ACADEMIC ASSOCIATES

STUDY GUIDE

MATHEMATICS METHODS

YEAR 11 ATAR COURSE – UNITS 1 & 2

REVISED EDITION

Gregory Hine

ACADEMIC ASSOCIATES

WACE STUDY GUIDE

MATHEMATICS METHODS

YEAR 11 ATAR COURSE

Gregory Hine

ACADEMIC GROUP
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First published 2014
Second Edition 2020
Reprnted 2021, 2022

National Library of Australia ISBN 978-1-87691-885-9

Printed in Western Australia on paper supporting responsible forestry.

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About the Author
Dr Gregory Hine has been a mathematics educator at several Western Australian schools and universities, teaching a range of mathematics pathways and has been an ATAR exam marker.

Acknowledgements
- The W.A. School Curriculum and Standards Authority with permission to use extracts from the Mathematics Methods syllabus
- My sincere gratitude is expressed to my wife, Cam, and my children, Mikayla and Jonathon. Your ongoing love and support is breathtaking.

CONTENTS

FOREWORD

The purpose of this text is to assist Year 11 students with their preparation for tests and examinations in the new Mathematical Methods course for Western Australia.

The ***Syllabus Checklist*** indicates to students which skills they must have acquired and the objectives they need to meet under each of the major headings of the course.

The ***Worked Examples*** are presented in a detailed manner, with brief notes and explanations being used to amplify the understanding for the particular question. Some of these worked examples could be used in the written notes that students are permitted to take into an examination.

The ***Problems to Solve*** section in each chapter provides students with a broad range of questions without the repetitive nature of problems usually associated with a course textbook.

The ***Trial Tests*** are an additional component to this text, and these allow students to familiarise themselves with examination questions. Suggested times are given for these tests, and students should be encouraged to adhere to these times to prepare properly for final examinations. Fully worked solutions are provided for students to receive immediate, accurate and useful feedback on their performance.

About the Units

In ***Unit 1***, students review the basic algebraic concepts and techniques required for a successful introduction to the study of functions and calculus. Simple relationships between variable quantities are reviewed, and these are used to introduce the key concepts of a function and its graph. The study of probability and statistics begins with a review of the fundamentals of probability together with the concepts of conditional probability and independence. The study of the trigonometric functions commences with a consideration of the unit circle using degrees, the trigonometry of triangles, and the application of trigonometry. Students are then introduced to radian measure, the graphs of the trigonometric functions, and the application of trigonometric functions.

In ***Unit 2***, students are introduced to exponential functions, their properties and graphs. Arithmetic and geometric sequences and their applications are explored and their recursive definitions applied. Rates and average rates of change are introduced, and this is followed by an examination of the key concept of the derivative as an 'instantaneous rate of change'. These concepts are reinforced numerically (by calculating difference quotients), geometrically (as slopes of chords and tangents), and algebraically. This first calculus topic concludes with differentiating polynomial functions, using simple applications of the derivative to sketch curves, determining slopes and equations of tangents, calculating instantaneous velocities, and solving optimisation problems.

Dr Gregory Hine, Ph.D.

FUNCTIONS I

1

Syllabus Checklist

By the end of this chapter, you should be able to:

☐ Lines and linear relationships

- determine the coordinates of the mid-point between two points
- determine an end-point given the other end-point and the mid-point
- examine examples of direct proportion and linearly related variables
- recognise features of the graph of $y = mx + c$, including its linear nature, its intercepts and its slope or gradient
- determine the equation of a straight line given sufficient information; including parallel and perpendicular lines
- solve linear equations, including those with algebraic fractions and variables on both sides

☐ Quadratic relationships

- examine examples of quadratically related variables
- recognise features of the graphs of $y = x^2$, $y = a(x - b)^2 + c$, and $y = a(x - b)(x - c)$, including their parabolic nature, turning points, axes of symmetry and intercepts
- solve quadratic equations, including the use of quadratic formula and completing the square
- determine the equation of a quadratic given sufficient information
- determine turning points and zeros of quadratics and understand the role of the discriminant
- recognise features of the graph of the general quadratic $y = ax^2 + bx + c$

☐ Powers and polynomials

- recognise features of the graphs of $y = x^n$ for $n \in N$, including shape, and behaviour as $x \to \infty$ and $x \to -\infty$
- identify the coefficients and the degree of a polynomial
- expand quadratic and cubic polynomials from factors
- recognise features and determine equations of the graphs of $y = x^3$, $y = a(x - b)^3 + c$ and $y = k(x - a)(x - b)(x - c)$, including shape, intercepts and behaviour as $x \to \infty$ and $x \to -\infty$
- factorise cubic polynomials in cases where a linear factor is easily obtained
- solve cubic equations using technology, and algebraically in cases where a linear factor is easily obtained

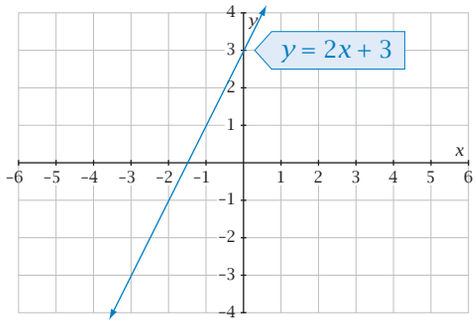
☐ Functions

- understand the concept of a function as a mapping between sets and as a rule or a formula that defines one variable quantity in terms of another
- use function notation; determine domain and range; recognise independent and dependent variables
- understand the concept of the graph of a function
- examine translations and the graphs of $y = f(x) + a$ and $y = f(x - b)$
- examine dilations and the graphs of $y = cf(x)$ and $y = f(dx)$
- recognise the distinction between functions and relations and apply the vertical line test

FORMULAE AND DEFINITIONS

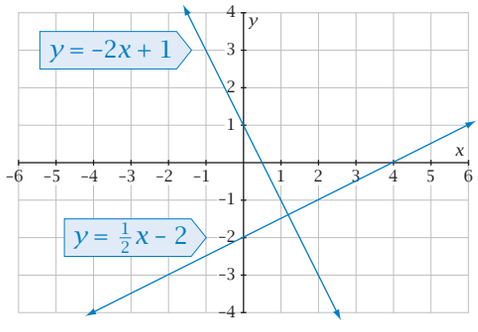
Linear Equations

Standard linear equations are of the form $y = mx + c$. The two main characteristics of linear equations are the gradient, m, and the y intercept, c.



The gradient, or steepness of the straight line, is represented by m. If the gradient of a linear function is not given, it can be measured by calculating $\frac{Rise}{Run}$ since it is constant. If two points on a straight line are given, (x_1, y_1) and (x_2, y_2) the gradient of that line can be determined by using the formula: $\frac{y_2 - y_1}{x_2 - x_1}$

- Graphs that have a positive gradient go from bottom left to top right.
- Graphs that have a negative gradient go from top left to bottom right.



Linear functions may also be expressed in some alternative ways:

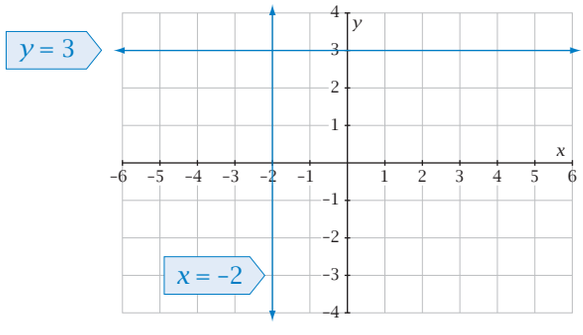
i) $Ax + By = C$

ii) $Ax + By + C = 0$

iii) $\frac{x}{a} + \frac{y}{b} = 1$, where a and b are the intercepts on the x and y axes respectively.

By rearranging the coefficients these equations can be written in the form $y = mx + c$.

- Horizontal lines have a y intercept and zero gradient.
- Vertical lines have an x intercept and an undefined gradient.
- You may apply the formula above for determining gradient to prove m for the following straight lines.

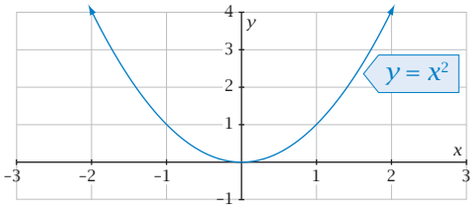


Quadratic Equations

Quadratic functions have the general form $y = ax^2 + bx + c$, where $a \neq 0$, and b and c are constants.

They may also be expressed in the form $y = a(x - b)(x - c)$ or $y = a(x - b)^2 + c$.

The simplest quadratic function is $y = x^2$, shown below in the Cartesian plane. The shape of this function is referred to as parabolic.



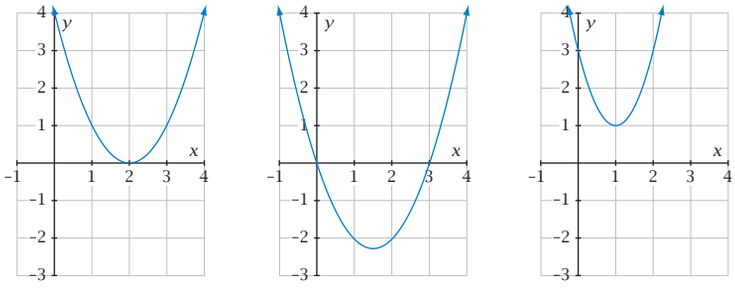
If a quadratic is in general form, then we can determine the equation of the line of symmetry using the formula $x = \frac{-b}{2a}$, then substituting this x value into the original equation for the corresponding y value. These two coordinates combine to give the turning point of the parabola.

For example, the equation $y = x^2 + 2x + 3$ yields $x = -1$ as the equation of line of symmetry. When this x value is substituted into the original equation, $y = 2$. Hence, the turning point of this parabola is $(-1, 2)$.

The x intercepts can be obtained by substituting the coefficients a, b, and c into the quadratic formula:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

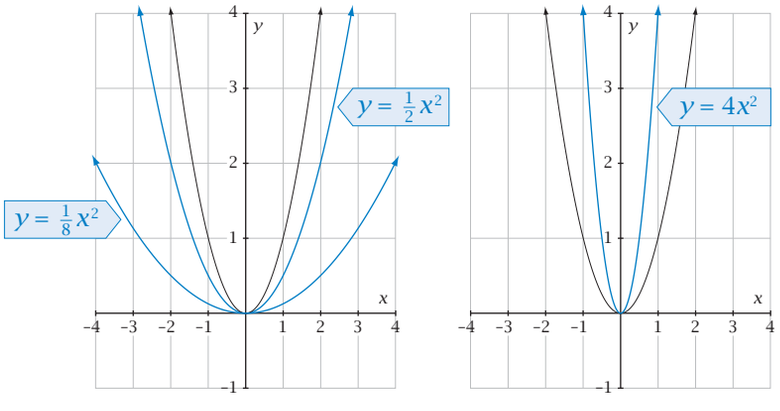
- If a quadratic function has two x intercepts, then $b^2 - 4ac > 0$.
- If a quadratic function has one x intercept, then $b^2 - 4ac = 0$.
- If a quadratic function has no x intercepts, then $b^2 - 4ac < 0$.



$b^2 - 4ac = 0$ $\quad$ $b^2 - 4ac > 0$ $\quad$ $b^2 - 4ac < 0$

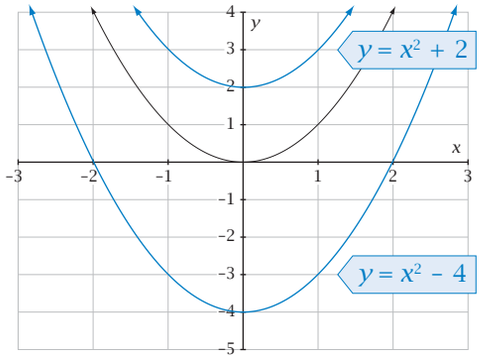
If we modify the simple equation $y = x^2$, some interesting geometric transformations (dilation, vertical translation, horizontal translation and reflection) emerge.

Dilation:
Starting from the simple quadratic equation $y = x^2$, we alter the overall steepness of the parabola with the inclusion of coefficient *'a'*. Consider the effects of this coefficient in the equation $y = ax^2$, where $a = \frac{1}{2}$, $\frac{1}{8}$ and 4.

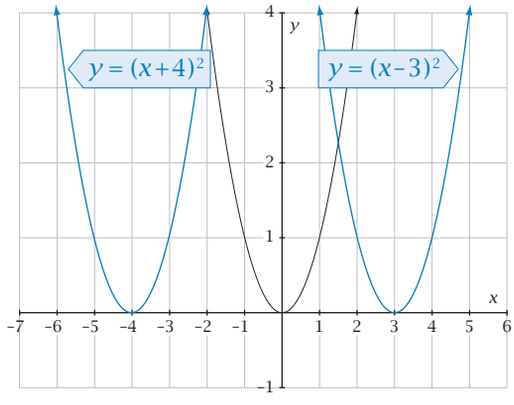


Thus, if $0 < a < 1$ the shape of the parabola will become flatter. Also, if $a > 1$ the shape of the parabola will become narrower.

Vertical Translation:
Consider the quadratic equation $y = x^2 + 2$. Displayed in the Cartesian plane with basic parabola $y = x^2$, this equation has been translated vertically upwards 2 units. Also shown is $y = x^2 - 4$, which has been translated vertically downwards 4 units. Hence, the constant value *'c'* in the equation $y = x^2 + c$ determines the overall vertical translation from the original parabola $y = x^2$.

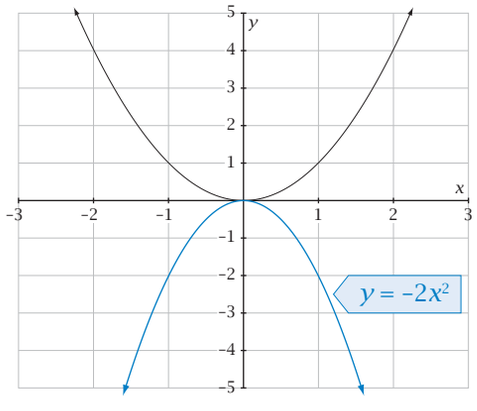


Horizontal Translation:
Consider the equation $y = (x+4)^2$. Displayed in the Cartesian plane with basic parabola $y = x^2$, this equation has been translated horizontally 4 units to the left. Also shown is $y = (x-3)^2$, which has been translated horizontally 3 units to the right. Hence, the constant value *'c'* in the equation $y = (x+c)^2$ determines the overall horizontal translation from the original parabola $y = x^2$.



Reflection:
The substitution of a negative scalar quantity for *'a'* in equation $y = ax^2$ will reflect original parabola $y = x^2$ in the x axis.

A graph with a negative *'a'* coefficient will yield a graph with a maximum point, and the curvature of the graph is concave down. A graph with a positive *'a'* coefficient will yield a graph with a minimum point, and the curvature of the graph is concave up.



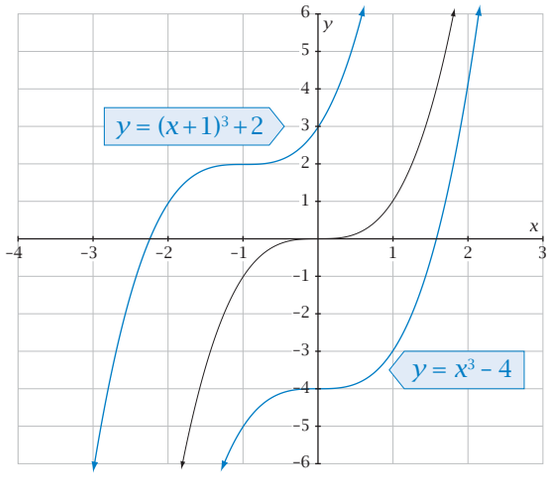
Quadratic equations may be comprised of some or all transformative properties, and is referred to as being in turning point form. This form is generally expressed as $y = a(x-b)^2 + c$, where $a \neq 0$. The turning point of this parabola is (b, c) and its line of symmetry is given by $x = b$.

Quadratic equations may also be expressed in factorised form $y = a(x - b)(x - c)$. This form is most commonly used for determining the x intercepts of a parabolic function.

For any of the three expressed forms of parabolic functions, the y intercept can be obtained by substituting $x = 0$. For the turning point and factored forms, the x intercepts can be determined by substituting the $y = 0$.

Cubic Functions

The general form of a cubic is $y = ax^3 + bx^2 + cx + d$, where $a \neq 0$, and b, c, and d are constants. The simplest cubic function is $y = x^3$. It is shown in the Cartesian plane on the following page. Note that its turning point, x intercept and y intercept occur at (0, 0). Also, the same properties of vertical and horizontal translation from quadratic functions can be easily applied to cubics.



A cubic function may have one or two turning points. These functions can also be written in factorised form, where $y = a(x - b)(x - c)(x - d)$, or in turning point form, where $y = a(x - b)^3 + c$.

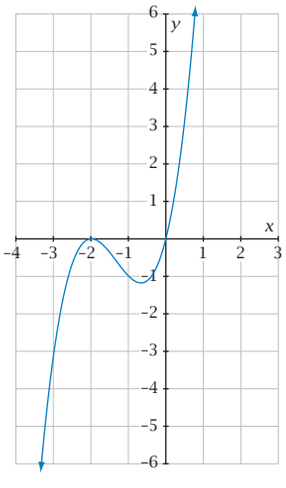
The behaviour of cubic functions as $x \to \pm\infty$ can be determined by examining the first term, ax^3, in $y = ax^3 + bx^2 + cx + d$.

More specifically,

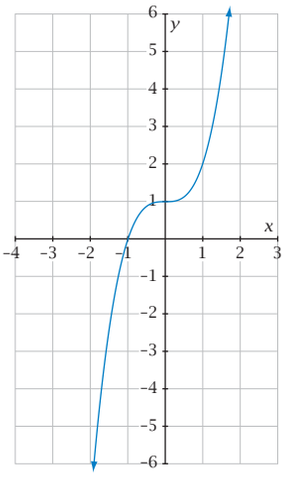
for $a > 0$, as $x \to +\infty$, $y \to +\infty$, and as $x \to -\infty$, $y \to -\infty$.

for $a < 0$, as $x \to +\infty$, $y \to -\infty$, and as $x \to -\infty$, $y \to +\infty$.

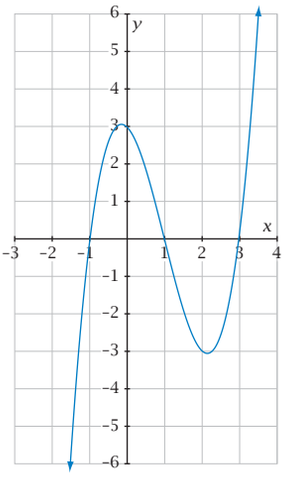
A cubic function may intersect the x axis at one, two or three points. The graphs below show an example of each.



$y = x(x+2)^2$



$y = x^3 + 1$



$y = (x+1)(x-1)(x-3)$

Function notation

Any mathematical relationship or rule that takes an input value and assigns it to an output value is called a function. For example, consider the relationship $y = 2x + 3$. Written in function notation, this relationship is $f(x) = 2x + 3$. This means that every value of x 'put into' this equation will be doubled and have three added to it.

Algebraic solution of equations

If we are asked for the solution of two functions $g(x)$ and $f(x)$, we simply equate the two functions and algebraically solve for the variable x. Then this solution can be substituted into either equation for the corresponding y variable.

There are other instances where we are asked to find the solution of a more complex equation. These can include quadratic and cubic functions.

With quadratic functions that are either in the form $0 = a^2x^2 - b^2$ or $0 = x^2 + bx + c$, the most common method used to find the solution is to first factorise the expression, and then apply the null factor law.

The expression $a^2x^2 - b^2$ is known as the difference of two squares.
It can be factorised to $(ax - b)(ax + b)$. If $(ax - b)(ax + b) = 0$, then $ax - b = 0$ and $ax + b = 0$.
Hence, $x = \pm \frac{b}{a}$.

Consider $0 = 9x^2 - 4$.
Now $0 = (3x - 2)(3x + 2)$, *and*
$3x - 2 = 0$ *and* $3x + 2 = 0$.
Therefore $x = \pm \frac{2}{3}$.

And since the solutions we have found are the x intercepts of the equation, we can write them as $(\frac{2}{3}, 0)$ and $(-\frac{2}{3}, 0)$.

The expression $x^2 + bx + c$ can be factorised into the form $(x + a)(x + b)$ so it can be solved.

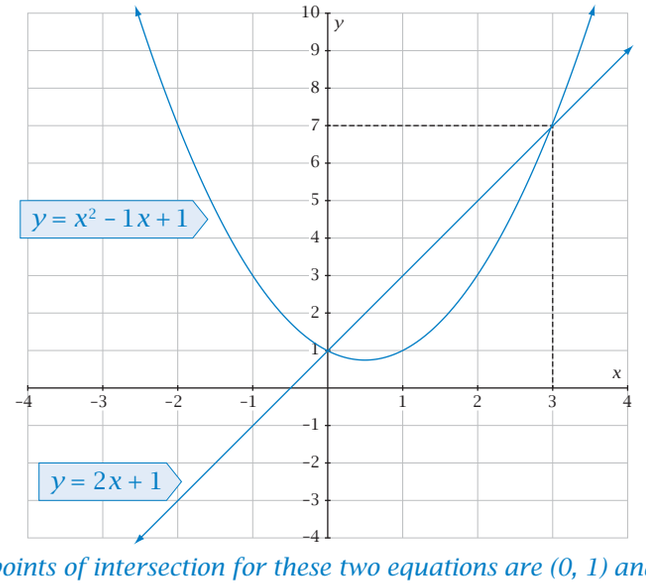
For example, consider $0 = x^2 + 7x + 12$.
This can be factorised to $0 = (x + 3)(x + 4)$
which leads us to $0 = x + 3$ or $0 = x + 4$

Hence, $x = -3$ and $x = -4$, and the x intercepts to the equation overall are (–3, 0) and (–4, 0).

Graphic solution of equations

To graphically determine where the solution of multiple functions lies, the following steps must be followed:

- Draw each of the functions on a neatly constructed axis.
- Locate the point(s) of intersection and read these coordinates from the graph accurately.
- Show dotted lines on your graph to indicate how you located your solutions.
- Check your solution(s) with the aid of CAS.



The points of intersection for these two equations are (0, 1) and (3, 7)

Worked Examples

1.1 Determine the gradient and y intercept of the following linear equations;

(a) $y = -2x + 6$

Since this equation is in gradient–intercept form, we can simply read the values from the equation. Hence, the gradient is –2 and the y intercept is (0, 6).

(b) $2x - 3y = 10$

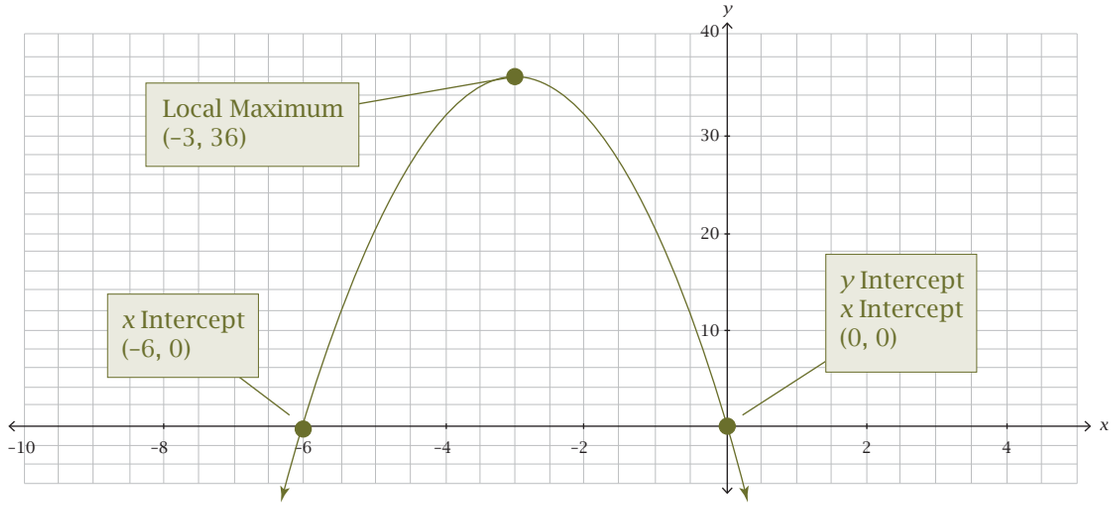
This equation is not in gradient-intercept form, but it can easily be arranged algebraically.

So if
$$2x - 3y = 10$$
$$2x - 10 = 3y$$
$$3y = 2x - 10$$
$$\therefore\ y = \frac{2x}{3} - \frac{10}{3}$$

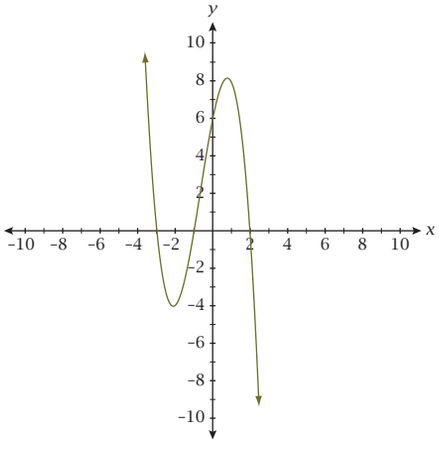
Now the gradient is $\frac{2}{3}$ *and the y intercept is* $(0, -\frac{10}{3})$.

1.2 Find the equation of the quadratic function graphed below:

(a)



(b) From the given graph, determine the equation of the displayed function:



(a) The function has x-intercepts of –6 and 0, and can therefore be expressed in the form:

$$y = a(x + 6)(x - 0) \qquad \text{or } y = ax(x + 6)$$

N.B. The 'a' in these equations is required because the dilation factor has not been determined (nor is it safe to assume that $a = 1$)

Substituting (–3, 36) into the equation will solve for the value of a.

$36 = a(-3)(-3 + 6)$
$36 = -9a$
$-4 = a$

$\therefore$ the equation of this quadratic function is $y = -4x(x + 6)$ or $y = -4x^2 - 24x$

(b) This function is cubic in nature, and will therefore be of the form $y = a(x - b)(x - c)(x - d)$. Given that the x intercepts are (–3, 0), (–1, 0) and (2, 0), these values can be substituted straight into the equation. So, $y = a(x+3)(x+1)(x-2)$. A quick check to confirm the y intercept (i.e. $x = 0$) reveals that this point is (0, 6), and that the value of a is –1. Algebraically, $6 = a(0+3)(0+1)(0-2)$. Therefore, $y = -1(x+3)(x+1)(x-2)$, usually written as y = –(x + 3)(x + 1)(x – 2).

1.3 For the points A (2, 6) and B (3, 4) find the midpoint of AB.

Substitute the given coordinates into the midpoint formula and simplify:

$$\left(\frac{2 + 3}{2}, \frac{6 + 4}{2}\right)$$

$$= \left(\frac{5}{2}, 5\right)$$

1.4 Given that the midpoint of CD is $(-\frac{1}{2}, \frac{1}{2})$ and point D has coordinates (2, 3), determine the coordinates of point C.

Substituting the x-coordinates into the midpoint formula gives:

$$-\frac{1}{2} = \frac{2 + x_2}{2}$$
$-1 = 2 + x_2$ (Simplifying)
$-3 = x_2$

Substituting the y-coordinates into the midpoint formula gives:

$$\frac{1}{2} = \frac{3 + y_2}{2}$$
$1 = 3 + y_2$
$-2 = y_2$

Putting x and y together gives point C as (–3, –2).

1.5 Find the equation of the straight line passing through (2, 5) and

(a) parallel to the line $y = 4x - 1$
(b) perpendicular to the line $y = -\frac{1}{3}x + 2$

(a) Because parallel lines have the same gradient, the equation will be of the form:

$y = 4x + c$

Substituting the point (2, 5) and simplifying will give the value of c.

$5 = 4(2) + c$
$5 = 8 + c$
$-3 = c$
$\therefore y = 4x - 3$

(b) Because the gradient of the line given is $-\frac{1}{3}$ the gradient of the new line will be 3 and the equation will be of the form:

$y = 3x + c$

Substituting the point (2, 5) and simplifying will give the value of c.

$5 = 3(2) + c$
$5 = 6 + c$
$-1 = c$
$\therefore y = 3x - 1$

1.6 Given the following cubic expression $f(x) = x^3 - 4x^2 + 1x + 6$ can be factorised into $f(x) = (x-2)(ax^2 + bx + c)$, determine the coefficients a, b, and c. Hence, fully factorise this cubic function.

If $f(x) = f(x)$, *then* $x^3 - 4x^2 + 1x + 6 = (x-2)(a\,x^2 + b\,x + c)$

Expand the parentheses to obtain $x^3 - 4x^2 + 1x + 6 = a\,x^3 + b\,x^2 + c\,x - 2a\,x^2 - 2b\,x - 2c$

Rearrange the like terms $x^3 - 4x^2 + 1x + 6 = a\,x^3 + b\,x^2 - 2a\,x^2 - 2b\,x + c\,x - 2c$

Compare coefficients of terms with similar degree

So if $x^3 = a\,x^3$

$1 = a$

Now $-4x^2 = b\,x^2 - 2a\,x^2$

$-4x^2 = b\,x^2 - 2(1)x^2$

$-2x^2 = b\,x^2$

Thus $-2 = b$

And $1x = -2b\,x + c\,x$

$1x = -2(-2)x + c\,x$

$3x = c\,x$

Thus $3 = c$

Hence $x^3 - 4x^2 + 1x + 6 = (x-2)(x^2 - 2x - 3)$

$= (x-2)(x-3)(x+1)$

1.7 If $f(x) = x^2 - 3x + 3$ and $g(x) = -2x + 1$, determine

(a) $f(2)$

(b) $g(-2)$

(c) $f(m)$

(d) the value(s) of x that make $f(x) = 1$

(e) the value(s) of p that make $f(p) = g(p) + 4$

(a) $f(2) = 2^2 - 3(2) + 3$ *Substitute the value of 2 for x in f(x)*

$= 4 - 6 + 3$ *Use BIMDAS and simplify*

$= 1$

(b) $g(-2) = -2(-2) + 1$ *Substitute the value of -2 for x in g(x)*

$= 4 + 1$

$= 5$

(c) *If* $f(x) = x^2 - 3x + 3$ *Substitute the value of m for x in f(m)*

Then $f(m) = m^2 - 3m + 3$

(d) *If* $f(x) = x^2 - 3x + 3$

then $1 = x^2 - 3x + 3$ *Substitute the value of 1 for x in f(x)*

i.e. $0 = x^2 - 3x + 2$ *Factorise*

i.e. $0 = (x-1)(x-2)$

therefore $x = 1, \quad x = 2$

(e) *Let* $f(p) = g(p) + 4$

Thus $p^2 - 3p + 3 = -2p + 1 + 4$

$p^2 - 1p - 2 = 0$

$(p-2)(p+1) = 0$

And $p = 2, \; p = -1$

PROBLEMS TO SOLVE

CHAPTER 1: FUNCTIONS I

1. Determine the midpoint of AB for the following:

   (a) A (–1, 5) B (3, 3)

   (b) A (7, –2) B (–5, 10)

   (c) A (3, 5) B (5, –7)

   (d) A (4, 5) B (5, 4)

   (e) A (–4, 2) B (–3, 4)

   (f) A (–8, –2) B (11, 7)

2. Find the equation of the straight line passing through (3, 2) and:

   (a) parallel to the line $y = 4x - 3$

   (b) perpendicular to the line $y = 5x + 1$

3. The line $2x - 3y = 12$ intersects the x-axis at A and the y-axis at B. Find the equation of the line containing both the origin and point C, the midpoint of AB.

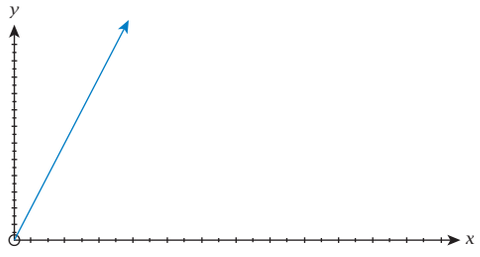
4. One endpoint of a line has coordinates (4, 11) and the midpoint of the line segment has coordinates (2, 14). Find the coordinates of the other endpoint.

5. Find the value(s) of k if $(k + 2)x + 3y = 4$ is perpendicular to $kx - 5y = 0$.

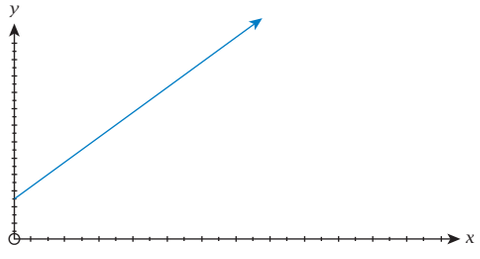
6. Find the value(s) of k if $kx - 2y = 3$ is parallel to $3x + (k + 7)y = -1$.

7. Which of the following represent a directly proportional relationship between x and y?

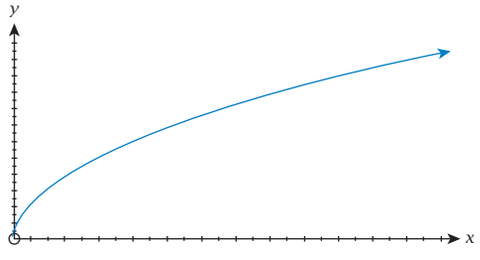
(a)



(b)



(c)



(d)

x	0	1	2
y	2	5	8

(e)

x	0	1	2
y	0	4	8

(f)

x	2	3	4
y	3	4.5	6

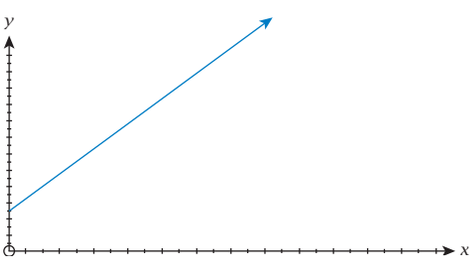
(g) $y = 4x$

(h) $C = \pi \times D$

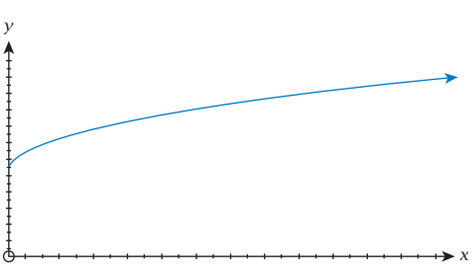
(i) $s = u + at$

8. Which of the following represent a relationship of linear partial variation between x and y?

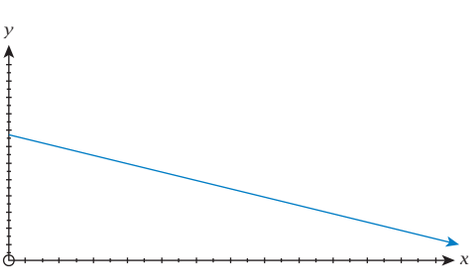
(a)



(b)



(c)



(d)

x	0	1	2
y	2	6	10

(e)

x	1	2	3
y	7	5	3

(f)

x	3	4	5
y	9	16	25

(g) $y = -3x + 7$

(h) $s = u + at$

(i) $P = 4l$

9. For the following functions, state the Domain and Range:

(a) $y = -4x + 1$

(b) $y = (x + 1)^2 - 3$

(c) $y = 2x^3 - 3x + 9$

(d) $y = \sqrt{x - 2} - 4$

(e) $y = -(x - 7)^2 + 5$

(f) $y = -\sqrt{x + 3} + 1$

10. Fully factorise the following cubic functions, showing each as the product of three linear factors

(a) $y = x^3 - 2x^2 - 5x + 6$

(b) $y = x^3 + 3x^2 - 6x - 8$

(c) $y = x^3 - 13x + 12$

(d) $y = 2x^3 - 7x^2 + 2x + 3$

(e) $y = 3x^3 + 8x^2 + 3x - 2$

(f) $y = 2x^3 - 11x^2 + 17x - 6$

11. Determine which of the following are functions. For those which are, indicate whether they are one-to-one or many-to-one.

(a) $\{(1, 2), (2, 3), (3, 4), (4, 5)\}$

(b) $\{(2, 0), (1, 0), (2, -1), (3, 4)\}$

(c) $y = x^2 - 2x + 8$

(d) $y = -\frac{7}{2}x + 1$

(e) $y = 6$

(f) $y = 2\sqrt{x + 1}$

(g) $x = 2$

(h) $y = x^2 + 4$

12. Which of the following equations represent linear functions?

(a) $y = \dfrac{1+x}{x}$

(b) $f(x) = 2x - 9$

(c) $2x + 5y = -6$

(d) $g(x) = x^2 + 3x - 18$

(e) $y = \dfrac{1+x}{3}$

(f) $x = 3y$

13. Determine the gradient of the straight line passing through:

(a) (3, 4) and (12, 16)

(b) (–2, 3) and (4, 6)

(c) (5, –10) and (–3, 15)

(d) (–4, –4) and (–8, 10)

14. What is the gradient and y intercept of the following equations?

(a) $y = 4x - 5$

(b) $2y = 3x + 7$

(c) $6x + 4y = -1$

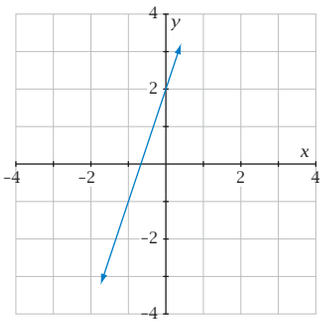
15. Write the equation of the line with the following characteristics:

(a) gradient = –2, y intercept = (0, 4)

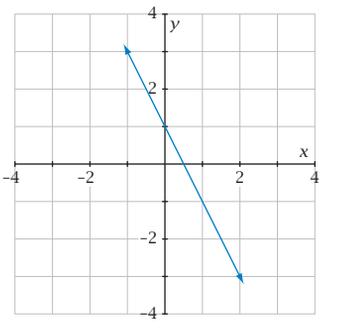
(b) gradient = 4, x intercept = (–5, 0)

16. From the graphs below, determine the equation of the straight line.

(a)



(b)



17. Write the equation of the line that passes through the point (–2, 4) and is parallel to the equation $y = 3x + 2$.

18. What is the equation of the line passing through the point (–5, 3) and is perpendicular to $2y = 10x - 6$?

19. State which, if any, of the points $A(-7, 16)$ and $B(4, 6)$ that lie on the line $3x + 4y = 36$.

20. A line passes through the point with coordinates (1, 2) and has a gradient of 2. If it passes through the point with coordinates (5, k) find the value of k.

21. On a neatly constructed Cartesian plane, represent the following linear equations:

(a) $y = 3x - 2$

(b) $y = \frac{-3x}{2} + 4$

(c) $2x + 3y = 10$

22. Listed below are 5 linear equations.
Prove that three of these equations are parallel to each other.

Line A: $3x + 4y = 6$ *Line B:* $y = \frac{4x}{3} + 2$ *Line C:* $6y - 12 = 8x$

Line D: $\frac{3y - 4x}{3} = 2$ *Line E:* $3x - 6y = -4$

23. Listed below are five equations.
Out of these, find the 3 equations that represent quadratic functions.

Equation A: $y^2 = x(x+3)$ *Equation B:* $\sqrt{y} = (x - 5)$ *Equation C:* $y+2 = x^2-3x$

Equation D: $2y = \frac{x^2 + 2}{x^2}$ *Equation E:* $2(y - 1) = \frac{-4x(2x+1)}{8}$

24. Produce graphs of the following quadratic equations on a neatly constructed Cartesian plane:

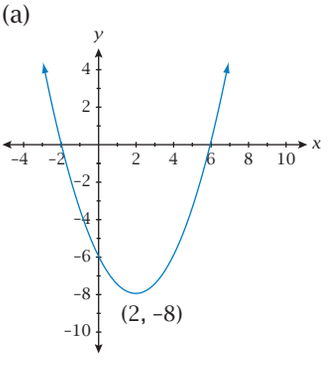
(a) $y = x^2 + 4$

(b) $y = (x + 3)^2 - 2$

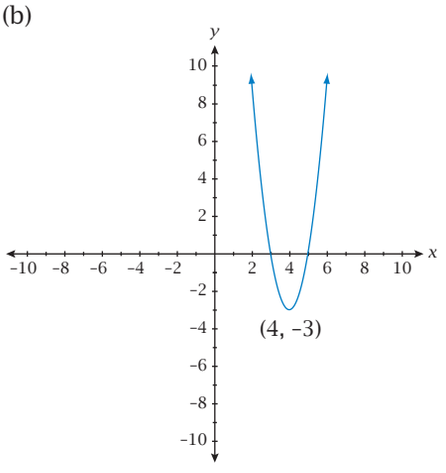
(c) $y = -2x^2 - 5$

25. From the following graphs, determine the quadratic equation:

(a)



(b)



26. Beginning with the equation $y = x^2$, write the new equation after the following transformations have been applied:

(a) shift vertically upwards 4 units

(b) shift horizontally 2 units left, and shift vertically 3 units

(c) reflect in the x axis, dilate with scale factor 2 parallel to the y-axis, and shift horizontally 5 units right

27. For the following quadratic equations, determine i) the equation of the line of symmetry, ii) the location and nature of the turning point and iii) where the graph cuts the y axis.

(a) $y = -3(x - 1)^2 + 2$

(b) $y = (x + 4)(x - 6)$

(c) $y = 2x^2 + 4x - 1$

28. Beginning with the equation $y = x^3$, write the new equation after the following transformations have been applied:

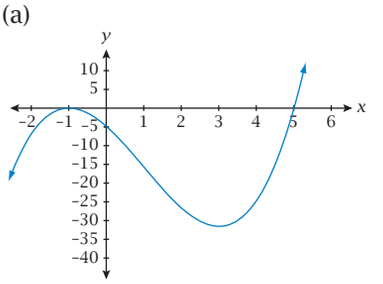
(a) shift vertically downwards 3 units

(b) reflect in the x axis, and shift horizontally 4 units right

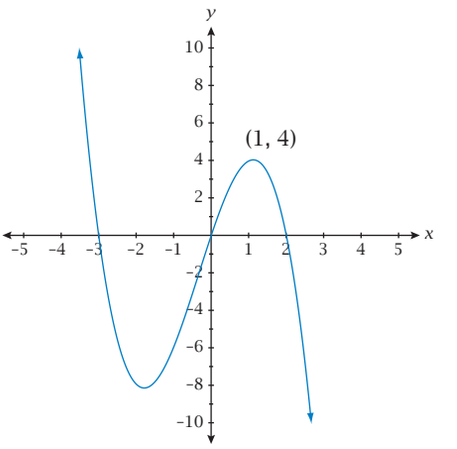
(c) dilate with scale factor $\frac{1}{2}$ parallel to the y-axis, shift horizontally 3 units left, and shift vertically upwards 1 unit.

29. Determine the equations of the cubic functions below.

(a)



(b)



30. Solve the following quadratic equations by factorising:

(a) $0 = x^2 + 7x + 12$

(b) $0 = x^2 - 6x + 9$

(c) $0 = 3x^2 + 6x$

(d) $0 = x^2 - 25$

31. Using the quadratic formula, determine how many solutions each of these equations has:

(a) $0 = x^2 + 2x - 24$

(b) $0 = x^2 + 4x + 4$

(c) $0 = x^2 + 1x + 2$

(d) $0 = 2x^2 + 3x - 5$

32. Use a calculator to find the roots of the following two equations to two decimal equations:

(a) $0 = 0.3x^2 - 4.7x + 8.7$

(b) $0.01x^2 + 0.25x - 0.6 = 0$

33. James sells cars for a living. He is paid a weekly wage of \$550 plus a 2% commission on the value of all cars he sells.

(a) Write a rule for finding James' earnings $\$E$ in a week when he sells cars of value $\$V$.

(b) How much will James earn in a week when he sells cars worth \$60 000?

(c) What value of cars would James have to sell to earn a wage of \$3050?

34. Jack's Phone Service operates for different rates on weekdays and weekends. During the week, the company charges \$15 for a call-out fee and \$5.50 for each hour of work completed by the repairman. On weekends, the call-out fee is \$20 and \$6.50 is charged per hour.

(a) Write a linear equation for the amount charged during a weekday.

(b) Write a linear equation for the amount charged during a weekend.

(c) On Thursday, 5 hours work was completed at Mrs. Jones' house. How much was she charged overall?

(d) On Sunday, 2.5 hours work was done at the local bowling alley.
What was the fee?

(e) For how long did the repairman spend working on Saturday if he charged $36.50 in fees?

35. If $f(x) = 4x - 17$ and $g(x) = x^2 - 3x - 5$ then find

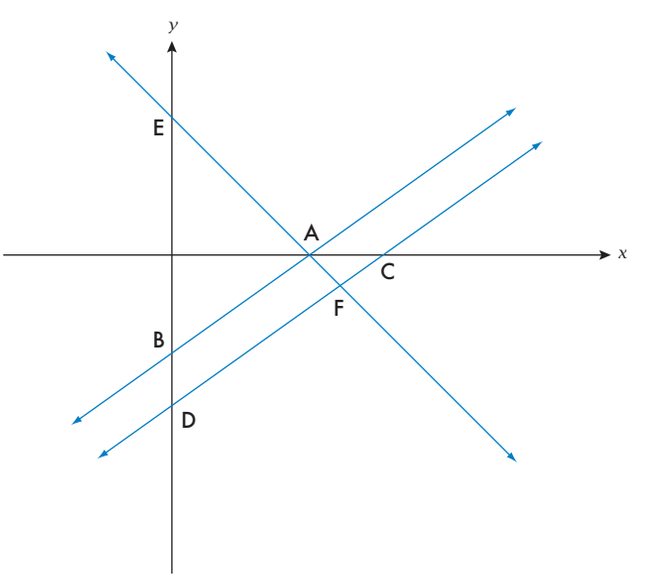
(a) $f(4) - f(-3)$

(b) $g(-3)$

(c) x so that $f(x) = 103$

(d) x so that $f(x) = g(x)$

36.



The three lines drawn above have their equations given below.
However, they are not necessarily in order.

(a) $y = 2x - 6$

(b) $y - 2x = -8$

(c) $x + y = 3$

Determine the coordinates of A, B, C, D, E and F.

37. A marketing company has found that if they sell their product over a certain time period, t (in months) the overall profit is defined by the equation:

$$C(t) = -t^2 + 10t + 2 \qquad (t > 0).$$

(a) During what time period will this product yield a profit for the company?

(b) During what time period will this product yield a loss for the company?

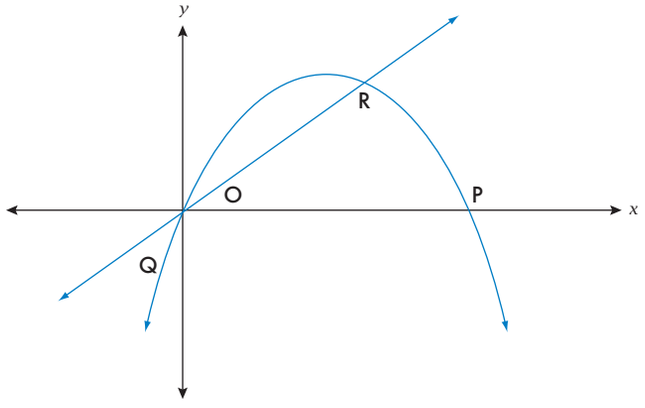
38. A frisbee is thrown straight upwards from the ground. The equation defining its movement is $h = 53.9t - 4.9t^2$, where h is height in metres and t is time elapsed in seconds.

(a) After how many seconds will the frisbee strike the ground?

(b) When is the frisbee at a height of 88.2 metres?

(c) When is the frisbee at its maximum height?

39. The curve $y = x(6 - x)$ cuts the x axis at the origin and at the point P.
A rough sketch of this parabola and a linear function is provided below.



(a) State the coordinates of P, which lies on the x axis.

(b) Given that the point $Q(-1, q)$ lies on the curve, find the value of q.

(c) Find the equation of the line of symmetry and hence the coordinates of the turning point of the curve.

(d) A line cuts the curve at the origin and at the point R. Given that the x coordinate of R is 4 find the y coordinate of R.

40. If $f(x) = x^2 + 7x - 8$, find

(a) $f(5)$

(b) $f(-2)$

(c) x when $f(x) = 0$

(d) $f(k - 1)$, and simplify the expression obtained.

41. The number of coloured crayons sold varies inversely with the price of each crayon. Three thousand crayons are sold at the price of \$0.20 each.

(a) Determine the constant of proportionality, k.

(b) Find the number of crayons that could be sold for \$0.15 each

(c) Find the price of crayons when 5,000 are sold.

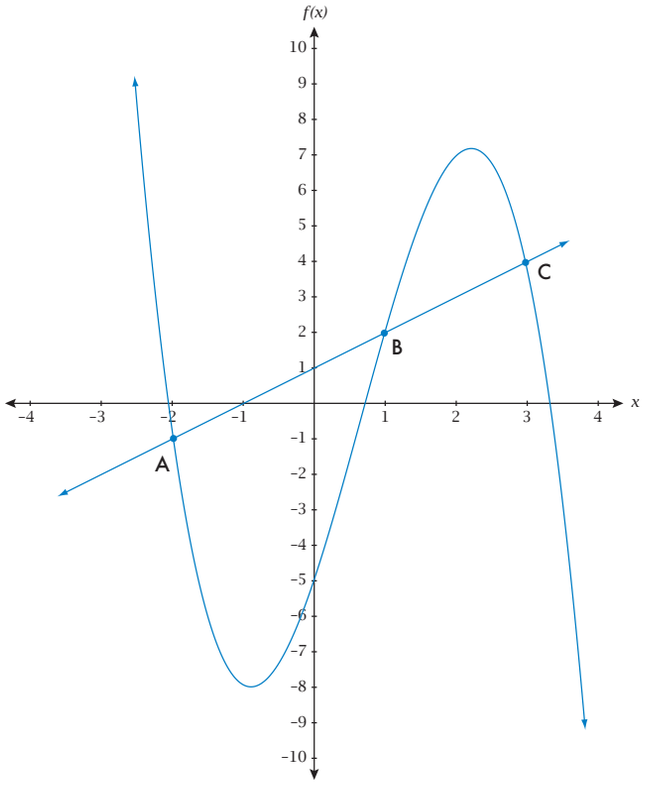
42. Consider the graph of $y = x(x-1)(x+2)$ then find

(a) For what range of values of k does the equation $x(x-1)(x+2) = k$ have three solutions?

(b) For what values of x is $x(x-1)(x+2)$ negative?

(c) For what values of x is $x(x-1)(x+2)$ positive?

43. The graphs of $f(x) = -x^3 + 2x^2 + 6x - 5$ and $g(x) = x + 1$ are drawn below.



Determine the coordinates of the points marked A, B and C, the points of intersection of the two graphs.

44. If $h(x) = x^3 + 2x^2 - 3x - 6$, show that $h(-3) = h(0) = h(1)$.

45. For a triangle the base length is five more than double the height. If the area of the triangle is 450 square centimetres, find the base length and height of the triangle.

2 FUNCTIONS II

Syllabus Checklist

By the end of this chapter, you should be able to:

☐ Inverse proportion
- examine examples of inverse proportion
- recognise features and determine equations of the graphs of $y = \frac{1}{x}$ and $y = \frac{a}{x-b}$, including their hyperbolic shapes and their asymptotes.

☐ Graphs of relations
- recognise features and determine equations of the graphs of $x^2 + y^2 = r^2$ and $(x-a)^2 + (y-b)^2 = r^2$, including their circular shapes, their centres and their radii
- recognise features of the graph of $y^2 = x$, including its parabolic shape and its axis of symmetry

☐ Powers and polynomials
- recognise features of the graphs of x^n for $n \in N$, $n = -1$ and $n = ½$, including shape, and behaviour as $x \to \infty$ and $x \to -\infty$.

FORMULAE AND DEFINITIONS

A *relation* is a set of ordered pairs, e.g. {(1,2), (1,3), (1,4)}.

You will have noticed how in the above example the first member corresponds (or maps onto) more than one second member. This feature distinguishes relations from functions.

One quick method used to determine whether a graphed set of ordered pairs represents a relation is the *vertical line test*. At any point (other than a stationary point) draw a vertical line through the graph. If the line cuts the graph at more than one place, the graph represents a relation.

Circular Equations

A common type of relation is a *circle*. A circle represents a set of points which are at a fixed distance (i.e. the radius) from a given point. One example of a circular equation is $x^2 + y^2 = 9$, graphed in Figure 1. This means that the circle represented by this equation has a centre of (0, 0) and a radius of 3.

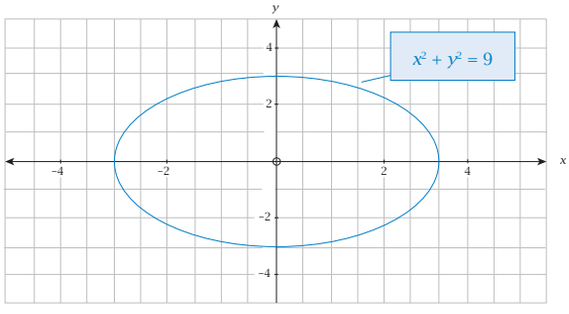


Figure 1: $x^2 + y^2 = 9$

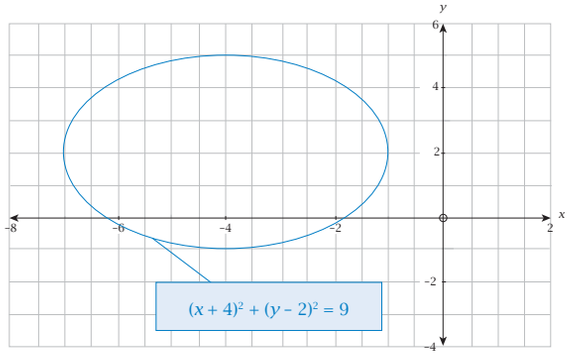


Figure 2: $(x + 4)^2 + (y - 2)^2 = 9$

If we were to move this circle 4 units left and two units up the new equation would be $(x + 4)^2 + (y - 2)^2 = 9$, which has been graphed in Figure 2.

The Cartesian equation of a circle with centre (a, b) and radius r is $(x - a)^2 + (y - b)^2 = r^2$.

When expanded this equation is written: $x^2 + y^2 - 2ax - 2by + a^2 + b^2 - r^2 = 0$.

In this form note that:

- The coefficients of the x^2 and the y^2 terms are identical
- The terms can only involve x^2, y^2, x, y, and a constant (and any of these last 3 could be 0).

The relation $y^2 = x$

Another common type of relation is $y^2 = x$, which is found within the conics family. Below is the graph of this relation. It can be thought of both as a 'sideways' quadratic (of form $y = x^2$) or the square root function $y = \pm\sqrt{x}$ showing both the positive and negative components.

Interestingly all of the rules for the function $y = x^2$ apply for the relation $y^2 = x$ (just remember to switch y for x).

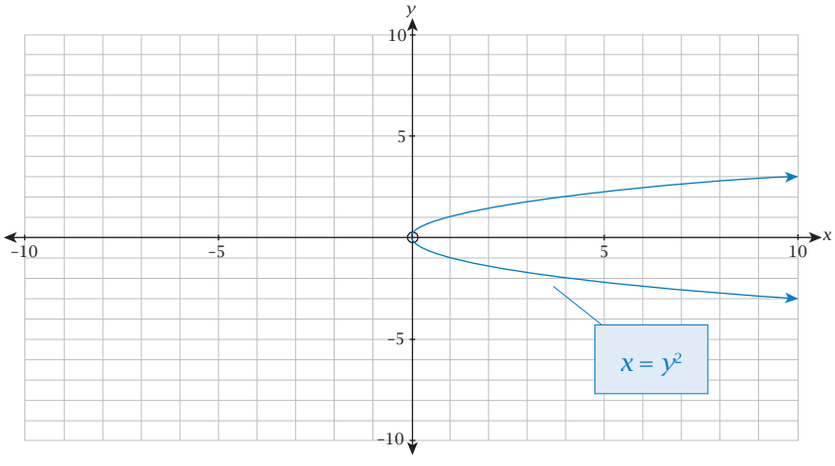


Figure 3: $y^2 = x$

Square Root Function

The general form of the square root function is $y = \sqrt{x}$, and this is an increasing function with a vertex at (0, 0) (see Figure 4). While dilation transformations of $y = \sqrt{x}$ will not change the vertex (see Figure 5), translation transformations will (see Figure 6).

