asset reaches a certain value, the asset is considered to be no longer of any worth. This is called its **scrap value**. At that point, the asset will be sold or sent for recycling.



WORKED EXAMPLE 12 Using the n th term rule for a depreciating asset

A plumber purchases new equipment for a total of \$80 000. The value of the equipment is depreciated by \$7500 per year. The equipment is considered to have a scrap value of \$10 000.



- a. Determine the rule for the n th term of the sequence defined, where n represents the number of years that have lapsed and t_n represents the value of the equipment.
- b. Calculate the future value of the equipment after 5 years.
- c. Calculate the number of years before the equipment reaches its scrap value.

THINK

- a. 1. Calculate the value of the equipment after 1 year.
- 2. Define t_1 and d .
- 3. Construct the n th term rule. Expand and simplify.
- 4. Write the answer.
- b. 1. Define the substituted value.
- 2. Write the n th term rule.
- 3. Substitute the value $n = 5$ into the rule.
- 4. Simplify.
- 5. Write the answer.
- c. 1. Define the scrap value.
- 2. Write the n th term rule.
- 3. Determine n .
- 4. Write the answer. Round up due to rounding conventions required in the question.

WRITE

Value of equipment after
1 year = $80\,000 - 7500 = 72\,500$
Hence, $t_1 = 72\,500$ and $d = -7500$.

$$\begin{aligned} t_n &= t_1 + (n - 1)d \\ &= 72\,500 + (n - 1) - 7500 \\ &= 72\,500 - 7500(n - 1) \\ &= 72\,500 - 7500n + 7500 \\ &= 80\,000 - 7500n \end{aligned}$$

$$\therefore t_n = 80\,000 - 7500n$$

Let $n = 5$.

$$\begin{aligned} t_n &= 80\,000 - 7500n \\ t_5 &= 80\,000 - 7500(5) \\ &= 80\,000 - 37\,500 \\ &= 42\,500 \end{aligned}$$

The value of the equipment at the end of 5 years is \$42 500.

Let $t_n = 10\,000$.

$$\begin{aligned} t_n &= 80\,000 - 7500n \\ 10\,000 &= 80\,000 - 7500n \\ 7500n &= 70\,000 \\ n &= 9.33 \end{aligned}$$

It takes 10 full years for the equipment to reach its scrap value.

4.4.3 Modelling practical situations

There are many situations involving arithmetic sequences that model practical situations. Use recursion or the n th term rule to generate terms of sequences in such situations. Algebraic methods will also be useful in various contexts.



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WORKED EXAMPLE 13 Using the n th term rule to model practical situations

You are considering two different doctors for virtual consultations through their booking app.

- Doctor Lily charges \$60 every 10 minutes and has a \$8 booking administration fee.
- Doctor Sneha charges \$62 every 10 minutes with no administration fee.

Determine which doctor would be more affordable and for how long.

THINK

1. Define n .
2. Calculate the cost to see Doctor Lily after one 10-minute appointment.
3. Define t_1 and d .
4. Construct the n th term rule for Doctor Lily. Expand and simplify.
5. Calculate the cost to see Doctor Sneha after one 10-minute appointment.
6. Define t_1 and d .
7. Construct the n th term rule for Doctor Sneha. Expand and simplify.
8. Use simultaneous equations through substitution to solve for n .
9. Write the n value.
10. Write the answer.

WRITE

Let n be the number of 10-minute appointments for each doctor.

Doctor Lily:

Charge for Doctor Lily after one 10-minute appointment = $60 + 8 = 68$.

Hence, $t_1 = 68$ and $d = 60$.

$$\begin{aligned}t_n &= t_1 + (n - 1)d \\ &= 68 + (n - 1)60 \\ &= 68 + 60n - 60 \\ &= 60n + 8 \quad (1)\end{aligned}$$

Doctor Sneha:

Charge for Doctor Sneha after one 10-minute appointment = 62.

Hence, $t_1 = 62$ and $d = 62$.

$$\begin{aligned}t_n &= t_1 + (n - 1)62 \\ &= 62 + (n - 1)62 \\ &= 62n \quad (2)\end{aligned}$$

Let (1) = (2)

$$60n + 8 = 62n$$

$$2n = 8$$

$$\therefore n = 4$$

At $n = 4$, both doctors charge the same amount.

This is equivalent to a 40-minute appointment. It is cheaper to see Doctor Sneha for any time shorter than 40 minutes, as her rates start with a lower cost progressing linearly towards $n = 4$. Doctor Lily is cheaper to see for any time longer than 40 minutes, as after $n = 4$ her rate will not progress as quickly as Doctor Sneha's.

Exercise 4.4 Using arithmetic sequences to model and analyse practical situations

4.4 Exercise

4.4 Exam questions **on**

Simple familiar

1, 2, 3, 4, 5, 6, 7,
8, 9

Complex familiar

10, 11, 12

Complex unfamiliar

13, 14, 15

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- Access sample responses
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Simple familiar

- WE11** A small personal simple interest loan of \$3800 is taken out for a duration of 4 years at an interest rate of 5.25% p.a.
 - Determine the rule for the n th term of the sequence defined, where n represents the number of years that have lapsed and t_n represents the amount owed in dollars.
 - Calculate the amount owed at the end of 4 years.
 - Use the developed rule to calculate the amount owed at the end of 6 years, if the loan continues to be taken out at the same rate.
 - Determine how many full years it would take for the amount owed to double.
- A taxi company charges a flag fall of \$20 and a rate of \$0.90/km.
 - Write the rule for calculating the cost (t_n) of taking a taxi for a trip of n km.
 - If the distance from the airport to the city is 15 km, determine the total charge for a taxi ride.
- Emily is using a mobile phone plan that includes specific charges for her calls. Each call incurs a flag fall charge of \$0.75, in addition to a per-minute rate of \$0.20 for every minute of the call.
 - Determine the rule for the n th term of the sequence defined, where n represents the number of minutes for a single call and t_n represents the amount that the call costs.
 - Calculate the cost of a 5-minute call.
 - If Emily makes two calls, one lasting 3 minutes and the other lasting 7 minutes, determine the total cost for both calls combined.
 - With a budget of \$10 for calls this month, determine how many 4-minute calls Emily can make without exceeding her budget.
- A graph is drawn to show the growth of an investment of \$6000 at 4.8% p.a. simple interest for each year for 5 years.
 - Explain why this will be a linear graph.
 - Without drawing the graph, calculate the value of the gradient.
 - Describe whether this situation is a linear model of growth or decay.
- Mark has \$5500 to invest at
 - 3% p.a.
 - 3.5% p.a.
 - 3.75% p.a.
 - For each of the interest rate options, determine the rule for the n th term of the sequence, where n represents the number of years that have lapsed and t_n represents the amount of dollars invested.

b. Complete the following table.

Year	0	1	2	3	4	5
3%	\$5500		\$5830		\$6160	\$6325
3.5%		\$5692.50	\$5885	\$6077.50		\$6462.50
3.75%	\$5500	\$5706.25		\$6118.75	\$6325	

- c. Sketch the three graphs of the investment options on the same axes.
 d. Compare and comment on the gradient of the graphs.

6. New exercise equipment was purchased for a gym for \$25 000. The flat rate depreciation for the equipment is \$1750 each year. Complete the following table to calculate the future value of the equipment at the end of 5 years.



Time, n (years)	Depreciation (\$)	Future value (\$)
0	0	25 000
1		
2		
3		
4		
5		

7. **WE12** An airline purchases an airplane for \$60 million. The aeroplane depreciates by \$4.5 million each year. The airplane is considered to have a scrap value of \$24 million.
- Calculate the future value of the airplane after 5 years.
 - Calculate the number of years before the airplane reaches its scrap value.
8. The rule $t_n = 50\,000 - 6000n$ gives the value, t_n , of a car when it is n years old.
- Plot a graph of t_n against n , for $n = 0, 1, 2, 3, 4, 5$.
 - Use your graph to determine after how many years (to the nearest year) the car reaches a scrap value of \$0.
9. Calculate the annual amount of depreciation in an asset that depreciates
- from \$20 000 to \$4000 in 4 years
 - from \$175 000 to \$50 000 in 10 years
 - from \$430 000 to \$299 500 in 9 years.

Complex familiar

10. **WE13** Russell and Tina are considering two different taxi companies for their travels. Company A charges \$2 per kilometre with a \$2 flag fall charge. Company B charges \$1.50 per kilometre with a \$4 flag fall charge. Determine what distance each company is cheaper to travel with.



11. Terry and Sarah are comparing simple interest loans. Terry takes out a personal simple interest loan of \$3 200 at an interest rate of 5% p.a. Sarah takes out a personal simple interest loan of \$3 500 at an interest rate of 4% p.a. Determine when they both will be owing the same amount and how much that would be.
12. A retail store has noticed that they are selling a consistent number of units each month. After 4 months, the inventory decreases to 1100 units, and after 8 months, it drops to 900 items. If the store wants to maintain at least 600 items in stock, determine for how many months from when demand began to decrease they can continue to sell items at the same rate as they are currently selling them.

Complex unfamiliar

13. Linh invested \$1500 in an insurance company bond that offers 12% p.a. simple interest, provided she holds the bond for 5 years. However, the insurance company allows early withdrawals, paying only 6% p.a. simple interest on any withdrawn amount over the period it was invested. The remaining balance in the bond continues to earn the original agreed interest rate.
Linh needed \$700 for repairs to her car 2 years after she has invested the money but chose to leave the rest of her investment in the bond for the full 5 years. Determine the total amount of money Linh will have at the end of 5 years.
14. Lucy attends a musical instrument auction and inspects two trumpets of the same brand and model. One trumpet is 5 years old and is priced at \$2500, while the other trumpet is 8 years old and is priced at \$400. Lucy assumes that both trumpets depreciate at the same rate and that an arithmetic model can be used to calculate the original value of the trumpets. Determine the original value of the trumpets and evaluate the reasonableness of your solution.
15. The following shows the depreciating value of Computer A.

Age (years)	Computer A Value (\$)
0	4500
1	3900
2	3300
3	2700

Victor knows that the value of Computer B after 6 terms is \$400 more than Computer A after 4 terms. If both computers depreciate at the same rate, determine the age of Computer B when it has a scrap value of \$100.



Fully worked solutions for this chapter are available online.

LESSON

4.5 Review

4.5.1 Summary

Hey students! Now that it's time to revise this chapter, go online to:



Access the chapter summary



Review your results



Practise exam questions

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4.5 Exercise

learnon

4.5 Exercise

4.5 QCAA questions **on**

Simple familiar

1, 2, 3, 4, 5, 6, 7, 8,
9, 10, 11, 12

Complex familiar

13, 14, 15, 16

Complex unfamiliar

17, 18, 19, 20

These questions are even better in jacPLUS!

- Receive immediate feedback
- Access sample responses
- Track results and progress



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Simple familiar

- Demonstrate that the following sequences are arithmetic. State t_1 and d .
 - 3, 6, 9, 12, 15, ...
 - 1, 6, 11, 16, 21, ...
 - 5, 2, -1, -4, -7, ...
- MC** Determine which one of the sequences below is generated by the following.

$$t_1 = -5, d = -6$$

- 5, -6, -7, -8, ...
 - 6, -1, 4, 9, ...
 - 5, -11, -17, -23, ...
 - 5, 1, 7, 13, ...
- The following table shows the first 6 terms of an arithmetic sequence.

n	t_n
1	2
2	9
3	
4	23
5	
6	37

- Determine the two missing values.
- Plot the graph of the sequence in part **a**. Describe this arithmetic sequence as modelling linear growth or linear decay, and justify your response.

4. **MC** For the arithmetic sequence, $-1, 1, 3, 5, 7, \dots$ the value of t_1 , the value of d and the rule for the sequence are given respectively as:
- A. $t_1 = -1, d = 2, t_n = -3 + 2n$.
 B. $t_1 = -1, d = 2, t_n = -3 - n$.
 C. $t_1 = 1, d = -1, t_n = -3 + 2n$.
 D. $t_1 = 2, d = -1, t_n = -3 + 2n$.
5. **MC** The 43rd term of the arithmetic sequence $-7, 2, 11, 20, 29, \dots$ is:
- A. -327 . B. -243 . C. 371 . D. 380 .
6. **MC** The 1st term of an arithmetic sequence is 5.3 and 7th term is -1.3 . The value of the common difference is:
- A. -0.5 B. -1.1 C. -0.8 D. -1.5
7. **MC** If Wai Keet had $\$63$ in his bank account and earned 9% p.a. over 3 years, the simple interest earned would be:
- A. $\$17.01$ B. $\$5.67$ C. $\$22.68$ D. $\$80.01$
8. **MC** Maclay invested $\$160$ in a bank for 6 years, earning 8% simple interest each year. At the end of 6 years, he will receive in total:
- A. $\$76.80$ B. $\$768$ C. $\$928$ D. $\$236.80$
9. **MC** Sharlene was charged $\$16$ for a taxi ride that cost $\$3.50$ per kilometre of travel and a $\$2$ flag-fall fee. Determine the distance she travelled in kilometres.
- A. 2 B. 3 C. 4 D. 5
10. **MC** Identify which one of the following tables shows straight-line depreciation.

A.

Age (years)	Value (\$)
New (0)	4000
1	3600
2	3240
3	2916
4	2624
5	2362

B.

Age (years)	Value (\$)
New (0)	4000
1	3600
2	3200
3	2800
4	2400
5	2000

C.

Age (years)	Value (\$)
New (0)	4000
1	3600
2	3300
3	3100
4	3000
5	2950

D.

Age (years)	Value (\$)
New (0)	4000
1	3000
2	2500
3	1500
4	1000
5	500

11. **MC** A restaurant purchased a new industrial-grade dishwasher for \$8500. The dishwasher depreciates according to a straight-line model at \$400 each year. Determine how many years will have lapsed when it reaches a scrap value of \$200.

- A. 18
- B. 19
- C. 20
- D. 21



12. **MC** A photocopier was purchased for \$5400. It depreciates at a rate of \$122.20 per million pages copied. If 8 million pages were printed in the first year, the value of the copier at the end of that year would be:

- A. \$4422.40
- B. \$3123.56
- C. \$2154.89
- D. \$2322.44

Complex familiar

13. Richard runs a small business where his monthly revenue amounts (in dollars) follow an arithmetic sequence as shown below for each consecutive month:

$$a, 90, b, 70, \dots$$

Determine the recurrence rule for Richard's situation and calculate the amount of revenue he generated in his business in the eighth month. Evaluate the success of his business.

14. **a.** The 4th term of an arithmetic sequence is 10, and the 9th term is 5. Determine the 15th term of this sequence.
- b.** The 2nd term of an arithmetic sequence is 15, and the 6th term is 30. Determine the 20th term of this sequence.
15. Nick is considering two different gyms for a fitness program through their membership plans.
- Gym A charges \$45 per month and has a one-time registration fee of \$15.
 - Gym B charges \$50 per month with no registration fee.
- If a person plans to sign up for a fitness program for 6 months or more, determine which gym would be more affordable and at what point this gym would become more expensive than the other gym. Provide algebraic justification.
16. Anna manages a bookstore. Due to a declining interest in physical books, she sells a consistent number of books each month without replenishing her stock. After 5 months, the collection decreases to 2350 books, and after 10 months, it drops to 2100 books. If Anna plans to transition her business online when the inventory reaches 1800 books, determine for how many more months she can continue selling books at the current rate before reaching this threshold.

Complex unfamiliar

17. Determine the value of the 6th term in the following arithmetic sequence.

$$x - 3, x + 3, 3 - 2x, \dots$$

18. Stacey invested money in a simple interest account. After 1 year there was \$848 in the account and after 4 years there was \$992.00. Calculate the interest rate per annum for this situation.

19. At the beginning of 2012, Ashley purchased an equipment van for \$45 100. The van depreciates according to the straight-line method at \$2700 per year. After 5 years, Ashley decided to upgrade to a newer model and is considering selling the old van to a buyer who has agreed to purchase it for \$33 825. Ashley claims that this is a good deal and that he would be getting 5% more than the value of the van according to the depreciation schedule. Evaluate the reasonableness of the claim.

20. At 7:00 pm, Jimmy, a forensic scientist, recorded the temperature of a deceased individual, finding it to be 33 °C. A subsequent measurement at 10:00 pm indicated a temperature of 31.5 °C. Given that the average human body temperature is typically around 36.5 °C, determine the approximate time of death, assuming that the cooling rate of the body follows an arithmetic sequence.

4.5 Past QCAA exam questions

Question 1 (1 mark)

Source: QCAA 2023 General Mathematics, Paper 1, Section 1, Q9; © State of Queensland (QCAA) 2024

Determine the 6th term of the arithmetic sequence that begins 3, 9, ...

- A. 21
- B. 33
- C. 45
- D. 729

Question 2 (3 marks)

Source: QCAA 2024 General Mathematics, Paper 1, Section 2, Q16; © State of Queensland (QCAA) 2024

The number of seats in each row of a theatre forms the terms of the arithmetic sequence

$$t_{n+1} = t_n + 8, \text{ where } t_1 = 25.$$

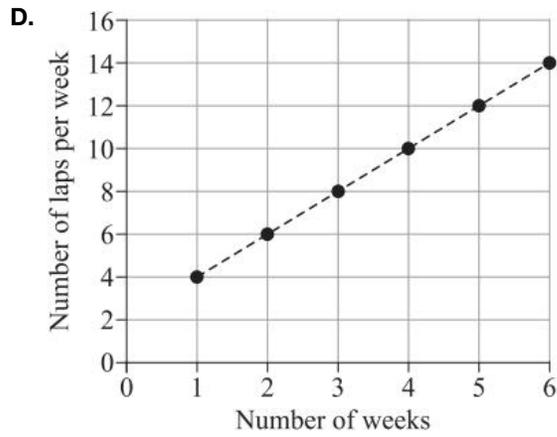
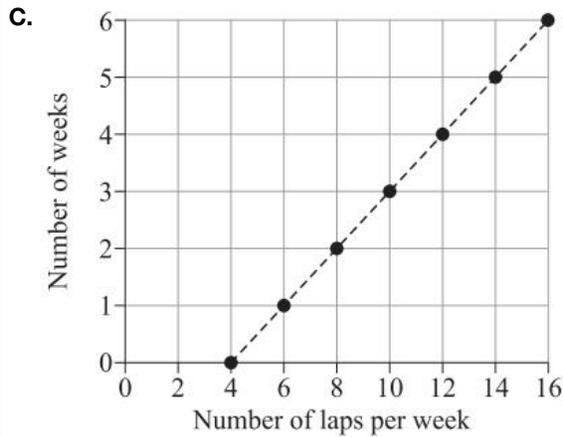
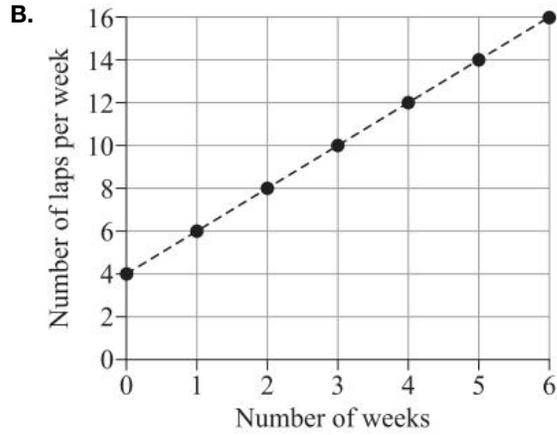
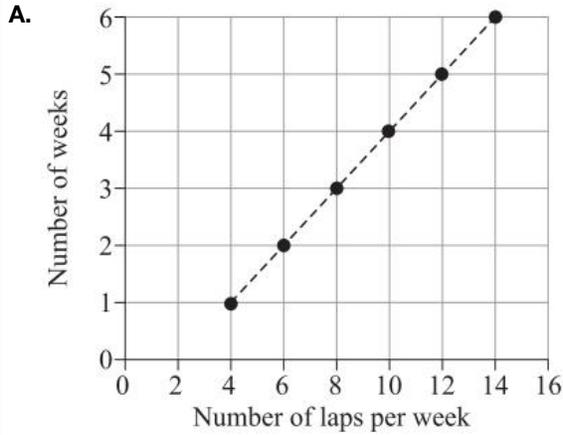
- a. How many seats are in the second row of the theatre? [1 mark]
- b. Complete the table and then calculate the total number of seats in the first four rows of the theatre. [2 marks]

Row	1	2	3	4
Number of seats				

Question 3 (1 mark)

Source: QCAA 2022 General Mathematics, Paper 1, Section 1, Q4; © State of Queensland (QCAA) 2024

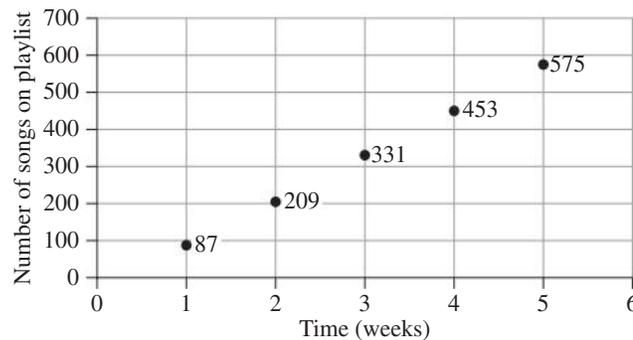
A swimmer has a weekly training routine to improve their fitness as modelled by the recursive function $T_{n+1} = T_n + 2$, where T_n is the number of laps they swim in week n and $T_1 = 4$. Which graph best represents the swimmer's routine?



Question 4 (4 marks)

Source: QCAA 2022 General Mathematics, Paper 1, Section 2, Q18; © State of Queensland (QCAA) 2024

The number of songs on a person's playlist, n , in each week since joining a music streaming service, t , forms an arithmetic sequence, as shown by the graph.



Use the arithmetic sequence to predict the number of songs on this person's playlist 25 weeks after joining the streaming service.

Question 5 (7 marks)

Source: QCAA 2022 General Mathematics, Paper 2, Section 1, Q6; © State of Queensland (QCAA) 2024

The first three lines in a pattern have the equations given. Their slopes form the terms of one sequence and their y -intercepts form the terms of another sequence. Each sequence is either arithmetic or geometric.

Line 1: $y = -0.8x + 1.2$

Line 2: $y = 0.4x + 2.7$

Line 3: $y = -0.2x + 4.2$

Determine the coordinates of the point where Line 5 in the pattern intersects Line 1.

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Answers

Chapter 4 The arithmetic sequence

4.2 Using recursion to generate an arithmetic sequence

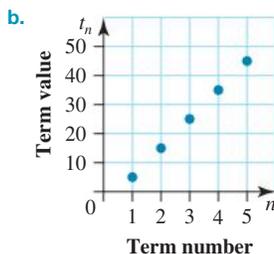
4.2 Exercise

- Arithmetic $d = 3$
 - Arithmetic $d = 5$
 - Arithmetic $d = 6$
 - Arithmetic $d = -1$
 - Arithmetic $d = -3$
 - Arithmetic $d = -5$
- Arithmetic $t_1 = 4, d = 4$
 - Arithmetic $t_1 = 5, d = -4$
 - Not arithmetic
 - Not arithmetic
 - Arithmetic $t_1 = 15, d = -5$
 - Arithmetic $t_1 = 6, d = -5$
- $t_1 = 2, d = 3$
 - $t_1 = 4, d = 5$
 - $t_1 = 10, d = -6$
 - $t_1 = 30, d = -10$
 - $t_1 = 8, d = 4$
 - $t_1 = -7, d = -3$
- $-3.6, 1.6, 6.8, 12, 17.2, 22.4$
 - $-0.4, -1.2, -2, -2.8, -3.6, -4.4$
- 4 hours. Handyman A is cheaper to hire for jobs shorter than 4 hours; handyman B is cheaper to hire for jobs longer than 4 hours. The model is limited to work that is required within a day. If the hours worked will exceed this timeframe an additional call out charge is required unless work is conducted beyond a workday. The model does not account for addition call out charges.
- 50, 55, 60, 65
 - This model demonstrates linear growth as there is a common difference between terms and this value is positive.
- 62 cm^3
 - 2 cm^3
- $t_1 = -5, t_{n+1} = t_n + 3, f = 1$
 - $t_1 = 3, t_{n+1} = t_n + 6, f = 9$
 - $t_1 = 10, t_{n+1} = t_n - 10, f = 0$
 - $t_1 = 3, t_{n+1} = t_n + 4, f = 7, g = 11$
 - $f = 25, g = 10, t_1 = 25, t_{n+1} = t_n - 5$
 - $f = 14, g = 6, t_1 = 14, t_{n+1} = t_n - 4$
- Arithmetic $d = 6$
 - $t_1 = 12, d = 6$
 - 48

- Arithmetic $d = -4$
 - $t_1 = 80, t_{n+1} = t_n - 4$
 - 60 degrees
 - The model is reasonable as it adheres to the trending pattern of the arithmetic sequence presented. The use of the rule may not consider various external factors that could influence the temperature.

11. a.

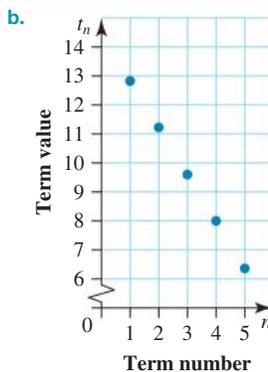
Term number	1	2	3	4	5
Term value	5	15	25	35	45



- This model demonstrates linear growth as there is a positive common difference between terms. This is also indicated by the increase in term values as demonstrated by the graph.

12. a.

Term number	1	2	3	4	5
Term value	12.8	11.2	9.6	8	6.4



- This model demonstrates linear decay as there is a negative common difference between terms. This is also indicated by the decreasing term values as demonstrated by the graph.

- \$21
- $t_1 = 12, t_{n+1} = t_n + 4$
- $t_1 = 8, t_{n+1} = t_n - 1$
- $t_1 = 6, d = 4, t_5 = 22$
- 40 L
- When $z = 2, y = 16, x = 7$.
When $z = 3, y = 14, x = 2$.

4.3 Using the rule for the n th term of an arithmetic sequence to make predictions

4.3 Exercise

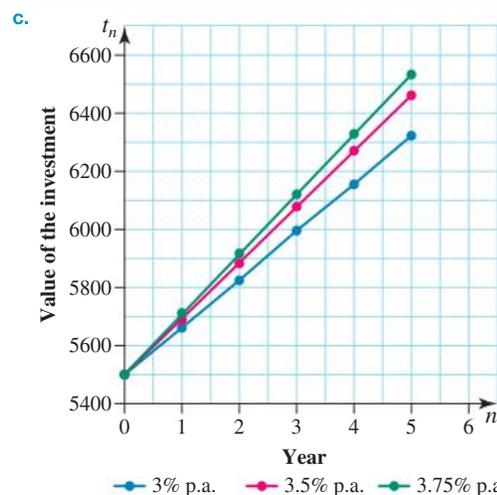
- $t_n = 3n - 1$
 - $t_n = 8n - 5$
 - $t_n = -3n + 11$
 - $t_n = 2n - 6$
 - $t_n = -3n - 7$
 - $t_n = 2n + \frac{3}{2}$
- 2, 5, 8, 11, 14
 - 10, 8, 6, 4, 2
 - 5, -1, 3, 7, 11
 - 6, 8.5, 11, 13.5, 16
 - 5, 8, 11, 14, 17
 - $\frac{1}{3}, 1, \frac{5}{3}, \frac{7}{3}, 3$
- $t_{12} = 23$
 - $t_{27} = -14$
 - $t_{23} = 108$
 - $t_{30} = -68$
 - $t_{22} = 13$
 - $t_{11} = -55$
- $t_1 = 2$
 - $t_1 = 3$
 - $t_1 = 28$
 - $t_1 = 17$
 - $t_1 = -100$
 - $t_1 = 20$
- $d = 13$
 - $d = 7$
 - $d = 18$
 - $d = -7$
 - $d = 7$
 - $d = -13$
- 3
 - 14
 - 7
 - 7
 - 8
 - 5
- $t_n = 10n + 13$
- $t_n = 8.5 + 3.5n$
 - 358.5 m
- \$72 500
 - 10 years
- 103
 - 1777
 - 60
 - 217
 - 152
 - 85
- 3
- t_4 - verify the solution and comment on reasonableness.
- $t_n = 9n - 39$
- $\theta = 76^\circ$
- $t_{100} = 301$

4.4 Using arithmetic sequences to model and analyse practical situations

4.4 Exercise

- $t_n = 3800 + 199.5n$
 - \$4598
 - \$4997
 - 20 years
- $t_n = 20 + 0.9n$
 - \$33.50
- $t_n = 0.75 + 0.2n$
 - \$1.75
 - \$3.50
 - 6 four-minute calls
- Simple interest follows an arithmetic sequence with the same amount being added to each successive term. A linear graph follows the same pattern.
 - $\frac{4.8}{100} \times 6000 = 288$. The gradient is \$288.
 - As the interest of \$288 is added to the initial investment each year, this is a linear model of growth.
- $t_n = 5500 + 165n$
 - $t_n = 5500 + 192.5n$
 - $t_n = 5500 + 206.25n$

Year	0	1	2	3	4	5
3%	5500	5665	5830	5995	6160	6325
3.5%	5500	5692.50	5885	6077.50	6270	6462.50
3.75%	5500	5706.25	5912.50	6118.75	6325	6531.25



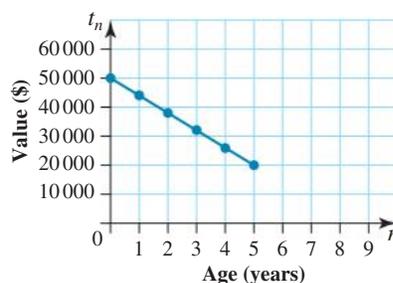
- d. The gradients of the graphs are \$165, \$192.50 and \$206.25, so as the gradients increase, the graphs get steeper.

6.

Time, n (years)	Depreciation (\$)	Future value (\$)
0	0	25 000
1	1750	23 250
2	1750	21 500
3	1750	19 750
4	1750	18 000
5	1750	16 250

7. a. \$ 37.5 million b. 9 years

8. a.



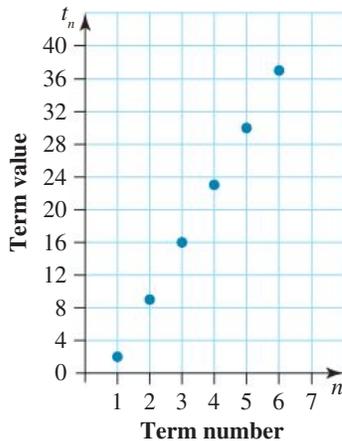
- b. 8 years
9. a. \$4000/year b. \$12 500/year c. \$14 500/year
10. Company A is cheaper any distance lower than 4 km as they start with a lower cost progressing linearly towards $n = 4$ and Company B is cheaper any distance greater than 4 km as after $n = 4$ their rate will not progress as quickly after this distance.

11. They will both owe \$5600 at 15 years.
12. The store can sell for 14 months if it wants to maintain at least 600 items in stock.
13. Linh will be left with \$1364 at the end of 5 years accounting for the fact that she needed to pay \$700 for repairs to her car.
14. The original values of both trumpets are calculated at \$6000. This may provide a rough guide as to the trumpets' original value. Other factors may influence the actual original values of the trumpets, such as their condition. They would be subject to a valuation process. If their judged values differed from the values in the arithmetic model, then the model could be deemed inaccurate.
15. Computer B will be 10 years old when it is at a scrap value of \$100.

4.5 Review

4.5 Exercise

1. a. $t_1 = 3, d = 3$
 b. $t_1 = 1, d = 5$
 c. $t_1 = 5, d = -3$
2. C
3. a. 16 and 30
 b. Linear growth



4. A
5. C
6. B
7. A
8. D
9. C
10. B
11. D
12. A

13. $t_1 = 100, t_{n+1} = t_n - 10, 30\,000$. There seems to be a decline in the monthly revenue amounts for Richard's business. This does not seem to indicate success as the amount he is earning is becoming less every month. Richard's business generated 30 000 in the eighth month.

14. a. $t_{15} = -1$ b. $t_{20} = 82\frac{1}{2}$
15. Gym B would be more affordable until 3 months initial charges are lower. Both would break even at 3 months then Gym A would be more affordable due to the lower rate. Gym A is more affordable for any memberships lasting 6 months or more.
16. The bookstore can continue for 16 months until it reaches the threshold rate of 1800 books.
17. 25
18. The interest rate is 6% p.a.
19. There is a 4.93% difference between the depreciated value and the agree purchasers' value. Ashley should take the offer since the price would actually be lower according to the depreciation projection at that time. His claim is reasonable as he is getting around 5% more from this deal in comparison to the depreciation schedule of the vehicle.
20. Approximate time of death is 12:00 pm as it had been 10 hours since normal body temperature.

4.5 Past QCAA exam questions

1. B
2. a. 33
 b.

Row	1	2	3	4
Number of seats	25	33	41	49

There are 148 seats in total in the first 4

3. D
4. 3015
5. $(-8, 7.6)$

5 The geometric sequence

LESSON SEQUENCE

5.1 Overview	200
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Answers	228

Fully worked solutions for this chapter are available online.

EXAM PREPARATION

Access exam-style questions in every lesson, available online and access past QCAA exam questions in every review.

Resources

 Solutions	Solutions — Chapter 5
 Exam questions	Exam question booklet — Chapter 5
 Digital documents	Learning matrix — Chapter 5 Chapter summary — Chapter 5

LESSON

5.1 Overview

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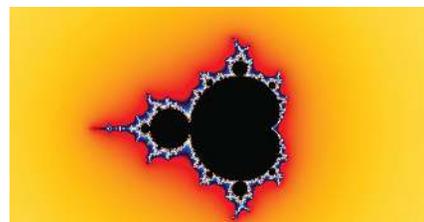
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5.1.1 Introduction

Nature exhibits many patterns that are not composed of regular shapes such as rectangles or circles but are made up of intricate patterns that are repeated over and over. A fern leaf, for example, is made up of many smaller leaves that all have the same shape as the bigger leaf. The smaller leaves, in turn, are made up of even smaller identical leaves. Snowflakes are composed similarly, with each snowflake made up of smaller snowflakes.



Discovered in 1980 by Benoit Mandelbrot, the Mandelbrot set is one of the most intricate and beautiful geometrical patterns in mathematics. The Mandelbrot set can be represented by images that capture many of the qualities people find fascinating about mathematics. Although it is generated by repeating a simple formula, its patterns are infinitely complex. If you select any portion of the Mandelbrot set and magnify it, you will see that no detail is lost — the magnified shape is as intricate and even contains parts that look like copies of the original. This notion of ‘worlds within worlds’ appeals to the philosopher in all of us.

The Mandelbrot set’s behaviour exhibits connections to geometric sequences in terms of growth rates, scaling patterns and the distance between points in certain cases. The set’s complexity arises from how these properties interact in the context of complex dynamics.

5.1.2 Syllabus links

Lesson	Lesson title	Syllabus links
5.2	Using recursion to generate a geometric sequence	<ul style="list-style-type: none">○ Use recursion to generate a geometric sequence.○ Display the terms of a geometric sequence in both tabular and graphical form and demonstrate that geometric sequences can be used to model exponential growth and decay in discrete situations.
5.3	Using the rule for the n th term of a geometric sequence to make predictions	<ul style="list-style-type: none">○ Use the rule for the nth term of a geometric sequence.<ul style="list-style-type: none">• $t_n = t_1 r^{(n-1)}$ where t_n is nth term, t_1 is first term, n is term number and r is common ratio.
5.4	Using geometric sequences to model and analyse practical situations	<ul style="list-style-type: none">○ Use geometric sequences to model and analyse practical situations involving geometric growth and decay (use of logarithms not required), e.g. modelling the growth of a bacterial population that doubles in size each hour, calculating the value of an item using the diminishing-value method of depreciation.

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LESSON

5.2 Using recursion to generate a geometric sequence

SYLLABUS LINKS

- Use recursion to generate a geometric sequence.
- Display the terms of a geometric sequence in both tabular and graphical form and demonstrate that geometric sequences can be used to model exponential growth and decay in discrete situations.

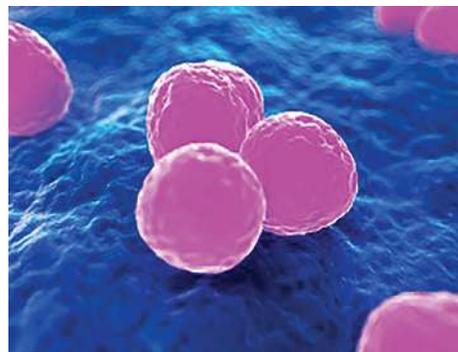
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5.2.1 Geometric sequences

A scientist observed that the cells in a bacterial infection doubled every hour. There were 150 infected cells when the scientist started her observations. She recorded the number of infected cells for 6 hours and collected the following data:

150, 300, 600, 1200, 2400, 4800, 9600, ...

This is an example of a geometric sequence, as each term is found by multiplying the previous term by the same number. In this case each term is multiplied by 2 to obtain the next term.



A **geometric sequence** is a sequence in which the ratio between any two successive terms in the sequence is the same. The next term in the sequence can be determined by multiplying the previous term by a fixed value. This fixed value is known as the common ratio and is referred to as r .

As with arithmetic sequences in the previous chapter, the first term of a geometric sequence is referred to as t_1 and the n th term is t_n , where n is the term number. The successive terms are referred to as $t_1, t_2, t_3, t_4, \dots, t_n$.

The sequence above is an example of a geometric sequence because the ratio between successive terms is 2. The common ratio, r , is calculated by dividing any term in the geometric sequence by the preceding term. The following are all expressions of the common ratio.

$$r = \frac{t_2}{t_1} = \frac{t_3}{t_2} = \frac{t_4}{t_3} \dots = \frac{t_{n+1}}{t_n}$$

The value of the common ratio will also determine the behaviour of the geometric sequence. This will be explored further in this chapter and can also be understood graphically.

Geometric sequences can be defined by providing the first term of the sequence and the common ratio.

For example, the sequence in the example above would be defined as:

$$t_1 = 150, r = 2.$$

WORKED EXAMPLE 1 Identifying a geometric sequence

Determine if the following sequences are geometric. State t_1 and r .

- a. 2, 10, 50, 250, 1250, ...
- b. -2, -6, 18, 54, -162, ...
- c. 4, 2, 1, 0.5, 0.25, ...

THINK

- a. 1. Calculate the ratio between successive terms in the sequence.

2. Identify if the ratio between successive terms is constant. If so, then the sequence can be called geometric.
- b. 1. Calculate the ratio between successive terms in the sequence.

2. Identify if the ratio between successive terms is constant. If so, then the sequence can be called geometric.
- c. 1. Calculate the ratio between successive terms in the sequence.

WRITE

$$\begin{aligned} \text{a. } \frac{t_2}{t_1} &= \frac{10}{2} \\ &= 5 \\ \frac{t_3}{t_2} &= \frac{50}{10} \\ &= 5 \\ \frac{t_4}{t_3} &= \frac{250}{50} \\ &= 5 \\ \frac{t_5}{t_4} &= \frac{1250}{250} \\ &= 5 \end{aligned}$$

There is a common ratio between successive terms. Therefore, this sequence is geometric; hence, $t_1 = 2$, $r = 5$.

$$\begin{aligned} \text{b. } \frac{t_2}{t_1} &= \frac{-6}{-2} \\ &= 3 \\ \frac{t_3}{t_2} &= \frac{18}{-6} \\ &= -3 \end{aligned}$$

There is no common ratio between successive terms as the ratio is not constant. Therefore this sequence is not geometric; $t_1 = -2$, but r does not exist.

$$\begin{aligned} \text{c. } \frac{t_2}{t_1} &= \frac{2}{4} \\ &= \frac{1}{2} \\ \frac{t_3}{t_2} &= \frac{1}{2} \\ \frac{t_4}{t_3} &= \frac{0.5}{1} \\ &= \frac{1}{2} \\ \frac{t_5}{t_4} &= \frac{0.25}{0.5} \\ &= \frac{1}{2} \end{aligned}$$

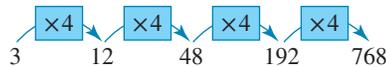
2. Identify if the ratio between successive terms is constant. If so, then the sequence can be called geometric.

There is a common ratio between successive terms. Therefore, this sequence is geometric; hence, $t_1 = 4$, $r = \frac{1}{2}$ or 0.5.

5.2.2 Recursion and the geometric sequence

As discovered in the previous chapter, recursion or the recurrence relation allows for the use of previous terms to calculate subsequent terms and thus form a sequence of terms.

Consider the geometric sequence for which $t_1 = 3$ and $d = 4$.



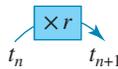
The generated sequence is 3, 12, 48, 192, 768, ...

Each successive term in the sequence can be calculated by multiplying the common ratio by the previous term. The entire sequence can be generated if t_1 and r are defined.

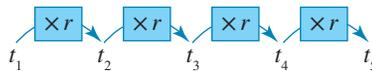
The recurrence relation for the above sequence can be defined as

$$t_1 = 3, t_{n+1} = t_n \times 4$$

Notice that t_1 is clearly defined and r is represented in the recurrence relation. Remember that t_n denotes the n th term in a sequence. Therefore, t_{n+1} represents the term immediately following t_n , or the next term in the sequence, which can be illustrated visually as shown.



The generated arithmetic sequence will then follow this general pattern.



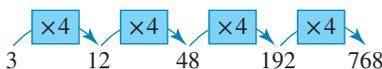
Recurrence relation for a geometric sequence

$$t_1 = \text{the first term}, t_{n+1} = t_n \times r$$

where r is the common ratio.

The following diagram demonstrates how each term is generated by the recurrence relation for the example above. As can be seen, each successive term is generated by multiplying the common ratio by the previous term.

$$t_1 = 3, t_{n+1} = t_1 \times 4$$



It is worthwhile observing the significant growth between term values due to the multiplicative effect. Unlike an arithmetic sequence, which exhibits a linear growth rate, a geometric sequence demonstrates exponential growth.

This is seen when each term is identified as a constant multiple of the previous term, leading to rapid increase in value as the sequence progresses.

The geometric sequence will be visualised graphically, and various applications of exponential growth and decay will be explored further in this chapter.

WORKED EXAMPLE 2 Generating terms in a geometric sequence using recursion

Generate the first 5 terms of each geometric sequence using the following recursion rules.

a. $t_1 = 2, r = 3$

b. $t_1 = -4, t_{n+1} = t_n \times 5$

c. $t_1 = 10, r = -2$

THINK

- a. To generate the terms of the geometric sequence recursively, start with $t_1 = 2$ and use $r = 3$ to determine each subsequent term.
- b. The recursive relation indicates that each term is derived from the previous term by multiplying by 5, beginning with $t_1 = -4$.
- c. To generate the terms of the geometric sequence recursively, start with $t_1 = 10$ and use $r = -2$ to determine each subsequent term.

WRITE

- a. The first 5 terms are 2, 6, 18, 54, 162.
- b. The first 5 terms are -4, -20, -100, -500, -2500.
- c. The first 5 terms are 10, -20, 40, -80, 160.

WORKED EXAMPLE 3 Defining t_1 and r for geometric sequences

Define t_1 and r for the following geometric sequences.

a. 3, 18, 108, 648, 3888, ...

b. 2, -8, 32, -128, 512, ...

c. 256, 64, 16, 4, 1, ...

THINK

- a. Since the sequence is defined as geometric, only one pair of terms is required to calculate r . Define t_1 and r .
- b. Since the sequence is defined as geometric, only one pair of terms is required to calculate r . Define t_1 and r .
- c. Since the sequence is defined as geometric, only one pair of terms is required to calculate r . Define t_1 and r .

WRITE

- a. $\frac{t_2}{t_1} = \frac{18}{3}$
 $= 6$
 $\therefore t_1 = 3, r = 6$
- b. $\frac{t_2}{t_1} = \frac{-8}{2}$
 $= -4$
 $\therefore t_1 = 2, r = -4$
- c. $\frac{t_2}{t_1} = \frac{64}{256}$
 $= \frac{1}{4}$ or 0.25
 $\therefore t_1 = 256, r = 0.25$

The recursive rule will be used in the next lesson as a proof to derive the formula for calculating the n th term of a geometric sequence. This approach not only solidifies our understanding of geometric sequences but also connects concepts in a meaningful way.

5.2.3 Using a scientific calculator to generate terms of a geometric sequence

A scientific calculator can be used to generate the terms of a geometric sequence.

WORKED EXAMPLE 4 Using a scientific calculator to generate a geometric sequence

A geometric sequence is described by the recurrence relation $t_1 = -8$, $t_{n+1} = t_n \times 0.80$. Use a scientific calculator to generate the first 6 terms of this sequence.

THINK

- a. 1. Type the first term of the sequence into the calculator screen and then press Enter =.
2. Type $\times 0.80$ then press Enter = 5 times.

WRITE

- a. -8

-8
-6.4
-5.12
-4.096
-3.2768
-2.62144

3. Write the answer.

The first 6 terms are

-8, -6.4, -5.12, -4.096, -3.2768, -2.62144

5.2.4 Tabular and graphical forms of geometric sequences

Tables of values

When considering the graphical form of a geometric sequence, it is often helpful to first draw a table of values that contains the term number of the sequence, n , in the top row and the term value, t_n , in the bottom row.

Continuing with our generated sequence 3, 12, 48, 192, 768, ..., its tabulated form will be as follows.

Term number (n)	1	2	3	4	5
Term value (t_n)	3	12	48	192	768

The coordinates for this sequence are (1, 3), (2, 12), (3, 48), (4, 192) and (5, 768). The data from this table can be used to generate coordinates that can be plotted on a graph to represent this geometric sequence.

Graphs of geometric sequences

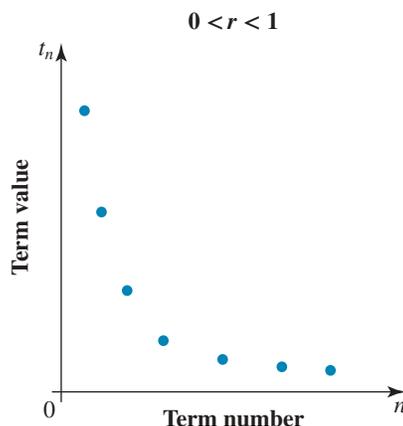
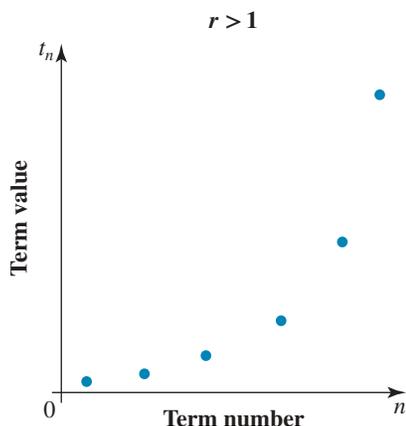
When graphs of a geometric sequence are drawn, the term number, n , is the explanatory variable, so it appears on the x -axis of the graph. The term value, t_n is the response variable, so it appears on the y -axis of the graph.

Because there is a common ratio between the terms of a geometric sequence, the relationship between the terms is exponential. This means that when the terms of a geometric sequence are graphed, the points resemble a curve depending on the magnitude of the common ratio.

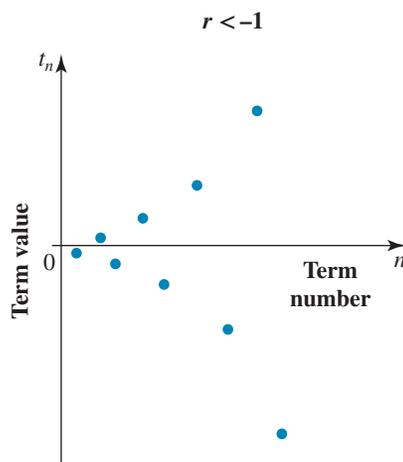
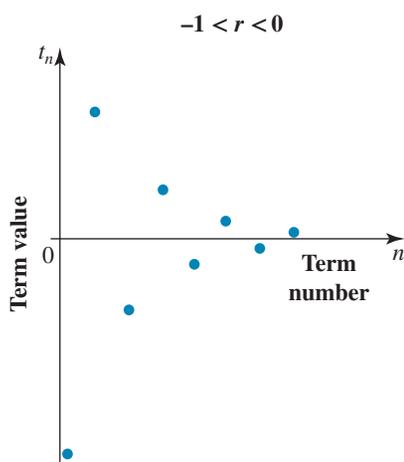
The shape of the graph of a geometric sequence depends on the value of r .

- When $r > 1$, the value of the terms increases at an exponential rate. If the first term is positive, as shown in the example below left, the terms will increase positively and exponentially. If the first term is negative, as shown in the example below right, the terms will grow negatively in magnitude at an exponential rate. These are both examples of **exponential growth**.

- When $0 < r < 1$, the value of the terms decreases at an exponential rate and converge towards 0. If the first term is positive, as shown in the example below left, the terms will decrease positively and exponentially. If the first term is negative, as shown in the example below right, the terms will decrease negatively in magnitude at an exponential rate. These are both examples of **exponential decay**.



- When $-1 < r < 0$, the values of the terms oscillate on either side of 0 but converge towards 0.
- When $r < -1$, the values of the terms oscillate on either side of 0 and move away from the starting value at an exponential rate.



WORKED EXAMPLE 5 Creating a table of values and graphing of a geometric sequence

A geometric sequence is described as follows.

$$t_1 = 5, r = 2$$

- Create a table of values showing the term number and term value for the first 5 terms of the sequence.
- Plot the graph of the sequence.
- Describe this geometric sequence as modelling exponential growth or decay and justify your response.

THINK

- Set up a table with the term numbers in the top row and the term values in the bottom row.

WRITE

a.	Term number	1	2	3	4	5
	Term value					

2. Use the t_1 and r values to determine the first 5 values of the sequence.

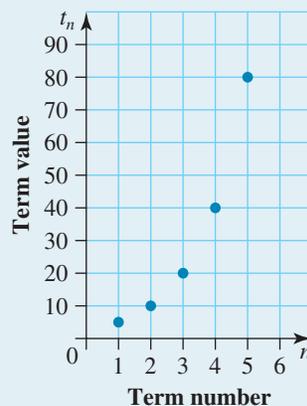
$$\begin{aligned}t_1 &= 5 \\t_2 &= 5 \times 2 = 10 \\t_3 &= 10 \times 2 = 20 \\t_4 &= 20 \times 2 = 40 \\t_5 &= 40 \times 2 = 80\end{aligned}$$

3. Complete the table with the calculated values.

Term number	1	2	3	4	5
Term value	5	10	20	40	80

- b. 1. Use the table of values to identify the points to be plotted.
2. Plot the points on the graph.

- b. The points to be plotted are (1, 5), (2, 10), (3, 20), (4, 40), and (5, 80).



- c. Observe the nature of the geometric sequence, in particular the value of its common ratio.

- c. This geometric sequence is modelling exponential growth as the common ratio is greater than 1. This is also indicated by the increase in term values as demonstrated by the graph.

5.2.5 Applying geometric sequences to contexts

Recursion can also be used in practical contexts to model exponential growth or decay, which are often expressed by geometric sequences.

WORKED EXAMPLE 6 Applying geometric sequences to contexts

A phone starts with a fully charged battery of 20 000 mAh. Due to a fault in a software upgrade, the phone loses 15% of its current battery charge every hour. Determine the first hour in which the phone will drop to below 50% of its original battery charge.

THINK

1. This is an example of a geometric sequence. Define t_1 .
2. Define r . To reduce by 15%, calculate the original battery life minus 15%. Note that $r \neq 0.15$.
3. Determine 50% of the phone's battery life.

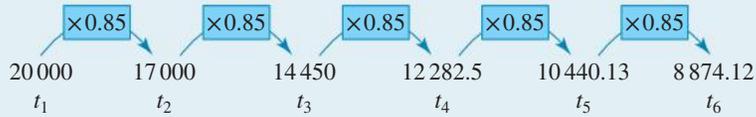
WRITE

$$t_1 = 20\,000$$

$$\begin{aligned}\text{Decrease by 15\%} \\ 100\% - 15\% &= 85\% \\ r &= 0.85\end{aligned}$$

10 000 mAh is the target amount.

4. Use recursion to generate the sequence by multiplying each term by 0.85.



5. Write the answer.

The first hour in which the battery life drops to below 10 000 mAh is t_6 , which is the fifth hour.

Exercise 5.2 Using recursion to generate a geometric sequence

learn on

5.2 Exercise

5.2 Exam questions

Simple familiar

1, 2, 3, 4, 5, 6, 7, 8, 9, 10

Complex familiar

11, 12, 13, 14

Complex unfamiliar

15, 16, 17

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Simple familiar

- WE1** Determine whether the following sequences are geometric. State t_1 and r .

a. 10, 13, 16, 19, 22, ...	b. 6, 30, 150, 750, 3750, ...	c. 128, 32, 8, 2, 0.5, ...
d. 1, 4, 9, 16, 25, ...	e. 3, 6, 24, 144, 1152, ...	f. 4, -28, 196, -1372, 9604, ...
- Calculate missing values in the following geometric sequences.

a. 1, 6, c , 216, 1296, ...	b. 2, 10, d , 250, 1250, ...	c. 3, 12, a , b , 768, ...
d. g , -6, h , -24, 48, ...	e. p , q , s , 300, 1500, ...	f. x , y , z , -135, 405, ...
- WE2** Use recursion to generate the first 5 terms in each of the following:

a. $t_1 = 3, r = 2$	b. $t_1 = -1, t_{n+1} = t_n \times 6$	c. $t_1 = 8, r = -3$
d. $t_1 = 5, t_{n+1} = t_n \times 5$	e. $t_1 = -16, r = 0.75$	f. $t_1 = 50, t_{n+1} = t_n \times -0.4$
- WE3** Define t_1 and r for the following geometric sequences.

a. 2, 8, 32, 128, 512, ...	b. 32, 16, 8, 4, 2, ...	c. 2, -3, 4.5, -6.75, 10.125, ...
d. 64, 16, 4, 1, 0.25, ...	e. 2, 10, 50, 250, 1250, ...	f. 10, -2, 0.4, -0.08, 0.016, ...
- WE4** A geometric sequence is described by the recurrence relation $t_1 = 10\,000, t_{n+1} = t_n \times 1.02$. Use a scientific calculator to generate the first 6 terms of this sequence.
- WE5** A geometric sequence is described as follows.

$$t_1 = 10, r = 5$$
 - Create a table of values showing the term number and term value for the first 5 terms of the sequence.
 - Plot the graph of the sequence from the table of values in part a.
 - Describe this geometric sequence as modelling exponential growth or decay, and justify your response.
- A geometric sequence is described as follows.

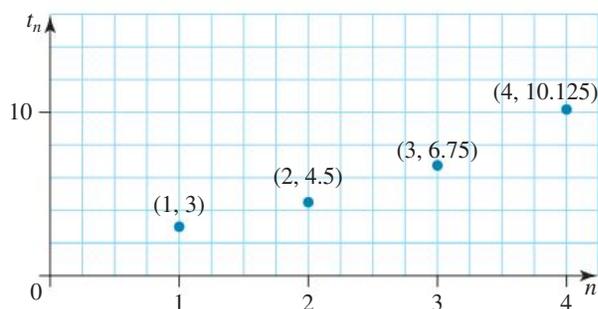
$$t_1 = 64, r = \frac{1}{2}$$
 - Create a table of values showing the term number and term value for the first 5 terms of the sequence.
 - Plot the graph of the sequence from the table of values in part a.
 - Describe this geometric sequence as modelling exponential growth or decay, and justify your response.

8. A geometric sequence is described by the following recurrence relation.

$$t_1 = 1.5, t_{n+1} = t_n \times 3$$

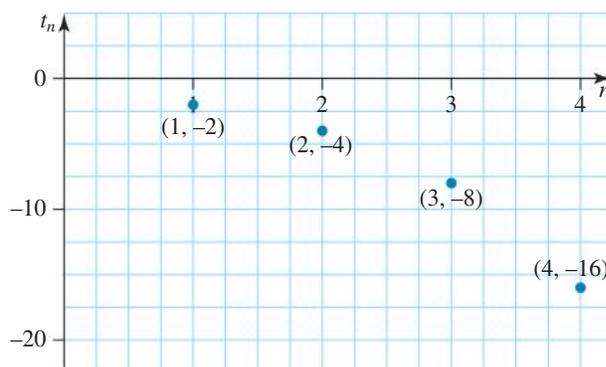
- Create a table of values showing the term number and term value for the first 5 terms of the sequence.
 - Plot the graph of the sequence from the table of values in part a.
 - Describe this geometric sequence as modelling exponential growth or decay, and justify your response.
9. Determine the values of t_1 and r for the geometric sequence that is modelled by the following table and graph. Evaluate the reasonableness of your solution.

Term number (n)	1	2	3	4
Term value (t_n)	3	4.5	6.75	10.125



10. Determine the recurrence relation for the geometric sequence that is modelled by the following table and graph. Evaluate the reasonableness of your solution.

Term number (n)	1	2	3	4
Term value (t_n)	-2	-4	-8	-16



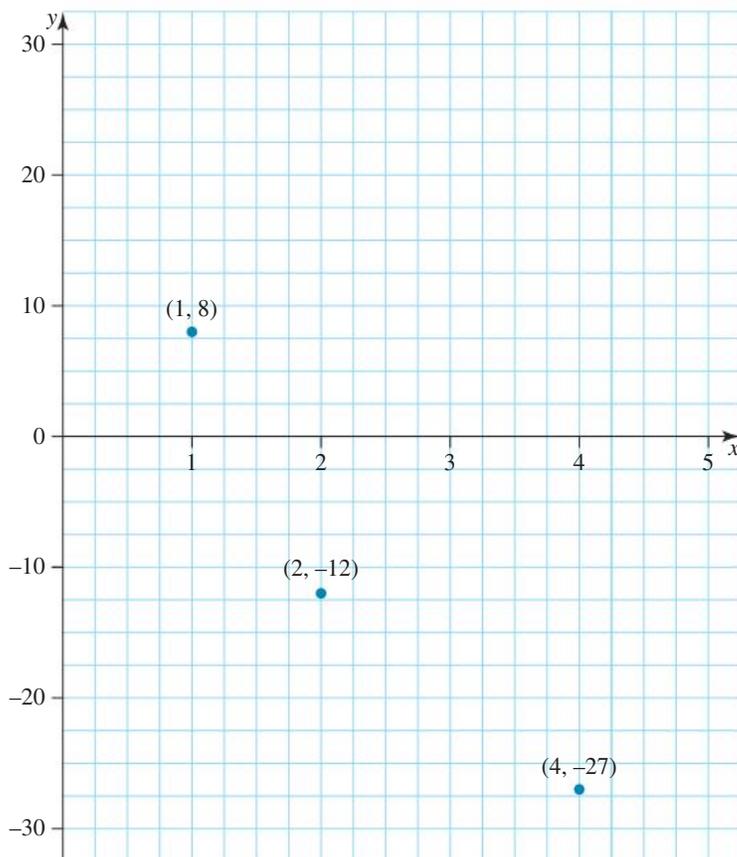
Complex familiar

11. Johnny has a baseball card with a valued market price that follows a geometric sequence for each consecutive year as below.

$$5, a, 45, \dots$$

Determine the valued market price for Johnny's baseball card in its fifth year.

12. **WE6** A company has saved \$20 000 throughout the year to donate to charities at Christmas time. Each day of December, they donate 15% of the account balance to a different nominated charity. They stop donating when the account balance drops below \$4000. Determine how many days they continue donating.
13. Determine the t_1 and r values of the geometric sequence shown in the graph and hence calculate the coordinates of the 3rd term.



14. A recurrence relation is defined as

$$t_1 = 2, t_{n+1} = t_n \times 3$$

If the 3rd term of this sequence is the same value as the 4th term of another geometric sequence with $t_1 = 144$, determine the recurrence relation of this geometric sequence.

Complex unfamiliar

15. Determine the 5th term in the following geometric sequence.

$$x - 10, x, 3x, \dots$$

16. A geometric sequence is graphed such that each coordinate forms the hypotenuse length of a right-angled triangle with the origin. A sequence for the areas of these triangles, measured in square units, is as follows.

$$2, 12, 54, 216, \dots$$

Determine the recurrence relation for the geometric sequence that represents the hypotenuse length of the triangles.

17. The first 3 terms of the following recurrence relation are graphed.

$$t_1 = 4, r = 2$$

If line segments are drawn between all of the coordinates, determine the area bound by the line segments and the x -axis.

Fully worked solutions for this chapter are available online.

LESSON

5.3 Using the rule for the n th term of a geometric sequence to make predictions

SYLLABUS LINKS

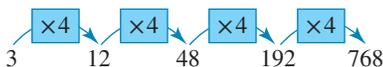
- Use the rule for the n th term of a geometric sequence.
 - $t_n = t_1 r^{(n-1)}$ where t_n is n th term, t_1 is first term, n is term number and r is common ratio.

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5.3.1 Developing the rule for the n th term of a geometric sequence

The previous lesson explored geometric sequences and demonstrated how these quantities can be defined using recursion. However, this method can become cumbersome, particularly when calculating terms that are further along in the sequence.

Consider the geometric sequence for which $t_1 = 3$ and $r = 4$.



Remember that by recursion, the entire sequence can be defined and generated as shown.

$$t_1 = 3, t_{n+1} = t_n \times 4$$

$$t_1 = 3$$

$$t_2 = t_1 \times r \quad t_2 = 3 \times 4 = 12$$

$$t_3 = t_2 \times r \quad t_3 = 12 \times 4 = 48$$

$$t_4 = t_3 \times r \quad t_4 = 48 \times 4 = 192$$

$$t_5 = t_4 \times r \quad t_5 = 192 \times 4 = 768$$

This system of equations can assist in producing a more efficient rule for calculating the n th term of a geometric sequence. By way of substitution the below equations can be generated with respect to each term but with t_1 and r expressed on the right-hand side.

$$\begin{array}{lll}
 t_1 = 3 & & \\
 t_2 = 3 \times 4 & t_2 = t_1 \times r & t_2 = t_1 \times r^1 \\
 t_3 = 3 \times 4 \times 4 & t_3 = t_1 \times r \times r & t_3 = t_1 \times r^2 \\
 t_4 = 3 \times 4 \times 4 \times 4 & t_4 = t_1 \times r \times r \times r & t_4 = t_1 \times r^3 \\
 t_5 = 3 \times 4 \times 4 \times 4 \times 4 & t_5 = t_1 \times r \times r \times r \times r & t_5 = t_1 \times r^4
 \end{array}$$

Notice that in the final system of equations, each term contains t_1 and powers of r . Also notice that the exponent for the r value in each equation is one less than the term number in its equation. This pattern will give rise to the following equation.

$$t_n = t_1 r^{n-1}$$

Remember that n represents the term number. This rule can be applied to the example above, since it is known that $t_1 = 3$ and $r = 4$.

$$t_n = 3 \times 4^{n-1}$$

For example, if $n = 5$, then the 5th term is:

$$\begin{aligned}
 t_5 &= 3 \times 4^{5-1} \\
 &= 3 \times 4^4 \\
 &= 768
 \end{aligned}$$

Therefore, the 5th term is 768, as verified in the geometric sequence above.

The n th term rule for a geometric sequence

$$t_n = t_1 r^{n-1}$$

where t_n is the n th term, t_1 is the first term and r is the common ratio.

WORKED EXAMPLE 7 Determining the rule for a geometric sequence

Determine the rules that represent the following geometric sequences.

- a. 7, 28, 112, 448, 1792, ... b. 8, -4, 2, -1, $\frac{1}{2}$, ...

THINK

- a. 1. Determine the values of t_1 and r .
Since the sequence is defined as geometric, only one pair of terms is required to calculate r .

2. Substitute the values for t_1 and r into the formula for geometric sequences.

WRITE

$$\begin{aligned}
 \text{a. } t_1 &= 7 \\
 r &= \frac{t_2}{t_1} \\
 &= \frac{28}{7} \\
 &= 4 \\
 t_n &= t_1 r^{n-1} \\
 &= 7 \times 4^{n-1}
 \end{aligned}$$

- b. 1.** Determine the values of t_1 and r .
 Since the sequence is defined as geometric, only one pair of terms is required to calculate r .

$$\begin{aligned} \text{b. } t_1 &= 8 \\ r &= \frac{t_2}{t_1} \\ &= \frac{-4}{8} \\ &= -\frac{1}{2} \end{aligned}$$

- 2.** Substitute the values for t_1 and r into the formula for geometric sequences.

$$\begin{aligned} t_n &= t_1 r^{n-1} \\ &= 8 \times \left(-\frac{1}{2}\right)^{n-1} \end{aligned}$$

The n th term rule enables any term of a geometric sequence to be determined provided that the values of t_1 and r are known.

WORKED EXAMPLE 8 Determining the term value in a geometric sequence

Determine the 12th term of the following geometric sequence.

2, 10, 50, 250, 1250, ...

THINK

- Determine the value of t_1 .
- Determine the value for r . Since the sequence is defined as geometric, only one pair of terms is required to calculate r .
- Use the rule $t_n = t_1 \times r^{n-1}$ to determine the 12th term.
- Write the answer.

WRITE

$$\begin{aligned} t_1 &= 2 \\ r &= \frac{t_2}{t_1} \\ r &= \frac{10}{2} \\ &= 5 \\ t_n &= t_1 \times r^{n-1} \\ t_{12} &= 2 \times 5^{12-1} \\ &= 97\,656\,250 \end{aligned}$$

The value of the 12th term is 97 656 250.

5.3.2 Using algebraic techniques with the n th term rule for a geometric sequence

WORKED EXAMPLE 9 Solving for unknowns using the n th term rule

For the following geometric sequences, determine:

- the first term given $t_6 = 128$ and $r = 2$.
- the common ratio given that the first term is 10 and the 4th term is -270 .

THINK

1. Substitute the given values into the n th term rule for a geometric sequence.

WRITE

- a. Since $t_6 = 128$ and $r = 2$,

2. Solve for t_1 .

$$t_n = t_1 \times r^{n-1}$$

$$t_6 = t_1 \times 2^{6-1}$$

$$128 = t_1 \times 2^5$$

$$t_1 \times 2^5 = 128$$

3. Write the answer.

$$\therefore t_1 = 4$$

b. 1. Substitute the given values into the n th term rule for a geometric sequence.

b. Since $t_1 = 10$ and $t_4 = -270$,

2. Solve for r .

$$t_n = t_1 \times r^{n-1}$$

$$t_4 = 10 \times r^{4-1}$$

$$-270 = 10 \times r^3$$

$$10 \times r^3 = -270$$

$$r^3 = -27$$

3. Write the answer.

$$\therefore r = -3$$

WORKED EXAMPLE 10 Using simultaneous equations to determine a term value

The 2nd term of a geometric sequence is 8 and the 5th term is 512. Determine the 10th term of this sequence.

THINK

1. It is known that $t_2 = 8$, and for this term $n = 2$. Substitute these values into the t_n rule.

2. It is known that $t_5 = 512$, and for this term $n = 5$. Substitute these values into the t_n rule.

3. Solve the two equations simultaneously by eliminating t_1 to determine r . Divide equation [2] by equation [1].

4. To determine t_1 , substitute the value of r into either of the two equations.

5. Write the rule.

6. To determine the 10th term, let $n = 10$.

7. Write the answer.

WRITE

$$t_n = t_1 \times r^{n-1}$$

$$t_2 = t_1 \times r^1 \\ = 8$$

$$t_5 = t_1 \times r^4 \\ = 512$$

$$t_1 \times r^1 = 8 \quad [1]$$

$$t_1 \times r^4 = 512 \quad [2]$$

$$\frac{t_1 \times r^4}{t_1 \times r} = \frac{512}{8} \quad [2] \div [1]$$

$$r^3 = 64$$

$$\sqrt[3]{r^3} = \sqrt[3]{64} \\ r = 4$$

Substituting $r = 4$ into equation [1]:

$$t_1 \times 4 = 8$$

$$t_1 = 2$$

$$t_n = 2 \times 4^{n-1}$$

$$t_{10} = 2 \times 4^9 \\ = 524\,288$$

The 10th term in the sequence is 524 288.

WORKED EXAMPLE 11 Using trial and error to determine the term number in a geometric sequence

The first three terms of a geometric sequence are 2, 6 and 18.
Which term number would be the first to exceed 1 000 000 in this sequence?

THINK

- Determine the rule for the sequence.
- Develop the equation to be solved.
- Trial various values of n .

WRITE

$$t_1 = 2 \text{ and } r = 3$$

$$t_n = t_1 \times r^{n-1}$$

$$t_n = 2 \times 3^{n-1}$$

$$2 \times 3^{n-1} = 1\,000\,000$$

Let $n = 8$, $2 \times 3^7 = 4374$ (too small)
 Let $n = 15$, $2 \times 3^{14} = 9\,565\,938$ (too large)
 Let $n = 12$, $2 \times 3^{11} = 354\,294$ (too small)
 Let $n = 14$, $2 \times 3^{13} = 3\,188\,646$ (too large)
 Let $n = 13$, $2 \times 3^{12} = 1\,062\,882$
 The 13th term is the required term.

Exercise 5.3 Using the rule for the n th term of a geometric sequence to make predictions

learn **on**

5.3 Exercise

5.3 Exam questions **on**

Simple familiar

1, 2, 3, 4, 5, 6, 7, 8,
9

Complex familiar

10, 11, 12, 13

Complex unfamiliar

14, 15, 16

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Simple familiar

1. **WE7** Determine the rules that represent the following geometric sequences.

a. 2, 6, 18, 54, 162, ...

b. $9, 3, 1, \frac{1}{3}, \frac{1}{9}, \dots$

c. 5, -10, 20, -40, 80, ...

d. 4, 6, 9, 13.5, 20.25, ...

e. 25, 10, 4, 1.6, 0.64, ...

f. $4, -3, \frac{9}{4}, -\frac{27}{16}, \frac{81}{64}, \dots$

2. Determine the first five terms of the geometric sequences defined by the following rules.

a. $t_n = 1 \times 4^{n-1}$

b. $t_n = 3 \times (-3)^{n-1}$

c. $t_n = 25 \times \left(\frac{2}{5}\right)^{n-1}$

d. $t_n = 64 \times \left(\frac{1}{4}\right)^{n-1}$

e. $t_n = 5 \times 2.2^{n-1}$

f. $t_n = 10 \times (-0.6)^{n-1}$

3. **WE8** For each of the geometric sequences given, determine:

- a. the 12th term of the sequence 1, 5, 25, 125, 625, ...
- b. the 14th term of the sequence 2, -8, 32, -128, 512, ...
- c. the 11th term of the sequence 3, 7.5, 18.75, 46.875, 117.1875, ...
- d. the 9th term of the sequence 128, 16, 2, 0.25, 0.03125, ...
- e. the 10th term of the sequence 32, 24, 18, 13.5, 10.125, ...
- f. the 8th term of the sequence 20, -4, 0.8, -0.16, 0.032, ...

4. For each of the given rules, determine the recursive relation that defines the geometric sequences.

- a. $t_n = 2 \times 6^{n-1}$
- b. $t_n = 81 \times \left(\frac{1}{3}\right)^{n-1}$
- c. $t_n = 4 \times (-5)^{n-1}$
- d. $t_n = 1 \times 3.5^{n-1}$
- e. $t_n = 200 \times \left(\frac{1}{10}\right)^{n-1}$
- f. $t_n = 16 \times (-0.5)^{n-1}$

5. **WE9a** Determine the first term in each of the following geometric sequences.

- a. $t_8 = 4374$ and $r = 3$
- b. $t_6 = 0.0625$ and $r = \frac{1}{4}$
- c. $t_5 = 162$ and $r = -3$
- d. $t_4 = 62.5$ and $r = 2.5$
- e. $t_{10} = 0.000\,025\,6$ and $r = \frac{1}{5}$
- f. $t_6 = -0.7776$ and $r = -0.6$



6. **WE9b** Determine the common ratio for each of the following geometric sequences.

- a. The first term is 1 and the 3rd term is 16.
- b. The first term is 32 and the 4th term is 4.
- c. The first term is 5 and the 6th term is -160.
- d. The first term is 8 and the 5th term is 40.5.
- e. The first term is 50 and the 4th term is 0.4.
- f. The first term is 15 and the 6th term is -0.036 45.

7. Piper is recording a new album and plans to layer the vocals to create a richer sound. For her first track, Piper records one vocal line. For each subsequent track, the number of vocal lines is multiplied by a factor of 2. Determine the number of vocal lines in track 5.

8. The decay of radioactive material is modelled as a geometric sequence where $r = \frac{1}{2}$. If there are 20 million radioactive atoms initially present in a sample, calculate the 7th term of the sequence.



9. The takings at a new cinema are recorded each month. If the takings each month continue to follow a geometric sequence, determine:

- a. a rule for the takings in month n
- b. the takings in month 9.

Month number	Takings
1	\$10 000
2	\$8500
3	\$7225



Complex familiar

10. **WE10** Determine the value of the term specified in each of the following geometric sequences.
- The 2nd term of a geometric sequence is 6 and the 5th term is 162. Determine the 10th term.
 - The 2nd term of a geometric sequence is 6 and the 5th term is 48. Determine the 12th term.
 - The 2nd term of a geometric sequence is 2 and the 5th term is 16. Determine the 16th term.
 - The 4th term of a geometric sequence is -32 and the 7th term is -256 . Determine the 14th term.
 - The 4th term of a geometric sequence is -192 and the 7th term is $-12\,288$. Determine the 12th term.
 - The 3rd term of a geometric sequence is 36 and the 6th term is -972 . Determine the 12th term.
11. The first term of a geometric sequence is 1.2, and the common ratio is 1.1. Determine the first term to exceed 20.
12. **WE11** Evaluate the following.
- The first three terms of a geometric sequence are 5, 12.5 and 31.25. Determine which term would be the first to exceed 50 000.
 - The first three terms of a geometric sequence are 3.2, 9.6 and 28.8. Determine which term would be the first to exceed 1 000 000.
 - The first three terms of a geometric sequence are 5.1, 20.4 and 81.6. Determine which term would be the first to exceed 100 000.
 - The first three terms of a geometric sequence are 4.3, 9.46 and 20.812. Determine which term would be the first to exceed 500 000.

13. A small town is renowned for spreading rumours. All of its citizens are aware in a short time of any new rumours. The spread of a rumour can be summarised in the following table.

Day	Number of citizens in the know
1	1
2	6
3	36

If the number of citizens who have been told the rumour each day continues to follow a geometric sequence, determine on which day all 4230 citizens will know of the rumour.



Complex unfamiliar

14. A geometric sequence has a first term of 5. If the product of the 2nd and 4th terms equals 2025, determine the value(s) of r .
15. If it is known that $t_1 = 10$, $t_n = -80$ and $t_{n+3} = 640$ for a geometric sequence, determine the value of n .
16. There is a difference of 72 between the 4th and 6th terms and a difference of 144 between the 5th and 7th terms of an increasing geometric sequence. Determine the first 5 terms of this sequence.

Fully worked solutions for this chapter are available online.

LESSON

5.4 Using geometric sequences to model and analyse practical situations

SYLLABUS LINKS

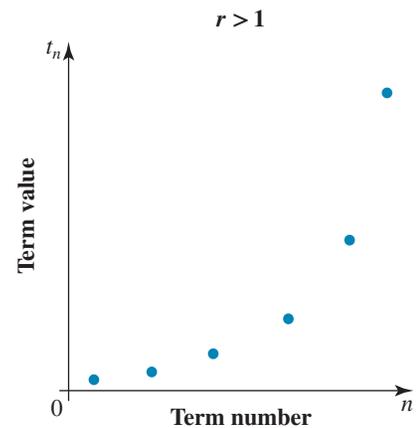
- Use geometric sequences to model and analyse practical situations involving geometric growth and decay (use of logarithms not required), e.g. modelling the growth of a bacterial population that doubles in size each hour, calculating the value of an item using the diminishing-value method of depreciation.

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5.4.1 Geometric growth models

Throughout this chapter, it is apparent that geometric sequences can exhibit either growth or decay in successive terms, depending on the value of r . This lesson will explore how geometric sequences can be used to model and analyse practice situations where exponential growth applies. This means that subsequent terms in the geometric sequence will increase at a given rate and therefore $r > 1$. The graph below shows a curve that grows exponentially, with increasing r values resulting in steeper growth. Geometric sequences can be applied to any situation where exponential growth occurs, for example finance, biology and population dynamics.

It is important to note that when percentages are involved, a percentage increase of x per cent between terms indicates that the value of the common ratio r will be $1 + \frac{x}{100}$.



WORKED EXAMPLE 12 Developing and using a geometric growth model

A city produced 100 tonnes of rubbish in the year 2022. Forecasts suggest that this may increase by 2% each year. If these forecasts are true:

- develop a model that determines the city's rubbish production over time
- determine the city's rubbish output in 2026
- determine in which year the amount of rubbish will reach 120 tonnes.

THINK

- This is an example of exponential growth; therefore, a geometric sequence can be used. Determine the first term, t_1 .
- Determine the common ratio, r . The amount of rubbish increases by 2%, that is the original amount plus an extra 2%. Note that $r \neq 0.02$.

- Determine which term is represented by the amount of rubbish for the year 2022.

WRITE/DISPLAY

a. $t_1 = 100$

Increase by 2%
 $100\% + 2\% = 102\%$
 $r = 1.02$

$$t_n = t_1 \times r^{n-1}$$

$$\therefore t_n = 100 \times 1.02^{n-1}$$

- Year 2022 is the first term, so $n = 1$. Year 2023 is the second term, so $n = 2$. Year 2026 is the fifth term, so $n = 5$.

2. Use $t_n = t_1 \times r^{n-1}$ to determine the amount of rubbish collected in the fifth year.

$$\begin{aligned} t_5 &= 100 \times 1.02^{5-1} \\ &= 100 \times 1.0824 \\ &= 108.24 \end{aligned}$$

3. Write the answer.

The amount of rubbish produced in the fifth year, or 2026, will be 108.24 tonnes.

- c. 1. Use $t_n = t_1 \times r^{n-1}$ and $t_n = 120$.

c. $100(1.02)^{n-1} = 120$

2. Try various values of n .

Let $n = 10$, $100 \times (1.02)^9 = 119.51$

Let $n = 11$, $100 \times (1.02)^{10} = 121.90$

3. Write the answer.

During the 11th year, that is, during 2032, the rubbish will exceed 120 tonnes.

WORKED EXAMPLE 13 Developing a model from a geometric sequence

The number of cells of a micro-organism after each process of cell division can be summarised as follows.

1, 2, 4, 8, 16, ...

If the number of cells after each division continues to follow a geometric sequence, determine:

- a rule to model this situation
- the number of cells at 12 divisions
- the number of divisions that have occurred if there are 1024 cells.

THINK

1. Define t_1 and r .
 2. Determine the rule.
 3. Write the answer.
1. Since divisions occur from the second term onward, $n = 13$ is required.
 2. Determine t_{13} .

WRITE

$t_1 = 1$ and $r = 2$.

$$t_n = t_1 \times r^{n-1}$$

$$t_n = 1 \times 2^{n-1}$$

$$\therefore t_n = 2^{n-1}$$

Let $n = 13$.

$$t_n = 2^{n-1}$$

$$t_{13} = 2^{13-1}$$

$$= 2^{12}$$

$$= 4096$$

There are 4096 cells after 12 divisions.

Let $t_n = 1024$.

$$t_n = 2^{n-1}$$

$$1024 = 2^{n-1}$$

Let $n = 8$, $2^7 = 128$

Let $n = 9$, $2^8 = 256$

Let $n = 10$, $2^9 = 512$

Let $n = 11$, $2^{10} = 1024$

10 divisions have occurred if there are 1024 cells.

3. Write the answer.

- c. 1. Define t_n .

2. Write the rule.

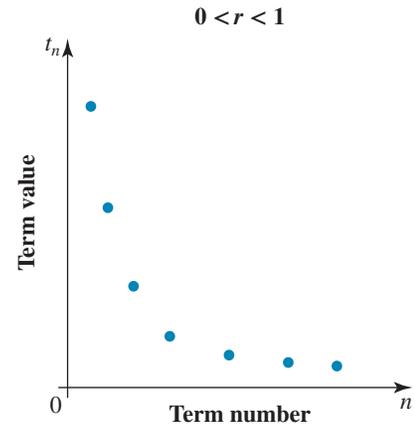
3. Determine n using trial and error.

4. Write the answer. Since divisions occur from the second term onward, consider the solution.

5.4.2 Geometric decay models

For a geometric sequence to exhibit decay, the common ratio must fall within the range of $0 < r < 1$. This is logical, as multiplying a value by a common ratio between 0 and 1 will always decrease a value. Furthermore, this reduction occurs at an exponential rate, similar to the behaviour observed in geometric growth models.

It is important to note that when percentages are involved, a percentage decrease of x per cent between terms indicates that the value of the common ratio r will be $1 - \frac{x}{100}$.



WORKED EXAMPLE 14 Developing and using a geometric decay model

The population of a town is decreasing by 10% each year. The population recorded in the first year was 10 000.

- Develop a model that determines the population of the town over time.
- Determine the town's population in the third year.
- Determine the first year in which the population drops to less than half of its original count.



THINK

1. This is an example of a geometric sequence. Define t_1 .
2. Define the common ratio, r . The population decreases by 10%, that is the original population minus 10%. Note that $r \neq 0.1$.
3. Develop the model that determines the population of the town over time.

WRITE

a. $t_1 = 10\,000$

Decrease by 10%
 $100\% - 10\% = 90\%$
 $r = 0.9$

$$t_n = t_1 r^{n-1}$$

$$\therefore t_n = 10\,000 \times (0.9)^{n-1}$$

1. Use the model to determine the population in the third year.

b. Let $n = 3$.

$$t_3 = 10\,000 \times (0.9)^{3-1}$$

$$= 10\,000 \times (0.9)^2$$

$$= 8100$$

The town's population in the third year is 8100.

2. Write the answer.

1. Use the model and $t_n = 5000$.

c. $10\,000 \times (0.9)^{n-1} = 5000$

2. Try various values of n .

Let $n = 7$, $10\,000 \times (0.9)^6 = 5314.41$

Let $n = 8$, $10\,000 \times (0.9)^7 = 4782.97$

3. Write the answer.

The first year in which the population will drop to less than half of its original count is the eighth year.

5.4.3 Diminishing value depreciation

The previous chapter explored the straight-line method of depreciation.

The other method of depreciation used is the diminishing value or reducing balance method of depreciation. In this method, the value of the item depreciates each year by a percentage of its current value. Under such depreciation, the value of the item never actually becomes zero.

