nelson maths.

11

Mathematical Methods

Units **1** & **2**

Lachlan Grierson
Sam Hodgson
Anna Wethereld

Series editor
Stephen Swift



QLD

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Contributing authors: Ross Brodie, Regina Edwards, Stephen Swift

LEARNING DISCOVERY

TRIGONOMETRIC FUNCTIONS



Many naturally occurring events follow a cyclic pattern, meaning that they repeat themselves after a cycle; for example, a town's daily temperature over a year, the height of tides in the ocean over a day, even the amount of road traffic outside a school. Such phenomena can be modelled by trigonometric functions, such as the sine function.

QLD

Nelson

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9780170484909

Preface

Nelson Maths 11–12 QLD is a new QCE mathematics series that is backed by research into the science of learning. The design and structure of the series has been informed by teacher advice and evidence-based pedagogy, with the focus on preparing QCE students for their exams and maximising their learning achievement.

- Using **backwards learning design**, this series has been built by analysing past QCE exam questions and ensuring that all theory and examples are precisely mapped to the QCAA syllabus and exams.
- To reduce the **cognitive load** for 21st century learners, explanations are clear and concise using the technique of **chunking** text with accompanying diagrams and infographics.
- The exercise structure of **Recap**, **Mastery** and **Exam practice** leads students from procedural fluency to **higher-order thinking** using the learning techniques of **interleaving**.
- **Exam practice** includes exam-style questions and in the Year 12 book past QCE exam questions with **examiner advice and feedback.**
- The cumulative structure of Exercise **Recaps** and **Cumulative examinations** in every chapter is built on the learning and memory techniques of **spacing** and **retrieval,** as recommended by the new syllabus.

About the authors

Lachlan Grierson has taught Mathematics and Information Technology in schools and at TAFE for over 20 years, most recently at Emerald and Warwick Christian Colleges. He was a QCAA Panellist for Information Processing Technology on the Gold Coast, and recently wrote topic tests for *Nelson Maths 7–10*.

Sam Hodgson teaches at Brisbane State High School. He has taught Mathematical Methods and marked its external exam extensively since it was introduced in 2019. Working at a large academic school, he also teaches Specialist Mathematics, is involved in Fast-Track Mathematics programs and coordinates the Year 10 Mathematical Methods Preparation course.

Anna Wethereld has been a leader in schools, curriculum, assessment and teacher training for over 25 years, including at Brisbane State High School, QCAA and Queensland College of Teachers. She was a panellist for Mathematics C (Specialist Maths) for Brisbane North, a Head of Mathematics and Deputy Principal. As the state's Mathematics Curriculum Manager at NESA, Anna led Mathematics K–12 in NSW and coordinated the development of its new Years 11–12 syllabuses.

Series editor **Stephen Swift** taught Mathematics, Science and Computing in urban and regional schools, and was Head of Mathematics at Wellington Point State High School. He was the lead author of *Nelson QMaths 7–12* and *Nelson Senior Maths 11–12* (Australian Curriculum).

Contributing authors

Amy Hawke and **Tara Kuhn** wrote many of the worksheets.
Deborah Da Cruz, **Scott Smith** and **Cherylanne Saywell** created the video tutorials.
Joel Speranza created the *Exam question analysis* videos.
Tom Sprenger, **Tania Eastcott** and **Elizabeth Nabhan** wrote the topic tests.
With thanks to **Jeremy Hunter** and **George Higgins** for reviewing the text.

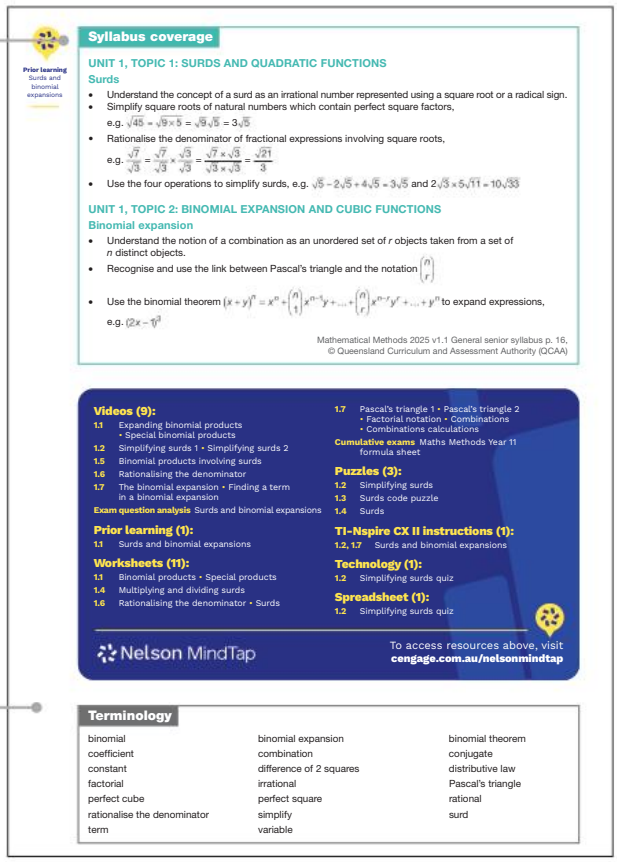
Syllabus reference grid

Topics and subtopics	*Nelson Maths 11 Mathematical Methods QLD* chapter	
UNIT 1: SURDS, ALGEBRA, FUNCTIONS AND PROBABILITY		
1 Surds and quadratic functions		
Surds	1	Surds and binomial expansions
Quadratic functions	2	Quadratic functions
2 Binomial expansion and cubic functions		
Binomial expansion	1	Surds and binomial expansions
Cubic functions	3	Functions and relations
3 Functions and relations		
Introduction to functions and relations	3	Functions and relations
Graphs of relations	6	Further functions and relations
Reciprocal functions	6	Further functions and relations
4 Trigonometric functions		
Circular measure and radian measure	4	Trigonometric functions
Introduction to trigonometric functions	4	Trigonometric functions
5 Probability		
Language of events and sets	5	Probability
Conditional probability and independence	5	Probability
UNIT 2: CALCULUS AND FURTHER FUNCTIONS		
1 Exponential functions		
Indices and index laws	8	Exponential functions
Introduction to exponential functions	8	Exponential functions
2 Logarithms and logarithmic functions		
Logarithms and logarithmic laws	9	Logarithms
Logarithmic functions	11	Logarithmic functions
3 Introduction to differential calculus		
Rates of change and the concept of derivatives	7	Rates of change and the derivative
	10	Applications of derivatives
4 Applications of differential calculus		
Graphical applications of derivatives	10	Applications of derivatives
5 Further differentiation		
Differentiation rules	12	Differentiation rules

About this book

At the beginning of each chapter

Syllabus coverage and extracts are shown at the front of the chapter, along with a listing of **Nelson MindTap** chapter resources.



Prior learning
Surds and binomial expansions

Syllabus coverage

UNIT 1, TOPIC 1: SURDS AND QUADRATIC FUNCTIONS

Surds

- Understand the concept of a surd as an irrational number represented using a square root or a radical sign.
- Simplify square roots of natural numbers which contain perfect square factors,
  e.g. $\sqrt{45} = \sqrt{9 \times 5} = \sqrt{9}\sqrt{5} = 3\sqrt{5}$
- Rationalise the denominator of fractional expressions involving square roots,
  e.g. $\frac{\sqrt{7}}{\sqrt{3}} = \frac{\sqrt{7}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{7} \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}} = \frac{\sqrt{21}}{3}$
- Use the four operations to simplify surds, e.g. $\sqrt{5} - 2\sqrt{5} + 4\sqrt{5} = 3\sqrt{5}$ and $2\sqrt{3} \times 5\sqrt{11} = 10\sqrt{33}$

UNIT 1, TOPIC 2: BINOMIAL EXPANSION AND CUBIC FUNCTIONS

Binomial expansion

- Understand the notion of a combination as an unordered set of r objects taken from a set of n distinct objects.
- Recognise and use the link between Pascal's triangle and the notation $\binom{n}{r}$
- Use the binomial theorem $(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \ldots + \binom{n}{r}x^{n-r}y^r + \ldots + y^n$ to expand expressions,
  e.g. $(2x-1)^3$

Mathematical Methods 2025 v1.1 General senior syllabus p. 16,
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Videos (9):
1.1 Expanding binomial products • Special binomial products
1.2 Simplifying surds 1 • Simplifying surds 2
1.5 Binomial products involving surds
1.6 Rationalising the denominator
1.7 The binomial expansion • Finding a term in a binomial expansion
Exam question analysis Surds and binomial expansions

Prior learning (1):
1.1 Surds and binomial expansions

Worksheets (11):
1.1 Binomial products • Special products
1.4 Multiplying and dividing surds
1.6 Rationalising the denominator • Surds
1.7 Pascal's triangle 1 • Pascal's triangle 2 • Factorial notation • Combinations • Combinations calculations
Cumulative exams Maths Methods Year 11 formula sheet

Puzzles (3):
1.2 Simplifying surds
1.3 Surds code puzzle
1.4 Surds

TI-Nspire CX II instructions (1):
1.2, 1.7 Surds and binomial expansions

Technology (1):
1.2 Simplifying surds quiz

Spreadsheet (1):
1.2 Simplifying surds quiz

Nelson MindTap
To access resources above, visit **cengage.com.au/nelsonmindtap**

Terminology previews the keywords and phrases from within the chapter.

Terminology

binomial	binomial expansion	binomial theorem
coefficient	combination	conjugate
constant	difference of 2 squares	distributive law
factorial	irrational	Pascal's triangle
perfect cube	perfect square	rational
rationalise the denominator	simplify	surd
term	variable	

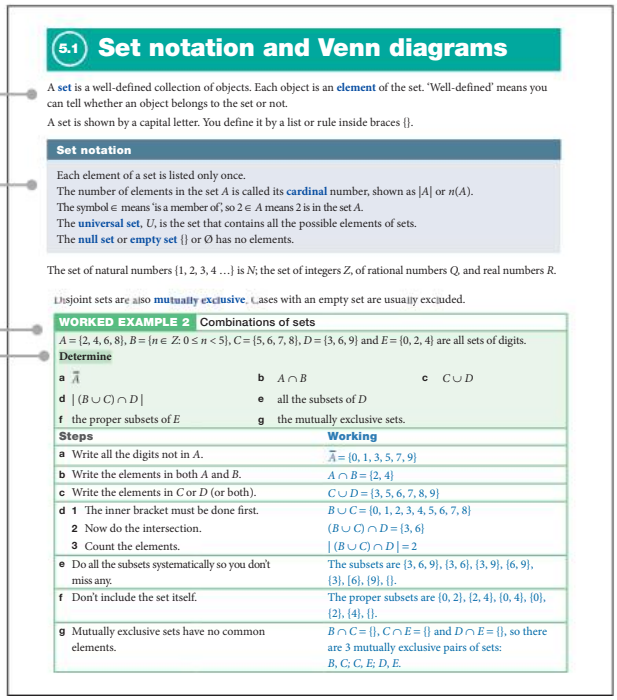
In each chapter

Important words and phrases are printed in blue and listed in the **Glossary and index** at the back of the book.

Important facts and formulas are highlighted in a shaded box.

Worked examples are explained clearly step-by-step, with the mathematical working shown on the right-hand side.

Cognitive verbs are highlighted in light green. See QCAA's glossary of cognitive verbs for more verbs and details.



5.1 Set notation and Venn diagrams

A **set** is a well-defined collection of objects. Each object is an **element** of the set. 'Well-defined' means you can tell whether an object belongs to the set or not.
A set is shown by a capital letter. You define it by a list or rule inside braces {}.

Set notation

Each element of a set is listed only once.
The number of elements in the set A is called its **cardinal** number, shown as $|A|$ or $n(A)$.
The symbol $\in$ means 'is a member of', so $2 \in A$ means 2 is in the set A.
The **universal set**, U, is the set that contains all the possible elements of sets.
The **null set** or **empty set** {} or Ø has no elements.

The set of natural numbers {1, 2, 3, 4 …} is N; the set of integers Z, of rational numbers Q, and real numbers R.

Disjoint sets are also **mutually exclusive**. Cases with an empty set are usually excluded.

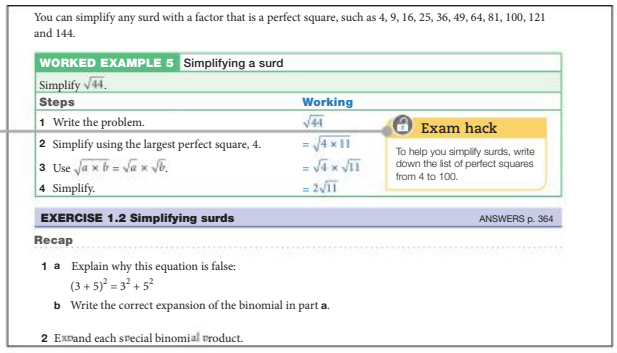
WORKED EXAMPLE 2 Combinations of sets

$A = \{2, 4, 6, 8\}$, $B = \{n \in Z: 0 \leq n < 5\}$, $C = \{5, 6, 7, 8\}$, $D = \{3, 6, 9\}$ and $E = \{0, 2, 4\}$ are all sets of digits.
Determine

a $\overline{A}$ b $A \cap B$ c $C \cup D$
d $|(B \cup C) \cap D|$ e all the subsets of D
f the proper subsets of E g the mutually exclusive sets.

Steps	Working
a Write all the digits not in A.	$\overline{A} = \{0, 1, 3, 5, 7, 9\}$
b Write the elements in both A and B.	$A \cap B = \{2, 4\}$
c Write the elements in C or D (or both).	$C \cup D = \{3, 5, 6, 7, 8, 9\}$
d 1 The inner bracket must be done first.	$B \cup C = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$
2 Now do the intersection.	$(B \cup C) \cap D = \{3, 6\}$
3 Count the elements.	$\|(B \cup C) \cap D\| = 2$
e Do all the subsets systematically so you don't miss any.	The subsets are {3, 6, 9}, {3, 6}, {3, 9}, {6, 9}, {3}, {6}, {9}, {}.
f Don't include the set itself.	The proper subsets are {0, 2}, {2, 4}, {0, 4}, {0}, {2}, {4}, {}.
g Mutually exclusive sets have no common elements.	$B \cap C = \{\}$, $C \cap E = \{\}$ and $D \cap E = \{\}$, so there are 3 mutually exclusive pairs of sets: B, C; C, E; D, E.

Exam hacks highlight valuable exam hints and common student errors.



You can simplify any surd with a factor that is a perfect square, such as 4, 9, 16, 25, 36, 49, 64, 81, 100, 121 and 144.

WORKED EXAMPLE 5 Simplifying a surd

Simplify $\sqrt{44}$.

Steps	Working
1 Write the problem.	$\sqrt{44}$
2 Simplify using the largest perfect square, 4.	$= \sqrt{4 \times 11}$
3 Use $\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$.	$= \sqrt{4} \times \sqrt{11}$
4 Simplify.	$= 2\sqrt{11}$

Exam hack
To help you simplify surds, write down the list of perfect squares from 4 to 100.

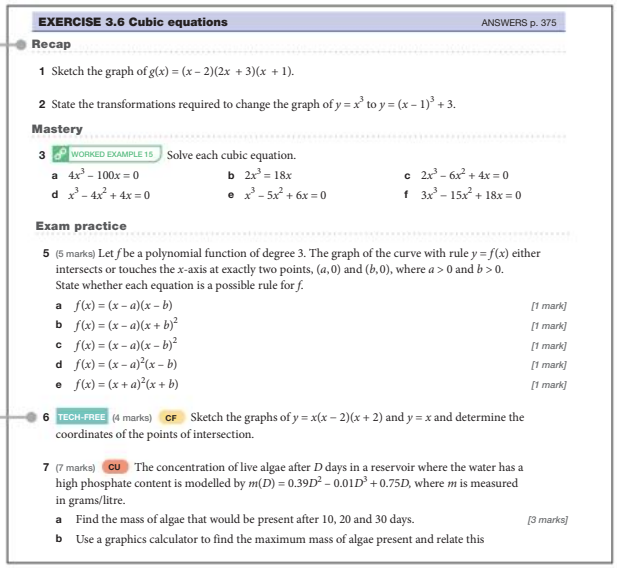
EXERCISE 1.2 Simplifying surds ANSWERS p. 364

Recap

1 a Explain why this equation is false:
$(3 + 5)^2 = 3^2 + 5^2$
b Write the correct expansion of the binomial in part **a**.

2 Expand each special binomial product.

In each chapter exercise, **Recap** questions revise skills from the previous exercise (useful for lesson starters), **Mastery** questions provide skill practice linked to worked examples, while **Exam practice** applies learned skills to exam-style problems (including past QCE questions in the Year 12 book), including multiple choice, short response, problem-solving, and simple and complex questions.



EXERCISE 3.6 Cubic equations ANSWERS p. 375

Recap

1 Sketch the graph of $g(x) = (x - 2)(2x + 3)(x + 1)$.

2 State the transformations required to change the graph of $y = x^3$ to $y = (x - 1)^3 + 3$.

Mastery

3 WORKED EXAMPLE 15 Solve each cubic equation.
a $4x^3 - 100x = 0$ b $2x^3 = 18x$ c $2x^3 - 6x^2 + 4x = 0$
d $x^3 - 4x^2 + 4x = 0$ e $x^3 - 5x^2 + 6x = 0$ f $3x^3 - 15x^2 + 18x = 0$

Exam practice

5 (5 marks) Let f be a polynomial function of degree 3. The graph of the curve with rule $y = f(x)$ either intersects or touches the x-axis at exactly two points, $(a, 0)$ and $(b, 0)$, where $a > 0$ and $b > 0$.
State whether each equation is a possible rule for f.
a $f(x) = (x - a)(x - b)$ [1 mark]
b $f(x) = (x - a)(x + b)^2$ [1 mark]
c $f(x) = (x - a)(x - b)^2$ [1 mark]
d $f(x) = (x - a)^2(x - b)$ [1 mark]
e $f(x) = (x + a)^2(x + b)$ [1 mark]

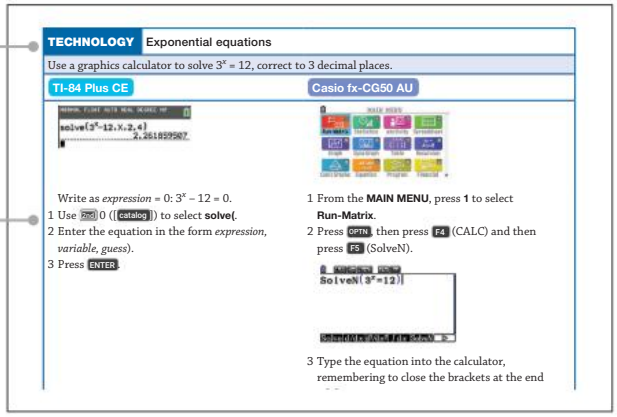
6 TECH-FREE (4 marks) CF Sketch the graphs of $y = x(x - 2)(x + 2)$ and $y = x$ and determine the coordinates of the points of intersection.

7 (7 marks) CU The concentration of live algae after D days in a reservoir where the water has a high phosphate content is modelled by $m(D) = 0.39D^2 - 0.01D^3 + 0.75D$, where m is measured in grams/litre.
a Find the mass of algae that would be present after 10, 20 and 30 days. [3 marks]
b Use a graphics calculator to find the maximum mass of algae present and relate this

Complex familiar **CF** and complex unfamiliar **CU** questions are clearly tagged and colour-coded.

Technology promotes ICT in the classroom, using spreadsheets and the internet.

Graphics calculator instructions are provided for the TI-84 Plus CE, Casio fx-CG50AU and TI-Nspire CXII calculators.



TECHNOLOGY Exponential equations

Use a graphics calculator to solve $3^x = 12$, correct to 3 decimal places.

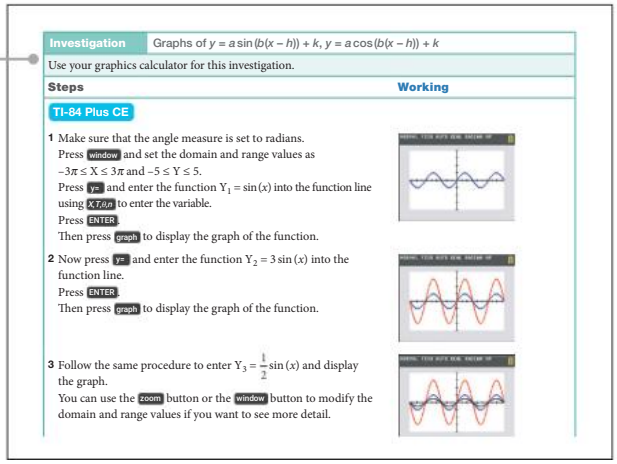
TI-84 Plus CE

Write as *expression* = 0: $3^x - 12 = 0$.
1 Use 2nd 0 ([catalog]) to select **solve(**.
2 Enter the equation in the form *expression*, *variable*, *guess*).
3 Press ENTER.

Casio fx-CG50 AU

1 From the **MAIN MENU**, press **1** to select **Run-Matrix**.
2 Press OPTN, then press F4 (CALC) and then press F5 (SolveN).
3 Type the equation into the calculator, remembering to close the brackets at the end

Investigations explore the syllabus in more detail, providing ideas for modelling activities and assessment tasks.



Investigation Graphs of $y = a\sin(b(x - h)) + k$, $y = a\cos(b(x - h)) + k$

Use your graphics calculator for this investigation.

Steps	Working

TI-84 Plus CE

1 Make sure that the angle measure is set to radians.
Press window and set the domain and range values as $-3\pi \le X \le 3\pi$ and $-5 \le Y \le 5$.
Press y= and enter the function $Y_1 = \sin(x)$ into the function line using X,T,θ,n to enter the variable.
Press ENTER.
Then press graph to display the graph of the function.

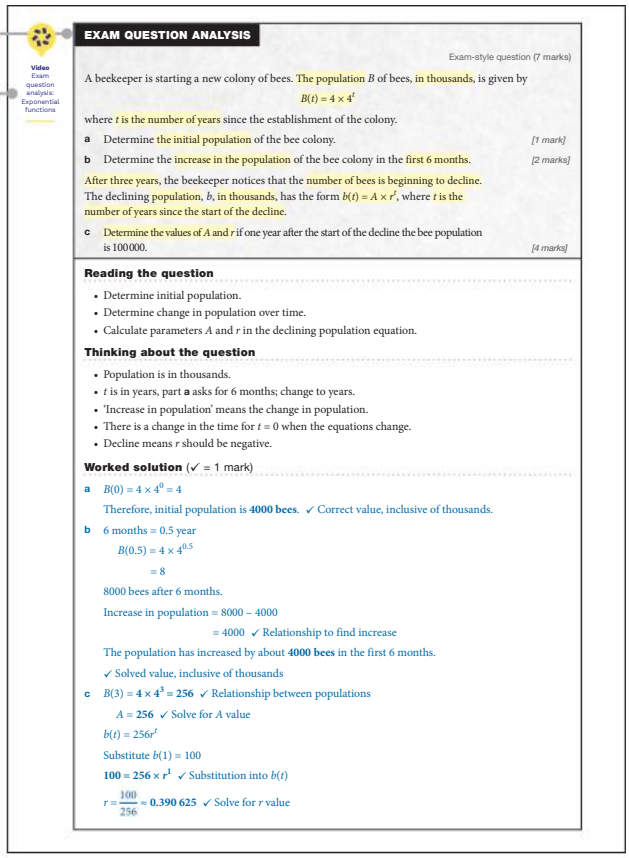
2 Now press y= and enter the function $Y_2 = 3\sin(x)$ into the function line.
Press ENTER.
Then press graph to display the graph of the function.

3 Follow the same procedure to enter $Y_3 = \frac{1}{2}\sin(x)$ and display the graph.
You can use the zoom button or the window button to modify the domain and range values if you want to see more detail.

At the end of each chapter

Exam question analysis leads students through a QCE exam-style question that exemplifies the chapter, discusses how to approach the question, provides advice on interpreting the question, highlights keywords, describes common student errors and presents a fully worked solution and marking key.

Each exam question analysis is accompanied by a video walkthrough explained by Joel Speranza.



EXAM QUESTION ANALYSIS

Video: Exam question analysis: Exponential functions

Exam-style question (7 marks)

A beekeeper is starting a new colony of bees. The population B of bees, in thousands, is given by

$$B(t) = 4 \times 4^t$$

where t is the number of years since the establishment of the colony.

a Determine the initial population of the bee colony. *[1 mark]*

b Determine the increase in the population of the bee colony in the first 6 months. *[2 marks]*

After three years, the beekeeper notices that the number of bees is beginning to decline. The declining population, b, in thousands, has the form $b(t) = A \times r^t$, where t is the number of years since the start of the decline.

c Determine the values of A and r if one year after the start of the decline the bee population is 100 000. *[4 marks]*

Reading the question

- Determine initial population.
- Determine change in population over time.
- Calculate parameters A and r in the declining population equation.

Thinking about the question

- Population is in thousands.
- t is in years, part **a** asks for 6 months; change to years.
- 'Increase in population' means the change in population.
- There is a change in the time for $t = 0$ when the equations change.
- Decline means r should be negative.

Worked solution (✓ = 1 mark)

a $B(0) = 4 \times 4^0 = 4$

Therefore, initial population is **4000 bees**. ✓ Correct value, inclusive of thousands.

b 6 months = 0.5 year

$B(0.5) = 4 \times 4^{0.5}$

$= 8$

8000 bees after 6 months.

Increase in population = 8000 − 4000

= 4000 ✓ Relationship to find increase

The population has increased by about **4000 bees** in the first 6 months.

✓ Solved value, inclusive of thousands

c $B(3) = 4 \times 4^3 = 256$ ✓ Relationship between populations

$A = 256$ ✓ Solve for A value

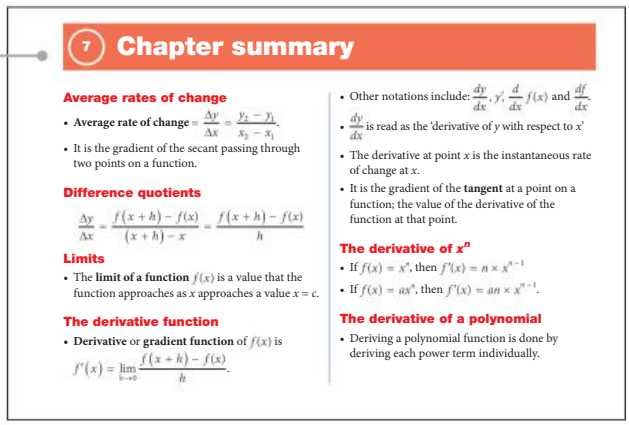
$b(t) = 256r^t$

Substitute $b(1) = 100$

$100 = 256 \times r^1$ ✓ Substitution into $b(t)$

$r = \frac{100}{256} \approx 0.390\,625$ ✓ Solve for r value

Chapter summary for easy reference.



7 Chapter summary

Average rates of change

- **Average rate of change** $= \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$.
- It is the gradient of the secant passing through two points on a function.

Difference quotients

$$\frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h}$$

Limits

- The **limit of a function** $f(x)$ is a value that the function approaches as x approaches a value $x = c$.

The derivative function

- **Derivative** or **gradient function** of $f(x)$ is $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.
- Other notations include: $\frac{dy}{dx}$, y', $\frac{d}{dx} f(x)$ and $\frac{df}{dx}$.
- $\frac{dy}{dx}$ is read as the 'derivative of y with respect to x'
- The derivative at point x is the instantaneous rate of change at x.
- It is the gradient of the **tangent** at a point on a function; the value of the derivative of the function at that point.

The derivative of x^n

- If $f(x) = x^n$, then $f'(x) = n \times x^{n-1}$
- If $f(x) = ax^n$, then $f'(x) = an \times x^{n-1}$.

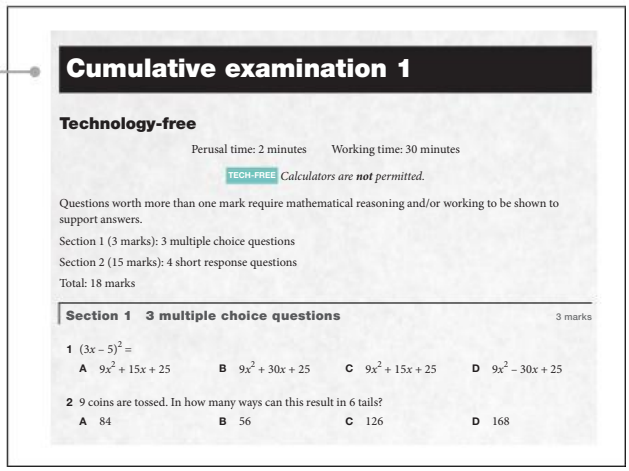
The derivative of a polynomial

- Deriving a polynomial function is done by deriving each power term individually.

Cumulative examinations 1 and **2** are mini exams based on the format of the QCE examinations 1 and 2, with about 50% of questions focusing on the chapter in which they appear and 50% on previous chapters, including complex familiar and unfamiliar questions.

Cumulative examination 1 contains technology-free questions in multiple choice and short response format: 30 minutes, 18 marks.

Cumulative examination 2 contains technology-active questions in short response format: 30 minutes, 18 marks.



Cumulative examination 1

Technology-free

Perusal time: 2 minutes Working time: 30 minutes

TECH-FREE *Calculators are **not** permitted.*

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

Section 1 (3 marks): 3 multiple choice questions

Section 2 (15 marks): 4 short response questions

Total: 18 marks

Section 1 3 multiple choice questions 3 marks

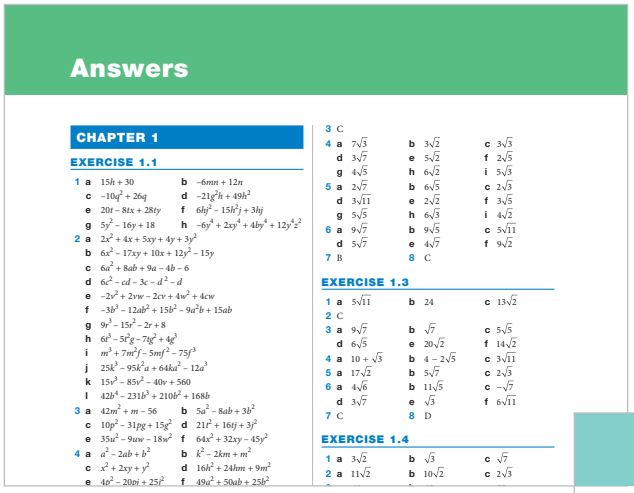
1 $(3x - 5)^2 =$

A $9x^2 + 15x + 25$ **B** $9x^2 + 30x + 25$ **C** $9x^2 + 15x + 25$ **D** $9x^2 - 30x + 25$

2 9 coins are tossed. In how many ways can this result in 6 tails?

A 84 **B** 56 **C** 126 **D** 168

At the end of the book



Answers

CHAPTER 1

EXERCISE 1.1

1 a $15h + 30$ b $-6mn + 12n$
c $-10q^2 + 26q$ d $-21g^2h + 49h^2$
e $20t - 8tx + 28ty$ f $6hj^2 - 15h^2j + 3hj$
g $5y^2 - 16y + 18$ h $-6y^3 + 2xy^4 + 4by^4 + 12y^4z^2$

2 a $2x^2 + 4x + 5xy + 4y + 3y^2$
b $6x^2 - 17xy + 10x + 12y^2 - 15y$
c $6a^2 + 8ab + 9a - 4b - 6$
d $6c^2 - cd - 3c - d^2 - d$
e $-2v^2 + 2vw - 2cv + 4w^2 + 4cw$
f $-3b^3 - 12ab^2 + 15b^2 - 9a^2b + 15ab$
g $9r^3 - 15r^2 - 2r + 8$
h $6t^3 - 5t^2g - 7tg^2 + 4g^3$
i $m^3 + 7m^2f - 5mf^2 - 75f^3$
j $25k^3 - 95k^2a + 64ka^2 - 12a^3$
k $15v^3 - 85v^2 - 40v + 560$
l $42b^4 - 231b^3 + 210b^2 + 168b$

3 a $42m^2 + m - 56$ b $5a^2 - 8ab + 3b^2$
c $10p^2 - 31pg + 15g^2$ d $21t^2 + 16tj + 3j^2$
e $35u^2 - 9uw - 18w^2$ f $64x^2 + 32xy - 45y^2$

4 a $a^2 - 2ab + b^2$ b $k^2 - 2km + m^2$
c $x^2 + 2xy + y^2$ d $16h^2 + 24hm + 9m^2$
e $4o^2 - 20oi + 25i^2$ f $49u^2 + 50ab + 25b^2$

3 C
4 a $7\sqrt{3}$ b $3\sqrt{2}$ c $3\sqrt{3}$
d $3\sqrt{7}$ e $5\sqrt{2}$ f $2\sqrt{5}$
g $4\sqrt{5}$ h $6\sqrt{2}$ i $5\sqrt{3}$
5 a $2\sqrt{7}$ b $6\sqrt{5}$ c $2\sqrt{3}$
d $3\sqrt{11}$ e $2\sqrt{2}$ f $3\sqrt{5}$
g $5\sqrt{5}$ h $6\sqrt{3}$ i $4\sqrt{2}$
6 a $9\sqrt{7}$ b $9\sqrt{5}$ c $5\sqrt{11}$
d $5\sqrt{7}$ e $4\sqrt{7}$ f $9\sqrt{2}$
7 B 8 C

EXERCISE 1.3

1 a $5\sqrt{11}$ b 24 c $13\sqrt{2}$
2 C
3 a $9\sqrt{7}$ b $\sqrt{7}$ c $5\sqrt{5}$
d $6\sqrt{5}$ e $20\sqrt{2}$ f $14\sqrt{2}$
4 a $10 + \sqrt{3}$ b $4 - 2\sqrt{5}$ c $3\sqrt{11}$
5 a $17\sqrt{2}$ b $5\sqrt{7}$ c $2\sqrt{3}$
6 a $4\sqrt{6}$ b $11\sqrt{5}$ c $-\sqrt{7}$
d $3\sqrt{7}$ e $\sqrt{3}$ f $6\sqrt{11}$
7 C 8 D

EXERCISE 1.4

1 a $3\sqrt{2}$ b $\sqrt{3}$ c $\sqrt{7}$
2 a $11\sqrt{2}$ b $10\sqrt{2}$ c $2\sqrt{3}$

Answers (with worked solutions provided on Nelson MindTap for teachers to allocate to students).



Glossary and index

1:1 *See* **one-to-one**.
1:n *See* **one-to-many**.
acceleration The rate of change or derivative of velocity with respect to time. (p. 211)
amplitude Distance from the average position to the extreme values of a trigonometric function. (p. 123)
and Mathematically, *A and B* means both *A* and *B* are true or both happen. (p. 158)
arc A portion of the circumference of a circle. (p. 108)
asymptote A function or relation characteristic: a horizontal line $y = a$ such that as $x \to \pm\infty$, $y \to a$; or a vertical line $x = a$ such that as $x \to a$, $y \to \pm\infty$. (p. 124)
average rate of change The rate of change of a function between 2 points on the function (x_1, y_1) and (x_2, y_2), given by $\frac{y_2 - y_1}{x_2 - x_1}$, which is also the gradient of the line passing through both points. (p. 212)
average speed The total distance travelled by an object divided by the elapsed time to travel that distance. (p. 211)
average velocity The displacement of an object divided by the total time for the displacement. (p. 211)
axis of symmetry A straight line such that a function or relation is symmetrical about the line. (p. 50)
base The value a in a term of the form a^n, the number or variable that has a power. (p. 243)
binomial An algebraic expression with 2 terms, such as $(x - 2)$. (p. 3)
binomial expansion Expansion of expressions of the type $(x + y)^n$ to produce $n + 1$ terms beginning with x^n. (p. 17)
binomial theorem The theorem that gives the expansion of
concave up A U-shaped curve with its inside above the curve. (p. 50)
conditional probability The probability of A given that B is true, shown as $P(A|B)$. (p. 166)
conjugate A binomial term with the opposite sign to the given term. The conjugate of $\sqrt{5} - 2$ is $\sqrt{5} + 2$. (p. 14)
constant A fixed number such as 2 or π. (p. 83) *See also* **variable**.
continuous A function whose graph has no breaks or values at which it is undefined. (p. 71)
cosine (cos) For $0° \le \theta \le 90°$, the ratio of the adjacent side to the hypotenuse of the right-angled triangle containing θ; for any angle, the x-coordinate of the trigonometric point $P(x, y)$ with angle θ on a unit circle. (p. 109)
cost The price of an item, particularly before sale. (p. 294) *See also* **marginal cost**.
cubic A polynomial of degree 3. (p. 87)
cycle One complete repetition of the pattern of a function. (p. 123)
decay factor The factor by which a quantity decreases in unit time. (p. 250)
decay function An exponential function $y = Ar^x$, where $0 < r < 1$; r is the decay factor. (p. 250)
degree (of a polynomial) The highest power of the variable in a polynomial. (p. 83)
dependent events Events A, B that are not independent; $P(A|B) \neq P(A)$. (p. 169) *See also* **conditional probability**, **independent events**.
dependent variable A variable whose value depends on that of another; the second variable of an ordered pair (x, y) of a

Glossary and index

A combined **Glossary and index**.

Nelson MindTap

Nelson MindTap is an online learning space that provides students with interactive learning and assessment experiences. Margin links in the student book signpost multimedia student resources found on Nelson MindTap*.

Videos
Skillsheets
Worksheets

For students:

- **Engage** with the **online eBook** by adding notes, highlights, bookmarks and using the **Search** and **Read Aloud** (in Australian voice) functions.
- **Watch videos** featuring expert teacher advice to unpack worked examples and deepen your understanding.
- **Revise** using pre-chapter **prior learning sheets, worksheets** and **skillsheets** to practise your skills and build your confidence.
- **Navigate** your own learning path, accessing the content and support as you need it.

For teachers:

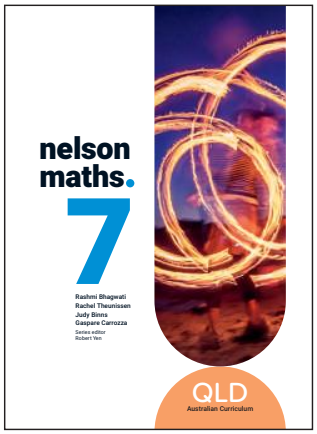
- **Access topic tests, teaching plans** and **worked solutions** to each exercise set.
- **Use course customisation** to tailor content and assign tasks to groups of students or the whole class.
- **Integrate** content directly within your school's LMS for ease of access.
- Help build your students' exam readiness with **Cognero Assess** – a test bank containing hundreds of questions and answers to create, assign or export formative and summative tests.

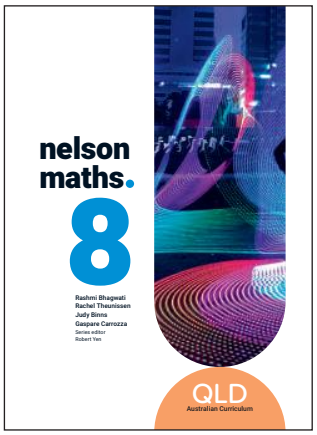
* Complimentary access to these resources is only available to schools that use this book as part of a class set, book hire or booklist. Not available for single purchases. Alternatively, teacher access can be purchased as a separate subscription for schools. Contact your Cengage Learning Consultant for information about access and conditions.

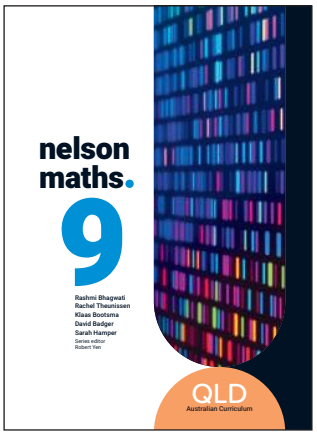
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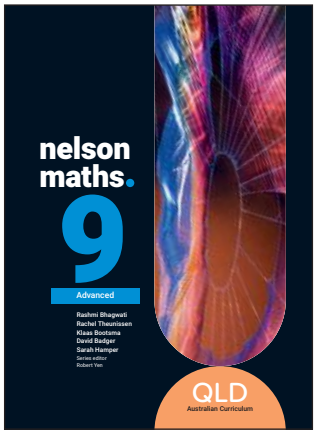
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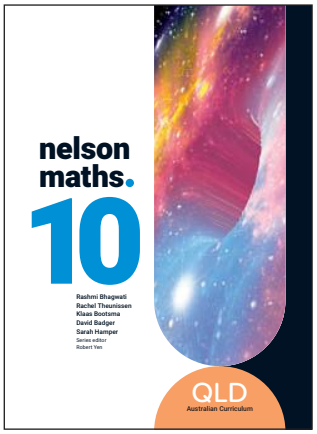
Nelson Maths 7–12 QLD series

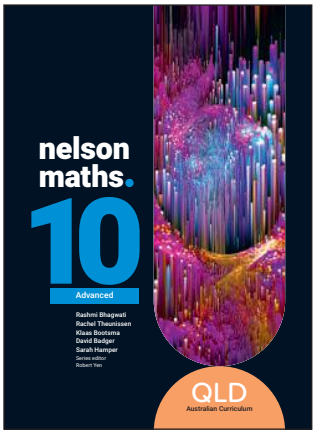


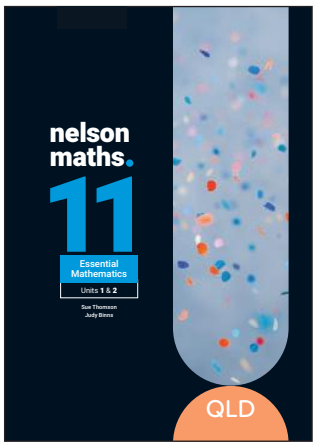


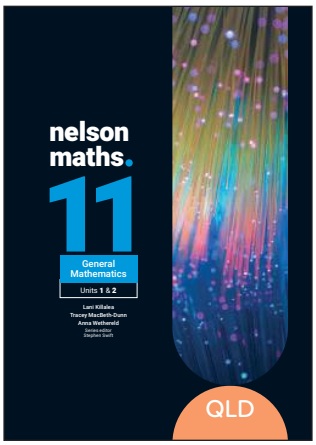


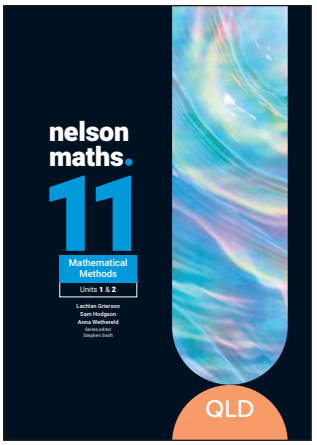


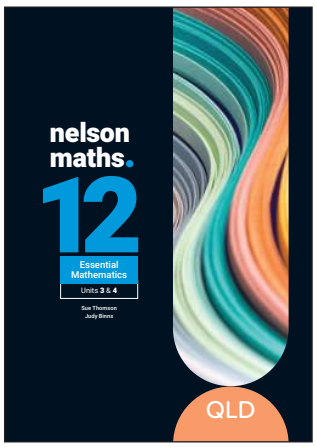


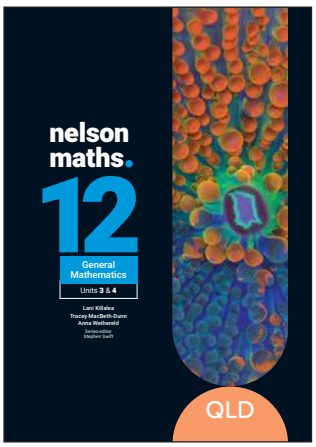


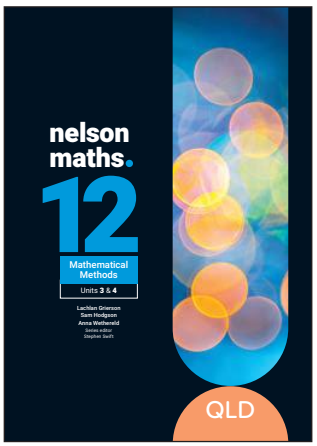












List of symbols and abbreviations

$=$	is equal to
$\equiv$	is identical to
$\neq$	is not equal to
$\approx$	is approximately equal to
$<$	is less than
$>$	is greater than
$\leq$	is less than or equal to
$\geq$	is greater than or equal to
$\in$	is an element of
$\cup$	union
$\cap$	intersection
$\pm$	plus or minus
π	pi ≈ 3.14
$\subseteq$	is a subset of
$\subset$	is a proper subset of
$\notin$	is not an element of
$\sum$	the sum of
$\ldots$	and so on
$\{\}, \varnothing$	the null set
U	the universal set
R	the real numbers
R^+	the positive real numbers
N	the natural numbers

$^\circ$	degree
∞	infinity
$\Rightarrow$	implies
$\Leftrightarrow$	if and only if
Δ	the discriminant
$\sqrt{\ }$	square root, radical sign
$\overline{A}$, A'	not A, the complement of A
:	such that
S	sample space
$P(A)$	the probability of A
$P(A\|B)$	the probability of A given B
$n(A)$, $\|A\|$	the cardinality (number of elements) of set A
$1:1$	one-to-one
$1:n$	one to n
$n:1$	n to one
$n:n$	n to n
LHS	left-hand side
RHS	right-hand side
m	gradient of a straight line

c	y-intercept of a straight line
(x, y)	Cartesian coordinates, ordered pair
${}^nC_r$, $\binom{n}{r}$	the number of combinations of r objects from n objects
$n!$	factorial n
$\lim_{h \to 0}$	the limit as $h \to 0$
$\to$	gets closer to
$\frac{dy}{dx}$, $f'(x)$	the derivative of y, $f(x)$
wrt	with respect to
s	displacement
v	speed, velocity
a	acceleration
$\log_{10}$	common logarithm
$\log_a b$	the logarithm of b to the base a
cos	cosine
sin	sine
tan	tangent

List of mathematical verbs

A glossary of 'doing words' commonly found in mathematics problems

Cognitive verbs are highlighted in light green. See QCAA's glossary of cognitive verbs for more verbs and details.

analyse: study and state in detail the relationship of parts of a situation

apply: use knowledge or a procedure in a given situation

calculate: find a numerical value

classify: identify and sort into a category

comment: express an opinion or judgement about a statement or calculation

communicate: transmit information to others

compare: state similarities and differences and their significance

consider: take into account, think carefully

construct: draw an accurate diagram or logically arrange items or ideas

convert: change from one form to another

deduce: use logic and reason to prove something

define: give the meaning of or identify in exact terms

demonstrate: show to be correct

describe: state the features of a situation, object, pattern, event, etc.

determine: find the answer or make a decision using evidence

discuss: give reasons based on evidence for and against a position or proposal

establish: introduce and develop a result

evaluate: find the value of or state the application, strengths and limitations of a solution

examine: investigate the details and assumptions of a situation

expand: remove brackets or change a product to an extended sum of terms

explain: state the meaning in logical detail

explore: examine or state the details and assumptions of a situation

factorise: convert to factors or change a sum of terms to a product of factors

formulate: give mathematical expression to an idea or situation

hence find/prove: find an answer or prove a result using previous answers or information supplied

identify: state the type, name or distinguishing feature of an item or situation

interpret: state a conclusion or trend from given information

investigate: establish facts, trends or conclusions from collected information

isolate a variable: express a formula or equation with one variable only, usually on the left-hand side (LHS)

justify: give reasons or evidence for an answer or conclusion

modify: change to accommodate different information

obtain: find an answer or conclusion

prove: use logical steps to establish the truth of a result, statement or assumption

predict: use given information or results to guess what will happen under different conditions

rationalise the denominator: To simplify a fraction involving a surd by making its denominator rational (that is, not a surd)

recall: remember (and state)

recognise: use knowledge to identify features of a situation

show that: in questions where the answer is given, to use mathematical reasoning to prove that the answer is true

simplify: reduce the size of numbers in a fraction, or reduce the size of an algebraic expression

sketch: draw a diagram that shows the general shape and includes relevant features

solve: find the answer or explanation for a problem, particularly the values of variables in equations or inequalities

substitute: replace a variable by a number to calculate an answer

test: check that a statement or result is correct

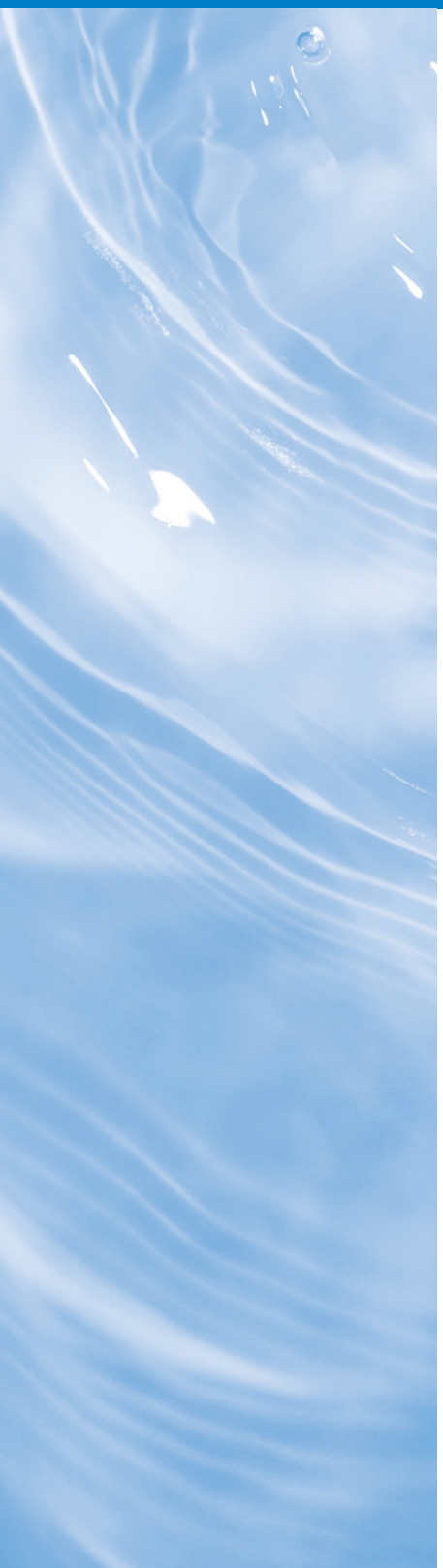
translate (to mathematical form): express a situation as mathematical relationships

verify: check a solution or result, usually referring to the given situation

write/state: give the answer, formula or result without showing any working or explanation (This usually means that the answer can be found mentally, or in one step)

CHAPTER

1 SURDS AND BINOMIAL EXPANSIONS



Syllabus coverage

Nelson MindTap chapter resources

Terminology

1.1 Binomial products

1.2 Simplifying surds

Rational and irrational numbers

1.3 Adding and subtracting surds

1.4 Multiplying and dividing surds

1.5 Binomial products involving surds

1.6 Rationalising the denominator

1.7 Pascal's triangle and the binomial theorem

Pascal's triangle

Binomial expansions and ${}^{n}C_{r}$

The binomial theorem

Combinations

Exam question analysis

Chapter summary

Cumulative examination 1

Cumulative examination 2

Prior learning
Surds and binomial expansions

Syllabus coverage

UNIT 1, TOPIC 1: SURDS AND QUADRATIC FUNCTIONS

Surds

- Understand the concept of a surd as an irrational number represented using a square root or a radical sign.
- Simplify square roots of natural numbers which contain perfect square factors,
  e.g. $\sqrt{45} = \sqrt{9 \times 5} = \sqrt{9}\sqrt{5} = 3\sqrt{5}$
- Rationalise the denominator of fractional expressions involving square roots,
  e.g. $\frac{\sqrt{7}}{\sqrt{3}} = \frac{\sqrt{7}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{7} \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}} = \frac{\sqrt{21}}{3}$
- Use the four operations to simplify surds, e.g. $\sqrt{5} - 2\sqrt{5} + 4\sqrt{5} = 3\sqrt{5}$ and $2\sqrt{3} \times 5\sqrt{11} = 10\sqrt{33}$

UNIT 1, TOPIC 2: BINOMIAL EXPANSION AND CUBIC FUNCTIONS

Binomial expansion

- Understand the notion of a combination as an unordered set of r objects taken from a set of n distinct objects.
- Recognise and use the link between Pascal's triangle and the notation $\binom{n}{r}$
- Use the binomial theorem $(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \ldots + \binom{n}{r}x^{n-r}y^r + \ldots + y^n$ to expand expressions,
  e.g. $(2x-1)^3$

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Videos (9):

1.1 Expanding binomial products • Special binomial products
1.2 Simplifying surds 1 • Simplifying surds 2
1.5 Binomial products involving surds
1.6 Rationalising the denominator
1.7 The binomial expansion • Finding a term in a binomial expansion
Exam question analysis Surds and binomial expansions

Prior learning (1):

1.1 Surds and binomial expansions

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Terminology

binomial	binomial expansion	binomial theorem
coefficient	combination	conjugate
constant	difference of 2 squares	distributive law
factorial	irrational	Pascal's triangle
perfect cube	perfect square	rational
rationalise the denominator	simplify	surd
term	variable	

1.1 Binomial products

The **distributive law** says that for any numbers a, b and c:

$a(b + c) = ab + ac$

The term outside the brackets is multiplied by each term inside the brackets.

Consider the expression: $(a + b)(c + d)$

$(a + b)$ is called a **binomial** because it has 2 terms.
$(a + b)(c + d)$ is called a **binomial product** because it is 2 binomials multiplied together.

Expanding using the distributive law: $(a + b)(c + d) = a(c + d) + b(c + d)$

Expand the brackets in both terms: $= ac + ad + bc + bd$

Progressively removing brackets using the distributive law is *expanding* an algebraic expression.
The expression can then be *simplified* by collecting like terms.

The acronym **FOIL** shows the order.

FOIL method for $(a + b)(c + d)$

First terms, **O**uter terms, **I**nner terms, **L**ast terms

$$\underset{\text{F}}{} \quad \underset{\text{O}}{} \quad \underset{\text{I}}{} \quad \underset{\text{L}}{}$$
$$(a + b)(c + d) = ac + ad + bc + bd$$

WORKED EXAMPLE 1 Expanding expressions

Expand and simplify each expression.

a $4(3x - 5)$ **b** $-2(6a - 3)$ **c** $(2c - 3)(4c + 5)$ **d** $(2x - 1)(3x + 4y - 5)$

Steps		Working
a	**1** Write the expression.	$4(3x - 5)$
	2 Expand using the distributive law.	$= 4 \times 3x + 4 \times (-5)$
	3 Simplify terms.	$= 12x - 20$
b	**1** Write the expression.	$-2(6a - 3)$
	2 You can use brackets around negatives.	$= (-2) \times 6a + (-2) \times (-3)$
	3 Simplify terms.	$= -12a + 6$
c	**1** Write the expression.	$(2c - 3)(4c + 5)$
	2 Expand using FOIL or the distributive law.	$= 2c(4c + 5) + (-3)(4c + 5)$
		$= 2c \times 4c + 2c \times 5 + (-3) \times 4c + (-3) \times 5$
	3 Simplify terms.	$= 8c^2 + 10c - 12c - 15$
	4 Collect like terms.	$= 8c^2 - 2c - 15$
d	**1** Write the expression.	$(2x - 1)(3x + 4y - 5)$
	2 Expand using the distributive law.	$= 2x(3x + 4y - 5) + (-1)(3x + 4y - 5)$
	3 Expand again and simplify.	$= 6x^2 + 8xy - 10x - 3x - 4y + 5$
		$= 6x^2 + 8xy - 13x - 4y + 5$

Videos
Expanding binomial products
Special binomial products

Worksheets
Binomial products
Special products

Special binomial products

Perfect squares

$$(a + b)^2 = a^2 + 2ab + b^2$$

Difference of 2 squares

$$(a + b)(a - b) = a^2 - b^2$$

WORKED EXAMPLE 2 Special binomial products

Expand and simplify each expression.

a $(2x + 1)^2$ **b** $(y - 3)^2$ **c** $(3a + 4)(3a - 4)$ **d** $(x + 5 + y)(x - 5 - y)$

Steps	Working
a 1 Write the expression.	$(2x + 1)^2$
2 Use perfect squares.	$= (2x)^2 + 2(2x \times 1) + (1)^2$
3 Simplify terms.	$= 4x^2 + 4x + 1$
b 1 Write the expression.	$(y - 3)^2$
2 Use perfect squares.	$= y^2 + 2(-3 \times y) + (-3)^2$
3 Simplify terms.	$= y^2 - 6y + 9$
c 1 Write the expression.	$= (3a + 4)(3a - 4)$
2 Use the difference of 2 squares.	$= (3a)^2 - (4)^2$
	$= 9a^2 - 16$
d 1 Write the expression.	$(x + 5 + y)(x - 5 - y)$
2 Use the difference of 2 squares.	$= (x + (5 + y))(x - (5 + y))$
	$= (x)^2 - (5 + y)^2$
3 Use perfect squares on the 2nd term.	$= x^2 - (5^2 + 2(5 \times y) + y^2)$
	$= x^2 - 25 - 10y - y^2$
	$= x^2 - 10y - y^2 - 25$

Perfect cubes

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

WORKED EXAMPLE 3 Perfect cubes

Expand $(2x - 3)^3$.

Steps	Working
Method 1: Using a perfect square	
1 Write the expression.	$(2x - 3)^3$
2 Write with a perfect square.	$= (2x - 3)(2x - 3)^2$
	$= (2x - 3)(4x^2 - 12x + 9)$
3 Multiply out.	$= 8x^3 - 24x^2 + 18x - 12x^2 + 36x - 27$
4 Collect like terms.	$= 8x^3 - 36x^2 + 54x - 27$
Method 2: Using a perfect cube	
1 Write the expression.	$(2x - 3)^3$
2 Use the cubic identity.	$= (2x)^3 + 3(2x)^2(-3) + 3(2x)(-3)^2 + (-3)^3$
3 Simplify.	$= 8x^3 - 36x^2 + 54x - 27$

EXERCISE 1.1 Binomial products

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Mastery

1 WORKED EXAMPLE 1 Expand and simplify each expression.

a $5(3h + 6)$ **b** $-3n(2m - 4)$ **c** $-3q(4q - 8) + 2q(q + 1)$

d $7h(-3g^2 + 7h)$ **e** $4t(5 - 2x + 7y)$ **f** $3hj(2j - 5h + 1)$

g $5y(y - 2) - 6(y - 3)$ **h** $-2y^4(3 - x - 2b - 6z^2)$

2 Expand and simplify each expression.

a $(x + y)(2x + 4 + 3y)$ **b** $(2x - 3y)(3x - 4y + 5)$ **c** $(2a - 1)(3a + 4b + 6)$

d $(3c + d)(2c - d - 1)$ **e** $-(2v - 4w)(v + w + c)$ **f** $-3b(b + a)(b + 3a - 5)$

g $(3r - 4)(3r + 2)(r - 1)$ **h** $(2t - g)(t + g)(3t - 4g)$ **i** $(m + 5f)^2(m - 3f)$

j $(5k - 2a)^2(k - 3a)$ **k** $5(3v + 7)(v - 4)^2$ **l** $7b(2b + 1)(b - 4)(3b - 6)$

3 Expand and simplify each expression.

a $(7m - 8)(6m + 7)$ **b** $(a - b)(5a - 3b)$ **c** $(2p - 5g)(5p - 3g)$

d $(3t + j)(7t + 3j)$ **e** $(5u + 3w)(7u - 6w)$ **f** $(8x - 5y)(8x + 9y)$

4 WORKED EXAMPLE 2 Expand and simplify each expression.

a $(a - b)^2$ **b** $(k - m)^2$ **c** $(x + y)^2$

d $(4h + 3m)^2$ **e** $(2p - 5j)^2$ **f** $(7a + 5b)^2$

g $(a - b)(a + b)$ **h** $(x + y)(x - y)$ **i** $(g + 3h)(g - 3h)$

j $(y - 6)(y + 6)$ **k** $(3q - 4)(3q + 4)$ **l** $(4r - 5s)(4r + 5s)$

5 Expand and simplify each expression.

a $(x + y + 1)(x + y - 1)$ **b** $(a + 2b + 4)(a + 2b - 4)$

c $(3t + s + 2)(3t - s - 2)$ **d** $(4w + 5 + z)(4w - z - 5)$

6 WORKED EXAMPLE 3 Expand and simplify each expression.

a $(x + a)^3$ **b** $(x - 5)^3$ **c** $(3x - 2)^3$

d $(3v - 4w)^3$ **e** $(2m + 3n)^3$ **f** $(x^2 - 3y^2)^3$

7 Expand and simplify each expression.

a $(a + b)(a^2 - ab + b^2)$ **b** $(m + n)(m^2 - mn + n^2)$ **c** $(a - b)(a^2 + ab + b^2)$

d $(x - y)(x^2 + xy + y^2)$ **e** $(p + 5)(p^2 - 5p + 25)$ **f** $(2 + 3k)(4 - 6k + 9k^2)$

Exam practice

8 $(m - 3n)(m + 4n) =$

A $m^2 - 7mn + 12n^2$ **B** $m^2 + 7mn - 12n^2$ **C** $m^2 + mn - 12n^2$ **D** $m^2 - mn - 12n^2$

9 $(x - y)(x + y) =$

A $x^2 - 2xy + y^2$ **B** $x^2 - y^2$ **C** $x^2 + y^2$ **D** $x^2 - 2xy - y^2$

10 (4 marks) Expand and simplify each expression.

a $(k + 2m)(k + 7m)$ *[1 mark]*

b $(3x - 5y)(4x + 3y)$ *[1 mark]*

c $(2d + 3e)^2$ *[1 mark]*

d $(5f + 4g)(5f - 4g)$ *[1 mark]*

11 (4 marks) Expand and simplify each expression.

a $(3p + 4r)^3$ *[2 marks]*

b $(4x^2 - 3y^2)^2$ *[2 marks]*

Videos
Simplifying surds 1
Simplifying surds 2

Puzzle
Simplifying surds

Technology
Simplifying surds quiz

Spreadsheet
Simplifying surds quiz

1.2 Simplifying surds

Rational and irrational numbers

A **rational number** is a number that can be written as a fraction $\frac{a}{b}$ where a and b are both integers but $b \neq 0$.

For example, $2.45 = \frac{245}{100}$, $3 = \frac{3}{1}$ and $-1.833333\ldots = -\frac{11}{6}$ are rational numbers.

All integers, fractions, terminating and recurring decimals, and percentages are rational numbers.

An **irrational number** is a number that *cannot* be written as a fraction $\frac{a}{b}$ where a and b are both integers.

For example, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ and π are irrational numbers because they cannot be written as a fraction. As a decimal, their digits run infinitely but without recurring.

A **surd** is an irrational number that is a root, such as $\sqrt{2}$.

WORKED EXAMPLE 4 Rational and irrational numbers

State whether each number is rational or irrational.

a $\sqrt{5}$ **b** $\frac{7}{5}$ **c** -0.059 **d** $\sqrt[3]{64}$

Steps	Working
a 1 5 is not a square number, so $\sqrt{5}$ is a surd.	$\sqrt{5}$
2 Evaluate with a calculator.	$= 2.236\,067\,97\ldots$
3 Decimal digits do not end or repeat.	$\sqrt{5}$ is irrational.
b $\frac{7}{5}$ is already a fraction.	$\frac{7}{5}$
	$\frac{7}{5}$ is rational.
c 1 -0.059 is a decimal.	-0.059
2 Express this as a fraction.	$= -\frac{59}{1000}$
	-0.059 is rational.
d 1 64 is a perfect cube.	$\sqrt[3]{64}$
2 The cube root of 64 is 4.	$= 4$
3 4 is an integer.	$\sqrt[3]{64}$ is rational.

Simplifying surds

$\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$

You can simplify any surd with a factor that is a perfect square, such as 4, 9, 16, 25, 36, 49, 64, 81, 100, 121 and 144.

WORKED EXAMPLE 5 Simplifying a surd

Simplify $\sqrt{44}$.

Steps	Working
1 Write the problem.	$\sqrt{44}$
2 Simplify using the largest perfect square, 4.	$= \sqrt{4 \times 11}$
3 Use $\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$.	$= \sqrt{4} \times \sqrt{11}$
4 Simplify.	$= 2\sqrt{11}$

Exam hack

To help you simplify surds, write down the list of perfect squares from 4 to 100.

EXERCISE 1.2 Simplifying surds

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Recap

1 **a** Explain why this equation is false:
$(3 + 5)^2 = 3^2 + 5^2$

b Write the correct expansion of the binomial in part **a**.

2 Expand each special binomial product.

a $(a + b)^2$ **b** $(a + b)(a - b)$

Mastery

3 WORKED EXAMPLE 4 Select the number that is irrational.

A $4\frac{2}{5}$ **B** $\sqrt[3]{-8}$ **C** $-\sqrt{7}$ **D** $0.\dot{6}$

4 WORKED EXAMPLE 5 Simplify each surd.

a $\sqrt{147}$ **b** $\sqrt{18}$ **c** $\sqrt{27}$
d $\sqrt{63}$ **e** $\sqrt{50}$ **f** $\sqrt{20}$
g $\sqrt{80}$ **h** $\sqrt{72}$ **i** $\sqrt{75}$

5 Simplify each surd.

a $\sqrt{28}$ **b** $\sqrt{180}$ **c** $\sqrt{12}$
d $\sqrt{99}$ **e** $\sqrt{8}$ **f** $\sqrt{45}$
g $\sqrt{125}$ **h** $\sqrt{108}$ **i** $\sqrt{32}$

6 Simplify each surd.

a $\sqrt{567}$ **b** $\sqrt{405}$ **c** $\sqrt{275}$
d $\sqrt{175}$ **e** $\sqrt{112}$ **f** $\sqrt{162}$

Exam practice

7 Select the largest number.

A π **B** $\sqrt{10}$ **C** $\frac{2}{3}$ **D** 120%

8 Which number is a surd?

A $\sqrt{9}$ **B** $\sqrt{81}$ **C** $\sqrt{160}$ **D** $\sqrt{225}$

Puzzle
Surds code puzzle

1.3 Adding and subtracting surds

With algebraic terms, you can add or subtract like terms. You can do the same with surds.

WORKED EXAMPLE 6 Adding and subtracting surds 1

Simplify $4\sqrt{2} - 3\sqrt{2} + 5\sqrt{2}$.

Steps	Working
1 Write the expression.	$4\sqrt{2} - 3\sqrt{2} + 5\sqrt{2}$
2 Simplify by gathering like terms.	$= 6\sqrt{2}$

You cannot simplify the expression $\sqrt{2} + \sqrt{3}$ because $\sqrt{2}$ and $\sqrt{3}$ are not like terms.

WORKED EXAMPLE 7 Adding and subtracting surds 2

Simplify $5\sqrt{3} + \sqrt{2} - 2\sqrt{3}$.

Steps	Working
1 Write the expression.	$5\sqrt{3} + \sqrt{2} - 2\sqrt{3}$
2 Group like terms.	$= 5\sqrt{3} - 2\sqrt{3} + \sqrt{2}$
3 Simplify.	$= 3\sqrt{3} + \sqrt{2}$

WORKED EXAMPLE 8 Adding and subtracting unsimplified surds 1

Simplify $\sqrt{20} - \sqrt{5}$.

Steps	Working
1 Write the expression.	$\sqrt{20} - \sqrt{5}$
2 Simplify $\sqrt{20}$.	$= \sqrt{4} \times \sqrt{5} - \sqrt{5}$
	$= 2\sqrt{5} - \sqrt{5}$
3 Simplify.	$= \sqrt{5}$

WORKED EXAMPLE 9 Adding and subtracting unsimplified surds 2

Simplify $\sqrt{27} + \sqrt{75}$.

Steps	Working
1 Write the expression.	$\sqrt{27} + \sqrt{75}$
2 Simplify each surd.	$= \sqrt{9} \times \sqrt{3} + \sqrt{25} \times \sqrt{3}$
3 Simplify.	$= 3\sqrt{3} + 5\sqrt{3}$
	$= 8\sqrt{3}$

EXERCISE 1.3 Adding and subtracting surds

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Recap

1 Simplify each surd.

a $\sqrt{275}$ **b** $\sqrt{576}$ **c** $\sqrt{338}$

2 Which number is not a surd?

A $\sqrt{18}$ **B** $\sqrt{77}$ **C** $\sqrt{169}$ **D** $\sqrt{200}$

Mastery

3 WORKED EXAMPLE 6 Simplify each expression.

a $4\sqrt{7} + 5\sqrt{7}$ **b** $3\sqrt{7} - 2\sqrt{7}$ **c** $3\sqrt{5} + 2\sqrt{5}$

d $9\sqrt{5} - 3\sqrt{5}$ **e** $9\sqrt{8} + \sqrt{8}$ **f** $9\sqrt{8} - 2\sqrt{8}$

4 WORKED EXAMPLE 7 Simplify each expression.

a $3\sqrt{4} + 2\sqrt{4} + \sqrt{3}$ **b** $3\sqrt{4} - \sqrt{4} - 2\sqrt{5}$ **c** $2\sqrt{11} + 7\sqrt{11} - 2\sqrt{99}$

5 WORKED EXAMPLE 8 Simplify each expression.

a $4\sqrt{50} - 3\sqrt{2}$ **b** $2\sqrt{28} + \sqrt{7}$ **c** $2\sqrt{27} - 4\sqrt{3}$

6 WORKED EXAMPLE 9 Simplify each expression.

a $2\sqrt{6} + 3\sqrt{6} - \sqrt{6}$ **b** $19\sqrt{5} - 4\sqrt{5} - 2\sqrt{20}$ **c** $3\sqrt{7} - 6\sqrt{7} + 2\sqrt{7}$

d $4\sqrt{7} + \sqrt{28} - 3\sqrt{7}$ **e** $1\sqrt{12} + 2\sqrt{3} - \sqrt{27}$ **f** $7\sqrt{11} + \sqrt{44} - \sqrt{99}$

Exam practice

7 Simplify $\sqrt{8} + \sqrt{32}$.

A $5\sqrt{2}$ **B** $5\sqrt{8}$ **C** $6\sqrt{2}$ **D** $\sqrt{40}$

8 Simplify $\sqrt{108} - \sqrt{27}$.

A $4\sqrt{27}$ **B** $6\sqrt{3}$ **C** $\sqrt{81}$ **D** $3\sqrt{3}$

1.4 Multiplying and dividing surds

When simplifying surds, you used the property $\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$.

To multiply surds, use the same property written in reverse:

Multiplying surds

$$\sqrt{a} \times \sqrt{b} = \sqrt{a \times b}$$

WORKED EXAMPLE 10 Multiplying surds 1

Simplify $2\sqrt{3} \times 4\sqrt{5}$.

Steps	Working
1 Write the expression.	$2\sqrt{3} \times 4\sqrt{5}$
2 Group the integers and surds separately.	$= 2 \times 4 \times \sqrt{3} \times \sqrt{5}$
3 Simplify using $\sqrt{a} \times \sqrt{b} = \sqrt{a \times b}$.	$= 8\sqrt{15}$

Worksheet Multiplying and dividing surds

Puzzle Surds

WORKED EXAMPLE 11 Multiplying surds 2

Simplify $3\sqrt{54} \times 12\sqrt{6}$.

Steps	Working
1 Write the expression.	$3\sqrt{54} \times 12\sqrt{6}$
2 Simplify $\sqrt{54}$.	$= 3\sqrt{9 \times 6} \times 12\sqrt{6}$
	$= 3 \times 3\sqrt{6} \times 12\sqrt{6}$
3 Group the integers and surds separately.	$= 3 \times 3 \times 12 \times \sqrt{6} \times \sqrt{6}$
4 Simplify.	$= 108 \times 6$
	$= 648$

Dividing surds

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$$

WORKED EXAMPLE 12 Dividing surds 1

Simplify $\frac{12\sqrt{10}}{3\sqrt{5}}$.

Steps	Working
1 Write the expression.	$\frac{12\sqrt{10}}{3\sqrt{5}}$
2 Simplify integer and surd parts separately using $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$.	$= \frac{12}{3} \times \sqrt{\frac{10}{5}}$
	$= 4 \times \sqrt{2}$
3 Simplify.	$= 4\sqrt{2}$

WORKED EXAMPLE 13 Dividing surds 2

Simplify $\frac{\sqrt{95}}{\sqrt{19}}$.