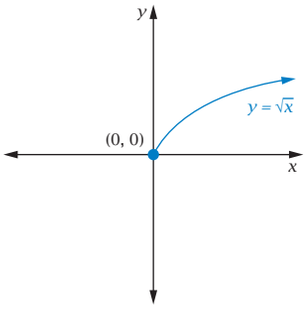


Figure 4

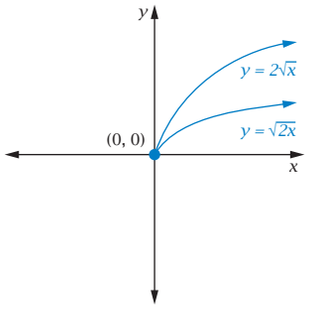


Figure 5

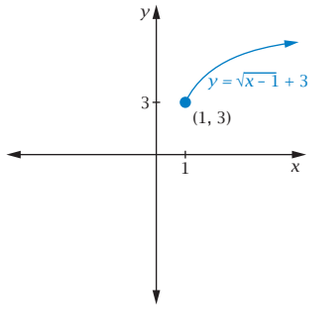


Figure 6

The considerations of Domain and Range for reciprocal functions, circular equations, and conic sections $y^2 = x$ and the square root function are tabulated below

Function/Relation	Domain	Range
Reciprocal Function	All x except vertical asymptote	All y except horizontal asymptote
Conic Section $y^2 = x$	All x but consider turning point	All y
Circular Equation	$r - a \le x \le r + a$	$r - b \le y \le r + b$
Square Root Function	All x but consider vertex	All y but consider vertex

Inverse proportion (or Inverse variation)

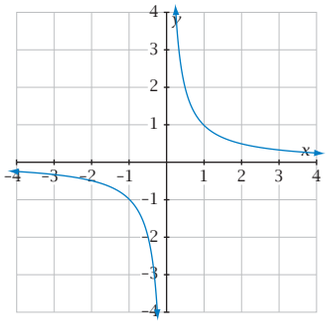
If one quantity, y, is inversely proportional to another quantity, x, then $y \propto \frac{1}{x}$. Furthermore, we can say that $y = \frac{k}{x}$, or that $xy = k$, where k is a constant. This means that as the y quantity increases, the x quantity decreases. Similarly, the opposite for both variables is true. From our previous notes, it is evident that reciprocal functions are similar to those found in inverse proportion. It should be noted that in practical applications of inverse proportion the variables assume non-negative values and the graph is restricted to the first quadrant.

Reciprocals

The general form of the reciprocal equation is written as $y = \frac{c}{x}$, where c is a constant. It can also be expressed as $xy = c$.

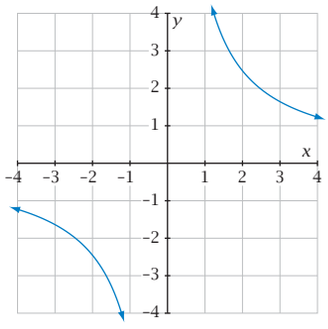
- If $c > 0$ then the graph of the function will lie in the first and third quadrants.
- If $c < 0$ then the graph of the function will lie in the second and fourth quadrants.

Some examples of reciprocal functions are shown below.



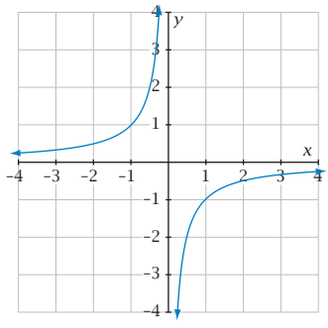
$y = \frac{1}{x}$

Figure 7



$y = \frac{5}{x}$

Figure 8



$y = -\frac{1}{x}$

Figure 9

Both of the graphs above have a horizontal asymptote at $y = 0$, and a vertical asymptote at $x = 0$.

Horizontal and vertical translations can be applied to reciprocal functions in much the same manner as they can to quadratic and cubic functions.

Horizontal translations take the form $y = \frac{c}{x+a}$.

- If $a > 0$, then the function will be translated to the left *'a'* units.
- If $a < 0$, then the function will be translated to the right *'a'* units.

The vertical asymptote will also have shifted horizontally and now has the equation $x = -a$.

Vertical translations take the form $y = \frac{c}{x} + b$.

- If $b > 0$, then the function will be translated vertically upwards *'b'* units.
- If $b < 0$, then the function will be translated vertically downwards *'b'* units.

The horizontal asymptote will also have shifted vertically and now has the equation $y = b$.

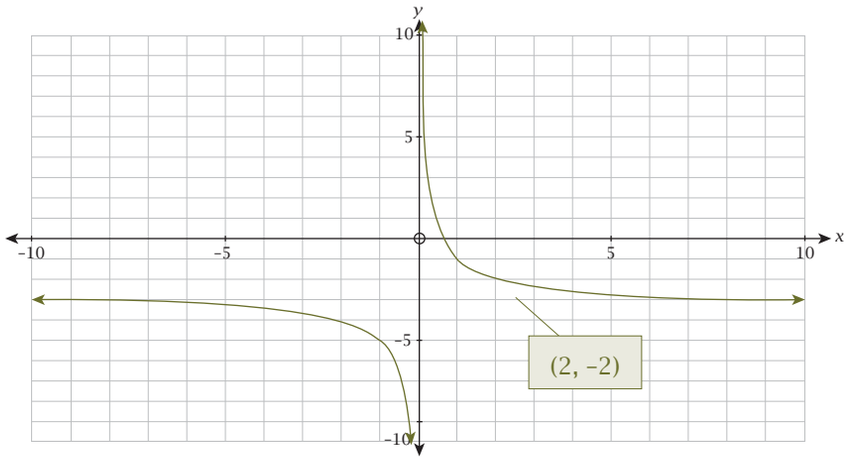
Other functions to consider include $y = \sqrt{x}$, $y = \sqrt[3]{x}$ and $y = \frac{1}{x^2}$

These functions all behave in the same manner as those already discussed in this chapter when constants are substituted into their original form.

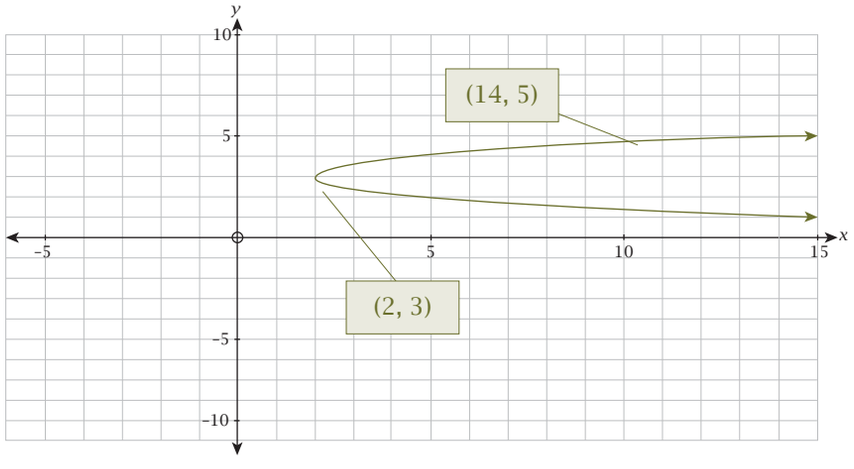
Worked Examples

2.1 From the given graphs, determine the equation of the displayed function or relation:

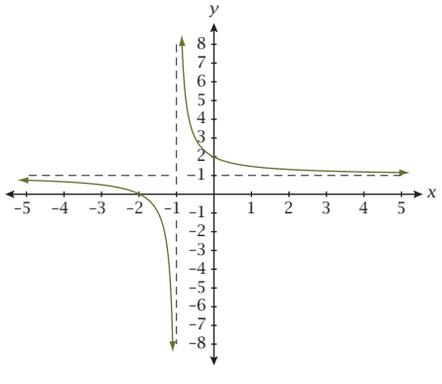
(a)



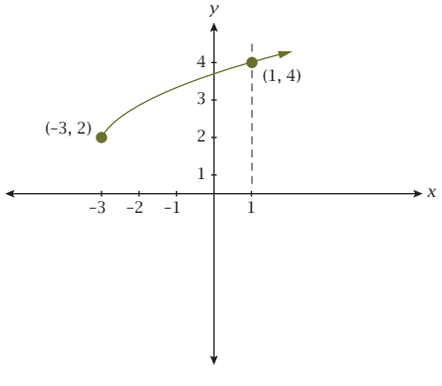
(b)



(c)



(d)



(a) This function is reciprocal in nature due to the hyperbolic curve, horizontal asymptote, and vertical asymptote. Because the vertical asymptote is along the line $x = 0$ the general equation would be of the form $y = \frac{a}{x} + c$. The horizontal asymptote is at $y = -3$, so $c = -3$. Finally we obtain the value of a by substituting the given point $(2, -2)$ into the equation $y = \frac{a}{x} - 3$. Solving gives $a = 2$, and thus $y = \frac{2}{x} - 3$.

(b) The shape of this relation is a transformation of the form $y^2 = x$, due to the parabolic shape of the graph. The turning point has been moved 2 units horizontally and 3 units vertically which would yield $a(y - 3)^2 + 2 = x$. Finally we obtain the value of a by substituting the given point $(14, 5)$ into the equation. Solving gives $a = 3$, and thus $3(y - 3)^2 + 2 = x$.

(c) This function is reciprocal in nature, given the hyperbolic curve, horizontal asymptote and vertical asymptote. The general equation for this function is $y = \frac{a}{x+b} + c$. Because the vertical asymptote has equation $x = -1$, the value of $b = 1$. Because the horizontal asymptote has equation $y = 1$, $c = 1$.
We must now substitute a point to ascertain the value of a; namely, $(0, 2)$.
So, $2 = \frac{a}{0+1} + 1$, and solving for a gives the value of a to be 1. Thus, $y = \frac{1}{x+1} + 1$.

(d) The shape of this graph is a transformation of $y = \sqrt{x}$, due to the 'half parabola' with a vertex at $(-3, 2)$. The vertex has been moved 3 units horizontally and 2 units vertically which would give $y = a\sqrt{x+3} + 2$. Substituting the given point $(-3, 2)$ into the equation would give $a = 1$. Therefore the equation is $y = \sqrt{x+3} + 2$.

2.2 Starting with the given 'base' equations, translate each 3 units vertically up and 2 units horizontally left. Write the transformed equation.

(a) $x^2 + y^2 = 4$

(b) $y^2 = x$

(c) $y = \frac{1}{x}$

(a) $(x + 2)^2 + (y - 3)^2 = 4$

(b) $(y - 3)^2 + 2 = x$

(c) $y = \frac{1}{x-2} + 3$

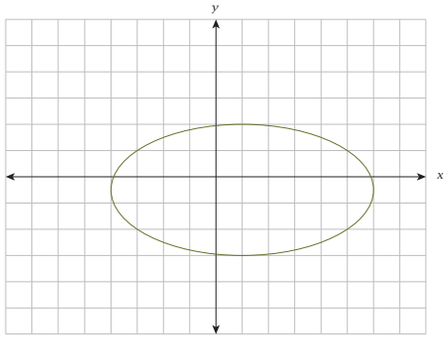
2.3 Determine the centre and radius for the circle with equation: $x^2 + y^2 + 2x - 4y - 15 = 0$

$x^2 + 2x + y^2 - 4y - 15 = 0$	Reorder the terms
$(x + 1)^2 - 1 + (y - 2)^2 - 4 - 15 = 0$	Complete the square for both the x terms and y terms
$(x + 1)^2 + (y - 2)^2 - 20 = 0$	Collect like terms
$(x + 1)^2 + (y - 2)^2 = 20$	Move constant to the RHS

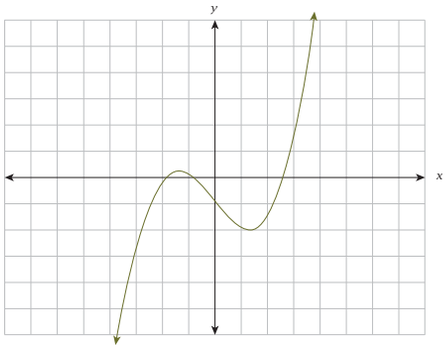
$\therefore$ circle centre $(-1, 2)$ and radius $\sqrt{20}$ units.

2.4 Apply the 'vertical line test' to each of the following graphs, and conclude whether the graph represents a relation or a function.

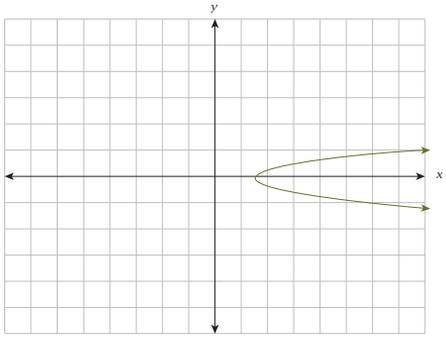
(a)



(b)



(c)



(a) A vertical line will intersect the graph in two places everywhere (except the end points). Therefore, this graph represents a relation.

(b) At no point on this graph does a vertical line intersect the graph twice. Therefore this graph represents a function.

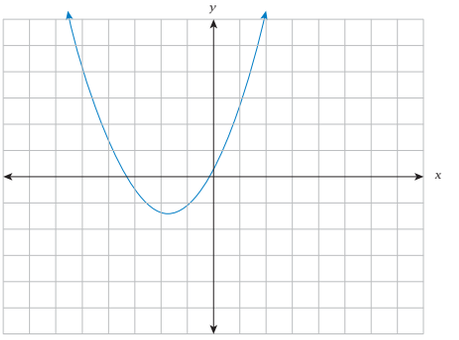
(c) Except for the turning point, a vertical line will intersect the graph twice. Therefore, this graph represents a relation.

PROBLEMS TO SOLVE

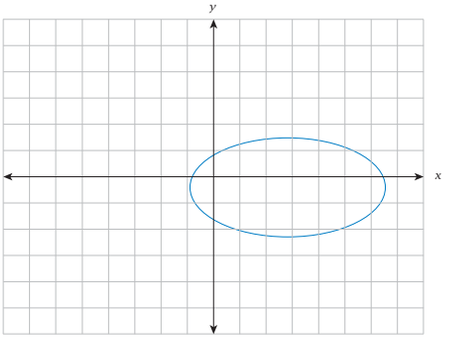
CHAPTER 2: FUNCTIONS II

1. Apply the 'vertical line test' to each of the following graphs, and conclude whether the graph represents a relation or a function.

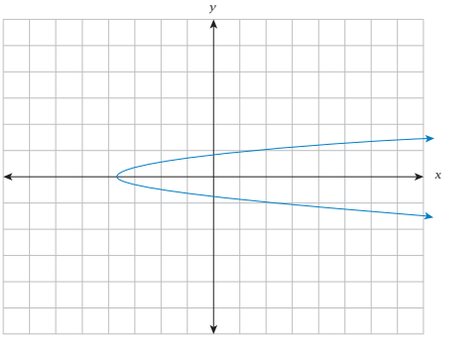
(a)



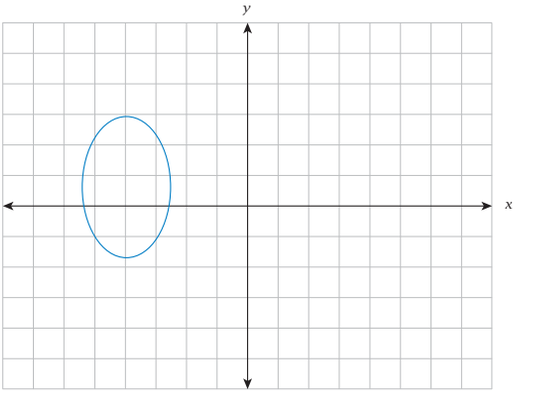
(b)



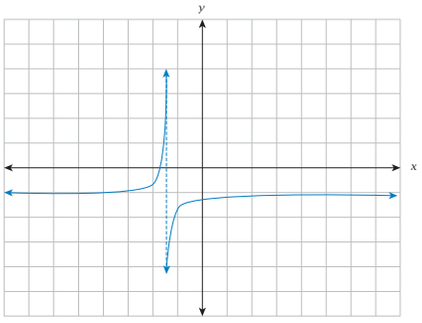
(c)



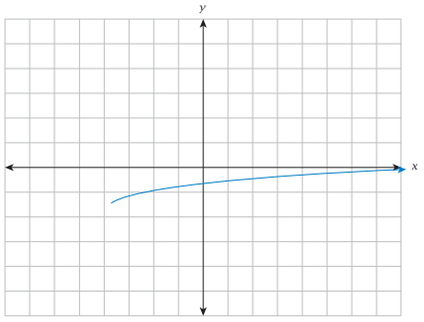
(d)



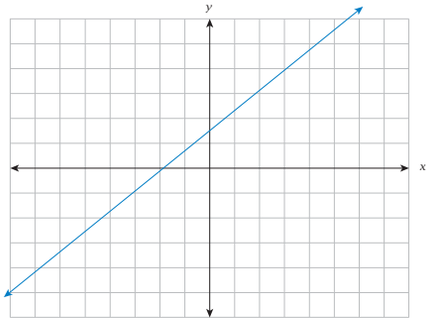
(e)



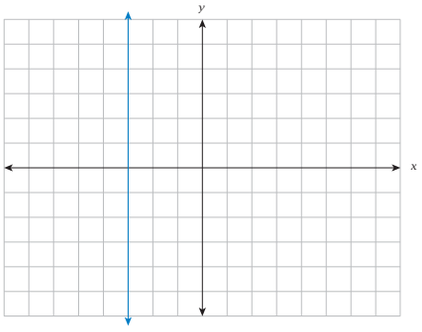
(f)



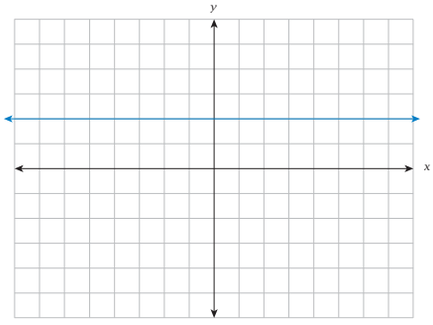
(g)



(h)



(i)



2. From the following identify which are relations:

(a) $x^2 + y^2 = 10$

(b) $3y = 2x^2 + 9$

(c) {(1,1), (1,2), (2,4)}

(d) $xy = 4$

(e) $y = -\frac{3}{x-2}$

(f) $x = -3$

(g) $y^2 - 2y + 1 =$

(h) $(x + 1)^2 + y^2 = 5$

(i) {(0,1), (1,2), (2,3)}

(j) $y^2 = x$

(k) $y = \sqrt{x} - 1$

(l) $y = 4$

3. Each of the following circles has centre (0, 0). Provide the radius for each.

(a) $x^2 + y^2 = 16$

(b) $x^2 + y^2 = 25$

(c) $x^2 + y^2 = 20$

(d) $2x^2 + 2y^2 = 200$

(e) $4x^2 + 4y^2 = 36$

(f) $7x^2 + 7y^2 = 49$

4. Find the equations of the circles determined by the following conditions (use form $x^2 + y^2 = a^2$):

(a) centre (0, 0), radius 5

(b) centre (0, 0), $r = \sqrt{5}$

(c) centre (2, 1), radius 6

(d) centre (–4, –3), radius $\sqrt{3}$

5. Which of the following are <u>not</u> circles?

(a) $x^2 + y^2 - 4x + 2y = 8$

(b) $x^2 + y^2 = 9$

(c) $x^2 + y^2 + 100 = 0$

(d) $x^2 + 2y^2 + 2x - 4y = 4$

(e) $2x^2 + 2y^2 = 50$

(f) $x^2 + y^2 - 3xy = 20$

6. Starting with the reciprocal equation $y = \frac{1}{x}$ apply the given transformations to produce a new equation. Write this new equation and state its domain and range.

(a) Left 2 units

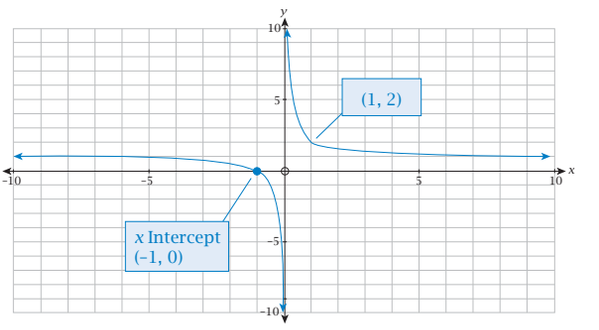
(b) Down 3 units

(c) Up 2 units, right 4 units

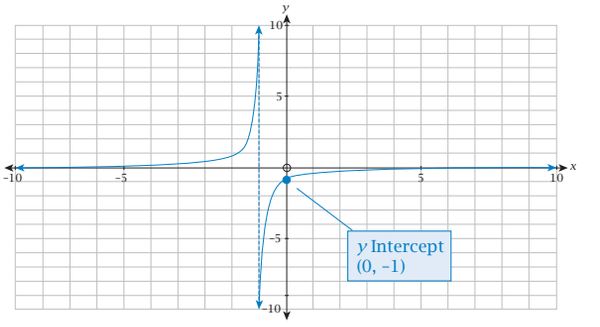
(d) Down 4 units, left 6 units

7. Determine the equation of the reciprocal functions graphed below:

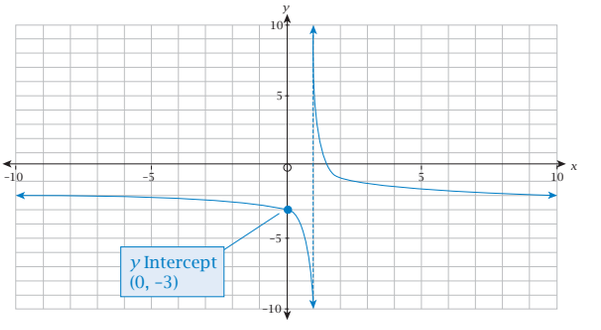
(a)



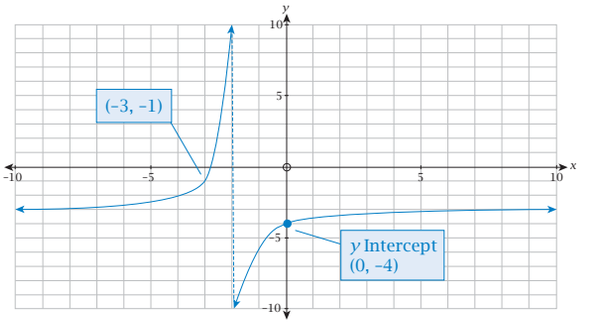
(b)



(c)



(d)



8. From the tables below determine whether or not a relationship of inverse variation exists between the variables.

(a)

x	1	2	3	4	5
y	10	5	$\frac{10}{3}$	$\frac{5}{2}$	2

(b)

x	1	2	3	4	5
y	6	3	2	$\frac{4}{3}$	$\frac{1}{3}$

(c)

x	1	2	3	4	5
y	20	10	$\frac{20}{3}$	5	4

(d)

x	1	2	3	4	5
y	-1	$\frac{1}{2}$	$-\frac{1}{3}$	$\frac{1}{4}$	$-\frac{1}{5}$

9. For each of the following situations write an equation that connects the variables. Use k to represent the constant of proportionality in each situation

(a) N varies inversely as x

(b) The amount of time spent studying for an examination (t) is inversely proportional to the amount of time spent sleeping (s).

10. For each of the following situations write an equation that connects the variables:

(a) The variable x varies inversely as y, and when $x = 3$, $y = 4$.

(b) The Volume (V) of a certain gas is inversely proportional to the surrounding pressure (P). From observations it is known that when 5 units of pressure are exerted, 4 units of volume result.

11. For each of the following conic sections, state the equation of the line of symmetry. Hence, or otherwise, determine the coordinates of the turning point.

(a) $y^2 = x$

(b) $y^2 - 4 = x$

(c) $y^2 + 7 = x$

(d) $(y - 2)^2 = x$

(e) $(y + 8)^2 = x$

(f) $(y - 1)^2 + 3 = x$

12. Find the Cartesian equation of a circle with centre (2, –1) and radius 4 units. If each of the following points lie on the circumference of the circle, determine the value of a, b, c and d.

(a) $(6, a)$

(b) $(2, b)$, $b > 0$

(c) $(c, -1)$, $c > 0$

(d) $(d, -5)$, $d > 0$

13. For each circular equation state the centre and radius

(a) $x^2 + y^2 = 9$

(b) $x^2 - 2x + y^2 = 5$

(c) $16x^2 + 16y^2 = 4$

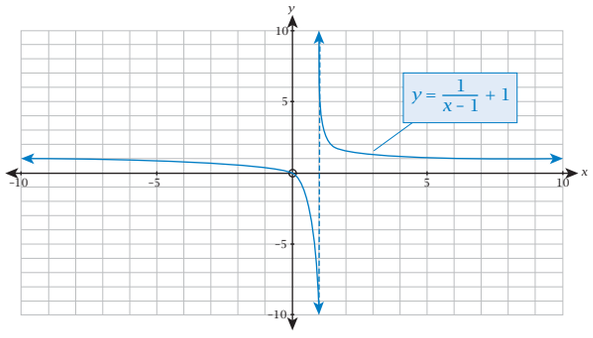
(d) $(x + 4)^2 + (y - 1)^2 = 64$

(e) $x^2 + y^2 - 6x + 4y + 4 = 0$

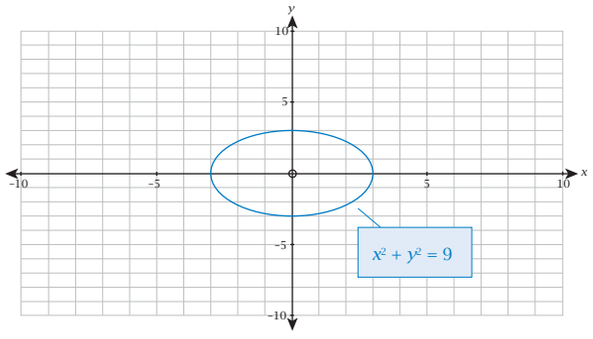
(f) $4(x + 4)^2 + 4(y - 1)^2 = 18$

14. State the domain and range for the following functions or relations:

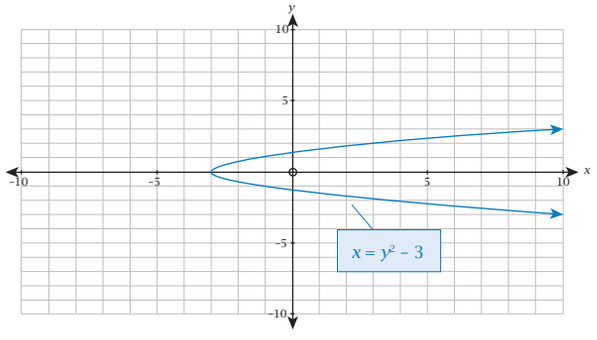
(a)



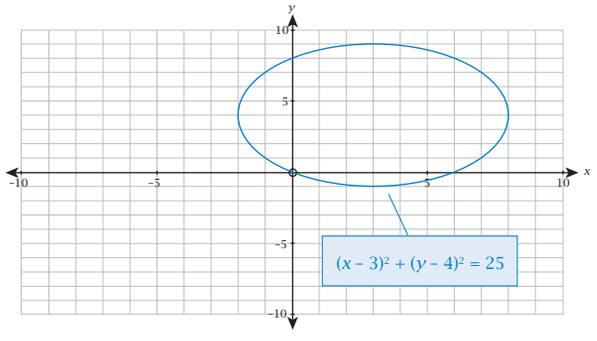
(b)



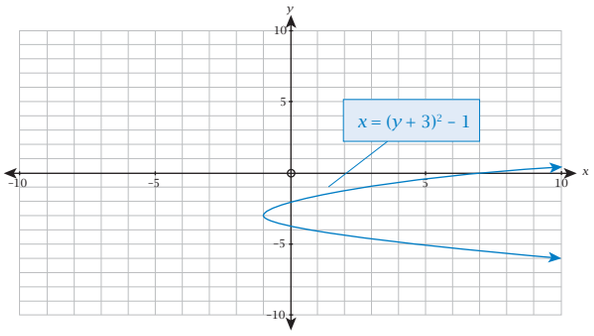
(c)



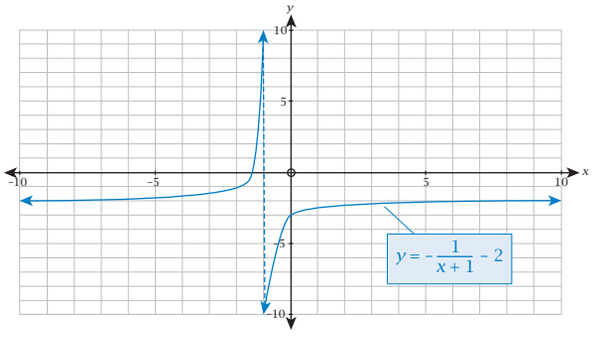
(d)



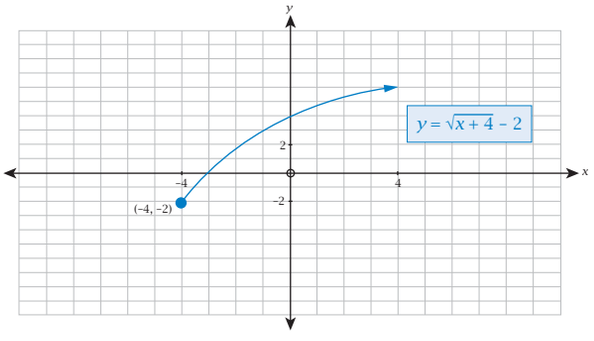
(e)


y
10
5
−10
−5
5
10
x
−5
−10
$x = (y + 3)^2 - 1$

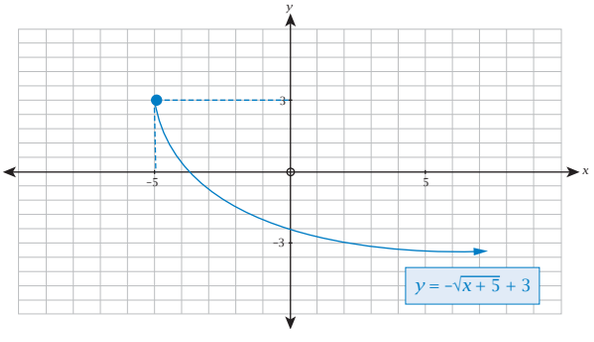
(f)


y
10
5
−10
−5
5
10
x
−5
−10
$y = -\frac{1}{x+1} - 2$

(g)


y
2
−4
4
x
−2
(−4, −2)
$y = \sqrt{x+4} - 2$

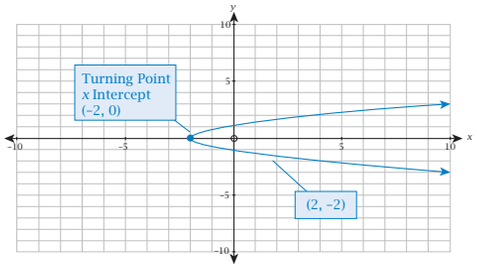
(h)


y
3
−5
5
x
−3
$y = -\sqrt{x+5} + 3$

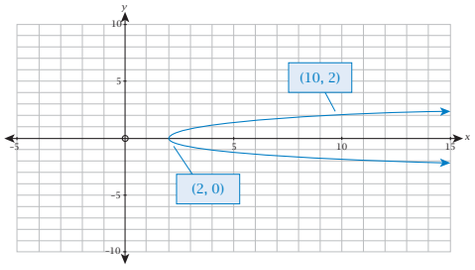
15. Match the given equations to the correct graph. Then, determine the value of the unknown constant(s) in each equation.

(i) $ay^2 - 2 = x$ (ii) $by^2 + 2 = x$ (iii) $y = \frac{c}{x+d} + e$

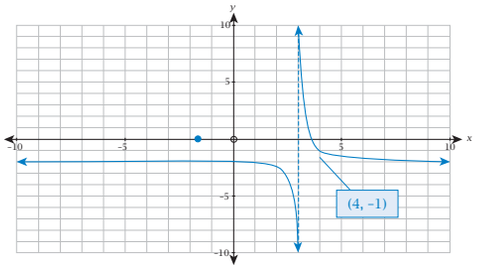
(iv) $y = \frac{-f}{x+g} + h$ (v) $(x - i)^2 + (y - j)^2 = 25$ (vi) $(x - k)^2 + (y - l)^2 = 16$



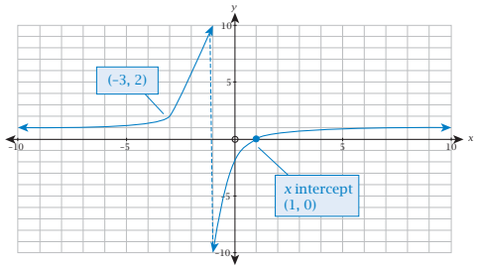
A



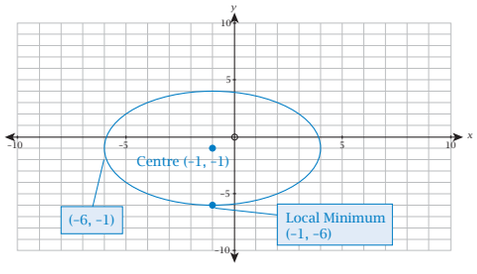
B



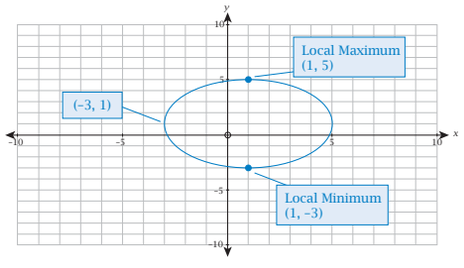
C



D



E



F

16. AB is the diameter of a circle whose equation is $x^2 + y^2 - 2x - 2y - 3 = 20$. If A is the point (–2, –3), find the coordinates of B.

17. M varies inversely as N, and $M = 15$ when $N = 3$.

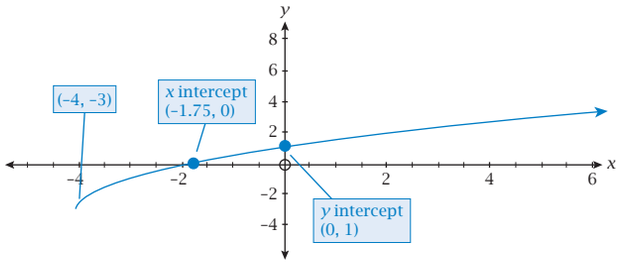
(a) Find the constant of proportionality, k, and hence determine the rule relating M and N.

(b) Use your graphics calculator to plot the graph governed by the rule found in part a.

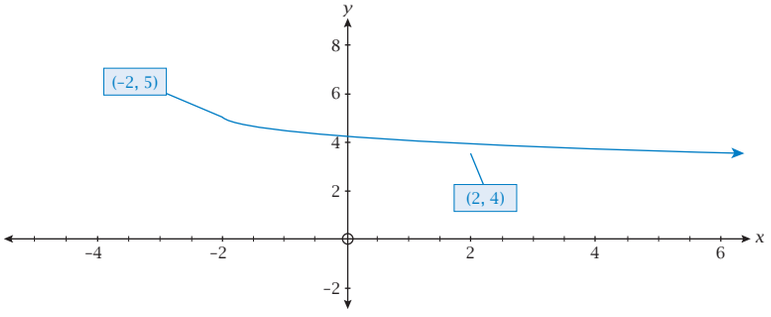
(c) From your rule or graph, find the value of N when $M = 20$.

18. Determine the equation of the graphed functions below:

(a)



(b)



TRIGONOMETRY I

3

Using Trigonometry to find distances and angles (in degrees and radians) in geometric figures in two and three dimensions is the focus of this section.

Radians are introduced because of their direct association with arc length,

Syllabus Checklist

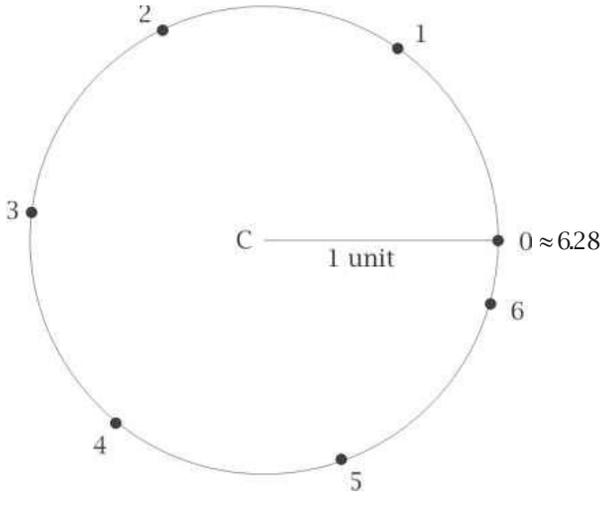
On the completion of this chapter you should be able to:

☐ Circular measure and radian measure
- define and use radian measure and understand its relationship with degree measure
- calculate lengths of arcs and areas of sectors and segments in circles

☐ Cosine and sine rules
- review sine, cosine and tangent as ratios of side lengths in right-angled triangles
- understand the unit circle definition of cos θ, sin θ and tan θ and periodicity using degrees
- examine the relationship between the angle of inclination of a line and the gradient of that line
- establish and use the cosine and sine rules, including consideration of the ambiguous case and the formula $Area = \frac{1}{2}bc\ sin\ A$ for the area of a triangle

THE RADIAN CONCEPT – MEASURING ANGLES IN A NEW WAY

Imagine getting a bendy ruler and wrapping it around a circle of radius 1 unit with the scale on the ruler the same as the radius as shown below.

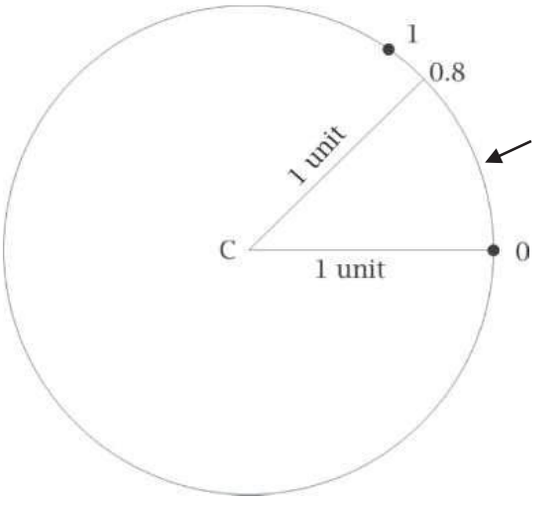


Now because $r = 1$

$$\begin{aligned} C &= 2\pi r \\ &= 2\pi \times 1 \\ &= 2\pi \\ &\approx 6.28\ \textit{units}\ (2\text{ d.p.}) \end{aligned}$$

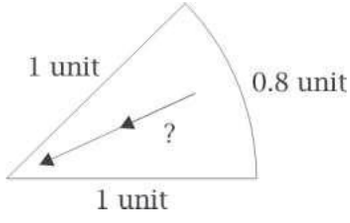
So ≈ 6.28 units will coincide with the starting point of 0.

Now consider an arc length of 0.8. This is where 0.8 on the ruler meets the circumference of the circle. The sector formed is as shown below.



The arc length of 0.8 units is now associated with the size of the angle subtended at the centre.

i.e.

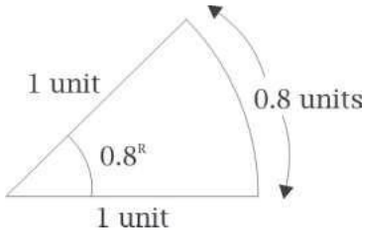


The association between the sector angle and the arc length when the radius is one unit is said like this:

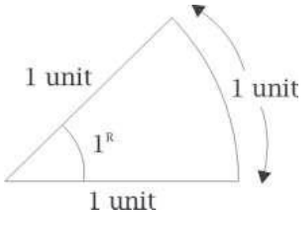
> An arc length of 0.8 units in a sector of radius 1 unit forms an angle at the centre which is 0.8 Radians in size.

0.8 Radians is notated as 0.8^R

The complete sector is now:



Obviously an angle of 1R is formed by an arc length at 1 unit, i.e.:



> The big question now is
> "What size in degrees is an angle of 1^R?"

Well

An arc length of 1 unit forms an angle of 1^R at the centre

An arc length of 2 units forms an angle of 2^R at the centre

An arc length of 3 units forms an angle of 3^R at the centre

An arc length of 4 units forms an angle of 4^R at the centre

An arc length of 5 units forms an angle of 5^R at the centre

An arc length of 6 units forms an angle of 6^R at the centre

An arc length of 6.28 units forms an angle of 6.28^R at the centre

An arc length of units 2π forms an angle of $2\pi^R$ at the centre

So an arc length of 2π is around the whole unit circle once, which makes an angle of $2\pi^R$ at the centre.

This means that

$$1 \text{ revolution} = 360^\circ = 2\pi^R$$

A proportion table leads to :-

$$2\pi^R = 360^\circ$$
$$\pi^R = 180^\circ$$
$$\frac{\pi^R}{2} = 90^\circ$$
$$\frac{\pi^R}{6} = 30^\circ$$
$$\frac{\pi^R}{3} = 60^\circ$$
$$\frac{\pi^R}{4} = 45^\circ \quad \text{etc}$$

Also if

$$\pi^R = 180^\circ$$

then $1^R = \frac{180^\circ}{\pi}$

i.e. $1^R = 57.2957795^\circ$ 7 d.p.

or $1^R = \left(\frac{180}{\pi}\right)^\circ$ exactly

and $1^\circ = \left(\frac{\pi}{180}\right)^R$ exactly

Degrees and Radians Conversion formulae

Consider an angle of size θ° which has a size in radians of θ^R.

If $1^\circ = \left(\frac{\pi}{180}\right)^R$ then θ° must have a size of $\left(\frac{\pi}{180^\circ} \times \theta^\circ\right)^R$

So $\theta^R = \left(\frac{\pi}{180^\circ} \times \theta^\circ\right)^R = \left(\frac{\pi\theta^\circ}{180^\circ}\right)^R$

And $\theta^\circ = \left(\frac{180}{\pi} \times \theta^R\right)^\circ = \left(\frac{180\theta^R}{\pi}\right)^\circ$

are the conversion formulae between degrees and radians.

Arc length in Radians

The arc length formula in a sector of radius r units and central angle of θ° is:

$$l = \frac{\theta^\circ}{360^\circ} \times 2\pi r$$
$$= \frac{\theta^\circ \pi r}{180^\circ} \quad \text{Divide 2 into 360}$$
$$= \left(\frac{\pi\theta^\circ}{180^\circ}\right) r \quad \text{Bracket everything except r}$$
$$= \theta^R r$$

So $l = \theta^R r$ or $l = r\theta$ where θ must be in radians if the ^{R} is left off

Then the new arc length formula is

$$l = r\theta$$

Area of a Sector in Radians

The area of a sector formula which has a radius of r units and central angle θ° is:

$$\begin{aligned} A_s &= \frac{\theta^\circ}{360^\circ} \times \pi r^2 \\ &= \frac{1}{2}\left(\frac{\pi\theta^\circ}{180^\circ}\right) r^2 \\ &= \frac{1}{2}\theta^R r^2 \\ &= \frac{1}{2} r^2 \theta \end{aligned}$$

Then the new sector area formula is:

$$A_S = \frac{1}{2} r^2 \theta$$ Where θ is in radians

Radian mode on your calculator

In Rad mode all inputted angles are presumed to be in radians and hence all calculated angles will also be in radians.

$$\text{So if } \cos\theta = \frac{2^2 + 3^2 - 4^2}{2 \times 2 \times 3}$$
$$\text{then } \cos\theta = -0.25$$
$$\theta = \cos^{-1}(-0.25)$$
$$\theta = 1.8235^R \quad (4 \text{ d.p.})$$
$$\text{and } \sin\left(1.8235^R\right) = 0.9682 \quad (4 \text{ d.p.})$$

Conversion summary

To change θ° into θ^R multiply by $\frac{\pi}{180}$

To change θ^R into θ° multiply by $\frac{180}{\pi}$

Examples

1) Change $\frac{5\pi^R}{12}$ to degrees. Replace θ by 180°

$$= \frac{5 \times 180^\circ}{12}$$
$$= 75^\circ$$

2) Change 140° to radians as an exact value.

$$\frac{140^\circ}{1} \times \frac{\pi}{180^\circ}$$
$$= \frac{7\pi}{9}$$

Divide 140° and 180° both by 20°

3) Change 140° to radians to 3 d.p.

$$\frac{140^\circ}{1} \times \frac{\pi}{180^\circ}$$
$$= 2.443^R \quad 3 \text{ d.p.}$$

4) Change 4.5^R to degrees as an exact value.

$$4.5^R = 4.5 \times \frac{180^\circ}{\pi}$$
$$= \left(\frac{810}{\pi}\right)^\circ$$

5) Change 4.5^R to degrees correct to 3 d.p.

$$4.5^R = 4.5 \times \frac{180^\circ}{\pi}$$
$$= 257.831^\circ \quad \text{3 d.p.}$$

A final wrap-up

An alternative formula for the area of sector A_S is developed as follows

$$A_S = \frac{1}{2} r^2 \theta^R \quad \text{but} \quad l = r\theta^R$$
$$\text{So} \quad \theta^R = \frac{l}{r}$$

$$\text{then } A_S = \frac{1}{2} r^2 \times \frac{l}{r} \quad \text{cancel r's}$$
$$= \frac{1}{2} r l$$

$$A_S = \frac{1}{2} r l$$

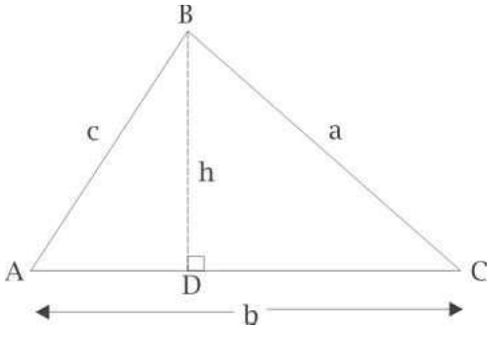
i.e. Area of a sector = half of the radius times the arc length.

Note that if $l = r = \theta^R$, $\theta^R = \frac{l}{r}$, and because l and r obviously have the same units it shows that θ^R is actually just a ratio without units. So radians as a unit don't really exist!

Radian is just a convenient name for a ratio.

New Area of a Triangle Formula

Consider Δ ABC labelled in the usual way as shown with a perpendicular drawn from B to meet AC at D.



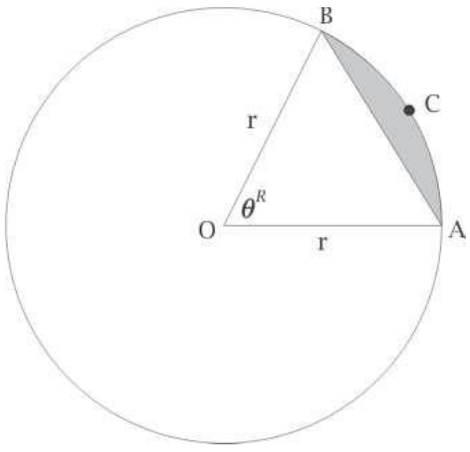
$$\text{Area} = \frac{1}{2} bh \qquad \text{and } \sin A = \frac{h}{c}$$
$$\therefore h = c \sin A$$
$$= \frac{1}{2} b \times c \sin A \quad \text{substitute}$$
$$= \frac{1}{2} b\, c \sin A$$
$$\text{Similarly Area} = \frac{1}{2} a\, c \sin B$$
$$= \frac{1}{2} b\, c \sin A$$

i.e. $$\text{Area} = \frac{1}{2} a\, b \sin C = \frac{1}{2} a\, c \sin B = \frac{1}{2} b\, c \sin A$$

Note that the known angle is always included between its two adjacent sides (i.e. S.A.S must be known) in order to use the formula. Using the sine rule or cosine rule may help to determine the size of the included angle.

Area of a Segment Formula

Consider the segment ABC (shaded) in a circle of radius r and sector angle θ^R as shown.



Then the area of the segment

$$= \text{Area Sector OACB} - \text{Area Triangle OAB}$$
$$= \frac{1}{2} r^2 \theta - \frac{1}{2} r.\, r.\sin\theta$$
$$= \frac{1}{2} r^2 \theta - \frac{1}{2} r^2 \sin\theta$$
$$= \frac{1}{2} r^2 (\theta - \sin\theta)$$

The area of a segment is then:

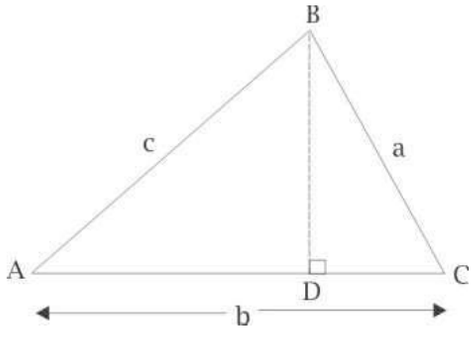
$$\text{Area}_{seg} = \frac{1}{2} r^2 (\theta - \sin\theta)$$

Where θ must be in radians and your calculator be in radian mode.

The Sine Rule

For ΔABC as shown:

$$\sin A = \frac{BD}{c} \qquad \sin C = \frac{BD}{a}$$
$$\therefore BD = c \sin A \qquad \therefore BD = a \sin C$$
$$\therefore c \sin A = a \sin C$$
$$\text{or } \frac{a}{\sin A} = \frac{c}{\sin C} \text{ which also equals } \frac{b}{\sin B}$$



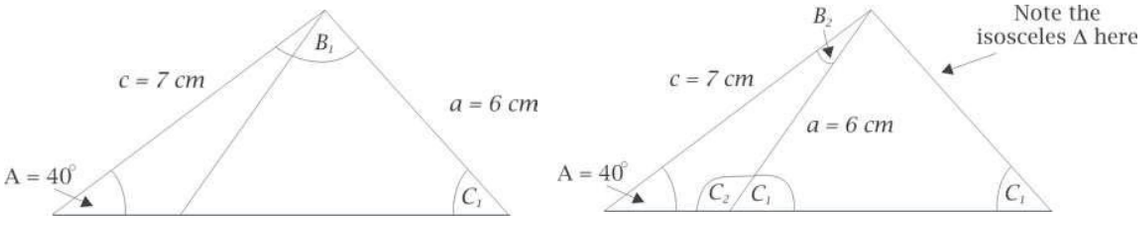
The sine rule is then $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

which is used to find missing angles and sides. You will need to understand that the sine rule is used if (i) any two angles and one side are known (ii) any two sides and the angle NOT between them are known.

The **Ambiguous Case of the Sine Rule** may occur when the rule is being used to find the size of an unknown angle. This is because when two sides and one angle of a triangle are known it may be possible to draw two different sized triangles, as the given information doesn't specify a unique triangle.

Example

If $\angle A = 40^\circ$, a = 6 cm and c = 7 cm, then the two possible triangles will be



Then for both triangles $\dfrac{a}{\sin A} = \dfrac{c}{\sin C}$

$$\frac{6}{\sin 40^\circ} = \frac{7}{\sin C}$$

$$\sin C = \frac{7 \sin 40^\circ}{6}$$

$$= 0.7499 \quad \text{(4 d.p.)}$$

For the triangle with A, B_1, C_1

$\angle C_1 = \text{inv sin } (0.7499)$
$= 48.58^\circ$

and $\angle B_1 = 180^\circ - 48.58^\circ - 40^\circ$
$= 91.42^\circ$

For the triangle with A, B_2, C_2

$\angle C_2 = 180^\circ - \angle C_1$
$= 180^\circ - 48.58^\circ$
$= 131.42^\circ$

and $\angle B_2 = 180^\circ - 131.42 - 40^\circ$
$= 8.58^\circ$

A second triangle will exist when

i) the above procedure is followed and the last angle found is positive

or ii) it is noticed that the given angle (i.e. $\angle$ A above) is opposite the smaller of the two given sides (except when $\angle$ C is found to be 90°).

The Cosine Rule

For the previous ΔABC

$$\cos A = \frac{AD}{c} \qquad \sin A = \frac{BD}{c}$$

$$\therefore AD = c \cos A \qquad \therefore BD = c \sin A$$

$$BD^2 = c^2 \sin^2 A$$

$$DC = b - AD$$

$$= b - c \cos A$$

$$(DC)^2 = (b - c \cos A)^2$$

$$= b^2 - 2bc \cos A + c^2 \cos^2 A$$

$$a^2 = (BD)^2 + (DC)^2$$

$$= c^2 \sin^2 A + b^2 - 2bc \cos A + c^2 \cos^2 A$$

$$= b^2 + c^2 \sin^2 A + c^2 \cos^2 A - 2bc \cos A$$

$$= b^2 + c^2 (\sin^2 A + \cos^2 A) - 2bc \cos A$$

$$= b^2 + c^2 - 2bc \cos A \qquad \text{as} \quad \sin^2 A + \cos^2 A = 1$$

So:

$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$b^2 = a^2 + c^2 - 2ac \cos B$$
$$c^2 = a^2 + b^2 - 2ab \cos C$$

is the cosine rule for finding a side length when S.A.S. is known

When the side lengths are known i.e. S.S.S. then either of the above three cosine rule formulae can be solved for the cosine of the angle

i.e. if $a^2 = b^2 + c^2 - 2\,b\,c\cos A$

then $2\,b\,c\cos A = b^2 + c^2 - a^2$

and

$$\cos A = \frac{b^2 + c^2 - a^2}{2\,b\,c}$$
$$\cos B = \frac{a^2 + c^2 - b^2}{2\,a\,c}$$
$$\cos C = \frac{a^2 + b^2 - c^2}{2\,a\,b}$$

is the cosine rule for finding an angle when S.S.S is known

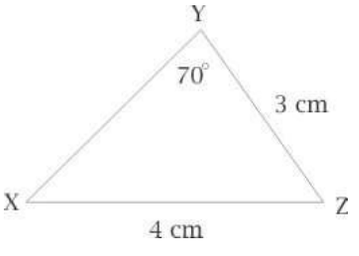
Note 1

If you have the choice of using the sine rule or the cosine rule to find an angle always use the cosine rule as then you avoid the possibility of the ambiguous case.

Note 2

In practice when using the sine and cosine rules and the area of a triangle formula you should be able to write down the equation by mentally substituting the values into the relevant formula without writing the general rule first.

Example: Find the unknown length, angles and area of this triangle.



Then $\frac{4}{\sin 70^\circ} = \frac{3}{\sin X}$

gives $\sin X = \frac{3 \sin 70^\circ}{4}$

$X = 44.8^\circ$

(no second triangle possible in this case)

$Z = 180^\circ - 70^\circ - 44.8^\circ$

$= 65.2^\circ$

$z^2 = 3^2 + 4^2 - 2 \times 3 \times 4 \times \cos 65.2^\circ$

$z = 3.86$ cm 2 d.p.

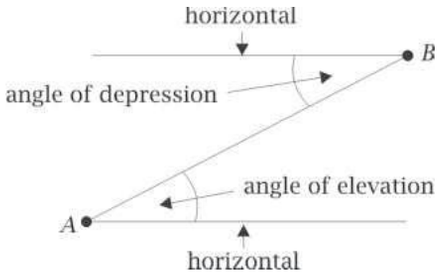
$$\text{Area} = \frac{1}{2} \times 3 \times 4 \times \sin 65.2^\circ$$
$$= 5.45 \text{ cm}^2$$

or $$\text{Area} = \frac{1}{2} \times 3 \times 3.86 \times \sin 70^\circ$$
$$= 5.44 \text{ cm}^2$$

(this difference shows that I should have used more decimal places in my calculations)

Angles of Elevation/Depression and Bearings

- When an object B is higher than an object A then B will be seen on an angle of elevation from A, and A will be seen on an angle of depression from B.