This type of depreciation is an example of exponential decay. A graph depicting the value over time is non-linear, showing a downward-falling curve that never actually reaches a zero value.



tlvd-12136

WORKED EXAMPLE 15 Modelling reducing balance depreciation using a rule

The purchase price of a yacht is \$15 000. The value of the yacht depreciates by 10% p.a. Calculate (correct to the nearest \$1) the future value of the yacht after 8 years.



THINK

1. This is an example of a geometric sequence.
Define t_1 .
2. Define r . To reduce by 10%, calculate the original value minus 10%. Note that $r \neq 0.1$.
3. Determine the rule.
4. Since depreciation occurs from the second term onward, $n = 9$ is required. Determine the future value.
5. Write the answer.

WRITE

$$t_1 = 15\,000$$

$$\begin{aligned} \text{Decrease by 10\%} \\ 100\% - 10\% = 90\% \\ r = 0.9 \end{aligned}$$

$$\begin{aligned} t_n &= t_1 \times r^{n-1} \\ \therefore t_n &= 15\,000 \times 0.9^{n-1} \end{aligned}$$

Let $n = 9$.

$$\begin{aligned} t_n &= 15\,000 \times 0.9^{n-1} \\ t_9 &= 15\,000 \times 0.9^{9-1} \\ &= 15\,000 \times 0.9^8 \\ &= 6457.01 \end{aligned}$$

The future value of the yacht after 8 years is \$6457.01.

Exercise 5.4 Using geometric sequences to model and analyse practical situations

5.4 Exercise

5.4 Exam questions **on**

Simple familiar

1, 2, 3, 4, 5, 6, 7, 8, 9

Complex familiar

10, 11, 12

Complex unfamiliar

13, 14, 15

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Simple familiar

- WE12** A farmer harvests 4 tonnes of lucerne in his first year of production. In his business plan, he has estimated an annual increase of 6% on his lucerne harvest.
 - Develop a model that determines the farmer's harvest production over time.
 - Determine how many tonnes of lucerne the farmer should harvest in his seventh year of production according to this plan.
- The cost of petrol was increasing at a rate of 4.5% due to inflation. In 2018 petrol cost 145 cents per litre. Determine the projected cost of petrol in 2022. Round to the nearest litre.
- WE13** Chris and Lisa own a bee colony that starts with 256 bees. Every month, the number of bees grow according to the following geometric sequence.

256, 384, 576, 864, ...

If the population of bees continues to follow this growth pattern, determine:

- a rule to model the situation
 - the population of bees at 6 months
 - the number of months it will take for the bee population to reach over 5000 bees.
- WE14** The promoters of Fleago flea treatment assert that continued application of the treatment will reduce the number of fleas on a dog by 15% each week. At the end of week 1, Fido the dog has 200 fleas left on him and his owner continues to apply the treatment.
 - Develop a model that determines the number fleas on Fido over time.
 - Determine how many fleas Fido would be expected to have on him at the end of the fourth week.
 - A radioactive substance starts with an initial mass of 500 grams. Each hour, the mass of the substance decreases due to radioactive decay and can be modelled by the following geometric sequence.

500, 400, 320, 256, ...

If the decay continues to follow this pattern, determine:

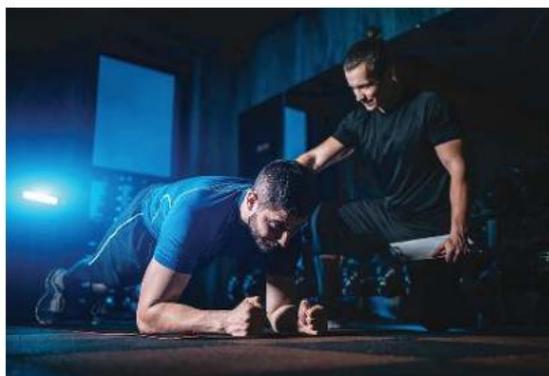
- a rule to model the situation
- the mass of the substance at 5 hours
- the number of hours it will take for the mass of the substance to drop below 100 grams.



6. **WE15** Calculate the future value after 7 years of a power generator purchased for \$800 000 that depreciates at a rate of 10% p.a. (Give your answer correct to the nearest \$1000.)
7. A company exported \$300 000 worth of manufactured goods in its first year of production. According to the business plan of the company, this amount should increase each year by 7.5%.
 - a. Develop a model that determines the worth of manufactured goods over time.
 - b. Determine how much the company would be expected to export in its fifth year.
8. A plumber has tools and equipment valued at \$18 000. If the value of the equipment depreciates by 30% each year, calculate the value of the equipment after 3 years.
9. A yacht is valued at \$950 000. The value of the yacht depreciates by 22% p.a. Calculate the amount that the yacht will depreciate in value over the first 5 years (correct to the nearest \$1000).

Complex familiar

10. An accountant purchased a computer for \$6000. The value of the computer depreciates by 33% p.a. When the value of the computer falls below \$1000, it is written off and a new one is purchased. Determine how many years it will take for the computer to be written off.
11. Coach Joey is planning a calorie-cutting phase for his client Nick, who is preparing for a bodybuilding contest. Nick's daily maintenance calories start at 3000 per week, with a goal of reducing this to 1800 calories over a 4-week period before the contest. Joey's advice is to implement a gradual 7% reduction in calorie intake each week leading up to the 4-week reduction block. Determine the earliest week Nick can compete.



12. Shenghao is building a ramp with several timber support beams. The first of the beams is 0.8 metres long and each successive beam is 3% longer than the previous one. Shenghao will stop producing support beams once he produces a beam at least 2 metres in length. Determine how many support beams are required for this task.

Complex unfamiliar

13. In a laboratory experiment, Kyanu discovers a new radioactive substance. The amount of the substance, measured in grams, is monitored month by month and expressed as a geometric sequence as below. The first term of the sequence represents the number of grams in the first month.

$$x^4, x^3, 9, x, \dots$$

Determine how many grams of the substance is present in the seventh month.

14. A fitness program is launched to help athletes improve their performance. In the second month Shao's strength is measured at 54 kg, and in the fifth month his strength is measured at 68 kg. Shao's measurements adhere to a geometric sequence. The program promises that athletes' improvement will increase by 10% of their current strength measurements each month. Evaluate the reasonableness of this claim.
15. The value of Ethan's classic car decreases by \$15 000 over 5 years. It is known that the car depreciates at a rate of 10% per annum. Determine the original price of the car rounded to the nearest \$100.

Fully worked solutions for this chapter are available online.

LESSON

5.5 Review

5.5.1 Summary

Hey students! Now that it's time to revise this chapter, go online to:

 Access the chapter summary

 Review your results

 Practise exam questions

Find all this and MORE in jacPLUS 



5.5 Exercise

learnon

5.5 Exercise

5.5 QCAA questions 

Simple familiar

1, 2, 3, 4, 5, 6, 7, 8,
9, 10, 11, 12

Complex familiar

13, 14, 15, 16

Complex unfamiliar

17, 18, 19, 20

These questions are even better in jacPLUS!

- Receive immediate feedback
- Access sample responses
- Track results and progress



Find all this and MORE in jacPLUS 

Simple familiar

- MC** Determine which of the following is a geometric sequence.
A. $\{2, 4, 6, \dots\}$ B. $\{2, 4, 16, \dots\}$ C. $\{2, 20, 40, \dots\}$ D. $\{2, 4, 8, \dots\}$
- MC** A geometric sequence is described as follows.

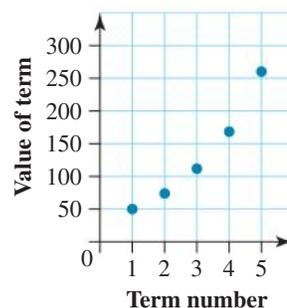
$$t_1 = 5, r = 3$$

Determine which of the following sequences is described by this recurrence relation.

- A. 3, 15, 75, 225, 1125, ... B. 5, 15, 45, 135, 405, ...
C. 5, 8, 11, 14, 17, ... D. 5, 9, 11, 15, ...
- MC** The 10th term of the geometric sequence 4, 12, 36, ... is:
A. 78 732 B. 177 147 C. 236 196 D. 786 432
- MC** The 3 consecutive terms of a geometric sequence are 5, y , 20. The value of y is:
A. 5 B. 12.5 C. 20 D. 10
- MC** Determine the common ratio of the geometric sequence $\frac{1}{8}, \frac{1}{2}, 2, \dots$
A. $\frac{1}{8}$ B. $\frac{1}{2}$ C. 2 D. 4
- MC** Determine the rule for the geometric sequence 7, 21, 63, ...
A. $7 \times 3^{n-1}$ B. 7×3^n C. 7×3 D. $3 \times 7^{n-1}$

7. **MC** Determine the common ratio for the geometric sequences where the first term is 4 and the 4th term is -500 .
- A. 5 B. -5 C. 10 D. -20
8. **MC** A 500-g packet of chocolate costs \$2.50 at the beginning of the year. Assuming inflation averages 2.8% per annum over each of the next 3 years, determine how much the chocolate will cost in three years.
- A. \$2.64 B. \$2.72 C. \$2.79 D. \$2.87
9. The diameter of a cell is 15 micrometres in its first hour. It regularly feeds on nutrients such that its diameter grows 4% every hour.
- a. Develop a model that determines the cell diameter over time.
b. Determine the diameter of the cell in the sixth hour.

10. **MC** The first five terms of a sequence are plotted on the graph. Determine which of the following could describe the sequence.



- A. Arithmetic sequence with $t_1 = 50$ and $d = 25$
B. Arithmetic sequence with $t_1 = 50$ and $d = 0.5$
C. Geometric sequence with $t_1 = 50$ and $r = 0.5$
D. Geometric sequence with $t_1 = 50$ and $r = 1.5$

11. **MC** A new engraving machine was bought for \$30 000. If depreciation is calculated at a rate of 27% reducing balance, the value at the end of 5 years is:
- A. \$21 900 B. \$15 987 C. \$11 670.11 D. \$6219.21
12. Will is the operations manager for a football stadium. Since attendance records began, crowds have been decreasing by 3% each year. In the first year that records were kept, the number of attendees was 63 000 for the season.
- a. Develop a model that determines the attendance over time.
b. Determine how many people attended in the fifth year.

Complex familiar

13. Sarah owns a chocolate factory and begins with 25 000 chocolate bars. Unbeknownst to her, the air conditioning in her storage room is broken, causing her chocolate bars to melt. Every hour 20% of the unmelted bars melt and can no longer be sold. When she realises the air conditioning is broken, there are only 5000 unmelted bars left in the storage room. Determine how long the machine was left running.
14. Corina is a pâtissier who purchased a rotating rack oven for \$50 000 for her fine dining restaurant in the first year of opening. If the depreciation rate of this oven is 15% per year, determine how many years she can use it before its value falls below \$10 000.
15. Tanya is a property tycoon who purchases a house for \$1.1 million in the first year. She projects that its annual growth rate will be 8% per annum. Her real estate agent promises that her property will double in value in 10 years. Using this projection, determine how long it will take for her property to double in value and evaluate the reasonableness of the real estate agent's claim.
16. Charna is a wildlife conservationist studying the growth of a population of endangered tortoises. She observes that in the 3rd month of the population growth, there are 12 tortoises, and in the 6th month there are 96 tortoises. Charna is interested in predicting how many tortoises will be in the population by the end of the season. Determine the tortoise population in the 10th month if the population is growing exponentially.

Complex unfamiliar

17. Marius maps out the ideal trajectory that his golf ball needs to take on a Cartesian plane using a geometric sequence in metres, where the x -axis represents the term number and the y -axis represents the value of each term. If the first term is 1 and the common ratio is 2, determine the shortest path that his golf ball would take from the 4th term to the 7th term.
18. In a chemistry lab, Daniel is conducting an experiment on the growth of crystal formations over time. He models the mass of the crystals over the first four days as a geometric sequence:

$$x, 3x, 9x, 27x, \dots$$

During his presentation, he points out that the mean mass of the crystals for these four days is 20 grams. Determine the mass of the crystals on the sixth day.

19. In a project management meeting, Paul, the lead engineer, is analysing the growth of a new engineering product. He notes that the projected sales figures for the first three quarters form a geometric sequence, with the sum of first and third quarter amounting to 78 units. The first quarter's sales are expected to start at 3 units. Determine the sum of the projected sales for the first five quarters to better understand the product's growth trajectory.
20. Quintus is a cryptographer tasked with developing a new encryption method. While analysing patterns in sequences to strengthen his algorithms, he graphs two mathematical sequences. The x -axis represents the term number and the y -axis represents the value of each term. He determines that both sequences intersect at the coordinates (2, 2) and (4, 8). Quintus knows that one of the sequences is arithmetic and the other is geometric. Determine the rule for the n th term of both sequences.

5.5 Past QCAA exam questions

Question 1 (1 mark)

Source: QCAA 2023 General Mathematics, Paper 1, Section 1, Q6; © State of Queensland (QCAA) 2024

In January 2022, 40 fish were released into a new dam that has the capacity to support 10 000 fish. It is predicted that the dam will reach its capacity in January 2030 if the fish population doubles every year.

Which sequence rule models the prediction?

- A. $t_n = t_1 r^{(n-1)}$, where $t_1 = 40, r = 2, n = 8$ B. $t_n = t_1 r^{(n-1)}$, where $t_1 = 40, r = 2, n = 9$
C. $t_n = t_1 + (n-1)d$, where $t_1 = 40, d = 2, n = 8$ D. $t_n = t_1 + (n-1)d$, where $t_1 = 40, d = 2, n = 9$

Question 2 (1 mark)

Source: QCAA 2023 General Mathematics, Paper 1, Section 1, Q15; © State of Queensland (QCAA) 2024

The resale value of a boat shows geometric decay.

Years after purchase	0	1	2
Resale value (\$)	50 000	40 000	32 000

Determine the resale value four years after purchase.

- A. \$16 000 B. \$20 480
C. \$22 000 D. \$25 600

Answers

Chapter 5 The geometric sequence

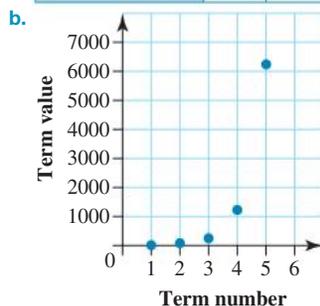
5.2 Using recursion to generate a geometric sequence

5.2 Exercise

1. a. Not geometric
b. Geometric $t_1 = 6, r = 5$
c. Geometric $t_1 = 128, r = 0.25$
d. Not geometric
e. Not geometric
f. Geometric $t_1 = 4, r = -7$
2. a. $c = 36$
b. $d = 50$
c. $a = 48, b = 192$
d. $h = 12, g = 3$
e. $p = 2.4, q = 12, s = 60$
f. $x = 5, y = -15, z = 45$
3. a. 3, 6, 12, 24, 48
b. -1, -6, -36, -216, -1296
c. 8, -24, 72, -216, 648
d. 5, 25, 125, 625, 3125
e. -16, -12, -9, -6.75, -5.0625
f. 50, -20, 8, -3.2, 1.28
4. a. $t_1 = 2, r = 4$
b. $t_1 = 32, r = 0.5$
c. $t_1 = 2, r = -1.5$
d. $t_1 = 64, r = 0.25$
e. $t_1 = 2, r = 5$
f. $t_1 = 10, r = -0.2$
5. 10 000, 10 200, 10 404, 10 612.08, 10 824.32, 11 040.81

6. a.

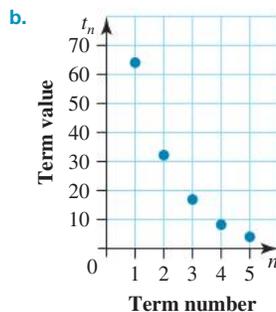
Term number	1	2	3	4	5
Term value	10	50	250	1250	6250



c. Exponential growth

7. a.

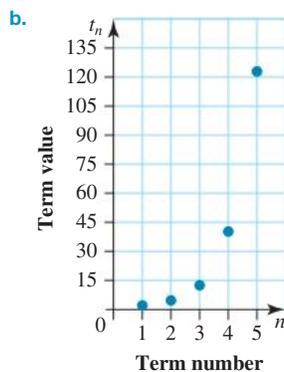
Term number	1	2	3	4	5
Term value	64	32	16	8	4



c. Exponential decay

8. a.

Term number	1	2	3	4	5
Term value	1.5	4.5	13.5	40.5	121.5



c. Exponential growth

9. $t_1 = 3, r = 1.5$
10. $t_1 = -2, t_{n+1} = t_n \times 2$
11. $t_1 = 5, r = 3, \$405$ in the fifth year
12. 10 days
13. $t_1 = 8, r = -1.5, (3, 18)$
14. $t_1 = 144, t_{n+1} = t_n \times 0.5$
15. $t_1 = 5, r = 3, 405$
16. $t_1 = 4, t_{n+1} = t_n \times 3$
17. 18 units²

5.3 Using the rule for the n th term of a geometric sequence to make predictions

5.3 Exercise

1. a. $t_n = 2 \times 3^{n-1}$
b. $t_n = 9 \times \left(\frac{1}{3}\right)^{n-1}$
c. $t_n = 5 \times (-2)^{n-1}$
d. $t_n = 4 \times \left(\frac{3}{2}\right)^{n-1}$
e. $t_n = 25 \times \left(\frac{2}{5}\right)^{n-1}$
f. $t_n = 4 \times \left(\frac{-3}{4}\right)^{n-1}$
2. a. 1, 4, 16, 64, 256
b. 3, -9, 27, -81, 243
c. 25, 10, 4, $\frac{8}{5}, \frac{16}{25}$

- d. $64, 16, 4, 1, \frac{1}{4}$
 e. 5, 11, 24.2, 53.24, 117.128
 f. 10, -6, 3.6, -2.16, 1.296
3. a. $t_{12} = 48\ 828\ 125$ b. $t_{14} = -134\ 217\ 728$
 c. $t_{11} = 28\ 610.22949$ d. $t_9 = 0.00\ 000\ 762\ 9$
 e. $t_{10} = 0.00\ 000\ 023\ 8$ f. $t_8 = -0.00\ 256$
4. a. $t_1 = 2, t_{n+1} = t_n \times 6$
 b. $t_1 = 81, t_{n+1} = t_n \times \frac{1}{3}$
 c. $t_1 = 4, t_{n+1} = t_n \times -5$
 d. $t_1 = 1, t_{n+1} = t_n \times 3.5$
 e. $t_1 = 200, t_{n+1} = t_n \times \frac{1}{10}$
 f. $t_1 = 16, t_{n+1} = t_n \times -0.5$
5. a. $t_1 = 2$ b. $t_1 = 64$ c. $t_1 = 2$
 d. $t_1 = 4$ e. $t_1 = 50$ f. $t_1 = 10$
6. a. $r = -4$ or 4 b. $r = 0.5$
 c. $r = -2$ d. $r = -1.5$ or 1.5
 e. $r = 0.2$ f. $r = -0.3$
7. 16 vocal lines
 8. 312 500 atoms
 9. a. $t_n = 10\ 000 \times 0.85^{n-1}$ b. \$2724.91
10. a. 39 366 b. 6144 c. 32 768
 d. -32 768 e. -12 582 912 f. -708 588
11. 31st term
 12. a. 12th b. 13th c. 9th d. 16th
 13. 6
 14. $r = 3$ or -3
 15. $n = 4$
 16. 3, 6, 12, 24, 48

5.4 Using geometric sequences to model and analyse practical situations

5.4 Exercise

1. a. $t_n = 4 \times (1.06)^{n-1}$ b. 5.67 tonnes
 2. 173 cents per litre
 3. a. $t_n = 256 \times (1.5)^{n-1}$
 b. 2916 bees
 c. 8 months
 4. a. $t_n = 200 \times (0.85)^{n-1}$
 b. 122 fleas
 5. a. $t_n = 500 \times (0.8)^{n-1}$
 b. 163.84 g
 c. It takes 8 hours for mass of the substance to drop below 100 g.
 6. \$382 600
 7. a. $t_n = 300\ 000 \times (1.075)^{n-1}$
 b. \$400 640.74
 8. \$382 600
 9. \$274 000
 10. 5 years
 11. 11 weeks

12. 32
 13. 0.11 g
 14. 7.99%, not reasonable
 15. \$36 600

5.5 Review

5.5 Exercise

1. D
 2. B
 3. A
 4. D
 5. D
 6. A
 7. B
 8. B
 9. a. $t_n = 15 \times (1.04)^{n-1}$
 b. 18.25 micrometers
 10. D
 11. D
 12. a. $t_n = 63\ 000 \times (0.97)^{n-1}$
 b. 55 773 people
 13. 8 hours
 14. 10 years
 15. \$2 198 905.09, reasonable
 16. 1536 tortoises
 17. 56.08 units
 18. 486 grams
 19. 2343
 20. $t_n = 3n - 4$ and $t_n = 1 \times 2^{n-1}$

5.5 Past QCAA exam questions

1. B
 2. B
 3. \$50 620
 4. A
 5. Sample responses are available in the worked solutions in the online resources.

6 Earth geometry and time zones

LESSON SEQUENCE

6.1 Overview	232
6.2 Position on Earth	233
6.3 Distances on Earth's surface	244
6.4 Time zones	250
6.5 Review	257
Answers	264

Fully worked solutions for this chapter are available online.

EXAM PREPARATION

Access exam-style questions in every lesson, available online and access past QCAA exam questions in every review.

Resources

 Solutions	Solutions — Chapter 6
 Exam questions	Exam question booklet — Chapter 6
 Digital documents	Learning matrix — Chapter 6 Chapter summary — Chapter 6

LESSON

6.1 Overview

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6.1.1 Introduction

For thousands of years, the people of each land observed celestial patterns to track time and mark significant events, such as seasons, ceremonies and travel. The rising of stars, phases of the Moon and planetary movements were key indicators used across various communities, including Aboriginal and Torres Strait Islander Peoples.

In this chapter, you will study Earth geometry and time zones to understand what determines time differences across the globe, which is crucial for communication, navigation and worldwide coordination.



6.1.2 Syllabus links

Lesson	Lesson title	Syllabus links
6.2	Position on Earth	<ul style="list-style-type: none"> <input type="radio"/> Understand the meaning of great circles. <input type="radio"/> Understand the meaning of angles of latitude and longitude (in decimal degrees, and degrees and minutes) in relation to the equator and the prime meridian respectively. <input type="radio"/> Locate positions on Earth's surface given latitude and longitude, e.g. using a globe, map, GPS and other digital technologies. <input type="radio"/> State latitude and longitude for positions on Earth's surface, e.g. investigating a map of Australia and locating boundary positions for Aboriginal peoples' and Torres Strait Islander peoples' language groups, Australian landmarks or local land boundaries.
6.3	Distances on Earth's surface	<ul style="list-style-type: none"> <input type="radio"/> Calculate angular distance and distance between two places on Earth on the same meridian. <ul style="list-style-type: none"> • $D = 111.2 \times \text{angular distance}$ where D is distance in kilometres <input type="radio"/> Calculate angular distance and distance between two places on Earth on the same parallel of latitude. <ul style="list-style-type: none"> • $D = 111.2 \cos(\theta) \times \text{angular distance}$ where D is distance in kilometres and θ is latitude <input type="radio"/> Solve practical problems involving latitude, longitude, angular distance and distance.
6.4	Time zones	<ul style="list-style-type: none"> <input type="radio"/> Understand the meaning of Greenwich Mean Time (GMT), International Date Line and Coordinated Universal Time (UTC). <input type="radio"/> Understand the link between longitude and time. <input type="radio"/> Determine the number of degrees of longitude for a given time difference. <input type="radio"/> Calculate time differences between two places on Earth. <input type="radio"/> Solve practical problems involving time zones, making allowances for daylight saving where necessary, e.g. seasonal time systems used by Aboriginal peoples and Torres Strait Islander peoples, making phone calls, broadcasting events, travelling, preparing an itinerary.

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LESSON

6.2 Position on Earth

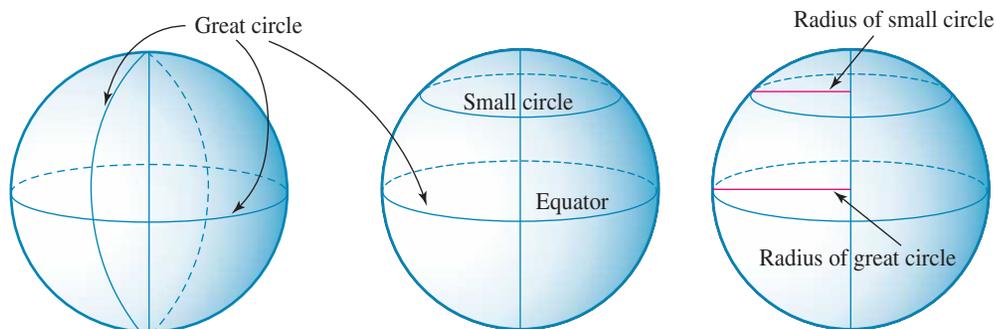
SYLLABUS LINKS

- Understand the meaning of great circles.
- Understand the meaning of angles of latitude and longitude (in decimal degrees, and degrees and minutes) in relation to the equator and the prime meridian respectively.
- Locate positions on Earth's surface given latitude and longitude, e.g. using a globe, map, GPS and other digital technologies.
- State latitude and longitude for positions on Earth's surface, e.g. investigating a map of Australia and locating boundary positions for Aboriginal peoples' and Torres Strait Islander peoples' language groups, Australian landmarks or local land boundaries.

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6.2.1 Great circles and small circles

Earth geometry treats Earth as a sphere, from which 2-dimensional circles can be drawn. There are two circle groups: great circles and small circles. **Great circles** are the largest circles that can be drawn on any given sphere, such as the equator of Earth. They divide a globe into two equal halves. **Small circles** are those smaller circles that run parallel to the equator.



Great circles and small circles.

- **Great circles are the largest circles that can be drawn on any given sphere. Great circles of Earth include the equator and the meridians that connect the poles. Their radius is approximately 6371 km.**
- **Small circles run parallel to the Equator. Their radius is less than 6371 km.**

WORKED EXAMPLE 1 Calculating the circumference of great and small circles

- Calculate the circumference of the great circle joining the North and South poles, given the average radius of Earth is 6371 km. Round to the nearest km.
- Calculate the circumference of the Tropic of Capricorn, given that the radius of Earth at that location is 5843 km. Round to the nearest km.

THINK

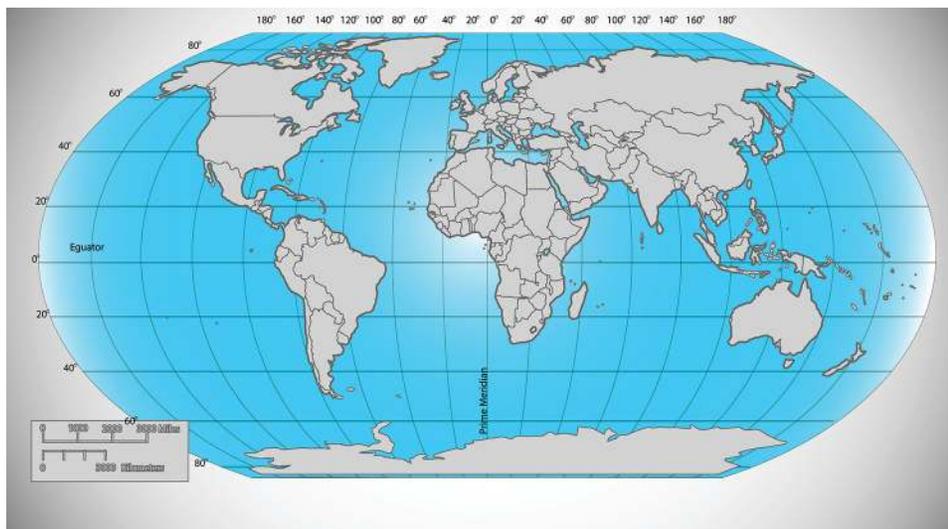
- Write the formula for circumference of a circle.
 - Substitute the radius for r .
 - Calculate the circumference.
 - Round to the nearest km.
- Write the formula for circumference of a circle.
 - Substitute the radius for r .
 - Calculate the circumference.
 - Round to the nearest km.

WRITE

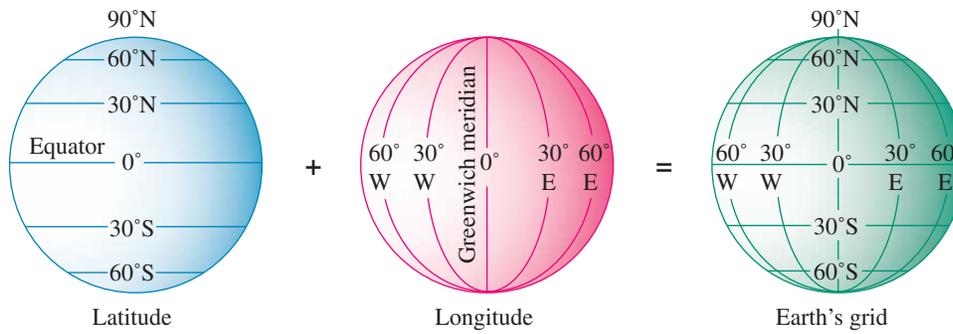
- $C = 2\pi r$
 $C = 2 \times \pi \times 6371 \text{ km}$
 $C = 40030.17359 \text{ km}$
The great circles of Earth are approximately 40 030 km in distance.
- $C = 2\pi r$
 $C = 2 \times \pi \times 5843 \text{ km}$
 $C = 36712.65175 \text{ km}$
The Tropic of Capricorn is approximately 36 713 km in distance.

6.2.2 Latitude and longitude

The image of Earth shown is similar to the Cartesian coordinate system. Like the Cartesian system, there is a system of reference points on Earth. As Earth is a sphere, the great circles and small circles on the surface of Earth are the grid lines. The horizontal axis (the x -axis) is represented by the equator, and the vertical axis (the y -axis) is represented by the Greenwich Meridian.

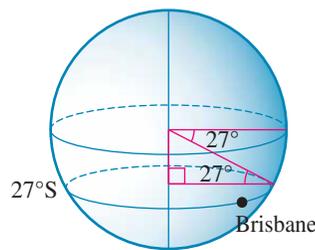


Using these two axes, it is possible to determine the coordinates of a point on Earth's surface and therefore describe the location of that point.



Parallels of latitude

Small circles parallel to the equator are called **parallels of latitude**. These small circles are used to describe how far north or south of the equator a place is located. For example, Brisbane lies close to the small circle 27°S. This means that Brisbane subtends a 27° angle at the centre of Earth and is south of the equator.

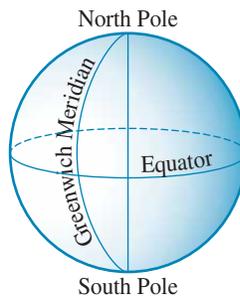


The maximum latitude for any point on Earth is 90°N or 90°S. The North and South poles lie at these points. All places on the equator have a latitude of 0°, neither North nor South.

Meridians of longitude

To locate a place on the globe in an east–west direction, the line of reference is the **Greenwich Meridian**. The Greenwich Meridian is half a great circle running from the North to the South Pole.

The Greenwich Meridian is named after Greenwich, a suburb of London.



Places that lie on the Greenwich Meridian have a longitude of 0°. All other places on the globe can be described as being east or west of the Greenwich Meridian. The half great circle passing through the poles on which a place lies is called a **meridian of longitude**.

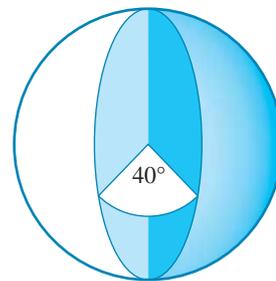
Each meridian of longitude is identified by the angle measured from the centre of Earth between it and the Greenwich Meridian and by whether it is east or west of Greenwich.

The maximum longitude for any point on Earth is 180° west or 180° east.

The meridian of longitude that lies at 180° east or 180° west is directly opposite the Greenwich Meridian and is called the **International Date Line** (see the map on the next page). On either side of the International Date Line the day changes. (This is explained in more detail later in the chapter.)

For the convenience of some small island nations and Russia, the International Date Line is bent so as not to pass through them.

World maps or globes are drawn with both parallels of latitude and meridians of longitude shown, which can be used to locate cities. Each point is then given a pair of coordinates: the parallel of latitude it lies on, followed by the meridian of longitude. For example, the coordinates of Brisbane are $(27^\circ\text{S}, 153^\circ\text{E})$.



Latitude and Longitude

- **Latitude and longitude are universally adopted measurements of location.**
- **Latitude is a measure of how many degrees north or south of the equator you are.**
E.g. Brisbane is 27°S equator.
- **Longitude is a measure of how many degrees east or west of Greenwich you are.**
E.g. Brisbane is 153°E of Greenwich.
- **Coordinates are the pairing of the latitude and longitude to provide an exact location,**
E.g. Brisbane is $(27^\circ\text{S}, 153^\circ\text{E})$.

WORKED EXAMPLE 2 Locating positions on Earth's surface given latitude and longitude

Identify the major cities closest to each of the following locations using the map at right.

a. $30^\circ\text{S}, 30^\circ\text{E}$

b. $30^\circ\text{N}, 120^\circ\text{E}$

c. $45^\circ\text{N}, 75^\circ\text{W}$

THINK

- Look for the city closest to the intersection of the 30°S parallel of latitude and the 30°E meridian of longitude.
- Look for the city closest to the intersection of the 30°N parallel of latitude and the 120°E meridian of longitude.
- Look for the city closest to the intersection of the 45°N parallel of latitude and the 75°W meridian of longitude.

WRITE

- Johannesburg
- Shanghai
- Montreal

WORKED EXAMPLE 3 Stating latitude and longitude in degrees for positions on Earth's surface

Write down the approximate coordinates of each of the following cities using the map at right.

a. Singapore

b. Perth

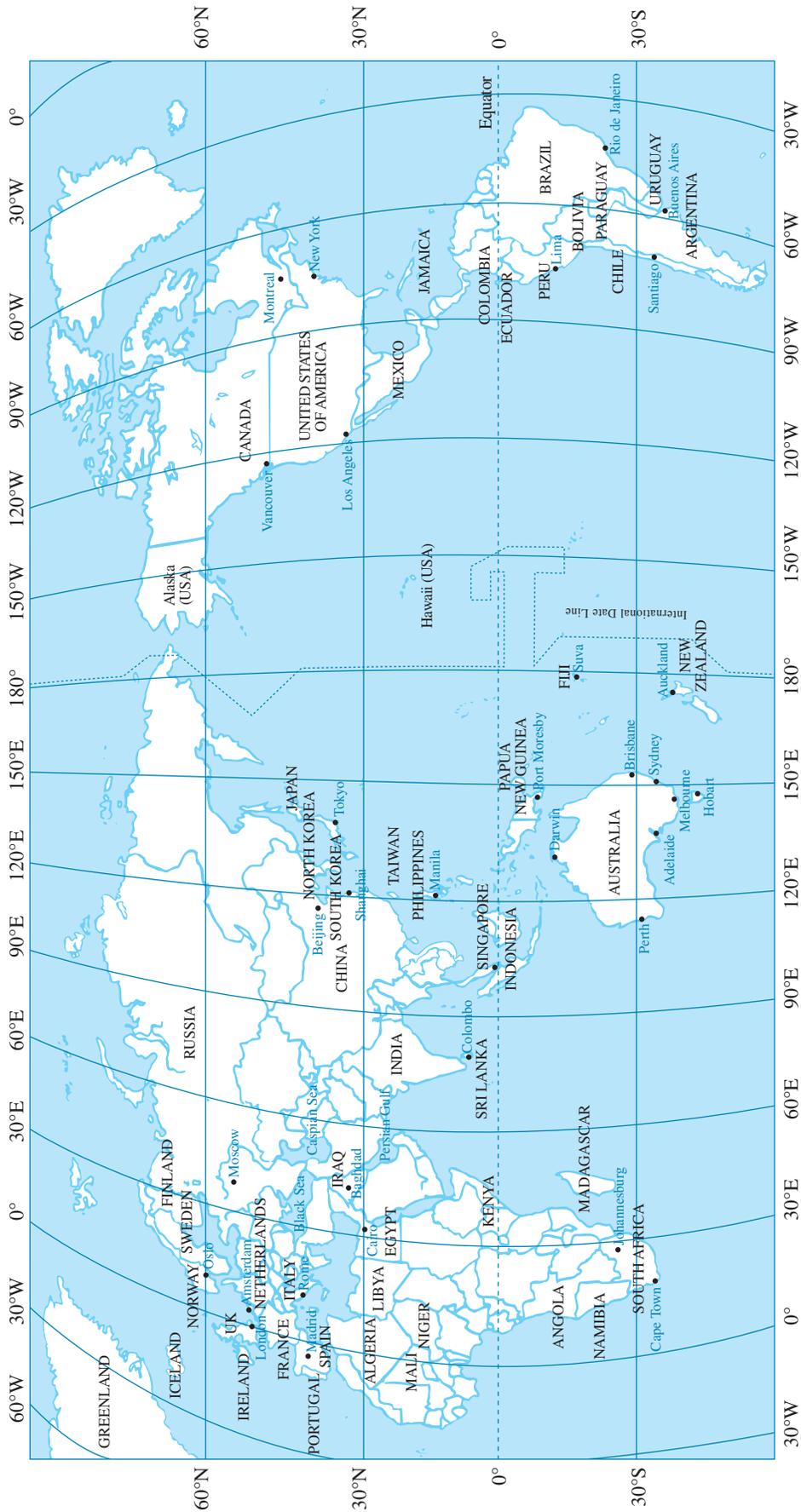
c. Los Angeles

THINK

- Use the parallels of latitude drawn to estimate the latitude.
 - Use the meridians of longitude drawn to estimate the longitude.
- Use the parallels of latitude drawn to estimate the latitude.
 - Use the meridians of longitude drawn to estimate the longitude.
- Use the parallels of latitude drawn to estimate the latitude.
 - Use the meridians of longitude drawn to estimate the longitude.

WRITE

- $1^\circ\text{N}, 104^\circ\text{E}$
- $32^\circ\text{S}, 115^\circ\text{E}$
- $35^\circ\text{N}, 118^\circ\text{W}$



6.2.3 Degrees, minutes and seconds

When you look at a protractor, one degree seems like a small unit of measurement. However, when applied to a great circle the size of Earth's circumference, that one degree of measurement can be over 100 km across.

Thus, there is a need for units of measurement smaller than a degree. Some maps will use decimal degrees, and some will use minutes and seconds.

Degrees, minutes and seconds

- 1 degree ($^{\circ}$) = 60 minutes ($'$)
- 1 minute ($'$) = 60 seconds ($''$)

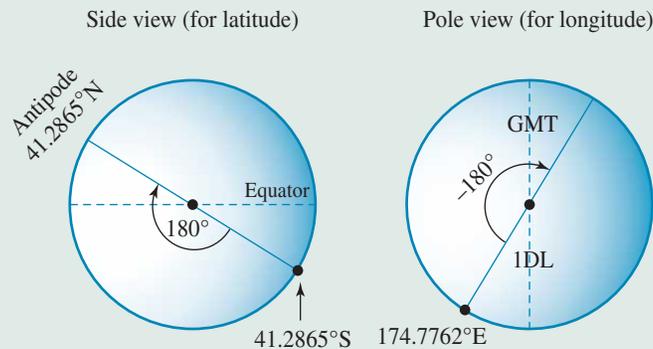
For example, the approximate location of Brisbane is (27°S , 153°E), but the exact location can be expressed in:

- decimal degrees (27.47°S , 153.03°E)
- degrees, minutes and seconds ($27^{\circ}28'\text{S}$, $153^{\circ}02'\text{E}$).



Activity: Exploring Earth!

You need to use mapping software such as Google Earth to answer the questions below.



1. What city am I, and what are my coordinates?
 - a. My latitude is the same as Cairo, Egypt, and my longitude is the same as New York City, USA.
 - b. My latitude is the same as Tokyo, Japan and my longitude is the same as Cape Town, South Africa.
 - c. Explore using mapping software to determine more cities with the same latitude or same longitude.
2. The [Map of Indigenous Australia | AIATSIS corporate website](#) shows the approximate boundaries of different Aboriginal and Torres Strait Islander groups. Use the map and a complementary mapping software to identify the northern, southern, eastern and western coordinates of the land on which your school resides.

If you could draw a straight line through the centre of Earth from one city, the other point where the line emerges is its antipode. Australians (and New Zealanders) are often called Antipodeans by people in the Northern Hemisphere, particularly in Europe and the United Kingdom. This is because Australia and New Zealand are geographically located on the opposite side of Earth from much of Europe.

3. The coordinates of **Wellington, New Zealand**, are (41.2865°S , 174.7762°E). Determine the coordinates and the nearest city located on the exact opposite side of Earth (the antipode) from Wellington.

WORKED EXAMPLE 4 Determining latitude and longitude in degrees and minutes for positions on Earth's surface

The following map shows the latitude and longitude locations of Badu and Moa Island.



Source: Data based on QSpatial, State of Queensland (Department of Natural Resources, Mines and Energy, Department of Environment and Science), <http://qdsatial.information.qld.gov.au/catalogue/>; Geoscience Australia.

- a. Write the approximate coordinates of each of the following places in degrees and minutes.
 - i. Rugged point
 - ii. Matu (Whale) Island
- b. Identify the location closest to:
 - i. ($10^{\circ}11'S$, $142^{\circ}23'E$)
 - ii. ($10^{\circ}16'S$, $142^{\circ}17'E$)

THINK

- a. i. 1. Use the parallels of latitude drawn to estimate the latitude.
2. Use the meridians of longitude drawn to estimate the longitude.
- ii. 1. Use the parallels of latitude drawn to estimate the latitude.
2. Use the meridians of longitude drawn to estimate the longitude.
- b. i. Estimate where the $10^{\circ}11'S$, parallel of latitude would lie. Estimate where the $142^{\circ}23'E$ meridian of longitude would lie. Look for the location closest to the intersection of those two lines.
- ii. Estimate where the $10^{\circ}16'S$ parallel of latitude would lie. Estimate where the $142^{\circ}17'E$ meridian of longitude would lie. Look for the location closest to the intersection of those two lines.

WRITE

- a. i. ($10^{\circ}11'S$, $142^{\circ}08'E$)
- ii. ($10^{\circ}16'S$, $142^{\circ}05'E$)
- b. i. Poid
- b. ii. Pabi point

Digital technology**Converting between decimal degrees and degrees, minutes and seconds using technology**

Your calculator can be used to convert between decimal degrees and degrees, minutes, seconds.

- a. Convert 142.75° to degrees, minutes, seconds.
b. Convert $19^{\circ}10'$ to decimal degrees.

a. Type 142.75
press =
Press "DMS" or  to convert.



b. Type 19  10 
(The symbols may be the same)
press =
press "S – D" or "F – D" to convert to decimals.

**Exercise 6.2 Position on Earth****learnon****6.2 Exercise****6.2 Exam questions** **Simple familiar**

1, 2, 3, 4, 5, 6, 7,
8, 9, 10, 11, 12, 13,
14, 15, 16, 17

Complex familiar

–

Complex unfamiliar

–

These questions are
even better in jacPLUS!

- Receive immediate feedback
- Access sample responses
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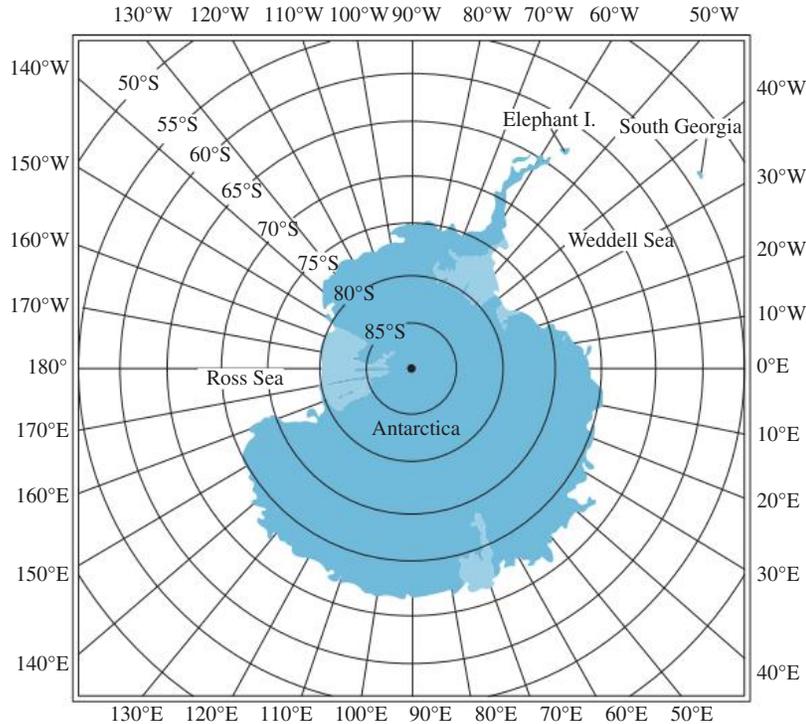


Find all this and MORE in jacPLUS 

Simple familiar

1. For two points along the equator (Location A and Location B), determine if they form a great circle or a small circle.

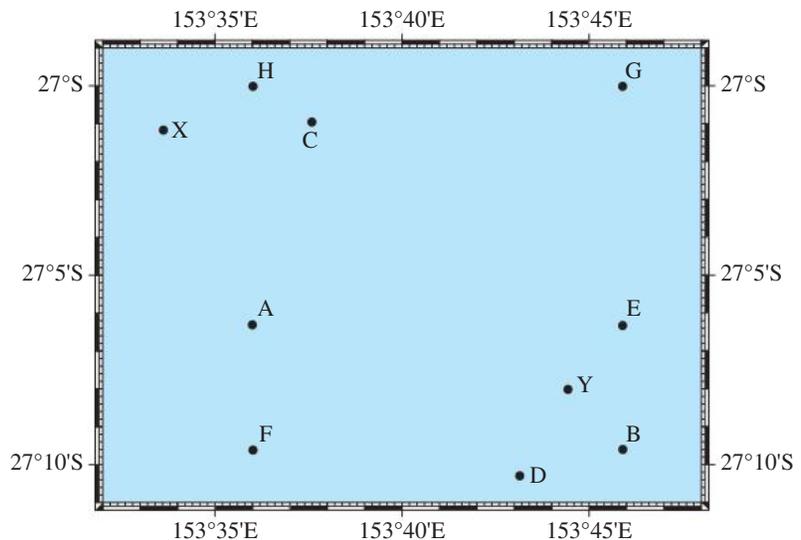
2. If a plane was circumnavigating Antarctica at 75°S , determine if the plane's route would form a great circle or a small circle.



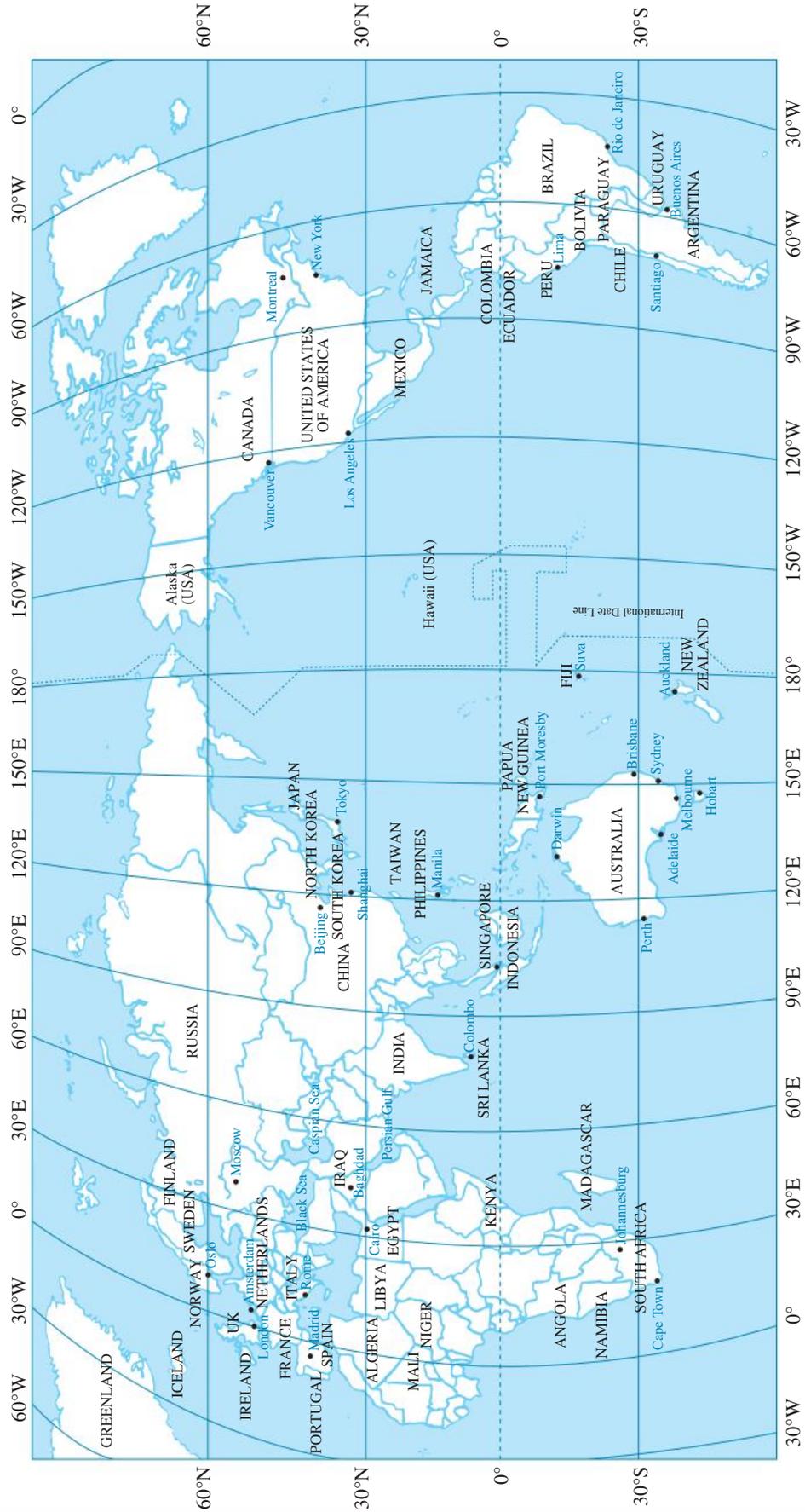
3. If one location is at 45°N and the other is at 45°S but on the same longitude, determine whether they lie on a great circle or a small circle.
4. Two locations are both at 60°N but are separated by 90° of longitude. Determine if they form a great circle or a small circle.
5. **WE1** Calculate the length of a great circle on Mars, where the radius is 3390 km. Give your answer rounded to the nearest km.
6. Calculate the length of a great circle on the Moon, where the radius is 1740 km. Give your answer rounded to the nearest km.
7. Draw a sphere. On it mark lines representing:
- 0° longitude
 - 20°E longitude
 - 50°W longitude.
8. Draw a sphere. On it mark:
- 0° latitude
 - 15° latitude North
 - the Tropic of Capricorn (23.5°S).

9. **WE4** Use the grid to determine the positions of the following points.

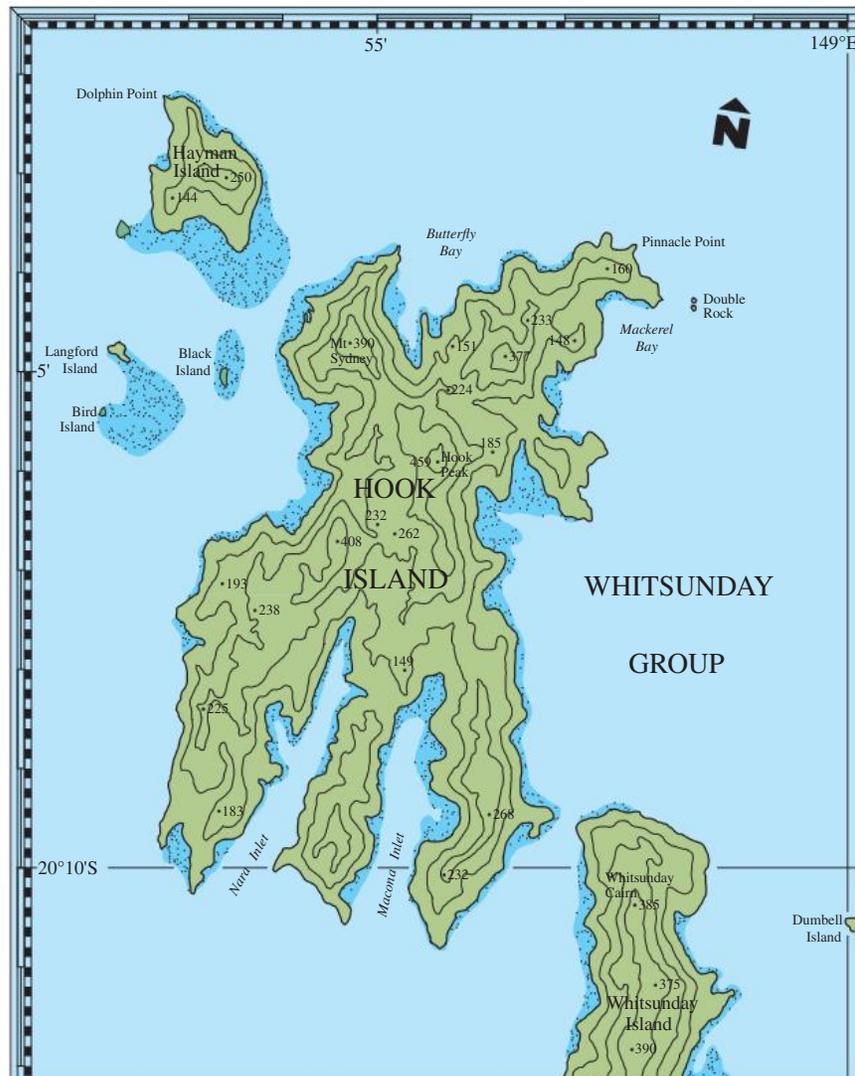
- F
- G
- H
- C
- X
- Y.



Use the following world map for questions 10 to 13.



10. **WE2** Write down the name of the city closest to each of the following pairs of coordinates.
- a. 30°N, 30°E b. 30°N, 120°E c. 15°S, 135°E d. 45°N, 75°W e. 50°N, 0°
11. Write down the name of the city closest to each of the following pairs of coordinates.
- a. 37°S, 175°E b. 35°N, 140°E c. 40°N, 115°E d. $22\frac{1}{2}^{\circ}$ S, 43°W e. 60°N, 11°E
12. **WE3** Determine the approximate latitude and longitude of each of the following major cities or islands.
- a. Melbourne b. New York c. Jamaica d. Johannesburg e. Rome
13. Determine the approximate latitude and longitude of each of the following major cities or islands.
- a. Buenos Aires b. Baghdad c. Moscow d. Singapore e. Suva
14. Plot the following positions on a grid.
- a. (27°S, 153°45'E) b. (27°6.3'S, 153°40'E)
 c. (27°7.4'S, 153°39.8'E) d. (27°10'S, 153°42.7'E)
15. Use the map of the Whitsunday Group to name the feature at:
- a. (20°5'S, 148°54.7'E) b. (20°5.1'S, 148°53.4'E) c. (20°3.8'S, 148°57.8'E)



16. Convert each of the following coordinates to decimal degrees.

a. (27°S, 153°45'E)

b. (27°6.3'S, 153°40'E)

c. (27°7.4'S, 153°39.8'E)

d. (27°10'S, 153°42.7'E)

17. Convert each of the following to degrees, minutes and seconds.

a. (37.7749°N, 122.4194°W)

b. (33.8688°S, 151.2093°E)

c. (51.5074°N, 0.1278°W)

d. (40.7128°N, 74.0060°W)

e. (23.5505°S, 46.6333°W)

Fully worked solutions for this chapter are available online.

LESSON

6.3 Distances on Earth's surface

SYLLABUS LINKS

- Calculate angular distance and distance between two places on Earth on the same meridian.
 - $D = 111.2 \times \text{angular distance}$ where D is distance in kilometres
- Calculate angular distance and distance between two places on Earth on the same parallel of latitude.
 - $D = 111.2 \cos(\theta) \times \text{angular distance}$ where D is distance in kilometres and θ is latitude
- Solve practical problems involving latitude, longitude, angular distance and distance.

Source: General Mathematics Senior Syllabus 2024 © State of Queensland (QCAA) 2024; licensed under CC BY 4.0.

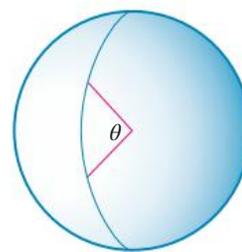
In this section, we shall consider distances between points on the same great circle, or the same small circle.

6.3.1 Points on the same great circle

A meridian is a circle of longitude passing through a given place on Earth's surface and the hemispherical poles. All meridians of longitude are half great circles. The equator is also a great circle. This means that measuring the distance between two points on the same meridian of longitude, or between two points on the equator, involves calculating the length of an arc of a great circle.

Consider a meridian of longitude on Earth's surface with two points on it. The angular distance between them will be the difference between their latitudes.

The angular distance is calculated by subtracting the latitudes of the points if both are on the same side of the equator and adding the latitudes if on opposite sides of the equator.



Hemispheres and angular distance

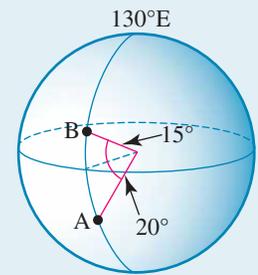
When two locations are in the same hemisphere (both North or both South) the angular distance can be calculated by subtracting the latitudes. However, if the two locations are in different hemispheres (one North, one South) the angular distance crosses the line of axis (the equator). In that case, you must add the latitudes.

Same side of the equator = subtract the latitudes

Opposite sides of the equator = add the latitudes

WORKED EXAMPLE 5 Determining angular distance

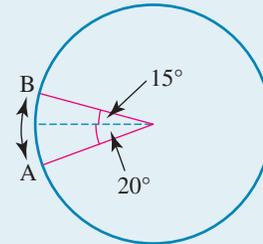
The coordinates of A are (20°S, 130°E) and the coordinates of B are (15°N, 130°E). Determine the angular distance between the two points.



THINK

1. Draw a circle with the important information.

WRITE



2. A and B are on opposite sides of the equator on the same meridian (130°E), so add the latitudes.

$$\begin{aligned}\text{Angular distance} &= 20^\circ + 15^\circ \\ &= 35^\circ\end{aligned}$$

Earlier, we discovered that the distance around a great circle on Earth is 40 030 km. This also represents an angular distance of 360°. Therefore:

$$\text{If } 360^\circ = 40\,030 \text{ km,}$$

$$\text{then } 1^\circ = 111.2 \text{ km.}$$

Thus, we have calculated that an angular distance of 1° on a great circle represents a distance of 111.2 km. Using this information, we can calculate the distance between two locations on a great circle.

Distance on a great circle

To calculate the distance (in kms) between two places on Earth, on the same meridian or equator, use the formula:

$$D = 111.2 \times \text{angular distance}$$

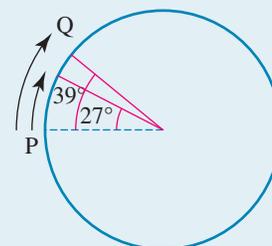
WORKED EXAMPLE 6 Calculating distance on a great circle

P and Q are two points on Earth's surface with coordinates (27°N, 30°W) and (39°N, 30°W) respectively. Calculate the distance between P and Q (to the nearest kilometre).

THINK

1. P and Q have the same longitude and therefore are on the same great circle. Draw a circle with the important information.

WRITE



2. Calculate the angular distance, PQ. As both P and Q are in the same hemisphere, subtract the angles.

$$\begin{aligned} \text{Angular distance} &= 39^\circ - 27^\circ \\ &= 12^\circ \end{aligned}$$

3. Calculate the distance in kilometres using $D = 111.2 \times \text{angular distance}$. Round the answer to the nearest kilometre.

$$\begin{aligned} \text{Distance} &= 111.2 \times 12 \\ &= 1334 \text{ km} \end{aligned}$$

6.3.2 Points on the same small circle

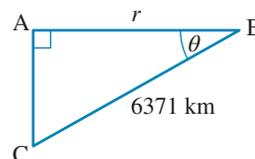
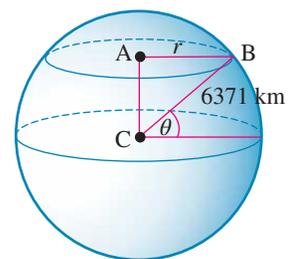
Points on the same parallel of latitude (other than the equator) lie on the circumference of a small circle. To determine the distance between two such points, we first need to know the radius of the small circle. To calculate the radius of a small circle, we use our knowledge of right-angled trigonometry.

To calculate the radius of latitude AB, we need to know the angular distance of the latitude; and Earth's radius.

Here is the same information represented as a right-angled triangle.

We have an angle (θ) and the hypotenuse length (6371), and we are trying to calculate the adjacent side.

$$\begin{aligned} \cos(\theta) &= \frac{\text{adjacent}}{\text{hypotenuse}} \\ &= \frac{r}{6371} \\ \therefore r &= 6371 \cos(\theta). \end{aligned}$$



Therefore, the radius of any parallel of latitude is proportional to the equator; adjusted by the $\cos(\theta)$.

Distance on a small circle

The distance between two places on the same parallel of latitude are closer together than two places on a great circle. To adjust accordingly, we use the formula:

$$D = 111.2 \times \cos(\theta) \times \text{angular distance}$$

where θ is the latitude of the two places. E.g. Brisbane is 27°S , so calculations on that latitude would require $D = 111.2 \times \cos(27^\circ) \times \text{angular distance}$.

When we travel between two cities, it makes sense to take the most direct route. For example, if we are flying from Cairns to Tokyo, a north-bound flight out of Cairns travels approximately 6000 km to reach Tokyo. If the flight out of Cairns travelled in the opposite direction (south), it would travel approximately 34 000 km to reach Tokyo.

It is possible for two locations to be 180° apart (antipodes). In this case, you could travel in either direction to reach your destination, and the distance would be the same. When calculating the distance between two locations, you may need to evaluate the reasonableness of the distance calculated to ensure that there is not a shorter route.

6.3 Exercise

6.3 Exam questions

Simple familiar

1, 2, 3, 4, 5, 6, 7, 8,
9, 10, 11, 12, 13, 14

Complex familiar

15, 16

Complex unfamiliar

17, 18

These questions are even better in jacPLUS!

- Receive immediate feedback
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- Track results and progress



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Simple familiar

- WE5** Two points, A and B, on Earth's surface are at $(30^\circ\text{N}, 25^\circ\text{W})$ and $(20^\circ\text{S}, 25^\circ\text{W})$. Calculate the angular distance between A and B.
- In each of the following, calculate the angular distance between the pairs of points given.
 - $(70^\circ\text{N}, 150^\circ\text{E})$ and $(30^\circ\text{N}, 150^\circ\text{E})$
 - $(40^\circ\text{N}, 25^\circ\text{W})$ and $(40^\circ\text{S}, 15^\circ\text{W})$
 - $(64^\circ\text{N}, 0^\circ)$ and $(7^\circ\text{S}, 0^\circ)$
 - $(42^\circ\text{S}, 97^\circ\text{W})$ and $(21^\circ\text{S}, 97^\circ\text{W})$
 - $(20^\circ\text{S}, 60^\circ\text{E})$ and $(20^\circ\text{S}, 0^\circ)$
- The city of Durban is at approximately $(30^\circ\text{S}, 30^\circ\text{E})$ and Cairo is at $(30^\circ\text{N}, 30^\circ\text{E})$. Calculate the angular distance between Durban and Cairo.
- WE6** P and Q are two points on Earth's surface with coordinates $(45^\circ\text{N}, 10^\circ\text{W})$ and $(15^\circ\text{N}, 10^\circ\text{W})$ respectively. Calculate the distance between P and Q to the nearest kilometre.
- Calculate the distance between each of the following points to the nearest kilometre.
 - A $(10^\circ\text{N}, 45^\circ\text{E})$ and B $(25^\circ\text{S}, 45^\circ\text{E})$
 - C $(75^\circ\text{N}, 86^\circ\text{W})$ and D $(60^\circ\text{S}, 86^\circ\text{W})$
 - E $(46^\circ\text{S}, 52^\circ\text{W})$ and F $(7^\circ\text{S}, 52^\circ\text{W})$
 - G $(34^\circ\text{N}, 172^\circ\text{E})$ and H $(62^\circ\text{S}, 172^\circ\text{E})$
- WE7** Freetown $(8^\circ\text{N}, 13^\circ\text{W})$ and Eyl $(8^\circ\text{N}, 50^\circ\text{E})$ are two cities on the same small circle.
 - Calculate the angular distance between Freetown and Eyl.
 - Calculate the shortest distance between them, to the nearest 100 km.
- Determine the shortest distance (in km) between the following places. Give your answers to the nearest kilometre.
 - $40^\circ\text{S}, 130^\circ\text{E}$ and $40^\circ\text{S}, 159^\circ\text{E}$
 - $70^\circ\text{N}, 15^\circ\text{E}$ and $70^\circ\text{N}, 100^\circ\text{E}$
 - $50^\circ\text{S}, 66^\circ\text{W}$ and $50^\circ\text{S}, 106^\circ\text{W}$
 - $80^\circ\text{S}, 67^\circ\text{W}$ and $80^\circ\text{S}, 89^\circ\text{W}$
 - $20^\circ\text{S}, 150^\circ\text{E}$ and $20^\circ\text{S}, 54^\circ\text{W}$
 - $30^\circ\text{N}, 28^\circ\text{E}$ and $30^\circ\text{N}, 39^\circ\text{W}$
- The Tropic of Cancer is at latitude $23\frac{1}{2}^\circ\text{N}$ and the Tropic of Capricorn is at latitude $23\frac{1}{2}^\circ\text{S}$. Calculate the distance between these two tropics along the same great circle in kilometres (correct to the nearest km).
- The city of Kingston is approximately at $(18^\circ\text{N}, 76^\circ\text{W})$. Naloung is at approximately $(18^\circ\text{N}, 104^\circ\text{E})$.
 - Calculate the angular distance between Kingston and Naloung.
 - Calculate the distance between Kingston and Naloung, to the nearest kilometre.
- The city of Osaka is at $(37^\circ\text{N}, 135^\circ\text{E})$ and Alice Springs is at $(23^\circ\text{S}, 135^\circ\text{E})$. Calculate the distance between Osaka and Alice Springs, to the nearest kilometre.
- M and N are two points on Earth's surface with coordinates $(56^\circ\text{N}, 122^\circ\text{W})$ and $(3^\circ\text{S}, 122^\circ\text{W})$. Calculate the distance, MN, correct to the nearest 100 km.
- Two Antarctic research stations, Casey (110°E) and Dumont d'Urville (140°E) lie on the parallel of latitude that is the Antarctic circle (66.5°S) . Calculate the shortest distance between these two research stations, to the nearest kilometre.

13. Determine the distance (in km) between two places on the equator which have a difference in their longitudes of 35° .
14. Determine the distance in kilometres from each of the following places to the closer pole. Indicate which pole (North or South) is the closer.
- | | |
|--|---|
| a. Warwick (28°S , 152°E) | b. Vancouver (49°N , 123°W) |
| c. St Moritz (46°N , 10°E) | d. Thursday Island (10°S , 142°E) |

Complex familiar

15. A shipping company wants to transport goods from Point A (37.7749°S , 122.4194°E) near Esperance to a location at Point B (37.7749°S , 140.4194°E) near Mount Gambier. However, due to rough seas across the Great Australian Bight, the ship must detour exactly halfway through the route. Determine how many kilometres (to two decimal places) the ship has already travelled before making this detour.
16. A GPS satellite system is trying to locate lost luggage packed with a tracking device. The system determines that the luggage is 695 km away from a satellite at ($27^\circ00'\text{S}$, $80^\circ11'\text{E}$) in the Southern Indian Ocean. Another satellite is located at ($35^\circ45'\text{S}$, $80^\circ11'\text{E}$), directly south along the same longitude. The luggage lies 278 km away from this second satellite. Determine the coordinates of the luggage.



Complex unfamiliar

17. An airplane travels from Santiago, Chile (33.4°S , 70.6°W) to Bogotá, Colombia (4.7°N , 70.6°W). The flight usually takes 6 hours. On this particular trip, the plane travels 10 per cent slower for the first 3 hours of the flight due to turbulence. Determine the speed needed in the second half, to ensure on-time arrival.
18. Mount Kosciuszko, located at ($36^\circ27'\text{S}$, $148^\circ16'\text{E}$) is Australia's highest peak, at an elevation of 2228 m above sea level. The nearest campground, Ngarigo, is due east of the summit, at the lesser altitude of 1227 m above sea level. The straight line diagonal from the mountain peak to the campground is 9 km. Determine the coordinates of the campground.



Fully worked solutions for this chapter are available online.

LESSON

6.4 Time zones

SYLLABUS LINKS

- Understand the meaning of Greenwich Mean Time (GMT), International Date Line and Coordinated Universal Time (UTC).
- Understand the link between longitude and time.
- Determine the number of degrees of longitude for a given time difference.
- Calculate time differences between two places on Earth.
- Solve practical problems involving time zones, making allowances for daylight saving where necessary, e.g. seasonal time systems used by Aboriginal peoples and Torres Strait Islander peoples, making phone calls, broadcasting events, travelling, preparing an itinerary.

Source: General Mathematics Senior Syllabus 2024 © State of Queensland (QCAA) 2024; licensed under CC BY 4.0.

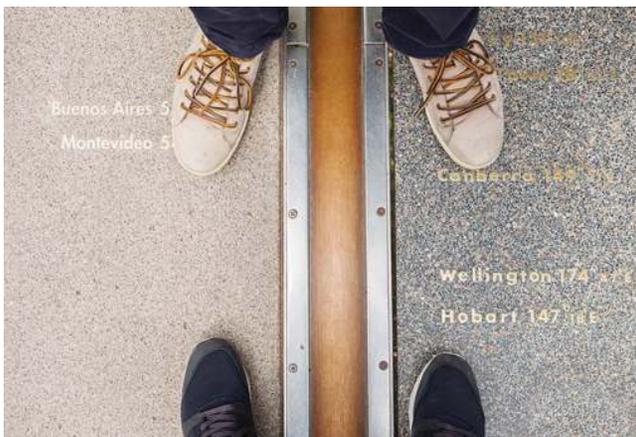
Earth takes 24 hours to rotate on its axis. This means that while one part of Earth is facing the Sun and experiencing daylight, it will be 12 hours until the countries on the opposite side of Earth face the Sun again. To simplify the partitioning of this rotation, Earth is divided into time zones.

6.4.1 GMT, IDL and UTC

int-6911

The time on the Greenwich Meridian is known as **Greenwich Mean Time (GMT)** and time zones are referenced as being ahead of (+), or behind (–), GMT. Being ahead of Greenwich (in the eastern hemisphere) means seeing the Sun ahead of, or before the people of Greenwich. Being behind Greenwich (in the western hemisphere) means seeing the Sun behind, or after the people of Greenwich. GMT was established over 150 years ago, and scientific equipment has since found a more precise reference point.

Coordinated Universal Time (UTC) is the modern, globally recognised reference point for time widely embraced by most countries and is the effective successor to GMT in everyday usage and common applications. UTC is within about one second of 0° longitude and has been adjusted several times in response to checks and balances to maintain the accuracy of the clocks that follow it.



Earth's 24 time zones

The circles of Earth comprise 360° and it takes 24 hours for Earth to rotate fully on its axis. Therefore, $\frac{360^\circ}{24 \text{ h}} = 15^\circ$ means that Earth rotates approximately 15° every hour.

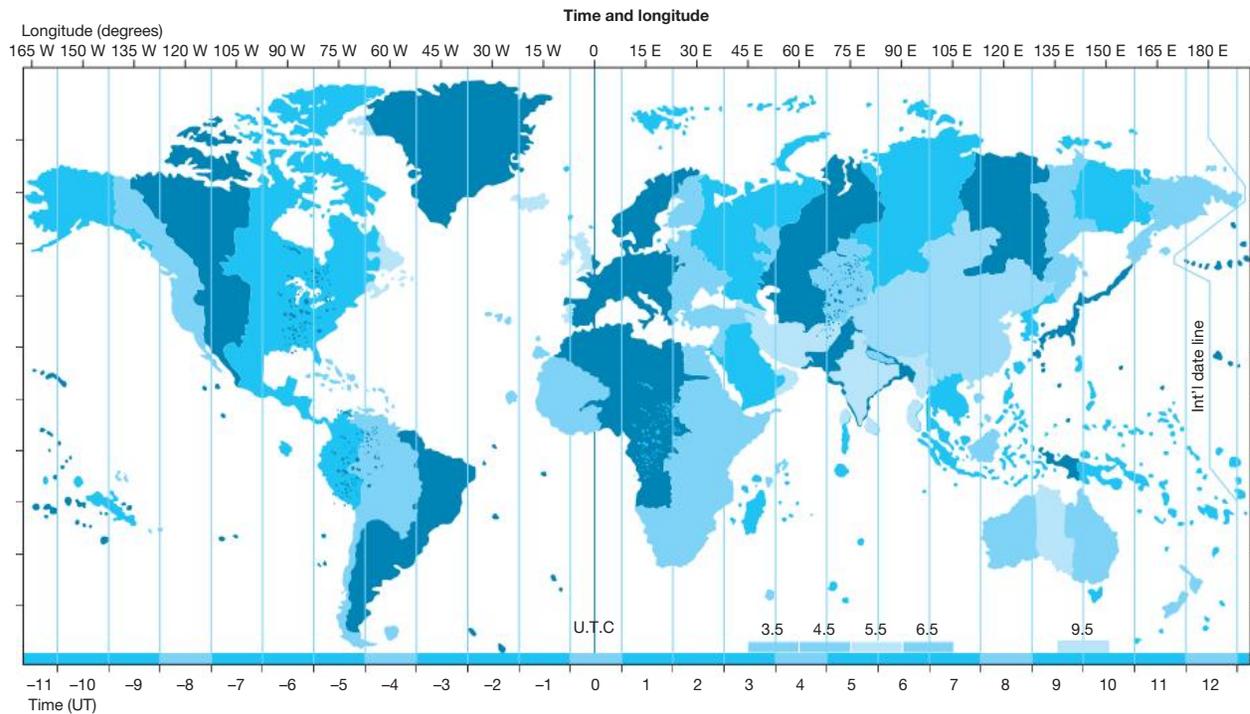
Time zones

360° Earth rotation = 24 hours of time

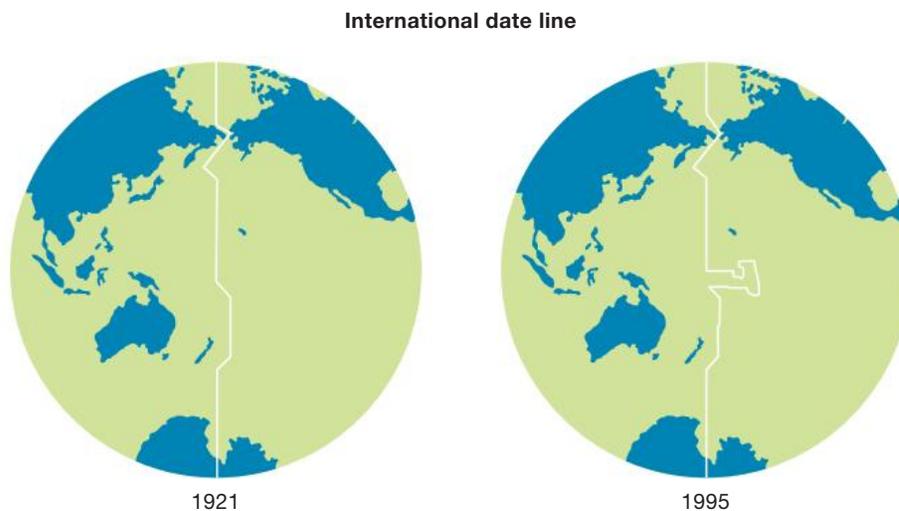
(÷24) 15° of longitude = 1 hour of time

(÷15) 1° of longitude = 4 minutes of time

Thus, the 24 different time zones of Earth can be represented by each 15° meridian. The time as calculated using the longitude of a place is called the *standard time*. Time zones are calculated to approximate all the standard times within a region, but governments can choose to adjust these time zones.



The **International Date Line (IDL)** is 12 hours ahead of Greenwich when travelling east and 12 hours behind when travelling west, so this totals 24 hours, or one day. Therefore, the day is always different on either side of the International Date Line. The International Date Line zigzags to accommodate the official international borders of countries and island nations.



WORKED EXAMPLE 8 Determining time zones

Determine the time zones of the following locations, with reference to UTC.

- New York City ($40^{\circ}42'N, 75^{\circ}0'W$)
- London ($51^{\circ}30'N, 0^{\circ}W$)
- Tokyo ($35^{\circ}41'N, 139^{\circ}41'E$)
- Sydney ($33^{\circ}52'S, 151^{\circ}12'E$)
- Cairo ($30^{\circ}3'N, 31^{\circ}14'E$)

THINK

- Identify the longitude of the place, since longitude determines the time zone.
 - Calculate the time zones.
- Identify the longitude of the place, since longitude determines the time zone.
 - Calculate the time zones.
- Identify the longitude of the place, since longitude determines the time zone.
 - Calculate the time zones.
- Identify the longitude of the place, since longitude determines the time zone.
 - Calculate the time zones.
- Identify the longitude of the place, since longitude determines the time zone.
 - Calculate the time zones.

WRITE

New York longitude = $75^{\circ}0'W$ of the Greenwich Meridian.

Thus, New York is behind (-) UTC.

$$\frac{75^{\circ}0'}{15} = 5 \text{ hours}$$

New York is UTC -5.

London longitude = $0^{\circ}W$ of the Greenwich Meridian.

Thus, London is UTC +0.

Tokyo longitude = $139^{\circ}41'E$ of the Greenwich Meridian.

Thus, Tokyo is ahead (+) of UTC.

$$\frac{139^{\circ}41'}{15} = 9.312 \text{ hours}$$

Tokyo is UTC +9.

Sydney longitude = $151^{\circ}12'E$ of the Greenwich Meridian.

Thus, Sydney is ahead (+) of UTC.

$$\frac{151^{\circ}12'E}{15} = 10.08 \text{ hours}$$

Sydney is UTC +10.

Cairo longitude = $31^{\circ}14'E$ of the Greenwich Meridian.

Thus, Cairo is ahead (+) of UTC.

$$\frac{31^{\circ}14'}{15} = 2.08 \text{ hours}$$

Cairo is UTC +2.

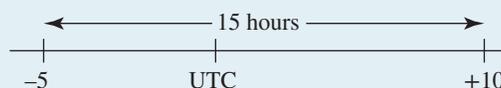
WORKED EXAMPLE 9 Using time zones to calculate time difference

Brisbane is UTC + 10 while New York is UTC - 5. Calculate the time difference between Brisbane and New York.

THINK

- Represent the information as a number line, with UTC at the centre.
- State the difference and which city is ahead in time.

WRITE



$$\begin{aligned} \text{Time difference} &= 10 - (-5) \\ &= 15 \end{aligned}$$

Brisbane is 15 hours ahead of New York.

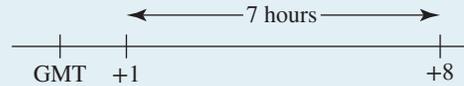
WORKED EXAMPLE 10 Using time zones to calculate time in a different location

Perth is UTC + 8 and Cape Town is UTC + 1. Determine the time in Perth when it is 11:00 am in Cape Town.

THINK

1. Represent the information as a number line, with UTC at the centre.

WRITE



$$\begin{aligned}\text{Time difference} &= 8 - 1 \\ &= 7 \text{ hours}\end{aligned}$$

Perth is 7 hours ahead of Cape Town.

2. Add the time difference to the time in Cape Town to calculate the time in Perth.

$$\begin{aligned}\text{Time in Perth} &= 11.00 \text{ am} + 7 \text{ hours} \\ &= 6.00 \text{ pm, same day}\end{aligned}$$



tivd-12195

WORKED EXAMPLE 11 Using coordinates to calculate time in a different location

Calculate the time in Los Angeles (34°N, 120°W) when it is 8:00 am on Wednesday in Sydney (33°S, 150°E).

THINK

1. Calculate the difference in longitudes.
2. Convert the difference in longitudes into hours by dividing by 15° for each hour.
3. Subtract the time difference from the time in Sydney.

WRITE

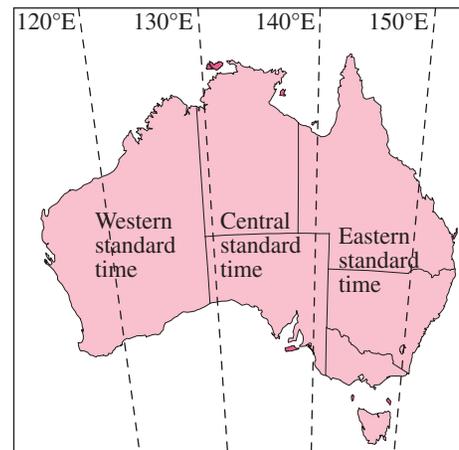
$$\begin{aligned}\text{Longitude difference} &= 150^\circ + 120^\circ \\ &= 270^\circ\end{aligned}$$

$$\begin{aligned}\text{Time difference} &= \frac{270^\circ}{15^\circ} \\ &= 18 \text{ hours}\end{aligned}$$

$$\begin{aligned}\text{Time in Los Angeles} &= 8.00 \text{ am Wednesday} - 18 \text{ hours} \\ &= 2.00 \text{ pm Tuesday}\end{aligned}$$

It is important to note that, for convenience, places that have almost the same longitude have the same time. An example of this is Australia's time zones, where all of Queensland, New South Wales, Victoria and Tasmania are in the same standard time zone, although there is a difference of 12° in longitude between the easternmost and westernmost points in this zone.

These calculations can then be used to calculate the arrival and departure times for domestic and international travel.



WORKED EXAMPLE 12 Determining the degrees of longitude for a given time difference

Blake is based in Perth (UTC + 8) and works remotely from their office headquarters in Sydney (UTC + 10). Determine approximately how many degrees of longitude separate these two locations.