Steps	Working
1 Write down the expression.	$\frac{\sqrt{95}}{\sqrt{19}}$
2 Use the formula.	$= \sqrt{\frac{95}{19}}$
3 Simplify.	$= \sqrt{5}$

EXERCISE 1.4 Multiplying and dividing surds

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Recap

1 Simplify each expression.

a $\sqrt{8} + \sqrt{2}$ **b** $5\sqrt{3} - \sqrt{48}$ **c** $\sqrt{63} - 2\sqrt{7}$

2 Simplify each expression.

a $\sqrt{50} + \sqrt{72}$ **b** $6\sqrt{2} + \sqrt{32}$ **c** $\sqrt{75} - \sqrt{27}$

Mastery

3 WORKED EXAMPLE 10 Simplify each expression.

a $4\sqrt{2} \times 5\sqrt{2}$ **b** $3\sqrt{7} \times 2\sqrt{7}$ **c** $3\sqrt{5} \times 2\sqrt{5}$

d $9\sqrt{11} \times 3\sqrt{11}$ **e** $9\sqrt{2} \times 1\sqrt{2}$ **f** $9\sqrt{3} \times 2\sqrt{3}$

4 WORKED EXAMPLE 11 Simplify each expression.

a $4\sqrt{2} \times 5\sqrt{18}$ **b** $3\sqrt{7} \times 2\sqrt{28}$ **c** $3\sqrt{5} \times 2\sqrt{45}$

d $9\sqrt{11} \times 3\sqrt{8}$ **e** $9\sqrt{2} \times 1\sqrt{50}$ **f** $9\sqrt{12} \times 2\sqrt{3}$

5 WORKED EXAMPLE 12 Simplify each expression.

a $\dfrac{12\sqrt{20}}{3\sqrt{5}}$ **b** $\dfrac{121\sqrt{33}}{11\sqrt{11}}$ **c** $\dfrac{15\sqrt{21}}{3\sqrt{7}}$

6 WORKED EXAMPLE 13 Simplify each expression.

a $\dfrac{\sqrt{128} + \sqrt{50}}{\sqrt{200}}$ **b** $\dfrac{\sqrt{98} \times \sqrt{2}}{\sqrt{36}}$ **c** $\dfrac{\sqrt{32}}{\sqrt{72}}$

Exam practice

7 Simplify $\dfrac{2\sqrt{15}}{\sqrt{3}}$.

A $2\sqrt{3}$ **B** $2\sqrt{5}$ **C** $7\sqrt{3}$ **D** 10

8 Simplify $3\sqrt{2} \times 2\sqrt{6}$.

A $18\sqrt{2}$ **B** $12\sqrt{6}$ **C** $12\sqrt{3}$ **D** $\sqrt{72}$

9 Simplify $10\sqrt{40} \div 5\sqrt{2}$.

A $4\sqrt{5}$ **B** $2\sqrt{5}$ **C** $50\sqrt{80}$ **D** 10

10 (15 marks) CF Simplify each expression.

a $\dfrac{3\sqrt{5} \times 8\sqrt{2}}{6\sqrt{40}}$ *[2 marks]*

b $\dfrac{3\sqrt{12} \times 8\sqrt{6}}{4\sqrt{18}}$ *[2 marks]*

c $\dfrac{5\sqrt{8} \times 4\sqrt{90}}{20\sqrt{24}}$ *[2 marks]*

d $\dfrac{4\sqrt{5}}{2\sqrt{15} \times 5\sqrt{27}}$ *[3 marks]*

e $\dfrac{10\sqrt{672} \times 3\sqrt{12}}{5\sqrt{28} \times 2\sqrt{18}}$ *[3 marks]*

f $\dfrac{8\sqrt{16} \times 3\sqrt{10}}{4\sqrt{5} \times 6\sqrt{8}}$ *[3 marks]*

Video
Binomial products involving surds

1.5 Binomial products involving surds

You can use the same algebraic techniques for expanding binomial products to expand binomial expressions involving surds.

WORKED EXAMPLE 14 Binomial products with surds

Expand $(\sqrt{3}+6)(\sqrt{3}-2)$.

Steps	Working
1 Write the expression.	$(\sqrt{3}+6)(\sqrt{3}-2)$
2 Expand the brackets.	$=\sqrt{3}\times\sqrt{3}+\sqrt{3}\times(-2)+6\times\sqrt{3}+6\times(-2)$
3 Simplify each term.	$=3-2\sqrt{3}+6\sqrt{3}-12$
4 Collect like terms.	$=4\sqrt{3}-9$

WORKED EXAMPLE 15 Perfect square with surds

Expand $(\sqrt{2}+5)^2$.

Steps	Working
1 Write the expression.	$(\sqrt{2}+5)^2$
2 Use $(a+b)^2=a^2+2ab+b^2$.	$=(\sqrt{2})^2+2\times\sqrt{2}\times5+5^2$
3 Simplify terms.	$=2+10\sqrt{2}+25$
4 Add the constants.	$=10\sqrt{2}+27$

EXERCISE 1.5 Binomial products involving surds

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Recap

1 Simplify each expression.

a $\sqrt{112}$ **b** $2\sqrt{12}-4\sqrt{3}+\sqrt{8}$ **c** $\sqrt{343}\times3\sqrt{7}$

2 Simplify each expression.

a $\dfrac{\sqrt{3}\times\sqrt{18}}{\sqrt{12}}$ **b** $\dfrac{\sqrt{6}\times\sqrt{12}}{\sqrt{27}\times2\sqrt{3}}$ **c** $\dfrac{\sqrt{3}\times\sqrt{12}}{\sqrt{48}}$

Mastery

3 WORKED EXAMPLE 14 Simplify each expression.

a $\sqrt{3}\left(5-\sqrt{2}\right)$ **b** $(3+\sqrt{7})(\sqrt{7}-2)$ **c** $\sqrt{5}\left(3\sqrt{5}+2\right)$

d $\left(\sqrt{3}-1\right)\left(\sqrt{6}+5\right)$ **e** $\left(9-\sqrt{3}\right)\left(2+\sqrt{3}\right)$ **f** $\left(11+2\sqrt{3}\right)\left(7-\sqrt{3}\right)$

4 WORKED EXAMPLE 15 Simplify each expression.

a $\left(3\sqrt{2}+5\right)^2$ **b** $\left(\sqrt{7}-\sqrt{3}\right)^2$ **c** $(1+7\sqrt{11})^2$

d $(\sqrt{3}+4)^2$ **e** $\left(9+\sqrt{7}\right)\left(9-\sqrt{7}\right)$ **f** $\left(\sqrt{10}+\sqrt{5}\right)\left(\sqrt{10}-\sqrt{5}\right)$

Exam practice

5 Simplify $\left(\sqrt{5}-3\right)\left(2\sqrt{5}+\sqrt{3}\right)$.

A $10-8\sqrt{15}$ **B** $10+\sqrt{15}-6\sqrt{5}-3\sqrt{3}$

C $50+\sqrt{15}-2\sqrt{15}-\sqrt{9}$ **D** $2\sqrt{16}$

6 Simplify $\left(7\sqrt{3}+1\right)\left(4\sqrt{2}+1\sqrt{3}\right)$.

A $28\sqrt{3}+29+\sqrt{3}$ **B** 29

C $44\sqrt{6}+21$ **D** $28\sqrt{6}+21+4\sqrt{2}+\sqrt{3}$

7 Simplify $\left(\sqrt{7}-2\sqrt{11}\right)\left(3\sqrt{7}+3\sqrt{11}\right)$.

A $308+14\sqrt{77}$ **B** $-45-3\sqrt{77}$

C $4\sqrt{7}+1\sqrt{11}$ **D** -67

1.6 Rationalising the denominator

If $\sqrt{2}\approx 1.4142$, what is the value of $\frac{3}{\sqrt{2}}$? Fractions containing surds in the denominator can be difficult to work with. When approximating the value of $\frac{3}{\sqrt{2}}$, it is difficult to mentally divide by 1.4142.

You can overcome this by making the denominator **rational** (that is, *not* a surd).

Numbers that have irrational denominators, such as $\frac{1}{\sqrt{5}}, \frac{3}{2\sqrt{7}}, \frac{\sqrt{3}}{\sqrt{2}}, \frac{5\sqrt{7}}{\sqrt{3}}$ can be rewritten with a rational denominator by multiplying both the numerator and denominator by the surd that appears in the denominator. This method is called **rationalising the denominator**.

Video Rationalising the denominator

Worksheets Rationalising the denominator

Surds

Rationalising the denominator

To rationalise the denominator of the surd $\frac{a}{\sqrt{b}}$, multiply by $\frac{\sqrt{b}}{\sqrt{b}}=1$.

WORKED EXAMPLE 16 Rationalising the denominator

Simplify $\frac{\sqrt{5}}{\sqrt{3}}$ by rationalising its denominator.

Steps	Working
1 Write the expression.	$\frac{\sqrt{5}}{\sqrt{3}}$
2 Multiply the numerator and denominator by $\sqrt{3}$.	$=\frac{\sqrt{5}}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}$
3 Simplify.	$=\frac{\sqrt{15}}{3}$

With the above example, it is easier to approximate $\frac{\sqrt{5}}{\sqrt{3}}$ by using $\frac{\sqrt{15}}{3}$ and mentally dividing $\sqrt{15}\approx 3.8729$ by 3 than by dividing $\sqrt{5}\approx 2.2360$ by $\sqrt{3}\approx 1.7320$. This is why writing surds with rational denominators is preferred.

WORKED EXAMPLE 17 Rationalising a denominator of the form $a\sqrt{b}$

Rationalise the denominator of $\frac{4}{7\sqrt{5}}$.

Steps	Working
1 Write the expression.	$\frac{4}{7\sqrt{5}}$
2 Multiply by $\frac{\sqrt{5}}{\sqrt{5}} = 1$.	$= \frac{4}{7\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$
3 Simplify.	$= \frac{4\sqrt{5}}{7 \times 5}$
	$= \frac{4\sqrt{5}}{35}$

A **binomial denominator** has 2 terms.

Rationalising a binomial denominator

To rationalise the denominator of the surd $\frac{a}{b+\sqrt{c}}$, multiply by $\frac{b-\sqrt{c}}{b-\sqrt{c}} = 1$.

$b - \sqrt{c}$ is called the **conjugate** surd of $b + \sqrt{c}$.

WORKED EXAMPLE 18 Rationalising a binomial denominator

Simplify each surd by rationalising its denominator.

a $\frac{2}{4-\sqrt{3}}$ **b** $\frac{\sqrt{21}}{\sqrt{7}+\sqrt{3}}$

Steps	Working
a 1 Write the expression.	$\frac{2}{4-\sqrt{3}}$
2 Multiply the numerator and denominator by $4 + \sqrt{3}$, the conjugate of $4 - \sqrt{3}$.	$= \frac{2}{4-\sqrt{3}} \times \frac{4+\sqrt{3}}{4+\sqrt{3}}$
3 Expand the denominator using difference of 2 squares $(a + b)(a - b) = a^2 - b^2$.	$= \frac{2(4+\sqrt{3})}{4^2-(\sqrt{3})^2}$
4 Simplify.	$= \frac{8+2\sqrt{3}}{16-3}$
	$= \frac{8+2\sqrt{3}}{13}$
b 1 Write the expression.	$\frac{\sqrt{21}}{\sqrt{7}+\sqrt{3}}$
2 Multiply by $\frac{\sqrt{7}-\sqrt{3}}{\sqrt{7}-\sqrt{3}}$.	$= \frac{\sqrt{21}}{\sqrt{7}+\sqrt{3}} \times \frac{\sqrt{7}-\sqrt{3}}{\sqrt{7}-\sqrt{3}}$
3 Expand the denominator using difference of 2 squares $(a + b)(a - b) = a^2 - b^2$.	$= \frac{\sqrt{21}(\sqrt{7}-\sqrt{3})}{(\sqrt{7})^2-(\sqrt{3})^2}$

4 Simplify.

$$= \frac{\sqrt{147} - \sqrt{63}}{7 - 3}$$
$$= \frac{\sqrt{49}\sqrt{3} - \sqrt{9}\sqrt{7}}{4}$$
$$= \frac{7\sqrt{3} - 3\sqrt{7}}{4}$$

Multiplying the denominator by the conjugate gives a difference of 2 squares. The surds are squared and become rational, so the denominator is rationalised.

EXERCISE 1.6 Rationalising the denominator

ANSWERS p. 364

Recap

1 Simplify each expression.

a $\sqrt{567}$ **b** $2\sqrt{45} - 4\sqrt{5} + \sqrt{20}$ **c** $\sqrt{128} \times 3\sqrt{2}$

2 Simplify each expression.

a $(\sqrt{6} + \sqrt{12})(\sqrt{27} - 2\sqrt{3})$ **b** $\dfrac{\sqrt{3} \times \sqrt{18}}{\sqrt{2}}$ **c** $(\sqrt{3} + \sqrt{12})\sqrt{48}$

Mastery

3 WORKED EXAMPLE 16 Simplify each surd by rationalising the denominator.

a $\dfrac{\sqrt{3}}{\sqrt{5}}$ **b** $\dfrac{\sqrt{7}}{\sqrt{3}}$ **c** $\sqrt{\dfrac{5}{2}}$

4 WORKED EXAMPLE 17 Simplify each surd by rationalising the denominator.

a $\dfrac{4}{7\sqrt{5}}$ **b** $\dfrac{2}{3\sqrt{7}}$ **c** $\dfrac{3}{2\sqrt{9}}$

5 WORKED EXAMPLE 18 Simplify each surd by rationalising the denominator.

a $\dfrac{2}{8 - \sqrt{7}}$ **b** $\dfrac{7}{4 + \sqrt{5}}$ **c** $\dfrac{12}{9 - \sqrt{15}}$

d $\dfrac{\sqrt{5} + \sqrt{3}}{\sqrt{5} - \sqrt{3}}$ **e** $\dfrac{2\sqrt{5}}{\sqrt{7} - \sqrt{5}}$ **f** $\dfrac{\sqrt{3} + 1}{\sqrt{6} + 1}$

6 Simplify each expression.

a $\sqrt{\dfrac{2}{5}}$ **b** $\sqrt{\dfrac{5}{3}}$ **c** $\sqrt{\dfrac{7}{2}}$ **d** $\sqrt{\dfrac{3}{7}}$

e $\dfrac{4}{2\sqrt{7}}$ **f** $\dfrac{5}{3\sqrt{5}}$ **g** $\dfrac{3}{5\sqrt{3}}$ **h** $\dfrac{8}{7\sqrt{2}}$

i $\dfrac{5}{1 + \sqrt{5}}$ **j** $\dfrac{10}{2 - \sqrt{7}}$ **k** $\dfrac{9}{3 + \sqrt{11}}$ **l** $\dfrac{2}{4 - \sqrt{13}}$

Exam practice

7 Rationalising the denominator of the surd $\dfrac{3}{\sqrt{5}}$ gives

A $\dfrac{3\sqrt{5}}{5}$ **B** 3 **C** $\dfrac{3\sqrt{5}}{\sqrt{5}}$ **D** $\dfrac{3\sqrt{5}}{25}$

8 Rationalising the denominator of the surd $\frac{3}{2\sqrt{3}}$ gives

A $\frac{1}{6}$ **B** $\frac{1}{2}$ **C** $\frac{\sqrt{3}}{2}$ **D** $\frac{\sqrt{3}}{6}$

9 Rationalising the denominator of the surd $\frac{2+\sqrt{3}}{2\sqrt{3}}$ gives

A $\frac{5}{2}$ **B** $\frac{2\sqrt{3}+3}{6}$ **C** $\frac{5}{6}$ **D** $\frac{2\sqrt{3}+3}{24}$

10 (2 marks) Simplify $\frac{\sqrt{5}+\sqrt{2}}{3+\sqrt{10}}$.

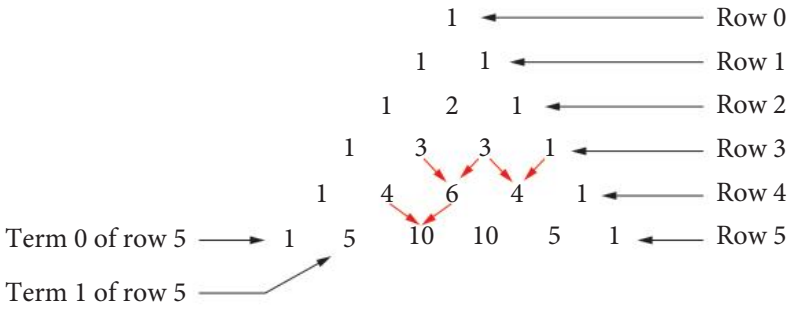
Video
The binomial expansion

Worksheets
Pascal's triangle 1
Pascal's triangle 2
Factorial notation

1.7 Pascal's triangle and the binomial theorem

Pascal's triangle

Pascal's triangle is a triangular pattern of numbers, named after the French mathematician Blaise Pascal (1623–1662).



Each row has one more number than the one above it.

The 1st and last term in each row is 1.

For the other terms, each term is the sum of the closest 2 numbers in the row above it.

As shown in the diagram, 3 + 3 = 6, 4 + 6 = 10.

The rows are numbered from 0, so the top row is Row 0 and contains just one term, 1.

The terms in each row are also numbered from 0, so in Row 5, term 0 is 1, term 1 is 5, term 2 is 10, and so on.

WORKED EXAMPLE 19 Pascal's triangle

Write the 6th and 7th rows of Pascal's triangle.

Steps	Working
1 Write the 5th row from the diagram above.	1 5 10 10 5 1
2 Put 1s at the ends for the new 6th row.	1 1
3 Add diagonally from above.	1 5 10 10 5 1 1 6 15 20 15 6 1
4 Repeat for the 7th row.	1 7 21 35 35 21 7 1

Binomial expansions and ${}^nC_r$

Examine these **binomial expansions**.

$$(a+b)^0 = 1$$
$$(a+b)^1 = a+b$$
$$(a+b)^2 = a^2+2ab+b^2$$
$$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$$
$$(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$$
$$(a+b)^5 = a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$$

Notice that the **coefficients** of the terms in each row are the values of Pascal's triangle.

$$(a+b)^0 = 1$$
$$(a+b)^1 = 1 \quad 1$$
$$(a+b)^2 = 1 \quad 2 \quad 1$$
$$(a+b)^3 = 1 \quad 3 \quad 3 \quad 1$$
$$(a+b)^4 = 1 \quad 4 \quad 6 \quad 4 \quad 1$$
$$(a+b)^5 = 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$$

Let the values (or coefficients) of Pascal's triangle be represented by the variable ${}^nC_r$, where n is the row number and r is the position number. For example,

${}^0C_0 = 1$	row 0, position 0
${}^3C_1 = 3$	row 3, position 1
${}^3C_3 = 1$	row 3, position 3
${}^4C_2 = 6$	row 4, position 2
${}^5C_4 = 5$	row 5, position 4

Notice also these general rules about the coefficients:

${}^nC_0 = 1$	The 1st term in each row is always 1.
${}^nC_n = 1$	The last term in each row is always 1.
${}^nC_1 = n$	The 2nd term in each row is always the row number, for example, ${}^4C_1 = 4$.
${}^nC_{n-1} = n$	The 2nd-last term in each row is always the row number, for example, ${}^4C_3 = 4$.
${}^nC_r = {}^nC_{n-r}$	Each row of Pascal's triangle is symmetrical, the rth and rth-last terms are the same; for example, ${}^5C_2 = {}^5C_3 = 10$.
${}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$	The sum of 2 consecutive terms in Pascal's triangle equals the term directly below them on the next row.

What is ${}^{11}C_3$, the value in row 11, position 3 of Pascal's triangle? You could continue creating the triangle up to row 11, but there is a formula for calculating ${}^nC_r$, which is also written as $\binom{n}{r}$.

${}^nC_r$ formula

$${}^nC_r = \binom{n}{r} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r(r-1)(r-2)\ldots\times 3\times 2\times 1},$$

or ${}^nC_r = \binom{n}{r} = \dfrac{n!}{r!(n-r)!}$

where $n! = 1\times 2\times 3\times\ldots\times n$ and $0! = 1$.
$n!$ is called n **factorial**.

WORKED EXAMPLE 20 Calculating ${}^nC_r$

Evaluate ${}^{11}C_3$.

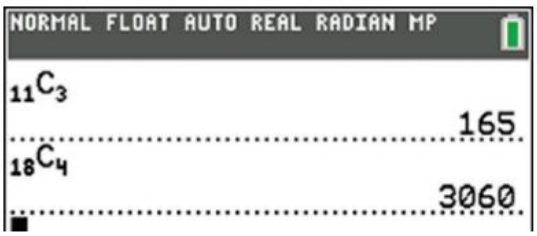
Steps	Working
Using the first formula.	
1 Because $r = 3$, write 3 terms in the denominator from 3 and 3 terms in the numerator from 11.	${}^nC_r = \binom{n}{r} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r(r-1)(r-2)\ldots\times 3\times 2\times 1}$ ${}^{11}C_3 = \frac{11\times 10\times 9}{3\times 2\times 1}$
2 Simplify.	$= \frac{990}{6}$ $= 165$
Using the factorial formula.	
1 For ${}^{11}C_3$, write 11! in the numerator and 3! with (11 – 3)! in the denominator.	${}^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$ ${}^{11}C_3 = \frac{11!}{3!\times 8!}$
2 Enter into a calculator using the $x!$ key and simplify.	$= 165$

TECHNOLOGY Calculating ${}^nC_r$

A graphics calculator has a ${}^nC_r$ function.

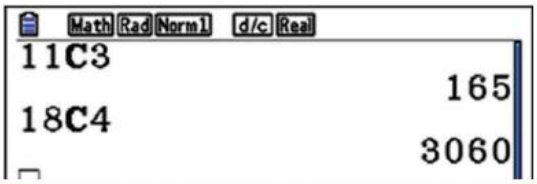
TI-84 Plus CE

Press MATH and use the cursor to go across to **PROB**.
Press **3**, put in 11, use the cursor to go to the second space, put in 3 and press ENTER.
Now do ${}^{18}C_4$.



Casio fx-CG50 AU

Choose **1 Run Matrix** and press OPTN.
Press F6 F3 to get **PROB** and put in 11.
Press F3 to get **C** (*nCr*), put in 3 and press EXE.
Now do ${}^{18}C_4$.



TI-Nspire
Surds and binomial expansions

The binomial theorem

Now study the binomial expansions again:

$$(a+b)^0 = 1$$
$$(a+b)^1 = a+b$$
$$(a+b)^2 = a^2+2ab+b^2$$
$$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$$
$$(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$$
$$(a+b)^5 = a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$$

Reading left-to-right, the powers of a are decreasing from n down to 0, while the powers of b are increasing from 0 up to n. Also, the powers of a and b in every term must add up to n.

Using the coefficients of Pascal's triangle, the binomial expansions can be written like this:

$$(a+b)^0 = 1$$
$$(a+b)^1 = a+b$$
$$(a+b)^2 = a^2 + {}^2C_1a^1b^1 + b^2$$
$$(a+b)^3 = a^3 + {}^3C_1a^2b^1 + {}^3C_2a^1b^2 + b^3$$
$$(a+b)^4 = a^4 + {}^4C_1a^3b^1 + {}^4C_2a^2b^2 + {}^4C_3a^1b^3 + b^4$$
$$(a+b)^5 = a^5 + {}^5C_1a^4b^1 + {}^5C_2a^3b^2 + {}^5C_3a^2b^3 + {}^5C_4a^1b^4 + b^5$$

The general formula for $(x+y)^n$ is called the **binomial theorem**.

The binomial theorem

$$(x+y)^n = x^n + {}^nC_1x^{n-1}y + \ldots + {}^nC_rx^{n-r}y^r + \ldots + y^n, \text{ or}$$
$$(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \ldots + \binom{n}{r}x^{n-r}y^r + \ldots + y^n$$

WORKED EXAMPLE 21 Binomial expansions

Expand each expression using the binomial theorem.

a $(x+y)^{12}$ **b** $(x-y)^8$ **c** $(2a-3b)^5$

Steps	Working
a 1 Find the coefficients.	${}^{12}C_0 = 1$, ${}^{12}C_1 = 12$, ${}^{12}C_2 = 66$, ${}^{12}C_3 = 220$, ${}^{12}C_4 = 495$, ${}^{12}C_5 = 792$ ${}^{12}C_6 = 924$, ${}^{12}C_7 = 792$, ${}^{12}C_8 = 495$, ${}^{12}C_9 = 220$, ${}^{12}C_{10} = 66$, ${}^{12}C_{11} = 12$, ${}^{12}C_{12} = 1$
2 Write the expansion with decreasing powers of x and increasing powers of y.	$(x+y)^{12} = x^{12} + 12x^{11}y + 66x^{10}y^2 + 220x^9y^3 + 495x^8y^4 + 792x^7y^5 + 924x^6y^6 + 792x^5y^7 + 495x^4y^8 + 220x^3y^9 + 495x^2y^{10} + 66xy^{11} + y^{12}$
b 1 Write as $(x+y)^8$.	$(x-y)^8 = (x+(-y))^8$
2 Find the coefficients.	${}^8C_0 = 1$, ${}^8C_1 = 8$, ${}^8C_2 = 28$, ${}^8C_3 = 56$, ${}^8C_4 = 70$, ${}^8C_5 = 56$, ${}^8C_6 = 28$, ${}^8C_7 = 8$, ${}^8C_8 = 1$
3 Write the expansion with y replaced by $(-y)$.	$(x-y)^8 = x^8 + 8x^7(-y) + 28x^6(-y)^2 + 56x^5(-y)^3 + 70x^4(-y)^4 + 56x^3(-y)^5 + 28x^2(-y)^6 + 8x(-y)^7 + (-y)^8$
4 Simplify the powers of $(-y)$.	$= x^8 - 8x^7y + 28x^6y^2 - 56x^5y^3 + 70x^4y^4 - 56x^3y^5 + 28x^2y^6 - 8xy^7 + y^8$
c 1 Write as $(x+y)^5$.	$(2a-3b)^5 = (2a+(-3b))^5$
2 Find the coefficients.	$1, 5, 10, 10, 5, 1$
3 Write the expansion with y replaced by $(-3b)$.	$(2a-3b)^5 = (2a)^5 + 5(2a)^4(-3b) + 10(2a)^3(-3b)^2 + 10(2a)^2(-3b)^3 + 5(2a)(-3b)^4 + (-3b)^5$
4 Simplify the powers of $(-3b)$.	$= 32a^5 + 5(16a^4)(-3b) + 10(8a^3)(9b^2) + 10(4a^2)(-27b^3) + 10a(81b^4) - 243b^5$
5 Simplify the terms.	$= 32a^5 - 240a^4b + 7208a^3b^2 - 1080a^2b^3 + 810ab^4 - 243b^5$

Powers of a negative number alternate in sign. This makes the signs alternate in parts **b** and **c** above.

Video
Finding a term in a binomial expansion

WORKED EXAMPLE 22 Finding a term in a binomial expansion

For the expansion of $(x + y)^{17}$, **determine**

a the first 3 terms **b** the term that contains x^{10}.

Steps	Working
a 1 Write the first 3 terms using ${}^nC_r$.	$(x + y)^{17} = {}^{17}C_0x^{17} + {}^{17}C_1x^{16}y + {}^{17}C_2x^{15}y^2 + \ldots$
2 Simplify.	$= x^{17} + 17x^{16}y + 136x^{15}y^2 + \ldots$
b 1 The term with x^{10} has y^7.	${}^{17}C_7x^{10}y^7$
2 Work out the coefficient.	$= 19\,448x^{10}y^7$

Worksheets
Combinations

Combinations Calculations

Combinations

If you flip 4 coins and count the number of heads, the possible events are 0 heads, 1 head, 2 heads, 3 heads and 4 heads. This table shows the different number of ways of obtaining each event.

Number of heads	Possible ways	Number of ways
0	TTTT	1
1	HTTT, THTT, TTHT, TTTH	4
2	HHTT, THHT, TTHH, HTHT, HTTH, THTH, THHT	6
3	HHHT, HHTH, HTHH, THHH	4
4	HHHH	1

Note that for 4 coins, the number of possible ways is the 4th row of Pascal's triangle.

${}^4C_0 = 1$: the number of ways of flipping 0 heads from 4 coins

${}^4C_1 = 4$: the number of ways of flipping 1 head from 4 coins

${}^4C_2 = 6$: the number of ways of flipping 2 heads from 4 coins

${}^4C_3 = 4$: the number of ways of flipping 3 heads from 4 coins

${}^4C_4 = 1$: the number of ways of flipping 4 heads from 4 coins

Combinations

A selection of r objects from n objects is called a **combination**.

The number of combinations of r objects from n objects is ${}^nC_r$ or $\binom{n}{r}$.

WORKED EXAMPLE 23 Applying combinations

Five friends are going to the beach in a car and they can all drive.

Determine the number of different pairs that can travel in the 2 front seats.

Steps	Working
1 You need the combinations of 2 from 5.	${}^5C_2 = 10$
2 Write the answer.	There are 10 different pairs of friends who can travel in the front.

EXERCISE 1.7 Pascal's triangle and the binomial theorem

ANSWERS p. 365

Recap

1 Simplify each expression.

a $\sqrt{343}$ **b** $2\sqrt{12} - 4\sqrt{3} + \sqrt{108}$ **c** $\sqrt{28} \times 3\sqrt{7}$

2 Simplify each expression.

a $(\sqrt{5} + \sqrt{20})(\sqrt{48} - 2\sqrt{3})$ **b** $\dfrac{\sqrt{3} \times \sqrt{4}}{-\sqrt{5}}$ **c** $\dfrac{\sqrt{49}}{\sqrt{3} + \sqrt{12}}$

Mastery

3 WORKED EXAMPLE 19 Construct Pascal's triangle up to row 12 and paste it inside the front cover of your maths workbook or save it to your device for future reference and the rest of this exercise.

4 WORKED EXAMPLE 20 Use Pascal's triangle to evaluate each coefficient.

a ${}^{7}C_{3}$ **b** ${}^{8}C_{5}$ **c** ${}^{9}C_{4}$ **d** ${}^{6}C_{2}$

e ${}^{10}C_{6}$ **f** $\binom{9}{6}$ **g** $\binom{8}{7}$ **h** $\binom{7}{4}$

i $\binom{10}{3}$ **j** $\binom{9}{2}$

5 Use a formula to evaluate each coefficient.

a ${}^{13}C_{5}$ **b** ${}^{16}C_{12}$ **c** ${}^{21}C_{3}$ **d** ${}^{15}C_{6}$

e ${}^{14}C_{7}$ **f** $\binom{18}{5}$ **g** $\binom{22}{4}$ **h** $\binom{13}{6}$

i $\binom{20}{7}$ **j** $\binom{30}{4}$

6 Use Pascal's triangle to find the value of n in each equation.

a ${}^{n}C_{2} = 6$ **b** ${}^{n+1}C_{3} = {}^{n}C_{2}$ **c** ${}^{n}C_{2} = 45$

d $\binom{n+1}{7} = \binom{n}{6}$ **e** $\binom{n}{2} = 66$ **f** $\binom{n}{5} = \binom{n}{11}$

7 WORKED EXAMPLE 21 Expand each power of a binomial.

a $(t - v)^{8}$ **b** $(n - t)^{12}$ **c** $(a + q)^{15}$

d $(r - u)^{14}$ **e** $(q + g)^{11}$ **f** $(n + 4c)^{5}$

g $(y - 5a)^{6}$ **h** $(p - 3q)^{7}$ **i** $(m + 2k)^{8}$

j $(k - 3e)^{9}$ **k** $(3v + 2n)^{4}$ **l** $(3n - 4m)^{4}$

m $(3m - 4g)^{5}$ **n** $(2m + 3c)^{7}$ **o** $(3x - 2y)^{7}$

8 WORKED EXAMPLE 22 Find the first 3 terms of each binomial expansion.

a $(q - x)^{9}$ **b** $(n + y)^{20}$ **c** $(2g - v)^{12}$

d $(e - 3m)^{20}$ **e** $(4m + 5u)^{7}$

9 Find the term containing x^{12} in the expansion of $(3x - 2y)^{16}$.

10 Find the term containing m^8 in the expansion of $(2m + 5n)^{12}$.

11 WORKED EXAMPLE 23 In how many ways can you choose 8 books from 12 books?

12 Find the number of ways to select groups of 12 hats from 20 hats.

Exam practice

13 Expanding $(x - y)^9$ gives

A $x^9 - 9x^8y + 36x^7y^2 - 86x^6y^3 + 128x^5y^4 - 128x^4y^5 + 86x^3y^6 - 36x^2y^7 + 9xy^8 - y^9$

B $x^9 - 9x^8y + 36x^7y^2 - 84x^6y^3 + 126x^5y^4 - 126x^4y^5 + 84x^3y^6 - 36x^2y^7 + 9xy^8 - y^9$

C $x^9 - 9x^8y + 36x^7y^2 - 81x^6y^3 + 126x^5y^4 - 126x^4y^5 + 81x^3y^6 - 36x^2y^7 + 9xy^8 - y^9$

D $x^9 - 9x^8y + 39x^7y^2 - 84x^6y^3 + 156x^5y^4 - 156x^4y^5 + 84x^3y^6 - 39x^2y^7 + 9xy^8 - y^9$

14 Expanding $(3x + 2y)^8$ gives

A $6561x^8 + 34\,992x^7y - 81\,468x^6y^2 + 108\,864x^5y^3 - 90\,270x^4y^4 + 48\,834x^3y^5 - 16\,128x^2y^6 + 3072xy^7 - 256y^8$

B $1x^8 + 8x^7y + 28\,x^6y^2 + 56x^5y^3 + 70\,x^4y^4 + 56\,x^3y^5 + 28\,x^2y^6 + 8\,xy^7 + 1y^8$

C $6561x^8 + 34\,992x^7y + 81\,468x^6y^2 + 108\,864x^5y^3 + 90\,270x^4y^4 + 48\,834x^3y^5 + 16\,128x^2y^6 + 3072xy^7 + 256y^8$

D $6561x^8 + 34\,992x^7y + 81\,648x^6y^2 + 108\,864x^5y^3 + 90\,720x^4y^4 + 48\,384x^3y^5 + 16\,128x^2y^6 + 3072xy^7 + 256y^8$

15 What are the first 3 terms in the expansion of $(3p - 4q)^6$?

A $729p^1q^5 - 5832p^2q^4 + 19\,440p3^3q^3$

B $729p^6 - 5832p^5q + 19\,440p^4q^2$

C $729p^6 - 5823p^5q + 14\,940p^4q^2$

D $729p^6 - 5382p^5q + 19\,440p^4q^2$

16 (1 mark) A simple lotto game uses the numbers 1 to 40, with 6 numbers being chosen. How many different combinations of 6 numbers are possible?

17 (1 mark) A car has a radio with 6 preset buttons that can be tuned to different radio stations. In a city with 12 radio stations, how many different selections of those stations can be chosen for the preset buttons?

18 (1 mark) The family banquet menu at a Chinese restaurant has 15 dishes. You can select 6 dishes from the menu for your banquet at a special price. How many different combinations are possible for the banquet?

19 (1 mark) A student council of 5 students is to be selected from 12 nominations. How many possible councils are there?

20 (1 mark) A normal hand in the card game Poker has 5 cards. How many possible hands are there if there are 52 cards in a deck?

21 (4 marks) CU Find the value of n, given that ${}^{n+2}C_4 = 2 \times {}^nC_4$.

22 (4 marks) CU A badminton competition of 8 teams has 14 rounds in the season before the finals. The top 3 teams play off for the title of badminton champion team for the year. What is the probability of correctly choosing the 3 finalists at the start of the season?

EXAM QUESTION ANALYSIS

Exam-style question (6 marks)

On one row of Pascal's triangle, the value of $\binom{n}{6}$ is 8 times the value of $\binom{n}{4}$.

a Write the formulas for $\binom{n}{6}$ and $\binom{n}{4}$. *[2 marks]*

b Determine the value of n. *[4 marks]*

Reading the question

- This question relates 2 values on one row of Pascal's triangle.
- It requires the formula for $\binom{n}{r}$.
- The question asks for the row number n to be found.

Thinking about the question

- Use the formula for both numbers.
- Write an equation.
- Solve to find n.

Worked solution (✓ = 1 mark)

a $\binom{n}{6} = \dfrac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{6 \times 5 \times 4 \times 3 \times 2 \times 1}$ ✓

$\binom{n}{4} = \dfrac{n(n-1)(n-2)(n-3)}{4 \times 3 \times 2 \times 1}$ ✓

b The relationship is $\binom{n}{6} = 8 \times \binom{n}{4}$.

$\dfrac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{1 \times 2 \times 3 \times 4 \times 5 \times 6} = 8 \times \dfrac{n(n-1)(n-2)(n-3)}{1 \times 2 \times 3 \times 4}$ ✓

Dividing both sides by common factors $\dfrac{n(n-1)(n-2)(n-3)}{1 \times 2 \times 3 \times 4}$

$\dfrac{(n-4)(n-5)}{5 \times 6} = 8$

$(n-4)(n-5) = 8 \times 5 \times 6$

$n^2 - 9n + 20 = 240$

$\boldsymbol{n^2 - 9n - 220 = 0}$ ✓

$(n-20)(n+11) = 0$

$\boldsymbol{n = 20}$ **or** $\boldsymbol{n = -11}$**, but** $\boldsymbol{n}$ **must be a positive integer.** ✓

Thus, $\boldsymbol{n = 20}$. ✓

Check: $\binom{20}{6} = 38\,760$

$\binom{20}{4} = 4845$

$8 \times \binom{20}{4} = 8 \times 4845 = 38\,760$

Video Exam question analysis: Surds and binomial expansions

1 Chapter summary

Binomial products

- The FOIL method (First terms, Outer terms, Inner terms, Last terms) can be used to expand **binomial** products like $(2x - y)(3x + 2y)$.
- **Perfect squares:** $(a + b)^2 = a^2 + 2ab + b^2$
- **Difference of 2 squares:** $(a + b)(a - b) = a^2 - b^2$
- **Perfect cubes:** $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

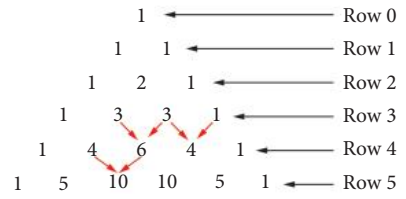
Surds

- A **rational number** is a number that can be written as a fraction $\frac{a}{b}$ where the numerator and denominator a and b are both integers but $b \neq 0$.
- All integers, fractions, terminating and recurring decimals, and percentages are rational numbers.
- An **irrational** number is a number that *cannot* be written as a fraction $\frac{a}{b}$.

  $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ and π are irrational numbers.
- A **surd** is an irrational number that is a root, such as $\sqrt{2}$.
- You can simplify any surd with a factor that is a perfect square, such as 4, 9, 16, 25, 26, 49, 64, 81, 100, 121, 144.

  $\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$
- You can add and subtract like surds:
  e.g. $\sqrt{2} - 4\sqrt{2} = -3\sqrt{2}$
- Multiplying surds: $\sqrt{a} \times \sqrt{b} = \sqrt{a \times b}$
- Dividing surds: $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$
- To rationalise the denominator of the surd $\frac{a}{\sqrt{b}}$, multiply by $\frac{\sqrt{b}}{\sqrt{b}} = 1$.
- To rationalise the denominator of the surd $\frac{a}{b + \sqrt{c}}$, multiply by $\frac{b - \sqrt{c}}{b - \sqrt{c}} = 1$.
- $b - \sqrt{c}$ is called the **conjugate** surd of $b + \sqrt{c}$.

Pascal's triangle



- **Pascal's triangle** is formed by adding pairs on one row to get the value below them.

  The rows are numbered from 0, so the top row is Row 0 and contains just one term, 1.

  The terms in each row are also numbered from 0, so in Row 5, term 0 is 1, term 1 is 5, term 2 is 10, and so on.
- The coefficients of Pascal's triangle are represented by ${}^nC_r$, where n is the row number and r is the position number.

${}^nC_0 = 1$	The 1st term in each row is always 1.
${}^nC_n = 1$	The last term in each row is always 1.
${}^nC_1 = n$	The 2nd term in each row is always the row number, e.g. ${}^4C_1 = 4$.
${}^nC_{n-1} = n$	The 2nd-last term in each row is always the row number, e.g. ${}^4C_3 = 4$.
${}^nC_r = {}^nC_{n-r}$	Each row of Pascal's triangle is symmetrical, so the rth and rth-last terms are the same, e.g. ${}^5C_2 = {}^5C_3 = 10$.
${}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$	The sum of 2 consecutive terms in Pascal's triangle equals the term directly below them on the next row.

${}^nC_r$ formula

- ${}^nC_r = \binom{n}{r} = \dfrac{n(n-1)(n-2)\ldots(n-r+1)}{r(r-1)(r-2)\ldots\times 3\times 2\times 1}$, or

  ${}^nC_r = \binom{n}{r} = \dfrac{n!}{r!(n-r)!}$

  where $n! = 1\times 2\times 3\times\ldots\times n$ and $0! = 1$.
- $n!$ is called n **factorial.**

The binomial theorem

- $(x+y)^n = x^n + {}^nC_1x^{n-1}y + \ldots + {}^nC_rx^{n-r}y^r + \ldots + y^n$

  or

  $(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \ldots + \binom{n}{r}x^{n-r}y^r + \ldots + y^n$
- In each term, the powers of x and y add up to n.

Combinations

- A selection of r objects from n objects is called a **combination**.
- The number of combinations of r objects from n objects is ${}^nC_r$ or $\binom{n}{r}$.
- ${}^5C_2 = 10$: the number of ways of selecting 2 people from 5 people
- ${}^4C_2 = 6$: the number of ways of flipping 2 heads from 4 coins

Worksheet
Maths Methods Year 11 formula sheet

Cumulative examination 1

Technology-free

Perusal time: 2 minutes Working time: 30 minutes

TECH-FREE *Calculators are **not** permitted.*

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

Section 1 (3 marks): 3 multiple choice questions

Section 2 (15 marks): 4 short response questions

Total: 18 marks

Section 1 3 multiple choice questions

3 marks

1 $(3x - 5)^2 =$

A $9x^2 + 15x + 25$ **B** $9x^2 + 30x + 25$ **C** $9x^2 + 15x + 25$ **D** $9x^2 - 30x + 25$

2 9 coins are tossed. In how many ways can this result in 6 tails?

A 84 **B** 56 **C** 126 **D** 168

3 Rationalise the denominator of $\frac{\sqrt{3}}{2 - \sqrt{3}}$.

A $\sqrt{3}$ **B** $2\sqrt{3} - 3$ **C** $4\sqrt{3} - 3$ **D** $2\sqrt{3} + 3$

Section 2 4 short response questions

15 marks

4 (4 marks) Expand each expression.

a $(y + 8)^2$ *[1 mark]*

b $(2c + 5d)(2c - 5d)$ *[1 mark]*

c $(4g + 3h)(2g - 5h)$ *[2 marks]*

5 (3 marks) Simplify each expression.

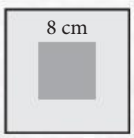
a $10\sqrt{3} - 3\sqrt{3} - 12\sqrt{3}$ *[1 mark]*

b $2\sqrt{48} \times 3\sqrt{12}$ *[1 mark]*

c $28\sqrt{5} \div \sqrt{980}$ *[1 mark]*

6 (4 marks) Simplify $\frac{3}{\sqrt{3}} - \frac{1}{2\sqrt{2}} + \frac{2}{\sqrt{12}}$.

7 (4 marks) An artist exhibits a square painting with a frame around it. The gap between the painting and the edge of the frame is a consistent 8 cm, as shown in the diagram.



The ratio of the gap to the width of the painting is $1 : \frac{1 + \sqrt{5}}{2}$.

Find the total area of the framed painting and simplify the expression.

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Cumulative examination 2

Technology-active

Perusal time: 2 minutes Working time: 30 minutes

*Calculators are **permitted**.*

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

Section 1 (3 marks): 3 multiple choice questions

Section 2 (15 marks): 4 short response questions

Total: 18 marks



Worksheet
Maths Methods Year 11 formula sheet

Section 1 3 multiple choice questions

3 marks

1 $\binom{20}{8} =$

A 75 582 **B** 77 520 **C** 125 970 **D** 167 960

2 $(\sqrt{3} - \sqrt{2})^3 =$

A $5 - 2\sqrt{6}$ **B** $\sqrt{3} - 11\sqrt{2}$ **C** $7\sqrt{3} - \sqrt{2}$ **D** $9\sqrt{3} - 11\sqrt{2}$

3 $(4d - m)^2 - (4d - m)(4d + m) =$

A $8dm$ **B** $32d^2 - 8dm$ **C** $2m^2 - 8dm$ **D** $32d^2 - 2m^2$

Section 2 4 short response questions

15 marks

4 (6 marks)

a Expand and simplify $(a + b)^6$. *[3 marks]*

b Hence show that $\left(3 + \sqrt{2}\right)^6 = 3707 + 2610\sqrt{2}$. *[3 marks]*

5 (3 marks)

a Write a numerical expression for the number of ways of flipping 5 heads from 10 coins. *[2 marks]*

b Calculate the number in part **a**. *[1 mark]*

6 (4 marks) Evaluate the terms that contain 3^{14} and 3^{13} in the expansion of the binomial $(3 - \sqrt{5})^{20}$ in scientific notation, correct to 4 significant figures.

7 (2 marks) Simplify $\binom{n}{n-1}$.

CHAPTER

2 QUADRATIC FUNCTIONS



Syllabus coverage

Nelson MindTap chapter resources

Terminology

2.1 Factorising quadratic expressions
- Perfect squares and the difference of 2 squares
- Grouping
- Monic quadratics
- The cross and decomposition methods

2.2 Quadratic equations
- Solving by factorising
- Solving by completing the square
- Solving by graphing

2.3 The quadratic formula and discriminant

2.4 Graphing linear functions (Revision)

2.5 Graphing quadratic functions
- Transformations of quadratic functions
- Sketching parabolas

2.6 Modelling with linear and quadratic functions

Exam question analysis

Chapter summary

Cumulative examination 1

Cumulative examination 2

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Syllabus coverage

UNIT 1, TOPIC 1: SURDS AND QUADRATIC FUNCTIONS

Quadratic functions

- Recognise and determine features of the graphs of $y = x^2$, $y = ax^2 + bx + c$, $y = a(x - h)^2 + k$ and $y = a(x - x_1)(x - x_2)$, including their parabolic nature, turning points, axes of symmetry and intercepts.
- Solve quadratic equations algebraically using factorisation, the quadratic formula (both exact and approximate solutions), completing the square and using technology.
- Sketch the graphs of quadratic functions, with and without technology.
- Use the discriminant to determine the number of solutions to a quadratic equation.
- Determine turning points and zeros of quadratic functions, with and without technology.
- Model and solve problems that involve quadratic functions, with and without technology.

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Prior learning
Quadratic functions

Videos (16):

2.1 Factorising quadratic expressions 1 • Factorising quadratic expressions 2 • Factorising quadratic expressions (Advanced)
2.2 Quadratic equations by factorising • Solving quadratic equations
2.3 The quadratic formula • Solving quadratic equations • The discriminant
2.4 Graphing linear equations • Gradient (slope) of a line • Finding the equation of a line • Gradient and y-intercept of a line
2.5 Graphing the parabola $y = a(x - b)^2 + c$
2.6 Applying linear functions • Linear modelling • Applying quadratic functions
Exam question analysis Quadratic functions

Prior learning (1):

2.1 Quadratic functions

Worksheets (21):

2.2 Completing the square
2.3 Quadratic formula • Quadratic equations • Problems involving quadratic equations
2.4 Graphing linear functions 1 • Graphing linear functions 2 • Linear graphs • Straight-line equations • Equations of a line • $y = mx + c$ • Finding the equation of a line • x- and y-intercepts
2.5 Sketching quadratic functions • Graphing quadratic functions • Graphing quadratics • Quadratic functions • Graphing parabolas 2 • Features of a parabola • A page of parabolas • Parabolas
Cumulative exams Maths Methods Year 11 formula sheet

Puzzles (5):

2.2 Factorising quadratic equations
2.3 The quadratic formula • Quadratic equations puzzle
2.4 Linear functions code puzzle
2.5 Matching parabolas with their equations

TI-Nspire CX II instructions (1):

2.2, 2.5 Quadratic functions

Technology (5):

2.1 Factorising trinomials
2.3 The quadratic formula
2.5 Investigating parabolas 1 • Investigating parabolas 2 • Curve sketcher

Spreadsheets (6):

2.1 Factorising trinomials
2.3 The quadratic formula
2.4 Graphing $y = mx + c$
2.5 Investigating parabolas 1 • Investigating parabolas 2 • Curve sketcher

Nelson MindTap

To access resources above, visit **cengage.com.au/nelsonmindtap**

Terminology

axis of symmetry	concave down	concave up	difference of 2 squares
dilation	discriminant	domain	factorising
function	gradient	highest common factor	monic
null factor law	parabola	perfect squares	quadratic
range	root	trinomial	turning point
variable	zero		

Videos
Factorising quadratic expressions 1

Factorising quadratic expressions 2

Factorising quadratic expressions (Advanced)

Technology
Factorising trinomials

Spreadsheet
Factorising trinomials

2.1 Factorising quadratic expressions

Factorising is the opposite of **expanding**. To factorise, an expression is rewritten with brackets as a product of factors. For example, $25x + 10 = 5(5x + 2)$, where the factors are 5 and $5x + 2$.

Factorising using the highest common factor (HCF)

1. Identify the **highest common factor** which divides into all the **terms**.
2. Write the HCF in front of the brackets.
3. Inside the brackets, write each of the original terms divided by the HCF.

WORKED EXAMPLE 1 Factorising using the HCF

Factorise each expression.

a $6xy - 9y^2$ **b** $3(x + 6) - 2a(x + 6)$ **c** $12x^2 + 18x - 6$

Steps	Working
a 1 Identify the HCF.	HCF $= 3y$
2 Write the HCF in front of the brackets. Inside the brackets, write each of the original terms divided by the HCF.	$6xy - 9y^2 = 3y(2x - 3y)$
	$-3y$ is also correct as the HCF. It gives the answer $-3y(3y - 2x)$. Both answers are correct.
b 1 Identify the HCF.	HCF $= (x + 6)$
2 Write the HCF in front of the brackets. Inside the brackets, write each of the original terms divided by the HCF.	$3(x + 6) - 2a(x + 6) = (x + 6)(3 - 2a)$
c 1 Identify the HCF.	HCF $= 6$
2 Inside the brackets, write each of the original terms divided by the HCF.	$12x^2 + 18x - 6 = 6(2x^2 + 3x - 1)$
	Check that the expression inside the brackets cannot be simplified further.

Exam hack

Check that there are no common factors of the terms inside the brackets. If there are, factorise again!

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Perfect squares and the difference of 2 squares

Perfect squares

$$a^2 + 2ab + b^2 = (a + b)^2$$
$$a^2 - 2ab + b^2 = (a - b)^2$$

A **quadratic** of the form $ax^2 + bx + c$ can be factorised as a **perfect square**. Look for a middle term of the correct form. It needs to be double the product of the square roots of the first and last terms.

WORKED EXAMPLE 2 Factorising as a perfect square

Factorise each expression.

a $x^2 + 10x + 25$ **b** $4k^2 - 12k + 9$

Steps	Working
a 1 Write the expression in the form $a^2 + 2ab + b^2$. $a = 1$ and $b = 5$	$x^2 + 10x + 25 = (1x)^2 + 2(1)(5)x + (5)^2$
2 Use the perfect square identity.	$= (x + 5)^2$
b 1 Write the expression in the form $a^2 - 2ab + b^2$. $a = 2$, $b = 3$. Take care with the signs.	$4k^2 - 12k + 9 = (2k)^2 - 2(2)(3)k + (3)^2$
2 Use the perfect square identity.	$= (2k - 3)^2$

WORKED EXAMPLE 3 Factorising as a difference of 2 squares

Factorise each expression.

a $w^2 - 9$ **b** $g^2 - 7$ **c** $16m^2 - 25$

Steps	Working
a 1 Write the expression in the form $a^2 - b^2$.	$w^2 - 9 = (w)^2 - (3)^2$
2 Use the difference of 2 squares identity.	$= (w + 3)(w - 3)$
b 1 Write the expression in the form $a^2 - b^2$.	$g^2 - 7 = (g)^2 - (\sqrt{7})^2$
2 Use the difference of 2 squares identity.	$= (g + \sqrt{7})(g - \sqrt{7})$
c 1 Write the expression in the form $a^2 - b^2$.	$16m^2 - 25 = (4m)^2 - (5)^2$
2 Use the difference of 2 squares identity.	$= (4m + 5)(4m - 5)$

Grouping

For 4 terms with no common factor between all the terms, try factorising by grouping.

Factorising by grouping

1. Identify 2 pairs of terms that have a common factor.
2. Make sure the pairs factorise so the brackets of each are the same.
3. Use the expression in brackets as a new common factor to factorise again.

WORKED EXAMPLE 4 Factorising using grouping

Factorise each expression.

a $x^2 - 4x + 28 - 7x$ **b** $4x^3 - 16x^2 + 4x - 16$

Steps	Working
a 1 Separate the expression into pairs that have a common factor.	$x^2 - 4x + 28 - 7x = (x^2 - 4x) + (-7x + 28)$
2 Factorise each pair individually.	$= x(x - 4) - 7(x - 4)$
3 Factorise again using $(x - 4)$ as the new common factor.	$= (x - 4)(x - 7)$
b 1 Separate the expression into pairs that have a common factor.	$4x^3 - 16x^2 + 4x - 16 = (4x^3 - 16x^2) + (4x - 16)$
2 Factorise each pair individually.	$= 4x^2(x - 4) + 4(x - 4)$
3 Factorise again using $(x - 4)$ as the new common factor.	$= (x - 4)(4x^2 + 4)$
4 Check if there are any more common factors – factorise again!	$= 4(x - 4)(x^2 + 1)$

Monic quadratics

A **monic** quadratic is a quadratic $ax^2 + bx + c$ where the coefficient of x^2 is 1; that is, $a = 1$.

To factorise $x^2 + bx + c$, find the values of p and q so that the sum $p + q = b$ and the product $pq = c$.

$(x + p)(x + q) = x^2 + (p + q)x + pq$

WORKED EXAMPLE 5 Monic quadratics

Factorise $x^2 - x - 20$.

Steps	Working
1 Write the quadratic.	$x^2 - x - 20$
2 Find 2 numbers with product -20 and sum -1.	$4 \times (-5) = -20$, $4 + (-5) = -1$
3 Write the answer.	$x^2 - x - 20 = (x + 4)(x - 5)$

The cross and decomposition methods

The common methods for non-monic quadratics are the **cross** and **decomposition** methods.

Cross method

1. Draw a cross and write factors of the first term, ax^2, on ends of the cross on the left.
2. Write factors of the last term, c, on the ends on the right.
3. Multiply each diagonal and add the results. Does it equal the middle, bx?
4. If it does, the top makes one factor and the bottom the other.
5. If it doesn't, try different factors of c until you've done them all.
   If it still doesn't work, try a new cross with different factors of ax^2.

WORKED EXAMPLE 6 The cross method

Factorise $4x^2 + 17x - 15$ using the cross method.

Steps	Working
1 Write factors of $4x^2$ on the left of the cross.	$2x$, $2x$
2 Write factors of -15, say -3×5, on the right.	$2x$, -3, $4x$, $2x$, 5
Cross-multiply and add. $10x + (-6x) = 4x$.	
$4x \neq 17x$. Try again.	
3 Try 3 and -5, 15 and -1, -15 and 1.	$2x$, ~~-3~~, ~~3~~, ~~15~~, ~~-15~~, $4x$, $-4x$, $28x$, $-28x$, $2x$, ~~5~~, ~~-5~~, ~~-1~~, ~~1~~
Nothing works!	
4 Set up a new cross with different factors of $4x^2$ on the left.	x, $4x$
5 Try different factors of -15 on the right. Stop when you get the right middle term.	x, ~~3~~, ~~-3~~, 5, $12x$, $-12x$, $17x$, $4x$, ~~-5~~, ~~-5~~, -3
5 and -3 works.	
6 Write the answer.	$4x^2 + 17x - 15 = (x + 5)(4x - 3)$

In a non-monic quadratic $ax^2 + bx + c$, consider $(ax + p)(x + r)$.

$(ax + p)(x + r) = ax^2 + arx + px + pr$

$= ax^2 + (p + ar)x + pr$

Let $ar = q$, so $r = \frac{q}{a}$.

Then $(ax + p)(x + r) = ax^2 + (p + q)x + p\left(\frac{q}{a}\right)$.

Comparing this to $ax^2 + bx + c$,

$p + q = b$ and $p\left(\frac{q}{a}\right) = c$; that is, $pq = ac$.

So to factorise $ax^2 + bx + c$ using the decomposition method, find 2 values p and q so that the sum $p + q = b$ and the product $pq = ac$.

Decomposition method

1. Multiply the coefficients of the first and last terms, ac.
2. Find values of p and q that add to give the middle term, b, and multiply to give ac, so $p + q = b$ and $pq = ac$.
3. Decompose bx to give $px + qx$.
4. Use grouping to factorise.

WORKED EXAMPLE 7 Decomposition method

Factorise $4x^2 + 17x - 15$ using the decomposition method.

Steps	Working
1 Multiply a and c.	$ac = -60$
2 Write some values of p and q that give a product of -60 until you get a sum of $+17$.	$-1, 60$ $-2, 30$ $-3, 20 \rightarrow$ that's it!
3 Decompose $17x$ into $-3x$ and $20x$.	$4x^2 + 17x - 15$ $= 4x^2 - 3x + 20x - 15$
4 Use grouping to factorise.	$= x(4x - 3) + 5(4x - 3)$
5 Use the common factor to get the answer.	$= (4x - 3)(x + 5)$

Exam hack

Choose the method that suits you, unless the question asks you to use a particular method.

The cross method is often quicker, but the decomposition method always works.

EXERCISE 2.1 Factorising quadratic expressions

ANSWERS p. 365

Mastery

1 WORKED EXAMPLE 1 Factorise each expression.

a $6a + 8$ **b** $-14x - 7y$ **c** $m^2 + 4m$

d $2x^2 - 4x$ **e** $a^2b + ab$ **f** $36ab^2 + 48a^2b$

2 WORKED EXAMPLE 2 Factorise each expression.

a $49b^2 - 56b + 16$ **b** $x^2 + 8xy + 16y^2$ **c** $4b^2 - 20bc + 25c^2$

3 WORKED EXAMPLE 3 Factorise each expression.

a $n^2 - 49$ **b** $100 - 81g^2$ **c** $81v^2 - 121w^2$

4 WORKED EXAMPLE 4 Factorise each expression.

a $2(a + b) + x(a + b)$ **b** $3a^2(x + y) - 17(x + y)$ **c** $(2 + a) - 3(2 + a)^2$

d $ac + bc - a - b$ **e** $5gi - 2gj - 5hi + 2hj$ **f** $km^2 - 2nh^2 - 2nm^2 + kh^2$

5 WORKED EXAMPLE 5 Factorise each monic quadratic using sum and product.

a $v^2 + 6v - 16$ **b** $x^2 - 15x - 16$ **c** $y^2 + 8y - 20$

d $m^2 + 7m + 12$ **e** $p^2 - 11p + 24$

6 WORKED EXAMPLE 6 Factorise each quadratic using the cross method.

a $2z^2 - 11z + 12$ **b** $6p^2 + 5p - 6$

7 WORKED EXAMPLE 7 Factorise each quadratic using the decomposition method.

a $3k^2 - 11k + 10$ **b** $2n^2 + 3n - 35$

8 Factorise each expression.

a $3x^2 - 11x + 10$ **b** $x^2 + 6x - 16$ **c** $7x^2 + x - 6$

d $2x^2 - 11x + 12$ **e** $4x^2 - 13x - 12$ **f** $8x^2 + 18x + 9$

g $5x^2 + 36x + 7$ **h** $3x^2 - 14x - 49$ **i** $7x^2 - 52x - 32$

Exam practice

9 $12x^3 - 18x^2 =$

A $x^2(12x - 18)$ **B** $6x(2x^2 - 3x)$ **C** $-6x^2(2x - 3)$ **D** $6x^2(2x - 3)$

10 $3x^2 + 2x - 15x - 10 =$

A $3x^2 - 13x - 10$ **B** $x(3x + 2) - 5(3x - 2)$ **C** $(3x + 2)(x - 5)$ **D** $x(3x + 2) - 5(3x + 2)$

11 $16w^2 - 9 =$

A $(4w + 3)(4w - 3)$ **B** $(4w + 9)(4w - 9)$ **C** $(4w + 3)(4w + 3)$ **D** $(4w - 3)(4w - 3)$

12 $4z^2 - 12z + 9 =$

A $(2w - 3)(2w + 3)$ **B** $(4w - 6)(w - 1)$ **C** $(2w + 3)(2w + 4)$ **D** $(2w - 3)^2$

13 (1 mark) Factorise $2d^2 - 9d - 5$.

14 (1 mark) Factorise $k^2 + 8 - 9k$.

15 (1 mark) Factorise $-2s^2 + 24s - 22$.

16 (2 mark) CF Factorise and simplify the expression $(x + z)^2 - (2x + z)^2$.

17 (2 mark) CF Factorise and simplify the expression $(x + 4)^2 - 12(x + 4) + 36$.

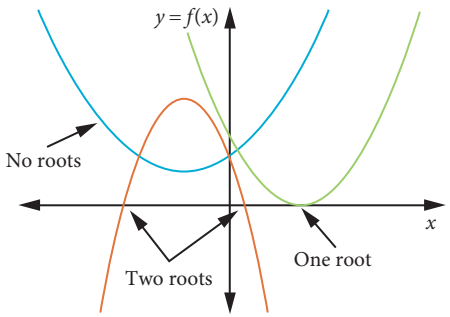
2.2 Quadratic equations

A quadratic equation has the standard (general) form $ax^2 + bx + c = 0$ where $a \neq 0$.

A **solution** or **root** of an equation is a value of the variable that makes the equation true.

Quadratic equations have 0, 1 or 2 solutions. A root of an equation is also a value of x where the graph crosses the x-axis; that is, an x-intercept (where $y = 0$), as shown in the graph.

The graphs of quadratics are all the same shape. The shape is called a **parabola**.



Videos
Quadratic equations by factorising

Solving quadratic equations

Worksheet
Completing the square

Puzzle
Factorising quadratic equations

Solving by factorising

Null factor law

If $ab = 0$, then either $a = 0$, $b = 0$ or $a = b = 0$.

The **null factor law** can be used to solve factorised quadratic equations.

WORKED EXAMPLE 8 Solving quadratic equations by factorising

Solve each quadratic equation.

a $(x - 3)(x + 4) = 0$ **b** $5p^2 + 2p - 10 = 2p^2 + 14p - 1$ **c** $3x^2 - 17x - 6 = 0$

Steps		Working
a **1**	Write the equation.	$(x - 3)(x + 4) = 0$
2	Apply the null factor law.	$(x - 3) = 0$ or $(x + 4) = 0$
3	Solve the resulting equations.	$x = 3$ or $x = -4$
b **1**	Write the equation.	$5p^2 + 2p - 70 = 2p^2 + 14p - 7$
2	Rearrange to standard form.	$3p^2 - 12p - 63 = 0$
3	Take out the common factor.	$3(p^2 - 4p - 21) = 0$
4	Factorise the quadratic.	$3(p - 7)(p + 3) = 0$
5	Solve using the null factor law.	$p = 7$ or $p = -3$
c **1**	Write the equation.	$3x^2 - 17x - 6 = 0$
2	Factorise the quadratic.	$(x - 6)(3x + 1) = 0$
3	Apply the null factor law.	$x - 6 = 0$ or $3x + 1 = 0$
4	Write the answers	$x = 6$ or $x = -\frac{1}{3}$

Exam hack

The null factor law can only be used when the quadratic equation equals 0.

Solving by completing the square

If a quadratic equation cannot be factorised, try **completing the square**. This involves making part of the equation into $x^2 \pm 2px + p^2 = (x \pm p)^2$.

1. Write the equation in standard form $ax^2 + bx + c = 0$.
2. Put the constant term on the right-hand side: $ax^2 + bx = -c$.
3. If necessary, divide the equation to make the coefficient of x^2 equal 1.
4. Halve the coefficient of x to get $\frac{b}{2}$.
5. Add the square of $\frac{b}{2}$ to both sides of the equation.
6. Factorise the perfect square on the LHS.
7. Take the square root of both sides and write equations for the positive and negative roots.
8. Solve the equations and state the solutions.

WORKED EXAMPLE 9 Completing the square

Solve $3p^2 = 12 - 15p$ by completing the square.

Steps	Working
1 Write the equation in standard form.	$3p^2 + 15p - 12 = 0$
2 Move the constant to the RHS.	$3p^2 + 15p = 12$
3 Divide by 3 to make $1p^2$.	$p^2 + 5p = 4$
4 Half the coefficient of p: 5.	$a = \frac{5}{2}$
5 Square $\frac{5}{2}$ and add to both sides.	$p^2 + 5p + \frac{25}{4} = 4 + \frac{25}{4}$
6 Factorise the LHS. Simplify the RHS.	$\left(p + \frac{5}{2}\right)^2 = \frac{41}{4}$
7 Take the square roots.	$p + \frac{5}{2} = \pm\frac{\sqrt{41}}{2}$
8 Solve the equations.	$p = \frac{-\sqrt{41} - 5}{2}$ or $p = \frac{\sqrt{41} - 5}{2}$

Exam hack

Remember to write the ± in front of the $\sqrt{\ }$.

Solving by graphing

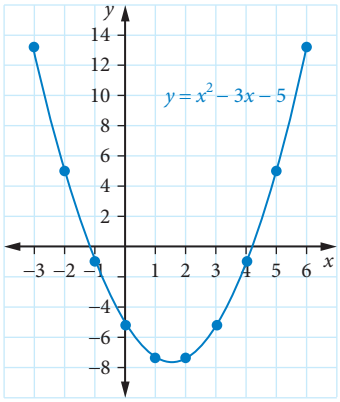
WORKED EXAMPLE 10 Solving quadratic equations graphically

Use a graph to find the roots of the quadratic $x^2 - 3x - 5 = 0$.

Steps	Working
1 Write the equation.	$x^2 - 3x - 5 = 0$
2 Express as a quadratic function.	$y = x^2 - 3x - 5$
3 Construct a table of values.	

x	−3	−2	−1	0	1	2	3	4	5	6
y	13	5	−1	−5	−7	−7	−5	−1	5	13

4 Use the values in the table to sketch the function, joining points with a smooth curve.



5 The roots of the quadratic are the values of x where $y = 0$ (the x-intercepts).

6 Estimate where $y = 0$.

7 Write the result.

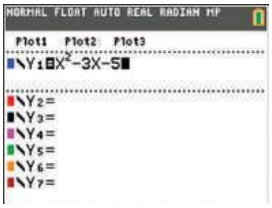
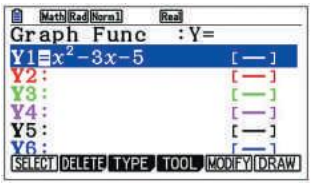
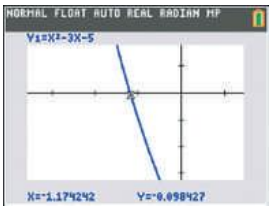
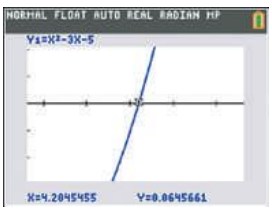
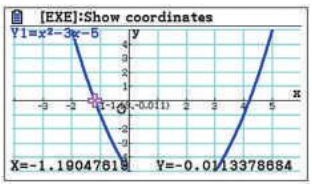
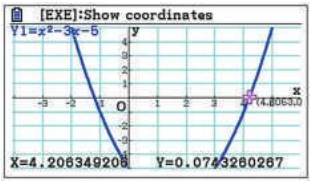
The roots of $x^2 - 3x - 5 = 0$ are $x \approx -1.2$ and $x \approx 4.2$.

A graphics calculator can be used to solve $x^2 - 3x - 5 = 0$.

TI-Nspire Quadratic functions

WORKED EXAMPLE 11 Solving quadratic equations using a graphics calculator

Use your graphics calculator to find the approximate roots of $x^2 - 3x - 5 = 0$.

Steps	Working	
	TI-84 Plus CE	**Casio fx-CG50 AU**
1 Enter the function $f(x) = x^2 - 3x - 5$ into the device.	Press y= and enter $Y_1 = x^2 - 3x - 5$ using X, T, θ, n, press ENTER and **graph**. 	Choose **5 Graph** and enter $Y_1 = x^2 - 3x - 5$ using X, T, θ, press EXE and F6. 
2 Use the zoom function to find the key points.	Use Window or Zoom and **Zbox** to find the part needed. As you scroll across the graph using **Trace** (F4), the value of Y changes from negative to positive and vice versa. The value of one root is between $x = -1.2121...$ and $x = -1.1742...$  Repeat to find the other root $4.1666... < x < 4.2045...$ Therefore, $x \approx -1.2$ or $x \approx 4.2$. 	Use V-Window (SHIFT F3) or Zoom (SHIFT F2) and **BOX** to find the part needed. As you scroll across the graph using **Trace** (q), the value of Y changes from negative to positive and vice versa. The value of one root is between $x = -1.2301...$ and $x = -1.1904...$  Repeat to find the other root $4.1666... < x < 4.2063...$ Therefore, $x \approx -1.2$ or $x \approx 4.2$. 

EXERCISE 2.2 Quadratic equations

ANSWERS p. 366

Recap

1 The factorisation of the expression $3x^2 + 6x + 3$ is

A $3x(x + 2) + 3$ **B** $3(x + 1)^2$ **C** $3(x + 1)(x - 1)$ **D** $(x + 1)^2$

2 Factorise the expression $ax^3 + 3x^2 - 8ax - 24$.

Mastery

3 WORKED EXAMPLE 8 Solve each quadratic equation. Factorise where necessary.

a $2x(x - 12) = 0$ **b** $(x + 2)(x - 5) = 0$ **c** $x^2 - 8x = 0$

d $x^2 + 9x + 14 = 0$ **e** $x^2 - 8x + 15 = 0$ **f** $x - 4 = \frac{5}{x}$

g $x^2 - 6x = -9$ **h** $2x^2 + 5x + 2 = 0$ **i** $3x^2 + 10x + 3 = 0$

4 Solve each quadratic equation by factorisation.

a $3x^2 + 5x - 12 = 0$ **b** $3x^2 + 15x + 12 = 0$ **c** $4x^2 - 28x - 32 = 0$

d $16(x^2 - 1) = 14x - 19$ **e** $(2m - 1)(1 + 3m) = 4$ **f** $6x^2 - 5x - 50 = 5x^2 - 5x - 1$

g $x^2 - 2x + 2 = \frac{1 + 3x}{4}$ **h** $16x = \frac{2x - 54}{x - 4}$ **i** $\frac{2(x + 2)}{15} - \frac{2}{x - 2} = 1$

5 WORKED EXAMPLE 9 Solve each quadratic equation by completing the square. Express answers in exact form.

a $k^2 + 4k - 5 = 0$ **b** $v^2 + 2v - 35 = 0$ **c** $j^2 = 9j - 8$

d $x^2 = x + 132$ **e** $m^2 = \frac{8}{9} - \frac{2m}{3}$ **f** $b^2 + 4b - 6 = 0$

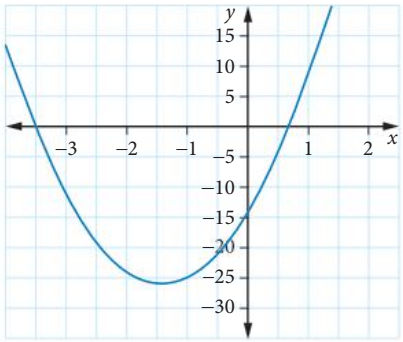
g $a^2 = 1 - a$ **h** $z^2 = \frac{3z}{2} + 2$ **i** $v^2 = 3(3v - 1)$

j $8x(x - 1) + 40 = 7x(x + 1)$ **k** $2x^2 + x - 6 = 0$ **l** $3c^2 = 10 - c$

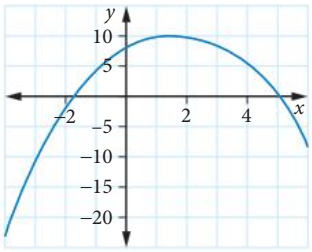
m $2m(m - 3) + 1 = 0$ **n** $7r^2 = 12r - 3$ **o** $h(5h - 7) = 1\frac{1}{2}$

6 WORKED EXAMPLE 10 For the graph of each quadratic function, use the graph to estimate the roots of the function correct to one decimal place.