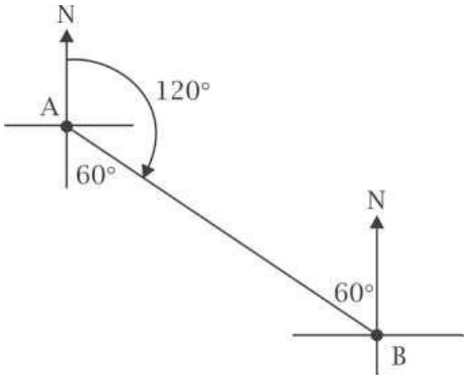
The angles of elevation and depression are always measured from the horizontal.

Notice these angles are alternate and so **angle of elevation = angle of depression**.

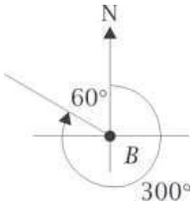
- A bearing is a direction in the horizontal plane specified as an angle which is always measured "clockwise" from due north.

Example

Consider that place B is on a bearing of 120° from place A. This is drawn as



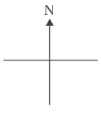
The bearing of place A from place B (called the back bearing) will be 300°.



TWO DIMENSIONAL BEARING PROBLEMS

The diagram associated with a bearing problem should be

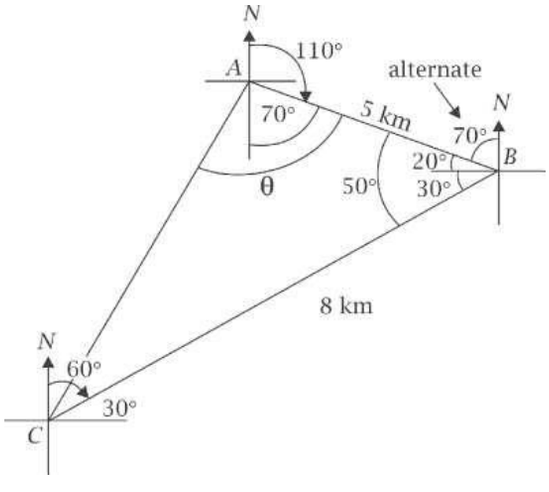
i) drawn large and roughly to scale

ii) have a  on each vertex

iii) used to find missing angles around each vertex leading to the sizes of some of the internal angles of the triangle (alternate angles are common here).

Example

Heather, a CALM ranger in an observation tower at A, recorded a bush fire at B, 5 km away on a bearing of 110°. John in a tower at C, observed the fire 8 km away on a bearing of 060°. Find the distance and bearing of tower C from tower A.



$\angle ABC = 20^\circ + 30^\circ$

$= 50^\circ$

$AC^2 = 5^2 + 8^2 - 2 \times 5 \times 8 \times \cos 50^\circ$

$AC = 6.13$ km

The distance from A to C = 6.13 km

$$\cos\theta = \frac{6.13^2 + 5^2 - 8^2}{2 \times 6.13 \times 5}$$

$\theta = 91.33^\circ$

Bearing of C from A = 110° + 91.33°

= 201.33°

THREE DIMENSIONAL PROBLEMS

Problems involving both bearings and angles of elevation/depression are 3D in nature. These are best tackled by:

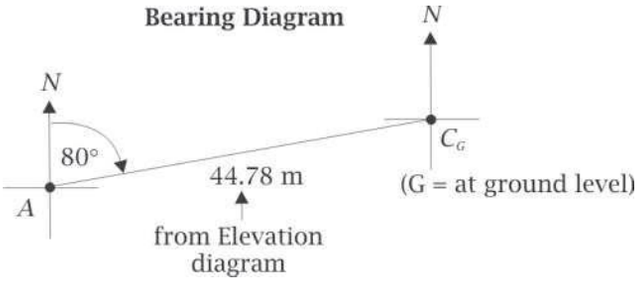
i) drawing a 2D diagram as before, showing the bearing angles and ground distances, then by

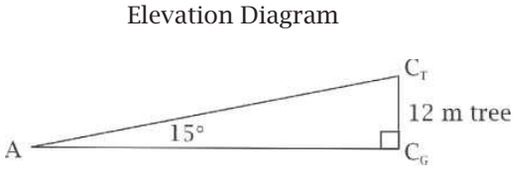
ii) drawing separate right angled triangles to show heights and angles of elevation/depression.

This method is safe and avoids drawing difficult and time consuming 3D diagrams.

Example

Consider that the sentence below is the first of a larger problem.
A 12 m tree is observed at C on a bearing of 080° from A and at an angle of elevation of 15°.
You would start by drawing the Bearing Diagram and then the Elevation Diagram. The ground distance AC_G found from the Elevation Diagram would then be transferred to the Bearing Diagram.





$$\tan 15^\circ = \frac{12}{AC_G}$$

$$AC_G = \frac{12}{\tan 15^\circ}$$

$$= 44.785 \text{ m}$$

transfer to bearing diagram

Keep AC_G to 3 or more decimal places in order to preserve the accuracy of the final answer.

Angles between planes and the line of greatest slope.

Consider any flat piece of paper with a straight line WXY drawn on it. Now consider a straight dotted line *AB* drawn to intersect WY at 90° at the point *X* (Fig. 1). If the paper is now folded along dotted line *AB* then two flat planes are formed which will be at an angle θ to each other where θ = ∠ WXY (Fig. 2).

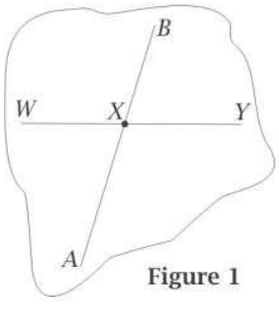


Figure 1

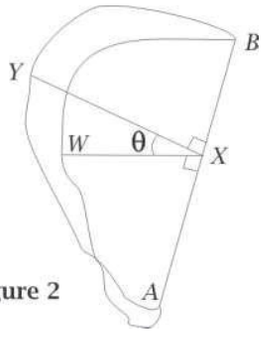


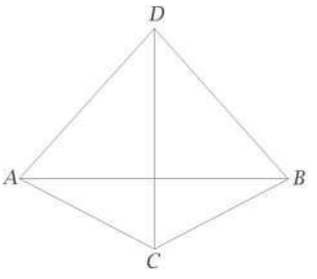
Figure 2

The angle ∠ WXY = θ is the angle between the two planes and it is vital to comprehend that **both** rays of this angle, *WX* and *YX* meet *AB* at 90°.

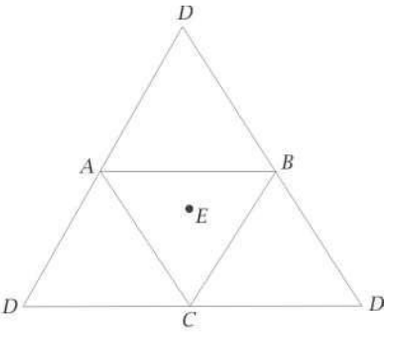
When finding angles between planes, draw in rays like *YX* and *WX* which meet another line in both planes, like *AB*, at 90°.

By unfolding a 3D shape to its net, the required 90° angles are then more easily seen.

Consider the regular tetrahedron shape of side length 1 unit and its net.



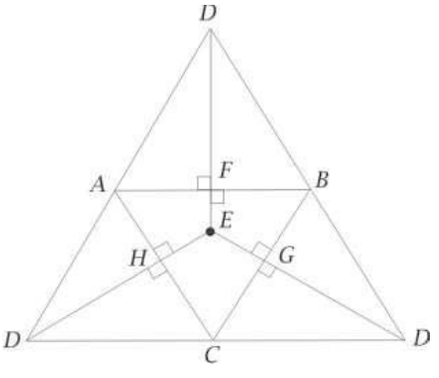
3D shape



net

If the point *E* is vertically below *D* then the 3 lines *DE* can be drawn meeting *AB* at *F*, *BC* at *G* and *CA* at *H*.

So



The angles between the slanting triangular faces and the base ABC are then ∠ DFE, ∠ DGE and ∠ DHE when the net is folded back to its 3D shape.

Because of the regular nature of this shape the angles between the planes are all the same size.

- **Under certain conditions the angle between two planes is associated with the line of greatest slope.**

  Look again at Fig. 2. If WX and AB are both in the same horizontal plane, then YX will be the line of greatest slope. Any other line in the same plane as YX and AB, but not parallel to YX, will have a smaller slope with respect to the horizontal plane.

- **3D problems involving shapes such as prisms and pyramids should have**

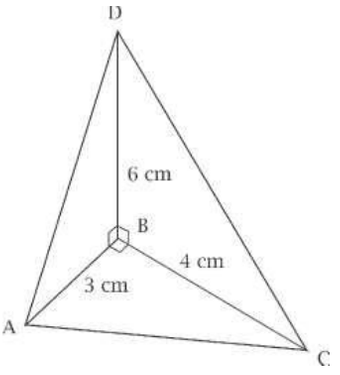
  i) a big, clear, well-labelled diagram;

  ii) all right angles identified and clearly marked;

  iii) newly found measurements put progressively on the diagram; and

  iv) some triangles drawn in 2D to clarify the situation.

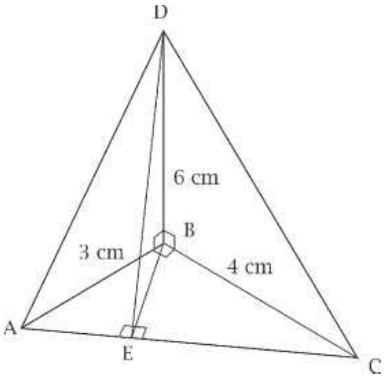
Example: Consider the 3D shape shown
Find the angle between Δ ABC and Δ ADC.

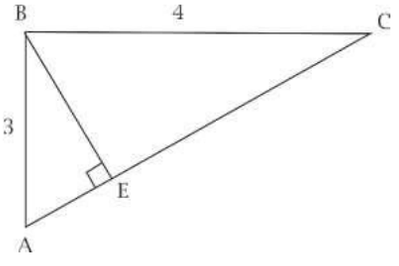


First find point E on AC such that BE ⊥ AC and DE ⊥ AC.

If BE is found then the required angle ∠ DEB easily follows.

BE can be found by a number of methods
one of which is by considering sin ∠ *BCA*.





$$AC = \sqrt{3^2 + 4^2}$$
$$= 5 \text{ cm}$$
$$\sin \angle BCA = \frac{3}{5} = \frac{BE}{4}$$
$$\therefore BE = \frac{3 \times 4}{5}$$
$$BE = 2.4 \text{ cm}$$

$$\tan \angle DEB = \frac{6}{2.4}$$
$$\angle DEB = 68.1986^\circ$$

So the angle between ΔABC and ΔADC is 68.20°

Worked Examples

Trigonometry I

3.1 (a) Change $\frac{13\pi}{15}$ to degrees.

(b) Change 165° to radians in exact form.

(c) Change 243° to radians 3 d.p.

(d) Change 2.35^R to degrees 3 d.p.

(a) $\frac{13\pi}{15}$
$= \frac{13 \times 180^\circ}{15}$
$= 156^\circ$

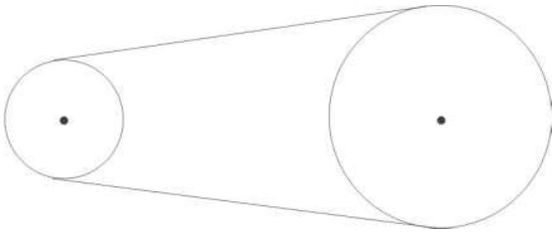
(b) 165°
$= \frac{165^\circ}{1} \times \frac{\pi}{180^\circ}$
$= \frac{11\pi}{12}$

(c) 243°
$= \frac{243^\circ}{1} \times \frac{\pi}{180^\circ}$
$= 4.241^R$

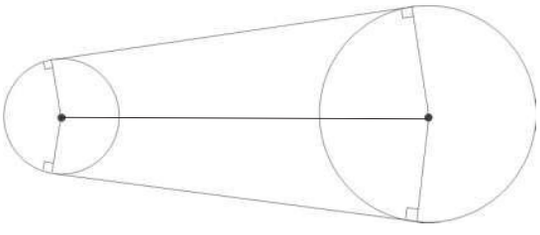
(d) 2.35^R
$= 2.35 \times \frac{180^\circ}{\pi}$
$= 134.645^\circ$

3.2 A belt fits tightly around two circular pulleys whose radii are 8 cm and 15 cm and whose centres are 25 cm apart. Find the length of the belt.

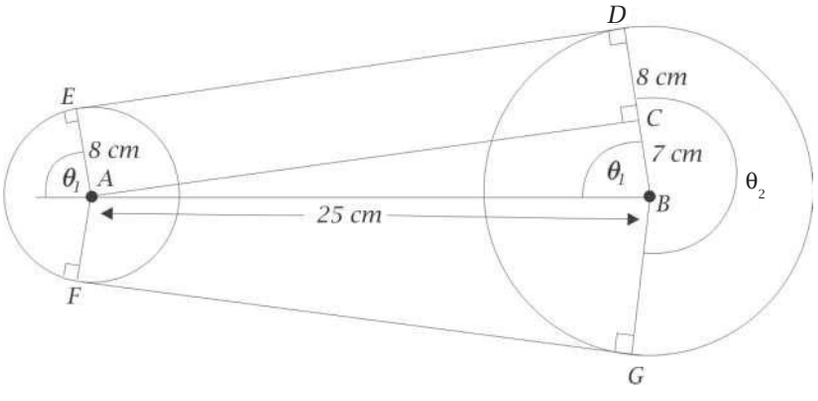
A big clear diagram is essential here. The diagram that you first draw will look like the one at right. Then you should join the centres and add the four radii which meet the tangential belt at 90°



i.e.



As angles are usually found from triangles it is important to realise that one more line needs to be drawn to form a right angled triangle. This is shown as AC below.



$$\text{Then } \cos\theta_1 = \frac{7}{25}$$

$$\text{In Rad mode } \theta_1 = 1.287^R$$

$$\text{Obtuse } \angle EAF = 2\theta_1$$
$$= 2.574^R$$

$$\text{minor arc } EF = r.2\theta_1$$
$$= 8(2.574)$$
$$= 20.592\,cm$$

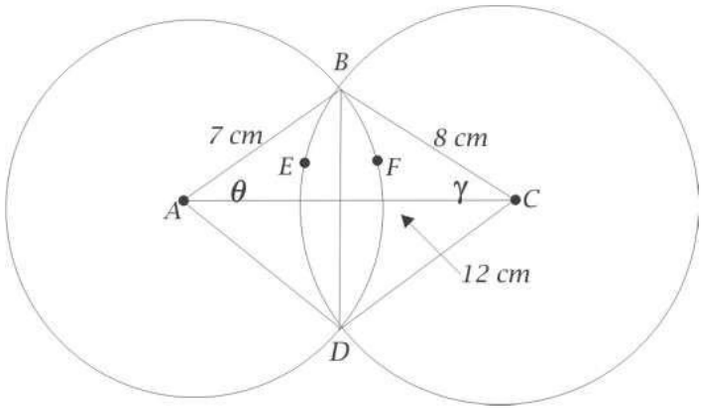
$$ED = AC = \sqrt{25^2 - 7^2}$$
$$= 24\,cm$$

$$\text{Relfex } \angle DBG = 2\pi - 2\theta_1$$
$$\theta_2 = 3.7092^R$$

$$\text{major arc } DG = r\,\theta_2$$
$$= 15(3.7092)$$
$$= 55.638\,cm$$

$$\therefore \text{Total belt length} = 24 + 20.592 + 24 + 55.638$$
$$= 124.23\ cm$$

3.3 Two circles whose radii are 7 cm and 8 cm have centres separated by 12 cm. Find the area common to both circles.



$Let \angle BAC = \theta,\ \cos\theta = \dfrac{7^2 + 12^2 - 8^2}{2.7.12}$

$\theta = 0.695^R$

$\angle BAD = 2\theta$

$= 1.390^R$

$Let \angle BCA = \gamma,\ \cos\gamma = \dfrac{8^2 + 12^2 - 7^2}{2.8.12}$

$\gamma = 0.595^R$

$\angle BCD = 2\gamma$

$= 1.190^R$

Area $\Delta BAD = \frac{1}{2} . 7 . 7 . \sin 1.390$

$= 24.101\ cm^2$

Area $\Delta BCD = \frac{1}{2} . 8 . 8 . \sin 1.190$

$= 29.708\ cm^2$

Area sector BADF $= \frac{1}{2} . 7^2 (1.390)$

$= 34.055\ cm^2$

Area sector BCDE $= \frac{1}{2} . 8^2 (1.19)$

$= 38.080\ cm^2$

$\therefore$ *Area segment BFD* $= 34.055 - 24.101$

$= 9.954\ cm^2$

Area segment BED $= 38.080 - 29.708$

$= 8.372\ cm^2$

$\therefore$ *Area common to both circles BEDF* $= 9.954 + 8.372$

$= 18.33\ cm^2$

Using the segment formula $\frac{1}{2} r^2 (\theta - \sin\theta)$ *twice would have simplified the above working.*

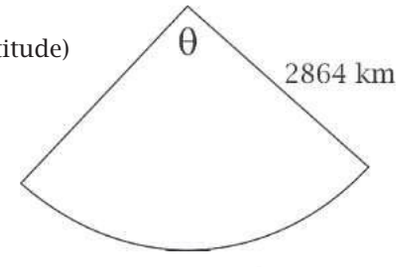
3.4 A plane takes off from (65° N, 35° E) and flies 7000 km West. Give the new position of the plane if the radius of the earth is considered to be 6350 km.

radius of small circle (i.e. along 65th parallel of latitude)

$r = R\cos\theta_{lat}$

$= 6350\cos 65^\circ$

$= 2684\ km$



7000 km

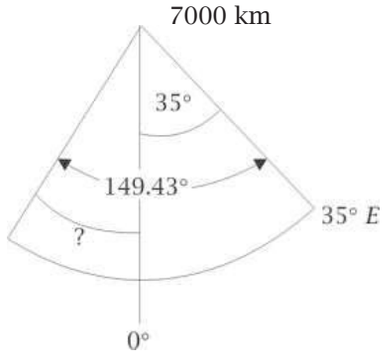
$l = r\theta$

$7000 = 2684\theta$

$\theta = 2.608^R$

$= 2.608 \times \dfrac{180^\circ}{\pi}$

$= 149.45^\circ$



new longitude $= 149.43^\circ - 35^\circ$

$= 114.43^\circ$

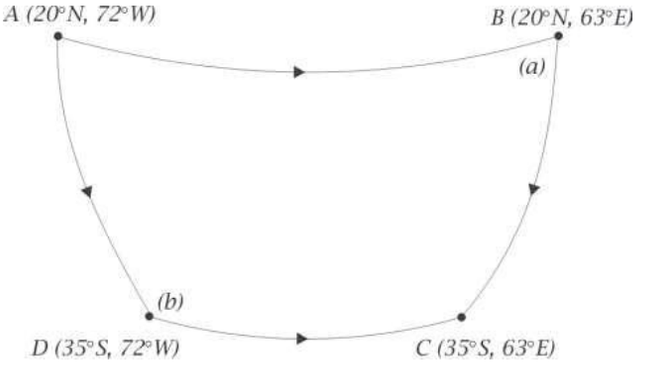
$\therefore$ new position of plane is 65° N, 114.43° W

3.5 Find the distance between A (20°N , 72°W) and C (35°S , 63°E) by

(a) first going East and then South

(b) first going South and then East.

The positions of B and D follow from the positions of A and C.



(a)

$A \to B?$ $r = R\cos\theta_{lat}$

$= 6350\cos 20^\circ$

$= 5967\,km$

$\theta_{long\ change} = 72^\circ + 63^\circ$

$= 135^\circ$

$= 2.356^R$

$\therefore dist_{A\to B} = r\theta$

$= 5967 \times 2.356$

$= 14058\,km$

$B \to C?$

$\theta_{lat\,change} = 20^\circ + 35^\circ$

$= 55^\circ$

$= 0.960^R$

$\therefore dist_{B\to C} = 6350 \times 0.960$

$= 6096\,km$

$\therefore dist_{A\to C}\ via\ B = 14058 + 6096$

$= 20154\,km$

(b) $dist_{A\to D} = dist_{B\to C}$

$= 6096\,km$

$D \to C?$ $r = R\cos\theta_{lat}$

$= 6350\cos 35^\circ$

$= 5202\,km$

$\theta_{long\ change} = 2.356^R$ *as before*

$\therefore dist_{D\to C} = 5202 \times 2.356$

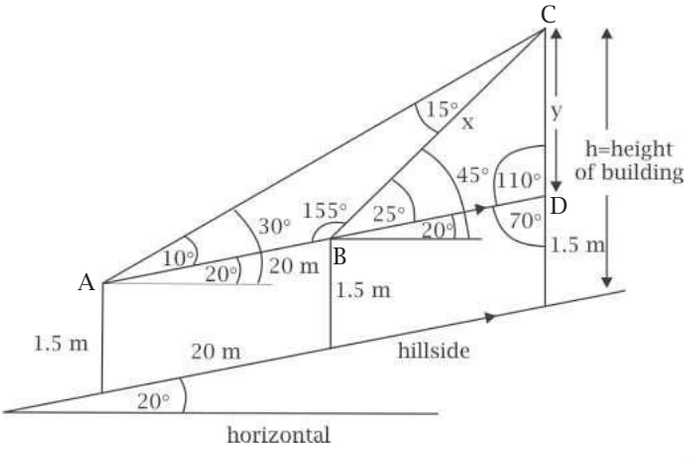
$= 12256\,km$

$\therefore dist_{A\to C}\ via\ D = 12256 + 6096$

$= 18352\,km$

3.6 A maths student stood on a hillside which is at 20° to the horizontal and observed the top of a building straight up the hill at an angle of elevation of 30°. The student then walked 20 m directly towards the building and observed the top of the building at an angle of elevation of 45°. If the student's eye level is 1.5 m from the ground find the height of the building.

A big clear diagram is very important with all original information carefully placed on it together with all newly found details.



In Δ ABC

$$\frac{x}{\sin 10^\circ} = \frac{20}{\sin 15^\circ}$$

$$x = \frac{20\sin 10^\circ}{\sin 15^\circ}$$

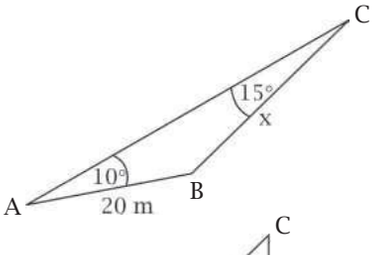
$$x = 13.4185\ m$$

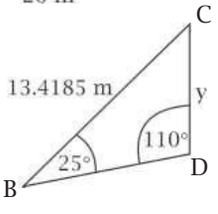
In Δ BCD

$$\frac{13.4185}{\sin 110^\circ} = \frac{y}{\sin 25^\circ}$$

$$y = \frac{13.4185\sin 25^\circ}{\sin 110^\circ}$$

$$y = 6.0348$$





$h = 6.0348 + 1.5 = 7.5348$

The building is 7.53 m high.

3.7 Mark sees a tree 100 m away on a bearing of 150° and at an elevation of 10°. Diane is on a bearing of 080° from Mark and views the same tree at an elevation of 7.7°. Find:

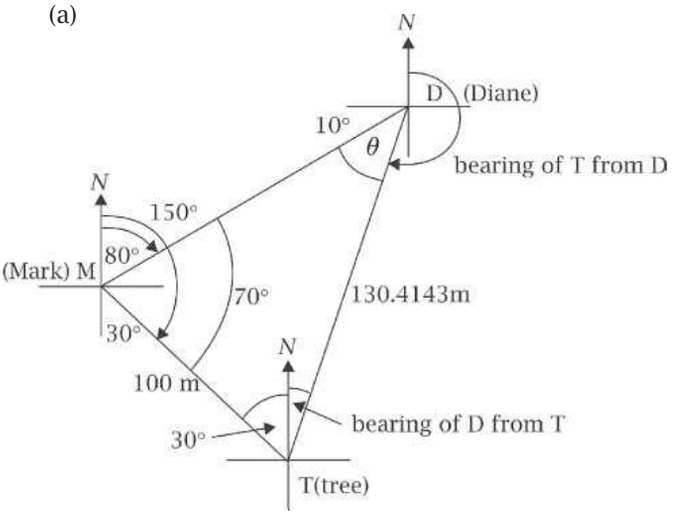
(a) how far Diane is from the tree

(b) the bearing that Diane would use to get to the tree

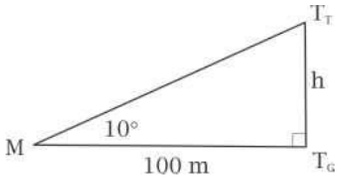
(c) the bearing from the tree to Diane

(d) the distance between Mark and Diane.

(a)



First determine the height of the tree.

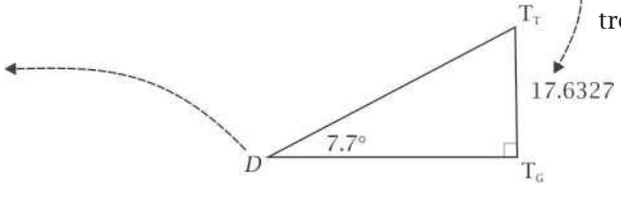


$$\tan 10^\circ = \frac{h}{100}$$

$$h = 100 \tan 10^\circ$$

$$h = 17.6327$$

Now find the distance from Diane to the tree.



$$\tan 7.7^\circ = \frac{17.6327}{DT_G}$$

$$DT_G = \frac{17.6327}{\tan 7.7^\circ}$$

$$DT_G = 130.4143\,m$$

Diane is 130.4143 m from the tree.

(b) *Let* $\angle MDT = \theta$

$$\frac{100}{\sin\theta} = \frac{130.4143}{\sin 70^\circ}$$

$$\sin\theta = \frac{100 \sin 70^\circ}{130.4143}$$

$$\theta = 46.0994^\circ \text{ (no second case possible)}$$

$\therefore$ *Bearing of the tree from Diane* $= 270^\circ - 10^\circ - 46.10^\circ$

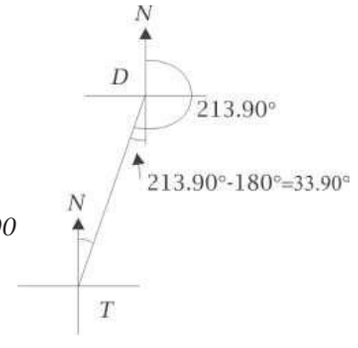
$= 213.90^\circ$

(c) *From the tree to Diane, the bearing is* $213.90^\circ - 180^\circ = 33.90^\circ$.

(d) *The Sine or Cosine Rule now gives MD.*

First $\angle MTD = 30^\circ + 33.90^\circ$

$= 63.90^\circ$

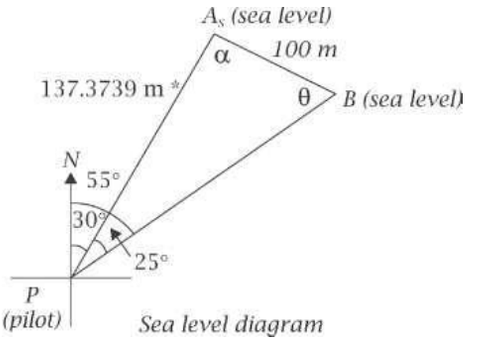


$MD^2 = 100^2 + 130.4143^2 - 2 \times 100 \times 130.4143 \cos 63.90$

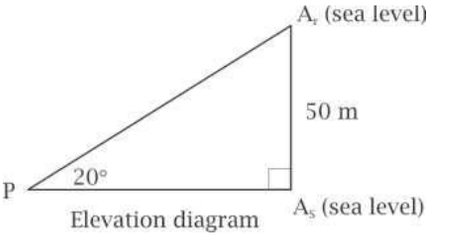
$MD = 124.6315$ (4 d.p.)

Mark and Diane are 124.63 m apart.

3.8 The top of a vertical cliff face is 50 m above the water. A pilot on a boat observes Andrew on the cliff on a bearing of 030° and at an elevation of 20°. Brodie is also on the cliff 100 m away from Andrew. The pilot sees Brodie on a bearing of 055°. What is his angle of elevation to Brodie?



Sea level diagram



Elevation diagram

$$\tan 20^\circ = \frac{50}{PA_S}$$
$$PA_S = 137.3739 \text{ m}$$

** calculated and added to diagram*

$$\frac{137.3739}{\sin\theta} = \frac{100}{\sin 25^\circ}$$
$$\sin\theta = \frac{137.3739 \sin 25^\circ}{100}$$
$$\theta_1 = 35.4904^\circ$$
$$\alpha_1 = 180^\circ - 35.4904^\circ - 25^\circ$$
$$= 119.5096^\circ$$

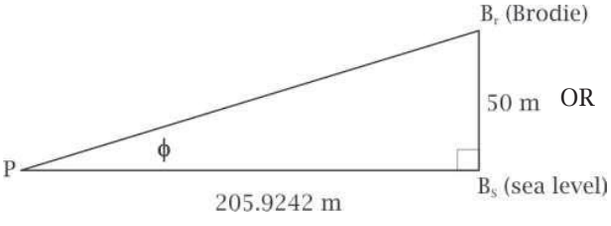
OR

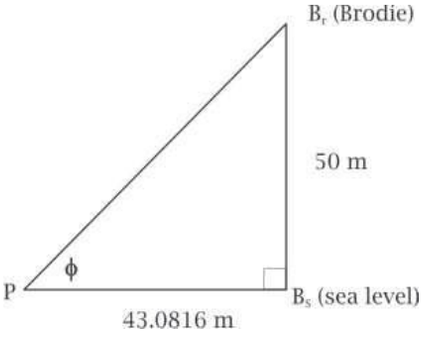
$$\theta_2 = 180^\circ - 35.4904^\circ \text{ (ambiguous case)}$$
$$= 144.5096^\circ$$
$$\alpha_2 = 180^\circ - 144.5096^\circ - 25^\circ$$
$$= 10.4904^\circ$$

$$\frac{PB}{\sin 119.5096^\circ} = \frac{100}{\sin 25^\circ}$$
$$PB = \frac{100 \sin 119.5096^\circ}{\sin 25^\circ}$$
$$PB = 205.9242 \text{ m}$$

OR

$$\frac{PB}{\sin 10.4904^\circ} = \frac{100}{\sin 25^\circ}$$
$$PB = 43.0816 \text{ m}$$





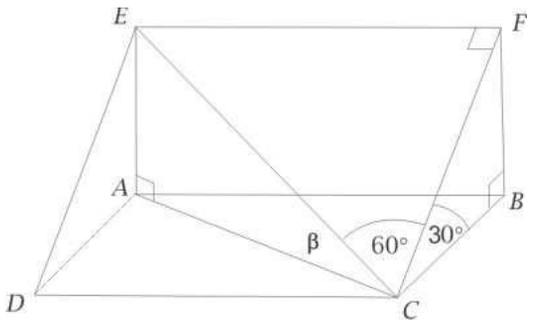
$$\tan\phi = \frac{50}{205.9242}$$
$$\phi = 13.6477^\circ$$

$$\tan\phi = \frac{50}{43.0816}$$
$$\phi = 49.2508^\circ$$

The elevation of Brodie from the pilot is either 13.65° or 49.25°.

3.9 A flat hillside makes an angle of 30° to the horizontal. Kylie cannot walk straight up the hill and so chooses to walk along a line which is at 60° to the line of greatest slope. At what angle with the horizontal is Kylie walking?

A hillside is modelled by a wedge shape which is itself half a rectangular prism.



It should be clearly noted that
$\angle$ EAC, $\angle$ EFC and $\angle$ FBC are all 90°.

The angle $\beta = \angle$ ECA is the one to find.
Let AE = BF = 1 m (or h if you want complete generality – the answer is the same)

Δ FCB: $\sin 30^\circ = \frac{BF}{FC}$

$\sin 30^\circ = \frac{1}{FC}$

$FC = \frac{1}{\sin 30^\circ}$

$\sin\beta = \frac{AE}{EC}$

$\sin\beta = \frac{1}{EC}$

$EC = \frac{1}{\sin\beta} = \frac{1}{\sin 30^\circ \cos 60^\circ}$

Δ EFC: $\cos 60^\circ = \frac{FC}{EC}$

$EC = \frac{FC}{\cos 60^\circ}$

$= FC \div \cos 60^\circ$

$= \frac{1}{\sin 30^\circ} \div \cos 60^\circ$

$= \frac{1}{\sin 30^\circ \cos 60^\circ}$

$\therefore \sin\beta = \sin 30^\circ \cos 60^\circ$

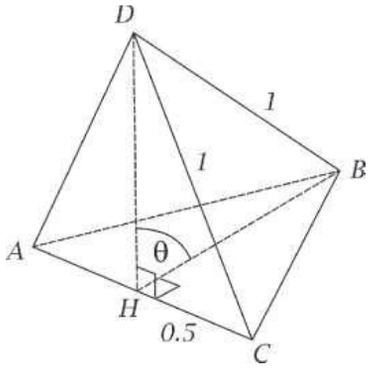
$\beta = 14.4775^\circ$

Kylie will be walking up a line which is at 14.48° with the horizontal.

Note: If $\angle$ FCB = θ $\angle$ FCE = α and $\angle$ECA= β then the above result can be generalised to $\sin\beta = \sin\theta \cos\alpha$ which links the three angles θ, α and β.

3.10 Find the angle between adjacent planes of a regular tetrahedron (see page 5).

While this problem can be solved using only right angled triangles, it is easier and quicker to include the cosine Rule.



Let each side length be 1 unit.
Let H be the midpoint of AC.
Join DH and BH.
Let $\angle$ DHB = θ

By Pythagoras

$$DH^2 = 1^2 - 0.5^2$$

$$= 0.75$$

$$DH^2 = \sqrt{0.75}$$

$$DH = BH$$

$$\cos\theta = \frac{DH^2 + BH^2 - DB^2}{2 \times DH \times BH}$$

$$\cos\theta = \frac{0.75 + 0.75 - 1}{2 \times \sqrt{0.75} \times \sqrt{0.75}}$$

$$= \frac{0.5}{1.5} = \frac{1}{3}$$

$$\theta = 70.5288^\circ$$

The angle between adjacent planes of a regular tetrahedron is 70.53°.

PROBLEMS TO SOLVE

CHAPTER 3: TRIGONOMETRY I

1.

(a) Express 480° in radians correct to 2 decimal places.

(b) Change $\frac{7\pi}{3}$ to degrees.

(c) Write 255° as an angle in radians in terms of π.

2. The area of a sector AOB, in a circle centre O and radius 4 cm is $\frac{16\pi}{3}$ cm^2.
Find the size of $\angle$ AOB

(a) in radians in terms of π;

(b) in degrees,

(c) in radians to 2 decimal places.

3. An arc of length 15 cm in a circle of radius 8 cm subtends an angle of θ at the centre.

(a) Find the exact size of θ in radians.

(b) Find the area of the sector.

(c) A forgetful maths student once got mixed up and found the area of the above sector in a similar way to the area of a triangle i.e. $\frac{1}{2} \times 15 \times 8$. Comment on his answer.

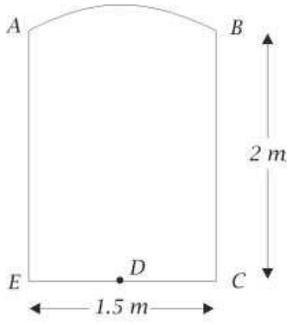
(d) Show that the area of a sector with angle θ, radius r and arc length l is in fact $\frac{rl}{2}$.

4. Three identical circles of radius 2 cm are placed on the table so that each circle touches the other two. Find the area enclosed by the circles correct to 3 d.p.

5. The centres of two circles of radii 20 cm and 15 cm are 27 cm apart. Find the size of the area common to both circles correct to 1 decimal place.

6. In a certain part of the world the radius of the earth is close to 6335 km. A person stands on top of a vertical 100 m cliff and observes the sun setting. How far is the sea horizon (to the nearest 100 m) from this person?

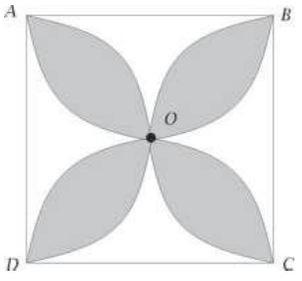
7. A new house is to have a front door opening as shown. The top is an arc of a circle, centre D. Find



(a) the perimeter of the opening;

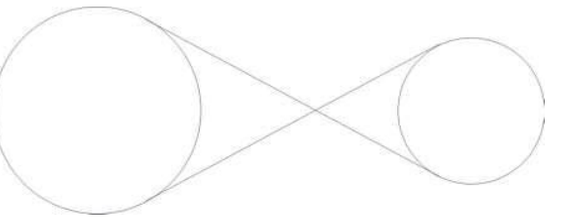
(b) the area of the opening.

8. In the figure shown, ABCD is a square with 1 m sides. AOD, BOC, AOB and COD are semi circular arcs. Find the shaded area.

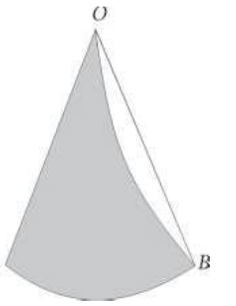


9. Find the length of the crossed belt passing around two pulleys of radii 20 cm and 30 cm and whose centres are 70 cm apart.

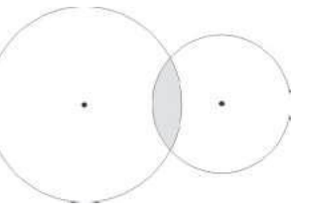
(Hint: Similar triangles and simultaneous equations may be appropriate.)



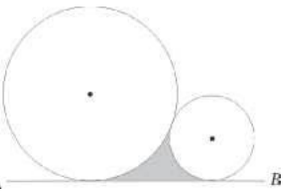
10. In the diagram $\angle AOB = 42.4^\circ$, the circular arc AB has centre O and radius 6 cm. If the circular arc OB has radius 5 cm find the area of the shaded region.



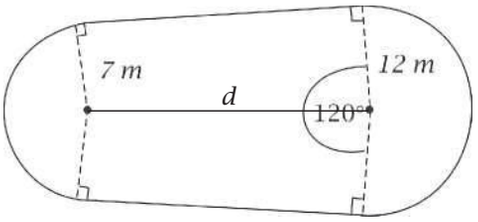
11. Two circles as shown have radii of 15 cm and 8 cm and centres that are 17 cm apart. Find the perimeter of the shaded portion to the nearest centimetre.



12. Two circles touch each other and each touches line AB as shown. If the circles have radii of 15 cm and 8 cm, find the shaded area.



13. The diagram shows a swimming pool which is to have a cover made for it. Find the surface area of the cover correct to 2 decimal places.



14. PQR is a right angled triangle with PQ = 6 cm, QR = 10 cm, and $\angle$ Q = 90°. A circle of radius 6 cm is drawn with centre Q. Find the area of that part of Δ PQR which is outside the circle, correct to one decimal place.

15. An aircraft leaves an airfield A and flies 150 km on a bearing of 072° to a point B. It then changes course and flies 90 km on a bearing of 132° to point C. Calculate the bearing and distance if the aircraft then returns directly to A.

16. Two straight ladders, one of which is twice as long as the other, lean against a vertical wall and meet the wall at the same height from the floor. If the shorter ladder makes an angle of 60° with the floor, determine to the nearest degree the angle that the longer ladder makes with the floor.

17. A plane is flying south at an altitude of 1.5 km when the navigator sees a town directly ahead at an angle of depression of 21°. At the same time she sees another town due east at an angle of depression of 18°. How far apart are the towns?

18. If Δ ABC has an area of 3.75 cm^2 and a = 5 cm and c = 3 cm, find the length of b if $\angle$ B is obtuse.

19. Mohammed needs to run a pipe from the dam to his house, so that he can fill his swimming pool. The house and the dam are hidden from each other by trees, so that he cannot measure the distance and bearing directly. If he walks 668 m from the dam on a bearing of 121.33°, the house is then 1205 m away on a bearing of 34.78°. Find the distance and bearing of Mohammed's house from the dam (assuming level ground).

20. A ship sails from port at 12 midday and sails at a steady speed of 15 km/hr on a bearing of 220° for 3 hours. It then changes course to a bearing of 290° and stays on that bearing until 5 pm.

(a) How far West of the port is the ship at 5 pm?

(b) How far South of the port is the ship at 5 pm?

(c) At this point the captain discovers a sick crewperson on board and decides that the ship must return to port. What bearing should the captain set for a direct return journey and how long will it take to reach the port if the ship keeps to its original speed of 15 km/hr?

21. The lengths of the hands of a clock are 15 cm and 20 cm. Find the distance between the tips of the hands when the time is 8.20 am.

22. ABCD is a triangular pyramid with apex D, 7 cm vertically above A. The base Δ ABC is isosceles with AB = AC and BC = 4 cm. If $\angle$ DCA = 50° find:

(a) the size of $\angle$ DBA

(b) the length of AB

(c) the size of $\angle$ CAB

(d) the angle that Δ CDB makes with Δ ABC.

23. A ship is sailing north at a constant speed of 10 km/hr. At 9.00 pm a lighthouse is observed on a bearing of 040°, while at 1.00 am the same lighthouse is observed on a bearing of 155°.

(a) Calculate to the nearest 0.1 km, the distance between the lighthouse and the ship at 1.00 am.

(b) Find whether the ship will be due west of the lighthouse before or after 11.30 pm.

24. A pole 8 m long stands on horizontal ground but leans at an angle of θ from the vertical towards the sun. When the sun's elevation is 40.5°, the pole casts a shadow 9 m long on the ground. Find θ correct to one decimal place.

25. A plot of land has the shape of a quadrilateral ABCD where AB = 70 m, CD = 36 m, AD = 80 m, diagonal BD = 65 m and $\angle$ C = 82°. Find:

(a) the length of BC

(b) the area of the land to the nearest m^2.

26. Jason sights an aeroplane due north at an angle of elevation of 15°. Lisa notes at the same instant that the plane has a bearing of 300°. If Lisa is 5 km from Jason and on a bearing of 045°, find:

(a) the altitude of the plane

(b) the elevation of the plane as seen by Lisa.

27. ABCD is a cyclic quadrilateral where AB = 22 cm, BC = 20 cm, $\angle$ ADC = 130° and $\angle$ BCD = 95°. Find:

(a) the length of diagonal AC

(b) the length of side AD

(c) the radius of circle.

28. Tyrone (who is 2 m tall) stands due west of a street light. His shadow is 3 m long. When he walks 20 m due south, his shadow lengthens to 7 m. How high is the light to the nearest 0.01 m?

29. A hill is inclined at an angle of θ to the horizontal. A track makes an angle of 60° to the angle of greatest slope and an angle of 15° with the horizontal. Find θ to the nearest degree.

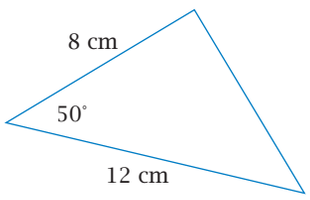
30. The elevation of the top of a radio mast from a point due south of the mast is 70.2° and from a point due east of it is 58.2°. If the distance between the two observation points is 200 m, find the height of the mast.

31. A mathematically minded ant observes the angle of elevation to the top of a letterbox due North to be 32°. It then moves 5.6 m due East and this time the ant observes the angle of elevation to be 10°. Find the height of the letterbox correct to 2 decimal places. (Hint: determine the two unknown sides of the base triangle in terms of the height of the letterbox "h".)

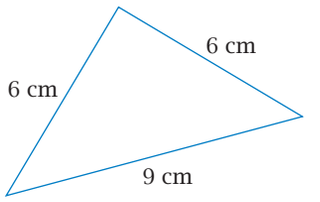
32. In Δ ABC consider that the values of A, c and b are known. Find the formula for tan C in terms of A, c and b by first drawing in the perpendicular from B to meet AC at D.

33. Find the area of the following triangles.

(a)



(b)



34. A submerged, stationary submarine at a depth of 200 metres detects a ship on the water's surface due north with an angle of elevation of 8°. Ten minutes later the ship is still due north but its angle of elevation is 6.5°.

(a) Find the initial direct distance the ship is from the submarine.

(b) Find the distance travelled by the ship in the 10 minutes.

(c) Find the speed of the ship in kilometres per hour.

35. The *Jolly Roger* leaves Port *P* on a bearing of 068° and heads directly towards an island which is 63 km away. After taking 2 hours and 15 minutes to reach the island, the *Jolly Roger* changes course to a bearing of 143° and maintains its previous speed. At the same time as the *Jolly Roger* leaves the island, the *Cutty Spark* departs from Port *P* in order to intercept the *Jolly Roger* in 3 hours' time. At what speed and in what direction should the *Cutty Spark* set out in order to successfully intercept the *Jolly Roger*?

36. A light plane flies from point *A* in a north-easterly direction for 300 km to point *B*.

(a) Make a sketch of this flight and determine how far North of *A* is the plane when it reaches *B*.

The plane then turns onto a true bearing of 170° and flies for a further 700 km to point C.

(b) On what bearing should the plane now fly to return to A and how long will it take to reach A from C if it can fly at 190 km/h?

37. Find the smallest angle of a triangle with sides of length 23 cm, 30 cm and 33 cm. (Give your answer correct to 2 decimal places.)

38. In ΔABC, $a = 7$ cm, $b = 9$ cm and $A = 50°$.

Find the size(s) of B, C and c (correct to one decimal place).

39. An isosceles triangle has two 10 cm sides and a 12 cm side. What is the size of the largest angle in the triangle?

40. A 21 m tall mobile phone aerial is supported by three guy wires attached 4 m below the top of the aerial. Each wire is 18 m long and anchored at the corners of an equilateral triangle on level ground.

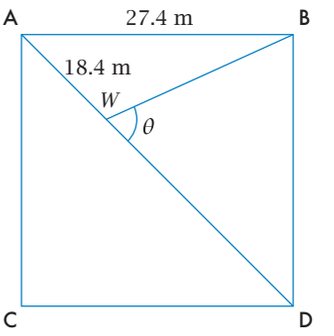
(a) At what angles do the wires meet the ground?

(b) How far are the base of the wires from the base of the aerial?

(c) What is the area of the equilateral triangle formed by the anchor points?

(d) What angle do the wires form where they meet?

41. The piping in a reticulation system of a front garden forms a square of side 27.4 m as marked on the diagram below. These pipes ensure that the whole front garden receives an appropriate amount of water. The garden's well, *W*, lies on the square's diagonal, such that the distance *AW* is 18.4 metres.

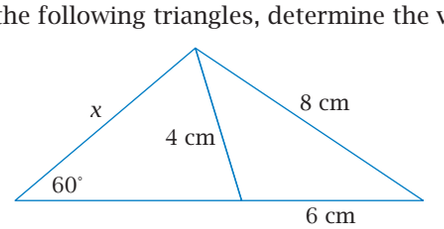


(a) Find the distance *BW*, the distance from the well to a corner of the reticulation's piping.

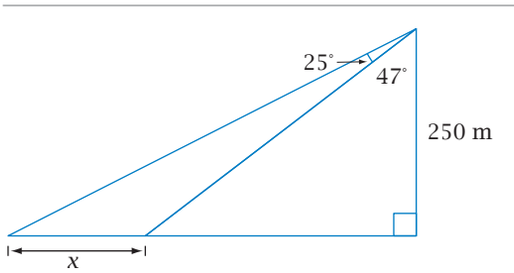
(b) Find the angle θ as marked on the diagram.

42. For the following triangles, determine the value of the unknown pronumeral(s).

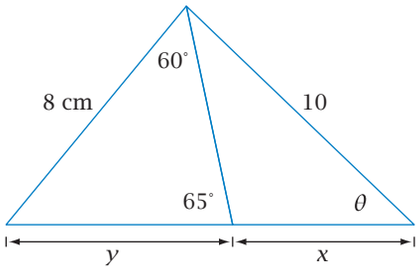
(a)



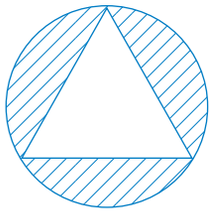
(b)



(c)



43. An equilateral triangle is circumscribed by a circle of radius 5 cm. Find the shaded area correct to 2 decimal places.



44. Three towns Allenstown, Braddock and Cambridge have positions as follows: Cambridge is 10 km from Allenstown and on a bearing of 155°, Allenstown and Braddock are 7 km apart and Braddock is on a bearing of 300° from Cambridge. Find the bearing of Braddock from Allenstown.

45. Chris enjoys off-road driving. One day she travels at 120 km/h for 45 minutes in a direction of 115° and then 140 km/h for 15 minutes in a direction of 080°. Find:

(a) the distance that Chris is from her starting position,

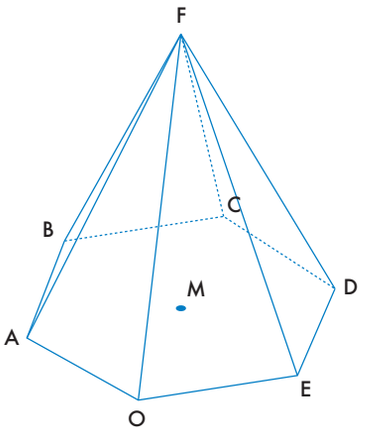
(b) the bearing of Chris from her starting position.

46. BD is a TV broadcasting tower. A and C are observers who report: A and B are 200 m apart;

- A is 300 m from C;
- $\measuredangle BAC$ is 40°;
- The angle of elevation of D from C is 20°

How high is the TV broadcasting tower?

47. $OABCDEF$ is a hexagonal pyramid. The base being a regular hexagon with side length 12 cm. The height of the pyramid is 15 cm. M is the centre of the base of the pyramid.



Determine the following:

(a) angle MOE

(b) side length OF

(c) angle FOM

(d) the angle each triangular face makes with the base of the pyramid

TRIGONOMETRY II

4

A thorough understanding of the trigonometric functions is an important foundation for the successful study of mathematics at higher levels. In this unit special emphasis is given to periodicity, amplitude and phase, which are illustrated by graphs wherever possible.

Syllabus Checklist

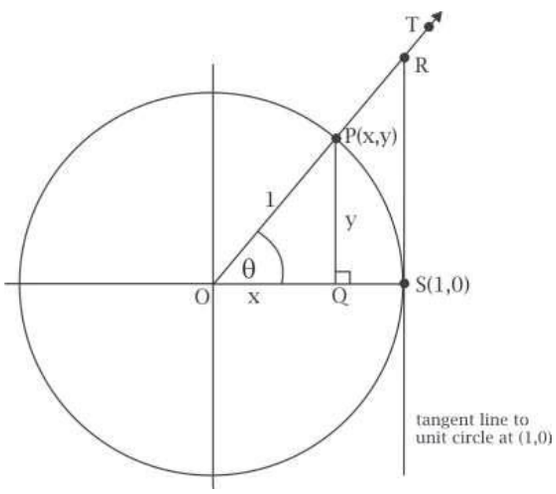
On the completion of this chapter you should be able to:

☐ Trigonometric functions

- understand the unit circle definition of sin θ, cos θ and tan θ and periodicity using radians
- recognise the exact values of sin θ, cos θ and tan θ at integer multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$
- recognise the graphs of $y = \sin x$, $y = \cos x$, and $y = \tan x$ on extended domains
- examine amplitude changes and the graphs of $y = a \sin x$ and $y = a \cos x$
- examine period changes and the graphs of $y = \sin bx$, $y = \cos bx$ and $y = \tan bx$
- examine phase changes and the graphs of $y = \sin(x - c)$, $y = \cos(x - c)$ and $y = \tan(x - c)$
- examine the relationships $\sin\left(x + \frac{\pi}{2}\right) = \cos x$ and $\cos\left(x - \frac{\pi}{2}\right) = \sin x$
- prove and apply the angle sum and difference identities
- identify contexts suitable for modelling by trigonometric functions and use them to solve practical problems
- solve equations involving trigonometric functions using technology, and algebraically in simple cases

The Unit Circle Definitions of sin θ, cos θ and tan θ

- The ray OT has a polar angle θ which meets the unit circle at P (x,y) and the tangent line at R as shown.



Then we define

$PQ = y = \sin\theta$
$OQ = x = \cos\theta$
$RS = \tan\theta$

It is important to understand that

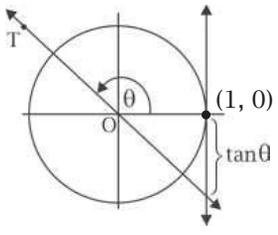
i) sin θ is always a y distance (i.e. a vertical distance)

ii) cos θ is always an x distance (i.e. a horizontal distance)

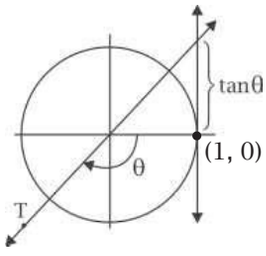
iii) tan θ is a vertical distance and when θ = 90°, 270°, –90°, etc, tan θ is undefined (OT will not intersect the tangent line because OT||RS).

iv) the tangent line is only drawn at (1,0) so for 90° < θ < 270° the ray OT has to be reflected back about the origin O for the intersection to occur

So



and



The cartesian plane is divided into quadrants

Second Quadrant	First Quadrant
Third Quadrant	Fourth Quadrant

and the signs of sin θ, cos θ and tan θ are as follows.

	First Quadrant	Second Quadrant	Third Quadrant	Fourth Quadrant
sin θ	+	+	-	-
cos θ	+	-	-	+
tan θ	+	-	+	-

This is summarised as

Only sin θ positive S	All positve A
Only tan θ positive T	Only cos θ positive C

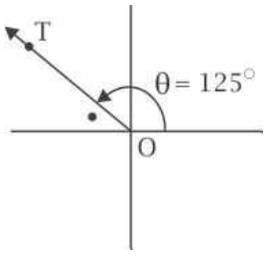
or just as

S	A
T	C

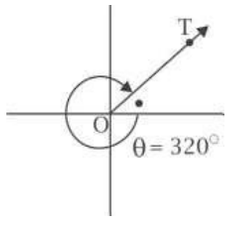
- **Reference Angle** Any polar angle has an associated angle called the Reference Angle (R.A.).

The Reference Angle for any angle θ is the acute angle that the ray OT makes with the x axis and it is always positive.

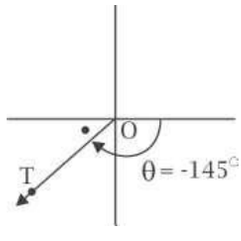
Example:



RA marked with a dot
= 180° – 125°
= 55°



RA = 360° – 320°
= 40°



RA = 180° – 145°
= 35°

.

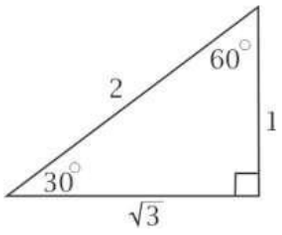
Exact Values of the Special Angles 30°, 45° and 60°

- Because $\sin 30° = \frac{1}{2} = \frac{\text{opp}}{\text{hyp}}$, we have

and $\sin 30^\circ = \frac{1}{2}$ $\quad \sin 60^\circ = \frac{\sqrt{3}}{2}$

$\cos 30^\circ = \frac{\sqrt{3}}{2}$ $\quad \cos 60^\circ = \frac{1}{2}$

$\tan 30^\circ = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$ $\quad \tan 60^\circ = \sqrt{3}$



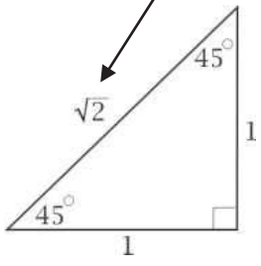
Using Pythagorean Theorem

Also because $\tan 45^\circ = 1 = \frac{1}{1} = \frac{\text{opp}}{\text{adj}}$

and $\sin 45^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\cos 45^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\tan 45^\circ = 1$

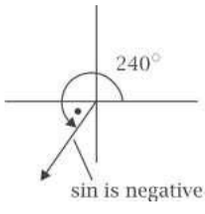


All these exact values follow from the two triangles above (which are easily sketched).