THINK

1. Calculate the time difference.
2. Convert this time difference to degrees (using $15^\circ = 1$ hour).

WRITE

$$10 - 8 = 2 \text{ hours difference}$$
$$2 \text{ hours} \times 15^\circ = 30^\circ$$

Perth and Sydney are approximately 30° apart.



tlvd-12196

WORKED EXAMPLE 13 Using coordinates to calculate travel across time zones

A plane leaves London (50°N , 0°) at 9:00 am Sunday, London time and flies to Sydney (33°S , 150°E). The flight takes 20 hours. Calculate the time in Sydney when the plane arrives.

THINK

1. Calculate the longitude difference between Sydney and London.
2. Use $15^\circ = 1$ hour to calculate the time difference.
3. Sydney is 10 hours ahead of London. Calculate the time in Sydney when the plane is departing London by adding the time difference.
4. Add the flying time to calculate the time when the plane lands.

WRITE

$$\text{Longitude difference} = 150^\circ - 0^\circ$$
$$= 150^\circ$$

$$\text{Time difference} = 150 \div 15$$
$$= 10 \text{ hours}$$

When the plane leaves London at 9:00 am (London time):

$$\text{Time in Sydney} = 9:00 \text{ am Sunday} + 10 \text{ hours}$$
$$= 7:00 \text{ pm Sunday}$$

Plane arrives at

$$7:00 \text{ pm Sunday} + 20 \text{ hours}$$
$$= 3:00 \text{ pm Monday}$$

More challenging examples will require you to allow for daylight saving time. When daylight saving time applies, we add one hour to the standard time at that location. For example, Sydney's time zone is usually UTC + 10. However, during daylight savings, Sydney moves forward one hour to UTC + 11. Similarly, Ottawa is usually UTC - 5. However, during daylight savings they move forward one hour to UTC - 4.

In Worked Example 13, the time that the plane arrives in Sydney will be 3:00 pm Monday AEST (Australian Eastern Standard Time) or 4:00 pm Monday during daylight saving time (AEDT).

6.4 Exercise

6.4 Exam questions

Simple familiar

1, 2, 3, 4, 5, 6, 7,
8, 9, 10, 11, 12, 13,
14, 15, 16

Complex familiar

17, 18, 19

Complex unfamiliar

20

These questions are even better in jacPLUS!

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- Track results and progress



Find all this and MORE in jacPLUS

Simple familiar

- WE8** Determine the time zones of the following locations, with reference to UTC.

 - Perth (32°S , 116°E)
 - Sao Paulo (24°S , 47°W)
 - Johannesburg (26°S , 28°E)
 - Buenos Aires (35°S , 58°W)
 - Auckland (37°S , 175°E)
- WE9** The time zone for New Zealand is GMT + 12, whereas for Turkey it is GMT + 2. Calculate the time difference between New Zealand and Turkey.
- Calculate the time difference between each of the following locations.

 - Tokyo GMT + 9 and New York GMT – 5
 - Los Angeles UTC – 8 and Dakar UTC – 1
 - Rio De Janeiro GMT – 3 and Perth GMT + 8
 - Hawaii UTC – 11 and Fiji UTC + 11
- WE10** Brisbane is GMT + 10 and San Francisco is GMT – 8. Determine the time in San Francisco when it is 5:00 pm on Tuesday in Brisbane.
- Calculate:

 - the time in Perth (GMT + 8) when it is 10:00 pm in Brisbane (GMT + 10)
 - the time in Washington (UTC – 5) when it is 4:00 am Saturday in Brisbane (UTC + 10)
 - the time in Auckland (GMT + 12) when it is 7:00 am Wednesday in Johannesburg (GMT + 2)
 - the time in Sydney (UTC + 10) when it is 6:00 am Tuesday in Salt Lake City (UTC – 7)
 - the time in Adelaide (GMT + 9.5) when it is 8:15 pm Sunday in the Cook Islands (GMT – 10).
- Jane is in Townsville (GMT + 10) and wants to telephone her friend in Paris (GMT) at 7:00 pm Friday, Paris time. Determine the time at which she must call from Townsville.
- Carl is holidaying in Hawaii (UTC – 11). If he wants to call his parents in Rockhampton (UTC + 10) at 8:00 pm, on Wednesday, Rockhampton time, determine the time at which he must call from Hawaii.
- Neville is in Brisbane (GMT + 10). He wants to set his video recorder to tape the Superbowl, which is being played in Atlanta (GMT – 5) and televised live in Brisbane. The Superbowl is due to begin at 7:00 pm on Sunday, Atlanta time. Determine the day and time Neville will need to set his video to begin taping.
- Sydney is GMT + 10.

 - Determine the time in Sydney when it is 12 noon GMT.
 - When daylight saving time is applied, describe the time zone in Sydney as compared to GMT.
 - During daylight saving time, determine the time in Sydney when it is:
 - 4:00 am Monday GMT
 - 9:00 pm Thursday GMT.



10. **WE11** Beijing is at approximately $(40^{\circ}\text{N}, 120^{\circ}\text{E})$. Rome is at approximately $(40^{\circ}\text{N}, 15^{\circ}\text{E})$. Calculate the difference in standard time between Beijing and Rome.
11. Melbourne is at UTC + 10 and Los Angeles is at UTC – 8. Calculate the time difference between Melbourne and Los Angeles when:
- both cities are on standard time
 - Melbourne has daylight saving time and Los Angeles is on standard time
 - Los Angeles has daylight saving time and Melbourne is on standard time.
12. Calculate the time difference in standard time between each of the following cities.
- Mumbai $(19^{\circ}\text{N}, 73^{\circ}\text{E})$ and Casablanca $(23^{\circ}\text{N}, 82^{\circ}\text{W})$
 - Tokyo $(36^{\circ}\text{N}, 140^{\circ}\text{E})$ and Adelaide $(23^{\circ}\text{S}, 134^{\circ}\text{E})$
 - Miami $(26^{\circ}\text{N}, 80^{\circ}\text{W})$ and Seattle $(47^{\circ}\text{N}, 122^{\circ}\text{W})$
13. **MC** At a point on Earth's surface, the coordinates are $(45^{\circ}\text{N}, 135^{\circ}\text{W})$. The standard time at this point would be
- GMT – 3.
 - GMT + 3.
 - GMT – 9.
 - GMT + 9.
14. **MC** It is 11:00 am Tuesday at a point X, with coordinates $(32^{\circ}\text{S}, 90^{\circ}\text{W})$. Determine the time at point Y, with coordinates $(51^{\circ}\text{N}, 120^{\circ}\text{E})$, if daylight saving time applies at Y.
- 9:00 pm Monday
 - 10:00 pm Monday
 - 11:00 pm Tuesday
 - 2:00 am Wednesday
15. **WE12** New York City is in the Eastern Standard Time Zone (UTC – 5) and London is in the Greenwich Mean Time Zone (UTC + 0). Calculate the difference in degrees of longitude between these two cities, assuming each time zone represents 15° of longitude.
16. Tokyo operates in Japan Standard Time (UTC + 9) and Los Angeles is in Pacific Standard Time (UTC – 8). Determine how many degrees of longitude separate these two cities based on their time zones.

Complex familiar

17. **WE13** A plane leaves Sydney $(32^{\circ}\text{S}, 150^{\circ}\text{E})$ at 2:00 pm on Tuesday. If it is an 18-hour flight to Los Angeles $(33^{\circ}\text{N}, 120^{\circ}\text{W})$, determine the time at which the plane will touch down in Los Angeles.
18. A plane leaves Perth $(32^{\circ}\text{S}, 120^{\circ}\text{E})$ on an 8-hour flight to Cape Town $(33^{\circ}\text{S}, 15^{\circ}\text{E})$ at 3:00 pm Wednesday.
- Determine the time at which the plane will arrive in Cape Town.
 - The return flight leaves Cape Town at 5:00 pm Saturday. Determine the time at which it will arrive in Perth if the flight time is still 8 hours.
19. A flight leaves Melbourne $(40^{\circ}\text{S}, 150^{\circ}\text{E})$ at 5:00 pm Tuesday on an 18-hour flight to Frankfurt $(50^{\circ}\text{N}, 15^{\circ}\text{E})$. Calculate the time of arrival in Frankfurt if it is:
- daylight saving time in Melbourne only
 - daylight saving time in Frankfurt only.

Complex unfamiliar

20. Sophie is travelling on a research trip. She departed from Port Macquarie $(31^{\circ}40'\text{S}, 152^{\circ}54'\text{E})$, and her destination is Perth $(31^{\circ}40'\text{S}, 115^{\circ}51'\text{E})$. The average speed of the light plane was 620 km/h. When Sophie landed, the local time in Perth was 10:30 am on the same day she departed. Determine what time it was in Port Macquarie (not during daylight savings) when she departed.

Fully worked solutions for this chapter are available online.

LESSON

6.5 Review

6.5.1 Summary

Hey students! Now that it's time to revise this topic, go online to:



Access the chapter summary



Review your results



Practise exam questions

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6.5 Exercise

learnon

6.5 Exercise

6.5 QCAA questions **on**

Simple familiar

1, 2, 3, 4, 5, 6, 7,
8, 9, 10, 11, 12, 13,
14, 15, 16

Complex familiar

17, 18

Complex unfamiliar

19, 20

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- Receive immediate feedback
- Access sample responses
- Track results and progress



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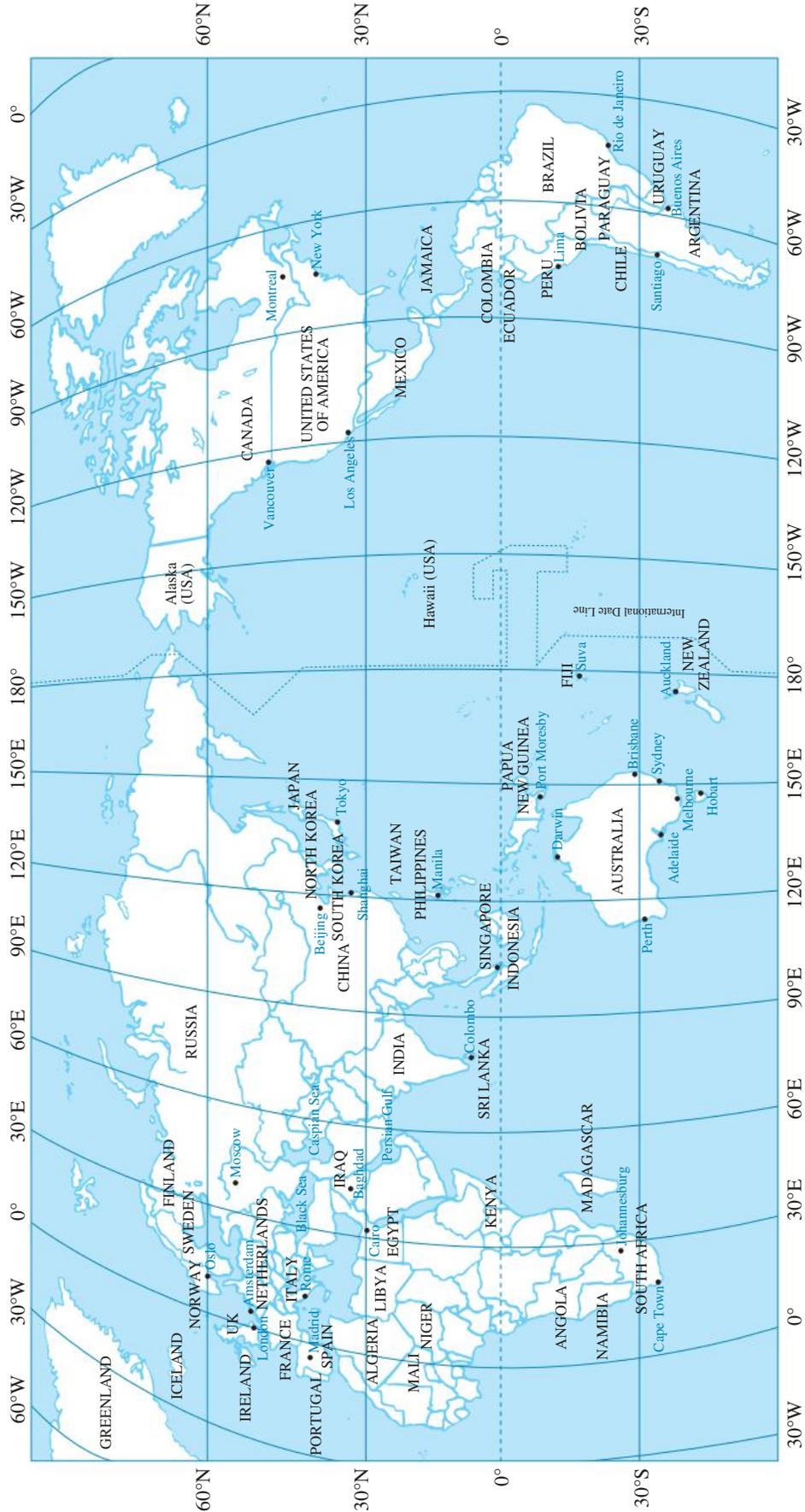


Simple familiar

- Sydney and Cape Town are both 33.5° south of the equator. Determine if they are on the same great circle or small circle.
 - Determine if the following places are on the same great circle or small circle: Indonesia ($0^\circ 0' N$ $127^\circ 53' E$) and Ecuador ($0^\circ 0' N$ $91^\circ 35' W$).
- Calculate the circumference of a great circle that lies on the surface of a sphere with a radius of 9 km. (Give your answer rounded to one decimal place.)
- Calculate the distance between the North Pole and the South Pole in kilometres.



Use the following world map for questions 5 and 6.



4. A sphere has a diameter of 45 cm. Calculate the distance on a great circle between the poles on this sphere. Give your answer rounded to the nearest centimetre.
5. Identify the cities at each of the following locations.
 - a. (14°N, 121°E)
 - b. (12°S, 76°W)
 - c. (33°S, 71°W)
6. Determine the approximate coordinates of each of the following cities.
 - a. Madrid
 - b. Singapore
 - c. Hobart
7. The points X and Y on Earth's surface have coordinates (32°N, 120°E) and (26°S, 120°E). Calculate the angular distance between X and Y.
8. A ship gives its coordinates as (56°N, 14°W) and is sailing to a port at (40°N, 14°W).
 - a. Calculate the angular distance through which the ship must sail to reach port.
 - b. Calculate the distance the ship must sail, to the nearest kilometre.
9. Santiago has approximate coordinates (33°S, 70°W) and Santo Domingo has approximate coordinates (18°N, 70°W). Calculate the distance between Santiago and Santo Domingo, to the nearest kilometre.
10. Calculate the shortest distance between the following points (to the nearest kilometre).
 - a. X(40°N, 120°E) and Y(40°N, 70°W)
 - b. A(23°S, 0°E) and B(23°S, 180°E)
11. Calculate the time difference between each of the following pairs of cities.
 - a. Sydney (UTC + 10) and Istanbul (UTC + 2)
 - b. Perth (GMT + 8) and New York (GMT - 3)
 - c. Ottawa (UTC - 5) and Fiji (UTC + 12)



12. Ann is on a skiing holiday in Winnipeg, Canada (UTC - 6). She needs to call her parents at 7:30 pm on Tuesday night, Brisbane time (UTC + 10). Determine the time at which she should make the call in Winnipeg.
13. Kingston, Jamaica, is at approximately (18°N, 75°W) and Oslo, Norway, is at approximately (60°N, 15°E). Calculate the time:
 - a. in Oslo when it is 5:00 am in Kingston
 - b. in Kingston when it is 5:00 pm in Oslo.

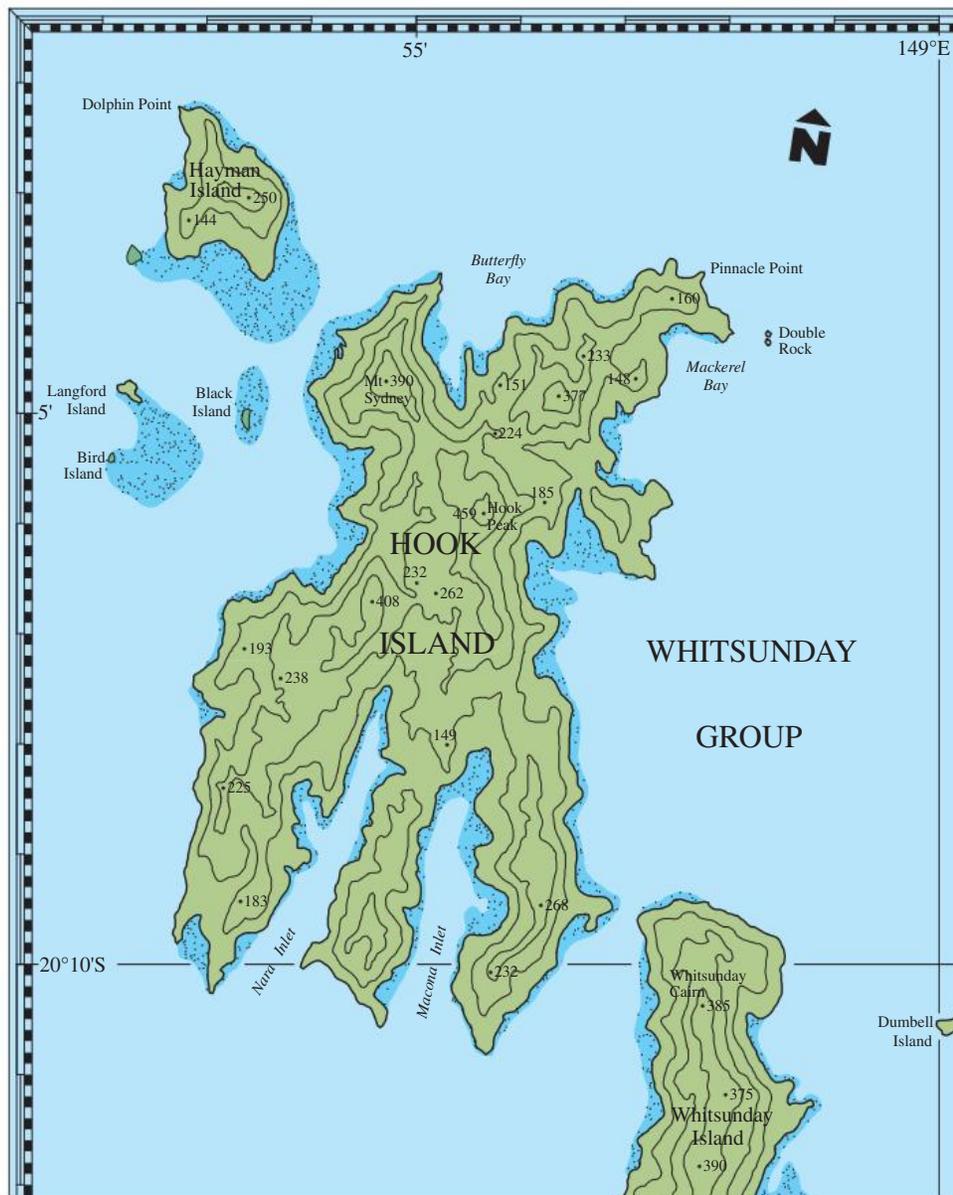
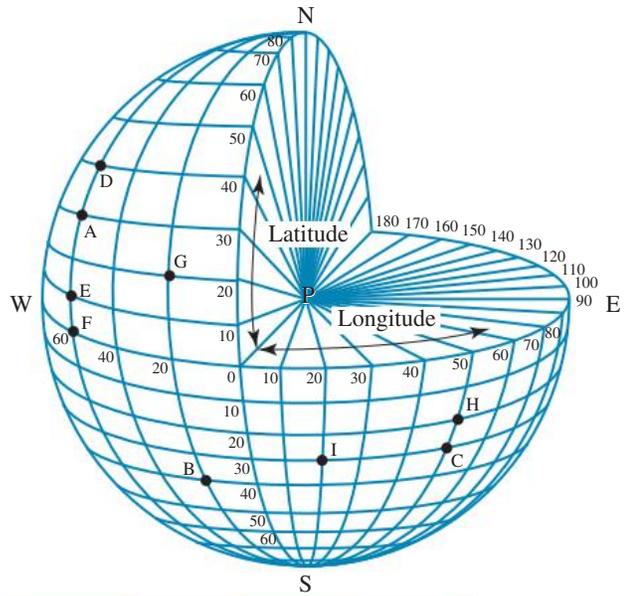


14. The diagram shown represents Earth.

- a. Give the position of A, B, C and D.
- b. Name a point on the equator.

15. Use the following map of the Whitsunday Group to give the position of:

- a. Dolphin Point on Hayman Island
- b. the entrance to Nara Inlet.

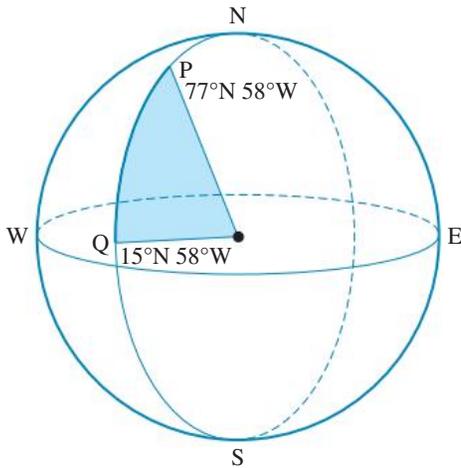


16. Calculate the distances travelled in the following situations.
- An aeroplane travels along the equator from a point of longitude 39°W to 102°E .
 - A ship travels from a point P: 11°N , 19°W to point Q: 18°S , 19°W .

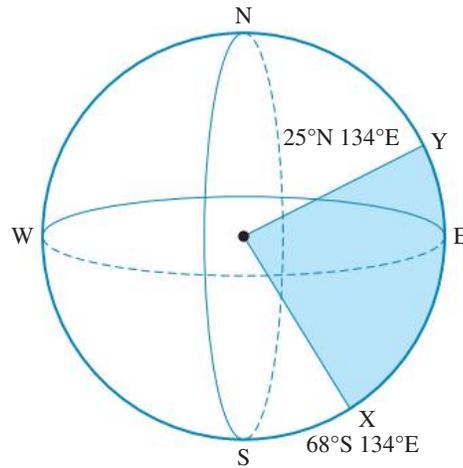
Complex familiar

17. A plane is flying from Munich (48°N , 15°E) to New York (41°N , 75°W). The flight departs Munich at 6:00 pm and takes 7 hours. Calculate the time of arrival in New York.
18. Calculate the great circle distances between the two points shown on each sphere, representing Earth (radius 6371 km). Give your answer to the nearest kilometre.

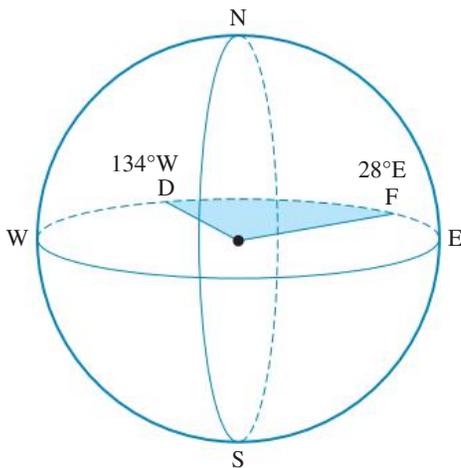
a.



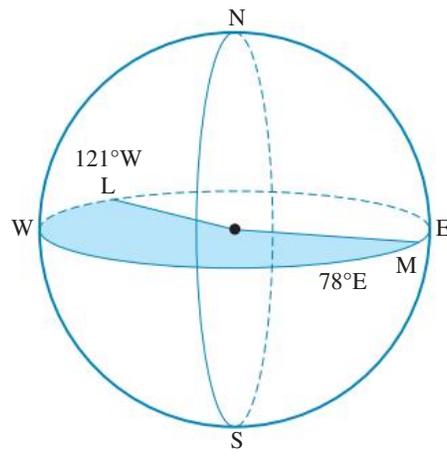
b.



c.



d.



Complex unfamiliar

19. The cities of Mecca in Saudi Arabia and Ha Long in Vietnam lie on the same parallel of latitude (21°N). They are 6955 km apart. If Mecca is at 40°E , determine the coordinates of the more eastern city, Ha Long.
20. The towns of Yarrawonga and Griffith lie on the same meridian and are 190 km apart. If Griffith, the southernmost city, lies on a latitude of 34.3°S , determine the latitude of Yarrawonga.

6.5 Past QCAA exam questions

Question 1 (3 marks)

Source: QCAA 2023 General Mathematics, Paper 1, Section 2, Q16; © State of Queensland (QCAA) 2024

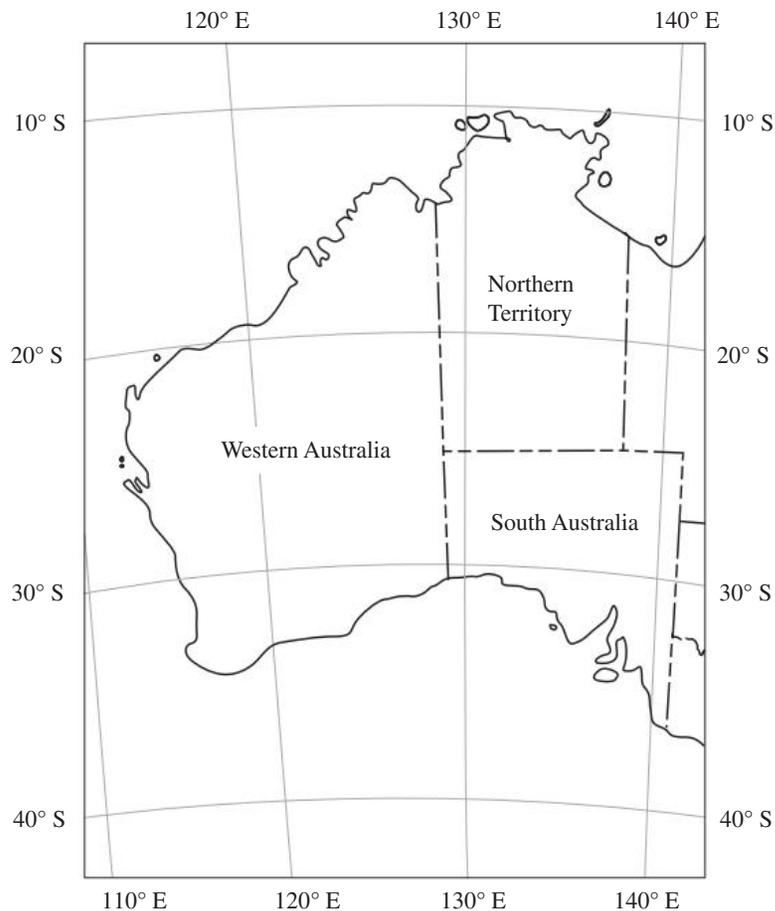
If it is 2:00 am local time in town A($30^{\circ}\text{N } 90^{\circ}\text{W}$), calculate the local time in town B($26^{\circ}\text{S } 120^{\circ}\text{E}$).

Question 2 (4 marks)

Source: QCAA 2023 General Mathematics, Paper 1, Section 2, Q18; © State of Queensland (QCAA) 2024

The locations of three spaces research sites in Australia are listed in the table.

Site	Coordinates
A	$32^{\circ}\text{S } 116^{\circ}\text{E}$
B	$32^{\circ}\text{S } 136^{\circ}\text{E}$
C	$12^{\circ}\text{S } 136^{\circ}\text{E}$

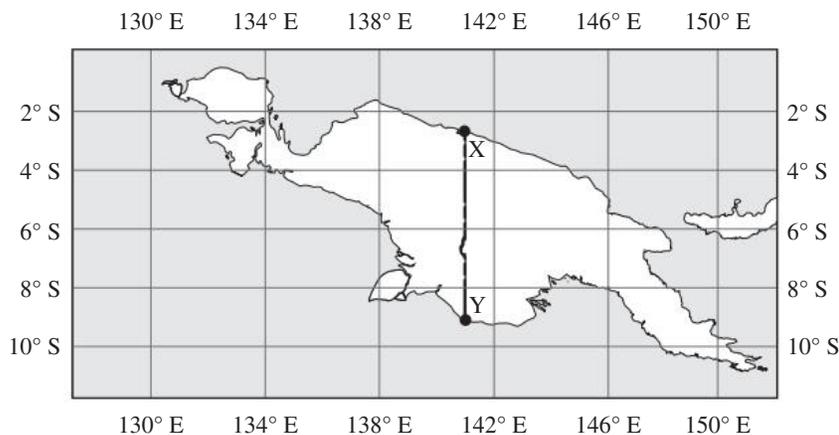


- Which site is closest to the equator? [1 mark]
- Name the state or territory in which site A is located. [1 mark]
- Determine and explain which two sites are in the same standard time zone. [2 marks]

Question 3 (5 marks)

Source: QCAA 2023 General Mathematics, Paper 1, Section 2, Q25; © State of Queensland (QCAA) 2024

The map shows a land border from X to Y.



- State the latitude and longitude of X to the nearest degree. [1 mark]
- State the latitude and longitude of Y to the nearest degree. [1 mark]
- Calculate the distance between X and Y in kilometres. [3 marks]

Question 4 (1 mark)

Source: QCAA 2022 General Mathematics, Paper 1, Section 1, Q14; © State of Queensland (QCAA) 2024

A rugby fan in Perth (Australia) plans to watch a live match played in New Zealand next winter. The time zone for Perth is UTC+8. The time for New Zealand is UTC+13 in winter and UTC+12 in summer.

If the match is played at 6:30 pm New Zealand time, what time will match be viewed in Perth?

- A. 1:30 pm B. 2:30 pm C. 10:30 pm D. 11:30 pm

Question 5 (1 mark)

Source: QCAA 2022 General Mathematics, Paper 1, Section 1, Q15; © State of Queensland (QCAA) 2024

The actual distance between two cities has been correctly calculated as 556 km. The latitude and longitude respectively of these two cities could be

- A. 2°N 104°W and 3°S 104°W B. 2°N 104°W and 3°N 104°W
C. 25°N 150°E and 30°S 150°E D. 25°N 145°E and 25°N 150°E

Fully worked solutions for this chapter are available online.

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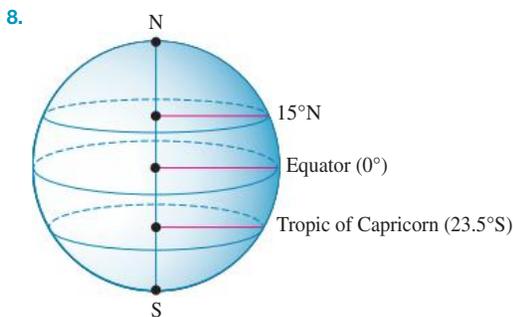
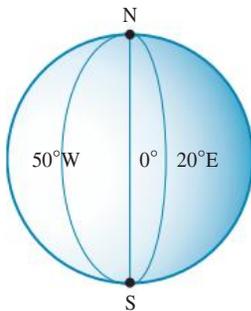
Answers

Chapter 6 Earth geometry and time zones

6.2 Position on Earth

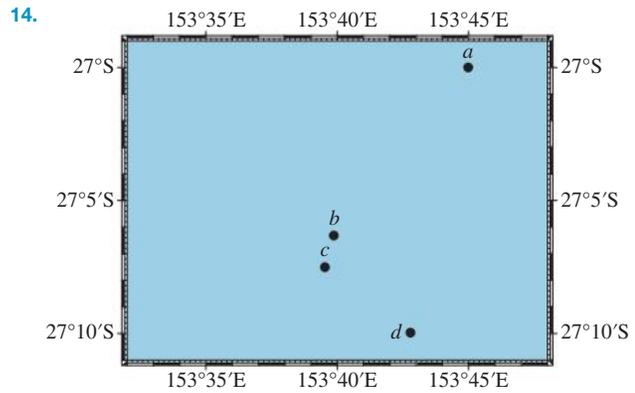
6.2 Exercise

- Great circle
- Small circle
- Great circle
- Small circle
- 21 300 km
- 10 933 km
-



- $27^{\circ}9.6'S$, $153^{\circ}36'E$
 - $27^{\circ}S$, $153^{\circ}45.9'E$
 - $27^{\circ}S$, $153^{\circ}36'E$
 - $27^{\circ}0.9'S$, $153^{\circ}37.6'E$
 - $27^{\circ}1.2'S$, $153^{\circ}33.6'E$
 - $27^{\circ}8'S$, $153^{\circ}44.5'E$
- $30^{\circ}N$, $30^{\circ}E$ Cairo
 - $30^{\circ}N$, $120^{\circ}E$ Shanghai
 - $15^{\circ}S$, $135^{\circ}E$ Darwin
 - $45^{\circ}N$, $75^{\circ}W$ Montreal
 - $50^{\circ}N$, 0° London
- $37^{\circ}S$, $175^{\circ}E$ Auckland
 - $35^{\circ}N$, $140^{\circ}E$ Tokyo
 - $40^{\circ}N$, $115^{\circ}E$ Beijing
 - $22\frac{1}{2}^{\circ}S$, $43^{\circ}W$ Rio de Janeiro
 - $60^{\circ}N$, $11^{\circ}E$ Oslo
- Melbourne ($38^{\circ}S$, $145^{\circ}E$)
 - New York ($40^{\circ}N$, $75^{\circ}W$)
 - Jamaica ($18^{\circ}N$, $76^{\circ}W$)

- Johannesburg ($26^{\circ}S$, $28^{\circ}E$)
 - Rome ($42^{\circ}N$, $12^{\circ}E$)
- Buenos Aires ($35^{\circ}S$, $57^{\circ}W$)
 - Baghdad ($33^{\circ}N$, $44^{\circ}E$)
 - Moscow ($55^{\circ}N$, $40^{\circ}E$)
 - Singapore ($2^{\circ}N$, $104^{\circ}E$)
 - Suva ($18^{\circ}S$, $178^{\circ}E$)



- Mt Sydney
 - Black Island
 - Pinnacle Point
- $27^{\circ}S$, $153.75^{\circ}E$
 - $27.105^{\circ}S$, $153.6^{\circ}E$
 - $27.123^{\circ}S$, $153.663^{\circ}E$
 - $27.16^{\circ}S$, $153.7116^{\circ}E$
- $37^{\circ}46'29.64''N$, $122^{\circ}25'9.84''W$
 - $33^{\circ}52'7.68''S$, $151^{\circ}12'33.48''E$
 - $51^{\circ}30'26.64''N$, $0^{\circ}7'40.08''W$
 - $40^{\circ}42'46.08''N$, $74^{\circ}0'21.60''W$
 - $23^{\circ}33'1.80''S$, $46^{\circ}37'59.88''W$

6.3 Distances on Earth's surface

6.3 Exercise

- 50°
- 40°
 - 10°
 - 71°
 - 21°
 - 60°
- 60°
- 3336 km
- 3892 km
 - 15 012 km
 - 4337 km
 - 10 675 km
- 63°
 - 7000 km (6937 km)
- 2470 km
 - 3233 km
 - 2859 km
 - 425 km
 - 16 301 km
 - 6452 km
- 5226 km
- 180°
 - 19 036 km
- 6672 km
- 6600 km
- 1330 km
- 3892 km
- 6894 km to South Pole
 - 4559 km to North Pole

- c. 4893 km to North Pole
- d. 8896 km to South Pole
- 15. 791 km
- 16. The luggage must be at $(33^{\circ}15'S, 80^{\circ}11'E)$.
- 17. 776.732 km
- 18. The campground is at $(36^{\circ}27'S, 148^{\circ}20'E)$.

6.4 Time zones

6.4 Exercise

1. a. Perth, UTC + 8
b. Sao Paulo, UTC - 3
c. Johannesburg, UTC + 2
d. Buenos Aires UTC - 4
e. Auckland, UTC + 12
2. 10 h
3. a. 14 h b. 7 h c. 11 h d. 22 h
4. 11:00 pm Monday
5. a. 8:00 pm
b. 1:00 pm Friday
c. 5:00 pm Wednesday
d. 11:00 pm Tuesday
e. 3:45 pm Monday
6. 5:00 am Saturday
7. 11:00 pm Tuesday
8. 10:00 am Monday
9. a. 10:00 pm, same day
b. GMT + 11
c. i. 3:00 pm Monday
ii. 8:00 am Friday
10. 7 h
11. a. 18 h b. 19 h c. 17 h
12. a. 10 h 20 min b. 24 min c. 2 h 48 min
13. C
14. D
15. 75°
16. 255° crossing UTC or 105° crossing IDL
17. 2:00 pm Tuesday Los Angeles local time
18. a. 4:00 pm Wednesday
b. 8:00 am Sunday
19. a. 1:00 am Wednesday
b. 3:00 am Wednesday
20. 6:51 am

6.5 Review

6.5 Exercise

1. a. Small circle b. Great circle
2. 56.5 km
3. 20016 km
4. 71 cm

5. a. Manila b. Lima c. Santiago
6. a. $(41^{\circ}N, 3^{\circ}W)$ b. $(1^{\circ}N, 104^{\circ}E)$ c. $(43^{\circ}S, 147^{\circ}E)$
7. 58°
8. a. 16° b. 1779 km
9. 5671 km
10. a. 14481 km b. 18425 km
11. a. 8 h b. 11 h c. 17 h
12. 3:30 am Tuesday
13. a. 11:00 am the same day
b. 11:00 am the same day
14. a. A $(30^{\circ}N, 60^{\circ}W)$; B $(40^{\circ}S, 20^{\circ}W)$;
C $(30^{\circ}S, 50^{\circ}E)$; D $(40^{\circ}N, 60^{\circ}W)$
b. F
15. a. $20^{\circ}2.2'S, 148^{\circ}52.7'E$
b. $20^{\circ}10'S, 148^{\circ}53.7'E$
16. a. 15679 km b. 3225 km
17. 7:00 pm the same day
18. a. 6894 km
b. 10341 km
c. 18014 km
d. 22128 km
19. $(21^{\circ}N, 107^{\circ}E)$
20. $32.6^{\circ}S$

6.5 Past QCAA exam questions

1. 4:00 pm
2. a. Site C
b. Western Australia
c. Sites B and C
3. a. $(3^{\circ}S, 141^{\circ}E)$ b. $(9^{\circ}S, 141^{\circ}E)$ c. 667.2 km
4. A
5. A

UNIT

4 Investing and networking

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7 Compound interest loans and investments

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Fully worked solutions for this chapter are available online.

EXAM PREPARATION

Access exam-style questions in every lesson, available online and access past QCAA exam questions in every review.

Resources

 Solutions	Solutions — Chapter 7
 Exam questions	Exam question booklet — Chapter 7
 Digital documents	Learning matrix — Chapter 7 Chapter summary — Chapter 7

LESSON

7.1 Overview

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7.1.1 Introduction

Most people will borrow and invest money at some stage. When entering the workforce, they may need a car, house, or other goods but lack the funds to buy them outright. They borrow, believing they can repay over time. Investing helps money grow independently, aiding in planning for home loan deposits and retirement. Given the likelihood of facing borrowing and investing decisions, it's essential to understand financial mathematics.



7.1.2 Syllabus links

Lesson	Lesson title	Syllabus links
7.2	Using a recurrence relation to model a compound interest loan or investment	<ul style="list-style-type: none">○ Use a recurrence relation to model a compound interest loan or investment.<ul style="list-style-type: none">• $A_{n+1} = rA_n$ where A_{n+1} is total amount at the beginning of the $(n + 1)$th period, A_n is total amount at the beginning of the nth period and $r = 1 + i$ where i is interest rate per compounding period
7.3	Using the compound interest formula	<ul style="list-style-type: none">○ Use the compound interest formula to model a compound interest loan or investment.<ul style="list-style-type: none">• $A = P(1 + i)^n$ where A is total amount, P is principal, i is interest rate per compounding period and n is number of compounding periods○ Solve practical problems involving compound interest loans or investments, including determining the total amount of the loan or investment, total interest, principal, interest rate per year and per compounding period, and the effect of the interest rate and number of compounding periods on the total amount.
7.4	Interest rates and the effect of changing the rate and periods per year	<ul style="list-style-type: none">○ Use the compound interest formula to model a compound interest loan or investment.<ul style="list-style-type: none">• $A = P(1 + i)^n$ where A is total amount, P is principal, i is interest rate per compounding period and n is number of compounding periods○ Solve practical problems involving compound interest loans or investments, including determining the total amount of the loan or investment, total interest, principal, interest rate per year and per compounding period, and the effect of the interest rate and number of compounding periods on the total amount.
7.5	The effective annual interest rate	<ul style="list-style-type: none">○ Calculate the effective annual rate of interest, $i_{\text{effective}}$ and use the results to compare interest paid on loans or investments when interest is paid or charged for different compounding periods, including daily, monthly, quarterly and six-monthly.<ul style="list-style-type: none">• $i_{\text{effective}} = (1 + i)^k - 1$ where i is interest rate per compounding period and k is number of compounding periods per year

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LESSON

7.2 Using a recurrence relation to model a compound interest loan or investment

SYLLABUS LINKS

- Use a recurrence relation to model a compound interest loan or investment.
- $A_{n+1} = rA_n$ where A_{n+1} is total amount at the beginning of the $(n + 1)$ th period, A_n is total amount at the beginning of the n th period and $r = 1 + i$ where i is interest rate per compounding period

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7.2.1 Recurrence relations

Compound interest can be described as a recurrence relation, where the amount invested or borrowed follows a geometric sequence for each consecutive payment period.

The amount

Let the starting amount be A_n . Then the amount at the start of the next compounding period is A_{n+1} . The recurrence relation rule for this sequence will be:

$$A_{n+1} = rA_n$$

where

A_n is the total amount at the beginning of the n th period

A_{n+1} is the total amount at the beginning of the $(n + 1)$ th period

r is $1 + i$, where i = interest rate per compounding period.

Consider an investment of \$2500 invested for 3 years at an interest rate of 5.5% p.a. compounded annually. Use the recurrence relation $A_{n+1} = 1.055A_n$, $A_0 = 2500$, to calculate the amount in the account at the end of each year.

A_n (\$)	Calculation	A_{n+1} (\$)
2500	2500×1.055	2637.50
2637.50	2637.50×1.055	2782.56
2782.56	2782.56×1.055	2935.60

WORKED EXAMPLE 1 Using a recurrence relation to model a compound interest investment

A person invests \$5000 in a financial institution at 6% p.a. compounded annually.

- Using A_0 to represent the initial investment, write this investment as a recurrence relation.
- Complete the table to show the growth of the investment over 3 years.

A_n (\$)	Calculation	A_{n+1} (\$)
5000		

- Determine the balance in the account at the end of 3 years.
- Calculate the amount of interest made in 3 years.

THINK

- Write the recurrence relation rule.
 - Write the rule for r .
 - Substitute the values for A_0 and r .
- b. Apply the recurrence relation rule.

- c. Write the answer.
- Subtract the initial investment from the total amount at the end of 3 years.
 - Write the answer.

WRITE

a. $A_{n+1} = rA_n$

$$\begin{aligned} r &= 1 + i \\ &= 1 + \frac{6}{100} \\ &= 1.06 \end{aligned}$$

$$A_{n+1} = 1.06A_n, A_0 = 5000$$

b.

A_n (\$)	Calculation	A_{n+1} (\$)
5000	1.06×5000	5300
5300	1.06×5300	5618
5618	1.06×5618	5955.08

- c. There was \$5955.08 in the account at the end of 3 years.
- d. Interest = total amount – initial investment
 $= 5955.08 - 5000$
 $= 955.08$
 The interest is \$955.08.

7.2.2 The effect of changing the compounding period

When interest is compounded more frequently on an investment, the amount of interest earned is larger since it is being added to the account more frequently.

WORKED EXAMPLE 2 Changing the compounding period

The person in Worked example 1 invests \$5000 in a different financial institution at 6% p.a. compounded monthly.

- Using A_0 to represent the initial investment, write this investment as a recurrence relation.
- Use your calculator to calculate the amount in the account after 1 year.
- Determine the amount of interest made in 1 year.
- Calculate how much more interest was made in 1 year when the interest was compounded monthly instead of each year.

THINK

- a. 1. Write the recurrence relation rule.
2. Calculate i . Write the rule for r and calculate r .
3. Substitute the values for A_0 and r .
- b. 1. On your calculator screen, type 5000, then press Enter. Type $\times 1.005$, then press Enter 12 times.
2. Write the answer.
- c. 1. Subtract the initial investment from the total amount at the end of 1 year.
2. Write the answer.
- d. Determine the difference between annual and monthly interest.

WRITE

- a. $A_{n+1} = rA_n$
- $$i = \frac{6}{12} = 0.5\%$$
- $$r = 1 + i$$
- $$= 1 + \frac{0.5}{100}$$
- $$= 1.005$$
- $$A_{n+1} = 1.005A_n, A_0 = 5000$$
- b. 5000
- $$5000 \times 1.005 = 5025$$
- $$5025 \times 1.005 = 5050.13$$
- $$5050.13 \times 1.005 = 5075.38$$
- $$\vdots$$
- $$5281 \times 1.005 = 5308.39$$
- There was \$5308.39 in the account at the end of the first year.
- c. Interest = total amount – initial investment
- $$= 5308.39 - 5000$$
- $$= 308.39$$
- The interest is \$308.39.
- d. When the interest was compounded monthly instead of annually, the amount of interest increased by $\$308.39 - \$300 = \$8.39$.

Exercise 7.2 Using a recurrence relation to model a compound interest loan or investment

learnon
7.2 Exercise
7.2 Exam questions **on**
Simple familiar

 1, 2, 3, 4, 5, 6, 7,
8, 9, 10, 11, 12, 13,
14, 15

Complex familiar

16, 17

Complex unfamiliar

–

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Simple familiar

1. **WE1** Brynn invests \$8500 in a financial institution at 12% p.a. compounded annually.
- a. Using A_0 , to represent the initial investment, write this investment as a recurrence relation.
- b. Complete table to show the growth of the investment over 3 years.
- c. Determine the amount of money in the account at the end of the 3 years.
- d. Determine the amount of interest made in 3 years.

A_n (\$)	Calculation	A_{n+1} (\$)
8500		

2. **WE2** Brynn invests \$8500 in a different financial institution at 12% p.a. compounded monthly.
- Using A_0 to represent the initial investment, write this investment as a recurrence relation.
 - Using your calculator and the iterative method, calculate the amount in the account after 1 year.
 - Determine the amount of interest made in 1 year.
 - Determine how much more interest was made in 1 year when the interest was compounded monthly instead of each year.
3. Sally invested \$2575 in a savings account at an interest rate of 8.25% p.a. compounded every 6 months.
- Using A_0 , to represent the initial investment, write this investment as a recurrence relation.
 - Calculate the total value of Sally's investment at the end of 3 years.
4. Su Fen was left an inheritance by her grandmother of \$34 000. She decided to invest it for 18 months in a bank account that offered 8% p.a. compounded monthly.
- Using A_0 to represent the initial investment, write this investment as a recurrence relation.
 - Determine how much interest Su Fen will earn on her investment in 5 months.
5. **MC** A person invested \$1500 at 5% p.a. compounded annually. The amount of interest, correct to the nearest cent, earned on the investment at the end of 3 years is closest to:
- A. \$225 B. \$236 C. \$1575 D. \$1736

The following information relates to questions 6–10.

Josh borrowed money from a finance company that offered a 'No deposit, no payment' for 2 years option. The interest accrued was compounded annually. The amount he owed after n years, A_n , can be modelled by the recurrence relation:

$$A_{n+1} = 1.08A_n, A_0 = 6000$$

6. **MC** The annual percentage compound interest rate for this account is:
- A. 1.08% B. 8% C. 80% D. 92%
7. **MC** The interest earned in the first year is:
- A. \$480 B. \$6480 C. \$4800 D. \$5520
8. **MC** The balance at the end of the first year is:
- A. \$480 B. \$6480 C. \$4800 D. \$5520
9. **MC** The balance of the loan at the start of the second year is:
- A. \$6000 B. \$5480 C. \$480 D. \$6480
10. **MC** The interest earned during the second year is closest to:
- A. \$600 B. \$480 C. \$518 D. \$318
11. **MC** If \$11 000 is invested for 5 years at 7% p.a. with interest compounding annually and the amount in the account at the end of the third year is \$13 475.47, then the interest earned in the fourth year is closest to:
- A. \$943 B. \$625 C. \$650 D. \$677

12. Julia invested \$10 000 for 4 years at an interest rate of 6% p.a. with interest compounded annually. Complete the table by calculating the values **A**, **B**, **C**, **D**, **E** and **F**.

A_n (\$)	Interest (\$)	A_{n+1} (\$)
10 000	A % of 10 000 = 600	10 600
B	6% of C = 636	D
11 236	6% of 11 236 = 674.16	11 910.16
11 910.16	6% of 11 910.16 = 714.61	E
F	6% of 12 624.77 = 757.49	13 382.26

13. Owen invested \$9500 for 4 years at an interest rate of 5.4% p.a. with interest compounded annually. Complete the table by calculating the values **A**, **B**, **C**, **D**, **E** and **F**.

A_n (\$)	Interest (\$)	A_{n+1} (\$)
9500	5.4% of 9500 = 513	10 013
10 013	5.4% of 10 013 = A	B
C	5.4% of 10 553.70 = 569.90	11 123.60
11 123.60	5.4% of 11 123.60 = 600.67	D
E	5.4% of 11 724.27 = 633.11	F

14. Ahmed invested \$2700 for 4 years at an interest rate of 4.5% p.a. with interest compounded monthly. Complete the table shown to determine the value of the investment at the end of 5 months.

A_n (\$)	A_{n+1} (\$)
2700	
2710.13	

15. A person invests \$2000 in a financial institution at 4.5% p.a. compounded quarterly. Calculate the value of the investment after 2 years. Write your answer correct to the nearest cent.

Complex familiar

16. Owen has some medical bills, totalling \$300, that must be paid but he has no access to any money or credit. He decides to apply for a payday loan from a lending company, PayNow. The company charges 10% interest per week. Owen intends to pay back the loan and interest accrued when he receives his pay in 4 weeks.

- Using A_0 , A_n and A_{n+1} , write a recurrence relation that models this situation where n is the number of weeks after the due date.
- Calculate how much interest Owen will need to pay if he pays the full amount on his next pay day.



17. Georgie borrows \$4500 from a bank with interest compounded monthly. After 1 month, Georgie's bank has added \$66 of interest to her loan. Calculate the annual rate of interest.

Fully worked solutions for this chapter are available online.

LESSON

7.3 The compound interest formula

SYLLABUS LINKS

- Use the compound interest formula to model a compound interest loan or investment.
 - $A = P(1 + i)^n$ where A is total amount, P is principal, i is interest rate per compounding period and n is number of compounding periods
- Solve practical problems involving compound interest loans or investments, including determining the total amount of the loan or investment, total interest, principal, interest rate per year and per compounding period, and the effect of the interest rate and number of compounding periods on the total amount.

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7.3.1 The compound interest formula

We have seen in previous lessons that we can write the value of a compounding investment as a recurrence relation. We can also use a compound interest rule to solve problems involving compound interest loans and investments.

Calculating compound interest

$$A = P(1 + i)^n$$

where

A = total amount (\$)

P = the principal (or present value) (\$)

i = interest rate per compounding period (expressed as a decimal)

n = number of compounding periods.

To find the amount of interest earned, I , the principal value is subtracted from the total amount.

$$I = A - P$$

WORKED EXAMPLE 3 Compound interest calculated annually

A sum of \$5000 is invested at 4% p.a. compounded annually. Use the compound interest formula to calculate the value of this investment after 5 years. Write your answer correct to the nearest cent.

THINK

1. Determine the number of compounding periods, n .
2. Determine i .
3. Substitute the values P , n and i into the compound interest formula.
4. Answer the question.

WRITE

$$n = 1 \times 5 = 5$$

$$i = 0.04$$

$$\begin{aligned} A &= P(1 + i)^n \\ &= 5000(1 + 0.04)^5 \\ &= 5000(1.04)^5 \\ &= 6083.26 \end{aligned}$$

The total amount of the investment after 5 years is \$6083.26.

7.3.2 Calculating compound interest with different compounding periods

To calculate compound interest with different compounding periods, you must first calculate how many periods there are and the interest rate. For example, when compounding monthly, the interest per annum needs to be divided by 12 for the monthly interest. The number of periods must also be in months.



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WORKED EXAMPLE 4 Compound interest calculated monthly

\$4500 is invested for 3 years at 6% p.a. compounded monthly. Using the compound interest formula determine the total amount of interested earned. Write your answer correct to the nearest cent.

THINK

1. Determine the number of compounding periods, n .
2. As the interest is compounded monthly, divide the interest rate (expressed as a decimal) by 12.
3. Substitute the values P , n and i into the compound interest formula.
4. Write the formula for interest. Calculate the interest.
5. Write the answer.

WRITE

$$n = 12 \times 3 = 36$$

$$i = \frac{0.06}{12} \\ = 0.005$$

$$A = P(1 + i)^n \\ = 4500(1 + 0.005)^{36} \\ = 4500(1.005)^{36} \\ = 5385.06$$

$$I = A - P \\ I = 5385.06 - 4500 \\ = 885.06$$

The total interest after 3 years is \$885.06.

WORKED EXAMPLE 5 Compound interest calculated half yearly

Use the compound interest formula to determine the principal required to yield a total amount of \$6842.85 after compounding every 6 months at a rate of 8% p.a. for 4 years. Write your answer correct to the nearest dollar.

THINK

1. Determine the number of compounding periods, n .
2. As the interest is compounded every 6 months, divide the interest rate (expressed as a decimal) by 12.
3. Substitute the values A , n and i into the compound interest formula and solve for P .
4. Answer the question.

WRITE

$$n = 2 \times 4 = 8$$

$$i = \frac{0.08}{2} \\ = 0.04$$

$$A = P(1 + 0.04)^n \\ 6842.85 = P(1 + 0.04)^8 \\ 6842.85 = P \times 1.04^8 \\ 6842.85 = P \times 1.368... \\ P = \frac{6842.85}{1.368...} \\ P = 5000.00$$

The principal of the investment is \$5000.00.

Exercise 7.3 The compound interest formula

7.3 Exercise

7.3 Exam questions **on**

Simple familiar

1, 2, 3, 4, 5, 6, 7, 8,
9, 10, 11

Complex familiar

12, 13, 14, 15

Complex unfamiliar

16

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Simple familiar

- WE3** Use the compound interest formula to calculate the total amount of each of the following investments at the end of the given time periods with interest compounded annually. Write your answers correct to the nearest cent.
 - \$4000 at 5% p.a. for 3 years
 - \$8000 at 3% p.a. for 5 years
 - \$18 000 at 8% p.a. for 4 years
 - \$11 500 at 5.5% p.a. for 3 years
- Use the compound interest formula to calculate the amount of compound interest earned on the following investments with interest rates compounded annually for the times shown. Write your answers correct to the nearest cent.
 - \$4655 at 4.55% p.a. for 3 years
 - \$12344 at 6.35% p.a. for 6 years
 - \$3465 at 2.015% p.a. for 8 years
 - \$365 000 at 7.65% p.a. for 20 years
- WE4** A principal of \$3800 is compounded monthly at 3% p.a. for 4 years. Use the compound interest formula to determine the total amount of interest earned. Write your answer correct to the nearest cent.
- Calculate the total amount of each of the following investments.
 - \$960 for 1 year at 4.50% p.a. with interest compounded 6-monthly
 - \$7500 for $3\frac{1}{2}$ years at 5.6% p.a. with interest compounded quarterly
 - \$152 000 for $2\frac{1}{2}$ years at 7.2% p.a. with interest compounded 6-monthly
 - \$14 000 for 4 years at 9% p.a. with interest compounded monthly
- MC** A sum of \$5000 is invested for 2 years at the rate of 4.75% p.a., compounded quarterly. The interest paid on this investment, to the nearest dollar, is:
A. \$475 **B.** \$495 **C.** \$1900 **D.** \$5475
- WE5** Using the compound interest formula calculate the principal required to yield each of the following total amounts. Interest is compounded annually. Write your answers correct to 2 decimal places.
 - \$15 000 after compounding at a rate of 5.25% p.a. for 8 years
 - \$22 500 after compounding at a rate of 7.15% p.a. for 10 years
 - \$1000 after compounding at a rate of 1.25% p.a. for 2 years
- Carla is to invest \$45 000 at 9.2% p.a. for 5 years with interest compounded 6-monthly. Calculate the total amount of the investment at the end of 5 years. Write your answer correct to the nearest cent.
- Vicky invests \$30 000 in a 1-year fixed deposit at an interest rate of 6% p.a., with interest compounding monthly. Calculate the total amount of the investment upon maturity. Write your answer correct to the nearest cent.

9. **MC** If \$14 000 is invested for $3\frac{1}{2}$ years at 5.75%, compounding fortnightly, the amount of interest that will accrue will be closest to:
- A. \$8435 B. \$2113 C. \$3117 D. \$17 117
10. \$1500 is invested for 2 years into an account paying 8% p.a. Calculate the balance in the account if
- interest is compounded yearly
 - interest is compounded quarterly
 - interest is compounded monthly
 - interest is compounded weekly
 - Compare your answers to parts a–d.
11. Calculate the interest generated from an account that pays compound interest at a rate of:
- 7% p.a. if \$2600 is invested for 3 years (compounded monthly)
 - 8% p.a. if \$3500 is invested for 4 years (compounded monthly)
 - 11% p.a. if \$960 is invested for $5\frac{1}{2}$ years (compounded fortnightly)
 - 7.3% p.a. if \$2370 is invested for 5 years (compounded weekly)
 - 15.25% p.a. if \$4605 is invested for 2 years (compounded daily).

Complex familiar

12. Determine the principal that will grow to \$15 600 in 4 years, if interest is added quarterly at 8.5%.
13. Calculate the principal that will grow to:
- \$3000 in 4 years, if interest is compounded 6 monthly at 9.5% p.a.
 - \$2000 in 3 years, if interest is compounded quarterly at 9% p.a.
 - \$5600 in $5\frac{1}{4}$ years, if interest is compounded quarterly at 8.7% p.a.
 - \$10 000 in $4\frac{1}{4}$ years, if interest is compounded monthly at 15% p.a.
14. A loan of \$2500 is charged interest at the rate of 6.5% p.a., compounded half yearly, calculate the interest charged over 3 years.
15. **MC** After 10 compounding periods, an investment has grown to \$8500 in an account with interest paid at 6.5% p.a. and compounded half-yearly. The amount of the original loan is closest to:
- A. \$6175 B. \$4775 C. \$8880 D. \$3445

Complex unfamiliar

16. Venya borrows some money to buy a new car. She is offered a compound interest loan at 10.5% p.a. compounding monthly. She repays the loan after 5 years and discovers that she has paid \$18 881.58 in interest. Determine how much Venya borrowed to buy her car.



Fully worked solutions for this chapter are available online.

LESSON

7.4 Interest rates and the effect of changing the rate and periods per year

SYLLABUS LINKS

- Use the compound interest formula to model a compound interest loan or investment.
 - $A = P(1 + i)^n$ where A is total amount, P is principal, i is interest rate per compounding period and n is number of compounding periods
- Solve practical problems involving compound interest loans or investments, including determining the total amount of the loan or investment, total interest, principal, interest rate per year and per compounding period, and the effect of the interest rate and number of compounding periods on the total amount.

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7.4.1 Using technology to compare the effect on investments of changing of interest rates

A graph of an investment for different compound interest rates can assist in comparing the effect changing interest rates can have on a loan or investment.

For compound interest graphs, the interest rate affects the slope of the curve. An increase in interest rates increases the value of the investment, which causes the curve to move away from the horizontal axis more quickly. This is an example of exponential growth.



tvd-12185

WORKED EXAMPLE 6 Comparing interest rates using technology

Using technology, show on one set of axes how each of the following compound interest rates affect the future value of \$4000 compounding yearly for 10 years. Hence, describe the effect increasing the compound interest rate has on the investment, using calculations.

- 4%
- 6%
- 8%
- 10%

THINK

- Using the technology of your choice, insert a formula to calculate the future value of \$4000 each year for 10 years for each of the interest rates.

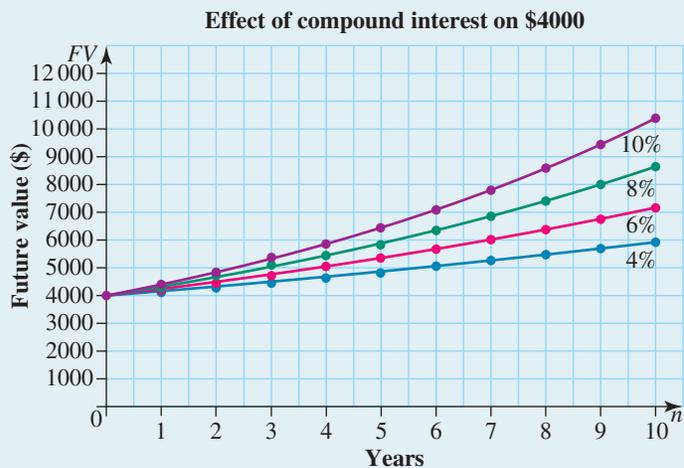
WRITE

	A	B	C	D	E
1	Year	4%	6%	8%	10%
2	0	4000	4000	4000	4000
3					
4	Formula	$= 1.04*b2$	$= 1.06*c2$	$= 1.08*d2$	$= 1.1*e2$

2. Determine the future value of \$4000 each year for 10 years for each of the compound interest rates.

	A	B	C	D	E
1	Year	4%	6%	8%	10%
2	0	4000.00	4000.00	4000.00	4000.00
3	1	4160.00	4240.00	4320.00	4400.00
4	2	4326.40	4494.40	4665.60	4840.00
5	3	4499.46	4764.06	5038.85	5324.00
6	4	4679.43	5049.91	5441.96	5856.40
7	5	4866.61	5352.90	5877.31	6442.04
8	6	5061.28	5674.08	6347.50	7086.24
9	9	5263.73	6014.52	6855.30	7794.87
10	8	5474.28	6375.39	7403.72	8574.36
11	9	5693.25	6757.92	7996.02	9431.79
12	10	5920.98	7163.39	8635.70	10374.97

3. Using the values from Step 2, construct a graph.



4. Answer the question.

Increasing the compound interest rate has the effect of increasing the future value of the investment each year. This is shown on the graph as the curves becoming steeper.

7.4.2 Using the compound interest formula to determine the interest rate

The compound interest formula $A = P(1 + i)^n$ can be used to determine the interest rate required to reach a certain total amount or future value in a set number of years.

WORKED EXAMPLE 7 Using the compound interest formula to determine the interest rate

Calculate the interest rate per annum that would enable an investment of \$4000 to grow to \$4200 over 3 years if the interest is compounded every 6 months. Give your answer correct to 2 decimal places.

THINK

1. Write the values of A , P and n .

WRITE

$$A = 4200$$

$$P = 4000$$

$$n = 3 \times 2 = 6$$

2. Write the compound interest rule and substitute the known values.

$$A = P(1 + i)^n$$

$$4200 = 4000(1 + i)^6$$

3. Rearrange the equation.

$$4200 = 4000(1 + i)^6$$

$$\frac{42}{40} = (1 + i)^6$$

$$(1 + i)^6 = 1.05$$

4. Use the $\sqrt[y]{x}$ button on your calculator to determine the value of i .

$$(1 + i) = \sqrt[6]{1.05}$$

$$i = \sqrt[6]{1.05} - 1$$

$$i = 0.00816\dots$$

5. Multiply i by 100 to obtain the interest rate per compounding period.

$$100 \times i = 100 \times 0.00816\dots$$

$$= 0.816\dots$$

6. Multiply the interest rate per compounding period by the number of compounding periods per year to obtain the annual interest rate.

$$\text{Annual interest rate} = 0.816\dots \times 2$$

$$= 1.63\dots$$

7. Write the answer.

The annual interest rate is 1.63% p.a.

Exercise 7.4 Interest rates and the effect of changing the rate and periods per year

learn **on**

7.4 Exercise

7.4 Exam questions **on**

Simple familiar

1, 2, 3, 4, 5, 6, 7, 8, 9, 10

Complex familiar

11, 12, 13, 14

Complex unfamiliar

15, 16

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Simple familiar

1. James has \$8000 to invest at either 4% p.a., 6% p.a. or 8% p.a. compounding annually.

a. Complete the following table to show the interest that he would earn over 5 years.

No. of years	1	2	3	4	5
Interest (4%)					
Interest (6%)					
Interest (8%)					

b. Show this information in graph form.

2. **WE6** Using technology, show on one set of axes how the following compound interest rates affect the future value of \$5000 compounding yearly for 8 years. Hence, describe the effect increasing the compound interest rate has on the investment, using calculations.

- a. 5%
- b. 10%
- c. 15%

3. Petra has \$4000 to invest at 6% p.a.

- a. Complete the following table to show the future value of the investment at the end of each year if interest is compounded:
 - i. annually.
 - ii. six-monthly.

No. of years	1	2	3	4	5
Annually					
Six-monthly					

b. Show this information in graphical form.

4. **WE7** Calculate the interest rate per annum that would enable an investment of \$1000 to grow to \$1200 over 2 years if the interest is compounded every quarter. Give your answer correct to 2 decimal places.

5. Determine the interest rate per annum (correct to 2 decimal places) that would enable an investment of \$10 000 to grow to \$15 500 over 3 years if interest is compounded 6-monthly.

6. **MC** The minimum interest rate per annum required for an investment of \$5400 to grow to at least \$8000 in 3 years if compounded quarterly would be:

- A. 3.33%
- B. 3.4%
- C. 8.67%
- D. 13.32%

7. A bank offers an interest rate of 5% p.a. compounded quarterly. After 2 years, the investment grows to \$1210. Calculate the initial principal amount (P) invested.

8. An investment of \$1500 grows to \$2000 in 3 years with interest compounded monthly. Determine the annual interest rate.

9. You invest \$5000 in a savings account that offers an interest rate of 6% p.a. compounded annually. Calculate how much money will be in the account after 5 years.

10. A loan is taken at an interest rate of 8% p.a. compounded daily. After 3 years, the loan amount grows to \$2500. Determine the initial principal amount.

Complex familiar

11. Tai has \$20 000 to invest at 4.75% compound interest for 2 years. She can choose from the following compounding periods:

- Option A: six-monthly
- Option B: monthly
- Option C: yearly.

Using technology, calculate the total amount of the investment after each compounding period. Hence, state which option Tai should choose.

12. An investment grows from \$2400 to \$5300 in 5 years and 3 months with interest compounded quarterly. Determine the annual interest rate, correct to two decimal places.

13. A principal of \$5000 grows to \$7500 in 6 years, compounding monthly. Determine the interest rate per compounding period. Give your answer correct to 4 decimal places.
14. Yasmin plans to borrow \$10 000 for 5 years. She is considering two options:
- Loan 1: 6.8% interest, compounded monthly
 - Loan 2: 7% interest, compounded quarterly
- Justify which loan Yasmin should choose.

Complex unfamiliar

15. Determine the time required for Peter to save up for a \$5000 car given that he has \$3500 now and chooses to invest it at an annual interest rate of 12% compounded monthly.



16. After selling their house, Mr and Mrs Dengate have \$61 800. They plan to invest it at 6% p.a., with interest compounded annually. Determine when the value of their investment will first exceed \$100 000.



Fully worked solutions for this chapter are available online.

LESSON

7.5 The effective annual interest rate

SYLLABUS LINKS

- Calculate the effective annual rate of interest, $i_{\text{effective}}$, and use the results to compare interest paid on loans or investments when interest is paid or charged for different compounding periods, including daily, monthly, quarterly and six-monthly.
- $i_{\text{effective}} = (1 + i)^k - 1$ where i is interest rate per compounding period and k is number of compounding periods per year

Source: General Mathematics Senior Syllabus 2024 © State of Queensland (QCAA) 2024; licensed under CC BY 4.0.

7.5.1 Effective annual rate of interest

A financial institution may advertise a **nominal interest rate** at 3.5% p.a. compounding monthly. In reality, the investor would not receive 3.5% per year on the money invested because the interest is compounding. The actual percentage return on the investment is more than 3.5% per year because the interest calculated each consecutive period is based on a higher principal.

The **effective interest rate** or actual interest rate represents the actual percentage return per year on an investment. It could also be considered as the simple interest rate that would produce the same return as the nominated compound interest rate.

The same applies to the cost of a loan. Money offered at 5% p.a. compounding quarterly will be at a higher interest rate than 5% due to the compounding factor.

Effective interest rate

$$i_{\text{effective}} = (1 + i)^k - 1$$

where

$i_{\text{effective}}$ is the effective annual interest rate

i is the interest rate per compounding period

k is the number of compounding periods per year.

WORKED EXAMPLE 8 Calculating effective rate of interest

An interest rate is quoted at 3.5% p.a. compounding monthly. Determine the effective interest rate. Give your answer correct to 2 decimal places.

THINK

1. Write the effective interest rate formula.
2. Write the values for i and k .
3. Substitute the values into the formula and calculate.
4. Write the answer correct to 2 decimal places.

WRITE

$$i_{\text{effective}} = (1 + i)^k - 1$$

$$i = \frac{0.035}{12} = 0.002917\dots, k = 12$$

$$i_{\text{effective}} = (1 + 0.002917)^{12} - 1 = 0.035567\dots$$

The effective interest rate is 3.56%.

The interest rate in Worked example 8 is 3.5% p.a. compounding monthly, which is equivalent to a simple interest rate of 3.56% p.a. The best way to compare interest rates is to determine the effective interest rate in each case.



WORKED EXAMPLE 9 Comparing effective rates of interest

Jack has the choice of investing money in three different funds:

- 3.9% compounding quarterly
- 3.95% p.a. simple interest
- 3.85% p.a. compounding daily.

Compare the effective interest rates and hence determine which investment would provide him with the greatest return.

THINK

1. Calculate the effective interest rate for each option. Write the effective interest rate formula.
2. Write the values for i and k for option 1.
3. Substitute the values into the formula and calculate.
4. The simple interest rate is the effective interest rate.
5. Write the values for i and k for option 3.
6. Substitute the values into the formula and calculate.
7. Compare the interest rates and write the answer.

WRITE

$$i_{\text{effective}} = (1 + i)^k - 1$$

Option 1:

$$i = \frac{0.039}{4} = 0.00975, k = 4$$

$$\begin{aligned} i_{\text{effective}} &= (1 + 0.00975)^4 - 1 \\ &= 0.039574 \\ &= 3.96\% \text{ p.a.} \end{aligned}$$

Option 2:

The effective interest rate for option 2 is 3.95% p.a.

Option 3:

$$i = \frac{0.0385}{365} = 0.000105\dots, k = 365$$

$$\begin{aligned} i_{\text{effective}} &= (1 + 0.000105\dots)^{365} - 1 \\ &= 0.0392\dots \\ &= 3.92\% \text{ p.a.} \end{aligned}$$

The effective interest rate for option 1 is 3.96% p.a.

The effective interest rate for option 2 is 3.95% p.a.

The effective interest rate for option 3 is 3.92% p.a.

The option that would provide Jack with the greatest return is 3.9% p.a. compounding quarterly.

Exercise 7.5 The effective annual interest rate

7.5 Exercise

7.5 Exam questions **ON**

Simple familiar

1, 2, 3, 4, 5, 6, 7, 8

Complex familiar

9, 10, 11, 12

Complex unfamiliar

13, 14, 15

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Simple familiar

1. **WEB** Tamara's bank offered her an interest rate of 4% p.a. compounding quarterly. Determine the equivalent effective interest rate.
2. Tamara's bank offered her an interest rate of 4% p.a. compounding daily, determine the effective interest rate that this would represent.
3. Calculate the simple interest rate equivalent to a rate of 3.75% p.a. compounding 6-monthly.
4. **WE9** Freya has the choice of investing in two different saving accounts.
 - 4.2% p.a. compounding quarterly
 - 4.175% p.a. compounding monthlyCompare the effective interest rates and determine which would provide Freya with the greatest return.
5. An interest rate of 4.5% p.a. is compounded quarterly on a loan. Determine the equivalent effective interest rate.

The following information relates to questions 6–8.

William takes a personal loan of \$14 000 to buy a car. The interest rate is 9% p.a. compounding monthly. The loan is to be paid back in 2 years.

6. **MC** The total interest that William will pay on this loan is closest to:
A. \$2520.00 B. \$2749.80 C. \$3244.51 D. \$3978.03
7. **MC** The monthly instalment is closest to:
A. \$700 B. \$688 C. \$514 D. \$499
8. **MC** The effective annual interest rate is:
A. 8.11% B. 8.67% C. 9% D. 9.38%

Complex familiar

9. If the nominal interest rate is 6% per year compounded monthly, determine the effective annual interest rate.
10. A loan is offered with an annual nominal interest rate of 8% compounded monthly. Determine the monthly effective interest rate.
11. You invest \$10 000 at an interest rate of 4% per year compounded quarterly. Determine how much interest, to the nearest dollar, you will earn after 5 years.
12. An investment offers an annual effective interest rate of 12%. Determine the equivalent monthly effective interest rate.

Complex unfamiliar

13. Patrick was offered an investment rate of 4.97% p.a. compounding daily or 5% p.a. compounding monthly. Patrick's friend suggests that the second option (5% p.a. compounding monthly) will be better. Evaluate the reasonableness of this suggestion.



14. Marilyn can invest in two different funds:

- 6% p.a. simple interest
- 5.75% p.a. compounding monthly

She is inclined to choose the 6% p.a. simple interest because this is a higher value. State whether this is a wise choice. Justify your answer.

15. Mr Freeburn and Mr McGrath are having a discussion about the best loan option for their colleague. The options are as follows.

- Option A: 5.2% p.a. compounding monthly
- Option B: 5.1% p.a. compounding daily

Mr McGrath believes Option B will be the best option, whereas Mr Freeburn believes Option A is better because it compounds less frequently. Evaluate the reasonableness of these suggestions and determine which is the better loan option.



Fully worked solutions for this chapter are available online.

LESSON

7.6 Review

7.6.1 Summary

Hey students! Now that it's time to revise this chapter, go online to:



Access the chapter summary



Review your results



Practise exam questions

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7.6 Exercise

learnon

7.6 Exercise

7.6 QCAA questions **on**

Simple familiar

1, 2, 3, 4, 5, 6, 7, 8,
9, 10, 11, 12

Complex familiar

13, 14, 15, 16

Complex unfamiliar

17, 18, 19, 20

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Simple familiar

The following information relates to questions 1–4.

Genevieve sold her car for \$4300 and decided to invest it in a savings account. The interest accrued was compounded annually. The amount in the account after n years, A_n , can be modelled by the recurrence relation

$$A_{n+1} = 1.04A_n, A_0 = 4300.$$

- MC** The annual percentage compound interest rate for this account is:
A. 4% p.a. B. 1.04% p.a. C. 104% p.a. D. 4.3% p.a.
- MC** The interest earned in the first year is:
A. \$4472 B. \$430 C. \$172 D. \$104
- MC** The balance at the end of the first year is:
A. \$104 B. \$172 C. \$4300 D. \$4472
- MC** The balance at the end of the second year is closest to:
A. \$4651 B. \$4650 C. \$4644 D. \$4472
- MC** An investment of \$6000 was placed in an account for 3 years at 4.25% p.a. compounded annually. Determine how much more money would be collected if the investment was compounded quarterly.
A. \$13.36 B. \$32.97 C. \$46.33 D. \$52.29
- MC** An interest rate of 4.5% p.a. compounding monthly is equivalent to an effective interest rate of:
A. 4.50% p.a. B. 4.55% p.a. C. 4.57% p.a. D. 4.59% p.a.

Use the following information to answer questions 7–9

An amount of \$12 000 is invested for a period of 9 months at 3% p.a. compounded monthly. The compound interest formula to calculate the future value of an investment over a period of time is:

$$A = P(1 + i)^n$$

7. **MC** The value of n in the formula would be:
 A. $\frac{9}{12}$ B. 1 C. 3 D. 9
8. **MC** The value of i in the formula would be:
 A. $\frac{9}{12}$ B. 0.025 C. 0.0025 D. 0.03
9. **MC** The value of A at the end of the time period would be closest to:
 A. \$12 030 B. \$12 273 C. \$12 070 D. \$15 657
10. A sum of \$10 000 is invested in an account that pays 7% p.a. compounded quarterly. Use a recurrence relation to describe this investment.
11. Warren wishes to invest \$10 000 for a period of 5 years (assume 365 days = 1 year). The following investment alternatives are suggested to him.
- simple interest at 9% p.a.
 - compound interest at 8% p.a. with interest compounded annually.
 - compound interest at 7.8% p.a. with interest compounded 6-monthly.
 - compound interest at 7.7% p.a. with interest compounded daily.
- Determine the best investment for Wayne.
12. Lars is considering investing \$10 495 at 5.50% p.a. compounded 6 monthly for 6 years. Complete the table below to show the value of his investment for the first 4 compounding periods.

n	Future value (\$)
1	
2	
3	
4	

Complex familiar

13. Sula invested in a share portfolio that earned \$750 in interest over 3 years at a rate of 3% compounded monthly. Determine the principal amount invested in the share portfolio.



14. Peta is considering investing \$17 580 at 4.80% p.a. compounding monthly for 4 years. Peta draws up a table to determine how the investment account will grow in the first 6 months. Peta shows the table to a friend who notices a mistake in the table. Use mathematical reasoning to determine and explain Peta's mistake in the table below. Calculate the correct value of Peta's investment at the end of the 6 months.

n		Future value (\$)
1	$17\,580 \times 1.048$	18 423.84
2	$18\,423.84 \times 1.048$	19 308.18
3	$19\,308.18 \times 1.048$	20 234.98
4	$20\,234.98 \times 1.048$	21 206.26
5	$21\,206.26 \times 1.048$	22 224.16
6	$22\,224.16 \times 1.048$	23 290.92

15. Lara has invested money in an account where interest is compounded half-yearly. She has been trying to determine the total amount she will have in her account at the end of the investment term. To do this Lara has used the following formula.

$$A = 8000(1.025)^{10}$$

Determine the annual interest rate that Lara has been offered and the number of years she plans to invest her money.

16. Mai has decided to borrow \$10 000 at 3.25% compound interest for 3 years. She can choose the following compounding periods:
- Option A: six-monthly
 - Option B: monthly
 - Option C: yearly.

Using technology, calculate the balance of the loan after 3 years for each option. Hence, state which option Mai should choose.

Complex unfamiliar

17. Ben invests \$5000 for 5 years. Each year, immediately after the yearly interest is added to the investment, Ben deposits an additional \$150. The table shows the future value of his investment for the first 2 years.

Year	0	1	2	3	4	5
Future value (\$A)	5000	5387.50	5793.41			

Ben's friend, Jerry, tells Ben that it doesn't make a difference to the value of the investment if Ben deposits the money at the beginning of the year or at the end of the year after the yearly interest is added to the investment. Evaluate the reasonableness of Jerry's comment and determine how much extra interest Ben will earn after 5 years with the better method.

18. Shivani has saved \$5000 from her part-time job and decides to invest it for at least 3 years. Her parents help her to find the best investment options and come up with the following choices.

- i. A local business promising a return of 3.5% compounded annually, with an additional 2% bonus on the total sum paid at the end of the 3-year period
- ii. A building society paying a fixed interest rate of 4.3% compounded monthly
- iii. A venture capitalist company guaranteeing a return of 3.9% compounded daily (assume 365 days = 1 year)



Assuming each option is equally secure, determine using mathematical reasoning the best option for how Shivani should invest her money.

19. You have a savings account with an interest rate of 3.2% p.a. compounded quarterly. Your bank offers a change to your account with the following two options:

- Option 1: Increase the interest rate by 0.6% p.a. while keeping quarterly compounding.
- Option 2: Keep the current 3.2% interest rate but switch to monthly compounding.

Determine which option is more profitable.

20. Sean and Keri are saving for their wedding. They deposit \$10 000 into a savings account that offers an annual interest rate of 4% p.a. compounding quarterly. They intend to leave the money in the account for 6 years (which is when they plan to get married).

After 3 years, Sean and Keri find their dream wedding venue. The venue requires a deposit of \$1500 to secure the booking, so they withdraw this amount from their savings account. A year after the deposit is paid, their bank notifies them that the interest rate is changing to 5% p.a. compounding semi-annually. Determine how much money they will have available for their wedding in 6 years.



7.6 Past QCAA exam questions

Question 1 (4 marks)

Source: QCAA 2023 General Mathematics, Paper 1, Section 2, Q19; © State of Queensland (QCAA) 2024

Ngarra compares two investment options and decides option A will provide the better return.

- Option A: 5.60% p.a. compounding monthly
- Option B: 5.62% p.a. compounding quarterly

Use the effective annual rate of interest formula to evaluate the reasonableness of Ngarra's decision.

Question 2 (4 marks)

Source: QCAA 2022 General Mathematics, Paper 1, Section 2, Q17; © State of Queensland (QCAA) 2024

An investment of \$50 000 that compounds interest monthly is modelled by the recurrence relation $A_{n+1} = 1.00375A_n$ where $A_0 = 50\,000$.

- a. What would be the advertised interest rate per annum, compounding monthly? **[2 marks]**
- b. How many months would it take for the value of the investment to exceed \$51 000? **[2 marks]**

Question 3 (1 mark)

Source: QCAA 2021 General Mathematics, Paper 1, Section 1, Q15; © State of Queensland (QCAA) 2024

Which of the following investment options gives the best return?

- A. 5.93% p.a. compounding daily
- B. 5.95% p.a. compounding monthly
- C. 5.97% p.a. compounding quarterly
- D. 5.99% p.a. compounding six-monthly

Question 4 (5 marks)

Source: QCAA 2020 General Mathematics, Paper 1, Section 2, Q25; © State of Queensland (QCAA) 2024

A financial institution offers two investment options:

Option 1: 7% p.a. compounding quarterly

Option 2: 6.8% p.a. compounding monthly

Use the effective interest rate formula to determine the option that will provide the better return.

Question 5 (3 marks)

Source: Jacaranda Exam-style question

Two parents have \$25 000 that they plan to invest in their children's university fund. They invest at 5% p.a., with interest compounded monthly. Determine in what year the value of their investment will first exceed \$60 000.

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Answers

Chapter 7 Compound interest loans and investments

7.2 Using a recurrence relation to model a compound interest loan or investment

7.2 Exercise

1. a. $A_{n+1} = 1.12A_n$, $A_0 = 8500$

b.