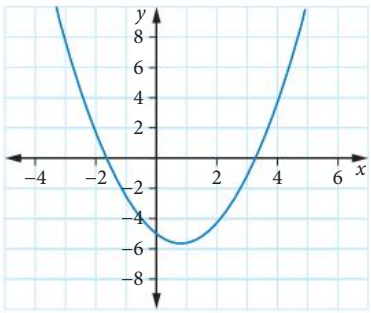
a



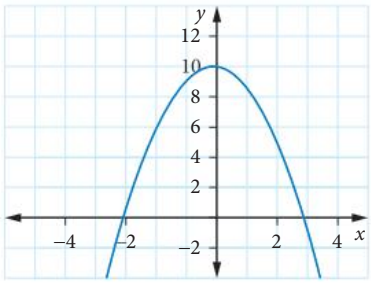
b



c



d



7 Graph each quadratic equation to find its approximate roots.

a $y = x^2 + x - 12$ **b** $y = 5 - 4x - x^2$ **c** $y = x^2 - 4x - 6$

d $y = 3x + 7 - 2x^2$ **e** $y = 2x^2 + 3x - 8$ **f** $y = \frac{x^2}{2} - 4x - 7$

8 WORKED EXAMPLE 11 Use a graphics calculator to graph each quadratic equation and find its approximate roots.

a $y = 2x^2 - 5x - 10$ **b** $f(x) = 8 - \frac{x^2}{2} - 4x$ **c** $y = 5x^2 - 14x - 11$

d $y = 6x + 31 - 3x^2$ **e** $y = 4x^2 - 8x - 26$ **f** $f(x) = \frac{x^2}{5} + 7x - 15$

Exam practice

9 The solutions of $(x + 3)(x - 5) = 0$ are

A $x = -3, x = -5$ **B** $x = 3, x = -5$ **C** $x = -3, x = 5$ **D** $x = 3, x = 5$

10 The solutions of $x^2 = 6x + 16$ are

A $x = 2, x = 8$ **B** $x = -2, x = 8$ **C** $x = 2, x = -8$ **D** $x = -2, x = -8$

11 The roots of the equation $2x^2 + 5x + 3 = 0$ are

A $x = -3$ **B** $x = -1$ **C** $x = \frac{3}{2}, x = 1$ **D** $x = \frac{-3}{2}, x = -1$

12 The solution(s) of $x^2 + 10x + 25 = 0$ is

A $x = -5$ **B** $x = 5$ **C** $x = \pm 5$ **D** $x = 10, x = 5$

13 (1 mark) Solve $6x^2 + 19x - 7 = 0$ using your graphics calculator.

14 (3 marks) Solve $5x^2 - 2x = 7$.

15 (3 marks) Solve $x^2 = -6x - 1$ by completing the square.

16 (4 mark) CF The length of a rectangular lawn is 0.9 m more than its width, and its area is 16.2 m^2. Write an equation and solve it to find the dimensions of the lawn.

17 (5 mark) CU A square room is 3 m high. 10 L of paint is needed to paint all the walls and ceiling in the room. 1 L of the paint covers 16 m^2. Write an equation and find the length of the room.

Videos
The quadratic formula

Solving quadratic equations

Worksheets
Quadratic formula

Quadratic equations

Problems involving quadratic equations

2.3 The quadratic formula and discriminant

The quadratic formula

For any quadratic equation $ax^2 + bx + c = 0$ where $a \neq 0$, its solutions are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where $b^2 - 4ac \geq 0$.

You can prove this formula by completing the square of $ax^2 + bx + c = 0$.

WORKED EXAMPLE 12 Using the quadratic formula

Solve each quadratic equation using the quadratic formula. Express solutions in both exact form and approximated to 2 decimal places.

a $3x^2 = 4x + 12$ **b** $5x^2 - 6x + 3 = 0$

Steps	Working
a 1 Write in standard form, $ax^2 + bx + c = 0$.	$3x^2 - 4x - 12 = 0$
2 Identify a, b and c.	$a = 3$, $b = -4$ and $c = -12$
3 Substitute into the quadratic formula and simplify.	$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $= \dfrac{-(-4) \pm \sqrt{(-4)^2 - 4 \times 3 \times (-4)}}{2 \times 3}$ $= \dfrac{4 \pm \sqrt{16 + 144}}{6}$ $= \dfrac{4 \pm \sqrt{160}}{6}$ $= \dfrac{4 \pm 4\sqrt{10}}{6}$ $= \dfrac{2(1 \pm \sqrt{10})}{3}$
4 Write the solutions exactly and to 2 decimal places.	$x = \dfrac{2(1 + \sqrt{10})}{3}$ and $\dfrac{2(1 - \sqrt{10})}{3}$ ≈ 2.77 and -1.44
b 1 Write the equation in standard form.	$5x^2 - 6x + 3 = 0$
2 Identify a, b and c.	$a = 5$, $b = -6$ and $c = 3$
3 Substitute into the quadratic formula and simplify. If you encounter the square root of a negative number, stop! There are no real solutions to this quadratic equation; that is, there are no square roots of negative numbers in the real number system.	$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $= \dfrac{-(-6) \pm \sqrt{(-6)^2 - 4 \times 5 \times 3}}{2 \times 5}$ $= \dfrac{6 \pm \sqrt{36 - 60}}{10}$ $= \dfrac{6 \pm \sqrt{-14}}{6}$ There is no square root of -14. Therefore, there are no real solutions to this quadratic equation.

When solving $5x^2 - 6x + 3 = 0$, you get $\sqrt{b^2 - 4ac} = \sqrt{-16}$ using the quadratic formula.

There is no real number whose square is -16.

Quadratic equations will have at most 2 solutions. The number of solutions depends on the value of the **discriminant**, $\Delta = b^2 - 4ac$.

Puzzles
The quadratic formula

Quadratic equations puzzle

Technology
The quadratic formula

Spreadsheet
The quadratic formula

Video
The discriminant

The discriminant

For any quadratic equation $ax^2 + bx + c = 0$, the discriminant is $\Delta = b^2 - 4ac$.

- If $\Delta > 0$, the quadratic equation has 2 distinct real solutions.
  - If Δ is a perfect square number, $\sqrt{\Delta}$ is an integer and the solutions are rational.
  - If Δ is not a perfect square number, the solutions are irrational.
- If $\Delta = 0$, the quadratic equation has one real solution.
- If $\Delta < 0$, the quadratic equation has no real solutions.

WORKED EXAMPLE 13 The discriminant

a Show that the quadratic equation $-4x^2 + 2x + 5 = 0$ has 2 real solutions.

b **Determine** the value(s) of k for which the quadratic equation $3x = 6x^2 + k$ has one solution.

Steps		Working
a	**1** Write in standard form $ax^2 + bx + c = 0$.	$-4x^2 + 2x + 5 = 0$
	2 Identify a, b and c.	$a = -4$, $b = 2$ and $c = 5$
	3 Substitute into the discriminant and simplify.	$\Delta = b^2 - 4ac$ $= 2^2 - 4 \times (-4) \times 5$ $= 84$
	4 Interpret the discriminant for the solutions.	$\Delta > 0$, so $-4x^2 + 2x + 5 = 0$ has 2 real solutions.
b	**1** Write the equation in standard form.	$6x^2 - 3x + k = 0$
	2 Identify a, b and c.	$a = 6$, $b = -3$ and $c = k$
	3 Substitute into the discriminant and simplify.	$\Delta = b^2 - 4ac$ $= (-3)^2 - 4 \times 6 \times k$ $= 9 - 24k$
	4 Make $\Delta = 0$ for one solution.	$9 - 24k = 0$
	5 Solve for k.	$24k = 9$ $k = \frac{9}{24}$ $= \frac{3}{8}$

EXERCISE 2.3 The quadratic formula and discriminant

ANSWERS p. 367

Recap

1 Solve $(2x + 3)(x - 9) = 0$.

2 Solve $x^2 + 12x + 14 = 0$ by completing the square.

Mastery

3 WORKED EXAMPLE 12 Solve each quadratic equation using the quadratic formula. Where appropriate, give both exact and approximate answers correct to 2 decimal places.

a $x^2 + 14x + 45 = 0$ **b** $x^2 = 7x + 294$ **c** $3f^2 + f - 14 = 0$ **d** $x^2 + 4x = 7$

e $x^2 - x - 5 = 0$ **f** $t^2 = 9t + 18$ **g** $x^2 = 13x + 140$ **h** $2d^2 + 3d = 6$

4 Solve each equation for x, using the most appropriate method.

a $x^2 + 6x - 10 = 0$

b $18x^2 + 15x = 12$

c $x^2 - 4x - 13 = 0$

d $x^2 - 6.146x + 9.328 = 0$

e $1 - \frac{x}{2} + \frac{x^2}{24} = 0$

f $\frac{1}{x+1} - \frac{4}{x} = \frac{1}{x+3}$

g $x^2 + 29x + 6 = 0$

h $3x^2 - 5.6x + 0.56 = 0$

5 WORKED EXAMPLE 13 Use the discriminant to determine whether each equation has 2 real roots, one real root or no real roots.

a $3x^2 + 5x - 1 = 0$

b $x^2 - 4x + 4 = 0$

c $2x^2 - 3x + 5 = 0$

d $-x^2 + 5x - 8 = 0$

e $-3x^2 - 6x - 3 = 0$

f $-x^2 - 2x + 5 = 0$

Exam practice

6 TECH-FREE The discriminant of $2x^2 - 5x - 4$ is

A -56 **B** -1 **C** 57 **D** 64

7 TECH-FREE The solutions of $3x^2 + 5x + 1 = 0$ are

A $x = \frac{-5 \pm \sqrt{13}}{6}$ **B** $x = -5 \pm \sqrt{13}$ **C** $x = \frac{5 \pm \sqrt{13}}{2}$ **D** $x = \frac{-5 \pm \sqrt{13}}{3}$

8 The solutions of $-5x^2 - 8x - 3 = 0$ are

A $x = -0.6, x = -1$ **B** $x = -1, x = 0.6$ **C** $x = -5, x = 2$ **D** $x = -1.2, x = -2$

9 TECH-FREE (2 marks) Determine the number of solutions of $x^2 - 4x + 9 = 0$.

10 TECH-FREE (1 mark) Show that $6x^2 + 1.5x - 5 = 0$ has 2 real solutions.

11 (2 marks) Find the roots of the equation $5x^2 = 4x + 116$.

12 TECH-FREE (4 marks) CF Show that the equation $x^2 - gx + (g - 1) = 0$ has rational solutions for all rational values of g.

13 TECH-FREE (5 marks) CF Hans can walk 1 km/h faster than Oscar. He completes a 20 km hike 1 hour before him. Write an equation and solve it to find their walking speeds.

14 (5 marks) CU A bath toy bought for $\$x$ is sold for \$24, making x% profit. What was the cost of the toy?

15 (5 marks) CU The total resistance R (in ohms) of two resistors R_1 and R_2 in parallel is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

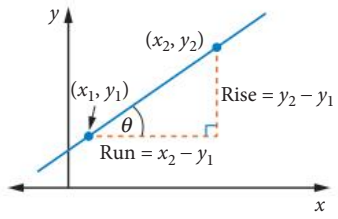
A particular resistor is placed in parallel with one 20 ohms greater in resistance.
The total resistance is 24 ohms. What are the resistances of the two resistors?

2.4 Graphing linear functions (Revision)

A **linear** function has the form $y = mx + c$.

The graph is a straight line with **gradient** m and ***y*-intercept** c.

The gradient (or **slope**) measures the 'steepness' of the line. This is the amount that y changes (rise) every time x increases by 1 (run).



The gradient between two points (x_1, y_1) and (x_2, y_2) is

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

A linear function (or equation) with:

- a positive gradient 'moves up' as the x values increase
- a negative gradient 'moves down' as the x values increase
- a zero gradient stays flat (horizontal) as the x values increase.

WORKED EXAMPLE 14 Calculating the gradient

Find the gradient of a line that

a passes through (2, 5) and (5, −7)

b intersects the x-axis at −2 and the y-axis at 3.

Steps	Working
a 1 Identify values.	Put $(x_1, y_1) = (2, 5)$, $(x_2, y_2) = (5, -7)$ into the gradient formula.
2 Write the gradient equation.	$m = \frac{y_2 - y_1}{x_2 - x_1}$
3 Substitute in the values.	$= \frac{-7 - 5}{5 - 2}$
4 Evaluate.	$m = -4$
b 1 Identify known values.	Let $(x_1, y_1) = (-2, 0)$, $(x_2, y_2) = (0, 3)$.
2 Write the gradient equation.	$m = \frac{y_2 - y_1}{x_2 - x_1}$
3 Substitute in the values and evaluate.	$= \frac{3 - 0}{0 - (-2)}$ $= 1.5$

Any linear equation can be rearranged into these forms.

The **general form** is $ax + by + c = 0$, where a, b and c are integers.

The **gradient–intercept form** is $y = mx + c$.

The **gradient–point form** is $y - y_1 = m(x - x_1)$, where (x_1, x_2) is a point on the line.

The **intercept form** is $\frac{x}{a} + \frac{y}{b} = 1$, where a is the x-intercept and b is the y-intercept.

WORKED EXAMPLE 15 Finding equations of straight lines

Determine the equation of a straight line that

a passes through $(-1, 4)$ and has a gradient of 3

b has a gradient of -5 and y-intercept of 7

c passes through the points $(3, 8)$ and $(6, 14)$

d has an x-intercept of -1 and a y-intercept of -9.

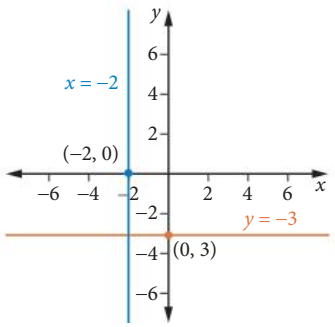
Steps	Working
a 1 Identify known values.	$m = 3, (x_1, y_1) = (-1, 4)$
2 Write the gradient-point form.	$y - y_1 = m(x - x_1)$
3 Substitute in the values.	$y - 4 = 3(x + 1)$
4 Simplify and express in general form.	$y - 3x - 7 = 0$
b 1 Identify known values.	$m = -5, c = 7$
2 Write the gradient–intercept form.	$y = mx + c$
3 Substitute in the values.	$y = -5x + 7$
4 Write in general form.	$y + 5x - 7 = 0$
c 1 Identify known values.	$(x_1, y_1) = (3, 8), (x_2, y_2) = (6, 14)$
2 Find the gradient.	$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{14 - 8}{6 - 3} = 2$
3 Write the gradient-point form.	$y - y_1 = m(x - x_1)$
4 Substitute in the values.	$y - 8 = 2(x - 3)$
5 Express in general form.	$y - 2 - 2x = 0$
d 1 Identify known values.	$a = -1, b = -9$
2 Write the intercept form.	$\frac{x}{a} + \frac{y}{b} = 1$
3 Substitute in the values, simplify and express in general form.	$\frac{x}{-1} + \frac{y}{-9} = 1$ $\frac{-9x - y}{9} = 1$ $-9x - y - 9 = 0$

Exam hack

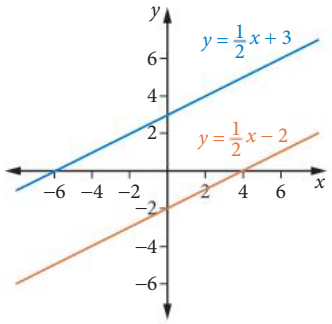
Using $y - y_1 = m(x - x_1)$ is often the fastest way to find the equation.

A **horizontal** line has zero gradient and is represented by $y = c$, where c is the y-intercept.

A **vertical** line has undefined gradient and is represented by $x = d$, where d is the x-intercept.

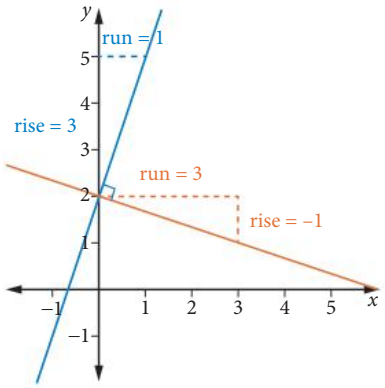


Parallel lines have the same gradient, so $m_1 = m_2$.



Perpendicular lines intersect at a right angle (90°).

Their gradients are negative reciprocals of each other, meaning that they multiply to −1.



So, their gradients are related as $m_1 m_2 = -1$, or $m_2 = -\frac{1}{m_1}$.



Exam hack

When finding equations of perpendicular lines, use $m_2 = -\frac{1}{m_1}$.

WORKED EXAMPLE 16 Finding equations of parallel and perpendicular lines

a **Determine** the equation of a line parallel to $2y + 13x - 9 = 0$ and passing through $(1, 3)$.

b **Determine** the equation of a line perpendicular to $5x - 9y - 1 = 0$ and passing through $(6, 2)$.

Steps	Working
a **1** Rearrange to the form $y = mx + c$.	$2y + 13x - 9 = 0$
	$y = -6.5x + 4.5$
2 Identify values.	$m = -6.5$
3 Write the gradient-point form.	$y - y_1 = m(x - x_1)$
4 Substitute in the values.	$y - 3 = -6.5(x - 1)$
5 Simplify and express in general form.	$y + 6.5x - 9.5 = 0$
b **1** Write in the form $y = mx + c$.	$y = \frac{5}{9}x - 1$, so $m_1 = \frac{5}{9}$. Line is perpendicular and the point is $(6, 2)$.
2 Write m and a point for the new line.	$m_2 = -\frac{1}{m_1} = -\frac{9}{5}$, $(x_1, y_1) = (6, 2)$
3 Write the gradient–point form.	$y - y_1 = m(x - x_1)$
4 Substitute in the values.	$y - 2 = -\frac{9}{5}(x - 6)$
5 Simplify.	$y = -1.8x + 12.8$

Sketching linear graphs

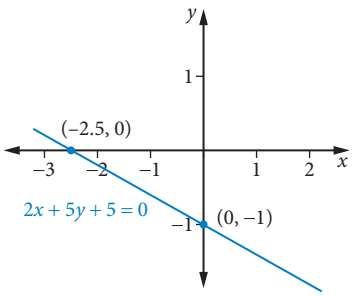
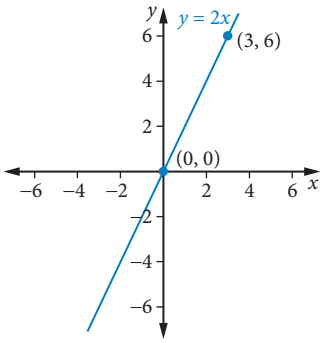
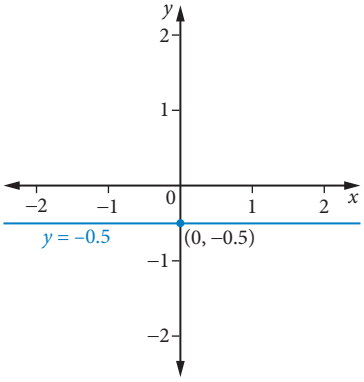
To sketch a graph of a linear equation in any form:

1. find the y-intercept by letting $x = 0$ and solving for y
2. find the x-intercept or another point using a different value of x
3. draw a straight line through the 2 **intercepts**
4. label the intercepts and the line.

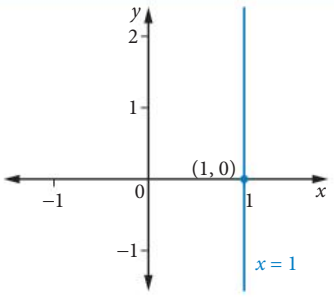
WORKED EXAMPLE 17 Sketching linear functions

Sketch the graph of each linear equation.

a $2x + 5y + 5 = 0$ **b** $y = 2x$ **c** $y = -0.5$ **d** $x = 1$

Steps	Working
a 1 Find the y-intercept.	$0 + 5y + 5 = 0$ The y-intercept is $(0, -1)$.
2 Find the x-intercept.	$2x + 0 + 5 = 0$ The x-intercept is $(-2.5, 0)$.
3 Sketch the graph, labelling all important points.	
b 1 Find the y-intercept.	$y = 2x = 2 \times 0 = 0$ The y-intercept is $(0, 0)$.
2 The x-intercept is the same, so choose another value of x.	At $x = 3$, $y = 6$. The point $(3, 6)$ is on the line.
3 Sketch the graph.	
c The equation will be parallel to the x-axis at $y = -0.5$. Sketch the graph.	

d The equation will be parallel to the y-axis at $x = 1$. Sketch the graph.



EXERCISE 2.4 Graphing linear functions (Revision)

ANSWERS p. 367

Recap

1 The solutions for the equation $3t^2 - t - 2 = 0$ are

A $(3t + 2)(t - 1)$ **B** $t = -1, t = 2$ **C** $t = -\frac{2}{3}, t = 1$ **D** $3(t + 2)(t - 1)$

2 The values of k for which $x^2 + 2x + k$ has 2 real solutions are

A $k < -1$ **B** $k = 1$ or -1 **C** $k \geq -1$ **D** $k < 1$

Mastery

3 Identify whether each equation is a linear equation.

a $x(x + 4) = 12$ **b** $5p - 3 = 6p + 9$ **c** $2d + 3 - 8d = \frac{1}{d}$

4 WORKED EXAMPLE 14 Find the gradient of a line that

a passes through $(3, 5)$ and $(5, 13)$ **b** passes through $(-1, -1)$ and $(-4, -4)$

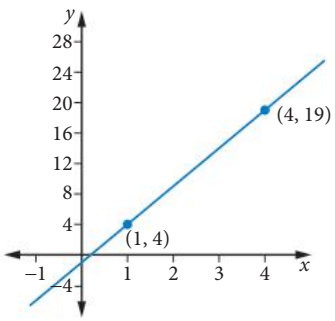
5 Find the gradient and y-intercept of the line represented by each linear equation.

a $y = -3x - 4$ **b** $2x - y + 5 = 0$ **c** $y = x$

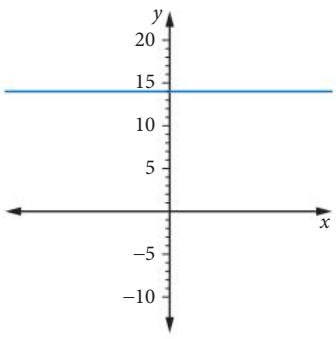
d $\frac{x}{8} + \frac{y}{9} = 1$ **e** $y = 3$ **f** $15 - 3x + 2y = 0$

6 Write the equation of each line in general form $ax + by + c = 0$.

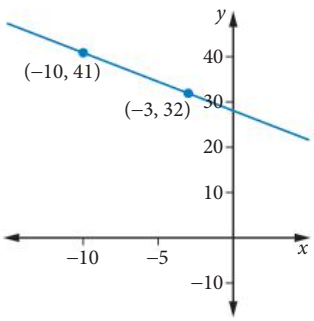
a



b



c



7 WORKED EXAMPLE 15 Find the equation of the line

a passing through $(1, 5)$ with slope 6

b passing through $(3, -1)$ and having slope -1

c horizontal and passing through $(1, 4)$

d passing through the points $(-5, -4)$ and $(-1, -6)$

e vertical and passing through $(6, -3)$

f that cuts the axes at $(-3, 0)$ and $(0, 2)$.

9780170484909

8 WORKED EXAMPLE 16 Find the equation of the line

a with gradient $-\frac{3}{4}$ and y-intercept -3

b that passes through $(3, 2)$ and is parallel to the x-axis

c that is parallel to the y-axis and passes through $(-1, 5)$

d with x-intercept 6 and y-intercept 2

e that passes through $(1, -7)$ and is perpendicular to the y-axis

f that is perpendicular to $y = -\frac{1}{2}x + 2.5$ and passes through $(-3, 6.5)$.

9 WORKED EXAMPLE 17 Sketch the graph of each linear equation. Label the x- and y-intercepts.

a $y = 3.5x + 14$ **b** $0 = -0.3x - 6 - y$ **c** $y = -4$

d $x = 5$ **e** $y + \frac{2x}{9} - 6 = 0$ **f** $y = 0.5x - 4$

Exam practice

10 TECH-FREE The gradient of the line passing through $(2, 3)$ and $(4, 7)$ is

A $\frac{1}{2}$ **B** $\frac{2}{3}$ **C** 2 **D** 3

11 TECH-FREE The equation of a line parallel to the x-axis and passing through $(3, 5)$ is

A $x = 3$ **B** $x = 5$ **C** $y = 3$ **D** $y = 5$

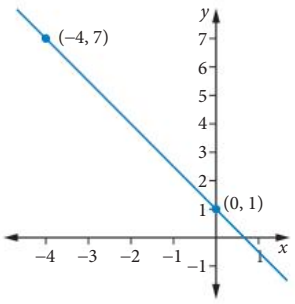
12 TECH-FREE What is the equation of a line with a slope of 4 and passing through $(0, -1)$?

A $y - 4x + 1 = 0$ **B** $y - 4x - 1 = 0$ **C** $y + 4x + 1 = 0$ **D** $y + 4x - 1 = 0$

13 TECH-FREE The graph represents the line

A $y = -1.5x + 1$ **B** $y = -1.5x - 1$

C $y = 1.5x + 1$ **D** $y = 1.5x - 1$



14 TECH-FREE (2 marks) What is the equation of the line passing through $(12, 9)$ and $(17, 11)$?

15 (2 marks) Draw a graph of the line $\frac{x}{4} + \frac{y}{9} = 6$. Label the axes intercepts.

16 (2 marks) Draw a graph of $15y + 3x = 0$. Label the axes intercepts.

17 (2 marks) The equations of 6 lines are as follows:

L_1: $5x + 6y + 4 = 0$ L_2: $7x + 3y - 8 = 0$ L_3: $2x + 5y + 2 = 0$

L_4: $8x + 3y - 1 = 0$ L_5: $5x + 6y - 5 = 0$ L_6: $5x - 2y + 9 = 0$

a Which lines are parallel? *[1 mark]*

b Which lines are perpendicular? *[1 mark]*

18 (8 marks) CF The vertices of a triangle are $(2, 3)$, $(5, 4)$ and $(5, -2)$.

a Draw a sketch to represent this information. *[3 marks]*

b Find the equations of the sides of the triangle. *[3 marks]*

c Determine whether the points $(4, 2)$ and $(3, -1)$ are inside or outside the triangle. *[2 marks]*

19 (4 marks) CF Show that the triangle with vertices $(1, -3)$, $(3, 2)$ and $(6, -5)$ is right-angled.

20 (4 marks) CU It costs \$65 for 5 people to have a small banquet, and \$80 for 8 people. The cost of the banquet is worked out to pay the fixed costs plus 'so-much a head'. What would be the cost of a similar banquet for 15 people?

Worksheets
Sketching quadratic functions
Graphing quadratic functions
Graphing quadratics
Quadratic functions
Graphing parabolas 2
Features of a parabola
A page of parabolas
Parabolas

Puzzle
Matching parabolas with their equations

Technology
Investigating parabolas 1
Investigating parabolas 2
Curve sketcher

Spreadsheets
Investigating parabolas 1
Investigating parabolas 2
Curve sketcher

2.5 Graphing quadratic functions

A **quadratic function** has the general form $y = ax^2 + bx + c$.
The shape of its graph is a parabola.

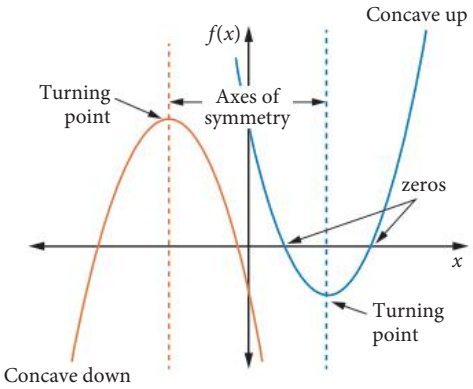
A parabola is symmetrical about a vertical axis passing through its **turning point**. This axis is called the **axis of symmetry**.

The equation for the axis of symmetry is $x = \frac{-b}{2a}$.

The turning point is also at $x = \frac{-b}{2a}$.

If a is positive, then y is **concave up**, like $\cup$, and has a **minimum**. The slope of the graph changes from negative to positive at the turning point.

If a is negative, then the parabola is **concave down**, like $\cap$, and has a **maximum**. The slope of the graph changes from positive to negative at the turning point.



A quadratic function has up to 2 **zeros** (roots, x-intercepts).

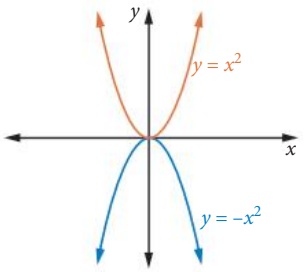
- If $\Delta > 0$, the quadratic function has 2 zeros.
- If $\Delta = 0$, the quadratic function has one zero.
- If $\Delta < 0$, the quadratic function has no zeros.

Transformations of quadratic functions

The graph of any quadratic function is a transformation of the basic parabola $y = x^2$, whose turning point is the origin, $(0, 0)$, and axis of symmetry is $x = 0$. (the y-axis)

Reflecting in the x-axis

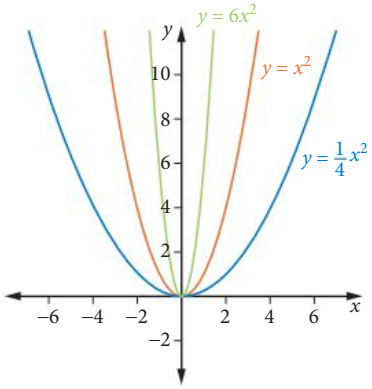
The graph of $y = -x^2$ is obtained by reflecting the graph of $y = x^2$ in the x-axis.



Dilation: making the parabola wider or narrower

The size of a in $y = ax^2 + bx + c$ is the **dilation** factor.

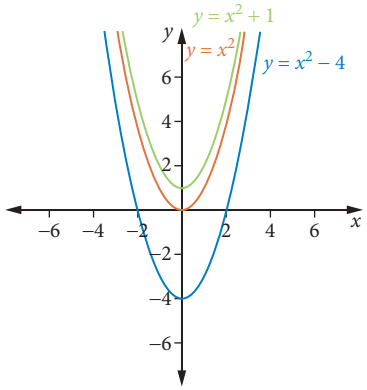
- If $a > 1$, every y value is dilated away (stretched) from the x-axis. This causes the graph to appear narrower, such as $y = 6x^2$ in the diagram.
- If $0 < a < 1$, every y value is dilated towards (compressed) the x-axis. This causes the graph to appear wider, such as $y = \frac{1}{4}x^2$ in the diagram.
- If *a is negative*, the graph is reflected in the x-axis and dilated.



Vertical translation: moving the graph up or down

In the form $y = ax^2 + k$:

- if $k > 0$, the graph is moved vertically up k units
- if $k < 0$, the graph is moved vertically down k units
- the turning point is $(0, k)$.

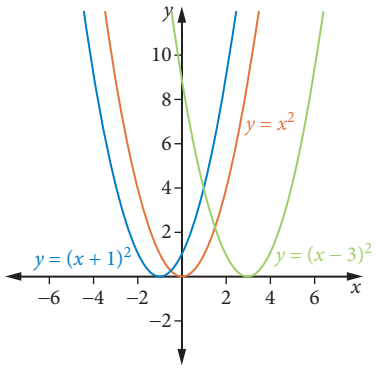


Horizontal translation: moving the graph left or right

In the form $y = a(x - h)^2$:

- if $h > 0$, the graph is moved to the right by h units
- if $h < 0$, the graph is moved to the left h units
- the turning point is $(h, 0)$.

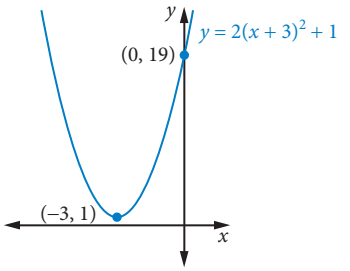
> If h is negative, this will show as a positive number in the form $y = a(x - h)^2 + k$ and give a translation of h units to the left.



Video
Graphing the parabola $y = a(x - b)^2 + c$

WORKED EXAMPLE 18 Using transformations to sketch a graph

Sketch the graph of $y = 2(x + 3)^2 + 1$.

Steps	Working
1 Identify the key values from the function.	$a = 2, h = -3, k = 1$
2 Consider how the graph of $y = x^2$ has transformed to become $y = 2(x + 3)^2 + 1$.	As a is positive the parabola is concave up (a minimum). The graph is vertically dilated by a factor of 2 from the x-axis. It has moved up 1 unit and to the left 3 units. The new turning point is $(-3, 1)$.
3 Find the y-intercept.	When $x = 0$, $y = 2 \times 3^2 + 1 = 19$.
4 Sketch the graph, labelling the key points.	

Sketching parabolas

There are three forms of the equation of a parabola:

- general form: $y = ax^2 + bx + c$
- intercept form: $y = a(x - x_1)(x - x_2)$
- turning point form: $y = a(x - h)^2 + k$.

When sketching a graph, show the general shape and important points.

Sketching parabolas

To sketch a graph of a parabola find these key features and label them clearly on the graph:

- y-intercept (when $x = 0$)
- x-intercepts, if they exist (when $y = 0$)
- turning point.

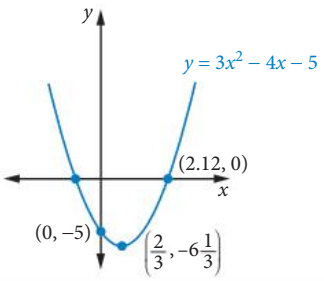
The general form of a parabola

$y = ax^2 + bx + c$

- The x value of the turning point is $x = \frac{-b}{2a}$.
- The y-intercept is c.
- The x-intercepts are found by substituting $y = 0$ and solving the quadratic equation.

WORKED EXAMPLE 19 Sketching quadratics in general form

Sketch the graph of $y = 3x^2 - 4x - 5$.

Steps	Working
1 The equation can't be factorised. Find the zeros using the quadratic formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $= \frac{4 \pm \sqrt{76}}{6}$ $= \frac{2 \pm \sqrt{19}}{3}$
2 State the x-intercepts.	$y = 0$ at $x \approx -0.79$ and $x \approx 2.12$.
3 Find the turning point at $x = \frac{-b}{2a}$.	$a = 3, b = -4$. $x = \frac{-(-4)}{2 \times 3} = \frac{4}{6} = \frac{2}{3}$ The turning point is at $x = \frac{2}{3}$. When $x = \frac{2}{3}$, $y = -6\frac{1}{3}$, so the turning point is $\left(\frac{2}{3}, -6\frac{1}{3}\right)$.
4 Identify the shape of the graph.	$a = 3$, so the graph is concave up, with a minimum turning point.
5 State the y-intercept.	When $x = 0$, $y = -5$.
6 Sketch the graph.	

Exam hack

Try to factorise first to find the zeros.

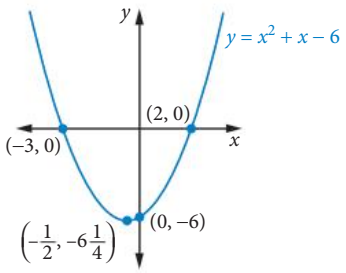
The intercept form of a parabola

$y = a(x - x_1)(x - x_2)$, where x_1 and x_2 are the x-intercepts.

- By symmetry, the turning point must be halfway between the x-intercepts.

WORKED EXAMPLE 20 Sketching quadratics in intercept form

Sketch the graph of $y = x^2 + x - 6$.

Steps	Working
1 Factorise if possible.	$y = x^2 + x - 6$ $= (x + 3)(x - 2)$
2 State the x-intercepts.	$y = 0$ at $x = -3$ and $x = 2$.
3 Use symmetry instead of the formula to find the turning point.	The turning point is at $x = \frac{-3 + 2}{2} = -\frac{1}{2}$.
4 State the turning point.	When $x = -\frac{1}{2}$, $y = 6\frac{1}{4}$, so the turning point is $\left(-\frac{1}{2}, 6\frac{1}{4}\right)$.
5 Identify the shape of the graph.	$a = 1$, so the graph is concave up with a minimum turning point.
6 Find the y-intercept.	When $x = 0$, $y = -6$.
7 Sketch the graph.	

The turning point form of a parabola

$y = a(x - h)^2 + k$, where (h, k) is the turning point.
The position of the y-intercept and the turning point will tell you if there are any zeros (x-intercepts).

WORKED EXAMPLE 21 Sketching quadratics in turning point form

Sketch the graph of $y = \frac{-(x - 1)^2}{3} - 2$.

Steps	Working
1 Write in the exact turning point form.	$y = -\frac{1}{3}(x - 1)^2 - 2$ $h = 1$ and $k = -2$. The turning point is $(1, -2)$.
2 Identify the shape of the graph.	$a = -\frac{1}{3}$, so the graph is concave down with a maximum turning point.

3 Find the y-intercept.

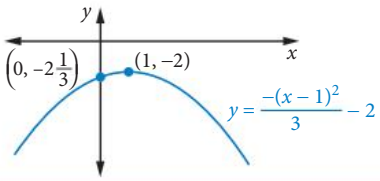
$y = -\frac{1}{3}(0-1)^2 - 2 = -2\frac{1}{3}$

The y-intercept is $\left(0, -2\frac{1}{3}\right)$.

4 Determine if there are any zeros.

The turning point is a maximum below the x-axis so there are no zeros.

5 Sketch the graph.

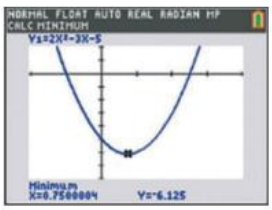


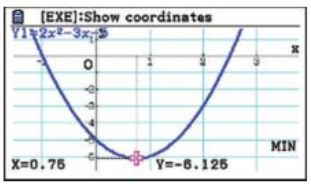
TI-Nspire
Quadratic functions

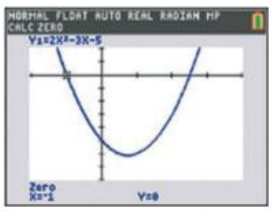
WORKED EXAMPLE 22 Using a graphics calculator to graph quadratics

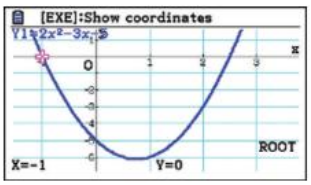
Sketch the graph of $y = 2x^2 - 3x - 5$, using your calculator to identify all key points.

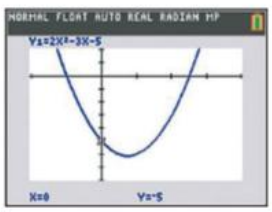
Steps	Working	
	TI-84 Plus CE	**Casio fx-CG50 AU**
1 Enter the function $f(x) = 2x^2 - 3x - 5$ into the device.	Press y= and enter $Y_1 = 2x^2 - 3x - 5$ using X, T, θ, n, press ENTER and Graph.	Choose **5** Graph and enter $Y_1 = 2x^2 - 3x - 5$ using X, T, θ, press EXE and F6.
2 Use the zoom function to find the key points.	Use Window or Zoom and **Zbox** to find the part needed. Use calc (2nd Trace) to find the zeros, minimum and y-intercept. Use the left and right cursor controls to set left, right and a guess in-between to find the minimum and zeros. Use **value** and set X = 0 to find the y-intercept.	Use V-Window (SHIFT F3) or **Zoom** (SHIFT F2) and **BOX** to find the part needed. Use **G-Solv** (SHIFT F5) to find the zeros, minimum and y-intercept. Press ▶ to get the second zero (ROOT).

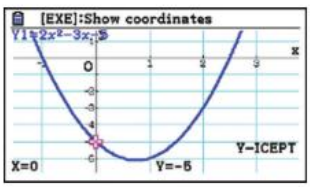












Exam hack

To earn full marks in an exam, don't just copy a graph from your calculator. Label all important points when you sketch a graph.

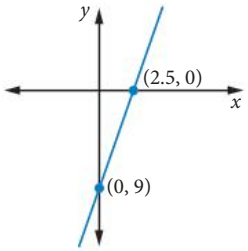
Use your calculator whenever possible to check your answers.

EXERCISE 2.5 Graphing quadratic functions

ANSWERS p. 368

Recap

1 This graph represents the line



A $y = -3.6x + 9$ **B** $y = -3.6x - 9$

C $y = 3.6x + 9$ **D** $y = 3.6x - 9$

2 Find the equation of the line passing through $(-2, 4)$ and $(2, 10)$.

Mastery

3 WORKED EXAMPLE 18 Describe the changes to $y = x^2$ to become each parabola.

a $y = -4x^2 + 2$ **b** $y = \frac{1}{2}(x + 3)^2$ **c** $y = \frac{-2(x - 4)^2}{5} - 7$

4 WORKED EXAMPLE 19 Sketch the graph of each quadratic function.

a $y = 9x^2 + 12x - 1$ **b** $y = -12x^2 - 12x - 2$

c $y = 8x^2 - 16x + 7$ **d** $y = -2x^2 + 12x - 17$

5 WORKED EXAMPLE 20 Sketch the graph of each quadratic function.

a $y = (x - 1)(x - 4)$ **b** $y = -2(x + 2)(x + 5)$

c $y = (2x - 1)(x + 3)$ **d** $y = (2 - x)(3x + 5)$

6 WORKED EXAMPLE 21 Sketch the graph of each quadratic function.

a $y = (x + 1)^2 - 2$ **b** $y = 3(x - 2)^2 + 3$ **c** $y = 9 + (x - 1)^2$

d $y = 1 - 2(x + 3)^2$ **e** $y = 12 - 4(x + 2)^2$ **f** $y = -2 - 4(x - 4)^2$

7 Sketch the graph of each quadratic function.

a $y = x^2 - 3x - 4$ **b** $y = 15 - x^2 - 2x$ **c** $y = 3x - 2x^2 + 20$

d $y = 14x - 4x^2 - 6$ **d** $y = x^2 - 2x + 2$ **e** $y = 2x^2 - 8x + 5$

8 Sketch the graph of each quadratic function and state its range of y values.

a $y = x^2 + 6x + 19, -4 < x < 1$ **b** $y = 4x^2 - 4x - 3, -1 < x \leq 1$

c $y = 2x^2 + 8x + 5, -4 < x \leq 1$ **d** $y = 8 - 10x - 3x^2, -3 < x \leq 2$

e $y = -4x^2 - 8x - 1$ **f** $y = 4x - 2x^2 - 6$

g $y = 4x^2 - 20x + 41$ **h** $y = -3x^2 - 24x - 5$

i $y = 2x^2 + 5x + 3$ **j** $y = 3x^2 - 8x - 3$

k $y = -x^2 - 8x - 15, -6 < x < 1$ **l** $y = 12x - 3x^2 + 15, -2 < x \leq 7$

9 WORKED EXAMPLE 22 Use your graphics calculator to find the key points of each quadratic graph.

a $y = x^2 + 6x + 2$ **b** $y = 9 - 2x^2 - 5x$

c $y = 23x - 4x^2 - 1$ **d** $y = 7 - 5x + 2x^2$

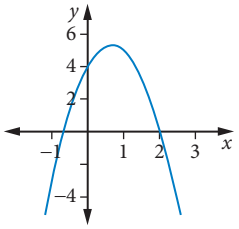
Exam practice

10 TECH-FREE (6 marks) Match each quadratic function with its correct sketch.

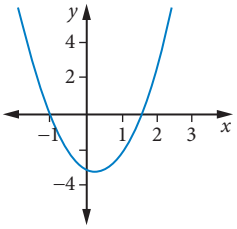
a $y = x^2 + 5$ **b** $y = -x^2 + 8x$ **c** $y = -3x^2 + 4x + 4$

d $y = 2x^2 - x - 3$ **e** $y = 2x^2 + 9x + 5$ **f** $y = -6x^2 + 7x + 3$

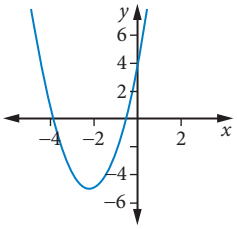
A



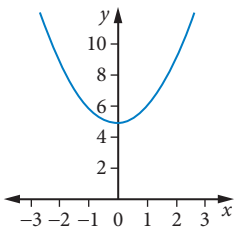
B



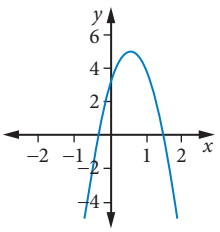
C



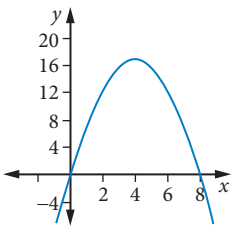
D



E



F



11 TECH-FREE The turning point of the function $y = x^2 + 6x + 10$ is

A $x = -3$ **B** $y = 1$ **C** $(0, 10)$ **D** $(-3, 1)$

12 TECH-FREE The zeros for the function $y = 2x^2 + 5x + 3$ are

A $(x + 1)(2x + 3)$ **B** $(0, 3)$ **C** $(-1, 0), (-3, 0)$ **D** $(-1, 0), (-1.5, 0)$

13 TECH-FREE (2 marks) For the equation $y = 2x^2 + 4x + 21$, identify the turning point and its type.

14 (4 marks) Sketch the graph of $y = x^2 + 6x + 2$.

15 (4 marks) CF Show that the turning point of $y = ax^2 + bx + c$ occurs at $\left(\frac{-b}{2a}, \frac{4ac - b^2}{4a}\right)$.

16 (6 marks) CU Find the equation of each parabola in the form $y = ax^2 + bx + c$.

a y–intercept of –6 and turning point at $(2, -2)$ *[2 marks]*

b y–intercept of –3 and x–intercepts of –1 and 2 *[2 marks]*

c x–intercepts of –2 and 3 and a maximum of 5 *[2 marks]*

17 (4 marks) CU Find the distance between the turning points of $y = 10x - x^2$ and $2y = 75 + 10x - x^2$.

2.6 Modelling with linear and quadratic functions

Many real-world problems can be solved using linear or quadratic equations.

You should identify and name the variable to express a problem algebraically. Then translate the information into an equation.

WORKED EXAMPLE 23 Modelling with linear equations

Matt's mum told him that if he paid the deposit on a skateboard, she would pay the rest. Matt paid a deposit of one-fifth of the price and left his mother $48 to pay. Write an equation to find the price of the skateboard and **calculate** how much Matt paid.

Steps	Working
1 Identify the variable.	Let C = cost of the skateboard.
2 Form the equation.	cost = deposit + remaining amount $C = \frac{1}{5}C + 48$
3 Solve the equation.	$\frac{4}{5}C = 48$ $C = \$60$
4 Deduct $48 from $60 to find out how much Matt paid.	$\$60 - \$48 = \$12$ (or $\frac{1}{5} \times \$60 = \$12$) The skateboard cost $60 and Matt paid $12.



Videos
Applying linear functions
Linear modelling
Applying quadratic functions

Applications involving quadratic relationships

The meanings of the key features are essential when using quadratics in practical situations.

- The x-intercepts occur when the y value is zero, e.g. zero height.
- The y-intercept occurs when the x value is zero, e.g. time zero.
- The turning point occurs when the y value is a maximum or minimum, e.g. maximum temperature.

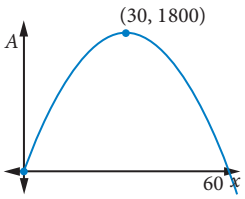
WORKED EXAMPLE 24 Minimum and maximum problems

A farmer wants to set up a rectangular pen with 120 m of temporary fencing against an existing straight fence.

Use a model to find the dimensions the farmer should use to get the maximum area for their sheep.

Verify and **evaluate** the usefulness of the model.

Steps	Working
1 Draw a diagram and choose variables.	Let the width = x m. 
2 Find the other dimension.	perimeter = length + $2x = 120$ length = $(120 - 2x)$ m
3 Write the area.	area, $A = x(120 - 2x) = 2x(60 - x)$
4 Find the zeros.	$A = 0$ at $x = 0$, $x = 60$
5 Find the turning point.	midpoint of zeros, $x = 30$ Since $-2x^2$, the area formula is negative, so there is a maximum turning point. When $x = 30$, $A = 1800$. Maximum point at $(30, 1800)$.

6	Sketch the graph to check.	
7	State the answer.	The farmer should choose 30 m wide by 60 m length parallel to the existing fence to get the maximum area.
8	Verify the model.	length of fencing used = (2×30) m + 60 m = 120 m area = 30 m × 60 m = 1800 m^2
9	Check the areas of pens with lengths close to 60 m.	61 m long: width = 29.5 m, area = 29.5 m × 61 m = 1799.5 m^2 59 m long: width = 30.5 m, area = 30.5 m × 59 m = 1799.5 m^2
10	State your findings.	The model uses all the fencing and small changes to the dimensions reduces the area, so it is a useful model.

WORKED EXAMPLE 25 Solving simultaneous linear equations

A group consisting of 3 adults and 5 children pay $152 to see a movie, and another group of 5 adults and 2 children pay the same total amount. What are the prices for an adult ticket and a child ticket?

Steps	Working
1 Identify the variables.	Let a = the adult ticket price. Let c = the child ticket price.
2 Form the equations.	$3a + 5c = 152$ [1] $5a + 2c = 152$ [2]
3 Solve the simultaneous equations. HINT: Use either substitution or elimination.	Using elimination: $5[1] - 3[2] \rightarrow 19c = 304$ $c = 16$
4 Substitute c into [1] to find a.	$3a + 5(16) = 152$ $3a = 72$ $a = 24$
5 Check the solution by substituting into [2]. State the answer.	$5(24) + 2(16) = 152$ ✓OK The adult ticket costs $24 and the child ticket costs $16.

There are many things in nature that vary according to a quadratic function. Only 3 points are needed to determine a quadratic function. If the turning point is known, then only one other point is needed.

WORKED EXAMPLE 26 Modelling a quadratic function

A function passes through (1, 8), (3, 16) and (5, 20). Find a quadratic model for the points, **verify** that it is correct and **evaluate** the reasonableness of the method.

Steps	Working
1 Write a general quadratic function.	Let $y = ax^2 + bx + c$.
2 Use the values to write simultaneous equations.	When $x = 1$, $y = a + b + c = 8$. [1]
	When $x = 3$, $y = 9a + 3b + c = 16$. [2]
	When $x = 5$, $y = 25a + 5b + c = 12$. [3]
3 Eliminate c.	[2] − [1]: $8a + 2b = 8$ [4]
	[3] − [2]: $16a + 2b = 4$ [5]
4 Solve for a, b.	[5] − [4]: $8a = -4$
	$a = -\frac{1}{2}$
	Substitute into [4]: $8 \times \left(-\frac{1}{2}\right) + 2b = 8$
	$-4 - 2b = 8$
	$-2b = 12$
	$b = 6$
5 Find c.	Substitute into [1]: $-\frac{1}{2} + 6 + c = 8 \Rightarrow c = 2\frac{1}{2}$
6 Write the function.	$y = -\frac{1}{2}x^2 + 6x + 2\frac{1}{2}$
7 Check values.	When $x = 1$, $y = -\frac{1}{2} + 6 + 2\frac{1}{2} = 8$ ✓OK
	When $x = 3$, $y = -\frac{1}{2} \times 9 + 6 \times 3 + 2 = 16$ ✓OK
	When $x = 5$, $y = -\frac{1}{2} \times 25 + 6 \times 5 + 2\frac{1}{2} = 20$ ✓OK
8 Write the answer.	The function $y = -\frac{1}{2}x^2 + 6x + 2\frac{1}{2}$ passes through the points, so using a quadratic is very useful for 3 points.

WORKED EXAMPLE 27 Solving simultaneous equations involving linear and quadratic functions

Find the length of the line cut off $y = 2x + 2$ by the graph of $y = 2x^2 - 3x - 5$.

Steps	Working
1 Solve the simultaneous equations to find the points of intersection.	$y = 2x + 2$ [1]
	$y = 2x^2 - 3x - 5$ [2]
	[1] = [2]: $2x + 2 = 2x^2 - 3x - 5$
	$2x^2 - 5x - 7 = 0$
	$(x + 1)(2x - 7) = 0$
	$x = -1$ or $3\frac{1}{2}$
2 Find the y values.	When $x = -1$, $y = 2 \times (-1) + 2 = 0$.
	When $x = 3\frac{1}{2}$, $y = 2 \times 3\frac{1}{2} + 2 = 9$.

3 Write the points of intersection.

The graphs intersect at $(-1, 0)$ and $(3\frac{1}{2}, 9)$.

4 Find the distance between the points of intersection.

$$d = \sqrt{\left(3\frac{1}{2} - (-1)\right)^2 + (9 - 0)^2}$$

$$= \sqrt{\left(4\frac{1}{2}\right)^2 + 81}$$

$$= \sqrt{\frac{405}{4}}$$

$$= \frac{\sqrt{405}}{2}$$

$$= \frac{9\sqrt{5}}{2}$$

5 Write the answer.

The length cut off is $= \frac{9\sqrt{5}}{2}$.

EXERCISE 2.6 Modelling with linear and quadratic functions

ANSWERS p. 371

Recap

1 Solve the quadratic equation $5r^2 + 3r + 4 = 0$.

2 The solutions to the quadratic $x^2 + 7x + 3 = 0$ are

A $x = \frac{7 \pm \sqrt{37}}{2}$ **B** $x = 7, x = 3$ **C** no solutions **D** $x = -6.54, x = 0.46$

Mastery

3 If the sum of 2 consecutive integers is 35, what are the 2 numbers?

4 WORKED EXAMPLE 23 Julie's daughter is one-third of Julie's age. In 11 years' time, Julie will be twice as old as her daughter.

a Write an equation to find Julie's current age.

b Determine how old Julie is currently.

5 A $5\,m^3$ load of concrete was used to put a path around the outside of a square lawn at the botanical gardens. The path is 10 cm thick and 1 m wide.

a Draw a diagram of the situation.

b Write an equation to represent the situation.

c Find the dimensions of the lawn.

6 WORKED EXAMPLE 24 Longer fence sections need fewer posts, but the rails need to be thicker to provide rigidity. The labour and materials cost for paling fence panels of length x metres is about $\$(22x^3 - 110x^2)$ per panel. There is also a payment of about $\$210x$ per panel for transport and equipment use/hire for a job in an average suburb with standard front gates.

a What is the cost of a paling fence per panel, with panels of length x m?

b What is the panel length (to the nearest 0.1 m) for the lowest cost?

c What is the lowest possible cost to fence a 20 m by 30 m block?

7 WORKED EXAMPLE 25 Riley and Jonno go on a hiking trip. Riley starts off carrying the pack with the tent and sleeping bags, and Jonno starts with the one containing the food and cooking gear. After they have walked 15 km, they swap packs for a change and discover that Jonno's original pack is twice as heavy. They transfer 6 kg of food from Riley's new pack to Jonno's new pack, so that when they have completed the final 25 km of the hike, the total product of weight and distance will be the same for each person.

a Write an equation to find the original weight of each pack.

b Find the original weights.

8 WORKED EXAMPLE 26 A quadratic function passes through (0, 50), (10, 30) and (20, 60). Determine its equation.

9 A quadratic function has values of 20, 30 and 40 at $x = 2$, $x = 3$ and $x = 6$ respectively. Find the function.

10 WORKED EXAMPLE 27 Find the length of the line $2x - y - 5 = 0$ cut off by the curve $f(x) = 3x - x^2 + 1$.

11 Find the length of the line $y = x - 2$ cut off by the curve $f(x) = 3x^2 + 3x - 7$.

12 The cost of making calculator chips in a factory decreases according to the equation $c = \$850 - 0.016\,995n$ per chip, where n is the number of chips, up to 50 000 chips. Over 50 000 chips, the cost per chip then remains constant. The initial cost of setting up the masks for the chips is \$21 000 000. The chips are sold for \$2 each. Find an expression for the total cost of making n chips and determine the minimum number that must be sold to make a profit. Is there is a maximum profit?

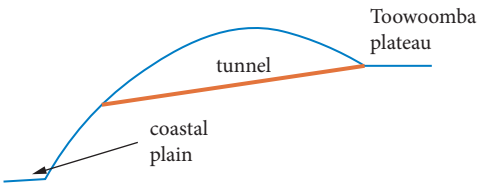
13 Resistance wire is used in heating elements. When a current is passed through the wire, its resistance increases with the temperature, further raising the temperature. A particular heating element using Nichrome 60 wire has a resistance of 80 Ω (ohms) at 0°C, 83 Ω at 300°C and 85.5 Ω at 600°C. Model the situation with a quadratic function to find the resistance at the operating temperature of 1200°C. Verify and evaluate the model.

Exam practice

14 (3 marks) CF The water solubility of a particular compound changes with temperature. Maximum amount dissolved (in g/L) $= 25 - T + 0.0125T^2$, where T is measured in °C. Sketch a graph of this relationship, and hence find the temperature with the minimum solubility.

15 (5 marks) CF The height of a ball thrown upwards at 25 m/s is approximately given by the equation $h = 2 - 5t^2 + 25t$. Find the turning point and intercepts, and hence the time taken to reach the maximum height and the time taken to reach the ground. Verify the result.

16 (4 marks) CU The height of the escarpment at the edge of a plateau is modelled by $h = \dfrac{x(16\,000 - x)}{200\,000}$, from $x = 0$ at the base of the escarpment to $x = 10\,000$ at the start of the plateau. The steepest gradient that a railway line can have is 1 in 40.



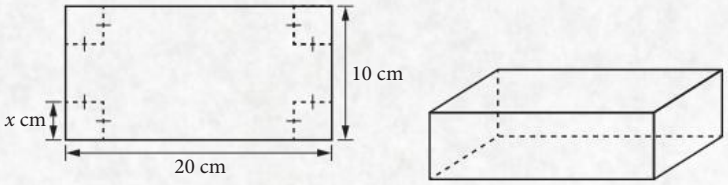
It is proposed to build a railway tunnel from the edge of the plateau to the other side of the escarpment as shown in red. Then the line will zigzag down to the coastal plain. How far horizontally and vertically from the edge of the plain will the tunnel emerge? Give your answers to the nearest metre. Verify and evaluate the model.

Video Exam question analysis: Quadratic functions

EXAM QUESTION ANALYSIS

Exam-style question (6 marks)

Nikky is making a small box to hold a gift from a rectangular sheet of cardboard. The dimensions of the cardboard are 10 cm × 20 cm. To fold up the sides, square corners of width x cm are cut out. The external faces of the gift box, including the base, will then be painted.



a Determine an expression for the total surface area, A cm^2, to be painted. *[1 mark]*

b Explain why this expression is only valid for $0 < x < 5$. *[1 mark]*

c Sketch the graph of A vs x for $0 < x < 5$. Label the endpoints with their coordinates. *[2 marks]*

d If the cost of paint is \$1.50 per square centimetre and \$246 was spent painting the box, what is the width of the squares? *[2 marks]*

Reading the question

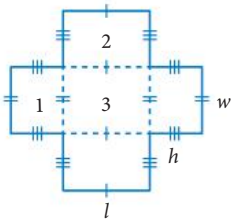
- The question contains a useful diagram that identifies the variable.
- Key values are given: 10, 20, \$1.50/cm^2 and \$246.
- Intervals are given which affect parts **b** and **c**.

Thinking about the question

- Identify what needs to be painted.
- Use the diagram and variable provided to create expressions for l, w and h.
- Use the interval provided.
- The cost to paint the box needs to be used with the cost/cm^2.

Worked solution (✓ = 1 mark)

a Know: $w = 10 - 2x$, $l = 20 - 2x$, $h = x$



$A = 2hw + 2hl + lw$

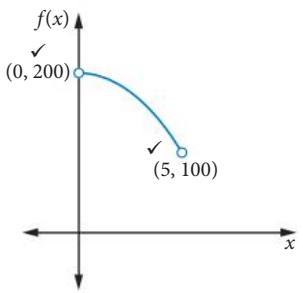
$= 2x(10 - 2x) + 2x(20 - 2x) + (20 - 2x)(10 - 2x)$

$= 20x - 4x^2 + 40x - 4x^2 + 200 - 40x - 20x + 4x^2$

$\mathbf{A = 200 - 4x^2}$ ✓

b **If $x = 0$ then no corners would be cut out. If $x = 5$ there would be no side left and the cardboard would be cut in half.** ✓

c



d $A = \dfrac{246}{1.50} = 164\text{ cm}^2$

$\mathbf{164 = 200 - 4x^2}$ ✓

$x^2 = \dfrac{36}{4}$

$x = \pm\sqrt{9}$

$x = 3$ as you can't have a negative length.

The width of the squares cut out of the corners is 3 cm. ✓

2 Chapter summary

Factorising quadratic expressions

- To factorise an expression, look for the **highest common factor** and place it outside the brackets.
- Perfect squares: $(a + b)^2 = a^2 + 2ab + b^2$
- Difference of 2 squares: $(a + b)(a - b) = a^2 - b^2$
- When you have 4 terms to factorise and there is no common factor, consider grouping the terms into pairs.
- When factorising **monic** and non-monic quadratics, look for the SUM, PRODUCT pattern.

Quadratic equations

- In a quadratic equation, the highest power on the variable is 2.
- Quadratic equations have the standard form $ax^2 + bx + c = 0$ where $a \neq 0$.
- Quadratic equations have at most 2 solutions or **roots**.
- Quadratic equations can be solved by factorising, completing the square, graphing and using the quadratic formula.
- The process of changing the form of a quadratic equation from $x^2 + 2ax + a^2 = (x + a)^2$ or $x^2 - 2ax + a^2 = (x - a)^2$ is called completing the square.

The quadratic formula and discriminant

- For any quadratic equation $ax^2 + bx + c = 0$ where $a \neq 0$, its solutions are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \text{ where } b^2 - 4ac \geq 0.$$

- The **discriminant** is $\Delta = b^2 - 4ac$.
- If $\Delta > 0$, the quadratic equation has 2 distinct real solutions.
  - If Δ is a perfect square number, $\sqrt{\Delta}$ is an integer and the 2 solutions are rational.
  - If Δ is not a perfect square number, the 2 solutions are irrational (surds).
- If $\Delta = 0$, the quadratic equation has one real solution.
- If $\Delta < 0$, the quadratic equation has no real solutions.

Graphing linear functions (Revision)

- The general form is $ax + by + c = 0$, where a, b and c are integers.
- The gradient–intercept form is $y = mx + c$, where m is the gradient and c is the y-intercept.
- gradient $= m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$
- The gradient–point form is $y - y_1 = m(x - x_1)$, where m is the gradient and (x_1, y_1) is a point on the line.
- The intercept form is $\frac{x}{a} + \frac{y}{b} = 1$, where a is the x-intercept and b is the y-intercept.
- A horizontal line has zero gradient and is represented by $y = c$.
- A vertical line has undefined gradient and is represented by $x = d$.
- Parallel lines have the same gradient, so $m_1 = m_2$.
- Perpendicular lines have gradients that are negative reciprocals of each other,

  so $m_1 m_2 = -1$ or $m_2 = -\frac{1}{m_1}$.
- To sketch a graph of a linear equation in any form:
  - find the y-intercept by letting $x = 0$ and solving for y
  - find the x-intercept by letting $y = 0$ and solving for x
  - draw a straight line through the 2 intercepts
  - label the intercepts and the line.

Graphing quadratic functions

- A quadratic function has the general form $y = ax^2 + bx + c$.
- The shape of its graph is called a **parabola**. It has a vertical axis of symmetry running through its centre and passing through the turning point.
- The turning point will be either a **maximum** or a **minimum** and occurs at $x = \frac{-b}{2a}$.
- If a is positive, then y is concave up, like $\cup$, and has a minimum.
- If a is negative, then y is concave down, like $\cap$, and has a maximum.
- The x-intercepts, if they exist, are called **zeros** or roots.
- When compared to $y = x^2$, the graph of $y = a(x - h)^2 + k$
  - is dilated by the factor a away from or towards the x-axis
  - is reflected in the x-axis if a is negative
  - is translated vertically by k units
  - is translated horizontally by h units
  - has its turning point at (h, k).
- Quadratic relationships can be expressed in three different forms:
  - general form: $y = ax^2 + bx + c$
  - intercept form: $y = a(x - x_1)(x - x_2)$ where x_1 and x_2 are the x-intercepts
  - turning point form: $y = a(x - h)^2 + k$ where (h, k) is the turning point
- By symmetry, the turning point must be halfway between the x-intercepts.
- To sketch a graph of a parabola, find these key features and label them clearly on the graph:
  - y-intercept (when $x = 0$)
  - x-intercepts, if they exist (when $y = 0$)
  - turning point.

Modelling with linear and quadratic functions

- To express a problem algebraically, the variable must first be identified and named, and then the information must be translated into an equation.
- Consider the meaning of the key features of the parabola and how they may relate to the situation you are exploring.
  - The x-intercepts occur when the y value is zero, e.g. zero height.
  - The y-intercept occurs when the x value is zero, e.g. time zero.
  - The turning point occurs when the y value is a maximum or minimum, e.g. maximum temperature.

Cumulative examination 1

Technology-free

Perusal time: 2 minutes Working time: 30 minutes

TECH-FREE *Calculators are **not** permitted.*

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

Section 1 (3 marks): 3 multiple choice questions

Section 2 (15 marks): 5 short response questions

Total: 18 marks



Worksheet
Maths Methods Year 11 formula sheet

Section 1 3 multiple choice questions

3 marks

1 The solutions of the equation $0 = (x - 1)(2x - 3)$ are

A $x = 1, x = 3$ **B** $x = 1, x = \frac{3}{2}$ **C** $x = 2, x = -\frac{3}{2}$ **D** $x = -1, x = -3$

2 The expansion of $(3x + 2)(2x - 3)$ is

A $6x^2 - 5x + 6$ **B** $6x^2 - 13x - 6$ **C** $6x^2 - 5x - 6$ **D** $6x^2 - 13x + 6$

3 Simplify $\sqrt{75}$.

A $5\sqrt{3}$ **B** $3\sqrt{25}$ **C** $15\sqrt{5}$ **D** $10\sqrt{3}$

Section 2 5 short response questions

15 marks

4 (2 marks) Factorise $3x^3 + 6x^2 + 3x + 6$.

5 (2 marks) Find the equation of the line that passes through the points $(0, 0.25)$ and $(4, 5)$.

6 (3 marks) Expand and simplify $\sqrt{3}\left(\sqrt{2} + \sqrt{6}\right)$.

7 (4 marks) Show that $3\sqrt{2} - 1$ is one of the roots of the equation $x^2 + 2x - 17 = 0$.

8 (4 marks) Sketch the graph of $y = -x^2 + 2x + 3$, showing all key points, and state the range of the graph.

Worksheet
Maths
Methods
Year 11
formula sheet

Cumulative examination 2

Technology-active

Perusal time: 2 minutes Working time: 30 minutes

*Calculators are **permitted**.*

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

Section 1 (3 marks): 3 multiple choice questions

Section 2 (15 marks): 5 short response questions

Total: 18 marks

Section 1 3 multiple choice questions

3 marks

1 When expressed with a rational denominator, $\frac{1}{\sqrt{5}-\sqrt{2}}$ is

A $\frac{\sqrt{5}+\sqrt{2}}{8}$ **B** $\frac{\sqrt{5}-\sqrt{2}}{8}$ **C** $\frac{\sqrt{5}-\sqrt{2}}{3}$ **D** $\frac{\sqrt{5}+\sqrt{2}}{3}$

2 ${}^{11}C_6 =$

A $\frac{11!}{6!5!}$ **B** $\frac{6!}{11!5!}$ **C** $\frac{5!}{6!11!}$ **D** 330

3 $4x^2 - x - 3 =$

A $(4x-3)(x+1)$ **B** $(4x+3)(x-1)$ **C** $(4x-1)(x+3)$ **D** $(4x+1)(x-3)$

Section 2 5 short response questions

15 marks

4 (2 marks) Determine the exact values of the solution(s) for the equation $(x+1)(x-3)=2$.

5 (2 marks) Find the solutions of $3x^2 - 3x - 5 = 0$, expressing your answers in both exact and approximate form to 3 decimal places where appropriate.

6 (3 marks) A mixed ultimate frisbee team of 6 players is selected from 5 males and 7 females. How many possible teams can be formed that contain 3 female and 3 male players?

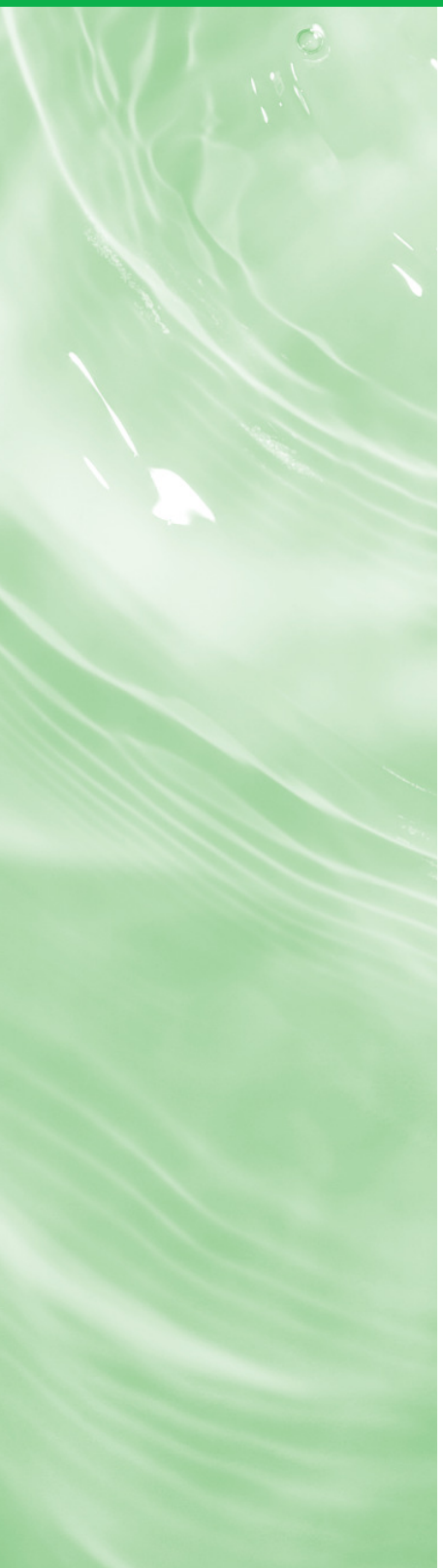
7 (4 marks) Write the rule for the expansion of $(a+b)^4$ and hence expand $(3x+1)^4$.

8 (4 marks) The length of a closed box is 3 times its height, and the box is 4 cm wide. If the total surface area is 88 cm^2, find the dimensions of the box.

9780170484909

CHAPTER

3 FUNCTIONS AND RELATIONS



Syllabus coverage

Nelson MindTap chapter resources

Terminology

3.1 Relations
- Set notation
- Interval notation
- Types of relations

3.2 Functions
- Function notation
- Vertical line test for functions
- Domain and range

3.3 Piecewise functions

3.4 Polynomials

3.5 Graphing cubic functions
- The graph of $y = x^n$
- Cubic functions

3.6 Cubic equations

3.7 Modelling with cubic functions
- Simultaneous equations

Exam question analysis

Chapter summary

Cumulative examination 1

Cumulative examination 2

Prior learning
Functions and relations

Syllabus coverage

UNIT 1, TOPIC 3: FUNCTIONS AND RELATIONS

Introduction to functions and relations

- Understand the concept of a relation as a mapping between sets, a graph and as a rule or a formula that defines one variable quantity in terms of another.
- Recognise the distinction between functions and relations and use the vertical line test to determine whether a relation is a function.
- Recognise and use function notation, domain and range, and independent and dependent variables.
- Recognise and use piecewise functions as a combination of multiple sub-functions with restricted domains.
- Model and solve problems that involve piecewise functions with and without technology.