- Exact values of angles of magnitude larger than 90° are found from reference angles and the quadrants as follows:

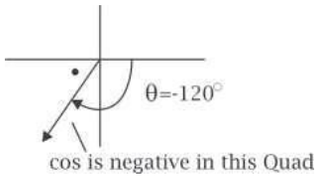
Example:

1. $\sin 240^\circ$
   $= -\sin 60^\circ$
   $= -\frac{\sqrt{3}}{2}$



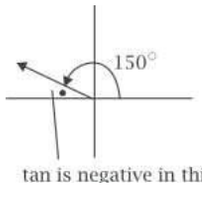
- $= RA = 240^\circ - 180^\circ$
  $= 60^\circ$

2. $\cos(-120^\circ)$
   $= -\cos 60^\circ$
   $= -\frac{1}{2}$



- $= RA = 180^\circ - 120^\circ$
  $= 60^\circ$

3. $\tan 150^\circ$
   $= -\tan 30^\circ$
   $= -\frac{\sqrt{3}}{2}$



- $= RA = 30^\circ$

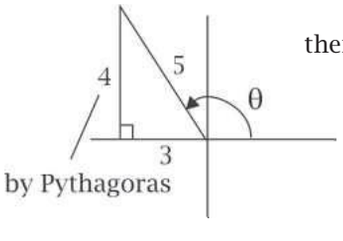
- Other exact value questions work like the following examples.

Example:

1. If $90^\circ < \theta < 180^\circ$ and $\cos\theta = -\frac{3}{5}$ find the exact value of

   a) $\sin\theta$

   b) $\tan\theta$



then a) $\sin\theta = \frac{4}{5}$

b) $\tan\theta = -\frac{4}{3}$

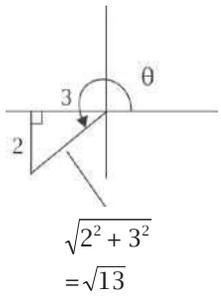
The signs come from

S	A
T	C

2. If $180^\circ < \theta < 270^\circ$ and $\tan\theta = \frac{2}{3}$ find the exact value of

a) $\cos\theta$

b) $\sin\theta$



then a) $\cos\theta = -\frac{3}{\sqrt{13}}$

b) $\sin\theta = -\frac{2}{\sqrt{13}}$

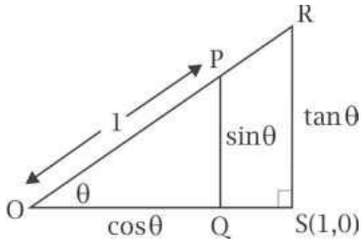
The signs come from

S	A
T	C

The key to these questions is the sketch showing all the information.

Trigonometric Relationships

- The unit circle diagram which defined the meanings of sinθ, cosθ and tanθ is again drawn as:

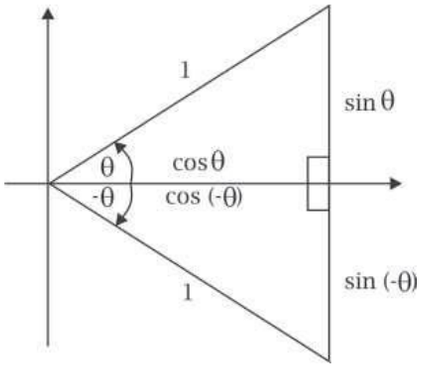


By the Pythagorean Theorem $\sin^2\theta + \cos^2\theta = 1$ and

by similar triangles $\frac{\tan\theta}{\sin\theta} = \frac{1}{\cos\theta}$

i.e. $\tan\theta = \frac{\sin\theta}{\cos\theta}$

- Parity relationships follow from the diagram below:



So $\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

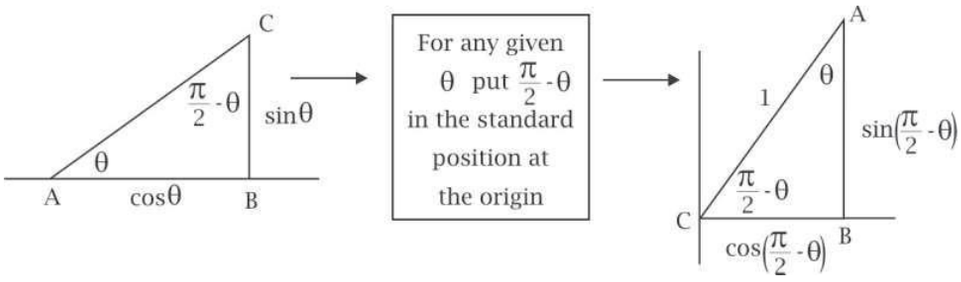
$\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)}$

$= \frac{-\sin\theta}{\cos\theta}$

These relationships can be shown to work for angle x in any quadrant

ie. $\therefore \tan(-\theta) = -\tan\theta$

- Complementary relationships



then for BC in both triangles

$$\sin\theta = \cos\left(\frac{\pi}{2}-\theta\right)$$

and for AB in both triangles

$$\cos\theta = \sin\left(\frac{\pi}{2}-\theta\right)$$

Also

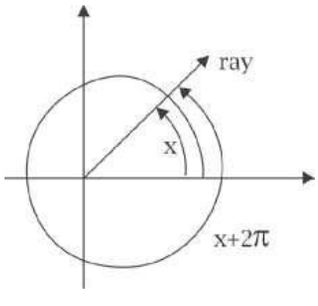
$$\tan\left(\frac{\pi}{2}-\theta\right) = \frac{\sin\left(\frac{\pi}{2}-\theta\right)}{\cos\left(\frac{\pi}{2}-\theta\right)}$$

$$= \frac{\cos\theta}{\sin\theta}$$

i.e. $$\tan\left(\frac{\pi}{2}-\theta\right) = \frac{1}{\tan\theta}$$

Periodicity Relationships

- If the ray for x is rotated by 2π the new ray coincides with the original ray and the diagram is:



So

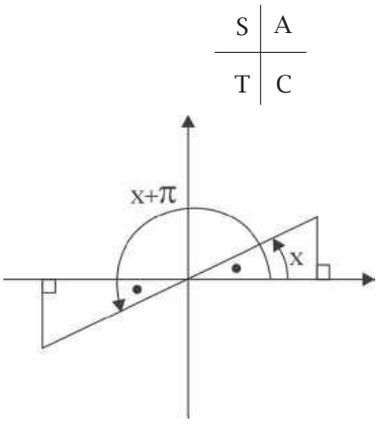
$$\sin(\theta + 2\pi) = \sin\theta$$
$$\cos(\theta + 2\pi) = \cos\theta$$
$$\tan(\theta + 2\pi) = \tan\theta$$

These relationships can be generalised:

$$\sin(\theta + 2n\pi) = \sin\theta$$
$$\cos(\theta + 2n\pi) = \cos\theta$$
$$\tan(\theta + 2n\pi) = \tan\theta$$

for any integer n.

- When x is rotated by π the new ray in the third quadrant will have a reference angle of x, and the signs are found from



So

$$\sin(\theta+\pi) = -\sin\theta$$
$$\cos(\theta+\pi) = -\cos\theta$$
$$\tan(\theta+\pi) = \tan\theta$$

- Phase relationships as really applies to all of the above identities are easily shown to be true from the study of the graphs of sin θ, cos θ and tan θ which is covered in the next section.

However consider

$$\cos\left(\theta-\frac{\pi}{2}\right) = \cos\left(-\left(\frac{\pi}{2}-\theta\right)\right) = \cos\left(\frac{\pi}{2}-\theta\right)$$

parity

from earlier

$$\therefore \cos\left(\theta-\frac{\pi}{2}\right) = \sin\theta$$

$$\sin\left(\theta-\frac{\pi}{2}\right) = \sin\left(-\left(\frac{\pi}{2}-\theta\right)\right) = -\sin\left(\frac{\pi}{2}-\theta\right)$$

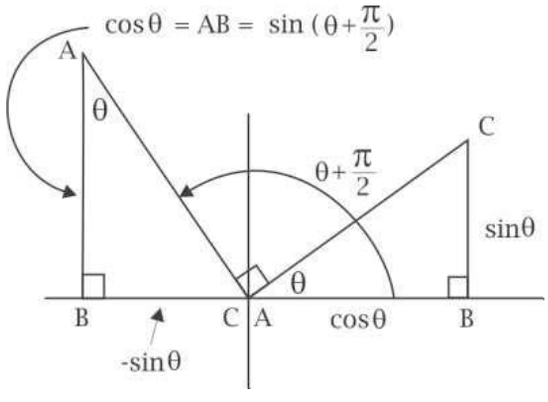
$$\therefore \sin\left(\theta-\frac{\pi}{2}\right) = -\cos\theta$$

From the diagram $\sin\left(\theta+\frac{\pi}{2}\right) = \cos\theta$

and $\cos\left(\theta+\frac{\pi}{2}\right) = -\sin\theta$

$$\tan\left(\theta+\frac{\pi}{2}\right) = \frac{\sin\left(\theta+\frac{\pi}{2}\right)}{\cos\left(\theta+\frac{\pi}{2}\right)} = \frac{\cos\theta}{-\sin\theta}$$

$$\tan\left(\theta+\frac{\pi}{2}\right) = -\frac{1}{\tan\theta}$$



Graphs of Trigonometric Functions – Basics

- The functions y = a sin b θ and y = a cos b θ are sinusoidal in shape when θ is plotted along the x axis. You should know that

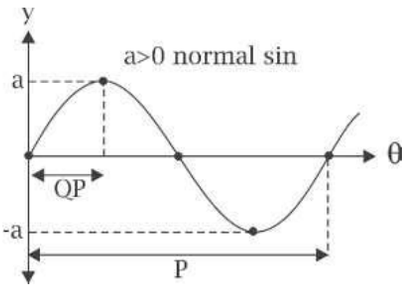
i) Amplitude = | a |. By definition the amplitude is positive.

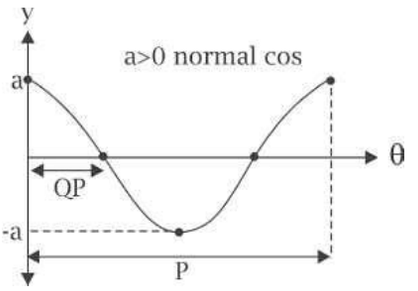
ii) Period $P = \frac{360^\circ}{|b|} = \frac{2\pi}{|b|}$

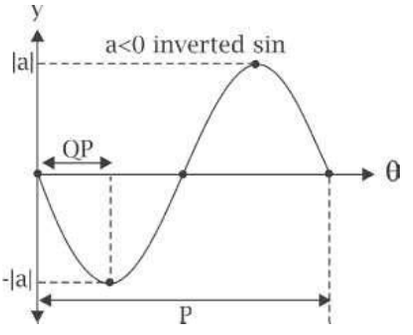
iii) If a < 0 the basic shape is reflected in the horizontal axis (inverted).

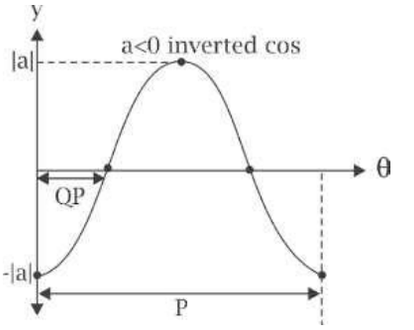
iv) Each period P can be divided into four quarter periods $QP = \frac{P}{4}$.

The basic shapes are:









- The function $y = a \tan b\theta$ has the basic shape shown below where

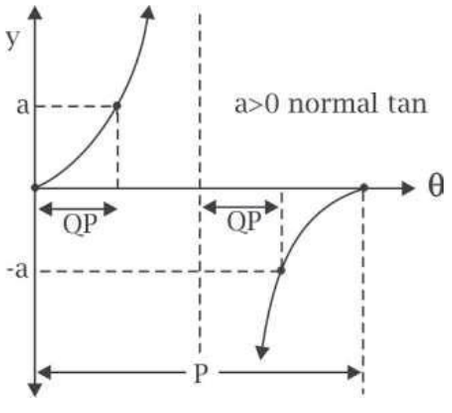
  i) Amplitude is not defined.

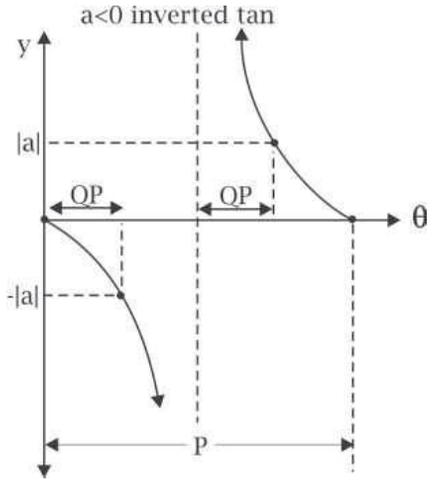
  ii) Period $P = \frac{180^\circ}{|b|} = \frac{\pi}{|b|}$

  iii) Quarter Period $QP = \frac{P}{4}$.

  iv) $a > 0$ gives a "normal" graph
  $a < 0$ gives an inverted graph

  iv) At $\theta = \frac{P}{2}$ (and odd multiples of $\frac{P}{2}$) there is a vertical asymptote.





It can be seen from the graphs that

when	$a > 0$ and $x = QP$ then $y = a$ and	This all swaps
when	$x = 3$ QP then $y = -a$	around for $a < 0$

which means that "a" affects the steepness of the curve.

GRAPHS OF TRANSLATED TRIG FUNCTIONS

If in the basic functions $y = a \sin b\theta$, $y = a \cos b\theta$ and $y = a \tan b\theta$, θ is replaced by $(\theta - m)$ and y is replaced by $y - n$,

then we have $y - n = a \sin b(\theta - m)$
or $y = a \sin b(\theta - m) + n$

$y - n = a \cos b(\theta - m)$ or $y = a \cos b(\theta - m) + n$

$y - n = a \tan b(\theta - m)$ or $y = a \tan b(\theta - m) + n$

These new functions are simply the original ones translated according to values of m and n as follows.

i) if $m > 0$ the basic function is shifted right m units

ii) if $m < 0$ the basic function is shifted left m units

iii) if $n > 0$ the basic function is shifted up n units

iv) if $n < 0$ the basic function is shifted down n units

Examples:

1. $y = 3 \sin 2(x + \frac{\pi}{3}) + 5$ is $y = 3 \sin 2x$ translated $\frac{\pi}{3}$ units left and 5 units up.

2. $y = -2 \tan(3x - \frac{\pi}{2}) - 4$

$= -2 \tan 3(x - \frac{\pi}{6}) - 4$ is $y = -2 \tan 3x$ translated $\frac{\pi}{6}$ units right and 4 units down.

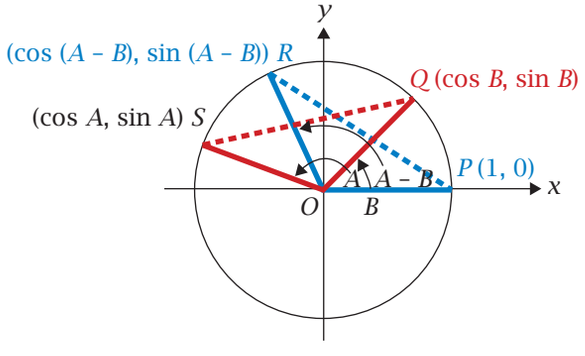
It is very important to remember that to get the horizontal shift the coefficient of θ must be factorised to give $b(\theta - m)$, giving the horizontal shift of m.

COSINE OF A SUM OR DIFFERENCE

$\cos(A - B) = \cos A \cos B + \sin A \sin B$

$\cos(A + B) = \cos A \cos B - \sin A \sin B$

Difference Identity for Cosine



- Point Q is on the unit circle, so the coordinates of Q are $(\cos B, \sin B)$.

  The coordinates of S are $(\cos A, \sin A)$.

  The coordinates of R are $(\cos(A - B), \sin(A - B))$.

  $m \angle SOQ = A - B$

- Since the central angles SOQ and POR are equal, $PR = SQ$.

  Using the distance formula, we have:

  $\sqrt{[\cos(A - B) - 1]^2 + [\sin(A - B) - 0]^2} = \sqrt{(\cos A - \cos B)^2 + (\sin A - \sin B)^2}$

- Square both sides and clear parentheses:

  $\cos^2(A - B) - 2\cos(A - B) + 1 + \sin^2(A - B)$
  $= \cos^2 A - 2 \cos A \cos B + \cos^2 B + \sin^2 A - 2 \sin A \sin B + \sin^2 B$

  Rearrange the terms:

  $[\cos^2(A - B) + \sin^2(A - B)] - 2\cos(A - B) + 1$
  $= [\cos^2 A + \sin^2 A] + [\cos^2 B + \sin^2 B] - 2 \sin A \sin B - 2 \cos A \cos B$

$\sin^2 \theta + \cos^2 \theta = 1$ for any value of θ.

$$1 - 2\cos(A - B) + 1 = 1 + 1 - 2\sin A \sin B - 2\cos A \cos B$$
$$2 - 2\cos(A - B) = 2 - \sin A \sin B + \cos A \cos B$$

Subtract 2, then divide by –2:

$$\cos(A - B) = \sin A \sin B + \cos A \cos B$$

Sum Identity for Cosine

- To find a similar expression for $\cos(A + B)$ rewrite $A + B$ as $A - (-B)$ and use the identity for $\cos(A - B)$.

$$\begin{aligned}\cos(A + B) &= \cos[A - (-B)] \\ &= \cos A \cos(-B) + \sin A \sin(-B) \\ &\qquad \text{Cosine difference identity} \\ &= \cos A \cos B + \sin A(-\sin B) \\ &\qquad \text{Negative angle identities} \\ &= \cos A \cos B - \sin A \sin B\end{aligned}$$

SINE OF A SUM OR DIFFERENCE

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

Sum and Difference Identities for Sine

- We can use the cosine sum and difference identities to derive similar identities for sine and tangent.

$$\begin{aligned}\sin(A + B) &= \cos[90^\circ - (A + B)] \qquad \text{Cofunction identity} \\ &= \cos[(90^\circ - A) - B] \\ &= \cos(90^\circ - A)\cos B + \sin(90^\circ - A)\sin B \\ &\qquad \text{Cosine difference identity} \\ &= \sin A \cos B + \cos A \sin B \\ &\qquad \text{Cofunction identities}\end{aligned}$$

- $$\begin{aligned}\sin(A - B) &= \sin[A + (-B)] \\ &= \sin A \cos(-B) + \cos A \sin(-B) \\ &\qquad \text{Sine sum identity} \\ &= \sin A \cos B - \cos A \sin B \\ &\qquad \text{Negative-angle identities}\end{aligned}$$

TANGENT OF A SUM OR DIFFERENCE

Sum and Difference Identities for Tangent

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

- Replace B with $-B$ and use the fact that $\tan(-B) = -\tan B$ to obtain the identity for the tangent of the difference of two angles.

- Use the cosine sum and difference identities to derive similar identities for sine and tangent.

$$\tan(A + B) = \frac{\sin(A + B)}{\cos(A + B)} \qquad \text{Fundamental identity}$$

$$= \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B} \qquad \text{Sum identities}$$

$$= \frac{\dfrac{\sin A \cos B + \cos A \sin B}{1}}{\dfrac{\cos A \cos B - \sin A \sin B}{1}} \cdot \frac{\dfrac{1}{\cos A \cos B}}{\dfrac{1}{\cos A \cos B}}$$

Multiply numerator and denominator by 1.

- $\tan(A+B) = \dfrac{\dfrac{\sin A\cos B}{\cos A\cos B} + \dfrac{\cos A\sin B}{\cos A\cos B}}{\dfrac{\cos A\cos B}{\cos A\cos B} - \dfrac{\sin A\sin B}{\cos A\cos B}}$ Multiply

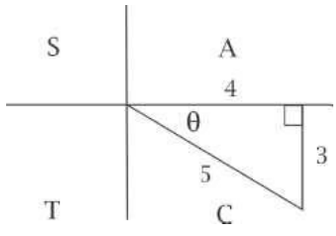
$= \dfrac{\dfrac{\sin A}{\cos A} + \dfrac{\sin B}{\cos B}}{1 - \dfrac{\sin A}{\cos A}\cdot\dfrac{\sin B}{\cos B}}$ Simplify

$= \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$ Fundamental identity

Worked Examples

Trigonometry II

4.1 If $\tan\theta = -\frac{3}{4}$ and $\sin\theta$ is negative, find the exact value of $\sin 2\theta$ without the use of your calculator.



If $\tan\theta < 0$ and $\sin\theta < 0$, θ must lie in quadrant 4.

$\sin 2\theta = 2\sin\theta\cos\theta$

$= \frac{2}{1}\left(\frac{-3}{5}\right)\frac{4}{5}$

$\therefore \sin 2\theta = -\frac{24}{25}$

$\sin\theta = -\frac{3}{5}$ Hyp $= \sqrt{3^2+4^2}$

$= 5$

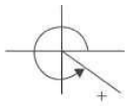
$\cos\theta = \frac{4}{5}$

4.2 Simplify $\dfrac{\cos(2\pi - A)\tan(\pi - A)\tan\left(3\pi - \frac{A}{2}\right)}{\sin\left(A - \frac{\pi}{2}\right)}$

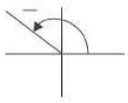
These related angle types can be tackled using the appropriate compound angle formulae. Alternatively and more quickly you can process parts of these types by:

- imagining that A is an angle in the first quadrant say 20° and using A = 20° find the quadrant that the related angle e.g. $(2\pi - A)$ lies in, in this case $2\pi - A = 340°$ (easier to think in degrees).
- the quadrant then determines the sign, so $\cos(2\pi - A)$ will be positive because cos is positive in the fourth quadrant
- lastly you will need to remember that sin, cos and tan of functions involving $\pi, 2\pi, 3\pi$ etc stay as they are, whereas these functions of $\frac{\pi}{2}, \frac{3\pi}{2}$ etc change over to cos, sin and cot* respectively.

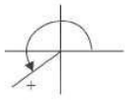
So $\cos(2\pi - A) = +\cos A$ $\quad 360^\circ - 20^\circ = 340^\circ$



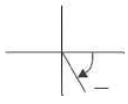
$\tan(\pi - A) = -\tan A$ $\quad 180^\circ - 20^\circ = 160^\circ$



$\tan\left(\frac{3\pi}{2} - A\right) = +\cot A$ $\quad 270^\circ - 20^\circ = 250^\circ$



$\sin\left(A - \frac{\pi}{2}\right) = -\cos A$ $\quad 20^\circ - 90^\circ = -70^\circ$



$$\therefore \frac{\cos(2\pi - A)\,\tan(\pi - A)\,\tan\left(\frac{3\pi}{2} - A\right)}{\sin\left(A - \frac{\pi}{2}\right)}$$

$$= \frac{\cos A \,.(-\tan A)\,.\cot A}{-\cos A}$$

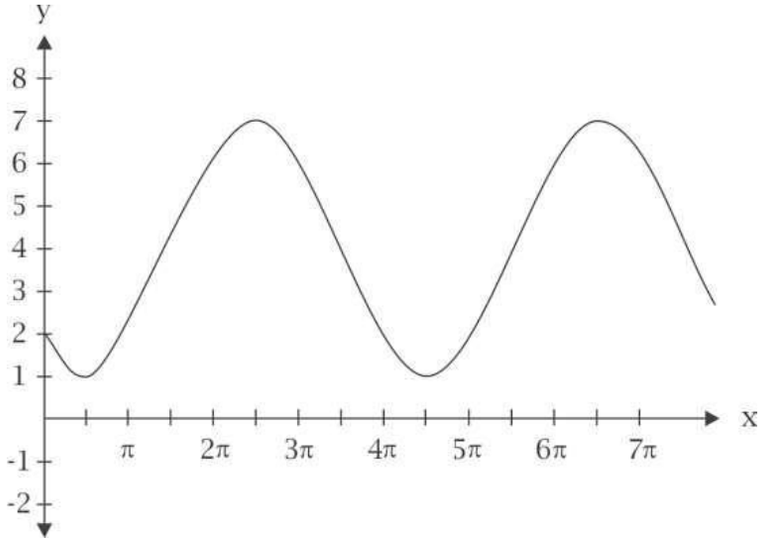
$$= \frac{\cos A \,.\left(\frac{-\sin A}{\cos A}\right).\frac{\cos A}{\sin A}}{-\cos A}$$

$= 1$ for all angles A

*$\cot A = \frac{1}{\tan A} = \frac{\cos A}{\sin A}$

4.3 Find an equation for each of the following.

(a)



Amplitude $= \frac{7-1}{2}$ y shift $= 7 - 3$

$= 3$ $= 4$

Each horizontal unit $= \frac{\pi}{2}$

Period $= 8$ units $b = \frac{2\pi}{4\pi}$

$= 8 \times \frac{\pi}{2}$ $b = \frac{1}{2}$

$= 4\pi$

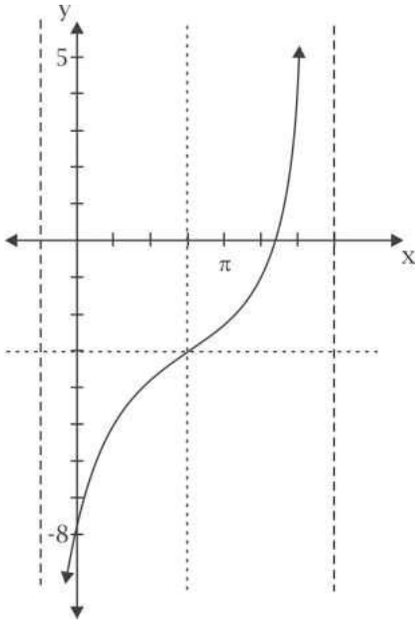
Students are encouraged to draw in a vertical dotted line at $x = \frac{\pi}{2}$ and a horizontal dotted line at $y=4$. i.e. at the positions of the horizontal and vertical shifts

Consider the function as inverted cos shifted 1 unit $= \frac{\pi}{2}$ right

$$\therefore\ y = -3\cos\frac{1}{2}\left(x - \frac{\pi}{2}\right) + 4$$

or $y = -3\cos\left(\frac{x}{2} - \frac{\pi}{4}\right) + 4$ is the required equation.

(b)



Each horizontal unit $= \frac{\pi}{4}$

Consider function as normal tan shifted right $\frac{3\pi}{4}$

Then vertical shift is 3 down.

Draw in dotted lines at $x = \frac{3\pi}{4}$ and $y = -3$. Imagine that these dotted lines are pretend y and x axes.

Middle position between the asymptote at $-\frac{\pi}{4}$ and $\frac{3\pi}{4}$ is $\frac{\pi}{4}$ where $y = -5$

$\therefore$ a value $= (-3) - (-5) = 2$

Period $= 2\left(\frac{3\pi}{4} - -\frac{\pi}{4}\right)$ $P = \frac{\pi}{b}$

$= 2\pi$ $2\pi = \frac{\pi}{b}$

$\therefore\ y = 2\tan\frac{1}{2}\left(x - \frac{3\pi}{4}\right) - 3$ $b = \frac{\pi}{2\pi}$

or $y = 2\tan\left(\frac{x}{2} - \frac{3\pi}{8}\right) - 3$ $b = \frac{1}{2}$

4.4 Find the exact value of

(a) $\cos 15^\circ$

(b) $\sin 75^\circ$

(c) $\tan \dfrac{7\pi}{12}$

(a) $\cos 15^\circ = \cos (45^\circ - 30^\circ)$

$= \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ$

$= \dfrac{\sqrt{2}}{2} \cdot \dfrac{\sqrt{3}}{2} + \dfrac{\sqrt{2}}{2} \cdot \dfrac{1}{2}$

$= \dfrac{\sqrt{6} + \sqrt{2}}{4}$

(b) $\sin 75^\circ = \sin (45^\circ + 30^\circ)$

$= \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ$

$= \dfrac{\sqrt{2}}{2} \cdot \dfrac{\sqrt{3}}{2} + \dfrac{\sqrt{2}}{2} \cdot \dfrac{1}{2}$

$= \dfrac{\sqrt{6} + \sqrt{2}}{4}$

(c) $\tan \dfrac{7\pi}{12} = \tan \left(\dfrac{\pi}{3} + \dfrac{\pi}{4} \right) = \dfrac{\tan \dfrac{\pi}{3} + \tan \dfrac{\pi}{4}}{1 - \tan \dfrac{\pi}{3} \tan \dfrac{\pi}{4}}$

$= \dfrac{\sqrt{3} + 1}{1 - \sqrt{3} \cdot 1}$

$= \dfrac{\sqrt{3} + 1}{1 - \sqrt{3}} \cdot \dfrac{1 + \sqrt{3}}{1 + \sqrt{3}}$

$= \dfrac{\sqrt{3} + 3 + 1 + \sqrt{3}}{1 - 3} = \dfrac{4 + 2\sqrt{3}}{-2}$

$= -2 - \sqrt{3}$

PROBLEMS TO SOLVE

CHAPTER 4: TRIGONOMETRY II

1. Determine the smallest positive angle θ as an exact value in radians for which $\cos\theta = -\frac{\sqrt{3}}{2}$

2. Find the exact values of:
   (a) $\sin\frac{\pi}{3}$
   (b) $\tan^2\frac{\pi}{6}$
   (c) $2\cos\frac{\pi}{4}$
   (d) $\tan(-240^\circ)$

3. Solve $3\cos^2 x - 7\cos x + 2 = 0$ over $0 \le x \le 2\pi$

4. Find the exact values of:
   (a) $\tan\left(\frac{4\pi}{3}\right)$
   (b) $\cos^2\left(\frac{-13\pi}{6}\right)$
   (c) $\sin^2\left(\frac{\pi}{37}\right) + \cos^2\left(\frac{\pi}{37}\right)$

5. Solve these equations.
   (a) $\tan 3x = 2$, $-90^\circ \le x \le 90^\circ$
   (b) $2\cos^2 x + \cos x - 1 = 0$, $-\pi \le x \le \pi$

6. If $\sin x = \frac{7}{25}$ and $0^\circ \le x \le 90^\circ$, without using a calculator evaluate $\sin (x + 45^\circ)$.

7. In ΔPQR, $PQ = \sqrt{7}$, $QR = 3$, $\angle PQR = 90^\circ$ and $\angle PRQ = \alpha$. Evaluate the following exactly:

(a) PR

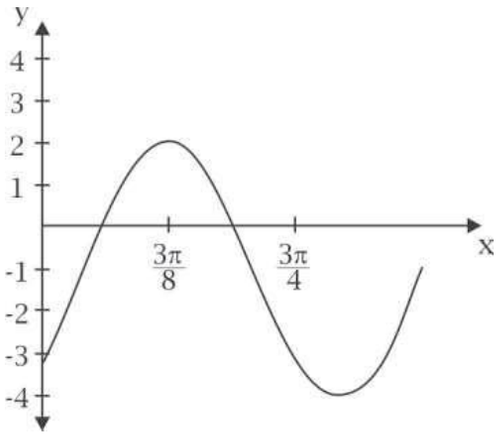
(b) $\sin \alpha$

(c) $\sin (90^\circ - \alpha)$

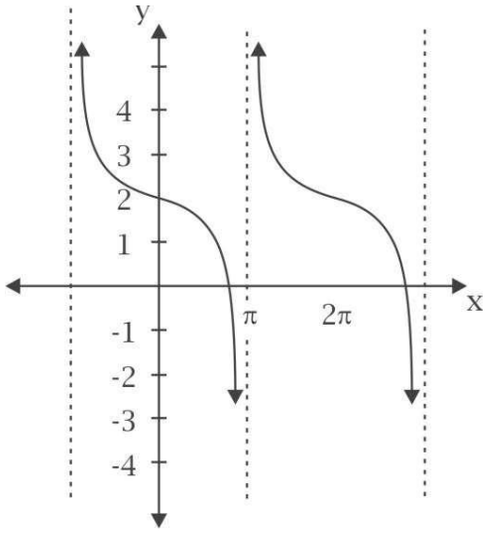
(d) $\cos\left(\frac{3\pi}{2} + \alpha\right)$

8. Find the equations of the following curves.

(a)



(b)



9. Solve the following. Where possible leave your answers as exact values.

(a) $2 \cos 3x + \sqrt{3} = 0$ $\quad -\pi \leq x \leq \pi$

(b) $4 \sin 2\theta + 4\sin\theta - 6\cos\theta = 3$ $\quad 0 \leq x \leq \pi$

10. The height of the tide above mean sea level at a certain point has been modelled by the equation

$$h(t) = 4 \sin \frac{\pi t}{6} \text{ m}$$

where *t* is the number of hours after midnight on a particular day.

(a) Neatly sketch the graph for $h(t) = 4 \sin \frac{\pi t}{6}$ for $0 \leq t \leq 24$.

(b) When was the high tide?

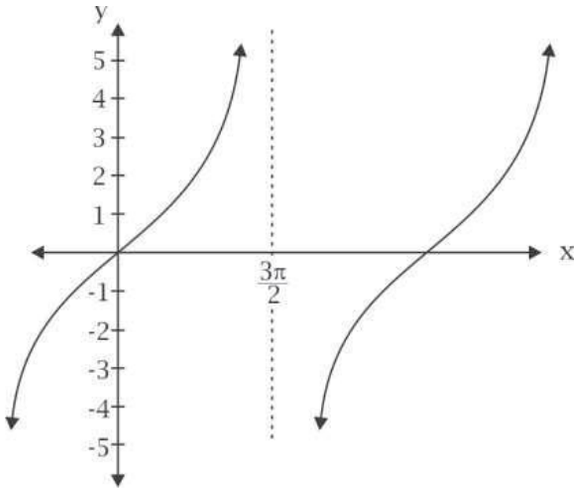
(c) What was the height of the tide above mean sea level at high tide?

(d) What was the height of the tide at 8 am?

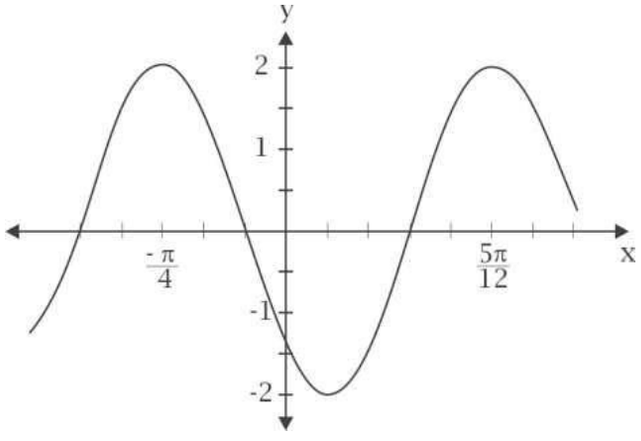
(e) A ship can only enter port when there is a depth of 5 metres of water above low tide. Between what times could a ship enter or leave port?

11. Write the equations for the following graphs.

(a)



(b)



12. Find the exact value of

(a) $\cos 105^\circ$

(b) $\cos \frac{7\pi}{12}$

(c) $\cos 105^\circ \cos 45^\circ + \sin 105^\circ \cos 45^\circ$

(d) $\sin 105^\circ$

(e) $\sin \frac{\pi}{12}$

(f) $\sin 40^\circ \cos 160^\circ - \cos 40^\circ \sin 160^\circ$

(g) $\tan \frac{\pi}{12}$

(h) $\frac{\tan 30^\circ + \tan 15^\circ}{1 - \tan 30^\circ \tan 15^\circ}$

(i) $\tan 150^\circ$

5 COUNTING TECHNIQUES

Syllabus Checklist

On the completion of this chapter you should be able to:

☐ Combinations

- understand the notion of a combination as a set of r objects taken from a set of n distinct objects
- use the notation $\binom{n}{r}$ and the formula $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ for the number of combinations of r objects taken from a set of n distinct objects
- expand $(x + y)^n$ for small positive integers n
- recognise the numbers $\binom{n}{r}$ as binomial coefficients (as coefficients in the expansion of $(x + y)^n$)
- use Pascal's triangle and its properties

COMBINATIONS

A combination involves choosing a set of objects from a set of n distinct objects. For instance: How many ways can we select 2 different vowels from all the possible vowels (i.e. from {a, e, i, o, u})?

a, e e, i i, o o, u

a, i e, o i, u

a, o e, u

a, u

There are 10 possible selections or *Combinations* of 2 vowels from the set of 5 vowels.

You would have noticed that in the second column the pair (*e, a*) was not listed. As a selection (*a, e*) is the same as (*e, a*) because the order is not important.

When choosing r objects from n objects, we need to divide the number of permutations (or arrangements) of r objects from n objects by $r!$ possible combinations (or selections).

This can be represented in the following formula:

$${}^{n}C_{r} = \frac{n!}{(n - r)!r!}$$

It is common to represent ${}^{n}C_{r}$ as $\binom{n}{r}$

PASCAL'S TRIANGLE AND BINOMIAL COEFFICIENTS

The numbers in each row of Pascal's Triangle correspond to

Row 1 $(a + b)^0 = 1$ = 1

Row 2 $(a + b)^1 = 1a + 1b$ = 1 1

Row 3 $(a + b)^2 = 1a^2 + 2ab + 1b^2$ = 1 2 1

Row 4 $(a + b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$ = 1 3 3 1

The final line can also be written as $\binom{3}{0}a^3 + \binom{3}{1}a^2b + \binom{3}{2}ab^2 + \binom{3}{3}b^3$

You might even consider the sum of the coefficients in each row as a power of 2.
For instance

Row 1: $1 = 2^0$

Row 2: $1 + 1 = 2 = 2^1$

Row 3: $1 + 2 + 1 = 4 = 2^2$

Note how: (i) The powers of a decrease by one each time

(ii) The powers of b increase by one each time

(iii) The term(s) with the largest coefficients is in the middle

(iv) The coefficients are symmetrical about the middle term(s)

(v) The sum of the coefficients in row n is 2^{n-1}

We can generalise that:

$$(a + b)^n = \binom{n}{0}a^nb^0 + \binom{n}{1}a^{n-1}b^1 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots \binom{n}{n}a^0b^n$$

Worked Examples

5.1 (a) Write $\binom{6}{2}$ as a factorial and evaluate

(b) Place $\frac{8 \times 7 \times 6 \times 5}{4 \times 3 \times 2 \times 1}$ into $\binom{n}{r}$ form and evaluate

(c) Solve for x in the following:

(i) $\binom{x}{4} = \frac{6!}{4!2!}$

(ii) $\binom{x}{2} = 10$

(a) $\binom{6}{2} = \frac{6!}{(6 - 2)!2!}$ (Substitute values into the formula $\binom{n}{r} = \frac{n!}{(n - r)!r!}$)

$= \frac{6!}{4!2!}$ (Evaluate)

$= 15$

(b) $\frac{8 \times 7 \times 6 \times 5}{4 \times 3 \times 2 \times 1} = \frac{8!}{4!4!}$ (Multiply numerator and denominator by 4! to make a better 'fit' to the formula)

$= \binom{8}{4}$ (Simplify into the form $\binom{n}{r}$ and evaluate)

$= 70$

(c) (i) $\frac{x!}{(x-4)!2!} = \frac{6!}{4!2!}$ (Compare LHS and RHS of the equation to formula $\frac{n!}{(n-r)!r!}$)

$\therefore x = 6$ (Conclude with correct value of x)

(ii) $\frac{x!}{(x-2)!2!} = 10$

$\frac{x!}{(x-2)!} = 10 \times 2!$ (Rearrange by multiplying RHS of equation by 2!, and simplify)

$\frac{x!}{(x-2)!} = 20$

$\frac{x(x-1)(x-2)!}{(x-2)!} = 20$ (Expand numerator of LHS using factorial until denominator is reached)

$x(x-1) = 20$ (Simplify by cancelling down $(x-2)!$)

$x^2 - x - 20 = 0$ (Create a quadratic trinomial and prepare to solve for x)

$(x-5)(x+4) = 0$ (Factorise LHS)

$x = 5, x = -4$ (Using Null Factor Law, solve for values of x)

$\therefore x = 5$ (Accept $x = 5$ as x cannot be negative)

5.2 Using Pascal's Triangle determine the first three terms of:

(a) $(c + 3d)^5$

(b) $(a + b)^8$

(a) $(c + 3d)^5$

$= c^5 + 5c^4(3d) + 10c^3(3d)^2$ (Use Pascal's Triangle to determine coefficients of first three terms, and as the powers of c decrease by one each term, the powers of $3d$ increase by one each term)

$= c^5 + 25c^4d + 90c^3d^2$ (Simplify)

(b) $(a + b)^8$

$= a^8 + 8a^7b + 28a^6b^2$ (Use Pascal's Triangle to determine coefficients of first three terms, and as the powers of a decrease by one each term, the powers of b increase by one each term). Simplify.

5.3 There are 12 burrows leading out of a rabbit warren, but a hunter has only 4 traps. He sets all of his traps in the entrance of 4 different burrows. How many choices of 4 burrows has he?

$\binom{12}{4}$ The farmer has a choice of 4 burrows from a possible 12 burrows. Using combination notation, express in form $\binom{n}{r}$ and evaluate.

$= \frac{12!}{(12-4)!4!}$

$= \frac{12!}{8!4!}$

$= 495$

PROBLEMS TO SOLVE

CHAPTER 5: COUNTING TECHNIQUES

1. Write the following as factorials and evaluate:

   (a) $\binom{4}{2}$

   (b) $\binom{5}{1}$

   (c) $\binom{10}{5}$

   (d) $\binom{7}{4}$

   (e) $\binom{6}{3}$

   (f) $\binom{9}{4}$

2. Place the following into $\binom{n}{r}$ form:

   (a) $\dfrac{9 \times 8 \times 7}{3 \times 2 \times 1}$

   (b) $\dfrac{11 \times 10}{2 \times 1}$

   (c) $\dfrac{7 \times 6 \times 5}{3 \times 2 \times 1}$

   (d) $\dfrac{8}{1}$

   (e) $\dfrac{5 \times 4}{2 \times 1}$

   (f) $\dfrac{10 \times 9 \times 8}{3 \times 2 \times 1}$

3. Solve for x:

(a) $\binom{x}{3} = \frac{8!}{5!3!}$

(b) $\binom{x}{6} = \frac{x!}{3!6!}$

(c) $\binom{7}{x} = \frac{7!}{5!2!}$

(d) $\binom{x}{x-1} = 8$

(e) $\binom{x}{x-2} = 45$

(f) $\binom{x}{3} = 56$

4. Complete the following definitions:

(a) ${}^{n}C_{1} = \binom{\ }{\ } = \frac{n!}{(n - \_\_)!\_\_!} = n$

(b) ${}^{n}C_{0} = \binom{\ }{\ } = \frac{n!}{(\_\_)!0!} = 1$

(c) ${}^{n}C_{n} = \binom{\ }{\ } = \frac{n!}{(n - n)!\_\_!} = 1$

(d) ${}^{n}C_{n-1} = \binom{\ }{\ } = \frac{n!}{(n - (\_\_))!(\_\_)!} = n$

5. Give the first three terms of each expansion in simplified form:

(a) $(m + n)^5$

(b) $(x + 2y)^6$

(c) $(3a - b)^4$

(d) $(2x - 3y)^5$

(e) $\left(x + \frac{1}{x}\right)^7$

(f) $\left(\frac{2}{a} - \frac{b}{3}\right)^3$

6. How many 4-lettered words can be selected from the English alphabet?

7. In the game of Poker, a hand consists of 5 cards. How many different hands can be dealt from a standard deck of 52 cards?

8. An examination paper contains 10 questions, and candidates are instructed to answer 7 questions. How many different ways can this examination be completed?

9. An AFL football squad is comprised of 40 players, yet the coaching staff must select a team of 25 players for any given game. How many different teams can be chosen for a game?

10. In a shipment of 12 cars, 2 are known to have defective brakes. How many different selections of 4 cars are possible if:

(a) a car dealer is unaware of the fault?

(b) a car dealer knows of the fault and wishes to choose 4 roadworthy cars?

11. Some of the numbers forming Pascal's Triangle are arranged in the table below:

1	1	1	1	1	1	1	1
1	2	3	4	5	6	7	8
1	3	6	10	15	21	28	36
1	4	10	20	35	56	84	120
1	5	15	35	70	126	210	330
1	6	21	56	126	252	x	
1	7	28	84	210	462		y
1	8	36	120	330	792		z

(a) Write each of the terms in the outlined box within the table in the form $\binom{n}{r}$

(b) Find values of n and r that are consistent with the table and that satisfy $x = \binom{n}{r}$

(c) hence or otherwise determine the values of x, y and z.

12. In the expression of $(c + d)^9$, what is the indicial value of k in the term that contains:

(a) $c^8 d^k$

(b) $c^k d^4$

(c) $c^{2k} d^k$

(d) $c^k d^{2k}$

(e) $c^k d^{k-5}$

PROBABILITY AND SETS

6

Syllabus Checklist

By the end of this chapter, you should be able to:

☐ Review of the fundamentals of probability
- review probability as a measure of 'the likelihood of occurrence' of an event
- review the probability scale: $0 \leq P(A) \leq 1$ for each event A, with $P(A) = 0$ if A is an impossibility and $P(A) = 1$ if A is a certainty
- review the rules: $P(\overline{A}) = 1 - P(A)$ and $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- use relative frequencies obtained from data as estimates of probabilities

☐ Conditional probability and independence
- understand the notion of a conditional probability and recognise and use language that indicates conditionality
- use the notation $P(A|B)$ and the formula $P(A \cap B) = P(A|B)P(B)$
- understand the notion of independence of an event A from an event B, as defined by $P(A|B) = P(A)$
- establish and use the formula $P(A \cap B) = P(A)P(B)$ for independent events A and B, and recognise the symmetry of independence
- use relative frequencies obtained from data as estimates of conditional probabilities and as indications of possible independence of events

☐ Language of events and sets
- review the concepts and language of outcomes, sample spaces, and events, as sets of outcomes
- use set language and notation for events, including:
  a. $\overline{A}$ (or A') for the complement of an event A
  b. $A \cap B$ and $A \cup B$ for the intersection and union of events A and B respectively
  c. $A \cap B \cap C$ and $A \cup B \cup C$ for the intersection and union of the three events A, B and C respectively
  d. recognise mutually exclusive events.
- use everyday occurrences to illustrate set descriptions and representations of events and set operations

5.1 CORE THEORY

i) Set notation

A set is the collective noun used in mathematics to describe the objects found in a group. These objects are called elements.

If set Z contains the letters b, c, d, e, and f, then we can write this set as Z = {b, c, d, e, f}. These "curly" brackets are always used to list the elements of a set.

We can also state that this set has five elements. This can be expressed either by writing n(Z) = 5 or | Z | = 5. Also, it is true that

- b is an element of Z, or $b \in Z$
- r is not an element of Z, or $r \notin Z$
- If set Y is {b, c, d}, then Y is a subset of Z, or $Y \subset Z$
- The intersection ($\cap$) of multiple sets refers to the set of elements that are common to all sets. For example, if C = {5, 6, 7, 8} and D = {6, 7, 8, 10) then $C \cap D$ = {6, 7, 8}.

- If multiple sets have no intersection, then these sets are said to be disjoint.
- The union (∪) of multiple sets refers to the addition of elements from those sets without repetition. For example, if G = {10, 11, 12} and H = {12, 13} then G ∪ H = {10, 11, 12, 13}.
- If a set has no elements, it is referred to as an empty set. This is expressed using the symbol Ø or a set of empty brackets { }.
- The complement of a set consists of all those elements not in the set. The complement of A can be denoted as A′ or $\overline{A}$.
- Sets of numbers are usually written in ascending order.
- Sets of letters are usually written in alphabetical order.

ii) Probability

To calculate the theoretical probability of an event happening use the formula:

$$\text{Probability of event happening} = \frac{\textit{Number of favourable outcomes}}{\textit{Total number of outcomes}}$$

- The probability, P, of an event happening must lie between the values $0 \leq P \leq 1$.
- 0 represents an event never happening; 1 represents the certainty of an event happening.
- In set notation, $\overline{A}$ is the complement of A. Concerning probability, P ($\overline{A}$) is the complement of P(A). The complement can be calculated through: P ($\overline{A}$) = 1 – P (A).

Compound Events
There are two different types of compound events:

- P(A and B) or P(A ∩ B) refers to both events occurring simultaneously.
- P(A or B) or P(A ∪ B) refers to either one event or both events occurring.

Addition Rule Principles
If n(A ∪ B) = n(A) + n(B) – n(A ∩ B), then P(A ∪ B) = P(A) + P(B) – P(A ∩ B).

Conditional Probability
This principle applies to situations where you are asked to find the probability of an event happening **given that** another event has already occurred. To work out a problem involving conditional probability, always determine the given event first (this becomes the total number of possible outcomes).

The notation for finding the probability of Event X happening given that Event Y has happened is: P (X | Y). Another way of reading this could be the probability of X given Y.

The formula for finding conditional probability is:

$$P(X|Y) = \frac{P(X \cap Y)}{P(Y)}$$

Independent Events
An event A is independent of event B if the probability that A occurring is not influenced by whether B has or has not occurred.

Two events A and B are independent if:

$$P(A \cap B) = P(A) \times P(B)$$

$$P(A|B) = P(A)$$

Mutually Exclusive Events
Two events are mutually exclusive if both events cannot occur at the same time.

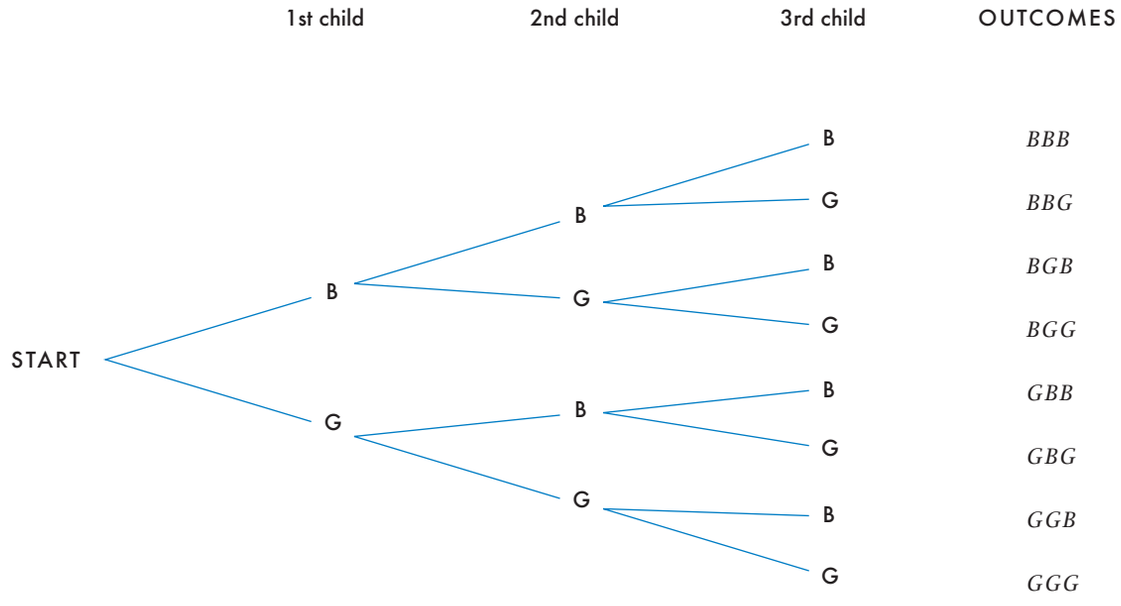
If sets A and B are mutually exclusive then:

- $P(A \cap B) = 0$
- $P(A \cap B) = P(A) \times P(B)$

Expression of Answers
- Answers may be given as fractions, decimals or percentages.
- Recurring decimals should not be rounded unless the question specifies that rounding is required.
- It is the convention to simplify fractions (e.g. $^{12}/_{20} = {}^{3}/_{5}$)

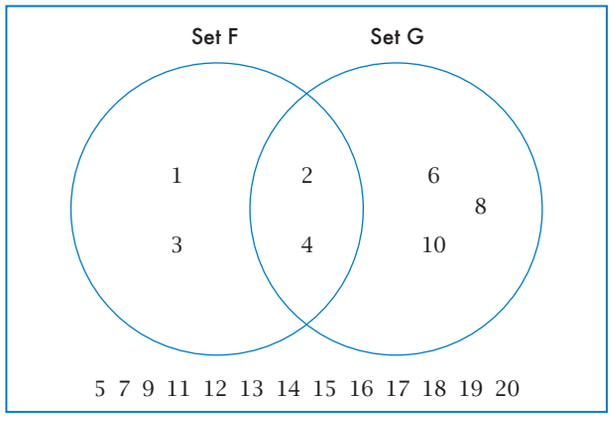
iii) Tree diagrams

Tree diagrams list the total number of possible outcomes of an event, or the sample space. Consider the tree diagram below which shows the possible gender combinations for a family of three children.

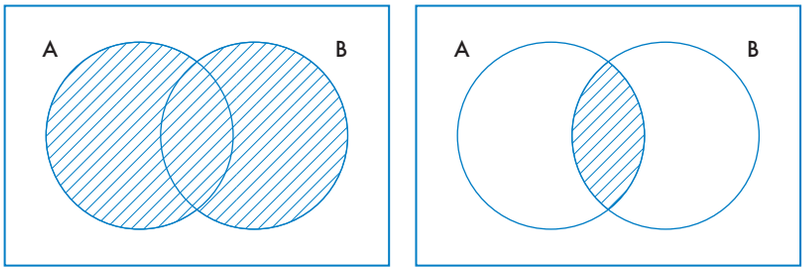


iv) Venn Diagrams

Venn diagrams show the entire universal set within the boundaries of a rectangle. Many sets can be represented simultaneously within the same Venn diagram. If we have Set F comprising {1, 2, 3, 4}, Set G consisting of {2, 4, 6, 8, 10}, and the Universal Set including all positive integers from 0 to 20, this information would make the Venn diagram as such:

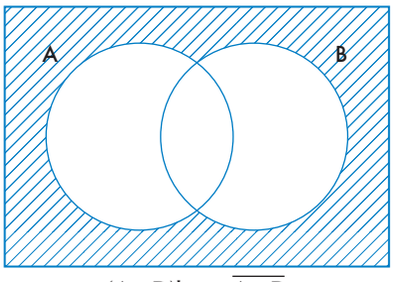


It is helpful to recognise how unions and intersections can be used to 'shade' in parts of a Venn diagram. Some more common examples include those below:

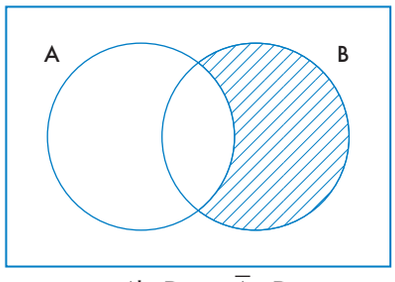


$A \cup B$

$A \cap B$



$(A \cup B)'$ or $\overline{A \cup B}$



$A' \cap B$ or $\overline{A} \cap B$

v) Two-Way Tables

Another way of representing data is through the use of a two-way table. Such a table can be generated through the occurrence or preference of two events. To set up a two-way table consider the following steps:

- Write the headings of P(A), P′(A) and Total across the top row (A is the first event)
- Write the headings of P(B), P′(B) and Total down the first column (B is the second event)
- Fill in the given probabilities from the question and determine the missing values.
- The sum of all probabilities must equal 1.