A_n (\$)	Calculation	A_{n+1} (\$)
8500	$1.12 \times 8500 = 9520$	9520
9520	$1.12 \times 9520 = 10\,662.40$	10 662.40
10 662.40	$1.12 \times 10\,662.40 = 11\,941.89$	11 941.89

c. \$11 941.89

d. \$3441.89

2. a. $A_{n+1} = 1.01A_n$, $A_0 = 8500$

b. \$9578.01

c. \$1078.01

d. \$58.01

3. a. $A_{n+1} = 1.04125A_n$, $A_0 = 2575$

b. \$3281.76

4. a. $A_{n+1} = 1.00667A_n$, $A_0 = 34\,000$

b. \$1148.55

5. B

6. B

7. A

8. B

9. D

10. C

11. A

12. A = 6

B = 10 600

C = 10 600

D = 11 236

E = 12 624.77

F = 12 624.77

13. A = 540.70

B = 10 553.70

C = 10 553.70

D = 11 724.27

E = 11 724.27

F = 12 357.38

14.

A_n (\$)	A_{n+1} (\$)
2700	$1.00375 \times 2700 = 2710.13$
2710.13	$1.00375 \times 2710.13 = 2720.29$
2720.29	$1.00375 \times 2720.29 = 2730.49$
2730.49	$1.00375 \times 2730.49 = 2740.73$
2740.73	$1.00375 \times 2740.73 = 2751.01$

15. \$2187.25

16. a. $A_{n+1} = 1.10A_n$, $A_0 = 300$

b. \$139.23

17. 17.6%

7.3 The compound interest formula

7.3 Exercise

1. a. \$4630.50

c. \$24 488.80

b. \$9274.19

d. \$13 503.78

2. a. \$664.76

c. \$599.58

b. \$5515.98

d. \$1 229 312.85

3. \$483.85

4. a. \$1003.69

c. \$181 402.12

b. \$9111.56

d. \$20 039.67

5. B

6. a. \$9961.26

b. \$11 278.74

c. \$975.46

7. \$70 555.25

8. \$31 850.33

9. C

10. a. \$1749.60

b. \$1757.49

c. \$1759.47

d. \$1759.97

e. More frequent compounding periods increase the final balance.

11. a. \$605.57

b. \$1314.91

c. \$795.82

d. \$1043.28

e. \$1642.73

12. \$11 143.24

13. a. \$2069.61

c. \$3564.10

b. \$1531.33

d. \$5307.05

14. \$528.87

15. A

16. \$27 500

7.4 Interest rates and the effect of changing the rate and periods per year

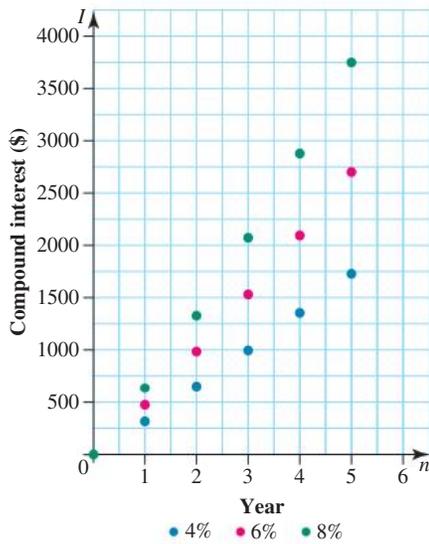
7.4 Exercise

1. a. See the table at the bottom of the page*

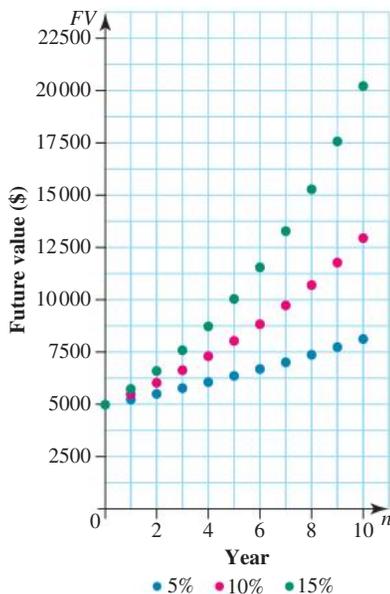
*1. a.

No. of years	1	2	3	4	5
Interest (4%)	\$320.00	\$652.80	\$998.91	\$1358.87	\$1733.22
Interest (6%)	\$480.00	\$988.80	\$1528.13	\$2099.82	\$2705.80
Interest (8%)	\$640.00	\$1331.20	\$2077.70	\$2883.91	\$3754.62

b. Compound interest earned on \$8000



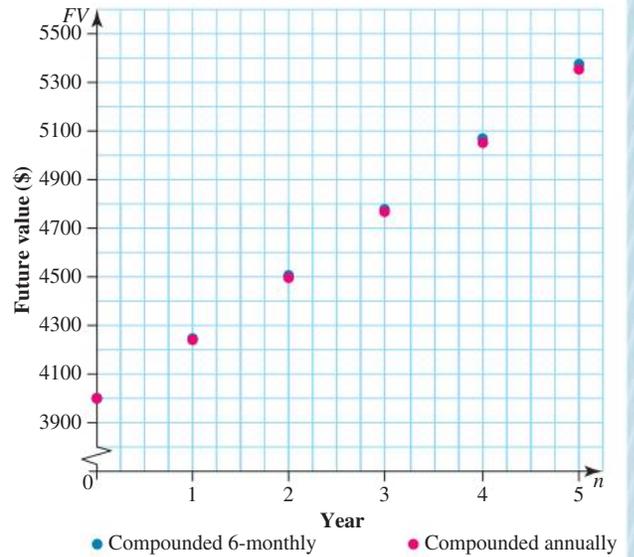
2. a-c. Effect of compound interest on \$5000



Increasing the compound interest rate means that the investment grows more quickly over time.

3. a. See the table at the bottom of the page*

b. Petra's investment



4. 9.22% p.a.
5. 15.16% p.a.
6. D
7. \$1095.95
8. 9.63%
9. \$6691.13
10. \$1966.62
11. Monthly: \$21 989.05; 6-monthly: \$21 968.77; yearly: \$21 945.13. Tai should choose option B.
12. 15.38%
13. 0.5647%
14. Loan 1
15. 36 months
16. 9 years

7.5 The effective annual interest rate

7.5 Exercise

1. 4.06% p.a.
2. 4.10% p.a.
3. 3.79% p.a.
4. 4.27% p.a. effective rate
4.26% p.a. effective rate
So 4.2% p.a. compounding quarterly is better.
5. 4.58% p.a.
6. B
7. A
8. D
9. 6.17%
10. 0.667%
11. \$2202

*3. a.

No. of years	1	2	3	4	5
Annually	4240.00	4494.40	4764.06	5049.91	5352.90
Six-monthly	4243.60	4502.04	4776.21	5067.08	5375.67

12. 0.9489%
13. 5.095% effective rate, 5.12% effective rate, so choose 5% p.a. compounding monthly.
14. 5.75% p.a. compounding monthly is equivalent to a 5.90% p.a. effective interest rate. In this case, when comparing the effective interest rates, the 6% p.a. simple interest is the better option, as it will create a greater return on Marilyn's investment.
15. Option A

7.6 Review

7.6 Exercise

1. A
2. C
3. D
4. A
5. A
6. D
7. D
8. C
9. B
10. $A_{n+1} = 1.0175A_n$; $A_0 = 10\,000$
11. compound interest at 7.7% p.a. with interest compounded daily.

12.

n	Future value (\$)
1	10 783.61
2	11 080.16
3	11 384.86
4	11 697.95

13. \$7974.48
14. The interest is compounded monthly; therefore, the interest rate needs to be divided by 12. The final amount is \$18 006.16.
15. 5% p.a., 5 years
16. Option A: \$11 015.48
Option B: \$11 022.66
Option C: \$11 007.03
Mai should choose option B.
17. Ben has made an extra \$477 in interest.
Yes, because adding \$150 before the interest is calculated increases Ben's final investment value by \$39.17 over 5 years; hence, the overall amount of interest added would be greater.
18. Option i: \$5654.46
Option ii: \$5687.14
Option iii: \$5620.56
Option ii is the best option.
19. Option 1 is more profitable.
20. \$11 220.13

7.6 Past QCAA exam questions

1. A sample response can be found online in the worked solution.
2. a. 4.5% b. 6 months
3. B
4. Option 1
5. In the 18th year.

8 Perpetuities, and present and future value of annuities

LESSON SEQUENCE

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Fully worked solutions for this chapter are available online.

EXAM PREPARATION

Access exam-style questions in every lesson, available online and access past QCAA exam questions in every review.

Resources

 Solutions	Solutions — Chapter 8
 Exam questions	Exam question booklet — Chapter 8
 Digital documents	Learning matrix — Chapter 8 Chapter summary — Chapter 8

LESSON

8.1 Overview

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8.1.1 Introduction

An ordinary **annuity** is a series of equal payments that occur at equal time intervals — usually monthly, quarterly, biannually or annually. The interest in an annuity is compounded at the end of each period before the payment is made or received.

Superannuation is a common example of an annuity. In many employment situations, the employer makes contributions (currently set at 12% of an employee's wage) into an account on a regular basis. The interest on the investment compounds. The annuity is usually set aside for the employee's working life and is used to fund their retirement. This type of annuity is also referred to as a superannuation fund, or 'super fund'.

House mortgages (home loans) and car loans are other examples of annuities. In these cases, the regular and equal monthly payments are subtracted from the loan amount, which is compounded each month along with the interest.



8.1.2 Syllabus links

Lesson	Lesson title	Syllabus links
8.2	Modelling the present value of an ordinary annuity with a recurrence relation	<ul style="list-style-type: none"> ○ Use a recurrence relation to model the present value of an ordinary annuity, e.g. reducing balance loan or retirement pension with periodic payments where interest is calculated before the periodic payment is made. <ul style="list-style-type: none"> • $A_{n+1} = rA_n - d$ where A_{n+1} is total amount at the beginning of the $(n + 1)$th period, A_n is total amount at the beginning of the nth period, d is periodic payment and $r = 1 + i$ where i is interest rate per compounding period
8.3	The present value annuity formula	<ul style="list-style-type: none"> ○ Use the present value annuity formula to model the present value of an ordinary annuity, e.g. reducing balance loan or retirement pension with periodic payments where interest is calculated before the periodic payment is made. <ul style="list-style-type: none"> • $A_{PV} = d \left(\frac{1 - (1 + i)^{-n}}{i} \right)$ where A_{PV} is total amount, d is periodic payment, i is interest rate per compounding period and n is number of compounding periods ○ Solve practical problems involving the present value of an ordinary annuity, including determining the total amount of the annuity, periodic payment, total payments and total interest.
8.4	Modelling the future value of an ordinary annuity with a recurrence relation	<ul style="list-style-type: none"> ○ Use a recurrence relation to model the future value of an ordinary annuity, e.g. compound interest investment with periodic payments where interest is calculated before the periodic payment is made. <ul style="list-style-type: none"> • $A_{n+1} = rA_n + d$ where A_{n+1} is total amount at the beginning of the $(n + 1)$th period, A_n is total amount at the beginning of the nth period, d is periodic payment and $r = 1 + i$ where i is interest rate per compounding period
8.5	The future value annuity and perpetuity formulae	<ul style="list-style-type: none"> ○ Use the future value annuity formula to model the future value of an ordinary annuity, e.g. compound interest investment with periodic payments where interest is calculated before the periodic payment is made. <ul style="list-style-type: none"> • $A_{FV} = d \left(\frac{(1 + i)^n - 1}{i} \right)$ where A_{FV} is total amount, d is periodic payment, i is interest rate per compounding period and n is number of compounding periods ○ Solve practical problems involving the future value of an ordinary annuity, including determining the total amount of the annuity, periodic payment, total payments and total interest. ○ Use the perpetuity formula, $A = \frac{d}{i}$ where A is total amount, d is periodic payment and i is interest rate per compounding period. ○ Solve practical problems involving perpetuities, including determining the total amount of the perpetuity, periodic payment and interest rate per compounding period.

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LESSON

8.2 Modelling the present value of an ordinary annuity with a recurrence relation

SYLLABUS LINKS

- Use a recurrence relation to model the present value of an ordinary annuity, e.g. reducing balance loan or retirement pension with periodic payments where interest is calculated before the periodic payment is made.
- $A_{n+1} = rA_n - d$ where A_{n+1} is total amount at the beginning of the $(n + 1)$ th period, A_n is total amount at the beginning of the n th period, d is periodic payment and $r = 1 + i$ where i is interest rate per compounding period

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8.2.1 Ordinary annuities

An ordinary annuity is a series of equal payments made at the end of time periods, where interest is accrued before the payment. Examples of ordinary annuities are reducing balance loans and retirement pensions.

The payments may be made weekly, monthly, quarterly or yearly.

A recurrence relation can be used to model the present value of an ordinary annuity.

Ordinary annuity modelled by a recurrence relation

$$A_{n+1} = rA_n - d$$

where,

A_{n+1} is total amount at the beginning of the $(n + 1)$ th period

A_n is total amount at the beginning of the n th period

d is periodic payment

$r = 1 + i$ where i is interest rate per compounding period

WORKED EXAMPLE 1 Modelling a reducing balance loan

Jeffrey borrows \$3500 at a rate of 6.5% p.a. (interest compounded monthly) which is to be paid back with monthly instalments of \$711.42.

- Write a recurrence relation to describe this situation.
- Use your calculator to determine the number of monthly repayments needed to pay off this loan.

THINK

1. State the initial values for A_0 , r and d .

WRITE

a. $A_0 = 3500$

$$r = 1 + \frac{6.5}{12}$$

$$= 1.00542$$

$$d = 711.42$$

2. Write the general form of the recurrence relation.

$$A_{n+1} = rA_n - d$$

3. Substitute in the values for A_0 , r and d .

$$A_{n+1} = 1.00542A_n - 711.42, A_0 = 3500$$

b. 1. On a calculator screen, type 3500, press ENTER/EXE and then type $\times 1.00542 - 711.42$. Continue to press ENTER/EXE until the output is less than 0. Count the number of iterations of the recurrence relation until the balance is less than 0.

3500	3500
$3500 \times 1.00542 - 711.42$	2807.55
$2807.55 \times 1.00542 - 711.42$	2111.35
$2111.35 \times 1.00542 - 711.42$	1411.37
$1411.37 \times 1.00542 - 711.42$	707.60
$707.60 \times 1.00542 - 711.42$	0.02

2. Write the answer.

After 5 months the account balance reads 0.02, so it would take 5 months to pay off the loan.

Note: As the above calculations have been rounded to 2 decimal places, this can affect the final balance. If there was an underpayment of 2 cents, the final payment would be adjusted by 2 cents.

8.2.2 Paying off a reducing balance loan using a table

The process of paying off a loan by regular payments over a period of time can be shown in a table (often referred to as an amortisation table). The process can be tracked in the table on a step-by-step basis by following the payments made, the interest and reduction in the principal. For each payment period, a table provides your loan balance, interest charges on your loan and the amount of principal that you pay off.

Consider a table for Worked example 1.

- The *scheduled payment* of \$711.42 remains fixed.
- The *interest rate* of 0.542% per month remains fixed.
- The *interest* is calculated on the previous month's balance.
- The *payment off the principal* is the scheduled payment – interest for that month.
- The *balance of the loan* is the previous month's balance – payment off the principal.
- In the table, n represents the *beginning* of the n th period, that is payment period n .

n	Payment (d)	Interest (\$)	Principal reduction (\$)	Balance of loan (A_n)
0	0.00	0.00	0.00	3500.00
1	711.42	$0.00542 \times 3500 = 18.97$	$711.42 - 18.97 = 692.45$	$3500 - 692.45 = 2807.55$
2	711.42	$0.00542 \times 2807.55 = 15.22$	$711.42 - 15.22 = 696.20$	$2807.55 - 696.20 = 2111.35$
3	711.42	$0.00542 \times 2111.35 = 11.44$	$711.42 - 11.44 = 699.98$	$2111.35 - 699.98 = 1411.37$
4	711.42	$0.00542 \times 1411.37 = 7.65$	$711.42 - 7.65 = 703.77$	$1411.37 - 703.77 = 707.60$
5	711.42	$0.00542 \times 707.60 = 3.84$	$711.42 - 3.84 = 707.58$	$707.60 - 707.58 = 0.02$

Note: The table above can be arranged differently. Another way to present the information is shown below.

n	Principal, A_n (\$)	Interest, I (\$)	Repayment, d (\$)	Balance $A_n + I - d$ (\$)
0	3500	0.00	0.00	3500.00
1	3500	$0.00542 \times 3500 = 18.97$	$711.42 - 18.97 = 692.45$	$3500 - 692.45 = 2807.55$
2	2807.55	$0.00542 \times 2807.55 = 15.22$	$711.42 - 15.22 = 696.20$	$2807.55 - 696.20 = 2111.35$
3	2111.35	$0.00542 \times 2111.35 = 11.44$	$711.42 - 11.44 = 699.98$	$2111.35 - 699.98 = 1411.37$
4	1411.37	$0.00542 \times 1411.37 = 7.65$	$711.42 - 7.65 = 703.77$	$1411.37 - 703.77 = 707.60$
5	707.60	$0.00542 \times 707.60 = 3.84$	$711.42 - 3.84 = 707.58$	$707.60 - 707.58 = 0.02$

WORKED EXAMPLE 2 Completing a loan payment table

Mimi takes out a loan of \$1500 at 9% p.a. to buy a new computer. The loan is to be paid back monthly in equal instalments of \$382.06. The incomplete table for this loan is below.

- Determine the interest rate per payment period.
- Calculate how much of the second payment of \$382.06 is interest.
- Determine the balance of the loan after two payments have been made.
- Calculate the principal reduction after three payments have been made.
- Determine the total interest paid on this loan.
- Calculate the amount of the final payment to ensure a balance of zero at the end of the loan period.

Payment period n	Payment (d)	Interest (\$)	Principal reduction (\$)	Balance of loan (A_n)
0	0.00	0.00	0.00	1500.00
1	382.06	11.25	370.81	1129.19
2	382.06		373.59	
3	382.06	5.67		379.21
4	382.06	2.84	379.22	-0.01

THINK

- Write the value of the annual interest rate.
 - Calculate the monthly interest rate.
 - Write the answer
- Calculate 0.75% of 1129.19 as this is the balance of the loan at the end of the first payment period.
 - Write the answer.

WRITE

- Annual interest rate = 9%

$$\text{Monthly interest rate} = \frac{9}{12} = 0.75\%$$

The interest rate per time period is 0.75%.
- $$0.0075 \times 1129.19 = 8.47$$

The interest component of the second payment is \$8.47.

- | | |
|--|--|
| <p>c. 1. The balance of the loan after two payments have been made is the previous balance – balance reduction</p> <p>2. Write the answer.</p> | <p>c. Previous balance – balance reduction
 $= 1129.19 - 373.59$
 $= 755.60$</p> <p>The balance of the loan after two payments have been made is \$755.60.</p> |
| <p>d. 1. Principal reduction = payment – interest</p> <p>2. Write the answer.</p> | <p>d. $382.06 - 5.67 = 376.39$</p> <p>The principal reduction after three payments is \$376.39.</p> |
| <p>e. 1. The interest paid is the total of the interest.</p> <p>2. Write the answer.</p> | <p>e. $11.25 + 8.47 + 5.67 + 2.84 = 28.23$</p> <p>The total interest paid is \$28.23.</p> |
| <p>f. The last payment results in an overpayment of 1 cent.</p> | <p>f. The last payment should be \$382.05.</p> |

8.2.3 The effect of changing the interest rate

Most home loans are not fixed interest rate loans, which means the interest rate will fluctuate for the life of the loan. The interest rate has a significant effect on the amount of interest paid. An increase in the interest rate will increase the time required to pay off the loan.



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WORKED EXAMPLE 3 Determining the amount of interest paid

Briony wants to borrow \$2400 to redecorate her living room and checks out her two local banks for their terms. Bank with Me charges 15.4% p.a. compounded monthly with monthly repayments of \$217.07. Better Bank charges 14.2% p.a. compounded monthly with monthly repayments of \$282.69.

- Calculate the number of months it would take to pay off the Bank with Me loan.
- Calculate the number of months it would take to pay off the Better Bank loan.
- Determine the amount of interest paid on each loan.

THINK

1. Write the rule for the recurrence relation for a reducing balance loan.
2. Write the values for r and d .
3. Substitute the values into the rule.

WRITE

$$\begin{aligned}
 \text{a. } A_{n+1} &= rA_n - d, A_0 = 2400 \\
 d &= 217.07, \\
 r &= 1 + \frac{15.4}{12} = 1.01283 \\
 A_{n+1} &= 1.01283 A_n - 217.07
 \end{aligned}$$

4. Use the iterative function on your calculator to determine how many months until the loan is paid off.

Note: For most calculators, you can type in the equation, substituting $A_n = 2400$. After pressing ENTER/EXE, type the equation again but substitute A_n for ANS on your calculator. This will use your previous answer each time in place of A_n .

2400	2400
	2213.72
	2025.05
	1833.97
	1640.43
	1444.40
	1245.86
	1044.78
	841.11
	634.83
	425.91
	214.30
	-0.02

5. Count the number of iterations.

- b. Write the rule for the recurrence relation for a reducing balance loan.

Write the values for r and d .

Substitute the values into the rule.

Use the iterative function on your calculator to determine how many months until the loan is amortised.

It will take 12 months to pay off the loan with Bank with Me.

b. $A_{n+1} = rA_n - d, A_0 = 2400$

$$d = 282.69,$$

$$r = 1 + \frac{\frac{14.2}{12}}{100} = 1.01183$$

$$A_{n+1} = 1.01183 A_n - 282.69$$

2400	2400
	2145.70
	1888.40
	1628.05
	1364.62
	1098.07
	828.37
	555.48
	279.36
	-0.03

Count the number of iterations.

- c. For Bank with Me, multiply the monthly repayment of \$217.07 by 12 and subtract \$2400.

For Better Bank, multiply \$282.69 by 9 and subtract \$2400.

Write the answer.

It will take 9 months to pay off the loan with Better Bank.

Bank with Me

$$217.07 \times 12 = 2604.84$$

$$2604.84 - 2400 = 204.84$$

Better Bank

$$282.69 \times 9 = 2544.21$$

$$2544.21 - 2400 = 144.21$$

The interest paid for the Bank with Me loan is \$204.84.

The interest paid for the Better Bank loan is \$144.21.

Exercise 8.2 Modelling the present value of an ordinary annuity with a recurrence relation

learnon

8.2 Exercise

8.2 Exam questions on

Simple familiar

1, 2, 3, 4, 5, 6, 7, 8,
9, 10, 11, 12

Complex familiar

13, 14, 15, 16

Complex unfamiliar

17, 18, 19

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Simple familiar

- WE1** Susie borrows \$20 540 at a rate of 6% p.a. (interest compounded monthly) and pays the loan back in monthly instalments of \$4169.82.
 - Write a recurrence relation to describe this situation.
 - Use your calculator to determine the number of monthly repayments needed to pay off this loan. Give your answer to the nearest whole number.
- Frankie borrows \$800 with interest paid at a rate of 7.5% p.a. (interest compounded monthly) and each month he pays back \$150 from his part-time job into the loan.
 - Write a recurrence relation to describe this situation.
 - Use your calculator to determine the balance of his loan after 4 months.
- WE2** Ahmet borrows \$2400 at 9% p.a., compounded monthly, to buy a new laptop. The loan is to be paid back monthly in instalments of \$489.64. Following is the incomplete amortisation table for this loan.

Payment period n	Payment (d)	Interest (\$)	Principal reduction (\$)	Balance of loan (A_n)
0	0.00	0.00	0.00	2400.00
1	489.64		471.64	1928.36
2	489.64	14.46	475.18	1453.18
3	489.64	10.90	478.74	
4	489.64	7.31		492.11
5	489.64	3.69	485.95	

- Determine the interest rate per payment period.
 - Calculate how much of the first payment of \$489.64 is interest.
 - Determine the balance of the loan after three payments have been made.
 - Determine the principal reduction after 4 payments have been made.
 - Determine the total interest paid on this loan.
 - Determine the balance left at the end of 5 months to pay out the loan.
- MC** Whitney took out a loan for \$15 000 for a new car at 17% p.a. The contract required that she repay the loan over 4 years with monthly instalments of \$432.83. After 6 months Whitney still owes:
 - \$13 866.69
 - \$14 779.67
 - \$14 556.22
 - \$13 630.31



5. A loan of \$34 000 is taken out over 6 years at a rate of 6.5% p.a. (interest debited monthly) and is to be paid back monthly with \$571.54 instalments. Complete the following table for the third payment.

Payment period n	Payment (d)	Interest (\$)	Principal reduction (\$)	Balance of loan (A_n)
0	0.00	0.00	0.00	34 000.00
1	571.54	184.17	387.37	33 612.63
2	571.54	182.07	389.47	33 223.16
3	571.54			

6. Lauren wanted to buy a new road bike so she took out an \$8000 loan over 4 years at a rate of 9% p.a. (interest debited quarterly) and is to be paid back quarterly with instalments of \$600.93.



Payment period n	Payment (d)	Interest (\$)	Principal reduction (\$)	Balance of loan (A_n)
0	0.00	0.00	0.00	8000
1	600.93			
2	600.93			
3	600.93			

- a. Write a recurrence relation for this situation.
- b. Complete the following table for the first three payments and state how much interest she has paid and how much she owes at the end of this time.
7. Su Fen borrows \$550 000 at 4.5% p.a. to buy a house, compounded monthly. The loan is to be paid back monthly in instalments of \$3800. Complete the following amortisation table for the first three repayments.

Payment period n	Payment (d)	Interest (\$)	Principal reduction (\$)	Balance of loan (A_n)
0				
1				
2				
3				

8. **WE3** Carlos wants to borrow \$1300 for a new motorcycle and looks at the online deals provided by two lending institutions. Finance Me charges 18.4% p.a. compounded monthly with monthly repayments of \$228.44 and Have it Now charges 16.6% p.a. compounded monthly with monthly repayments of \$336.32.



- a. Calculate the number of months it would take to pay off the Finance Me loan.
- b. Calculate the number of months it would take to pay off the Have it Now loan.
- c. Determine the total interest that would be paid with each loan.

9. Cassandra needs to borrow \$3000 to furnish her apartment before she starts her new job and makes an appointment with her two local banks to discuss what they can offer. The first bank charges 10.2% p.a. compounded monthly with monthly repayments of \$389.49 and the second bank charges 8% p.a. compounded monthly with monthly repayments of \$612.05. Cassandra needs to budget very carefully each month and is inclined to go with the smaller monthly repayment of \$389.49. Calculate how much Cassandra would save if she could pay the larger amount each month.



10. Mary is saving for her retirement and begins with an initial deposit of \$10 000 in her retirement account. The account earns 4% annual interest, and she withdraws \$500 each year to cover living expenses. Determine how much money will be left in her account after 4 years.
11. Jared is saving money for his child's college tuition. He initially deposits \$5000 into a savings account in a bank that earns 3% p.a. compounding monthly. Each month the bank charges \$54 in fees and charges. Determine how much money will remain in the account at the end of 6 months.
12. Tom has set up an emergency fund with an initial balance of \$2000. The fund earns 5% annual interest, compounding biannually. Every 6 months he withdraws \$150 to cover unforeseen expenses. Calculate the amount remaining in the fund after 2 years.

Complex familiar

13. Oscar borrows \$13 000, taken out over 2 years and to be repaid in monthly instalments. If interest is calculated at a rate of 5% and the monthly instalment is \$570.33, determine the amount of interest Oscar has paid in 4 months.
14. A reducing balance loan can be modelled by the recurrence relation

$$A_{n+1} = 1.0075A_n - 600.35, A_0 = 45\,000.$$

Interest is accrued monthly and is to be paid off in monthly instalments. Determine the amount of interest paid after the third instalment.

15. Ivona took out a loan for \$25 000 over 6 years to set up her hydroponic nursery. The interest rate was set at 8.5% p.a. (interest debited monthly), which is to be paid back monthly with instalments of \$444.46. After 6 months, Ivona receives a small business grant and can afford to increase her payments by \$50 per month. Determine how much Ivona will owe after 12 months.



16. Jenny decides to borrow \$6500 for a trip to London and takes a personal loan at an interest rate of 11.5% debited fortnightly, with fortnightly repayments of \$98.72 over 3 years. Determine how much Jenny still owe after 6 fortnights.



Complex unfamiliar

17. Marcus took out a loan of \$1800 to buy a new sofa, with the loan having an interest rate of 14% p.a. The loan will be repaid with equal monthly payments of \$191.75.
Determine when the loan will be paid out and how much will need to be added to the last payment to pay out the loan.
18. Stephanie borrows \$3600 from a bank at an interest rate of 12.75% p.a. She can pay it off in 6 months with fortnightly payments of \$286.52 or monthly payments of \$622.51.
Determine which is the better option. Use mathematical reasoning to explain your answer.
19. Harry borrows \$2700 from a bank at an interest rate of 10.5% p.a. He can pay it off in 12 months with quarterly payments of \$719.87 or monthly payments of \$238.00. By paying off the loan with more frequent payments, determine how much money Harry saved in interest.

Fully worked solutions for this chapter are available online.

LESSON

8.3 The present value annuity formula

SYLLABUS LINKS

- Use the present value annuity formula to model the present value of an ordinary annuity, e.g. reducing balance loan or retirement pension where interest is calculated before the periodic payment is made.
 - $A_{PV} = d \left(\frac{1 - (1 + i)^{-n}}{i} \right)$ where A_{PV} is total amount, d is periodic payment, i is interest rate per compounding period and n is number of compounding periods
- Solve practical problems involving the present value of an ordinary annuity, including determining the total amount of the annuity, periodic payment, total payments and total interest.

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8.3.1 Present value annuity formula

In Lesson 8.2 a recurrence relation was used to model the present value of an ordinary annuity. In this section, the present value annuity formula can be used to determine the present value of a loan or a pension where interest is calculated before the periodic payment is made.

The present value annuity formula

$$A_{PV} = d \left(\frac{1 - (1 + i)^{-n}}{i} \right)$$

where

A_{PV} is the total amount

d is the periodic payment

i is the interest rate per compounding period

n is the number of compounding periods.

WORKED EXAMPLE 4 Determining the present value of a loan

A loan is to be repaid in full with three annual instalments of \$5000. If interest is charged at 6% compounded annually, determine the total value of the annuity.

THINK

1. Write the present value annuity formula.
2. Write the values for d , i and n .
3. Substitute the values into the formula and calculate the answer.
4. Write the answer.

WRITE

$$A_{PV} = d \left(\frac{1 - (1 + i)^{-n}}{i} \right)$$

$$d = 5000, i = 0.06, n = 3$$

$$\begin{aligned} A_{PV} &= 5000 \left(\frac{1 - (1 + 0.06)^{-3}}{0.06} \right) \\ &= 5000 \left(\frac{1 - 1.06^{-3}}{0.06} \right) \\ &= 13\,365.06 \end{aligned}$$

The total value of the annuity is \$13 365.06.

If the value of the annuity, the interest rate charged and the number of compounding periods are known, the periodic payment can be calculated.



tlvd-12188

WORKED EXAMPLE 5 Determining the periodic payment

A small pension fund of \$80 000 pays interest of 4.5% p.a. calculated monthly. If the pension fund will be paid out in 6 years, determine the payment due each month for the 6 years, to the nearest cent.



THINK

1. Rearrange the present value annuity formula to make d the subject.
2. Write the values of A_{PV} , i and n .
3. Substitute the values into the formula and calculate the answer.
4. Write the answer.

WRITE

$$A_{PV} = d \left(\frac{1 - (1 + i)^{-n}}{i} \right)$$

$$d = \frac{A_{PV}}{\left(\frac{1 - (1 + i)^{-n}}{i} \right)}$$

$$A_{PV} = 80\,000, i = \frac{0.045}{12} = 0.003\,75, n = 72$$

$$\begin{aligned} d &= \frac{A_{PV}}{\left(\frac{1 - (1 + i)^{-n}}{i} \right)} \\ &= \frac{80\,000}{\left(\frac{1 - (1.003\,75)^{-72}}{0.003\,75} \right)} \\ &= 1269.92... \end{aligned}$$

The monthly payment is \$1269.92, to the nearest cent.

8.3.2 Determining the total amount repaid

A loan of a large sum of money, such as to purchase a home, usually takes many years to repay. It is not uncommon for home loans to take 20 or 30 years to repay.

When interest is calculated monthly over such a long period of time, the amount of money required to pay off such a loan can be more than the initial loan.

WORKED EXAMPLE 6 Determining the total amount repaid

A loan of \$120 000 is paid off at 9% p.a. reducible interest over a period of 25 years. The monthly repayment is \$1007.04. Calculate the total repayments on this loan and the total amount of interest paid.

THINK

1. Calculate the number of repayments by multiplying the number of years by 12.
2. Multiply the monthly repayments by the number of repayments.
3. Calculate the total amount of interest paid.
4. Answer the question.

WRITE

$$\begin{aligned}\text{Number of repayments} &= 25 \times 12 \\ &= 300\end{aligned}$$

$$\begin{aligned}\text{Total repayments} &= 1007.04 \times 300 \\ &= 302\,112.00\end{aligned}$$

$$\begin{aligned}\text{Total interest paid} &= \text{total repayments} - \text{loan amount} \\ &= 302\,112 - 120\,000 \\ &= 182\,112\end{aligned}$$

The total repayments on this loan are \$302 112 and the total amount of interest paid is \$182 112.

Exercise 8.3 The present value annuity formula

learn on

8.3 Exercise

8.3 Exam questions

Simple familiar

1, 2, 3, 4, 5, 6, 7, 8,
9, 10, 11, 12, 13, 14

Complex familiar

15, 16

Complex unfamiliar

17, 18

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Simple familiar

1. A couple borrow \$70 000 on a reducing balance loan for 10 years. The interest rate is 7.5% p.a. and their monthly payment is \$840 per month. Calculate for the first month, correct to the nearest cent:
 - a. the interest paid
 - b. the amount that the principal is reduced by
 - c. the balance owing.
2. **WE4** A loan is to be repaid in full with five annual instalments of \$7500. If interest is charged at 8% compounded annually, determine the total value of the loan.
3. A scholarship is paid to the recipient in monthly payments of \$500 for the 3 years of their undergraduate degree. If the fund attracts interest of 3.5% p.a. calculated monthly, determine the total value of the annuity.

4. **WE5** A pension fund of \$110 000 pays interest of 7.5% p.a. calculated monthly. If the pension fund will be paid out in 10 years, determine the payment due each month for the 10 years.
5. A loan of \$250 000 with interest of 5.5% p.a. calculated quarterly is to be paid back in 8 years. Determine the quarterly repayment.
6. Mr and Mrs Devcich borrow \$80 000 to top up their home loan. The interest rate is 6.5% p.a. and their monthly payment is \$850. Write your answers to the following correct to the nearest cent.



- a. Calculate the interest for the second month of the loan.
- b. Calculate the balance owing at the end of the second month.
7. **MC** The first month's interest on a \$60 000 reducing balance loan at 12% p.a. is:
- A. \$12 B. \$600
C. \$7200 D. \$6000

8. **WE6** The repayments on a loan of \$105 000 at 8% p.a. reducible interest over 25 years are \$810.41 per month. Calculate the total repayments on this loan and the total amount of interest paid.
9. **MC** A \$95 000 reducing balance loan at 8% p.a. over a 15-year term has monthly payments of \$907.87. The total amount of interest paid on this loan, to the nearest dollar, is:

- A. \$49 167 B. \$68 416 C. \$87 978 D. \$98 152
10. The repayment on a loan of \$50 000 at 7.5% p.a. over a 15-year term is \$463.51 per month. Write your answers to the following correct to the nearest cent.
- a. Calculate the interest for the first month of the loan and the balance owing at the end of the first month.
- b. Calculate the amount by which the balance has reduced in the first month.
- c. Calculate the interest for the second month of the loan and the balance at the end of the second month.
- d. Calculate the reduction in the loan balance during the second month.

11. Nerida wants to buy a car. She needs to borrow \$22 000. The interest rate is 10.5% p.a. compounded monthly for a personal loan. She wants to pay off the loan in 3 years. Calculate her monthly repayments. Write your answer correct to the nearest cent.



12. A couple decide to borrow \$440 000 to buy an apartment. The interest rate is 5.75% p.a. compounded monthly for a home mortgage. They want to pay off the loan in 15 years. Determine their monthly repayments. Write your answer correct to the nearest cent.

13. The Taylors borrow \$140 000 over 20 years at 9% p.a.
- a. The monthly repayment on this loan is \$1259.62. Calculate the total made in repayments.
- b. The Taylors attempt to pay the loan off quickly by increasing their monthly payment to \$1500. The loan is then paid off in 161 months. Calculate the total repayments made under this plan.
- c. How much will the Taylors save by increasing each monthly payment?

14. **MC** The Nguyen family borrows \$250 000 on a reducing balance loan for 20 years at 5.25% p.a. with monthly payments of \$1684.61. Identify the calculation that determines the balance owing, A , at the start of the second month.

A. $A = 250\,000 \times \left(1 + \frac{5.25}{100}\right) + 1684.61$

B. $A = 249\,409.14 \times \left(1 + \frac{5.25}{100}\right) + 1684.61$

C. $A = 250\,000 \times \left(1 + \frac{5.25}{1200}\right) - 1684.61$

D. $A = 249\,409.14 \times \left(1 + \frac{5.25}{1200}\right) - 1684.61$

Complex familiar

15. Mr and Mrs Chakraborty are borrowing \$100 000 on a reducing balance loan to purchase a motor home. The interest rate charged by the bank is 7% p.a. with a payment of \$1161.08 per month over a 10-year term. Mr and Mrs Chakraborty have another option to borrow the money against their existing home loan. The balance owing on their home is \$45 850. They have 3.5 years left to pay back the loan in full, with an interest rate of 5.85% p.a. and monthly repayments of \$1175. Adding \$100 000 on to their home loan will extend the loan to 16 years based on the existing monthly repayments.



Determine whether it would be a better option for Mr and Mrs Chakraborty to borrow the money against their existing home loan. Support your answer using mathematical reasoning.

16. The Smith and Jones families each take out a \$50 000 reducing balance loan at 9.5% p.a. compounded monthly. The Smith family repays the loan at \$500 per month and the Jones family repays the loan at \$750 per month. After 5 months the Smith family receives a \$15 000 inheritance. On their 6th payment, they put the entire amount onto their loan as an additional payment (i.e. they pay \$15 000 plus \$500). Determine which family owes less money and by how much after one year.



Complex unfamiliar

17. Ghan applies to borrow \$8000 from a financial institution at 4.5% p.a. for 5 years. Ghan is offered a simple interest loan. Interest is calculated for the entire term of the loan and then the amount owing (principal + interest) is repaid in equal monthly instalments. Another lender offers Ghan a reducing balance loan at the same interest rate (4.5% p.a.) and term (5 years) with monthly repayments of \$149.14 (to the nearest cent). Using technology, determine how much Ghan would owe with each option and explain which loan is the most cost-effective option for Ghan.
18. Helda is borrowing \$15 000 on a reducing balance loan at 4.75% p.a. After budgeting she has \$450 left for the monthly payments. If she takes out the loan for 10 years, her monthly payments are \$157.27. Using technology, determine how much time and money Helda will save if she uses all the money at her disposal to pay back the loan.

Fully worked solutions for this chapter are available online.

LESSON

8.4 Modelling the future value of an ordinary annuity with a recurrence relation

SYLLABUS LINKS

- Use a recurrence relation to model the future value of an ordinary annuity, e.g. compound interest investment with periodic payments where interest is calculated before the periodic payment is made.
 - $A_{n+1} = rA_n + d$ where A_{n+1} is total amount at the beginning of the $(n + 1)^{\text{th}}$ period, A_n is total amount at the beginning of the n^{th} period, d is periodic payment and $r = 1 + i$ where i is interest rate per compounding period

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8.4.1 Recurrence relations

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The future value of an annuity can be modelled using a recurrence relation. In the previous sections, we considered annuities that were decreasing in value. In this section we will study annuities that are increasing in value, for example a compound interest investment.

Consider, \$100 compounded monthly by 1% (that is, multiplied by 1.01) would generate the following terms in the sequence:

$$\begin{aligned} &100 \\ &100 \times 1.01 = 101 \\ &101 \times 1.01 = 102.01 \\ &102.01 \times 1.01 = 103.03 \\ &\text{Geometric sequence: } 100, 101, 102.01, 103.03 \end{aligned}$$

A recurrence relation representing the value of an annuity at the end of each payment period will be of the following form.

Value of an annuity at the end of each payment period

$$A_{n+1} = rA_n + d$$

where

A_{n+1} is the total amount at the beginning of the $(n + 1)^{\text{th}}$ period

A_n is the total amount at the beginning of the n^{th} period

d is the periodic payment,

$r = 1 + i$ where i is the interest rate per compounding period.

WORKED EXAMPLE 7 Modelling a future annuity with recursion

At the start of each year \$1000 is invested at 5.75% p.a. compounded annually.

- Set up a recurrence relation that represents the value of the annuity after each payment is made.
- Using the recurrence relation from part a, determine the value of the annuity at the start of the payment period for the first 4 years. Write your answers correct to the nearest cent.

THINK

- Identify the components of the recurrence relation.
 - Substitute the components into the recurrence relation. *Note:* A_1 is the opening balance, at the start of the first period. The interest is compounded at the end of the first period.
- Substitute $n = 1$ and $A_1 = 1000$ into the recurrence relation to calculate the value of the annuity after the second payment is made at the start of the second period.
 - Calculate term A_2 .
 - Substitute $n = 2$ and $A_2 = 2057.50$ into the recurrence relation to calculate the value of the annuity after the third payment is paid.
 - Calculate term A_3 .
 - Substitute $n = 3$ and $A_3 = 3175.81$ into the recurrence relation to find the value of the annuity after the fourth payment is made.
 - Calculate term A_4 .
 - Answer the question.

WRITE

a. $A_1 = 1000$,
 $r = \left(1 + \frac{5.75}{100}\right) = 1.0575$ $d = 1000$

$$A_{n+1} = rA_n + d,$$

$$A_{n+1} = 1.0575 \times A_n, A_1 = 1000$$

b. $n = 1, A_1 = 1000$
 $A_{1+1} = A_2 = 1.0575 \times 1000 + 1000$

$$A_2 = 2057.50$$

$$n = 2, A_2 = 2057.50$$

$$A_{2+1} = A_3 = 1.0575 \times 2057.50 + 1000$$

$$A_3 = 3175.81$$

$$n = 3, A_3 = 3175.81$$

$$A_{3+1} = A_4 = 1.0575 \times 3175.81 + 1000$$

$$A_4 = 4358.42$$

The values of the annuity for the first 4 years are \$1000, \$2057.50, \$3175.81 and \$4358.42.

8.4.2 Using technology to determine the future value of an annuity

Technology can be used to determine the future value of an annuity by using formulas to perform the repetitive calculations required.

WORKED EXAMPLE 8 Using technology to determine the future value of an annuity

Using a spreadsheet or scientific calculator, determine the future value of an annuity of \$500 that is invested every 6 months at 5.75% p.a. for 6 years.

THINK

- Determine the value of r . (Note that the investment is compounded twice a year.)
- Set up a spreadsheet as shown.
- Use a spreadsheet function to complete the table.
Note: Every 6 months (twice per year) for 6 years equates to 12 payments.

WRITE

$$r = \left(1 + \frac{5.75}{200} \right) = 1.02875$$

	A	B	C
1	Payment period, n	Payment (\$)	Future value of annuity (\$)
2	1	500.00	500.00

	A	B	C
	Formulas		$= c2*1.02875 + b3$ (starting from cell c3)
1	Payment period, n	Payment (\$)	Future value of annuity (\$)
2	1	500.00	500.00
3	2	500.00	1014.38
4	3	500.00	1543.54
5	4	500.00	2087.92
6	5	500.00	2647.94
7	6	500.00	3224.07
8	7	500.00	3816.76
9	8	500.00	4426.49
10	9	500.00	5053.76
11	10	500.00	5699.05
12	11	500.00	6362.90
13	12	500.00	7045.83

- Answer the question.

After 6 years (on maturity) the future value of the annuity is \$7045.83.

3. Use a spreadsheet function to complete the table.

	A	B	C
		Formula	= c2*1.014375 + b3 (starting from cell c3)
1	Payment period, n	Payment (\$)	Future value of annuity (\$)
2	1	250.00	250.00
3	2	250.00	503.59
4	3	250.00	760.83
5	4	250.00	1021.77
6	5	250.00	1286.46
7	6	250.00	1554.95
8	7	250.00	1827.30
9	8	250.00	2103.57
10	9	250.00	2383.81
11	10	250.00	2668.08
12	11	250.00	2956.43
13	12	250.00	3248.93
14	13	250.00	3545.63
15	14	250.00	3846.60
16	15	250.00	4151.90
17	16	250.00	4461.58
18	17	250.00	4775.71
19	18	250.00	5094.37
20	19	250.00	5417.60
21	20	250.00	5745.47
22	21	250.00	6078.07
23	22	250.00	6415.44
24	23	250.00	6757.66
25	24	250.00	7104.80

4. Answer the question.

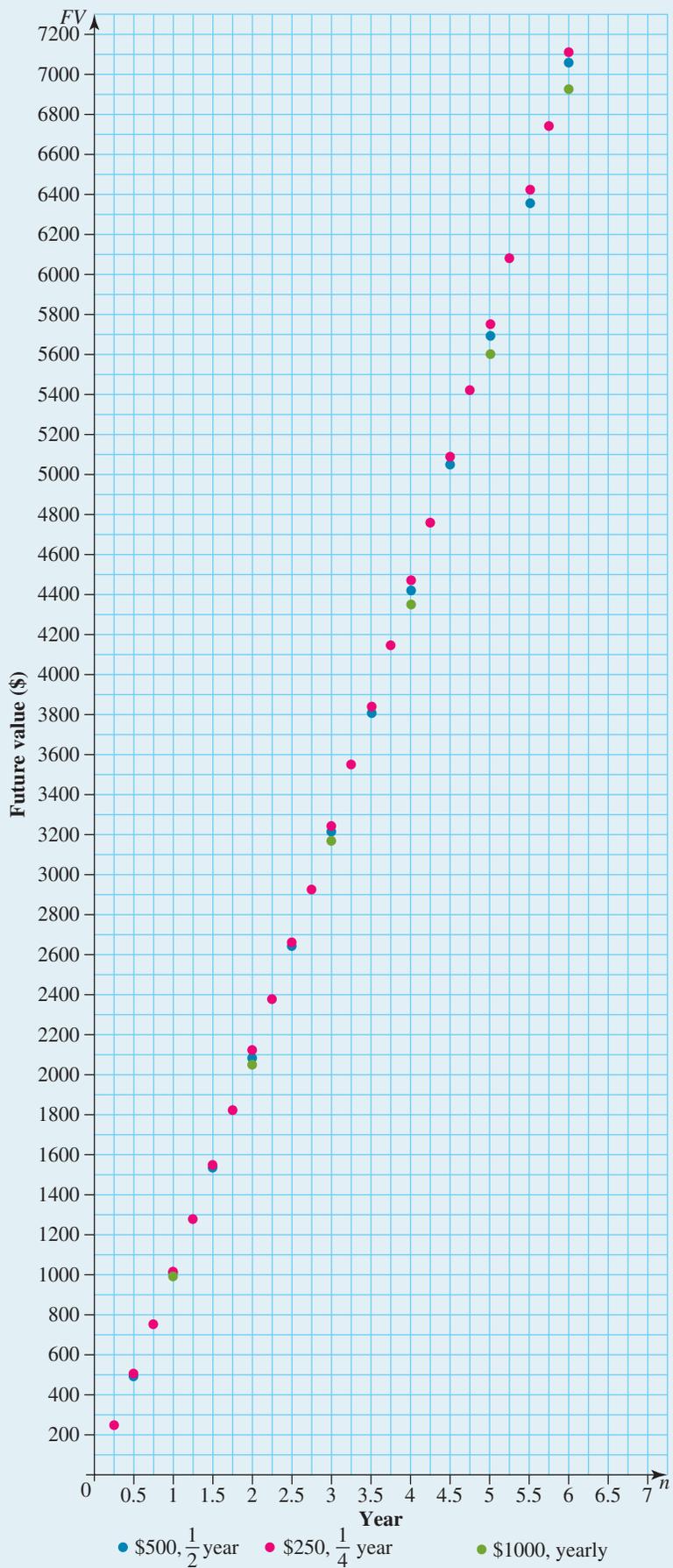
After 6 years (on maturity), the future value of the annuity is \$7104.80.

- b. 1. Determine the value of the annuity of \$1000 invested each year at 5.75% p.a. compounded annually. (Note that this is also done in Worked example 7.)

b.

	A	B	C
1		Formula	= c2*1.0575+ b3 (starting from cell c3)
2	Payment period, n	Payment (\$)	Future value of annuity (\$)
3	1	1000.00	2057.50
4	2	1000.00	3175.81
5	3	1000.00	4358.42
6	4	1000.00	5609.02
7	5	1000.00	6931.54

2. Using the values obtained from part a and Worked examples 7 and 8, construct a graph.



- c. 1. Compare the future values of the annuity for each of the 3 situations.

2. Comment on the findings.

- c. i. For \$1000 invested each year, the future value after 6 years is \$6931.54.
 ii. For \$500 invested half-yearly, the future value after 6 years is \$7045.83.
 iii. For \$250 invested quarterly, the future value after 6 years is \$7104.80.

If payments are more frequent, the future value of the annuity increases. In this example the yearly payment amount did not change, but the payments were made with different frequencies. Making 4 payments per year resulted in a future value of \$7104.80, which is \$173.26 more than the result from a \$1000 payment once a year and \$58.97 more than the result from a \$500 payment twice a year.

Exercise 8.4 Modelling the future value of an ordinary annuity with a recurrence relation

learn **on**

8.4 Exercise

8.4 Exam questions **on**

Simple familiar

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14

Complex familiar

15, 16, 17, 18, 19, 20

Complex unfamiliar

21, 22

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Simple familiar

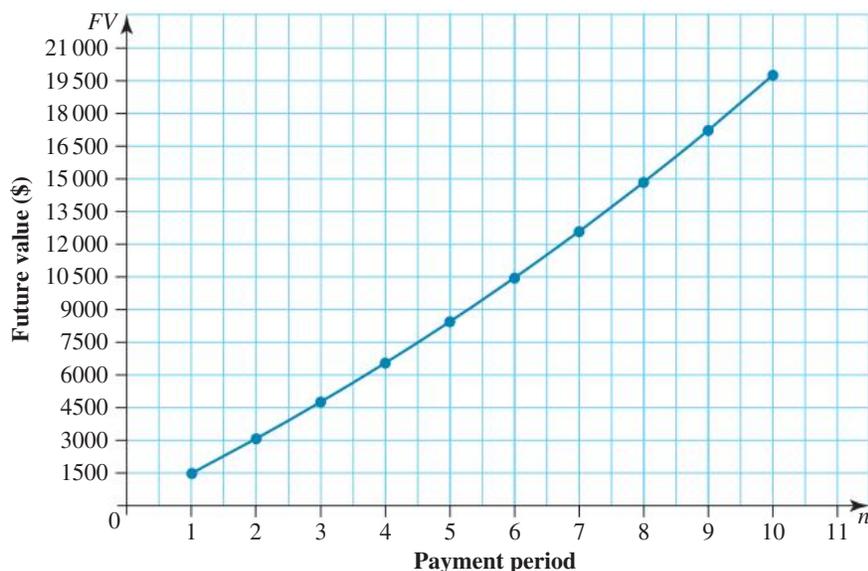
1. **WE7** At the end of each year, \$200 is invested at 6% p.a. compounded annually.
- Set up a recurrence relation to represent the value of the annuity at the end of each payment.
 - Using the recurrence relation from part a, find the value, in dollars, of the annuity after the first 3 payment periods. Write your answers correct to the nearest cent.
2. The following recurrence relation represents the future value of an annuity.

$$A_{n+1} = 1.025A_n + 750, A_1 = 750$$

- Determine the interest rate per payment period.
- Write down the regular payment amount, in dollars.
- Using the recurrence relation, find the future value of the annuity after the first 3 payments, correct to the nearest cent.



3. Set up recurrence relations for each of the following annuities.
- \$1500 invested each year at 8% p.a.
 - \$600 invested every 6 months at 4% p.a.
 - \$450 invested every month at 7.5% p.a.
 - \$600 invested quarterly at 9.5% p.a.
4. **MC** Identify the recurrence relation that represents monthly \$500 payments on a \$25 000 car loan at 6% p.a. compounded monthly.
- $A_{n+1} = 1.06A_n - 500, A_0 = 25\ 000$
 - $A_{n+1} = 1.005A_n - 500, A_0 = 25\ 000$
 - $A_{n+1} = 1.005A_n + 500, A_0 = 25\ 000$
 - $A_{n+1} = 1.06A_n, A_0 = 25\ 000$
5. **MC** The recurrence relation $A_{n+1} = 1.02A_n + 500, A_1 = 500$ represents the future value of an annuity with regular payments made 4 times a year. Determine which one of the following statements is correct.
- The annual interest rate is 2%.
 - The interest rate per payment is 1.02%.
 - The future value of the annuity after the 3rd payment period is \$1530.20.
 - The future value of the annuity after the 2nd payment period is \$510.00.
6. **WEB** Using technology, determine the future values of the following annuities.
- \$1000 invested each year for 5 years at 4% p.a. compounded annually
 - \$500 invested every 6 months for 4 years at 5% p.a. compounded half-yearly
 - \$700 invested every 4 months for 3 years at 4.5% p.a. compounded every 4 months
7. **MC** The future value of an annuity of \$250 invested each month for 2 years at 5.5% p.a. compounded monthly has a future value, correct to the nearest dollar, of:
- \$6347
 - \$6328
 - \$7052
 - \$8342
8. Each of the following recurrence relations determines the future value of an annuity. For each relation, find the future value after the first 3 payments, correct to the nearest cent.
- $A_{n+1} = 1.005A_n + 100, A_1 = 100$
 - $A_{n+1} = 1.0004A_n - 1650, A_1 = 250\ 000$
 - $A_{n+1} = 1.015A_n + 2000, A_1 = 3000$
9. The same sum of money is invested each year. The following graph shows the future value of an annuity for the first 10 years.



Complex familiar

15. Using technology, determine the future value of each of the following annuities after the final payment has been made. (For part **c**, assume 26 fortnights = 1 year, 26 weeks = $\frac{1}{2}$ year.)
- \$900 invested every 6 months for 4 years at 6% p.a. compounded every 6 months
 - \$5000 invested every year for 10 years at 8% p.a. compounded annually
 - \$100 invested every fortnight for 1 year at 5% p.a. compounded fortnightly.
16. A payment of \$2000 is invested each year for 12 years at 7% p.a. compounded annually.
- Write down the calculations that determine the future value of the annuity after the first and second payments.
 - Using technology, determine the future value of the annuity each year until maturity. Write your answers correct to the nearest cent.
17. Use technology to find the future value of \$500 invested each year for 10 years at the following interest rates, compounded annually. Hence, explain the effect that increasing interest rates has on the future value of an annuity.
- 4%
 - 5%
 - 6%
18. The following recurrence relation finds the future value of an annuity.

$$A_{n+1} = 1.02A_n + x, A_1 = x$$

If $A_3 = \$1530.20$, show that $x = \$500$

19. The following calculation finds the future value of an annuity after the third payment when interest has been added and the payment made.

$$FV = 1.012 \times 503 + 250$$

Using mathematical reasoning, determine the original value of the annuity.

20. Monthly mortgage payments of \$1358 are paid towards a loan of \$200 000 that is compounded monthly at 3.25%. The following calculation shows the amount in dollars, A , of the balance of the loan after each payment.

$$A = 200\,000(1 + i)^1 - 1358$$

Using technology, determine the balance of the loan after the first 12 months.

Complex unfamiliar

21. The future values of an annuity after the first 3 payment periods are as follows.

$$\$650, \$1326, \$2029.04$$

Determine the future value of the annuity after the fourth and fifth payments.

22. Anna has a super fund and decides to increase her fortnightly contribution to \$800. The amount in her super fund after interest is added and the next payment is made is represented by the following recurrence relation:

$$A_{n+1} = rA_n + 800, A_1 = 125\,700$$

If the annual interest rate is 7%, determine the balance in Anna's super fund after she makes the next 3 payments. Write your answer correct to the nearest cent.



Fully worked solutions for this chapter are available online.

LESSON

8.5 The future value annuity and perpetuity formulae

SYLLABUS LINKS

- Use the future value annuity formula to model the future value of an ordinary annuity, e.g. compound interest investment with periodic payments where interest is calculated before the periodic payment is made.
 - $A_{FV} = d \left(\frac{(1+i)^n - 1}{i} \right)$ where A_{FV} is total amount, d is periodic payment, i is interest rate per compounding period and n is number of compounding periods
- Solve practical problems involving the future value of an ordinary annuity, including determining the total amount of the annuity, periodic payment, total payments and total interest.
- Use the perpetuity formula, $A = \frac{d}{i}$ where A is total amount, d is periodic payment and i is interest rate per compounding period.
- Solve practical problems involving perpetuities, including determining the total amount of the perpetuity, periodic payment and interest rate per compounding period.

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8.5.1 Calculating the future value of an annuity using a formula

The future value of an annuity can be calculated using the following formula.

Future value of an annuity formula

$$A_{FV} = d \left(\frac{(1+i)^n - 1}{i} \right)$$

where,

A_{FV} is the total amount

d is the periodic payment

i is the interest rate per compounding period

n is the number of compounding periods.

WORKED EXAMPLE 10 Determining the future value of an investment

Using the annuity future value formula, calculate the future value of \$500 invested every 6 months for 10 years at 6% p.a. compounded 6-monthly. Write your answer correct to the nearest cent.

THINK

1. Identify the components of the annuity future value formula.

WRITE

$$\begin{aligned}d &= 500 \\n &= 2 \times 10 = 20 \\i &= \frac{6}{200} = 0.03\end{aligned}$$

2. Substitute the values into the formula.

$$\begin{aligned}A_{FV} &= d \left(\frac{(1+i)^n - 1}{i} \right) \\&= 500 \left(\frac{(1+0.03)^{20} - 1}{0.03} \right) \\&= 13\,435.19\end{aligned}$$

3. Answer the question

In 10 years the future value of the annuity is \$13 435.19.

Rearranging the annuity formula enables us to calculate the payment amount required to obtain a specific annuity future value.

WORKED EXAMPLE 11 Determining the periodic payment

Calculate the annual contribution needed to obtain \$50 000 in 10 years at 4.5% p.a. with interest compounded annually.

THINK

1. Identify the key components of the formula.
2. Rearrange the formula for the future value of an annuity to make d the subject. Substitute the values into the formula.

WRITE

$$\begin{aligned}A_{FV} &= 50\,000, n = 10 \\i &= \frac{4.5}{100} = 0.045\end{aligned}$$

$$\begin{aligned}A_{FV} &= d \left(\frac{(1+i)^n - 1}{i} \right) \\d &= A_{FV} \left(\frac{i}{(1+i)^n - 1} \right) \\&= 50\,000 \left(\frac{0.045}{(1+0.045)^{10} - 1} \right) \\&= 4068.94\end{aligned}$$

3. Answer the question.

Annual regular payments of \$4068.94 are needed to obtain \$50 000 in 10 years at 4.5% interest compounded annually.

8.5.2 Perpetuities

A **perpetuity** is a special type of annuity where an amount of money is invested to provide regular payments that continue indefinitely. Scholarships and bursaries are forms of perpetuities.

The investment amount remains the same for a perpetuity as the funds paid out must never exceed the interest earned on the invested money.

Perpetuity formula

$$A = \frac{d}{i}$$

where,

A is the total amount

d is the periodic payment

i is the interest rate per compounding period.

WORKED EXAMPLE 12 Using the perpetuity formula

A wealthy businesswoman decides to set up a scholarship fund to assist a disadvantaged young person in her community. She invests in a perpetuity that is guaranteed an interest rate of 6% p.a. Determine the amount to be invested in this perpetuity to provide a regular monthly income of \$500.

THINK

1. Write the perpetuity formula
2. Write the values of d and i .
3. Substitute the values of d and i into the perpetuity formula.
4. Write the answer.

WRITE

$$A = \frac{d}{i}$$

$$d = 500, i = \frac{6}{12} = 0.005$$

$$A = \frac{500}{0.005} = 100\,000$$

An investment of \$100 000 is required to provide a monthly income of \$500.



tlvd-12190

WORKED EXAMPLE 13 Using the perpetuity formula to determine interest rate

Genevieve's grandparents set up a perpetuity to help pay for her tertiary education. They invest \$125 000 at an interest rate of 4.5% p.a., with the interest rate for this perpetuity compounded quarterly.

- a. Calculate the amount Genevieve can pay off her university fees each quarter.
- b. Her grandparents want Genevieve to receive \$2000 each quarter and so decide to shop around for a better interest rate. Calculate the interest rate that is needed to ensure a quarterly payment of \$2000.

THINK

- a. 1. Write the perpetuity formula.
2. Rearrange the formula to make d the subject.

WRITE

$$a. A = \frac{d}{i}$$

$$d = A \times i$$

3. Write the values of A and i .

$$A = 125\,000, i = \frac{4.5}{4} = 0.01125$$

4. Substitute the values of A and i into the perpetuity formula.

$$d = 125\,000 \times 0.01125 = 1406.25$$

5. Write the answer.

Genevieve will be able to pay \$1406.25 off her university fees each quarter.

b. 1. Write the perpetuity formula.

$$A = \frac{d}{i}$$

2. Rearrange formula to make i the subject.

$$i = \frac{d}{A}$$

3. Write the values of A and d .

$$A = 125\,000, d = 2000$$

4. Substitute the values of A and d into the perpetuity formula.

$$i = \frac{2000}{125\,000} = 0.016$$

5. Convert the decimal interest rate from quarterly to yearly, then to a percentage.

$$\text{Interest rate} = 0.016 \times 100 \times 4 = 6.4\%$$

6. Write the answer.

The interest rate required is 6.4% p.a.

Exercise 8.5 The future value annuity and perpetuity formulae **learn on**

8.5 Exercise

8.5 Exam questions **on**

Simple familiar

1, 2, 3, 4, 5, 6, 7, 8,
9, 10, 11

Complex familiar

12, 13

Complex unfamiliar

14, 15

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- WE10** Using the future value annuity formula, calculate the future value of each of the following annuities on maturity. Write your answers correct to the nearest cent.
 - \$400 invested at the end of every 6 months for 12 years at 12% p.a. with interest compounded 6-monthly
 - \$1000 invested at the end of every quarter for 5 years at 8% p.a. with interest compounded every quarter
 - \$2500 invested at the end of each quarter at 7.2% p.a. for 4 years with interest compounded quarterly
 - \$1000 invested at the end of every month for 5 years at 6% p.a. with interest compounded monthly
- MC** Tracey invests \$500 into a fund at the end of each year for 20 years. The fund pays 12% p.a. interest, compounded annually. The total amount of interest that Tracey earns on this fund investment is
 - \$1200.
 - \$4323.15.
 - \$4823.23.
 - \$26026.22.
- WE11** Calculate the amount of each annual contribution needed to obtain each of the following amounts, correct to the nearest cent.
 - \$25 000 in 5 years at 5% p.a., with interest compounded annually
 - \$100 000 in 10 years at 7.5% p.a., with interest compounded annually
 - \$500 000 in 40 years at 8% p.a., with interest compounded annually

4. Thomas has the goal of saving \$400 000 for his retirement in 25 years. If the best interest rate that Thomas can obtain is 10% p.a., with interest compounded annually, calculate the amount of each annual contribution that Thomas will need to make.
5. **WE12** A college alumnus decides to set up a scholarship fund for a gifted musician at his old college. He invests in a perpetuity that is guaranteed an interest rate of 7.5% p.a., compounded every 6 months. Determine the amount to be invested in this perpetuity to provide a regular 6-monthly income of \$3000.
6. The director of a theatre uses some of an inheritance to create a drama scholarship for talented young actors. She invests in bonds that pay 8% p.a. and secures a payment each year of \$22 000. How much did she invest in the bonds?
7. Use the perpetuity formula to calculate the payment as specified in each of the following.
- \$150 000 invested at 7.2%, compounded monthly
 - \$350 000 invested at 12% p.a. compounded 6-monthly and paid out annually
8. Use the perpetuity formula to calculate the interest rate p.a. required for each of the following.
- \$200 000 provides \$3500 p.a. with interest compounded annually
 - \$600 000 provides \$25 000 every 6 months with interest compounded biannually
 - \$350 000 provides \$500 per fortnight with interest calculated fortnightly
9. A sum of money was invested each year at 4% p.a. for 5 years. Upon maturity the value of the annuity was \$3249.78. Calculate the sum of money invested.
10. **WE13** A local Lions Club donates \$100 000 to the football and netball club to be invested and hence provide an ongoing source of money to provide uniforms and sports shoes to junior players. The club invests in a perpetuity paying an interest rate of 5.5% p.a., with the interest rate for this perpetuity compounded quarterly.
- Calculate the amount the club receives each quarter.
 - The club need \$3000 each quarter and so decide to shop around for a better interest rate. Calculate the interest rate that is needed to ensure a quarterly payment of \$3000.
11. Jacinta's financial adviser has told her that if she plans on retiring in 25 years, she will need \$500 000 more in her retirement fund to live comfortably. The interest rate on her retirement fund is 6% p.a. and she intends to make monthly payments.
- Show that if Jacinta's monthly contribution is \$721.51, correct to the nearest cent, the future value of the annuity in 25 years will be \$500 000.
 - Jacinta has received an inheritance and decides to invest a lump sum into her retirement fund, with the same monthly 6% p.a. compounded interest. Calculate the value of that inheritance if the amount in her retirement fund is still \$500 000 after 25 years, without any monthly payments.

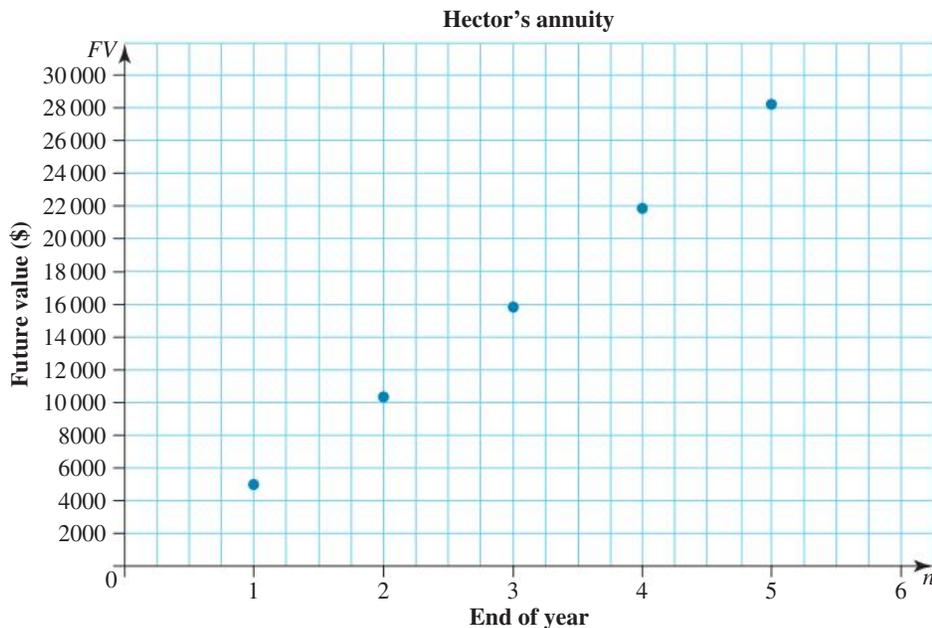


Complex familiar

12. Emerton makes monthly contributions of \$950 into a compounding investment at 5% p.a. compounded monthly. His investment will mature in 10 years. Emerton's friend Gilroy decides to invest a sum of money at the same interest rate and conditions as Emerton. If the future value of Gilroy's investment is the same as Emerton's, determine how much Gilroy invests. Justify your explanation using calculations.



13. After speaking to his financial adviser, Hector wants \$275 000 more in his superannuation upon retirement in 25 years. His super fund is expected to grow at 6% p.a. Hector makes regular and equal annual contributions to his fund. The following graph shows the future value of Hector's annuity for the first 5 years, after speaking to his financial adviser.



The value of Hector's annuity after the second payment is \$10 300, correct to the nearest dollar. If Hector's annuity continues to grow at the same rate compounded annually, the value of the annuity after 24 years will be \$254 078, correct to the nearest dollar. Calculate the future value of the annuity on maturity, correct to the nearest dollar.

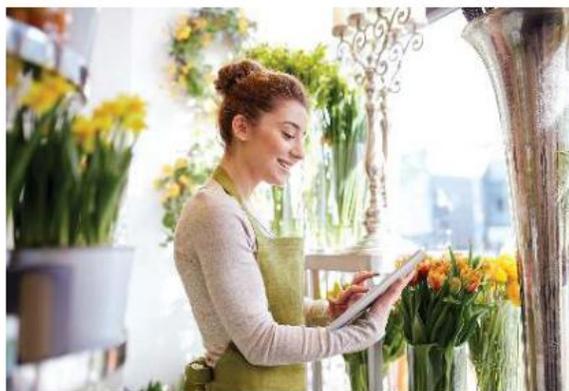
Complex unfamiliar

14. Samantha invests \$500 every 6 months for 5 years into an annuity at 8% p.a. with interest compounded every 6 months.

Samantha's financial institution offers to increase her rate to 9% p.a. To calculate the future value of the annuity, Samantha finds the future value of \$1 at 4% and at 5% and then finds the average. Determine the future value of Samantha's annuity:

- using her method
- using the future value annuity formula.

Comment on your results.



15. Lisa is planning to save for her wedding, which is 10 years away. She decides to make biannual contributions into a saving accounts that earns an interest rate of 6% per year. Lisa wants to have \$50 000 in the account by the time of her wedding. If Lisa reaches her target, determine how much of it is from interest.

Fully worked solutions for this chapter are available online.

5. **MC** Elena needed to borrow \$450 to purchase a dining table. The yearly interest rate was 3.75%, compounded monthly. The loan was to be paid back in five equal monthly instalments. Her monthly repayments were:

A. \$90.85
B. \$37.50
C. \$139.27
D. \$38.91



6. **MC** If the interest rate on a loan is 7.5% p.a., compounded and debited fortnightly, then the growth rate (r) would be closest to:

A. 0.002 B. 0.003 C. 1.002 D. 1.003

7. **MC** The future value of \$500 invested every 6 months compounding biannually at 4% p.a. after 2 years is:

A. \$1506.38 B. \$2060.80 C. \$3415.60 D. 4001.38

8. **MC** The future value of \$300 invested every 3 months for 1 year at 6% p.a. compounded quarterly is:

A. \$1227.27 B. \$1010.49 C. \$655.32 D. \$2114.36

9. **MC** If \$1000 is invested every year for 5 years at 3% p.a. compounded annually, the future value of the annuity after the third payment, correct to the nearest dollar, is:

A. \$1093 B. \$2060 C. \$3091 D. \$3184

10. **MC** Identify the recurrence relation that represents \$1000 invested every 3 months at 7.5% p.a. compounded quarterly.

A. $A_{n+1} = 1.075A_n + 1000, A_1 = 1000$
B. $A_{n+1} = 1.01875A_n + 1000, A_1 = 1000$
C. $A_{n+1} = 1.075A_n + 250, A_1 = 250$
D. $A_{n+1} = 1.01875A_n + 250, A_1 = 250$

11. Grei invests \$200 into a fund at the end of each fortnight for 15 years. The fund pays 8% p.a. interest compounded fortnightly (assume 26 fortnights = 1 year). Determine the total amount of interest, correct to the nearest dollar, that Grei earns on this investment.



12. Calculate the amount of each annual contribution, correct to the nearest dollar, needed to obtain \$150 000 in 12 years at 6% p.a. compounded annually.

Complex familiar

13. Jamal borrows \$1300 with interest paid at a rate of 4.5% p.a. (interest compounded monthly) and each month he makes a repayment of \$188.51.
Determine the amount of interest Jamal has paid after 4 months.
14. A couple borrow \$25 000 on a reducing balance loan for 8 years. The interest rate is 6.35% p.a. and their monthly payment is \$332.81 per month.
Calculate the amount that the principal is reduced by for the first month, correct to the nearest cent.
15. A payment of \$1500 is being invested at the end of every year at 6% p.a. for 4 years. Complete the table that shows the future value of the annuity every year for 4 years. Write your answers correct to the nearest cent.

Payment period n	Payment amount (\$)	Future value (\$)
1	1500	1500
2	1500	
3	1500	
4	1500	

16. Abdul invests \$3100 per quarter into a savings account earning 2.8% p.a. compounding quarterly. After 3 years, he receives a gift from his grandparents of \$5000 and invests that in the savings account and continues with his regular contributions of \$3100 per quarter. Determine how much is in this investment at the end of 5 years.

Complex unfamiliar

17. Suki borrows \$4000 on a reducing balance loan to pay for new kitchen appliances. The interest rate is 5.50% p.a. with monthly payments of \$176.38 for 2 years.
After her second payment, Suki is offered a lower interest rate of 5.20% p.a. with monthly payments of \$175.89. Determine how much Suki saves, correct to the nearest cent.



18. A sound system in a nightclub costs \$5500. The DJ has taken out a personal loan on a reducing balance with an interest rate of 8.75% p.a. to pay for the sound system. His monthly payments are \$250.64 for 2 years. The table shows the balance owing after 12 payments. Complete the table by finding the interest and principal reduction for the ninth month.



Month	Balance owing at start of month (\$)	Interest (\$)	Principal reduction (\$)	Balance owing (\$)
1	5500.00	40.10	210.54	5289.46
2	5289.46	38.57	212.07	5077.39
3	5077.39	37.02	213.62	4863.78
4	4863.78	35.47	215.17	4648.60
5	4648.60	33.90	216.74	4431.86
6	4431.86	32.32	218.32	4213.53
7	4213.53	30.72	219.92	3993.62
8	3993.62	29.12	221.52	3772.10
9	3772.10			3548.96
10	3548.96	25.88	224.76	3324.20
11	3324.20	24.24	226.40	3097.80
12	3097.80	22.59	228.05	2869.75

19. According to Jett's financial adviser, over the next 15 years Jett will need to save \$275 000 in his super fund, which grows at 6% p.a. compounded monthly. At the start of the 15 years, Jett decides to sell his rental investment property and invest this money. Determine how much, in dollars, Jett needs to sell his property for so that he will have the same amount of money as his super fund upon maturity. Write your answer correct to the nearest dollar.
20. In 10 years, Billie would like to go on the trip of a lifetime and spend 12 months visiting every continent. Her travel agent suggests that she will need at least \$85 000. Billie's budget allows her to save \$800 each month. She invests this money at 5% p.a. compounded monthly. Billie makes her monthly payments for 5 years, but then a change in her circumstances means that she can no longer save \$800 each month. After her change in circumstances, Billie can make monthly payments of \$200 at the same interest rate of 5% p.a. The value of Billie's annuity, correct to the nearest cent, after 58 payments is \$82 897.11. Using a recurrence relation, show that Billie will have to postpone her trip by 2 months.

8.6 Past QCAA exam questions

Question 1 (1 mark)

Source: QCAA 2023 General Mathematics, Paper 1, Section 1, Q.11; © State of Queensland (QCAA) 2024

An annuity with an initial zero balance has \$500 deposited at the end of every month. The annuity earns 4.8% p.a. interest, compounding monthly. At the end of the fourth month, the balance is closest to

- A. \$2002 B. \$2008 C. \$2012 D. \$2014

Question 2 (1 mark)

Source: QCAA 2023 General Mathematics, Paper 1, Section 1, Q.12; © State of Queensland (QCAA) 2024

A reducing balance loan with an initial balance of \$6000 is modelled by the recurrence relation

$$A_{n+1} = \left(1 + \frac{0.03}{12}\right) A_n - 400, \text{ where } n \text{ is the number of months.}$$

The loan balance at the end of 2 months is closest to

- A. \$5100 B. \$5200 C. \$5215 D. \$5230

Question 3 (4 marks)

Source: QCAA 2023 General Mathematics, Paper 1, Section 2, Q.17; © State of Queensland (QCAA) 2024

Terome paid a \$50 000 deposit on a house valued at \$570 000 and borrowed the remainder as a reducing balance loan at 6.6% p.a. compounding monthly. Determine the monthly repayment required to pay off the loan over 25 years.

Question 4 (5 marks)

Source: QCAA 2023 General Mathematics, Paper 2, Section 1, Q.7; © State of Queensland (QCAA) 2024

Five years ago, a retiree invested \$100 000 in a compound interest account earning 3.8% p.a. compounding monthly. They now intend to use the balance of the account to begin a perpetuity that will return 4% p.a. compounding annually and pay them \$6000 each year.

Provide advice to the retiree about whether their compound interest investment is large enough to finance the perpetuity.

Question 5 (5 marks)

Source: QCAA 2022 General Mathematics, Paper 1, Section 2, Q.25; © State of Queensland (QCAA) 2024

A couple borrow money to complete home renovations. Their bank has loaned the amount at 2.4% p.a. compounding monthly with repayments of \$993.14 each month for 15 years.

- a. Determine the amount of money borrowed. **[3 marks]**
b. Write a recurrence relation for the amount owing after n months. **[2 marks]**

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Answers

Chapter 8 Perpetuities, and present and future value of annuities

8.2 Modelling the present value of an ordinary annuity with a recurrence relation

8.2 Exercise

- $A_{n+1} = 1.005A_n - 4169.82$, $A_0 = 20\,540$
 - Five repayments
- $A_{n+1} = 1.00625A_n + 150$, $A_0 = 800$
 - \$1425.84
- 0.75%
 - \$18
 - \$974.44
 - \$482.33
 - \$54.36
 - \$6.16
- D
- See the table at the bottom of the page*
- $A_{n+1} = 1.0225A_n - 600.93$, $A_0 = 8000$
 - See the table at the bottom of the page*
After 3 payments, Lauren has paid \$511.38 in interest and still owes \$6708.58.
- See the table at the bottom of the page*
- 6 months
 - 4 months
 - Finance Me interest = \$70.64
Have it Now interest = \$45.28
- \$55.67
- \$9575.35
- \$4749.44
- \$1584.75
- \$203.74
- \$1006.57
- \$21\,363.00

- \$6074.62
- 10 payments, with 37 cents added to his last payment.
- More frequent payments reduce the amount of interest, so paying each fortnight is the better option.
- \$23.48

8.3 The present value annuity formula

8.3 Exercise

- \$437.50
 - \$402.50
 - \$69\,597.50
- \$29\,945.33
- \$17\,063.53
- \$1305.72
- \$9709.63
- \$431.08
 - \$79\,164.41
- B
- Total amount paid: \$243\,123, total interest paid: \$138\,123
- B
- Interest = \$312.50, balance owing = \$49\,848.99
 - \$151.01
 - Interest = \$311.56, balance owing = \$49\,697.04
 - \$151.95
- \$715.05
- \$3653.80
- \$302\,308.80
 - \$241\,500
 - \$60\,808.80
- C
- If they can afford the monthly payments of \$2336.08, Mr and Mrs Chakraborty should borrow the money for the motor home on a reducing balance loan at 7% p.a.

*5.

Payment period (n)	Payment (\$)	Interest (\$)	Principal reduction (\$)	Balance of loan (\$)
0	0.00	0.00	0.00	34\,000.00
1	571.54	184.17	387.37	33\,612.63
2	571.54	182.07	389.47	33\,223.16
3	571.54	179.96	391.58	32\,831.58

*6. b.

Payment period (n)	Payment (\$)	Interest (\$)	Principal reduction (\$)	Balance of loan (\$)
0	0.00	0.00	0.00	8000
1	600.93	180	420.93	7579.07
2	600.93	170.53	430.40	7148.67
3	600.93	160.85	440.08	6708.58

*7.

Payment period (n)	Payment (\$)	Interest (\$)	Principal reduction (\$)	Balance of loan (\$)
0	0.00	0	0	550\,000
1	3800	2062.50	1737.50	548\,262.50
2	3800	2055.98	1744.02	546\,518.48
3	3800	2049.44	1750.56	544\,767.93

16. At the end of 12 months, the Jones family owes \$45 559.98 and the Smith family owes \$32 967.36. The Jones' owe \$12 592.62 more than the Smiths.

Jones family			
Month	Principal (\$)	Interest (\$)	Balance owing (\$)
1	50 000	395.83	49 645.83
2	49 645.83	393.02	49 288.86
3	49 288.86	390.20	48 929.07
4	48 929.07	387.35	48 566.42
5	48 566.42	384.48	48 200.91
6	48 200.91	381.59	47 832.50
7	47 832.50	378.67	47 461.17
8	47 461.17	375.73	47 086.90
9	47 086.90	372.77	46 709.68
10	46 709.68	369.78	46 329.46
11	46 329.46	366.77	45 946.24
12	45 946.24	363.74	45 559.98

Smith family			
Month	Principal (\$)	Interest (\$)	Balance owing (\$)
1	50 000	395.83	49 895.83
2	49 895.83	395.01	49 790.84
3	49 790.84	394.18	49 685.02
4	49 685.02	393.34	49 578.36
5	49 578.36	392.50	49 470.85
6	49 470.85	391.64	49 362.50
7	49 362.50	390.77	49 253.53
8	49 253.53	389.89	49 144.04
9	49 144.04	389.00	49 034.04
10	49 034.04	388.10	48 923.54
11	48 923.54	387.19	48 812.54
12	48 812.54	386.27	48 701.07

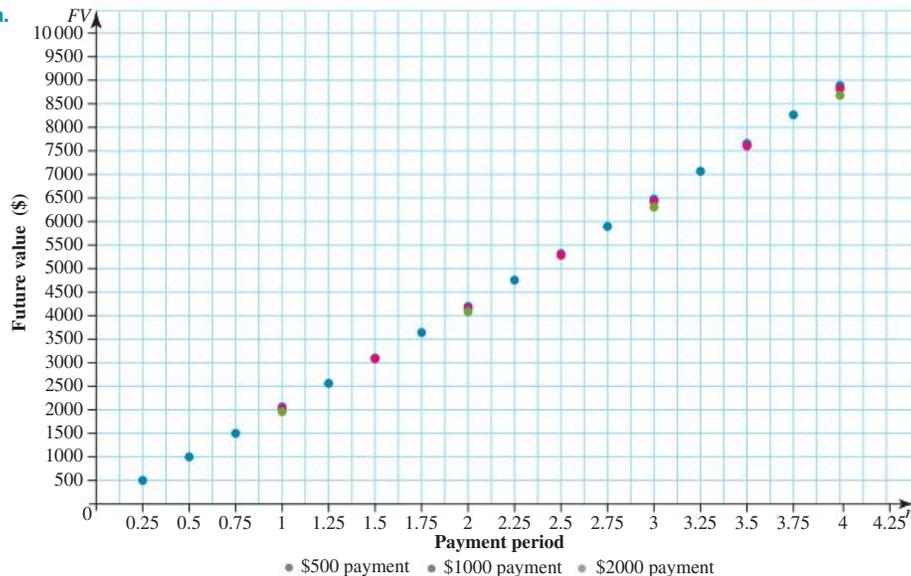
17. Ghan will pay \$851.60 less with the reducing balance loan.
 18. Helda should increase her monthly payments to \$450, as she will save \$2755.64 in interest and repay the loan 7 years earlier.