UNIT 1, TOPIC 2: BINOMIAL EXPANSION AND CUBIC FUNCTIONS

Cubic functions

- Identify the coefficients and the degree of a polynomial.
- Expand quadratic and cubic polynomials from factors.
- Recognise and determine features of the graphs of $y = x^3$, $y = a(x - h)^3 + k$ and $y = a(x - x_1)(x - x_2)(x - x_3)$, including shape, intercepts, and behaviour as $x \to \infty$ and $x \to -\infty$
- Solve cubic equations using technology, and algebraically in cases where the equation is factorised.
- Sketch the graphs of cubic functions, with or without technology.
- Model and solve problems that involve cubic functions, with and without technology.

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Videos (3):
3.2 Domain and range
3.5 The intercept form of a cubic curve
Exam question analysis Functions and relations

Prior learning (1):
3.1 Functions and relations

Worksheets (7):
3.2 Functions and relations • Function notation • Algebra 5
3.5 Cubic functions • Graphing cubics • Graphing cubics 2
Cumulative exams Maths Methods Year 11 formula sheet

TI-Nspire CX II instructions (1):
3.3, 3.7 Functions and relations

Spreadsheet (1):
3.5 Polynomials

Nelson MindTap

To access resources above, visit **cengage.com.au/nelsonmindtap**

Terminology

continuous
discrete
hybrid function
many-to-many (n : n)
piecewise function
relation
cubic
domain
independent variable
many-to-one (n : 1)
point of inflection
set
degree of a polynomial
element
interval
one-to-many (1 : n)
polynomial
vertical line test
dependent variable
function
leading term
one-to-one (1 : 1)
range

3.1 Relations

Set notation

A **set** is a collection of objects or things with a common attribute, such as the collection of numbers {1, 3, 5}. Each individual object or thing within the set is called an **element**. The elements may be listed or described by a rule. Sets are usually shown in curly braces { } with elements separated by commas. For example, $A = \{1, 3, 5\}$ is a set where A is the name of the set and 1, 3 and 5 are its elements.

Interval notation

You can show a range of numbers over an **interval** using only the end values and brackets.

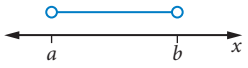
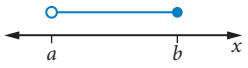
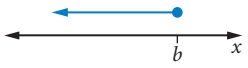
Use square brackets [ ] to show that end values are included.

Use round brackets () to show that end values are not included.

Use both to show that one end is included and the other isn't.

On a number line, a closed circle (•) indicates that the value is included and an open circle (○) indicates that the value is not included.

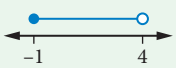
This table shows the same set of numbers written using set notation, **interval notation** and graphed on a number line.

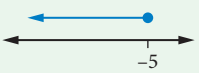
Set notation	Interval notation	Diagram
$\{x: a < x < b\}$	(a, b)	
$\{x: a \leq x \leq b\}$	$[a, b]$	
$\{x: a < x \leq b\}$	$(a, b]$	
$\{x: x > a\}$	(a, ∞)	
$\{x: x \leq b\}$	$(-\infty, b]$	

Always use round brackets for infinity in interval notation.

WORKED EXAMPLE 1 Using set and interval notation

Describe each set of numbers using set notation and interval notation.

a 

b 

c 

Steps	Working
a The interval is from −1 (included) to 4 (excluded).	Set notation: $\{x: -1 \leq x < 4\}$ Interval notation: $[-1, 4)$
b The interval is from $-\infty$ to −5 (included).	$\{x: x \leq -5\}$ $(-\infty, -5]$
c The first interval is less than −5 (excluded). The second interval is from 0 (included) to 3 (excluded).	$\{x: x \leq -5, 0 \leq x < 3\}$ $(-\infty, -5) \cup [0, 3)$ *Hint:* The symbol $\cup$ (union) is used to indicate that two intervals are combined.

A **relation** is a set of **ordered pairs** or couples (x, y), where x and y are usually both in the set of real numbers. Relations can be shown by a list of ordered pairs, a rule, a **mapping** or a graph. A mapping shows ordered pairs connected by arrows from the x values to the y values.

Examples of relations are:

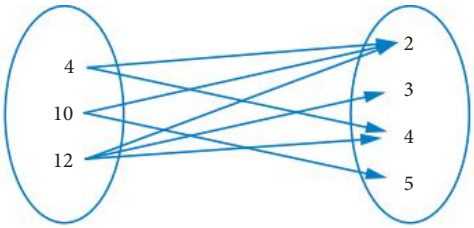
As a list:

$G = \{(1, 2), (-5, 3), (9, 12)\}$

As a rule:

$H = \{(x, y); y = 2x + 3\}$

As a mapping:



In a set of ordered pairs the first element, or x value, is called the **independent variable** and the second element, or y value, is called the **dependent variable**.

The set of values of the independent variable, x, is called the **domain**.

The set of values of the dependent variable, y, is called the **range**.

If a domain is not specifically stated, it is generally assumed that it consists of the set of real numbers (R).

- The domain for $y = 2x + 5$ is the set of all real numbers. $\{(x, y): y = 2x + 5, x \in R\}$.
- The domain for $y = \dfrac{7}{x - 2}$ is all real numbers except 2 as $x \neq 2$. $\{(x, y): y = \dfrac{7}{x - 2}, x < 2 \cup x > 2\}$.
- The domain for $y = \sqrt{x}$ can only include non-negative real numbers, as $\sqrt{x}$ only has meaning when $x \geq 0$. $\{(x, y): y = \sqrt{x}, x \geq 0\}$.

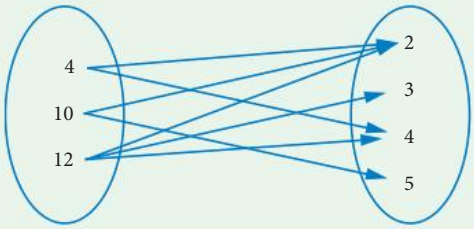
WORKED EXAMPLE 2 Ordered pairs and mapping

The mapping shows the relation N.

a Write N as a set of ordered pairs.

b **Represent** N as a table.

c State the domain and range of N.



Steps	Working
a Write the ordered pairs.	$N = \{(4, 2), (4, 4), (10, 2), (10, 5), (12, 2), (12, 3), (12, 4)\}$
b Write the ordered pairs in a table.	

x	4	4	10	10	12	12	12
y	2	4	2	5	2	3	4

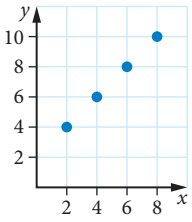
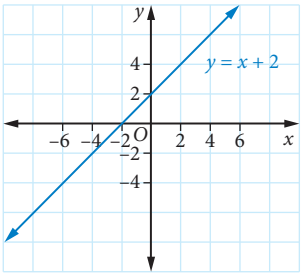
c 1 The domain is the set of the first numbers.	domain $= \{4, 10, 12\}$
2 The range is the set of the second numbers.	range $= \{2, 3, 4, 5\}$

Relations can be represented as graphs. Relations can be **discrete**, where only certain values are allowed, or **continuous**, where all values of a variable are possible within a given interval.

WORKED EXAMPLE 3 Graphing relations

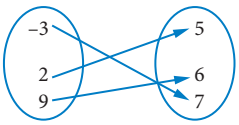
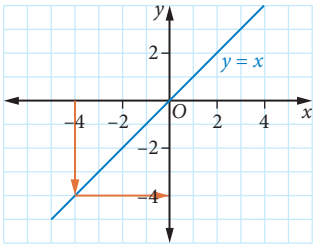
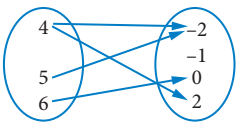
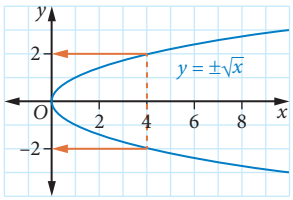
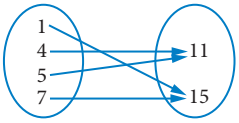
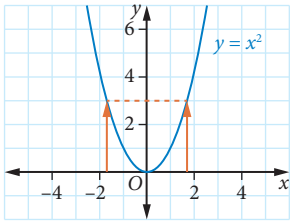
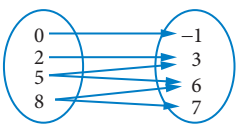
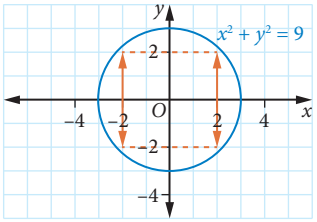
Sketch the graph of each relation.

a $\{(x, y): y = x + 2, x \in \{2, 4, 6, 8\}\}$ **b** $\{(x, y): y = x + 2, x \in R\}$

Steps	Working
a Sketch the graph. It is a discrete relation and points are only plotted where x exists (2, 4, 6 and 8).	
b Sketch the graph. It is a continuous relation as x is all the real numbers. The line is continuous. The arrowheads show that the line continues infinitely in both directions.	

Types of relations

There are 4 types of relations, which are classified by the correspondence between the ordered pair coordinates.

One-to-one (1:1)	Each x value has a unique y value.	$\{(-3, 7), (2, 5), (9, 6)\}$  −3 maps to 7 only	
One-to-many (1:*n*)	An x value maps onto more than one y value.	$\{(4, -2), (4, 2), (5, -2), (6, 0)\}$  4 maps to −1 and 2	
Many-to-one (*n*:1)	More than one x value maps onto the same y value.	$\{(1, 15), (4, 11), (5, 11), (7, 15)\}$  1 and 7 both map to 15	
Many-to-many (*n*:*n*)	More than one x value maps onto more than one y value.	$\{(0, -1), (2, 3), (5, 3), (5, 6), (8, 6), (8, 7)\}$  5 maps to 3 and 6, 8 also maps to 6	

EXERCISE 3.1 Relations

ANSWERS p. 372

Mastery

1 WORKED EXAMPLE 1 Represent each interval on a number line.

a $[-1, 4)$ **b** $(-1, 7)$ **c** $(-\infty, 8)$

d $[5, 17]$ **e** $[1, 4) \cup [6, \infty)$ **f** $(-6, 2) \cup (3, 5]$

2 Represent each interval in question **1** in set notation.

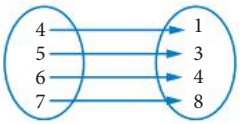
3 Represent each set in interval notation.

a $\{x: -1 \leq x\}$ **b** $\{x: -5 \leq x < 5\}$ **c** $\{x: x \geq -1\}$

d $\left\{x: \frac{\pi}{3} < x \leq \pi\right\}$ **e** $\{y: 0 < y < \infty\}$ **f** $\{z: z < 3\} \cup \{z: z \geq 4\}$

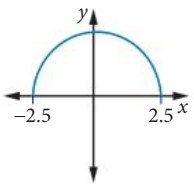
4 WORKED EXAMPLE 2 State the domain and range of each relation.

a

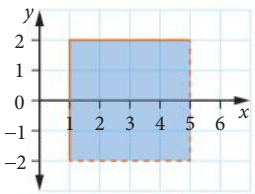


b $\{(x, y): y = x + 2, x \in R\}$ **c** $\{x: x \geq -1\}$

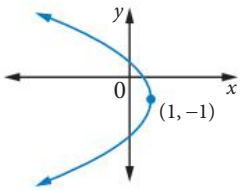
d



e



f



Solid lines around regions *include* values and use ≥ or ≤; dashed lines *exclude* values and use < or >.

5 WORKED EXAMPLE 3 Sketch each relation.

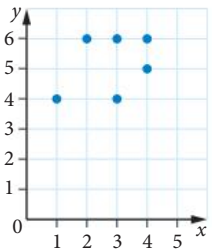
a $y = 10x - 5, x \in \{1, 2, 3, 4, 5\}$ **b** $y = 2x^2 + 2, x \in [-1, 2)$

c $\{(x, y): y = (5 - x)^2 + 3\}$ **d** $\{(x, y): y = x(x + 1)(x + 2), -2 \leq x < 0\}$

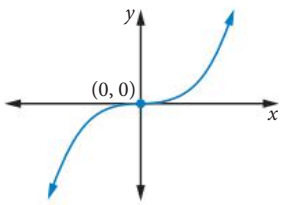
6 For each relation, identify the type of relation: one-to-one, one-to-many, many-to-one or many-to-many.

a $\{(1, 5), (3, 8), (4, 1)\}$ **b** $\{(x, y): y = x^2 + 3, x \in R\}$ **c** $y = \pm\sqrt{x - 2}$

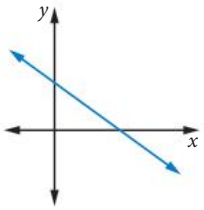
d



e



f



Exam practice

7 Given the function $f(x) = \frac{1}{x}$, what is the domain of $f(x)$?

A all real numbers **B** all real numbers except $x = 0$

C positive real numbers **D** negative real numbers

8 Convert $\{x: x > -3\}$ into interval notation.

A $(-\infty, -3)$ **B** $(-3, \infty)$ **C** $[-3, \infty)$ **D** $(-\infty, -3]$

9 Which relation represents a $1:n$ relationship?

A $\{(1,2), (2,3), (3,4), (4,5)\}$

B $\{(5,6), (5,7), (6,8), (7,9)\}$

C $\{(10,11), (12,13), (14,15), (16,17)\}$

D $\{(8,19), (10,20), (12,21), (13,22)\}$

10 Express the set $\{x: -\pi < x \leq 2\pi\}$ in interval notation.

A $(-\pi, 2\pi]$ **B** $[-\pi, 2\pi)$ **C** $(-\pi, 2\pi)$ **D** $[-\pi, 2\pi]$

11 What is the range of the function $f(x) = x^2 - 4x + 5$?

A all real numbers **B** $y \geq 0$ **C** $y \leq 1$ **D** $y \geq 1$

12 What type of relation is described by the statement 'each student is assigned a unique student ID number'?

A $1:n$ **B** $n:1$ **C** $1:1$ **D** $n:1$

13 (2 marks) CF If the range of $g(x) = 4 - 2x$ is $2 < g(x) \leq 7$, what is its domain?

3.2 Functions

Worksheets
Functions and relations
Function notation
Algebra 5

Functions

- A **function** $y = f(x)$ is a relation that is one-to-one or many-to-one.
- Every number in the domain of a function maps to only one number in the range.
- Each x value is in only one ordered pair.

Function notation

Functions are usually denoted with lower case letters such as f, g or h.

'$f(x)$', which is read as 'f of x', is called **function notation**.

It is used to replace '$y =$' and is short for 'a function of x'.

$f(3)$ is the value of $f(x)$ when $x = 3$.

Functions are often specified using set notation or function notation.

Set notation	Function notation
$\{(x,y): y = 2x + 3, x \in R\}$ Rule: $y = 2x + 3$ Domain: R	$f: R \to R, f(x) = 2x + 3$ Rule: $f(x) = 2x + 3$ Domain: R

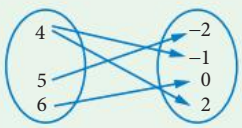
WORKED EXAMPLE 4 Relations and functions

Which of these relations are functions?

a $\{(3,5), (4,5), (5,8), (7,9), (7,10)\}$

b $\{(2,3), (3,4), (4,4), (5,4), (6,5)\}$

c



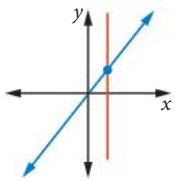
d $f(x) = 3x^4 - 5$

e $(x, \sqrt{x})$, where x is positive.

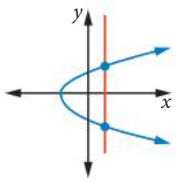
Steps	Working
a 1 Write the ordered pairs and check if any pairs have the same x value.	$\{(3,5), (4,5), (5,8), (7,9), (7,10)\}$
2 Write the conclusion.	It is not a function as there are 2 ordered pairs with the same x value.
b 1 Write the ordered pairs and check if any pairs have the same x value.	$\{(2,3), (3,4), (4,4), (5,4), (6,5)\}$
2 Write the conclusion.	It is a function because each x value only occurs once.
c 1 Write the ordered pairs and check if any pairs have the same x value.	$\{(4, -1), (4,2), (5,2), (6,0)\}$
2 Write the conclusion.	It is not a function as there are 2 ordered pairs with the same x value.
d 1 Do any ordered pairs have the same x value?	$3x^4 - 5$ is unique for any x value.
2 Write the conclusion.	It is a function.
e 1 Do any ordered pairs have the same x value? If the function was $\pm\sqrt{x}$ there would be multiple ordered pairs with the same x value.	$\sqrt{x}$ is taken to mean the positive square root, so each x value has a unique y value.
2 Write the conclusion	It is a function.

Vertical line test for functions

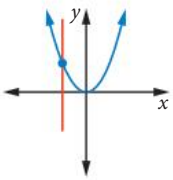
You can usually tell from the graph of a relation if it is a function by using the **vertical line test**. If a vertical line drawn intersects the graph only once, then the graph is a function. If the vertical line intersects the graph more than once, then the graph is not a function.



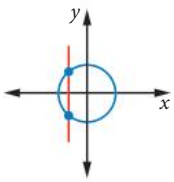
a function



not a function



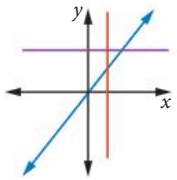
a function



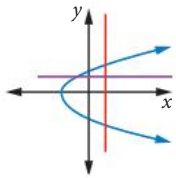
not a function

Horizontal and vertical lines can also be used to classify relations.

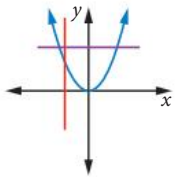
A line crossing once indicates 'one'. A line crossing more than once indicates 'many'.



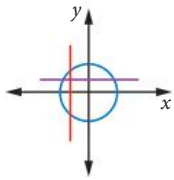
one-to-one



one-to-many



many-to-one

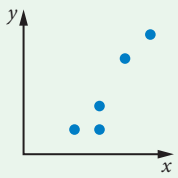


many-to-many

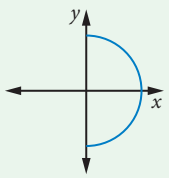
WORKED EXAMPLE 5 Using graphs to classify relations and functions

Classify each relation (1:1, 1:n, n:1, n:n) and state whether it is a function.

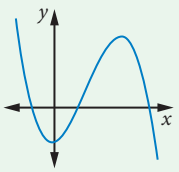
a



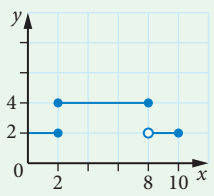
b

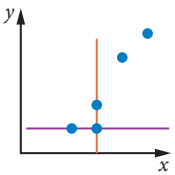
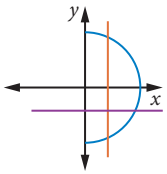
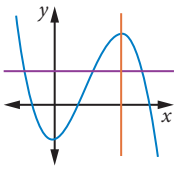


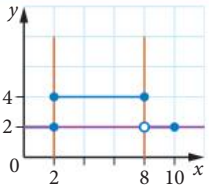
c



d



Steps	Working
a 1 Draw the graph and use the horizontal and vertical line tests.	
2 Classify the relation.	The horizontal line cuts the graph more than once. The vertical line cuts the graph more than once. The relation is n:n (many-to-many).
3 State whether the relation is a function.	The graph is not a function (it fails the vertical line test).
b 1 Draw the graph and use the horizontal and vertical line tests.	
2 Classify the relation.	The horizontal line cuts the graph once. The vertical line cuts the graph more than once. The relation is 1:n (one-to-many).
3 State whether the relation is a function.	The graph is not a function (it fails the vertical line test).
c 1 Draw the graph and use the line tests.	
2 Classify the relation.	The horizontal line cuts the graph more than once. The vertical line cuts the graph once. The relation is n:1.
3 State whether the relation is a function.	The graph is a function.

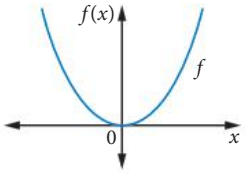
d	**1** Draw the graph and use the line tests.	
	2 Classify the relation.	The horizontal line cuts the graph more than once. At $x = 2$, the vertical line cuts the graph more than once as the circles are solid, which means they are included. (At $x = 8$, the vertical line cuts the graph once as only one of the circles is solid.) The relation is $n:n$.
	3 State whether the relation is a function.	The graph is not a function.

Video
Domain and range

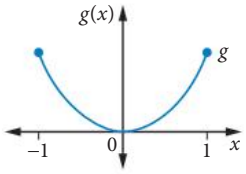
Domain and range

If the domain of a function is unspecified, it is generally assumed that it will consist of all real numbers where the function has meaning. This is called the **implied domain** or **natural domain**.

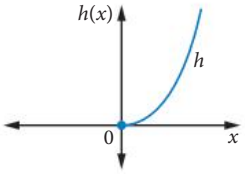
Restrictions may be placed on the values of the variable. A restricted domain generally affects the range.



$f(x) = x^2, x \in R$



$g(x) = x^2, -1 \leq x \leq 1$



$h(x) = x^2, x \geq 0$

The functions $g(x)$ and $h(x)$ have restrictions placed on the domain. Notice that at the ends of the graphs of $g(x)$ and $h(x)$ there are solid circles, which indicates the values are included.

You can use set or interval notation for domains and ranges.

WORKED EXAMPLE 6 Finding the domain and range of a function

Determine the domain and range of each function.

a $g(x) = x^2 - 3x + 2$ **b** $h(x) = \sqrt{x + 1}$ **c** $f(x) = \dfrac{1}{x - 5}$

Steps	Working
a 1 Identify the nature of the quadratic.	As a is positive, the graph will be a minimum.
2 Find the turning point.	$x = \dfrac{-b}{2a} = 1.5$ $g(1.5) = -0.25$ The minimum is at $(1.5, -0.25)$.
3 State the domain and range.	The domain is $x \in R$. The range is $y \geq -0.25$.
b 1 Identify where the function is defined.	$x + 1$ cannot be negative. $x + 1 \geq 0$ $x \geq -1$
2 Consider the possible range.	The function can only be positive (because of the square root sign), so $y \geq 0$.
3 Write the answer.	The domain is $x \geq -1$ and the range is $y \geq 0$.

c **1** Identify where the function is defined. — x cannot equal 5.

2 Consider the possible range. — As the function is the reciprocal of $(x - 5)$, the function can be any value other than 0.

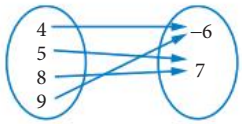
3 Write the answer. — The domain is all real numbers except $x = 5$. The range is all real numbers except $f(x) = 0$.

EXERCISE 3.2 Functions

ANSWERS p. 372

Recap

1 The mapping shows the relation H.



a Write H as a set of ordered pairs.

b Represent H as a table.

c State the domain and range of H.

d Is the mapping discrete or continuous?

2 Describe this inequality using interval notation.



Mastery

3 WORKED EXAMPLE 4 State whether each of the following relation is a function.

a $\{(4,1), (5,7), (-1,-6), (0,2), (3,8)\}$

b $\{(-3,7), (2,5), (6,-3), (8,5), (4,7)\}$

c $\{(11,-1), (11,4), (11,0), (11,-3)\}$

d $\{(7,16), (-2,-2), (10,22), (-5,-8), (10,0)\}$

e $y = 4x - 11$

f $xy = 1$

g $5x^2 = 2y$

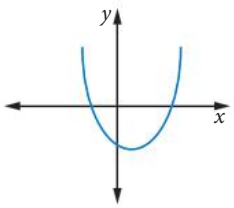
h $2y^2 = 5x$

i $f(x) = 7x - 10$

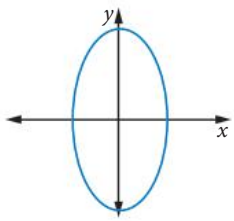
j $f(x) = 10$

4 WORKED EXAMPLE 5 For each graph of a relation, classify the relation and state if it represents a function.

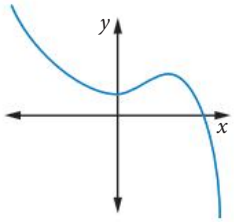
a



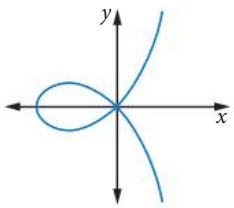
b



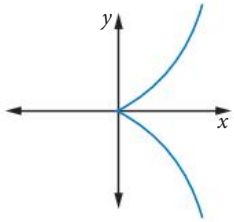
c



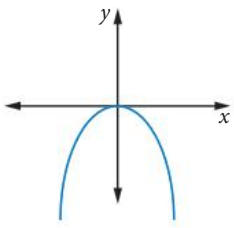
d



e



f



5 WORKED EXAMPLE 6 What is the domain and range of each function?

a $y = 12 - x$

b $y = -\sqrt{6 - x}$

c $y = 3x^2 + 2$

d $y = \frac{3}{x}$

e $y = 4 - 1.5x^2$

f $y = \frac{1}{\sqrt{x + 1}} - 4$

6 What is the range of $f(x) = 2x^2 + 13x - 15$?

7 Sketch each function.

a $f(x) = 5x - 3, x \geq 0$ **b** $h(x) = \sqrt{4 - x}, x \geq -2$ **c** $g(x) = x(x + 1), -1 \leq x \leq 2$

d $y(x) = 2 - x, (-1, \infty)$ **e** $r(x) = 5, (-2, 4)$ **f** $w(x) = \sqrt{4 - x^2}, (-\infty, 0)$

Exam practice

8 If $g(x) = 3x - 1$, $g(2) = ?$

A 4 **B** 5 **C** 7 **D** 8

9 Which relation can be classified as a function?

A $\{(2,1), (3,1), (3,5), (4,6)\}$ **B** $\{(-1,2), (0,4), (1,3), (4,6)\}$

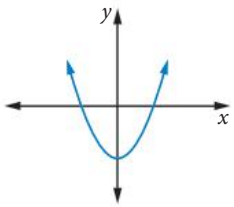
C $\{(-1,2), (0,6), (1,3), (1,6)\}$ **D** $\{(0,1), (1,1), (1,2), (2,1)\}$

10 Which relation can be classified as a function?

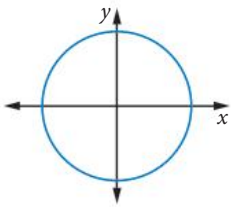
A $1:n$ **B** $1:1$ **C** $1:1$ and $n:1$ **D** $1:1$ and $1:n$

11 Which relation can be classified as a function?

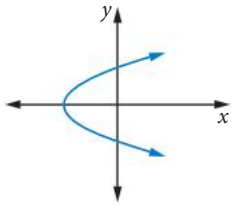
A



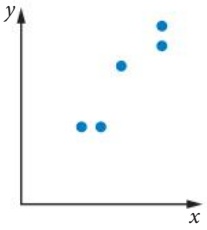
B



C



D



12 (4 marks) State whether each relation can be classified as a function.

a $y = -0.34x - 5.2$ *[1 mark]*

b $y - x^2 + x - 1 = 0$ *[1 mark]*

c $f(x) = \dfrac{1}{3x + 0.5}$ *[1 mark]*

d $\sqrt{y} = x^2 + 2$ *[1 mark]*

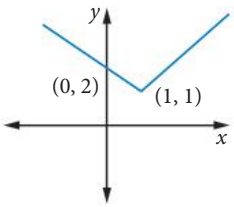
13 (2 marks) What is the domain and range of $h(x) = \dfrac{1}{x + 12}$?

14 (2 marks) CF The linear function $g(x) = 3 - 2x$ has range $-5 < g(x) \leq 9$. What is its domain?

3.3 Piecewise functions

A **piecewise** or **hybrid function** has different rules for different parts of the domain, as shown by the example and graph on the right.

$$f(x) = \begin{cases} 2 - x, & x \leq 1 \\ x, & x > 1 \end{cases}$$



Sketching a piecewise function is like sketching 2 or more functions with restricted domains on the same graph.

If the branches do not join, you need to indicate which endpoints are included or excluded by using solid and open circles, similar to that used with number lines.

WORKED EXAMPLE 7 Piecewise functions

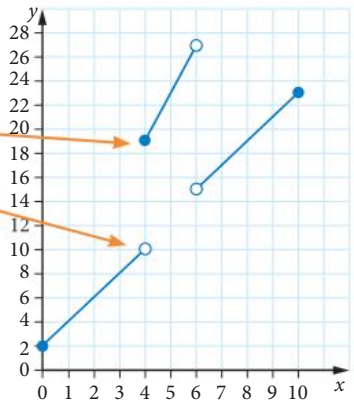
$$f(x) = \begin{cases} 2x + 2, & x < 4 \\ 4x + 3, & 4 \leq x < 6 \\ 2x + 3, & x > 6 \end{cases}$$

a **Evaluate** $f(3), f(5)$ and $f(9)$.

b **Sketch** the graph of $f(x)$ for $0 \leq x \leq 10$.

Steps	Working
a **1** Identify which rule of the function to use for each value. $3 < 4$, so use $2x + 2$.	$f(3) = 2 \times 3 + 2 = 8$
2 $4 \leq 5 < 6$, so use $4x + 3$.	$f(5) = 4 \times 5 + 3 = 23$
3 $9 > 6$, so use $2x + 3$.	$f(9) = 2 \times 9 + 3 = 21$
b Identify the type of graph of the rule of each branch. Then determine appropriate points (including endpoints), to sketch that rule over the given domain.	
1 $2x + 2$ is a straight line.	$f(0) = 2, f(3) = 8$ and at $x = 4, 2 \times 4 + 2 = 10$
2 $4x + 3$ is a straight line.	$f(4) = 19, f(5) = 23$ and at $x = 6, 4 \times 6 + 3 = 27$
3 $2x + 3$ is a straight line.	At $x = 6, 2 \times 6 + 3 = 15, f(7) = 17$ and $f(10) = 2 \times 10 + 3 = 23$
4 Sketch the graph, using open circles for the interval ends that are not included.	

Make sure you show the endpoints as included (•) or not included (○).

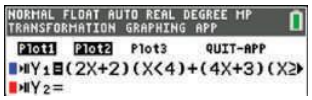


TI-Nspire instructions
Functions and relations

You can use your calculator to verify your piecewise graph.

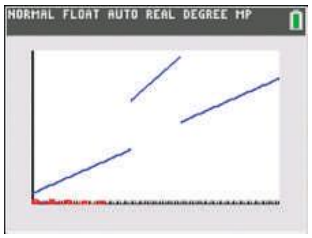
TI-84 Plus CE

Use y= to enter the function in Y_1 as (2X+2)(X<4)+(4X+3)(X≥4)(X<6) +(2X+3)(X>6).



The inequality signs are in the test menu (2nd math)

Set the window to $0 \le x \le 10$, $0 \le y \le 28$ and draw the graph.



Casio fx-CG50 AU

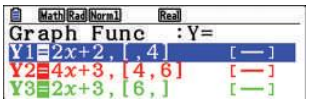
Choose **5 Graph**, enter the function parts as

Y1 = 2X+2,[,4]

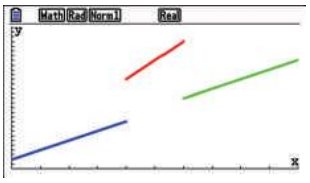
Y2 = 4X+3,[4,6]

Y3 = 2X+3[6,]

Square brackets are given by SHIFT [+] and SHIFT −



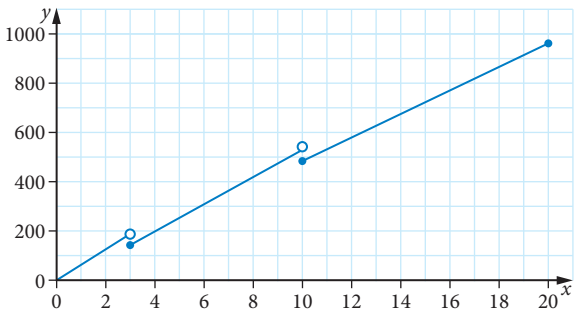
Set the V-Window to $0 \le x \le 10$, $0 \le y \le 28$ and draw the graph. You can change all the parts to the same colour using **FORMAT** (SHIFT 5).



The function in the above example has a **discontinuity** at $x = 6$, where the function doesn't have a value. Discontinuities are not common in real-life piecewise functions. Piecewise functions are often about money, such as prices and taxation. Some are just step graphs, which have horizontal sections.

WORKED EXAMPLE 8 Practical piecewise application

Screened topsoil costs \$65/m^3 for 0 to 2 m^3, \$54/m^3 for quantities from 3 m^3 to 9 m^3 and \$48/m^3 for 10 m^3 or more. **Construct** a function for the total prices of quantities up to 20 m^3 and sketch the graph.

Steps	Working
1 Choose names of variables.	Let total price of x m^3 be $P(x)$.
2 Where does the cost change?	Cost changes at 3 m^3 and 10 m^3.
3 Write the function using price = cost × quantity	$P(x) = \begin{cases} 65x, & 0 < x < 3 \\ 54x, & 3 \le x < 10 \\ 48x, & x \ge 10 \end{cases}$
4 Draw the graph. 1st section from (0, 0) to (3, 195), 2nd section from (3, 162) to (10, 540), 3rd section from (10, 480) to (20, 960).	

Exam hack

Always check if the end values on each piece of your graph are included or excluded.

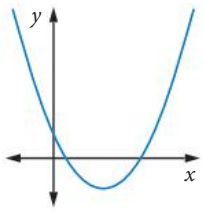
EXERCISE 3.3 Piecewise functions

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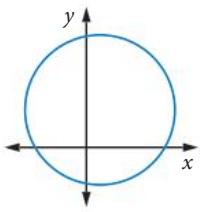
Recap

1 Which graph(s) shows a function?

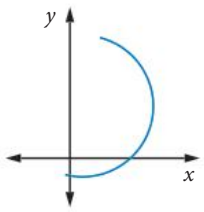
I



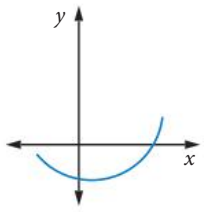
II



III



IV



A I **B** II and III **C** I and IV **D** All of them

2 Identify whether each relation can be classified as a function.

a $\{(3,5), (6,7), (4,1), (5,7), (3,1), (7,7)\}$

b $\{(1,6), (-1,6), (3,6), (5,6), (2,6)\}$

c $\{(7,1), (7,3), (7,9), (7,4), (7,13)\}$

d $\{(1,6), (2,5), (4,5), (3,8), (-1,-5)\}$

e $y = 7x + 8$

f $x^2 + y^2 = 36$

g $y = 16x^2$

h $x = 4y^2$

i $3x + 7y = 15$

j $y = 0$

Mastery

3 WORKED EXAMPLE 7

$$f(x) = \begin{cases} 2x + 4, & x \leq 6 \\ 3x - 5, & x > 6 \end{cases}$$

a Find the values of $f(5)$, $f(6)$ and $f(7)$.

b Sketch the graph of $f(x)$ for $0 \leq x \leq 10$.

4 $f(x) = \begin{cases} x + 1, & x < 3 \\ 2x - 2, & 3 \leq x < 5 \\ 8 - x, & x \geq 5 \end{cases}$

a Find the values of $f(2)$, $f(3)$, $f(4)$, $f(5)$ and $f(7)$.

b Sketch the graph of $f(x)$ for $0 \leq x \leq 12$.

5 $f(x) = \begin{cases} 3x^2 + 4, & x \leq 3 \\ 8x + 7, & x > 3 \end{cases}$

a Find the values of $f(1)$, $f(3)$ and $f(5)$.

b Sketch the graph of $f(x)$ for $0 \leq x \leq 10$.

6 $f(x) = \begin{cases} 3 + \sqrt{4 - (x-2)^2}, & 0 \leq x < 4 \\ 3, & 4 \leq x \leq 7 \\ 3 - \sqrt{4 - (x-9)^2}, & 7 < x \leq 1 \end{cases}$

a Find the values of $f(2)$, $f(6)$ and $f(8)$.

b Sketch the graph of $f(x)$ for $0 \leq x \leq 11$.

7 WORKED EXAMPLE 8 Royal Gala apples cost \$5.50/kg for up to 2 kg, \$4.50/kg for 2 to 5 kg and \$3.50/kg for 5 kg or more. Construct a function for the price for 0 to 12 kg and draw a graph.

8 The price of a taxi trip in Brisbane during the day (7 am to 7 pm) depends on the distance of the trip and the time spent stationary (waiting at traffic lights, at intersections, etc.). You also pay a flagfall (starting) cost of \$2.90 and a booking fee of \$1.50 if you book a taxi. The cost is \$2.17/km and \$0.81/minute for the time spent waiting. On suburban streets, the average speed of a taxi is about 45 km/h and about 5% of travel time is spent waiting in light traffic. In heavy traffic, the average speed drops to about 30 km/h and about 25% of the time is spent waiting.

a Show that $C = 4.4 + 2.224d$ is the cost of a booked taxi trip of d km in light traffic.

b Show that $C = 4.4 + 2.575d$ is the cost of a booked taxi trip of d km in heavy traffic.

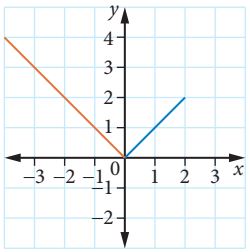
c Draw a graph showing both functions on the same axes.

Exam practice

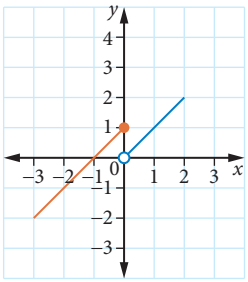
9 TECH-FREE $f(x) = \begin{cases} -2x, & x \leq -1 \\ 0.5x, & -1 < x < 2 \end{cases}$

Which graph best represents this function?

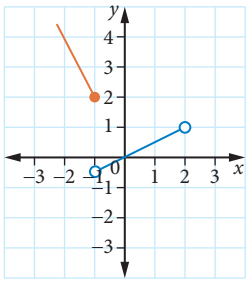
A



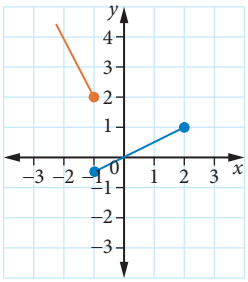
B



C



D



10 TECH-FREE What is the domain of $f(x)$ in Question **9**?

A $(-\infty, 2]$ **B** $(-\infty, 2)$ **C** R **D** $[0, \infty)$

11 (15 marks) The terminal velocity of a falling person is approximately 54 m/s without a parachute and about 5–8 m/s with a parachute, depending on the parachute design.
It takes a parachutist about 12 seconds to reach 54 m/s after jumping. It takes about 3.5 seconds to slow to 8 m/s after opening the parachute. A skydiver jumps from a plane and opens her parachute 35 seconds later. She reaches the ground 1 minute after jumping from the plane.

a Construct a function for her velocity t seconds after jumping, assuming that she accelerates and decelerates at constant rates. *[5 marks]*

It is more realistic to model her speed as $56(1 - 0.75^t)$ after jumping from the plane and $8 + 46 \times 0.25^t$ in the t seconds after opening the parachute.

b Construct a function for her velocity t seconds after jumping, assuming that she accelerates and decelerates in accordance with the above. *[2 marks]*

c Use both functions to find her speed 3, 5 and 10 seconds after jumping and comment on the difference between the functions. *[3 marks]*

d Find her speed 1, 2, 3 and 4 seconds after opening her parachute and comment on the difference between the functions. *[3 marks]*

e Draw graphs of both functions on the same axes and comment on the differences. *[2 marks]*

12 (10 marks) CF Transfer duty is a tax on buying a house. The rate for home purchases in Queensland are lower than for investment houses. For home purchase it is:

- for prices up to $350 000, it is $1 for each $100 or part thereof
- for prices over $350 000 up to $540 000, it is $3500 plus $3.50 for every $100 or part thereof over $350 000
- for prices over $540 000 up to $1 000 000, it is $10 150 plus $4.50 for every $100 or part thereof over $540 000
- for prices over $1 000 000, it is $30 850 plus $5.75 for every $100 or part thereof over $1 000 000.

a Construct a function for home transfer duty. *[4 marks]*

b Calculate the home transfer duty for properties bought for $300 000, $600 000 and $900 000. *[3 marks]*

c Construct a graph for the home transfer duty for properties worth up to $1 200 000. *[3 marks]*

3.4 Polynomials

A **polynomial** is an algebraic expression in which the powers of the variable are positive integers.

Polynomials

A polynomial $P(x)$ is of the form:
$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2x^2 + a_1x^1 + a_0$
where the coefficients $a_0, a_1, a_2 \ldots a_{n-1}, a_n$ are real numbers and n is a positive integer (or 0).
a_0 is a **constant** and called the constant term.
The **degree of a polynomial** is the highest power of the variable.
The term that contains the highest power is called the **leading term**, $a_n x^n$, and the coefficient of the leading term, a_n, is called the leading coefficient.
A polynomial is called a **monic polynomial** if the leading coefficient $a_n = 1$.

Examples of polynomials are:

3	A polynomial of degree 0. It could also be written as $3x^0$.
$4x - 8$	A polynomial of degree 1, called a **linear polynomial**.
$3x^2 + 4\sqrt{5}x + 1$	A polynomial of degree 2, called a **quadratic polynomial**.
$-\frac{3}{7}x^3 - 4x^2 + 9x + 4$	A polynomial of degree 3, called a **cubic polynomial**.
$x^4 - 6x$	A polynomial of degree 4, called a **quartic polynomial**. This is also a monic polynomial because the coefficient of x^4 is 1.

WORKED EXAMPLE 9 Polynomial expressions and their features

For each expression, **determine** whether it is a polynomial, giving reasons.
If it is a polynomial, state its degree, leading term and identify the coefficients.

a $-2x + 3x^2$ **b** $x^3 - \frac{5}{x} + x^2$ **c** $x^2 + 3\sqrt{x} - 10$ **d** $(x + 1)(x - 3)(2x + 4)$

Steps	Working
a 1 Write the expression with powers of x in descending order.	$3x^2 - 2x$
2 Check the powers are positive integers or 0.	This is a polynomial.
3 Write the degree, leading term and coefficients.	degree = 2 leading term = $3x^2$ coefficients = 3, –2
b 1 Write the expression with powers of x in descending order.	$x^3 + x^2 - 5x^{-1}$
2 Check the powers are positive integers or 0.	The power of the last term is a negative integer. This is not a polynomial.
c 1 Write the expression with powers of x in descending order.	$x^2 + 3x^{\frac{1}{2}} - 10$
2 Check the powers are positive integers or 0.	The second term has a fractional power. This is not a polynomial.
d 1 Write the expression.	$(x + 1)(x - 3)(2x + 4)$
2 Expand and simplify the expression with powers of x in descending order.	$= (x + 1)(2x^2 + 4x - 6x - 12)$ $= 2x^3 - 2x^2 - 12x + 2x^2 - 2x - 12$ $= 2x^3 - 14x - 12$
3 Check the powers are positive integers or 0.	This is a polynomial.
4 Write the degree, leading term and coefficients.	degree = 3 leading term = $2x^3$ coefficients = 2, –14, –12

WORKED EXAMPLE 10 Substituting values into polynomials

a If $P(x) = 2x^2(3x - 1)^2$, **calculate** $P(-2)$.

b Given that $P(x) = 2x^2 + 5x - 3$, find x when $P(x) = 4$.

Steps	Working
a 1 Write the expression.	$P(x) = 2x^2(3x - 1)^2$
2 Substitute $x = -2$ into $P(x)$ and evaluate.	$P(-2) = 2(-2)^2(3(-2) - 1)^2$ $= 392$
b 1 Substitute in $P(x) = 4$.	$2x^2 + 5x - 3 = 4$
2 Solve for x.	$2x^2 + 5x - 7 = 0$ $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $= \frac{-5 \pm \sqrt{5^2 - 4 \times 2 \times (-7)}}{2 \times 2}$ $= \frac{-5 \pm \sqrt{81}}{4}$ $x = 1$ or $x = -\frac{7}{2}$

EXERCISE 3.4 Polynomials

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Recap

The following function relates to Questions 1 and 2.

$$f(x) = \begin{cases} 3x - 5, & x \leq 4 \\ 2x + 1, & x > 4 \end{cases}$$

1 Evaluate $f(2)$, $f(4)$ and $f(6)$.

2 Sketch the function for $0 \leq x \leq 7$.

Mastery

3 WORKED EXAMPLE 9 State whether each expression is a polynomial. Justify your decision.

a $-3x^4 - 3x + 5 + \frac{1}{x^2}$ **b** $5x^2 - 2x + 8$

c $x^2 + 2^x + 7$ **d** $2x - 1$

e $10x^4 + 9x^2 - 2\sqrt{x} + 1$ **f** $2x^2 - 0.75x + 8$

g $\frac{1}{x - 1} + x^2$ **h** $3x^5 - 2x^2 + \sqrt{x^2 - 2x + 1}$

4 For each polynomial, state its leading term and degree.

a $3 - x^2 - x^4 - 5x$ **b** $3x^7 - x^5 + 4x^4 - 2x$

c $5x + 2$ **d** $t^9 - 9t^7 + t^5 + 15t^3$

e 5 **f** $7y^3 - y^{10}$

5 For the polynomial $P(x) = 2x^6 - 3x^5 + \frac{x^4}{4} - 2x^2 + 7$, state

a the degree of $P(x)$ **b** the coefficient of x^4

c the constant term **d** the coefficient of x^2.

6 Expand and simplify each expression.

a $(x + 3)(x + 2)$ **b** $a(a - 1)(a + 7)$

c $3m^3(m - 2)^2$ **d** $(3x + 1)(2x - 3)(x - 1)$

e $-(2x - 3)^3$ **f** $y^2(y + 2)(y - 2)$

7 WORKED EXAMPLE 10 If $S(x) = 2x^4 - x^3 + 5x^2 + 6x - 4$, evaluate

a $S(3)$ **b** $S(0)$ **c** $S(-2)$ **d** $S(\sqrt{2})$

8 If $P(x) = 2x^4 - x^2 + 7$, show that $P(x) = P(-x)$.

9 Use your calculator to evaluate $P(-6)$ when $P(x) = -3x^5 + 8x^4 - 4x^2 - 8$.

10 For the polynomial $Q(x) = (a+2)x^4 + (b-1)x^3 + (c-3)x^2 + d + 4$, find the value(s) of

a b if the coefficient of x^3 is 5

b c if x^2 has a coefficient of $-\frac{1}{3}$

c a if the leading term has a coefficient of 5

d a and b if the polynomial has degree 2

e d if the constant is -2

f a if $Q(x)$ is monic.

11 Given $f(x) = x^3 + 2x^2 + x$, find

a $f(x)$ when $x = -2$

b x when $f(x) = 0$

c $f(x)$ when $x = a + 2$

d $f(-2)\,f(x+h)$

Exam practice

12 For the polynomial $P(x) = x^4 - 5x^3 - 6x^2 + 3$, the degree is

A 1 **B** 2 **C** 3 **D** 4

13 For the polynomial $P(x) = x^4 - 5x^3 - 6x^2 + 3$, the constant term is

A -6 **B** -5 **C** 1 **D** 3

14 TECH-FREE Given $f(x) = (x-2)(x+2)$, $f(1)$ is

A 9 **B** 3 **C** -3 **D** -9

15 Which function is a polynomial?

A $f(x) = 3x^2 - 2x + 5$

B $g(x) = \sqrt{x^3 - 4x + 7}$

C $h(x) = \dfrac{1}{x^2 + 3x + 2}$

D $k(x) = (x+2)^{-2}$

16 TECH-FREE (5 marks) CF $P(x) = (2a+3)x^3 + (b-4)x^2 + c + 7$. Find the value of a, b and c if

a $P(x)$ is monic *[1 mark]*

b the coefficient of x^2 is 5 *[1 mark]*

c the constant is 3 *[1 mark]*

d $P(x)$ has degree 2 *[1 mark]*

e the leading term has a coefficient of 6. *[1 mark]*

17 TECH-FREE (4 marks) CF If $P(x) = 3x^3 - 2x^2 + nx + 5$ and $P(2) = 19$, then find the value of n.

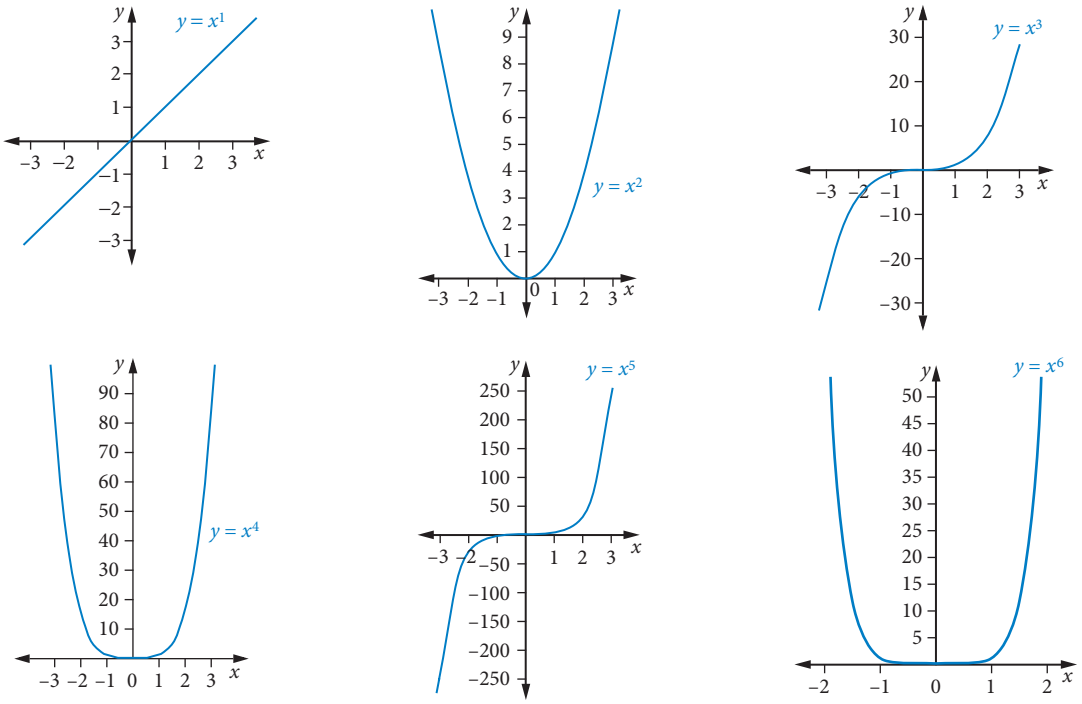
18 (2 marks) CU If $f(x) = 3x^3 + bx^2 - 2x$ and $f(-2) = 2f(1)$, find b.

19 (4 marks) CU Find the value of A, B and C if $4x^2 - 16x + 9 = A(x+B)^2 + C$.

3.5 Graphing cubic functions

The graph of $y = x^n$

The graphs of the simple polynomial $y = x^n$ with degrees $n = 1$ to 6 are shown below. Note that the vertical scales are different so the general shape of each can be seen.



The graphs with an even degree (4 and 6) approximate the shape of a parabola. The graph with degree 5 approximates the shape of the cubic curve with degree 3.

Features of $y = x^n$	Even powers of n	Odd powers of n
Shape	Like a parabola, concave up	Like a cubic curve $y = x^3$
Turning point	Minimum point	Point of inflection: where graph changes from concave down (∩) to concave up (∪)
Symmetry	The y-axis is an axis of symmetry.	Rotational symmetry about $(0, 0)$
Domain	R	R
Range	$y \geq 0$	R

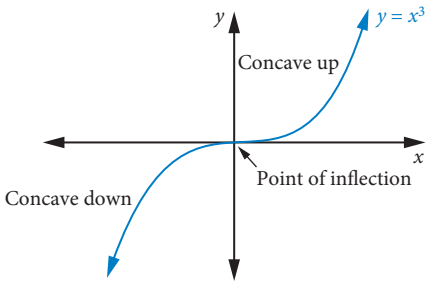
Cubic functions

A **cubic** function is a polynomial of degree 3 and has the general form $y = ax^3 + bx^2 + cx + d$, where a, b, c and d are constants and $a \neq 0$.

The centre of the graph of $y = x^3$ at $(0, 0)$ is the **point of inflection** where the graph changes concavity: concave down (∩) to concave up (∪).

As $x \to \infty$, $y \to \infty$ and as $x \to -\infty$, $y \to -\infty$.

Cubic graphs can be reflected, dilated and translated in a similar way to quadratics.



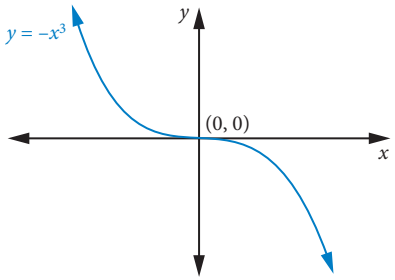
Worksheets
Cubic functions
Graphing cubics
Graphing cubics 2

Spreadsheet
Polynomials

Reflecting in the *x*-axis

The graph of $y = -x^3$ is obtained by reflecting the graph of $y = x^3$ (downwards) in the x-axis.

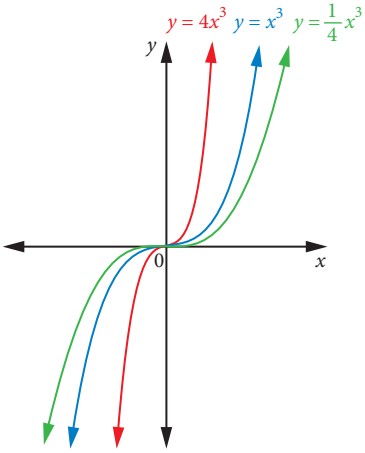
- As $x \to \infty$, $y \to -\infty$ and as $x \to -\infty$, $y \to \infty$.
- The graph changes from concave up to concave down at $(0, 0)$ going from left to right on the graph.



Dilation: making the cubic wider or narrower

If $y = ax^3$ (where $a > 0$), the value of a indicates a vertical dilation by a scale factor of a.

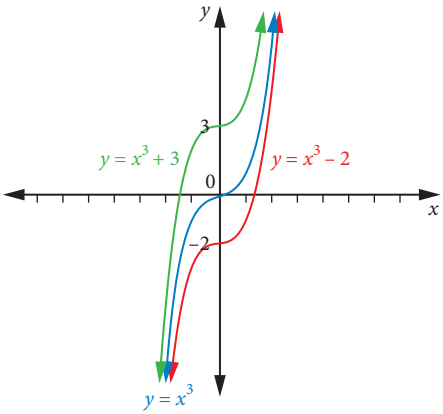
- If $a > 1$, every y value is dilated away from the x-axis. This causes the graph to appear narrower (the red graph, for example).
- If $0 < a < 1$, every y value is dilated towards the x-axis. This causes the graph to appear wider (the green graph, for example).
- If $y = -ax^3$, the graph is reflected in the x-axis and dilated by a factor of a.



Vertical translation: moving the graph up or down

If $y = x^3 + k$, the value of k indicates a vertical translation of k units.

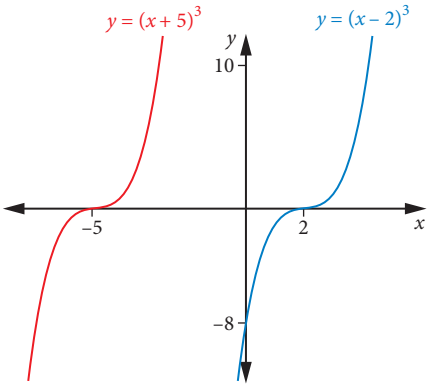
- If $k > 0$, the graph is moved vertically up k units.
- If $k < 0$, the graph is moved vertically down k units.
- The point of inflection is at $(0, k)$.



Horizontal translation: moving the graph left or right

If $y = (x - h)^3$, the value of h indicates a horizontal translation of h units.

- If $h > 0$, the graph is moved to the right by h units.
- If $h < 0$, the graph is moved to the left by h units.
- The point of inflection is at $(h, 0)$.



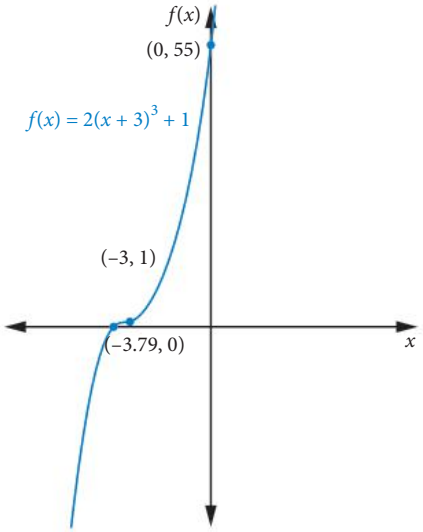
There can be multiple transformations of the basic cubic function.

For instance, compared with $y = x^3$, the graph of $y = a(x - h)^3 + k$:

- is dilated by the scale factor a from the x-axis
- is reflected in the x-axis if a is negative
- is translated vertically by k units
- is translated horizontally by h units
- has its point of inflection at (h, k).

WORKED EXAMPLE 11 Using transformations to sketch a cubic graph

Sketch the graph of $f(x) = 2(x + 3)^3 + 1$.

Steps	Working
1 Identify the key values of the function.	$a = 2, h = -3, k = 1$
2 Consider how the graph of $f(x) = x^3$ has changed to become $f(x) = 2(x + 3)^3 + 1$.	The graph is vertically stretched by a factor of 2 from the x-axis. It has moved up 1 unit and to the left 3 units. The new point of inflection is $(-3, 1)$.
3 Find the y-intercept (when $x = 0$).	$f(0) = 55$
4 Find the x-intercept (when $y = 0$).	$0 = 2(x + 3)^3 + 1$ $-\frac{1}{2} = (x + 3)^3$ $x = \sqrt[3]{-\frac{1}{2}} - 3$ ≈ -3.79
5 Sketch the graph, labelling the key points and the graph.	

A cubic function may be written in 3 different forms:

- turning point form: $y = a(x - h)^3 + k$, with point of inflection at (h, k)
- intercept or factorised form: $y = a(x - x_1)(x - x_2)(x - x_3)$ with x-intercepts at x_1, x_2 and x_3
- general form: $y = ax^3 + bx^2 + cx + d$.

At this stage, you are not required to graph cubics in general form without the use of technology.

When you are asked to *sketch* a cubic graph, show the general shape and label important points.

Sketching cubic functions

To sketch a graph of a cubic function, find and label clearly on the graph:

- the y-intercept
- the x-intercepts (1, 2 or 3) if you can find them
- the point of inflection.

Sketching from turning point form

For the form $y = a(x - h)^3 + k$:

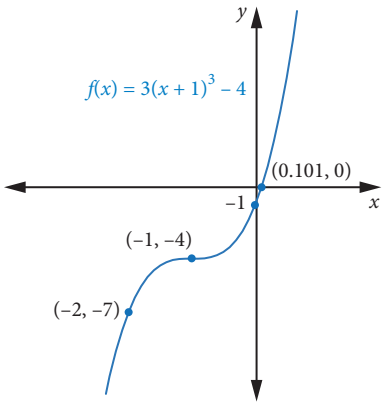
- the point of inflection is (h, k), also called the **stationary point**
- the graph changes concavity at the point of inflection
- if $a > 0$, the graph behaves like $y = x^3$; it starts below the x-axis and increases
- if $a < 0$, the graph behaves like $y = -x^3$; it starts above the x-axis and decreases
- find the x-intercept by substituting $y = 0$ and solving
- find the y-intercept by substituting $x = 0$ and solving.

When sketching the graph, also consider how the points $(-1, -1)$ or $(1, 1)$ have been transformed.

WORKED EXAMPLE 12 Sketching cubic functions in turning point form

Sketch the graph of each cubic function.

a $f(x) = 3(x + 1)^3 - 4$ **b** $y = 3 - 0.5(x - 2)^3$

Steps	Working
a 1 Identify the h and k values to find the point of inflection.	$f(x) = 3(x + 1)^3 - 4$ $h = -1$ and $k = -4$. Point of inflection is $(-1, -4)$.
2 Identify the shape of the graph.	The graph will look similar to $y = x^3$, translated 1 unit to the left and down 4 units, and dilated by a factor of 3 from the x-axis.
3 Find the y-intercept.	$f(0) = 3(0 + 1)^3 - 4$ $= -1$
4 Find the x-intercept.	$0 = 3(x + 1)^3 - 4$ $\sqrt[3]{\frac{4}{3}} - 1 = x$ $x \approx 0.101$
5 State the effect on $(-1, -1)$ and $(1, 1)$.	$(-1, -1) \rightarrow (-2, -7)$ and $(1, 1) \rightarrow (0, -1)$
6 Sketch the graph.	

b **1** Identify the h and k values to find the point of inflection.

$y = 3 - 0.5(x - 2)^3 = -0.5(x - 2)^3 + 3$

The point of inflection is $(2, 3)$.

2 Identify the shape of the graph.

$a < 0$, so the graph is reflected downwards and translated 2 units to the right and up 3 units, and vertically dilated by a factor of 0.5.

3 Find the y-intercept.

At $x = 0$, $y = 7$.

4 Find the x-intercept.

$0 = 3 - 0.5(x - 2)^3$

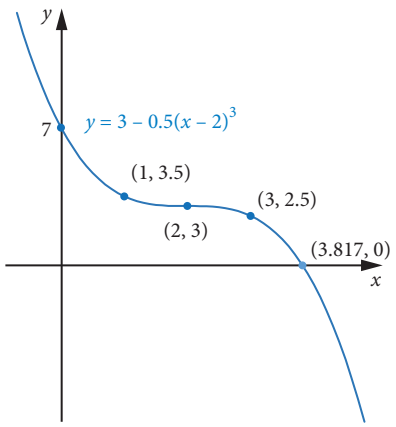
$x = \sqrt[3]{\frac{3}{0.5}} + 2$

≈ 3.817

5 State the effect on $(-1, -1)$ and $(1, 1)$.

$(-1, -1) \rightarrow (1, 3.5)$ and $(1, 1) \rightarrow (3, 2.5)$

6 Sketch the graph, labelling important points.



Sketching from intercept (factorised) form

A cubic of the form $y = a(x - x_1)(x - x_2)(x - x_3)$ has zeros at x_1, x_2 and x_3.

When a cubic has 3 factors, its graph has a maximum point between two x-intercepts and a minimum point between the others.

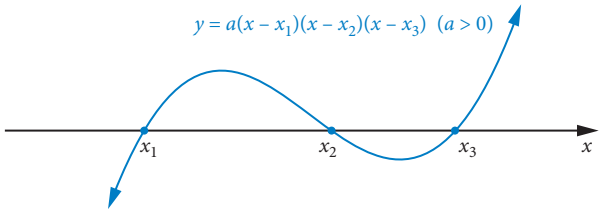
For $a > 0$, as $x \to -\infty$, $y \to -\infty$; as $x \to \infty$, $y \to \infty$.

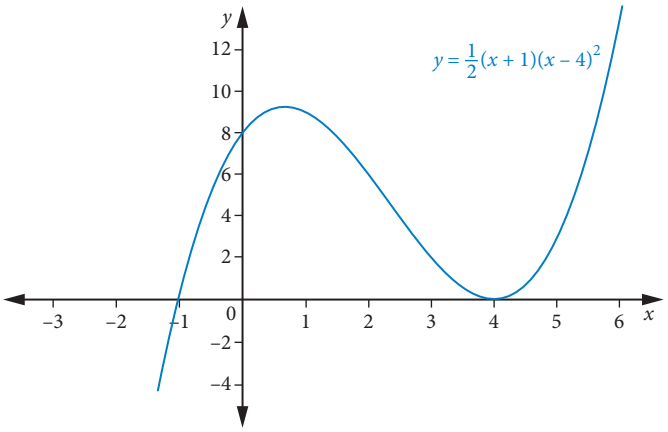
If $a < 0$, the graph will be reflected in the x-axis.

For $a < 0$, as $x \to -\infty$, $y \to \infty$ and as $x \to \infty$, $y \to -\infty$.

When a cubic has a square factor, then its graph touches the x-axis at the zero of that factor.

For example, $y = \frac{1}{2}(x + 1)(x - 4)^2$ touches the x-axis at $x = 4$ and cuts the x-axis at -1. The x-intercept at 4 is also a turning point.





$(x - 4)^2$ is called a repeated or double factor. The repeated factor identifies which zero is also a turning point.

Although the turning point on the x-axis must be identified when sketching the graph, the other turning point cannot yet be located without technology.

When sketching a cubic in intercept form, it is sufficient to identify the x- and y-intercepts and to use the sign of the leading coefficient a to identify the behaviour and shape of the graph. You should also check the signs between the zeros to check the shape.

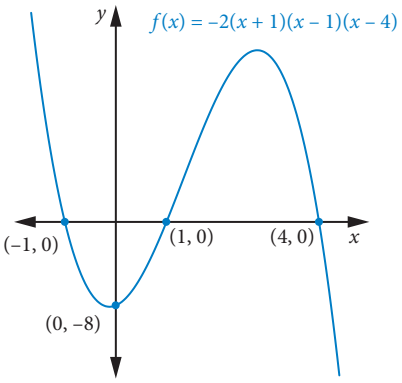
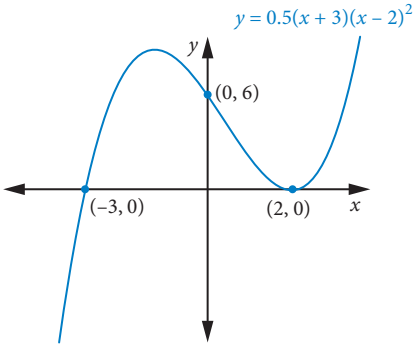
When a cubic has a triple factor and is of the form $y = a(x - h)^3$, you can sketch the function as a simple cubic $y = ax^3$ with its point of inflection translated horizontally h units to $(h, 0)$.

Video
The intercept form of a cubic curve

WORKED EXAMPLE 13 Sketching cubic functions in intercept form

Sketch the graph of each function.

a $f(x) = -2(x + 1)(x - 1)(x - 4)$ **b** $y = 0.5(x + 3)(x - 2)^2$

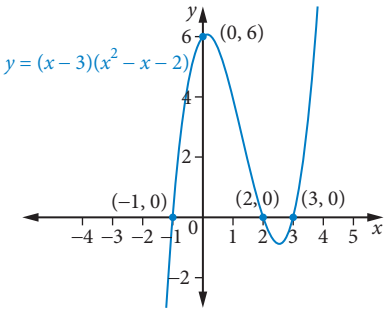
Steps	Working
a 1 State the zeros.	The graph has zeros at $x = -1$, 1 and 4.
2 Find the y-intercept.	$f(0) = -8$
3 Use the sign of the leading coefficient a to determine the shape of the curve.	$a = -2 < 0$, so the graph will be an 'upside-down' cubic curve.
	$f(x)$ has a minimum between -1 and 1 and a maximum between 1 and 4.
4 Sketch the graph, showing important points.	
b 1 State the zeros.	The graph has zeros at $x = -3$ and 2.
2 Find the y-intercept.	At $x = 0$, $y = 6$.
3 State the shape of $f(x)$.	$a = 0.5 > 0$. y has a maximum between -3 and 2. $(x - 2)^2$ is a double factor, so y has a minimum at $(2, 0)$.
4 Sketch the graph, showing important points.	

Exam hack

Do not show the maximum or minimum points as halfway between the zeros, as they are not halfway and you do not know exactly where they are.

WORKED EXAMPLE 14 Cubic functions with a linear and quadratic factor

Sketch the graph of $f(x) = (x - 3)(x^2 - x - 2)$.

Steps	Working
1 Factorise if possible.	$f(x) = (x - 3)(x^2 - x - 2)$ $= (x - 3)(x - 2)(x + 1)$
2 State the zeros.	The graph has zeros at $x = -1$, 2 and 3.
3 Find the y-intercept.	$f(0) = 6$
4 State the shape of $f(x)$.	$a = 1 > 0$. $f(x)$ has a maximum between -1 and 2, and a minimum between 2 and 3.
5 Sketch the graph, showing important points.	

EXERCISE 3.5 Graphing cubic functions

ANSWERS p. 374

Recap

1 Evaluate $g(2)$ if $g(x) = 3x^4 - 4x^3 - 5$.

A 11 **B** 16 **C** 21 **D** 35

2 For the polynomial $P(x) = 3x^7 - 2x^4 + \frac{x^3}{5} - x^2 + 1$, state

a the degree of $P(x)$ **b** the coefficient of x^4

c the constant term **d** the coefficient of x^3.

Mastery

3 WORKED EXAMPLES 11, 12 Sketch the graph of each function.

a $f(x) = (x - 1)^3 + 3$ **b** $y = 4 - (x + 2)^3$ **c** $f(x) = 3 - (x - 1)^3$

d $y = -3 - 0.4(x + 2)^3$ **e** $f(x) = 2(x - 4)^3 + 1$ **f** $y = 2 + 3(x + 1)^3$

4 State the range of each function.

a $y = (x + 2)^3 - 3$ for $-4 \leq x \leq 1$ **b** $f(x) = 2 - (x + 1)^3$ for $-3 \leq x < 3$

c $y = (x - 2)^3 - 3$ for $-2 < x \leq 5$ **d** $y = 1 - (x + 1)^3$ for $-4 \leq x \leq 4$

e $f(x) = -1 - 0.4(x - 2)^3$ for $-3 < x < 5$ **f** $y = 2(x - 3)^3 + 5$ for $-1 \leq x \leq 4$

5 WORKED EXAMPLES 13, 14 Sketch the graph of each function.

a $f(x) = (x - 3)(x + 2)(x + 1)$ **b** $y = -(x + 1)(x - 4)(x - 5)$

c $f(x) = (5 - x)(5 + x)(x + 1)$ **d** $y = (x - 2)(x + 2)(x - 2)$

e $f(x) = 0.5(x + 2)(x - 3)(x + 1)$ **f** $y = -2(x + 1)^2(2x - 5)$

Exam practice

6 TECH-FREE (6 marks) Match each function with its graph.

a $f(x) = (x - 2)^3 + 3$

b $y = (x + 2)(x - 1)(2x - 5)$

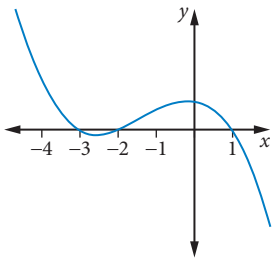
c $y = 4 - 2(x - 1)^3$

d $f(x) = (x + 3)(x + 2)(1 - x)$

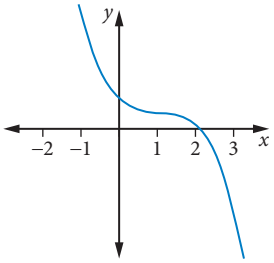
e $y = (3x + 5)(x + 3)(x - 2)$

f $f(x) = -1 - 0.5(x + 2)^3$

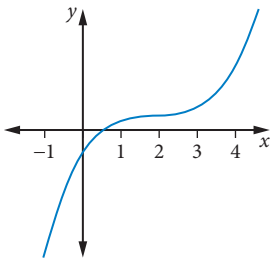
A



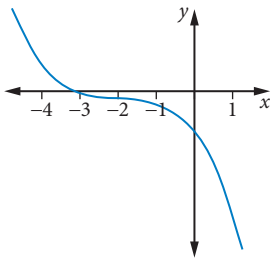
B



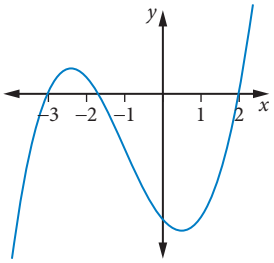
C



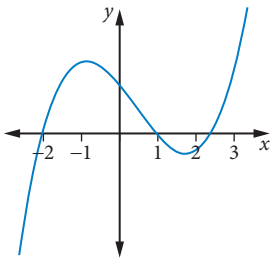
D



E



F



7 (2 marks) Graph the function $y = (3x + 6)^3 - 2$.