In table form, this is represented as:

	P(A)	P′(A)	Total
P(B)			
P′(B)			
Total			

Worked Examples

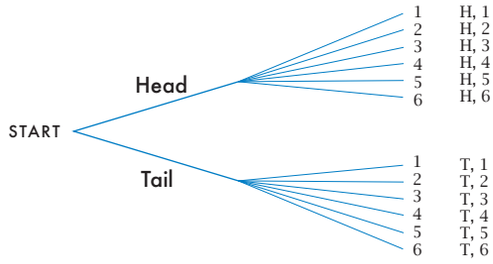
6.1 A fair coin and a fair die are tossed. Draw a tree diagram to represent this sample space, and hence, find the probability that

(a) a tail and a 6 show

(b) a tail shows

(c) a head or an odd number show

A tree diagram is required to obtain the sample space.



(a) A tail and a six occurs only once in this sample space.
Thus, the probability associated with this event is $\frac{1}{12}$

(b) There are six ways a tail can occur.
Therefore, $\frac{6}{12} = \frac{1}{2}$

(c) $\frac{9}{12} = \frac{3}{4}$

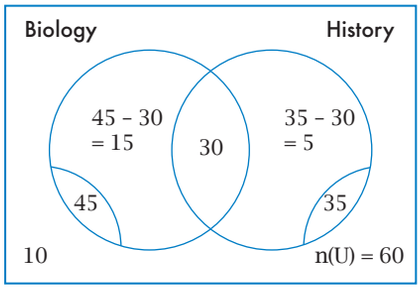
6.2 In a certain year group of 60 Year 12 students, the number of students studying History is 35, the number of students studying Biology is 45, and the number of students taking both classes is 30. Use a Venn diagram to determine the probability that a randomly selected student:

(a) does not study History

(b) does not study Biology

(c) studies Biology or History

A Venn diagram is needed to represent the given data.



(a) The number of students not studying History is represented by all those students outside of the History circle. This involves $15 + 10 = 25 \rightarrow \frac{25}{60} = \frac{5}{12}$

(b) This probability represents the number of students not studying Biology = 5 + 10 = 15
So $P(B') = \frac{15}{60} = \frac{1}{4}$

(c) This probability is the complement of the probability of all students who take History or Biology, or the number of students outside the two circles. Hence, $\frac{10}{60} = \frac{1}{6}$. (The set (B ∪ H) = 50, leaving a remainder of 10 students who sat neither subject.)

6.3 If $P(A \cap B) = 0.4$, $P(A) = 0.5$ and if events A and B are independent determine:

(a) $P(A|B)$

(b) $P(A'|B)$

If events A and B are independent then:

$$\begin{aligned} P(A \cap B) &= P(A) \times P(B) \\ 0.4 &= 0.5 \times P(B) \\ P(B) &= \frac{0.4}{0.5} \\ P(B) &= 0.8 \end{aligned}$$

Using conditional probability formula

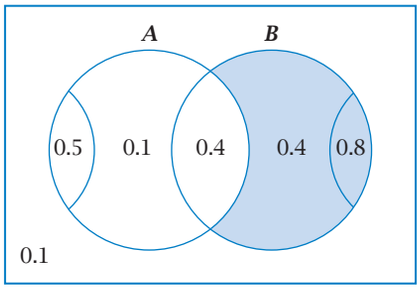
$$\begin{aligned} P(A|B) &= \frac{P(A \cap B)}{P(B)} \\ P(A|B) &= \frac{0.4}{0.8} \\ P(A|B) &= 0.5 \end{aligned}$$

If events A and B are independent then

$$P(A'|B) = \frac{P(A' \cap B)}{P(B)}$$
$$P(A'|B) = \frac{0.4}{0.8}$$
$$P(A'|B) = 0.5$$

To assist in calculating $P(A' \cap B)$ use a Venn Diagram.

$P(A' \cap B)$



6.4 At a local university 500 students were enrolled in a particular course. Within this course 329 students took Chemistry, 186 students took Physics, and 295 students took Calculus. It was also known that the following numbers of students took 2 courses:

Chemistry and Physics : 83 students
Chemistry and Calculus : 217 students
Physics and Calculus : 63 students

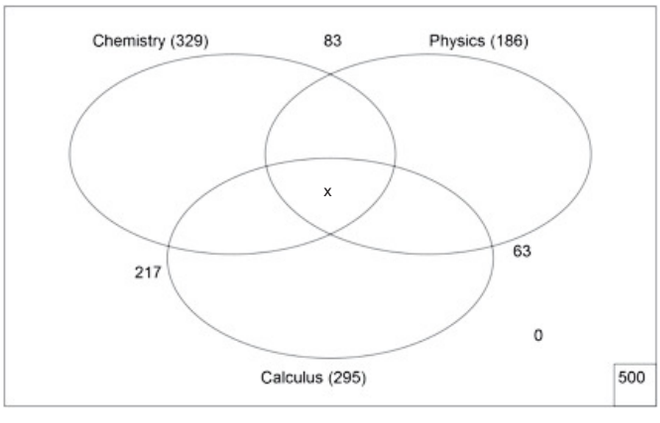
Determine the number of students who took:

(a) Chemistry, Physics and Calculus

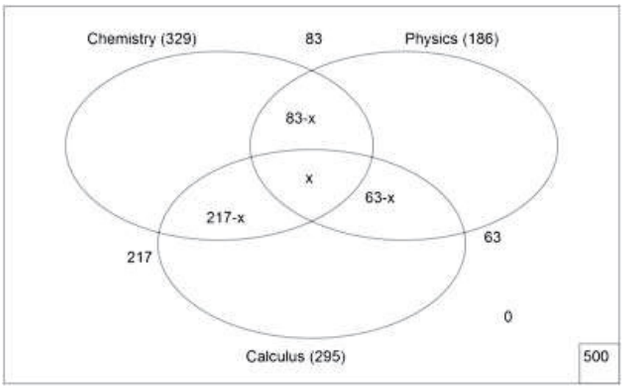
(b) Only Chemistry

(c) Exactly one of the classes: Chemistry, Physics or Calculus

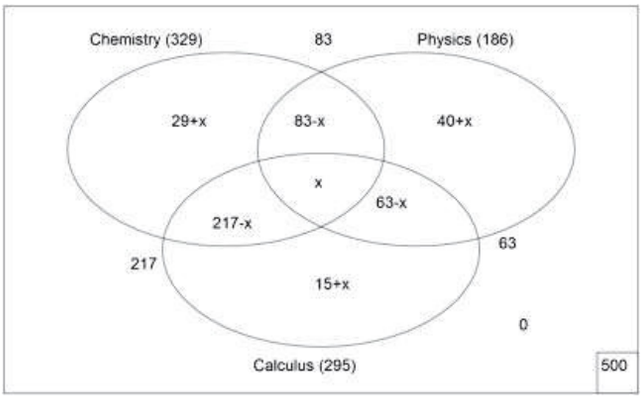
Step 1. Place all of the known information into a Venn Diagram. Note how there are no 'non-participants'; said another way, all of the 500 students took at least one of the three classes offered, leaving $n(A \cup B \cup C)' = 0$. Denote the middle region as x.



Step 2. Express the 6 unknown regions in terms of x. For instance, the intersection of Chemistry and Physics has 83 students, with x students taking Chemistry, Physics and Calculus. To obtain the number of students taking only Chemistry and Physics (and not Calculus) subtract x from 83 i.e. 83 – x. Repeat this procedure for the remaining 2 intersecting regions.



Step 3. Express the number of students taking only one subjects offered in terms of x. For instance, those students taking only Chemistry can be expressed by 329 – [(83 – x) + x + (217 – x)] which is equivalent to 29 + x. Repeat this procedure for the remaining 2 'only' regions.

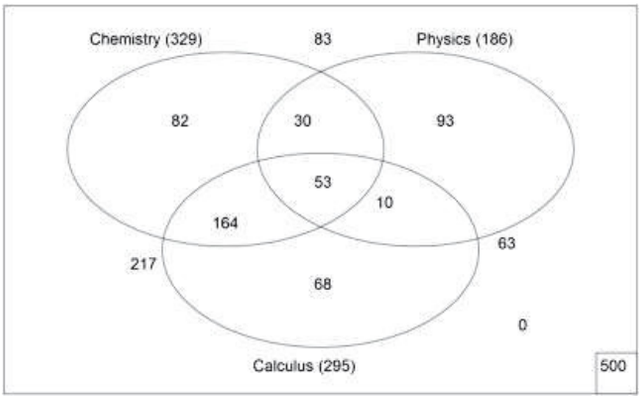


Step 4. Add all of the 8 regions (including the region 'outside' the circles) and sum them to the number in the Universal Set.

In other words, 29 + x + 83 – x + x + 217 – x + 15 + x + 63 – x + 40 + x + 0 = 500.

Solving gives x = 53

Step 5. Redisplay the Venn Diagram after determining the number of students within each region (use x = 53).



Now the questions can be answered:

(a) The number of students who took Chemistry, Physics and Calculus is the 'middle' value, 53

(b) The number of students who only did Chemistry is 82. Note that these students are within the Chemistry set but are not in any other set.

(c) Those students who study exactly one of Chemistry, Physics and Calculus can be found by adding the 3 'only' regions in the Venn Diagram. Hence, 82 + 68 + 93 = 243 students studied exactly one subject.

PROBLEMS TO SOLVE

CHAPTER 6: PROBABILITY AND SETS I

1. The table below shows the results of 100 votes at a local warehouse to poll opinions about whether to take industrial strike action. There are three times that workers can commence their shift; morning, afternoon and night. The 100 workers polled gave a response of yes, no or undecided concerning whether to strike or not.

	YES	NO	UNDECIDED	TOTAL
MORNING	24	21	10	
AFTERNOON		3		
NIGHT	11	6	3	
TOTAL				100

(a) Complete the table above using the following information. The Afternoon Shift workers who responded 'yes' represented $\frac{2}{3}$ of the Morning Shift workers who responded 'yes'.

(b) For a randomly surveyed person, determine the probability that the person:

(i) worked Afternoon Shift and was in favour of striking;

(ii) worked the Morning Shift:

(iii) was not in favour of going on strike;

(iv) was a Night Shift worker given that the person was in favour of striking;

(v) was not in favour of taking industrial action if the person worked on Morning Shift;

(vi) was not on Night Shift if the person did not want to go on strike;

(vii) was either not in favour or undecided about striking given the person worked afternoons.

2. A famous musical group is playing one concert in the park. There is a choice of two seating packages, the Diamond Package or the Green Package. Patrons selecting the Diamond Package will be provided with chairs and may either buy food at the concert or bring their own. Patrons selecting the Green Package may bring their own chairs or picnic rug to sit on and bring their own food or buy their food at the venue. Programmes are sold at the venue. Patrons who buy their food are given a programme for free. Therefore, patrons need to decide whether to select the Diamond or Green package, whether to bring or buy their food and whether or not to buy a programme.

(a) Draw a clearly labelled tree diagram to show all the possibilities.

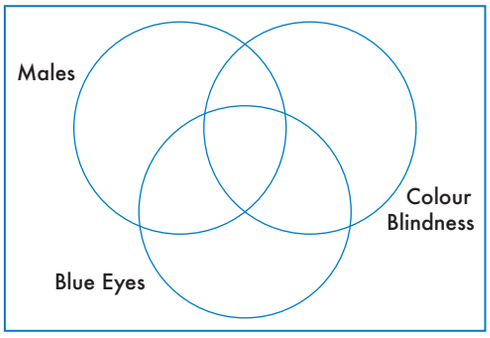
(b) If all the possibilities are equally likely, what is the probability that a patron chooses the Green Package?

(c) Given a patron has a programme, what is the probability that they got it for free?

(d) What is the probability that patrons choose to bring their own food?

3. In a Human Biology class of 30 students, the teacher discovers that four of the twenty boys and one of the girls are colour blind. Eight of the boys, including one which is colour blind, have blue eyes. Of the six girls with blue eyes, five are not colour blind.

(a) Use the Venn Diagram below to represent the above information.



If one of these thirty students is chosen at random, determine the probability of the chosen student being

(b) a person with blue eyes

(c) a colour blind person with blue eyes

(d) a male with blue eyes who is not colour blind

(e) a male given that they are not colour blind

(f) a male given that they do not have blue eyes

4. For a Universal Set of {0,1,2,3,4,5,6,7,8,9}, with $A = \{1,3,5,7,9\}$, $B = \{2,4,6,8\}$ and $C = \{2,3,5,7\}$, list the elements of:

(a) $A \cup B$

(b) $B \cap C$

(c) $\overline{B}$

(d) $A \cap \overline{C}$

(e) $A \cap B \cap C$

5. A new pharmacy assistant types three prescriptions for three patients based on an e-mail sent to her by the pharmacist. Unfortunately, during the process, the e-mail was deleted accidently. To avoid looking incompetent, the assistant sent the prescriptions out randomly to the patients.

(a) Draw an appropriate sample space.

(b) Use the sample space to find the probability that:

(i) all prescriptions end up with the correct patients.

(ii) no prescriptions end up the correct patients.

6. A statistician for Champion Records questioned teenagers on what Compact Disc they owned. Of 105 people questioned, 78 owned a Hip-Hop CD, 57 owned R & B and 73 owned Pop Rock. Twenty nine people owned CDs produced by all 3, while 10 had R & B and Hip-Hop but not Pop Rock. Seven only owned Hip-Hop, while 12 had R & B and Pop Rock but not Hip-Hop CDs.

(a) Draw a Venn diagram to represent this information.

(b) How many teenagers did not own any of the CDs?

(c) How many teenagers owned only Pop Rock CDs?

(d) Determine the probability that a teenager selected at random did not own an R & B CD?

7. Papa Joe's Pizza Parlour sells pizza in two styles, thick or thin crust. Customers have the choice of four toppings – cheese, pineapple, ham and tomato. From these four toppings a customer can choose two toppings to make a pizza; however, thick crust is never served with cheese, and thin crust is never served with ham.

(a) Draw a tree diagram to illustrate the possible outcomes.

(b) If every pizza topping combination is equally likely to be chosen, find the probability that a customer buys a pizza

(i) with ham and pineapple as the toppings

(ii) with pineapple as one of the choices of topping

(iii) which does not have tomato as a topping

(iv) with cheese on it given that the pizza was thin crust.

8. In a survey, 500 males and females were asked at what age they planned to start a family. The information from the survey is displayed in the table below.

	Age "20s"	Age "30s"	Over age 40	Total
Male	21	115		197
Female	20	212		
Total				

(a) Complete the table of values in the spaces provided.

(b) How many females planned to have children by the age of 40?

(c) What is the probability that a person will have started a family by 40 years of age given that he is male?

9. A student conducted a survey in the Fremantle area with 150 participants. The survey sought to research the preferred AFL and WAFL teams of people with the following results:

WAFL Team	AFL Team			
	Dockers	Eagles	Other	Total
East Freo.	40	9	17	66
South Freo.	8	22	11	41
Other	18	12	13	43
Total	66	43	41	150

Determine the probability that a person selected at random from those surveyed:

(a) Supported South Fremantle and the Dockers?

(b) Supported South Fremantle or supported the Eagles?

(c) Did not support South Fremantle?

(d) Did not support the Eagles or South Fremantle?

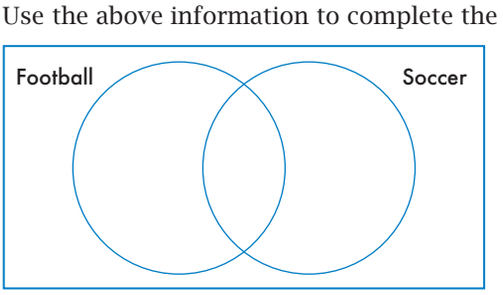
(e) Supported the Dockers given they supported South Fremantle?

(f) Barracked for East Fremantle or South Fremantle if they did not support the Dockers?

10. A group of 120 Year 10 students were asked whether they were liked playing football or soccer. The survey showed:

60% liked Football
35% liked Soccer
and 32.5% liked both Football and Soccer.

(a) Use the above information to complete the Venn diagram below



(b) How many students did Soccer but not Football?

(c) What is the probability that a student selected at random from the group liked both sports, given they liked at least one of the sports?

(d) What percentage of students liked neither sport?

11. A wardrobe has 3 shelves – the top, middle and bottom – and contained a mixture of 100 shirts and jumpers. 45% of all the clothing items were shirts. There are 20 jumpers on the middle shelf, and a total of 30 articles of clothing occupy the top shelf. Half of the clothing items on the bottom shelf are shirts, the whole of this shelf having 30 fewer clothing items than the middle shelf.

(a) Use this information to complete the table on the following page.

	Shirts	Jumpers	Total
Top			
Middle			
Bottom			
Total			

(b) What fraction, in simplest form, of all the clothing items were on the bottom shelf?

(c) What percentage of those clothing items on the top shelf were shirts?

12. The universal set, U, and the two sets A and B contained within it are such that

$|A \cap B| = 35$ $|A| = 77$ $|B| = 51$ $|U| = 100$

Determine

(a) $|\overline{A}|$

(b) $|\overline{B}|$

(c) $|\overline{A \cup B}|$

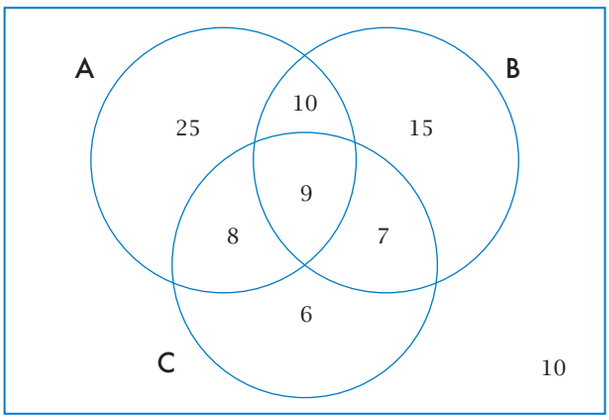
(d) $|\overline{A \cap B}|$

13. 90 Year 10 pupils at a certain school were asked whether they liked hot chips or hash browns for recess. 18 said they liked both, 40 said they liked only hash browns and 4 said they liked neither.

(a) Put this information into a two-way table.

(b) Use the table to find out how many pupils liked hot chips for recess.

14. The numbers in the Venn Diagram below indicate the number of insects in each of the sets A, B and C. If one insect is chosen at random from the universal group, U, determine:



n(U) =

(i) $P(A)$

(ii) $P(C)$

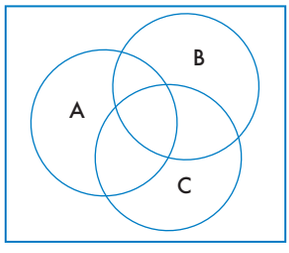
(iii) $P(A \cup B)$

(iv) $P(A \cap B \cap C)$

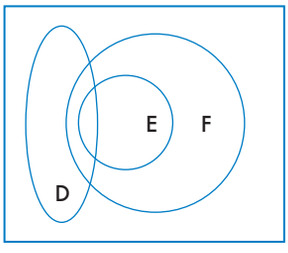
(v) $P(A \mid (B \cap C))$

(vi) $P(A \mid (\overline{B \cap C}))$

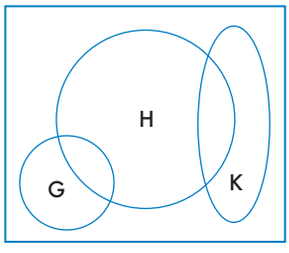
15. Shade the regions, as defined, on the Venn diagrams below:



(a) (A ∩ C) ∪ B



(b) (D ∩ F) ∩ E'



(c) (H ∩ K') ∪ G

16. (a) Write down the set F of Fibonacci numbers less than 35.

(b) Write down the set P of prime numbers less than 35.

(c) Hence, verify that n(P ∪ F) = n(P) + n(F) – n(P ∩ F)

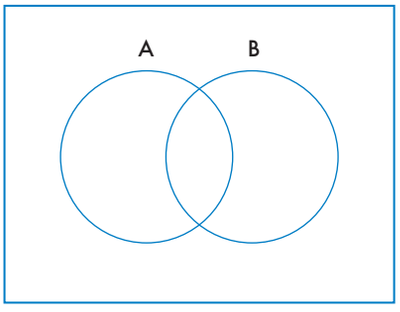
(d) What is the probability that if:

(i) an element is chosen from P, it will also be in F?

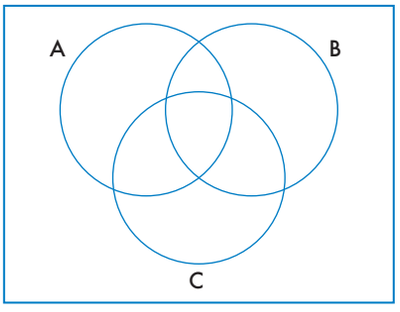
(ii) an element is chosen from F, it will also be in P?

(iii) a whole number is chosen between 2 and 17 inclusive, it will be a prime Fibonacci number?

17. Shade the following Venn diagrams according to the area indicated.

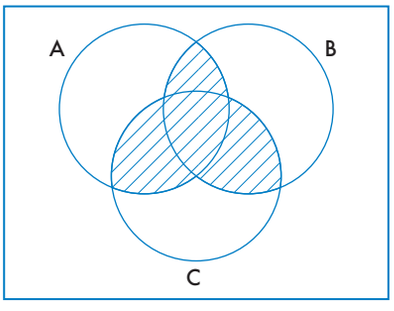


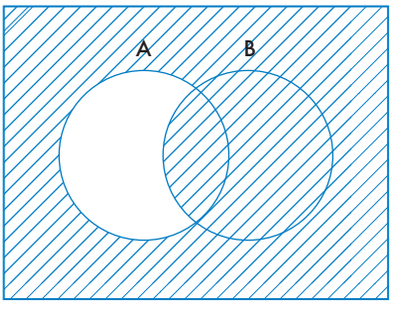
$A' \cap B'$



$A' \cap (B \cup C)$

18. Label the shaded areas of the following Venn diagrams.





19. If $P(A \cap B') = x$ $P(A' \cap B) = y$

$P(A \cap B) = z$ $P(A|B) = 0.25$

$P(B|A) = 0.2$ $P(A \cup B') = 0.6$

Determine the values of x, y and z.

20. Given $P(A) = 0.5$ $P(A \cap B \cap C) = 0.1$

$P(B) = 0.6$ $P(A \cap B' \cap C') = 0.1$

$P(C) = 0.6$ $P(A' \cap B \cap C') = 0.2$

$P(A' \cap B' \cap C) = 0.1$

$P(A \cap B \cap C') = x$

$P(A \cap B' \cap C) = y$

$P(A' \cap B \cap C) = z$

$P(A \cup B \cup C) = 1$

Find:

(a) x, y and z.

(b) $P((B' \cap A) \cup C)$

(c) $P(A|B)$

(d) Are events A and B independent? Support your answer with reasoning.

21. The probability of Alex achieving an A grade in Maths, Physics and Chemistry are 0.2, 0.3 and 0.9 respectively. If the grades are independent events determine the probability that Alex receives:

(a) all A grades

(b) no A grades

(c) exactly two A grades.

22. Mr and Mrs Brown decide to go shopping. The probability that only Mr Brown goes shopping is 0.3 and is identical to the probability of only Mrs Brown going shopping. The probability they both go shopping is 0.09.

(a) Are the events 'Mr Brown going shopping' and 'Mrs Brown going shopping' independent? Give reasons for your answer.

(b) What is the probability that neither Mr or Mrs Brown go shopping?

23. If the probabilities that A and B will die within a year are x and y respectively, determine the probability in terms of x and/or y that only one of them will be alive at the end of the year (assume that these events are independent).

24. One hundred people were surveyed regarding their relaxation activities. Forty eight watched TV, fifty eight exercised and sixty two read a good book. Twenty eight people read a book and watched TV, forty two read a book and exercised, while eight watched TV and exercised only.

The probability that a person does not get involved in any of the three activities is 0.1. Thirty three people surveyed participated in exactly one activity.

(a) How many people participated in all three activities?

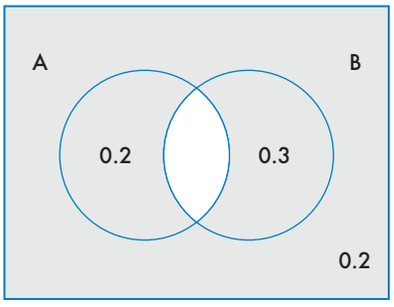
If a person is selected at random determine the probability that he/she participates in:

(b) exactly two activities

(c) all three activities, given they participate in at least one

(d) the exercise or reading a book activity

25. Given the probabilities in the Venn Diagram below, determine whether or not the events A and B are independent.



26. Given that events A and B are independent, show that $P(A|B) = P(A)$ if $P(A|B) = 0.52$ and $P(B) = 0.43$.

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

27. Show that the following events are independent:

Event A: A heart is drawn from a pack of playing cards

Event B: A queen is drawn from a pack of playing cards

28. A and B are two mutually exclusive events with $P(A) = \frac{2}{5}$ and $P(A \cup B) = \frac{7}{10}$. Find

(a) $P(B)$

(b) $P(B')$

(c) $P(A' \cap B')$.

7 EXPONENTS AND EXPONENTIAL FUNCTIONS

Syllabus Checklist

By the end of this chapter, you should be able to:

☐ Indices and the index laws
- review indices (including fractional and negative indices) and the index laws
- use radicals and convert to and from fractional indices
- understand and use scientific notation and significant figures

☐ Exponential functions
- establish and use the algebraic properties of exponential functions
- recognise the qualitative features of the graph of $y = a^x$ $(a > 0)$, including asymptotes, and of its translations ($y = a^x + b$ and $y = a^{x-c}$)
- identify contexts suitable for modelling by exponential functions and use them to solve practical problems
- solve equations involving exponential functions using technology, and algebraically in simple cases

Rounding to a Certain Number of Significant Figures

The accuracy of a number is determined by the number of significant figures it has been rounded to. For instance 13 165 mL has 5 significant figures, and can be rounded to:

13 170 mL (4 significant figures)
13 200 mL (3 significant figures)
13 000 mL (2 significant figures)
or 10 000 mL (1 significant figure)

You will have noticed that that the measurement is more precise with a greater number of significant figures (s.f.).

There are five general rules to follow when rounding, and some extra rules when rounding after addition/subtraction and multiplication/division. These rules and an illustrative example for each are tabulated below.

Case	Explanation	Example
1	All non-zero digits are significant	249.7 has 4 s.f.
2	Zeroes between non-digits are significant	7.0001 has 5 s.f.
3	Final zeroes in decimals are significant	38.40 has 4 s.f.
4	Zeroes to the left of the first non-zero digit in decimals are not significant	0.10785 has 5 s.f.
5	Final zeroes in non-decimals may be significant	91 500 may have 3, or 5 s.f. on certain conditions
Addition/Subtraction	The answer must contain no more significant figures than the least precise measurement	16.7 (3 s.f.) + 2.719 (4 s.f.) = 19.4 (3 s.f.)
Multiplication/Division	The answer should have as many significant figures as the least precise measurement	21.9 (3 s.f.) 527.04 (5 s.f.) = 11 500 (3 s.f.)

Scientific Notation and Standard Form

Scientific Notation is used to write very large or very small numbers more concisely.

To convert a number to scientific notation:

(1) Place a decimal point between the first and second digits
(2) Multiply this number by the correct power of 10

For instance (i) 897 400 becomes 8.974×10^5
(ii) 0.00326 becomes 3.26×10^{-3}

Indices

The index laws can prove to be instrumental when solving equations or simplifying expressions. These laws are as follows:

1. $a^x . a^y = a^{x+y}$
2. $a^x \div a^y = a^{x-y}$
3. $(a^x)^y = a^{xy}$
4. $(ab)^x = a^x b^x$
5. $(a/b)^x = \dfrac{a^x}{b^x}$
6. $a^0 = 1$
7. $a^{-x} = \dfrac{1}{a^x}$
8. $a^{\frac{x}{y}} = \sqrt[y]{a^x}$

where a is known as the base and x is known as the index, power or exponent.

Law 7 is important as it is the convention to express answers with positive indices.

Law 8 is important as it is the convention to present answers without radical signs.

Exponential Functions

In its general form, the exponential function is written $y = a^x$, where $a > 0$. This equation has a horizontal asymptote at $y = 0$, and a y intercept at (0, 1).

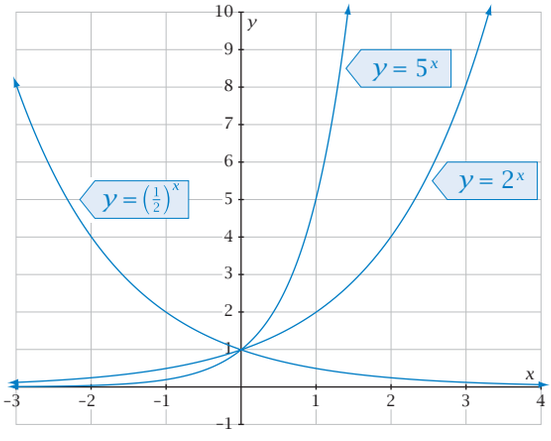
- If $0 < a < 1$ then the graph becomes steeper and as y increases, x decreases.
- If $a > 1$ then the graph becomes steeper and as y increases, x increases.

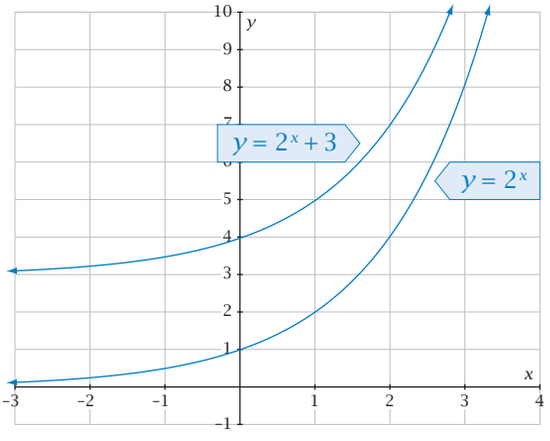
A few examples in the Cartesian plane below illustrate these characteristics.

Consider adding a constant *'b'* to the function, where $y = a^x + b$.

- If $b > 0$, then the function will be translated vertically upwards *'b'* units.
- If $b < 0$, then the function will be translated vertically downwards *'b'* units.

The horizontal asymptote will also have shifted vertically and now has the equation $y = c$.





You may also consider how $y = ax$ is transformed to $y = ax + c$

- If $c > 0$ then $y = ax$ is translated horizontally left c units
- If $c < 0$ then $y = ax$ is translated horizontally right c units

Worked Examples

7.1 Simplify the following expressions:

(a) $= \dfrac{(2xy)^3(4x)^{-3}}{(-3y)^{-2}}$

$= \dfrac{2^3x^3y^3.\,4^{-3}x^{-3}}{(-3)^{-2}y^{-2}}$ *Remove brackets and apply index law* $(a^x)^y = a^{xy}$

$= \dfrac{8x^3y^3(-3^2)(y^2)}{4^3x^3}$ *Apply index law* $a^{-x} = \frac{1}{a^x}$

$= \dfrac{72x^3y^5}{64x^3}$ *Multiply numbers together, apply index law* $a^x \,.\, a^y = a^{x+y}$ *and simplify like terms.*

$= \dfrac{9y^5}{8}$

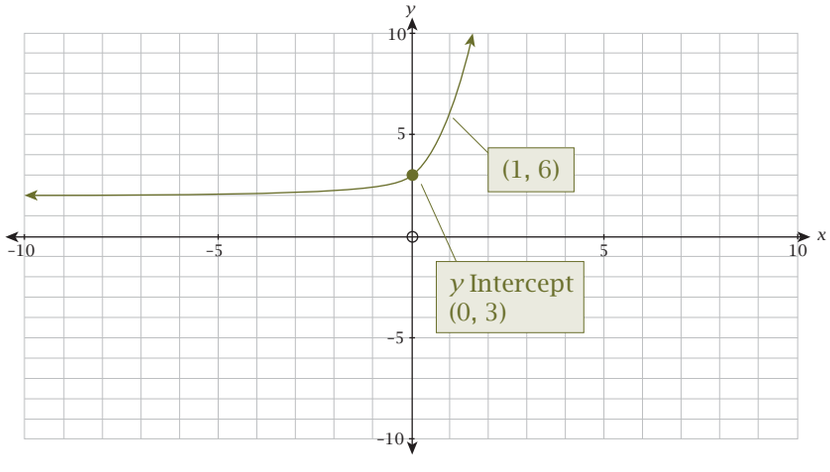
(b) $= \dfrac{y^5 + 2y^8}{y^3}$

$= \dfrac{y^3(y^2 + 2y^5)}{y^3}$ *Factorise the expression for like terms and simplify.*

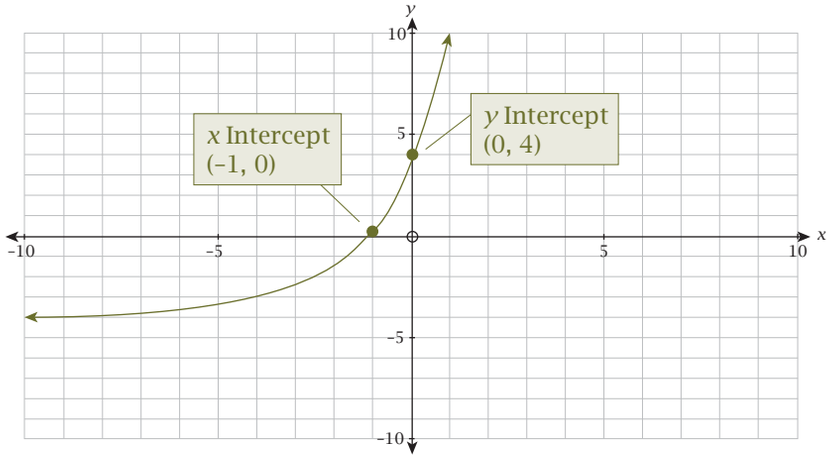
$= y^2 + 2y^5$

7.2 Determine the equation of each graphed exponential function

(a)



(b)



(a) Considering the horizontal asymptote at $y = 2$ and the shape of the curve, an exponential function is required. As the function has been translated vertically 2 units we can begin with $y = a^{x+c} + 2$, and then we can substitute two sets of coordinates to solve for a and c.

Sub. (1, 6) $6 = a^{1+c} + 2$
$6 - 2 = a^{1+c}$
$4 = a^{1+c}$
$4 = a^1 . a^c$ (Equation 1)

Sub. (0, 3) $3 = a^{0+c} + 2$
$3 - 2 = a^c$
$1 = a^c$ (Equation 2)

Sub. Equation 2 into Equation 1, i.e. $4 = a^1 . (1)$

$$4 = a$$

Sub $a = 4$ into $y = a^{x+c} + 2$ to give $y = 4^{x+c} + 2$

Also, looking at Equation 2 c must equal 0 following index laws i.e. $4^0 = 1 \therefore c = 0$

$$\therefore y = 4^x + 2$$

(b) Considering the horizontal asymptote at $y = -4$ and the shape of the curve, an exponential function is required.

Beginning with $y = a^{x+c} - 4$ we can substitute a coordinate to solve for a.

Sub. (–1, 0) $0 = a^{-1+c} - 4$
$4 = a^{c-1}$
$4 = a^c . a^{-1}$
$4 = \frac{4^c}{a}$ or $4a = a^c$ (Equation 1)

Now substitute the other given point:

Sub. (0, 4) $4 = a^{0+c} - 4$
$8 = a^c$
$8 = a^c$
$8 = a^c$ (Equation 2)

Because both of the equations are equal to a^c we can equate them for a solution

Let Equation 1 = Equation 2

$$4a = 8$$
$$a = 2$$

Sub. $a = 2$ into Equation 2 $\rightarrow 8 = 2^c \therefore c = 3$

$$\therefore y = 2^{x+3} - 4$$

7.3 Solve for x

(a) $3^{5x+3} = 9^x$

$3^{5x+3} = (3^2)^x$	(Change both bases to the lowest bases, i.e. 3)
$3^{5x+3} = 3^{2x}$	Apply Index Law 3 and remove brackets
$5x + 3 = 2x$	(As bases on both sides of the equation are alike, remove and drop the exponents)
$x = -1$	Solve

(b) $x^{\frac{3}{2}} = 64$

$x^{\frac{3}{2} \times \frac{2}{3}} = 64^{\frac{2}{3}}$	(Multiply each exponent by $\frac{2}{3}$)
$x = (2^6)^{\frac{2}{3}}$	(Simplify LHS exponents, and 'break down' 64 to lowest base)
$x = 2^{\frac{12}{3}} = 2^4 = 16$	(Expand brackets using Law 3 and evaluate)

(c) $3^{2x} - 12(3^x) + 27 = 0$

Let $3^x = y$

$y^2 - 12y + 27 = 0$	(Using the substitution, change the equation into a quadratic)
$(y - 3)(y - 9) = 0$	(Factorise the quadratic)
$y = 3, y = 9$	(Solve)
$3^x = 3, 3^x = 9$	(Resubstitute 3^x for y)
$\therefore x = 1, x = 2$	Solve for x

7.4 Worked Example 6.1(c) to be used with other WEs already in WACE Revision Guide chapter

$$\frac{5^{n-1} + 5^{n+1}}{5^n}$$

$= \dfrac{5^n.5^{-1} + 5^n.5^1}{5^n}$ Use Index Law 1 to simplify terms

$= \dfrac{5^n(5^{-1} + 5^1)}{5^n}$ Factorise

$= 5^{-1} + 5^1 = \dfrac{1}{5} + 5 = \dfrac{26}{5}$

PROBLEMS TO SOLVE

CHAPTER 7: EXPONENTS AND EXPONENTIAL FUNCTIONS

1. Convert the following numbers to scientific notation:

   (a) 294 500

   (b) 0.000496

   (c) 137

   (d) 0.000001724

   (e) 96

   (f) 0.194

   (g) 42.786

   (h) 0.0000000872

2. Convert the following from scientific notation to numbers:

   (a) 1.478×10^{-3}

   (b) 2.4976×10^{8}

   (c) 9.52×10^{-7}

   (d) 8.041×10^{5}

   (e) 7.18907×10^{7}

   (f) 5.4142×10^{6}

   (g) 4.591×10^{-1}

   (h) 3.207×10^{-4}

3. Perform the following calculations and round to an appropriate number of significant figures:

(a) $3.307 + 22.6$

(b) $14.91 - 3.68$

(c) $5.2587 + 11.96$

(d) $185.715 + 25.45$

(e) 12.1×3.57

(f) $27.4 \div 4.51$

(g) 9.8×2.373

(h) $100.49 \div 20.81$

4. Simplify the following expressing each with positive indices:

(a) $b^3 \times b^{-2} \times b^0$

(b) $(2x^3)^2$

(c) $\left(\frac{3m^2}{2n}\right)^3$

(d) $(2a^2b^{-2})^{-2}$

(e) $\frac{x^{\frac{3}{2}}}{x^{\frac{5}{4}}}$

(f) $\sqrt{x^2y^3}$

(g) $3^0 \times 3^{-4}$

(h) $\dfrac{x}{x^{-2}}$

(i) $(y^{-1})^{-1}$

(j) $3x^2 \times 2x^{-3}$

(k) $\dfrac{(ab^2)^3}{(a^{-2}b)^2}$

(l) $\dfrac{b^{-2}}{\sqrt[3]{b^6}}$

5. Evaluate the following.

(a) $8^{\frac{5}{3}}$

(b) 3^{-2}

(c) $\sqrt[4]{16^{-2}}$

(d) $0.01^{-\frac{3}{2}}$

(e) $\left(2\frac{1}{4}\right)^{0.5}$

6. Solve the following equations.

(a) $2^x = 16$

(b) $3^x = \dfrac{1}{27}$

(c) $4^x = \sqrt{8}$

(d) $27^x = 81$

(e) $2^{2x+1} = 32$

(f) $8^x = 0.25$

7. Solve the following equations.

(a) $x^{-2} = 36$

(b) $x^{\frac{4}{3}} = 16$

(c) $t^{\frac{1}{2}} = 4$

(d) $3m^{-1} = 5$

(e) $2x \times 3x^2 = 48$

(f) $(x + 1)^{\frac{2}{3}} = 64$

8. Simplify the following expressing each with positive indices.

(a) $\dfrac{b^{-3}c^2}{b^{-1}c^{-2}}$

(b) $\dfrac{(a^{\frac{-5}{2}})^2 b^2}{a^3 (b^{\frac{1}{4}})^6}$

(c) $cd^2 \times (c^2 d^{-1})^2$

(d) $\dfrac{mn^0 \times m^{-2}n}{m^4 n^{-2}}$

(e) $\dfrac{\sqrt[3]{a^2} \times \sqrt{a}}{\sqrt[3]{a^{-1}}}$

(f) $\dfrac{15b \times 6b^4}{9b^2}$

9. Evaluate

(a) $4^{0.6} \times 8^{0.3} \div 16^{0.4}$

(b) $\sqrt{6} \div \left(\dfrac{1}{8}\right)^{-\frac{1}{6}} \times 3^{0.2}$

10. Simplify

(a) $3^{n+2} \times 9^{2-n}$

(b) $(2x^n)^2 \times (2x^2)^{n+1}$

11. Factorise

(a) $\dfrac{b^3 + b^5}{b^2}$

(b) $\dfrac{3^3 - 3^{n+3}}{3^2}$

(c) $\dfrac{2^{n+2} + 8}{4}$

(d) $\dfrac{5^{n-1} + 5^{n+1}}{5^n}$

(e) $\dfrac{3 \times 2^n + 21}{35 + 5 \times 2^n}$

(f) $\dfrac{4^{n+1} - 2^{n+1}}{2^{n+2} - 2}$

12. Find x if $2^x \times 5^x = 0.001$

13. Starting with the 'base' exponential equation $y = 2^x$, list the transformations that have taken place to reach:

(a) $y = 2^x - 4$

(b) $y = -2^x + 1$

(c) $y = 2^{x-1} - 5$

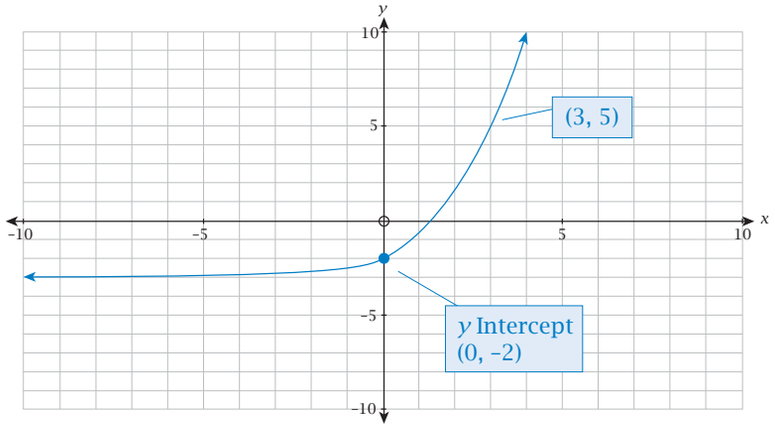
(d) $y = 2^{-x} + 3$

(e) $y = -2^{x+2}$

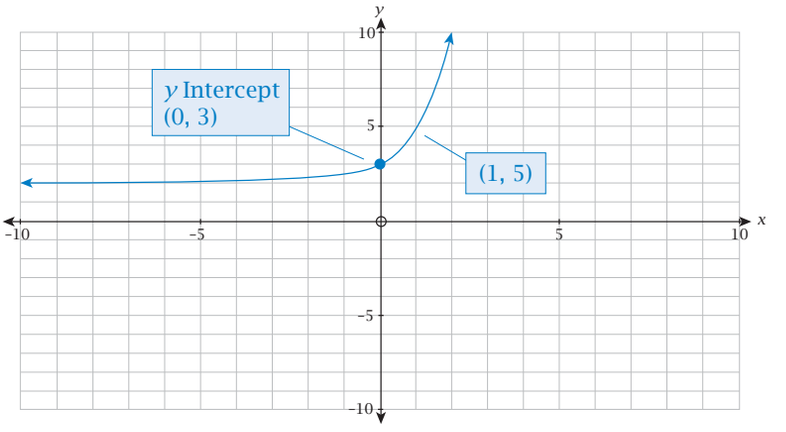
(f) $y = -2^{x+5} + 4$

14. Determine the equation of each graphed function below

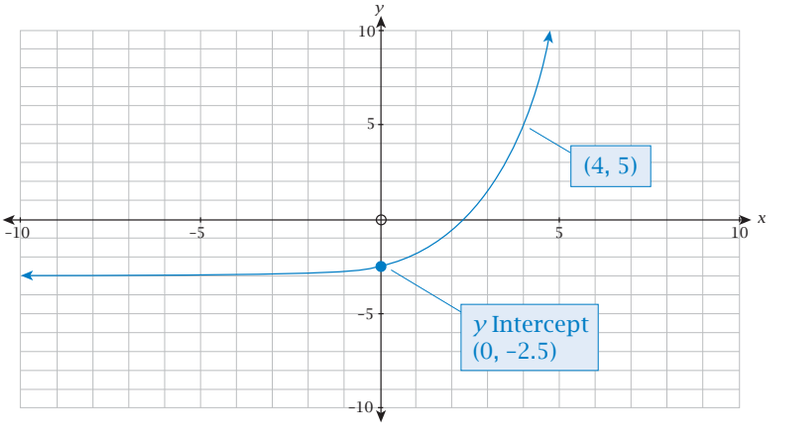
(a)



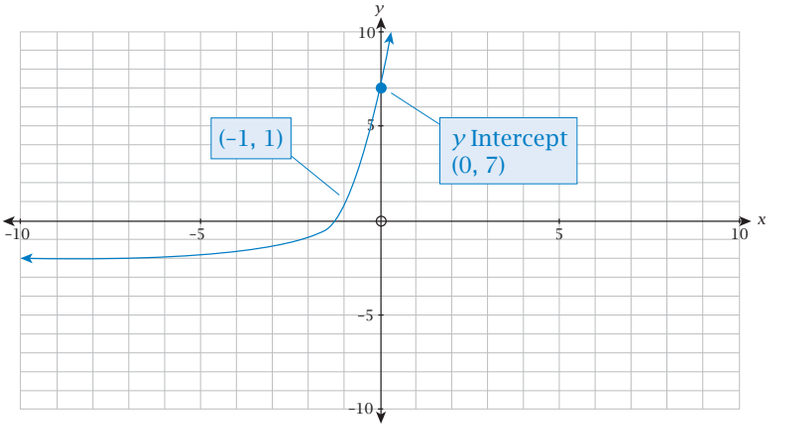
(b)


y
10
5
−5
−10
−10
−5
5
10
x
y Intercept
(0, 3)
(1, 5)

(c)


y
10
5
−5
−10
−10
−5
5
10
x
(4, 5)
y Intercept
(0, −2.5)

(d)


y
10
5
−5
−10
−10
−5
5
10
x
(−1, 1)
y Intercept
(0, 7)

15. Until it dries completely a cut piece of hardwood will shrink in size. Drying takes between 2 and 3 years. During this time the thickness, T, of a 100 mm cut-sized length of timber will follow the equation $T = 100(0.97)^n$, where n is the time in years.

(a) Find the thickness of a 100 mm cut-sized length of timber after 1 year.

(b) When will the thickness of this length of timber be 92 mm? (give your answer to the nearest month)

16. The projected population, P, of a city is given by

$$P = 150\,000(1.06)^t \qquad t \text{ years after 2010}$$

(a) Find the projected population for 2020.

(b) During which year will the population of this city first exceed 500 000?

17. Find the value(s) of x for which

(a) $3^x + 3^{x+1} = 36$ [Let $3^x = y$]

(b) $4^x - 4^{x-1} = 24$ [Let $4^x = y$]

(c) $3^{2x} - 2(3^x) - 3 = 0$ [Let $3^x = y$]

(d) $16^x + 4(4^x) = 32$ [Let $4^x = m$]

(e) $9^x - 4(3^x) + 3 = 0$ [Let $3^x = p$]

(f) $2(3^{2x}) - 8(3^x) + 6 = 0$ [Let $3^x = y$]

18. A particular mammal's population can be modelled at any time (t) in years using the formula:

$$P = 20(2^{3t})$$

(a) Is the mammal's population increasing or decreasing?

(b) What was the initial population of this mammal?

(c) What is the percentage increase/decrease in the mammal's population each year?

19. Simplify the following expressions:

(a) $\left(-\frac{2a^2}{b}\right)^{-1} \times (-2a^3b^{-1})^3$

(b) $\left(\frac{3m^2}{-n}\right)^5 \left(\frac{-n^3}{5m^4}\right)^2$

(c) $\sqrt{(4f^{-2}g^4)}$

(d) $(-2a^3b^{-1})^3 \times \left(\frac{3a^2}{b}\right)^2$

(e) $\sqrt[3]{64d^{12}e^{-24}}$

(f) $\dfrac{\left(5p^{\frac{3}{4}}q^5\right)^2}{10\sqrt{pq^2}}$

20. Consider the equations $f(x) = 3^x$ and $g(x) = 2x + 3$.
Use CAS to find both solutions for $3^x = 2x + 3$ correct to 2 decimal places.

21. Simplify the following expressions, leaving your answers with positive indices.

(a) $a^5 \times a^0 \times a^{-7}$

(b) $\dfrac{4x^{-3}y^5z}{12xy^7z^{-1}}$

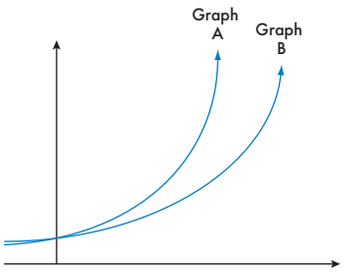
(c) $\dfrac{a^4 + a^6}{a^4}$

(d) $\dfrac{(2xy^2)^3}{xy^8}$

(e) $\dfrac{\sqrt{9x^{-2}}}{3x^2}$

(f) $\left(\dfrac{4a^2b}{5a^3b^2}\right)^{-2}$

22. Below on the Cartesian plane are two functions: $y = 3^x$ and $y = 4^x$. They are not labelled according to their rules, but simply as Graph A and Graph B. Identify each graph and support your assertion with justification.



SEQUENCES AND RECURSION

8

Syllabus Checklist

By the end of this chapter, you should be able to:

☐ Arithmetic sequences
- recognise and use the recursive definition of an arithmetic sequence: $t_{n+1} = t_n + d$
- develop and use the formula $t_n = t_1 + (n-1)d$ for the general term of an arithmetic sequence and recognise its linear nature
- use arithmetic sequences in contexts involving discrete linear growth or decay, such as simple interest
- establish and use the formula for the sum of the first n terms of an arithmetic sequence

☐ Geometric sequences
- recognise and use the recursive definition of a geometric sequence: $t_{n+1} = t_n r$
- develop and use the formula $t_n = t_1 r^{n-1}$ for the general term of a geometric sequence and recognise its exponential nature
- understand the limiting behaviour as $n \to \infty$ of the terms t_n in a geometric sequence and its dependence on the value of the common ratio r
- establish and use the formula $S_n = t_1 \frac{r^{n-1}}{r-1}$ for the sum of the first n terms of a geometric sequence
- use geometric sequences in contexts involving geometric growth or decay, such as compound interest

FORMULAE AND DEFINITIONS

Sequences

An arithmetic sequence is a progression of numbers sharing a common difference. To obtain the next term in an arithmetic sequence, simply add the common difference to the previous term.

Algebraically, an arithmetic sequence is generated with the following terms:

$$a,\ a+d,\ a+2d,\ a+3d,\ \ldots$$

It is therefore defined by the rule $t_n = a+(n-1)d$, where t_n is the nth term, a is the first term, n is the term number and d is the common difference.

This can be written in recursive form, where $t_{n+1} = t_n + d$, $t_1 = a$.

Arithmetic series

For an arithmetic sequence with first term a and common difference d, we can obtain the sum of n terms by using the formula:

$$S_n = \frac{n}{2}[2a+(n-1)d]$$

There is a special variation to this formula if we know the last term. Substituting ℓ for $a+(n-1)d$ the rule is then written as:

$$S_n = \frac{n}{2}[a+\ell].$$

Geometric sequences

A geometric sequence is a progression of numbers sharing a common ratio. To obtain the next term in a geometric sequence, simply multiply the previous term by the common ratio.

Algebraically, a geometric sequence is generated with the following terms:

$$a,\ ar,\ ar^2,\ ar^3,\ \ldots$$

It is therefore defined by the rule $t_n = ar^{n-1}$, where t_n is the n th term, a is the first term, n is the term number and r is the common ratio.

This can also be written in recursive form, where $t_n = t_1 r^{n-1}$.

To determine the common ratio, r, of a geometric sequence, simply divide $\frac{t_{n+1}}{t_n}$ (i.e. divide any term by the previous term)

Geometric series

For a geometric sequence with first term a and common ratio r, we can obtain the sum of n terms by using the formula:

$$S_n = t_1\frac{(1-r^n)}{(1-r)}, \quad r \neq 1$$

However, if r is not between –1 and 1 it is easier to use the formula in the form:

$$S_n = t_1\frac{(r^n-1)}{(r-1)}, \quad r \neq 1$$

Sum to Infinity

If $-1 < r < 1$ (that is, r lies between –1 and 1, $r \neq 0$ then the geometric sequence is said to be convergent.

In each case S_∞ approaches a limiting value.

As the terms of such a sequence gets smaller and smaller and eventually becomes negligible, we are able to calculate a value which represents the sum of all the terms of the sequence. We refer to this sum (the sum of an infinite number of terms) as the sum to infinity (∞) and it is calculated using the formula:

$$S_\infty = \frac{a}{1-r}$$

Growth and decay

Geometric growth and decay problems are modelled after exponential equations of the form $y = ka^x$.

Growth is characterised by an equation of the form $A = A_0 r^t$ for $r > 1$ where

A_0 = initial amount
r = 1 + growth rate, where growth rate is a decimal
t = time

Decay is characterised by an equation of the form $A = A_0 r^t$ for $r < 1$ where

A_0 = initial amount
r = 1 – decay rate
t = time

Finance

Simple interest is the amount of money earned over a period of time. The interest rate used is not dependent upon the length of time of investment.

$$\text{Simple Interest} = \frac{P \times R \times T}{100} \quad \text{where}$$

P = the principal amount invested originally
R = the rate of interest (as a whole number)
T = the amount of time the principal is invested for (in years)

Alternatively, simple interest can be calculated using

$$\text{Simple Interest} = P \times R \times T \quad \text{(N.B. } R \text{ must be expressed as a decimal)}$$

Compound interest is also the amount of money earned over a period of time. However, the interest rate is dependent upon the length of time of investment. Essentially, compounding occurs when interest earned is left in the account and further interest is paid on this interest. Compound interest can be calculated using the equations:

$A = P(1+r)^t$	where the interest is compounded annually
$A = P(1+\frac{r}{n})^{nt}$	where the interest is compounded n times per year
$I = P(1+\frac{r}{n})^{nt} - P$	where only the interest earned is required.

For both of these equations,

A = Amount in the account
P = Principal amount
r = rate of interest, expressed as a decimal (if a recurring decimal, do not round but use the fraction equivalent)
n = number of compounding periods per year
t = amount of time the principal amount is invested for in years.

Reducible Interest

When an amount of money is borrowed from an institution, the institution will loan this money for a period of time at a particular rate of interest. Over the time period, the money borrowed will be repaid in equal amounts at regular intervals. At each interval, the amount of interest to be repaid is calculated on how much is to be repaid. Thus, as the principal amount to be repaid decreases, so too does the amount of interest to be repaid. This model of repayment is called reducible interest.

Worked Examples

8.1 The first term of an arithmetic series is 7 and the fifth term is 17. Find

(a) The common difference

(b) The sum of the first 10 terms

(a) If $a = 7$, then $t_5 = a + (n-1)d = a + 4d = 17$. By substituting a, we can obtain $d = 2.5$

(b) Using the formula $S_n = \frac{n}{2}[2a + (n-1)d]$

we obtain $S_{10} = \frac{10}{2}[2(7) + (10-1)(2.5)]$

$= 182.5$

8.2 Determine the common ratio and the first term of a geometric progression whose second term is 16 and whose fourth term is 4. Also, state the first five terms of this progression.

If the second term, t_2, equals 16, that means algebraically $ar = 16$. Also, if the fourth term, t_4, equals 4, then $ar^3 = 4$.