8.4 Modelling the future value of an ordinary annuity with a recurrence relation

8.4 Exercise

- $A_{n+1} = 1.06A_n + 200, A_1 = 200$
 - First payment period: \$200, second payment: \$412, third payment: \$636.72
- 2.5%
 - \$750
 - First payment period: \$750, second payment: \$1518.75, third payment: \$2306.72
- $A_{n+1} = 1.08A_n + 1500, A_1 = 1500$
 - $A_{n+1} = 1.02A_n + 600, A_1 = 600$
 - $A_{n+1} = 1.00625A_n + 450, A_1 = 450$
 - $A_{n+1} = 1.02375A_n + 600, A_1 = 600$
- B
- C
- \$5416.32
 - \$4368.06
 - \$6691.53
- B
- \$301.50
 - \$246 899.38
 - \$7120.68
- \$1500
 - 6 payments
 - 6%
- A
- \$8684.53
 - i, ii, iii. See the figure at the bottom of the page*
 - If payments are more frequent, the future value of the annuity increases. Making 4 payments per year results in a future value of \$8880.38, which is \$195.85 more than the result from \$2000 once a year and \$66.54 more than the result from \$1000 twice a year.

*11. a.



12. a. i. \$1257.79 ii. \$1414.71 iii. \$1593.74
 b. Increasing the interest rate increases the future value of the annuity, with an approximate 5% increase with each 1% increase in interest rate.

13. a. Interest rate = 0.3% per payment period
 b. $A_{n+1} = 1.003A_n + 1000, A_1 = 1000$
 c. Fourth payment: \$4018.03, fifth payment: \$5030.08

14. a. \$400 000
 b. Sample responses can be found in the worked solutions in the online resources.
 c. This is an example of an annuity because there are regular and equal monthly payments.
 d. \$399 247.65

15. a. \$8003.10 b. \$72 432.81 c. \$2663.47

16. a. $A_1 = \$0 \times 1.07 + 2000 = \2000
 $A_2 = \$2000 \times 1.07 + 2000 = 4140.00$

b.	1st payment: \$2000.00	7th payment: \$17 308.04
	2nd payment: \$4140.00	8th payment: \$20 519.61
	3rd payment: \$6429.80	9th payment: \$23 955.98
	4th payment: \$8879.89	10th payment: \$27 632.90
	5th payment: \$11 501.48	11th payment: \$31 567.20
	6th payment: \$14 306.58	12th payment: \$35776.90

17. a. \$6003.05 b. \$6288.95
 c. \$6590.40
 Increasing the interest rate increases the future value of the annuity, with an approximate 4.8% increase for each 1% increase in interest rate.

18. Sample responses can be found in the worked solutions in the online resources.

19. \$250

20.

1st month: \$199 183.60	7th month: \$194 238.56
2nd month: \$198 364.99	8th month: \$193 406.56
3rd month: \$197 544.16	9th month: \$192 572.31
4th month: \$196 721.11	10th month: \$191 735.79
5th month: \$195 895.83	11th month: \$190 897.01
6th month: \$195 068.32	12th month: \$190 055.96

21. Fourth payment: \$2760.20, fifth payment: \$3520.61

22. \$129 124.36

8.5 The future value annuity and perpetuity formulae

8.5 Exercise

1. a. \$20 326.23 b. \$24 297.37 c. \$45 881.32
 d. \$69 770.03
 2. D
 3. a. \$4524.37 b. \$7068.59 c. \$1930.08
 4. \$4067.23
 5. \$80 000
 6. \$275 000

7. a. \$900 b. \$42 000

8. a. 1.75% b. 8.33% c. 3.7%

9. \$600

10. a. \$1375 b. 12%

11. a. Sample responses can be found in the worked solutions in the online resources.

b. $P = \frac{500\,000}{1.005^{300}} = 111\,983.30$; this means that \$111 983.20 invested now as a single sum investment would have the same future value as an annuity of \$721.51 invested each month for 25 years at 6% p.a. compounded monthly.

12. Emerson will have \$147 549.74 after 10 years. To have the same final amount as Emerson, Gilroy will need to invest a lump sum of \$89 550.77 (using $A = P \times (1 + i)^n$) with the same rate and term.

13. \$274 323

14. a. \$6146

- b. \$6144.10

The results from the two methods are similar but not the same (\$6146 compared to \$6144.10).

15. \$12 784.29

8.6 Review

8.6 Exercise

1. D
 2. B
 3. A
 4. D
 5. A
 6. D
 7. B
 8. A
 9. C
 10. B
 11. \$72 413
 12. \$8892
 13. \$561.32
 14. \$200.52

15.

Payment, n	Payment amount (\$)	Future value (\$)
1	1500	1500.00
2	1500	3090.00
3	1500	4775.40
4	1500	6561.92

16. \$71 515.14
 17. \$10.78
 18. Interest = \$27.50, principal reduction = \$223.14
 19. Jett will need to sell his rental for \$112 057 which is the present value.
 20. Sample responses can be found in the worked solutions in the online resources.

8.6 Past QCAA exam questions

1. C
2. D
3. \$3543.64
4. Sample responses are available in the worked solutions in the online resources.
5. a. \$150 000.29
b. $A_0 = 150\,000.29$
 $A_{n+1} = 1.002A_n - 993.14$

9 Graphs and networks

LESSON SEQUENCE

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Fully worked solutions for this chapter are available online.

EXAM PREPARATION

Access exam-style questions in every lesson, available online and access past QCAA exam questions in every review.

Resources

 Solutions	Solutions — Chapter 9
 Exam questions	Exam question booklet — Chapter 9
 Digital documents	Learning matrix — Chapter 9 Chapter Summary — Chapter 9

LESSON

9.1 Overview

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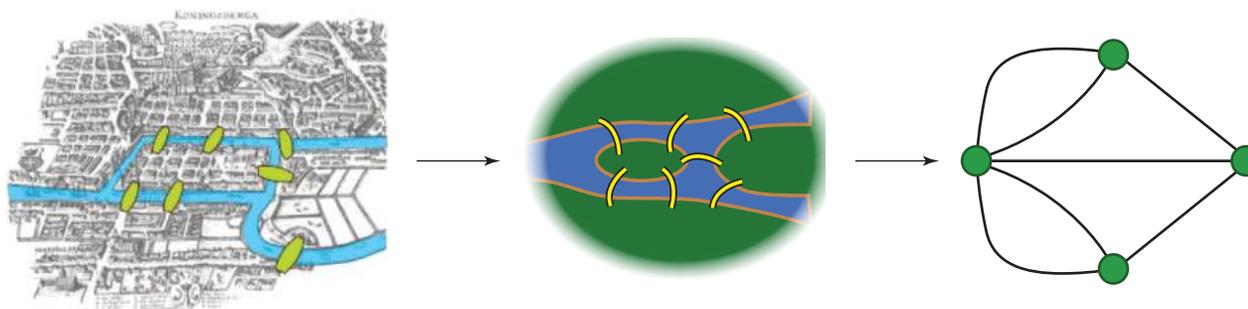
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9.1.1 Introduction

Just like matrices, networks are used to show how things are connected. This type of mathematics is credited to Leonhard Euler's 1736 exploration into a famous problem in mathematics: the Seven Bridges of Königsberg.

The city of Königsberg (now Kaliningrad, Russia) had seven bridges connecting various parts of the city over the Pregel River. The challenge was to find a walking path through the city that would cross each bridge exactly once.



Euler proved that such a walk was impossible by recreating the city layout into vertices (land masses) and edges (bridges). He showed that for such a walk to be possible, each vertex must have an even number of bridges connected to it, which wasn't the case in Königsberg. In proving this, he created a new field of mathematics. Euler originally called the new mathematics the 'geometry of position' but later applications of the mathematics led to it being called graph theory.

Modern-day applications include social media networks (exploring connections between people or groups); video game maps (exploring the most lucrative or most efficient paths) and traffic maps (to determine the best route).

9.1.2 Syllabus links

Lesson	Lesson title	Syllabus links
9.2	Graphs and associated terminology	<ul style="list-style-type: none"> <input type="radio"/> Understand the meaning of graph, vertex (node), edge (arc), loop, degree of a vertex, subgraph, simple graph, complete graph, bipartite graph, directed graph (digraph) weighted graph and network. <input type="radio"/> Construct a network diagram to represent practical situations, e.g. tracks connecting camp sites in a national park, a social network, a transport network with one-way streets, the results of a round-robin sporting competition.
9.3	The adjacency matrix	<ul style="list-style-type: none"> <input type="radio"/> Construct an adjacency matrix from a given graph or digraph. <input type="radio"/> Construct a graph or digraph from a given adjacency matrix.
9.4	Planar graphs	<ul style="list-style-type: none"> <input type="radio"/> Understand the meaning of planar graph and face. <input type="radio"/> Apply Euler's formula to solve problems relating to planar graphs. <ul style="list-style-type: none"> • $v + f - e = 2$ where v is number of vertices, f is number of faces and e is number of edges
9.5	Traversing graphs – walks, paths and trails	<ul style="list-style-type: none"> <input type="radio"/> Understand the meaning of walk, trail, path, open walk, open trail, open path, closed walk, closed trail (circuit), closed path (cycle), connected graph and bridge. <input type="radio"/> Solve practical problems to determine the shortest path between two vertices in a weighted graph (by trial-and-error methods only). <input type="radio"/> Understand the meaning of Eulerian trail, semi-Eulerian graph, Eulerian circuit and Eulerian graph, and the conditions for their existence. <input type="radio"/> Solve practical problems involving semi-Eulerian graphs and Eulerian graphs. <input type="radio"/> Understand the meaning of Hamiltonian path, semi-Hamiltonian graph, Hamiltonian cycle and Hamiltonian graph. <input type="radio"/> Solve practical problems involving semi-Hamiltonian graphs and Hamiltonian graphs (by trial-and-error methods only).

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LESSON

9.2 Graphs and associated terminology

SYLLABUS LINKS

- Understand the meaning of graph, vertex (node), edge (arc), loop, degree of a vertex, subgraph, simple graph, complete graph, bipartite graph, directed graph (digraph) weighted graph and network.
- Construct a network diagram to represent practical situations, e.g. tracks connecting camp sites in a national park, a social network, a transport network with one-way streets, the results of a round-robin sporting competition.

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As you will have noticed in previous years, it is a common practice to draw diagrams and other visual and graphic representations when solving many mathematical problems. In the branch of mathematics known as graph theory, diagrams involving points and lines are used as a planning and analysis tool for systems and connections. Applications of graph theory include business efficiency, transportation systems, design projects, building and construction, food chains, and communications networks. The graphs referred to in graph theory are different from the graph of a function and are called networks.

9.2.1 Parts of graphs and networks

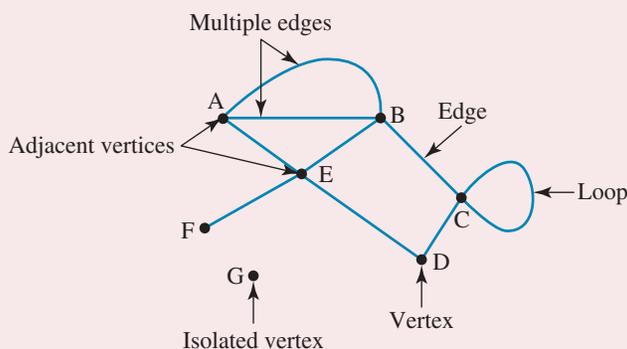
In this section, we focus on naming the parts of a graph and the different types of graphs. Let's start with defining a graph: A **graph** or **network** is a series of points and lines that can be used to represent the connections that exist in various settings.

The parts of a graph or network

- **Vertices (or nodes)** are the points in a network.
- **Edges (or arcs)** are the lines (straight or curved) that connect two points.

Associated terminology

- The graph has **multiple edges** if two or more edges connect the same pair of vertices.
- Two vertices connected by an edge are called **adjacent vertices**.
- A **loop** is an edge that connects a vertex to itself. If there are no arrows, you can travel in either direction.
- An **isolated vertex** is not connected to any other vertex.

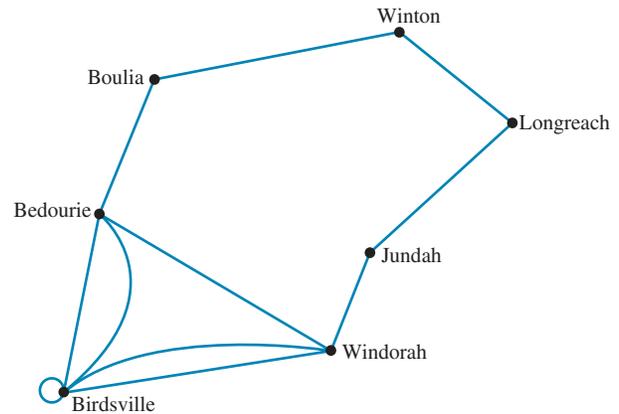


Consider the road map shown. In addition to the roads and towns seen, there is a scenic route that you can take around the greater Birdsville area, which does not use any of the existing major roads between the towns.



This map can be represented by the following graph or network.

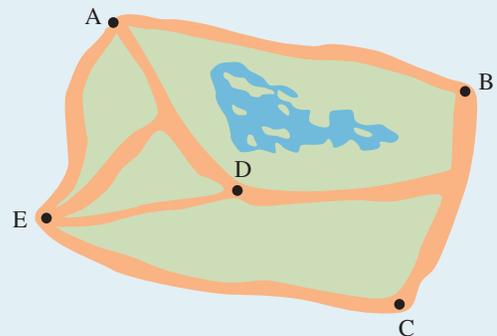
As there is more than one route connecting Birdsville to Windorah and Birdsville to Bedourie, they are each represented by an edge in the graph. In this case, we say there are multiple edges. Also, as it is possible to travel along a road from Birdsville that returns without passing through another town, this is represented by an edge. When this happens, the edge is called a loop.



tlvd-12191

WORKED EXAMPLE 1 Constructing a network diagram

The diagram represents a system of paths and gates in a large park. Draw a graph to represent the possible ways of travelling to each gate in the park.

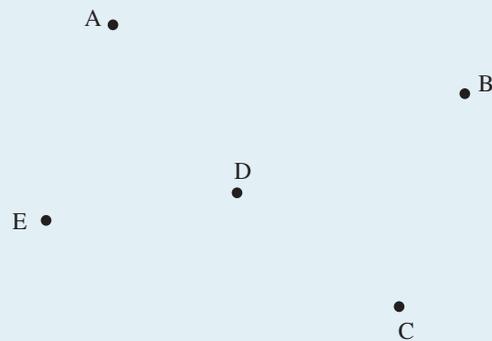


THINK

1. Identify, draw and label all possible vertices

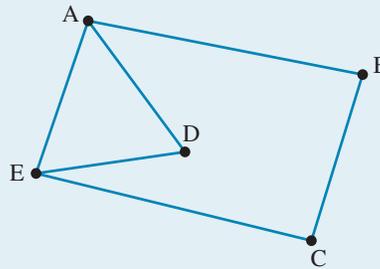
WRITE/DRAW

Represent each of the gates as vertices.



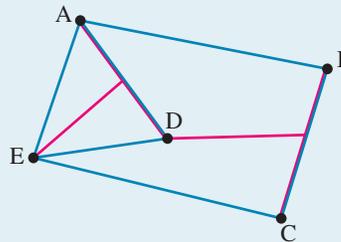
2. Draw edges to represent all the direct connections between the identified vertices.

Direct pathways exist for A–B, A–D, A–E, B–C, C–E and D–E.

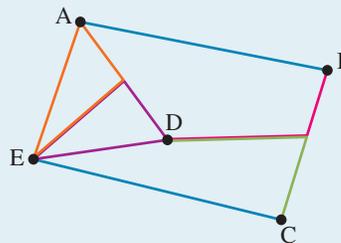


3. Identify all the other unique ways of connecting vertices.

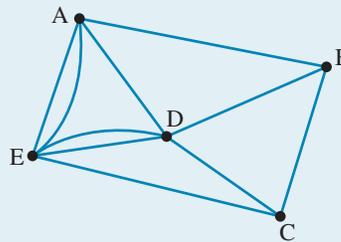
Other unique pathways exist for A–E, D–E, B–D and C–D.



4. Network diagrams are not a scale image of the paths. The edges are to represent the number of connections between points. Thus, there are two ways to travel between A to E (orange edges). There are two ways to travel between D to E (purple edges). There is a way to travel between B and D (pink edge). There is a way to travel between C and D (green edge).

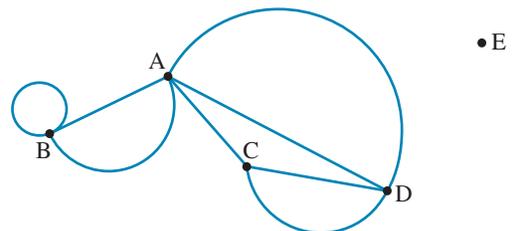


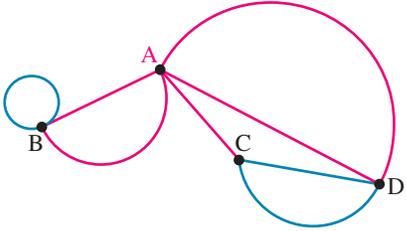
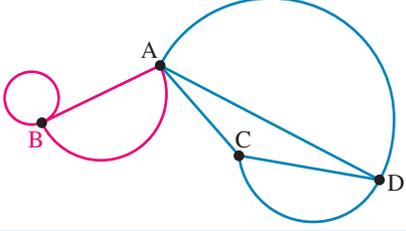
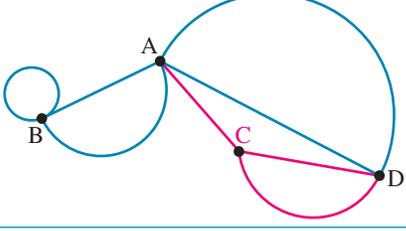
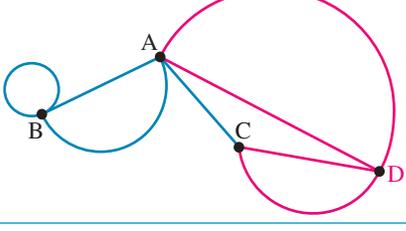
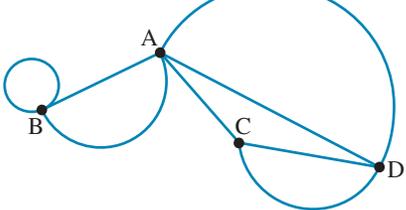
5. Draw the final graph.



9.2.2 The degree of a vertex

When analysing the situation that a graph is representing, it can often be useful to consider the number of edges that are directly connected to a particular vertex. This is referred to as the **degree** of the vertex and is given the notation $\text{deg}(V)$, where V represents the vertex. Undirected loops (loops without an arrow) are counted twice to represent both directions.



	<p>•E</p> <p>There are 5 edges directly connected to vertex A.</p>	<p>Deg(A) = 5</p>
	<p>•E</p> <p>There are 4 edges directly connected to vertex B, including an undirected loop (the undirected loop is counted twice).</p>	<p>Deg(B) = 4</p>
	<p>•E</p> <p>There are 3 edges directly connected to vertex C.</p>	<p>Deg(C) = 3</p>
	<p>•E</p> <p>There are 4 edges directly connected to vertex D.</p>	<p>Deg(D) = 4</p>
	<p>•E</p> <p>There are no edges directly connected to vertex E.</p>	<p>Deg(E) = 0</p>

Notice that the sum of the degrees in this graph is 16.

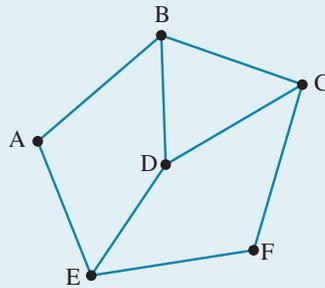
The total number of edges in the graph should always be half of the sum of the degrees. Half of 16 is 8, which is the number of edges in this network.

The degree of a vertex

- The degree of the vertex is the number of edges directly connected to the vertex. Undirected loops (loops without an arrow) are counted twice to represent both directions.

WORKED EXAMPLE 2 Showing that the number of edges is half the sum of the degrees of the vertices

For the graph in the diagram, show that the number of edges is equal to half the sum of the degrees of the vertices.



THINK

1. Identify the degree of each vertex.
2. Calculate the sum of the degrees for the graph.
3. Count the number of edges for the graph.
4. State the final answer.

WRITE/DRAW

$\deg(A) = 2$, $\deg(B) = 3$, $\deg(C) = 3$,
 $\deg(D) = 3$, $\deg(E) = 3$ and $\deg(F) = 2$
The sum of the degrees for the graph
 $= 2 + 3 + 3 + 3 + 3 + 2$
 $= 16$

The graph has the following edges:
A–B, A–E, B–C, B–D, C–D, C–F, D–E, E–F.
The graph has 8 edges.

The total number of edges in the graph is therefore
half the sum of the degrees.

9.2.3 Types of graphs

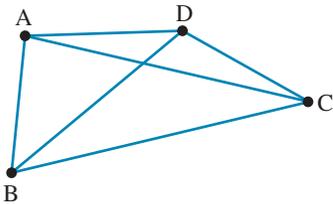
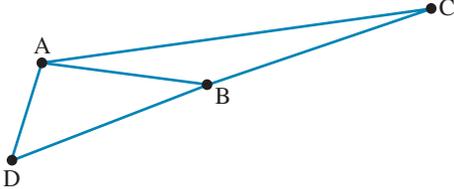
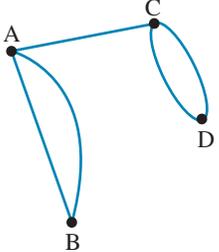
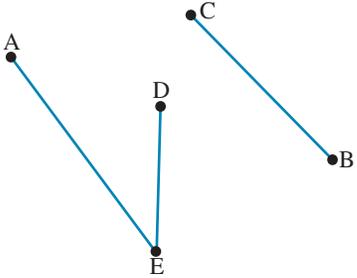
For network diagrams to represent practical situations, there need to be ways to display one-way streets, multiple connections of a social network and the results of a round-robin sporting competition. Each of these representations can be defined by the terms listed below.

Types of graphs

- **Simple graph:** A graph with no loops (edges that connect a vertex to itself) and no multiple edges (more than one edge connecting the same two vertices). Each pair of vertices is connected by at most one edge.
- **Complete graph:** A simple graph in which every vertex is connected to every other vertex by exactly one edge.
- **Connected graph:** A graph where there is a path between every pair of vertices, meaning you can get from any vertex to any other, possibly through other vertices. It does not require direct edges between all pairs, and multiple edges are allowed. (If multiple edges exist, the connected graph is no longer a simple graph).
- **Disconnected graph:** A graph that has at least one pair of vertices with no path connecting them. (There is an isolated vertex).
- **Directed graph (digraph):** A graph where edges have a direction, shown by arrows, indicating the direction of the connection between vertices.

- **Weighted graph:** A graph where each edge has a number (weight) associated with it, representing quantity (eg distance, time, cost).
- **Subgraph:** A graph that is made from part of the original graph. It uses some of the same vertices and edges, but doesn't add anything new.
- **Bipartite graph:** A graph where the vertices can be divided into two groups, and edges only connect vertices from one group to the other, not within the same group.

It is possible for some network diagrams and graphs to exemplify more than one graph type.

	<ul style="list-style-type: none"> <input checked="" type="checkbox"/> Simple: one edge at most between points. <input checked="" type="checkbox"/> Complete: has an edge connecting each vertex to all other vertices. <input checked="" type="checkbox"/> Connected graph: paths exist between each vertex. <input checked="" type="checkbox"/> Diagram: no arrows. <input checked="" type="checkbox"/> Weighted: no numbers.
	<ul style="list-style-type: none"> <input checked="" type="checkbox"/> Simple: one edge at most between points. <input checked="" type="checkbox"/> Complete: not all vertices are connected to each other by their own edge. C and D are not connected by an edge. <input checked="" type="checkbox"/> Connected graph: paths exist between each vertex. <input checked="" type="checkbox"/> Diagram: no arrows. <input checked="" type="checkbox"/> Weighted: no numbers.
	<ul style="list-style-type: none"> <input checked="" type="checkbox"/> Simple: multiple edges between A and B; and C and D. <input checked="" type="checkbox"/> Complete: not all vertices are connected to each other by their own edge. <input checked="" type="checkbox"/> Connected graph: paths exist between each vertex. <input checked="" type="checkbox"/> Diagram: no arrows. <input checked="" type="checkbox"/> Weighted: no numbers.
	<ul style="list-style-type: none"> <input checked="" type="checkbox"/> Simple: one edge at most between points. <input checked="" type="checkbox"/> Complete: not all vertices are connected to each other. <input checked="" type="checkbox"/> Connected graph: disconnected vertices. No path exists between A-E-D and B-C. <input checked="" type="checkbox"/> Diagram: no arrows. <input checked="" type="checkbox"/> Weighted: no numbers.

(continued)

(continued)

	<ul style="list-style-type: none"> <input checked="" type="checkbox"/> Simple: one edge at most between points. <input checked="" type="checkbox"/> Complete: not all vertices are connected to each other; e.g. E is not connected to D by their own edge. <input checked="" type="checkbox"/> Connected graph: paths exist between each vertex. <input checked="" type="checkbox"/> Diagraph: no arrows. <input checked="" type="checkbox"/> Weighted: numbers on edges.
	<ul style="list-style-type: none"> <input checked="" type="checkbox"/> Simple: one edge at most between points. <input checked="" type="checkbox"/> Complete: not all vertices are connected to each other by their own edge. <input checked="" type="checkbox"/> Connected graph: paths exist between each vertex. <input checked="" type="checkbox"/> Diagraph: arrows indicate direction. <input checked="" type="checkbox"/> Weighted: numbers on edges.
<div style="display: flex; justify-content: space-around; align-items: flex-end;"> <div style="text-align: center;"> <p>Graph</p> </div> <div style="text-align: center;"> <p>Subgraph</p> </div> </div>	<ul style="list-style-type: none"> <input checked="" type="checkbox"/> Subgraph: part of another graph.
	<ul style="list-style-type: none"> <input checked="" type="checkbox"/> Bipartite graph: subset A-B-C-D is connected to subset E-F-G-H.

9.2 Exercise

9.2 Exam questions **on**

Simple familiar

1, 2, 3, 4, 5, 6, 7, 8, 9, 10

Complex familiar

11, 12, 13

Complex unfamiliar

14, 15

These questions are even better in jacPLUS!

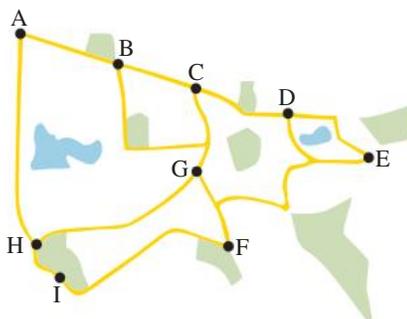
- Receive immediate feedback
- Access sample responses
- Track results and progress



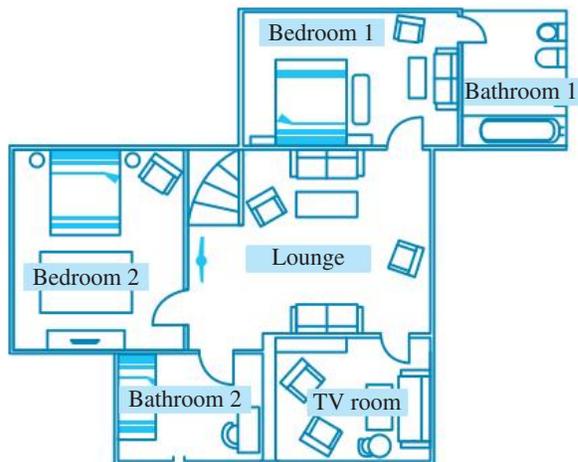
Find all this and MORE in jacPLUS

Simple familiar

1. **WE1** Draw a graph to represent the following tourist map.

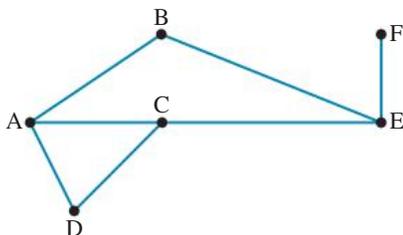


2. The diagram shows the plan of a floor of a house. Draw a graph to represent the possible ways of travelling between each room of the floor.

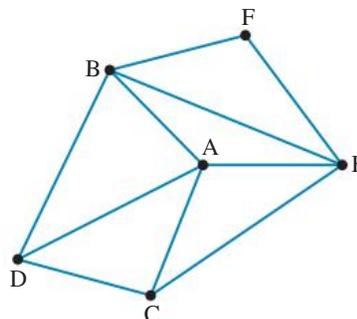


3. **WE2** For each of the following graphs, verify that the number of edges is equal to half the sum of the degrees of the vertices.

a.

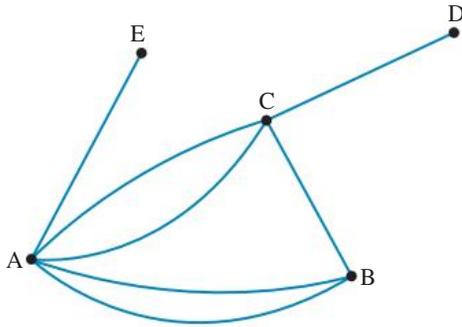


b.

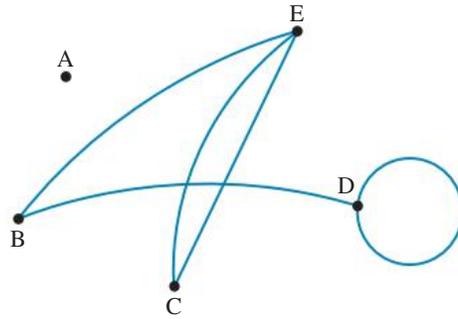


4. Identify the degree of each vertex in the following graphs.

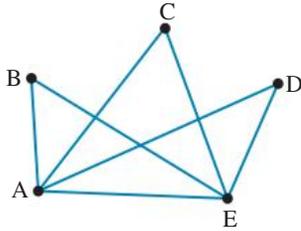
a.



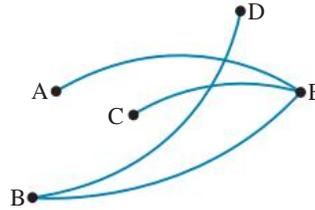
b.



c.

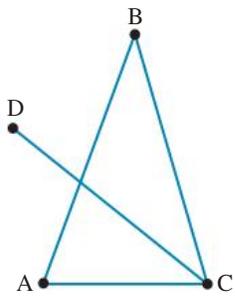


d.

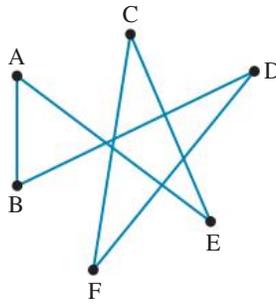


5. Complete the following table for the graphs shown.

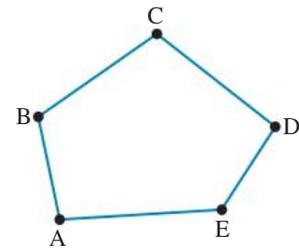
	Simple	Complete	Connected
Graph 1	Yes	No	Yes
Graph 2			
Graph 3			
Graph 4			
Graph 5			



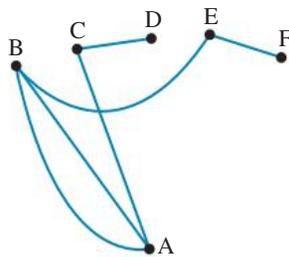
Graph 1



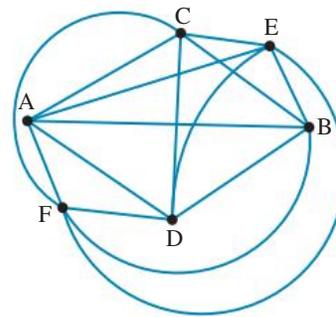
Graph 2



Graph 3



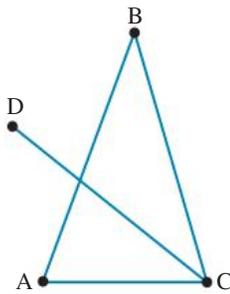
Graph 4



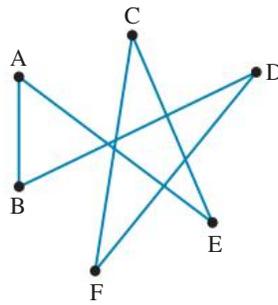
Graph 5

6. Use trial and error to construct diagraphs by adding arrows to each edge of the following graphs. The arrows should illustrate a one-way direction of travel that traverses every edge in the graph.

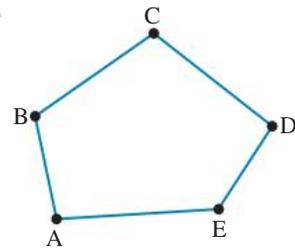
a.



b.



c.



7. In painting, primary colours can be combined to make secondary colours.

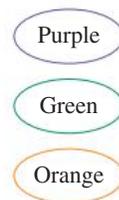


Complete the bipartite graph to show the combinations required.

Primary colours



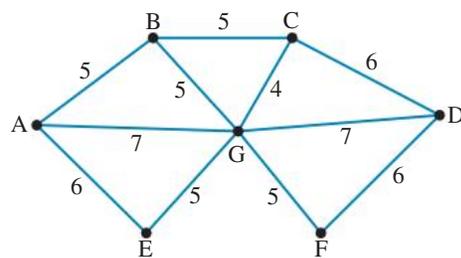
Secondary colours



8. Consider the weighted graph shown.

Identify the edge(s) with:

- a. the least weighting
- b. the greatest weighting.



9. Construct complete graphs made of 2, 3, 4, 5, 6 vertices and use those diagrams to add detail to the table. State the relationship between vertices, edges and degrees in terms of n .

Vertices	Total degrees	Edges
2		
3		
4		
5		
6		
n		

10. Draw a graph of:
- a simple, connected graph with 6 vertices and 7 edges
 - a simple, connected graph with 7 vertices and 7 edges, where 1 vertex has degree 3 and 5 vertices have degree 2
 - a simple, connected graph with 9 vertices and 8 edges, where 1 vertex has degree 8.

Complex familiar

11. Five teams have registered to play a round-robin sporting competition scheduled to start at 9 am on Saturday. Each game is assigned 45 minutes and only one game is played at each time. Construct a network diagram to represent the games to be played, and determine the finish time of the competition.
12. The diagram shows the map of some of the main suburbs of Beijing.



These suburbs of Beijing are due for an upgrade of their train network. Suburbs are connected to each other by one train line that crosses their shared boundary. It has been decided that the most connected suburbs will be upgraded ahead of the least connected suburbs. Construct a network diagram of this information and determine the order in which suburbs will be upgraded to the new train network.

13. The map shows some of the main highways connecting some of the states on the west coast of the United States.



The amount of \$3.45 billion is budgeted for the annual maintenance of these border crossing highways, shared equally among the highways. When a highway crosses a state border, the allocation for that highway maintenance budget is shared equally between the two states. Construct a network diagram of this information and determine the budget allocation that the state of Washington receives.

Complex unfamiliar

14. A graph currently has vertices $V = \{A, B, C, D, E\}$ and edges $E = \{(A, B), (A, B), (A, C), (B, D), (C, D), (D, E)\}$. A new vertex F is to be connected to vertex E only with an unknown number of connections between E and F , in a way that still satisfies the rule 'the total number of edges in the graph should always be half of the sum of the degrees'.
- Organise this information in a way to show how the number of connections between E and F will always satisfy the rule 'the total number of edges in the graph is half of the sum of the degrees' is valid.
15. Two complete graphs are known to have a difference of 11 edges in their networks. Determine the difference in the number of vertices in the networks.

Fully worked solutions for this chapter are available online.

LESSON

9.3 The adjacency matrix

SYLLABUS LINKS

- Construct an adjacency matrix from a given graph or digraph.
- Construct a graph or digraph from a given adjacency matrix.

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9.3.1 Constructing the adjacency matrix

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In Unit 2, you learned that when matrices are used to analyse a graph, the graph must first be converted into a square matrix. This matrix is called the adjacency matrix, and it is used to represent the information in a graph.

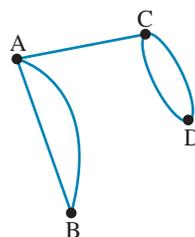
Adjacency matrix

Each column and row of an adjacency matrix corresponds to a vertex of the graph, and the numbers indicate how many edges are connecting them.

The following are the general steps that need to be taken to create an adjacency matrix:

Step 1: Start with an $n \times n$ matrix where n is the number of objects/places.

Step 2: Input the number of connections between objects in the respective element in the adjacency matrix.



$$\begin{bmatrix} 0 & 2 & 1 & 0 \\ 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

Graph

Adjacency Matrix

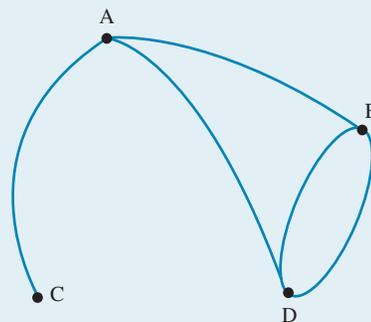


tlvd-12192

WORKED EXAMPLE 3 Constructing an adjacency matrix

The diagram shows the number of roads connecting four towns, A, B, C and D.

Construct an adjacency matrix to represent this information.



THINK

- Since there are four connecting towns, a 4×4 adjacency matrix needs to be constructed. Label the row and columns with the relevant towns A, B, C and D.

WRITE

$$\begin{matrix} & \begin{matrix} A & B & C & D \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \end{matrix} & \begin{bmatrix} - & - & - & - \\ - & - & - & - \\ - & - & - & - \\ - & - & - & - \end{bmatrix} \end{matrix}$$

2. There is one road connecting town A to town B, so enter 1 in the cell from A to B.

	A	B	C	D
A	—	—	—	—
B	①	—	—	—
C	—	—	—	—
D	—	—	—	—

3. There is also only one road between town A and towns C and D; therefore, enter 1 in the appropriate matrix positions. There are no loops at town A (i.e. a road connecting A to A); therefore, enter 0 in this position.

	A	B	C	D
A	0	—	—	—
B	1	—	—	—
C	1	—	—	—
D	1	—	—	—

4. Repeat this process for towns B, C and D. Note that there are two roads connecting towns B and D, and that town C only connects to town A.

	A	B	C	D
A	0	1	1	1
B	1	0	0	2
C	1	0	0	0
D	1	2	0	0

9.3.2 Characteristics of adjacency matrices

Adjacency matrices are square matrices with n rows and columns, where ‘ n ’ is equal to the number of vertices in the graph.

Column:	1	2	...	$n - 1$	n	Row
	0	2	...	1	0	1
	2	0	...	0	0	2
	\vdots	\vdots	...	\vdots	\vdots	\vdots
	1	0	...	0	2	$n - 1$
	0	0	...	2	0	n

The adjacency matrix of an undirected graph will be symmetrical around the leading diagonal.

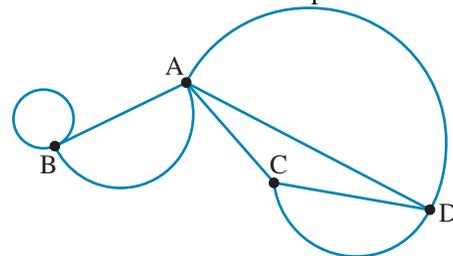
$$\begin{bmatrix} 0 & 2 & 1 & 0 \\ 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

In a directed graph, the edges have a direction associated with them, meaning the adjacency matrix will not necessarily be symmetrical.

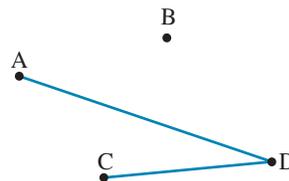
Any non-zero value in the leading diagonal will indicate the existence of a loop.

$$\begin{bmatrix} 0 & 2 & 1 & 2 \\ 2 & ① & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 2 & 0 & 2 & 0 \end{bmatrix}$$

The ‘1’ indicates that a loop exists at vertex B:

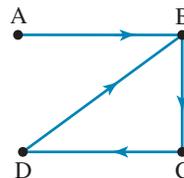


A row or column consisting of all zeros indicates an isolated vertex (a vertex that is not connected to any other vertex).

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$


When completing an adjacency matrix from a diagraph (directed graph) you typically complete it row by row.

$$\begin{matrix} & \begin{matrix} A & B & C & D \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{matrix}$$



WORKED EXAMPLE 4 Constructing a graph from an adjacency matrix

There are five friends on a social media site: Peta, Seth, Tran, Ned and Wen. The number of communications made between these friends in the last 24 hours is shown in the adjacency matrix below.

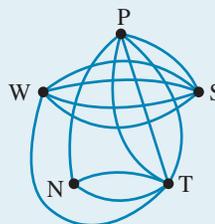
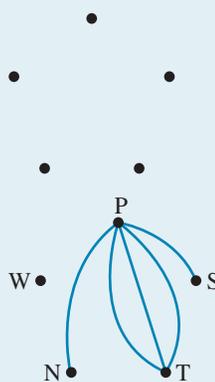
$$\begin{matrix} & \begin{matrix} P & S & T & N & W \end{matrix} \\ \begin{matrix} P \\ S \\ T \\ N \\ W \end{matrix} & \begin{bmatrix} 0 & 1 & 3 & 1 & 0 \\ 1 & 0 & 0 & 0 & 4 \\ 3 & 0 & 0 & 2 & 1 \\ 1 & 0 & 2 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 \end{bmatrix} \end{matrix}$$

Using the adjacency matrix, construct a diagram that shows the number of communications between the five friends.

THINK

- Since it is a 5×5 matrix, it indicates 5 vertices: P, S, T, N, W.
For ease of construction, vertices can be arranged in the shape of regular polygons. In this case, a 5-sided shape (pentagon).
- Starting with the top row (Vertex P)
Add 1 edge between P and S
Add 3 edges between P and T
Add 1 edge between P and N
- Repeat for all other rows, mindful not to repeat edges that are already drawn in.
- Check that the degree of each vertex in the diagram equals the sum of each row.

WRITE



$$\begin{aligned} \deg(P) &= 5, \text{ and } 0 + 1 + 3 + 1 + 0 = 5 \quad \checkmark \\ \deg(S) &= 5, \text{ and } 1 + 0 + 0 + 0 + 4 = 5 \quad \checkmark \\ \deg(T) &= 6, \text{ and } 3 + 0 + 0 + 2 + 1 = 6 \quad \checkmark \\ \deg(N) &= 3, \text{ and } 1 + 0 + 2 + 0 + 0 = 3 \quad \checkmark \\ \deg(W) &= 5 \text{ and } 0 + 4 + 1 + 0 + 0 = 5 \quad \checkmark \end{aligned}$$

Exercise 9.3 The adjacency matrix

9.3 Exercise

9.3 Exam questions **on**

Simple familiar
1, 2, 3, 4, 5, 6, 7, 8,
9, 10, 11

Complex familiar
12, 13, 14

Complex unfamiliar
15

These questions are even better in jacPLUS!

- Receive immediate feedback
- Access sample responses
- Track results and progress



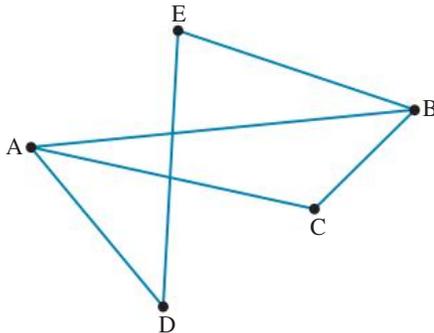
Find all this and MORE in jacPLUS

Simple familiar

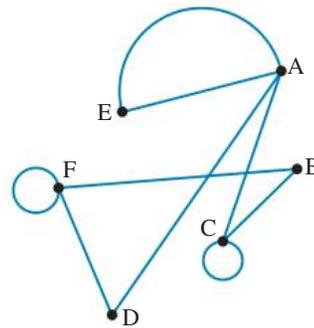
- MC** Select the best description of an adjacency matrix.
 - A. A 3D array representing graph edges.
 - B. A 2D array representing graph connections.
 - C. A list of edges.
 - D. A weighted graph.
- MC** In an adjacency matrix, identify what a value of 1 indicates.
 - A. No edge exists.
 - B. An edge exists.
 - C. The graph is directed.
 - D. The graph is weighted.
- MC** Identify how the presence of a loop in an adjacency matrix can be detected.
 - A. There is a row of zeros.
 - B. The matrix is asymmetrical.
 - C. The presence of a non-zero in the leading diagonal.
 - D. The leading diagonal will consist of ones only.

WE3 Construct adjacency matrices for the following graphs.

a.

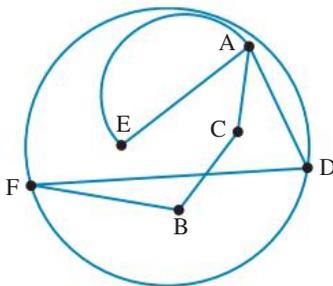


b.

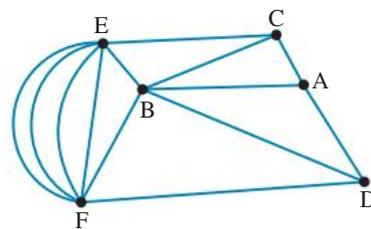


5. Construct adjacency matrices for the following graphs.

a.



b.



WE4 Draw graphs to represent the following adjacency matrices.

a.
$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

b.
$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

7. Draw graphs to represent the following adjacency matrices.

a.
$$\begin{bmatrix} 0 & 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

b.
$$\begin{bmatrix} 2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 1 & 0 \end{bmatrix}$$

8. Complete the following adjacency matrices.

a.
$$\begin{bmatrix} 0 & 0 & \\ 0 & 2 & 2 \\ 1 & & 0 \end{bmatrix}$$

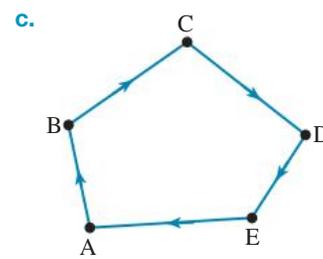
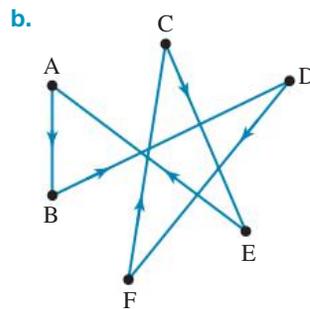
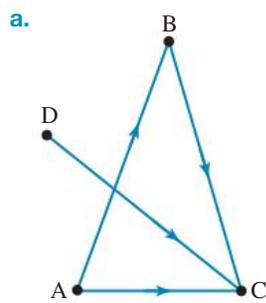
b.
$$\begin{bmatrix} 2 & 1 & & 0 \\ & 0 & & \\ 0 & 1 & 0 & 1 \\ & 2 & & 0 \end{bmatrix}$$

9. Complete the following adjacency matrices.

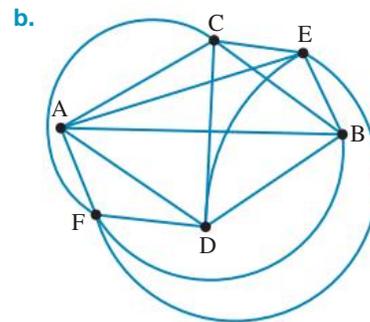
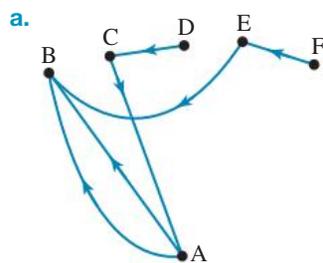
a.
$$\begin{bmatrix} 0 & & 1 & & 0 \\ 0 & 0 & & & 0 \\ & 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & & & 1 & 0 \end{bmatrix}$$

b.
$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ & 0 & 0 & 0 & 1 \\ & & & 0 & 0 \\ 0 & & & 0 & 1 \end{bmatrix}$$

10. Construct the adjacency matrices for each of the digraphs shown.



11. Construct the adjacency matrices for each of the graphs shown.



Complex familiar

12. The diagram shows the map of some of the main suburbs of Beijing.
Draw an adjacency matrix to represent the shared boundaries between the suburbs.



13. The map shows some of the main highways connecting some of the states on the west coast of the United States.
Draw an adjacency matrix to represent the number of highways between the states.



Draw an adjacency matrix to represent the number of highways between the states.

14. Maisie has discovered that the direct flight between Hanoi and Bangkok is sold out for her planned date of travel. Construct a digraph of this network and list all possible alternative routes that she can take that are inclusive of one stopover (where she stops at one other city).

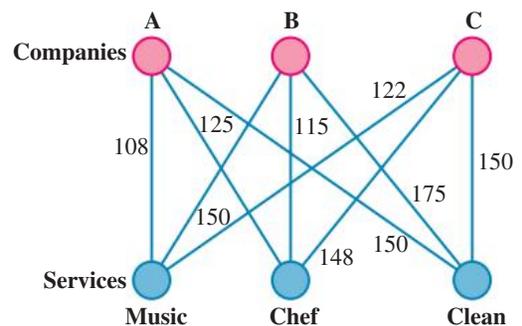


The table indicates the number of direct flights per day between key cities.

From: To:	Bangkok	Manila	Singapore	Kuala Lumpur	Jakarta	Hanoi	Phnom Penh
Bangkok	0	2	5	3	1	1	1
Manila	2	0	4	1	1	0	0
Singapore	5	4	0	3	4	2	3
Kuala Lumpur	3	1	3	0	0	3	3
Jakarta	1	1	4	0	0	0	0
Hanoi	1	0	2	3	0	0	0
Phnom Penh	1	0	3	3	0	0	0

Complex unfamiliar

15. The following weighted bipartite graph shows the cost per year (in thousands of dollars) of contracting three companies for their services. You must hire all three companies, and you must use all three services. Show graphical justification of the cheapest hiring combination.



Fully worked solutions for this chapter are available online.

LESSON

9.4 Planar graphs

SYLLABUS LINKS

- Understand the meaning of planar graph and face.
- Apply Euler's formula to solve problems relating to planar graphs.
 - $v + f - e = 2$ where v is number of vertices, f is number of faces and e is number of edges

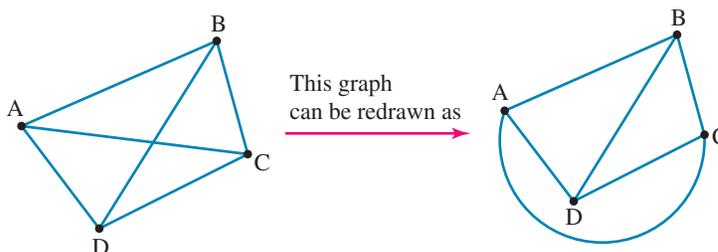
Source: General Mathematics Senior Syllabus 2024 © State of Queensland (QCAA) 2024; licensed under CC BY 4.0.

9.4.1 The necessity for planar graphs

Graphs can be drawn with intersecting edges. However, in many applications, intersections may be undesirable. Consider a graph of an underground railway network. In this case, intersecting edges would indicate the need for one rail line to be in a much deeper tunnel, which could add significantly to construction costs.

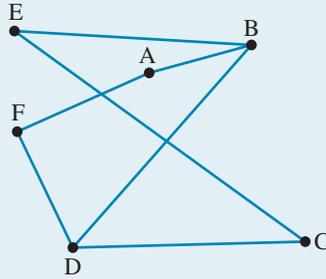


In some cases, it is possible to redraw graphs so that they have no intersecting edges. When a graph can be redrawn in this way, it is known as a **planar graph**. For example, in the graph shown below, it is possible to redraw one of the intersecting edges so that it still represents the same information.



WORKED EXAMPLE 5 Constructing a planar graph

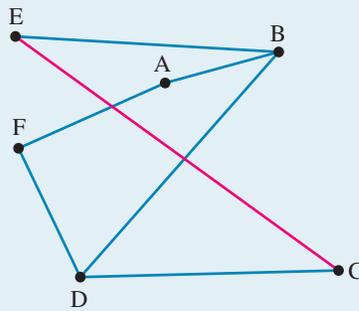
Redraw the graph so that it has no intersecting edges.



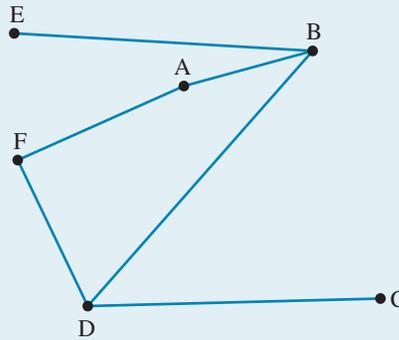
THINK

1. Identify any intersecting edges.

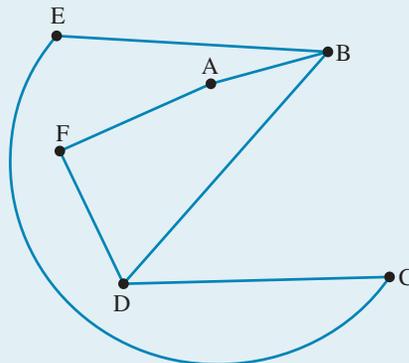
WRITE/DRAW



2. Redraw the diagram without the intersecting edge(s).

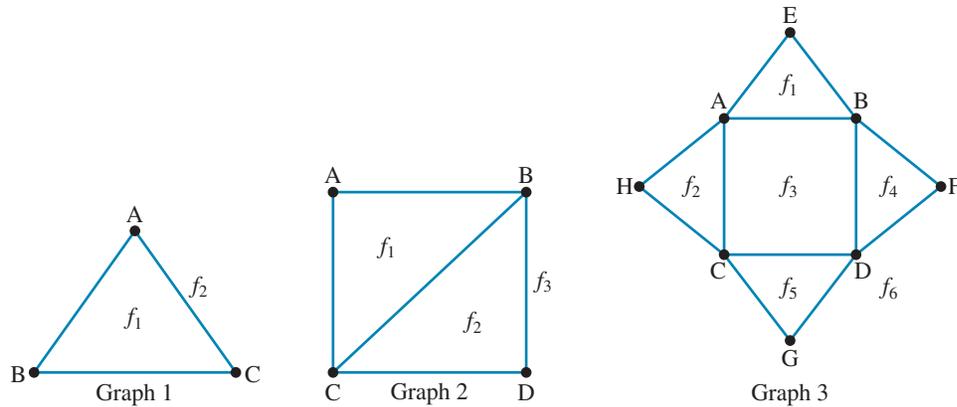


3. Redraw the remaining edge so that it does not intersect with the other edges.



9.4.2 Vertices, edges and faces

In all planar graphs, the edges and vertices create distinct areas referred to as **faces**. Each enclosed shape is counted a face, and the area outside the graph represents a face. Consider the following group of planar graphs.



Euler's formula for planar graphs states that for any **connected, planar** graph, there is a relationship between its vertices v , edges e and faces f :

Euler's formula

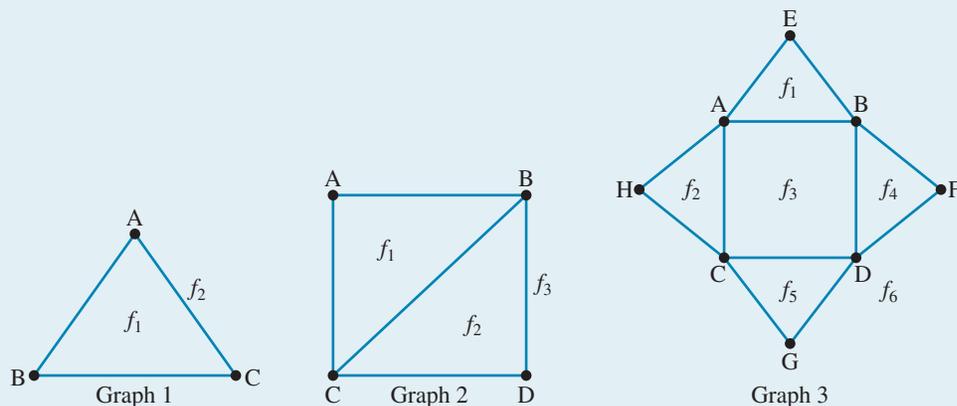
$$v + f - e = 2$$

where v is the number of vertices, e is the number of edges and f is the number of faces.

This is known as **Euler's polyhedron formula** or simply **Euler's formula**. It applies to graphs that can be drawn on a plane without edges crossing (i.e., planar graphs).

WORKED EXAMPLE 6 Identifying vertices, edges, faces.

Consider Graphs 1, 2, 3 seen above.



Show that the relationship between the vertices, edges and faces is $v + f - e = 2$.

THINK

- Graph 1 has 3 vertices, 3 edges, 2 faces.
- Substitute into Euler's formula $2 = 2$.

WRITE

$$v + f - e = 2$$

$$3 + 2 - 3 = 2$$

- | | |
|---|------------------|
| 3. Graph 2 has 4 vertices, 5 edges, 3 faces. | $v + f - e = 2$ |
| 4. Substitute into Euler's formula $2 = 2$. | $4 + 3 - 5 = 2$ |
| 5. Graph 3 has 8 vertices, 12 edges, 6 faces. | $v + f - e = 2$ |
| 6. Substitute into Euler's formula $2 = 2$. | $8 + 6 - 12 = 2$ |

WORKED EXAMPLE 7 Substituting and rearranging Euler's formula

Determine how many faces there will be for a connected planar graph of 7 vertices and 10 edges.

THINK

- Substitute the given values into Euler's formula.
- Solve the equation for the unknown value.
- State the final answer.

WRITE

$$v + f - e = 2$$

$$7 + f - 10 = 2$$

$$7 + f - 10 = 2$$

$$f = 2 - 7 + 10$$

$$f = 5$$

There will be 5 faces in a connected planar graph with 7 vertices and 10 edges.

Exercise 9.4 Planar graphs

learnon

9.4 Exercise

9.4 Exam questions on

Simple familiar

1, 2, 3, 4, 5, 6, 7, 8, 9, 10

Complex familiar

11, 12, 13

Complex unfamiliar

14, 15, 16

These questions are even better in jacPLUS!

- Receive immediate feedback
- Access sample responses
- Track results and progress

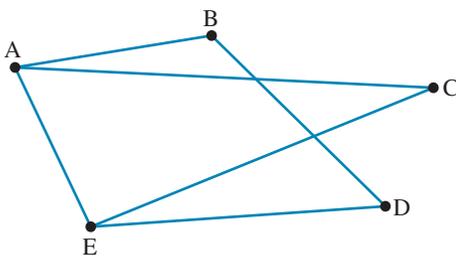


Find all this and MORE in jacPLUS

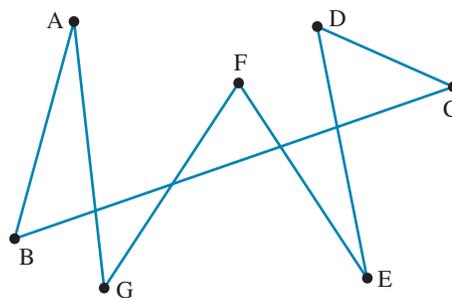
Simple familiar

1. **WE5** Redraw the following graphs so that they have no intersecting edges.

a.

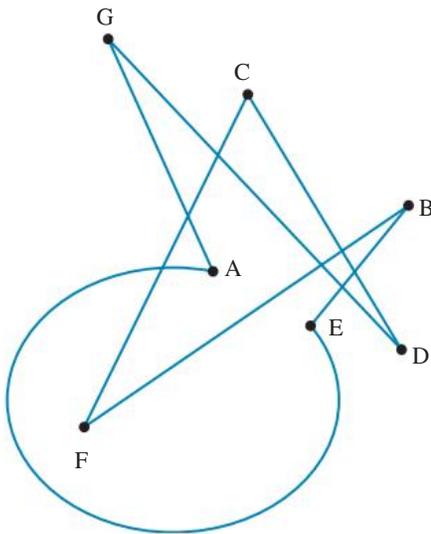


b.

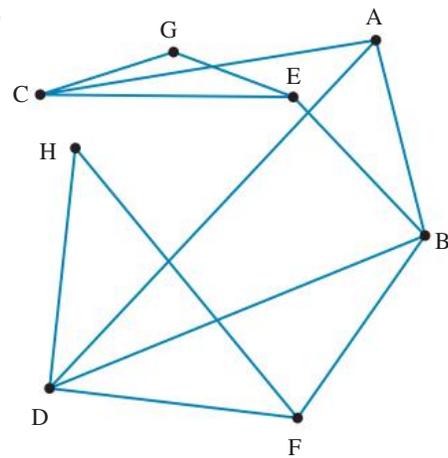


2. Redraw the following graphs to show that they are planar. The vertices are not fixed, and can be repositioned as needed to avoid intersecting lines.

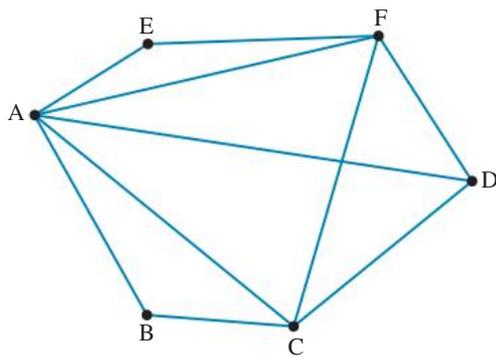
a.



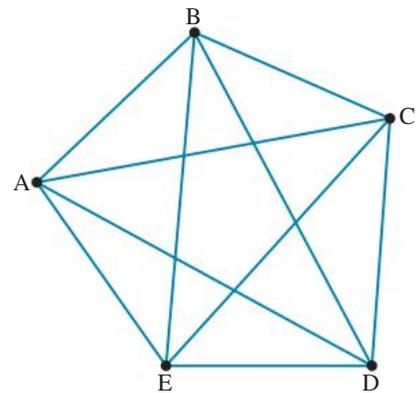
b.



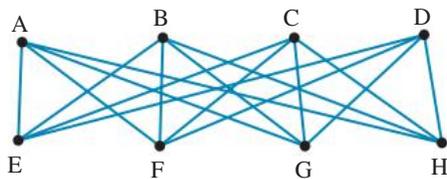
3. Determine which of the following graphs are not a planar graph and explain your reasoning.



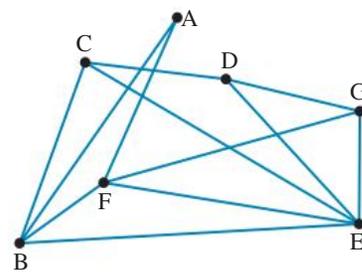
Graph 1



Graph 2



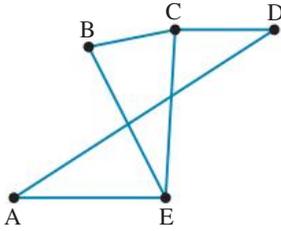
Graph 3



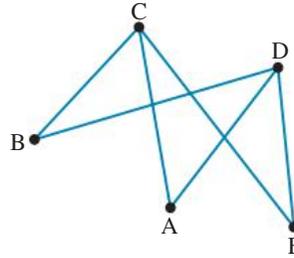
Graph 4

4. Redraw the following as planar graphs and identify the number of faces.

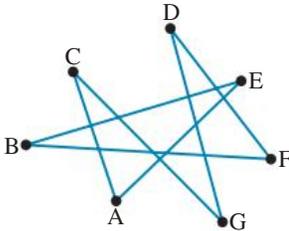
a.



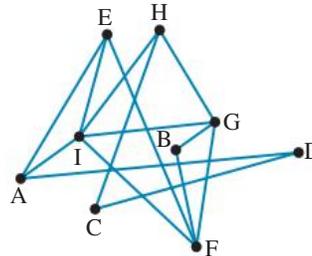
b.



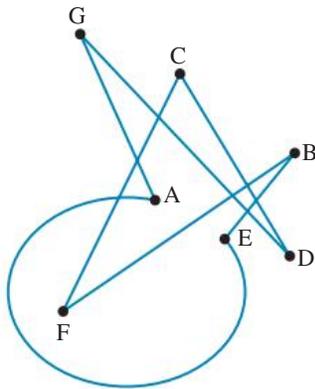
c.



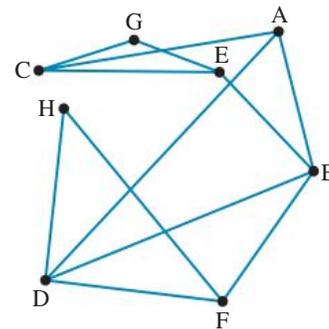
d.



5. **WE6** Redraw the following graphs as planar graphs and show that the relationship between vertices, edges and faces of the following graphs is $v + f - e = 2$.



Graph 1



Graph 2

6. **WE7** Determine how many faces there will be for a connected planar graph of:

a. 8 vertices and 10 edges

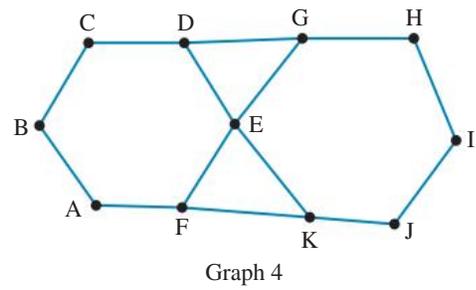
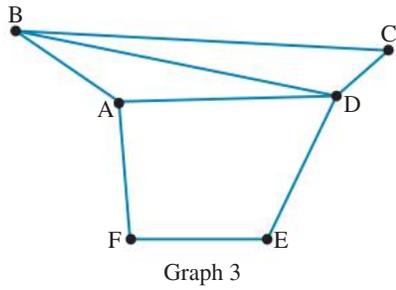
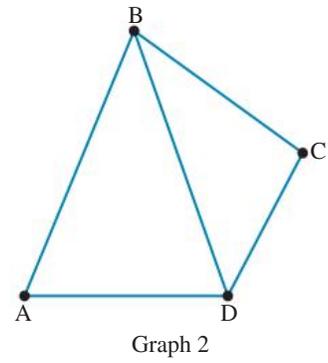
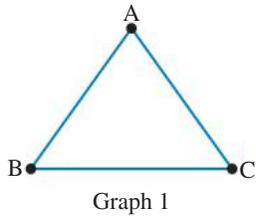
b. 11 vertices and 14 edges.

7. a. For a connected planar graph of 5 vertices and 3 faces, determine how many edges there will be.

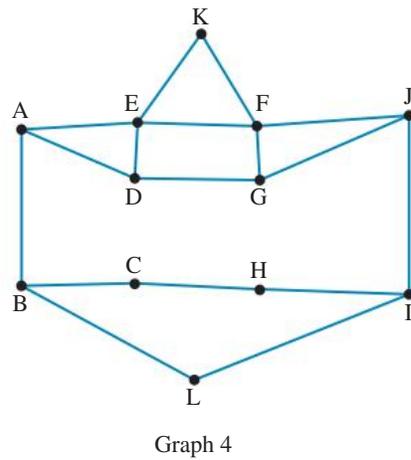
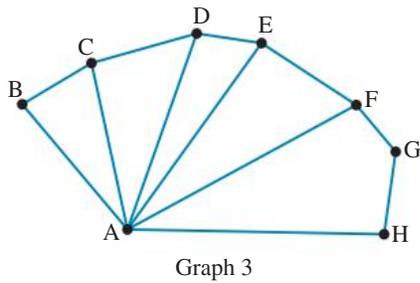
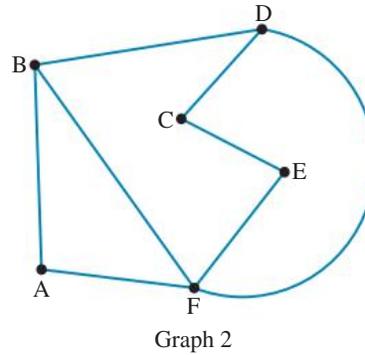
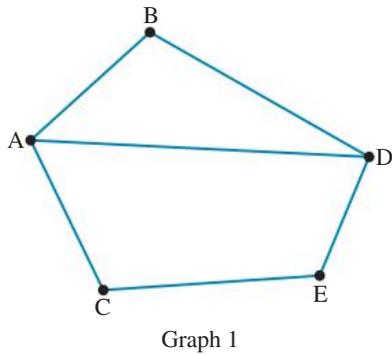
b. For a connected planar graph of 8 edges and 5 faces, determine how many vertices there will be.

8. a. Use the planar graphs shown to complete the table.

Graph	Total edges	Total sum of degrees
Graph 1		
Graph 2		
Graph 3		
Graph 4		



- b. State whether any pattern is evident from the table.
9. a. Use the planar graphs shown to complete the table.



Graph	Total vertices of even degree	Total vertices of odd degree
Graph 1		
Graph 2		
Graph 3		
Graph 4		

b. State whether any pattern is evident from this table.

10. Construct a connected planar graph with:

a. 6 vertices and 5 faces

b. 11 edges and 9 faces.

Complex familiar

11. Answer the following for this adjacency matrix.

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

a. Draw a graph that has no intersecting edges.

b. Identify the number of enclosed faces.

c. Identify the maximum number of additional edges that can be added to maintain a simple planar graph.

12. Answer the following regarding this adjacency matrix.

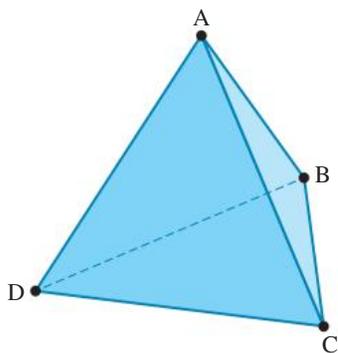
$$\begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

a. Draw a graph that has no intersecting edges.

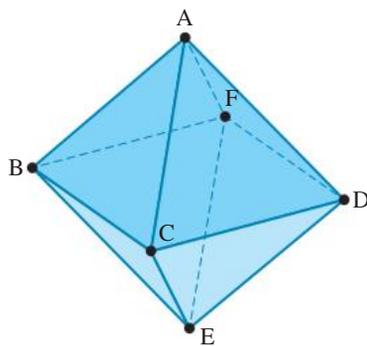
b. Identify the number of enclosed faces.

c. Identify the maximum number of additional edges that can be added to maintain a simple planar graph.

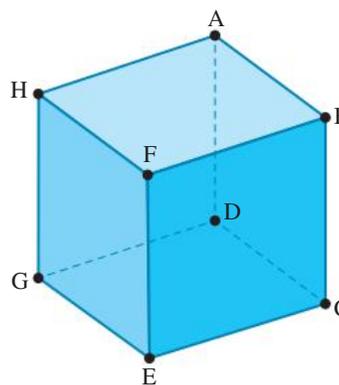
13. Represent the following 3D shapes as planar graphs.



Tetrahedron



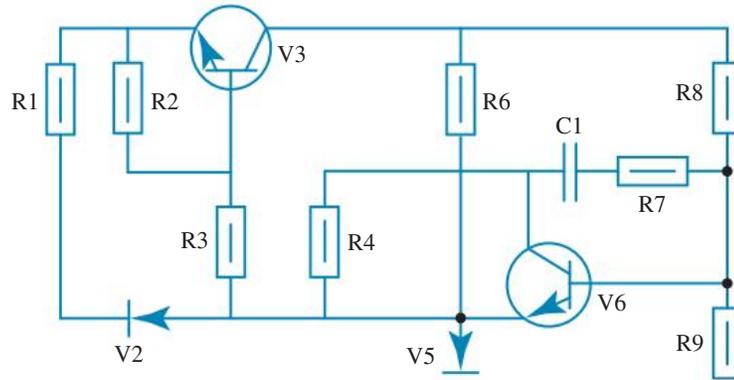
Octahedron



Cube

Complex unfamiliar

14. A section of an electric circuit board is shown in the diagram.



Explain whether it is possible to represent the circuit board as a planar graph.

15. The table displays the most common methods of communication for a group of people.

	Email	Facebook	SMS
Adam	Ethan, Liam	Ethan, Liam	Ethan
Michelle		Sophie, Emma, Ethan	Sophie, Emma
Liam	Adam		
Sophie		Michelle, Chloe	Michelle, Chloe
Emma	Chloe	Chloe, Ethan, Michelle	Chloe, Ethan
Ethan		Emma, Adam, Michelle	Emma
Chloe	Emma, Sophie	Emma, Sophie	Emma, Sophie

Identify the best person to introduce Chloe and Michelle.



16. Show that Euler's formula can be verified by a tetrahedron.

Fully worked solutions for this chapter are available online.

LESSON

9.5 Traversing graphs — walks, paths and trails

SYLLABUS LINKS

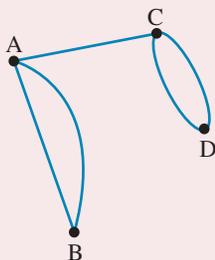
- Understand the meaning of walk, trail, path, open walk, open trail, open path, closed walk, closed trail (circuit), closed path (cycle), connected graph and bridge.
- Solve practical problems to determine the shortest path between two vertices in a weighted graph (by trial-and-error methods only).
- Understand the meaning of Eulerian trail, semi-Eulerian graph, Eulerian circuit and Eulerian graph, and the conditions for their existence.
- Solve practical problems involving semi-Eulerian graphs and Eulerian graphs.
- Understand the meaning of Hamiltonian path, semi-Hamiltonian graph, Hamiltonian cycle and Hamiltonian graph.
- Solve practical problems involving semi-Hamiltonian graphs and Hamiltonian graphs (by trial-and-error methods only).

Source: General Mathematics Senior Syllabus 2024 © State of Queensland (QCAA) 2024; licensed under CC BY 4.0.

In section 9.2.3, many graph types were discussed: simple, complete, connected, directed, weighted, subgraph and bipartite. In this section, we explore connected graphs further.

Connected graphs

- A **connected graph** has edges between each vertex and can comprise multiple edges. Each vertex is connected but does not need to be connected to all vertices. A disconnected graph may have vertices not connected to others (isolated vertex).
- A **bridge** is any edge that maintains the graph's connectivity. For example, vertex A and C are connected by one edge, and the removal of A-C would create a disconnect. Thus, the edge A-C is a bridge. Vertex A and B are connected by multiple edges, and the removal of one A-B edge will not create a disconnect. Thus, it is not a bridge.

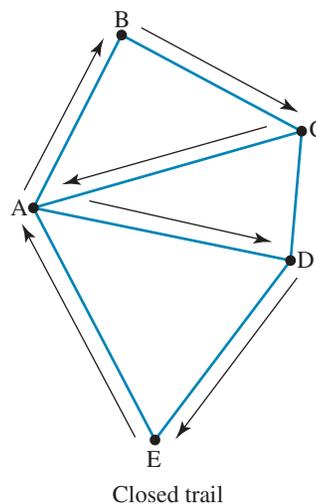
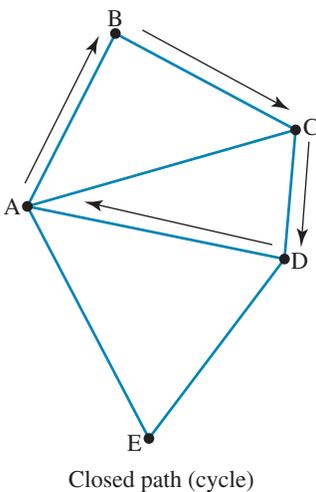
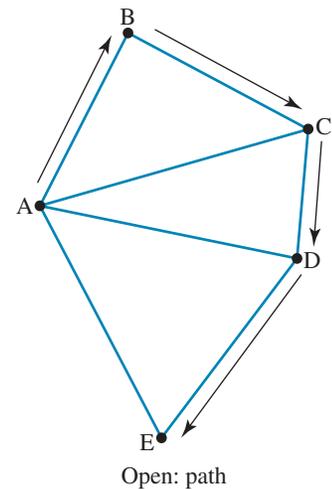
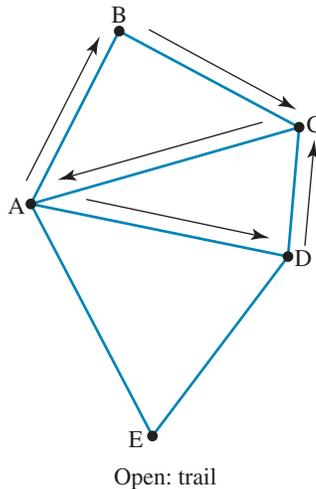
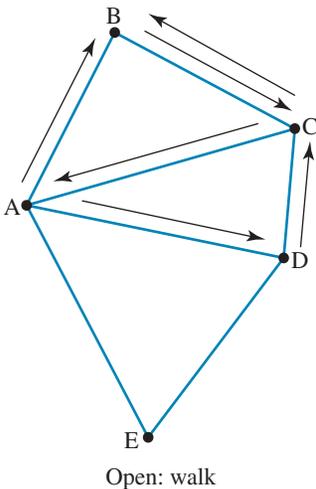


9.5.1 Defining walks, paths and trails

Many applications of graphs involve an analysis of movement around a network. These could include fields such as transport, communications or utilities, to name a few. Movement through a simple connected graph is described in terms of starting and finishing at specified vertices by travelling along the edges. This is usually done by listing the labels of the vertices visited in the correct order. In more complex graphs, edges may also have to be indicated, as there may be more than one connection between vertices. The definitions of the main terms used when describing movement across a network are as follows.

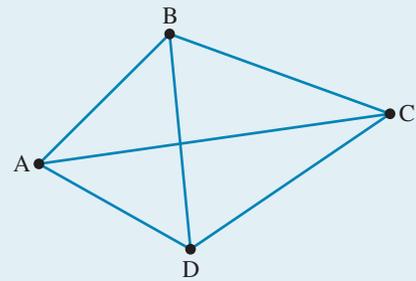
Traversing a graph

- A **walk** is any route through a network. There are no restrictions on how few or how many edges it can cover. A walk can repeat edges and can repeat vertices. For example, the student who repeatedly walks many school paths and corridors to reach classrooms.
- A **closed walk** starts and finishes at the same vertex. For example, the student starting and ending their day at their locker.
- An **open walk** starts and finishes at different vertices. For example, the student who goes straight to class and then finishes their day at their locker.
- A **trail** has a restriction in that no edges can be repeated.
- A **closed trail** starts and finishes at the same vertex, and no edges are repeated. This is commonly called a **circuit**.
- An **open trail** starts and finishes at different vertices, and no edges are repeated.
- A **path** has a restriction in that no vertices can be repeated. Starting and ending at the same vertex is considered a repeat.
- A **cycle** (or **closed path**) is the exception. It is a path that starts and ends at the same vertex, and no other vertices are repeated.
- An **open path** starts and finishes at different vertices, and no vertices are repeated.



WORKED EXAMPLE 8 Identifying different routes

In the following network, identify two different routes: one cycle and one closed trail.



THINK

1. For a cycle, identify a route that doesn't repeat a vertex apart from the start/finish.
2. For a closed trail, identify a route that doesn't repeat an edge and ends at the starting vertex.

WRITE

Cycle: ABDCA

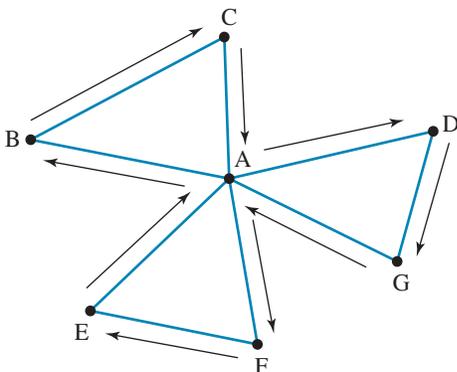
Closed trail: ADBCA

9.5.2 Eulerian and semi-Eulerian graphs

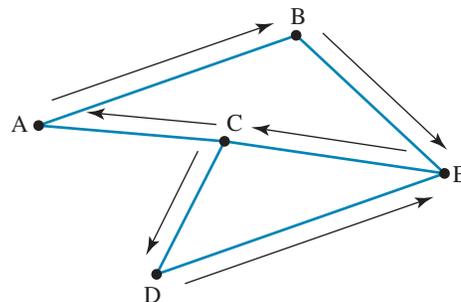
In some practical situations, all edges must be traversed. For example, council rubbish collections need to visit every street. They also start and end at the same location (the depot). However, it is inefficient to repeatedly drive streets already collected, so a network needs to be designed that covers all edges, once only. These graphs are named for the mathematician who discovered graph theory, Leonhard Euler.

Eulerian graphs

- A **Eulerian graph** is a graph that contains a **closed Eulerian trail**, also called a **Eulerian circuit**.
 - It starts and ends at the same vertex (closed, circuit).
 - It uses every edge exactly once (no repeated edges).
 - It can revisit vertices — only edges must not be repeated (trail).
 - A graph is Eulerian if every vertex has an even degree (i.e. an even number of edges connected to it). Example: $\text{deg}(A) = 6$, $\text{deg}(B) = 2$, $\text{deg}(C) = 2$ etc.
- A **semi-Eulerian graph** contains an **open Eulerian trail**.
 - It starts and ends at different vertices (open).
 - It still uses every edge exactly once.
 - Vertices can be revisited as many times as needed — but no edge is repeated (trail).
 - A graph is semi-Eulerian if exactly two vertices have an odd degree — the trail starts at one odd-degree vertex and ends at the other. Example: $\text{deg}(A) = 2$, $\text{deg}(B) = 2$, $\text{deg}(C) = 3$, $\text{deg}(D) = 2$, $\text{deg}(E) = 3$.