3.6 Cubic equations

If you can factorise a cubic equation, then you can solve it using the **null factor law**.

Null factor law for cubic equations

If $abc = 0$, then at least one of $a = 0$, $b = 0$, $c = 0$ is true.
For the cubic equation $(x - a)(x - b)(x - c) = 0$, the solutions are $x = a$, $x = b$ or $x = c$.

WORKED EXAMPLE 15 Solving cubic equations

Solve each cubic equation.

a $16x = x^3$ **b** $3(x - 3)^3 - 375 = 0$ **c** $-3x^3 + 33x^2 + 90x = 0$

Steps	Working
a 1 Group the variables and factorise.	$16x = x^3$
	$0 = x^3 - 16x$
	$x(x^2 - 16) = 0$
	$x(x - 4)(x + 4) = 0$
2 Use the null factor law.	$x = 0$, $x = 4$ or $x = -4$
b 1 Make $(x - 3)$ the subject.	$3(x - 3)^3 - 375 = 0$
	$(x - 3)^3 = 125$
2 Take the cube root of both sides and solve.	$x - 3 = \sqrt{125}$
	$x - 3 = 5$
	$x = 8$

Exam hack

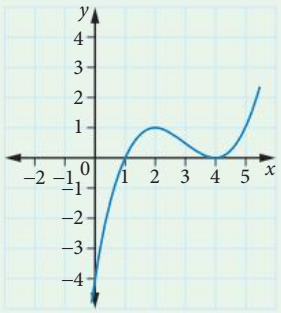
Never divide both sides by x, because x could be 0 and dividing by 0 is undefined.

c		
1	Take out the common factor.	$-3x^3 + 33x^2 + 90x = 0$
2	Factorise the quadratic.	$-3x(x^2 + 11x + 30) = 0$
		$-3x(x + 5)(x + 6) = 0$
3	Use the null factor law.	$x = 0, x = -5, x = -6$

If the cubic is not in a form that can be easily factorised, use your calculator to determine the solutions or zeros.

WORKED EXAMPLE 16 Finding a cubic equation from a graph

Determine the cubic equation of this graph.



Steps	Working
1 Identify the x-intercepts.	$(1,0), (4,0)$
2 Substitute the values into $y = a(x - x_1)(x - x_2)(x - x_3)$.	$y = a(x - 1)(x - 4)(x - 4)$
	$y = a(x - 1)(x - 4)^2$
	The factor $(x - 4)$ is repeated as the curve touches the x-axis rather than cuts the x-axis at $x = 4$.
3 Substitute a point on the graph into the equation to find a.	Substitute $(2, 1)$.
	$1 = a(2 - 1)(2 - 4)^2$
	$1 = 4a$
	$a = \frac{1}{4}$
4 Write the equation.	$y = \frac{1}{4}(x - 1)(x - 4)^2$

EXERCISE 3.6 Cubic equations

ANSWERS p. 375

Recap

1 Sketch the graph of $g(x) = (x - 2)(2x + 3)(x + 1)$.

2 State the transformations required to change the graph of $y = x^3$ to $y = (x - 1)^3 + 3$.

Mastery

3 WORKED EXAMPLE 15 Solve each cubic equation.

a $4x^3 - 100x = 0$ **b** $2x^3 = 18x$ **c** $2x^3 - 6x^2 + 4x = 0$

d $x^3 - 4x^2 + 4x = 0$ **e** $x^3 - 5x^2 + 6x = 0$ **f** $3x^3 - 15x^2 + 18x = 0$

4 Solve each cubic equation.

a $2(x + 3)^3 - 54 = 0$ **b** $4(8x - 2)^3 - 32 = 0$

c $-3(x + 1)^3 + 81 = 0$ **d** $5(3x - 2)^3 + 625 = 0$

5 Solve each cubic equation.

a $(x + 2)(x - 3)(x + 9) = 0$

b $0.5(x - 2)(x + 3)(x - 4) = 0$

c $3(x - 2)(2x + 4)(x - 5) = 0$

d $4(x - 1)(3x - 2)(x - 6) = 0$

6 Solve each cubic equation.

a $(x + 3)(x^2 - 5x - 36) = 0$

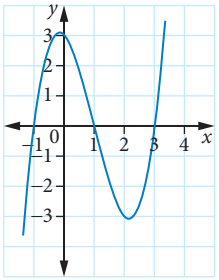
b $(2x - 8)(x^2 + 3x - 10) = 0$

c $3\left(\frac{1}{2}x - 3\right)(6x^2 - 7x - 20) = 0$

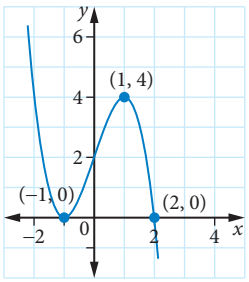
d $(x + 5)(x^2 - 4x - 21) = 0$

7 WORKED EXAMPLE 16 Find the cubic equation of each graph.

a



b



Exam practice

8 TECH-FREE What are the solutions of $3x^3 - 9x^2 + 6x = 0$?

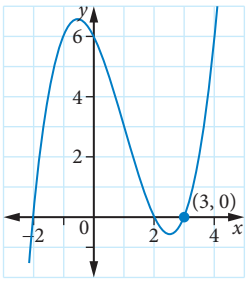
A $x = 0$ **B** $x = 2$ **C** $x = 0$ or $x = 2$ **D** $x = 1, x = 0$ or $x = 2$

9 TECH-FREE What is the point of inflection on the graph of $f(x) = 3(x + 2)^3 - 27$?

A $(0, 0)$ **B** $(-2, 0)$ **C** $(-2, -9)$ **D** $(-2, -27)$

10 TECH-FREE (1 mark) Find the x-intercept(s) of the cubic curve with equation $f(x) = (x + 1)^3 + 3$.

11 TECH-FREE (2 marks) Find the equation of this graph.



12 (3 marks) CF The graph of a cubic function has a point of inflection at $(-3, 2)$. Find its equation if it passes through the point $(1, -30)$.

13 (2 marks) CF The graph of a cubic function passes through the x-axis when $x = -1$, $x = 0$ and $x = 3$. If the graph also passes through the point $(-3, 36)$, find the equation of the function.

3.7 Modelling with cubic functions

Simultaneous equations

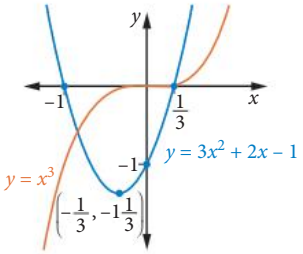
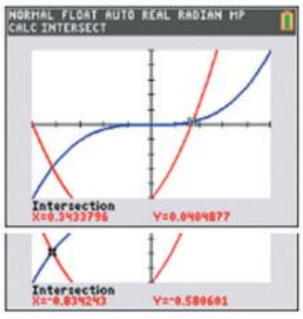
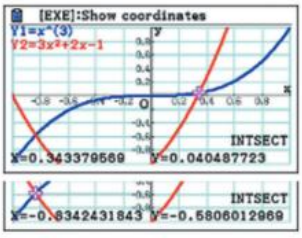
Many real-world relationships are written as functions. A situation with different relationships might give an equation using two functions.

If you want to find the temperature that a heated object reaches, you can find the temperature at which the heat lost is equal to the heat gained. If you want to find the price of a natural resource, you can find the price at which the demand and supply are equal.

These are **simultaneous equations**. You can solve two simultaneous equations using algebra or graphs. You can determine how many solutions there are by sketching the graphs. Technology can also help to solve the equations.

WORKED EXAMPLE 17 Solving simultaneous equations with cubic and quadratic functions

Solve $x^3 = 3x^2 + 2x - 1$, correct to 2 decimal places.

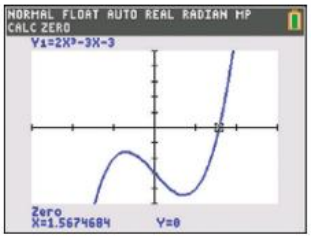
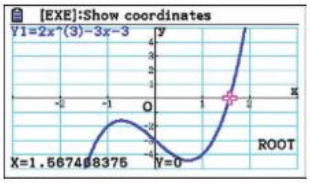
Steps	Working
1 Write the functions involved.	The functions are $y = x^3$ and $y = 3x^2 + 2x - 1$.
2 To sketch the functions, you need the zeros of the quadratic.	$y = 3x^2 + 2x - 1 = (3x - 1)(x + 1)$ has zeros at -1 and $\frac{1}{3}$. The turning point halfway between is $\left(-\frac{1}{3}, -1\frac{1}{3}\right)$.
3 Sketch the graphs on the same axes.	
4 Write the number of solutions.	There are two intersections, so there are two solutions.
5 Decide the solution method.	There is no simple linear factor, so use technology.
6 **TI-84 Plus CE** Use y= to enter the functions in Y_1 and Y_2, set the window to $-1 \le x \le 1$ and $-1 \le y \le 1$ and draw the graph. Use calc, 5: **intersect** twice to find the intersections. Use the cursor keys to set the left and right bounds and the estimate. 	**Casio fx-CG50 AU** Choose **5 Graph** and enter the functions in Y1 and Y2. Set the V-Window to $-1 \le x \le 1$ and $-1 \le y \le 1$ and press F6 to draw the graph. Use F5: **G-Solv** and F5: **INTSECT** to find the intersections. Use the cursor keys to swap between the intersections. 
7 Write the solutions.	$x \approx -0.83$ or $x \approx 0.34$

TI-Nspire instructions
Functions and relations

It may be simpler to solve an equation by finding the zero(s) of a *combined* function.

WORKED EXAMPLE 18 Solving simultaneous equations involving cubic and linear functions

Solve $6x + 2 = 2x^3 + 3x - 1$, correct to 2 decimal places.

Steps	Working
1 Rearrange to a single function.	$6x + 2 = 2x^3 + 3x - 1$ $2x^3 - 3x - 3 = 0$
2 Decide on the method.	There is no simple linear factor, so use technology.
3 **TI-84 Plus CE** Use y= to enter the function in Y_1, draw the **graph** and set the window approximately. Draw the **graph** again and use calc, 2: **zero** to find the zero(s). Use the cursor keys to set the left and right bounds and the estimate. 	**Casio fx-CG50 AU** Choose **5 Graph** and enter the functions in Y1. Press F6 to draw the graph and set V-Window approximately. Press F6 again, then F5 (G-Solv) and F1 (ROOT) to find the zero(s). 
4 Write the solutions.	$x \approx 1.57$

TI-Nspire instructions
Functions and relations

WORKED EXAMPLE 19 Solving equations involving cubic functions

Solve each equation.

a $6x = x^4$ **b** $(x - 2)(x^2 - 2x - 5) = 0$

Steps	Working
a 1 Rearrange to equal zero.	$x^4 - 6x = 0$
2 Factorise.	$x(x^3 - 6) = 0$
3 Use the null factor law.	$x = 0$ or $x^3 = 6$.
4 Write the solutions.	$x = 0$ or $x = \sqrt[3]{6}$
b 1 Use the null factor law.	$x + 2 = 0$ or $x^2 - 2x - 5 = 0$
2 Solve the equations.	$x = -2$ or $x = \frac{2 \pm \sqrt{24}}{4} = \frac{2 \pm 2\sqrt{6}}{4}$
3 Simplify and write the answers.	$x = -2$, $x = \frac{1 - \sqrt{6}}{2}$ or $x = \frac{1 + \sqrt{6}}{2}$

EXERCISE 3.7 Modelling with cubic functions

ANSWERS p. 376

Recap

Consider the equation $f(x) = 2(x + 3)^3 - 54 = 0$.

1 What are the possible values of x?

A $x = -3$ **B** $x = 0$ **C** $x = -6$ or $x = 0$ **D** $x = -3$ or $x = 6$

2 Find the domain and range of $f(x)$.

Mastery

3 WORKED EXAMPLES 17, 18 Solve each equation, correct to 2 decimal places where appropriate.

a $x^3 = 2x^2 + x - 15$ **b** $x^3 = x^2 + 6x - 5$

c $x^3 = 4x - 2x^2 + 6$ **d** $(x - 3)(x - 1)(x + 2) = x^2 - x - 2$

4 WORKED EXAMPLE 19 Solve each equation, correct to 2 decimal places where appropriate.

a $x^3 = x^2 + 4x$ **b** $(2x^2 - 5x + 3)(x + 2) = 0$ **c** $(x + 3)(x^2 + 4x - 2) = 0$

Exam practice

5 (5 marks) Let f be a polynomial function of degree 3. The graph of the curve with rule $y = f(x)$ either intersects or touches the x-axis at exactly two points, $(a, 0)$ and $(b, 0)$, where $a > 0$ and $b > 0$. State whether each equation is a possible rule for f.

a $f(x) = (x - a)(x - b)$ *[1 mark]*

b $f(x) = (x - a)(x + b)^2$ *[1 mark]*

c $f(x) = (x - a)(x - b)^2$ *[1 mark]*

d $f(x) = (x - a)^2(x - b)$ *[1 mark]*

e $f(x) = (x + a)^2(x + b)$ *[1 mark]*

6 TECH-FREE (4 marks) CF Sketch the graphs of $y = x(x - 2)(x + 2)$ and $y = x$ and determine the coordinates of the points of intersection.

7 (7 marks) CU The concentration of live algae after D days in a reservoir where the water has a high phosphate content is modelled by $m(D) = 0.39D^2 - 0.01D^3 + 0.75D$, where m is measured in grams/litre.

a Find the mass of algae that would be present after 10, 20 and 30 days. *[3 marks]*

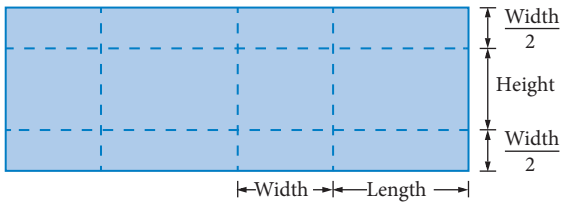
b Use a graphics calculator to find the maximum mass of algae present and relate this to the initial nutrient present in the water. *[3 marks]*

c State the domain of D for which the model is accurate. *[1 mark]*

8 (7 marks) CU A specimen of bacteria is smeared on a round agar plate of diameter 12 cm and left for testing. The area of the bacterial colony is given by $A = 6t^2 - t^3 + 16t + 0.5\,\text{cm}^2$, where t is the number of weeks after the culture is started.

Determine the percentage of the plate covered after 4 weeks and the maximum percentage covered. Evaluate the usefulness of the model.

9 (4 marks) CU A carton with overlapping flaps is made from a rectangular piece of cardboard as shown.



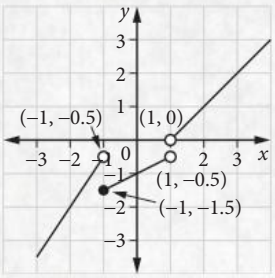
Find the dimensions of the carton with maximum volume for a sheet of cardboard 150 cm by 100 cm. Verify and evaluate the model used.

Video
Exam question analysis: Functions and relations

EXAM QUESTION ANALYSIS

Exam-style question (5 marks)

The graph of $f(x)$ is shown. Determine the equation of $f(x)$ and state its domain and range.



Reading the question

- $f(x)$ is a piecewise function.
- Key values are (i) line 1: $(-1, -0.5)$, (ii) line 2: $(-1, -1.5)$ and $(1, -0.5)$ and (iii) line 3: $(1, 0)$.
- The graph includes information about domain and range with included and excluded points.

Thinking about the question

- Find the equation of each section of the graph.
- Find the domain and range. Be careful of the included/excluded points.
- The complete equation must be stated.

Worked solution (✓ = 1 mark)

Line 1: Use the points $(-1, -0.5)$ and $(-2, -2)$ from the graph.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{-2 - (-0.5)}{-2 - (-1)}$$

$$= 1.5$$

$$y - y_1 = m(x - x_1)$$

$$y + 0.5 = 1.5(x + 1)$$

Line 1: $\boldsymbol{y = 1.5x + 1, x < -1}$ ✓

Line 2: Use $(-1, -1.5)$ and $(1, -0.5)$.

$$m = \frac{-0.5 - (-1.5)}{1 - (-1)}$$

$$= 0.5$$

$$y + 0.5 = 0.5(x - 1)$$

Line 2: $\boldsymbol{y = 0.5x - 1, -1 \leq x < 1}$ ✓

Line 3: Use $(1, 0)$ and $(2, 1)$.

$$m = \frac{1 - 0}{2 - 1}$$

$$= 1$$

$$y - 1 = 1(x - 2)$$

Line 3: $\boldsymbol{y = x - 1, 1 < x}$ ✓

The function $f(x) = \begin{cases} 1.5x + 1, & x < -1 \\ 0.5x - 1, & -1 \leq x < 1 \\ x - 1, & 1 < x \end{cases}$ ✓

From the graph, domain: $\boldsymbol{R \backslash 1}$ and range: $\boldsymbol{(-\infty, -0.5) \cup (0, \infty)}$. ✓

3 Chapter summary

Relations

- In **interval notation**, a square bracket [ ] is used to indicate a value that is included in an interval and a rounded bracket () indicates a value that is excluded.
- A **relation** is a set of ordered pairs such as $(3, -5)$. The relation can be shown by a list of ordered pairs, a rule, a mapping or part of the Cartesian plane.
- The first element, or x value, of the ordered pairs is called the **independent variable** and the second element, or y value, is called the **dependent variable**.
- The **domain** is the set of values of the independent variable, x, in the relation.
- The **range** is the set of values of the dependent variable, y, in the relation.
- Relations can be **discrete** or **continuous**.
- There are 4 types of relations:
  $1:1$ (one-to-one), $n:n$ (many-to-many), $n:1$ (many-to-one) and $1:n$ (one-to-many).

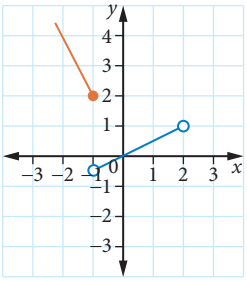
  Each of these rules describe how the domain (independent variable) relates to the range (dependent variable). For example, a $n:n$ relation is a mapping from more than one element in the domain to more than one element in the range (many-to-many), whereas a $1:1$ relation pairs elements from the domain and range uniquely (one-to-one).

Functions

- A **function** $y = f(x)$ is a $1:1$ or $n:1$ relation. Every number, or x value, in the domain of a function maps to only one number, or y value, in the range. Each x value is in only one ordered pair.
- The **vertical line test** identifies if a relation is a function. The horizontal and vertical line tests identify the type of relation.

Piecewise functions

- A **piecewise** or **hybrid function** has different rules for different parts of the domain.



$$f(x) = \begin{cases} -2x, & x \leq -1 \\ 0.5x, & -1 < x < 2 \end{cases}$$

Polynomials

- A **polynomial** is an algebraic expression in which the powers of the variable are positive integers or 0.
  $P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2 x^2 + a_1 x^1 + a_0$
- A polynomial is called a **monic** polynomial when the leading coefficient, a_n, is 1.

The graph of $y = x^n$

- The graph of a function of the form $y = x^n$, where n is a positive integer, has a turning point at $(0, 0)$.
- For even powers of n, the turning point is a minimum and the graph is concave up.
- For odd powers of n, the graph changes from concave down to concave up at the point of inflection.
- The graphs of even powers of n have an axis of symmetry on the y-axis.
- The domain is the set of real numbers R.
- The range of odd powers is R.
- The range of even powers is $y \geq 0$.

Graphing cubic functions

- A **cubic function** is a polynomial of degree 3 and has the general form $y = ax^3 + bx^2 + cx + d$, where a, b, c and d are constants and $a \neq 0$.
- If $a > 0$, then as $x \to \infty$, $y \to \infty$ and as $x \to -\infty$, $y \to -\infty$.
- Cubic graphs can be reflected, dilated and translated in a similar way to quadratics.
- The turning point form of a cubic relationship is $y = a(x - h)^3 + k$.
- The intercept (or factorised) form of a cubic relationship is $y = a(x - x_1)(x - x_2)(x - x_3)$.

Cubic equations

- If a cubic (or any polynomial) is expressed in factorised form, then it can be solved using the **null factor law**.

Worksheet
Maths Methods Year 11 formula sheet

Cumulative examination 1

Technology-free

Perusal time: 2 minutes Working time: 30 minutes

TECH-FREE *Calculators are **not** permitted.*

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

Section 1 (3 marks): 3 multiple choice questions

Section 2 (15 marks): 4 short response questions

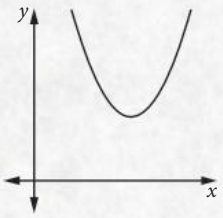
Total: 18 marks

Section 1 3 multiple choice questions

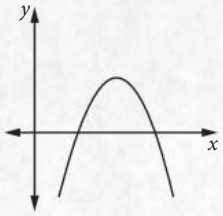
3 marks

1 Which diagram best represents $y = 2 - (x - 3)^2$?

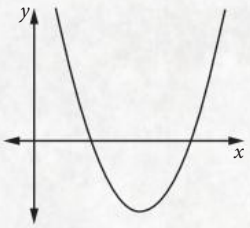
A



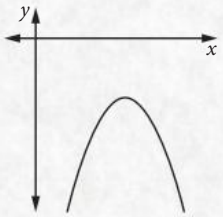
B



C



D



2 The function $g(x) = x^3$ is transformed to $h(x) = (x + 3)^3 - 4$ by a horizontal translation of –3 units followed by a vertical translation of –4 units. Which option best describes the directions of the translations?

A left 3 units and up 4 units

B right 3 units and up 4 units

C left 3 units and down 4 units

D right 3 units and down 4 units

3 What are the solutions of $3x^2 - 5x - 1$?

A $\frac{-5 \pm \sqrt{13}}{6}$ **B** $\frac{5 \pm \sqrt{13}}{6}$ **C** $\frac{-5 \pm \sqrt{37}}{6}$ **D** $\frac{5 \pm \sqrt{37}}{6}$

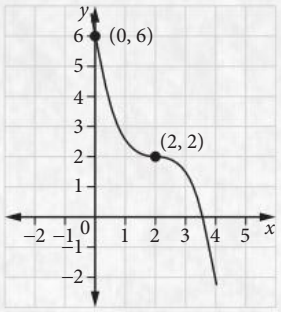
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Section 2 4 short response questions

15 marks

4 (3 marks) Determine the values of n for the equation $y = (n + 1)x^2 - nx - 1$ to have 2 real roots.

5 (4 marks) State the transformations required to change the graph of $y = x^3$ into the graph below.



6 (4 marks) Determine the equation of a parabola with a turning point at $(1, 2)$ and y-intercept at $(0, 4)$. Write your answer in the form $(x - h)^2 = 4a(y - k)$.

7 (4 marks) The transfer duty rates for investment property in Queensland are as follows:

- Up to \$5000: Nil
- Over \$5000, up to \$75 000: \$1.50 for every \$100 or part thereof over \$5000
- Over \$75 000, up to \$540 000: \$1050 plus \$3.50 for every \$100 or part thereof over \$75 000
- Over \$540 000, up to \$1 000 000: \$17 325 plus \$4.50 for every \$100 or part thereof over \$540 000
- Over \$1 000 000: \$38 025 plus \$5.75 for every \$100 or part thereof over \$1 000 000

Construct a model for duty rates on an investment property in Queensland and sketch the graph for prices up to \$1 200 000. Verify and evaluate the reasonableness of the model.

Worksheet
Maths Methods Year 11 formula sheet

Cumulative examination 2

Technology-active

Perusal time: 2 minutes Working time: 30 minutes

*Calculators are **permitted**.*

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

Section 1 (3 marks): 3 multiple choice questions

Section 2 (15 marks): 4 short response questions

Total: 18 marks

Section 1 3 multiple choice questions

3 marks

1 What is the range of $y = x^2 - 2$?

A $(-\infty, -2]$ **B** $(-\infty, \infty)$ **C** $[-2, \infty)$ **D** $[-2, 2]$

2 How many solutions are there for $3x^2 - 4x - 5 = 0$?

A none

B 1

C 2

D 3

3 Which of these types of relations is a function?

A $n : n$

B $1 : n$

C $1 : 1$ and $n : 1$

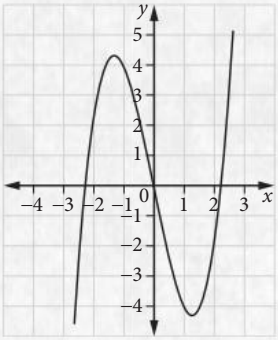
D $1 : 1$ and $1 : n$

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Section 2 4 short response questions

15 marks

4 (4 marks) The graph of $f(x) = x^3 - 5x$ is shown.



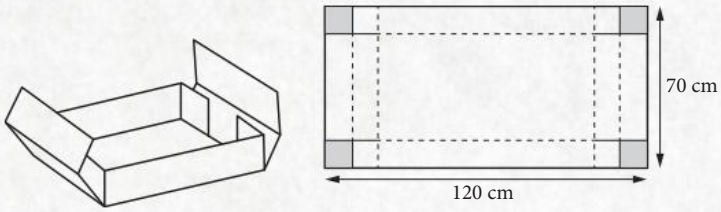
a Determine the x-intercepts. *[2 marks]*

b The coordinates of the turning points of the cubic are the solutions of the quadratic $3x^2 - 5$. Find the coordinates of the turning points, stating them as exact values. *[2 marks]*

5 (3 marks) The height of a ball thrown upwards at 25 m/s is approximately given by the equation $H(t) = 3 - 4t^2 + 20t$. Find the time taken for the ball to descend from its maximum height to the ground and the maximum height reached by the ball.

6 (4 marks) Find the distance between the turning points of $y = x - x^2 + 2.5$ and $y = 30 + 5x - 0.5x^2$.

7 (4 marks) A reinforced open box is to be constructed from a rectangular sheet of metal that is 120 cm × 70 cm. To construct the box, identical squares are to be cut from each corner before folding up each of the flaps. What size squares should be cut in each corner to ensure the resulting box holds the maximum volume and what is the maximum volume?



CHAPTER

4 TRIGONOMETRIC FUNCTIONS

Syllabus coverage
Nelson MindTap chapter resources
Terminology
4.1 Radian measure
4.2 Arcs and sectors of circles
4.3 The unit circle and exact trigonometric ratios
4.4 The trigonometric functions
4.5 Transformations of trigonometric functions
4.6 Trigonometric equations
4.7 Modelling with trigonometric functions
Exam question analysis
Chapter summary
Cumulative examination 1
Cumulative examination 2

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Syllabus coverage

UNIT 1, TOPIC 4: TRIGONOMETRIC FUNCTIONS

Circular measure and radian measure

- Define and use radian measure and understand its relationship with degree measure.
- Calculate lengths of arcs and areas of sectors in circles.

Introduction to trigonometric functions

- Understand the unit circle definition of $\cos(\theta)$, $\sin(\theta)$ and $\tan(\theta)$ and periodicity using radians.
- Understand and use the exact values of $\cos(\theta)$, $\sin(\theta)$ and $\tan(\theta)$ at integer multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$.
- Sketch the graphs of $y = \sin(x)$, $y = \cos(x)$ and $y = \tan(x)$ on extended domains.
- Recognise and determine the effect of the parameters a, b, h and k on the graphs of $y = a\sin(b(x - h)) + k$, $y = a\cos(b(x - h)) + k$, with and without technology.
- Sketch the graphs of $y = a\sin(b(x - h)) + k$, $y = a\cos(b(x - h)) + k$, with and without technology.
- Solve trigonometric equations, with and without technology, including the use of the Pythagorean identity $\sin^2(A) + \cos^2(A) = 1$
- Model and solve problems that involve trigonometric functions, with and without technology.

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Prior learning
Trigonometric functions

Videos (8):

4.1 Degrees and radians • Exact trigonometric values • The exact trigonometric ratios
4.2 Arc length
4.5 Amplitude and period • Graphing the cosine function
4.6 Trigonometric equations
Exam question analysis Trigonometric functions

Prior learning (1):

4.1 Trigonometric functions

Worksheets (13):

4.1 Converting degrees and radians • Radians
4.3 Angles of any magnitude • Radians of any magnitude
4.4 Sine and cosine curves
4.5 Sketching periodic functions: amplitude and period • Sketching periodic functions: phase and vertical shift • Trigonometric graphs
4.6 Trigonometric equations
4.7 Applying trigonometric functions • Trigonometric functions assignment • Trigonometric functions extension
Cumulative exams Maths Methods Year 11 formula sheet

Puzzles (3):

4.1 Exact values • Exact values 2
4.5 Trigonometric graphs match-up

TI-Nspire CX II instructions (1):

4.1, 4.5, 4.6, 4.7 Trigonometric functions

Nelson MindTap

To access resources above, visit **cengage.com.au/nelsonmindtap**

Terminology

amplitude	arc	cosine (cos)	degree
period	periodic function	phase shift	radian
reference angle	sector	sine (sin)	sinusoidal
standard triangles	tangent (tan)	trigonometric functions	unit circle

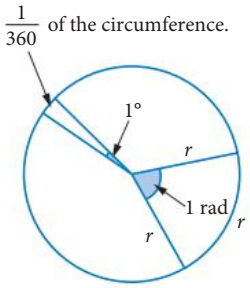
Video
Degrees and radians

Worksheets
Converting degrees and radians

Radians

4.1 Radian measure

An angle of 1 degree is $\frac{1}{360}$ of a complete revolution.



Angle size can also be measured in **radians**.

One radian is the angle at the centre of a circle opposite an **arc** of length, r, equal to the radius of the circle, as shown in the diagram.

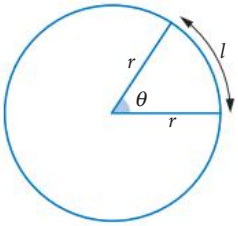
The abbreviation for radians is **rad**, but it is usually not used.

Any angle with no degree sign is assumed to be in radians.

As the angle size increases, the arc length increases in the same proportion.

An angle of 2 radians is opposite an arc of length $2r$.

An angle of 3 radians is opposite an arc of length $3r$, and so on.



So, for an angle of size θ radians, the length of the opposite arc is $l = \theta r$.

How many radians are in a complete revolution?

For a complete revolution, $l = C = 2\pi r$.

So, $2\pi r = \theta r$

$\theta = 2\pi$

As $\theta = 360°$ for a complete revolution, this means 2π rad = 360°, or π rad = 180°.

Radian facts

$$\pi = 180°$$

$$\frac{\pi}{2} = 90°, \frac{\pi}{3} = 60°, \frac{\pi}{6} = 30°, \frac{\pi}{4} = 45°$$

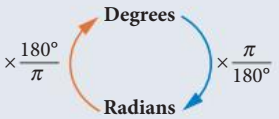
Since $\pi = 180°$, this means that:

$1° = \frac{\pi}{180°}$ radians, and

1 radian $= \frac{180°}{\pi}$ ← So, 1 radian ≈ 57.3°.

Converting between degrees and radians

- To convert from degrees to radians, multiply by $\frac{\pi}{180°}$.
- To convert from radians to degrees, if the expression contains π, use the relation $\pi = 180°$; otherwise multiply by $\frac{180°}{\pi}$.



Exam hack

If you forget these rules, always remember that $\pi = 180°$ to start the conversion.

9780170484909

WORKED EXAMPLE 1 Conversion of angles

Convert:

a $\frac{9\pi}{5}$ to degrees

b 2.07 rad to degrees

c 240° to radians (exact value)

d 110.25° to rad

Steps	Working
a Use $\pi = 180°$.	$\frac{9\pi}{5} = \frac{9 \times 180°}{5}$ $= 324°$
b 1 Use the conversion factor.	$2.07 \text{ rad} = 2.07 \times \frac{180°}{\pi}$
2 Evaluate and round off.	$\approx 118.60°$
Alternative method:	
Use $\pi = 180°$.	$2.07 \text{ rad} = 2.07 \times \frac{180°}{\pi}$
So, 1 rad $= \frac{180°}{\pi}$.	$\approx 118.60°$
c 1 Use the conversion factor.	$240° = 240° \times \frac{\pi}{180°}$
2 Leave in terms of π.	$= \frac{4\pi}{3}$
Alternative method:	
Use $\pi = 180°$.	$240° = 240° \times \frac{\pi}{180°}$
So, $1° = \frac{\pi}{180}$.	$= \frac{4\pi}{3}$
d 1 Use the conversion factor.	$110.25° = 110.25° \times \frac{\pi}{180°}$
2 Evaluate and round off.	≈ 1.92

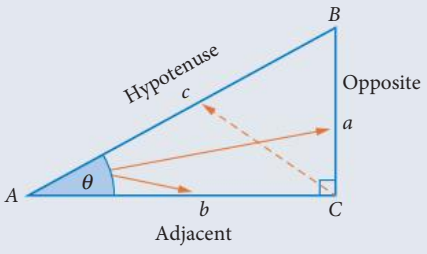
You should remember the trigonometric ratios **sine**, **cosine** and **tangent** for right-angled triangles. They are abbreviated to **sin**, **cos** and **tan**.

Trigonometric ratios

$$\sin(A) = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{a}{c}$$

$$\cos(A) = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{b}{c}$$

$$\tan(A) = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{a}{b}$$



The ratios can be remembered using the following mnemonic.

SOH **CAH** **TOA**

$\sin = \frac{O}{H}$ $\cos = \frac{A}{H}$ $\tan = \frac{O}{A}$

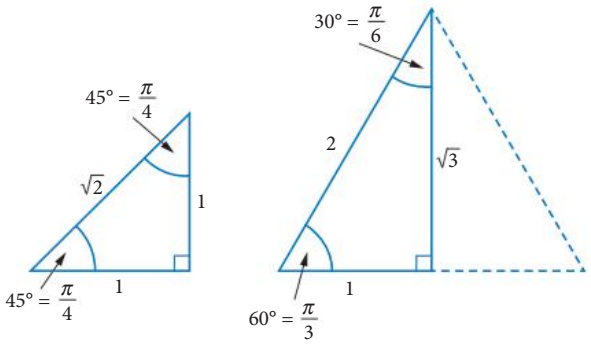
Say 'Soh-Car-Toe-ah'

You can use 2 **standard triangles** to remember the exact trigonometric ratios.

Standard triangles are a 45° isosceles right-angled triangle of side length 1 and a 30°–60° triangle, as shown below.

The 30°–60° triangle is half an equilateral triangle of side length 2.

You can work out the other side lengths by using Pythagoras' theorem.



Videos
Exact trigonometric values

The exact trigonometric ratios

Puzzles
Exact values

Exact values 2

Exact trigonometric ratios

Angle in degrees	30°	45°	60°
Angle in radians	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
sin	$\frac{1}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
cos	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{1}{2}$
tan	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

WORKED EXAMPLE 2 Exact trigonometric ratios

Determine the exact value of

a $\sin\left(\frac{\pi}{4}\right) \times \cos\left(\frac{\pi}{6}\right)$ **b** $\sin\left(\frac{\pi}{3}\right) + \cos\left(\frac{\pi}{4}\right)$ **c** $\tan^2\left(\frac{\pi}{6}\right)$

Steps	Working
a 1 Write the expression.	$\sin\left(\frac{\pi}{4}\right) \times \cos\left(\frac{\pi}{6}\right)$
2 Substitute exact values.	$= \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2}$
3 Evaluate.	$= \frac{\sqrt{6}}{4}$
b 1 Write the expression.	$\sin\left(\frac{\pi}{3}\right) + \cos\left(\frac{\pi}{4}\right)$
2 Substitute exact values.	$= \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}$
3 Evaluate.	$= \frac{\sqrt{3} + \sqrt{2}}{2}$

c **1** Write the expression. $\tan^2\left(\frac{\pi}{6}\right) = \left[\tan\left(\frac{\pi}{6}\right)\right]^2$

2 Substitute exact values. $= \left(\frac{1}{\sqrt{3}}\right)^2$

3 Evaluate. $= \frac{1}{3}$

Exam hack

When using your calculator to evaluate the trigonometric ratio of values given in radians, make sure that your calculator is set on **radian** mode, not **degree** mode. Forgetting to do this is a common exam error.

WORKED EXAMPLE 3 Calculator ratio values

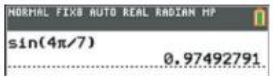
Calculate each trigonometric ratio correct to 2 decimal places.

a $\sin\left(\frac{4\pi}{7}\right)$ **b** $\cos(1.12)$

Steps

TI-84 Plus CE

a **1** Press 2nd MODE quit if necessary. Enter the expression as shown.



2 State the result. $\sin\left(\frac{4\pi}{7}\right) \approx 0.97$

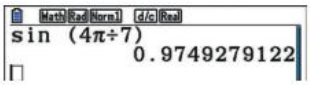
b **1** Enter the expression as shown.



2 State the result. $\cos(1.12) \approx 0.44$

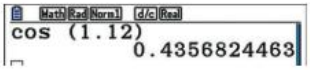
Casio fx-CG50 AU

Select the Run-Matrix page. Enter the expression as shown.



$\sin\left(\frac{4\pi}{7}\right) \approx 0.97$

Enter the expression as shown.



$\cos(1.12) \approx 0.44$

EXERCISE 4.1 Radian measure

ANSWERS p. 376

Mastery

1 WORKED EXAMPLE 1 Express each angle in radians, leaving your answers in terms of π.

a 60° **b** 45° **c** 90° **d** 135° **e** 120°

f 100° **g** 420° **h** 72° **i** 288°

2 Express each angle in radians, correct to 4 decimal places where necessary.

a 56° **b** 140° **c** 276° **d** 267.8° **e** 566.42°

3 Convert each angle from radians to degrees, correct to one decimal place where necessary.

a $\frac{2\pi}{3}$ **b** $\frac{5\pi}{6}$ **c** $\frac{7\pi}{4}$ **d** $\frac{6\pi}{11}$ **e** $\frac{11\pi}{5}$

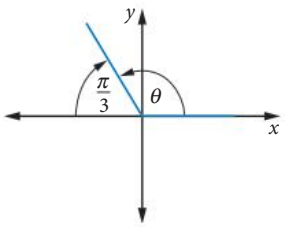
4 Convert each angle from radians to degrees, correct to one decimal place where necessary.

a 1 **b** 2.2 **c** 0.8 **d** 3.3 **e** 2.06

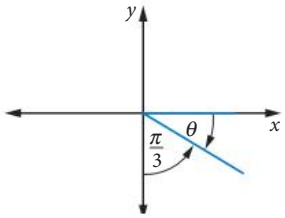
TI-Nspire Trigonometric functions

5 Find the value of θ in each diagram.

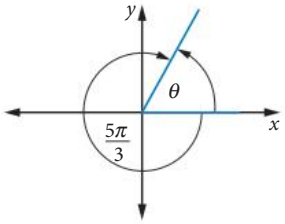
a



b



c



6 WORKED EXAMPLE 2 Find the exact value of each expression.

a $\cos\left(\frac{\pi}{3}\right) \times \sin\left(\frac{\pi}{4}\right)$

b $\tan\left(\frac{\pi}{3}\right) + \sin\left(\frac{5\pi}{6}\right)$

c $\sin^2\left(\frac{11\pi}{4}\right) + \cos^2\left(\frac{11\pi}{4}\right)$

d $\dfrac{3\cos\left(\frac{2\pi}{3}\right)}{\sin\left(\frac{\pi}{6}\right)}$

e $\dfrac{\cos\left(\frac{\pi}{4}\right)+1}{\sin\left(\frac{\pi}{4}\right)-1}$

f $\sin^2\left(\frac{2\pi}{3}\right) + \cos\left(\frac{\pi}{6}\right) + 1$

7 WORKED EXAMPLE 3 Calculate correct to 2 decimal places:

a $\cos\left(\frac{8\pi}{7}\right)$ **b** $\sin\left(\frac{4\pi}{9}\right)$ **c** $\cos\left(\frac{2\pi}{5}\right)$ **d** $\tan\left(\frac{\pi}{13}\right)$

8 Calculate correct to 2 decimal places:

a $\cos(1.31)$ **b** $\sin(0.41)$ **c** $\tan(2.04)$

d $\sin(1.63)$ **e** $\cos(0.18)$

9 Show that $\sin\left(\frac{\pi}{12}\right) = \sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)$.

10 Show that $\sin^2\left(\frac{\pi}{6}\right) + \cos^2\left(\frac{\pi}{6}\right) = 1$.

Exam practice

11 TECH-FREE $\tan(45°) =$

A $\frac{1}{\sqrt{2}}$ **B** 1 **C** $\sqrt{3}$ **D** $\frac{1}{\sqrt{3}}$

12 TECH-FREE $\cos(45°) =$

A $\frac{1}{2}$ **B** 1 **C** $\frac{\sqrt{3}}{2}$ **D** $\frac{1}{\sqrt{2}}$

13 TECH-FREE $\sin(60°) =$

A $\frac{\sqrt{3}}{2}$ **B** $\frac{1}{2}$ **C** $\frac{1}{\sqrt{3}}$ **D** $\sqrt{3}$

14 (3 marks) CU Solve $\sin(2x) = 3\cos(2x)$ for the domain $0° \leq x \leq 90°$, to the nearest minute.

4.2 Arcs and sectors of circles

In the previous section, you learned that there is a relationship between the angle θ at the centre of a circle of radius r and the length l of the arc opposite the angle.

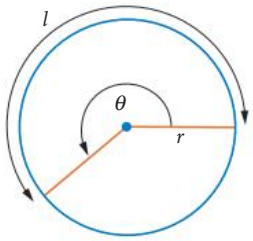
We can show this by starting with the formula for the circumference of a circle:

$$C = 2\pi r$$

$$\text{so } \frac{l}{C} = \frac{\theta}{2\pi}$$

$$\frac{l}{2\pi r} = \frac{\theta}{2\pi}$$

$$\therefore l = \theta r$$

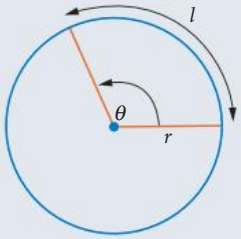


Arc length

The arc length opposite the angle θ at the centre of a circle is given by

$$l = r\theta$$

where θ is in radians.



WORKED EXAMPLE 4 Arc length

a Find the exact length of an arc that is opposite an angle of $\frac{5\pi}{6}$ in a circle of radius 10 cm.

b A circle has a radius of 12 cm. An arc measuring 25 cm is opposite the central angle θ.
Calculate the size of θ in radians, correct to 2 decimal places.

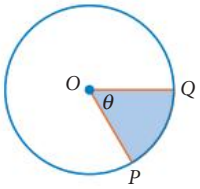
c A circle has a radius of 8 cm. **Calculate** the length of the arc that is opposite a central angle of 120°.

Steps		Working
a	**1** Write the formula.	$l = r\theta$
	2 Substitute known values.	$= 10 \times \frac{5\pi}{6}$
	3 Evaluate.	$= \frac{25\pi}{3}$
	4 State the result.	The length of the arc is $\frac{25\pi}{3}$ cm.
b	**1** Write the formula.	$l = r\theta$
	2 Substitute known values.	$25 = 12 \times \theta$
	3 Isolate θ.	$\theta = \frac{25}{12}$
	4 Evaluate.	$= 2.0833\ldots$
	5 State the result.	θ is approximately 2.08.

Video
Arc length

c 1	Convert the central angle to radians.	$120° = 120° \times \frac{\pi}{180°}$
2	Evaluate.	$= \frac{2\pi}{3}$
3	Write the formula.	$l = r\theta$
4	Substitute known values.	$= 8 \times \frac{2\pi}{3}$
5	Evaluate.	$= \frac{16\pi}{3}$
6	State the result.	The length of the arc is $\frac{16\pi}{3}$ cm.

For the area of a **sector**, consider the diagram on the right showing sector POQ shaded. The area of the sector is proportional to its angle at the centre of the circle.



So, $\frac{\text{area of sector}}{\text{area of circle}} = \frac{\theta}{2\pi}$

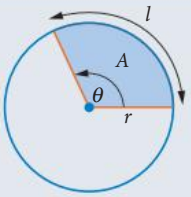
$$\frac{A}{\pi r^2} = \frac{\theta}{2\pi}$$

$$A = \frac{1}{2}r^2\theta$$

Area of a sector

For a circle with radius r and central angle θ (measured in radians):

$$\text{area of sector } A = \frac{1}{2}r^2\theta$$



WORKED EXAMPLE 5 Area of a sector

Determine the area of a sector of a circle if the arc length of the sector is 15 cm and the radius is 10 cm.

Steps	Working
1 First find θ, the central angle.	$l = r\theta$
	$\theta = \frac{l}{r}$
2 Substitute known values.	$= \frac{15}{10}$
3 Evaluate.	$= 1.5$ radians
4 Write the formula for the area of a sector.	$A = \frac{1}{2}r^2\theta$
5 Substitute known values.	$= \frac{1}{2} \times 10^2 \times 1.5$
	$= 75$
6 Write the result.	The area of the sector is 75 cm^2.

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WORKED EXAMPLE 6 Finding radius from sector area

The area of the sector of a circle formed by the central angle of $\frac{2\pi}{5}$ is $\frac{33\pi}{4}$ cm^2.

Calculate the radius of the circle, correct to one decimal place.

Steps	Working
1 Write the formula for the area of a sector.	$A = \frac{1}{2}r^2\theta$
2 Substitute known values.	$\frac{33\pi}{4} = \frac{1}{2} \times r^2 \times \frac{2\pi}{5}$
3 Isolate r^2.	$r^2 = \frac{165}{4}$
4 Evaluate.	$= 41.25$
5 Calculate r.	$r = 6.4226\ldots$
6 Round off and state the result.	The radius of the circle is about 6.4 cm.

Positive square root because radius is positive.

EXERCISE 4.2 Arcs and sectors of circles

ANSWERS p. 377

Recap

1 Express each angle in radians, leaving your answers in terms of π.

a 270° **b** 225° **c** 330° **d** 288°

2 Express each angle in radians correct to 4 decimal places.

a 47° **b** 220.75°

Mastery

3 WORKED EXAMPLE 4 For each arc of given radius, r, and central angle, θ, find the length of the arc (correct to 2 decimal places if necessary).

a $r = 5$ cm, $\theta = \frac{\pi}{3}$ **b** $r = 17$ mm, $\theta = \frac{5\pi}{6}$ **c** $r = 1.8$ m, $\theta = \frac{\pi}{2}$

d $r = 12$ cm, $\theta = \frac{2\pi}{3}$ **e** $r = 36$ mm, $\theta = 30°$ **f** $r = 10.9$ cm, $\theta = 125°42'$

4 For each arc length, l, and central angle, θ, calculate the length of the radius (correct to 2 decimal places if necessary).

a $l = 30$ mm, $\theta = \frac{\pi}{5}$ **b** $l = 21$ cm, $\theta = \frac{2\pi}{3}$ **c** $l = 60$ cm, $\theta = \frac{\pi}{4}$

d $l = 8.4$ m, $\theta = \frac{\pi}{7}$ **e** $l = 43$ cm, $\theta = \frac{6\pi}{13}$ **f** $l = 1.43$ m, $\theta = 0.85$

5 Calculate the length of an arc opposite an angle of 65° at the centre of a circle of radius 20 cm.

6 WORKED EXAMPLE 5 For each sector of given radius (r) and central angle (θ), calculate its area (correct to 2 decimal places if necessary).

a $r = 30$ mm, $\theta = \frac{\pi}{3}$ **b** $r = 8.4$ m, $\theta = 120°$ **c** $r = 1.43$ m, $\theta = 1.32$

7 WORKED EXAMPLE 6 For each sector area (A) and central angle (θ), calculate the radius of the sector (correct to 2 decimal places if necessary).

a $A = 20\ \text{cm}^2$, $\theta = 45°$ **b** $A = 8.4\ \text{m}^2$, $\theta = 0.74$

c $A = 4.3\ \text{cm}^2$, $\theta = 76°$ **d** $A = 1.43\ \text{cm}^2$, $\theta = 1.84$

8 A circle has a central angle of $\frac{7\pi}{6}$ opposite an arc measuring 33.27 cm. Calculate the radius of the circle and the area of the sector bounded by the arc.

9 The area of sector XOY of a circle with centre O and radius 18 cm is found to be $210\ \text{cm}^2$. Calculate

a the size of $\angle XOY$ **b** the length of arc XY.

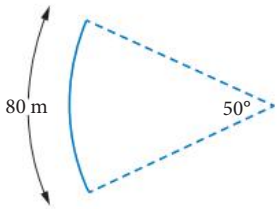
10 In a circle of radius 1.2 m, an arc is opposite an angle of 50° at the centre. Calculate

a the arc length

b the length of the chord that joins the endpoints of the arc.

11 The minute hand of a wall clock is 15 cm long. How far does the tip of the minute hand move in 18 minutes?

12 A curve on an athletics track is designed to be laid on the arc of a circle. If 80 m of track is required for a change of direction of 50°, what radius should be used?



13 A windscreen wiper is 48 cm long. The blade reaches from the endpoint of the wiper to a point that is 8 cm from the pivot point. Find the area (to the nearest cm^2) of the windscreen swept by the blade when the wiper moves through an angle of 140°.

14 A screen printer creates printed designs by rotating a circular drum over a screen that has been placed over the material on which the print is required. The drum has a diameter of 50 cm and the screen is 75 cm long. What fraction of the drum needs to be inked?

Exam practice

15 TECH-FREE The length of an arc opposite an angle of $\frac{\pi}{4}$ at the centre of a circle with radius 6 cm is

A $\frac{3\pi}{4}$ cm **B** $\frac{3\pi}{2}$ cm **C** 2π cm **D** 3π cm

16 What is the approximate central angle of an arc of length 10 cm in a circle of radius 8 cm?

A 46° **B** 72° **C** 80° **D** 144°

17 The area of a sector with a central angle of $\frac{\pi}{3}$ and a radius of 9 cm is approximately

A $4.7\ \text{cm}^2$ **B** $9.4\ \text{cm}^2$ **C** $18.8\ \text{cm}^2$ **D** $42.4\ \text{cm}^2$

4.3 The unit circle and exact trigonometric ratios

Using triangles, the definitions of sin, cos and tan only work for angles up to 90°.
You can use a circle of radius 1 (the **unit circle**) to extend the definitions to angles of any size.

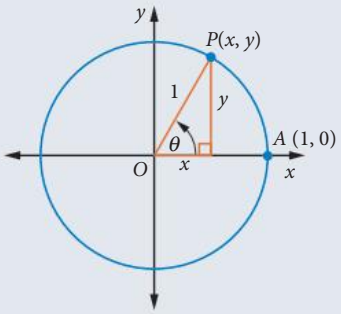
Trigonometric ratios using the unit circle

If $P(x, y)$ is a point on the unit circle found by rotating the point $A(1, 0)$ anticlockwise through an angle of θ, then:

$$\sin(\theta) = y$$

$$\cos(\theta) = x$$

$$\tan(\theta) = \frac{y}{x} = \frac{\sin(\theta)}{\cos(\theta)}$$

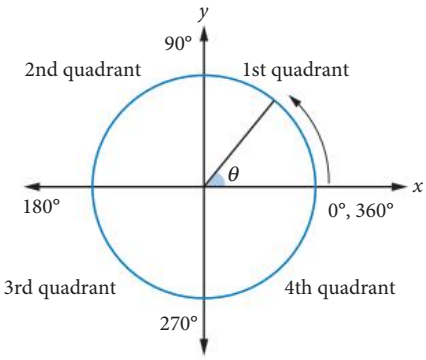


This means that for any angle θ, the values for sin, cos and tan can be found using the coordinates of $P(x, y)$ matching that angle.

The sine ratio is represented by the y-coordinate of P, so it is positive for angles in the 1st and 2nd quadrants, where $y > 0$.

The cosine ratio is represented by the x-coordinate of P, so it is positive for angles in the 1st and 4th quadrants, where $x > 0$.

The tangent ratio is represented by $\frac{y}{x}$, so it is positive for angles in the 1st and 3rd quadrants, where $\frac{y}{x} > 0$.



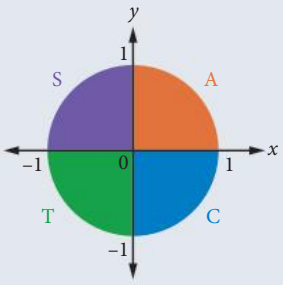
You need to remember which trigonometric ratios are positive in each quadrant.
One way to remember this is by picturing the unit circle diagram below using the mnemonic:

ASTC: 'All Stations To Central.'

ASTC: 'All–Sine–Tangent–Cosine.'

Signs of the trigonometric ratios

1st quadrant:	All ratios are positive (**A**).
2nd quadrant:	Sine only is positive (**S**).
3rd quadrant:	Tangent only is positive (**T**).
4th quadrant:	Cosine only is positive (**C**).

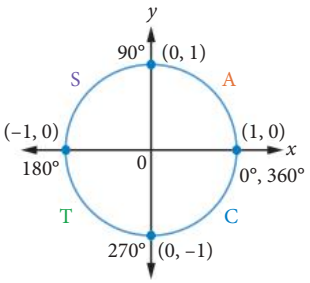


Because $P(x, y)$ moves around the unit circle, its x- and y-coordinates stay between −1 and 1, so the values of $\sin(\theta)$ and $\cos(\theta)$ are between 1 and −1. However, the value of $\tan(\theta)$ is unbounded and is undefined at 90° and 270° because $x = 0$ on the unit circle there and dividing by 0 for $\frac{y}{x}$ is undefined.

Worksheets
Angles of any magnitude
Radians of any magnitude

The table below shows the trigonometric values of the angles at the boundaries of the quadrants.

	$0° = 0$	$90° = \frac{\pi}{2}$	$180° = \pi$	$270° = \frac{3\pi}{2}$
sin	0	1	0	−1
cos	1	0	−1	0
tan	0	undefined $\left(\frac{1}{0}\right)$	0	undefined $\left(\frac{1}{0}\right)$

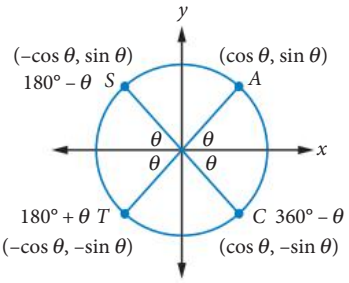


You can use symmetry within a circle to find trigonometric ratios for an angle in any quadrant and relate it back to an acute angle in the 1st quadrant.

From the diagram, you can see that the y-coordinate of point A is the same as for point S; therefore, $\sin(\theta) = \sin(180° - \theta)$.

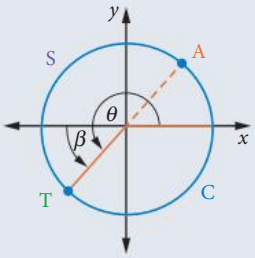
You can also see the x-coordinate of A is the same as for point C; therefore $\cos(\theta) = \cos(360° - \theta)$.

The x-coordinate of T has the same magnitude but would be negative in comparison to the x-coordinate of A; therefore, $\sin(180° + \theta) = -\sin(\theta)$.



Finding trigonometric ratios for an angle of any magnitude

For angles greater than 90°, the acute **reference angle** is always measured from 180° or 360° (π or 2π) on the x-axis.
In this diagram, β is the reference angle for θ, which is in the 3rd quadrant.



1. Draw the angle from the x-axis.
2. Calculate the acute reference angle β that corresponds to θ.
3. Use ASTC to determine the sign (+ or −) of the value of the trigonometric ratio.
4. Find the value of the required trigonometric ratio for β and include the sign.

WORKED EXAMPLE 7 Angles of any magnitude

a Simplify each expression if each variable is an acute angle.

i $\sin(360° - \beta)$ **ii** $\tan(180° + m)$

b **Determine** the exact value of

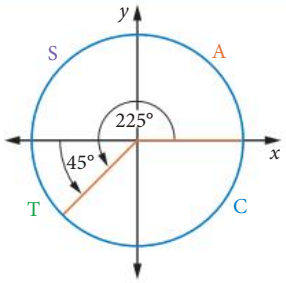
i $\sin(225°)$ **ii** $\cos\left(\frac{11\pi}{6}\right)$ **iii** $\tan\left(\frac{2\pi}{3}\right)$

Steps		Working
a i 1	State the quadrant.	$(360° - \beta)$ is in the 4th quadrant.
2	sin is negative in the 4th quadrant, so simplify.	$\sin(360° - \beta) = -\sin(\beta)$
ii 1	State the quadrant.	$(180° + m)$ is in the 3rd quadrant.
2	tan is positive in the 3rd quadrant, so simplify.	$\tan(180° + m) = \tan(m)$

b i 1 Draw a sketch.

Calculate the reference angle and mark it on your sketch.

Using ASTC, sin (225°) is negative.



2 Write the trigonometric ratio using the reference angle.

$\sin(225°) = -\sin(45°)$

3 Evaluate.

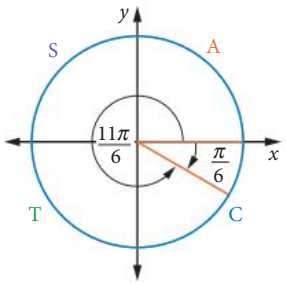
$= -\frac{1}{\sqrt{2}}$

$= \frac{\sqrt{2}}{2}$

ii 1 Draw a sketch.

Calculate the reference angle and mark it on your sketch.

Using ASTC, $\cos\left(\frac{11\pi}{6}\right)$ is positive.



2 Write the trigonometric ratio using the reference angle and evaluate.

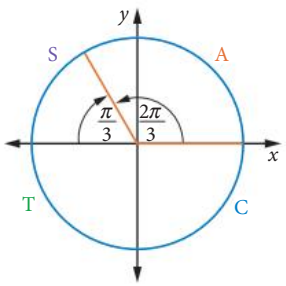
$\cos\left(\frac{11\pi}{6}\right) = \cos\left(\frac{\pi}{6}\right)$

$= \frac{\sqrt{3}}{2}$

iii 1 Draw a sketch.

Calculate the reference angle and mark it on your sketch.

Using ASTC, $\tan\left(\frac{2\pi}{3}\right)$ is negative.



2 Write the trigonometric ratio using the reference angle and evaluate.

$\tan\left(\frac{2\pi}{3}\right) = -\tan\left(\frac{\pi}{3}\right)$

$= -\sqrt{3}$

You need more than one revolution for trigonometric ratios of angles greater than 360° or 2π.

WORKED EXAMPLE 8 Angles greater than 360°

a In which quadrant is 850°?

b Find the exact value of $\sin\left(\frac{10\pi}{3}\right)$.

Steps	Working
a 1 Find the number of revolutions that are needed to reach 850°.	360° is one revolution. 720° is two revolutions.
2 Calculate the angle.	850° – 720° = 130°
3 State the result.	130° is in the 2nd quadrant, so 850° is in the 2nd quadrant.

b 1 Find the number of revolutions that are needed to reach $\frac{2\pi}{3}$.

2π is one revolution.

4π is 2 revolutions.

$\frac{10\pi}{3}$ is between 1 and 2 revolutions.

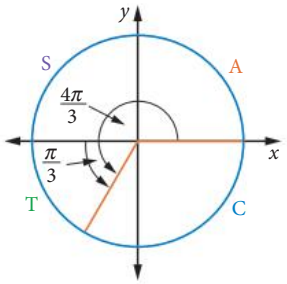
2 Calculate the equivalent angle under one revolution.

$\frac{10\pi}{3} - 2\pi = \frac{4\pi}{3}$

3 Draw a sketch.

Calculate the reference angle.

Using ASTC, $\sin\left(\frac{4\pi}{3}\right)$ is negative.



4 Evaluate the trigonometric ratio using the reference angle.

$\sin\left(\frac{4\pi}{3}\right) = -\sin\left(\frac{\pi}{3}\right)$

$= -\frac{\sqrt{3}}{2}$

5 State the result.

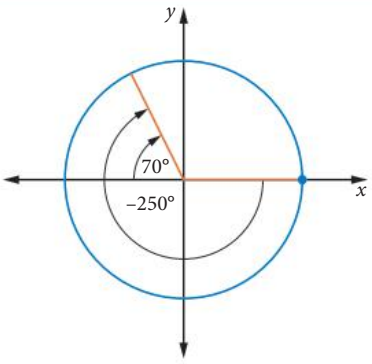
$\sin\left(\frac{10\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

You can change a negative angle to the positive equivalent to work out the trigonometric ratios.

WORKED EXAMPLE 9 Negative angles

a In which quadrant is –250°?

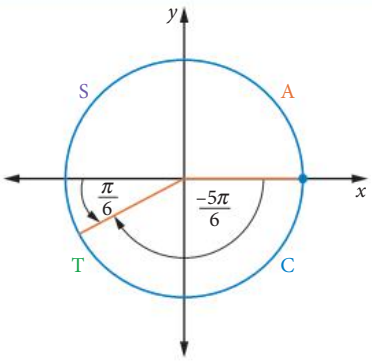
b **Calculate** the exact value of $\cos\left(-\frac{5\pi}{6}\right)$.

Steps	Working
a 1 Draw a sketch. A negative angle is measured clockwise from 0°.	
2 State the result.	–250° is in the 2nd quadrant.

b **1** Draw a sketch.

Calculate the reference angle.

Using ASTC, $\cos\left(-\frac{5\pi}{6}\right)$ is negative.



2 Evaluate the trigonometric ratio using the reference angle.

$$\cos\left(-\frac{5\pi}{6}\right) = -\cos\left(\frac{\pi}{6}\right)$$
$$= -\frac{\sqrt{3}}{2}$$

EXERCISE 4.3 The unit circle and exact trigonometric ratios

ANSWERS p. 377

Recap

1 Calculate the length of an arc opposite an angle of 55° at the centre of a circle of radius 15 cm.

2 Find the exact value of each expression.

a $\cos\left(\frac{\pi}{3}\right) \times \sin\left(\frac{3\pi}{4}\right)$ **b** $\sin\left(\frac{5\pi}{6}\right) + \cos\left(\frac{\pi}{3}\right)$

Mastery

3 WORKED EXAMPLE 7 Complete each statement if each variable is an acute angle.

a $\cos(\pi + \alpha) =$ **b** $\sin(360° - x) =$ **c** $\tan(180° + \theta) =$

d $\tan(360° + \beta) =$ **e** $\tan(2\pi - \alpha) =$ **f** $\sin(\pi + \alpha) =$

4 Find the exact value of each ratio.

a $\cos(120°)$ **b** $\sin\left(\frac{4\pi}{3}\right)$ **c** $\tan(135°)$

d $\cos(300°)$ **e** $\tan\left(\frac{7\pi}{6}\right)$ **f** $\cos(330°)$

5 Use the unit circle to find each ratio.

a $\sin(90°)$ **b** $\cos(180°)$ **c** $\tan(2\pi)$

d $\cos(270°)$ **e** $\tan(\pi)$ **f** $\sin(180°)$

6 WORKED EXAMPLE 8 Find the exact value of each ratio.

a $\sin\left(\frac{13\pi}{6}\right)$ **b** $\cos(510°)$ **c** $\tan(4\pi)$ **d** $\cos(450°)$

7 WORKED EXAMPLE 9 Find the exact value of each ratio.

a $\cos(-60°)$ **b** $\sin(-150°)$ **c** $\tan\left(-\frac{2\pi}{3}\right)$ **d** $\sin\left(-\frac{5\pi}{3}\right)$

8 **a** Draw a right-angled triangle in the Cartesian plane with θ in quadrant 1 and $\sin(\theta) = \frac{4}{5}$.

b Calculate the length of the other side of the triangle.

c Without actually calculating the value of θ, find the exact values of the other 2 trigonometric ratios using the information supplied.

9 Without actually calculating the value of θ, find the exact values of the other 2 trigonometric ratios using the information supplied.

a $\sin(\theta) = -\frac{1}{\sqrt{2}}$ and θ is in quadrant 4

b $\cos(\theta) = \frac{1}{\sqrt{2}}$ and θ is in quadrant 1

c $\cos(\theta) = -\frac{24}{25}$ and θ is in quadrant 3

d $\sin(\theta) = \frac{12}{13}$ and θ is in quadrant 2

10 Find the exact value of each expression.

a $\tan(135°) - \cos(135°)$

b $\sin(240°) + \sin(150°)$

c $\cos(120°) + \sin(240°)$

d $\sin^2(330°)$

e $\sin^2(300°) + \cos^2(300°)$

11 Simplify each expression.

a $\tan(\pi + x) + 3\tan(x)$

b $\cos(180° - x) + \cos(360° - x)$

c $\sin(180° + 5y) - 5\sin(5y)$

d $\cos(\pi + x) + 2\cos(2\pi - x)$

Exam practice

12 TECH-FREE $\sin\left(\frac{5\pi}{6}\right) =$

A $-\frac{\sqrt{3}}{2}$ **B** $-\frac{1}{2}$ **C** $\frac{1}{2}$ **D** $\frac{\sqrt{3}}{2}$

13 TECH-FREE $\cos\left(\frac{4\pi}{3}\right) =$

A $-\frac{\sqrt{3}}{2}$ **B** $-\frac{1}{2}$ **C** $\frac{1}{2}$ **D** $\frac{\sqrt{3}}{2}$

14 TECH-FREE $\tan\left(\frac{5\pi}{3}\right) =$

A $-\sqrt{3}$ **B** $-\frac{1}{\sqrt{3}}$ **C** $\frac{1}{\sqrt{3}}$ **D** $\sqrt{3}$

15 (4 marks) CF For each given trigonometric ratio, without actually calculating the value of θ, find the exact value of the other 2 trigonometric ratios using the information supplied.

a $\sin(\theta) = \frac{5}{13}$ and θ is in quadrant 2 *[2 marks]*

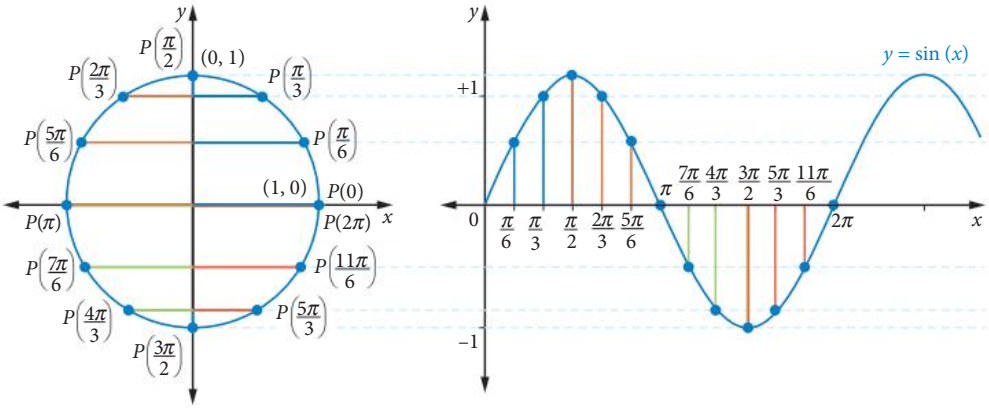
b $\cos(\theta) = \frac{3}{5}$ and θ is in quadrant 4. *[2 marks]*

4.4 The trigonometric functions

You can now treat the trigonometric ratios as **trigonometric functions**, $y = \sin(x)$, $y = \cos(x)$ and $y = \tan(x)$, and graph them on a Cartesian plane. You will use the unit circle and study how the x- and y-coordinates of $P(x, y)$ change as P moves around the circle.

For the graph of the sine function, draw a unit circle and divide it into multiples of $\frac{\pi}{6}$, i.e. 0, $\frac{\pi}{6}$, $\frac{2\pi}{6} = \frac{\pi}{3}$, $\frac{3\pi}{6} = \frac{\pi}{2}$, $\frac{4\pi}{6} = \frac{2\pi}{3}$, $\frac{5\pi}{6}$ and so on up to 2π. The y-coordinates or heights of these points represent the sine of the angles.

You can plot these heights to graph the sine function. Now, draw horizontal lines from these points on the circle to the y-axis. These are the heights for the sines of the angles because $\sin(\theta) = y$ on the unit circle.



On a new Cartesian plane, put multiples of $\frac{\pi}{6}$ on the x-axis.

Put points at the heights of $\sin(\theta)$ from the unit circle.

Join these points with a smooth curve to give you the graph of $y = \sin(x)$.

Because P moving on the circle repeats after one **cycle**, you can extend the graph by repeating the shape in both directions. Sine is a **periodic function**. It repeats after every 2π. This is called the **period** (T).

The value of $\sin(x)$ also moves between -1 and 1, and its **mean** or **central value** is 0.

The maximum height of the graph from the middle is called the **amplitude**.

Key features of the graph of $y = \sin(x)$

Domain: all real numbers

Range: $-1 \leq y \leq 1$

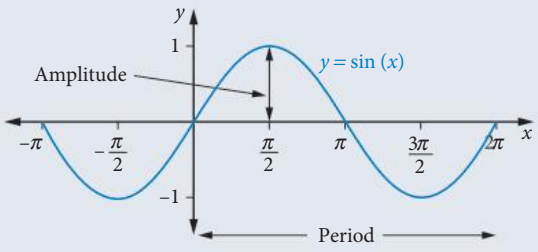
Period = 2π

Amplitude = 1

Zeros occur at … $-\pi$, 0, π, 2π …

Maxima occur at … $-\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}$ …

Minima occur at … $-\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2}$ …



Worksheet
Sine and cosine curves

The graph of the cosine function $y = \cos(x)$ can be drawn by graphing the x-coordinates or lengths of the points from the unit circle.

The x-coordinates or lengths of these points represent the cosine of the angles.

$\cos(0) = 1$, then it decreases until $\cos\left(\frac{\pi}{2}\right) = 0$ and so on.

You can graph these lengths to graph the cosine function.

Key features of the graph of $y = \cos(x)$

Domain: all real numbers

Range: $-1 \leq y \leq 1$

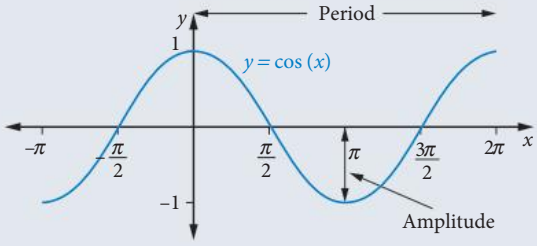
Period = 2π

Amplitude = 1

Zeros occur at … $-\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$ …

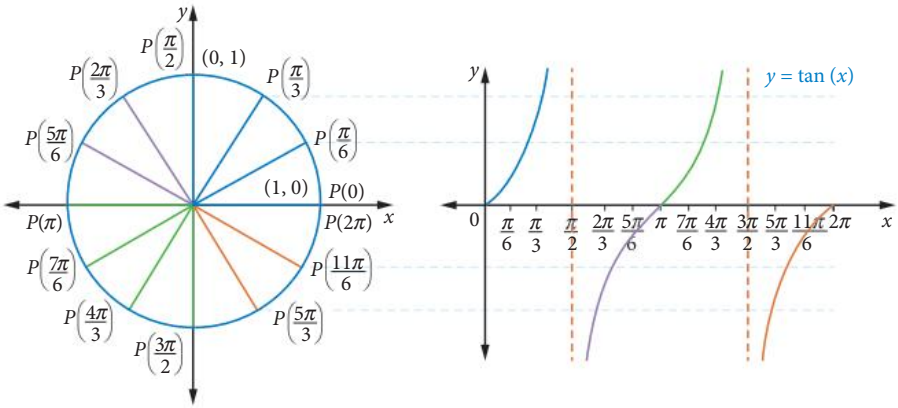
Maxima occur at … $-2\pi, 0, 2\pi, 4\pi$ …

Minima occur at … $-\pi, \pi, 3\pi, 5\pi$ …



The cosine graph has the same shape as the sine graph, but transposed along the x-axis.

You can also draw the graph of the tangent function from the unit circle. The value of tan is given by the height divided by the length, or $\frac{y}{x}$, of each point.



From the graph, the period of $y = \tan(x)$ is π. Amplitude does not apply.