Dividing these terms gives $\frac{t_4}{t_2} = \frac{ar^3}{ar} = \frac{4}{16}$

Giving $r^2 = \frac{1}{4}$

and hence $r = \pm\frac{1}{2}$

Substituting $r = \pm\frac{1}{2}$ into $ar = 16$, would yield the first term, a, as being ± 32.

Therefore, the two possible sets of first five terms could be:

$$-32, 16, -8, 4, -2 \text{ for } r = -\frac{1}{2} \quad \text{or} \quad 32, 16, 8, 4, 2 \text{ for } r = \frac{1}{2}.$$

8.3 The population of a certain country in 2002 was 20 million, and is increasing at 5% per year. Assuming the rate of population remains constant, determine

(a) the population after t years

(b) the population in 2006

(c) when the population will be double that of 2002

(a) If $t_n = ar^n$ then $P_t = 20\,000\,000\,(1.05)^t$.

(b) In 2006, $t = 4$. Therefore, $P_4 = 20\,000\,000\,(1.05)^4$

$= 24\,310\,125$ people.

(c) Let $P_t = 40\,000\,000$ and we need to find the time, t, when this occurs.

So, $40\,000\,000 = 20\,000\,000\,(1.05)^t$

$2 = 1.05^t$

By taking logs of both sides or using another solving method we obtain $t = 14.21$. Therefore, the population will reach 40 000 000 during 2016.

8.4 If the sum of the terms of the sequence 1, $2x$, $4x^2$ … is 7/8, find the value of x.

This question is saying that $S_\infty = \frac{7}{8}$

Thus, $\frac{a}{1-r} = \frac{7}{8}$, where $a = 1$ and $r = 2x$

Substituting gives $\frac{a}{1-2x} = \frac{7}{8}$

$1 = \frac{7}{8} - \frac{7x}{4}$

$\frac{7x}{4} = \frac{-1}{8}$

$x = \frac{-1}{14}$

8.5 Express $0.2\dot{5}$ as a fraction.

$0.2\dot{5} = 0.2 + \{0.05 + 0.005 + 0.0005 + \ldots\}$

The sum $0.05 + 0.005 + 0.0005 + \ldots$ forms an infinite geometric where $a = \frac{5}{100}$, $r = \frac{1}{10}$

$$0.2\dot{5} = \frac{2}{5} + \frac{\frac{5}{100}}{1 - \frac{1}{10}}$$

$$= \frac{2}{5} + \frac{\frac{5}{100}}{\frac{9}{10}}$$

$$= \frac{2}{5} + \frac{5}{90}$$

$$= \frac{23}{90}$$

PROBLEMS TO SOLVE

CHAPTER 8: SEQUENCES AND RECURSION

1. For the following sequences determine whether the progression is arithmetic, geometric or neither.

   (a) 2, 4, 6, 8, 10, 12, …

   (b) 19, 21, 23, 24, 26, …

   (c) 2, 6, 18, 54, 162, …

   (d) –2, 4, –8, 16, –32, …

   (e) 12.5, 14, 15.5, 16, 17.5, …

2. An arithmetic progression has a first term of 5 and a common difference of 4. Give the first five terms, and the 20th term of this sequence.

3. For the sequence 18, 42, 66, … find

   (a) the 6th term

   (b) the 16th term

   (c) the sum of the first 16 terms

4. A geometric progression has a first term of 3 and a common ratio of –2. Give the first five terms, and the 20th term of this sequence.

5. A geometric progression has a 6th term of 3072 and a 8th term of 49 152. Determine the first term, the common ratio and hence, the recursive formula for this progression.

6. Determine n if the sum of n terms of the sequence 21, 18, 15, … is 84

7. Determine the value of x if $30, x, 134$ are successive terms in a sequence which is

(a) Arithmetic

(b) Geometric

8. Find t_{10} of the geometric sequence $3.25, 6.5, 13, 26, \ldots$.

9. The first 3 terms of a geometric sequence are $2.25, 4.5$ and 9.
Find the first term which would exceed 1000.

10. The 3rd term of a geometric sequence is 0.9 and the 6th term is 7.2. Find the sum of the first 12 terms of the sequence.

11. If $-1.1, x, -4.4$ are consecutive terms of a geometric sequence, find x.

12. If $144, -108, x$ are consecutive terms of a geometric sequence, find x.

13. A real estate agent made \$112 000 profit after his first year of operation. The region around this company has an expected population growth for the next 12 years. The owner of the real estate business has predicted that his profit will increase by 6% each year for the next 12 years.

(a) How much profit will the real estate agent make after his seventh year in business?

(b) How much profit will the real estate agent have made in total after 13 years in business?

14. Jacaranda trees should increase in height by 9% each year under optimum conditions. If a batch of jacaranda trees measure 2.2 metres at the start of the first year,

(a) How high should they be at the end of the third year? (to the nearest cm)

(b) In which year should they exceed 5 metres in height?

15. The iron ore output from the Newman Mine for the first five years is given below:

2002	25 000 tonnes
2003	18 500 tonnes
2004	13 690 tonnes
2005	10 131 tonnes
2006	7 497 tonnes

(Output is given to the nearest tonne)

(a) Show that the sequence of output from the mine is geometric and state the common ratio.

(b) Find the expected sum of the iron ore output from the mine up and including the year 2010. Give your answer correct to the nearest tonne.

16. Determine the simple interest rate that would cause an investment to grow from $12 000 to $23 500 in 9 years.

17. Jeanette is reading a literary masterpiece of 700 pages. The first day she reads 100 pages, the second day, 80 pages and the third day she reads 64 pages. She continues with this sequence until she reads a minimum of 13 pages per day. After this she reads 13 pages per day until completion. Determine:

(a) how many pages she reads on the seventh day.

(b) how many pages she reads on the twentieth day.

(c) the total number of pages she has read by the end of the twentieth day.

(d) how many days she takes to read the complete masterpiece.

18. Peter organises a short-term loan of $12 000 to help him purchase a car. He plans to pay the loan off quickly to reduce the amount of interest he has to pay. The details of his loan are 12% interest p.a., with monthly repayments of $2 500.

(a) Complete the table below, showing details of how the loan progresses.

Month	Owing at start of month	Interest owing for month	Now Owing	Monthly Repayment	Owing at end of month
1	12 000			2500	
2					
3					
4					
5					
6					

(b) How many repayments did he have to make to repay the loan completely?

(c) What was his last repayment?

(d) How much money did Peter pay in interest?

19. A population of rats was released by accident on a deserted island, and initially their numbers increased exponentially at a rate of 50% per month. If the initial population was 80 rats,

(a) Give the exponential equation (defining rule) for this situation.

(b) Find the number of rats 1 year later.

(c) List 2 practical factors that would probably prevent the growth of this population continuing to increase exponentially.

(d) Graph the growth for the first 5 years.

20. The amount of deodorant spray left in a can (D_1) will be $D_1 = 0.63(0.85)^n$ units, after it has been used n times.

The amount of deodorant spray left in another can (D_2) will be $D_2 = 0.7(0.78)^n$ units, after it has been used n times.

(a) What is the amount of deodorant spray in each can before it has been used?

(b) Complete the table for D_1

n	0	1	2	3	10	25
D_1						

(c) Complete the table for D_2

n	0	1	2	3	10	25
D_2						

(d) By what percentage do the deodorants reduce each time they are used?

21. The following table shows \$1000 growing under Simple Interest conditions and Compound Interest conditions.

Date	Simple	Compound
Jan 1, 2004	1000	1000
Feb 1, 2004	1020	1010
March 1, 2004	1040	1020.10
April 1, 2004	1060	1030.30
May 1, 2004	1080	1040.60
June 1, 2004	*A*	*B*
June 1, 2014	3500	3468.74

(a) Calculate

i. *A*

ii. *B*

iii. the simple interest rate per month.

iv. the compound interest rate per month.

Exponential growth will always exceed linear growth if given enough time.

(b) When will the numbers in the "compound" column be larger than the numbers in the "simple" column for the first time?

22. Mr. and Mrs. Johnson have borrowed $180 000 to purchase an Italian sports car. They would like to repay their loan at $2500 per month. Below is a monthly summary for the first 18 months of the repayment of the loan.

Month	Balance at start of month	Interest	Repayment	Balance at end of month
1	180 000	900	2500	178 400
2	178 400	892	2500	176 792
3	176 792	883.96	2500	175 175.96
4	175 175.96	875.88	2500	173 551.84
5	173 551.84	867.76	2500	171 919.60
6	171 919.60	859.60	2500	170 279.20
7	170 279.20	851.39	2500	168 630.59
8	168 630.59	843.16	2500	166 973.75
9	166 973.75	834.86	2500	165 308.61
10	165 308.61	826.55	2500	163 635.16
11	163 635.16	818.17	2500	161 953.33
12	161 953.33	809.77	2500	160 263.10
13	160 263.10	801.32	2500	158 564.42
14	158 564.42	792.82	2500	156 857.24
15	156 857.24	784.28	2500	155 141.52
16	155 141.52	775.71	2500	153 417.23
17			2500	
18			2500	

(a) What is the monthly interest rate?

(b) What is the yearly interest rate?

(c) Complete the last 2 rows of the table (round your answers to the nearest cent).

(d) How much interest have the Johnsons paid over 18 months?

(e) Write a recursive formula to determine the amount owing at the start of each month.

23. Find the sum to infinity of the geometric series $72 - 48 + 32 - \ldots$

24. If the sum to infinity of a geometric series is 310 and the first term is 124, find the sum of the first 7 terms.

25. The sum of the first 3 terms of a geometric series is 37 while the sum of the infinite series is 64. Find the first 3 terms.

26. For the geometric series $40 + 8 + 1.6 + \ldots$, find the least value of n so that the difference between S_∞ and S_n is less than 0.1.

27. A rubber ball is dropped from an initial height of 50 m on to a flat horizontal surface. It rebounds from the surface to a vertical height of 3/5 of the height from which it just fell.

(a) How far will the ball have travelled by the time it hits the floor for the 7th time?

(b) How far will the ball travel before coming to rest?

28. A particular shirt loses a certain proportion of its coloured dye after each wash. With each wash the amount of dye removed is approximately 5% of the amount lost in the previous wash. During the first wash 90 mg of coloured dye is removed. Determine the quantity of coloured dye that will be removed on the

(a) 4th wash?

(b) nth wash?

(c) If the same washing process continues indefinitely and no extra coloured dye is added, what is the maximum amount of coloured dye that can be removed?

INTRODUCTION TO DIFFERENTIAL CALCULUS

9

Syllabus Checklist

By the end of this chapter, you should be able to:

☐ Rates of change

- interpret the difference quotient $\frac{f(x+h)-f(x)}{h}$ as the average rate of change of a function f
- use the Leibniz notation δx and δy for changes or increments in the variables x and y
- use the notation $\frac{\delta y}{\delta x}$ for the difference quotient $\frac{f(x+h)-f(x)}{h}$ where $y = f(x)$
- interpret the ratios $\frac{f(x+h)-f(x)}{h}$ and $\frac{\delta y}{\delta x}$ as the slope or gradient of a chord or secant of the graph of $y = f(x)$

☐ The concept of the derivative

- examine the behaviour of the difference quotient $\frac{f(x+h)-f(x)}{h}$ as $h \to 0$ as an informal introduction to the concept of a limit
- define the derivative $f'(x)$ as $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$
- use the Leibniz notation for the derivative: $\frac{dy}{dx} = \lim_{\delta x\to 0}\frac{\delta y}{\delta x}$ and the correspondence $\frac{dy}{dx} = f'(x)$ where $y = f(x)$
- interpret the derivative as the instantaneous rate of change
- interpret the derivative as the slope or gradient of a tangent line of the graph of $y = f(x)$

☐ Computation of derivatives

- estimate numerically the value of a derivative for simple power functions
- examine examples of variable rates of change of non-linear functions
- establish the formula $\frac{d}{dx}(x^n) = nx^{n-1}$ for non-negative integers n expanding $(x+h)^n$ or by factorising $(x+h)^n - x^n$

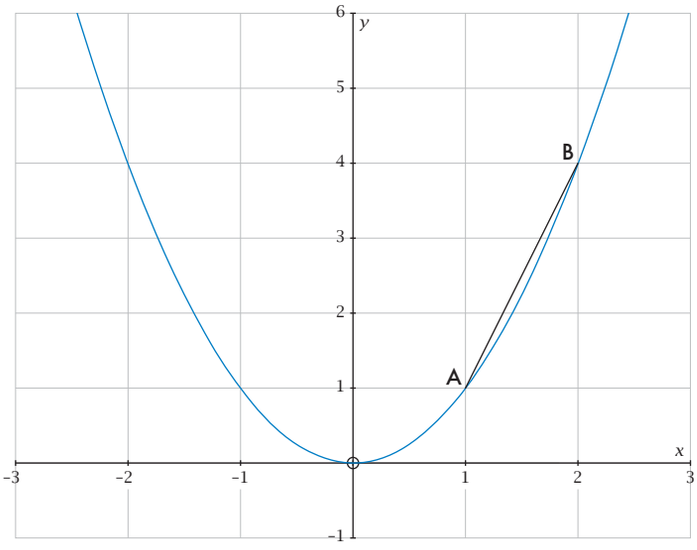
☐ Properties of derivatives

- understand the concept of the derivative as a function
- identify and use linearity properties of the derivative
- calculate derivatives of polynomials

FORMULAE AND DEFINITIONS

Theory

We can determine the rate of change (or gradient) of any function if we examine the graph of that function. Consider the function $y = x^2$ and the steps that follow:



i) Pick a point, A, on the function at (1, 1).

ii) Pick a second point, B, on the function at (2, 4)

iii) Determine the gradient between these two points by drawing a straight line and using:

$$m = \frac{\Delta y}{\Delta x}$$

The gradient of line segment AB is therefore 3.

iv) Pick a third point, C, on the function at (1.5, 2.25), and connect this point to A.

v) Repeat the process from step *iii).*

What should become evident is that as the 'moving' point approaches point A, the gradient will tend towards a certain value. In this case the gradient tends towards 2.

This process can also be proven algebraically using a limiting chord process:

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Using the function $f(x) = x^2$, we have $f(x+h) = (x+h)^2$.

Therefore

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}} \qquad \text{Factorise}$$

$$= \lim_{h \to 0} (2x+h)$$

And as $h \to 0$, the gradient function $\frac{dy}{dx} = 2x$.

At the point (1, 1), the gradient of $y = x^2$ is therefore $2(1) = 2$.

Some simple rules

Instead of differentiating functions by first principles each time, you can apply the following rules to expedite the process.

Original Function	Derivative Function
$f(x) = c$ ($'c'$ is a constant)	$f'(x) = 0$
$f(x) = mx$	$f'(x) = m$
$f(x) = x^n$	$f'(x) = nx^{n-1}$
$f(x) = ax^n$	$f'(x) = anx^{n-1}$

Other ways of writing a derivative function, $\frac{dy}{dx}$, include y', f', $f'(x)$, $\frac{df}{dx}$ and $\frac{d}{dx}f(x)$.

Other rules

The Second Derivative

The second derivative of any function can be found by differentiating that function twice.
In algebraic notation, the second derivative is expressed as $\frac{d^2y}{dx^2}$ or $\frac{d^2}{dx^2}(y)$ or $f''(x)$.

The Sum rule

If $y = f(x) \pm g(x)$ then $\frac{dy}{dx} = f'(x) \pm g'(x)$.

Worked Examples

9.1 Differentiate the following functions with respect to x:

(a) $y = 3x^2 + 2x - 1$

If $y = 3x^2 + 2x - 1$ *Use the general rule to differentiate all three terms.*

Then $\frac{dy}{dx} = 6x + 2$

(b) $f(x) = -4x^3(2x+3)$

$f'(x) = (2)(-4x^3) + (2x + 3)(-12x^2)$ *Use the product rule to differentiate*

$f'(x) = -8x^3 - 24x^3 - 36x^2$ *Expand the brackets*

$f'(x) = -32x^3 - 36x^2$ *Collect like terms and simplify*

(c) $y = \frac{2}{\sqrt[3]{x}}$

If $y = \frac{2}{\sqrt[3]{x}} = 2x^{-\frac{1}{3}}$ *Change the radical sign to a numeral value*

Then $\frac{dy}{dx} = (-\frac{1}{3})(2x^{-\frac{4}{3}})$ *Apply the general rule for differentiation*

$= -\frac{2}{3x^{\frac{4}{3}}}$ *Simplify and express with positive indices*

PROBLEMS TO SOLVE

CHAPTER 9: INTRODUCTION TO DIFFERENTIAL CALCULUS

1. Differentiate the following functions.

   (a) $y = 8x^2 + 5x - 7$

   (b) $f(x) = \frac{2}{x^2} + \frac{3}{x}$

   (c) $f(x) = (x^2 + 2x + 10)(-8x + x^2)$

   (d) $f(x) = \frac{1}{x} - \frac{5}{\sqrt{x}}$

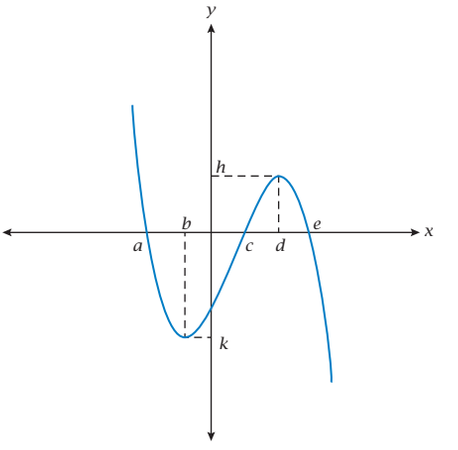
2. Find $\frac{d^2y}{dx^2}$ for the equations

   (a) $y = \frac{x^3}{2} + 4x^2 - 3x + 12$

   (b) $y = x^4 + x^3 + \frac{5}{x}$

3. Find the derivative of the function $f(x) = x^2 - 2$ using first principles.

4. Use the graph to answer the following.



   (a) Which points are roots of the polynomial?

(b) Which points have a gradient of zero?

(c) Between which points is the gradient positive?

(d) Where is the gradient negative?

5. Find the gradient using the limiting chord process on the curve $y = x^2 + 2$ at the point

(a) (1,3)

(b) (3,11)

6. Use the gradient function

$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$ to

(a) clearly show that the gradient function of $y = 3x^2$ is $6x$; and

(b) find the gradient function of $y = 5x^3$.

7. Find the gradient function of

(a) $y = 2x^4$

(b) $y = -5x^2$

(c) $y = 5x$

(d) $y = 4$

(e) $y = \frac{2}{x}$

(f) $y = 5\sqrt{x}$

(g) $y = \frac{4}{x^2}$

(h) $y = \frac{-2}{\sqrt{x}}$

(i) $y = \sqrt[3]{x}$

8. Differentiate with respect to x.

(a) $y = 3x + 2$

(b) $y = x^2 + 2x$

(c) $y = 4x^3 - x$

(d) $y = \frac{x^4}{2} + 3x^3 + 1$

(e) $y = x - \sqrt{x}$

(f) $y = x - \frac{2}{x}$

(g) $y = x^2 - 3 + \frac{2}{x^2}$

(h) $y = \frac{x^2}{4} + \frac{x}{2}$

9. Find

(a) $\frac{dm}{dt}$ if $m = 3t^2 - 2$

(b) $\frac{du}{dv}$ if $u = \frac{5}{v} + v$

(c) $\frac{dx}{du}$ if $x = \frac{3}{u^2}$

(d) $\frac{dp}{dn}$ if $p = 2\sqrt{n} + 1$

10. Find $\frac{d^2y}{dx^2}$

(a) $y = 3x^3 - 2x$

(b) $y = -\frac{2}{x} + 1$

(c) $y = x + \sqrt{x}$

(d) $y = \frac{1}{6}x^3 - 2x^2 + x - 2$

(e) $y = \frac{1}{3}x^{\frac{5}{3}} - 2$

11. Using first principles of differentiation, differentiate $f(x) = 2x^2 - 3x + 8$.

10 APPLICATIONS OF DIFFERENTIAL CALCULUS

Syllabus Checklist

By the end of this chapter, you should be able to:

☐ Applications of derivatives
- determine instantaneous rates of change
- determine the slope of a tangent and the equation of the tangent
- construct and interpret position-time graphs with velocity as the slope of the tangent
- recognise velocity as the first derivative of displacement with respect to time
- sketch curves associated with simple polynomials, determine stationary points, and local and global maxima and minima, and examine behaviour as $x \to \infty$ and $x \to -\infty$
- solve optimisation problems arising in a variety of contexts involving polynomials on finite interval domains

☐ Anti-derivatives
- calculate anti-derivatives of polynomial functions

FORMULAE AND DEFINITIONS

Applications

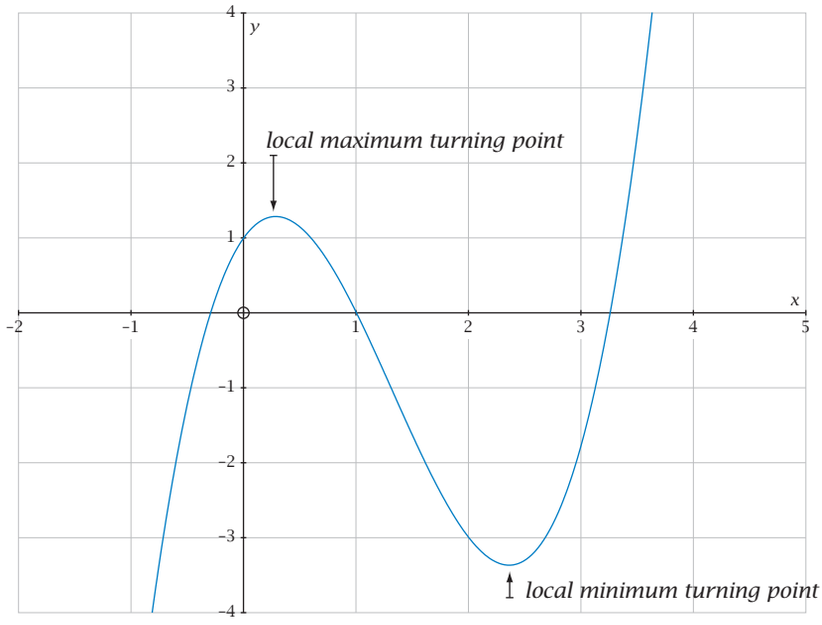
Determining tangent lines

To find the tangent line at a point (x_1, y_1) on the function $y = f(x)$, follow these steps:

1. Differentiate the function $f(x)$ to obtain the gradient function $f'(x)$.
2. Substitute (x_1) into the gradient function to determine the tangent's gradient at that point.
3. Substitute the value found in step 2, as well as the original point (x_1, y_1) into the equation of a straight line $y = mx + c$.
4. Calculate the value for 'c' and hence determine the equation of the tangent to a curve at a point.

Graph sketching

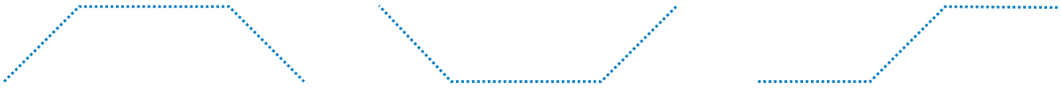
Using calculus techniques, graphs of polynomial functions can be accurately constructed.



On the graph of $f(x)$ are several stationary points; a minimum turning point and a maximum turning point.

These stationary points can be found by solving $f'(x) = 0$.
The two most common methods of determining the nature of stationary points; namely, the sign test and the second derivative test.

The sign test requires an examination of the slope of the gradient function either side of a stationary point(s) where $f'(x) = 0$.



this indicates a local maximum turning point

this indicates a local minimum turning point

this indicates a horizontal point of inflection

The second derivative test requires differentiating $f'(x)$ to obtain $f''(x)$. Then the nature of the stationary points can be determined.

If $f''(a) > 0$ a local minimum turning point exists at $x = a$.

If $f''(a) < 0$ a local maximum turning point exists at $x = a$.

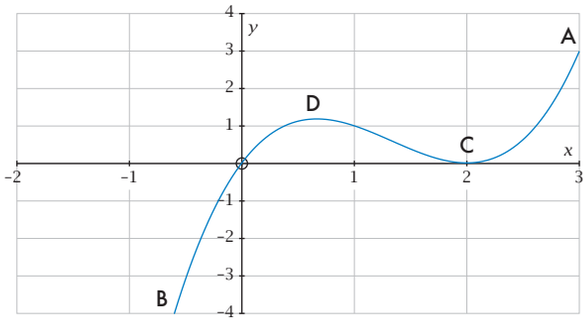
If $f''(a) = 0$ a turning point or a horizontal point of inflection exists at $x = a$.

At all points where $f''(a) > 0$ the function is displayed as 'concave up'.

At all points where $f''(a) < 0$ the function is displayed as 'concave down'.

A function is said to be increasing at all points on an interval if the gradient is positive, and decreasing if the gradient is negative. Expressed another way, if $f'(x) > 0$ then $f(x)$ is increasing at x. Conversely, if $f'(x) < 0$ then $f(x)$ is decreasing at x.

When a continuous function is clearly defined over a given interval it is possible to identify global extrema. Considering the graph below, there exists a global maximum at A and a global minimum at B. There also exists a local minimum at C and a local maximum at D.



N.B. Global extrema can only exist at turning points or end points.

Optimisation

The application of calculus techniques is a useful method for solving optimisation problems. There are a number of steps to follow when trying to determine a maximum or minimum value of a function.

1. Wherever possible, draw a diagram of the problem.
2. Define the quantity to be optimised and identify all variables to be used.
3. Create a formula for the quantity to be optimised in terms of the variables
4. Use relationships between the variables in the formula to reduce it to a formula containing only one variable.
5. Differentiate the expression, equate the derivative to zero and solve to obtain the location of the stationary point(s). Reject any values which do not fit the problem.
6. Determine the nature of the maximum or minimum point(s) using either the second derivative test or the sign test.
7. Substitute the relevant point back into the original formula for the optimal value.
8. Write the final answer in words, and expressed with correct units.

Worked Examples

10.1 Determine the equation of the tangent to the curve $g(x) = 2x^3 - 4x^2 + 5x + 1$ at the point where $x = 1$.

If	$g(x) = 2x^3 - 4x^2 + 5x + 1$	
then	$g'(x) = 6x^2 - 8x + 5$	*Differentiate to find the gradient function*
and	$g'(1) = 6(1)^2 - 8(1) + 5$	*Find the gradient of the curve at that point*
	$= 6 - 8 + 5$	*through substitution of x = 1*
	$= 3$	
Now	$y = mx + c$	*Begin with the equation of a straight line*
and	$y = 3x + c$	*Use the gradient obtained before*
Using (1, 4)	$4 = 3(1) + c$	*Substitute x = 1 into g(x) to obtain point for use here*
Thus	$c = 1$	
And hence	$y = 3x + 1$	*Determine the tangent equation to the curve at the point.*

10.2 An average aluminium can holds approximately 375mL of liquid. Determine the dimensions (i.e. radius and height) of such a can that would minimise the amount of metal used in the construction process.

If $\text{Volume} = \pi r^2 h$ *Use the equation for a cylinder's volume to rearrange*

then $375 = \pi r^2 h$ *and find an expression for h in terms of r*

and $\frac{375}{\pi r^2} = h$

$\text{Surface Area} = 2\pi r^2 + 2\pi r h$

$= 2\pi r^2 + 2\pi r\left(\frac{375}{\pi r^2}\right)$ *Substitute h to have an expression in terms of r*

$= 2\pi r^2 + \frac{750\pi r}{\pi r^2}$

$= 2\pi r^2 + \frac{750}{r}$

$\frac{dSA}{dr} = 4\pi r - \frac{750}{r^2}$ *Differentiate S.A. with respect to r*

max/min at $\frac{dSA}{dr} = 0$

So $0 = 4\pi r - \frac{750}{r^2}$ *Determine the max/min points of the function*

$\frac{750}{r^2} = 4\pi r$

$750 = 4\pi r^3$

$r^3 = \frac{750}{4\pi}$

Thus $r = 3.91$ (2 d.p.) *Solve for radius and hence, height*

$h = 7.82$ (2 d.p.)

10.3 For the equation $y = x^3 - 2x^2 + 1x - 6$, use Calculus methods to determine the nature and location of any stationary points on this curve.

If $y = x^3 - 2x^2 + 1x - 6$

then $\frac{dy}{dx} = 3x^2 - 4x + 1$ *First we differentiate to obtain the gradient function*

$0 = 3x^2 - 4x + 1$ *Find any points on the curve where the gradient equals zero*

$0 = (3x - 1)(x - 1)$

Thus $x = \frac{1}{3}$, $x = 1$ and the stationary points are

$(\frac{1}{3}, \frac{-158}{27})$ and $(1, -6)$ *Determine both x and y coordinates for stationary points*

Now $\frac{d^2x}{dy^2} = 6x - 4$ *Obtain the second derivative function*

and when $x = \frac{1}{3}, \frac{d^2x}{dy^2} = -2$ *Perform the second derivative test to determine the nature of each stationary point*

and when $x = 1, \frac{d^2x}{dy^2} = 2$

Hence $(\frac{1}{3}, \frac{-158}{27})$ is a maximum point and $(1, -6)$ is a minimum point.

10.4 Find the coordinates of any point(s) on the curve $y = x^3 - 4x^2 - 2x + 10$ where the gradient is equal to 1.

If $y = x^3 - 4x^2 - 2x + 10$

then $\frac{dy}{dx} = 3x^2 - 8x - 2$ this is the general value of the gradient

we want $3x^2 - 8x - 2 = 1$

that is $3x^2 - 8x - 3 = 0$

$x = 3$ or $x = -\frac{1}{3}$

The corresponding y values are found by substituting the x values into the equation of the curve.

The required points are $(3, -5)$ and $\left(-\frac{1}{3}, \frac{275}{27}\right)$

10.5 The displacement (x) of a body at any time (t) is given by equation $x = t^3 - 7t^2 + 8t + 1$. Find:

(a) the $v - t$ equation

(b) the $a - t$ equation

(c) the displacement when the body is at rest.

$x = t^3 - 7t^2 + 8t + 1$

(a) the velocity is found by getting the first derivative of the displacement therefore the $v - t$ (velocity – time) equation is given by $v = \frac{dx}{dt} = 3t^2 - 14t + 8$

(b) the $a - t$ equation is found by taking the next derivative hence $a = \frac{dx}{dt}\ 6t - 14$

(c) a body is a rest when $v = 0$

we need to solve $3t^2 - 14t + 8 = 0$

that is $(3t - 2)(t - 4) = 0$

that is $t = \frac{2}{3}$ or $t = 4$

the body is at rest on two occasions and the corresponding displacements can be found by substituting into the displacement equation

at $t = \frac{2}{3}$ $x = \frac{95}{27}$

at $t = 4$ $x = -15$

10.6 Find the distance travelled in the first five seconds of motion by a body whose displacement (x) from the origin at any time (t) is given by $x = t^3 + t^2 - 16t + 16$.

When considering distance we need to determine whether the body stops at any time because when this occurs the direction of motion may change.

$x = t^3 + t^2 - 16t + 16$
$v = 3t^2 + 2t - 16$
$v = 0$ when $3t^2 + 2t - 16 = 0$
that is $(3t + 8)(t - 2) = 0$
that is $t = 2$ (as $t < 0$ is unacceptable)

This now means that we are interested in the distance covered between $t = 0$ and $t = 2$ and again between $t = 2$ and $t = 5$.

at $t = 0$ $x = 16$ } distance covered
at $t = 2$ $x = -4$ } is 20 units

at $t = 2$ $x = -4$ } distance covered
at $t = 5$ $x = 86$ } is 90 units

The total distance travelled in the first five seconds is 20 + 90 = 110 units.

(*Note:* the displacement would by 86 – 16 = 70 units.)

10.7 A small manufacturing company produces and then sells x items. The respective cost and revenue functions are: $C(x) = 84 + 4x$ and $R(x) = 11x$. Find:

(a) a formula for the profit function, $P(x)$

(b) the marginal profit and comment on the result

(c) how many items must be sold to break even.

(a) $P(x) = R(x) - C(x)$
$= 11x - (84 + 4x)$
$= 7x - 84$

(b) $P'(x) = 7$

This implies that whenever one more item is produced and sold the profit will increase by 7. (Note this is only true since $P'(x)$ is constant.)

(c) We want $P(x) = 0$
that is $7x - 84 = 0$
that is $x = 12$

The production and sale of 12 items will result in breaking even.

10.8 Matthew has a small production company and determines that the cost function for the production of x items is $C(x) = 400 + 10x - 0.01x^2$ dollars. Find:

(a) $C'(x)$

(b) $C'(10)$ and comment on the result

(c) the average cost per item if Matthew was to produce 10 items.

$C(x) = 400 + 10x - 0.01x^2$

(a) $C'(x) = 10 - 0.02x$

(b) $C'(10) = 10 - 0.02(10)$
$= 9.80$

This tells us that the cost of producing the next item, at the time that 10 items have been produced, would be $9.80.

(c) Total cost of producing 10 items is $C(10)$

$C(10) = 400 + 10(10) - 0.01(100)$
$= \$499$

Average cost $= \dfrac{499}{10}$
$= \$49.90$ each

10.9 For the function $f(x) = x^3 - 9x^2 + 15x - 12$, find the value(s) of x for which $f(x)$ is increasing.

$f(x) = x^3 - 9x^2 + 15x - 12$
$f'(x) = 3x^2 - 18x + 15$
$f'(x) = 0$ when $3x^2 - 18x + 15 = 0$
that is $x^2 - 6x + 5 = 0$
that is $x = 1$ or $x = 5$

We need to consider three intervals, $x < 1$, $1 < x < 5$ and $x > 5$ and test to see whether $f'(x)$ is positive in any of these intervals.

(Remember, for $f(x)$ to be increasing we need $f'(x) > 0$.)

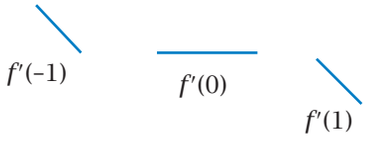
To test these intervals we need only to choose one point within the interval and test it. If this point gives us the desired result then all points in that interval will also.

Choose $x = 0$ (from the interval $x < 1$)
$f'(0)$ is positive and hence $f(x)$ is increasing for $x < 1$
Choose $x = 2$ (from the interval $1 < x < 5$)
$f'(2)$ is negative and hence $f(x)$ is decreasing for $1 < x < 5$
Choose $x = 6$ (from the interval $x > 5$)
$f'(6)$ is positive and hence $f(x)$ is increasing for $x > 5$

10.10 For the function $f(x) = x^3 (x - 8)$ use calculus to find the stationary points and points of inflection.

$f(x) = x^3 (x - 8)$
$f(x) = x^4 - 8x^3$ ——————— expand brackets
$f'(x) = 4x^3 - 24x^2$ ——————— calculate derivative
$= 4x^2 (x - 6)$ ——————— factorise
$0 = 4x^2 (x - 6)$ ——————— let $f'(x) = 0$ to solve for x
$f'(x) = 0$ when $x = 0$ or $x = 6$
$f''(x) = 12x^2 - 48x$ _____ calculate second derivative
$= 12x (x - 4)$ _____ factorise
$f''(x) = 0$ which implies that $x = 0$ could be a turning point or point of horizontal inflection.

We need to use the sign test



Hence at $x = 0$ we have a point of horizontal inflection.

$f''(6)$ is positive which implies that we have a minimum turning point at $x = 6$.

Other inflection points could occur if $f''(x) = 0$.

that is $12x (x - 4) = 0$
$x = 0$ or $x = 4$

We already have a horizontal inflection point at $x = 0$.

At $x = 4$ we have another inflection point.
Now $f(0) = 0$
$f(6) = -432$
$f(4) = -256$

In summary (0,0) is a horizontal inflection
(6,–432) is a minimum turning point
(4, –256) is a point of inflection

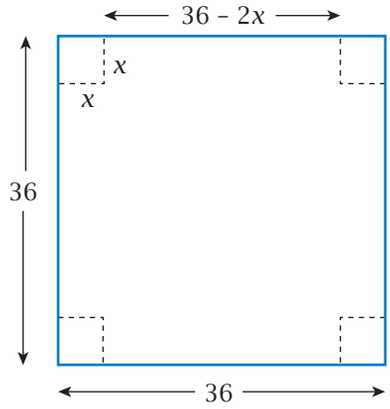
10.11 A square piece of sheet metal (side length 36 cm) is to have congruent squares cut from each corner so that when the sides are folded up it will form a lidless square based container. Find the length of the side of the square that should be removed from each corner so that the resultant container will have maximum volume.

Let x be the length of the side of the square to be removed.

The base of the container will be square with side lengths of $36 - 2x$.

The height of the container will be x.

The volume of the container is given by:



$V = (36 - 2x)(36 - 2x)x$

$= 1296x - 144x^2 + 4x^3$

$\frac{dV}{dx} = 1296 - 288x + 12x^2$

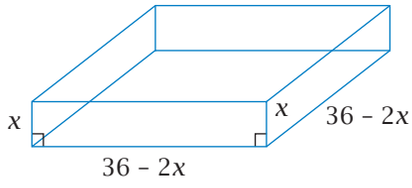
$\frac{dV}{dx} = 0$ when $x = 6$ or $x = 18$

We can reject $x = 18$ as this would not produce a container:

$\frac{d^2V}{dx^2} = -288 + 24x$

at $x = 6$

$\frac{d^2V}{dx^2}$ is negative which says that $x = 6$ is a maximum point.



$\frac{d^2V}{dx^2} = -288 + 24(6) = -144$ which is less than zero. This result implies a local maximum at $x = 6$.

To produce a maximum volume we would need to remove squares of side length 6 cm from each of the corners.

PROBLEMS TO SOLVE

CHAPTER 10: APPLICATIONS OF DIFFERENTIAL CALCULUS

1. Find the gradient of the given function at the given point.

   (a) $y = 2x^2 - 3x + 5$ (1, 1).

   (b) $g(x) = x^2 - \frac{1}{x}$ at (–2, 4)

2. (a) At what point(s) does the tangent to the curve given by $y = 3 + x^3$ have a gradient of 12?

   (b) Find the equation(s) of the tangent(s).

3. For the curve $f(x) = (3x + 4)(x^2 - 3x + 4)$ find x such that $f'(x) = 0$.

4. A shell is fired vertically upwards and its equation of motion is $s = 147t - 4.9t^2$ metres. What is the greatest height the shell reaches?

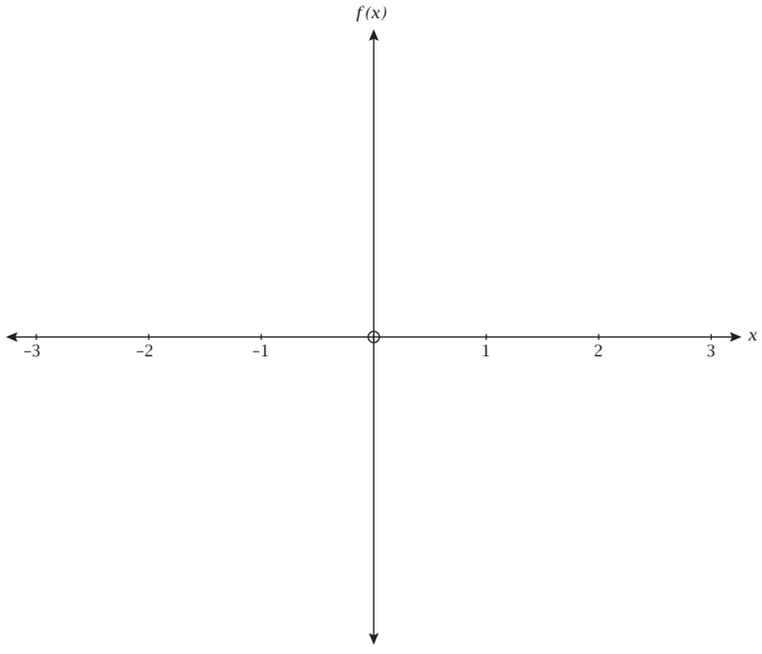
5. What are the dimensions of the largest rectangle with a perimeter of 70 cm, and an area of 306.25 cm^2?

6. Find the least area of sheet metal required to make an open baking dish of square base and vertical sides of capacity 2048 cm^3?

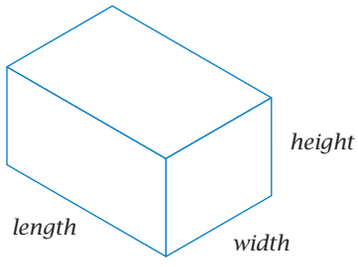
7. The line $y = 1 - 6x$ is parallel to the tangent of $y = x^3 + ax^2 - 5x + 1$ at $x = \frac{-1}{3}$. Find the value of a and the equation of the tangent at the indicated point.

8. A function has **all** the following features. Sketch a possible function on the axes provided.

- $f(-2) = f(0) = f(2) = 0$
- $f'(-1) = f'(1) = 0$
- $f(1) > 0$
- $f''(-1) > 0$



9. The frame of a rectangular box used to store steel fencing slats is to be made from 40 metres of piping.



The width of the box needs to be 75% of its height.

(a) Determine the length of the box in terms of h, the height of the box.

(b) Show clearly that the volume of the box is given as:

$$V(h) = \frac{15}{2}h^2 - \frac{21}{16}h^3$$

(c) Determine the maximum volume of the box, correct to one decimal place.

(d) Determine the width of the box, correct to two decimal places, that gives its maximum volume.

10. The curve $f(x) = ax^2 + bx - 3$, goes through the point (1, 0) and has a gradient of 2 at this point. Find a and b.

11. A fatal disease is caused by a particular bacteria increasing in such a way that the number present after t hours is given by :

$$N(t) = 5t^3 + 4t^2 + 300$$

Find:

(a) the number of bacteria present initially.

(b) the average rate of increase of bacteria in the first 10 hours.

(c) an expression for the instantaneous rate of change of bacteria.

(d) the rate that the colony of bacteria is increasing at

i. $t = 150$ minutes

ii. $t = 1$ day

12. The linear equation $y = x + 2$ is a tangent to the curve $y = ax^2 + bx + c$ at the point (0, 2). If (1, 0) is another point on the curve, find a, b and c.

13. Sketch the graph of a function satisfying all of the following conditions:

- $f'(x) = 0$ at $x = -2$ and $x = 3$
- $f'(x) < 0$ for $x < 0$
- $f'(x) = 0$ at $x = 0$ and $x = 4$
- $f'(x) > 0$ for $0 < x < 4$ and $x > 4$

14. Sketch the graphs of each of the following polynomials after finding the intercepts on both axes, and the stationary points. Check these graphs carefully with a graphics calculator.

(a) $f(x) = x^3 - x^2 - x - 1$

(b) $g(x) = -x^3 + 36x$

15. Find the equation of the tangent to the curve $y = x^3 - 2x^2 + x - 1$ at the point (2,1).

16. Show that the tangent at the y intercept is perpendicular to the tangent at the left most zero for the curve

$$y = 0.5(x + 1)(x - 1)(2x + 1)$$

17. The period of swing T seconds of a pendulum of length l metres is given by the formula

$$T = 2\pi\sqrt{\frac{l}{9.81}}$$

(a) Find the period when l is 60 cm.

(b) Find the rate of change of period with respect to the length when length is 1.5 m.

18. The power W in watts when a current I amps is flowing is given by

$$W = 240I - 9I^2$$

(a) Find W when I is 10 amps.

(b) Find the rate of change of W with respect to I when I is 15 amps.

19. An object moves in a straight line so that its displacement x metres from an origin O at time t seconds is given by the equation

$$x = t^3 - 3t^2 + 2t - 6$$

Find

(a) initial displacement

(b) velocity when $t = 2$

(c) acceleration when $t = 2$

(d) the velocity when the acceleration is zero

(e) when the object is at the origin.

20. The displacement of a body from an origin O at time t seconds is x metres where

$$x = 2t^3 - 15t^2 + 24t - 4$$

Find

(a) when the body is at rest

(b) the distance travelled in the first 5 seconds.

21. A particle moving along a line has displacement x metres at any time t seconds given by

$$x = 4 - 6.5t + 3t^2 - \frac{t^3}{3}$$

Find

(a) the acceleration when $t = 3$

(b) the values of t for which the particle has a speed of 1.5 m/s.

22. A manufacturer's cost of producing x items is

$$C(x) = 2000 + 100\sqrt{x} \text{ dollars}$$

Each item produced is sold for \$150. Find.

(a) the revenue function $R(x)$

(b) the profit function $P(x)$

(c) the profit if 10 items are sold

(d) the marginal cost function $C'(x)$

(e) the marginal profit function $P'(x)$

(f) $P'(10)$

23. Mass production of an electrical component is believed to reduce the price. The cost function for producing x items is given by

$$C(x) = 2500 - \frac{1000}{x+1} \text{ dollars}$$

If the revenue function is given by

$$R(x) = 40x$$

find

(a) the profit function $P(x)$

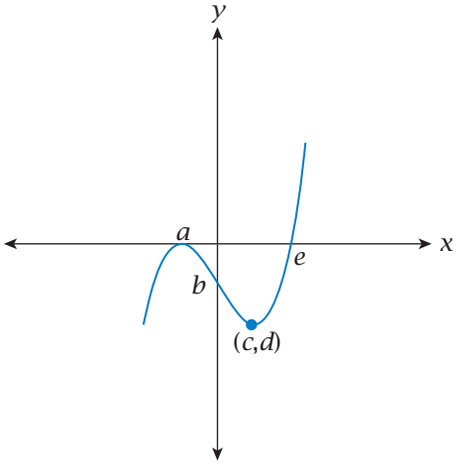
(b) (i) $P'(99)$

(ii) $C'(99)$

(iii) $R'(99)$

and comment on the meeting of each.

24. The graph shown is $y = f(x)$.



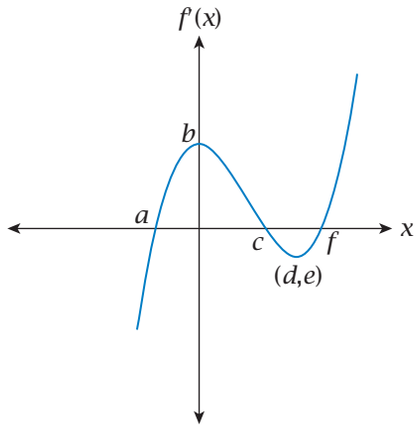
(a) For which x values is the function increasing?

(b) Give the location of any stationary points and state their nature.

(c) Give the location of any point of inflection.

(d) For which x values is the function concave downwards?

25. The graph shows the derivative function $y = f'(x)$.

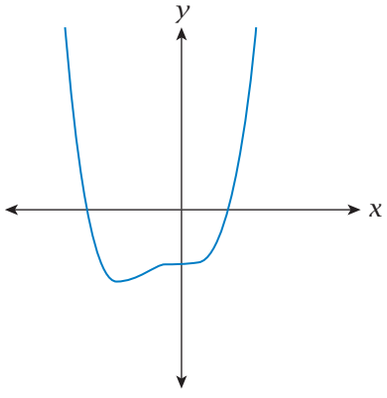


(a) For which x values is the function $y = f(x)$ increasing?

(b) Give the x values of the stationary points of $y = f(x)$. State the nature of any point given.

(c) Give the x values of any point of inflection of $y = f(x)$.

26. The curve $y = f(x)$ is shown.



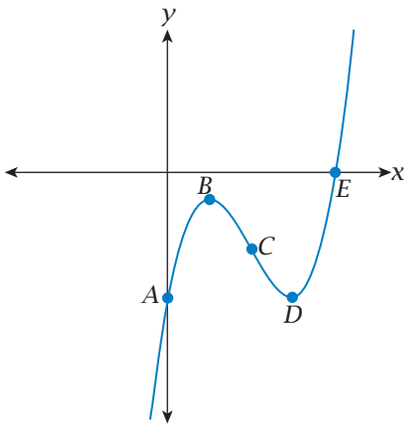
Sketch the curves

(a) $y = f'(x)$

(b) $y = f''(x)$

27. (a) Use your graphics calculator to display the graph of $y = x^3 - 6x^2 + 9x - 5$.

(b) Determine the coordinates of the points A to E.



Use calculus methods to find

(c) any stationary points and determine their nature

(d) any points of inflection

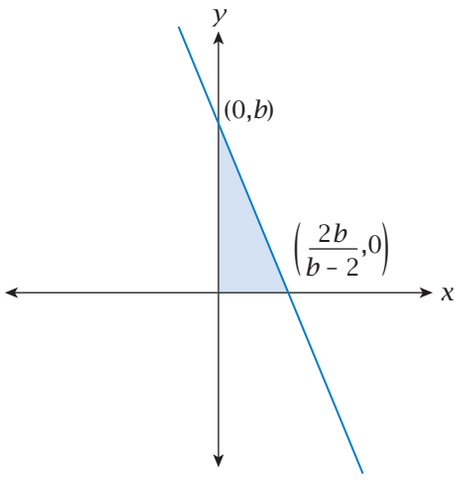
(e) graph the first derivative function y' and compare it with the graph of y

(f) graph the second derivative function y'' and compare it with the graphs of y and y'.

28. (a) Determine the coordinates and nature of stationary points for the curve with equation $y = 2x^3 - 6x^2 + 6x + 1$

(b) Draw a sketch of the function using the information from (a).

29. Find the least area possible for the shaded region if b varies ($b > 0$).



30. To manufacture x items costs a company $40x + 15000$ dollars. The company has set a sale price of $150 - 0.02x$ dollars per item. Find:

(a) the number of items that should be produced for maximum profit

(b) the price per item to achieve this profit.

31. The rejection rate for specialised computer chips is high. Currently 160 chips are produced weekly with a failure rate of 23%. Management wants to increase production by 10 per week every week. Workers say this will lead to an increase of 3% per week every week in the failure rate. How long should this new procedure last?

Total Produced	Rejected Rate	Number Rejected	Total Suitable
160	23%	36.8	123.2
170	26%	44.2	125.8

32. The function $y = x^2 + \frac{2}{x} - 1$ is defined for $0.5 \leq x \leq 2$.

Find

(a) relative (local) minimum

(b) relative (local) maximum

(c) global minimum

(d) global maximum.

33. Use your graphics calculator to investigate the position of the relative maximum for the equation

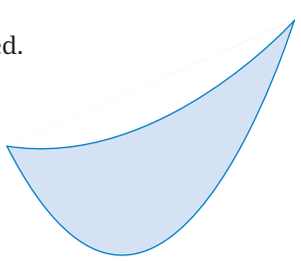
$$y = x(x - a)^2 \quad \text{for} \quad a > 0$$

34. An underground passage opening is in the shape shown shaded.
The top has equation

$$y = 0.25x^2 - x + 2$$

and the bottom has equation

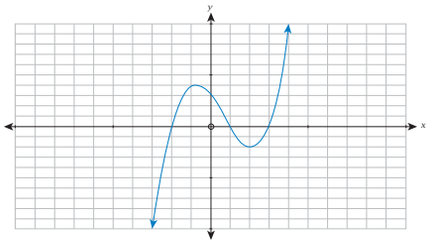
$$y = x^2 - 6x + 8.$$



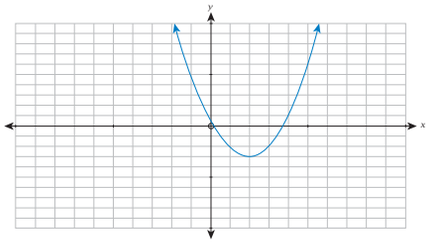
Find the widest part of the opening.
Distances are in metres.

35. Match one of the functions found in the left-hand column with the corresponding gradient function graphs in the right-hand column.

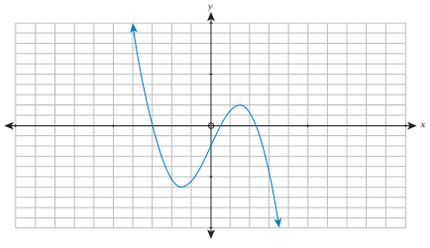
(a)



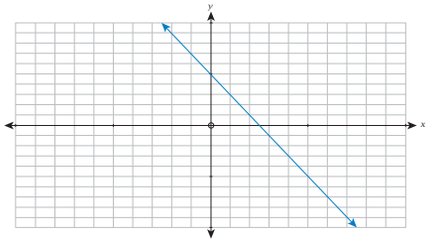
(b)



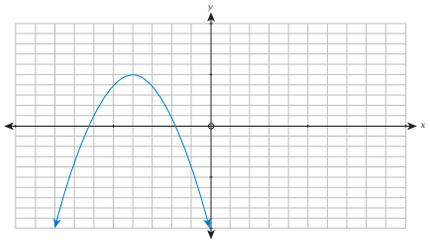
(c)



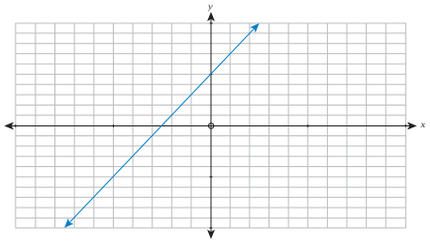
(d)



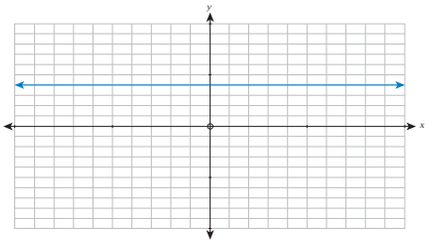
(e)



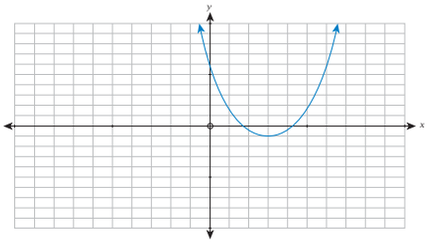
(f)



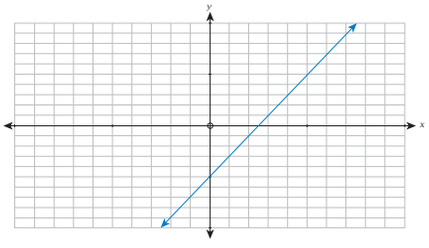
(i)



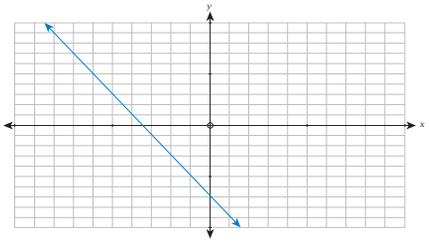
(ii)



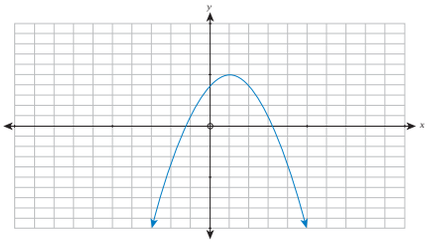
(iii)



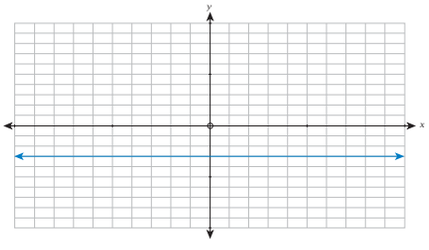
(iv)



(v)



(vi)



TRIAL TEST 1: FUNCTIONS I

Calculators NOT allowed
Time Allowed: 40 minutes
Total Marks: 38

1. (a) For the equation $x^2 + 9x - 15 = 0$,
use the discriminant to determine whether the equation has:

- two rational solutions
- two irrational solutions
- one solution
- no real solutions.

[2]

(b) Given the equation $y = x^2 + 6x - 7$

i. state the y intercept

ii. find the x intercepts

iii. determine the turning point

[4]

2. Circle the changes which should be made to the graph of $y = x^2$ to obtain the graph of $y = -2(x + 1)^2 - 4$

(i) dilated by a factor of 2 (made wider) or (made narrower)

(ii) is (inverted) or (not inverted)

(iii) (translated 1 unit to the left) or (translated 1 unit to the right)

(iv) (translated 4 units up) or (translated 4 units down)

[4]

3. Robbie, the cable man has just been quoted on two separate jobs. For the first, which is a 5 hour job, he quoted a fee of $285. For a 13 hour job, his quote was $645. His quotes always include a fixed call out fee and an hourly rate. Determine his hourly rate and the fee for a job taking 6½ hours.