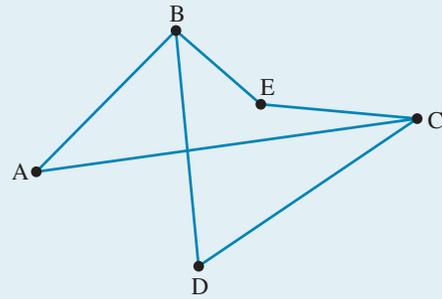
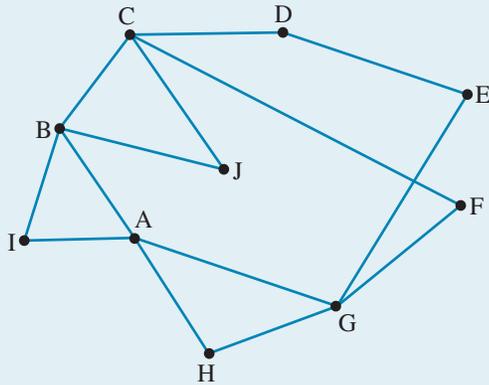
Eulerian graph



Semi-Eulerian graph

WORKED EXAMPLE 9 Eulerian and semi-Eulerian graphs

- a. Identify an Eulerian trail in Graph 1, if it exists.
- b. Identify a semi-Eulerian trail in Graph 2, if it exists.
- c. Add one edge to make Graph 2 Eulerian.



THINK

- a. 1. Count each vertex degree and the number of edges.
- 2. As every vertex is an even degree, so an Eulerian trail is possible. Any vertex can be your starting point. Check you have 13 dashes to represent the 13 edges.
- b. 1. Count each vertex degree, and the number of edges.
- 2. There are two vertices with an odd degree, so a semi-Eulerian trail is possible. You must start at one of the odd vertices. You will finish at the other odd vertex. Check you have 6 dashes to represent the 6 edges.
- c. Eulerian graphs require all edges to be even. To convert a semi-Eulerian to Eulerian, add 1 edge between the two odd vertices.

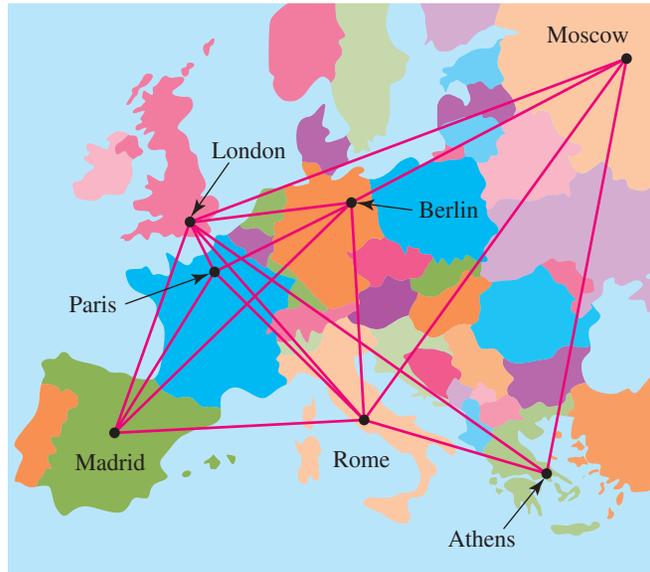
WRITE

- a. $\text{Deg}(A) = 4, \text{deg}(B) = 4, \text{deg}(C) = 4,$
 $\text{deg}(D) = 2, \text{deg}(E) = 2, \text{deg}(F) = 2,$
 $\text{deg}(G) = 4, \text{deg}(H) = 2$
 $\text{Deg}(I) = 2.$ Total of 26.
 13 edges
 A-B-C-D-E-G-F-C-J-B-I-A-G-H-A (other trails exist)
- b. $\text{Deg}(A) = 2, \text{deg}(B) = 3, \text{deg}(C) = 3,$
 $\text{deg}(D) = 2, \text{deg}(E) = 2.$ Total of 12.
 6 edges.
 B-A-C-D-B-E-C
 (other trails exist)
- c. An extra edge between B and C makes them $\text{deg}(B) = 4, \text{deg}(V) = 4.$ All even. Eulerian.

9.5.3 Hamiltonian and semi-Hamiltonian graphs

In other situations, it may be more practical if all vertices can be reached without using all of the edges of the graph. For example, if you wanted to visit a selection of the capital cities of Europe, you wouldn't need to use all the available flight routes shown in the diagram.

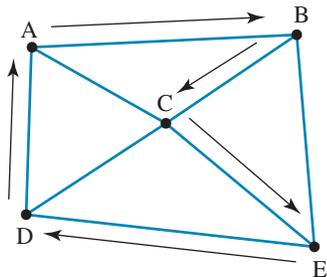
These graphs are named after the Irish mathematician Sir William Rowan Hamilton (1805–1865). He was a prominent figure in mathematical physics and algebra, introducing this concept in relation to a game he devised called the Icosian Game, which involved finding such a cycle on the vertices of a dodecahedron.



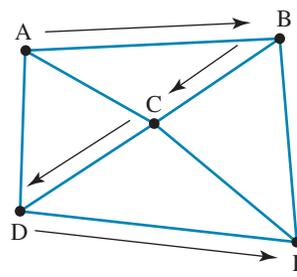
Hamiltonian graphs

- A **Hamiltonian graph** contains a **Hamiltonian cycle**.
 - It starts and ends at the same vertex (a cycle).
 - It visits every vertex exactly once.
 - No vertices are repeated (path), except the starting/ending one (which is the same).
 - Edges can be repeated, although in a standard Hamiltonian cycle they usually are not.
- A **semi-Hamiltonian graph** contains a **Hamiltonian path**.
 - It starts and ends at different vertices (open path).
 - It visits every vertex exactly once.
 - No vertices are repeated (path).

Hamiltonian cycles and semi-Hamiltonian cycles reach all vertices of a network once without necessarily using all of the available edges.



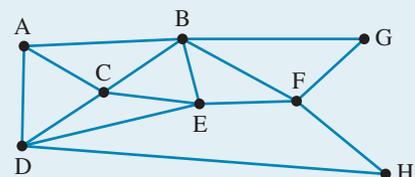
Hamiltonian cycle



Semi-Hamiltonian cycle

WORKED EXAMPLE 10 Identifying a semi-Hamiltonian graph

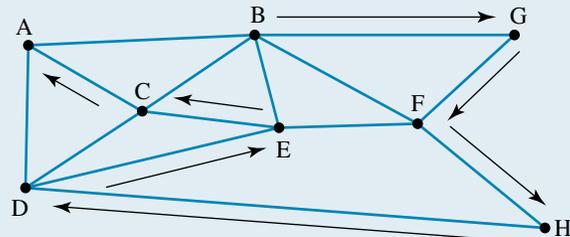
Identify a semi-Hamiltonian path in the following graph.



THINK

Trial-and-error a route that reaches all vertices once

WRITE/DRAW



Semi-Hamiltonian cycle: BGFHDECA
(other possibilities exist)

Exercise 9.5 Traversing graphs- walks, paths and trails

learn**on**

9.5 Exercise

9.5 Exam questions **on**

Simple familiar

1, 2, 3, 4, 5, 6, 7, 8,
9, 10, 11, 12, 13, 14

Complex familiar

15, 16, 17

Complex unfamiliar

18, 19

These questions are even better in jacPLUS!

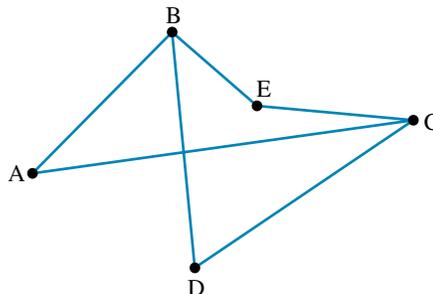
- Receive immediate feedback
- Access sample responses
- Track results and progress



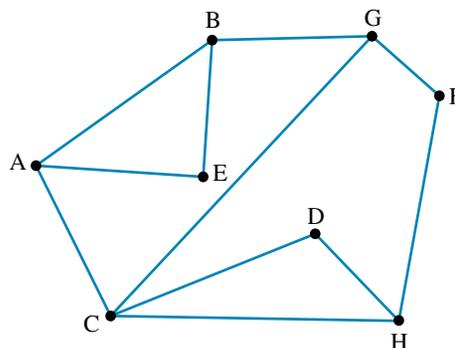
Find all this and MORE in jacPLUS

Simple familiar

1. **WE8** In the following network, identify two different routes: one cycle and one closed trail.

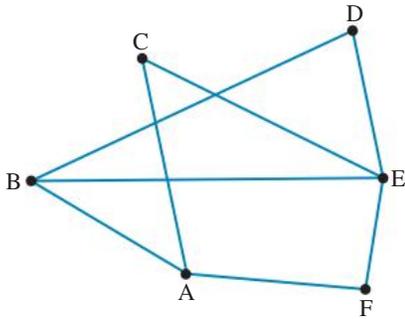


2. In the following network, identify three different routes: one path, one cycle and one closed trail.

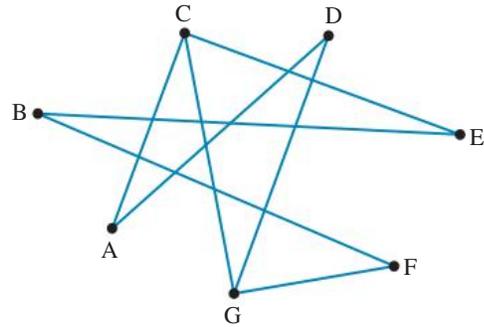


7. Identify a semi-Eulerian trail and a semi-Hamiltonian path in each of the following graphs.

a.

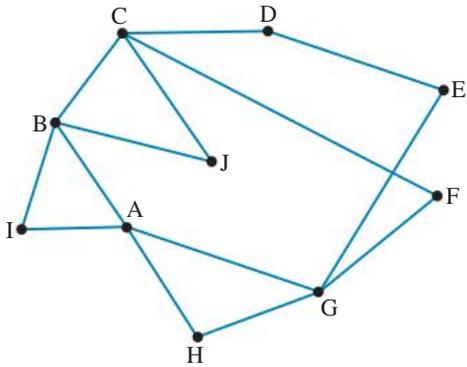


b.

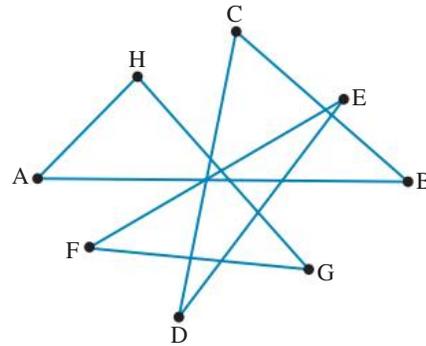


8. Identify an Eulerian trail and a Hamiltonian cycle in each of the following graphs, if they exist.

a.

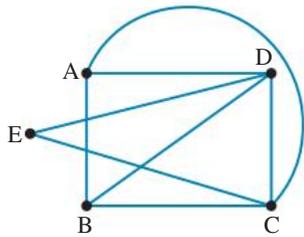


b.

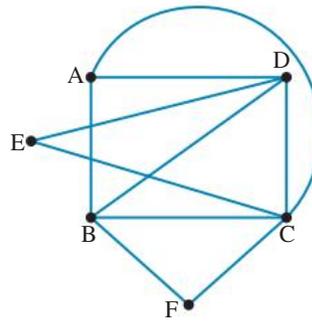


9. a. Identify which of the following graphs have a semi-Eulerian trail.

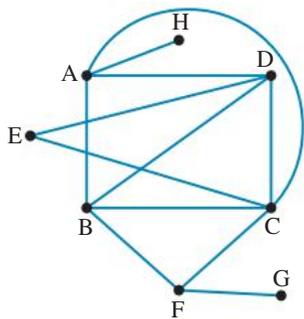
i.



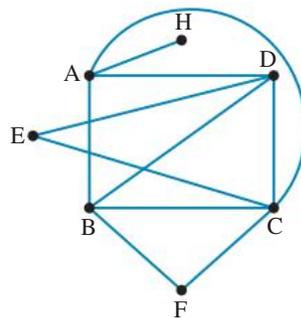
ii.



iii.



iv.

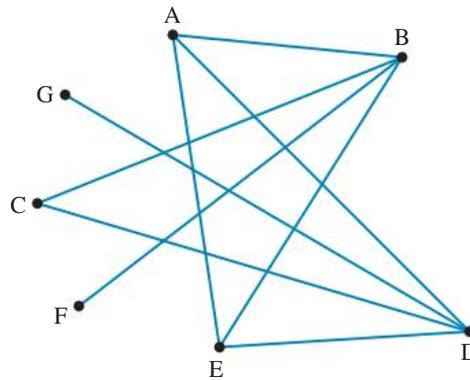


b. Identify the semi-Eulerian trails found.

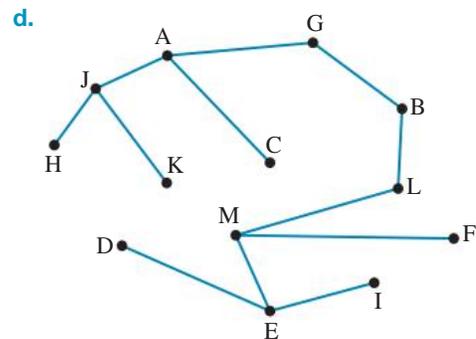
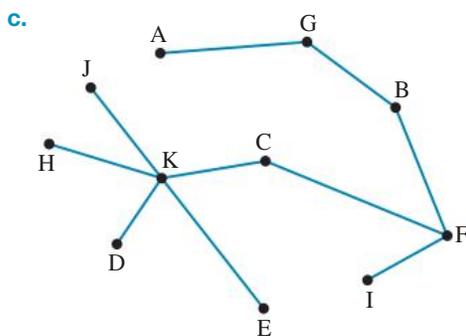
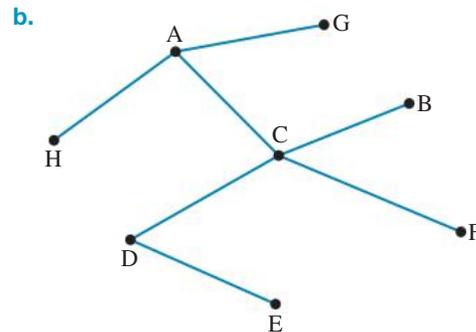
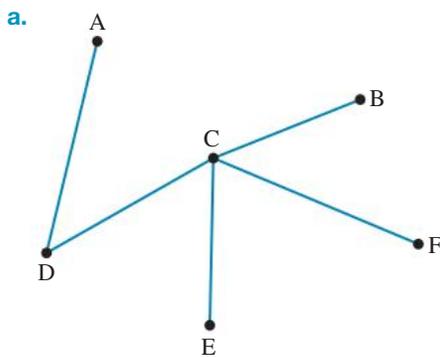
10. a. Identify which of the graphs from question 9 have a Hamiltonian cycle.

b. Identify the Hamiltonian cycles found.

11. a. Construct adjacency matrices for each of the graphs in question 9.
 b. Explain how these might assist with making decisions about the existence of Eulerian trails and semi-Eulerian trails, and Hamiltonian cycles and semi-Hamiltonian paths.
12. In the following graph, other than from G to F, identify 2 vertices between which you must add an edge in order to create a semi-Hamiltonian path that commences from the given vertex.
- a. G
 b. F

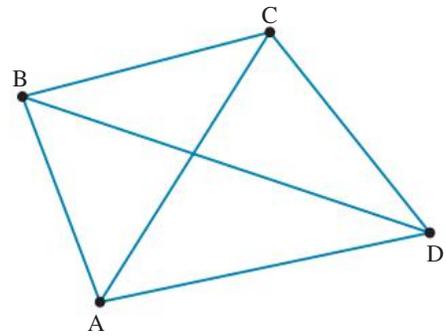


13. For each of the following graphs:
- identify a bridge that could disconnect each graph into two separate subgraphs, where vertex A is in one subgraph and vertex D is in the other
 - add the minimum number of edges to create a semi-Eulerian trail
 - identify the semi-Eulerian trail created.



14. a. Use the graph shown to complete the table to identify all of the Hamiltonian cycles commencing at vertex A.

	Hamiltonian cycle
1.	ABCDA
2.	
3.	
4.	
5.	
6.	

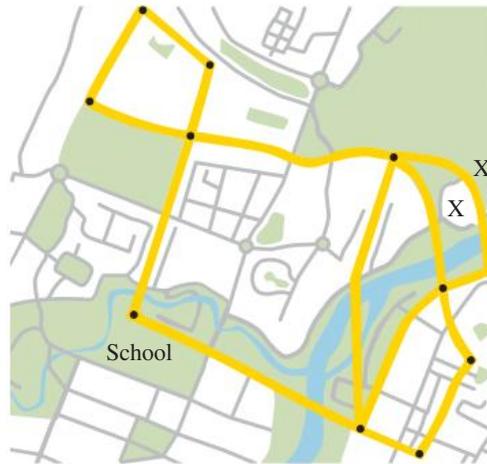


- b. Determine whether any other Hamiltonian cycles are possible.

Complex familiar

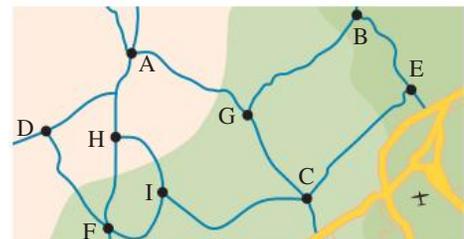
15. On the map shown, a school bus route is indicated in yellow. The bus route starts and ends at the school indicated.

Draw a graph to represent the bus route.

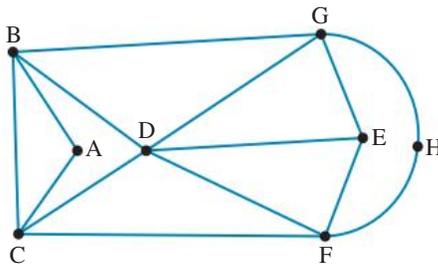


- a. Students can catch the bus at stops that are located at the intersections of the roads marked in yellow. Determine whether it is possible for the bus to collect students by driving down each section of the route only once. Explain your answer.
- b. If road works prevent the bus from travelling along the two sections indicated by the Xs, determine whether it will be possible for the bus to still collect students on the remainder of the route by travelling each section only once. Explain your answer.
16. The map of an orienteering course is shown. Participants must travel to each of the nine checkpoints along any of the marked paths.

- a. If participants must start and finish at A and visit every other checkpoint only once, identify two possible routes they could take.
- b. If participants can decide to start and finish at any checkpoint, and the paths connecting D and F, H and I, and A and G are no longer accessible, it is possible to travel the course by moving along each remaining path only once. Explain why and identify the two possible starting points.



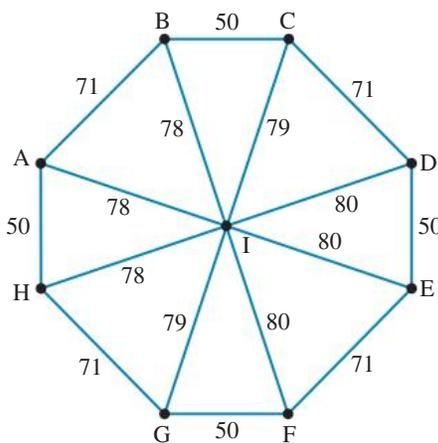
17. The graph shown outlines the possible ways a tourist bus can travel between eight locations.



- If vertex A represents the second location visited, list the possible starting points.
- If the bus also visited each location only once, determine which of the starting points listed in part a could not be correct.
- If the bus also needed to finish at vertex D, list the possible paths that could be taken.
- If instead the bus company decides to operate a route that travelled to each connection only once, identify the possible starting and finishing points.
- If instead the company wanted to travel to each connection only once and finish at the starting point, determine which edge of the graph would need to be removed.

Complex unfamiliar

- A logistics company is planning the delivery route for a driver who needs to visit 10 different warehouses spread across a city. Each warehouse is connected to exactly 3 other warehouses by roads. The roads are one way, meaning that the driver can only travel in one direction along each road. Determine how many roads the driver can avoid. Evaluate the reasonableness of your solution.
- Consider the weighted graph shown, which displays the time it takes to travel between different branches of a restaurant chain (A–H) and the head office (I). The CEO claims it is possible to start at the head office and visit every restaurant in the same work day. Evaluate the reasonableness of the CEO's claim.



Fully worked solutions for this chapter are available online.

LESSON

9.6 Review

9.6.1 Summary

Hey students! Now that it's time to revise this chapter, go online to:



Access the chapter summary



Review your results



Practise exam questions

Find all this and MORE in jacPLUS



9.6 Exercise

learnon

9.6 Exercise

9.6 QCAA questions **on**

Simple familiar

1, 2, 3, 4, 5, 6, 7,
8, 9, 10, 11, 12, 13,
14, 15

Complex familiar

16, 17, 18

Complex unfamiliar

19, 20

These questions are even better in jacPLUS!

- Receive immediate feedback
- Access sample responses
- Track results and progress



Find all this and MORE in jacPLUS



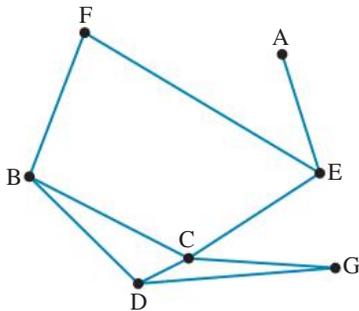
Simple familiar

1. **MC** The minimum number of edges in a connected graph with 8 vertices is:

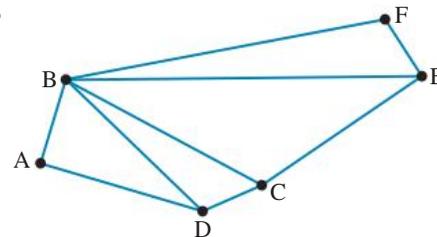
- A. 5. B. 6. C. 7. D. 8.

2. For each of the following graphs, verify that the number of edges is equal to half the sum of the degrees of the vertices.

a.



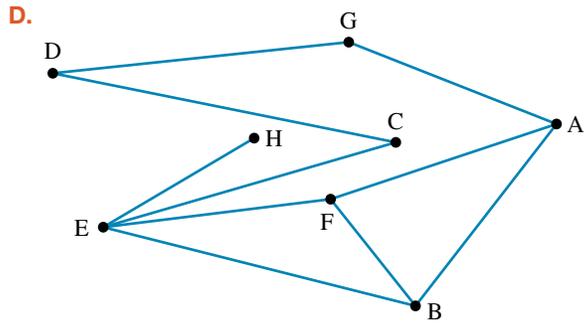
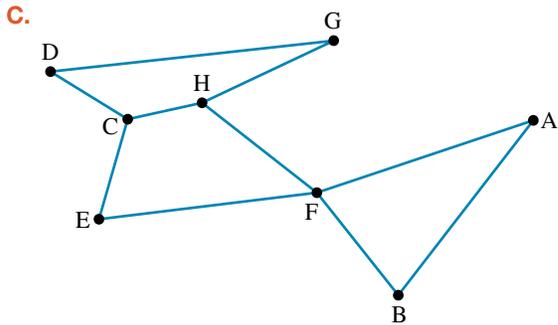
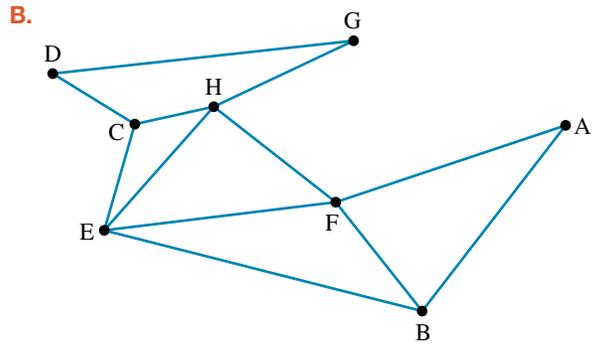
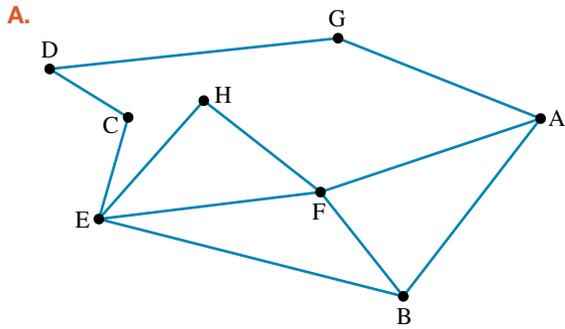
b.



3. **MC** A connected graph with 9 vertices has 10 faces. The number of edges in the graph is:

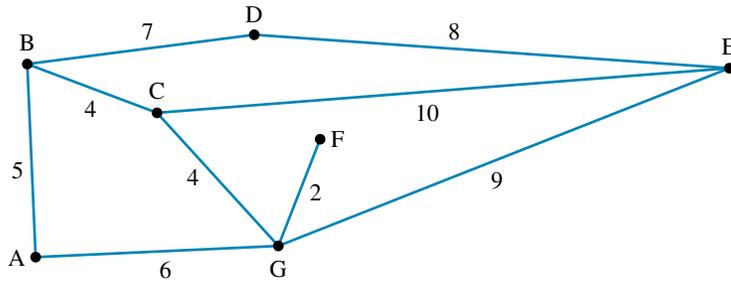
- A. 15 B. 16 C. 17 D. 18

4. **MC** Identify the graph that will not have a semi-Eulerian trail.



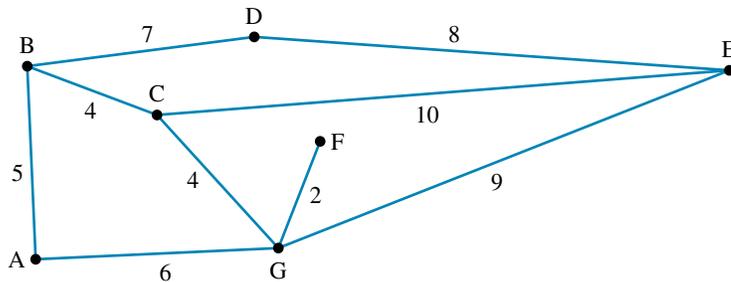
5. **MC** Determine which edge needs to be removed so that the edges in the weighted graph shown total 50.

- A.** CE **B.** FG **C.** AG **D.** AB



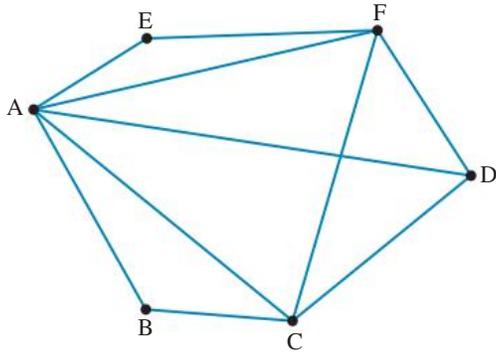
6. **MC** Determine the edge acting as a bridge so that vertex F does not become isolated.

- A.** CE **B.** FG **C.** AG **D.** AB

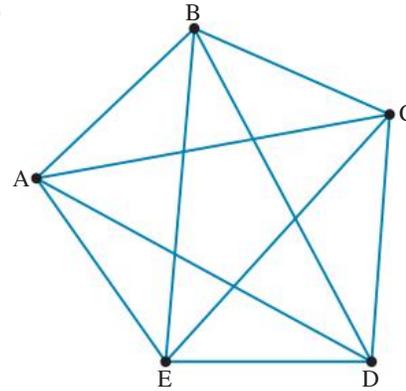


7. **MC** Identify the graph that most resembles a bipartite graph.

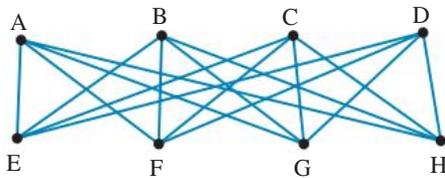
A.



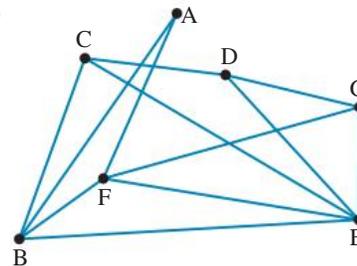
B.



C.



D.



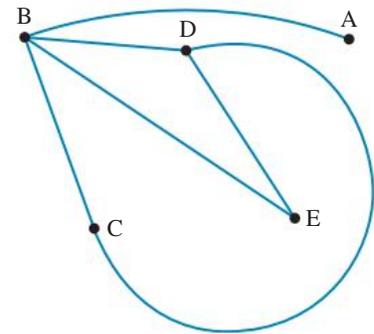
8. **MC** An Eulerian trail can be created in the following graph by adding an edge between the vertices:

A. A and D.

B. A and B.

C. A and C.

D. B and C.



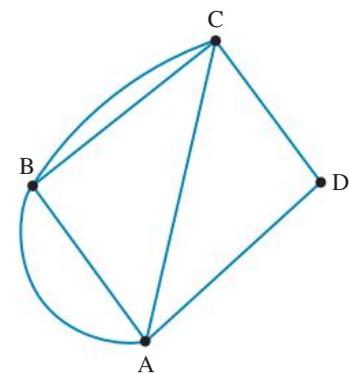
9. **MC** The adjacency matrix that represents the following graph is:

A.
$$\begin{bmatrix} 0 & 2 & 2 & 2 \\ 2 & 0 & 2 & 0 \\ 2 & 2 & 0 & 1 \\ 2 & 0 & 1 & 0 \end{bmatrix}$$

B.
$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

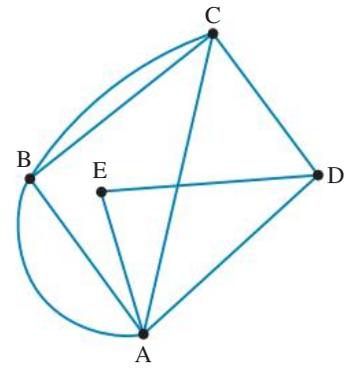
C.
$$\begin{bmatrix} 0 & 2 & 1 & 1 \\ 2 & 1 & 2 & 0 \\ 1 & 2 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

D.
$$\begin{bmatrix} 0 & 2 & 1 & 1 \\ 2 & 0 & 2 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$



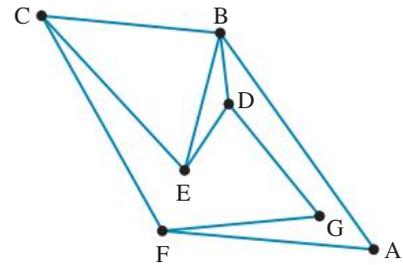
10. **MC** The number of faces in the following planar graph is:

- A. 6 B. 7 C. 8 D. 9



11. **MC** A Hamiltonian cycle for the following graph is:

- A. ABCEDGFA B. ABDGFCEA
 C. ABDGFCEDEBCFA D. ABDGFCECFA

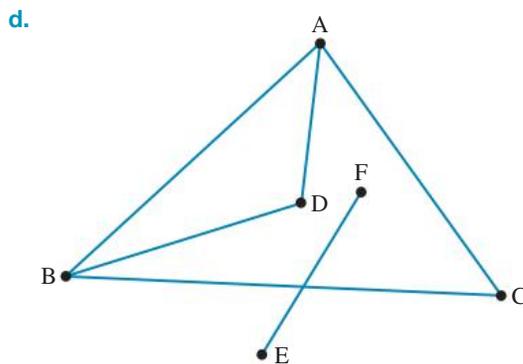
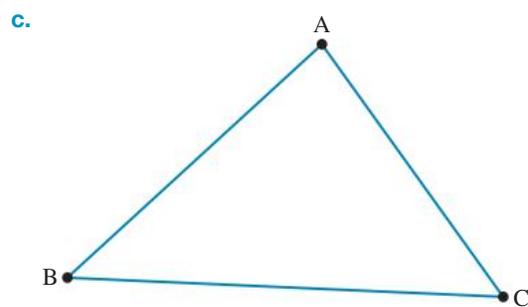
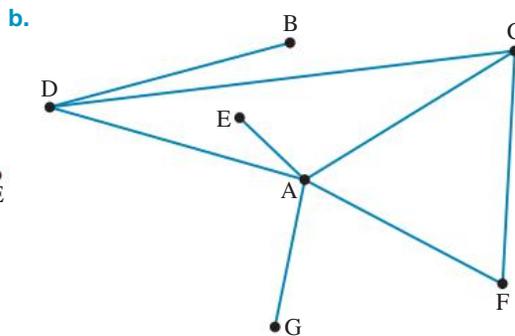
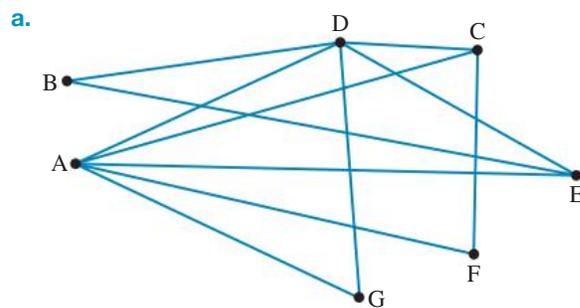


12. **MC** A complete graph with 7 vertices will have a total number of edges of:

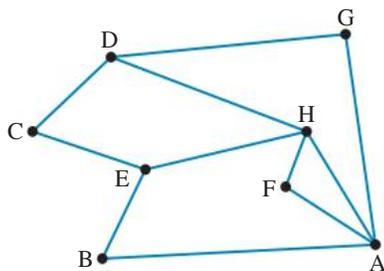
- A. 7 B. 8 C. 14 D. 21

13. Identify which of the following graphs are:

- i. simple
- ii. complete
- iii. planar.



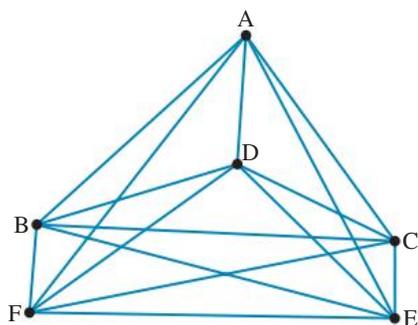
14. Complete this diagram by adding arrows to every edge to illustrate a one-way direction of travel.



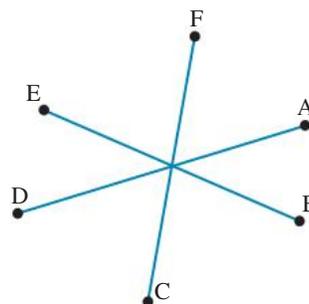
15. Identify which of the following graphs are:

- i. simple
- ii. complete
- iii. planar.

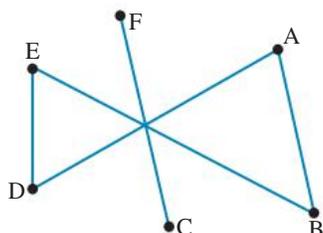
a.



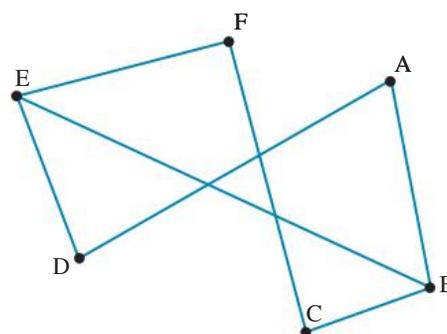
b.



c.



d.

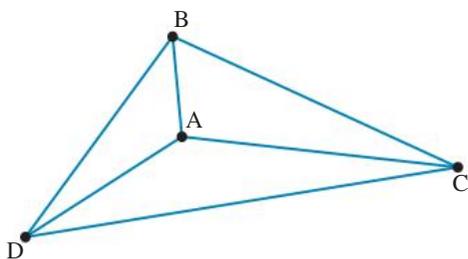


Complex familiar

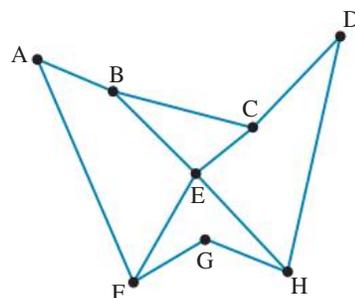
16. For each of the following graphs

- i. add the minimum number of edges to the following graphs in order to create a semi-Eulerian trail
- ii. state the semi-Eulerian trail created.

a.



b.



17. Complete the following adjacency matrices.

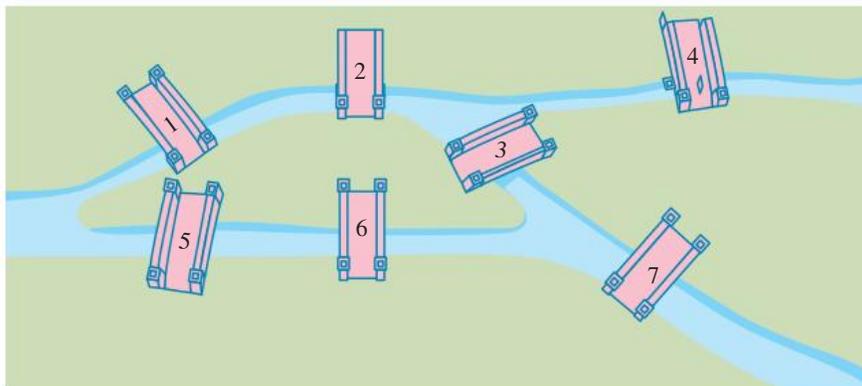
a.
$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ & 0 & & 0 \\ & 3 & 1 & \\ & & 1 & 0 \end{bmatrix}$$

c.
$$\begin{bmatrix} 0 & 1 & 3 & 1 & \\ 2 & 0 & & & 1 \\ & 3 & 0 & 2 & \\ & 1 & & 2 & 2 & 1 \\ & & 3 & & 3 & 1 \\ 2 & 0 & 1 & & & 0 \end{bmatrix}$$

b.
$$\begin{bmatrix} 0 & 1 & 2 & 1 \\ & 0 & 0 & 1 \\ & 2 & 0 & 2 \\ & & & 2 & 2 \\ 1 & 3 & & 0 \end{bmatrix}$$

d.
$$\begin{bmatrix} 0 & & & & 0 \\ 2 & 0 & 1 & & \\ 1 & 2 & 0 & 1 & 1 & 0 \\ 3 & & 0 & & & 1 \\ & 2 & 0 & 0 & & 3 \\ 1 & 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & & & 0 \end{bmatrix}$$

18. The European city of Königsberg (now called Kaliningrad) is set on the banks of the River Pregel. Seven bridges were arranged as shown to connect the mainland of the city to two large islands.



Bridges of Königsberg

For a long time the townspeople wondered it was possible to travel around the city in such a way that all seven bridges would only have to be crossed once each.

- Determine whether there is a way to cross all 7 bridges without crossing any bridge more than once.
- Explain why or why not.

Complex unfamiliar

19. Consider a city designed as a network of 9 major intersections (nodes) and 14 roads (edges) connecting them. The city's traffic planners want to set up a maintenance schedule that allows workers to cover every road **exactly once** without having to retrace any road. Your task is to determine whether this is possible by analysing the graph and suggesting additional edges as required.

- Intersections: A, B, C, D, E, F, G, H, I
- Roads: $A \leftrightarrow B, A \leftrightarrow C, A \leftrightarrow D, B \leftrightarrow C, B \leftrightarrow E, C \leftrightarrow D, C \leftrightarrow F, D \leftrightarrow G, E \leftrightarrow F, F \leftrightarrow G, F \leftrightarrow H, G \leftrightarrow H, H \leftrightarrow I, G \leftrightarrow I$.

20. Dirac's theorem provides an easy-to-check condition that guarantees the existence of a Hamiltonian cycle without needing to explicitly find the cycle.

Let G be a simple graph with n vertices, where $n \geq 3$. If every vertex in G has degree at least $\frac{n}{2}$, then G contains a Hamiltonian cycle.

Draw a network that satisfies this theorem, and as such, comprises a Hamiltonian cycle.

9.6 Past QCAA exam questions

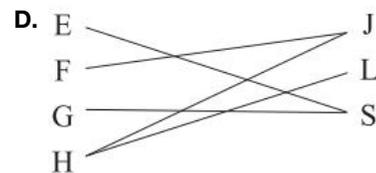
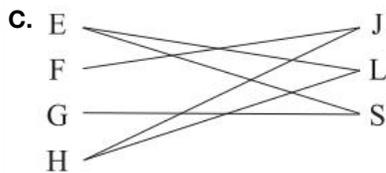
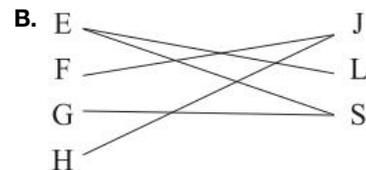
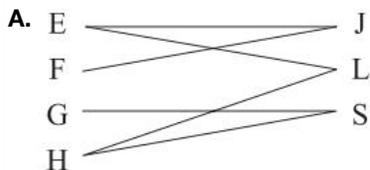
Question 1 (1 mark)

Source: QCAA 2023 General Mathematics, Paper 1, Section 1, Q.13; © State of Queensland (QCAA) 2024

Four athletes, Eoin (E), Fedir (F), Gede (G) and Hana (H), compete in three events: javelin (J), long jump (L) and sprints (S).

Athlete	Events
Eoin	L, S
Fedir	J
Gede	S
Hana	J, L

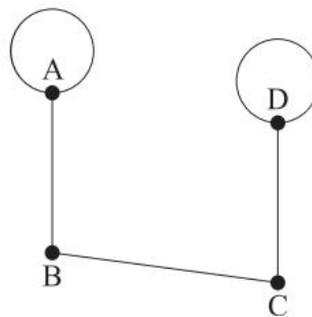
Which bipartite graph represents this information?



Question 2 (1 mark)

Source: QCAA 2022 General Mathematics, Paper 1, Section 1, Q.6; © State of Queensland (QCAA) 2024

This semi-Eulerian graph can be changed to an Eulerian graph by

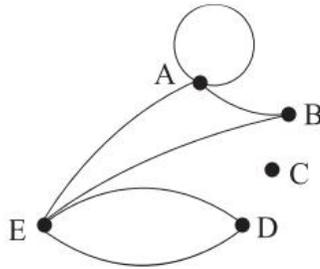


- A.** adding a loop to vertex B.
- B.** removing the loop at vertex A.
- C.** adding an edge between vertices A and D.
- D.** removing the edge between vertices B and C.

Question 3 (5 marks)

Source: QCAA 2023 General Mathematics, Paper 1, Section 2, Q.23; © State of Queensland (QCAA) 2024

A network graph is shown.

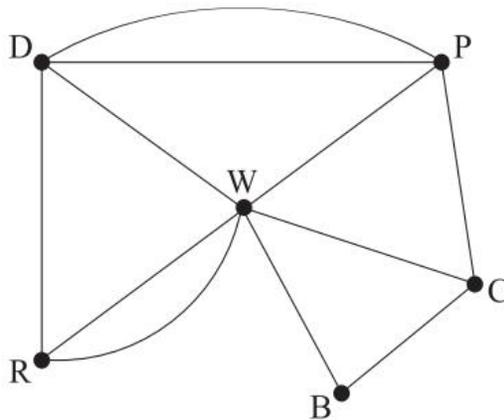


- a. State the degree of vertex E. [1 mark]
- b. State the number of edges joining D and E. [1 mark]
- c. Construct an adjacency matrix from the graph with the vertices in alphabetical order. [3 marks]

Question 4 (5 marks)

Source: QCAA 2022 General Mathematics, Paper 1, Section 2, Q.21; © State of Queensland (QCAA) 2024

The paths connecting various landmarks in a park are shown.



Key	
B	Bus stop
C	Coffee shop
D	Duck pond
P	Playground
R	Rose garden
W	Water feature

- a. Identify one cycle that passes through the rose garden and the playground. [1 mark]
- b. Identify whether the graph is Eulerian or semi-Eulerian. Justify your response. [2 marks]
- c. Construct an adjacency matrix from the graph, using the vertex order listed in the key. [2 marks]

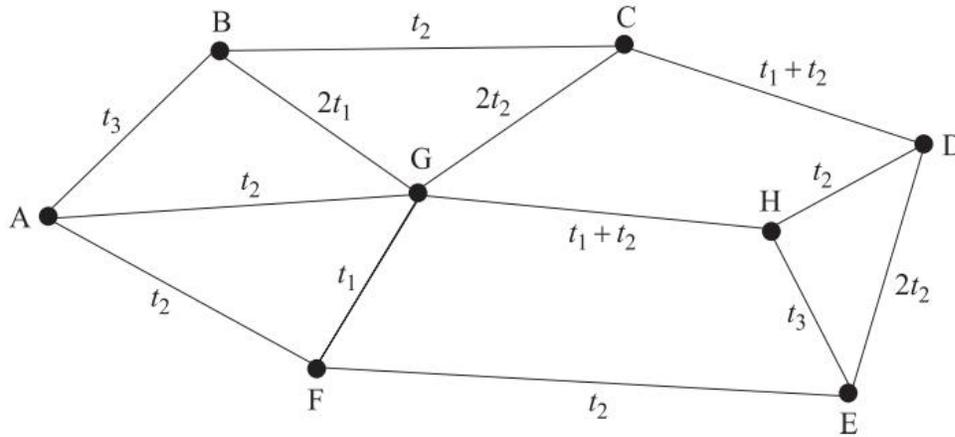
Question 5 (7 marks)

Source: QCAA 2023 General Mathematics, Paper 2, Section 1, Q.5; © State of Queensland (QCAA) 2024

At 9 : 00 am, a security guard begins their patrol of the eight work sites represented in the network diagram, starting and ending at site A. They drive at 40 km/h on the roads between sites and check every site once for 15 minutes.

The length (km) of each road corresponds to the terms of the arithmetic sequence $t_n = t_1 + 2(n - 1)$, where $t_1 = 1$.

Determine the earliest possible time the security guard can finish their patrol, and identify the route they must follow.



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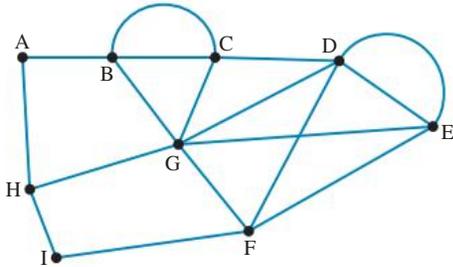
Answers

Chapter 9 Graphs and networks

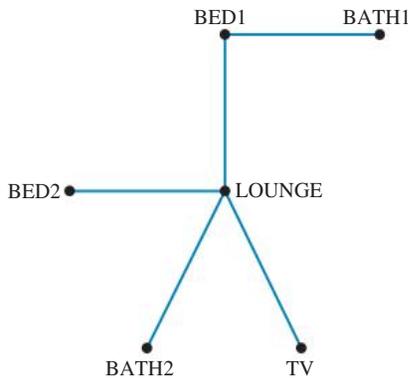
9.2 Graphs and associated terminology

9.2 Exercise

1.



2.



3. a. Edges = 7; $\text{deg}(A) = 3$; $\text{deg}(B) = 2$; $\text{deg}(C) = 3$; $\text{deg}(D) = 2$; $\text{deg}(E) = 3$; $\text{deg}(F) = 1$; degree sum = 14
- b. Edges = 10; $\text{deg}(A) = 4$; $\text{deg}(B) = 4$; $\text{deg}(C) = 3$; $\text{deg}(D) = 3$; $\text{deg}(E) = 4$; $\text{deg}(F) = 2$; degree sum = 20
4. a. $\text{deg}(A) = 5$; $\text{deg}(B) = 3$; $\text{deg}(C) = 4$; $\text{deg}(D) = 1$; $\text{deg}(E) = 1$
- b. $\text{deg}(A) = 0$; $\text{deg}(B) = 2$; $\text{deg}(C) = 2$; $\text{deg}(D) = 3$; $\text{deg}(E) = 3$
- c. $\text{deg}(A) = 4$; $\text{deg}(B) = 2$; $\text{deg}(C) = 2$; $\text{deg}(D) = 2$; $\text{deg}(E) = 4$
- d. $\text{deg}(A) = 1$; $\text{deg}(B) = 2$; $\text{deg}(C) = 1$; $\text{deg}(D) = 1$; $\text{deg}(E) = 3$

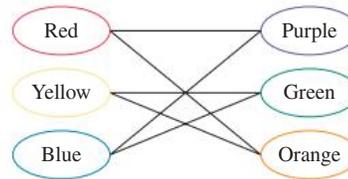
5.

Graph	Simple	Complete	Connected
Graph 1	Yes	No	Yes
Graph 2	Yes	No	Yes
Graph 3	Yes	No	Yes
Graph 4	No	No	Yes
Graph 5	Yes	Yes	Yes

6. Answers will vary; examples are provided.
- a. D-C-B-A-C
- b. A-B-D-F-C-E-A
- c. A-B-C-D-E-A

7. Primary colours

Secondary colours



8. a. C-G

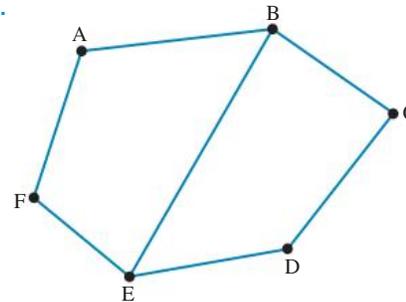
b. A-G and G-D

9.

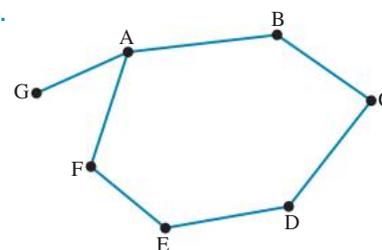
Vertices	Total degrees	Edges
2	1	2
3	3	6
4	6	12
5	10	20
6	15	30
n	$\frac{n(n-1)}{2}$	$n(n-1)$

10. Answers will vary. Possible answers are shown.

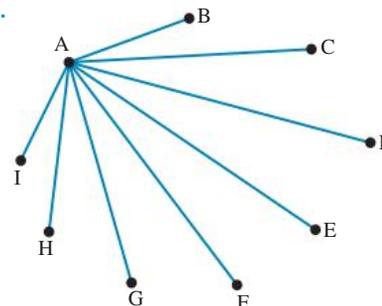
a.



b.



c.



11. 4:30 pm

12. Sample responses are provided in the worked solutions in the online resources.

13. \$225 000 000

14. Sample responses are provided in the worked solutions in the online resources.

15. 5

9.3 The adjacency matrix

9.3 Exercise

1. B

2. B

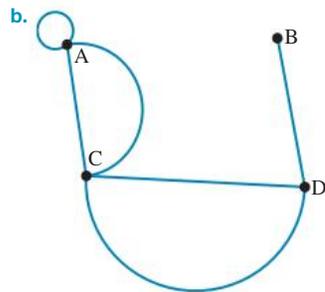
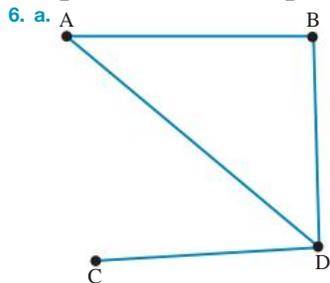
3. C

4. a.
$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

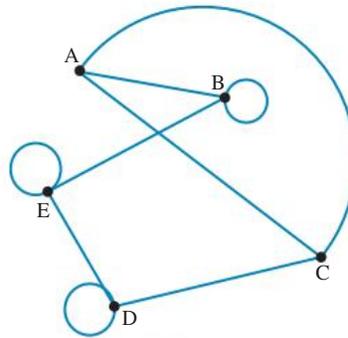
b.
$$\begin{bmatrix} 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

5. a.
$$\begin{bmatrix} 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 3 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 & 0 \end{bmatrix}$$

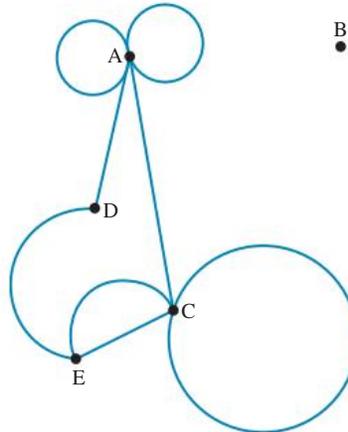
b.
$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 & 4 & 0 \end{bmatrix}$$



7. a.



b.



8. a.
$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

b.
$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix}$$

9. a.
$$\begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 1 & 0 \end{bmatrix}$$

b.
$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

10. a.
$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

b.
$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

c.
$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

11. a.
$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

b.
$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

12.

	Y	H	M	C	S	P
Y	0	1	0	1	0	0
H	1	0	1	1	1	0
M	0	1	0	0	1	1
C	1	1	0	0	1	0
S	0	1	1	1	0	1
P	0	0	1	0	1	0

13.

	Wa	O	Ca	I	N	A	M	U	Wy	Co	NM
Wa	0	2	0	1	0	0	0	0	0	0	0
O	2	0	1	1	0	0	0	0	0	0	0
Ca	0	1	0	0	2	3	0	0	0	0	0
I	1	1	0	0	0	0	2	2	0	0	0
N	0	0	2	0	0	0	0	2	0	0	0
A	0	0	3	0	0	0	0	0	0	0	2
M	0	0	0	2	0	0	0	0	1	0	0
U	0	0	0	2	2	0	0	0	1	1	0
Wy	0	0	0	0	0	0	1	1	0	1	0
Co	0	0	0	0	0	0	0	1	1	0	1
NM	0	0	0	0	0	2	0	0	0	1	0

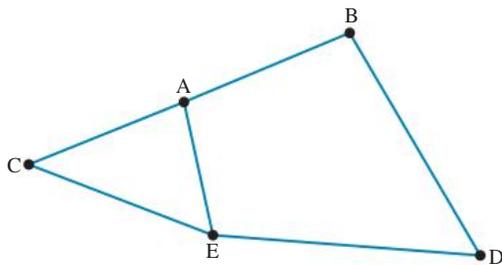
14. Diagraphs will vary, but Hanoi only has two outbound connections: Singapore or Kuala Lumpur. The possible alternative routes with only one stopover are:
 Hanoi → Kuala Lumpur → Bangkok
 Hanoi → Singapore → Bangkok

15. An adjacency matrix of the information can be read row-by-row to determine the diagraph that represents the cheapest service cost. Company A for music, company B for the chef and company C for cleaning

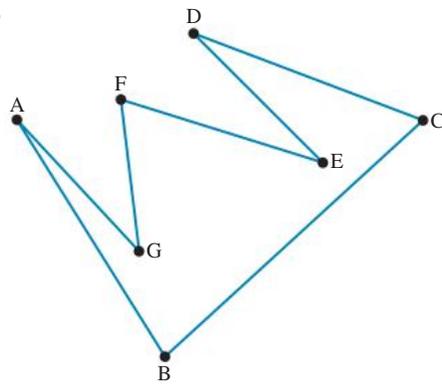
9.4 Planar graphs

9.4 Exercise

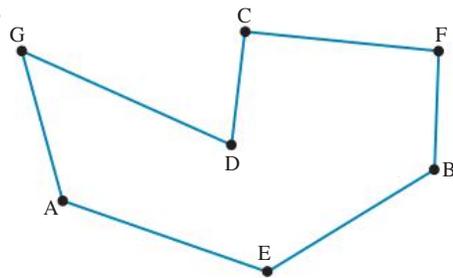
1. a.



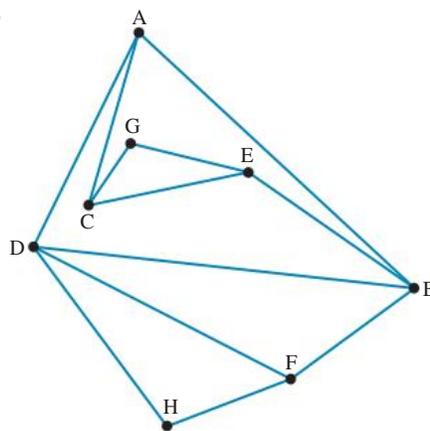
b.



2. a.



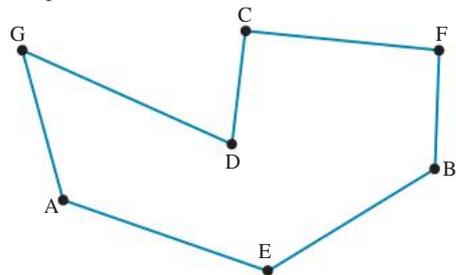
b.



3. Graphs 2 and 3 cannot be redrawn without intersecting lines.

4. a. 3 b. 3 c. 2 d. 7

5. Graph 1:



Vertices = 7, faces = 2, edges = 7
 $v + f - e = 2$

14. No; there will always be some intersecting edges.
 15. Sophie or Emma
 16. When drawn as a planar graph, a tetrahedron has 4 vertices, 4 faces and 6 edges.
 $v + f - e = 2$
 $4 + 4 - 6 = 2$
 $2 = 2$

9.5 Traversing graphs — walks, paths and trails

9.5 Exercise

1. Cycle: ABECA (others exist)
 Closed trail: BECDB (others exist)
 2. Path: ABGFHDC (others exist)
 Cycle: DCGFHD (others exist)
 Closed trail: AEBGFHDCA (others exist)
 3. a. Walk
 b. Walk, trail and path
 c. Walk, trail, path, cycle and closed trail
 d. Walk and trail
 4. a. MCHIJGFAED b. AEDBLKMC
 c. MDEAFGJIHCM d. FMCHIJGF
 5. E (the only other vertex with odd degree)
 6. a. A or C b. B or D
 7. a. Semi-Eulerian trail: AFEDBECAB; semi-Hamiltonian path: BDECAF
 b. Semi-Eulerian trail: GFBECDGAC; semi-Hamiltonian path: BECADGF
 8. a. Eulerian trail: AIBAHGFCJBCDEGA; Hamiltonian cycle: none exist
 b. Eulerian trail: ABCDEFGHA (others exist); Hamiltonian cycle: HABCDEFGH (others exist)
 9. a. Graphs **i**, **ii** and **iv**
 b. Graph **i**: ACDABDECB (others exist)
 Graph **ii**: CFBCEDBADCA (others exist)
 Graph **iv**: CFBCEDCADBAH (others exist)
 10. a. Graphs **i** and **ii** b. Graph **i**: CEDABC
 Graph **ii**: CEDABFC

11. a. i.
$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

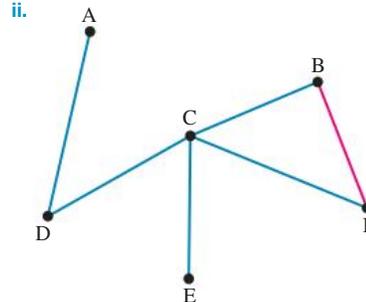
ii.
$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

iii.
$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

iv.
$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

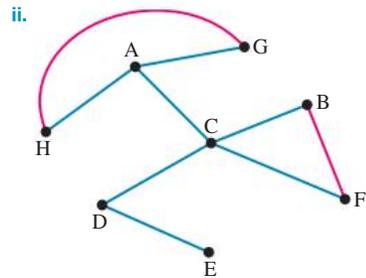
- b. The presence of semi-Eulerian trails and Eulerian trails can be identified by using the adjacency matrix to check the degree of the vertices. The presence of Hamiltonian and semi-Hamiltonian cycles can be identified by using the adjacency matrix to check the connections between vertices.

12. a. G to C b. F to E
 13. a. i. D–C, A–D (other solutions exist)



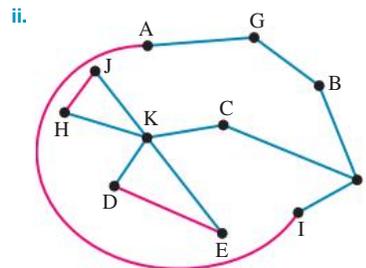
- iii. ADCBFCE or ADCFBCE

- b. i. A–C (other solutions exist)



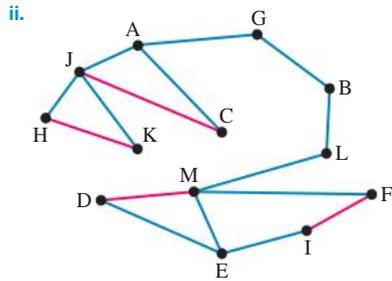
- iii. AHGACBFCDE or similar

- c. i. K–C (other solutions exist)



- iii. KDEKHJKCFIAGBF or similar

- d. i. A–G (other solutions exist)



iii. EDMEIFMLBG AJHKJCA or similar

14. a.

	Hamiltonian cycle
1.	ABCD A
2.	ABDCA
3.	ACBDA
4.	ACDBA
5.	ADBCA
6.	ADCBA

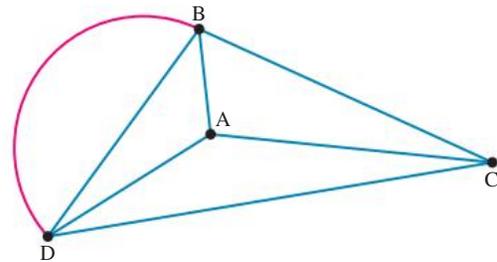
- b. Yes, commencing on vertices other than A
15. a. Yes, because the degree of each intersection or corner point is an even number.
 b. Yes, because the degree of each remaining intersection or corner point is still an even number.
16. a. The possible routes are ADHFICEBGA and AHDFICEBGA.
 b. Yes, because two of the checkpoints have odd degrees. The two possible starting points are H and C.
17. a. B, C, D, F or G
 b. B or C
 c. DCABGHFED or DBACFHGED as all nodes are used, 2nd stop is A and ends with D.
 d. D or E
 e. D to E
18. Let each warehouse be a vertex such that $v = [A, B, C, D, E, F, G, H, I, J]$. If each vertex has a degree of 3, sum total of degrees is 30. Therefore, the number of edges will be 15. The driver can complete the route by using a minimum of 10 roads, A-B-C-D-F-J-I-H-G-E-A, so there are 5 that would be unused. There may be other routes, eg A-B-E-G-J-F-C-D-I-H-A, but there will always be 5 unused roads.
19. Using trial and error, the shortest path is IA HGFEDCBI (or a similar variation of the same values).
 $78 + 50 + 71 + 50 + 71 + 50 + 71 + 50 + 78$
 $= 569 \text{ min or } 9 \text{ h } 29 \text{ min}$
 The CEO's claim is not reasonable.

9.6 Review

9.6 Exercise

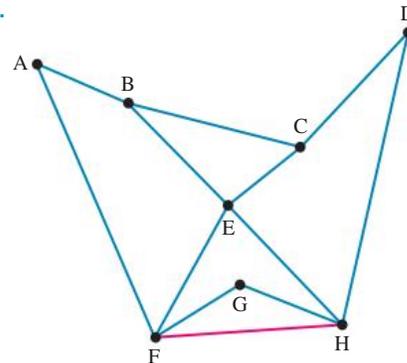
- C
- a. Edges = 9; degree sum = 18
 b. Edges = 9; degree sum = 18
- C
- D

- D
- B
- C
- A
- D
- B
- A
- D
- a. Simple, planar
 b. Simple, planar
 c. Simple, complete, planar
 d. Simple, planar
- E-C-D-G-A-H-F-A-B-E-H-D
- a. Simple, complete b. Simple, planar
 c. Simple, planar d. Simple, planar
- a. i. 3



ii. ABDBCADC

b. i.



ii. BAFEHGFHDCEBC

17. a.
$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 3 & 0 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$
- b.
$$\begin{bmatrix} 0 & 1 & 2 & 1 & 1 \\ 1 & 0 & 2 & 0 & 1 \\ 2 & 2 & 0 & 2 & 3 \\ 1 & 0 & 2 & 2 & 2 \\ 1 & 1 & 3 & 2 & 0 \end{bmatrix}$$
- c.
$$\begin{bmatrix} 0 & 2 & 1 & 3 & 1 & 2 \\ 2 & 0 & 3 & 1 & 1 & 0 \\ 1 & 3 & 0 & 2 & 3 & 1 \\ 3 & 1 & 2 & 2 & 2 & 1 \\ 1 & 1 & 3 & 2 & 3 & 1 \\ 2 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

$$d. \begin{bmatrix} 0 & 2 & 1 & 3 & 0 & 1 & 0 \\ 2 & 0 & 2 & 1 & 2 & 1 & 0 \\ 1 & 2 & 0 & 1 & 1 & 0 & 1 \\ 3 & 1 & 1 & 0 & 0 & 2 & 1 \\ 0 & 2 & 1 & 0 & 0 & 0 & 3 \\ 1 & 1 & 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & 3 & 2 & 0 \end{bmatrix}$$

18. a. No
- b. In this network, all 4 vertices have an odd degree. For an Eulerian or semi-Eulerian trail to exist 0 or 2 vertices of a connected network must have an odd numbered degree.
19. A Eulerian trail does not currently exist. An edge needs to be added between A and H and B and D, making all degrees even. The trail could be A-B-C-A-D-G-F-C-F-E-B-D-A-H-G-I-H
20. Answers may vary. Some examples include:
- A 3 vertex graph where each vertex has a degree of 2 or more.
 - A 4 vertex graph where each vertex has a degree of 2 or more.
 - A 5 vertex graph where each vertex has a degree of 3 or more.
 - A 6 vertex graph where each vertex has a degree of 3 or more.
 - A 7 vertex graph where each vertex has a degree of 4 or more.
 - A 8 vertex graph where each vertex has a degree of 4 or more.

9.6 Past QCAA exam questions

1. C
2. C
3. a. Degree = 4
b. Number of edges = 2
c. Sample responses are available in the worked solutions in the online resources.
4. a. A cycle must contain R and P
For example
RWPDR
PDRWBCP (etc.)
- b. This graph is semi-Eulerian.
Because it has two odd degree vertices and the remaining vertices are even degree.
- c. The required adjacency matrix is given as follows

$$\begin{array}{c} B \\ C \\ D \\ P \\ R \\ W \end{array} \begin{array}{c} B \\ C \\ D \\ P \\ R \\ W \end{array} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 2 & 1 & 1 \\ 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 1 & 1 & 1 & 1 & 2 & 0 \end{bmatrix}$$

5. Sample responses are available in the worked solutions in the online resources.

10 Networks and decision mathematics

LESSON SEQUENCE

10.1 Overview	398
10.2 Trees and minimum connector problems	400
10.3 Project planning and scheduling	410
10.4 Critical Path Analysis (CPA)	423
10.5 Flow networks	431
10.6 Assigning order and the Hungarian algorithm	443
10.7 Review	455
Answers	465

Fully worked solutions for this chapter are available online.

EXAM PREPARATION

Access exam-style questions in every lesson, available online and access past QCAA exam questions in every review.

Resources

-  **Solutions** Solutions — Chapter 10
-  **Exam questions** Exam question booklet — Chapter 10
-  **Digital documents** Learning matrix — Chapter 10
Chapter summary — Chapter 10

LESSON

10.1 Overview

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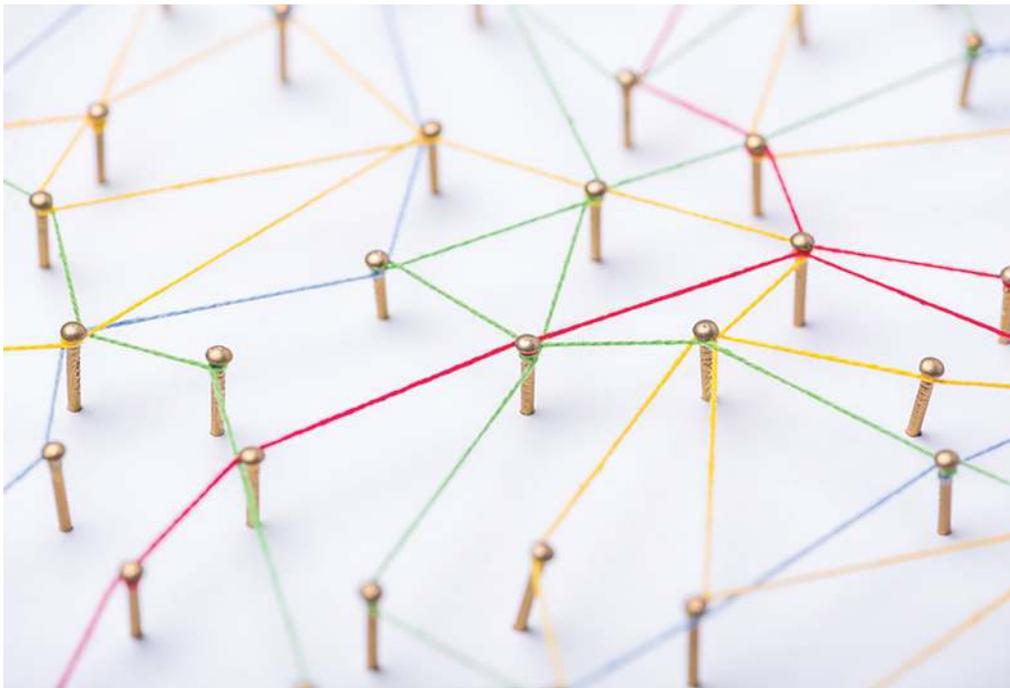
Answer questions and check results

Find all this and MORE in jacPLUS



10.1.1 Introduction

Whether you are building a skyscraper, managing a construction project, or working on various projects in your small business, critical path analysis is essential for staying on budget, meeting deadlines, and managing personnel efficiently. Critical path techniques can be done with a pen and paper, but as a project becomes larger and more complex, it becomes increasingly necessary to implement modern software solutions. These software tools can help streamline the process, making it easier to identify the critical path and manage project timelines effectively.



Although there are many types of software available to assist you in determining critical paths, you will still need to understand the basic mathematical concepts behind these techniques in order to correctly reach your project goals. This includes understanding how to calculate the earliest and latest start and finish times for each task, as well as how to identify and manage dependencies between tasks. By mastering these concepts, you can ensure that your projects are completed on time and within budget, while also making the most efficient use of your resources.

In this chapter, we will explore the fundamental principles of network and decision mathematics, providing you with the tools and knowledge you need to apply these techniques to real-world projects. Whether you are a student, a project manager, or a business owner, this introduction will help you develop the skills necessary to succeed in today's fast-paced and competitive environment.

10.1.2 Syllabus links

Lesson	Lesson title	Syllabus links
10.2	Trees and minimum connector problems	<ul style="list-style-type: none"> <input type="radio"/> Understand the meaning of tree, spanning tree and minimum spanning tree. <input type="radio"/> Determine a minimum spanning tree in a weighted connected graph. <input type="radio"/> Solve practical problems involving minimum spanning trees, e.g. minimising the length of cable needed to provide power from a single power station to substations in several towns.
10.3	Project planning and scheduling	<ul style="list-style-type: none"> <input type="radio"/> Construct a project network diagram (activity on arc) to represent the durations and interdependencies of activities that must be completed during the project (excluding dummy activities). <input type="radio"/> Use forward and backward scanning to determine the earliest starting time (EST) and latest starting time (LST) for each activity in the project.
10.4	Critical path analysis (CPA)	<ul style="list-style-type: none"> <input type="radio"/> Use ESTs and LSTs to locate the critical path/s for a project. <input type="radio"/> Use the critical path to determine the minimum time for a project to be completed. <input type="radio"/> Calculate float times for non-critical activities. <input type="radio"/> Solve small-scale practical problems involving critical path analysis.
10.5	Flow networks	<ul style="list-style-type: none"> <input type="radio"/> Understand the meaning of source node, sink node, cut, minimum cut and maximum flow. <input type="radio"/> Use a flow network diagram to identify a cut. <input type="radio"/> Determine the capacity of a cut. <input type="radio"/> Solve small-scale practical problems involving flow networks (up to 8 possible cuts), including determining the minimum cut and the maximum flow.
10.6	Assigning order and the Hungarian algorithm	<ul style="list-style-type: none"> <input type="radio"/> Use a bipartite graph and its tabular or matrix form to represent possible assignments for an allocation problem. <input type="radio"/> Determine the optimum (minimum and maximum) assignment/s for small-scale practical problems by inspection. <input type="radio"/> Use the Hungarian algorithm (3×3 up to 5×5 square matrices) to determine the optimum (minimum and maximum) assignment/s for larger practical problems.

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LESSON

10.2 Trees and minimum connector problems

SYLLABUS LINKS

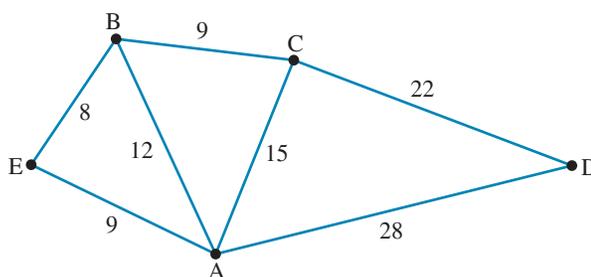
- Understand the meaning of tree, spanning tree and minimum spanning tree.
- Determine a minimum spanning tree in a weighted connected graph.
- Solve practical problems involving minimum spanning trees, e.g. minimising the length of cable needed to provide power from a single power station to substations in several towns.

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10.2.1 Weighted graphs

In many applications using graphs, it is useful to attach a value to the edges. These values could represent the length of the edge in terms of time or distance, or the costs involved with moving along that section of the path. Such graphs are known as **weighted graphs**.

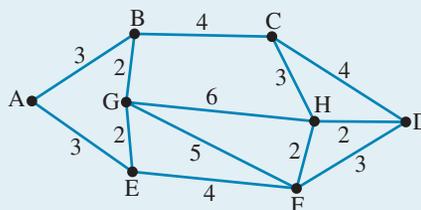
Weighted graphs can be particularly useful as analysis tools. For example, they can help determine how to travel through a network in the shortest possible time.



WORKED EXAMPLE 1 Identify the shortest distance of travel

The graph represents the distances in kilometres between eight locations.

Identify the shortest distance to travel from A to D that goes to all vertices.



THINK

1. Identify the semi-Hamiltonian paths that connect the two vertices.
2. Calculate the total distances for each path to determine the shortest.
3. State the final answer.

WRITE

Possible paths:

- a. ABGEFHCD
- b. ABCHGEFD
- c. AEGBCHFD
- d. AEFGBCHD
- e. AEFHGBCD

a. $3 + 2 + 2 + 4 + 2 + 3 + 4 = 20$

b. $3 + 4 + 3 + 6 + 2 + 4 + 3 = 25$

c. $3 + 2 + 2 + 4 + 3 + 2 + 3 = 19$

d. $3 + 4 + 5 + 2 + 4 + 3 + 2 = 23$

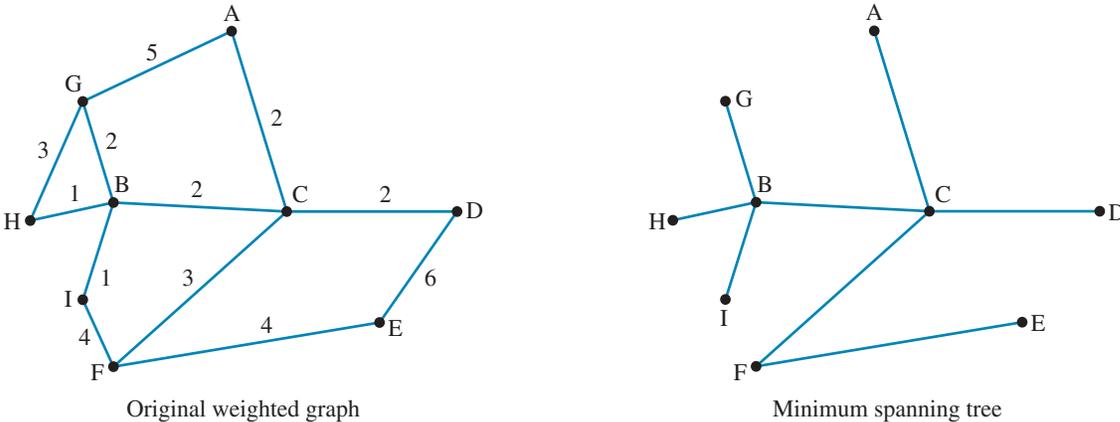
e. $3 + 4 + 2 + 6 + 2 + 4 + 4 = 25$

Path c (AEGBCHFD) is the shortest.

The shortest distance from A to D that travels to all vertices is 19 km.

10.2.2 Trees

A tree is a simple connected graph with no circuits. As such, any pairs of vertices in a tree are connected by a unique path, and the number of edges is always 1 less than the number of vertices. Spanning trees are sub-graphs (graphs that are formed from part of a larger graph) that include all of the vertices of the original graph. In practical settings, they can be very useful in analysing network connections. For example, a minimum spanning tree for a weighted graph can identify the lowest-cost connections to upgrade plumbing between buildings, roads between towns or changed flight paths between cities.



Trees

- **Trees** are simple, connected graphs with no circuits.
Edges = vertices – 1
- A **spanning tree** is a subgraph that can be found to exist within a larger network. Consider all the streets that exist in a town or city. A spanning tree would be the street(s) that connect your house to your friends' houses, ignoring the rest.
- A **minimum spanning tree** is a subgraph with weighted edges. A minimum spanning tree could be used to determine connections with the lowest cost or the shortest route.

The minimum spanning tree algorithm

Determining the minimum spanning tree within a network can be done through trial-and-error. However, a set of systematic steps can also help to determine minimum spanning trees. The set of steps was first discovered by a Czech mathematician Vojtěch Jarník in 1930, and later reported again by American computer scientist, Robert Prim (1957) and Dutch mathematician Edsger W. Dijkstra (1959). Their different countries and fields of mathematics means that the algorithm is sometimes labelled as belonging to one of them (Prim's algorithm, Jarník's algorithm), sometimes two, or it can be known as the combination of their names (DJP algorithm).

Regardless, all minimum spanning tree algorithms is a set of logical steps that can be used to identify the minimum spanning tree for a weighted connected graph.

The steps for the minimum spanning tree algorithm

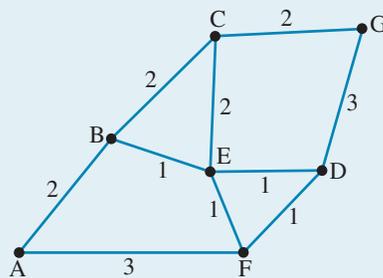
Step 1: Begin at a vertex with low weighted edges.

Step 2: Progressively select edges with the lowest weighting (unless they form a circuit).

Step 3: Continue until all vertices are included.

WORKED EXAMPLE 2 Determine a minimum spanning tree

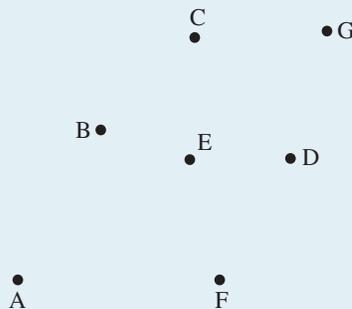
Identify the minimum spanning tree of the graph shown and determine the total weighted value of that tree.



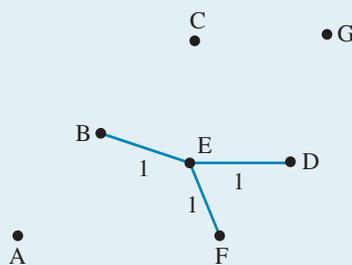
THINK

1. Draw the vertices of the graph.

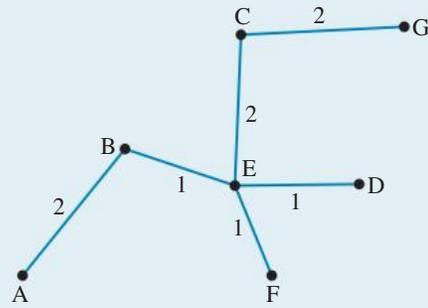
DRAW



2. Draw in any edges with the lowest weighting that do not complete a circuit. Identify the vertex with low weighted edges where you will start. This is vertex E.



3. Draw in any edges with the next lowest weighting that do not complete a circuit. Continue until all vertices are connected.



4. Add up the value of the branches to determine the total weighted value of the tree.

Total = 9

Exercise 10.2 Trees and minimum connector problems

learn on

10.2 Exercise

10.2 Exam questions **on**

Simple familiar

1, 2, 3, 4, 5, 6, 7, 8, 9

Complex familiar

10, 11, 12, 13

Complex unfamiliar

14, 15

These questions are even better in jacPLUS!

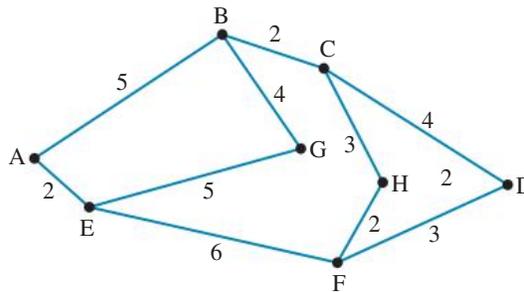
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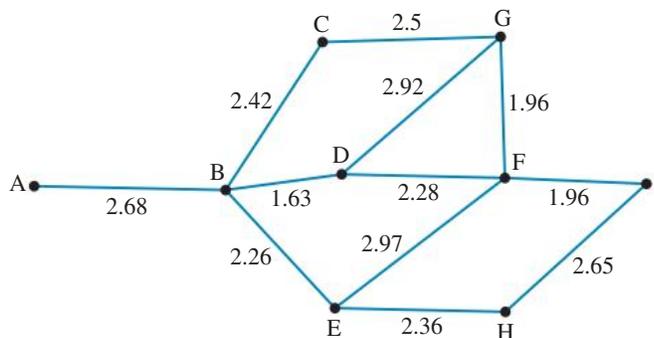
Find all this and MORE in jacPLUS

Simple familiar

1. **WE1** Use the graph to identify the shortest distance to travel from A to D that goes to all vertices.

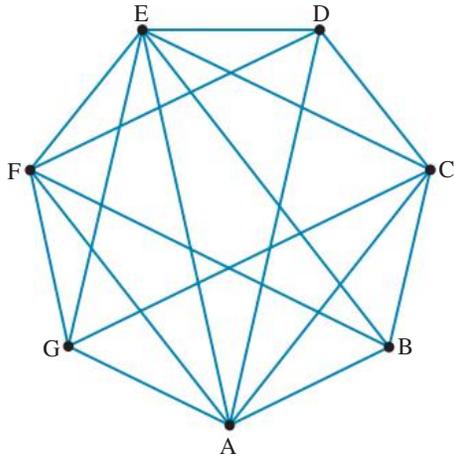


2. Use the graph to identify the shortest distance to travel from A to I that goes to all vertices.

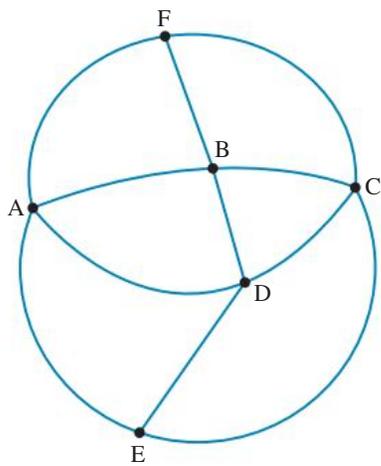


3. Draw three spanning trees for each of the following graphs.

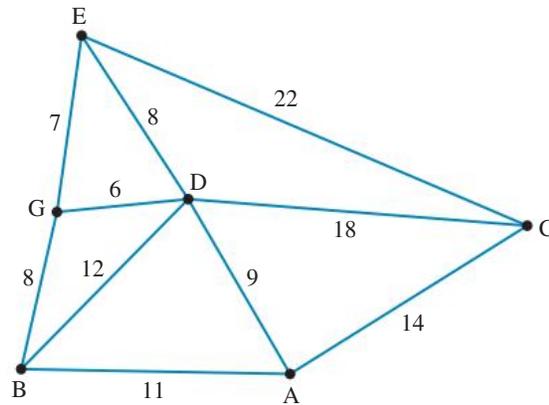
a.



b.