Key features of the graph of $y = \tan(x)$

Domain: all real numbers except odd multiples of $\frac{\pi}{2}$, that is $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$

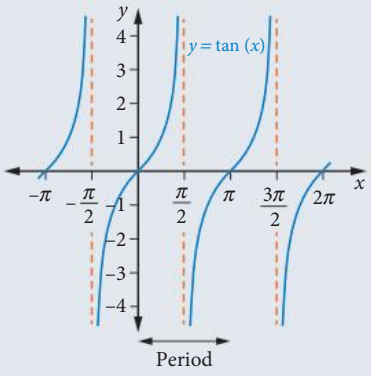
Range: all real numbers

Period = π

Zeros occur at … $-\pi, 0, \pi, 2\pi$ …

The graph has **asymptotes** at $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}$ …

The function is undefined at the asymptotes. Near the asymptotes the graph becomes very large or very small.



WORKED EXAMPLE 10 Graphs of trigonometric functions

Sketch the graph of each trigonometric function.

a $y = \cos(x)$ in the domain $-\pi \le x \le \pi$

b $y = \tan(x)$ in the domain $-\frac{\pi}{2} \le x \le \pi$.

Steps

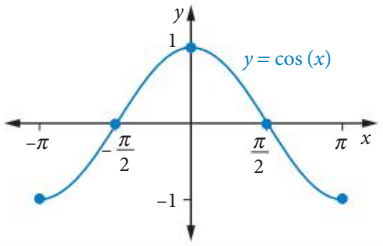
a 1 Construct a table of values for $-\pi \le x \le \pi$. You can limit the number of points you choose because you already know the shape of the graph $y = \cos(x)$.

2 Plot the points and join them with a smooth curve.

Working

$y = \cos(x)$

x	$-\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π
y	-1	0	1	0	-1



b 1 Construct a table of values for $-\frac{\pi}{2} \le x \le \pi$.

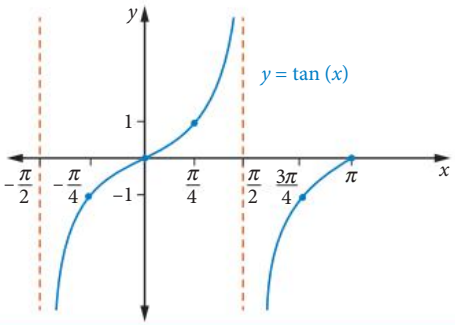
$\tan\left(\frac{\pi}{4}\right) = 1$, so it is convenient to use $x = -\frac{\pi}{4}$, $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$.

2 You know that there are asymptotes at $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$. Plot the points and join them with a smooth curve.

$y = \tan(x)$

x	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
y	ND	-1	0	1	ND	-1	0

ND = not defined



What happens to the shape of the graphs of trigonometric functions when they are translated vertically?

WORKED EXAMPLE 11 Vertical translation

Sketch the graph of $y = \sin(x) - 2$ in the domain $0 \le x \le 2\pi$.

Steps

1 Construct a table of values for $0 \le x \le 2\pi$.

$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$, so it is convenient to use values such as $x = \frac{\pi}{6}$ and $\frac{7\pi}{6}$.

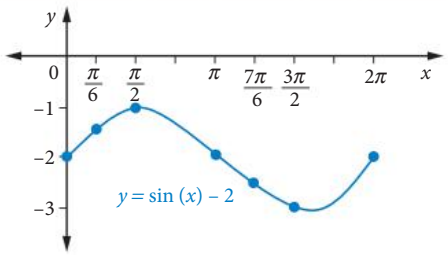
2 Plot the points and join them with a smooth curve.

This is the graph of $y = \sin(x)$ translated downwards 2 units.

Working

$y = \sin(x) - 2$

x	0	$\frac{\pi}{6}$	$\frac{\pi}{2}$	π	$\frac{7\pi}{6}$	$\frac{3\pi}{2}$	2π
y	-2	-1.5	-1	-2	-2.5	-3	-2



Translation of trigonometric functions

The graphs of $y = \sin(x) + k$, $y = \cos(x) + k$ and $y = \tan(x) + k$ are **translated** k units vertically compared with the graphs of $y = \sin(x)$, $y = \cos(x)$ and $y = \tan(x)$ respectively.

If $k > 0$, the graph is translated up.

If $k < 0$, the graph is translated down.

Reflection of trigonometric functions in the *x*-axis

The graphs of $y = -\sin(x)$, $y = -\cos(x)$ and $y = -\tan(x)$ are reflections in the x-axis of $y = \sin(x)$, $y = \cos(x)$ and $y = \tan(x)$ respectively.

EXERCISE 4.4 The trigonometric functions

ANSWERS p. 377

Recap

1 Find the exact value of each ratio.

a $\cos\left(\frac{13\pi}{6}\right)$ **b** $\sin\left(\frac{-2\pi}{3}\right)$ **c** $\cos\left(\frac{7\pi}{3}\right)$

d $\tan\left(\frac{-\pi}{4}\right)$ **e** $\sin\left(\frac{22\pi}{3}\right)$

2 Find the exact value of each expression.

a $\tan(135°) + \cos(135°)$ **b** $\sin(240°) - \sin(120°)$

c $\cos(120°) - \sin(210°)$ **d** $\cos^2(330°)$

Mastery

3 WORKED EXAMPLE 10 Sketch the graph of each trigonometric function in the domain $-\frac{3\pi}{2} \leq x \leq \frac{\pi}{2}$.

a $f(x) = \sin(x)$ **b** $f(x) = \cos(x)$ **c** $f(x) = \tan(x)$

4 WORKED EXAMPLE 11 Sketch the graph of each trigonometric function in the domain $0 \leq x \leq 2\pi$.

a $f(x) = \cos(x) - 5$ **b** $f(x) = \tan(x) + 3$ **c** $f(x) = \sin(x) + 4$

5 Sketch the graph of each trigonometric function in the domain $-2\pi \leq x \leq \pi$.

a $y = -\sin(x)$ **b** $y = -\tan(x)$ **c** $y = -\cos(x)$

6 Sketch the graph of each trigonometric function in the domain $0 \leq x \leq 2\pi$.

a $y = 5 - \cos(x)$ **b** $y = 3 - \sin(x)$ **c** $y = 1 - \tan(x)$

7 Sketch the graph of the function $f(t) = 3 - \cos(t)$ in the domain $0 \leq t \leq 2\pi$.

a Use the equation of the function to calculate the exact value of the function when $t = \frac{5\pi}{6}$.

b Use the graph to find the value(s) of t when $f(t) = 2.5$.

c Use the graph to find the approximate value of $f(t)$ when $t = 3$.

d Use the graph to find the approximate value of t when $f(t) = 2.8$.

9780170484909

8 The velocity (in m/s) of an object is given by $v = 5 + \sin(t)$ in the domain $0 \leq t \leq 4\pi$ where t is measured in seconds.

a Calculate its velocity at

i $\frac{7\pi}{6}$ s **ii** π s **iii** $\frac{\pi}{2}$ s

iv $\frac{\pi}{3}$ s **v** $\frac{\pi}{4}$ s **vi** $\frac{4\pi}{3}$ s

b Draw a graph of the velocity over the stated domain.

c Use the graph to find the time(s) when the velocity is 5.7 m/s.

Exam practice

9 TECH-FREE What is the amplitude of $y = 3\sin(x) - 4$?

A 3 **B** 4 **C** 6 **D** 8

10 TECH-FREE What is the mean of $y = 3\cos(x) + 4$?

A 3 **B** 4 **C** 6 **D** 8

11 TECH-FREE (2 marks) What are the minimum and maximum values of $y = 5\sin(x) + 4$?

12 TECH-FREE (2 marks) CF Determine the amplitude and range of the function $y = 2\sin\left(\frac{x}{12}\right) + 2$.

13 TECH-FREE (6 marks) CU The voltage of an AC current that is nominally A volts is given by

$$V = A\sqrt{2}\sin(100\pi t)$$

where V is in volts and t in seconds.
If $A = 240$, determine for V

a its period *[2 marks]*

b its maximum voltage *[2 marks]*

c its amplitude *[2 marks]*

Videos
Amplitude and period

Graphing the cosine function

Worksheets
Sketching periodic functions: amplitude and period

Sketching periodic functions: phase and vertical shift

Trigonometric graphs

Puzzle
Trigonometric graphs match-up

TI-Nspire
Trigonometric functions

4.5 Transformations of trigonometric functions

In Chapters 2 and 3, you compared $y = a(x - h)^2 + k$ to $y = x^2$ and $y = a(x - h)^3 + k$ to $y = x^3$.

The transformations produced were:

- a dilates (stretches) the graph by a factor of a from the x-axis
- when $a < 0$, the graph is reflected in the x-axis
- h translates the graph h units horizontally
- k translates the graph k units vertically.

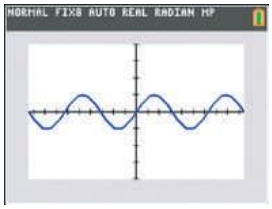
Investigation — Graphs of $y = a\sin(b(x - h)) + k$, $y = a\cos(b(x - h)) + k$

Use your graphics calculator for this investigation.

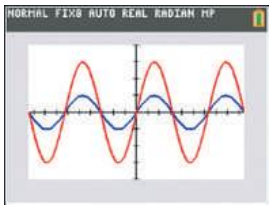
Steps | **Working**

TI-84 Plus CE

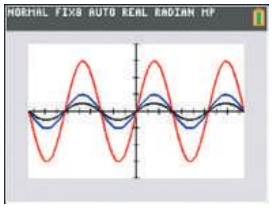
1 Make sure that the angle measure is set to radians.
Press window and set the domain and range values as $-3\pi \le X \le 3\pi$ and $-5 \le Y \le 5$.
Press y= and enter the function $Y_1 = \sin(x)$ into the function line using X,T,θ,n to enter the variable.
Press ENTER.
Then press graph to display the graph of the function.



2 Now press y= and enter the function $Y_2 = 3\sin(x)$ into the function line.
Press ENTER.
Then press graph to display the graph of the function.

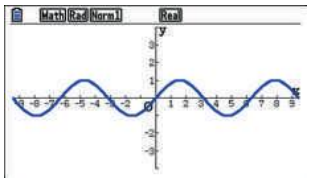


3 Follow the same procedure to enter $Y_3 = \frac{1}{2}\sin(x)$ and display the graph.
You can use the zoom button or the window button to modify the domain and range values if you want to see more detail.

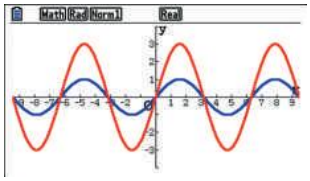


Casio fx-CG50 AU

1 Press MENU and select the **Graph** page.
Press SHIFT F3 V-Window and set the domain and range values as $-3\pi \le X \le 3\pi$ and $-5 \le Y \le 5$.
Enter the function $Y_1 = \sin(x)$ into the function line using X,θ,T to enter the variable.
Press EXE.
Then press F6 (DRAW) to display the graph of the function.



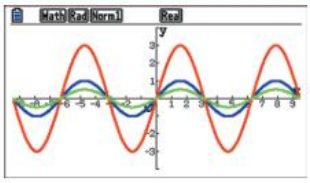
2 Now press EXIT and enter the function $Y_2 = 3\sin(x)$ into the function line.
Press EXE.
Then press F6 (DRAW) to display the graph of the function.



9780170484909

3 Follow the same procedure to enter $Y_3 = \frac{1}{2}\sin(x)$ and display the graph.

You can press SHIFT F2 zoom or press SHIFT F3 V-Window to modify the domain and range values if you want to see more detail.



Both calculators:

Use your graphics calculator to compare the following graphs to $y = \sin(x)$:

- $y = \sin(2x)$, $y = \sin(-2x)$, $y = \sin(0.5x)$ and $y = \sin(-0.5x)$.
- Then graph $y = \sin\left(x + \frac{\pi}{6}\right)$, $y = \sin\left(x - \frac{\pi}{6}\right)$, $y = \sin\left(x + \frac{\pi}{3}\right)$ and $y = \sin\left(x - \frac{\pi}{3}\right)$.
- Compare graphs of $\cos(x)$, etc. in the same way.

From your investigation, you should see that $\sin(x)$, $\cos(x)$ and $\tan(x)$ are all transformed through dilation, translation and reflection. $y = \sin(bx)$ and $y = \cos(bx)$ have the period changed by the factor $\frac{1}{b}$ to $\frac{2\pi}{b}$.

Graphs of $y = a\sin(b(x - h)) + k$, $y = a\cos(b(x - h)) + k$

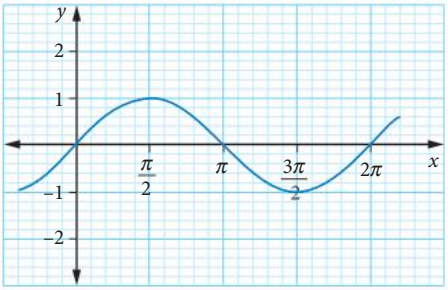
- The **amplitude** is the magnitude of a.
- If a is negative, the graph is flipped upside down (reflected in the line $y = k$).
- The **period** is $\frac{2\pi}{b}$.
- The horizontal translation is h units. This is also called the **phase shift**.
- The vertical translation is k units.
- The **mean value** is k.

The 'beginning' of the graphs is shifted from $(0, 0)$ to (h, k).

WORKED EXAMPLE 12 Transformations of trigonometric functions 1

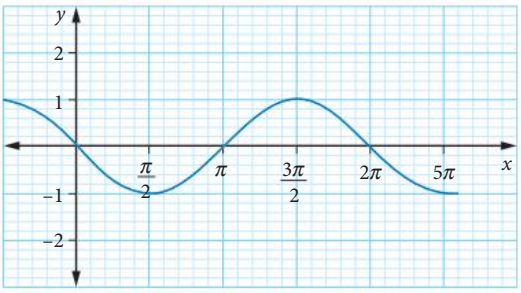
Transform the graph of $y = \sin(x)$ into $y = -2\sin\left(2x + \frac{\pi}{4}\right) + 2$.

Steps	Working
1 Write the equation in the form $y = a\sin(b(x - h)) + k$.	$y = -2\sin\left(2\left(x - \left(-\frac{\pi}{8}\right)\right)\right) + 2$
2 Start with $y = \sin(x)$.	$y = \sin(x)$



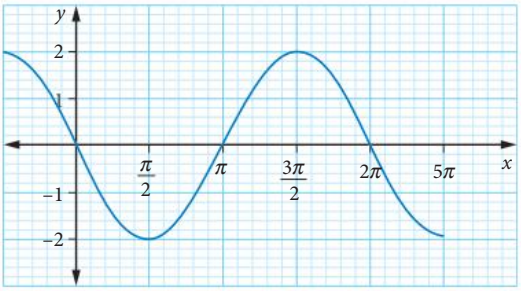
3 Reflect $y = \sin(x)$ in the x-axis.

$y = -\sin(x)$



4 Increase the amplitude of the function to 2.

$y = -2\sin(x)$

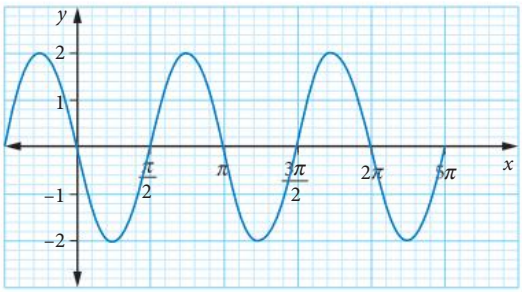


Exam hack

Change the equation to the correct form. Do the dilations and reflections *before* the translations.

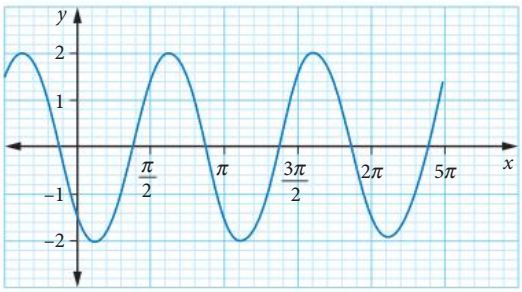
5 Change the period to $\frac{2\pi}{2} = \pi$.

$y = -2\sin(2x)$



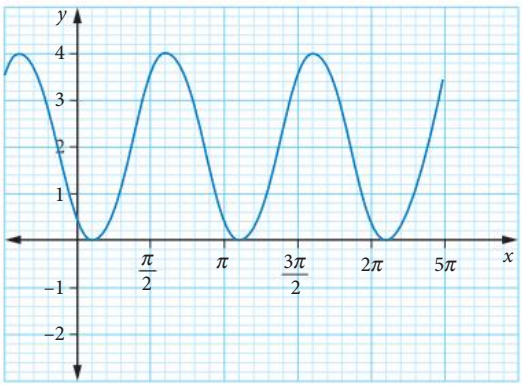
6 Translate the graph by $-\frac{\pi}{8}$ units, that is, $\frac{\pi}{8}$ units to the left.

$y = -2\sin\left(2\left(x - \left(-\frac{\pi}{8}\right)\right)\right)$



7 Translate the graph vertically by +2 units.

$y = -2\sin\left(2\left(x + \frac{\pi}{8}\right)\right) + 2$



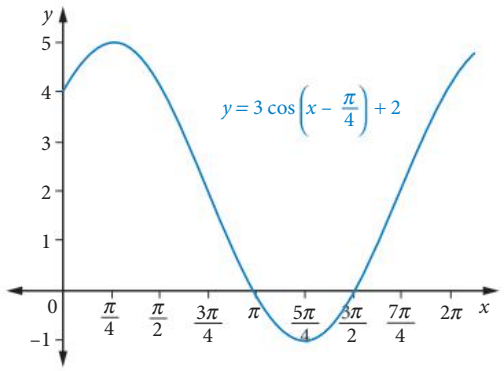
9780170484909

You can sketch the graphs using a 'grid' to work out where the graph is.

1. Arrange your equation to $y = a\sin(b(x - h)) + k$ or $y = a\cos(b(x - h)) + k$.
2. Write down a, b, h and k.
3. Sketch in dotted lines for the mean: $y = k$, the **maximum** and **minimum:** $y = k \pm a$
4. Work out the period $T = \frac{2\pi}{b}$.
5. Divide the period by 4. Label the x-axis in multiples of $\frac{T}{4}$.
6. Pencil in the turning points and zeros at these values.
7. Translate points horizontally by the phase shift h. Mark their new locations.
8. Join the new points with a smooth curve.
9. Label your graph.

WORKED EXAMPLE 13 Transformations of trigonometric functions 2

Sketch the graph of $y = 3\cos\left(x - \frac{\pi}{4}\right) + 2$ in the domain $0 \le x \le 2\pi$.

Steps	Working
1 List how the graph of $y = \cos(x)$ has been transformed.	The graph of $y = 3\cos\left(x - \frac{\pi}{4}\right) + 2$ has: • period = 2π • amplitude = 3 • phase shift = $\frac{\pi}{4}$ (to the right) • vertical translation = 2 units up
2 Draw the graph of $y = 3\cos\left(x - \frac{\pi}{4}\right) + 2$ by transforming the graph of $y = \cos(x)$.	

Alternative method: Using key points

1 Work out the changes.

The period does not change.

The graph is dilated vertically by the factor 3.

The graph is translated $\frac{\pi}{4}$ to the right and 2 up.

2 Do the transformations for the key values of $\cos(x)$.

$$(0, 1) \to (0, 3 \times 1) = (0, 3) \to \left(0 + \frac{\pi}{4}, 3 + 2\right) = \left(\frac{\pi}{4}, 5\right)$$

$$\left(\frac{\pi}{2}, 0\right) \to \left(\frac{\pi}{2}, 3 \times 0\right) = (0, 0) \to \left(\frac{\pi}{2} + \frac{\pi}{4}, 0 + 2\right) = \left(\frac{3\pi}{4}, 2\right)$$

$$(\pi, -1) \to (\pi, 3 \times -1) = (\pi, -3)$$

$$\to \left(\pi + \frac{\pi}{4}, -3 + 2\right) = \left(\frac{5\pi}{4}, -1\right)$$

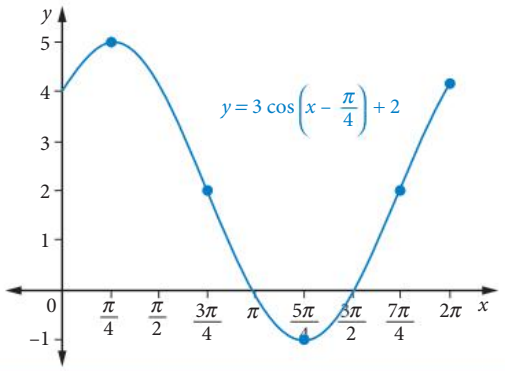
$$\left(\frac{3\pi}{2}, 0\right) \to \left(\frac{3\pi}{2}, 3 \times 0\right) = (0, 0) \to \left(\frac{3\pi}{2} + \frac{\pi}{4}, 0 + 2\right) = \left(\frac{7\pi}{4}, 2\right)$$

$$(2\pi, 1) \to (2\pi, 3 \times 1) = (2\pi, 3)$$

$$\to \left(2\pi + \frac{\pi}{4}, 3 + 2\right) = \left(\frac{9\pi}{4}, 5\right)$$

3 Put the key points on the axes.

Join the points with a smooth curve.



EXERCISE 4.5 Transformations of trigonometric functions

ANSWERS p. 379

Recap

1 Sketch the graph of each trigonometric function in the domain $0 \le x \le 2\pi$.

a $y = 3 - \cos(x)$ **b** $y = 2 - \sin(x)$ **c** $y = 5 - \tan(x)$

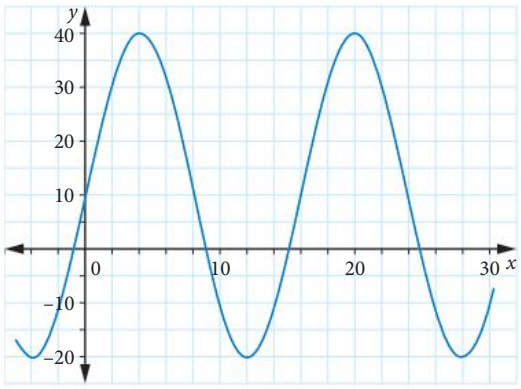
2 a Draw the graph of the function $f(t) = 4 - \cos(t)$ in the domain $0 \le t \le 2\pi$.

b Use the equation to calculate the function's exact value when $t = \frac{5\pi}{6}$.

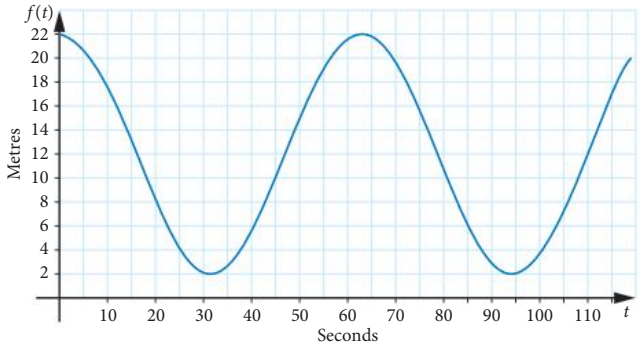
Mastery

3 Find the amplitude and the period of each function.

a



b



4 Identify the amplitude of each trigonometric function.

a $y = 3\cos(x)$ **b** $y = 7\sin(x)$ **c** $y = -2\sin(x)$

d $y = 2.8\cos(x)$ **e** $y = 5\cos(x) + 9$ **f** $y = 8\sin(x) - 6$

g $f(t) = 12 - 10\sin(t)$ **h** $f(x) = -15\cos(x) - 18$

5 Determine the maximum and minimum values for each trigonometric function for $0 \le x \le 2\pi$.

a $y = \sin(x)$ **b** $y = 3\sin(x)$ **c** $y = 4\cos(x)$

d $y = \sin(x) - 4$ **e** $y = \cos(x) + 3$ **f** $y = 5 - 2\sin(x)$

6 Calculate the period of each function.

a $y = \cos(x)$ **b** $y = \sin(2x)$ **c** $y = \tan(0.5x)$

d $y = -4\tan(5x)$ **e** $y = 3\cos\left(\frac{x}{2}\right)$ **f** $y = 7\sin\left(\frac{x}{3}\right)$

7 Draw a graph of each function for $0 \le x \le 2\pi$.

a $y = 0.5\cos(x)$ **b** $y = -4\sin(x)$

8 Draw a graph of each function for $0 \le x \le 2\pi$.

a $y = 5 + 2\sin(x)$ **b** $y = 3\cos(x) - 2$

9 Draw a graph of each function showing a full cycle. State the amplitude and the period.

a $y = 2\cos(3x)$ **b** $y = \frac{1}{2}\sin(2x)$ **c** $y = 4\sin\left(\frac{x}{2}\right)$

d $y = -3\cos\left(\frac{x}{3}\right)$ **e** $y = 5\cos\left(\frac{x}{4}\right)$ **f** $y = -\frac{1}{3}\sin(4x)$

10 Draw a graph of each function showing a full cycle. Check the graph with a graphics calculator. (Only do parts **c** and **f** if you want a challenge.)

a $y = \sin\left(x - \frac{\pi}{2}\right)$ **b** $y = \sin\left(x + \frac{\pi}{2}\right)$ **c** $y = \tan\left(x - \frac{\pi}{4}\right)$

d $y = \cos\left(x - \frac{\pi}{6}\right)$ **e** $y = \cos\left(x + \frac{\pi}{6}\right)$ **f** $y = \tan\left(x + \frac{\pi}{4}\right)$

11 WORKED EXAMPLES 12, 13 Draw a graph of each function showing a full cycle, and check with a graphics calculator.

a $y = 4\sin\left[2\left(x - \frac{\pi}{3}\right)\right] + 2$ **b** $y = -2\cos\left[3\left(x - \frac{\pi}{6}\right)\right] - 3$

c $y = 5\sin\left[4\left(x + \frac{\pi}{8}\right)\right] - 2$ **d** $y = 2\tan\left[2\left(x - \frac{\pi}{4}\right)\right] + 3$

12 Draw a graph of each function showing a full cycle, and check with a graphics calculator.

a $y = 2\sin\left(2x - \frac{\pi}{3}\right) + 3$ **b** $y = 3\cos(4x + \pi) - 2$

c $y = 4 - 3\sin\left(6x + \frac{3\pi}{2}\right)$ **d** $y = 5\cos\left(2x - \frac{3\pi}{4}\right)$

13 The tide height at a jetty is approximately periodic. The high tide of 2.4 m occurs at 11 am, followed by a low tide of 0.6 m at about 5 pm on the same day.

a Calculate the amplitude of the tide height.

b Calculate the period of the tide height.

14 The temperature, T, in °C measured by a weather station is modelled by $T = 24 - 3\cos\left(\frac{\pi t}{12}\right)$, where t is time in hours and $t = 0$ is midnight.

a Draw a graph of the temperature for the 24 hours after midnight.

b Use the graph to calculate when the maximum temperature occurs.

c When does the minimum temperature occur?

d At what time is the temperature 26°C?

15 The height of the tide, h, in metres above sea level at a pier is modelled by $h = 4\sin\left(\frac{\pi t}{6}\right) + 2$, where t is time in hours after midnight.

a Draw a graph of the tide height for $0 \leq t \leq 24$.

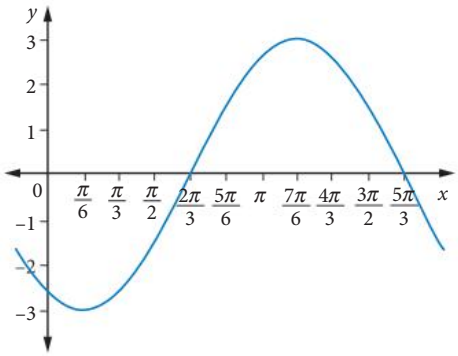
b Use the graph to calculate the height of the high tide.

c What was the height of the tide at 11:30 am?

d A ferry can only reach the pier when the tide is 3 m above sea level. When could the ferry reach the pier?

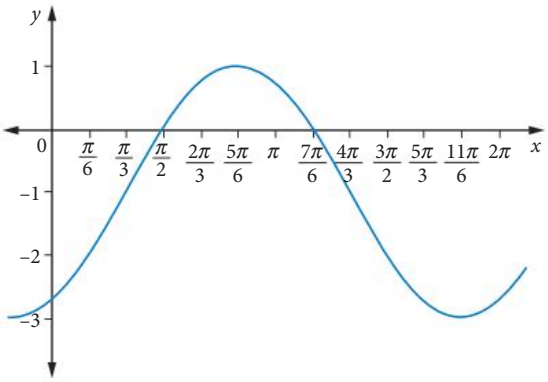
16 This graph is a function whose equation is in the form of $y = a\cos(x + c)$.

Determine the equation of the function.



17 The graph below is a function whose equation is in the form of $y = a\sin b(x + c) + d$.

Find the values of a, b, c and d and hence write the equation of the function.



Exam practice

18 What is the phase shift of $4\cos\left(3x + \frac{\pi}{2}\right) - 2$?

A $-\frac{3\pi}{2}$ **B** $-\frac{\pi}{6}$ **C** $\frac{\pi}{6}$ **D** $\frac{3\pi}{2}$

19 What is the vertical dilation factor of $4\cos\left(3x + \frac{\pi}{2}\right) - 2$?

A $\frac{1}{4}$ **B** $\frac{1}{3}$ **C** 3 **D** 4

20 What is the horizontal dilation factor of $4\cos\left(3x + \frac{\pi}{2}\right) - 2$?

A $\frac{1}{4}$ **B** $\frac{1}{3}$ **C** 3 **D** 4

21 (3 marks) CF Find, correct to 2 decimal places, the values of x in the interval $0 \le x \le 2\pi$ for which $3\cos\left(2x + \frac{\pi}{6}\right) = 2$.

22 (7 marks) CU The intensity of light from the Moon at t days after a full moon can be modelled by the function with the rule $y = 0.5\cos\left(\frac{\pi t}{14}\right) + 0.5$.

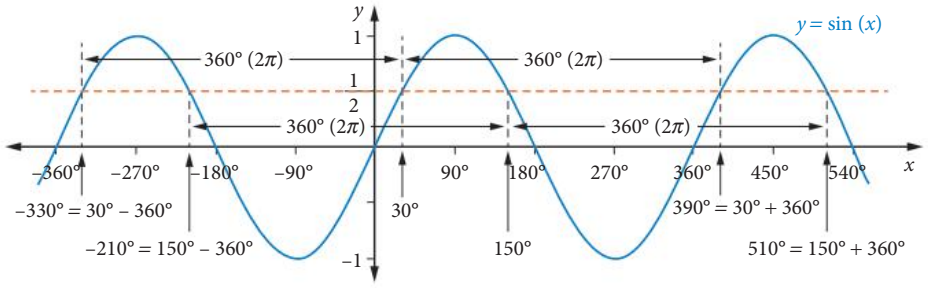
Find the maximum intensity and when it is 75% of the maximum.

Video
Trigonometric equations

Worksheet
Trigonometric equations

4.6 Trigonometric equations

Because trigonometric functions are cyclic and repeat themselves after a period, trigonometric equations have infinite solutions; for example, the solutions for $\sin(x) = 0.5$ are shown by the dotted lines on the graph below.



However, the number of solutions can be limited by the domain, such as $0 \le \theta \le 2\pi$.

A scientific calculator only gives one answer for each value of $\sin^{-1}$, $\cos^{-1}$ or $\tan^{-1}$. You need to work out the other answers needed. Always give an exact answer if you can.

WORKED EXAMPLE 14 Trigonometric equations

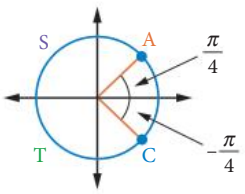
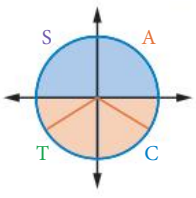
Solve each trigonometric equation.

a $\cos(x) = \frac{1}{\sqrt{2}}$ for $-\pi < x \le \pi$

b $4 + 4\sin(\theta) = 2$ for $0 \le \theta \le 2\pi$

c $2\cos(x) = \sqrt{3}$ for $-180° \le x \le 180°$

d $3\sin(x) + 1 = 0$ for $0 \le x \le 2\pi$

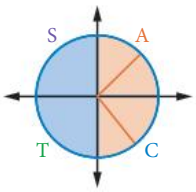
Steps	Working
a 1 Use the exact value for the acute (reference) angle.	$\cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$
2 Draw a sketch. Use the reference angle and ASTC to mark another solution.	
3 Write the solutions in the restricted domain.	$x = -\frac{\pi}{4}$ or $\frac{\pi}{4}$
b 1 Write the equation.	$4 + 4\sin(\theta) = 2$
2 Rearrange and simplify.	$4\sin(\theta) = -2$
3 Ignore the sign to get the reference angle, β.	$\sin(\theta) = -\frac{1}{2}$ $\beta = \frac{\pi}{6}$
4 Show where sine is negative.	
5 Use the reference angle and the domain ($0 \le \theta \le 2\pi$) to work out the answers.	$\theta = \pi + \frac{\pi}{6}$ and $\theta = 2\pi - \frac{\pi}{6}$
6 Simplify.	$\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$

c **1** Write the equation. $2\cos(x) = \sqrt{3}$

2 Rearrange and simplify. $\cos(x) = \frac{\sqrt{3}}{2}$

3 Find the reference angle, β. $\beta = 30°$

4 Draw a sketch.



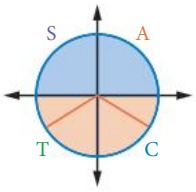
5 Use the reference angle and the domain. $x = -30°, 30°$

d **1** Write the equation. $3\sin(x) + 1 = 0$

2 Rearrange and simplify. $\sin(x) = -\frac{1}{3}$

3 The question does not ask for an approximation, so you need an exact reference angle. $0 \le \sin^{-1}\left(\frac{1}{3}\right) \le \frac{\pi}{2}$

4 Draw a sketch.



5 Use the reference angle and the domain ($0 \le \theta \le 2\pi$). $x = \pi + \sin^{-1}\left(\frac{1}{3}\right)$ or $2\pi - \sin^{-1}\left(\frac{1}{3}\right)$

In part **d** above, the answers to 4 decimal places would be $\pi + 0.3398 = 3.4814$, $2\pi - 0.3398 = 5.9433$.

You can use your graphics calculator to find approximate solutions for trigonometric equations.

TECHNOLOGY Trigonometric equations

Solve $2\sin(x) = \sqrt{3}$, $0 \le x \le 2\pi$, correct to 2 decimal places.

Rearrange the equation: $\sin(x) = \frac{\sqrt{3}}{2}$

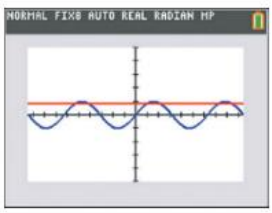
TI-84 Plus CE

Make sure that the angle measure is set to radians.

Press **y=** and enter the functions

$Y_1 = \sin(x)$

$Y_2 = \frac{\sqrt{3}}{2}$

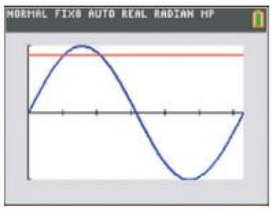


Then press **graph** to display the graphs of the functions.

Press **window** and change the window settings to

$0 \le X \le 2\pi$ and $-1 \le Y \le 1$

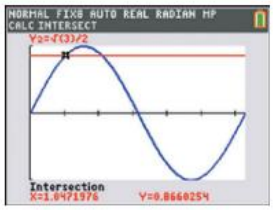
to make the intersections clearer.



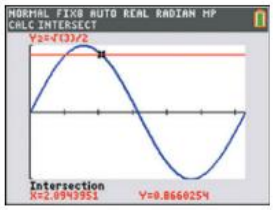
TI-Nspire
Trigonometric functions

Press 2nd trace calc and select **5: intersect**.

Move the cursor to position the cross on Y_1 close to the first point of intersection of the graphs. Press ENTER. Then move the cursor to position the cross on Y_2 close to the point of intersection of the graphs. Press ENTER. Finally move the cursor to position the cross close to your guess for the point of intersection of the graphs. Press ENTER.



Repeat the procedure to find the second intersection.

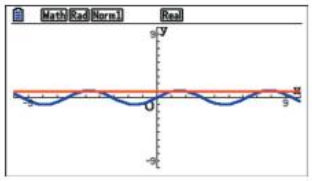


State the result.

$x \approx 1.05, 2.09$

Casio fx-CG50 AU

Press MENU and select the **Graph** page.



Enter the functions

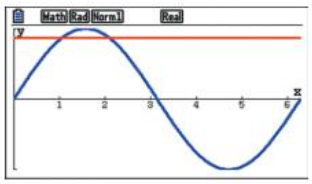
$Y_1 = \sin(x)$

$Y_2 = \frac{\sqrt{3}}{2}$

Press EXE.

Then press F6 (DRAW) to display the graph of the function.

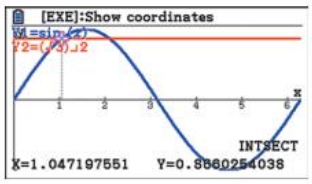
Press SHIFT F3 V-Window and set the domain and range values as $0 \le X \le 2\pi$ and $-1 \le Y \le 1$ to make the intersections clearer.

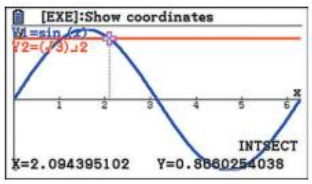


Press SHIFT F5 (G-Solv) and select F5 (INTSECT).

The first point of intersection is then displayed.

Press the right arrow button to display the second intersection.





State the result.

$x \approx 1.05, 2.09$

Use the same procedure to find the solutions to

a $y = \cos(x)$ and $y = x - 3$ in the domain $-\pi \le x \le \pi$

b $y = 2\sin(x) + 3$ and $y = 3x - 4$ in the domain $0 \le x \le \pi$

c $y = \frac{1}{2}\tan\left(x + \frac{\pi}{4}\right)$ and $y = 4 - 5x$ in the domain $0 \le x \le \pi$.

Solving trigonometric equations

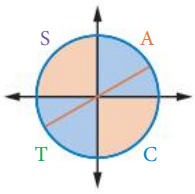
To solve equations of the form $\sin(x) = a$, $\cos(x) = a$ or $\tan(x) = a$:
1. use the sign of a to determine the quadrants
2. find the reference angle using either exact values or inverse functions
3. sketch the solutions on the unit circle or using a graph
4. examine the given domain to determine the required values.

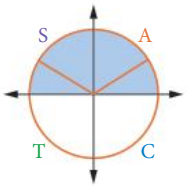
You sometimes need to use algebraic techniques to solve trigonometric equations.

WORKED EXAMPLE 15 Trigonometric equations 2

a **Solve** $\sin(2x) = 1.5\cos(2x)$ for $0 \le x \le 2\pi$ correct to 2 decimal places.

b **Solve** $2\sin^2(B) + 7\sin(B) - 4 = 0$, where $\frac{\pi}{2} \le B \le \frac{3\pi}{2}$.

Steps	Working
a 1 Write the equation.	$\sin(2x) = 1.5\cos(2x)$
2 Divide both sides by $\cos(2x)$.	$\frac{\sin(2x)}{\cos(2x)} = 1.5$
3 Simplify.	$\tan(2x) = 1.5$
4 Use ASTC.	tan > 0 in 1st and 3rd quadrants.
5 Draw a sketch.	
6 Use a calculator to find $\tan^{-1}$.	$\tan^{-1}(1.5) = 0.9827\ldots$
7 Domain is $0 \le x \le 2\pi$, so you need to find values of $2x$ in the domain $0 \le 2x \le 4\pi$.	$2x = 0.9827\ldots, \pi + 0.9827\ldots, 2\pi + 0.9827\ldots, 3\pi + 0.9827\ldots$
8 Evaluate.	$\approx 0.9827\ldots, 4.1243\ldots, 7.2659\ldots, 10.4075\ldots$
9 Solve for x and round off.	$x \approx 0.49, 2.06, 3.63, 5.20$
10 Compare with the domain for x.	The solutions are within the given domain.
b 1 The equation $2\sin^2(B) + 7\sin(B) - 4 = 0$ is a quadratic. This is easier to see if $\sin(B)$ is replaced by the variable x.	$2\sin^2(B) + 7\sin(B) - 4 = 0$ Let $x = \sin(B)$.
2 Replace $\sin(B)$ with x.	$2x^2 + 7x - 4 = 0$
3 Factorise.	$(2x - 1)(x + 4) = 0$
4 Solve for x.	$2x - 1 = 0$ or $x + 4 = 0$ $x = \frac{1}{2}$ or $x = -4$
5 Replace x with $\sin(B)$.	$\sin(B) = \frac{1}{2}$ or $\sin(B) = -4$

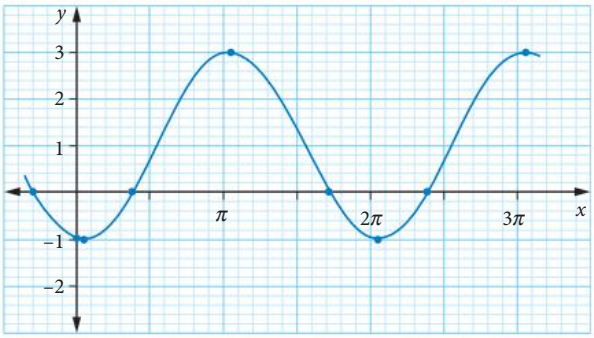
6	Discard the false solution.	$\sin(B) = 4$ has no solution (not between -1 and 1) $\sin(B) = \frac{1}{2}$ only.
7	Use ASTC.	$\sin > 0$ in 1st and 2nd quadrants.
8	Draw a sketch.	
9	Write the reference angle.	$\beta = \frac{\pi}{6}$
10	Use the reference angle and domain.	$B = \frac{\pi}{6}, \pi - \frac{\pi}{6}$
11	Simplify.	$B = \frac{\pi}{6}, \frac{5\pi}{6}$
12	Refer to the domain $\frac{\pi}{2} \leq B \leq \frac{3\pi}{2}$.	$B = \frac{5\pi}{6}$

EXERCISE 4.6 Trigonometric equations

ANSWERS p. 382

Recap

1 This graph has an equation in the form of $y = a\cos(x + h) + k$.



Determine the equation of the function.

2 Draw a graph of each function, showing a full cycle, and check with a graphics calculator.

a $y = 3\sin\left(3x - \frac{\pi}{4}\right) + 2$

b $y = 2\cos(3x + \pi) - 3$

Mastery

3 WORKED EXAMPLE 14 Solve each equation for the domain given.

a	$2\cos(x) = -2$	$-\pi \leq x \leq \pi$
b	$\sin(x) - 1 = 0$	$0 \leq x \leq 2\pi$
c	$1 - \tan(x) = 0$	$-2\pi \leq x \leq 0$
d	$6\cos(x) - 3 = 3$	$-\pi \leq x \leq 2\pi$
e	$10\cos(x) + 5 = -5$	$-\pi \leq x \leq \pi$
f	$\cos(x) + 8 = 8$	$0 \leq x \leq 3\pi$
g	$3\tan(x) + 3 = 0$	$-\pi \leq x \leq \pi$
h	$2\sin(x) - 1 = 0$	$-\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$

9780170484909

4 Solve each equation for $0° \le \theta \le 360°$.

a $\tan(\theta) = 1$ **b** $\sin(\theta) = \frac{1}{2}$ **c** $\cos(\theta) = -\frac{\sqrt{3}}{2}$ **d** $\sin(\theta) = -\frac{1}{\sqrt{2}}$

e $\cos(\theta) = 0$ **f** $\tan(\theta) = \sqrt{3}$ **g** $\sin(\theta) = \sin 50°$ **h** $\cos(\theta) = \cos 115°$

5 WORKED EXAMPLE 15 Solve each equation over the domain $0 \le x \le \pi$, correct to 4 decimal places.

a $3\sin(x) = 1$ **b** $4\cos(x) + 1 = 0$ **c** $5\tan(x) - 2 = 0$

d $9\cos(x) = 5$ **e** $7\sin(x) - 3 = 0$ **f** $3\tan(x) + 20 = 0$

6 Find all values of x in the domain $0 \le x \le \pi$ for which:

a $\sin(2x) = 0.5$ **b** $\cos(2x) = -1$ **c** $\sin(2x) = 1$

7 Solve each equation for $0° \le \beta \le 360°$.

a $\cos^2(\beta) = \frac{1}{4}$ **b** $\sin^2(\beta) = 1$

c $6\tan^2(\beta) = 2$ **d** $2\cos^2(\beta) = 1$

8 Solve each equation for the domain indicated.

a $2\sin^2(\theta) + \sin\theta = 0$, where $90° \le \theta \le 270°$

b $\cos^2(\theta) - \frac{1}{2}\cos\theta = 0$, where $180° \le \theta \le 360°$

c $\tan^2(\theta) = \tan(\theta)$, where $0 \le \theta \le 2\pi$

d $4\sin^2(\theta) + 2\cos^2(\theta) - 3 = 0$, where $-\pi \le \theta \le \pi$

9 Solve each equation for $0 \le \alpha \le 2\pi$, giving approximate answers where appropriate.

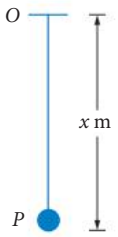
a $2\sin^2(\alpha) + 3\sin(\alpha) - 5 = 0$ **b** $2\cos^2(\alpha) + 5\cos(\alpha) - 3 = 0$

c $3\cos^2(\alpha) - 4\cos(\alpha) = 4$ **d** $6\sin^2(\alpha) + \sin(\alpha) - 1 = 0$

10 A particle P bobs up and down on the end of an elastic string that is fixed at O. Its distance below O is given by the rule

$$x = 0.34 + 0.04\sin(4t)$$

where x is in metres and t is the time in seconds since observation began.



a How far below O is the particle initially?

b How far below O is the particle at $t = 0.6$?

c What is the distance below O of the lowest point in the particle's path, and what is the value of t when the particle first reaches this point?

11 Find, correct to 2 decimal places, the 2 values of x closest to 0 for which $5\sin\left(2x - \frac{\pi}{6}\right) - 2 = 0$.

Exam practice

12 The solution(s) of $\sqrt{3}\sin(x) = 1.5$ between 0 and π are

A $\frac{\pi}{6}$ only **B** $\frac{\pi}{6}$ and $\frac{5\pi}{6}$ **C** $\frac{\pi}{3}$ only **D** $\frac{\pi}{3}$ and $\frac{2\pi}{3}$

13 (2 marks) Solve $2\cos(x) = -1$ for $0 \leq x < 2\pi$.

14 (3 marks) Solve $4\sin^2(x) + 2(\sqrt{3} - 1)\sin(x) = \sqrt{3}$ for $-\pi \leq x < \pi$.

15 (3 marks) CF Find the values of x between 0° and 360° for which $2\cos(x + 50°) + \sqrt{3} = 0$.

16 (4 marks) CU The temperature in a room varies over time in a sinusoidal fashion, with a minimum of 8°C at 6 am and a maximum of 20°C at 11 am.

What temperature is it at noon? At what other times during this day would it be the same?

Sinusoidal means the shape of the sine graph (or cosine graph).

Worksheets
Applying trigonometric functions

Trigonometric functions assignment

Trigonometric functions extension

4.7 Modelling with trigonometric functions

Various real-world situations have periodic or cyclic movement. These include ocean tides, pendulums, springs and the rotation of the Earth.

WORKED EXAMPLE 16 Models from data

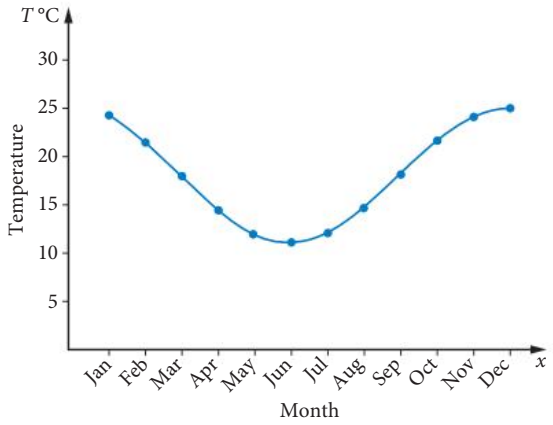
The table shows the average monthly temperature on the Sunshine Coast over a 12-month period.

x	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
T (°C)	24.1	21.5	18.2	14.5	11.9	11.2	12.1	14.7	17.8	21.8	23.8	25.2

a **Sketch** a graph to represent this data.

b Find the period and amplitude.

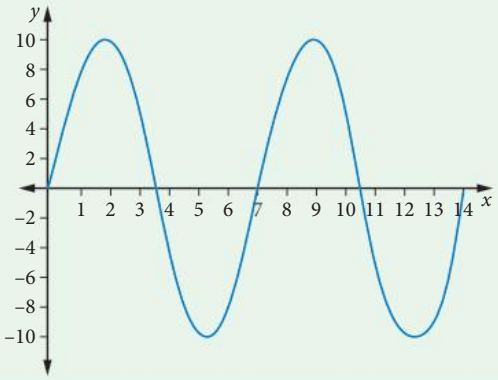
c Write an equation for the curve.

Steps	Working
a Plot the points and join them with a smooth curve.	
b **1** Describe the shape of the graph.	The graph approximates a cosine function over a year.
2 Use the maximum and minimum.	$a \approx \frac{25.2 - 11.2}{2} = 7$
3 State the answer.	The period is 12 months and the amplitude 7°C.

c 1 Choose the unit of time.	Let the time be in months, with January as 0.
2 Write the mean.	The mean temperature is about $\frac{11.2 + 25.2}{2} = 18.2°C$.
3 Write the function.	$T = a\cos(bt) + k$ where $a = 7$, $k = 18.2$
	Period $= \frac{2\pi}{b} = 12$
	$b = \frac{2\pi}{12} = \frac{\pi}{6}$
	$T = 7\cos\left(\frac{\pi}{6}t\right) + 12°C$, where $t = 0$ in January and t is in months.

WORKED EXAMPLE 17 Models from a graph

Construct a model for the periodic function shown here.



Exam hack

In a technology-active exam, graph your equation using a calculator to check your answer.

Steps	Working
1 Identify the periodic function.	The graph commences at the origin, so it is a sine function.
2 Write the general form of a sine function. You need to calculate the values of a, b, h and k.	$y = a\sin(b(x - h)) + k$
3 Determine the amplitude.	$a = 10$
4 Use the period to find b.	period $= \frac{2\pi}{b} = 7$
	$b = \frac{2\pi}{7}$
5 Write the values of h and k.	$h = 0$, $k = 0$
6 Use the values to write the equation.	$y = 10\sin\left(\frac{2\pi x}{7}\right)$

WORKED EXAMPLE 18 Making predictions from a model

In a harbour, high tides occur 12 h 20 min apart. A high tide occurs at 2 am on 3 October. Low tides occur halfway between high tides. The depth of water at the entrance bar is 2.8 m at low tide and 8.4 m at high tide.

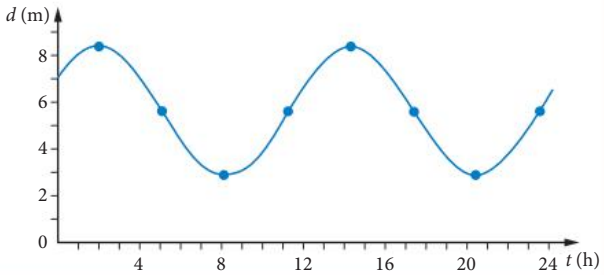
a **Sketch** a rough version of the tidal heights.

b **Construct** a sinusoidal model for the depth of the water, d, as a function of the time t hours since midnight on 3 October.

c **Determine** the depth of water at 4 pm on 5 October.

d Use a graphics calculator to find the times on 3 October at which a ship with a draught of 5 m can enter the harbour.

e **Evaluate** the reasonableness of your model.

Steps	Working
a Low tides occur halfway between high tides. High tides are 12 h 20 min apart so the next low tide will be 6 h 10 min after 2 am (i.e. at 8:10 am). The average depth will be halfway between the tides, so the first average depth will occur 3 h 5 min after 2 am (i.e. 5:05 am). The depth will be (8.4 + 2.8) ÷ 2 = 5.6 m. You can continue adding 3 h 5 min to the last times to get the following times and depths. Draw a sketch using the points found and knowledge of the shape of sine and cosine.	<table><tr><th>Time</th><th>d (m)</th></tr><tr><td>2:00 am</td><td>8.4</td></tr><tr><td>5:05 am</td><td>5.6</td></tr><tr><td>8:10 am</td><td>2.8</td></tr><tr><td>11:15 am</td><td>5.6</td></tr><tr><td>2:20 pm</td><td>8.4</td></tr><tr><td>5:25 pm</td><td>5.6</td></tr><tr><td>8:30 pm</td><td>2.8</td></tr><tr><td>11:35 pm</td><td>5.6</td></tr></table> 
b Since the graph starts near a maximum, it is more convenient to use cosine than sine.	
1 Write the cosine function.	$y = a\cos(b(x-h)) + k$
2 Find the amplitude.	$\text{amplitude} = \frac{8.4-2.8}{2}$ $a = 2.8$
3 Use the period to evaluate b.	$\text{period} = 12\text{ h }20\text{ min}$ $= 12\frac{1}{3}\text{ h}$ $= \frac{37}{3}\text{ h}$ $\frac{2\pi}{b} = \frac{37}{3}$ $b = \frac{6\pi}{37}$
4 Find the phase shift.	Maximum is 2 hours after $t = 0$. The graph has been shifted to the right, so the phase shift, $h > 0$. $h = 2$
5 Find the mean value.	$k = \frac{8.4 + 2.8}{2} = 5.6$
6 Choose variables.	Let $t = x$ and $d = y$.
7 Substitute into $y = a\cos(b(x-h)) + k$.	$d = 2.8\cos\left[\frac{6\pi}{37}(t-2)\right] + 5.6$ m, where t is in hours after midnight on 3 October.
c At 4 pm on 5 October, $t = 64$ h. Evaluate and round off.	$d = 2.8\cos\left[\frac{6\pi}{37}(64-2)\right] + 5.6$ ≈ 8.36 m

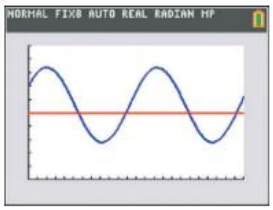
d **TI-84 Plus CE**

Make sure that the angle measure is set to radians. Press y= and enter the functions

$$Y_1 = 2.8\cos\left(\frac{6\pi x}{37} - \frac{12\pi}{37}\right) + 5.6$$

$$Y_2 = 5$$

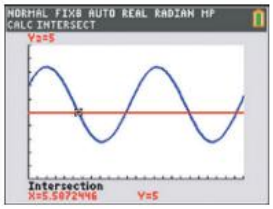
Then press graph to display the graphs of the functions.



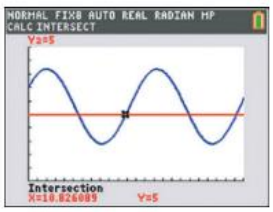
Press window and change the window settings to $0 \le X \le 24$ and $0 \le Y \le 10$

Press 2nd trace calc and select **5: intersect**.

Follow the procedure as outlined previously to locate the first point of intersection of the graphs.



Repeat the procedure to find the other intersections.



$x \approx 5.507, 10.83, 17.84, 23.16$

Both calculators:

Use the fact that t = hours after midnight to convert the x values into times.

$5.507 \approx 5{:}30\text{ am}$

$10.83 \approx 10{:}50\text{ am}$

$17.84 \approx 5{:}50\text{ pm}$

$23.15 \approx 11{:}10\text{ pm}$

Use the graph and the points of intersection to state the result.

The ship could enter from midnight to 5:30 am, from 10:50 am to 5:50 pm, and after 11:10 pm.

Casio fx-CG50 AU

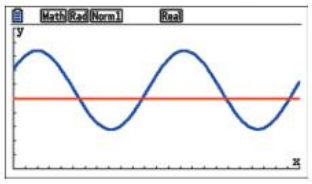
Press MENU and select the **Graph** page.

Enter the functions

$$Y_1 = 2.8\cos\left(\frac{6\pi x}{37} - \frac{12\pi}{37}\right) + 5.6$$

$$Y_2 = 5$$

Then press F6 (DRAW) to display the graph of the function.

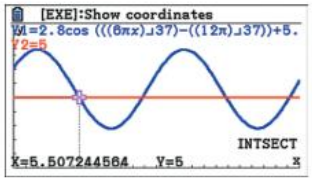


Press SHIFT F3 V-Window and change the window settings to:

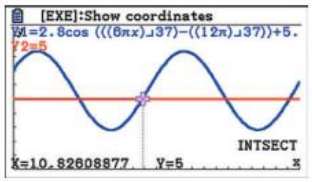
$0 \le X \le 24$ and $0 \le Y \le 10$

Press SHIFT F5 (G-Solv) and select F5 (INTSECT).

The first point of intersection is then displayed.



Press the right arrow button to display the next intersection and repeat until all intersections are found.



$x \approx 5.507, 10.83, 17.84, 23.16$

e Consider the application of the model.

Tides change height with the phases of the Moon, so the model could not be used for other dates.

EXERCISE 4.7 Modelling with trigonometric functions

ANSWERS p. 382

Recap

1 Solve each equation for $0° \leq \theta \leq 360°$.

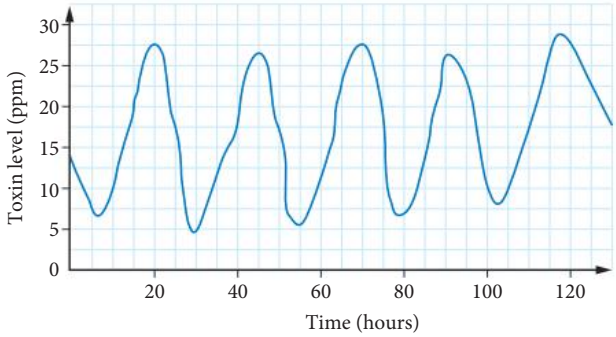
a $\tan(\theta) = -1$ **b** $\sin(\theta) = -\frac{1}{2}$ **c** $\cos(\theta) = \frac{\sqrt{3}}{2}$ **d** $\sin(\theta) = \frac{1}{\sqrt{2}}$

2 Solve each equation over the domain $0 \leq x \leq \pi$, correct to 4 decimal places.

a $2\sin(x) = 1$ **b** $4\cos(x) + 1 = 0$ **c** $5\tan(x) - 2 = 0$

Mastery

3 WORKED EXAMPLE 16 A factory discharges waste into a local stream and this is causing problems for wildlife in the area. The toxin levels in the stream were measured over a period of days. The toxin levels are recorded in parts per million (ppm). The first reading was taken at midnight and the graph of the readings is shown below.



a Does the toxin level appear to be a periodic function over time?

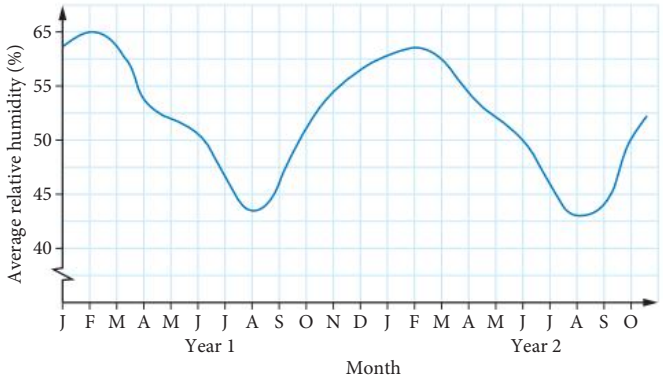
b At about what time does the toxin level seem to peak?

c When does the toxin level appear to be at its lowest?

d What is the period?

e What is the amplitude?

4 This graph shows the average relative humidity (%) for Brisbane over two consecutive years.

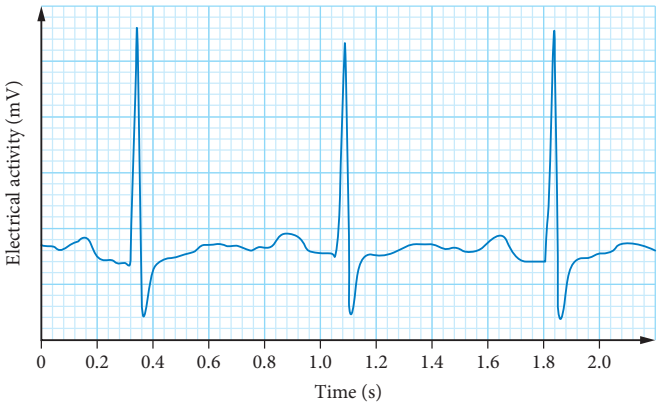


a When is Brisbane's most humid month?

b Calculate the period and the amplitude.

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5 The graph below shows the electrocardiogram (ECG) for an adult male. The peaks show when the ventricles of the heart contract.



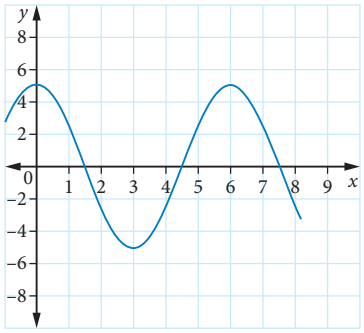
a Calculate the period.

b What does the period represent?

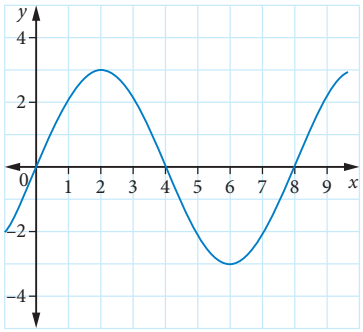
c How long does it take for the heart to beat 10 000 times?

6 WORKED EXAMPLE 17 Find an equation for each periodic function.

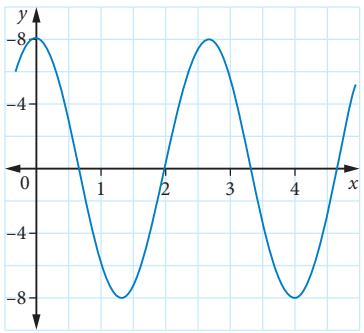
a



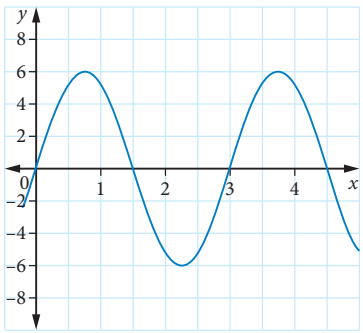
b



c



d

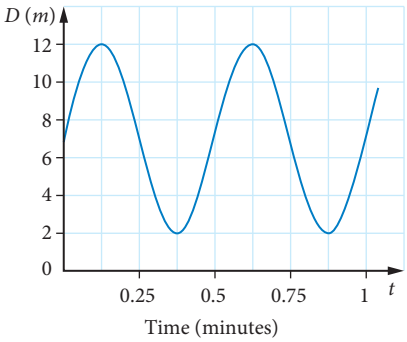


7 Juanita is lying on a tropical beach, enjoying the sound of the waves. She notices that the waves appear to roll up the beach at regular time intervals, and she is able to estimate the distance of the wave front from her toes over time. Juanita draws Cartesian axes in the sand and sketches the distance of the wave front from her toes against time in minutes.

She realises that the distance can be modelled by a sine curve

$$d = a\sin(kt) + d$$

where a, k and d are positive constants.



a State the maximum and minimum distances of the waves from her feet.

b How many waves wash up on the beach each hour?

c Find the values of a, k and d.

Her beach umbrella is stuck into the sand 4.5 m closer to the wave front than her toes and the waves are washing over its base.

d Calculate the percentage of time for which the base of the umbrella is in the water.

e Evaluate the reasonableness of her model.

8 WORKED EXAMPLE 18 The sales, S in 100s of units, of a seasonal product are modelled by

$$S = 54.8 + 32.5\cos\left(\frac{\pi t}{6}\right)$$

where t is the time in months ($t = 1$ is January and $t = 12$ is December).

a Draw a graph of the sales for a period of 12 months.

b Use the graph to determine the months for which sales exceed 6800 units.

c Is the model likely to be useful in the long term?

9 In an unusual meteorological investigation, the temperature, T °C, in a town in central Queensland was found to fluctuate approximately according to the rule

$$T = 25 + 6\sin(0.1\pi t)$$

where t is the number of hours after 10:00 am.

a Sketch a graph of the temperature fluctuations for a sufficient number of hours to be able to determine the maximum and minimum temperatures for that day and the next night.

b Use the graph to determine the maximum and minimum temperatures.

c When did they occur?

d At what time was the temperature

i 27°C? **ii** 20°C?

e Why was this unusual?

10 In Kolkata, India the highest mean monthly temperature is 29.45°C in June and the lowest is 18.3°C in December. Construct a model for the temperature throughout the year and graph it. Evaluate the reasonableness of the model.

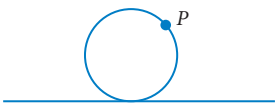
11 In Upernivik, Greenland, the average temperature varies between −23°C in January and 5°C in July. Find and graph a model for the average temperature and evaluate your model.

12 A large hoop is rolling along the ground. The height above the ground of a point P on the rim of the hoop is given by

$$y = 1.25 - 1.25\cos 2\pi t$$

where y is in metres and t is in seconds.

Find the first 3 times at which P is 2.00 m above the ground.
How good is this model?



13 The time between successive high tides at a pier is 12 h 20 min. The average depth of water is 4 m, but at low tide it is 3.1 m. Write an equation to model this relationship and use it to find the lengths of time that boats with a draught of 3.5 m and 4.5 m can use the pier.

Consider the reasonableness of the model.

14 An oscilloscope tracing signals from a microphone shows a sinusoidal curve. The amplitude relates to the volume of the sound while the frequency gives the pitch.

A violin string vibrates at 200 Hz (1 Hz is 1 vibration per second). The oscilloscope trace amplitude is 12 V.

a Find the period of the sound wave.

b Write an equation for the sound wave.

c Evaluate the reasonableness of the model.

Exam practice

Use this data to answer Questions 15 to 18.

The table shows the average monthly temperature for New York (USA) over a 12-month period.

x	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
T (°C)	0.5	2	6.5	12.5	17	22.5	24.5	24.5	20.5	14	9.5	3.5

15 What is the value of the amplitude of the equation used to model this data?

A 12 **B** 12.5 **C** $\frac{\pi}{6}$ **D** 25

16 By how many units is the curve vertically translated in the model for this data?

A 12 **B** 12.5 **C** $\frac{\pi}{6}$ **D** 25

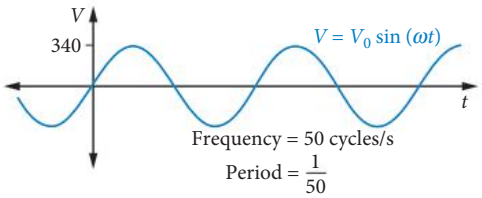
17 What is the equation for the model for this data?

A $f(t) = 12 - 12.5\cos\left(\frac{\pi}{6}x\right)$ **B** $f(t) = 12 + 12.5\cos\left(\frac{\pi}{6}x\right)$

C $f(t) = 12.5 - 12\cos\left(\frac{\pi}{6}x\right)$ **D** $f(t) = 12.5 + 12\cos\left(\frac{\pi}{6}x\right)$

18 (3 marks) Plot the data onto a graph and sketch the line of the equation that models it.

19 (6 marks) CF The ordinary power supply is a sinusoidal wave of alternating voltage, as shown.



The power of a stove element varies with time. This is also true for incandescent bulbs, which flicker accordingly. The instantaneous power in a stove element is given by the equation

$$\text{power} = \frac{V_0^2}{R}\sin^2(\omega t)$$

where $\omega = 100\pi$, voltage $V_0 = 340$ V (for a 240 V supply), t is measured in seconds, resistance R is in ohms (Ω) and power is in watts (W).

a Find the power for a 48 Ω stove element when $t = 0.001$. *[2 marks]*

b Find the maximum power. *[2 marks]*

c Find the first time maximum power occurs. *[2 marks]*

20 (5 marks) CU In the shotput event, when an athlete puts a shot, the horizontal distance travelled (d m) and height (h m) of the shot after t s are given by

$$d = [v\cos(\alpha)]t$$

$$h = -\frac{1}{2}gt^2 + [v\sin(\alpha)]t + r$$

where v = speed at which shot is released, r = height at which shot is released, g = acceleration due to gravity and α = angle at which shot is released.

a The Queensland state shotput title was claimed with a winning put of 18.45 m. Assuming $\alpha = 44°$, $g = 9.8\text{ m/s}^2$ and $r = 1.8$ m, at what speed did the shotput leave the winning competitor's hand? *[4 marks]*

b Other things being equal, what advantage does a tall shotput competitor have? *[1 mark]*

Video
Exam question analysis: Trigonometric functions

EXAM QUESTION ANALYSIS

Exam-style question (7 marks)

Solve the equation $4\sin(x)\cos(x) - 2\sqrt{2}\cos(x) + 2\sqrt{3}\sin(x) - \sqrt{6} = 0$ for $0 \le x < 2\pi$.

Reading the question

- It is a trigonometric equation.
- Both $\sin(x)$ and $\cos(x)$ are involved.
- There are square roots of 2, 3 and 6.
- There are 4 terms on the LHS.

Thinking about the question

- You need to separate the $\sin(x)$ and $\cos(x)$ parts.
- 4 terms suggests a grouping factorisation.
- The square root of 6 can be written as $\sqrt{2}\sqrt{3}$.

Worked solution (✓ = 1 mark)

$4\sin(x)\cos(x) - 2\sqrt{2}\cos(x) + 2\sqrt{3}\sin(x) - \sqrt{6} = 0$

$4\sin(x)\cos(x) - 2\sqrt{2}\cos(x) + 2\sqrt{3}\sin(x) - \sqrt{2}\sqrt{3} = 0$ ✓

$2\cos(x)(2\sin(x) - \sqrt{2}) + \sqrt{3}(2\sin(x) - \sqrt{2}) = 0$

$(2\sin(x) - \sqrt{2})(2\cos(x) + \sqrt{3}) = 0$ ✓

From the null factor law: $2\sin(x) - \sqrt{2} = 0$ or $2\cos(x) + \sqrt{3} = 0$ ✓

$\sin(x) = \frac{\sqrt{2}}{2}$ or $\cos(x) = -\frac{\sqrt{3}}{2}$ ✓

$\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ and $\sin(x)$ is positive in the 1st and 2nd quadrants,

so $x = \frac{\pi}{4}$ or $\pi - \frac{\pi}{4} = \frac{3\pi}{4}$. ✓

$\cos\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$ and $\cos(x)$ is negative in the 2nd and 3rd quadrants,

so $x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$ or $\pi + \frac{\pi}{6} = \frac{7\pi}{6}$. ✓

The solutions are $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{6}$ or $\frac{7\pi}{6}$. ✓

4 Chapter summary

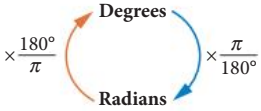
Radian facts

$\pi = 180°$

$\frac{\pi}{6} = 30°$ $\quad \frac{\pi}{4} = 45°$ $\quad \frac{\pi}{3} = 60°$ $\quad \frac{\pi}{2} = 90°$

Converting between degrees and radians

- If the expression contains π, use the relation $\pi = 180°$.



Exact trigonometric ratios

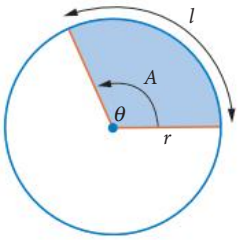
Angle in degrees	30°	45°	60°
Angle in radians	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
sin	$\frac{1}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
cos	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{1}{2}$
tan	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

Arc length and area of a sector

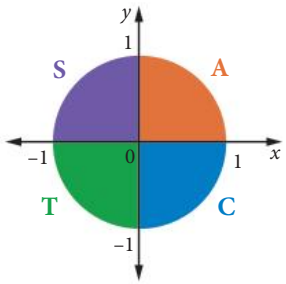
- Given that θ is in radians:

arc length $= l = r\theta$

area of sector $= A = \frac{1}{2}r^2\theta$



Signs of the trigonometric ratios

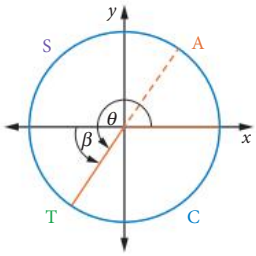


ASTC: 'All Stations To Central'
'All–Sine–Tangent–Cosine.'
1st quadrant: All ratios are positive (A).
2nd quadrant: Sine only is positive (S).
3rd quadrant: Tangent only is positive (T).
4th quadrant: Cosine only is positive (C).