[3]

4. Write the equation of the line that passes through the point (–3, 5) and is parallel to $2y = x + 4$.

[3]

5. The equations of six lines are shown below:

(a) $y = 2x - 3$

(b) $2y = 3 - x$

(c) $2x - 2y = 3$

(d) $2x + y + 1 = 0$

(e) $2x - 4y = -3$

(f) $y = x + 3$

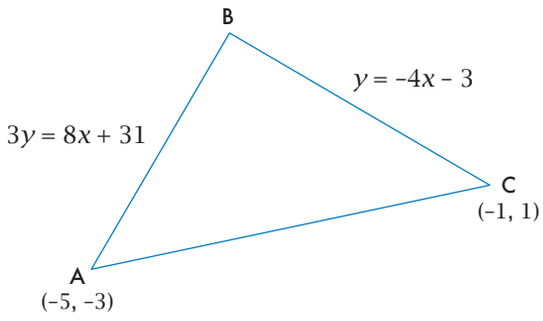
Determine which pair(s) of the above lines are:

i. parallel

ii. perpendicular

[5]

6. The diagram shows A (–5, –3), C (–1, 1),
side BC with equation $y = -4x - 3$ and
side AB with equation $3y = 8x + 31$



Find:

(a) the co-ordinates of point B

[1]

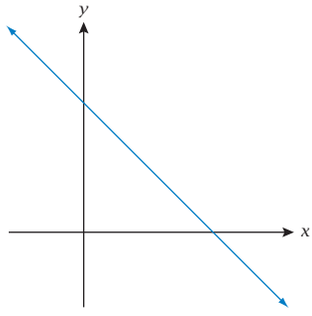
(b) the equation of median BD where D is the midpoint of side AC.

[4]

7. In the triangle ABC, A has co-ordinates (–6, –3) while B has co-ordinates (k, 5) and C has co-ordinates (8, 2). Find the value of k if angle ABC is a right angle.

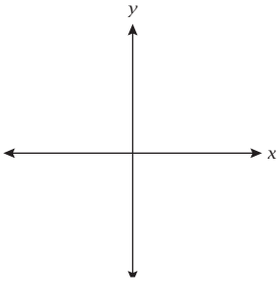
[6]

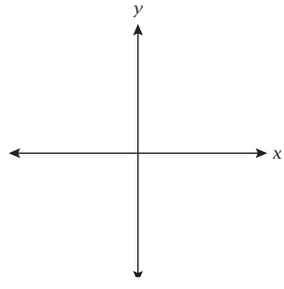
8. The graph shown is that of the linear function $y = mx + c$

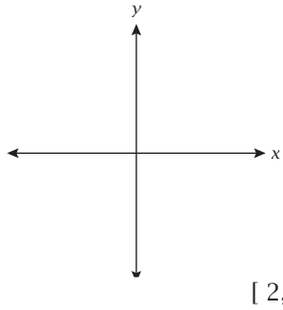


Make accurate sketches of each of the following:

(a) $y = mx - c$ (b) $y = -mx + c$ (c) $y = -(mx + c)$





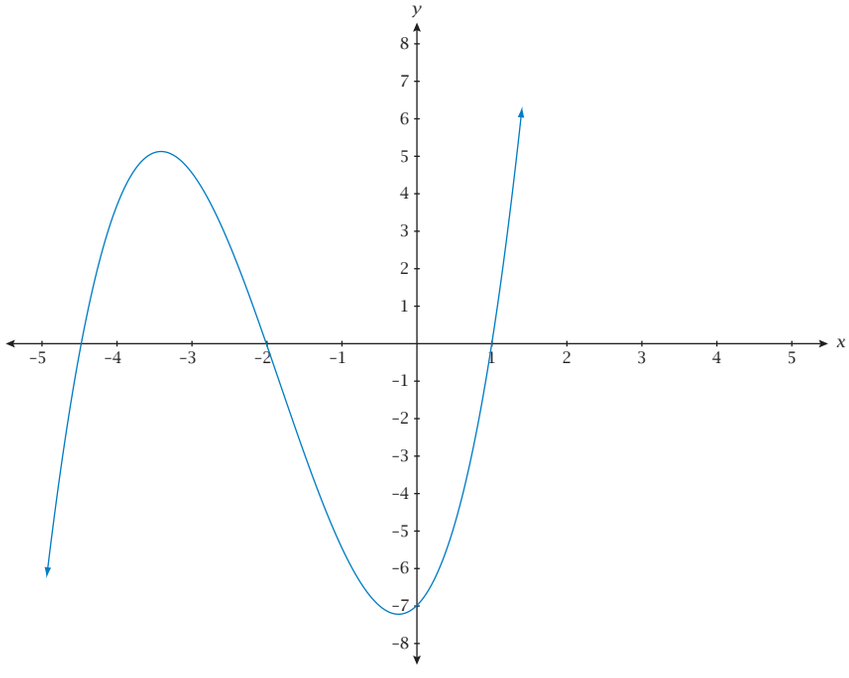


[2, 2, 2]

TRIAL TEST 2: FUNCTIONS II

Calculators NOT allowed
Time Allowed: 30 minutes
Total Marks: 29

1. Use the graph below to solve for x in the questions following.



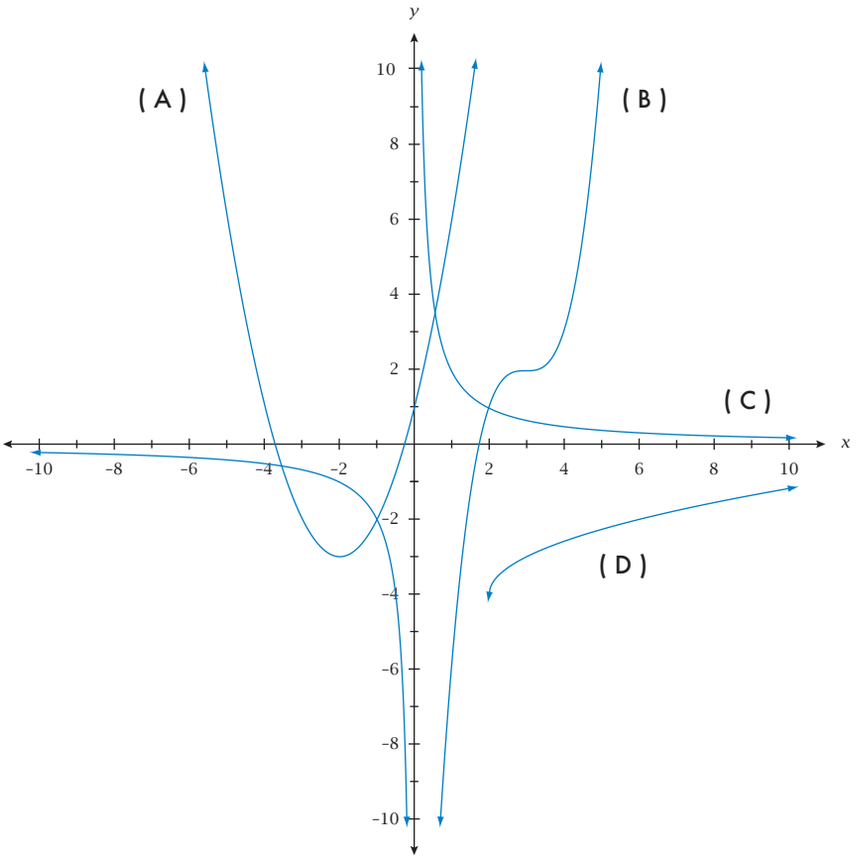
(a) $f(x) = 0$

[1]

(b) $f(x) \geq 0$

[1]

2. Match the graphs with the appropriate equation from the list below



(1) $y = (x - 2)^2 + 3$

(2) $y = (x - 2)^3 + 3$

(3) $y = 2^x$

(4) $y = \sqrt{x - 2} - 4$

(5) $y = \sqrt{x - 4} - 2$

(6) $y = x^3 + 2$

(7) $y = (x + 2)^2 - 3$

(8) $y = x^2 + 4x + 4$

(9) $y = \sqrt{x + 2} - 4$

(10) $y = \dfrac{1}{x}$

(11) $y = (x - 3)^3 + 2$

(12) $y = \dfrac{2}{x}$

Graph A = _____ [1]

Graph B = _____ [1]

Graph C = _____ [1]

Graph D = _____ [1]

3. For the quadratic equation $y = -2(x + 3)^2 - 4$ determine

(i) the equation of the line of symmetry

_____ [1]

(ii) the location and nature of the turning point

_____

_____ [2]

(iii) where the graph intersects the y axis, and

[1]

(iv) where the graph intersects the x axis.

[1]

4. Write the vertical and horizontal asymptotes of the following equations:

(a) $y = \dfrac{3}{x - 6}$

[1]

(b) $y = \dfrac{1}{4x - 3} + 2$

[2]

5. Describe how the graphs of each of the functions in list A can be transformed into the graphs of the corresponding functions in list B.

	A	**B**
(i)	$y = x^3$	$y = (x + 1)^3 - 2$

[2]

(ii)	$y = \dfrac{1}{x - 2}$	$y = \dfrac{1}{x + 3}$

[1]

(iii)	$y = 2^x$	$y = 3(2^x) + 1$

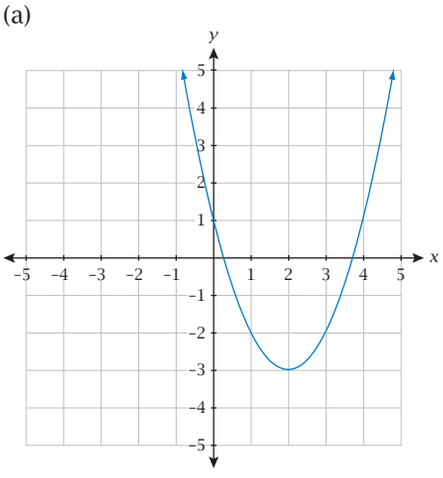
[2]

(iv)	$y = (x - 2)^3 + 3$	$y = x^3$

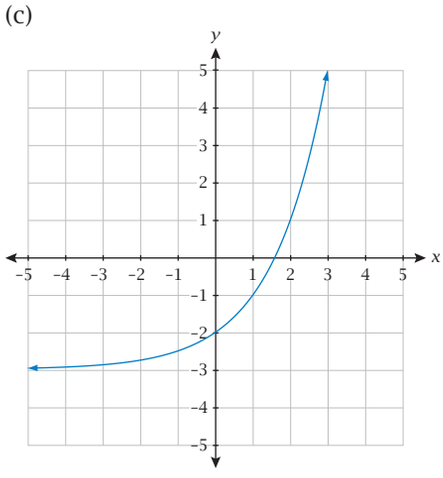
[2]

6. Find the equations of each of the functions shown in the graphs below.

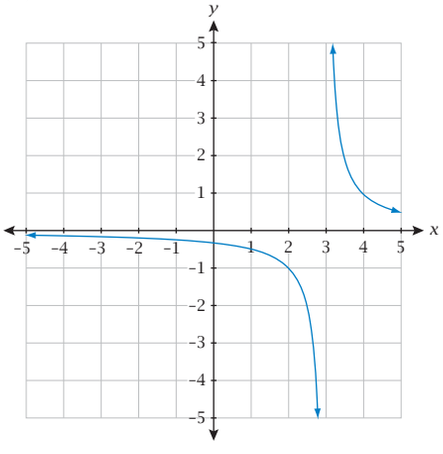
(a)



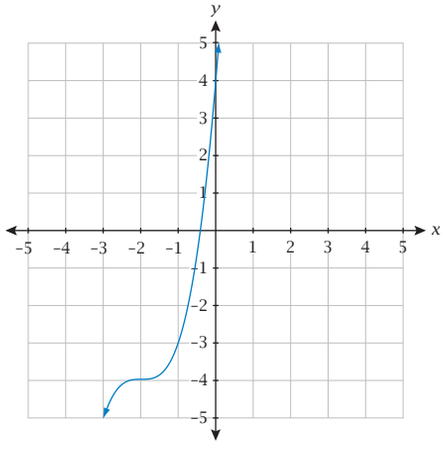
(c)



(b)



(d)



[2, 2, 2, 2]

TRIAL TEST 3:
SEQUENCES, RECURSION AND EXPONENTS

Calculators allowed
Time Allowed: 45 minutes
Total Marks: 43

1. $12 000 is invested and accrues interest at a rate of 3.2% per annum, compounded quarterly.

   (a) If no further deposits are made, how much will be in the account after 3 years?

   _____

   _____ [2]

   (b) How long will it take until there is $15 000 in the account?

   _____

   _____ [2]

2. 300 g of radioactive gas was released into the atmosphere due to a breakdown in the cooling mechanism of a nuclear power station. Given the rate of decay for the radioactive gas is 0.7% per year, how much would remain after:

   (a) i. 1 year?

   _____ [1]

   ii. 20 years?

   _____ [1]

   iii. 100 years?

   _____ [1]

   iv. n years?

   _____ [1]

   (b) The half-life of a radioactive substance is the time it takes for that substance to decay to half of its initial mass. Determine the half-life of this substance.

   _____

   _____ [2]

3. For the sequence 20, 34, 48, … determine

(a) whether this sequence is arithmetic or geometric, and then give the general formula

[2]

(b) the 21st term

[1]

(c) the sum of the first 15 terms

[2]

4. If a geometric sequence is defined recursively as $T_{n+1} = 3T_n$, with $T_1 = 14$, determine:

(a) the n th term

[3]

(b) the 5th term

[1]

(c) the first term that would exceed 1000.

[3]

5. Geoff was able to take control of his trust fund on his 21st birthday. He was told to spend the \$20 000 contained within the fund wisely. Geoff spent $\frac{3}{4}$ of it on a car, and the rest he invested. There were two options for Geoff to consider, Option A and Option B.

Option A: 7.25% Simple Interest
Option B: 7.25% Compounded annually

(a) Which option is better for Geoff after 4 years and by how much?

[2]

(b) If Option B allowed Geoff the choice to compound half-yearly, quarterly and daily, which one should he choose for maximum return? Show all working out for each of the three choices.

[3]

(c) Geoff's friend is trying to persuade him to invest his money in another bank. This friend mentioned that he started with \$2500 and after 2 years finished with \$2900. This investment was compounded annually. Should Geoff change his bank? Justify your answer, and find the interest rate this new bank offers.

[3]

6. Simplify the following expressions, writing your answer with positive indices.

(a) $\dfrac{(4a^2b^3)^3}{(-2ab^4)^3}$

[2]

(b) $\dfrac{6x^{-3}y^{-4}}{2x^{-4}y^{-2}}$

[3]

(c) $\dfrac{8m^3n^{-6}}{4m^2n^{-5}} \div \dfrac{3n^2}{(4m^4)^2}$

[4]

(d) $(8p^4q^2)^{-3} \times 16(p^5q^{10})^{\frac{2}{5}}$

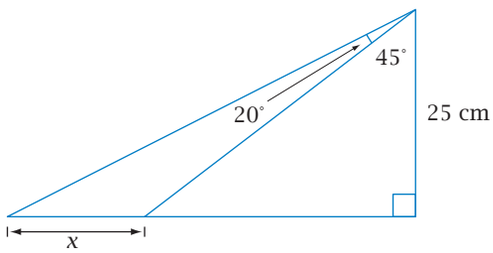
[4]

TRIAL TEST 4: TRIGONOMETRY I

Calculators allowed
Time Allowed: 60 minutes
Total Marks: 59

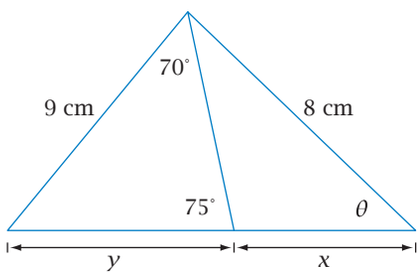
1. Determine the value of the unknown variables in the triangles below:

(a)



[2]

(b)



[3]

2. From a point P due south of a vertical tower, the angle of elevation of the top of the tower is 25° and from a point Q due east of the tower the angle of elevation is 35°. If the distance from P to Q is 50 metres, find the height of the tower.

[6]

3. A boat on a lake sails due west from a point A to a point C. During this trip it passes alongside a marker buoy B. The bearing and distance of B from a point P on the shore due south of A are 313° and 1.2 km respectively. The bearing of C from P is 299°. Find the distance BC correct to the nearest metre.

[5]

4. Three vertical radio masts AD, BE and CF stand in a straight line on level ground with their bases at A, B and C where $AB = BC$. The height of AD and CF are 40m and 30m respectively. If the elevation of E from D is 20° and the angle of depression of F from E is 30° find the distance between the two shorter masts. (Hint: let $AB = BC = x$)

[9]

5. Three towns Abbydale, Bonham and Connolly have positions as follows: Connolly is 10k m from Abbydale and on a bearing of 155°, Abbydale and Bonham are 7 km apart and Bonham is on a bearing of 300° from Connolly. Find the possible bearings of Bonham from Abbydale.

[7]

6. A box in the shape of a rectangular prism has base $ABCD$ with AB = 50cm and BC = 40cm. If the top $EFGH$ has E 30 cm vertically above D find the angle that ΔACE makes with the base. (Hint: let P be a special point on AC.)

[7]

7. Triangle PQR has area 75 cm^2, p = 15 cm and q = 10 cm. Calculate the size of angle Q, correct to the nearest degree.

[6]

8. If the perimeter of a sector of a circle is 30cm and the area is 54 cm^2, find the radius and the angle at the centre for each of the two possible sectors.

[9]

9. An equilateral triangle of side length 9cm has circles with centres at each of the vertices drawn to pass through the other two vertices. Find the area common to the three circles.

[5]

TRIAL TEST 5: TRIGONOMETRY II

Calculators allowed
Time Allowed: 30 minutes
Total Marks: 25

1. Give exact values for the following:

(a) $\cos(240^\circ)$

[1]

(b) $\sin\left(\frac{7\pi}{4}\right)$

[1]

(c) $\tan(45^\circ)$

[1]

(d) $\cos\left(\frac{5\pi}{6}\right)$

[1]

2. In the function $g(x) = a\sin(bx - c)$ a, b, and c are all positive constants.
In terms of a, b, and c determine:

(a) the amplitude of $g(x)$

[1]

(b) the period of $g(x)$

[2]

(c) the horizontal shift required to make $f(x) = a\sin(x)$ become $g(x) = a\sin(bx - c)$

[2]

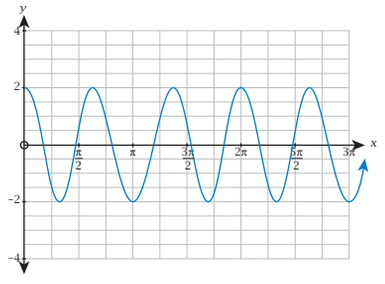
3. If $\sin A = \frac{3}{5}$ and $\cos B = \frac{5}{13}$ where $\frac{\pi}{2} < A < \pi$ and $0 < B < \frac{\pi}{2}$, what is the value of $\sin(A + B)$

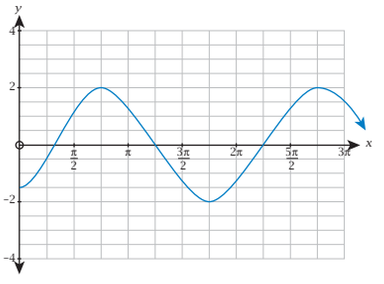
[4]

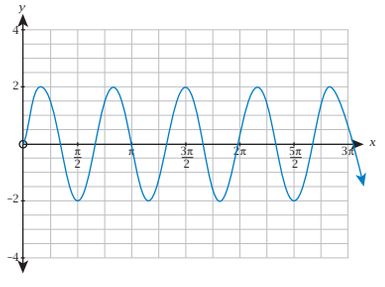
4. Match one of the equations (i)–(viii) to one of the graphs A, B, C, and D listed below.

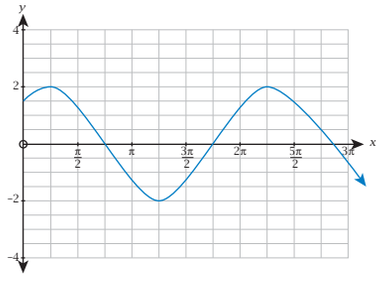
(i) $y = 2\sin x$ (ii) $y = 2\cos x$ (iii) $y = 2\cos 3x$ (iv) $2\cos(x - \frac{\pi}{4})$

(v) $y = 2\sin 2x$ (vi) $y = 2\cos 2x$ (vii) $y = 2\sin 3x$ (viii) $2\sin(x - \frac{\pi}{4})$

A 

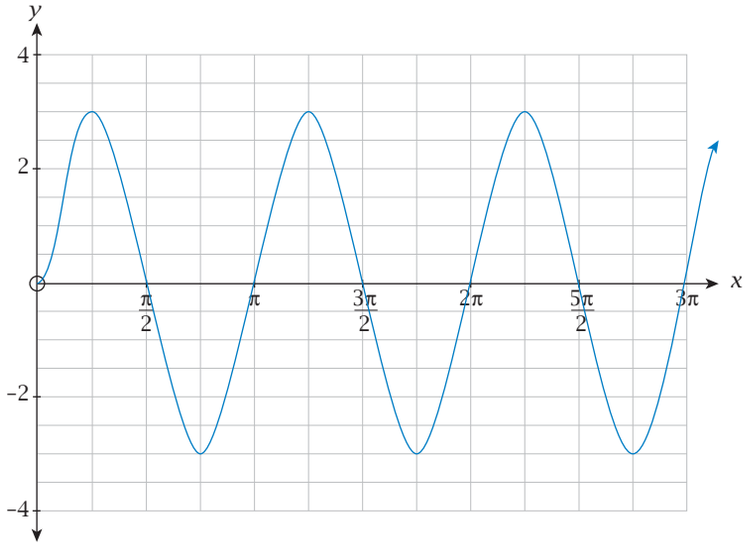
B 

C 

D 

[8]

5. If the graphed function below has equation of form $f(x) = a\cos(bx - c)$, determine the values of a, b, and c.



[4]

TRIAL TEST 6:
PROBABILITY, SETS AND COUNTING TECHNIQUES

Calculators allowed
Time Allowed: 45 minutes
Total Marks: 44

1. The W.A.C.A. has 15 000 members listed in its database. 78% of these members are male, and 65% of all males attended Day One of the recent Test Match. Only 10 000 members attended Day One. Each W.A.C.A. member was sent out a raffle ticket by post prior to the Test Match. The raffle prize was drawn at the end of the first day's play, and consisted of a bat signed by the players on both teams.

   Complete the table shown below.

	Attended the first day's play?		
	Yes	No	Totals
Male			
Female			
Total			

[2]

What is the probability the winner of the raffle prize is

(a) male

_____ [1]

(b) someone at the Test Match on the first day

_____ [1]

(c) a male not at the Test Match on the first day

_____ [1]

(d) a female given that the winner was not at the Test Match on the first day?

_____

_____ [2]

2. 3 subsets are drawn from the Universal set of positive integers less than 50.

Set *A* represents all the perfect squares, **Set *B*** represents all the factors of 24, and **Set *C*** represents the perfect cubes.

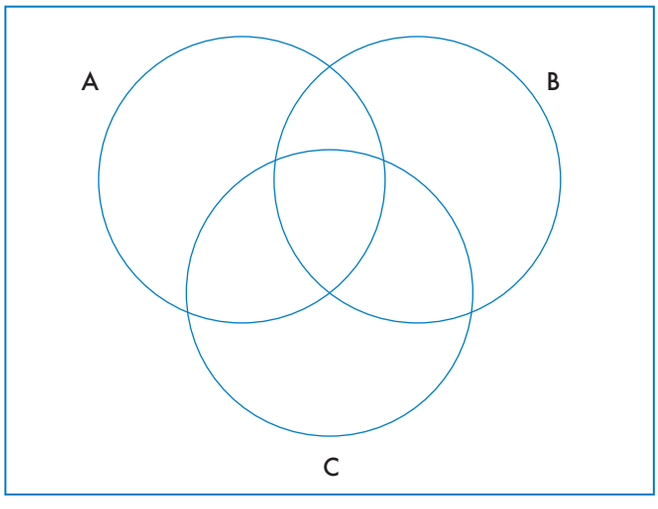
(a) List the elements of each of the following sets

Set *A* = _____

Set *B* = _____

Set *C* = _____ [3]

(b) Place the elements of sets A, B and C only in the Venn diagram below



[3]

(c) Complete:

i. $n(A \cap B \cap C)$

_____ [1]

ii. $n(\overline{A} \cap B \cap C)$

_____ [1]

(d) List the elements of the following sets:

i. $(A \cap B)$

_____ [1]

ii. $(B \cap C) \cup A$

_____ [1]

iii. $A' \cap (B \cup C)$

_____ [1]

(e) Calculate the following probabilities:

i. $P(A')$

[1]

ii. $P(A \cap B)$

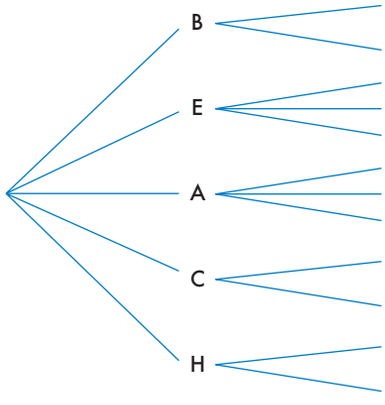
[1]

iii. $P(A \mid B)$

[1]

3. Four-letter arrangements are made from the letters of *BEACH*. These arrangements must meet two conditions; vowels can not follow each other, and consonants can not follow each other. Letters must not be repeated in each arrangement.

(a) Complete the tree diagram below



[1]

Find the probability that the arrangement formed

(b) has the consonant *B*

[1]

(c) does not have the consonant *C*

[1]

(d) has both the vowels *E* and *A*

[1]

(e) has *H* coming before *C*

[1]

(f) has a *C* given that it ends in a *B*

[2]

(g) starts with a vowel given it has no *H*

[2]

4. Solve for n

$$\binom{n}{13} = \binom{n}{8}$$

[2]

5. Use the binomial theorem to find the coefficient of x^6 in $(3 + 2x^2)^5$.

[4]

6. Two students, John and Mary, place their keys in a bag of similar keys containing 8 keys (a total of 10). John then selects a key followed by Mary.

(a) Show that the probability that Mary selects her key is the same as the probability that John selects his.

[3]

(b) Find the probability that at least one of them selects his/her own keys.

[2]

(c) Find the probability that John selects his if it is known that Mary selects her own key.

[3]

TRIAL TEST 7: DIFFERENTIATION

Calculators allowed
Time Allowed: 30 minutes
Total Marks: 30

1. The population, N, of cows being bred by a farmer, is modelled by

$$N = 50 + 45t - 7t^2 + \frac{t^3}{4},$$

where t is time in years.

(a) Find the number of cows at $t = 2$ and $t = 10$.

[2]

(b) Find the instantaneous population growth or decay when $t = 2$ and $t = 10$. Interpret your answers.

[2]

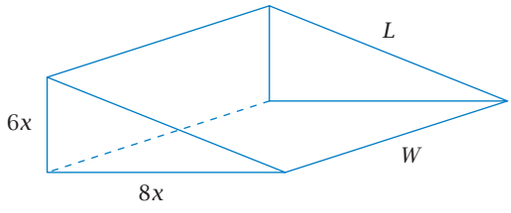
(c) Does the population of cows ever experience negative growth i.e. the population decreases? If so, over what time interval?

[2]

2. Find the equation of the tangent to the curve $y = \frac{1}{\sqrt{x}} + \frac{2}{x^3} - 2x$ at $x = 2$.

[5]

3. A piece of wire, 399 cm long, is used to make the 9 edges of the frame of a wedge as shown. The height and length of the wedge are $6x$ and $8x$ respectfully.



(a) Given that L is the length of the hypotenuse of the cross sectional right angled triangle, show that $L = 10x$

[1]

(b) Show that the width of the wedge, W, is given by $W = 133 - 16x$

[1]

(c) Show that the volume, V, of the wedge is given by $V = -384x^3 + 3192x^2$

[2]

(d) Use differentiation to find the dimensions of the frame that will maximise the volume of the wedge. State what the volume is.

[4]

4. Differentiate the following functions:

(a) $y = -4x^3 + 2x^2 + \frac{3}{x} - 7$

[2]

(b) $f(x) = 2(x + 1)^2$

[2]

(c) $f(x) = \frac{-4}{x^3} + \frac{1}{x}$

[2]

5. Using calculus techniques, determine the nature and location of all significant turning points and intercepts of the function $y = 2x^4 - 2x^2 - 4$. Hence, accurately sketch this function on the Cartesian plane, labelling all significant points.

[5]

TRIAL TEST 8: INTEGRATION

Calculators allowed

Time Allowed: 25 minutes

Total Marks: 25

1. Determine the exact area enclosed by the curve $y = x^2 - 5$ and the x axis. Use a sketch to assist you.

[3]

2. Find the antiderivative of the following expressions:

(a) $\frac{4}{x^3}$

[1]

(b) π

[1]

(c) $(2x - 1)^4$

[2]

(d) $\int \left(4x + x^2 - \frac{5}{3}\right) dx$

[2]

(e) $\int \frac{2\sqrt{x} + 5x}{2x} dx$

[2]

3. Determine $f(x)$ if $f'(x) = \frac{5}{\sqrt{x}} + 4x$ and $f(4) = 0$.

[3]

4. Determine the value of each of the following exactly:

(a) $\int_0^4 (4x + 1) dx$

[2]

(b) $\int_1^9 2\sqrt{x} + \frac{2x}{5} dx$

[2]

5. Calculate the area between $y = 2x^3 - 2x^2 - 4x$ and the x axis. Use a sketch.

[4]

6. Find the value of a, given that $\int_1^a 3x^2 + 2x \, dx = 34$.

[3]

A

ANSWERS

CHAPTER 1: *Functions I*

1. (a) (1, 4)
   (b) (1, 4)
   (c) (4, –1)
   (d) $\left(\frac{9}{2}, \frac{9}{2}\right)$
   (e) $\left(-\frac{7}{2}, 3\right)$
   (f) $\left(\frac{3}{2}, \frac{5}{2}\right)$

2. (a) $y = 4x - 10$
   (b) $y = -\frac{1}{5}x + \frac{13}{5}$ or $5y = -1x + 13$

3. $y = -\frac{2}{3}x$ or $3y = -2x$

4. (0, 17)

5. $k = -5$ or $k = 3$

6. $k = -1$ or $k = -6$

7. (a), (e), (f), (g), (h)

8. (a), (c), (d), (e), (g), (h)

9. (a) D: $\{x: x \in R\}$ R: $\{y: y \in R\}$
   (b) D: $\{x: x \in R\}$ R: $\{y: y \in R, y \geq -3\}$
   (c) D: $\{x: x \in R\}$ R: $\{y: y \in R\}$
   (d) D: $\{x: x \in R, x \geq 2\}$ R: $\{y: y \in R, y \geq -4\}$
   (e) D: $\{x: x \in R\}$ R: $\{y: y \in R, y \leq 5\}$
   (f) D: $\{x: x \in R, x \geq -3\}$ R: $\{y: y \in R, y \leq 1\}$

10. (a) $y = 2\sqrt{x + 4} - 3$
    (b) $y = -\frac{1}{2}\sqrt{x + 2} + 5$

11. (a) $y = (x - 1)(x + 2)(x - 3)$
    (b) $y = (x + 1)(x + 4)(x - 2)$
    (c) $y = (x - 3)(x + 4)(x - 1)$
    (d) $y = (2x + 1)(x - 1)(x - 3)$
    (e) $y = (3x - 1)(x + 2)(x + 1)$
    (f) $y = (2x - 1)(x - 2)(x - 3)$

12. (a) Yes; one-to-one
    (b) No
    (c) Yes; many-to-one
    (d) Yes; one-to-one
    (e) Yes; many-to-one
    (f) Yes; one-to-one
    (g) No
    (h) Yes; many-to-one

13. *b*, *c*, *e* and *f*.

14. (a) $\frac{4}{3}$
    (b) $\frac{1}{2}$
    (c) $-\frac{25}{8}$
    (d) $-\frac{7}{2}$

15. (a) $m = 4$; (0, –5)
    (b) $m = \frac{3}{2}$; (0, 3.5)
    (c) $m = -\frac{3}{2}$; (0, –0.25)

16. (a) $y = -2x + 4$
    (b) $y = 4x + 20$

17. (a) $y = 3x + 2$
    (b) $y = -2x + 1$

18. $y = 3x + 10$

19. $y = \frac{-1x}{5} + 2$

20. Point *B*

21. $k = 10$

22. check graphs on calculator

23. Line B, Line C, Line D

24. Line B, Line C, Line E

25. check graphs on calculator

26. *(a)* $y = \frac{1}{2}(x + 2)(x - 6)$

(b) $y = 3(x - 4)^2 - 3$

27. *(a)* $y = x^2 + 4$

(b) $y = (x + 2)^2 + 3$

(c) $y = -2(x - 5)^2$

28. *(a)* *i.* $x = 1$
ii. (1, 2), maximum
iii. (0, –1)

(b) *i.* $x = 1$
ii. (1, –25) minimum
iii. (0, –24)

(c) *i.* $x = -1$
ii. (–1, –3) minimum
iii. (0, –1)

29. *(a)* $y = x^3 - 3$

(b) $y = -(x - 4)^3$

(c) $y = \frac{1}{2}(x + 3)^3 + 1$

30. *(a)* $y = (x + 1)^2(x - 5)$

(b) $y = -x(x + 3)(x - 2)$

31. *(a)* $x = -3, -4$

(b) $x = 3$

(c) $x = 0, x = -2$

(d) $x = \pm 5$

32. *(a)* 2

(b) 1

(c) 0

(d) 2

33. *(a)* 2.14, 13.52 $x = 2.14, x = 13.52$

(b) –2.21, –27.21 $x = -2.21, x = -27.21$

34. *(a)* $E = 0.02V + 550$

(b) \$1750

(c) \$125 000

35. *(a)* Amount = $5.50x + 15$

(b) Amount = $6.50x + 20$

(c) \$42.50

(d) \$36.25

(e) 2.54 hours, or 2 hours 32 minutes

36. *(a)* 28

(b) 13

(c) 30

(d) $x = 3, x = 4$

37. A: (3,0) B: (0,–6) C: (4, 0)
D: (0,–8) E: (0,3) F: $\left(\frac{11}{3}, \frac{-2}{3}\right)$

38. *(a)* $0 \le t \le 10$

(b) $t > 10$

39. *(a)* $11s$

(b) $2s$ and $9s$

(c) $5.5s$

40. *(a)* (6, 0)

(b) (–1, –7)

(c) $x = 3$

(d) $y = 8$

41. *(a)* 52

(b) –18

(c) $x = -8, 1$

(d) $k^2 + 5k - 14$

42. *(a)* $k = 600$

(b) $N = 4000$

(c) \$0.12

43. *(a)* $-0.63 < k < 2.11$

(b) $x < -2,\ 0 < x < 1$

(c) $-2 < x < 0\ ,\ x > 1$

44. (–2, –1), (1, 2), (3, 4)

45. Proof.

46. $b = 45,\ h = 20$

CHAPTER 2: *Functions II*

1. (a) Function
(b) Relation
(c) Relation
(d) Relation
(e) Function
(f) Function
(g) Function
(h) Relation
(i) Function

2. (a), (c), (g), (h), (j), (l)

3. (a) 4
(b) 5
(c) $\sqrt{20}$ or $2\sqrt{5}$
(d) 10
(e) 3
(f) $\sqrt{7}$

4. (a) $x^2 + y^2 = 25$
(b) $x^2 + y^2 = 5$
(c) $(x - 2)^2 + (y - 1)^2 = 36$
(d) $(x + 4)^2 + (y + 3)^2 = 3$

5. (c), (d), (f)

6. (a) $y = \frac{1}{x + 2}$ D: $\{x: x \in R, x \neq -2\}$ R: $\{y: y \in R, y \neq 0\}$
(b) $y = \frac{1}{x} - 3$ D: $\{x: x \in R, x \neq 0\}$ R: $\{y: y \in R, y \neq -3\}$
(c) $y = \frac{1}{x - 4} + 2$ D: $\{x: x \in R, x \neq 4\}$ R: $\{y: y \in R, y \neq 2\}$
(d) $y = \frac{1}{x + 6} - 4$ D: $\{x: x \in R, x \neq -6\}$ R: $\{y: y \in R, y \neq -4\}$

7. (a) $y = \frac{1}{x} + 1$
(b) $y = -\frac{1}{x + 1}$
(c) $y = \frac{1}{x - 1} - 2$
(d) $y = \frac{-2}{x + 2} - 3$

8. (a) Yes
(b) No
(c) Yes
(d) No

9. (a) $N = \frac{k}{x}$
(b) $t = \frac{k}{s}$

10. (a) $y = \frac{12}{x}$
(b) $V = \frac{20}{P}$

11. (a) $y = 0$, (0, 0)
(b) $y = 0$, (–4, 0)
(c) $y = 0$, (7, 0)
(d) $y = 2$, (0, 2)
(e) $y = -8$, (0, –8)
(f) $y = 1$, (3, 1)

12. (a) $a = -1$
(b) $b = 3$
(c) $c = 6$
(d) $d = 2$

13. (a) (0, 0), $r = 3$
(b) (1, 0), $r = \sqrt{6}$
(c) (0, 0), $r = \frac{1}{2}$
(d) (–4, 1), $r = 8$
(e) (3, –2), $r = 3$
(f) (–4, 1), $r = \frac{3\sqrt{2}}{2}$

14. (a) D: $\{x: x \in R, x \neq 1\}$ R: $\{y: y \in R, y \neq 1\}$
(b) D: $\{x: -3 \leq x \leq 3\}$ R: $\{y: -3 \leq y \leq 3\}$
(c) D: $\{x: x \in R, x \geq -3\}$ R: $\{y \in R\}$
(d) D: $\{x: -2 \leq x \leq 8\}$ R: $\{y: -1 \leq y \leq 9\}$
(e) D: $\{x: x \in R, x \geq -1\}$ R: $\{y \in R\}$
(f) D: $\{x: x \in R, x \neq -1\}$ R: $\{y: y \in R, y \neq -2\}$

15. Graph Solutions
(i) – A (ii) – B (iii) – C (iv) – D (v) – E (vi) – F

Letter Solutions
$a = 1$, $b = 2$, $c = 1$, $d = -3$, $e = -2$, $f = 2$, $g = 1$, $h = 1$, $i = -1$, $j = -1$, $k = 1$, $l = 1$

16. (4, 5)

17. (a) $k = 45$, $M = \frac{45}{N}$
(b) graph
(c) $N = 2.25$

CHAPTER 3: *Trigonometry I*

1. *(a)* 8.38^R

(b) 420°

(c) $\frac{17\pi}{12}$

2. *(a)* $\angle AOB = \frac{2\pi}{3}$

(b) 120°

(c) 2.09^F

3. *(a)* $\theta = 1.875^R$

(b) Area = 60 cm^2

(c) Same answer as before!

4. Area = 0.645 cm^2

5. Common area = 120.1 cm^2

6. Distance = 35.6 km

7. *(a)* Perimeter = 7.03 m

(b) Area = 3.14 m^2

8. Area = 0.57 m^2

9. Belt length = 3.35 m

10. Area = 9.23 cm^2

11. Perimeter = 32 cm

12. Area = 49.87 cm^2

13. Area = 517.45 m^2

14. Area = 5.3 cm^2

15. Distance = 210 km
Bearing = 273.79°

16. 26°

17. 6.05 km

18. 7.74 cm

19. Distance = 1412.49 km
Bearing = 62.95°

20. *(a)* 57.12 km

(b) 24.21 km

(c) Bearing = 67.03°
Time = 4 hrs 8 min

21. 31.79 cm

22. *(a)* 50°

(b) 5.87 cm

(c) 39.81°

(d) 51.73°

23. *(a)* 28.4 km

(b) Before

24. 2.6°

25. *(a)* 59.36 m

(b) 3229 m^2

26. *(a)* 1.49 km

(b) 20.10°

27. *(a)* 17.84 cm

(b) 9.54 cm

(c) 11.65 cm

28. 8.32 m

29. 31°

30. 278.95 m

31. 1.03 m

32. $\tan C = \frac{c \sin A}{b - c \cos A}$

33. *(a)* 36.8 cm^2

(b) 17.9 cm^2

34. *(a)* 1437 m

(b) 332.3 m

(c) 1.99 km/hr

35. 23.8km/hr @ 84.5° T

36. *(a)* 212 km

(b) 3 hours 9 minutes @ 325°

37. 42.49°

38. $B = 80°,\ C = 50°,\ c = 7$ or
$B = 100°,\ C = 30°,\ c = 4.6$

39. $x = 73.74°$

40. *(a)* 71°
(b) 5.9 m
(c) 45.22 m²
(d) 33°

41. *(a)* 19.4m
(b) 92.9°

42. *(a)* $x = 4.47$
(b) $x = 501$
(c) $x = 4.50,\ y = 7.64,\ \theta = 40.9°$

43. 46.1 cm²

44. 245°T

45. *(a)* 120.35 km *(b)* 105°T

46. 71 m

47. *(a)* 60°
(b) 19.2
(c) 51.3°
(d) 55.3°

CHAPTER 4: *Trigonometry II*

1. $\frac{5\pi}{6}$

2. *(a)* $\frac{\sqrt{3}}{2}$
(b) $\frac{1}{3}$
(c) $\sqrt{2}$
(d) $-\sqrt{3}$

3. $x_1 = 1.23^R \quad x_2 = 5.05^R$

4. *(a)* $\sqrt{3}$
(b) 0.75
(c) 1

5. *(a)* $x_1 = 21.14° \quad x_2 = 81.14°$
$x_3 = -38.86°$

(b) $x_1 = \frac{\pi}{3} \quad x_2 = \pi$
$x_3 = \frac{-\pi}{3} \quad x_4 = -\pi$

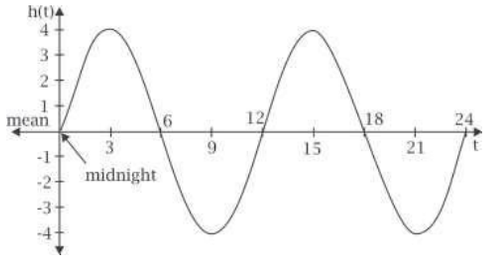
6. $\frac{31\sqrt{2}}{50}$

7. *(a)* $PR = 4$
(b) $\sin\alpha = \frac{\sqrt{7}}{4}$
(c) $\sin(90 - \alpha) = 0.75$
(d) $\cos\left(\frac{3\pi}{2} + \alpha\right) = \frac{\sqrt{7}}{4}$

8. *(a)* $y = -3\cos\left(2x + \frac{\pi}{4}\right) - 1$ or
$y = 3\sin\left(2x + \frac{\pi}{4}\right) - 1$
(b) $y = -\frac{1}{2}\tan\frac{1}{2}x + 2$

9. *(a)* $\pm\frac{5\pi}{18}, \pm\frac{7\pi}{18}, \pm\frac{17\pi}{18}$
(b) $\theta_1 = \frac{2\pi}{3}$
$\theta_2 = 0.85$
$\theta_3 = 2.29$

10. *(a)*



(b) 3 am and 3 pm
(c) 4 m
(d) 3.46 m below mean sea level
(e) between 12.29 and 5.31 am and
12.29 and 5.31 pm

11. *(a)* $y = 2\tan\frac{1}{3}x$
(b) $y = -2\cos\left(3x + \frac{\pi}{4}\right)$
or $y = -2\sin\left(3x + \frac{\pi}{4}\right)$

12. (a) $\frac{\sqrt{2}-\sqrt{6}}{4}$

(b) $\frac{\sqrt{2}-\sqrt{6}}{4}$

(c) $\frac{1}{2}$

(d) $\frac{\sqrt{6}+\sqrt{2}}{4}$

(e) $\frac{\sqrt{6}-\sqrt{2}}{4}$

(f) $\frac{-\sqrt{3}}{2}$

(g) $\frac{\sqrt{3}-1}{4}$

(h) 1

(i) $\frac{-\sqrt{3}}{3}$

CHAPTER 5: *Counting Techniques*

1. (a) $\frac{4!}{2!2!} = 6$

(b) $\frac{5!}{4!1!} = 5$

(c) $\frac{10!}{5!5!} = 252$

(d) $\frac{7!}{3!4!} = 35$

(e) $\frac{6!}{3!3!} = 20$

(f) $\frac{9!}{5!4!} = 126$

2. (a) $\binom{9}{3}$

(b) $\binom{11}{2}$

(c) $\binom{7}{3}$

(d) $\binom{8}{1}$

(e) $\binom{5}{3}$

(f) $\binom{10}{3}$

3. (a) $x = 8$

(b) $x = 9$

(c) $x = 2$

(d) $x = 8$

(e) $x = 10$

(f) $x = 8$

4. (a) $n, 1, 1, 1$

(b) $n, 0, n, 0$

(c) n, n, n

(d) $n, n-1, n-1, n-1$

5. (a) $m^5 + 5m^4n + 10m^3n^2$

(b) $x^6 + 12x^5y + 60x^4y^2$

(c) $81a^4 - 108a^3b + 54a^2b^2$

(d) $32x^5 - 240x^4y + 720x^3y^2$

(e) $x^7 + 7x^5 + 21x^3$

(f) $\frac{8}{a^3} - \frac{4b}{a^2} + \frac{2b^2}{3a}$

6. $\binom{26}{4} = 14\,950$

7. $\binom{52}{5} = 2\,598\,960$

8. $\binom{10}{7} = 120$

9. $\binom{40}{25} = 4.022 \times 10^{10}$

10. (a) $\binom{12}{4} = 495$

(b) $\binom{10}{4} = 210$

11. (a)

0	1	2	3	4	5	6	7
$\binom{0}{0}$	$\binom{1}{1}$	$\binom{2}{2}$	$\binom{3}{3}$	$\binom{4}{4}$	$\binom{5}{5}$	$\binom{6}{6}$	$\binom{7}{7}$
$\binom{1}{0}$	$\binom{2}{1}$	$\binom{3}{2}$	$\binom{4}{3}$	$\binom{5}{4}$	$\binom{6}{5}$	$\binom{7}{6}$	$\binom{8}{7}$
$\binom{2}{0}$	$\binom{3}{1}$	$\binom{4}{2}$	$\binom{5}{3}$	$\binom{6}{4}$	$\binom{7}{5}$	$\binom{8}{6}$	$\binom{9}{7}$
$\binom{3}{0}$	$\binom{4}{1}$	$\binom{5}{2}$	$\binom{6}{3}$	$\binom{7}{4}$	$\binom{8}{5}$	$\binom{9}{6}$	$\binom{10}{7}$
$\binom{4}{0}$	$\binom{5}{1}$	$\binom{6}{2}$	$\binom{7}{3}$	$\binom{8}{4}$	$\binom{9}{5}$	$\binom{10}{6}$	$\binom{11}{7}$
$\binom{5}{0}$	$\binom{6}{1}$	$\binom{7}{2}$	$\binom{8}{3}$	$\binom{9}{4}$	$\binom{10}{5}$	x	
$\binom{6}{0}$	$\binom{7}{1}$	$\binom{8}{2}$	$\binom{9}{3}$	$\binom{10}{4}$	$\binom{11}{5}$		y
$\binom{7}{0}$	$\binom{8}{1}$	$\binom{9}{2}$	$\binom{10}{3}$	$\binom{11}{4}$	$\binom{12}{5}$		z

(b) $x = \binom{11}{6}$ $n = 11, r = 6$

(c) $x = 462$, $y = \binom{13}{7} = 1716$, $z = \binom{14}{7} = 3432$

12. (a) $k = 1$

(b) $k = 5$

(c) $k = 3$

(d) $k = 3$

(e) $k = 7$

CHAPTER 6: *Probability and Sets*

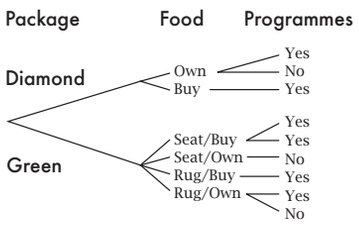
1. *(a)*

	Yes	No	Undecided	Total
Morning	24	21	10	55
Afternoon	16	3	6	25
Night	11	6	3	20
Total	51	30	19	100

(b) *i.* $\frac{4}{25}$ *ii.* $\frac{11}{20}$ *iii.* $\frac{3}{10}$ *iv.* $\frac{11}{51}$

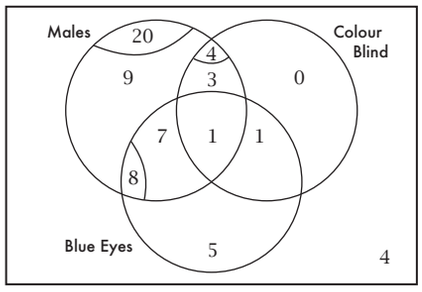
v. $\frac{21}{55}$ *vi.* $\frac{4}{5}$ *vii.* $\frac{9}{25}$

2. *(a)*



(b) $\frac{2}{3}$ *(c)* $\frac{1}{4}$ *(d)* $\frac{1}{3}$

3. *(a)*



(b) $\frac{14}{30}$ *(c)* $\frac{1}{15}$ *(d)* $\frac{7}{30}$ *(e)* $\frac{16}{25}$

(f) $\frac{3}{4}$

4. *(a)* $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

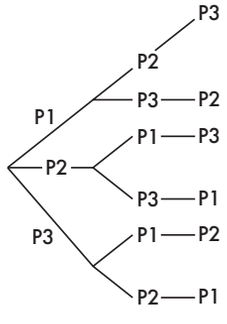
(b) $\{2\}$

(c) $\{0, 1, 3, 5, 7, 9\}$

(d) $\{1, 9\}$

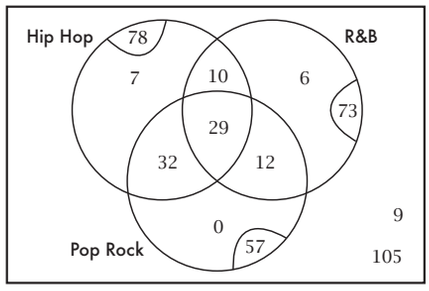
(e) Ø or $\{\ \}$

5. *(a)*



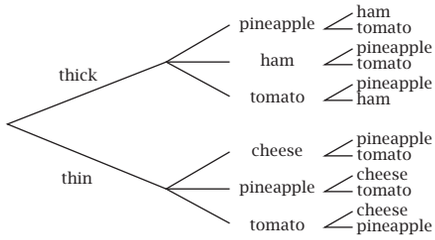
(b)(i) $\frac{1}{6}$ *(ii)* $\frac{1}{3}$

6. *(a)*



(b) 9 *(c)* 0 *(d)* $\frac{48}{105} = \frac{16}{35}$

7. *(a)*



(b) *i.* $\frac{1}{6}$ *ii.* $\frac{2}{3}$ *iii.* $\frac{1}{3}$ *iv.* $\frac{2}{3}$

8. *(a)*

	Age 55	Age 65	Over Age 65	Total
Male	21	115	61	197
Female	20	212	71	303
Total	41	327	132	500

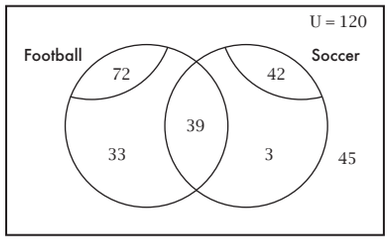
(b) 232 *(c)* $\frac{136}{197}$

9. *(a)* $\frac{8}{150} = \frac{4}{75}$ *(b)* $\frac{62}{150} = \frac{31}{75}$

(c) $\frac{109}{150}$ *(d)* $\frac{88}{150} = \frac{44}{75}$

(e) $\frac{8}{41}$ *(f)* $\frac{59}{84}$

10. *(a)*



(b) 3

(c) $\frac{39}{75} = \frac{13}{25}$

(d) $\frac{45}{120} \times 100 = 37.5\%$

11. *(a)*

	Shirts	Jumpers	Total
Top	5	25	30
Middle	30	20	50
Bottom	10	10	20
Total	45	55	100

(b) $\frac{1}{5}$ *(c)* $16.6\dot{}\%$

12. *(a)* 23 *(b)* 49 *(c)* 7 *(d)* 65

13. *(a)*

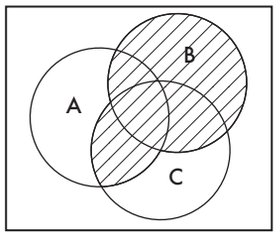
	Like Hash Browns	Dislike Hash Browns	Total
Like Hot Chips	18	28	46
Dislike Hot Chips	40	4	44
Total	58	32	90

(b) 46

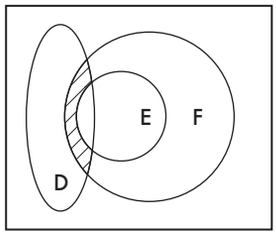
14. *(i)* $\frac{26}{45}$ *(ii)* $\frac{1}{3}$ *(iii)* $\frac{37}{45}$ *(iv)* $\frac{1}{10}$

(v) $\frac{9}{16}$ *(vi)* $\frac{43}{74}$

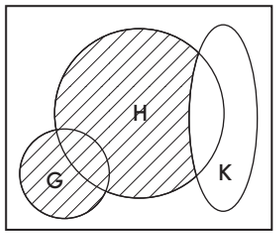
15. *(a)*



(b)



(c)



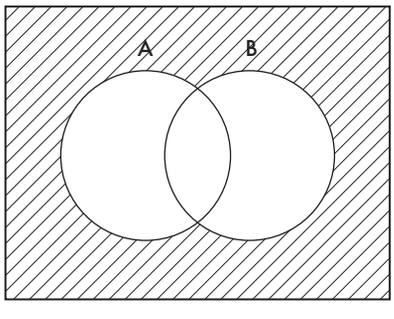
16. *(a)* $F = \{1, 2, 3, 5, 8, 13, 21, 34\}$

(b) $P = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31\}$

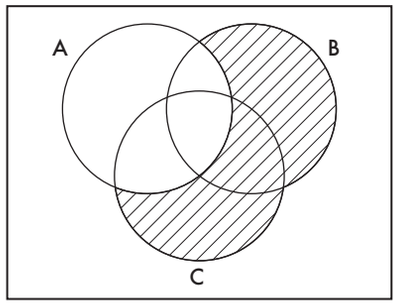
(c) Proof

(d) *i.* $\frac{4}{11}$ *ii.* $\frac{1}{2}$ *iii.* $\frac{4}{16}$

17. *(a)*



(b)



18. *(a)* $(B \cup C) \cap (A \cup B)$

(b) $(A \cap B) \cup A'$

19. $z = 0.1,\ x = 0.4,\ y = 0.3$

20. *(a)* $x = 0.1,\ y = 0.2,\ z = 0.2$

(b) 0.6

(c) $\frac{3}{8}$

(d) For independence

$P(A \cap B) = P(A) \times P(B)$

$0.1 \neq 0.5 \times 0.6$

$\therefore$ Events A and B are not independent.

21. *(a)* 0.054

(b) 0.056

(c) 0.348

22. *(a)* For independence

$P(MrB \cap MrsB) = P(MrB) \times P(MrsB)$

$0.09 \neq 0.39 \times 0.39$

$\therefore$ Events A and B are not independent.