4. A truck starts from the main distribution point at vertex A and makes deliveries at each of the other vertices before returning to A.

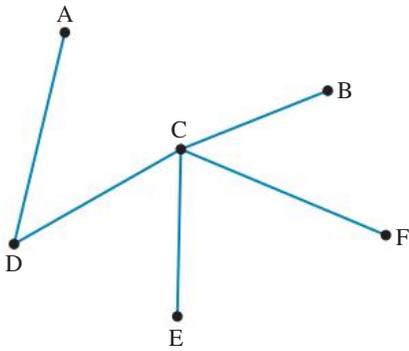


Identify the shortest route the truck can take.

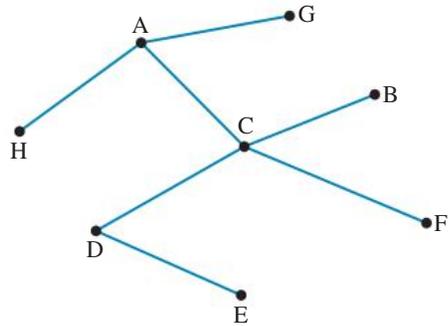
5. For the following trees:

- i. add the minimum number of edges to create a semi-Eulerian trail
- ii. identify the semi-Eulerian trail created.

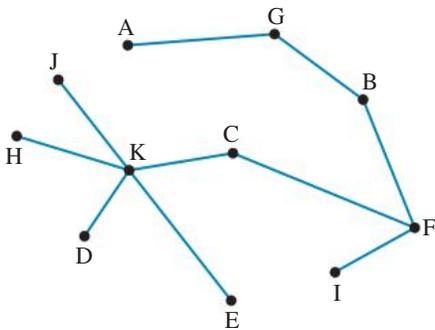
a.



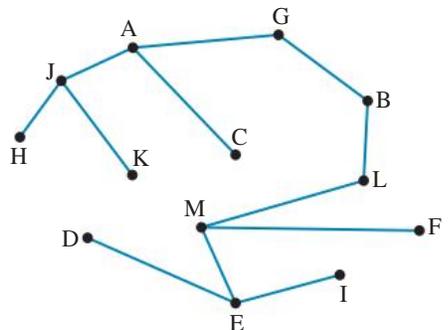
b.



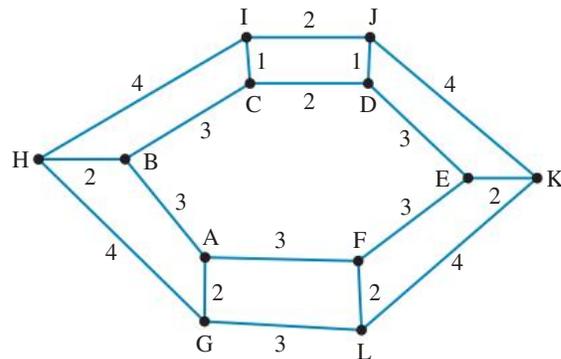
c.



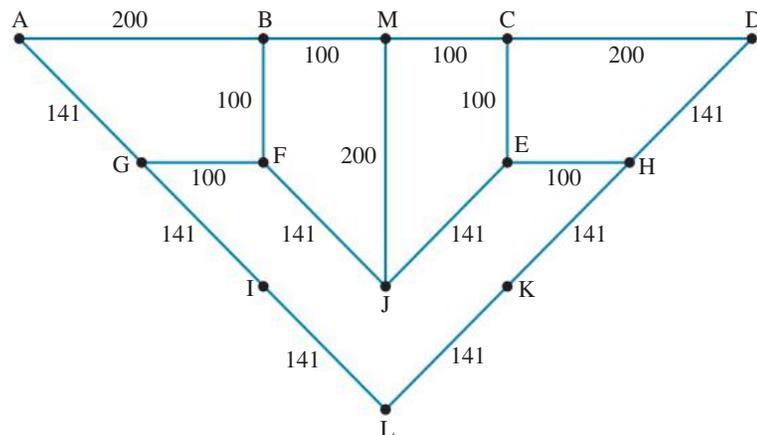
d.



6. WE2 Identify the minimum spanning tree of the graph shown.

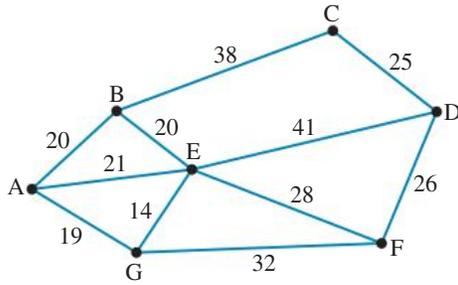


7. Identify the minimum spanning tree of the graph shown.

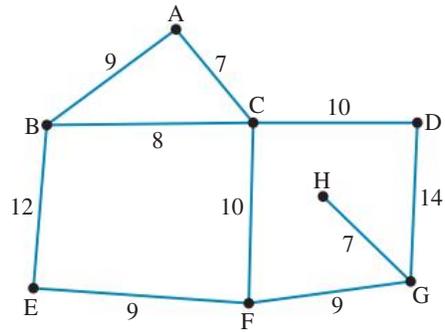


8. Identify the minimum spanning tree for each of the following graphs.

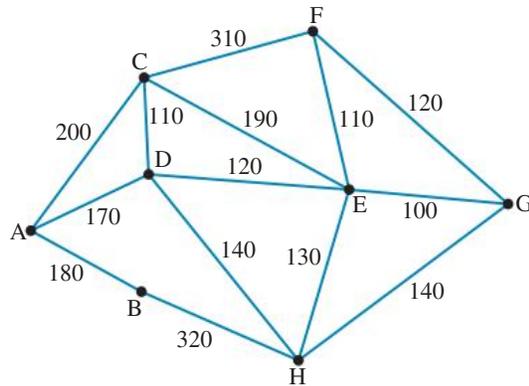
a.



b.



9. The weighted graph represents the costs incurred by a salesman when moving between the locations of various businesses.



- Determine the cheapest way of travelling from A to G.
- Determine the cheapest way of travelling from B to G.
- If the salesman starts and finishes at E, determine the cheapest way to travel to all vertices.

Complex familiar

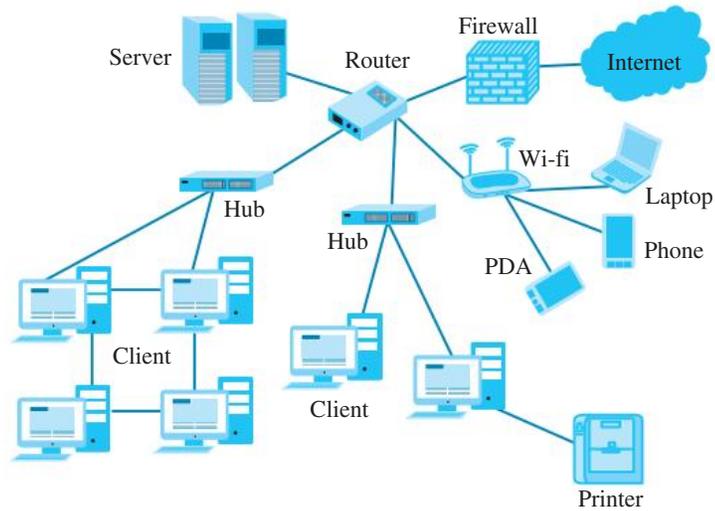
10. Part of the timetable and description for a bus route is shown in the table.

Bus stop	Description	Time
Bus depot	The northernmost point on the route	7.00 am
Northsea Shopping Town	Reached by travelling south-east along a highway from the bus depot	7.15 am
Highview Railway Station	Travel directly south along the road from Northsea Shopping Town	7.35 am
Highview Primary School	Directly east along a road from the railway station	7.40 am
Eastend Medical Centre	Continue east along the road from the railway station	7.55 am
Eastend Village	South-west along a road from the medical centre	8.05 am
Southpoint Hotel	Directly south along a road from Eastend Village	8.20 am
South Beach	Travel south-west along a road from the hotel	8.30 am

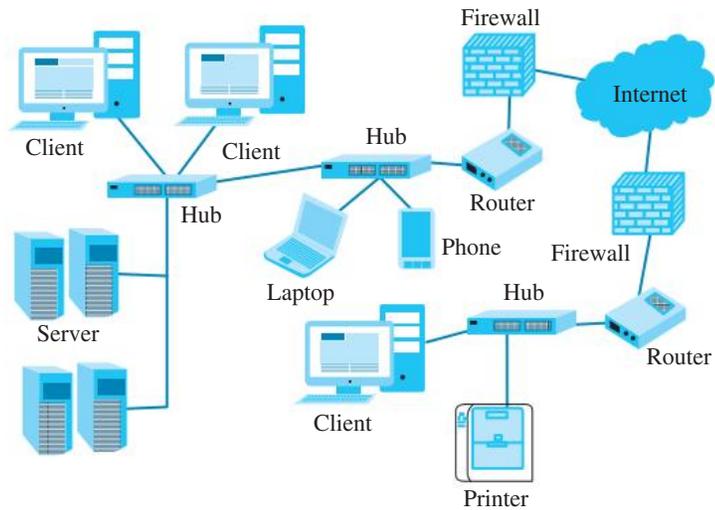
Draw a weighted graph to represent the bus route.

11. The diagrams show two options for the design of a computer network for a small business.

Option 1



Option 2

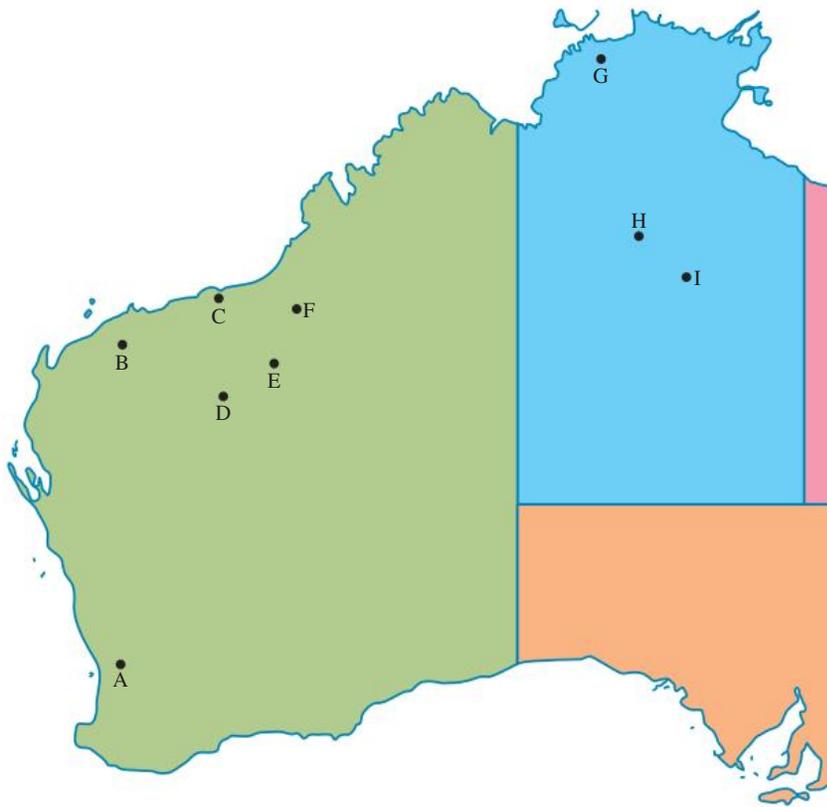


Information relating to the total costs of setting up the network is shown in the following table.

Connected to:	Server	Client	Hub	Router	Firewall	Wi-fi	Printer
Server			\$995	\$1050			
Client		\$845	\$355				\$325
Hub			\$365	\$395			\$395
Router	\$1050		\$395		\$395	\$395	
Laptop			\$295			\$325	
Phone			\$295			\$325	
PDA						\$325	
Internet					\$855		

Determine which option is cheaper. Justify your decision with weighted graphs for each option.

12. A mining company operates in several locations in Western Australia and the Northern Territory, as shown on the map.

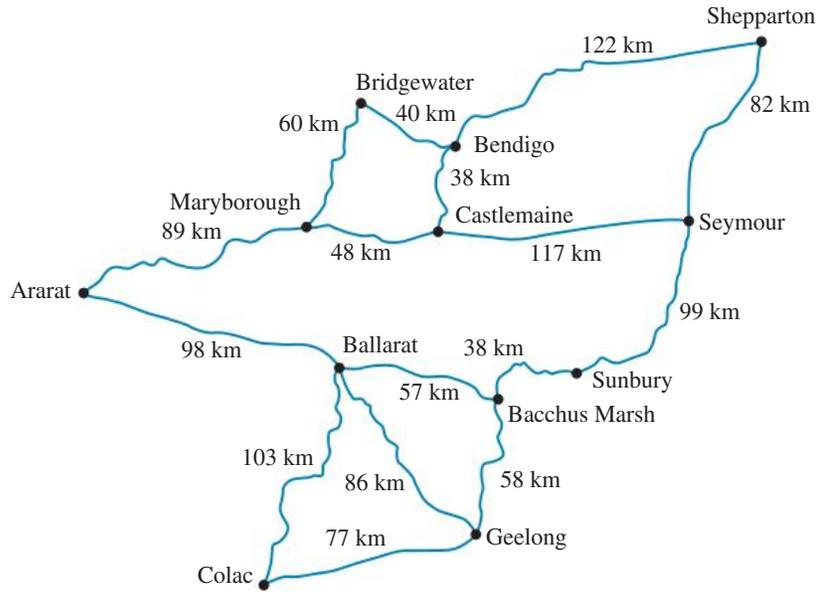


Flights operate between selected locations, and the flight distances (in km) are shown in the following table.

	A	B	C	D	E	F	G	H	I
A		1090		960			2600		2200
B	1090		360	375	435				
C		360							
D	960	375							
E		435							
F							1590	1400	
G	2600					1590		730	
H						1400	730		220
I	2200							220	

Draw the minimum spanning tree for the graph.

13. The organisers of the 'Tour de Vic' bicycle race are using the following map to plan the event.



If the organisers decide to use the minimum spanning tree as the course, determine the shortest possible distance if each location had to be reached at least once, starting anywhere and finishing anywhere.

Complex unfamiliar

14. A network graph has four vertices (A, B, C, D) and six edges with the following weights: $AB = x$, $AC = x + 1$, $AD = 5$, $BC = x + 2$, $BD = 7$ and $CD = 4$. The minimum spanning tree (MST) of this graph has a total weight of 15. Determine the value of x .
15. A wildlife sanctuary features nine key zones connected by various trails. The distance (in metres) of each trail between zones is shown on the table below.

Trails	Distance (m)
Entrance to Bird Aviary	80
Entrance to Reptile House	130
Bird Aviary to Petting Zoo	100
Bird Aviary to Safari Ride	75
Reptile House to Petting Zoo	90
Reptile House to Safari Ride	160
Petting Zoo to Aquarium	70
Safari Ride to Aquarium	120
Aquarium to Food Court	140
Safari Ride to Food Court	110
Food Court to Exit	60

The annual maintenance cost of each trail is \$250 per metre.

The sanctuary manager suggests that by removing some trails while still allowing access to all zones, at least \$150 000 can be saved annually. Evaluate whether the manager's belief is reasonable.

Fully worked solutions for this chapter are available online.

LESSON

10.3 Project planning and scheduling

SYLLABUS LINKS

- Construct a project network diagram (activity on arc) to represent the durations and interdependencies of activities that must be completed during the project (excluding dummy activities).
- Use forward and backward scanning to determine the earliest starting time (EST) and latest starting time (LST) for each activity in the project.

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10.3.1 Activity charts and networks

We all have daily tasks that need fitting into our schedules, and without organisation, we risk running out of time or double-booking ourselves. Major operations, like construction projects, also require efficient planning to ensure the right people and materials are in place at the right time. Poor planning results in wasted time and money.

To illustrate the benefits of planning, consider this simple example:

Blake needs to organise his morning routine to fit everything into seven minutes. He has three tasks: logging into school emails (1 minute), reading the emails (2 minutes) and eating breakfast (6 minutes). Clearly, he needs to multitask.

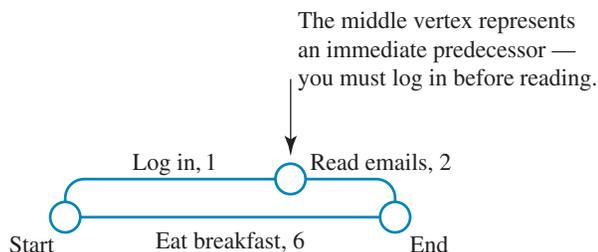
The following table shows how he might manage each minute.

Minute	Activity	Simultaneous activity
1	Eat breakfast	Log in to school email account
2	Eat breakfast	Read emails
3	Eat breakfast	Read emails
4	Eat breakfast	
5	Eat breakfast	
6	Eat breakfast	



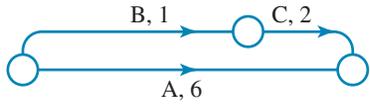
More complex activities require a much greater amount of planning and analysis. Another way of representing this information is in a network diagram. A network diagram can be used to represent the ‘flow’ of activities.

In the network diagram, the edges (or arcs) of our network represent the three activities of eating, logging in and reading. The left vertex represents the start of all activity, the right vertex represents the end of all activity, and the middle vertex indicates that the logging in must occur before the reading starts. In other words, logging in is the immediate predecessor of reading emails. Another way of representing this information is in an activity chart, which shows the durations (times) of interdependencies (predecessors) using a shorthand system of labelling each activity as a letter.



Activity letter	Activity	Predecessor	Time (min)
A	Eat breakfast	—	6
B	Logging in	—	1
C	Read emails	B	2

This activity chart also shows that activity B (logging in) is the immediate predecessor of activity C (reading), and that the other two activities (B and A) have no predecessors. Following is an alternative network diagram.



This network diagram also indicates a direction. The activities can only be undertaken in a certain sequence (C must follow B), so arrowheads are shown on the edges. Because of the implied direction, networks of this type are called **directed networks or digraphs**. The edges in a directed network represent a one-way path between the vertices.

WORKED EXAMPLE 3 Creating a simple network diagram

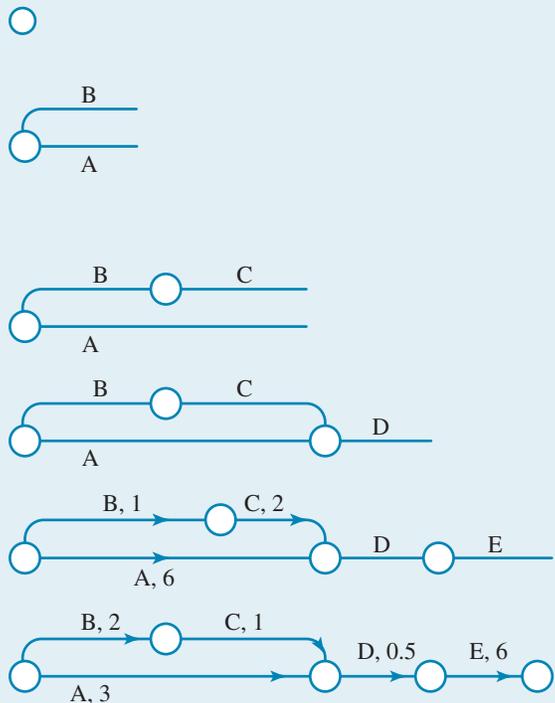
Prepare a network diagram from the activity chart below.

Activity letter	Activity	Predecessor	Time (min)
A	Cook the eggs	—	3
B	Toast the bread	—	2
C	Butter the toast	B	1
D	Put cooked egg on toast	A, C	0.5
E	Eat breakfast	D	6

THINK

1. Begin the network diagram by drawing the starting vertex.
2. The table shows that two activities have no predecessor, so they are the starting activities in the network diagram. To show this, draw two edges out from the first vertex in step
3. Activity B is a predecessor for activity C, so a new vertex needs to be drawn to link the two paths.
4. Activities A and C are the predecessor for activity D, so a new vertex needs to be drawn to link the two paths.
5. Activity D is a predecessor for activity E, so needs to be drawn to link the two paths.
6. When you have included all activities, add the finishing vertex. Add arrows to the network diagram to indicate the direction in which the activities flow.

DRAW



Lastly, write the times next to each activity.

Let us now extend the activity chart to a more complex set of activities for Blake’s morning routine.

WORKED EXAMPLE 4 Creating a complex network diagram

From the following activity chart, prepare a network diagram of Blake's morning schedule.

Activity letter	Activity	Predecessor	Time (min)
A	Prepare breakfast	—	4
B	Cook breakfast	A	2
C	Eat breakfast	B, E, G	6
D	Have shower	A	4
E	Get dressed	D	4
F	Brush teeth	C, H	2
G	Log in	A	1
H	Read email	B, E, G	2
		Total time	25

THINK

1. Begin the network diagram by drawing the starting vertex.
2. Examine the table looking for activities that have no predecessors. There must be at least one of these. This activity becomes the first edge and is labelled with its activity letter.
3. List all activities for which A is the immediate predecessor. This is provided in the table.

4. Add a vertex to the end of the edge for activity A, and an arrow indicating direction of flow.

5. Create one edge from this vertex for each of the listed activities. Label these edges with their activity letters.
Note: The end vertex for each of these activities is not drawn until either you are certain that it is not the immediate predecessor of any later activities, or all activities have been completed.

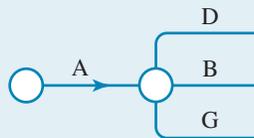
WRITE



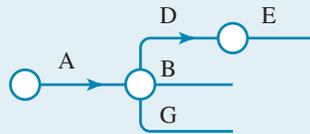
Activity A has no predecessors.



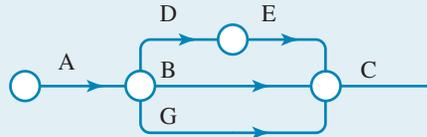
Activity B has A as an immediate predecessor.
Activity D has A as an immediate predecessor.
Activity G has A as an immediate predecessor.



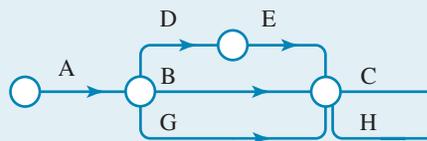
6. Repeat Step 3 for activity D. Since it is the *only* immediate predecessor of activity E, this can be added to the diagram. Otherwise, activity E could not be added yet.



7. Repeat Step 3 for activities B and G. They have no activities for which they are the only predecessors. Since activity C is preceded by all of B, G and E, join all the edges at a single vertex.

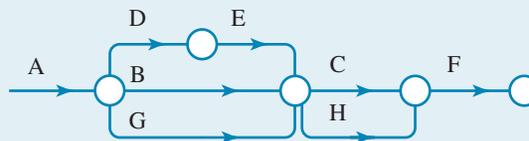


8. Add activity C after this joining vertex. Note that activity H is also preceded by all of B, G and H but *not* by activity C.

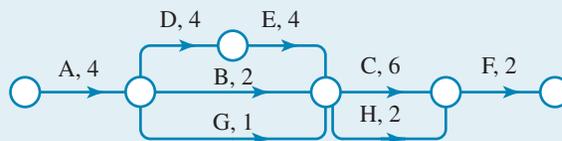


9. Determine whether activity C and H are independent of each other. Neither C nor H are predecessors of each other. Since they are independent, activity H starts from the same vertex as activity C.

10. The last activity is F, which has C and H as its immediate predecessors. Therefore join C and H with a vertex, then add an edge for F. Since F is the final activity, also add the end vertex.



11. Add the time required for each activity next to its letter.



Now that the tasks have been reduced to a network diagram, we can use the diagram to help Blake reduce the total time spent (duration) on all these tasks. If all the tasks were spread out in a straight line so that no tasks were completed at the same time, then his morning routine would take 25 minutes (see the activity chart). The diagram shows that some of Blake's tasks can take place at the same time. Let us investigate the time savings available.

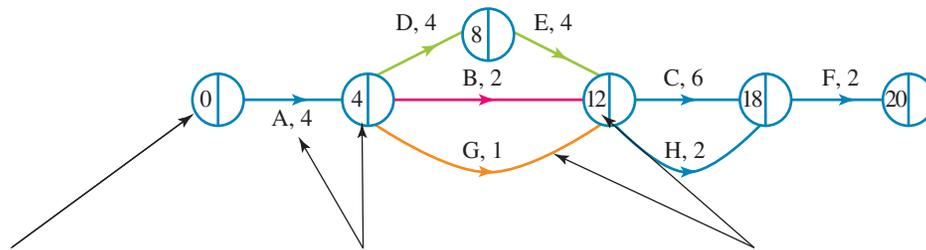
10.3.2 Forward scanning and earliest start time (EST)

Forward scanning is a key part of project management and scheduling. Forward scanning allows you to calculate the earliest start and finish times for each activity in a project. These values are used to determine how long the entire project will take and which activities must be completed on time to avoid delaying the project.

We will begin with how to determine **earliest start time (EST)**. In a network vertex or node, the EST is the number on the left of the node.



Let's consider Blake's morning routine activity network.



1. Set the earliest start time (EST) for the first activity.

The earliest start time for the first activity is set to 0.

2. Forward scan through the activity to calculate the earliest start time for the next activity.

Activity A takes 4 minutes, so the earliest start time for the next activity is set to 4.

3. When there are multiple predecessors, the EST is determined by the pathway that takes longest.

Activity G takes 1 minute, so the EST for the next activity would be 5.

Activity B takes 2 minutes, so the EST for the next activity would be 6.

Activities D and E take 8 minutes, so the EST for the next activity would be 12.

We cannot start the next activities (C and H) until ALL of G, B, D, E are complete, so we must wait for D and E before moving on. D and E are **critical** to the timeline of the project. Delays in E and E will affect the project timeline. G and B are not critical, so short delays in G and B may not affect the timeline.

Earliest start time

Forward scanning starts at the start vertex and moves forward through the network, adding the time of each edge (or arc) to the earliest start time of each previous vertex.

The **earliest start time (EST)** is the earliest that any activity can be started after all prior activities have been completed.

WORKED EXAMPLE 5 Forward scanning and earliest start time

Using all the activities listed in Blake's morning routine, determine the earliest completion time and hence identify those tasks that are critical to the timeline, and those that may be delayed without extending the completion time.

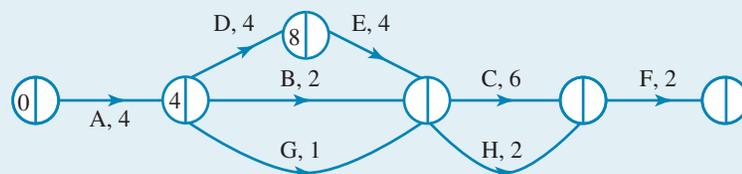
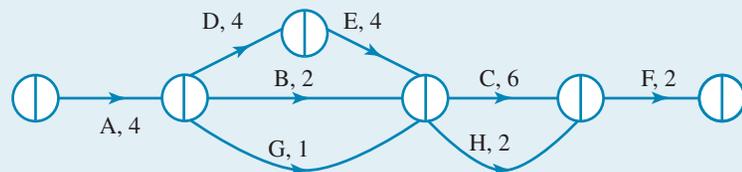
From the following activity chart, prepare a network diagram of Blake's morning schedule.

Activity letter	Activity	Predecessor	Time (min)
A	Prepare breakfast	—	4
B	Cook breakfast	A	2
C	Eat breakfast	B, E, G	6
D	Have shower	A	4
E	Get dressed	D	4
F	Brush teeth	C, H	2
G	Log in	A	1
H	Read email	B, E, G	2
		Total time	25

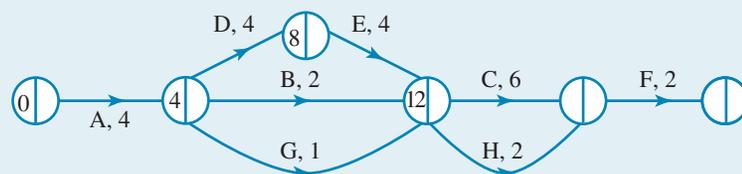
THINK

1. Draw a directed network with split nodes at each vertex. Include arrows to show the direction of Blake's morning routine.
2. Begin forward scanning. The earliest start time for the first three vertices in the path can be entered immediately.
3. Calculate the time values for the paths to the fourth vertex. As there are multiple time values that could be entered, enter the largest value into the left-hand side (LHS) of the vertex. This ensures all tasks are completed along these paths.
4. A and D and E take the greatest time, so they are critical to the timeline. B can be delayed. G can be delayed.
5. Repeat Step 3 for the next vertex. Note that calculations begin by using the time from the previous vertex (12 minutes). Again, select the higher time value and enter it into the vertex.

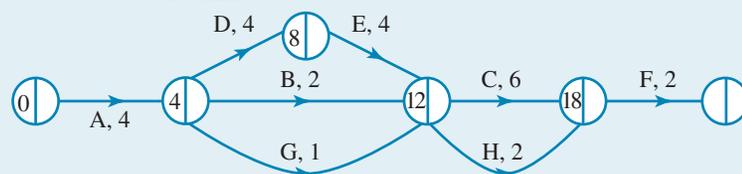
WRITE



$$\begin{aligned}
 A-D-E &= 4 + 4 + 4 \\
 &= 12 \text{ minutes} \\
 A-B &= 4 + 2 \\
 &= 6 \text{ minutes} \\
 A-G &= 4 + 1 \\
 &= 5 \text{ minutes}
 \end{aligned}$$



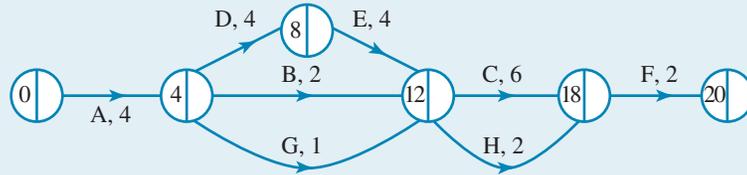
$$\begin{aligned}
 A-D-E-C &= 12 + 6 \\
 &= 18 \text{ minutes} \\
 A-D-E-H &= 12 + 2 \\
 &= 14 \text{ minutes}
 \end{aligned}$$



6. C takes the greatest time; it is critical to the timeline. H can be delayed.

7. There is only one path to the last activity (F). Add its time requirement to that of the previous vertex (18 minutes). F is critical to the timeline.

$A-C-F = 18 + 2$
 $= 20$ minutes
 Earliest completion time is 20 minutes.



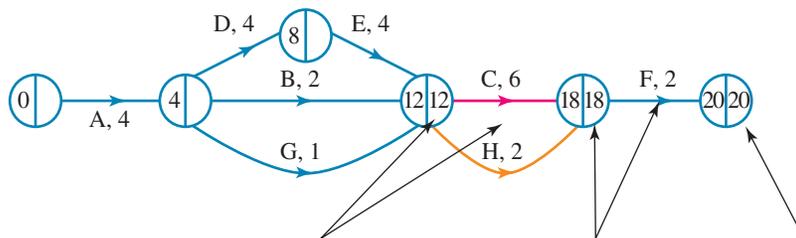
Earliest completion time is 20 minutes.

8. The time in the last vertex indicates the earliest completion time of the morning routine.

10.3.3 Backward scanning and latest start time (LST)

In the previous section, we encountered some activities that were critical to the project timeline, and others that could be delayed. In smaller networks, this can be done by inspection. With more complex projects requiring the coordination of many activities, it is necessary to record more information on the network diagrams and to display the information using charts. A procedure called backward scanning can assist.

In forward scanning, we record the earliest start time for an activity in the left-hand side of each vertex; in backward scanning, we record the latest start time in the right-hand side of each vertex — that is, the latest time that this activity can start without delaying the project.



3. When there are multiple predecessors, the LST is determined by the pathway that takes longest.

Activity H takes 2 minutes, so the LST for the previous activity would be $18 - 2 = 16$.

Activity C takes 6 minutes, so the LST for the previous activity would be $18 - 6 = 12$.

We have to start both activities C and H at the same time, so we must start both at the time that is earliest for all. The LST for C and H has to be 12, so that there is enough time for C to be completed.

C is **critical** to the timeline of the project. H is not critical, so short delays may not affect the timeline.

2. Backward scan through the activity to calculate the latest start time for the previous activity

Activity F takes 2 minutes, so the latest start time for the previous activity is set to 18.

$$\begin{aligned} \text{LST} &= 20 - 2 \\ &= 18 \end{aligned}$$

1. Set the latest start time (LST) for the last activity.

The latest start time for the last activity is equal to the EST.

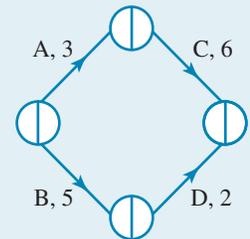
Latest start time

Backward scanning starts at the end vertex and moves backward through the network, subtracting the time of each edge from the earliest start time of each succeeding vertex.

Latest start time (LST) is the latest time for an activity (or activities) to start so that timelines are not delayed.

WORKED EXAMPLE 6 Backward scanning and latest start time

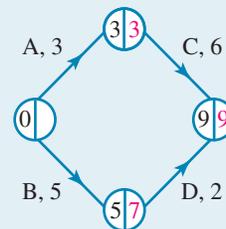
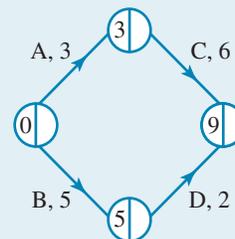
The network diagram has been constructed for a project manager. Use forward and backward scanning to clearly display the EST and LST.



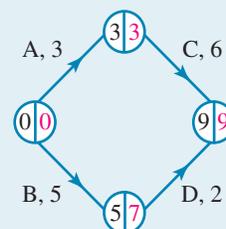
THINK

- Forward scan through the network and record the earliest start time for each activity in the left-hand side of the appropriate vertex. The beginning vertex is between paths A and B.
- Begin backward scanning. Start at the end vertex and trace backwards along all paths from this vertex.
- Subtract the times of the activities along each path from the earliest completion time 55 km and record the value in the right-hand side of the previous vertex. These values are the latest start times for the activities along the path.
- Repeat the process backwards through the diagram. Where two (or more) paths come together at a vertex (activities A and B), record the *smaller* value in the right-hand side of the vertex.

WRITE



Backtracking along path C: $9 - 6 = 3$
 The latest start time for activity C = 3
 Backtracking along path D: $9 - 2 = 7$
 The latest start time for activity D = 7.



Along path A: $3 - 3 = 0$
 Along path B: $7 - 5 = 2$
 Smaller value = 0

10.3 Exercise

10.3 Exam questions on

Simple familiar

1, 2, 3, 4, 5, 6, 7, 8, 9, 10

Complex familiar

11, 12

Complex unfamiliar

13, 14, 15

These questions are even better in jacPLUS!

- Receive immediate feedback
- Access sample responses
- Track results and progress



Find all this and MORE in jacPLUS

Simple familiar

1. WE3 From each of the following activity charts, prepare a network diagram.

a.

Activity	Immediate predecessor	Time (min)
A	—	3
B	—	5
C	A	2
Total time		10

b.

Activity	Immediate predecessor	Time (min)
D	—	2
E	D	1
F	D	3
G	E, F	4
Total time		10

2. WE4 From each of the following activity charts, prepare a network diagram.

a.

Activity	Immediate predecessor	Total time (min)
A	—	3
B	A	2
C	A	4
D	C	1
E	B	6
F	B	2
G	F	3
H	D, E, G	5
I	J, H	4
J	D, E, G	6
Total time		36

b.

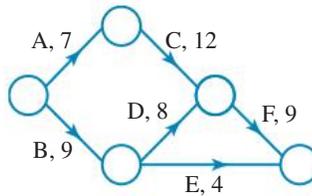
Activity	Immediate predecessor	Total time (min)
N	—	2
O	N	4
P	O, T	3
Q	P	5
R	—	6
S	N	4
T	S, Y	3
U	O, T	2
V	O, T	8
W	V	10
X	Y	2
Y	R	3
Z	U, X	5
Total time		57

3. **WE5** When a laptop computer is being assembled the following processes must be performed.

Activity letter	Activity	Predecessor	Time (min)
A	Install memory board	—	2
B	Test hard drive	A	20
C	Install hard drive	B, E	4
D	Install I/O ports	A	5
E	Install CD-ROM	D	3
F	Test CD-ROM	E	5
G	Install operating system	C, F	10
H	Test assembled computer	G	12
		Total time	61

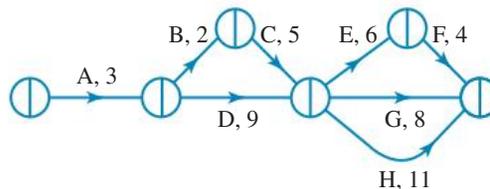
- From the activity chart, prepare a network diagram of the laptop assembly process.
- Determine the earliest completion time for all tasks to be completed.

Refer to the network diagram shown to answer questions 4 and 5. Times shown are in minutes.



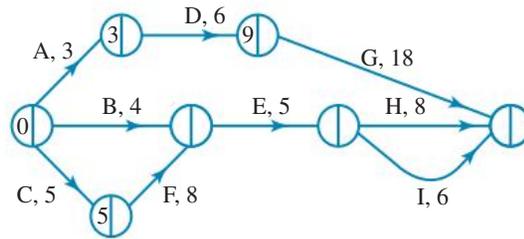
- MC** Determine which of the following statements is true.
 - Activity A is an immediate predecessor of F.
 - Activity D is an immediate predecessor of F.
 - Activity F must be done before activity D.
 - Activity F must be done before activity E.
- MC** The minimum time taken to complete all activities is:
 - 13 minutes
 - 21 minutes
 - 26 minutes
 - 28 minutes

6. **WE6** Refer to the network diagram shown.

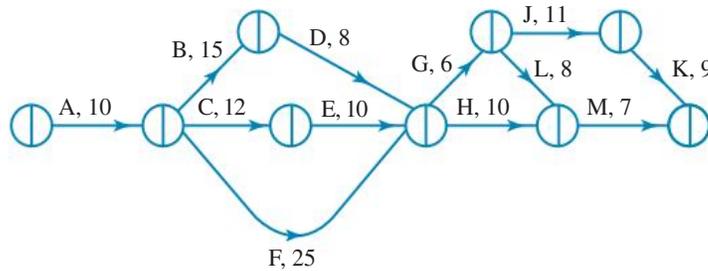


- Use forward and backward scanning to clearly display the earliest start time and latest start time.
- Determine the earliest completion time.
- Identify tasks that may be delayed without increasing the earliest completion time.

Refer to the network diagram shown to answer questions 7 to 9.



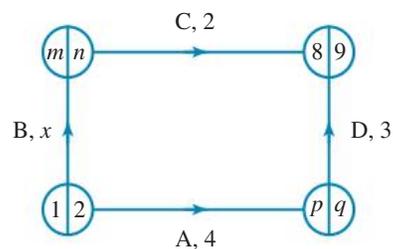
7. **MC** The number required in the left-hand side of the vertex after activities B and F is
A. 4. **B.** 13. **C.** 5. **D.** 8.
8. **MC** The number required in the left-hand side of the vertex after activity E is
A. 5. **B.** 9. **C.** 10. **D.** 18.
9. **MC** The earliest completion time for all tasks is
A. 27. **B.** 24. **C.** 21. **D.** 17.
10. **a.** Refer to the network diagram. Determine the earliest start time for each vertex shown.



- b.** Hence, determine the earliest completion time for the project.

Complex familiar

11. Determine the value of all pronumerals in the following network graph.



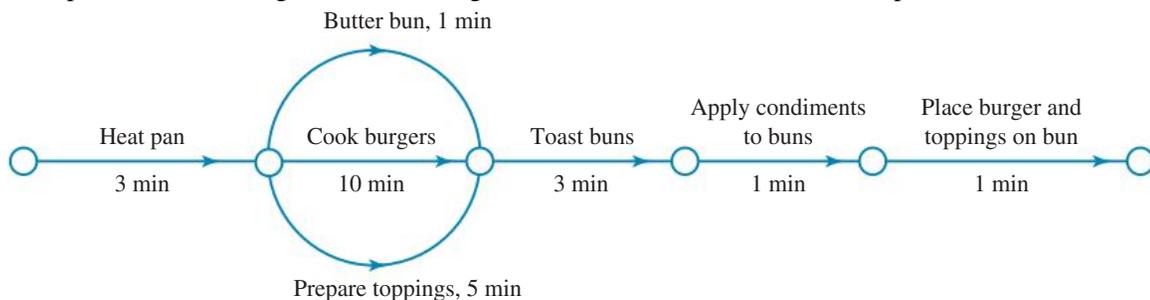
12. Following is a recipe for cheese and broccoli muffins.

Cheese and broccoli muffins	
	<p>Directions</p> <p>A — Chop broccoli (4 min). B — Steam/boil broccoli. While steaming, measure out your ingredients in separate bowls (8 min). <i>Note:</i> Steps C, D and E are being performed simultaneously as the broccoli is steaming. C — Measure cheese (2 min). D — Measure wet ingredients (2 min). E — Measure dry ingredients (2 min). F — Whisk wet ingredients (2 min). G — Remove broccoli from steamer (1 min). H — Combine broccoli, cheese and wet ingredients (3 min). I — Combine these with the dry ingredients (3 min). J — Preheat oven to 180 °C (10 min). K — Prepare muffin tins (2 min). L — Spoon mixture into prepared muffin tins (5 min). M — Place in hot oven for 25 min (25 min). N — Check with skewer in centre after 25 min to make sure the muffins are cooked through (1 min). O — Once cooked, remove from oven and allow to cool for 5 minutes (5 min).</p>
<p>Ingredients</p> <p>1.5 cups of raw broccoli 1.25 cups of cheese 0.75 cup of milk 2 large eggs 0.25 cup of vegetable oil 1.5 cups of self-raising flour Salt and pepper Cooking at 180 °C for 25 min</p>	

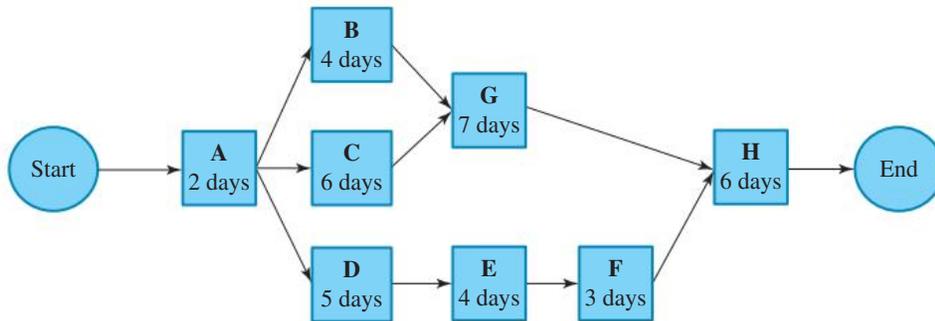
- Determine the minimum completion time for the entire recipe.
- Explain whether there would be a change to the minimum completion time if C, D and E each took 1 minute longer.
- Explain whether there would be a change to the minimum completion time if C, D and E each took 5 minutes longer.
- If step L took 9 minutes, explain what would happen to the project time.

Complex unfamiliar

13. The following network diagram describes the process of preparing a burger. John made an incorrect assumption when drawing the network diagram. Determine the new earliest completion time.



14. When completing a question, a student incorrectly transcribed the activities into the nodes instead of the edges as requested by the teacher. Recreate the network graph with activities as edges and determine the critical activities of this project.



15. You are responsible for overseeing a construction project. The table summarises the activities. The tradesperson responsible for Activity D has requested a week off work. Determine which project days the tradesperson can have off and which project days they must be on site.

Activity	Duration (days)	Predecessors
A	5	—
B	8	A
C	4	A
D	6	B
E	5	B
F	3	C
G	9	D, E
H	6	F
I	4	G, H



Fully worked solutions for this chapter are available online.

LESSON

10.4 Critical Path Analysis (CPA)

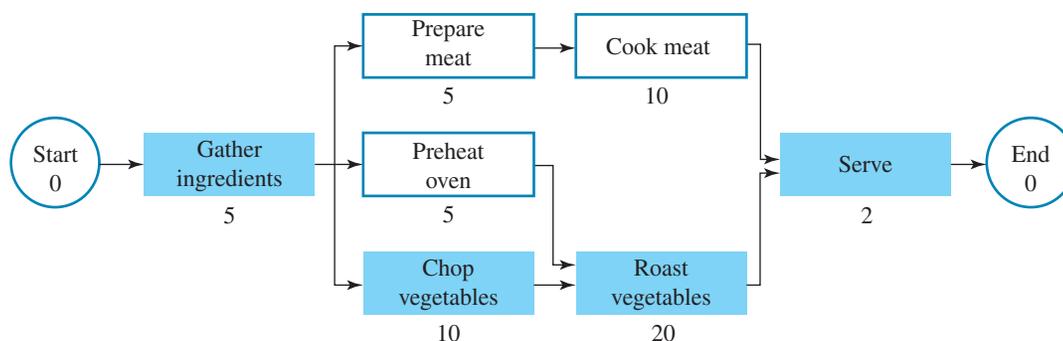
SYLLABUS LINKS

- Use ESTs and LSTs to locate the critical path/s for a project.
- Use the critical path to determine the minimum time for a project to be completed.
- Calculate float times for non-critical activities.
- Solve small-scale practical problems involving critical path analysis.

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10.4.1 Critical path

In the previous section you identified activities that were critical to the timeline of the project and activities that could be delayed. The combination of critical activities from project start to project end is known as the **critical path** of the project.



The network above shows some activities needed for meal preparation. The critical path is indicated in blue. For example, you should chop the vegetables before roasting them, and not the reverse order.

The timeline of the critical path is determined by those activities that have predecessors- where some activities must occur before the next can happen. For example, if it took 15 minutes to chop the vegetables, instead of 10, the entire project is delayed by 5 minutes.

Critical path

The path through the network that follows those activities that cannot be delayed without causing the entire project to be delayed is called the critical path.

Critical path analysis handles human resources and materials; the mathematics of network flow deal with the flow of resources such as water, oil, electricity or information through a given network. Imagine trying to maximise the amount of water that flows through a pipe system, determine the quickest way to shut down an oil leak in a gas field, or increase the amount of data flowing through a communications network. Problems such as these can be solved using the algorithms of network flow.

10.4.2 Float time

Float time determines the exact amount of time delay that can occur with any activity. A float time of zero suggests that the activity is critical to the project timeline. A float time greater than zero suggests the amount of time that an activity can be delayed without disrupting the project timeline. Float time can be calculated by considering the earliest start time from the latest finish time at any node and subtracting the activity length.

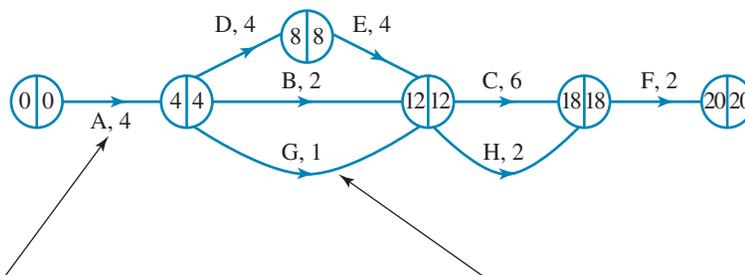
Float time

Float time is the maximum time an activity can be delayed without delaying the entire project.

Float time = LFT – EST – activity time

where the latest finish time (LFT) of an activity = EST of next activity.

Activities on the critical path will have a float time of zero.



Float time of activity A

Float time = LFT – EST – activity time

The latest time you can finish A is 4.
The earliest time you can start A is 0.
The activity length is 4.

$$\begin{aligned} \text{Float time} &= \text{LFT} - \text{EST} - \text{activity time} \\ &= 4 - 0 - 4 \\ &= 0 \end{aligned}$$

There is no float time for Activity A. It must be an activity of the critical path.

Float time of activity G

Float time = LFT – EST – activity time

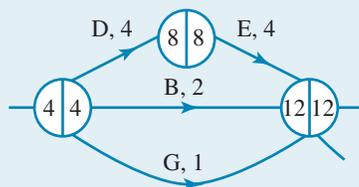
The latest time you can finish G is 12.
The earliest time you can start G is 4.
The activity length is 1.

$$\begin{aligned} \text{Float time} &= \text{LFT} - \text{EST} - \text{activity time} \\ &= 12 - 4 - 1 \\ &= 7 \end{aligned}$$

There is 7 minutes of float time for Activity G. It can be delayed by 7 minutes without disrupting the project timeline.

WORKED EXAMPLE 7 Calculating float time

Work out the float time for activity B.



THINK

1. The latest time you can finish B is 12 The earliest time you can start B is 4 The activity length is 2

WRITE

$$\begin{aligned} \text{Float time} &= \text{LFT} - \text{EST} - \text{activity time} \\ &= 12 - 4 - 2 \\ &= 6 \\ \text{The float time for activity B is 6 minutes.} \end{aligned}$$

WORKED EXAMPLE 8 Determining the critical path

Reggie is a project manager for a construction project. John, his boss, told him that he would receive a bonus if he could complete the construction in less than 9 months.

Phases	Completion times (weeks)
A-B	5
A-C	6
B-D	8
C-E	10
D-F	7
E-F	5
F-G	4
F-H	12
G-H	7

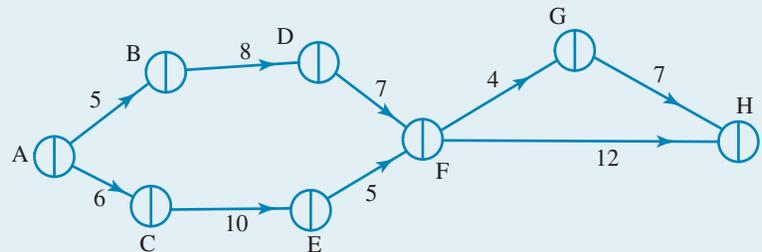
- Create a network diagram of this project.
- Perform forward and backward scans on all activities.
- Determine the critical path and completion time for the project.

THINK

- Draw a directed network with split nodes at each vertex. Include arrows to show the direction of the project.

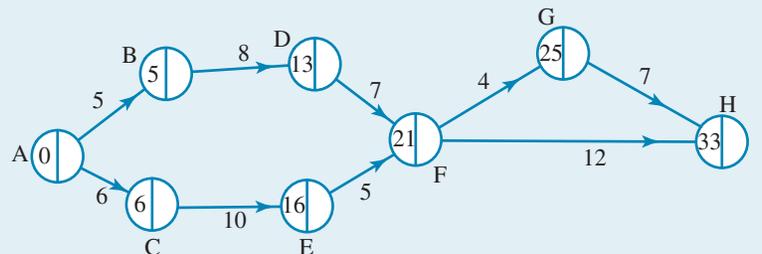
WRITE

a.

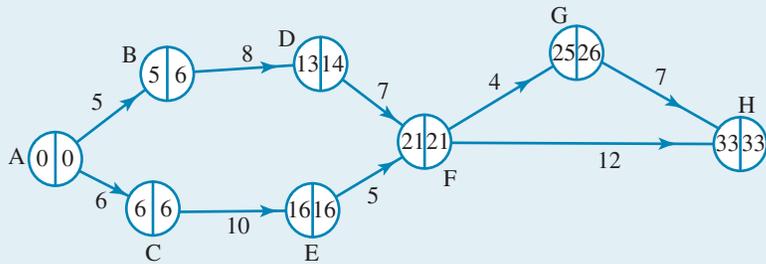


1. Begin forward scanning. Enter the earliest start time on the left-hand side of the node.

b.



2. Begin backward scanning. Enter the latest start time on the right-hand side of the node.



c. Determine the critical path. Hint: a critical path has a float time of zero, so at each node, the EST = LST (the numbers in each vertex are the same)

The earliest completion time is recorded in the last node.

c. The EST = LST at A–C–E–F–H

The critical path is A–C–E–F–H. The earliest time to complete the project is 33 weeks, which is less than 9 months.

Reggie will receive a bonus as long as activities A, C, E, F, H are not delayed by 6 or more weeks.

Exercise 10.4 Critical Path Analysis (CPA)

learn on

10.4 Exercise

10.4 Exam questions **on**

Simple familiar

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12

Complex familiar

13, 14, 15

Complex unfamiliar

16

These questions are even better in jacPLUS!

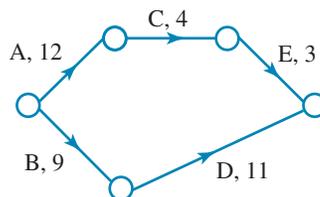
- Receive immediate feedback
- Access sample responses
- Track results and progress



Find all this and MORE in jacPLUS

Simple familiar

For questions 1 and 2, refer to the network diagram shown. Times are in minutes.

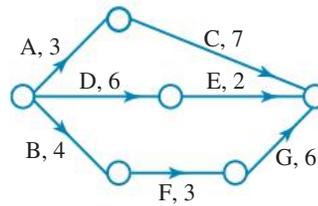


1. **WE7** Use forward and backward scanning to clearly display the critical path and to list any float times.

2. **MC** The earliest completion time for all tasks is:

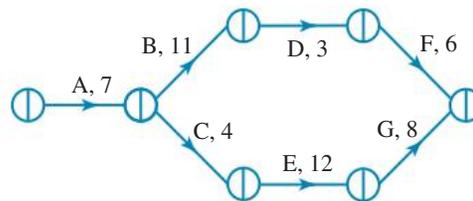
- A. 13 minutes.
- B. 17 minutes.
- C. 19 minutes.
- D. 20 minutes.

For questions 3 and 4, refer to the network diagram shown. Times are in hours.



3. **WE8** For the network diagram shown, use forward and backward scanning to clearly display the critical path and to list any float times for non-critical activities.
4. **MC** The earliest completion time for all tasks is:
A. 19 hours. **B.** 10 hours. **C.** 12 hours. **D.** 13 hours.

For questions 5 to 7, refer to the network diagram shown. Times are in days.

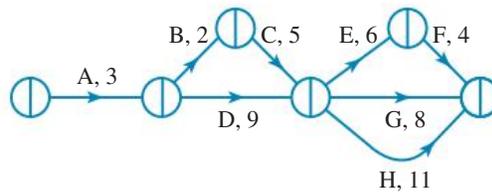


5. Complete the figure at right by forward and backward scanning and hence:
a. determine the earliest completion time
b. indicate the critical path.
6. **MC** The float time for activity D is:
A. 1 day. **B.** 2 days. **C.** 3 days. **D.** 4 days.
7. **MC** The latest start time for activity D is:
A. 18 days. **B.** 21 days. **C.** 22 days. **D.** 25 days.
8. The manufacturing of bicycles can be considered as a 7-step process.
 A — Collect all the parts — 12 minutes
 B — Paint frame — 35 minutes (requires A to be completed first)
 C — Assemble brakes — 16 minutes (requires A to be completed first)
 D — Assemble gears — 20 minutes (requires B to be completed first)
 E — Install brakes — 12 minutes (requires C to be completed first)
 F — Install seat — 5 minutes (requires C to be completed first)
 G — Final assembly — 18 minutes (requires D and E to be completed first)
- a.** Construct an activity chart.
b. Construct a network diagram.
c. Determine the earliest completion time using forward and backward scanning.
d. Determine the critical path.
e. Determine the amount of time saved, as a percentage, using the critical path approach versus completing each task sequentially.

9. **MC** In the bicycle manufacturing system described in question 8, activities with float time are:

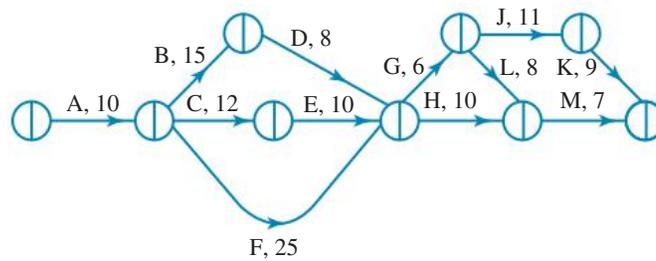
- A. A, B, C, D, E, F, G.
- B. A, B, C, D.
- C. C, E, F.
- D. C only.

10. Refer to the network diagram shown.



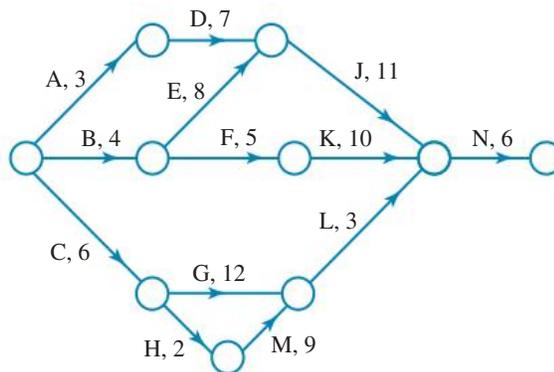
Determine the critical path for the network.

11. Refer to the network diagram shown.



- a. Determine the critical path.
- b. Determine which activities have float time.

12. For the network diagram shown:

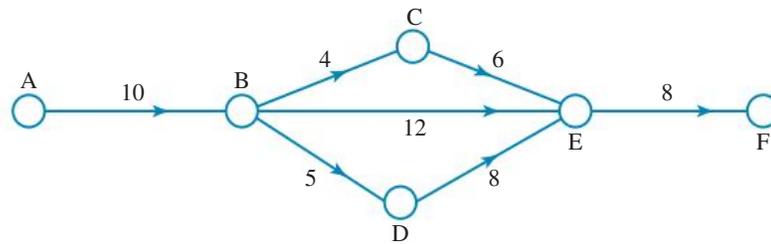


- a. complete a forward scan and hence determine the earliest completion time
- b. complete a backward scan and hence determine the critical path.

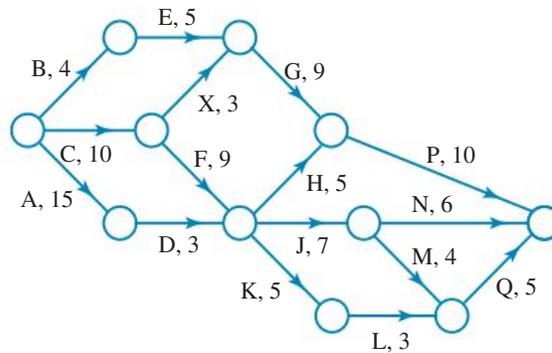
Complex familiar

13. Byron was assigned to create a computer network in his company. The table and network diagram below detail the process and times that Byron configured for each phase of the project. Byron's manager looked at the proposed plan and created amended reductions to the timetable.

Phases	Original completion times (weeks)	Time reduction (weeks)
A–B	10	0
B–C	4	2
B–D	5	1
B–E	12	2
C–E	6	2
D–E	8	1
E–F	8	2



- Determine the original critical path for this project.
 - Determine the original completion time for this project.
 - Determine the new critical path after making necessary reductions.
 - Determine the new completion time for this project after making appropriate reductions.
14. Consider the network diagram shown.

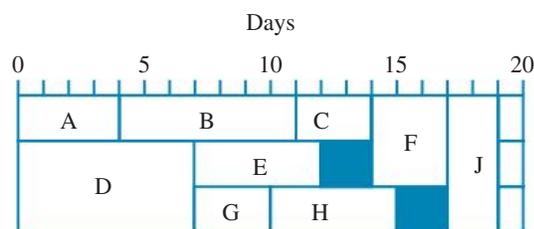


Determine the float time for activity X.

15. A method often used in business to display the critical path is a critical path chart, as shown.

The chart indicates that the activities A–B–C–F–J are the critical path. The chart works as follows. Activities immediately to the left are immediate predecessors. For example, A is the immediate predecessor of B, while D is the immediate predecessor of E and G.

The length of activity is read off the scale (days) at the top. For example, activity C is 3 days long.

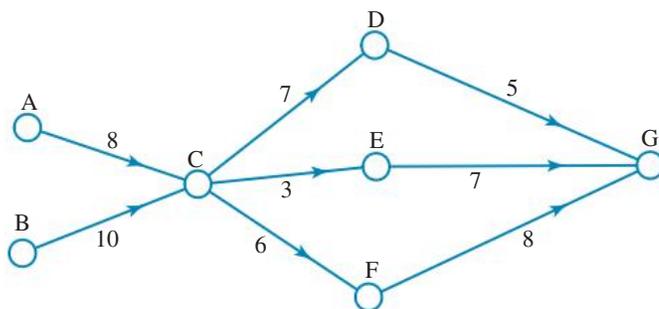


- Construct a network diagram.
- Determine the float times for each non-critical activity. (*Hint:* This can be determined directly from the critical path chart.)

Complex unfamiliar

16. The construction schedule for a pier in the seaside town of Bendalong is shown below. The network diagram corresponding to the construction table is also provided.

Phase	Time (weeks)	Cost (\$)
A–C	8	4 000
B–C	10	8 000
C–D	7	6 000
C–E	3	1 500
C–F	6	6 000
D–G	5	10 000
E–G	7	5 000
F–G	8	4 000



The construction company can offer some time savings to certain activities, for a cost.

Phase	Time reduction	Cost (\$)
C–D	1 week reduction	8 000
C–F	1 week reduction	7 000
D–G	1 week reduction	12 500
F–G	1 week reduction	5 000

Evaluate the offer made by the construction company to determine which, if any, time reductions are worth consideration.

Fully worked solutions for this chapter are available online.

LESSON

10.5 Flow networks

SYLLABUS LINKS

- Understand the meaning of source node, sink node, cut, minimum cut and maximum flow.
- Use a flow network diagram to identify a cut.
- Determine the capacity of a cut.
- Solve small-scale practical problems involving flow networks (up to 8 possible cuts), including determining the minimum cut and the maximum flow.

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10.5.1 Flow network diagrams

An application of networks used to analyse the flow of traffic is network flow. These usually involve directed networks where arrows show the direction of flow. An example follows.

A driver heads to work in the city each morning. He lives in an outer suburb and as he travels from his driveway through a few local streets, there is not much traffic. As he joins the road that connects his suburb to the next, the volume of traffic increases. On the four-lane freeway into the city, the flow of traffic becomes immense, with bumper to bumper cars and drivers changing lanes to determine the fastest route.



The driver's commute can be summarised in terms of traffic flow:

- his suburban street usually sees 1 car per minute
- the two-lane road usually sees 5 cars per minute
- the freeway usually sees 20 cars per minute and modelled as a flow network:

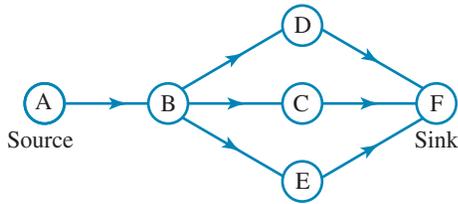


When you also consider all of the different routes that the driver can take to work, the traffic flow network becomes complex. Engineers use mathematical models of network flow to ensure smooth flow of traffic.

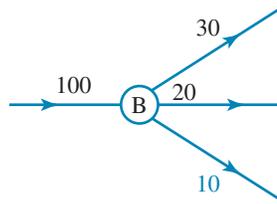
Source and sink

- **Network:** A network consists of nodes (vertices) and edges (links) between those nodes. Each edge has a capacity, representing the maximum amount of “flow” (such as data, water, etc.) that can pass through it.
- **Flow:** Flow represents how much material (or data, traffic, etc.) is being passed through the network from the source to the sink, constrained by the capacities of the edges.
- **The network's starting vertex(es) is called the source node. This is where all flows commence.**
- **The flow goes through the network to the end vertex(es) which is called the sink node.**

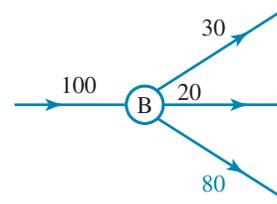
Consider the following figures.



All flow commences at A. It is therefore the source. All flow converges on F indicating it is the sink.



B has an inflow of 100. The flow capacity of the edges leaving B is $30 + 20 + 10 = 60$. The outflow is the minimum of 100 and 60, which is 60.



In this figure, B still has an inflow of 100 but now the capacity of the edges leaving B is $80 + 20 + 30 = 130$. The outflow from B is now 100.

The flow capacity of the network is the total flow possible through the entire network.

WORKED EXAMPLE 9 Create a flow network diagram

Consider the information presented in the following flow table. The water source node is Rockybank Reservoir (R) and the sink node is the wastewater collection point (C).

From	To	Inflow (kilolitres per minute)	Demand
Rockybank Reservoir (R)	Marginal Dam (M)	1000	—
Marginal Dam (M)	Freerange (F)	200	200
Marginal Dam (M)	Waterlogged (W)	200	200
Marginal Dam (M)	Dervishville (D)	300	300

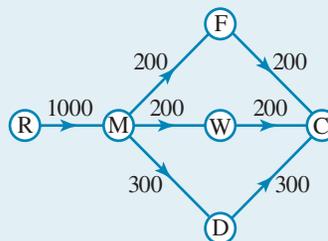
- Convert the information presented in the table into a network diagram, clearly indicating the direction and quantity of the flow.
- Determine the flow capacity of the network.
- Determine whether the flow through the network is sufficient to meet the demand of all the towns.

THINK

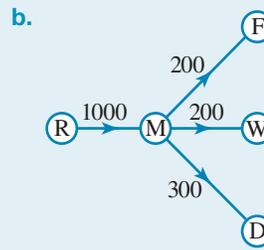
- Construct and label the required number of vertices. The vertices are labelled with the names of the source of the flow and the corresponding quantities are recorded on the edges. Make the final vertex the wastewater collection point (C).

WRITE

a.



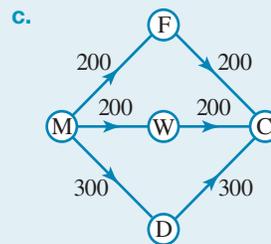
- b. 1. Examine the flow into and out of the Marginal Dam vertex. Record the smaller of the two at the vertex. This is the maximum flow through this point in the network.



Even though it is possible for the reservoir to supply 1000 kL/min (in theory), the maximum flow that the dam can pass on is 700 kL/min (the minimum of the inflow and the sum of the capacities of the edges leaving the dam).

Maximum flow is 700 kL/min.

2. In this case the maximum flow through Marginal Dam is also the maximum flow of the entire network.
- c. 1. Determine that the maximum flow through Marginal Dam meets the total flow demanded by the towns. Add all the values in the demand column of the table to get the demand.



Flow through Marginal Dam = 700 kL/min

Flow demanded = 200 + 300 + 200

= 700 kL/min

2. If the requirements of Step 1 are able to be met, then determine that the flow into each town is equal to the flow demanded by them.

By inspection of the table, all town inflows equal town demands (capacity of edges leaving the town vertices).

Consider what would happen to the system if Rockybank Reservoir continually discharged 1000 kL/min into Marginal Dam while its output remained at 700 kL/min. Such flow networks enable future planning. Future demand may change, the population may grow or a new industry that requires more water may come to one of the towns. Worked example 10 examines such a case.



t1vd-12140

WORKED EXAMPLE 10 Excess flow

Use the information contained in Worked example 9 for this worked example. A new dairy factory, Laitier (L) is to be set up on the outskirts of Dervishville. The factory will require 250 kL/min of water.

- Determine whether the original flow to Dervishville is sufficient.
- If the answer to part a is no, is there sufficient flow capacity into Marginal Dam to allow for a new pipeline to be constructed directly to the factory to meet their demand?
- Determine the maximum flow through the network if the new pipeline was constructed.

THINK

a. 1. Add the demand of the new factory to Dervishville's original flow requirements. If this value exceeds the flow into Dervishville then the new demand cannot be met.

2. The new requirements exceed the flow.

b. 1. Reconstruct the network including a new path for the factory after Marginal Dam.

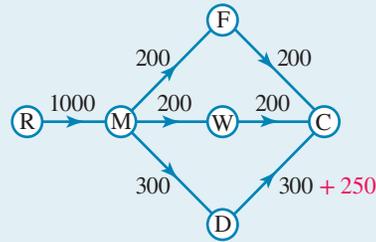
2. Add all outflow paths from M together to calculate the outflow of vertex M.

3. Determine if the flow is sufficient for a new pipeline to be constructed.

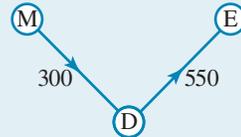
c. This answer can be gained from part b Step 2 above.

WRITE

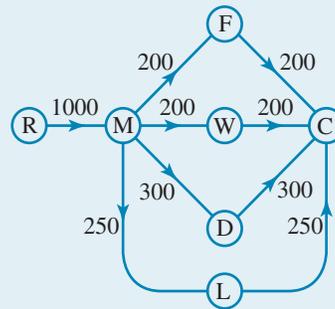
a.



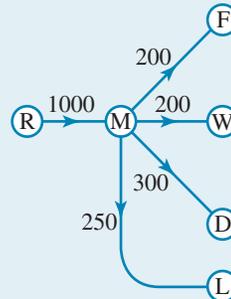
The present network is not capable of meeting the new demands.



b.



Marginal Dam inflow = 1000
 Marginal Dam outflow
 = 200 + 200 + 300 + 250
 = 950



There is an excess flow capacity of 300 into Marginal Dam, which is greater than the 250 demanded by the new factory. The existing flow capacity to Marginal Dam is sufficient.

c. The maximum flow through the new network is 950 kL/min.

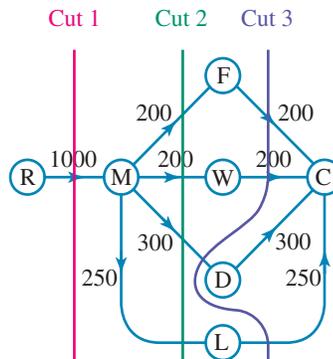
The maximum flow through most simple networks can be determined using these observational methods, but more complex networks require an algorithm or method that guarantees the maximum flow can be achieved every time.

10.5.2 The maximum-flow, minimum-cut theorem

Maximum-flow minimum-cut algorithms are frequently used to program computer networks. These algorithms can also be used to solve problems in electrical engineering, hydraulic design, airline timetable scheduling and many other technical areas. When applying the maximum-flow minimum-cut algorithm, it is necessary to ensure that the network meets the following three requirements:

- The network is weighted and directed; that is, there is an inflow and there is an outflow.
- All edges connected to the source are leaving the source.
- All edges connected to the sink are entering the sink.

Let's consider our town water supply again. If the pipe between the reservoir (R) and Marginal Dam (M) is damaged, the water supply is cut off. Let's call that Cut 1. There are many other cuts that can cut off the water supply to the wastewater collection system. Only three are shown here:



- The capacity of Cut 1 = capacity of pipe cut = 1000 kL/min
- The capacity of Cut 2 = capacity of four pipes cut = $200 + 200 + 300 + 250$
= 950 kL/min
- The capacity of Cut 3 = capacity of four pipes cut = $200 + 200 + 300 + 250$
= 950 kL/min

That Cut 2 and 3 have a total capacity of 950 kL/min confirms what we already knew from Worked example 10. Even though the reservoir has the capacity to supply 1000 kL/min, the pipes that branch off from it only have capacity to hold and supply a maximum of 950 kL/min.

Thus, if you have identified the minimum cut needed to stop the flow, you have also found the pipes/roads/edges that determine the maximum amount that can be supplied throughout the entire network.

Knowing the maximum flow is important for identifying bottlenecks in traffic, busy walkways in a school and town planning infrastructure that needs to be upgraded to cater for population growth.

Maximum flow, minimum cut

A cut is defined by the set of edges that you cross in order to separate the source from the sink.

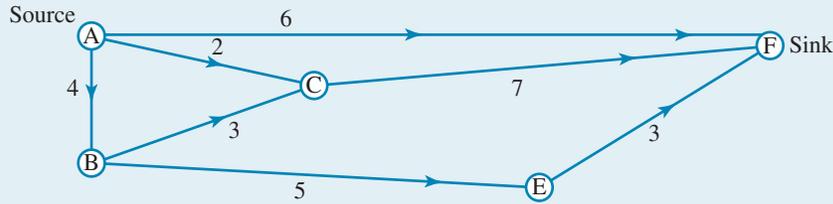
The capacity of a cut is the total capacity of all edges crossed in order to separate the source from the sink. It is the total of the capacities of the edges in the cut — that is, how much flow could pass through if those edges were used to their full capacity.

The minimum cut is the cut with the smallest possible capacity that, if removed, would completely block any flow from reaching the sink from the source. This cut effectively represents the bottleneck of the network. The minimum cut represents the maximum flow through any weighted network.

WORKED EXAMPLE 11 Calculating the capacity of a cut

For the weighted network below, calculate:

- the capacity of each cut set
- the capacity of the minimum cut set and maximum flow through the network.

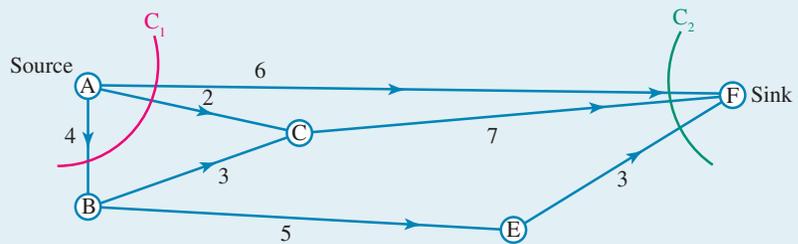


THINK

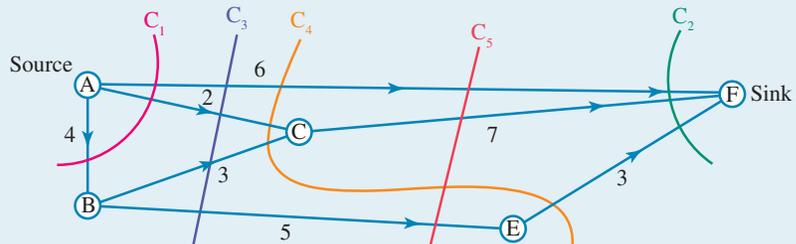
- Identify all possible cuts in the network.

WRITE

a.

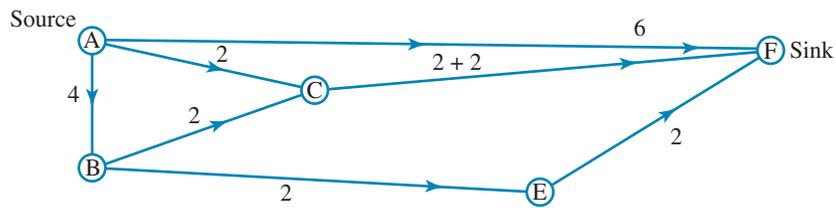


- It is easiest to start with a cut near the source (C_1) and one near the sink (C_2).
- Identify all other cuts C_3, C_4
- The capacity of all cut sets are
 $C_1 = 6 + 2 + 4 = 12$
 $C_2 = 6 + 7 + 3 = 16$
 $C_3 = 6 + 2 + 3 + 5 = 16$
 $C_4 = 6 + 2 + 3 + 3 = 14$
 $C_5 = 6 + 7 + 5 = 18$



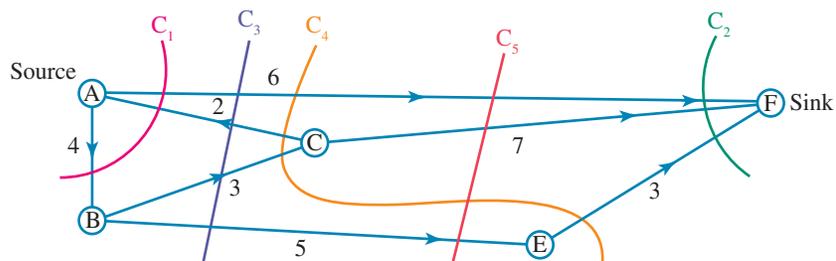
- Cut 1, nearest to the source, suggests that the source can only supply 12 units, despite later capacity of 18 units. Therefore, the maximum flow through this network is 12 units of flow.

Here is one model of those 12 units of flow from Worked example 11 travelling through the network:



The direction of the flow is a factor to determining the cut capacity. If we consider the previous example, with a variation of direction between A and C:

The edge between A and C (with a weighting of 2) receives all its supply from A-B, B-C. If the A-B or B-C supply is cut, then A-C receives 0.



The capacity of all cut sets are now:

$$C_1 = 6 + 0 + 4 = 10$$

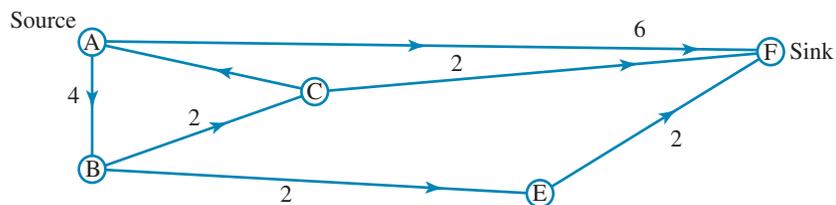
$$C_2 = 6 + 7 + 3 = 16$$

$$C_3 = 6 + 0 + 3 + 5 = 14$$

$$C_4 = 6 + 0 + 3 + 3 = 12$$

$$C_5 = 6 + 7 + 5 = 18$$

Here is one model of those 10 units of flow travelling through the network:



Exercise 10.5 Flow networks

10.5 Exercise

10.5 Exam questions **on**

Simple familiar

1, 2, 3, 4, 5, 6, 7, 8, 9

Complex familiar

10, 11, 12, 13, 14

Complex unfamiliar

15

These questions are even better in jacPLUS!

- Receive immediate feedback
- Access sample responses
- Track results and progress



Find all this and MORE in jacPLUS

Simple familiar

1. **WE9** Consider the information presented in the following flow table.

From	To	Flow capacity
A	B	100
A	C	200
B	C	50
C	D	250
D	E	300

- Convert the information into a network diagram, clearly indicating the direction and quantity of the flow.
- Determine the flow capacity.
- Determine whether the flow through the network is sufficient to meet the demand.

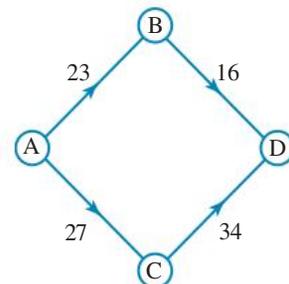
2. Consider the information presented in the following flow table.

From	To	Flow capacity
R	S	250
S	T	200
T	U	100
T	E	100
U	E	50

- Convert the information into a network diagram, clearly indicating the direction and quantity of the flow.
- Determine the flow capacity.
- Determine whether the flow through the network is sufficient to meet the demand.

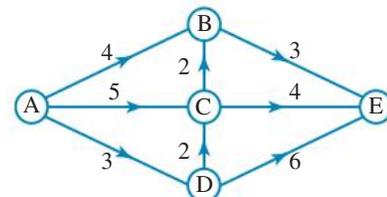
3. **WE10** For vertex B in the network shown, state:

- the inflow at B
- the edge capacities flowing out of B
- the outflow from B.

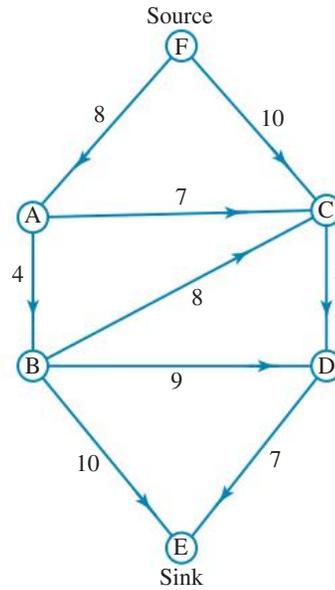


4. For vertex B in the network shown, state:

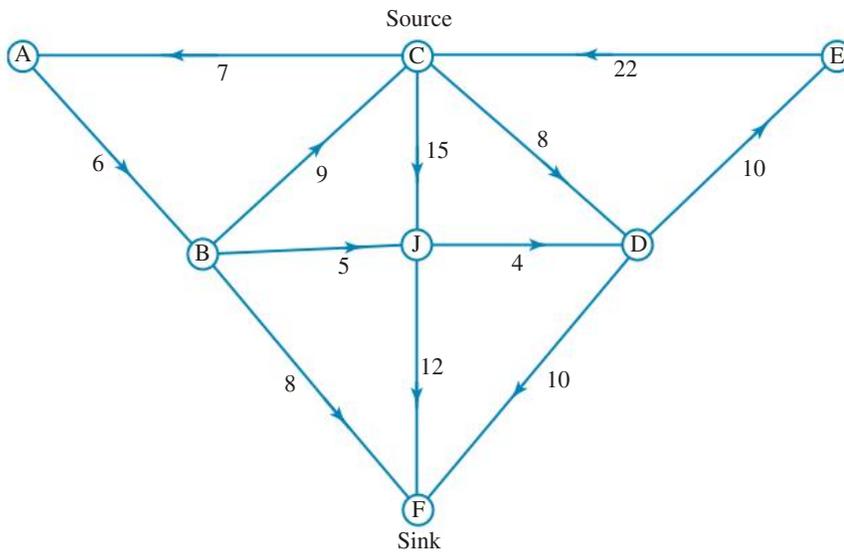
- the inflow at B
- the edge capacities flowing out of B
- the outflow from B.



5. **WE11** Refer to the weighted network shown and answer the following questions.



- a. Identify some cuts (maximum of 8) in this flow network diagram and calculate the capacity of each cut set.
 - b. Calculate the capacity of the minimum cut set and maximum flow through the network.
6. Consider the weighted network shown.



- a. Draw three different cut sets for the network, including the minimum cut set.
- b. Calculate the maximum flow for the network.
- c. Determine whether more than 1 minimum cut set has the same capacity. If so, determine how many there are.

7. Consider the information presented in the following flow table.

- Convert the information into a network diagram, clearly indicating the direction and quantity of the flow.
- Use the minimum cut, maximum flow theorem to determine whether the flow through the network is sufficient to meet the demand.