Finding trigonometric ratios for an angle of any magnitude

2nd quadrant	$\sin(180° - \theta) = \sin(\theta)$
	$\cos(180° - \theta) = -\cos(\theta)$
	$\tan(180° - \theta) = -\tan(\theta)$
3rd quadrant	$\sin(180° + \theta) = -\sin(\theta)$
	$\cos(180° + \theta) = -\cos(\theta)$
	$\tan(180° + \theta) = \tan(\theta)$
4th quadrant	$\sin(360° - \theta) = -\sin(\theta)$
	$\cos(360° - \theta) = \cos(\theta)$
	$\tan(360° - \theta) = -\tan(\theta)$

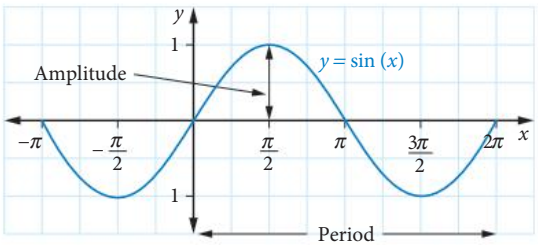
The reference angle is always measured from 180° or 360° (π or 2π).



1. Draw the angle from the x-axis.
2. Calculate the acute reference angle, β, that corresponds to θ.
3. Use ASTC to determine the sign (+ or –) of the value of the trigonometric ratio.
4. Find the value of the required trigonometric ratio for β and include the sign.

Trigonometric functions

- The graph of a **trigonometric function** repeats itself. The **period** is the length of one **cycle** of the function.
- The **amplitude** (a) of a periodic function is the maximum height of the graph from its mean value.



$y = \sin(x)$

Domain: all real numbers

Range: $-1 \leq y \leq 1$

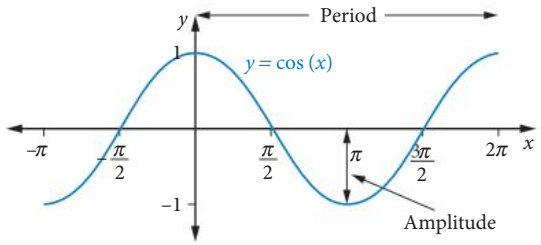
Period $= 2\pi$

Amplitude $= 1$

Zeros occur at … $-\pi$, 0, π, 2π …

Maxima occur at … $-\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}$ …

Minima occur at … $-\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}$ …



$y = \cos(x)$

Domain: all real numbers

Range: $-1 \leq y \leq 1$

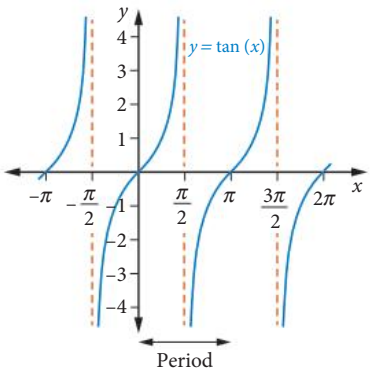
Period = 2π

Amplitude = 1

Zeros occur at … $-\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$ …

Maxima occur at … -2π, 0, 2π …

Minima occur at … $-\pi$, π, 3π …



$y = \tan(x)$

Domain: all real numbers except odd multiples of $\frac{\pi}{2}$

Range: all real numbers

Period = π

Zeros occur at … $-\pi$, 0, π …

Asymptotes at … $x = \pm\frac{\pi}{2}, x = \pm 3\frac{\pi}{2}$ …

Transformations of trigonometric functions

Graphs of $y = a\sin(b(x - h)) + k$, $y = a\sin(b(x - h)) + k$

- Amplitude = magnitude of a
- If a is negative, the graph is flipped upside down (reflected in the line $y = k$)
- Period = $\frac{2\pi}{b}$
- Horizontal translation of h units: the **phase shift**
- Vertical translation of k units
- Mean value = k

Solving trigonometric equations

To solve equations of the form $\sin(x) = a$, $\cos(x) = a$ or $\tan(x) = a$:

1. use the sign of a to determine the quadrants
2. find the reference angle using either exact values or inverse functions
3. sketch the solutions on the unit circle or using a graph
4. examine the given domain to determine the required values.

- Periodic phenomena can be modelled by the **sinusoidal** functions $y = \sin(x)$ and $y = \cos(x)$.

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Cumulative examination 1

Technology-free

Perusal time: 2 minutes Working time: 30 minutes

TECH-FREE *Calculators are **not** permitted.*

Section 1 (3 marks): 3 multiple choice questions

Section 2 (15 marks): 4 short response questions

Total: 18 marks

Section 1 3 multiple choice questions

3 marks

1 The area of a sector of a circle if the arc length is 4 cm and the radius is 3 cm is closest to

A $3\,\text{cm}^2$ **B** $6\,\text{cm}^2$ **C** $12\,\text{cm}^2$ **D** $18.85\,\text{cm}^2$

2 $\begin{pmatrix} 16 \\ 12 \end{pmatrix} =$

A 1520 **B** 1660 **C** 1820 **D** 2018

3 Factorise $12x^2 - 11x - 15$.

A $(2x + 3)(6x - 5)$ **B** $(8x + 5)(1.5x - 3)$ **C** $(4x + 3)(3x - 5)$ **D** $(6x + 1)(2x - 15)$

Section 2 4 short response questions

15 marks

4 (3 marks) Find the amplitude, period and range of $y = 3\cos(2(x - \pi)) + 1$.

5 (4 marks) Find the solutions of $5x^2 - 6x - 4 = 0$, simplifying answers if possible.

6 (3 marks) Sketch the graph of $f(x) = (x + 2)^3$.

7 (5 marks) Sketch the graph of $y = 6\cos\left(3x - \frac{\pi}{4}\right) + 4$, showing one full cycle.

Worksheet
Maths Methods Year 11 formula sheet

Worksheet
Maths Methods Year 11 formula sheet

Cumulative examination 2

Technology-active

Perusal time: 2 minutes Working time: 30 minutes

*Calculators are **permitted**.*

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

Section 1 (3 marks): 3 multiple choice questions

Section 2 (15 marks): 4 short response questions

Total: 18 marks

Section 1 3 multiple choice questions

3 marks

1 Express 0.43 rad in degrees, correct to one decimal place.

A 11.9° **B** 22.5° **C** 24.6° **D** 79.1°

2 $(\sqrt{3} - \sqrt{7})^3 =$

A $27 - 2\sqrt{21}$ **B** $\sqrt{3} - 18\sqrt{7}$ **C** $3\sqrt{3} - 3\sqrt{7}$ **D** $24\sqrt{3} - 16\sqrt{7}$

3 Find the exact value of $\tan\left(\frac{\pi}{3}\right)$.

A $\frac{\sqrt{3}}{2}$ **B** $\sqrt{3}$ **C** $\sqrt{2}$ **D** $\frac{\sqrt{3}}{3}$

Section 2 4 short response questions

15 marks

4 (4 marks) The length of a closed box is 1.625 times its width, and the box is 8 cm high. If the total surface area is 1095.25 cm^2, find the dimensions of the box.

5 (4 marks) The table shows the monthly average overnight minimum temperature for Melbourne.

m	0	1	2	3	4	5	6	7	8	9	10	11
T	15	14.5	12.7	10.6	8.2	6.5	6	6.7	8.3	10.4	12.9	14.3

where T is the temperature in degrees Celsius and m is the number of months since January.

a Draw a graph to represent this data. *[3 marks]*

b What is the amplitude? *[1 mark]*

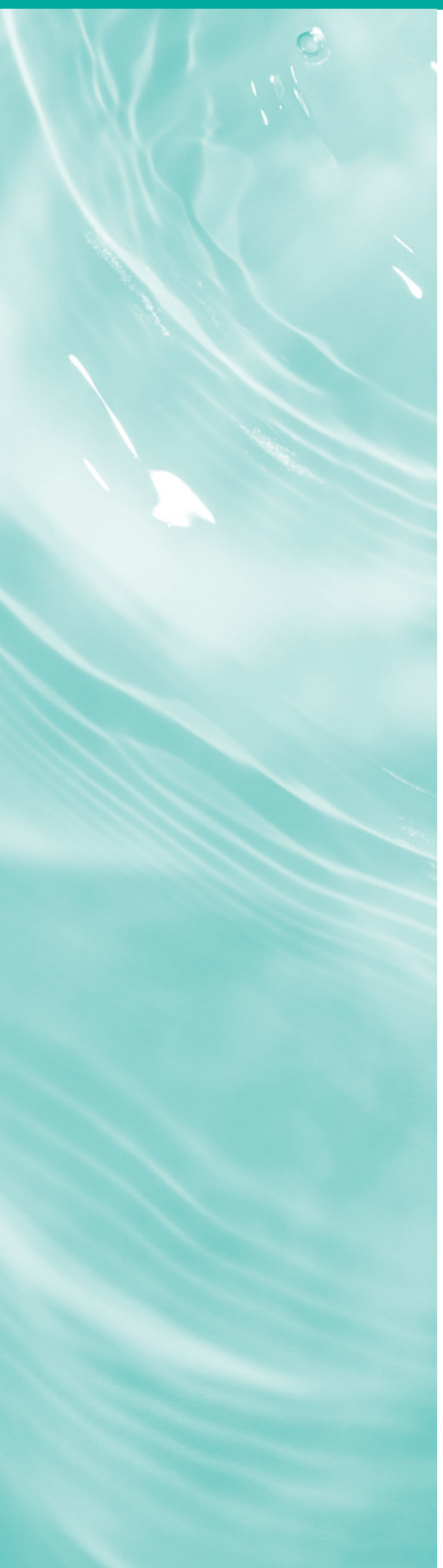
6 (3 marks) Solve $2x + 4 = 4x^3 + 3x + 2$, correct to 2 decimal places.

7 (4 marks) A clock stands on a shelf. The clock's centre, where the hands are attached, is 15 cm above the shelf. The minute hand is 10 cm long. Find the first two times after 5:30 am when tip is 10 cm above the shelf.

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CHAPTER

5 PROBABILITY



Syllabus coverage
Nelson MindTap chapter resources
Terminology
5.1 Set notation and Venn diagrams
Sets and probability
5.2 Probability rules
5.3 Conditional probability
5.4 Independent events
5.5 Relative frequency
Exam question analysis
Chapter summary
Cumulative examination 1
Cumulative examination 2



Prior learning
Probability

Syllabus coverage

UNIT 1, TOPIC 5: PROBABILITY

Language of events and sets

- Use the concepts and language of outcomes, sample spaces and events as sets of outcomes.
- Use set language and notation for events, including $\bar{A}$ or A' for the complement of an event A, $A \cap B$ for the intersection of events A and B, and $A \cup B$ for the union, and recognise mutually exclusive events.
- Use everyday occurrences to illustrate set descriptions and representations of events, and set operations, including the use of Venn diagrams.

Conditional probability and independence

- Use the rules $P(\bar{A}) = 1 - P(A)$ and $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
- Understand the notion of a conditional probability and recognise and use language that indicates conditionality.
- Use the notation $P(A|B)$ and the formula $P(A \cap B) = P(A|B)P(B)$ to solve problems.
- Understand and use the notion of independence of an event A from an event B, as defined by $P(A|B) = P(A)$.
- Use the formula $P(A \cap B) = P(A)P(B)$ for independent events A and B.
- Use relative frequencies obtained from data as point estimates of conditional probabilities and as indications of possible independence of events.
- Model and solve problems that involve probability, with and without technology.

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Videos (5):
5.1 Venn diagrams 1
5.2 Venn diagrams 2
5.3 Conditional probability
5.5 Experimental probability
Exam question analysis Probability

Prior learning (1):
5.1 Probability

Worksheets (10):
5.1 Set operations • Venn diagrams 1 • Venn diagrams 2
5.2 Dartboard probabilities • Theoretical probability
5.3 Conditional probability
5.4 Multi-stage problems
5.5 Basic probability assignment • Basic probability extension
Cumulative exams Maths Methods Year 11 formula sheet

Puzzles (5):
5.1 Venn diagrams matching activity
5.2 Matching probabilities
5.3 Conditional probability: Two-way tables
Chapter summary Probability find-a-word • Probability crossword

Nelson MindTap

To access resources above, visit
cengage.com.au/nelsonmindtap

Terminology

and	cardinality	complement	conditional probability
element	empty set	event	expected frequency
independent events	intersection	mutually exclusive	not
or	outcome	probability	random
relative frequency	sample space	set	subset
trials	union	universal set	Venn diagram

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5.1 Set notation and Venn diagrams

A **set** is a well-defined collection of objects. Each object is an **element** of the set. 'Well-defined' means you can tell whether an object belongs to the set or not.

A set is shown by a capital letter. You define it by a list or rule inside braces {}.

Set notation

Each element of a set is listed only once.
The number of elements in the set A is called its **cardinal** number, shown as $|A|$ or $n(A)$.
The symbol $\in$ means 'is a member of', so $2 \in A$ means 2 is in the set A.
The **universal set**, U, is the set that contains all the possible elements of sets.
The **null set** or **empty set** {} or $\emptyset$ has no elements.

The set of natural numbers {1, 2, 3, 4 …} is N; the set of integers Z, of rational numbers Q, and real numbers R.



Video Venn diagrams 1

Worksheets Set operations

Venn diagrams 1

Venn diagrams 2

Puzzle Venn diagrams matching activity

WORKED EXAMPLE 1 Set notation

a List the elements of these sets:

$A = \{x: x \text{ is a prime number less than } 15\}$

$B = \{y \in A: 3y = 12\}$

$C = \{n: n \text{ is a positive even number}\}$

$D = \{\text{days of the week}\}$

b State the cardinality of A, B, C and D.

Steps	Working
a 1 Prime numbers have exactly 2 factors.	$A = \{2, 3, 5, 7, 11, 13\}$
2 $y = \frac{12}{3} = 4$, but $4 \notin A$.	$B = \{\}$
3 Use … to show that a pattern continues.	$C = \{2, 4, 6, 8, 10, 12 \ldots\}$
4 Give the names.	$D = \{\text{Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday}\}$
b 1 There are 6 elements in A.	$\lvert A\rvert = 6$
2 B is the null set.	$\lvert B\rvert = 0$
3 Write a pairing of even numbers and counting numbers.	$2 \leftrightarrow 1, 4 \leftrightarrow 2, 6 \leftrightarrow 3, 8 \leftrightarrow 4, 10 \leftrightarrow 5 \ldots$
4 Write the cardinality of C.	C is an infinite set, so $\lvert C\rvert$ is ∞.
5 Count the days of the week.	$\lvert D\rvert = 7$

You can compare and combine the elements of different sets.

For $A = \{4, 5, 6\}$, $B = \{1, 3, 5\}$, $C = \{5, 3, 1\}$, $D = \{1, 2, 3, 5\}$, $E = \{3, 7, 9\}$, $U = \{1, 2 \ldots 10\}$

$B = C$	equal	B and C have the same elements.	$B = C$
$B \subseteq D$	**subset**	A set that is part or all of another set. Every element in B is in D.	$B \subseteq C$ and $B \subseteq D$
$B \subset D$	**proper subset**	A set that is part (and not all) of another set. B is a subset smaller than D.	
$A \cup B$	**union**	Elements belonging to sets A **or** B.	$A \cup B = \{1, 3, 4, 5, 6\}$
$A \cap B$	**intersection**	Elements belonging to sets A **and** B.	$A \cap B = \{5\}$
	disjoint	The sets have no common elements.	$A \cap E = \emptyset$
$\overline{A}$ or A'	**complement**	The elements not in A.	$\overline{A} = \{1, 2, 3, 7, 8, 9, 10\}$

Disjoint sets are also **mutually exclusive**. Cases with an empty set are usually excluded.

WORKED EXAMPLE 2 Combinations of sets

$A = \{2, 4, 6, 8\}$, $B = \{n \in Z: 0 \leq n < 5\}$, $C = \{5, 6, 7, 8\}$, $D = \{3, 6, 9\}$ and $E = \{0, 2, 4\}$ are all sets of digits.

Determine

a $\overline{A}$ **b** $A \cap B$ **c** $C \cup D$

d $|(B \cup C) \cap D|$ **e** all the subsets of D

f the proper subsets of E **g** the mutually exclusive sets.

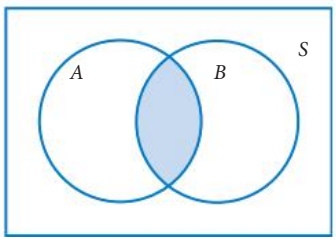
Steps	Working
a Write all the digits not in A.	$\overline{A} = \{0, 1, 3, 5, 7, 9\}$
b Write the elements in both A and B.	$A \cap B = \{2, 4\}$
c Write the elements in C or D (or both).	$C \cup D = \{3, 5, 6, 7, 8, 9\}$
d **1** The inner bracket must be done first.	$B \cup C = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$
2 Now do the intersection.	$(B \cup C) \cap D = \{3, 6\}$
3 Count the elements.	$\|(B \cup C) \cap D\| = 2$
e Do all the subsets systematically so you don't miss any.	The subsets are {3, 6, 9}, {3, 6}, {3, 9}, {6, 9}, {3}, [6], {9}, {}.
f Don't include the set itself.	The proper subsets are {0, 2}, {2, 4}, {0, 4}, {0}, {2}, {4}, {}.
g Mutually exclusive sets have no common elements.	$B \cap C = \{\}$, $C \cap E = \{\}$ and $D \cap E = \{\}$, so there are 3 mutually exclusive pairs of sets: B, C; C, E; D, E.

Sets and probability

In a probability situation:

- the set of all **outcomes** (the universal set) is called the **sample space** and is written as S
- an **event** is a subset of the sample space
- A **and** B means $A \cap B$: both A and B occur
- A **or** B means $A \cup B$: at least one of A or B occurs
- **not** A means $\overline{A}$ or A'.

In probability, $n(A)$ is often used for $|A|$, the number of elements in A.
When solving probability problems, you can use a **Venn diagram** to show events in a sample space. The sample space is shown by a rectangle and events are shown as circles or ovals. The intersection of two sets is shown by the overlapping part of their circles.



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WORKED EXAMPLE 3 Sample spaces

The 3-handed game of 1000 uses only the 9, 10, Jack, Queen, King and Ace of each suit (Hearts, Diamonds, Clubs and Spades) from a pack of playing cards. The order of cards in each suit is 9, J, Q, K, 10, A, with the Ace having the highest value. Each player is dealt 7 cards, with the remaining 3 going into the kitty.
You must have the King and Queen of a suit and lead one to declare trumps.

a **Use** the symbols 9, J, Q, K, 10, A and H, D, C and S to write the sample space.

b Write the event A = 'cards to declare trumps'.

c Write the event B = 'ace'.

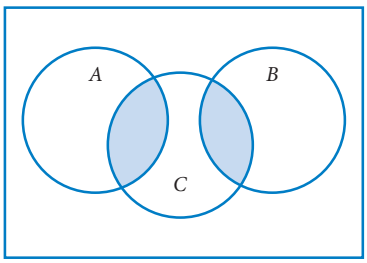
d Write the event C = 'hearts'.

e Write $n(S)$, $n(A)$ and $n(C)$.

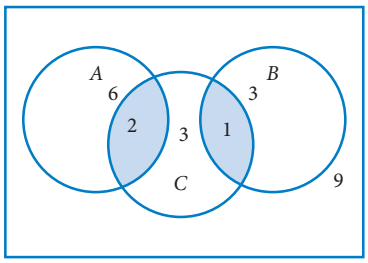
f Find $n(A \text{ and } C)$.

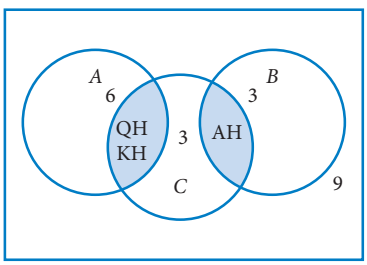
g Find $n(A \text{ or } B)$.

h **Construct** a Venn diagram showing A, B and C.

Steps	Working
a Write all possible outcomes.	S = {9H, 10H, JH, QH, KH, AH, 9D, 10D, JD, QD, KD, AD, 9C, 10C, JC, QC, KC, AC, 9S, 10S, JS, QS, KS, AS}
b Write all the kings and queens.	A = {QH, KH, QD, KD, QC, KC, QS, KS}
c Write the aces.	B = {AH, AD, AC, AS}
d Write the hearts.	C = {9H, 10H, JH, QH, KH, AH}
e Count the numbers of elements.	$n(S) = 24$, $n(A) = 8$ and $n(C) = 6$
f **1** $(A \text{ and } C) = A \cap C$	$A \cap C$ = {QH, KH}
2 Count the set.	$n(A \text{ and } C) = 2$
g **1** $(A \text{ or } B) = A \cup B$	$A \cup B$ = {QH, KH, AH, QD, KD, AD, QC, KC, AC, QS, KS, AS}
2 Count the set.	$n(A \text{ or } B) = 12$
h A and B are mutually exclusive, so show them with no intersection. C intersects with both A and B.	

Showing some of the outcomes or the numbers of elements in a Venn diagram may help you solve a problem. You could redraw the diagram in Worked example 3, as shown below.





EXERCISE 5.1 Set notation and Venn diagrams

ANSWERS p. 384

Mastery

1 WORKED EXAMPLE 1 List the elements of each set.

a $A = \{\text{even numbers between 13 and 33}\}$

b $B = \{n \in N: 21 < 2n - 1 < 51\}$

c $C = \{x = 2n + 1 : n \in N, 4 \leq n \leq 12\}$

d $D = \{\text{subsets of } \{1, 3, 5, 7\}\}$

e $F = \{\text{3-letter abbreviations of months of the year}\}$

2 State the cardinality of each set in question **1**.

3 How many elements does each set have?

a $A = \{\text{odd numbers between 2 and 20}\}$

b $B = \{x = 2n : n \in N : 8 \leq n \leq 16\}$

c $C = \{\text{result of tossing 2 coins}\}$

d $D = \{\text{possible numbers formed from the digits 7, 8, 9; no repeated digits}\}$

e $F = \{\text{numbers that are both even and odd}\}$

4 WORKED EXAMPLE 2 $A = \{a, b, c\}$, $B = \{b, d, e\}$, $C = \{m, p, r\}$, $D = \{p, r, t\}$ and $F = \{\text{non-vowels}\}$. Find

a U **b** $A \cap B$ **c** $C \cup D$ **d** $\overline{F}$

e $A \cup C$ **f** $A \cap C$ **g** $n(F)$ **h** any disjoint sets

i the subsets of D **j** the proper subsets of C.

5 $M = \{3, 7, 9\}$, $P = \{2, 4, 6, 8\}$, $Q = \{1, 2, 3, 4, 5\}$, $T = \{5, 6, 7, 8\}$ and $V = \{0, 3, 6, 9\}$. Find

a U **b** $M \cup P$ **c** $\overline{Q \cup T}$ **d** $\overline{Q \cap T}$

e $\overline{Q} \cup \overline{T}$ **f** $\overline{Q} \cap \overline{T}$ **g** $\overline{\overline{V}}$ **h** any disjoint sets

i the subsets of P **j** the proper subsets of T.

6 WORKED EXAMPLE 3 Two coins are tossed. Use H (heads) and T (tails) for each of the following.

a List the sample space.

b Write the event A 'the first coin is heads'.

c Write the event B 'the second coin is heads'.

d Write the event 'A and B'.

e Write the event 'A or B'.

f Write the event C 'at least one coin is tails'.

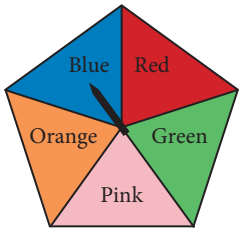
g Write the event D 'exactly one coin is tails'.

h Write the event E 'two tails'.

i Write the event F 'no tails'.

j Write the event 'not C'.

7 A spinner made in the shape of a regular pentagon is spun.
The equal-sized sectors are coloured blue, red, green, pink and orange.



a Write the sample space.

b Write the event 'a primary colour'.

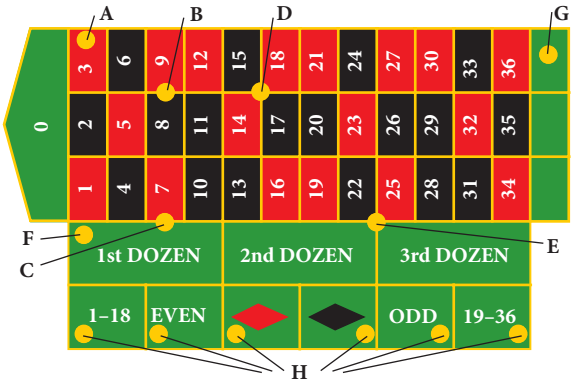
c Write the event 'not a primary colour'.

d Write the event 'black or white'.

e Write the event 'not black or white'.

f How many elements are there in each of events **b**–**e**?

8 A roulette wheel in a casino has numbers 0 to 36.
Bets are placed on a table marked as shown.
Eight different types of bets are possible. They are indicated in the diagram by the letters A to H, which show the positions at which chips would be placed to indicate the bets.



The bets are

A a single number from 0 to 36 —'straight up'

B 2 numbers next to each other horizontally or vertically (except 0) —'split' (either of the numbers the chip touches wins)

C a row of 3 numbers — 'street' (any of the 3 numbers in the row wins)

D a rectangle of 4 numbers — 'corner' (any of the 4 numbers the chip touches wins)

E 2 rows of 6 numbers — 'six line' (any of the numbers in the 2 rows wins)

F first, second or third dozen — 'dozen' (12 numbers, 4 rows)

G left, middle or right column — 'column' (12 numbers)

H odd/even, red/black, first/second half — 'even chances' (18 numbers)

For each type of bet, find the number of possible bets.

Exam practice

Use the following sets for Questions 9 to 11.

$A = \{3, 6, 9, 12, 15\}$, $B = \{2, 4, 6, 8, 10, 12, 14\}$, $C = \{7, 14\}$

9 Which of the following is true?

A $A \cap B = \{\}$

B $A \cup B = \{n \in N: 2 \leq n \leq 15\}$

C $A = \overline{B}$

D $A \cap C = \{\}$

10 (1 mark) Find $A \cup B$.

11 (1 mark) Find $A \cap B$.

12 (1 mark) List the sample space for Australian capital cities.

13 (3 marks) What is the sample space if you pick 2 marbles from a bag containing 2 red marbles and 3 green marbles? The order of the marbles is not noted. Remember to think of the marbles as being distinguishable.

14 (6 marks) One of each of white, black, pink and yellow snooker balls are placed with 2 red balls in a box. Two balls are taken out (in order) at random without replacement.

- **a** Use the symbols W, B, R, P and Y to write the sample space. *[1 mark]*
- **b** Write the event 'one of the balls is black'. *[1 mark]*
- **c** Write the event 'one of the balls is red'. *[1 mark]*
- **d** Write the event 'a black or a yellow ball is taken'. *[1 mark]*
- **e** Write the event 'a black and a yellow ball are drawn'. *[1 mark]*
- **f** Write the event 'both balls drawn are black'. *[1 mark]*

Video
Venn diagrams 2

Worksheets
Dartboard probabilities

Theoretical probability

Puzzle
Matching probabilities

5.2 Probability rules

Probability

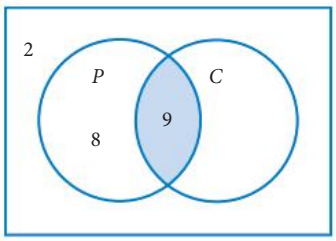
The **probability** of an event E in a **finite** sample space S with equally likely outcomes is

$$P(E) = \frac{n(E)}{n(S)}$$

Equally likely outcomes are described as being **random**.

WORKED EXAMPLE 4 Simple probability

From a class of 24 Maths Methods students, 9 study both Physics and Chemistry and 17 take Physics. Two students do not take either Physics or Chemistry. **Calculate** the probability that a randomly selected student takes Chemistry but not Physics.

Steps	Working
1 Identify the sample space.	$S = \{\text{students in the class}\}$
2 Choose names for events.	Let $P = \{\text{students doing Physics}\}$ and $C = \{\text{students doing Chemistry}\}$.
3 Draw a Venn diagram. Put in the known numbers. $n(\overline{P \cup C}) = 2$ $n(P \cap C) = 9$ and $n(P) = 17$, so $n(P \cap \overline{C}) = 8$.	
4 Find the number studying only Chemistry.	$n(C \cap \overline{P}) = 24 - (2 + 8 + 9)$ $= 5$
5 Calculate the required probability.	$P(C \cap \overline{P}) = \frac{n(C \cap \overline{P})}{n(S)}$ $= \frac{5}{24}$
6 Write the answer.	The probability that a student takes Chemistry but not Physics is $\frac{5}{24}$.

The empty set has no outcomes, so it cannot occur. It is **impossible** and has a probability of 0.
The sample space has all outcomes, so it must occur. It is **certain** and has a probability of 1.

Complementary events

The complement of A, 'not A', has a probability given by $P(\bar{A}) = 1 - P(A)$.

WORKED EXAMPLE 5 Complementary events

A box contains 3 red, 4 green and 5 yellow marbles. A marble is chosen at random.
What is the probability that it is not green?

Steps	Working
1 Write the probability of getting a green marble.	$P(\text{green}) = \frac{n(\text{green})}{n(S)} = \frac{4}{12} = \frac{1}{3}$
2 Use the rule for complementary events.	$P(\text{not green}) = 1 - P(\text{green}) = 1 - \frac{1}{3} = \frac{2}{3}$

The sum of the probabilities of all the outcomes in a sample space must be 1.

WORKED EXAMPLE 6 Combined probabilities

A die is weighted so that the probability of rolling a 6 is increased to $\frac{5}{16}$ and the probability of rolling a 1 decreases to $\frac{1}{16}$. The probabilities of rolling 2 to 5 are all the same. **Calculate** the probability of rolling:

a 3 **b** a number greater than 3 **c** a number less than 3

Steps	Working
a 1 Write the probability of {2, 3, 4 or 5}.	$P(\{2, 3, 4 \text{ or } 5\}) = 1 - P(\{1 \text{ or } 6\})$
2 Substitute the known values.	$= 1 - \left(\frac{1}{16} + \frac{5}{16}\right)$
3 Simplify.	$= \frac{5}{8}$
4 The probabilities of 2 to 5 are the same.	$P(3) = \frac{5}{8} \div 4$
5 Simplify.	$= \frac{5}{32}$
6 Write the answer.	The probability of a 3 is $\frac{5}{32}$.
b 1 Add the probabilities of each outcome.	$P(\{4, 5, 6\}) = \frac{5}{32} + \frac{5}{32} + \frac{5}{16}$
2 Simplify.	$= \frac{5}{8}$
3 Write the answer.	The probability of a number greater than 3 is $\frac{5}{8}$.
c 1 Add the probabilities of each outcome.	$P(\{1, 2\}) = \frac{1}{16} + \frac{5}{32}$ $= \frac{7}{32}$
2 Write the answer.	The probability of a number less than 3 is $\frac{7}{32}$.

EXERCISE 5.2 Probability rules

ANSWERS p. 384

Recap

1 **a** List the elements of the sets $A = \{3n + 1: n \in Z, 2 \leq n \leq 6\}$ and $B = \{\text{prime numbers between 20 and 40}\}$.

b What are the cardinalities of A and B?

2 $P = \{2, 3, 7, 11\}$, $Q = \{7, 11, 13, 17\}$ and $R = \{3, 5, 7, 13, 17\}$ are sets of prime numbers less than 20. Find:

a $\overline{P}$ **b** $Q \cap R$ **c** proper subsets of Q **d** $Q \cup R$

Draw a Venn diagram of the 3 sets.

Mastery

3 WORKED EXAMPLE 4 A rugby squad of 24 students has 5 students with a Pacific Island background. Three of those with Islander backgrounds are 17 years old. There are ten 17-year-olds in the team altogether. What is the probability that a randomly selected student from the squad is

a an Islander who is not 17?

b a non-Islander who is 17?

c neither an Islander nor 17?

4 WORKED EXAMPLE 5 A bag contains 5 red marbles, 8 green marbles and 7 black marbles. A marble is taken at random from the bag.

a What is the sample space?

b What is the probability of taking a black marble?

c What is the probability of taking a white marble?

d What is the probability of taking a green marble?

e Which is more likely to be taken: a green marble or a red marble?

5 A bag in the sports store contains 5 white hockey balls, 6 red cricket balls and 4 white cricket balls. A ball is taken at random.

a What is the probability that the ball is red?

b What is the probability that the ball is white?

c Which is more likely to be taken: a red ball or a white ball?

d What is the probability that a hockey ball is selected?

e What is the probability that a cricket ball is chosen?

6 A sack contained 6 venomous snakes and 5 harmless snakes. One snake found a hole in the sack and escaped.

a List the sample space for the escaped snake, using V for venomous and H for harmless. Assume that the snakes are equally able to escape.

b What is the probability that a venomous snake escaped?

c What is the probability that a harmless snake escaped?

d What is the probability that the bag now contains equal numbers of venomous and harmless snakes?

e What is the probability that the bag now contains 10 snakes?

7 WORKED EXAMPLE 6 A die is weighted so that the probability of each number rolled is proportional to the number. For example, a 6 has twice the probability of a 3.

a Find the probability of rolling each number of the sample space (that is, 1, 2, 3, 4, 5 and 6).

b What is the probability of an even number?

c What is the probability of a prime number?

d Find the probability of an odd number.

8 A baby born in Australia is selected at random. Assume that babies are equally likely to be born at any time of the year. Find the probability that the baby was born

a after 30 June **b** on the weekend **c** in July

d between midnight and 4 am **e** on a Monday.

Exam practice

9 A bucket with 5 white and 6 yellow tennis balls is shaken up, and 2 balls are taken out.

What is the probability that both are yellow?

A $\frac{2}{11}$ **B** $\frac{3}{11}$ **C** $\frac{5}{11}$ **D** $\frac{6}{11}$

10 (3 marks) Three horses A, B and C are in a race. A is twice as likely to win as B and B is twice as likely to win as C. Calculate their respective probabilities of winning, assuming that other horses in the race have a combined 79% chance of winning.

11 (6 marks) An Australian navy patrol vessel has 21 personnel, of whom 5 are women. Two of the women have served more than 5 years and 1 of these is an officer. There are 5 officers altogether on the vessel, not including the captain, who is a man. All except 1 of the officers have served more than 5 years and 16 of all the people on board have served more than 5 years.

Find the probability that a person randomly selected from the vessel is

a an officer (including the captain) *[1 mark]*

b a female officer *[1 mark]*

c a man who has served 5 years or less *[1 mark]*

d a woman who has served more than 5 years *[1 mark]*

e a female captain *[1 mark]*

f a man who has served more than 5 years. *[1 mark]*

12 (4 marks) CF A box contains 3 blue, 1 black and 2 red marker pens. Two pens are taken at random. Find the probabilities of 2 blue pens and different coloured pens.

13 (2 marks) CF Find the probability that the difference between the 2 numbers rolled on a pair of normal dice is less than 3.

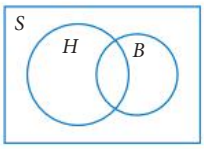
Video
Conditional probability

Worksheet
Conditional probability

Puzzle
Conditional probability: Two-way tables

5.3 Conditional probability

A team of medical researchers were interested in the probability of patients with a BMI over 30 developing heart disease. They drew a Venn diagram. The set of those with heart disease is H. The set of those with a BMI over 30 is B.



The intersection of the sets is $H \cap B$.

The second diagram shows *only* set B, the people with a BMI over 30.

In the restricted sample space B, H_1 is the set of people with heart disease.



In the new sample space B, the probability of having heart disease is $\frac{n(H_1)}{n(B)}$.

This is called the probability of having heart disease *given* a BMI over 30, written as $P(H|B)$.

But $H_1 = H \cap B$, so $P(H|B) = \frac{n(H \cap B)}{n(B)}$.

This can also be written as $P(H|B) = \frac{P(H \cap B)}{P(B)}$ because $P(H \cap B) = \frac{n(H \cap B)}{n(S)}$ and $P(B) = \frac{n(B)}{n(S)}$,

so $\frac{P(H \cap B)}{P(B)} = \frac{n(H \cap B)}{n(S)} \div \frac{n(B)}{n(S)} = \frac{n(H \cap B)}{n(B)}$.

Conditional probability

The **conditional probability** of A given B, is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

This is often written in the form $P(A \cap B) = P(A|B) \times P(B)$.

The definition means that the sample space is restricted to the given event.

WORKED EXAMPLE 7 Conditional probability of consecutive events

Two marbles are drawn from a bag containing 5 yellow and 4 red marbles without replacement.

Calculate the probability they are both red.

Steps	Working
1 Find the probability the first marble is red.	$P(\text{1st red}) = \frac{4}{9}$
2 Find the probability the second marble is *also* red. There are 3 reds left out of 8 marbles.	$P(\text{2nd red} \mid \text{1st red}) = \frac{3}{8}$
3 Use conditional probability (the second formula).	$P(\text{2nd red} \cap \text{1st red})$ $= P(\text{2nd red} \mid \text{1st red}) \times P(\text{1st red})$
4 Substitute values.	$= \frac{3}{8} \times \frac{4}{9}$ $= \frac{1}{6}$
5 Write the answer.	The probability that both are red is $\frac{1}{6}$.

9780170484909

Conditional probability is very important in medical research.

You can't ask 100 people to smoke heavily to find if there's a link between smoking and lung cancer, but you can use statistics to compare the probability of lung cancer for smokers and non-smokers.

WORKED EXAMPLE 8 Conditional probability from statistics

In 2022, 99 924 men died in Australia; 4855 died of lung cancer and of these 4370 were smokers. According to the ABS data, 12.6% of men of the average age of those that died from lung cancer were smokers when they died. Compare the probabilities of dying of lung cancer for smokers and non-smokers.

Steps	Working
1 Choose abbreviations.	Let S = smokers and L = lung cancer death.
2 What probabilities are needed?	$P(L\|S)$ = probability of a smoker dying of lung cancer $P(L\|\bar{S})$ = probability of a non-smoker dying of lung cancer
3 Find the probabilities needed for the calculations.	$P(L \cap S) = \frac{4370}{99\,924}$, $P(L \cap \bar{S}) = \frac{4855 - 4370}{99\,924} = \frac{485}{99\,924}$ $P(S) = 0.126$, $P(\bar{S}) = 0.874$
4 Find the probabilities.	$P(L\|S) = \frac{P(L \cap S)}{P(S)} = \frac{4370}{99\,924} \div 0.126 = 0.3471$ $P(L\|\bar{S}) = \frac{P(L \cap \bar{S})}{P(S)} = \frac{485}{99\,924} \div 0.874 = 0.005\,553$
5 Write the answer.	The probability of smokers dying from lung cancer is about 34.71% and for non-smokers about 0.55%.

Source: Data sourced from Australian Bureau of Statistics (2022), National Health Survey, Causes of Death, Smoking and vaping, accessed 3 Feb 2024

From the example, the probability of a smoker dying of lung cancer is about 63 times that of a non-smoker.

EXERCISE 5.3 Conditional probability

ANSWERS p. 385

Recap

1 There are 23 students in a class. Five of the 13 girls have blue eyes and there are 16 students who do not have blue eyes. What is the probability of randomly selecting a blue-eyed boy from the class?

2 A coin is chosen at random from a bag containing 4 20c coins, 6 $1 coins and 5 50c coins. What is the probability that it is a 50c coin?

Mastery

3 WORKED EXAMPLE 7 A bag contains 5 yellow beads and 7 red beads. Two beads are taken without replacement. Find the probability of drawing

- **a** 2 yellow beads
- **b** 2 red beads
- **c** one of each colour
- **d** all yellow beads, if 3 beads are taken without replacement.

4 A coin is weighted so that $P(H) = \frac{2}{3}$. If 'heads' appears, a number is randomly selected from 1 to 9. If 'tails' appears, a number is randomly selected from 1 to 5. Find the probability that an even number is selected.

5 Two normal dice are thrown. Find the probability that the sum is 10 or greater if

- **a** a 5 appears on the first die
- **b** a 5 appears on at least one of the dice.

6 WORKED EXAMPLE 8 A study of the causes of death of Australian women found that of 91 015 deaths, 21 669 were from heart attacks. Of those women who died from heart attacks, 7800 were classed as obese. A total of 14 562 of the 91 015 women studied were obese. Compare the probabilities of obese and non-obese women dying of heart attacks.

7 A comprehension quiz has 5 multiple choice questions with 4 choices for each.

Find the probability of randomly guessing

a all answers correctly

b 4 answers correctly

c 3 answers correctly

d no answers correctly.

8 At a Brisbane school, 4% of the boys and 1% of the girls are taller than 180 cm. Given that 60% of all students at the school are female, find the probability that the student is female if a randomly selected student is taller than 180 cm.

Exam practice

The following information relates to Questions 9 to 11.

A container has 20 beads, including 8 pearl beads and 6 clear beads. The container is shaken and 2 beads are taken out at random.

9 TECH-FREE What is the probability of getting a pearl bead and then a clear bead?

A $\frac{1}{10}$ **B** $\frac{12}{95}$ **C** $\frac{14}{40}$ **D** $\frac{48}{400}$

10 TECH-FREE (1 mark) What is the probability of getting 2 clear beads?

11 TECH-FREE (1 mark) What is the probability of getting 2 pearl beads?

12 (2 marks) CF 40% of the students surveyed about hair washing were boys. 10% of the boys and 70% of the girls used conditioner after shampooing their hair. Find the probability that a student uses conditioner.

13 (3 marks) CF A restricted pack of cards for the card game Euchre consists of the 7, 8, 9, 10, Jack, Queen, King and Ace of each of the 4 suits, together with a Joker. A hand consists of 5 cards.

What is the probability of being dealt 2 Jacks of the same colour and the Joker in the first 3 cards?

14 (5 marks) CU The probability that a motor accident due to driver inattention (for example, texting, talking, eating) is correctly attributed to this cause is 0.6. The probability that a motor vehicle accident not due to a driver's inattention is diagnosed incorrectly as being due to a driver's inattention is 0.2. About 30% of all motor vehicle accidents are due to driver inattention. What is the probability that an accident attributed to driver inattention is correctly attributed?

15 (5 marks) CU If one engine fails on a twin-engine aeroplane, it can continue to fly on only the second engine. Unfortunately, when it is flying on only the second engine, the increased load increases the probability that this engine will fail, so pilots will normally land as soon as possible if an engine fails. The probability of a single-engine failure on a flight from Cairns to Townsville is 0.1%.

The probability that both engines will fail is only 0.000 25%.

What is the probability of the second engine failing if the first has already failed?

5.4 Independent events

Independent actions do not affect each other. For instance, the outcome of tossing a fair coin is independent of what was tossed before. The probability of it landing on tails is always the same: $\frac{1}{2}$.

Mathematically, for A = 'previous coin toss is heads' and B = 'heads on this toss', $P(B) = P(B|A)$.

This is commonly used to define **independent events**. However, if $P(A) = 0$, then $P(B|A)$ is undefined because you can't divide by zero.

Some mathematicians prefer to use $P(A \cap B) = P(A) \times P(B)$ instead.

Independent and dependent events

Two events A and B are **independent events** if and only if $P(B) = P(B|A)$ or $P(A \cap B) = P(A) \times P(B)$.
Events that are not independent are called **dependent events**.



Worksheet
Multi-stage problems

You can see that $P(B|A) \times P(A) = P(A \cap B) = P(A|B) \times P(B)$, so for 2 independent events A and B, $P(A \cap B) = P(A) \times P(B)$, $P(A) = P(A|B)$ and $P(B) = P(B|A)$ are equivalent.

You can use the concept of independence to simplify some probability calculations.

WORKED EXAMPLE 9 Independent event calculations

A coin is tossed 3 times. What is the probability of 3 heads?

Steps	Working
1 State the conditions.	Each toss of the coin is independent.
2 Use the rule for independent events.	$P(\text{1st H} \cap \text{2nd H} \cap \text{3rd H})$
	$= P(\text{1st H} \cap \text{2nd H}) \times P(\text{3rd H})$
3 Use it again.	$= P(\text{1st H}) \times P(\text{2nd H}) \times P(\text{3rd H})$
4 Substitute values.	$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$
5 Calculate the result.	$= \frac{1}{8}$
6 Write the answer.	The probability of 3 heads from 3 tosses of a coin is $\frac{1}{8}$.



Exam hack

When calculating probabilities from combined events, you need to work out whether the events are dependent or independent. Then apply the appropriate rule.

The probability of independent and dependent events

$P(A \cap B) = P(A) \times P(B)$ for **independent** events A and B.
$P(A \cap B) = P(A|B) \times P(B)$ or $P(A \cap B) = P(B|A) \times P(A)$ for **dependent** events A and B.

WORKED EXAMPLE 10 Applying conditional probability

Melanie's brother died of Duchenne muscular dystrophy when he was 29. It is a sex-linked genetic disease carried on the X chromosome. Girls have two X chromosomes, one from each parent and boys have an X chromosome from their mother and a Y chromosome from their father. This means that the disease is carried by the mother but only affects boys.
Melanie does not know if she is a carrier or not.
Determine the probability that if she has a child, it will have the disease.

Steps	Working
1 What are the conditions for Melanie to have a child with muscular dystrophy?	For a child to have muscular dystrophy, Melanie must be a carrier and have a boy who inherits the gene.
2 Choose abbreviations.	Let M = a child with Duchenne muscular dystrophy, C = Melanie is a carrier and B = Melanie has a boy.
3 What genes must Melanie's mother have?	Melanie's mother has 2 X chromosomes, one of which has the muscular dystrophy gene.
4 What is the probability that Melanie is a carrier?	Melanie has one X from her mother, so $P(C) = \frac{1}{2}$.
5 What is the probability of Melanie having a baby that is a boy?	$P(B) = \frac{1}{2}$
6 What is the probability that Melanie is a carrier and has a boy?	Being a carrier and having a boy are independent, so $P(C \cap B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$
7 What is the probability of a child having muscular dystrophy if Melanie is a carrier?	A child gets an X chromosome from Melanie, so $P(M\|C \cap B) = \frac{1}{2}$
8 What is the probability of Melanie having a child with muscular dystrophy?	$P(M) = P(M\|C \cap B) \times P(C \cap B)$ $= \frac{1}{2} \times \frac{1}{4}$ $= \frac{1}{8}$
9 Write the answer.	Melanie has a probability of $\frac{1}{8}$ of having a child with Duchenne muscular dystrophy.

In some situations, it is easier to find a probability by working out the complement.

WORKED EXAMPLE 11 Using the complement of an event

What is the probability that at least 2 people in a group of 7 have the same birth month?
Answer correct to 3 decimal places.

Steps	Working
1 Write the complementary event.	The complementary event is 'they all have different birth months'.
2 Find the complementary probability.	$P(\bar{A}) = P(\text{1st birth month}) \times P(\text{2nd different}) \times P(\text{3rd different}) \times \ldots$
3 Use conditional probability.	$= \frac{12}{12} \times \frac{11}{12} \times \frac{10}{12} \times \frac{9}{12} \times \frac{8}{12} \times \frac{7}{12} \times \frac{6}{12}$
4 Evaluate.	$= 0.1114\ldots$
5 Find the required probability.	$P(\text{at least 2 the same}) = 1 - 0.1114\ldots = 0.8885\ldots$
6 Write the answer.	The probability that at least 2 people out of 7 have the same birth month is about 0.889.

EXERCISE 5.4 Independent events

ANSWERS p. 385

Recap

1 A carton of eggs contains 3 × 55 g eggs, 5 × 59 g eggs and 2 × 52 g eggs.
Find the probability of choosing two 55 g eggs at random.

2 A card is drawn randomly from a well-shuffled pack of cards, and the suit is noted. The card is then replaced in the pack, which is reshuffled. This is done 3 times.
What is the probability that all 3 cards will be clubs?

Mastery

3 WORKED EXAMPLE 9 The probability that a person wears glasses is about $\frac{1}{3}$ and the probability that a person is left-handed is about $\frac{1}{15}$. Find the probability that a person chosen at random is

- **a** left-handed and wears glasses
- **b** left-handed and does not wear glasses
- **c** right-handed and wears glasses.

4 Three yachts, Alpha, Bravo and Charlie, often race, and it is known that their probabilities of winning are $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{6}$ respectively. A contest involving 2 races is held.

- **a** Find the probability that Bravo wins no races.
- **b** Find the probability that Bravo wins both races.
- **c** Find the probability that Bravo wins 1 race.
- **d** What is the probability that Alpha wins 1 race?
- **e** What is the probability that Charlie wins 1 race?
- **f** What is the probability that one of the yachts wins both races?

5 Max is approaching his end-of-semester exam period and he knows from his many past performances that his chances of passing Maths, History and English are $\frac{3}{4}$, $\frac{3}{5}$ and $\frac{1}{2}$ respectively. Max's performance in any one subject has no impact on his performance in any other subject.

Calculate the probability that he passes

- **a** only 1 subject
- **b** at least Maths and English.

6 WORKED EXAMPLE 10 Haemophilia is a sex-linked recessive genetic disease carried on the X chromosome. Gina's uncle on her maternal grandmother's side has the disease. What is the probability of Gina having a child with the disease if there is no other information?

7 Two brands of a drug, A and Z, are currently used by equal portions of the population who have a particular condition. The probability of side effects being experienced by users of drug A is 1 in 300, while the probability for those using drug Z is 1 in 900.

Find the probability that a person selected at random will be

- **a** a drug A user who experiences no side effects
- **b** a drug Z user who experiences side effects.

8 Part of a Mathematics test has 3 multiple choice questions, each with 5 options. A student who did not study decides to guess the answers to the questions randomly. What is the probability of guessing

a the correct answer to a question

b an incorrect answer to a question

c 3 correct answers

d 3 incorrect answers

e the first answer correct and the next 2 answers incorrect

f 2 correct answers

g 2 incorrect answers?

9 WORKED EXAMPLE 11 What is the probability that at least 2 people out of 5 pick the same number at random from 1 to 15?

10 The probability that a man will live 10 more years is $\frac{1}{4}$, and the probability that his wife will live longer than 10 more years is $\frac{1}{3}$. Find the probability that

a both will be alive in 10 years

b at least one will be alive in 10 years

c neither will be alive in 10 years

d only the wife will be alive in 10 years

e only the husband will be alive in 10 years.

11 A fitness club reviewed its accounts and found that 12% of all accounts were in arrears (unpaid) while the other 88% were fully paid up. The review also revealed that 66% of the accounts belonged to married people. Only 4% of the accounts both were in arrears and belonged to married people. There were 500 accounts in all.

a Find the probability of an account not being in arrears and belonging to an unmarried person.

b Find the probability of an account being in arrears, given that the account belongs to an unmarried person.

c Is account status independent of marital status?

Exam practice

12 What is the probability of getting all 4s when 4 fair dice are rolled?

A $\frac{1}{1296}$ **B** $\frac{4}{1296}$ **C** $\frac{1}{24}$ **D** $\frac{1}{6}$

Refer to the following table for Questions 13 and 14.

The table shows how some Year 11 students felt about their weight.

	Underweight	Average	Overweight
Male	8	14	5
Female	6	16	9

13 Find $P(\text{thinks underweight}|\text{female})$.

A $\frac{6}{31}$ **B** $\frac{8}{27}$ **C** $\frac{6}{14}$ **D** $\frac{8}{14}$

14 Find $P(\text{thinks underweight}|\text{male})$.

A $\frac{6}{31}$ **B** $\frac{8}{27}$ **C** $\frac{6}{14}$ **D** $\frac{8}{14}$

15 (4 marks) From company records it is known that 80% of all applicants for a particular position have passed a standard literacy examination. According to the records, an applicant who fails the examination on the first try has a 90% chance of passing the exam on the second try. Three applicants take the examination.

a What is the probability that 2 will pass? *[1 mark]*

b What is the probability that at least one will pass? *[1 mark]*

c What is the probability that an applicant will fail the examination twice? *[1 mark]*

d If an applicant can take the exam only twice, what is the probability of them passing the exam in 1 or 2 trials? *[1 mark]*

16 (4 marks) CF What is the probability that at least 2 cards in a hand of 5 cards from a normal pack are the same (a pair)?

17 (6 marks) CF Girls outnumber boys 60% to 40% at a large school. Of the girls, 3% are classified as very tall, while 7% of the boys are classified as very tall. Find the probability that 2 very tall students chosen at random are a male and then a female.

What assumption do you need to make?

5.5 Relative frequency



Video
Experimental probability

Worksheets
Basic probability assignment

Basic probability extension

The equality of outcomes in throwing dice or tossing coins is so well-established that you think it is obvious. In cases where the outcomes are not well established, you *estimate* the probability using the data available.

For example, 1467 people out of 9861 surveyed said they were left-handed. This is $\frac{1467}{9861} \approx 0.15$.

This suggests that the probability of someone being left-handed is about 15%. The large number of people surveyed means this is a good estimate. It doesn't mean that 15 out of every 100 people *must* be left-handed. You would expect *about* 15 in every 100 but it could be any number. This is called a **point estimate** because only one value is involved in the estimate, not a range of values.

Relative frequency

Relative frequency (or **experimental probability**) is used to estimate probability from data.
The characteristic being estimated is called the **favourable outcome**.
The number of data items used is called the number of **trials**.
The estimated probability is given by

$$\text{relative frequency} = \frac{\text{frequency of favourable outcomes}}{\text{number of trials}}$$

The **expected frequency** of an outcome is given by

$$\text{expected frequency} = \text{probability} \times \text{number of trials}$$

WORKED EXAMPLE 12 Calculating relative frequency

In a survey of newspaper readers, 297 preferred *The Courier Mail*, 263 *The Australian*, 176 the *Australian Financial Review* and 104 another paper. **Calculate** the probability of choosing someone at random who prefers *The Courier Mail*.

Steps	Working
1 State the number of favourable outcomes.	*The Courier Mail* frequency = 297
2 State the number of trials.	number of trials = 297 + 263 + 176 + 104 = 840
3 Calculate the relative frequency.	relative frequency = $\frac{297}{840} = \frac{99}{280} \approx 0.354$
4 Write the answer.	The probability of choosing someone at random who prefers *The Courier Mail* is about 0.35 or 35%.

WORKED EXAMPLE 13 Making predictions from relative frequency

Weather bureau records show that the probability of rain in Tully in November is about 35% on any day. How many wet days would you expect in a school week in Tully in November?

Steps	Working
1 State the number of trials.	There are 5 school days in a week.
2 Use the formula.	expected frequency = probability × number of trials = 0.35 × 5 = 1.75
3 Round appropriately for the answer.	There will be about 2 wet days in a school week in Tully in November.

EXERCISE 5.5 Relative frequency

ANSWERS p. 385

Recap

1 You must inherit the sickle-cell trait from both parents to get sickle-cell disease. If Ada's maternal and paternal grandmothers both had brothers with sickle-cell disease, but her parents didn't, what is the probability that Ada will have sickle-cell disease?

2 The blood test for a disease affecting about 3% of people has an accuracy of 90%.

- **a** What is the probability of a false positive in the general population?
- **b** What is the probability of a true positive in the general population?
- **c** What is the probability of two successive false positives in the general population?

Mastery

3 WORKED EXAMPLES 12,13 A packaging machine fills 100 g packets of potato chips. As part of a quality assurance program, the contents of 40 packets are weighed. The results are shown in the table.

Weight of contents (g)	90–94	95–99	100–104	105–109	110–114
Number of packets	3	8	21	7	1

Estimate as a decimal the probability that the next packet filled by the machine will

- **a** contain less than 95 g of chips
- **b** contain at least 110 g of chips
- **c** be underweight (< 100 g)
- **d** A carton of chips holds 500 packets. How many underweight packets would you expect to find in a carton?

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4 People walking down George Street were asked how they had travelled to the city. The results are recorded in the table.

Main method of transport	Number of people
Taxi	18
Train	101
Bus	78
Car	86
Bicycle	6
Other	11

a Find, correct to 2 decimal places, the relative frequency for each method of transport.

b What is the probability, as a fraction, that the next person spoken to would have travelled by car?

c If you asked another 20 people, how many would you expect to have travelled by train?

d For 30 people, how many could you expect to travel by bus?

5 The reasons that Queensland drivers had their licences cancelled or were disqualified from holding a driver's licence during a particular year are shown in the table.

Reason for cancellation	Number of drivers
Alcohol related: • blood alcohol > 0.05% • refusal to provide breath sample • other	 3357 417 525
Dangerous driving	1589
Speeding	3792
Driving under disqualification	1516
Driving without a licence	1413
Demerit points suspension	3159
Others	2846
Total	**18 614**

Source: Data from Fairfax Indication/Rohan Thomson

About 300 disqualifications occurred during a 5-day blitz on Queensland highways. Estimate the number of cancellations as a result of

a speeding

b the accumulation of demerit points

c an alcohol-related cause.

6 The exact times shown on the phones of a group of students were checked against the actual time.

+10	0	0	+20	0	−10	−50	+40	+30	0	0	+20
0	0	0	0	−30	−10	0	0	+30	−20	+50	+50
−50	0	−30	0	+20	0	−10	−30	+10	0	0	−10
−20	+20	0	0	+10	0	−10	−20	0	0	+20	0

The approximate amounts they were out (in seconds) were:

a What was the number of trials?

b What was the frequency of the correct time (0)?

c What was the relative frequency of the correct time?

d What was the relative frequency of being 20 seconds fast (+20)?

e If you checked a group of 50 students at the school, how many would you expect to have the correct time?

7 Complications from measles are relatively common. Of 4000 cases in children in a recent outbreak, 54 developed complications of some sort, resulting in death for 3 children. In 5 cases the children survived but they suffered serious brain damage, and in 2 other cases the children became deaf.

Complications from the measles vaccination also occur. Of 50 000 vaccinations, 80 children had some complications. In 75 of these, it was a fever that subsided after a few days. In 1 case, the child suffered some loss of hearing, and in 4 cases the children developed mild cases of measles that did not result in any permanent loss of hearing.

a What is the relative frequency of complications from measles?

b What is the probability of complications from the vaccine?

c What is the probability of serious complications from measles?

d What is the probability of serious complications from the vaccine?

e If 1000 children were vaccinated, how many would you expect to suffer complications?

Exam practice

8 (1 mark) Of 63 people interviewed in a shopping centre, 18 said they were there to buy clothes. What is the probability of a shopper intending to buy clothes?

9 (2 marks) Traffic lights at intersections in the suburbs of Brisbane have average times of red for 66 seconds, yellow for 4 seconds and green for 50 seconds. What is the probability that you will get 4 reds in a row in a Brisbane suburb?

10 (2 marks) CF In a retirement home, there are 5 couples, 12 widows and 2 widowers over 80 years of age. Estimate the probability that a man will outlive his wife and state any assumptions you make.

11 (3 marks) CF Billie throws 10 tennis balls towards a bucket and counts the number of balls that make it into the bucket. She plays this game 100 times and her scores are listed below.

0 1 1 1 1,	2 3 3 3 3,	1 1 1 1 2,	2 2 2 2 2,	1 1 1 1 1,
4 4 5 5 2,	2 2 6 5 1,	2 2 5 5 2,	2 2 2 2 2,	2 2 4 6 6
2 2 3 3 3,	3 2 2 2 2,	2 0 0 0 3,	3 4 4 4 4,	0 2 2 3 3
3 3 3 3 0,	0 2 2 6 7,	2 3 3 3 3,	3 3 3 4 4,	4 4 6 6 9

If Billie plays this game for another 30 times, predict the number of times exactly 2 balls will land in the bucket.

12 (3 marks) CU At a high school, 32 of the teachers have no children at home, and of these, 19 live with a partner. At the same school, 6 teachers have children at home but do not live with a partner. Altogether, there are 63 teachers at the school. Estimate the number of teachers in a high school with 95 teachers who live with children and a partner.

EXAM QUESTION ANALYSIS

Exam-style question (6 marks)

The Custom Coffee Store blends coffee using a mixture of 4 types of coffee: Arabica, Robusta, Excelsa and Liberica. They use the flavours vanilla bean, cacao and chai. They use the spices cinnamon, nutmeg, allspice and cardamon.

They have two blends.

Blend 1: One variety of coffee, one flavour and one spice.

Blend 2: 2 varieties of coffee, 2 flavours and 2 spices.

40% of customers pick Blend 1. 25% of Blend 1 customers have cacao. 70% of Blend 2 customers have cacao. Calculate the probability that a customer with cacao in their coffee chose Blend 2.

Reading the question

- The question uses conditional probability.
- The spices are irrelevant.
- Cacao is the focus.

Thinking about the question

- $P(\text{Blend 2}|\text{cacao})$ needs $P(\text{cacao})$.
- $P(\text{cacao})$ is from cacao and Blend 1 with cacao and Blend 2.

Worked Solution (✓ = 1 mark)

$$P(\text{Blend 1} \cap \text{cacao}) = P(\text{cacao}|\text{Blend 1}) \times P(\text{Blend 1})$$
$$= 0.25 \times 0.4 = 0.1 \checkmark$$

$$P(\text{Blend 2} \cap \text{cacao}) = P(\text{cacao}|\text{Blend 2}) \times P(\text{Blend 2}) \checkmark$$
$$= 0.7 \times 0.6 = 0.42 \checkmark$$

$$P(\text{cacao}) = 0.1 + 0.42 = 0.52 \checkmark$$

$$P(\text{Blend 2}|\text{cacao}) = \frac{P(\text{Blend2} \cap \text{cacao})}{P(\text{cacao})} \checkmark$$
$$= \frac{0.42}{0.52}$$
$$= 0.807 \ldots$$

The probability that a customer with cacao in their coffee chose Blend 2 is **81%**.✓



Video Exam question analysis: Probability

Puzzles
Probability find-a-word

Probability crossword

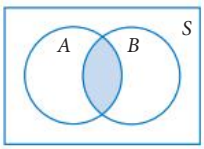
5 Chapter summary

Sets

- A **set** is a well-defined collection of objects. Each object is an **element** of the set.
- The number of elements in set A is called its **cardinal** number, shown as $|A|$ or $n(A)$.
- The **universal set**, U, is the set that contains all the elements of sets.
- The **null set** or **empty set** $\{\}$ or $\emptyset$ has no elements.
- A **proper subset** is smaller than the set it is a subset of.
- The **union** of two sets has the elements belonging to either or both sets.
- The **intersection** of two sets has elements belonging to both.
- **Disjoint** (**mutually exclusive**) sets have no common elements.
- The **complement** of set A, $\overline{A}$ or A', has all the elements not in A.

Probability

- The set of all outcomes is called the **sample space**, shown as S.
- An **event** is a collection of outcomes; it is a subset of the sample space.
- A **and** B means $A \cap B$: both A and B occur.
- A **or** B means $A \cup B$: at least one of A or B occurs.
- **Not** A means $\overline{A}$ or A'.
- A **Venn diagram** shows the sample space in a rectangle.
- The intersection of two sets is shown as the overlapping part of their circles.



- The **probability** of an event E with equally likely outcomes is $P(E) = \dfrac{n(E)}{n(S)}$.
- **Random** outcomes are equally likely.
- The **complement** of A, 'not A', has a probability given by $P(\overline{A}) = 1 - P(A)$.

Conditional probability

- The **conditional probability** of A given B is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

or $P(A \cap B) = P(A|B) \times P(B)$

Independent events

- Two events A and B are **independent** if and only if $P(B) = P(B|A)$ or $P(A \cap B) = P(A) \times P(B)$.
- For independent events A and B:

$$P(A \cap B) = P(A) \times P(B)$$

- For dependent events A and B:

$$P(A \cap B) = P(A|B) \times P(B)$$

or $P(A \cap B) = P(B|A) \times P(A)$

Relative frequency

- The number of data items used is called the number of **trials**.
- **Relative frequency** is used to estimate probability.
- The estimated probability is given by

$$\textbf{relative frequency} = \frac{\text{frequency of favourable outcomes}}{\text{number of trials}}$$

- The **expected frequency** of an outcome is given by

expected frequency = probability × number of trials

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Cumulative examination 1

Technology-free

Perusal time: 2 minutes Working time: 30 minutes

TECH-FREE *Calculators are **not** permitted.*

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

Section 1 (3 marks): 3 multiple choice questions

Section 2 (15 marks): 4 short response questions

Total: 18 marks

Worksheet
Maths Methods Year 11 formula sheet

Section 1 3 multiple choice questions

3 marks

1 Which of these situations does not constitute independent events?

A Heads when a coin is tossed 12 times.

B Left-handed people in a sample of 100 people.

C Number of reds when 2 marbles are drawn without replacement from a bag with 4 red marbles and 5 blue marbles.

D Number of red marbles when 2 are drawn with replacement from the same bag as in part **c**.

2 A company's daily expenditure (in hundreds of dollars) on staff meals follows the pattern of the 6th row of Pascal's triangle. What is the total cost for one week?

A \$300 **B** \$3200 **C** \$6200 **D** \$6400

3 The turning point of $f(x) = x^2 - 3x - 10$ is at

A $(-2, 5)$ **B** $(-1.5, -12.25)$ **C** $(-1.5, -7.75)$ **D** $(3, -1)$

Section 2 4 short response questions

15 marks

4 (3 marks) What is the area of a sector bounded by an arc of length 4 cm in a circle of radius 5 cm?

5 (4 marks) What is the probability of selecting at least 1 heart in 3 cards from a normal pack of playing cards? You may leave your answer as an expression.

6 (4 marks) Fifty rabbits were caught in traps at night in an area in western Queensland. They had small tags attached and were then released. A few weeks later, 85 rabbits were caught in the same area, and 12 were found to have been previously tagged. Estimate the population of rabbits in the area.

7 (4 marks) A monic cubic function passes through the points $(-2, -6)$ and $(2, 8)$ and has y-intercept -3. Determine its equation.

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Worksheet
Maths Methods Year 11 formula sheet

Cumulative examination 2

Technology-active

Perusal time: 2 minutes Working time: 30 minutes

*Calculators are **permitted**.*

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

Section 1 (3 marks): 3 multiple choice questions

Section 2 (15 marks): 4 short response questions

Total: 18 marks

Section 1 3 multiple choice questions

3 marks

1 37% of people in Brisbane have regular access to more than 1 car.
From 578 people in Brisbane, estimate the number with access to more than 1 car.

A 213 **B** 213.86 **C** 214 **D** 280

2 Simplify $\frac{3\sqrt{15}}{\sqrt{10}}$.

A $\frac{3\sqrt{6}}{2}$ **B** $\frac{3\sqrt{3}}{2}$ **C** $\frac{3\sqrt{5}}{2}$ **D** $\frac{3\sqrt{10}}{2}$

3 What is the probability of getting 3 6s and then a 1 when you roll a fair die 4 times?

A 0.000 77 **B** 0.0028 **C** 0.0039 **D** 0.023

Section 2 4 short response questions

15 marks

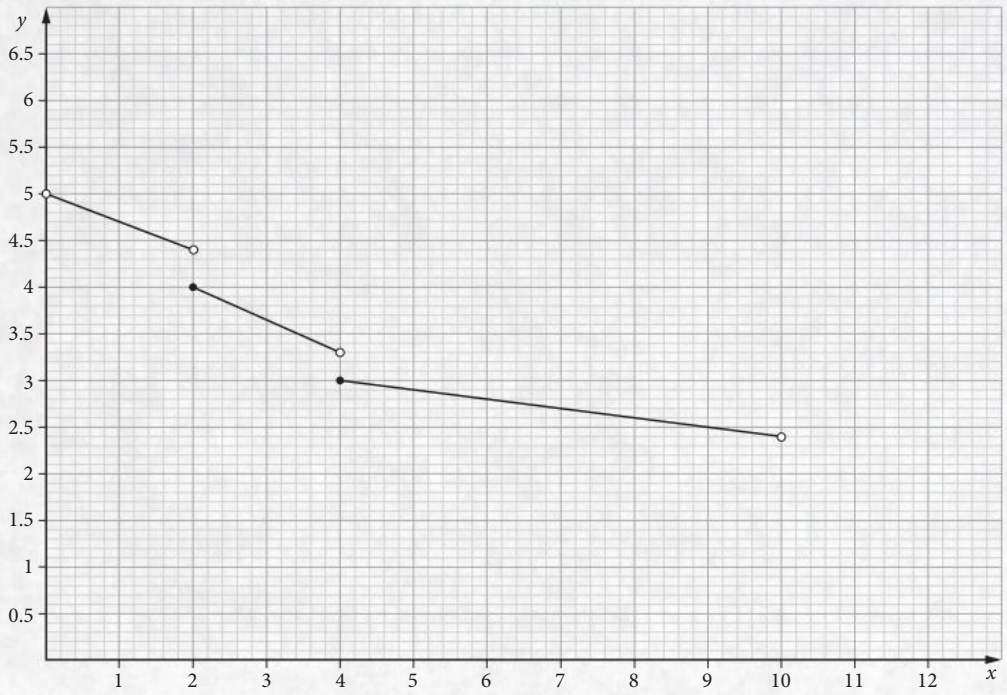
4 (4 marks) An ice-cream stand offers customers a choice of ice cream in a waffle cone or a paper cup. They can also have freeze-dried strawberry dust on their ice cream.

- 20% of customers choose a paper cup and 60% choose strawberry dust.
- 44% of customers choose a waffle cone with strawberry dust.

What is the probability a customer allergic to strawberries will choose a paper cup?

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5 (3 marks) Find the equation of the piecewise function $f(x)$ graphed below and state its domain.



6 (4 marks) Solve $8\sin^2(x) + 2\sin(x) = 3$ for $0 \leq x < 2\pi$, correct to 3 decimal places if necessary.

7 (4 marks) A medical researcher was investigating a possible genetic factor and heart disease. The task was complicated by the fact that some of the patients investigated were also obese, which is a known risk factor. Of 1400 people with heart disease, 800 were obese. Of these, 100 had the genetic factor present. 80 of the non-obese patients also had the genetic factor present. Of 1200 people without heart disease, 300 were obese and 28 of these had the genetic factor. 65 of the remaining 900 had the genetic factor present. About 30% of the population suffers from heart disease.

Compare the probabilities of heart disease for those with and without the genetic factor and comment on your result.

CHAPTER

6 FURTHER FUNCTIONS AND RELATIONS



Syllabus coverage
Nelson MindTap chapter resources
Terminology
6.1 Graphing circles
6.2 Graphing quadratic relations
6.3 Graphing square root functions
6.4 Graphing hyperbolas
6.5 Modelling with relations and reciprocal functions
Exam question analysis
Chapter summary
Cumulative examination 1
Cumulative examination 2

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Syllabus coverage

UNIT 1, TOPIC 3: FUNCTIONS AND RELATIONS

Graphs of relations

- Recognise and determine features of the graphs of $x^2 + y^2 = r^2$ and $(x - h)^2 + (y - k)^2 = r^2$, including their circular shapes, centres and radii.
- Recognise and determine features of the graph of $y^2 = x$, including its parabolic shape and axis of symmetry.
- Recognise and determine features of the graphs of $y = a\sqrt{x - h} + k$, including their shape, intercepts, and behaviour as $x \to \infty$ and $x \to -\infty$.
- Sketch the graphs of relations, with or without technology.
- Model and solve problems that involve relations, with and without technology.

Reciprocal functions

- Recognise features of the graphs of $y = \frac{1}{x}$ and $y = \frac{a}{x - h} + k$, including their hyperbolic shape, intercepts, asymptotes, and behaviour as $x \to \infty$ and $x \to -\infty$.
- Model and solve problems that involve reciprocal functions, with and without technology.
- Sketch the graphs of reciprocal functions, with or without technology.