(b) 0.31

23. $P(A') \times P(B) + P(A) \times P(B')$

$= (1 - x)y + x(1 - y)$

$= y - xy + x - xy$

$= x + y - 2xy$

24. *(a)* 21

(b) $\frac{36}{100}$

(c) $\frac{21}{90}$

(d) $\frac{78}{100}$

25. Yes A and B are independent. $P(A \cap B) = 0.3 = P(A) \times P(B)$

26. $P(A/B) \times P(B) = P(A \cap B)$

$0.52 \times 0.43 = P(A \cap B)$

$0.2236 = P(A \cap B)$

$P(A \cap B) = P(A) \times P(B)$

If A and B are independent then

$0.2236 = P(A) \times P(B)$

$P(A) = 0.52 = P(A/B)$

27. $P(A) = \frac{1}{4}$, $P(B) = \frac{1}{13}$

P(Queen of Hearts) $= \frac{1}{52}$

$P(A \cap B) = \frac{1}{4} \times \frac{1}{13} = \frac{1}{52}$

$\therefore$ A and B are independent events

28. *(a)* $\frac{3}{10}$

(b) $\frac{7}{10}$

(c) $\frac{3}{10}$

CHAPTER 7: *Exponents and Exponential Functions*

1. *(a)* 2.945×10^5

(b) 4.96×10^{-4}

(c) 1.37×10^2

(d) 1.724×10^{-6}

(e) 9.6×10^1 or 9.6×10

(f) 1.94×10^{-1}

(g) 4.2786×10^1 or 4.2786×10

(h) 8.72×10^{-8}

2. *(a)* 0.001478

(b) 249 760 000

(c) 0.000000952

(d) 804 100

(e) 71 890 700

(f) 5 414 200

(g) 0.4591

(h) 0.0003207

3. *(a)* 25.9

(b) 11.23

(c) 17.22

(d) 211.2

(e) 43.2

(f) 6.08

(g) 23

(h) 4.829

4. *(a)* b

(b) $4x^6$

(c) $\frac{27m^6}{8n^3}$

(d) $\frac{b^4}{4a^4}$

(e) $x^{\frac{1}{4}}$

(f) $xy^{\frac{3}{2}}$

(g) $\frac{1}{3^4}$ or $\frac{1}{81}$

(h) x^3

(i) y

(j) $\frac{6}{x}$

(k) a^7b^4

(l) $\frac{1}{b^4}$

5. *(a)* 32

(b) $\frac{1}{9}$

(c) $\frac{1}{4}$

(d) 1000

(e) $\frac{3}{2}$

6. *(a)* 4

(b) –3

(c) $\frac{3}{4}$

(d) $1\frac{1}{3}$

(e) 2

(f) $-\frac{2}{3}$

7. *(a)* $\frac{1}{6}$

(b) ±8

(c) 16

(d) $\frac{3}{5}$

(e) 2

(f) 511 or –513

8. (a) $\frac{c^4}{b^2}$

(b) $\frac{b^{\frac{1}{2}}}{a^8}$

(c) c^5

(d) $\frac{n^3}{m^5}$

(e) $a^{\frac{3}{2}}$

(f) $10b^3$

9. (a) $2^{0.5}$

(b) $3^{0.7}$

10. (a) 3^{6-n}

(b) $2^{n+3}\ x^{4n+2}$

11. (a) $b(1 + b^2)$

(b) $3(1 - 3^n)$

(c) $2^n + 2$

(d) $5\frac{1}{5}$

(e) $\frac{3}{5}$

(f) 2^n

12. -3

13. (a) Moved down 4 units

(b) Reflection in x-axis, move up 1 unit

(c) Move right 1 unit, move down 5 units

(d) Reflection in the y-axis, move up 3 units

(e) Reflection in x-axis, move left 2 units

(f) Reflection in x-axis, move left 5 units, move up 4 units

14. (a) $y = 2^x - 3$

(b) $y = 3^x + 2$

(c) $y = 2^{x-1} - 3$

(d) $y = 3^{x+2} - 2$

15. (a) 97 mm

(b) 2 years 9 months

16. (a) $\approx$ 268 627 people

(b) During 2031

17. (a) $x = 2$

(b) $x = \frac{5}{2}$

(c) $x = 1$

(d) $x = 1$

(e) $x = 0, x = 1$

(f) $x = 0, x = 1$

18. (a) Increasing

(b) 20

(c) 700%

19. (a) $\frac{4a^7}{b^2}$

(b) $\frac{-243m^2n}{25}$

(c) $\frac{2g^2}{f}$

(d) $\frac{-72a^{13}}{b^5}$

(e) $\frac{4d^4}{e^8}$

(f) $\frac{5pq^9}{2}$

20. $x = 1.69$ and $x = -1.39$

21. (a) $\frac{1}{a^2}$

(b) $\frac{z^2}{3x^4y^2}$

(c) $1 + a^2$

(d) $\frac{8x^2}{y^2}$

(e) $\frac{1}{x^3}$

(f) $\frac{25a^2b^2}{16}$

22. Graph A is $y = 4^x$ and

Graph B is $y = 3^x$ as $y = 4^x$ grows at a faster rate than $y = 3^x$ because 4 is greater than 3.

CHAPTER 8: *Sequences and Recursion*

1. (a) A

(b) N

(c) G

(d) G

(e) N

2. 5, 9, 13, 17, 21; 99

3. *(a)* 138

(b) 378

(c) 3168

4. 3, –6, 12, –24, 48; –1, 572, 864

5. $a = 3,\ r = 4$ or $a = -3,\ r = -4$.
$T_{n+1} = 4T_n\ ;\ T_1 = 3$ or $T_{n+1} = -4T_n\ ;\ T_1 = -3$

6. 7 or 8

7. *(a)* 82

(b) 63.4

8. 1664

9. $n = 10$

10. 921.375

11. 2.2 or –2.2

12. 81

13. *(a)* \$168 406.59

(b) \$2 114 799.42

14. *(a)* 2.85

(b) 11

15. *(a)* proof; $r = 0.74$

(b) 89 756

16. 10.6%

17. *(a)* 26

(b) 13

(c) 576 (573 if rounding is done after each day)

(d) 30 days

18. *(a)*

Month	Owing at Start of Month	Interest Owing for Month	Now Owing	Monthly Repayment	Owing at End of Month
1	12 000	120	12 120	2500	9620
2	9620	96.20	9716.20	2500	7216.20
3	7216.20	72.16	7288.36	2500	4788.36
4	4788.36	47.88	4836.24	2500	2336.24
5	2336.24	23.36	2359.60	2359.60	0
6					

(b) 5

(c) \$2359.60

(d) \$359.60

19. *(a)* $P = 80(1.5)^n$

(b) 10379

(c) Lack of fresh water, lack of food

(d) Graph

20. *(a)* $D_1 = 0.63$ units, $D_2 = 0.70$

(b)

n	0	1	2	3	10	25
D_1	0.63	0.54	0.46	0.39	0.33	0.28

(c)

n	0	1	2	3	10	25
D_1	0.70	0.55	0.43	0.33	0.26	0.20

(d) D_1 decreases by 15%,
D_2 decreases by 23%

21. *(a)* *i.* $A = \$1100$
ii. $B = \$1051$
iii. 2% per month
iv. 1% per month

(b) At the end of September 2014

22. *(a)* 0.5%

(b) 6%

(c)

17	153 417.23	767.09	2500	151 684.32
18	151 684.32	758.42	2500	149 942.74

(d) \$14 942.74

(e) $A_{n+1} = (A_n \times 1.005) - 2500,\ T_1 = 180\,000$

23. 43.2

24. 301.321984

25. 16, 12, 9

26. 4

27. (a) 193 m

(b) $\therefore$ 200 m

28. (a) 0.01125 mg

(b) $T_n = 90(0.05)^{n-1}$

(c) 94.74 mg

CHAPTER 9: *Introduction to Differential Calculus*

1. (a) $\frac{dy}{dx} = 16x + 5$

(b) $f'(x) = \frac{-4}{x^3} - \frac{3}{x^2}$

(c) $f'(x) = 4x^3 - 18x^2 - 12x - 80$

(d) $f'(x) = -\frac{1}{x^2} + \frac{5}{2\sqrt[3]{x}}$

2. (a) $3x + 8$

(b) $12x^2 + 6x + \frac{10}{x^3}$

3. Proof.

4. (a) $(a, 0), (c, 0), (e, 0)$

(b) $(b, k), (d, h)$

(c) (b, k) to (d, h)

(d) $x < b, x > d$

5. (a) 2

(b) 6

6. (b) $15x^2$

7. (a) $8x^3$

(b) $-10x$

(c) 5

(d) 0

(e) $\frac{-2}{x^2}$

(f) $\frac{2.5}{\sqrt{x}}$

(g) $\frac{8}{x^3}$

(h) $\frac{1}{\sqrt{x^3}}$

(i) $\frac{1}{3\sqrt[3]{x^2}}$

8. (a) 3

(b) $2x + 2$

(c) $12x^2 - 1$

(d) $2x^3 + 9x^2$

(e) $1 - \frac{1}{2\sqrt{x}}$

(f) $1 + \frac{2}{x^2}$

(g) $2x - \frac{4}{x^3}$

(h) $\frac{x}{2} + \frac{1}{2}$

9. (a) $6t$

(b) $-\frac{5}{v^2} + 1$

(c) $-\frac{6}{u^3}$

(d) $\frac{1}{\sqrt{n}}$

10. (a) $18x$

(b) $-\frac{4}{x^3}$

(c) $-\frac{1}{4\sqrt{x^3}}$

(d) $x - 4$

(e) $\frac{10}{27x^{\frac{1}{3}}}$

11. Proof.

CHAPTER 10: *Applications of Differential Calculus*

1. (a) 1

(b) -3.75

2. (a) (2, 11) and (-2, -5)

(b) $y = 12x - 13$ and $y = 12x + 19$

3. $x = \frac{10}{9}, 0$

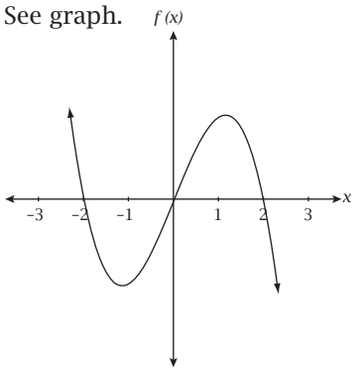
4. 1102.5 m

5. Length = Width = 17.5 cm

6. 768 cm^2

7. $a = 2, \quad y = -6x + \frac{67}{27}$ or $27y = -162x + 67$

8. See graph.



9. *(a)* $l = 10 - \frac{7h}{4}$

(b) Proof.

(c) 36.3 cm^3

(d) 2.86 cm

10. $a = -1$, $b = 4$

11. *(a)* 300

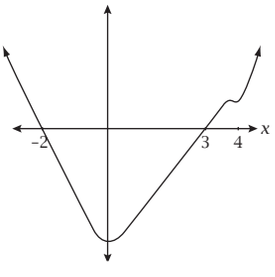
(b) 540 per hour

(c) $N'(t) = 15t^2 + 8t$

(d) *i.* 113.75
ii. 8832

12. $a = -3$, $b = 1$, $c = 2$

13.



14. Calculator

15. $y = 5x - 9$

16. $m_1 = -1$, $m_2 = 1$ $m_1 . m_2 = -1$ perpendicular

17. *(a)* 1.55 seconds

(b) 0.82 s/m

18. *(a)* 1500 watts

(b) –30 watts/amp

19. *(a)* 6 m to left of origin

(b) 2 m/s

(c) 6 m/s^2

(d) –1 m/s

(e) $t = 3$

20. *(a)* $t = 1, 4$

(b) 49 m

21. *(a)* 0 m/s^2

(b) $t = 1, 2, 4, 5$

22. *(a)* $R(x) = 150x$

(b) $P(x) = 150x - 2000 - 100\sqrt{x}$

(c) Loss $816.23

(d) $C'(x) = \frac{50}{\sqrt{x}}$

(e) $P'(x) = 150 - \frac{50}{\sqrt{x}}$

(f) $P'(10) = 134.19$

23. *(a)* $P(x) = 40x - 2500 + \frac{1000}{x+1}$

(b) *i.* $P'(99) = 39.9$
profit of 100th unit $39.90

ii. $C'(99) = 0.1$
cost of 100th unit $0.10

iii. $R'(99) = 40$
revenue of 100th unit $40

24. *(a)* $x < a$, $x > c$

(b) $(a, 0)$ relative max
(c, d) relative min

(c) $(0, b)$

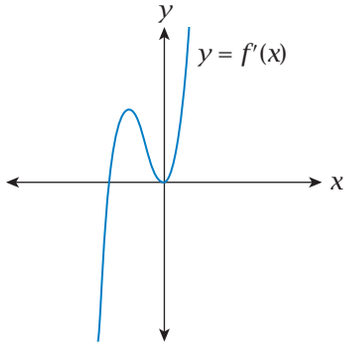
(d) $x < 0$

25. *(a)* $a < x < c$, $x > f$

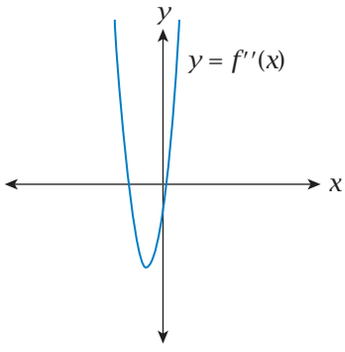
(b) $x = a$ rel min, $x = c$ rel max,
$x = f$ rel min

(c) $x = 0$, $x = d$

26. *(a)*



(b)

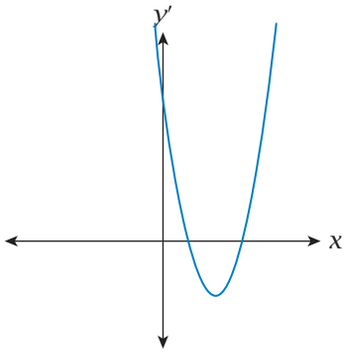


27. *(b)* A (0, –5) B (1, –1) C (2, –3)
D (3, –5) E (4.1, 0)

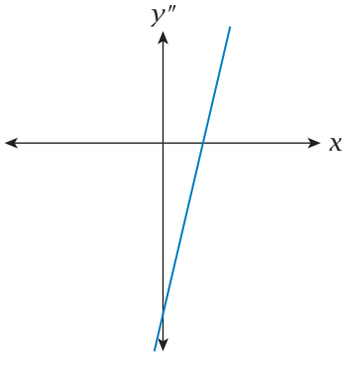
(c) B rel max, D rel min

(d) C

(e)

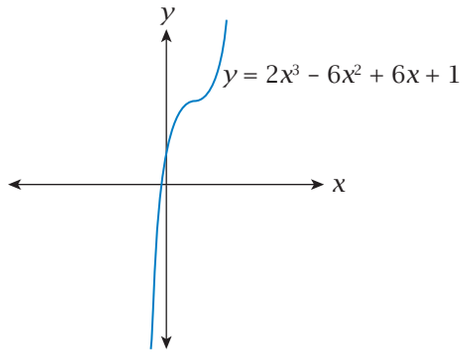


(f)



28. *(a)* (1, 3) horizontal point of inflection

(b)



29. 8 sq units

30. *(a)* 2750

(b) \$95

31. Produce 210 38% rejected 130.2 suitable

32. *(a)* 2 at (1, 2)

(b) no relative max

(c) 2 at (1, 2)

(d) 4 at (2, 4)

33. at $x = \frac{a}{3}$

34. $2\frac{1}{3}$ m

35. A–II, B–III, C–V, D–VI, E–IV, F–I

SOLUTIONS TO TRIAL TESTS

TRIAL TEST 1: *Functions I*

1. *(a)* Using the quadratic formula for $y = x^2 + 6x - 7$ we obtain:

$$\frac{-6 \pm \sqrt{6^2 - 4(1)(-7)}}{2(1)}$$ ✓

From the discriminant, we can see that there will be two rational solutions. ✓

(b) *i.* If $y = x^2 + 6x - 7$, then the y intercept is when $x = 0$.

So, $y = 0^2 + 6(0) - 7$
$= -7$ ✓

ii. We can factorise this expression and equate it to zero to find the x intercepts.

Thus, $0 = (x + 7)(x - 1)$
and hence $x = -7, 1$.

So, the x intercepts are $(-7, 0)$ and $(1, 0)$. ✓

iii. To obtain the turning point for this expression, we can complete the square.

If $y = x^2 + 6x - 7$
$= (x + 3)^2 - 9 - 7$
$= (x + 3)^2 - 16$ ✓

Thus, the turning point is $(-3, -16)$. ✓

2. *(i)* Narrower ✓

(ii) inverted ✓

(iii) translated to the left ✓

(iv) translated 4 units down ✓

3. $y = mx + c$,

so $C = mh + c$
$285 = 5m + c$ ①
and $645 = 13m + c$ ② ✓

Subtracting ① from ② gives

$8m = 360$
$m = 45$

Substituting into ① gives $c = 60$ ✓

Thus Cost $= 45(6.5) + 60$
$= \$352.50$ ✓

4. If $y = \frac{x}{2} + 2$

Then a line parallel to y would have gradient of $\frac{1}{2}$.

Thus, $y = \frac{x}{2} + c$ ✓

$-5 = \frac{1}{2}(-3) + c$

and $c = \frac{13}{2}$ ✓

So $y = \frac{x}{2} + \frac{13}{2}$ or $2y = x + 13$. ✓

5. The gradients of each of the lines are:

$A = 2$, $B = -\frac{1}{2}$, $C = 1$, $D = -2$,
$E = \frac{1}{2}$ and $F = 1$ ✓✓

(i) the parallel lines are C and F ✓

(ii) the perpendicular lines are A and B ✓ and also D and E ✓

6. *(a)* The co-ordinates of B can be found by finding the intersection of the lines $3y = 8x + 31$ and $y = -4x - 3$
B is the point $(-2, 5)$ ✓

(b) The midpoint of AC is $D(-3, -1)$ ✓

the gradient of $BD = \frac{5 - (-1)}{-2 - (-3)} = 6$ ✓

now $y = mx + c$
$5 = 6(-2) + c$
$c = 17$

The equation of the median is $y = 6x + 17$ ✓✓

7. If angle ABC is a right angle, then AB is perpendicular to BC

$m_{AB} = \frac{5 - (-3)}{k - (-6)} = \frac{8}{k + 6}$ ✓

$m_{BC} = \frac{5 - 2}{k - 8} = \frac{3}{k - 8}$ ✓

Since the lines are perpendicular,

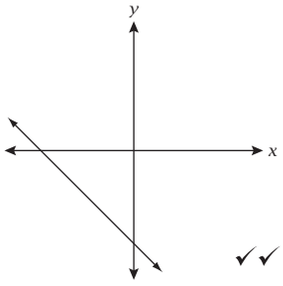
$$\frac{8}{k + 6} = -\frac{k - 8}{3}$$ ✓✓

$24 = -(k^2 - 2k - 48)$

$k^2 - 2k - 24 = 0$

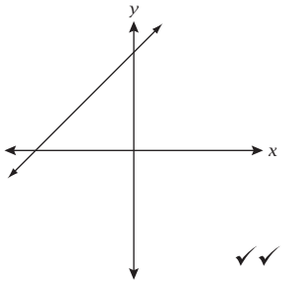
$k = -4$ or 6 ✓✓

8. *(a)* $y = mx - c$



✓✓

(b) $y = -mx + c$



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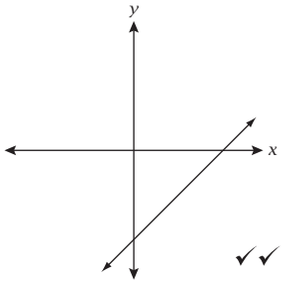
(c) $y = -(mx + c)$



✓✓

TRIAL TEST 2: *Functions II*

1. *(a)* Where $f(x) = 0$, the cubic function cuts the x axis. This happens at $x = -4.5, -2$ and 1. ✓

(b) Where $f(x) \geq 0$, the cubic function appears above the x axis. The parts of the domain where this happens are $-4.5 < x < -2$ and $x > 1$ ✓

2. Graph A is quadratic with a turning point of $(-2, -3)$. Therefore, graph 7. ✓

Graph B is cubic with a turning point of $(3, 2)$. Therefore, graph 11. ✓

Graph C is hyperbolic/reciprocal and passes through $(2, 1)$. Therefore, graph 12. ✓

Graph D is a square root function with turning point $(2, -4)$. Therefore, graph 4. ✓

3. *(i)* If $y = -2(x + 3)^2 - 4$, then the equation of the line of symmetry is $x = -3$ ✓

(ii) The turning point is $(-3, -4)$. This is a maximum point, due to the negative coefficient of the function. ✓✓

(iii) The y intercept is where $x = 0$.
So, $y = -2(0 + 3)^2 - 4$
$= -2(9) - 4$
$= -22$, thus $(0, -22)$. ✓

(iv) For x intercepts, use the quadratic formula for equation $y = -2x^2 - 12x - 22$

Thus,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{12 \pm \sqrt{-12^2 - 4(-2)(-22)}}{2(-2)}$$

The discriminant becomes $144 - 176 = -32$, and therefore there are no x intercepts. ✓

4. *(a)* $x - 6 = 0$
Thus $x = 6$ (vertical)
and $y = 0$ (horizontal). ✓

(b) $4x - 3 = 0$

Thus $x = \frac{3}{4}$ (vertical) ✓

and $y = 2$ (horizontal). ✓

5. *(i)* translate 1 unit left and 2 units down ✓✓

(ii) translate 5 units left ✓

(iii) dilate vertically by a factor of 3 and translate 1 unit up ✓✓

(iv) translate 2 units left and 3 units down ✓✓

6. *(a)* $y = (x - 2)^2 - 3$ ✓✓

(b) $y = \frac{1}{x - 3}$ ✓✓

(c) $y = 2^x - 3$ ✓✓

(d) $y = (x + 2)^3 - 4$ ✓✓

TRIAL TEST 3: *Sequences, Recursion and Exponents*

1. *(a)* $A = 12\,000(1.008)^{12}$
$= \$13\,204.06$ ✓✓

(b) $15\,000 = 12\,000(1.008)^n$
$1.25 = 1.008^n$ ✓
$n = 28$, and thus $n = 7$ years ✓

2. *(a)* *i.* $T_n = 300(0.993)^n$
$T_1 = 300(0.993)^1$
$= 297.9$ ✓

ii. $T_{20} = 260.68$ ✓

iii. $T_{100} = 148.61$ ✓

iv. $T_n = 300(0.993)^n$ ✓

(b) $150 = 300(0.993)^n$
$0.5 = 0.993^n$
$n = 98.67$ years ✓✓

3. *(a)* This sequence is arithmetic due to the common difference being 14. ✓

$T_n = a + (n - 1)d$
$= 20 + (n - 1)14$
$= 20 + 14n - 14$
$= 14n + 6$ ✓

(b) $T_{21} = 14(21) + 6$
$= 300$ ✓

(c) $S_n = \frac{n}{2}[2a + (n - 1)d]$

$S_{15} = \frac{15}{2}[2(20) + (15 - 1)\,14]$ ✓

$= 1770$ ✓

4. *(a)* If $T_{n+1} = 3T_n$, then

$T_{1+1} = 3T_1$ ✓

$T_2 = 3(14)$

$T_2 = 42$ ✓

$\frac{T_2}{T_1} = r$, so $\frac{42}{14} = 3$.

Therefore, $T_n = 14(3)^{n-1}$ ✓

(b) $T_5 = 14(3)^{5-1}$

$= 14(3)^4$

$= 1134$ ✓

(c) If $T_n = 14(3)^{n-1}$

then $1000 > 14(3)^{n-1}$ ✓

$\frac{1000}{14} > 3^{n-1}$

$71.43 > 3^{n-1}$

$4.89 > n$ ✓

$5 = n$ ✓

5. *(a)* $P = 5000,\ r = 1.0725,\ t = 4$

(Option A)

S.I. $= \frac{PRT}{100}$

$= 5000(1.0725)(4)$

$= \$1450$

Amount $= 5000 + 1450$

$= \$6450$ ✓

(Option B)

$A = P(1 + r)^t$

$= 5000(1.0725)^4$

$= \$6615.45$

Option B is the best choice by
$6615.45 - 6450 = \$165.45$ ✓

(b) Half-yearly

$A = 5000(1 + \frac{0.0725}{2})^{4(2)}$

$= 5000(1.03625)^8$

$= \$6647.93$ ✓

Quarterly

$A = 5000(1 + \frac{0.0725}{4})^{4(4)}$

$= 5000(1.018125)^{16}$

$= \$6664.81$ ✓

Daily

$A = 5000(1 + \frac{0.0725}{365})^{4(365)}$

$= 5000(1.0002)^{1440}$

$= \$6681.95$

Therefore he should compound daily for a maximum return. ✓

(c) $A = P(1 + r)^t$;
where $A = 2900,\ P = 2500,\ t = 2$ ✓

Use SOLVE mode (CAS)
$r = 0.077 \times 100$
$= 7.7\%$ ✓

Yes, Geoff should change his bank as this interest rate is higher than at his own bank. ✓

6. *(a)* $\frac{(4a^2b^3)^3}{(-2ab^4)^3}$

$= \frac{4^3a^6b^9}{-2^3a^3b^{12}}$ ✓

$= \frac{64a^3}{-8b^3}$

$= \frac{-8a^3}{b^3}$ ✓

(b) $\frac{6x^{-3}y^{-4}}{2x^{-4}y^{-2}}$

$= \frac{3x^4y^2}{x^3y^4}$ ✓✓

$= \frac{3x}{y^2}$ ✓

(c) $\frac{8m^3n^{-6}}{4m^2n^{-5}} \div \frac{3n^2}{(4m^4)^2}$

$= \frac{2mn^5}{n^6} \times \frac{4^2m^8}{3n^2}$ ✓✓

$= \frac{32m^9n^5}{3n^8}$ ✓

$= \frac{32m^9}{3n^3}$ ✓

(d) $(8p^4q^2)^{-3} \times 16(p^5q^{10})^{\frac{2}{5}}$

$= 8^{-3}p^{-12}q^{-6} \times 16p^2q^4$ ✓

$= \dfrac{16p^2q^4}{512p^{12}q^6}$ ✓✓

$= \dfrac{1}{32p^{10}q^2}$ ✓

TRIAL TEST 4: *Trigonometry I*

1. (a) $\tan 45^\circ = \dfrac{y}{25}$

$25 \tan 45^\circ = y$

$25 = y$ ✓

$\tan 65^\circ = \dfrac{x+25}{25}$

$25 \tan 65^\circ = x + 25$

$28.61 = x$ ✓

(b) $\dfrac{9}{\sin 75^\circ} = \dfrac{y}{\sin 70^\circ}$

$y = \dfrac{9 \sin 70^\circ}{\sin 75^\circ}$

$= 8.76$ ✓

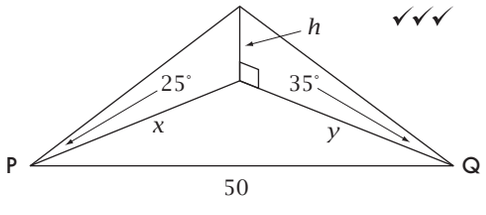
and $\dfrac{9}{\sin\theta} = \dfrac{8}{\sin 35^\circ}$

$\theta = 40.18^\circ$ ✓

finally, $\dfrac{x}{\sin 34.82^\circ} = \dfrac{8}{\sin 105^\circ}$

$x = 4.73$ ✓

2. A diagram helps to clarify the problem.



✓✓✓

$x^2 + y^2 = 50^2$

$\dfrac{h^2}{(\tan 25^\circ)^2} + \dfrac{h^2}{(\tan 35^\circ)^2} = 2500$ ✓

$h^2 = 376.21$

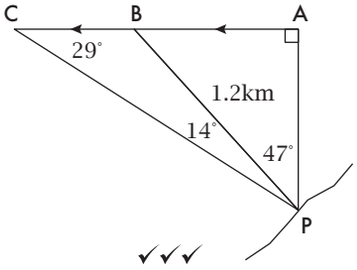
$h = 19.40\text{m}$ ✓✓

3. $360^\circ - 313^\circ = 47^\circ$

$360^\circ - 299^\circ = 61^\circ$

$61^\circ - 47^\circ = 14^\circ$

$90^\circ - 61^\circ = 29^\circ$



✓✓✓

$\dfrac{BC}{\sin 14^\circ} = \dfrac{1.2}{\sin 29^\circ}$ ✓

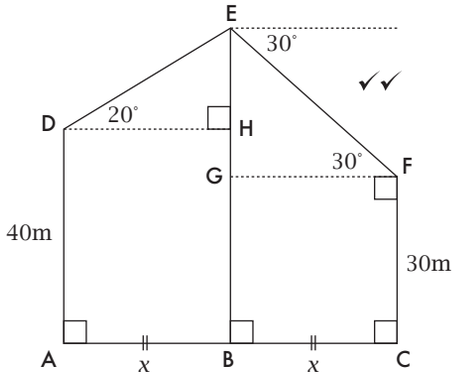
$BC = \dfrac{1.2 \sin 14^\circ}{\sin 29^\circ}$

$BC = 0.5988\text{km}$

The distance BC is 599m (nearest m) ✓

[5]

4.



✓✓

Let $AB = BC = x$ and G and H be the points as shown.

$\tan 20^\circ = \dfrac{EH}{x}$

$EH = x \tan 20^\circ$ ✓

$\tan 30^\circ = \dfrac{EG}{x}$

$EG = x \tan 30^\circ$ ✓

$BE = 40 + x \tan 20^\circ = 30 + x \tan 30^\circ$ ✓

$\therefore \quad 40 - 30 = x \tan 30^\circ - x \tan 20^\circ$

$10 = x(\tan 30^\circ - \tan 20^\circ)$

$x = \dfrac{10}{\tan 30^\circ - \tan 20^\circ}$ ✓✓

$x = 46.8647$ ✓

$AC = 2x$

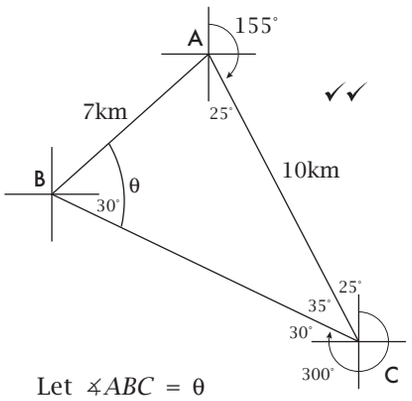
$= 2 \times 46.8647$

$= 93.7294$ ✓

So the distance between the two shorter masts is 93.73m.

[9]

5.



✓✓

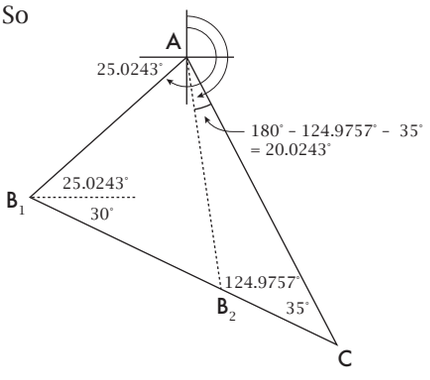
Let $\measuredangle ABC = \theta$

$$\frac{10}{\sin\theta} = \frac{7}{\sin 35^\circ} \checkmark$$

$$\sin\theta = \frac{10\sin 35^\circ}{7}$$

$\theta = 55.0243^\circ$ ✓ or $180^\circ - 55.0243^\circ$
$= 124.9757^\circ$ ✓

So



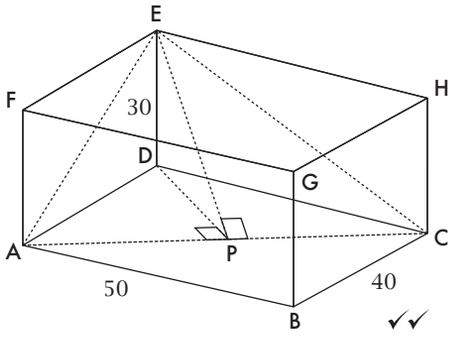
Bearing of B from A is either

$270^\circ - 25.0243^\circ$
$= 244.98^\circ$ ✓

or

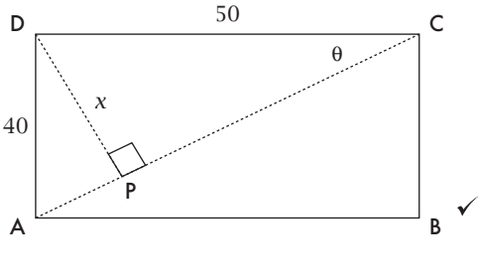
$155^\circ + 20.0243^\circ$
$= 175.02^\circ$ ✓

6.



✓✓

Let P be on AC such that $\measuredangle APD = 90^\circ$ and $\measuredangle APE = 90^\circ$.



✓

Let $\measuredangle ACD = \theta$

$$\tan\theta = \frac{40}{50}$$

$$\theta = 38.6598^\circ$$

$$\sin 38.6598^\circ = \frac{x}{50}$$

$$x = 50\sin 38.6598^\circ$$

$$x = 31.2347\text{cm} \checkmark\checkmark\checkmark$$

$$\tan\measuredangle EPD = \frac{30}{31.2347}$$

$$\therefore \measuredangle EPD = 43.84^\circ$$

$\therefore \Delta ACE$ makes an angle of 43.84° with the base. ✓ [7]

7. Two triangles possible $Q = 65^\circ$ or $Q = 180^\circ - 65^\circ = 115^\circ$ [6]

8. (All ticks ✓ are ½ mark for this question)

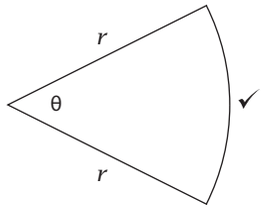
$$2r + l = 30 \checkmark$$

$$l = r\theta \checkmark$$

$$2r + r\theta = 30 \checkmark$$

$$r(\theta + 2) = 30 \checkmark$$

$$r = \frac{30}{\theta + 2} \text{ ①} \checkmark$$



✓

$$54 = \frac{1}{2} r^2\theta \checkmark$$

$$108 = r^2\theta$$

$$r^2\theta = 108 \text{ ②} \checkmark$$

Substitute ① into ② for r

$$\left(\frac{30}{\theta + 2}\right)^2 \theta = 108 \checkmark$$

$$\frac{30^2\theta}{(\theta + 2)^2} = 108 \checkmark$$

$$108(\theta + 2)^2 = 900\theta \checkmark$$

$$\theta^2 + 4\theta + 4 = \frac{900\theta}{108}$$

$$\theta^2 + 4\theta + 4 = 8\frac{1}{3}\theta \checkmark$$

$$\theta^2 - 4\frac{1}{3}\theta + 4 = 0$$

$$\theta_1 = 3^R \checkmark$$

$$r_1 = \frac{30}{3 + 2}$$

$$r_1 = 6\text{cm} \checkmark$$

$$\theta_2 = 1\frac{1}{3}{}^R \checkmark$$

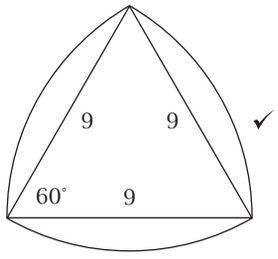
$$r_2 = \frac{30}{1\frac{1}{3} + 2}$$

$$r_2 = 9\text{cm} \checkmark$$

So first possible sector has a radius of 6cm and angle of 3^R. The second possible sector has a radius of 9cm and an angle of $1\frac{1}{3}^R$. ✓✓

[9]

9.



✓

Area of each segment

$$= \frac{1}{2}(9^2)\left(\frac{\pi}{3} - sin60^\circ\right)$$ (Area of a segment formula.)

$$= 7.3375\text{cm}^2$$ ✓✓

$$\text{Area } \Delta = \frac{1}{2} \times 9^2 \, sin60^\circ$$

$$= 35.0740$$ ✓

Required area

$$= 35.0740 + 3 \times 7.3375$$

$$= 57.09\text{cm}^2$$ ✓

TRIAL TEST 5: *Trigonometry II*

1. *(a)* $-\frac{1}{2}$ ✓

(b) $-\frac{1}{\sqrt{2}}$ ✓

(c) 1 ✓

(d) $-\frac{\sqrt{3}}{2}$ ✓

2. *(a)* a ✓

(b) $\frac{360}{b}$ ✓✓

(c) $\frac{c}{b}$ units right ✓✓

3. $\sin(A + B)$

$= \sin A\cos B + \cos A\sin B$ ✓

$= \frac{3}{5} \times \frac{5}{13} + \frac{4}{5} \times \frac{12}{13}$ ✓

$= \frac{3}{13} + \frac{48}{65}$ ✓

$= \frac{63}{65}$ ✓

4. A – (iii) B – (viii) C – (vii) D – (iv) [8 marks]

5. $a = 3, b = 2, c = \frac{\pi}{2}$ [4 marks]

TRIAL TEST 6: *Probability, Sets and Counting Techniques*

1.

	Attended the first day's play?		
	Yes	No	Totals
Male	7605	4095	11 700
Female	2395	905	3300
Total	10 000	5000	15 000

✓✓

(a) There are 11 700 males out of the possible 15 000. Therefore,

$$\frac{11\,700}{15\,000} = \frac{39}{50}$$ ✓

(b) 10 000 attended the First Day's play out of the 15 000 members. Therefore,

$$\frac{10\,000}{15\,000} = \frac{2}{3}$$ ✓

(c) 4095 males did not attend the First Day's play. Therefore,

$$\frac{4095}{15\,000} = \frac{273}{1000}$$ ✓

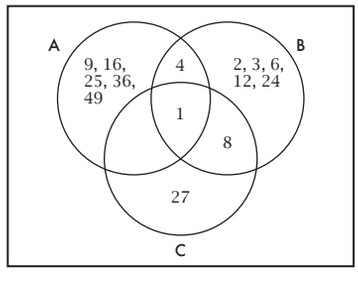
(d) Because we are dealing with those members who did not attend the First Day's play, we have a reduced sample space of 5000. Out of these, there are 905 females. Therefore,

$$\frac{905}{5000} = \frac{181}{1000}$$ ✓✓

[5]

2. *(a)* $A = \{1, 4, 9, 16, 25, 36, 49\}$, ✓
$B = \{1, 2, 3, 4, 6, 8, 12, 24\}$, ✓
$C = \{1, 8, 27\}$ ✓

(b)



✓✓✓

(c) *i.* The only element that intersects sets A, B and C is 1. Hence, $n = 1$. ✓

ii. The only element that intersects sets A', B and C is 8. Hence, $n = 1$. ✓

(d) *i.* $\{1, 4\}$ ✓

ii. $\{1, 4, 8, 9, 16, 25, 36, 49\}$ ✓

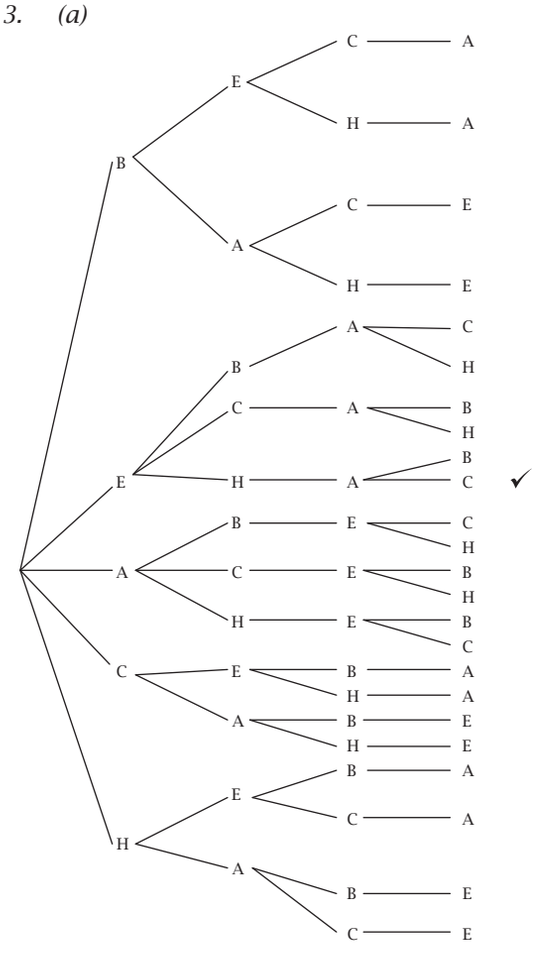
iii. $\{2, 3, 6, 8, 12, 24, 27\}$ ✓

(e) *i.* $\frac{42}{49} = \frac{6}{7}$ ✓

ii. $\frac{2}{49}$ ✓

iii. $\frac{2}{8} = \frac{1}{4}$ ✓

3. *(a)*



(b) $\frac{14}{24} = \frac{7}{12}$ ✓

(c) $\frac{8}{24} = \frac{1}{3}$ ✓

(d) 1 ✓

(e) $\frac{4}{24} = \frac{1}{8}$ ✓

(f) $\frac{2}{4} = \frac{1}{2}$ ✓✓

(g) $\frac{4}{8} = \frac{1}{2}$ ✓✓

4. $n = 13 + 8 = 21$ ✓✓

5. $(3 + 2x^2)^5$

the coefficient of $x^6 = \binom{5}{3} 3^2.2^3$

$= 720$ ✓✓✓✓

6. *(a)* $P(John) = \frac{1}{10}$ ✓

$P(Mary) = P(J\bar{M}) + P(\bar{J}M)$ ✓

$= \left(\frac{1}{10}\right)\left(\frac{1}{9}\right) + \left(\frac{8}{10}\right)\left(\frac{1}{9}\right)$ ✓

$= \frac{1}{10}$

(b) $\frac{17}{90}$ ✓✓

(c) $P(J / M) = \frac{P(J \cap M)}{P(M)}$ ✓

$= \frac{\frac{1}{90}}{\frac{1}{10}}$

$= \frac{1}{9}$ ✓✓

TRIAL TEST 7: *Differentiation*

1. *(a)* $N(2) = 50 + 45(2) - 7(2^2) + \frac{2^3}{4}$

$= 50 + 90 - 28 + 2$

$= 114$ ✓

$N(10) = 50 + 45(10) - 7(10^2) + \frac{10^3}{4}$

$= 50 + 450 - 700 + 250$

$= 50$ ✓

(b) $N'(t) = 45 - 14t + \frac{3t^2}{4}$

$N'(2) = 45 - 14(2) + \frac{3(2^2)}{4}$

$= 45 - 28 + 3$

$= 20$ ✓

$N'(10) = 45 - 14(10) + \frac{3(10^2)}{4}$

$= 45 - 140 + 75$

$= -20$ ✓

These values signify the rate of change of cows with respect to time at certain time intervals.

(c) Using CAS, and analysing both the function $N(t)$ and its derivative, $N'(t)$, the population of cows experiences negative growth over the domain $4.13 < t < 14.54$ (2d.p.). ✓✓

2. $y = \frac{1}{\sqrt{x}} + \frac{2}{x^3} - 2x$

at $x = 2$, $y = \frac{4 - 15\sqrt{2}}{4\sqrt{2}}$;

point $(2, y = \frac{4 - 15\sqrt{2}}{4\sqrt{2}})$ is on the curve ✓

$\frac{dy}{dx} = -\frac{1}{2x^{\frac{3}{2}}} - \frac{6}{x^4} - 2$ ✓

at $x = 2$, $\frac{dy}{dx} = -\frac{1}{2(2)^{\frac{3}{2}}} - \frac{6}{(2)^4} - 2$

$= -\frac{1}{4\sqrt{2}} - \frac{6}{16} - 2$

$= -\frac{1}{4\sqrt{2}} - \frac{3}{8} - 2$

$= -\frac{2}{8\sqrt{2}} - \frac{3\sqrt{2}}{8\sqrt{2}} - \frac{16\sqrt{2}}{8\sqrt{2}}$

$= -\frac{2}{8\sqrt{2}} - \frac{19\sqrt{2}}{8\sqrt{2}}$ ✓

So $y = \left(-\frac{2}{8\sqrt{2}} - \frac{19\sqrt{2}}{8\sqrt{2}}\right)x + c$

$\frac{4 - 15\sqrt{2}}{4\sqrt{2}} = \left(-\frac{2}{8\sqrt{2}} - \frac{19\sqrt{2}}{8\sqrt{2}}\right)(2) + c$

$\frac{4 - 15\sqrt{2}}{4\sqrt{2}} = -\frac{4}{8\sqrt{2}} - \frac{38\sqrt{2}}{8\sqrt{2}} + c$

$\frac{8 - 30\sqrt{2}}{8\sqrt{2}} = -\frac{4}{8\sqrt{2}} - \frac{38\sqrt{2}}{8\sqrt{2}} + c$

$\frac{12 + 8\sqrt{2}}{8\sqrt{2}} = c$ ✓

$\therefore y = \left(\frac{-2 - 19\sqrt{2}}{8\sqrt{2}}\right)x + \frac{12 + 8\sqrt{2}}{8\sqrt{2}}$ ✓

3. *(a)* $(6x)^2 + (8x)^2 = L^2$

$36x^2 + 64x^2 = L^2$

$100x^2 = L^2$

$\therefore 10x = L$ ✓

(b) $3W = 399 - (10x + 10x + 8x + 8x + 6x + 6x)$

$= 399 - 48x$

$W = 133 - 16x$ ✓

(c) Volume = Area of Base × Height

$= \frac{1}{2}(6x)(8x) \times (133 - 16x)$

$= -384x^3 + 3192x^2$ ✓✓

(d) $\frac{dV}{dx} = -1152x^2 + 6384x$ ✓

max./min. @ $\frac{dV}{dx} = 0$

$0 = -1152x^2 + 6384x$

$x = 5.54$ (2 d.p.) ✓

$\frac{d^2y}{dx^2} = -2304x + 6384$

$\frac{d^2y}{dx^2}$ @ $5.54 = -6380.16$ →

a maximum because $\frac{d^2y}{dx^2} < 0$ ✓

and Volume = $32\,675.51\text{cm}^3$ ✓

4. *(a)* $y = -4x^3 + 2x^2 + \frac{3}{x} - 7$

$y' = -12x^2 + 4x - \frac{3}{x^2}$ ✓✓

(b) $f(x) = 2(x + 1)^2$

$= 2(x + 1)(x + 1)$

$= 2[x^2 + 2x + 1]$

$= 2x^2 + 4x = 2$

$f'(x) = 4x + 4$ ✓✓

(c) $f(x) = \frac{-4}{x^3} + \frac{1}{x}$

$= -4x^{-3} + x^{-1}$

$f'(x) = 12x^{-4} - 1x^{-2}$

$= \frac{12}{x^4} - \frac{1}{x^2}$ ✓✓

5. $y = 2x^4 - 2x^2 - 4$

$= 2(x^4 - x^2 - 2)$

$= 2(x^2 - 2)(x^2 + 1)$

so if $0 = 2(x^2 - 2)(x^2 + 1)$

then $0 = (x^2 - 2)$ and $0 = (x^2 + 1)$

for the x intercepts

Thus, $x = \pm\sqrt{2}$ and the x intercepts are

$(\sqrt{2}, 0)$ and $(-\sqrt{2}, 0)$. ✓

The y intercept is at $(0, -4)$. ✓

$\frac{dy}{dx} = 8x^3 - 4x$;

stationary point(s) @ $\frac{dy}{dx} = 0$.

So $0 = 8x^3 - 4x$

$4x = 8x^3$

and $x = \pm\sqrt{\frac{1}{2}}$

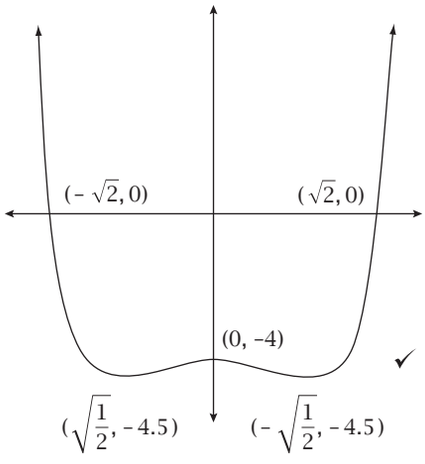
and the points are thus

$(\sqrt{\frac{1}{2}}, -4.5)$ and $(-\sqrt{\frac{1}{2}}, -4.5)$. ✓

$\frac{d^2y}{dx^2} = 24x^2 - 4$

Substituting $x = \pm\sqrt{\frac{1}{2}}$ gives a value of 8. ✓

Therefore both points are local minima.
Also, (0, –4) is a local maximum.



✓

TRIAL TEST 8: *Integration*

1. Area $= -\int_{-\sqrt{5}}^{\sqrt{5}} x^2 - 5dx$

$= -\left[\frac{x^3}{3} - 5x\right]_{-\sqrt{5}}^{\sqrt{5}}$ ✓

$= \left[\frac{5\sqrt{5}}{3} - 5\sqrt{5}\right] - \left[\frac{-5\sqrt{5}}{3} + 5\sqrt{5}\right]$ ✓

$= -\left(\frac{-20\sqrt{5}}{3}\right)$

$= \frac{20\sqrt{5}}{3}$ units^2 ✓

2. *(a)* $\int \frac{4}{x^3} dx$

$= \int 4x^{-3} dx$

$= \frac{4x^{-2}}{-2} + c$

$= -\frac{2}{x^2} + c$ ✓

(b) $\int \pi dx$

$= \pi x + c$ ✓

(c) $\int (2x - 1)^4 dx$

$= \frac{(2x - 1)^5}{(5)(2)} + c$

$= \frac{(2x - 1)^5}{10} + c$ ✓✓

(d) $\int 4x + x^2 - \frac{5}{3} dx$

$= \frac{4x^2}{2} + \frac{x^3}{3} - \frac{5}{3}x + c$ ✓

$= 2x^2 + \frac{x^3}{3} - \frac{5}{3}x + c$ ✓

(e) $\int \frac{2\sqrt{x} + 5x}{2x} dx$

$= \int \frac{2\sqrt{x}}{2x} + \frac{5x}{2x} dx$ ✓

$= \int x^{\frac{-1}{2}} + \frac{5}{2} dx$

$= 2x^{\frac{1}{2}} + \frac{5}{2}x + c$ ✓

3. $f'(x) = 5x^{\frac{-1}{2}} + 4x$

$f(x) = \int 5x^{\frac{-1}{2}} + 4x\, dx$

$= 10x^{\frac{1}{2}} + 2x^2 + c$ ✓

$f(4) = 10(4^{\frac{1}{2}}) + 2(4^2) + c$ ✓

$0 = 20 + 32 + c$

$-52 = c$

$\therefore f(x) = 10x^{\frac{1}{2}} + 2x^2 - 52$ ✓

4. *(a)* $\int_0^4 (4x + 1)dx$

$= \left[2x^2 + x\right]_0^4$ ✓

$= \left[2(4^2) + 4\right] - [0]$

$= 36$ ✓

(b) $\int_1^9 2\sqrt{x} + \frac{2x}{5} dx$

$= \left[\frac{4x^{\frac{3}{2}}}{3} + \frac{2x^2}{10}\right]_1^9$ ✓

$= \left[\frac{4(9)^{\frac{3}{2}}}{3} + \frac{2(9)^2}{10}\right] - \left[\frac{4(1)^{\frac{3}{2}}}{3} + \frac{2(1)^2}{10}\right]$

$= \frac{108}{3} + \frac{162}{10} - \frac{4}{3} - \frac{2}{10}$

$= 50\frac{2}{3}$ ✓

5. $y = 2x^3 - 2x^2 - 4x$

x intercepts @ $x = -1, 0, 2$ ✓

$\therefore \int_{-1}^{0} (2x^3 - 2x^2 - 4x)dx - \int_0^2 (2x^3 - 2x^2 - 4x)dx$ ✓

$= \left[\frac{x^4}{2} - \frac{2x^3}{3} - 2x^2\right]_{-1}^{0} - \left[\frac{x^4}{2} - \frac{2x^3}{3} - 2x^2\right]_0^2$ ✓

$= \left[(0) - \left(\frac{1}{2} + \frac{2}{3} - 2\right)\right] - \left[\left(8 - \frac{16}{3} - 8\right) - (0)\right]$

$= \frac{37}{6}$ units^2 ✓

6. If $\int_1^a 3x^2 + 2x\,dx = 34$

then $\left[\frac{3x^3}{3} + \frac{2x^2}{2}\right]_1^a = 34$ ✓

i.e. $\left[x^3 + x^2\right]_1^a = 34$

$a^3 + a^2 - 1 - 1 = 34$ ✓

$a^3 + a^2 - 36 = 0$

Using solve we can obtain $a = 3$ ✓

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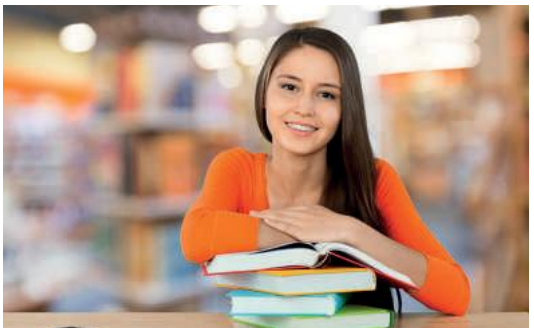

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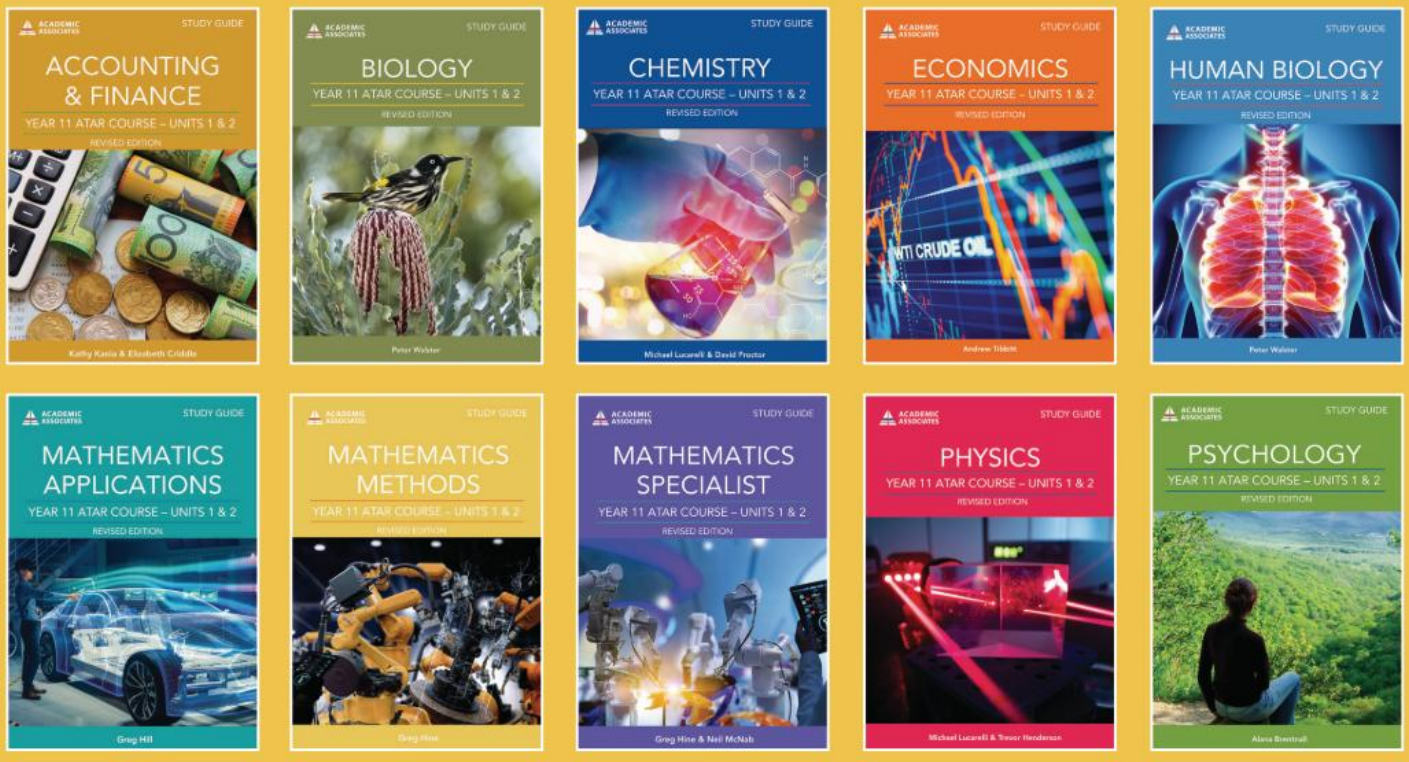

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