From	To	Flow capacity
M	N	20
M	Q	20
N	O	15
N	R	5
Q	R	10
O	E	12
R	E	12

8. Consider the information presented in the following flow table.

- Convert the information into a network diagram, clearly indicating the direction and quantity of the flow.
- Use the minimum cut, maximum flow theorem to determine whether the flow through the network is sufficient to meet the demand.
- Determine the flow capacity.

From	To	Flow capacity
D	F	8
D	G	8
G	H	5
G	J	3
F	H	2
F	J	6
J	E	8
H	E	8

9. Consider the tables below.

For each table:

- convert this information into new network diagrams
- calculate the new network flow capacities.

a.

From	To	Flow capacity
A	B	100
A	C	200
B	C	50
C	D	250
D	E	300
B	E	100

b.

From	To	Flow capacity
R	S	250
S	T	200
T	U	100
T	E	100
U	E	50
S	T	100

c.

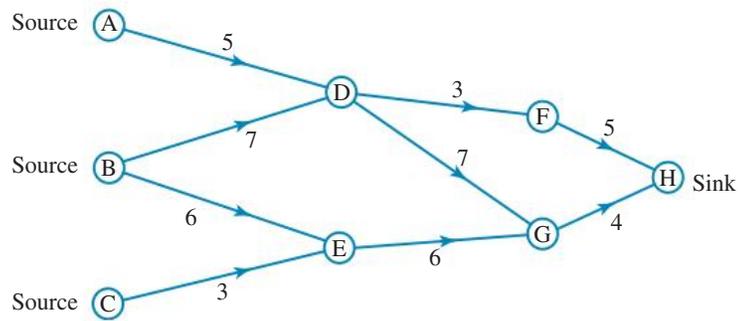
From	To	Flow capacity
M	N	20
M	Q	20
N	O	15
N	R	5
Q	R	10
O	E	12
R	E	12
N	E	5

d.

From	To	Flow capacity
D	F	8
D	G	8
G	H	5
G	J	3
F	H	2
F	J	6
J	E	8
H	E	8
D	E	10

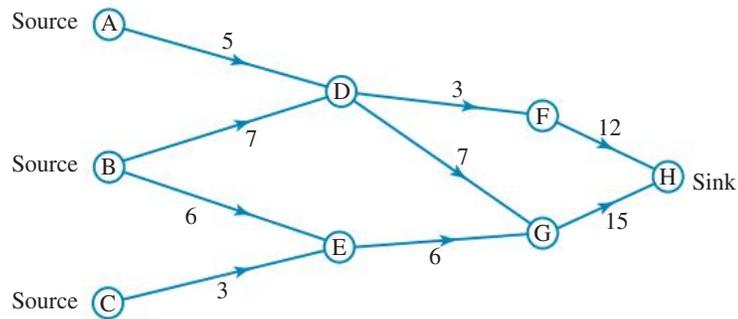
Complex familiar

10. Consider the network shown.



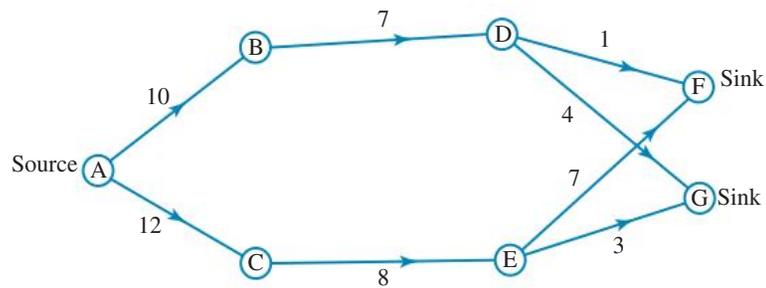
Determine the maximum flow in this network.

11. Consider the network shown.



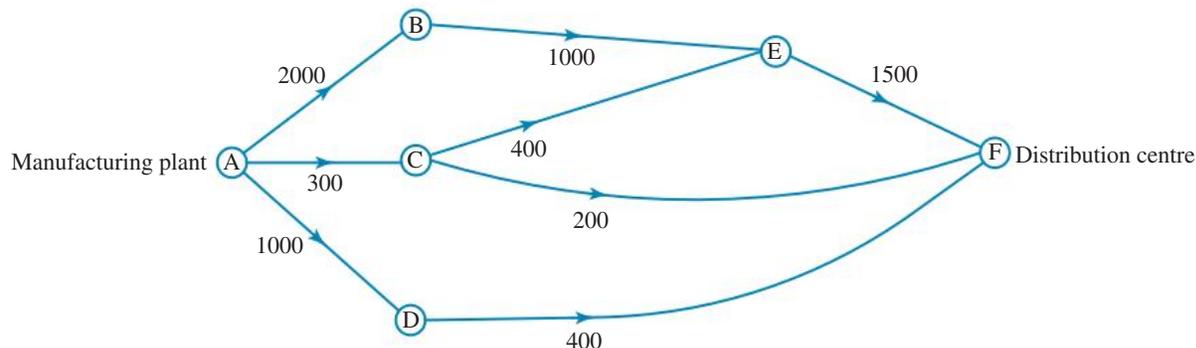
If there are an infinite number of sources in a directed network, can you apply the maximum-flow minimum-cut theorem? Explain.

12. The network shown has one source and two sinks.

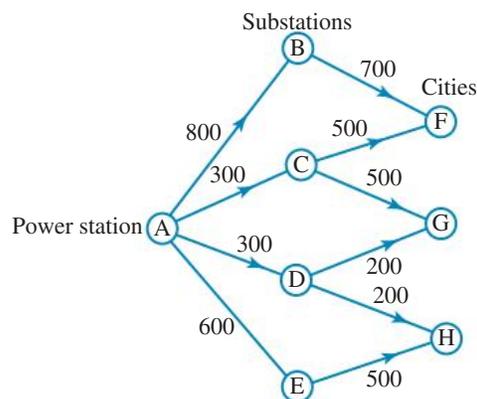


If there are an infinite number of sinks in a directed network, Explain if you apply the maximum-flow minimum-cut theorem.

13. A car manufacturing parts company (Rac Inc.) has to deliver various numbers of parts to their distribution centres around Australia and 1000 parts to their final distribution location in Europe, as shown in the network diagram.



- Determine whether it will be possible to achieve the goal according to the distribution network.
 - Calculate the number of parts that will be delivered to Europe.
 - If you have a surplus in part **b**, determine how you can reduce the number of parts to 1000. If you have a shortage in part **b**, determine how you can increase the number of parts to 1000. (If necessary, draw a new network diagram to show the change(s) required to meet the 1000-part limit.)
14. The network shown represents the electricity usage of three cities in megawatt-hours (MWh). The power station that supplies electricity to these cities can produce 2000 MWh.
- Determine the maximum number of hours the three cities are using.
 - If you can only change the distribution of electricity to the substations, explain how you can ensure that the three cities will receive a total of 2000 MWh. Draw a picture of your new network. (*Hint*: You can use any numbers for AB, AC, AD or AE to achieve your goal.)



Complex unfamiliar

15. The city of Solis is upgrading its electrical grid. Power is generated at a central power plant (P) and needs to be distributed to a high-demand industrial district (D). The power grid connects several substations, and each transmission line between substations has a maximum power limit, measured in megawatts (MW). This limit represents the maximum amount of power that can safely pass through each line. The following information describes the power grid:
- $P \rightarrow A$ has a limit of 40 MW.
 - $P \rightarrow B$ has a limit of 60 MW.
 - $A \rightarrow B$ has a limit of 20 MW.
 - $A \rightarrow C$ has a limit of 30 MW.
 - $B \rightarrow C$ has a limit of 30 MW.
 - $B \rightarrow D$ has a limit of 50 MW.
 - $C \rightarrow D$ has a limit of 40 MW.

Transmission line $A \rightarrow B$ was damaged during a recent storm and is no longer operational. The mayor claims that the industrial district (D) will have no alteration to power supply. Evaluate the reasonableness of the mayor's claim.

Fully worked solutions for this chapter are available online.

LESSON

10.6 Assigning order and the Hungarian algorithm

SYLLABUS LINKS

- Use a bipartite graph and its tabular or matrix form to represent possible assignments for an allocation problem.
- Determine the optimum (minimum and maximum) assignment/s for small-scale practical problems by inspection.
- Use the Hungarian algorithm (3×3 up to 5×5 square matrices) to determine the optimum (minimum and maximum) assignment/s for larger practical problems.

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10.6.1 Allocating and assigning tasks

Network flow problems involve, for example, the analysis of traffic flow or water flow in a particular direction. In many situations, it is important to analyse the flow through each path in the network and ensure that each edge is used to capacity. However, network flow problems also present in the form of allocation problems.

For example, a transport company has three large trucks (source) which deliver to four supermarkets (sinks), so that each supermarket gets the necessary supplies, regardless of the truck used. Each truck has to be allocated a supermarket to ensure the most cost or time effective means of delivering the goods.



WORKED EXAMPLE 12 Allocating tasks by inspection

A small business owner needs to hire three contractors for three different tasks to completed. Each contractor has provided a quote as follows (in tens of thousands of dollars).

	Task 1	Task 2	Task 3
A	40	70	30
B	20	80	35
C	35	65	40

- Use inspection to assign the tasks to the contractors in the most cost-efficient method.
- Use inspection to assign the tasks to the contractors in the least cost-efficient method.
- Determine the savings made.

THINK

- a. Determine the best value price for each task and note the contractor.

Task 1 can be completed for \$20 000 by contractor B

Task 2 can be completed for \$65 000 by contractor C

Task 3 can be completed for \$30 000 by contractor A.

- b. Determine the highest value price for each task and note the contractor.

Task 1 can be completed for \$40 000 by contractor A

Task 2 can be completed for \$80 000 by contractor B

Task 3 can be completed for \$40 000 by contractor C.

- c. By assigning the tasks in the most cost-efficient manner, the small business owner has saved

WRITE

	Task 1	Task 2	Task 3
A	40	70	30
B	20	80	35
C	35	65	40

All tasks can be assigned for a total cost of \$115 000.

	Task 1	Task 2	Task 3
A	40	70	30
B	20	80	35
C	35	65	40

All tasks can be assigned for a total cost of \$160 000.

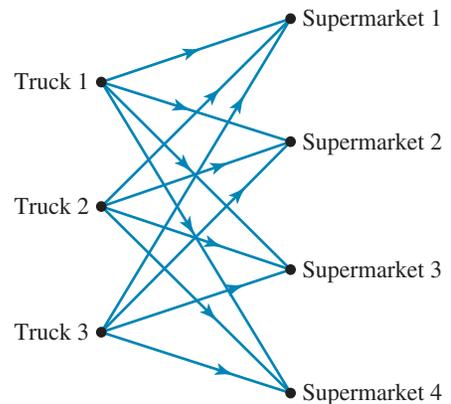
$$\$1\ 600\ 000 - \$115\ 000 = \$45\ 000.$$

For allocations that cannot be performed by inspection, bipartite graphs can be useful.

10.6.2 Bipartite graphs

In the supermarket truck scenario, the trucks can be considered to be the supply vertices and the supermarket the demand vertices. A bipartite graph can be split into two groups of vertices; supply and demand, where no two vertices in the same group share an edge.

Consider the bipartite graph shown which illustrates the situation mentioned above. In this case, each truck can deliver to all four supermarkets. There is no weighting assigned. If we factor in the distances each truck may need to travel, we can assign a weighting and determine efficiency.



Bipartite graph

- A bipartite graph is a graph whose set of vertices can be divided into two distinct subsets and where the vertices of one subset are connected to the vertices of the other. They are useful in assigning tasks between the two subsets of a weighted graph.

We will first look at how a bipartite graph can assist in an assignment problem where there are no weightings involved.

WORKED EXAMPLE 13 Bipartite graphs and possible assignments

The following table lists four customers and their preferred menu items offered at a pop-up bakery.

Customer	Menu items
Jackie	Croissant, pizza roll
Billy	Pizza roll, donut
Christine	Croissant, pizza roll, sausage roll
Wally	Sausage roll

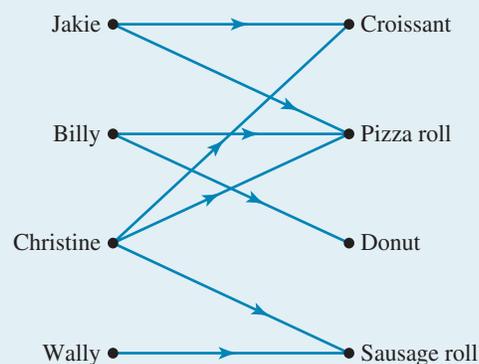
- Represent this information as a bipartite graph.
- Determine whether the following statements are true or false.
 - Jackie purchases more items than Christine.
 - Between Christine and Billy, all menu items are chosen.
 - Between Jackie and Billy, more menu items are chosen than by Christine.
- If each student was to receive one bakery item only, determine who should get what, so that all students are happy with their allocated bakery item.

THINK

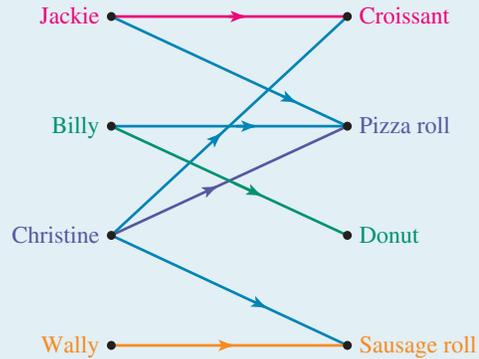
- List the students down the left-hand side and the menu items down the right-hand side. Link the students with the menu items according to the table.
- Determine the truth of statement 1.
 - Determine the truth of statement 2.
 - Determine the truth of statement 3.
- Start with the person/food with the least number of connections.
Wally only likes sausage rolls.
In allocating the sausage roll to Wally, Christine no longer has access to that food item.

WRITE

- Jackie purchases only 2 items, Christine purchases 3, so the statement is false.
Christine purchases a croissant, pizza roll and sausage roll. Billy purchases a pizza roll and a donut. Their choices include all 4 menu items. The statement is true.
Christine chooses 3 of the menu items and between Jackie and Billy 3 menu items are chosen.
The statement is false.



2. The donut only has one connection, to Billy.
3. Both Jackie and Christine both enjoy Croissants and Pizza Rolls, so either can be allocated those items.
The final allocation is:
Jackie – Croissant (or Pizza Roll)
Billy – Donut
Christine – Pizza roll (or croissant)
Wally – Sausage roll



10.6.3 Optimum assignments of weighted tasks

In an office in the city there are four employees and four tasks that need to be completed. Each person can do the task in a different amount of time. What is the best way for their manager to allocate these tasks, one per person, so that the time can be minimised? This is known as optimal allocation.

In the general case the jobs (in hours) and people can be put in an allocation matrix similar to the one below.

	Task 1	Task 2	Task 3	Task 4
Employee 1	10	4	9	6
Employee 2	8	11	10	7
Employee 3	6	8	7	9
Employee 4	8	5	3	9

For example, Employee 1 would take 10 hours to complete Task 1, 4 hours to complete Task 2, 9 hours to complete Task 3 and 6 hours to complete Task 4 and so on.

Optimum assignment of weighted tasks

If an assignment allocation cannot be performed by inspection, the following steps can be taken:

1. Row reduction
2. Column reduction
3. Further matrix adjustment

Steps 1 and 2 are interchangeable. Some problems require only Step 1, whereas others require you to progress through all three steps.

Worked example 14 demonstrates the process of using only a row reduction to determine the optimal allocation.

WORKED EXAMPLE 14 Reducing rows for optimal allocation

In an office in the city there are four employees and four tasks that need to be completed. The times taken by each employee to do the four jobs are given in the following table. Determine the optimal allocation and hence state the minimum time.

	Task A	Task B	Task C	Task D
Employee 1	10	4	9	6
Employee 2	8	11	10	7
Employee 3	6	8	7	9
Employee 4	8	5	3	9

THINK

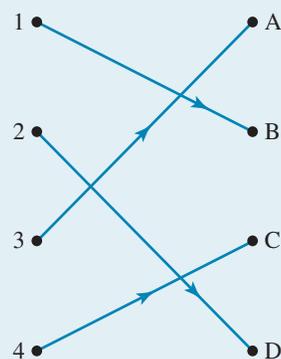
- Determine the best value time for each task and note the employee.
Employee 1 is matched to two tasks, which is not permitted. This allocation cannot be done by inspection and requires a row or column reduction.
- Row reduction.
Identify the smallest value in each row.
This is each employee's most efficient task.
Subtract that number from each other value in the row.
Note: there is one zero per row, and one zero per task. This tells us that a bipartite graph drawn from this information can allocate the four tasks.

WRITE

Task A, employee 3, 6 hours
Task B, employee 1, 4 hours
Task C, employee 4, 3 hours
Task D, employee 1, 6 hours.

	A	B	C	D	
1	10	4	9	6	= 4
2	8	11	10	7	= 7
3	6	8	7	9	= 6
4	8	5	3	9	= 3

	A	B	C	D
1	6	0	5	2
2	1	4	3	0
3	0	2	1	3
4	5	2	0	6



There is only one possible allocation.

- 1 → B
- 2 → D
- 3 → A
- 4 → C

4. To determine the time required for this optimum allocation, refer to the original matrix.
The minimum total for this allocation is 20 hours, assuming the tasks are successive (that Task A must be done before Task B, etc.).

	A	B	C	D
1	10	4	9	6
2	8	11	10	7
3	6	8	7	9
4	8	5	3	9

Task A, employee 3, 6 hours.
Task B, employee 1, 4 hours.
Task C, employee 4, 3 hours.
Task D, employee 2, 7 hours.

10.6.4 The Hungarian algorithm

Sometimes, even after completing the row and column reductions, tasks still can't be assigned. If that happens, you'll need to make additional changes to the matrix. The complete process is known as the **Hungarian algorithm**.

The Hungarian algorithm

If an assignment allocation cannot be performed by inspection or a row reduction, the following steps can be taken:

1. Perform a row reduction to achieve weightings of 0.
Determine the minimum number of horizontal and vertical lines needed to cover all the zeros.
 - a. If the number of lines = number of tasks, then you can draw a bipartite graph and allocate all tasks.
 - b. If the number of lines \neq number of tasks, then you are not yet able to allocate, and need additional steps.
2. Perform a column reduction to achieve weightings of 0.
Determine the minimum number of horizontal and vertical lines needed to cover all the zeros.
 - a. If the number of lines = number of tasks, then you can draw a bipartite graph and allocate all tasks.
 - b. If the number of lines \neq number of tasks, then you are not yet able to allocate, and need additional steps.
3. Perform additional matrix adjustment to the uncovered values.
This is done by determining the smallest number not covered by your lines.
 - a. Subtract that number from any matrix element not covered by a line.
 - b. Add that number to any matrix element covered by two lines. Determine the minimum number of lines needed to cover all the zeros.
 - c. If the number of lines = number of tasks, then you can draw a bipartite graph and allocate all tasks.
 - d. If the number of lines \neq number of tasks, then you are not yet able to allocate, you need to perform Step 3 again.

Steps 1 and 2 are interchangeable, and where some problems require only Step 1, others require you to progress through all three steps.



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WORKED EXAMPLE 15 Applying the Hungarian algorithm for optimal allocation

Four workers need to be allocated to four tasks. The time required for each worker for each task is summarised in the table below.

Use the Hungarian algorithm to minimise the time required to complete the tasks by allocating one job to each worker.

	Task 1	Task 2	Task 3	Task 4
Worker 1	80	81	67	90
Worker 2	75	35	47	90
Worker 3	9	67	3	84
Worker 4	6	7	96	21

THINK

1. Set up the matrix of employees against tasks.

WRITE

	T1	T2	T3	T4
W1	80	81	67	90
W2	75	35	47	90
W3	9	67	3	84
W4	6	7	96	21

2. Perform row reduction by locating the smallest value in each row and subtracting it from all numbers in that row.

The smallest number in Row 1 is 67.

The smallest number in Row 2 is 35.

The smallest number in Row 3 is 3.

The smallest number in Row 4 is 6.

Only 3 lines are required to cover the zeros.

This means that a bipartite graph drawn at this step will only allocate 3 tasks — e.g. Task 3 will be allocated to two workers.

An additional step is needed. (there are other line combinations, all of which are acceptable, all of which still only require 3 lines)

	T1	T2	T3	T4
W1	13	14	0	23
W2	40	0	12	55
W3	6	64	0	81
W4	0	1	90	15

3. Perform a column reduction by subtracting the smallest number in each column from all the numbers in the column.

The smallest number in Column 1 is 0.

The smallest number in Column 2 is 0.

The smallest number in Column 3 is 0.

The smallest number in Column 4 is 15.

	T1	T2	T3	T4
W1	13	14	0	8
W2	40	0	12	40
W3	6	64	0	66
W4	0	1	90	0

4. Again, only 3 lines are required to cover the zeros, So continue to the next step.
(there are other line combinations, all of which are acceptable, all of which still only require 3 lines)

5. The smallest uncovered number is 6. Subtract 6 from all uncovered elements and add it to all elements that are covered twice.

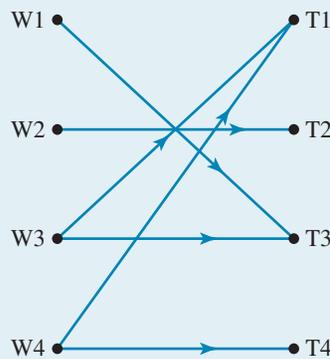
The smallest uncovered number is 6.

	T1	T2	T3	T4
W1	7	8	0	2
W2	40	0	18	40
W3	0	58	0	60
W4	0	1	96	0

6. Cover all zeroes with a minimum number of lines.
Now that there are four lines required to cover all the zeros, we know that the bipartite graph can create four allocations.

	T1	T2	T3	T4
W1	7	8	0	2
W2	40	0	18	40
W3	0	58	0	60
W4	0	1	96	0

7. Draw a bipartite graph, where the zeroes connect the employee to the tasks.



8. Starting at those workers and/or tasks with only 1 connection, we assign the tasks.

Worker 1 is allocated Task 3.
Worker 2 is allocated Task 2.
Worker 3 is allocated Task 1 (Task 3 is already allocated)
Worker 4 is allocated Task 4 (Task 1 is already allocated)

9. Refer to the original matrix to determine the minimum time for this allocation.

	Task 1	Task 2	Task 3	Task 4
Worker 1	80	81	67	90
Worker 2	75	35	47	90
Worker 3	9	67	3	84
Worker 4	6	7	96	21

$$67 + 35 + 9 + 21 = 132 \text{ hours.}$$

In Worked Example 15, the objective was to minimize the time. However, some problems require finding a maximum allocation instead. In these cases, all elements in the matrix are first subtracted from the largest element. Other large numbers are thus presented as having a small difference (possibly even a zero difference). After that, the process follows the same steps as in Worked Example 15.

10.6 Exercise

10.6 Exam questions **on**

Simple familiar

1, 2, 3, 4, 5, 6, 7

Complex familiar

8, 9, 10, 11, 12, 13

Complex unfamiliar

14, 15

These questions are even better in jacPLUS!

- Receive immediate feedback
- Access sample responses
- Track results and progress



Find all this and MORE in jacPLUS

Simple familiar

1. **WE12** The following employee task matrices represent times (in hours). Attempt an optimal allocation for the minimum time by inspection. State the minimum time.

a. $T_1 \quad T_2 \quad T_3$

$$E_1 \begin{bmatrix} 5 & 2 & 6 \end{bmatrix}$$

$$E_2 \begin{bmatrix} 1 & 3 & 4 \end{bmatrix}$$

$$E_3 \begin{bmatrix} 2 & 4 & 1 \end{bmatrix}$$

b. $T_1 \quad T_2 \quad T_3$

$$E_1 \begin{bmatrix} 8 & 4 & 8 \end{bmatrix}$$

$$E_2 \begin{bmatrix} 4 & 5 & 8 \end{bmatrix}$$

$$E_3 \begin{bmatrix} 7 & 6 & 6 \end{bmatrix}$$

2. **WE13** Five customers (Will, Penny, Fan, Roya and Su Yi) go to the local fish and chip shop for dinner and place the orders as shown in the table.

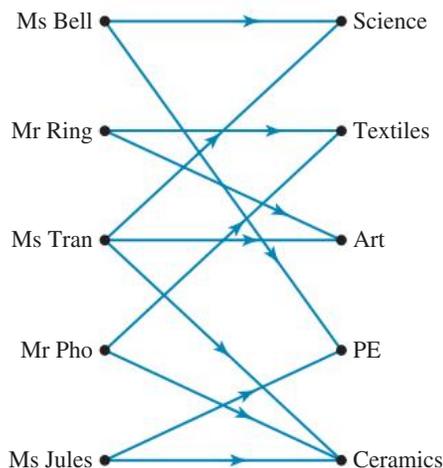
Diner	Dishes
Will	Fish, chips
Penny	Chips
Fan	Fish, potato cakes, dim sims
Roya	Potato cakes, dim sims
Su Yi	Chips, dim sims

- a. Represent this information as a bipartite graph
- b. Determine whether the following statements are true or false.
1. Will and Penny between them have more different items than Fan and Roya.
 2. Fan and Roya together have tried all the options.
 3. Roya and Su Yi between them have more variety than Penny and Fan.
 4. Penny and Fan between them have more variety than Roya and Su Yi.
- c. Determine whether it is possible to allocate the items such that each customer receives only one item from the menu.

3. Four visitors on a tour to Europe have a choice of five countries to visit. The countries are France, Germany, Italy, Spain and Ireland, they can visit as many of these countries as they wish. Sally decides to visit Italy and Germany, but not the others. Joe decides to spend all his time in France. Mike wants to see Germany, Spain and Ireland. Genevieve is keen to visit all of the countries on this trip.



- Explain why a bipartite graph is suitable to represent this information.
 - Draw a bipartite graph to represent the information.
 - Determine the degree of the vertex representing Mike.
4. Five teachers can teach a variety of five subjects as indicated by the bipartite graph.



- State whether the following statements are true or false.
- Ms Bell and Ms Tran can teach all five subjects between them.
 - Mr Ring and Ms Tran, in total, can teach more subjects than Ms Bell and Ms Jules.
 - Ms Bell and Ms Tran each teach the same number of subjects.
 - Ms Bell and Mr Pho, in total teach fewer subjects than Mr Ring and Ms Tran.
 - Ms Jules teaches fewer subjects than all other teachers.
5. **WE14** The matrix shown represents times in hours.

$$\begin{bmatrix} 5 & 4 & 8 & 4 \\ 10 & 5 & 7 & 6 \\ 6 & 7 & 8 & 9 \\ 5 & 9 & 4 & 6 \end{bmatrix}$$

Perform a row reduction on the matrix and attempt an optimal allocation for the minimum time. State the minimum time.

6. A shipping company has four ships that deliver fuel to four different oil rigs lying off shore. The times taken by each ship to do the four deliveries are given in the following table. Determine the optimal allocation and hence state the minimum time required to complete all four deliveries.

	Rig A	Rig B	Rig C	Rig D
Ship 1	16	14	20	13
Ship 2	15	16	17	16
Ship 3	19	13	13	18
Ship 4	22	26	20	24

7. **MC** Consider the following matrix.

$$\begin{bmatrix} 8 & 4 & 8 \\ 4 & 4 & 6 \\ 7 & 6 & 6 \end{bmatrix}$$

The total value of the optimal allocation is

- A. 9. B. 11. C. 14. D. 16.

Complex familiar

8. **WE15** Four workers need to be allocated to four tasks. The time required for each worker for each task is summarised in the table below. Use the Hungarian algorithm to minimise the time required to complete the tasks by allocating one job to each worker.

	Task 1	Task 2	Task 3	Task 4
Worker 1	7	10	10	5
Worker 2	11	10	10	8
Worker 3	5	10	7	4
Worker 4	6	9	9	7

9. Four delivery vans need to deliver to four different supermarkets. The distance of the four drivers from each of the four supermarkets are given in the table below. If the drivers take their loaded vans home in the evening before the delivery day, use the optimal allocation method to minimise the total distance travelled by the vans to reach all four supermarkets.

	S1	S2	S3	S4
Driver 1	7	25	21	6
Driver 2	13	31	8	16
Driver 3	23	19	16	15
Driver 4	22	29	24	10

10. A florist wishes to purchase peonies, roses and lilies for three bouquets from three different flower wholesalers. The peonies cost \$40, \$55 and \$60 from the three stores, the roses cost \$65, \$60 and \$70 and the lilies cost \$45, \$50 and \$40. Determine the optimal allocation for the flower order.



11. Perform an optimal allocation on the following matrices. by conducting in order each of the following until the allocation is complete. State the minimum value.

- i. Row reduction
- ii. Column reduction
- iii. The Hungarian algorithm

a. $\begin{bmatrix} 20 & 18 & 13 \\ 21 & 16 & 18 \\ 25 & 28 & 26 \end{bmatrix}$

b. $\begin{bmatrix} 20 & 30 & 40 & 50 \\ 60 & 20 & 30 & 60 \\ 50 & 40 & 50 & 20 \\ 10 & 70 & 40 & 60 \end{bmatrix}$

12. A large holiday park has four maintenance workers and four tasks that need one person to complete each morning.
The time it takes each of the workers to complete the four tasks is summarised in the following table.

	Task 1	Task 2	Task 3	Task 4
Worker 1	50	60	70	80
Worker 2	90	50	60	90
Worker 3	80	70	80	30
Worker 4	30	100	70	90

- a. Determine the optimal allocations and display these allocations using a bipartite graph.
b. Determine the total time required to complete all four tasks.
13. Four swimmers are to compete in a medley relay. During training each swimmer has swum each leg of the relay and their times have been recorded in the following table

	Leg 1	Leg 2	Leg 3	Leg 4
Mary	2	4	3	5
Jenny	3	5	3	4
Pauline	2	3	4	2
Jacinta	2	4	2	3

Determine the optimal allocation and the minimum time to complete the race.

Complex unfamiliar

14. The town of Harmony is organising an art gallery exhibition featuring five talented artists. Each artist has submitted a list of preferred locations within the gallery to display their artwork, along with the commission percentage that they will charge in each location. The gallery has five different display spaces, and each artist can only be assigned to one space, while each space can only host one artist.
The following table shows the commission rate for each artist to set up their artwork in the different display spaces:

	Display space 1	Display space 2	Display space 3	Display space 4	Display space 5
Artist A	12%	14%	13%	11%	15%
Artist B	10%	20%	15%	17%	18%
Artist C	13%	11%	14%	16%	19%
Artist D	14%	13%	12%	15%	17%
Artist E	15%	16%	17%	13%	11%

Artist C decides to withdraw from the exhibition and claims that the other artists are not financially disadvantaged by this. Evaluate the reasonableness of this claim.

15. If the optimal allocation for this matrix totals 32 hours, determine the value of x .

$$\begin{array}{c}
 \text{D} \qquad \text{E} \qquad \text{F} \\
 \left[\begin{array}{l}
 \text{A} \left[\begin{array}{l} x + 6 \text{ hours} \\ x + 3 \text{ hours} \\ x \text{ hours} \end{array} \right. \begin{array}{l} 2x + 3 \text{ hours} \\ 2x + 4 \text{ hours} \\ 2x + 1 \text{ hours} \end{array} \left. \begin{array}{l} x + 7 \text{ hours} \\ x + 5 \text{ hours} \\ x + 3 \text{ hours} \end{array} \right]
 \end{array}
 \right.
 \end{array}$$

Fully worked solutions for this chapter are available online.

LESSON

10.7 Review

10.7.1 Summary

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Access the chapter summary



Review your results



Practise exam questions

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10.7 Exercise

learnon

10.7 Exercise

10.7 QCAA questions **on**

Simple familiar

1, 2, 3, 4, 5, 6, 7, 8,
9, 10, 11, 12

Complex familiar

13, 14, 15, 16

Complex unfamiliar

17, 18, 19, 20

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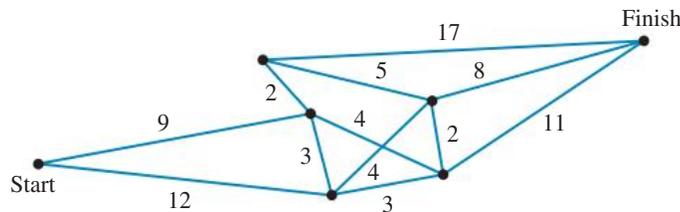


Find all this and MORE in jacPLUS



Simple familiar

1. a. Determine the shortest distance from start to finish in the following graph.



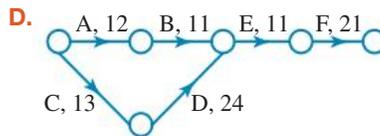
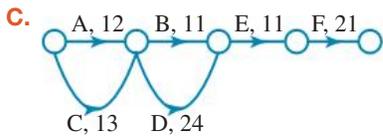
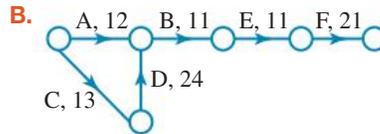
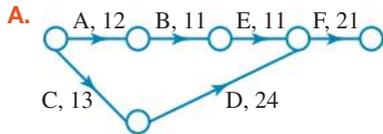
- b. Determine the total length of the shortest semi-Hamiltonian path from start to finish.
c. Draw the minimum spanning tree for this graph.

Questions 2 to 5 refer to the following table.

Activity	Time	Immediate predecessor
A	12	—
B	11	A
C	13	—
D	24	C
E	11	D, B
F	21	E

2. **MC** Using the table above, the activities that come before activity E are:
A. D and B. **B.** A, D and B. **C.** A, B, C and D. **D.** A, C and D.

3. **MC** The correct diagram for the table above is:



4. **MC** The earliest start time for activity E is:

A. 23

B. 24

C. 30

D. 37

5. **MC** The earliest completion time for the network is:

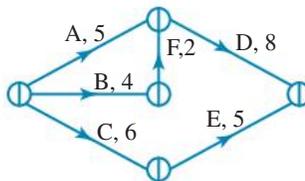
A. 56

B. 55

C. 68

D. 69

Questions 6 to 8 refer to the following diagram.



6. **MC** Identify which of the following is an immediate predecessor to E.

A. E

B. C

C. D

D. B

7. Determine the earliest finishing time for activities C and D.

8. Determine the latest starting time for activity E and the float time for activity C.

9. Consider the following matrix.

$$\begin{bmatrix} 15 & 11 & 16 & 6 \\ 13 & 5 & 2 & 12 \\ 4 & 2 & 11 & 14 \\ 12 & 7 & 12 & 10 \end{bmatrix}$$

a. **MC** Identify the row-reduced matrix for the matrix above.

A. $\begin{bmatrix} 9 & 5 & 10 & 0 \\ 11 & 3 & 2 & 10 \\ 2 & 0 & 9 & 12 \\ 5 & 0 & 5 & 5 \end{bmatrix}$

B. $\begin{bmatrix} 11 & 6 & 14 & 0 \\ 9 & 3 & 0 & 6 \\ 0 & 0 & 9 & 8 \\ 8 & 5 & 10 & 4 \end{bmatrix}$

C. $\begin{bmatrix} 9 & 5 & 10 & 0 \\ 11 & 3 & 0 & 10 \\ 2 & 0 & 9 & 12 \\ 5 & 0 & 5 & 3 \end{bmatrix}$

D. $\begin{bmatrix} 9 & 5 & 10 & 6 \\ 11 & 3 & 2 & 10 \\ 2 & 2 & 9 & 12 \\ 7 & 5 & 5 & 5 \end{bmatrix}$

b. **MC** After column reduction is performed to the matrix obtained in part a, the resultant matrix is:

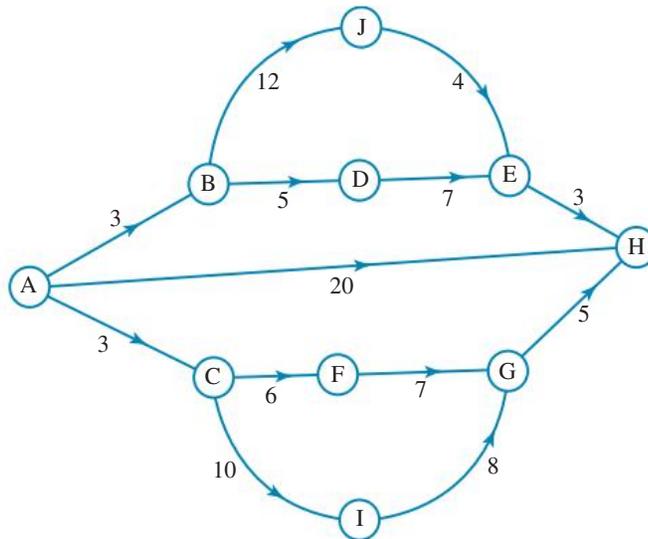
A.
$$\begin{bmatrix} 7 & 5 & 10 & 0 \\ 9 & 3 & 0 & 10 \\ 0 & 0 & 9 & 12 \\ 3 & 0 & 5 & 3 \end{bmatrix}$$

B.
$$\begin{bmatrix} 7 & 3 & 8 & 1 \\ 9 & 1 & 0 & 5 \\ 0 & 0 & 7 & 7 \\ 5 & 3 & 3 & 0 \end{bmatrix}$$

C.
$$\begin{bmatrix} 7 & 5 & 8 & 0 \\ 9 & 3 & 0 & 10 \\ 0 & 0 & 7 & 12 \\ 3 & 0 & 3 & 5 \end{bmatrix}$$

D.
$$\begin{bmatrix} 11 & 6 & 14 & 0 \\ 9 & 3 & 0 & 6 \\ 0 & 0 & 9 & 8 \\ 8 & 5 & 10 & 4 \end{bmatrix}$$

10. The network diagram shown is a representation of the processes required to build a tennis court (times in hours).



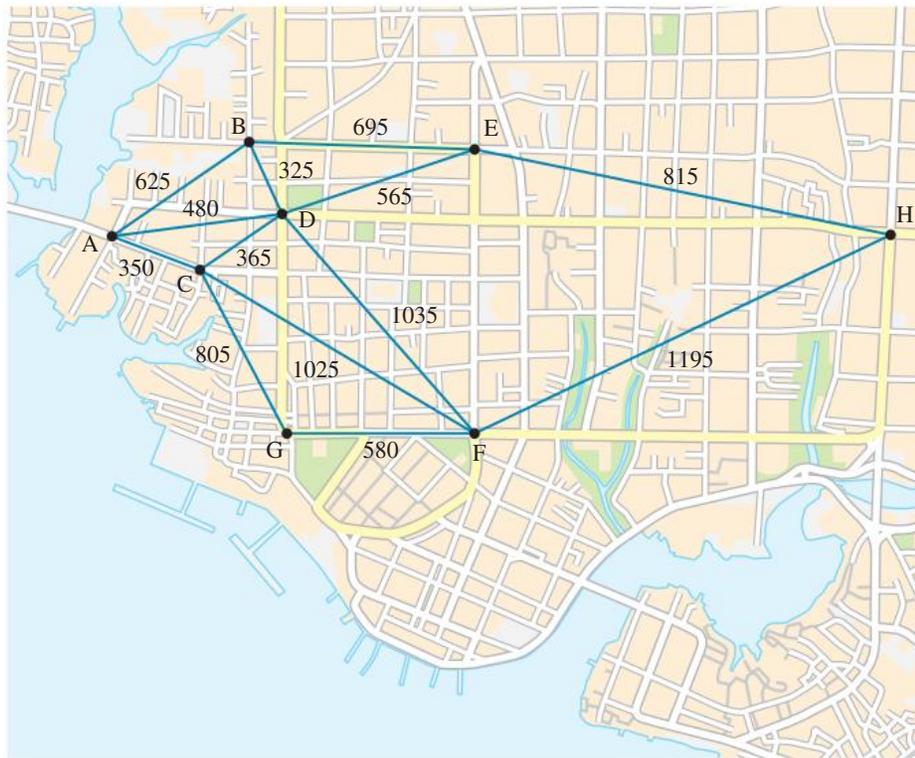
Determine the earliest completion time.

11. The flying distances between the capital cities of Australian mainland states and territories are listed in the following table.

	Adelaide	Brisbane	Canberra	Darwin	Melbourne	Perth	Sydney
Adelaide		2055	1198	3051	732	2716	1415
Brisbane	2055		1246	3429	1671	4289	982
Canberra	1198	1246		4003	658	3741	309
Darwin	3051	3429	4003		3789	4049	4301
Melbourne	732	1671	658	3789		3456	873
Perth	2716	4363	3741	4049	3456		3972
Sydney	1415	982	309	4301	873	3972	

- Draw a weighted graph to show this information.
- If technical problems are preventing direct flights from Melbourne to Darwin and from Melbourne to Adelaide, identify the shortest way of flying from Melbourne to Darwin.
- If no direct flights are available from Brisbane to Perth or from Brisbane to Adelaide, identify the shortest way of getting from Brisbane to Perth.
- Draw the minimum spanning tree for the graph and state its total distance.

12. The following diagram shows the streets in a suburb of a city with a section of underground tunnels shown in black. Weightings indicate distances in metres. The tunnels are used for utilities such as electricity, gas, water and drainage.
- If the gas company wishes to run a pipeline that minimises its total length but reaches each vertex, determine the total length required.
 - Draw a graph to show the gas lines.
 - If drainage pipes need to run from H to A, determine the shortest path they can follow, and calculate how long this path will be in total.
 - A single line of cable for a computerised monitoring system needs to be placed so that it starts at D and reaches every vertex once. Determine the minimum length possible and the path that the cable must follow.
 - If a power line has to run from D so that it reaches every vertex at least once and finishes back at the start, determine the path that it must take to be a minimum.



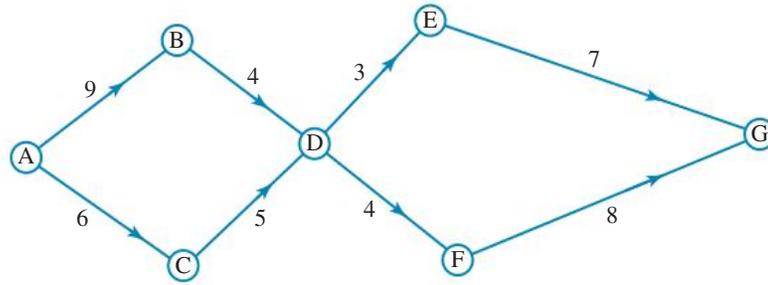
Complex familiar

13. A team of four — Barnie, Ruth, Shelley and Carlos have been selected, to play four holes of golf for a social event for their workplace. Each member of the team will play one of the holes. Their scores for their previous rounds of golf for these four holes have been summarised in the following table.

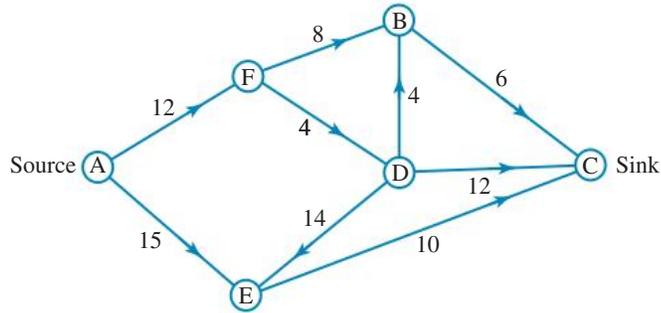
	Hole 1	Hole 2	Hole 3	Hole 4
Barnie	5	7	5	9
Ruth	6	10	10	7
Shelley	7	5	3	8
Carlos	7	8	8	9

State the optimal team for this event.

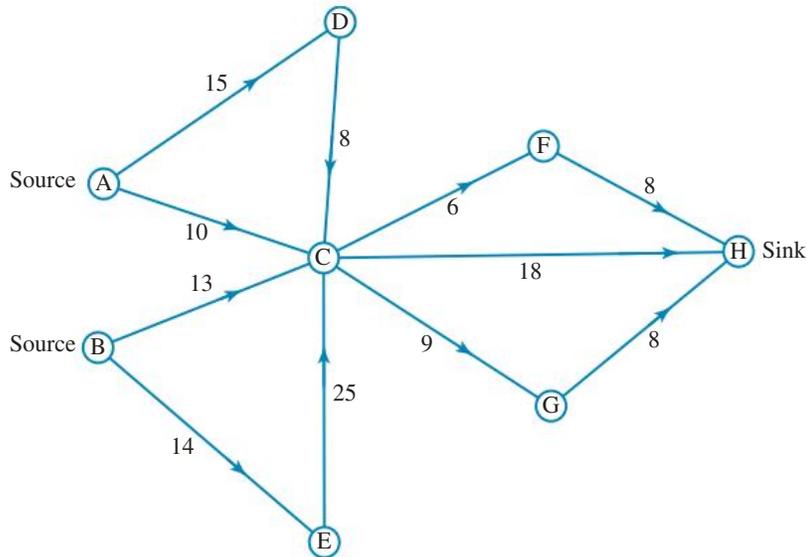
14. The earliest completion time for the project represented in the network below has to be reduced by 2 hours.



- a. Create a table to represent the different phases of the project and their times.
 b. Create a table showing how you can reduce the earliest completion time by 2 hours.
15. Determine whether the maximum flow would change if the capacity for EC was infinite. Explain why or why not.

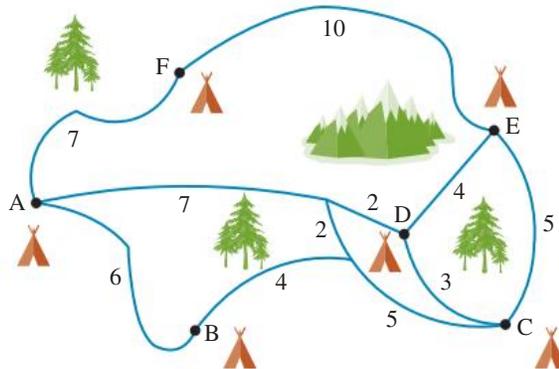


16. Calculate the capacity of the minimum cut set/maximum flow through the network.



Complex unfamiliar

17. A brochure for a national park includes a map showing the walking trails and available camping sites at the park.



If the main entrance to the park is situated at A, identify the shortest way to travel to each campsite and return to A.

18. A cruise ship takes passengers around Tasmania between the seven locations marked on the map.

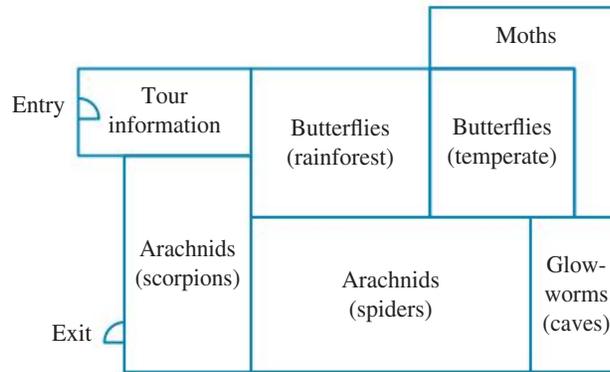


The sailing distances between locations are indicated in the table.

	Hobart	Bruny I.	Maria I.	Flinders I.	Devonport	Robbings I.	King I.
Hobart	—	65 km	145 km	595 km	625 km	—	—
Bruny I.	65 km	—	130 km	—	—	715 km	—
Maria I.	145 km	130 km	—	450 km	—	—	—
Flinders I.	595 km	—	450 km	—	330 km	405 km	465 km
Devonport	625 km	—	—	330 km	—	265 km	395 km
Robbings I.	—	715 km	—	405 km	265 km	—	120 km
King I.	—	—	—	465 km	395 km	120 km	—

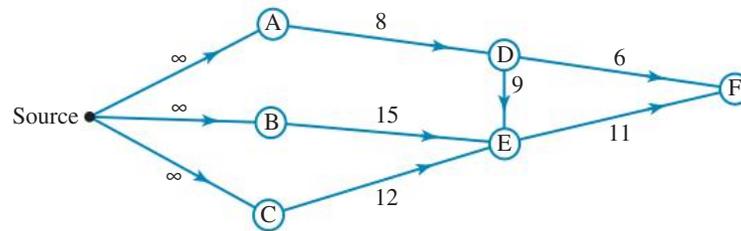
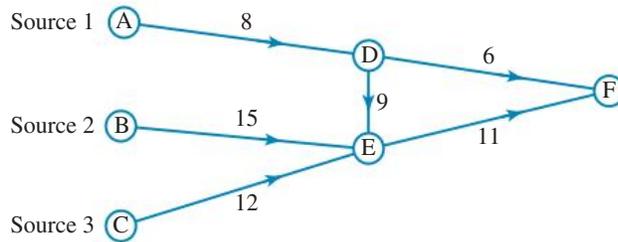
Identify the shortest route from Hobart to Robbings Island.

19. A Lepidoptera and Arachnid building is to be set up at the zoo. The floor plan is shown at right. The building is to be designed so that people can flow through in only one direction. Each doorway will open only one way and is designed to ensure that there is no mixing of the exhibits.



Determine the rate at which people should be admitted so that they can flow smoothly through the building.

20. Compare the following two networks.



If possible, make a general statement about multiple sources.

10.7 Past QCAA exam questions

Question 1 (1 mark)

Source: QCAA 2023 General Mathematics, Paper 1, Section 1, Q.3; © State of Queensland (QCAA) 2024

The duration, in minutes, of all activities in a project are shown.

Activity	P	Q	R	S	T	U	V
Duration	38	42	32	34	16	14	26

The critical path for the project is PRSV.

What is the earliest completion time for the project if it starts at 11:00 am?

- A. 12:30 pm
- B. 1:10 pm
- C. 1:30 pm
- D. 2:10 pm

Question 2 (1 mark)

Source: QCAA 2023 General Mathematics, Paper 1, Section 1, Q.8; © State of Queensland (QCAA) 2024

Activities P and Q are the critical activities for a project.

Activity	Duration	Prerequisite activity
P	3	—
Q	6	P

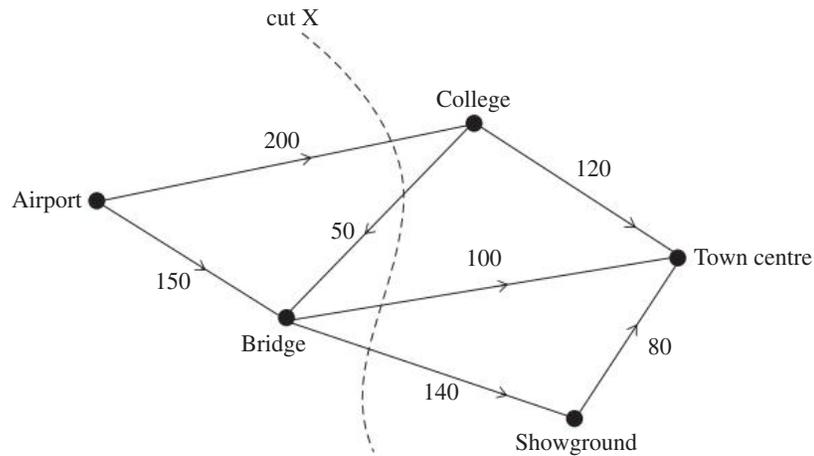
What are the earliest starting time (EST) and latest starting time (LST) for Activity Q?

	EST	LST
A.	3	3
B.	3	6
C.	6	6
D.	6	9

Question 3 (4 marks)

Source: QCAA 2023 General Mathematics, Paper 1, Section 2, Q.21; © State of Queensland (QCAA) 2024

The road network shows the number of vehicles per hour travelling from the airport to the town centre.



- Determine the capacity of cut X. [1 mark]
- Determine the maximum flow from the airport to the town centre. [2 marks]
- During a weather emergency, all roads to and from the bridge are closed to vehicles. Determine the maximum flow from the airport to the town centre during this time. [1 mark]

Question 4 (5 marks)

Source: QCAA 2023 General Mathematics, Paper 1, Section 2, Q.24; © State of Queensland (QCAA) 2024

Jed is preparing and serving a meal of curry and rice with naan bread. The table shows the duration and prerequisites for all the tasks they must complete.

Task	Task description	Duration (minutes)	Prerequisite
A	Assemble equipment	2	—
B	Boil rice	20	A
C	Prepare curry ingredients	6	A
D	Make naan bread	8	A
E	Simmer curry	40	C
F	Fry naan bread	4	D
G	Serve meal	2	B, E, F

- Construct a network diagram to show the sequence of tasks from start to finish, labelling all tasks and durations. Use forward and backward scanning to show the earliest and latest starting times for all tasks. [3 marks]
- Determine the critical activities and minimum completion time for preparing and serving the meal. [2 marks]

Question 5 (4 marks)

Source: QCAA 2022 General Mathematics, Paper 1, Section 2, Q.20; © State of Queensland (QCAA) 2024

The table summarises the distances in kilometres (km) between three flower stores and three delivery locations: A, B and C.

Use the Hungarian algorithm to determine the minimum total distance needed to deliver flowers to all locations if each store delivers flowers to only one location.

	A	B	C
Store 1	19	17	24
Store 2	15	14	22
Store 3	23	16	40

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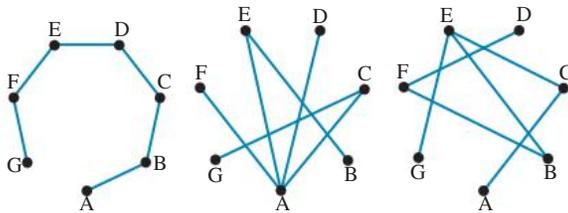
Answers

Chapter 10 Networks and decision mathematics

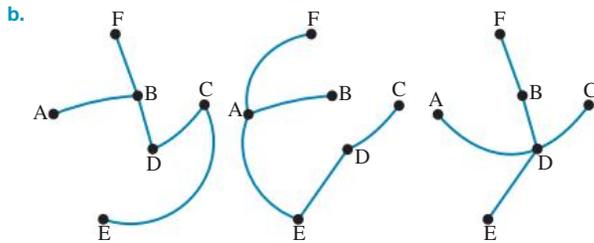
10.2 Trees and minimum connector problems

10.2 Exercise

1. 21
2. 20.78
3. a.



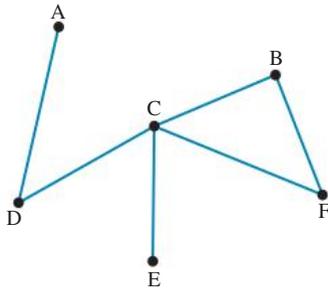
Other possibilities exist.



Other possibilities exist.

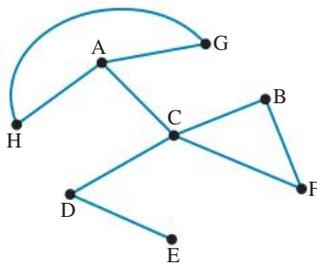
4. ABGEDCA or ACDEGBA (length 66)

5. a. i.



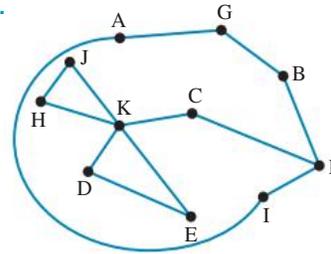
- ii. ADCBFCE or ADCFBCE

- b. i.



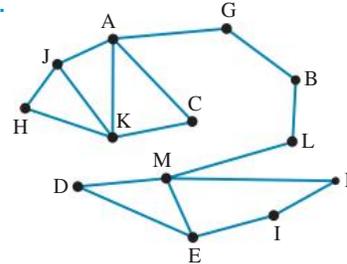
- ii. AHGACBFCDE or similar

- c. i.



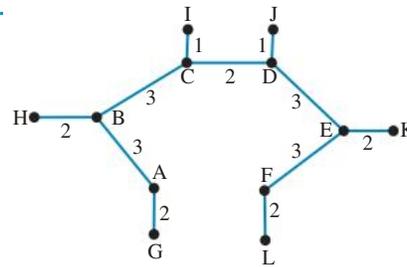
- ii. KDEKHJKCFIAGBF or similar

- d. i.

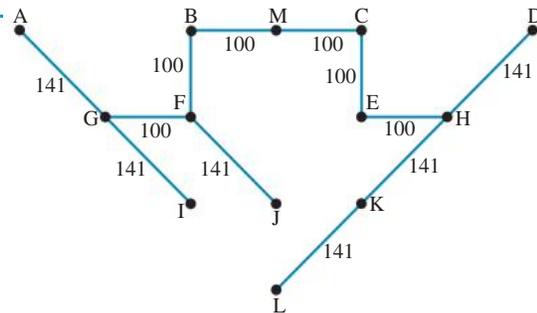


- ii. EDMEIFMLBGACKHJKA or similar

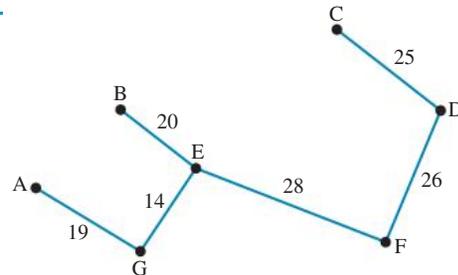
- 6.

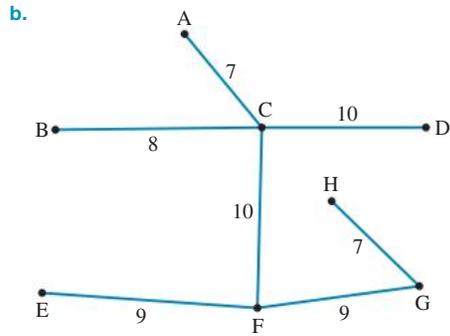


- 7.

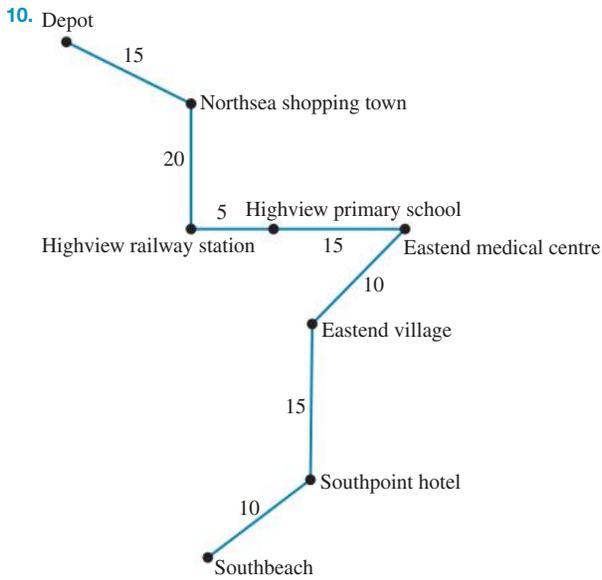


8. a.

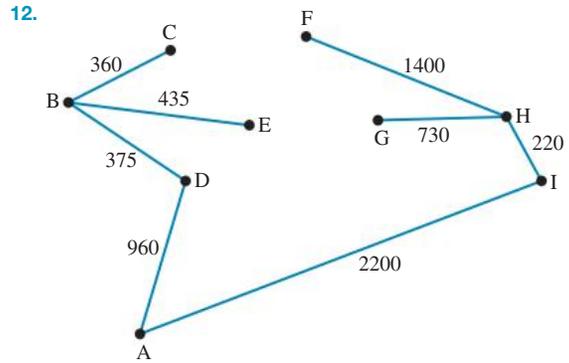




9. a. $ADEG = 390$
 b. $BHG = 460$
 c. $EGFCDABHE = 1440$



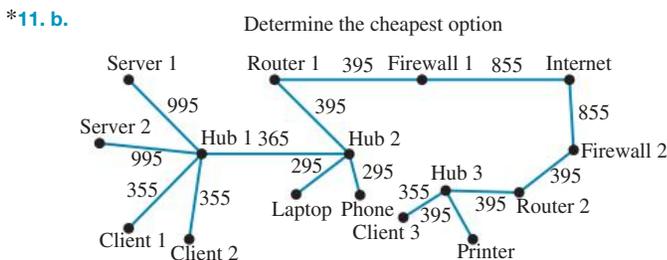
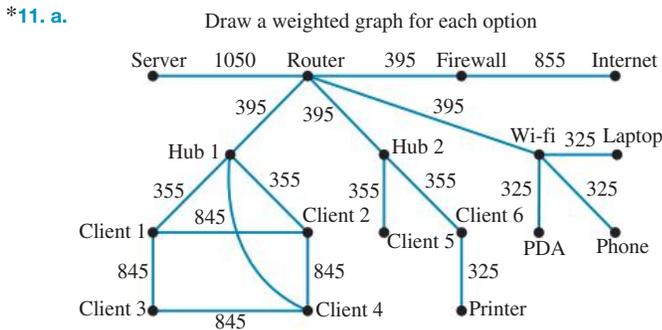
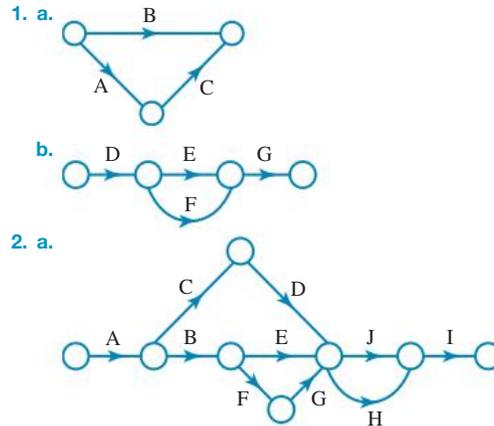
11. a. Draw a weighted graph for each option.
 See the figure at the bottom of the page*
 b. Determine the cheapest option.
 See the figure at the bottom of the page*

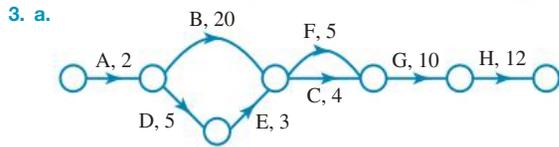
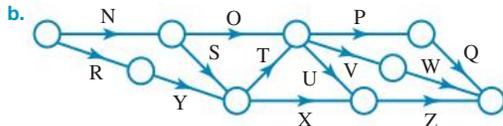


13. 859 km
 14. $x = 6$
 15. The manager's claim is unreasonable. The actual savings from removing unnecessary trails while keeping all locations connected is only \$100 000.

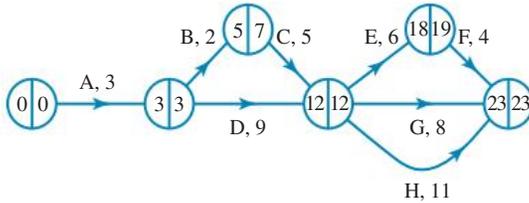
10.3 Project planning and scheduling

10.3 Exercise

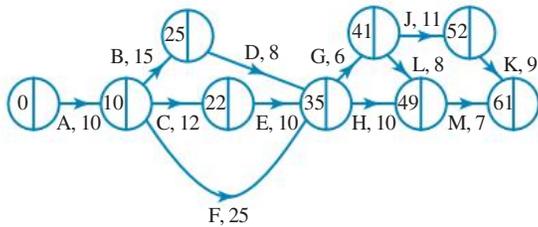




- b. 49 minutes
 4. B
 5. D
 6. a.



- b. 23 minutes
 c. B, C, E, F and G can be delayed.
 7. B
 8. D
 9. A
 10. a.



- b. 61 minutes
 11. $p = 5, q = 6, m = 6, n = 7, x = 5$
 12. a. The minimum completion time for the recipe is 60 minutes.
 b. If you add an extra 1 minute to C, D and E, they would form the new critical path because the path C–F would become 11 minutes, causing the earliest completion time to be 63 minutes.
 c. If you add an extra 5 minutes to C, D and E, they would form the new critical path because the path C–F would become 23 minutes, causing the earliest completion time to be 75 minutes.
 d. If you change L to 9 minutes, then K–L becomes the new critical path at 11 minutes, 1 more minute than J. The new earliest completion time becomes 61 minutes.

13. 15 min
 14. See the figure at the bottom of the page*
 The critical activities are A, C, G, H.
 15. The tradesperson can have days 1–12 off, must work on days 13–19, and can have days 20–32 off.

10.4 Critical Path Analysis (CPA)

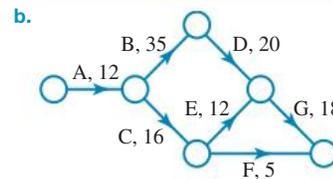
10.4 Exercise

1. Critical path = B – D
 Float times for the non-critical activities:
 A: 1 minute
 C: 1 minute
 E: 1 minute
 2. D
 3. Critical path = B–F–G

Float time for the non-critical activities:

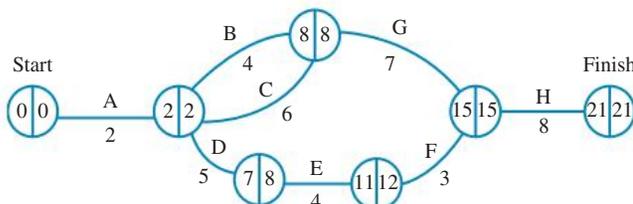
- A: 3 hours D: 5 hours
 C: 3 hours E: 5 hours
 4. D
 5. a. 31 days b. A–C–E–G
 6. D
 7. C
 8. a.

Activity	Immediate predecessor	Time
A	–	12
B	A	35
C	A	16
D	B	20
E	C	12
F	C	5
G	D, E	18

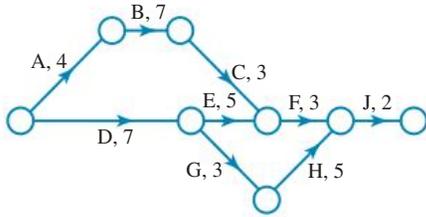


- b.
 c. 85 minutes
 d. A–B–D–G
 e. 28%
 9. C
 10. A–D–H = 23 minutes

*14.



11. a. A–F–G–J–K
 b. B, D, C, E, H, L and M
12. a. 29 minutes b. B–E–J–N
13. a. ABDEF
 b. 31 weeks
 c. ABDEF
 d. 27 weeks
14. 3 days
15. a.



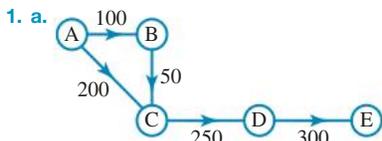
b. Float times

Activity	Times	EFT	EFT	Float time
D	7	0	9	2
E	5	7	14	2
G	3	7	12	2
H	5	10	17	2

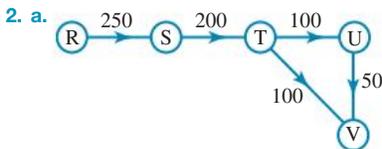
16. Sample responses can be found in the worked solutions in the online resources.

10.5 Network flow

10.5 Exercise



- b. 250
 c. This doesn't meet the demand.

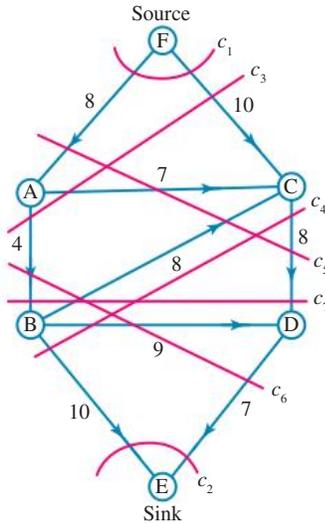


- b. 150
 c. This does meet the demand.

3. a. 23 b. 16 c. 16

4. a. 6 b. 3 c. 3

5. a. Identify the cuts in the network.



$$c_1 = 8 + 10 = 18$$

$$c_2 = 10 + 7 = 17$$

$$c_3 = 10 + 4 = 14$$

$$c_4 = 8 + 8 = 16$$

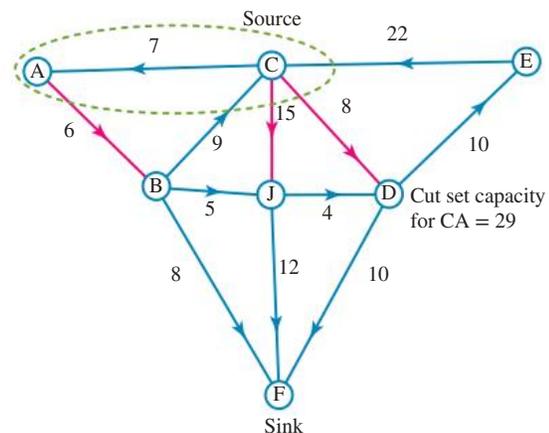
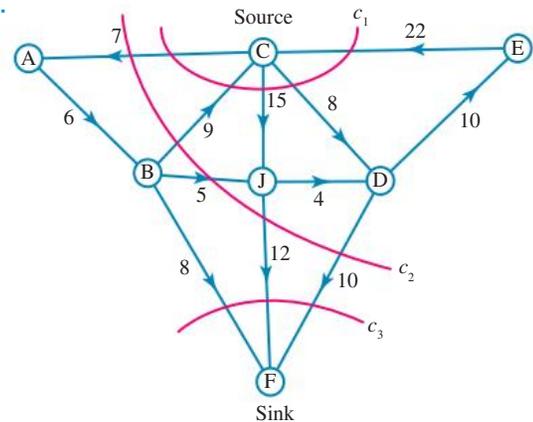
$$c_5 = 8 + 7 = 15$$

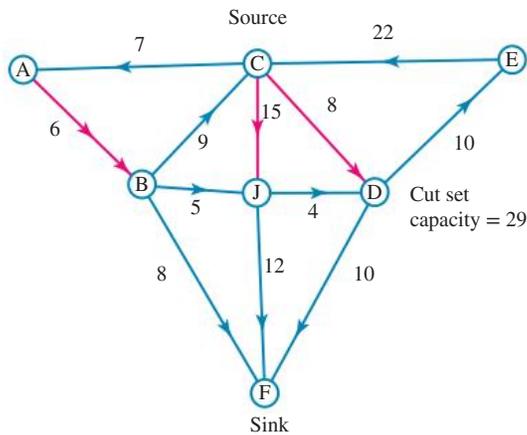
$$c_6 = 10 + 10 = 20$$

$$c_7 = 4 + 8 = 12$$

- b. The minimum cut set is $c_7 = 12$. Thus, the maximum flow through the network is 12.

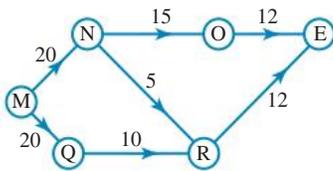
6. a.





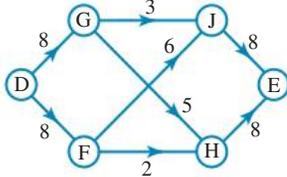
- b. Maximum flow = 29
 c. Yes, there are 2: CA, CDE

7. a.



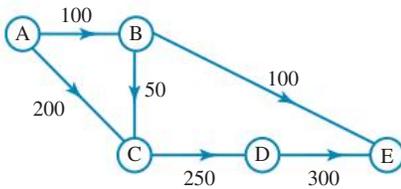
b. This does meet the demand.

8. a.



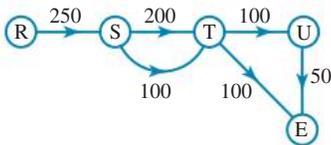
b. This doesn't meet the demand.

9. a. i.



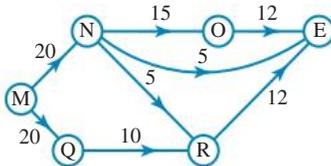
ii. 300

b. i.



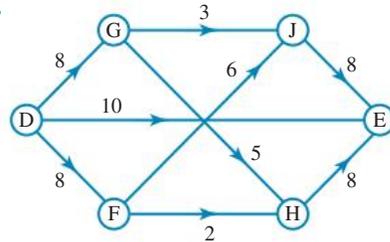
ii. 150

c. i.



ii. 29

d. i.



ii. 25

10. 7

11. Yes

12. Yes

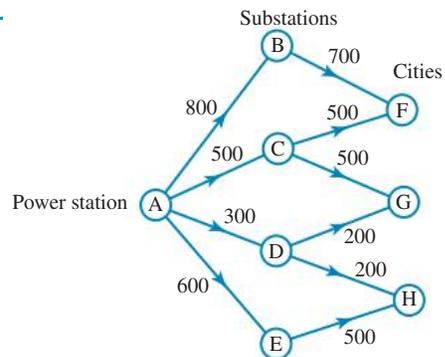
13. a. Yes

b. 1700. The maximum flow/minimum cut is equal to 1700 parts.

c. One option is to reduce the path BE to 300 parts, though there are many other options.

14. a. 1800 MW

b.



15. The mayor is correct that the flow remains at 90 MW.

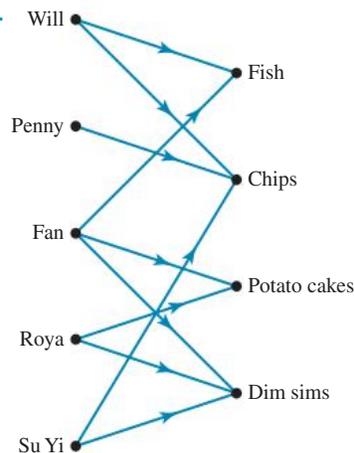
10.6 Assigning order and the Hungarian algorithm

10.6 Exercise

1. a. T1 – E2, T2 – E1, T3 – E3, 4 hours

b. T1 – E2, T2 – E1, T3 – E3, 14 hours

2. a.

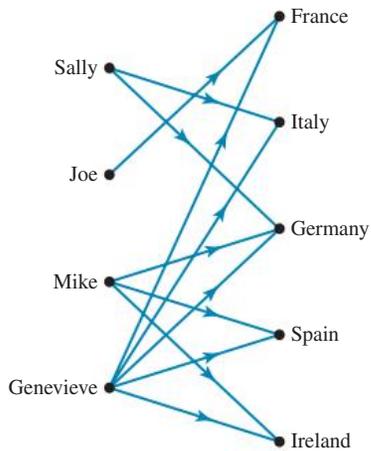


b. False, False, False, True

c. It is not possible to allocate one food item to each person as there are 4 items and 5 people. Someone will miss out.

3. a. The visitors can be considered the supply vertices and the countries the demand vertices. No two vertices in each of these groups share an edge, so a bipartite graph is suitable.

b.



c. 3

4. a. False b. True c. False
d. False e. False

5. $R1 \rightarrow C4$
 $R2 \rightarrow C2$
 $R3 \rightarrow C1$
 $R4 \rightarrow C3$
 $4 + 5 + 6 + 4 = 19$ hours.

6. $S1 \rightarrow D$
 $S2 \rightarrow A$
 $S3 \rightarrow B$
 $S4 \rightarrow C$
61 hours

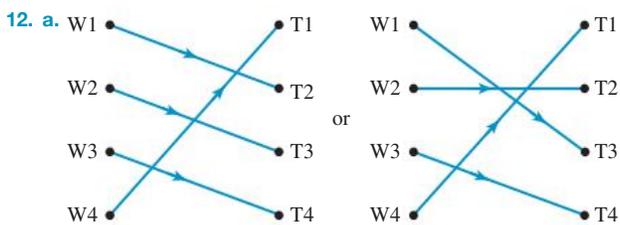
7. C

8. 28 hours

9. 44 km

10. \$140

11. a. 54 b. 90



b. 180 hours

13. One possible allocation:

$M \rightarrow L1$
 $Je \rightarrow L3$
 $P \rightarrow L2$
 $Ja \rightarrow L4$
11 minutes

14. Sample responses are available in the worked solutions in the online resources.

15. $x = 6$

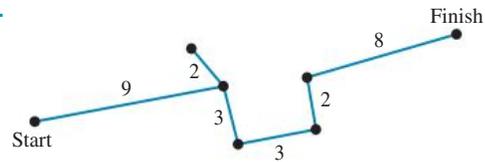
10.7 Review

10.7 Exercise

1. a. 23

b. 34

c.



2. C

3. D

4. D

5. D

6. B

7. Earliest finishing time: 6 minutes for activity C, 14 minutes for activity D.

8. Latest starting time for activity E: 9 minutes; float time for activity C: 3 minutes.

9. a. C

b. A

10.

Phases	Hours
AB	3
AC	3
AH	20
BJ	12
BD	5
CF	6
CI	10
DE	7
EH	3
FG	7
GH	5
IG	8
JE	4

ACIGH = 26 hours

11. a. See the figure at the bottom of the page*

b. Via Canberra (4661 km)

c. Via Sydney (4954 km)

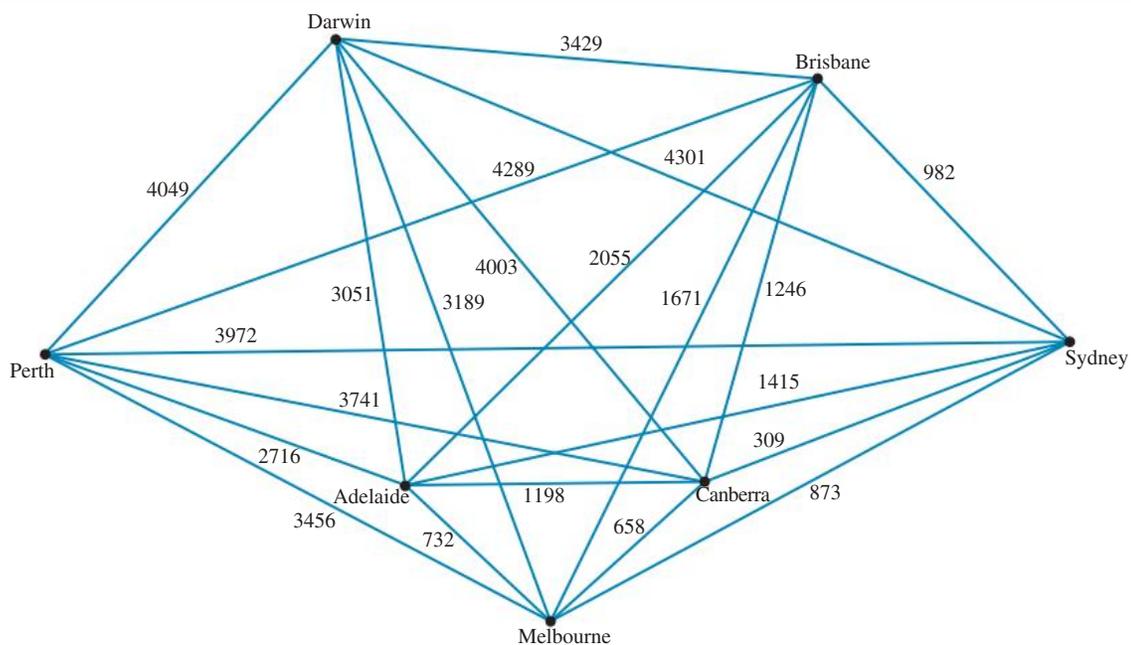
d. See the figure at the bottom of the page*

The total distance is 8448 km.

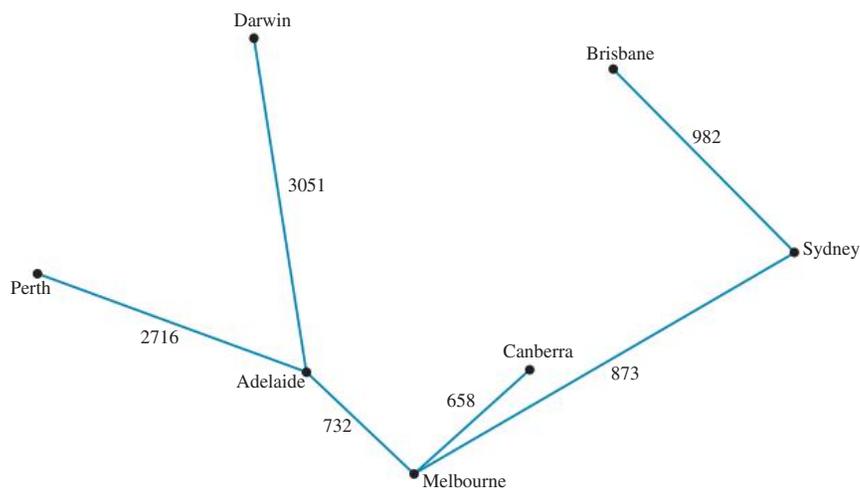
12. a. i. 3805 m

ii. See the figure at the bottom of the page*

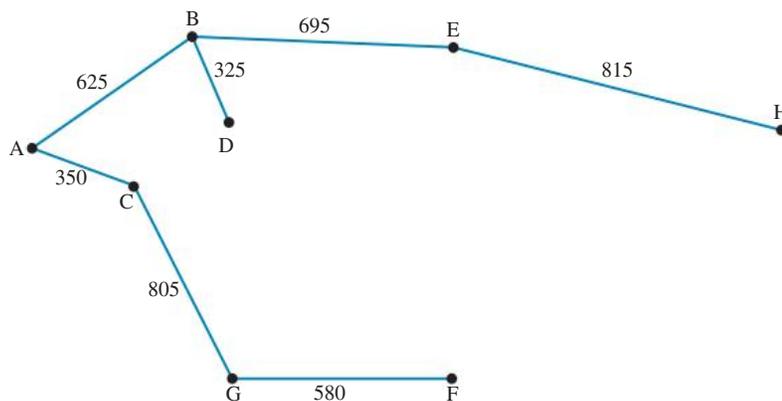
*11. a.



*11. d.



*12. a. ii.



- b. HEDA, 1860 m
 - c. 4905 m, DFGCABEH
 - d. DEHFGCABD, 5260 m
13. The minimum score is $5 + 7 + 3 + 8 = 23$
14. a. Earliest completion time = 25 hours ABDFG

Phase	Hours
AB	9
AC	6
BD	4
CD	5
DE	3
DF	4
EG	7
FG	8

- b. Earliest completion time = 25 hours ABDFG

Phase	Hours
AB	9
AC	6
BD	4
CD	5
DE	3
DF	4
EG	7
FG	$8 - 2 = 6$ crashed 2 hours

- 15. Yes, Sample response can be found in the worked solutions in the online resources.
- 16. 32
- 17. AFEDCBA (39)
- 18. Hobart–Bruny–Robbins (780 km)
- 19. 5 people per minute
- 20. Multiple sources and maximum flow are not affected when a dummy source with infinite flows enters the multiple sources.

10.7 Past QCAA exam questions

- 1. B
- 2. A
- 3. a. capacity = 440
 - b. 300 vehicles per hour
 - c. maximum flow 120 vehicles per hour
- 4. a. Sample responses can be found in the worked solutions in the online resources.
 - b. Sample responses can be found in the worked solutions in the online resources.
- 5. 55 km