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Prior learning
Further functions and relations

Videos (2):
6.4 Graphing hyperbolas
Exam question analysis Further functions and relations

Prior learning (1):
6.1 Further functions and relations

Worksheets (8):
6.1 Equations of circles
6.3 Square root functions
6.4 Graphing hyperbolas
- Graphing $y = \frac{10}{x}$
- Advanced graphs

6.5 Functions assignment
- Functions problems

Cumulative exams Maths Methods Year 11 formula sheet

Puzzles (2):
6.1 Circle equations
6.4 Matching graphs (Advanced)

TI-Nspire CX II instructions (1):
6.4 Further functions and relations

Technology (1):
6.4 Curve sketcher

Spreadsheet (1):
6.4 Curve sketcher

Nelson MindTap
To access resources above, visit
cengage.com.au/nelsonmindtap

Terminology

asymptote	circle	horizontal asymptote	hyperbola
mathematical model	modelling	quadratic relation	semicircle
square root function	vertex	vertical asymptote	

Worksheet
Equations of circles

Puzzle
Circle equations

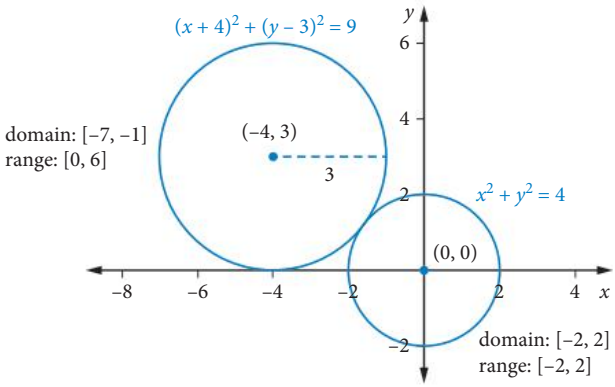
6.1 Graphing circles

A **circle** is a *many-to-many* $(n:n)$ relation.

Circles

Circles have the general equation $(x-h)^2+(y-k)^2=r^2$ with centre (h,k) and radius r.
A circle with its centre at the origin $(0,0)$ has the equation $x^2+y^2=r^2$.

The domain and range of a circle are obtained from the endpoints of the horizontal and vertical diameters.
A circle with centre (h,k) and radius r has the domain $[h-r,h+r]$ and range $[k-r,k+r]$.

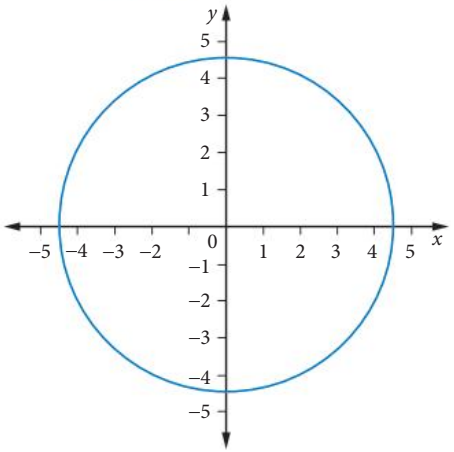


WORKED EXAMPLE 1 Sketching circles

Sketch each circle.

a $x^2+y^2=20$ and hence **sketch** $x^2+y^2>20$.

b $(x-2)^2+(y-3)^2=16$ and state the domain and range.

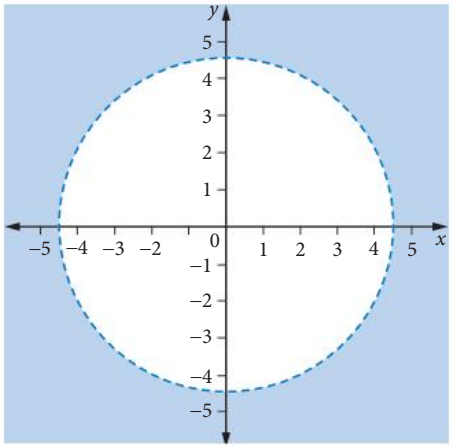
Steps	Working
a 1 Write in the form $x^2+y^2=r^2$.	$x^2+y^2=\left(\sqrt{20}\right)^2$
2 Simplify to find the radius.	$x^2+y^2=\left(\sqrt{4}\sqrt{5}\right)^2$ $=\left(2\sqrt{5}\right)^2$
3 Sketch the relation.	

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4 Evaluate the inequality – clearly identify what points are included or excluded.

The points on the line $x^2 + y^2 = 20$ are not included, so it is a dashed line.

The points are on the outside of the circle, so shade outside.



b 1 Identify the radius and (h, k).

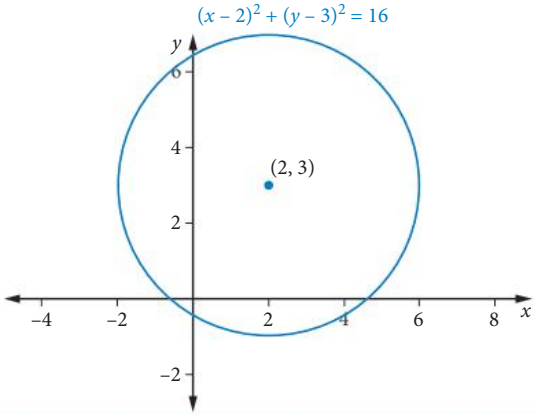
$r = 4$, $(h, k) = (2, 3)$

2 Identify the domain and range.

domain: $[-2, 6]$

range: $[-1, 7]$

3 Sketch the relation.



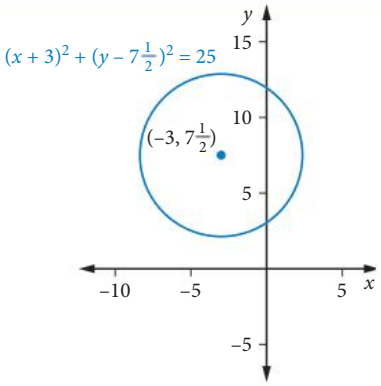
You need to convert circles in the form $x^2 + ax + y^2 + by + c = 0$ to general form by completing the squares for both x and y.

WORKED EXAMPLE 2 Sketching circles from expanded form

Sketch the relation $4x^2 + 24x + 4y^2 = 60y - 161$.

Steps	Working
1 Make the coefficients of x^2 and y^2 both 1.	$x^2 + 6x + y^2 = 15y - 40\frac{1}{4}$
2 Put the variable terms on the LHS.	$x^2 + 6x + y^2 - 15y = -40\frac{1}{4}$
3 Complete the squares.	$x^2 + 2 \times 3x + 3^2 + y^2 - 2 \times \frac{15}{2}y + \left(\frac{15}{2}\right)^2 = -40\frac{1}{4} + 9 + 56\frac{1}{4}$
4 Write as perfect squares and simplify.	$(x + 3)^2 + \left(y - 7\frac{1}{2}\right)^2 = 25$
5 Write in general form.	$(x + 3)^2 + \left(y - 7\frac{1}{2}\right)^2 = 5^2$
6 State the centre, radius, domain and range.	The centre is $\left(-3, 7\frac{1}{2}\right)$ and the radius is 5. domain: $[-8, 2]$ range: $[2.5, 12.5]$

7 Graph the relation.



When you write the equation of a circle with y as the **subject**, you get two **semicircles** $y = \pm\sqrt{r^2 - (x-h)^2} + k$.

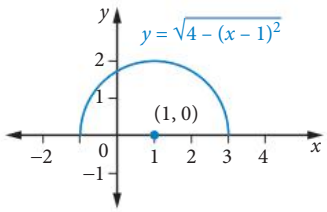
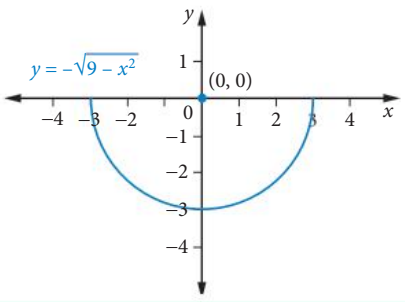
The equation for the upper half of the circle is given by the positive square root and for the lower half by the negative square root.

For circles with centres at $(0,0)$, the equation of the top semicircle is $y = \sqrt{r^2 - x^2}$.

WORKED EXAMPLE 3 Graphing semicircles

Graph each semicircle.

a $y = \sqrt{4 - (x - 1)^2}$ **b** $y = -\sqrt{9 - x^2}$

Steps	Working
a 1 Identify the radius, centre (h, k), domain and range.	$r = \sqrt{4} = 2$, centre $(1,0)$ domain: $[-1, 3]$ range: $[0, 2]$
2 Graph the relation.	
b 1 Identify the radius, centre (h, k), domain and range.	$r = \sqrt{9} = 3$, centre $(0,0)$ domain: $[-3, 3]$ range: $[-3, 0]$
2 Graph the relation.	

Circles are not functions but semicircles are functions.

EXERCISE 6.1 Graphing circles

ANSWERS p. 386

Mastery

1 WORKED EXAMPLE 1 Describe and sketch each circle.

a $x^2 + y^2 = 36$

b $(x + 3)^2 + (y - 5)^2 = 9$

c $(x + 2)^2 + (y + 1)^2 = 10$

d $(x - 6)^2 + (y - 4)^2 = 52$

2 Describe and sketch each relation.

a $x^2 + y^2 \leq 24$

b $(x - 1)^2 + (y + 2)^2 > 16$

c $(x + 4)^2 + (y - 3)^2 \geq 1$

d $(x + 2)^2 + (y + 5)^2 < 48$

3 WORKED EXAMPLE 2 Find the centre and radius of each circle.

a $x^2 + y^2 - 6x + 8y = 24$

b $x^2 + y^2 - 2x - 6y = 26$

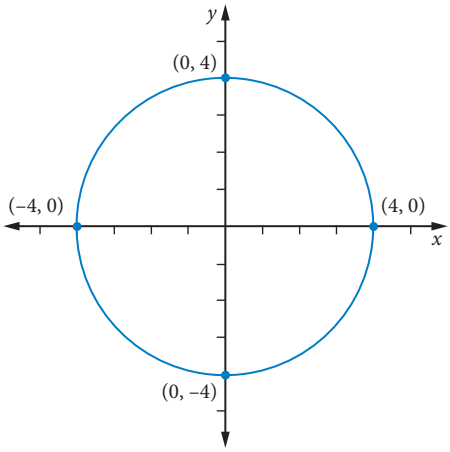
c $2x^2 + 2y^2 - 10x + 14y = 13$

d $x^2 + y^2 + 4x + 8y + 11 = 0$

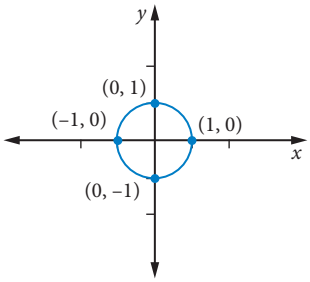
e $x^2 + y^2 + 8x - 4y = 61$

4 Find the equation of each circle.

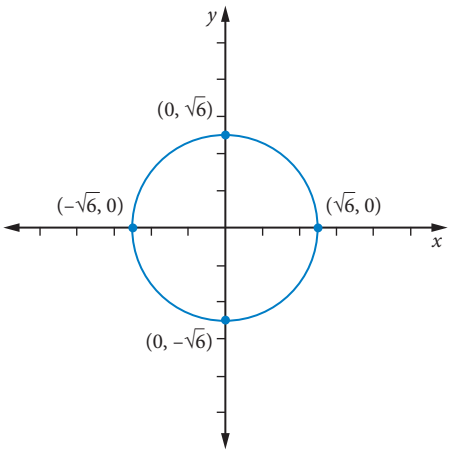
a



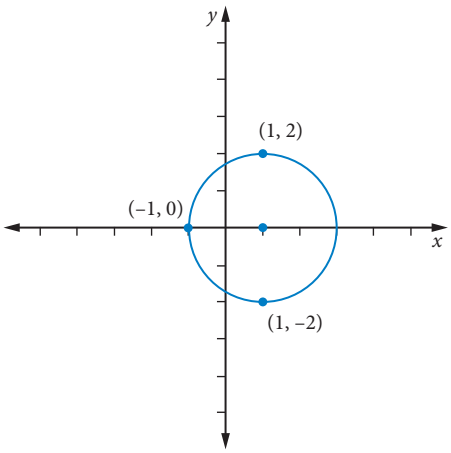
b



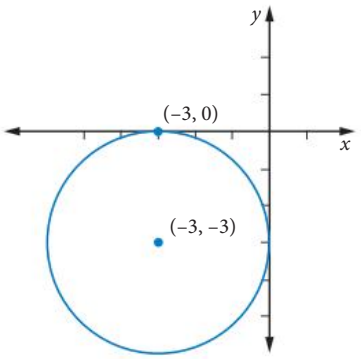
c



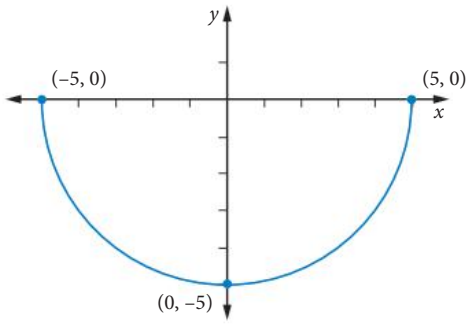
d



e



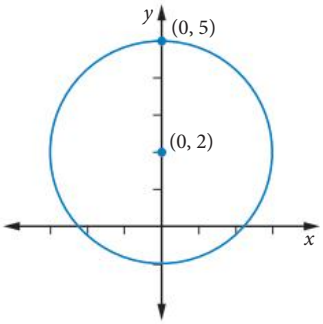
f



5 State the domain and range of each circle in Question **4**.

Exam practice TECH-FREE

This diagram relates to Questions 6 and 7.



6 The equation of this circle is

A $x^2 + y^2 = 9$ **B** $(x - 2)^2 + y^2 = 3$

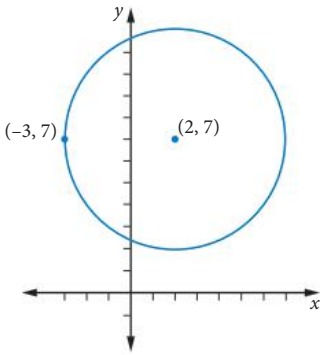
C $x^2 + (y - 2)^2 = 3$ **D** $x^2 + (y - 2)^2 = 9$

7 The range of this circle is

A $[-5, 5]$ **B** $[-1, 5]$

C R **D** $[0, 5]$

This diagram relates to Questions 8 and 9.



8 The equation of this circle is

A $(x - 2)^2 + (y - 7)^2 = 25$ **B** $(x + 2)^2 + (y + 7)^2 = 25$

C $(x - 2)^2 + (y - 7)^2 = 5$ **D** $(x + 2)^2 + (y + 7)^2 = 5$

9 The domain of this circle is

A $(-3, 7)$ **B** R

C $[-3, 7]$ **D** $[2, 12]$

10 (4 marks) Sketch the function $y = \sqrt{3 - x^2}$, and state its domain and range.

11 (3 marks) Find the centre and radius of $x^2 - 6x + y^2 = -2y - 6$.

12 (2 marks) CF Is $y = \pm\sqrt{7 - x^2}$ a function? Justify your answer.

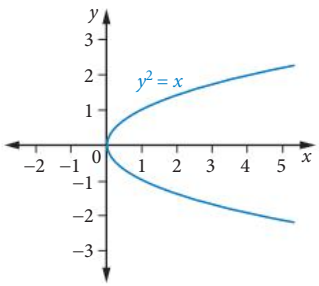
6.2 Graphing quadratic relations

The **quadratic relation** $y^2 = x$ is a *one-to-many* $(1:n)$ relation.

Its graph is a horizontal parabola with an axis of symmetry at the x-axis.

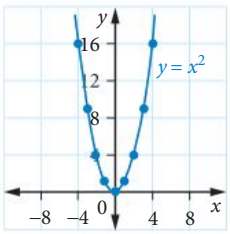
These quadratic relations are not functions.

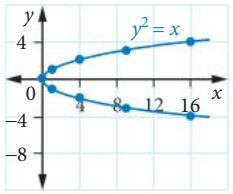
The ordered pairs of the quadratic relation $y^2 = x$ are the reverse of the ordered pairs of the **quadratic function** $y = x^2$. The graph of $y^2 = x$ is turned 90° clockwise from that of $y = x^2$.



x	−4	−3	−2	−1	0	1	2	3	4
y^2	16	9	4	1	0	1	4	9	16

y	−4	−3	−2	−1	0	1	2	3	4
x^2	16	9	4	1	0	1	4	9	16





Quadratic relations

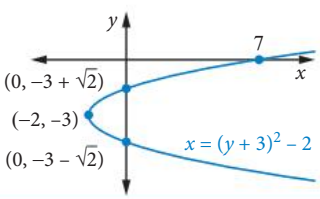
- **Intercept form:** A quadratic relation in the form $x = a(y - y_1)(y - y_2)$ has y-intercepts at $y = y_1$ and $y = y_2$. By symmetry, the turning point (**vertex**) is located halfway between y_1 and y_2.
- **Turning point form:** A quadratic relation in the form $x = a(y - k)^2 + h$ has a vertex at (h, k). The horizontal axis of symmetry passes through the vertex and has the equation $y = k$.
- If $a > 0$, the graph opens to the right; if $a < 0$, the graph is reflected in the y-axis and opens to the left.
- If $a > 1$, the graph is dilated away from the y-axis (stretched horizontally); if $0 < a < 1$, it is dilated towards the y-axis (compressed horizontally).

You can sketch a quadratic relation in a similar way to a quadratic function.

- Instead of finding zeros, find the y-intercepts.
- Instead of finding the y-intercept, find the x-intercept.
- Find the turning point.

WORKED EXAMPLE 4 Graphing quadratic relations

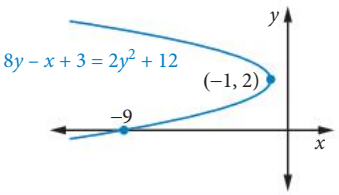
Sketch the graph of $x = (y + 3)^2 - 2$.

Steps	Working
1 State the vertex.	$h = -2, k = -3$ The vertex is at $(-2, -3)$.
2 Find the y-intercepts.	At $x = 0$, $(y + 3)^2 - 2 = 0$. $y + 3 = \pm\sqrt{2}$ $y = -3 \pm \sqrt{2}$
3 Find the x-intercept.	At $y = 0$, $x = 7$.
4 Sketch the graph.	

If the equation is in expanded form or if there are no y-intercepts, you need to complete the square.

WORKED EXAMPLE 5 Graphing quadratic relations from expanded form

Sketch the graph of $8y - x + 3 = 2y^2 + 12$, stating the domain and range.

Steps	Working
1 Rearrange.	$x = -2y^2 + 8y - 9$
2 Find any y-intercepts.	$-2y^2 + 8y - 9 = 0$ $\Delta = b^2 - 4ac$ $= 8^2 - 4 \times (-2) \times (-9) = -8$ so there are no y-intercepts.
3 Complete the square.	$x = -2(y^2 - 4y + 4) - 9 + 8$ $= -2(y - 2)^2 - 1$
4 State the vertex.	The vertex is at $(-1, 2)$.
5 Find the x-intercept.	At $y = 0$, $x = -9$.
6 State the domain and range.	domain: $[-\infty, -1]$ range: R
7 Sketch the graph.	

You need to know at least two pieces of information to find the equation of a quadratic relation.

WORKED EXAMPLE 6 Finding the equation of a quadratic relation

Determine the equation of the quadratic relation that passes through the point $(3, 6)$ and has its vertex at $(4, 1)$.

Steps	Working
1 Identify the information given and which general equation to use.	Known vertex $(4, 1)$ and point $(3, 6)$. Use $x = a(y - k)^2 + h$
2 Substitute in coordinates of the vertex.	$x = a(y - 1)^2 + 4$
3 Use $(3, 6)$ to determine the value of a.	$3 = a(6 - 1)^2 + 4$ $3 = a5^2 + 4$ $3 = 25a + 4$ $-1 = 25a$ $a = -\frac{1}{25}$
4 State the equation.	The equation is $x = -\frac{1}{25}(y - 1)^2 + 4$.

EXERCISE 6.2 Graphing quadratic relations

ANSWERS p. 387

Recap

1 What is the equation of the circle with centre at $(11, 13)$ and radius 5?

2 Sketch the equation $3x^2 + 3y^2 - 6x + 9y = 0$, stating the value for the radius.

Mastery

3 Explain why a quadratic relation is not a function.

4 WORKED EXAMPLES 4, 5 State the coordinates of the vertex and the domain and range for each quadratic relation.

a $x = \frac{1}{4}(y-3)^2 - 2$ **b** $x = 2(y-1)(y+3)$

5 Sketch the graph of each quadratic relation.

a $x = (y-2)^2 + 5$ **b** $x = (y+2)(y-4)$ **c** $x = -y^2 + 4$

d $y^2 = x + y + 6$ **e** $y - x = 5y - 2y^2 + 8$ **f** $2x - 5y = 4y^2 - 6 + 5y$

6 WORKED EXAMPLE 6 A quadratic relation has y-intercepts 2 and 6 and an x-intercept of 24. Find its equation.

Exam practice TECH-FREE

7 What is the equation of a quadratic relation that has y-intercepts at 2 and 4 and passes through the point $(3, 5)$?

A $(y-2)^2 = (x-4)$ **B** $(y-3)^2 = (x-5)$ **C** $x = (y-2)(y-4)$ **D** $x = 3(y-2)(y-4)$

8 What is the domain of the relation $(y-4)^2 = 2x + 1$?

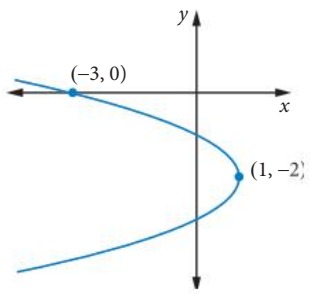
A $[\frac{1}{2}, \infty)$ **B** $[-\frac{1}{2}, \infty)$ **C** $(-\infty, -\frac{1}{2}]$ **D** $(-\infty, \frac{1}{2}]$

9 What are the coordinates of the vertex of $(y-5)(y+1) = x$?

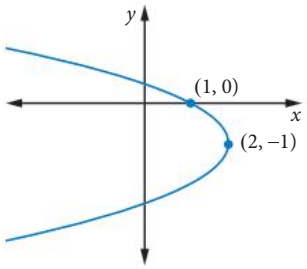
A $(-9, 2)$ **B** $(-1, 5)$ **C** $(5, -1)$ **D** $(9, -2)$

10 Which graph represents $(y+1)^2 = x - 2$?

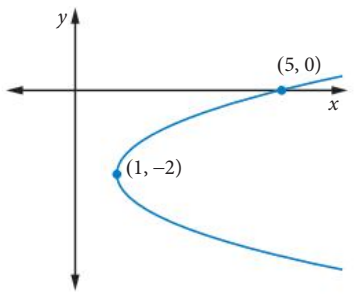
A



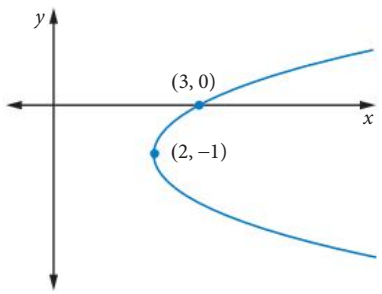
B



C



D



11 (2 marks) Write the equation $y^2 + 10y - 2x + 31 = 0$ in vertex form.

12 (2 marks) Sketch the graph of $x = (2y-4)(y+1)$.

13 (4 marks) CF A quadratic relation passes through the points $(-8, 0)$, $(-11, 2)$ and $(-23, -3)$. Find its equation.

Worksheet
Square root functions

6.3 Graphing square root functions

The **square root function** in its basic form is $y = \sqrt{x}$.

It is actually the relation $y^2 = x$ with the restriction that y must be positive.

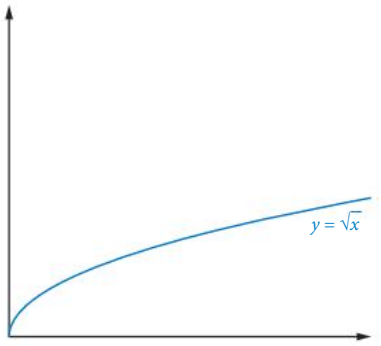
$$y^2 = x$$
$$y = \sqrt{x}$$

y is the positive square root of x.

Its graph is the top half of the horizontal parabola, so it is a function.

Its domain is $[0, \infty]$.

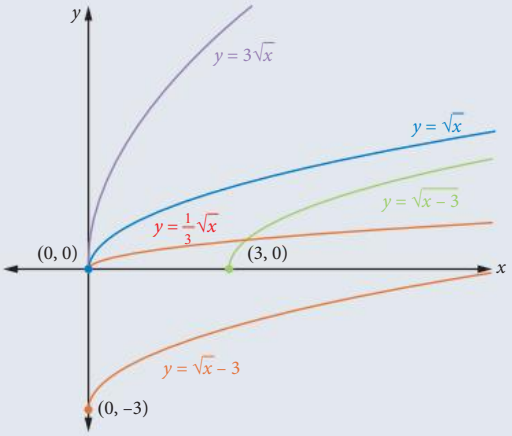
Its range is also $[0, \infty]$.



The square root function

The general form of the square root function is $y = a\sqrt{x - h} + k$, where the vertex (h, k) is also the start of the graph.

- a is the vertical dilation.
  - If $a > 1$, the graph is wider.
  - If $0 < a < 1$, the graph is narrower.
  - If $a < 0$, the graph is reflected in the x-axis.
- h is the horizontal translation h units (right if $h > 0$, left if $h < 0$)
- k is the vertical translation k units (up if $k > 0$, down if $k < 0$)



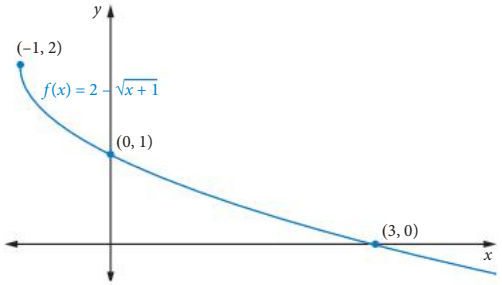
WORKED EXAMPLE 7 Sketching square root functions

Consider the function $f(x) = 2 - \sqrt{x + 1}$.

a **Sketch** the function.

b State the domain and range.

c List the transformations required to produce the graph of $f(x)$ from $f(x) = \sqrt{x}$.

Steps	Working
a 1 Rewrite $f(x)$ in the general form $y = a\sqrt{x - h} + k$.	$f(x) = 2 - \sqrt{x + 1}$ $= -\sqrt{x + 1} + 2$
2 Identify if there are any restrictions on the function.	$f(x)$ will only exist when $x + 1 \geq 0$ or $x \geq -1$.
3 Find the start coordinates.	When $x = -1$, $y = 2$ (or read (h, k) from the equation).
4 Find the x- and y-intercepts.	When $x = 0$, $y = 1$. When $y = 0$, $x = 3$.
5 Sketch the function.	

From the equation, note that the simple square root function is reflected in the x-axis.

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b State the domain and range.	domain: $[-1, \infty)$ range: $(-\infty, 2]$
c 1 Identify the a, h and k values.	$a = -1$, $h = -1$ and $k = 2$
2 List the transformations required.	The graph of $y = \sqrt{x}$ is • reflected in the x-axis • translated left 1 unit • translated up 2 units.

EXERCISE 6.3 Graphing square root functions

ANSWERS p. 387

Recap

1 What is the equation of a quadratic relation that has a vertex at $(-1, -2)$ and passes through the point $(3, 0)$?

A $(y + 2)^2 = (x + 1)$ **B** $(y - 2)^2 = (x - 1)$ **C** $x = y(y - 3)$ **D** $x = (y + 1)(y + 2)$

2 Sketch $3(y + 1)^2 = 6(x - 1)$.

Mastery

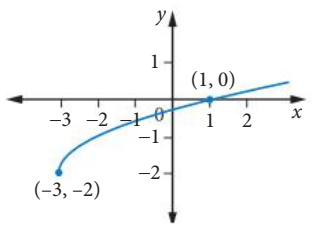
3 WORKED EXAMPLE 7 Sketch each square root function, labelling all key features and stating the domain and range.

a $y = \sqrt{x} + 5$ **b** $y = \sqrt{x - 4}$ **c** $y = \sqrt{x + 1} - 3$

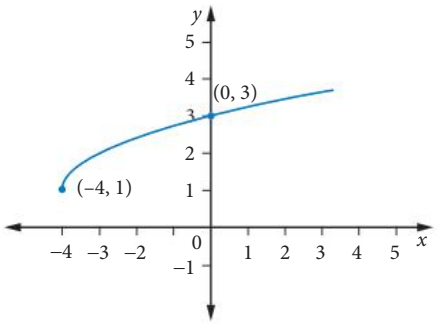
d $y = -\sqrt{x + 2} + 1$ **e** $y = -2\sqrt{5x} - 7$ **f** $y = \frac{1}{5}\sqrt{x - 6} + 2$

4 Determine the equation of each graph.

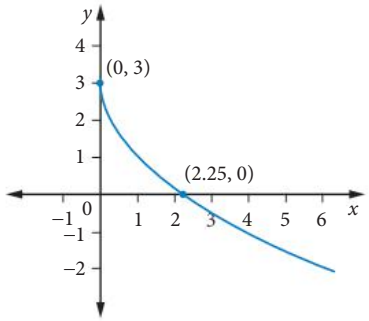
a



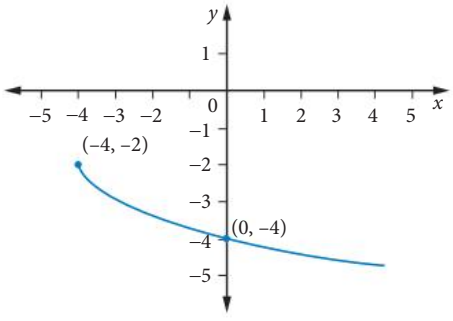
b



c



d



5 List the transformations required to produce the graph of $f(x) = 5 - \frac{1}{4}\sqrt{x - 9}$ from $f(x) = \sqrt{x}$.

Exam practice TECH-FREE

6 What is the domain of the function $y = 3 - \sqrt{x}$?

A $(-\infty, 0]$ **B** $(-\infty, 3]$ **C** $[0, \infty)$ **D** $[3, \infty)$

7 What is the range of the function $y = 3 - \sqrt{x}$?

A $(-\infty, 0]$ **B** $(-\infty, 3]$ **C** $[0, \infty)$ **D** $[3, \infty)$

8 What is the equation of the square root function that starts at the point $(3,5)$ and passes through $(4,4)$?

A $y = \sqrt{x-3} + 5$ **B** $y = -\sqrt{x-3} + 5$ **C** $y = -\sqrt{x+3} + 5$ **D** $y = \sqrt{x+3} - 5$

9 (2 marks) Sketch the graph of $f(x) = 3\sqrt{x+8} - 2$.

10 (4 marks) CU Sketch the function $f(x) = \sqrt{1-x}$ and list the transformations required to produce the graph of $f(x)$ from $f(x) = \sqrt{x}$.

Worksheets
Graphing hyperbolas

Graphing $y = \frac{10}{x}$

Advanced graphs

Puzzle
Matching graphs (Advanced)

Technology
Curve sketcher

Spreadsheet
Curve sketcher

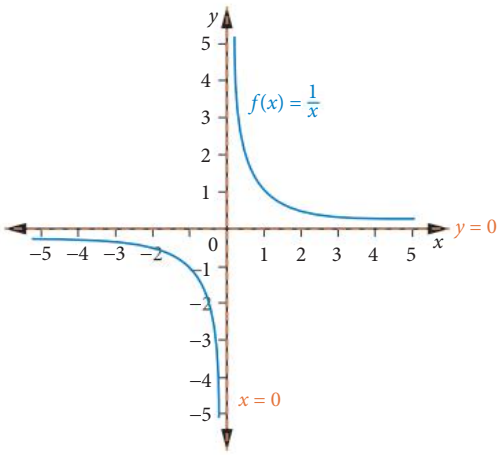
6.4 Graphing hyperbolas

A **hyperbola** is the graph of the function of the general form $f(x) = \frac{1}{x}$.

There are 2 key elements that can be observed from the function:

- At $x = 0$ the function is undefined.
- $y = 0$ is excluded from the range as there is no value that will make $f(x) = 0$.

The lines $x = 0$ and $y = 0$ are **asymptotes**.



The hyperbola

For the hyperbola $f(x) = \frac{1}{x}$:

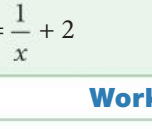
- The line $y = 0$ (the x-axis) is a **horizontal asymptote**. As $x \to \pm\infty$ (x approaches positive or negative ∞), $y \to 0$ (y approaches 0 but never reaches it).
- The line $x = 0$ (the y-axis) is a **vertical asymptote**. As $x \to 0$, $y \to \pm\infty$.
- Each of the points $(-1,-1)$ and $(1,1)$ is a vertex.
- The domain is $R\backslash\{0\}$; the real numbers except 0.
- The range is also $R\backslash\{0\}$.

The general form of a hyperbola is $y = \frac{a}{x-h} + k$.

- The centre is translated from $(0,0)$ to (h,k).
- The vertical asymptote is at $x = h$. As $x \to h$, $y \to \pm\infty$.
- The horizontal asymptote is at $y = k$. As $x \to \pm\infty$, $y \to k$.
- a is the dilation factor from the x-axis. If $a > 1$, the graph is stretched; if $0 < a < 1$, the graph is compressed.
- If $a < 0$, the graph is reflected in the x-axis.
- The domain is $R\backslash\{h\}$; the real numbers except h.
- The range is $R\backslash\{k\}$.

When sketching a hyperbola, follow these steps.

1. Find the asymptotes, $x = h$ and $y = k$.
2. Find the x- and y-intercepts.
3. Find a and identify whether the graph is reflected and dilated.
4. On the graph, draw in the asymptotes and mark the x- and y-intercepts. Label these features.
5. Using the asymptotes as the 'new' set of axes, sketch the function, ensuring you correctly adjust for a reflection if needed.
6. It is useful to show how the vertices of $y = \frac{1}{x}$ at $(-1, -1)$ and $(1, 1)$ have moved in the new function. This helps show any shift in scale due to the dilation factor.

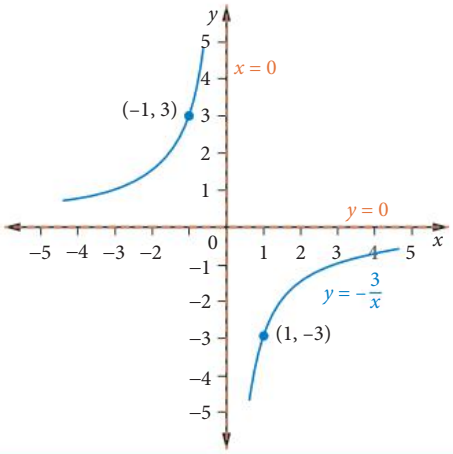


Video
Graphing hyperbolas

WORKED EXAMPLE 8 Sketching hyperbolas

Sketch each hyperbola.

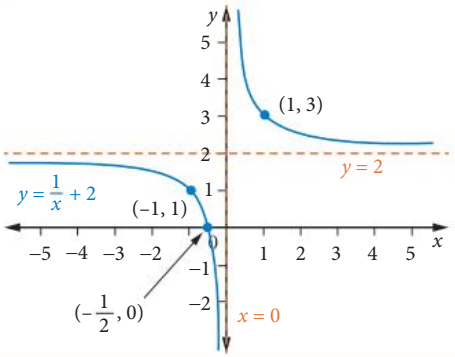
a $y = -\frac{3}{x}$ **b** $y = \frac{1}{x} + 2$ **c** $y = \frac{1}{x-4}$

Steps	Working
a 1 Identify h, k and the asymptotes.	$h = 0, k = 0$ Asymptotes at $x = 0$ and $y = 0$.
2 Find the x- and y-intercepts.	As there are asymptotes at $x = 0$ and $y = 0$, there are no intercepts.
3 Identify a and its effect.	$a = -3$. As $a < 0$, the graph is reflected in the x-axis and dilated by a factor of 3 away from the x-axis.
4 Consider the effect on the vertices of $\frac{1}{x}$ to become $-\frac{3}{x}$.	$(1, 1) \rightarrow (1, -3)$ and $(-1, -1) \rightarrow (-1, 3)$
5 Draw the asymptotes on the graph.	
6 Draw the graph, labelling all key features.	
b 1 Identify h, k and the asymptotes.	$h = 0, k = 2$ Asymptotes at $x = 0$ and $y = 2$.
2 Find the x- and y-intercepts.	No y-intercept as the function is undefined at $x = 0$. At $y = 0$, $x = -\frac{1}{2}$.
3 Identify a and its effect.	$a = 1$; no effect
4 Consider the effect on the vertices of $\frac{1}{x}$ to become $\frac{1}{x} + 2$.	$(1, 1) \rightarrow (1, 3)$ and $(-1, -1) \rightarrow (-1, 1)$

5 Draw the asymptotes on the graph.

6 Draw the graph, labelling all key features.

The hyperbola has been translated 2 units up.



c 1 Identify h, k and the asymptotes.

$h = 4, k = 0$

Asymptotes at $x = 4$ and $y = 0$.

2 Find the x- and y-intercepts.

No x-intercept as the asymptote is at $y = 0$.

At $x = 0$, $y = -\frac{1}{4}$.

3 Identify the a and its effect.

$a = 1$; no effect

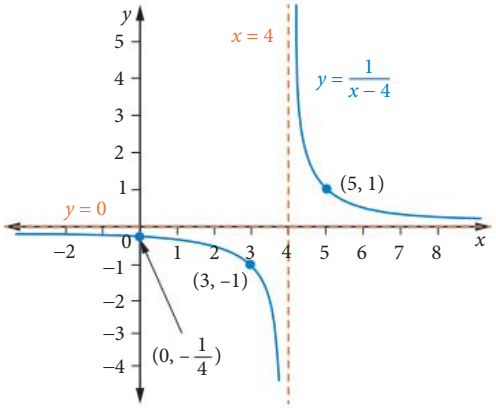
4 Consider the effect on the vertices of $\frac{1}{x}$ to become $y = \frac{1}{x-4}$.

$(1, 1) \rightarrow (5, 1)$ and $(-1, -1) \rightarrow (3, -1)$

5 Draw the asymptotes on the graph.

6 Draw the graph, labelling all key features.

The hyperbola has been translated 4 units to the right.

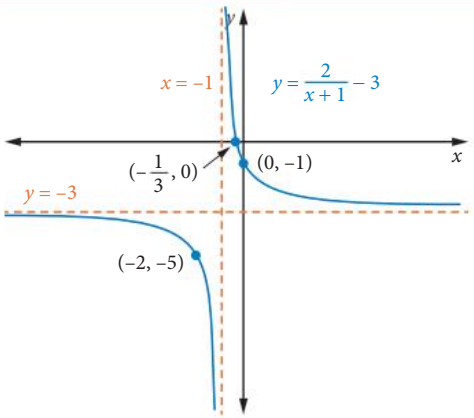


WORKED EXAMPLE 9 Using transformations to sketch hyperbolas

List the transformations for $y = \frac{1}{x}$ to become $y = \frac{2}{x+1} - 3$ and **sketch** it.

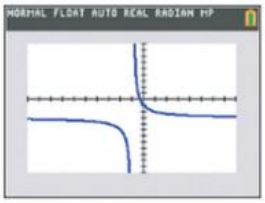
Steps	Working
1 Identify h and its effect.	$h = -1$. The graph is moved 1 unit left from $\frac{1}{x}$. The vertical asymptote is $x = -1$.
2 Identify k and its effect.	$k = -3$. The graph is moved 3 units down from $\frac{1}{x}$. The horizontal asymptote is $y = -3$.
3 Identify a and its effect.	$a = 2$. As $a > 1$, the graph is stretched vertically from $\frac{1}{x}$.
4 State the effect on $(-1, -1)$ and $(1, 1)$, doing dilations before translations.	$(-1, -1) \rightarrow (-2, 5)$ and $(1, 1) \rightarrow (0, -1)$

5 Sketch the graph, showing important points and asymptotes.

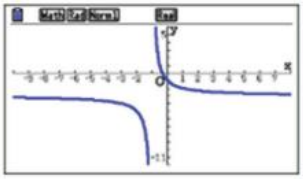


You can use your calculator to verify your graph.

TI-84 Plus CE



Casio fx-CG50AU



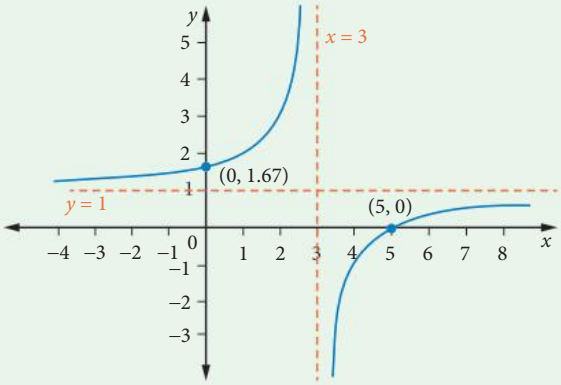
TI-Nspire
Further functions and relations

Exam hack

You must show working and give exact values for intercepts, points and so on in an exam, but check your graph and values using your calculator if you can.

WORKED EXAMPLE 10 Finding the equation of a hyperbola

Find the equation of the function graphed below.



Steps	Working
1 Identify the asymptotes and state the h and k values.	vertical asymptote: $x = 3$, so $h = 3$. horizontal asymptote: $y = 1$, so $k = 1$.
2 Write the general form of the hyperbola and substitute in the h and k values.	$y = \frac{a}{x-h} + k$ $y = \frac{a}{x-3} + 1$

3 To find the value of a, substitute a point from the graph.

The x-intercept is $(5, 0)$.

Choosing a point with whole numbers, if possible, will make it easier.

The graph is reflected in the x-axis so $a < 0$.

$$0 = \frac{a}{5-3} + 1$$
$$-1 = \frac{a}{2}$$
$$a = -2$$

4 State the equation.

The equation is $y = \frac{-2}{x-3} + 1$

EXERCISE 6.4 Graphing hyperbolas

ANSWERS p. 388

Recap

1 What is the domain and range of the equation $y = 2 + \sqrt{x}$?

A domain: $(-\infty, 0]$, range: $[2, \infty)$

B domain: $(-\infty, 2]$, range: $[0, \infty)$

C domain: $[0, \infty)$ range: $[2, \infty)$

D domain: $[2, \infty)$, range: $[0, \infty)$

2 Sketch $y = -\sqrt{2x} + 1$, labelling all key features.

Mastery

3 WORKED EXAMPLE 8 Sketch the graph of each hyperbola.

a $y = \frac{1}{2x}$ **b** $y = -\frac{1}{3x}$ **c** $f(x) = \frac{1}{x} - 1$

d $f(x) = 2 - \frac{1}{x}$ **e** $f(x) = \frac{1}{3-x}$ **f** $y =$

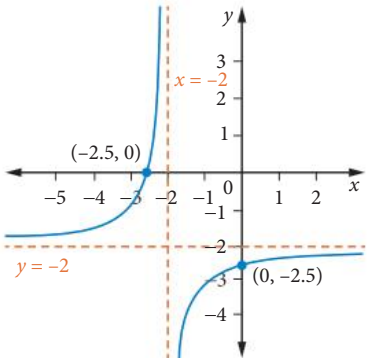
4 WORKED EXAMPLE 9 Sketch the graph of each hyperbola.

a $y = \frac{3}{4-x}$ **b** $f(x) = 2 + \frac{1}{2x}$ **c** $y = \frac{3}{x+4} + 2$ **d** $f(x) = 3 - \frac{1}{4(x-3)}$

5 State the transformations that need to be made to the graph of $y = \frac{1}{x}$ to become the graph of

a $y = \frac{4}{x} + 3$ **b** $y = \frac{2}{x-1}$ **c** $y = \frac{-6}{x+5} - 7$ **d** $y = \frac{2}{3} - \frac{0.5}{2.5x-1}$

6 WORKED EXAMPLE 10 Find the equation of this hyperbola.



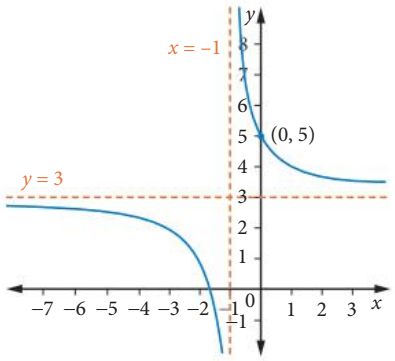
9780170484909

7 State the domain and range of the hyperbola in Question **6**.

8 A hyperbola has asymptotes of $x = 3$ and $y = 2$ and passes through $(4, 7)$.
Write a possible equation for the hyperbola and verify your solution.

Exam practice TECH-FREE

This diagram relates to Questions 9 and 10.



9 The equation of this hyperbola is

A $f(x) = 3 - \frac{2}{x+1}$ **B** $f(x) = 3 + \frac{2}{x+1}$

C $f(x) = 3 + \frac{2}{x-1}$ **D** $f(x) = 3 - \frac{2}{x-1}$

10 The domain and range of the hyperbola is

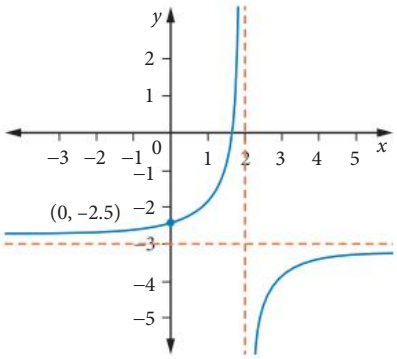
A domain: $R\backslash\{-1\}$, range: $R\backslash\{3\}$

B domain: $R\backslash\{1\}$, range: $R\backslash\{3\}$

C domain: $R\backslash\{-1\}$, range: $R\backslash\{-3\}$

D domain: $R\backslash\{1\}$, range: $R\backslash\{-3\}$

This diagram relates to Questions 11 and 12.



11 The asymptotes of this hyperbola are

A $h = 2, k = -3$ **B** $k = -2, h = 3$

C $x = 2, y = -3$ **D** $x = -2, y = 3$

12 The equation of this hyperbola is

A $f(x) = \frac{1}{x-2} - 3$ **B** $f(x) = \frac{1}{x-2} + 3$

C $f(x) = \frac{-1}{x-2} - 3$ **D** $f(x) = \frac{-1}{x+2} + 3$

13 (2 marks) A hyperbola has asymptotes at $x = -2$ and $y = 4$ and a y-intercept of 2.
Write a possible equation for the hyperbola and verify your solution.

14 (2 marks) State the transformations made to the graph of $y = \frac{1}{x}$ to become $y = \frac{-0.9}{x+4} + 2.6$.

15 (2 marks) Sketch the graph of the function $f(x) = \frac{4}{3-x} + 5$.

Worksheets
Functions assignment

Functions problems

6.5 Modelling with relations and reciprocal functions

EXERCISE 6.5 Modelling with relations and reciprocal functions

ANSWERS p. 390

Recap

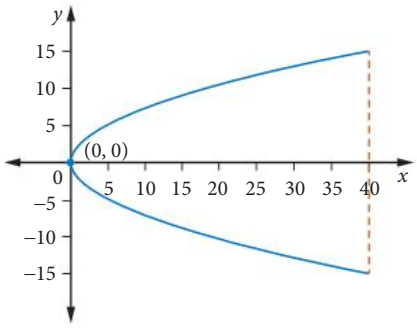
1 The asymptotes of the hyperbola $y = \frac{3}{x+2} + 4$ are

A $x = 2, y = 4$ **B** $x = -2, y = 4$ **C** $x = 4, y = -2$ **D** $x = 4, y = 2$

2 Sketch the graph of $f(x) = 2 - \frac{1}{x+1}$.

Exam practice

3 (3 marks) A student organising the spotlight to be used in a play notices that the reflector on the inside of the spotlight forms a sideways parabolic shape that focuses the light beam effectively when pointed directly horizontally at the stage. Assuming that the vertex of the reflector is located at (0, 0) and the depth is 40 cm and the diameter is 30 cm, as shown on the basic diagram of the spotlight, determine an equation for the spotlight reflector.

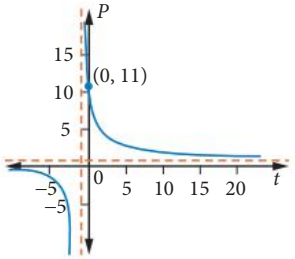


4 (3 marks) CF At a hockey competition, a player shoots for goal 7 m from the goal box. The path of the ball through the air can be modelled by the equation $y = \sqrt{x}$, where the shooting point is $x = 0$. The player scores a goal and it is assumed that there is no 'give' in the goal box net.

a Sketch a graph of the path of the ball through the air. *[2 marks]*

b State the domain of the shot. *[1 mark]*

5 (3 marks) CF An ecologist is studying the population (P) of a bird species in a forest over time in years (t). They collected data and created the following graph with asymptotes at $t = -1$ and $P = 1$.



a What is the initial population of birds? *[1 mark]*

b Create a mathematical model of the situation, stating any restriction on the domain and range. *[2 marks]*

6 (7 marks) CU A manufacturing company calculates the average cost ($\$C$) per item based on the number of items produced (x). This average cost can be modelled by the function

$$C(x) = \frac{200}{x-5} + 8$$

a What is the average cost when the company produces 100 items? *[1 mark]*

b Sketch the graph of the average cost related to the number of items produced. *[2 marks]*

c Identify the relevant domain and range, providing reasons to justify your answer. *[2 marks]*

d Explain what happens to the average cost as the number of items produced increases. *[1 mark]*

e Does the average cost per item have a minimum value in this context? *[1 mark]*

7 (4 marks) CU When a pebble is dropped into a pond it creates circular ripples on the surface of the pond. The first ripple can be represented by the equation $x^2 + y^2 = 9$. Five seconds later, the same ripple is represented by $x^2 + y^2 = 324$, where the measurements are in centimetres.

a What is the radius of each ripple? *[2 marks]*

b What is the speed at which the ripples are expanding outward? *[2 marks]*

8 (6 marks) CU A gardener wants to water a rectangular garden bed that is 6 m × 8 m. There is a tap located at the corner of the garden bed (0, 0) and he has a short hose with a sprinkler attached that luckily reaches the centre of the garden bed. The sprinkler head rotates, spraying water in a circular pattern. Find an equation to model the sprinkler's circular reach if the whole garden is watered and determine how much additional area to the garden will be watered.

EXAM QUESTION ANALYSIS

Exam-style question (3 marks)

The graph of $x = a(y - k)^2 + h$ passes through the point $(-5, 7)$ and has its vertex at $(-4, 3)$.

Determine the equation of the relation.

Reading the question

- The form of the equation to find is given.
- Key values are $(-5, 7)$ and vertex at $(-4, 3)$.

Thinking about the question

- Identify what variables have been given in the equation.
- Identify (h, k).
- Use the point provided to find a. Be careful with negative values.
- The complete equation must be stated.

Worked solution (✓ = 1 mark)

Known vertex: $(-4, 3) = (h, k)$

Use: $x = a(y - k)^2 + h$

$\boldsymbol{x = a(y - 3)^2 - 4}$ ✓

Known point $(-5, 7)$:

$-5 = a(7 - 3)^2 - 4$

$-5 = a(4^2) - 4$

$-1 = 16a$

$\boldsymbol{-\frac{1}{16} = a}$ ✓

The equation of the relation is $\boldsymbol{x = -\frac{1}{16}(y - 3)^2 - 4}$ ✓

Video Exam question analysis: Further functions and relations

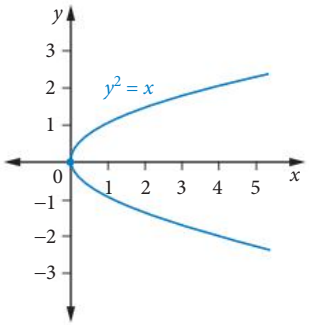
6 Chapter summary

Graphing circles

- The equation of a circle with radius r and centre $(0, 0)$ is $x^2 + y^2 = r^2$.
- The equation of a circle with radius r and centre (h, k) is $(x - h)^2 + (y - k)^2 = r^2$.
- It has the domain $[h - r, h + r]$ and range $[k - r, k + r]$.
- The equations for semicircles with radius r and centre at the origin are $y = \sqrt{r^2 - x^2}$ and $y = -\sqrt{r^2 - x^2}$.

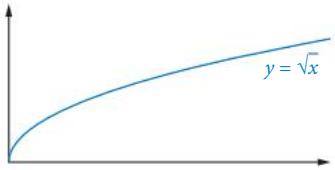
Graphing quadratic relations

- A **quadratic relation** in the form $x = a(y - y_1)(y - y_2)$ has y-intercepts at $y = y_1$ and $y = y_2$. The turning point (**vertex**) is located halfway between y_1 and y_2.
- A quadratic relation in the form $x = a(y - k)^2 + h$ has a vertex at (h, k).
- The horizontal axis of symmetry passes through the vertex and has the equation $y = k$.
- If $a > 0$, the graph opens to the right; if $a < 0$, the graph is reflected about the y-axis and opens to the left.
- If $a > 1$, the graph is dilated away from the y-axis (stretched horizontally); if $0 < a < 1$, it is dilated towards the y-axis (compressed horizontally).



Graphing square root functions

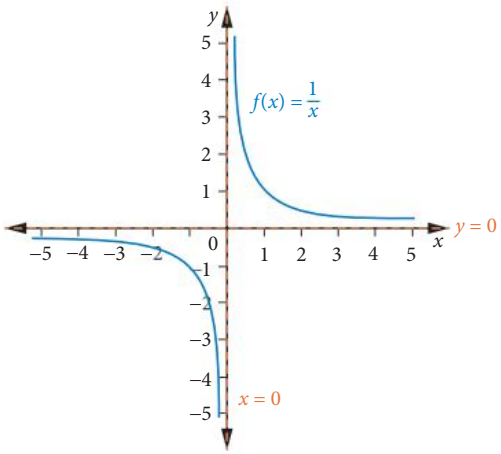
- The general form of the square root function is $y = a\sqrt{x - h} + k$, where (h, k) is the start of the graph, h is the horizontal translation, k is the vertical translation and a is the dilation factor.
- If $a > 1$, the graph is wider. If $0 < a < 1$, the graph is narrower. If $a < 0$, the graph is reflected in the x-axis.



Graphing hyperbolas

For the graph of $f(x) = \dfrac{1}{x}$

- the line $y = 0$ (the x-axis) is a **horizontal asymptote**
- the line $x = 0$ (y-axis) is a **vertical asymptote**
- each of the points $(-1, -1)$ and $(1, 1)$ is a **vertex**
- the domain is $R\backslash\{0\}$; the real numbers except 0
- the range is also $R\backslash\{0\}$.



For the graph of $y = \dfrac{a}{x - h} + k$

- the vertical asymptote is at $x = h$
- the horizontal asymptote is at $y = k$
- a is the dilation factor from the x-axis. If $a > 1$, the graph is stretched. If $0 < a < 1$, the graph is compressed. If $a < 0$, the graph is reflected in the x-axis.
- the domain is $R\backslash\{h\}$
- the range is $R\backslash\{k\}$.

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Cumulative examination 1

Technology-free

Perusal time: 2 minutes Working time: 30 minutes

TECH-FREE *Calculators are **not** permitted.*

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

Section 1 (4 marks): 4 multiple choice questions

Section 2 (14 marks): 4 short response questions

Total: 18 marks

Section 1 4 multiple choice questions

4 marks

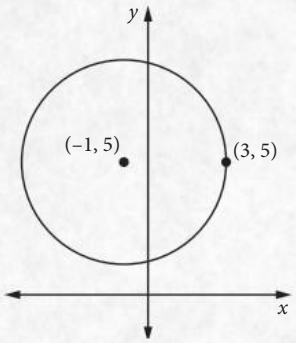
1 The equation of this circle is

A $(x-1)^2+(y-5)^2=16$

B $(x+1)^2+(y-5)^2=16$

C $(x-1)^2+(y-5)^2=4$

D $(x+1)^2+(y+5)^2=4$



2 Find the asymptotes of the graph of $f(x)=2+\frac{1}{2x}$.

A $x=2, y=0$ **B** $x=\frac{1}{2}, y=2$ **C** $x=0, y=-2$ **D** $x=0, y=2$

3 In a survey conducted at a music festival, it was found that of 200 attendees, 50 preferred pop music. What is the relative frequency of attendees who prefer pop music?

A 0.20 **B** 0.25 **C** 0.40 **D** 0.50

4 An angle of 60° is equivalent to

A $\frac{\pi}{6}$ **B** $\frac{\pi}{4}$ **C** $\frac{\pi}{3}$ **D** $\frac{\pi}{2}$

Worksheet
Maths Methods Year 11 formula sheet

Section 2 4 short response questions

14 marks

5 (3 marks) What is the equation of the square root function that starts at the point $(-1, 4)$ and passes through $(-3, 3)$?

6 (4 marks) Sketch the function $f(x) = 1 + \dfrac{2}{x-3}$.

Label the intercepts with their coordinates and any asymptotes with the appropriate equation.

7 (3 marks) Simplify $\dfrac{\sin\left(\dfrac{5\pi}{2}\right)\cos\left(\dfrac{3\pi}{4}\right)\sin\left(\dfrac{\pi}{4}\right)}{\sqrt{3}\cos\left(\dfrac{5\pi}{6}\right)}$.

8 (4 marks) The change in water level (above sea level) of an inlet is described by the function

$$h(t) = 0.5\cos\left(\frac{\pi t}{2}\right) + 0.5$$

where h is in metres and t is the time in hours after midnight.

a Sketch the graph of the function in the interval $0 \leq t \leq 12$. *[2 marks]*

b At what times after midnight is the water level at its maximum? *[2 marks]*

Cumulative examination 2

Technology-active

Perusal time: 2 minutes Working time: 30 minutes

*Calculators are **permitted**.*

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

Section 1 (4 marks): 4 multiple choice questions

Section 2 (14 marks): 4 short response questions

Total: 18 marks

Worksheet
Maths Methods Year 11 formula sheet

Section 1 4 multiple choice questions

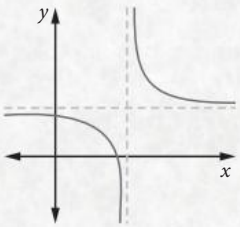
4 marks

1 What is the equation of a quadratic relation that has a vertex at $(2, 5)$ and passes through the point $(0, 6)$?

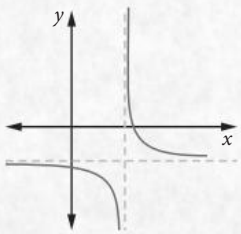
A $x = 2(y-2)^2 + 5$ **B** $x = -2(y-5)^2 + 2$ **C** $x = y(y-6)$ **D** $x = (y+1)^2$

2 Which of the following graphs most closely represents $y = \dfrac{1}{x-3} + 2$?

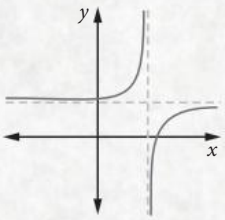
A



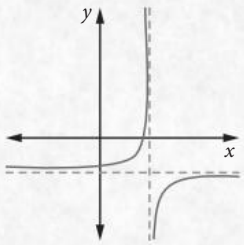
B



C



D



3 For two events, A and B, $P(A \cap B') = p$, $P(A) = 3p$ and $P(A \cap B) = 0.24$. The value of p is

A 0.48 **B** 0.36 **C** 0.24 **D** 0.12

4 The gradient of a line perpendicular to the line that passes through $(-2, 0)$ and $(0, -4)$ is

A -2 **B** $-\dfrac{1}{2}$ **C** $\dfrac{1}{2}$ **D** 4

Section 2 4 short response questions

14 marks

5 (2 marks) What transformations have been performed on $y = \sqrt{x}$ to obtain $y = -\frac{\sqrt{3x-1}}{2} + 3$?

6 (5 marks) The population of sloths in a particular forest varies according to the rule

$$n(t) = 1200 + 400\cos\left(\frac{\pi t}{3}\right)$$

where n is the number of sloths and t is the number of months after 1 March 2025.

a Find the period and amplitude of the function n. *[2 marks]*

b Find the maximum and minimum populations of sloths in this location. *[2 marks]*

c How many sloths will there be after 10 months? *[1 mark]*

7 (4 marks) In a talent show, there are 50 contestants, of whom 18 are magicians and 6 are performers who can both sing and perform magic tricks. All contestants either perform magic tricks or sing. Calculate the probability of randomly selecting a magician given that a singer is selected.

8 (3 marks) When a new species of plant is introduced to an ecosystem, it spreads rapidly, forming circular colonies on the forest floor. The first colony can be represented by the equation $x^2 + y^2 = 16$. Five months later, the same colony is represented by $x^2 + y^2 = 625$, where the measurements are in metres. What is the speed at which the colonies are expanding outwards?

CHAPTER

7 RATES OF CHANGE AND THE DERIVATIVE



Syllabus coverage
Nelson MindTap chapter resources
Terminology
7.1 Average rates of change
7.2 Instantaneous rates of change
7.3 Difference quotients
7.4 Limits
7.5 The derivative function
7.6 The derivative of x^n
7.7 The derivative of a polynomial
Exam question analysis
Chapter summary
Cumulative examination 1
Cumulative examination 2



Prior learning
Rates of change and the derivative

Syllabus coverage

UNIT 2, TOPIC 3: INTRODUCTION TO DIFFERENTIAL CALCULUS

Rates of change and the concept of derivatives

- Determine average rate of change in a variety of practical contexts.
- Use the rule $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ to determine the derivative of simple power functions and polynomial functions from first principles.
- Interpret the derivative as the instantaneous rate of change.
- Interpret the derivative as the gradient of a tangent line of the graph of $y = f(x)$
- Use the rule $\frac{d}{dx}x^n = nx^{n-1}$ for positive integers.
- Understand the concept of the derivative as a function.
- Recognise and use properties of the derivative $\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$
- Calculate derivatives of power and polynomial functions.

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Videos (3):

7.5 Differentiation from first principles
7.7 Displacement and velocity
Exam question analysis Rates of change and the derivative

Prior learning (1):

7.1 Rates of change and the derivative

Worksheets (12):

7.1 Rates of change 1 • Rates of change – Gradients of secants
7.2 Instantaneous rates of change
7.4 Limits
7.5 Finding derivatives from first principles • Differentiation from first principles
7.7 Gradient functions • Derivatives of polynomials • Derivatives of linear products • Derivatives of a sum of terms • Basic differentiation
Cumulative exams Maths Methods Year 11 formula sheet

TI-Nspire CX II instructions (1):

7.1, 7.2, 7.4 Rates of change and the derivative



Nelson MindTap

To access resources above, visit **cengage.com.au/nelsonmindtap**

Terminology

acceleration	average rate of change	average speed	average velocity
derivative	difference quotient	displacement	distance
first principles	flow rate	gradient	gradient function
instantaneous rate of change	limit	polynomial function	power rule
power term	rate	rate of change	secant
speed	tangent	velocity	

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7.1 Average rates of change

A **rate** is a ratio of 2 related quantities.

You see rates in everyday life such as $/kg, km/h, c/L.

Rate of change

The **rate of change** of a quantity with respect to a related quantity is given by

$$\text{rate of change} = \frac{\text{change in quantity 2}}{\text{change in quantity 1}}$$

The Greek capital letter Δ (delta) is used to represent the change in a quantity.

The rate of change of y wrt x is given by $\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$.

'with respect to' is often abbreviated to wrt.

Worksheets
Rates of change 1
Rates of change – Gradients of secants

The rate of change has the same formula as the **gradient** of a straight line, $m = \frac{y_2 - y_1}{x_2 - x_1}$.
The gradient is the rate of change in y values wrt x values.

WORKED EXAMPLE 1 Rate of change

Five people go to a market to buy strawberries. The mass of strawberries each person buys (in kg) and the cost (in $) is outlined in the table below.

Mass (kg)	0.365	0.122	0.439	0.512	0.222
Cost ($)	6.38	2.13	7.68	8.95	3.88

a **Calculate** the rate of change of cost wrt mass ($/kg) by comparing 2 strawberry purchases and **comment** on the results.

b Plot the data to show the relationship between cost and mass.

Steps

a Determine the change in quantity for each successive pair of strawberry purchases.

This rate is cost per mass.

State the rate of change.

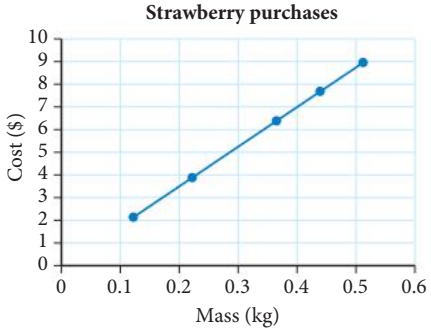
Comment on results.

Working

Mass (kg)	0.365	0.122	0.439	0.512	0.222
Cost ($)	6.38	2.13	7.68	8.95	3.88
Δ$		−4.25	5.55	1.27	−5.07
Δkg		−0.243	0.317	0.073	−0.290
RoC		17.490	17.508	17.397	17.483

Rates of change (RoC): 17.490, 17.508, 17.397 and 17.483. The rates of change are all approximately the same. Variation could be due to rounding of costs.

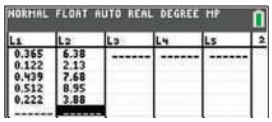
b Plot the points and draw a line of best fit.



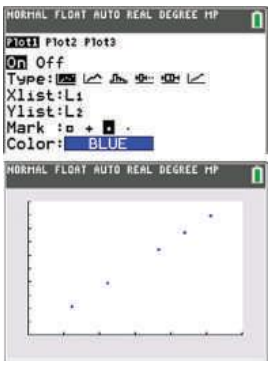
This worked example can also be completed using a graphics calculator.

TI-84 Plus CE

a



1. Press stat, then **1** to choose **1: Edit**.
2. Put the masses in L1, move across to L2 and put in the costs.



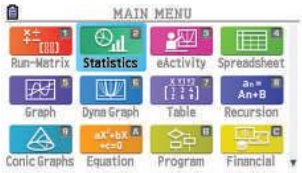
b

1. Press 2nd y= ([stat plot]) and **1:Plot1**.
2. Turn Plot1 On, move down, make the Type a scatterplot, XList: L1 (2nd 1) and YList: L2.
3. Set the window appropriately ($0 \le x \le 0.6$, $0 \le y \le 10$).
4. Make sure there is nothing in Y= and press graph.

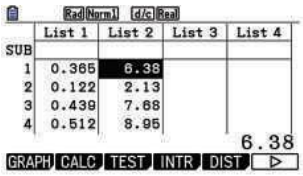


5. The graph looks linear, so fit a linear regression line.
6. Press stat, move across to CALC and press **4**: **LinReg(ax+b)**.
7. Set to XList, L1, YList: L2, Clear FreqList and choose, say Y1 (ALPHA trace 1) to store the equation.
8. Move down and press ENTER to calculate the line of best fit.

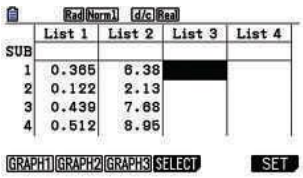
Casio fx-CG50 AU



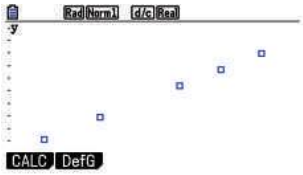
1. From the **MAIN MENU**, select **Statistics**.



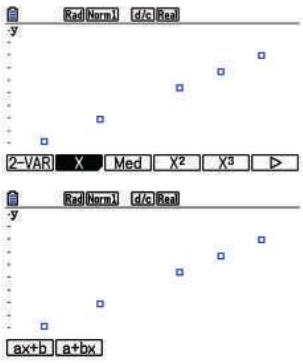
2. Enter the data, with Mass in the List 1 column, and Cost in the List 2 column.



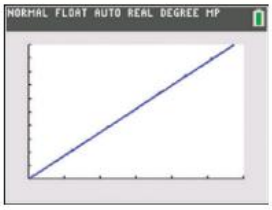
1. Press F1 (GRAPH) then F1 (GRAPH1) (or select the function key that you have previously set up for scatterplots).

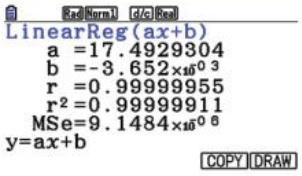


2. The relationship is approx. linear, so press F1 CALC to determine the regression relationship. Press F2 (X) to select linear regression, then Press F1 (ax+b) or F2 (a+bx) for the form of the equation you prefer.



9780170484909





9 Press graph again to see the line of best fit shown by Y1.

The r^2 value being close to 1 indicates that this is close to a perfect relationship.

Rates of change with respect to time

- **Displacement**: change in position from a starting point to an end point
- **Velocity**: rate of change of displacement of a moving object
- **Average velocity**: $\dfrac{\text{total displacement}}{\text{time taken}}$
- **Acceleration**: rate of change of velocity of a moving object

Displacement, velocity and acceleration have both a magnitude and direction, and hence can be positive or negative.

- **Distance**: length of journey from a starting point to an end point, not necessarily the same as displacement (which is the shortest distance between the 2 points)
- **Speed**: rate of change of distance of a moving object
- **Average speed**: $\dfrac{\text{total distance}}{\text{time taken}}$

Distance and speed only have a magnitude and hence can only be positive.

- **Flow rate**: rate of change of volume of a fluid

WORKED EXAMPLE 2 Average rate of change 1

The graph shows the distance of a cyclist from their home at various times on a trip to a store and back. The store is 15 km from the cyclist's home.

a During what times did the cyclist have the greatest speed?

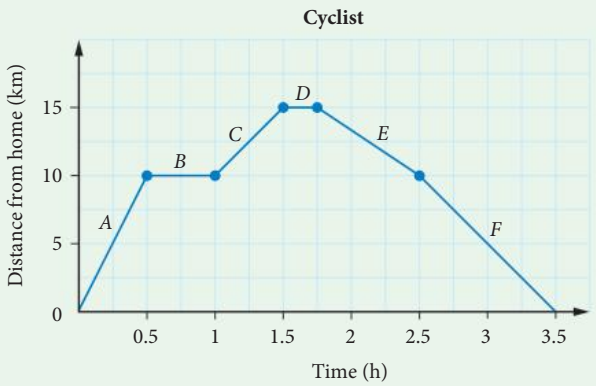
b **Calculate** the average speed for the trip to the store.

c **Identify** any times when the cyclist was stopped.

d **Calculate** the average speed for section *F*.

e **Calculate** the average speed for the whole trip.

f **Calculate** the average velocity for the whole trip.



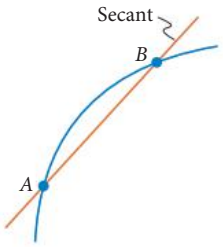
Steps	Working
a **1** Interpret the question.	Greatest speed means highest gradient, regardless of whether it is positive or negative.
2 Identify highest gradient sections.	Highest positive gradient is section *A* (20 km/h), while highest negative gradient is section *F* (10 km/h).
3 State the solution.	Therefore, the cyclist is fastest in the first half hour.

b **1**	State the formula.	$\text{AROC} = \frac{d_2 - d_1}{t_2 - t_1}$
2	Substitute in values.	$= \frac{15 - 0}{1.5 - 0}$
		$= 10$
3	State the solution with units.	Therefore, the average speed for the trip to the store was 10 km/h.
c **1**	Interpret the question.	The cyclist is stopped when AROC is 0.
2	State the solution.	This occurs between $t = 0.5$ and $t = 1$, as well as $t = 1.5$ and $t = 1.75$.
d **1**	State the formula.	$\text{AROC} = \frac{d_2 - d_1}{t_2 - t_1}$
2	Substitute in values.	$= \frac{10 - 0}{2.5 - 3.5}$
		$= \frac{10}{-1}$
		$= -10$
3	State the solution with units.	Therefore, the average speed for section F is 10 km/h. (Note: the negative indicates the direction the cyclist is travelling, which is back home.)
e **1**	State the formula.	$\text{average speed} = \frac{\text{total distance}}{\text{time taken}}$
2	Substitute in values.	$\text{average speed} = \frac{30}{3.5} = 8.57$
3	State the solution with units.	Therefore, the average speed for the trip is 8.57 km/h.
f **1**	State the formula.	$\text{average velocity} = \frac{\text{total displacement}}{\text{time taken}}$
2	Substitute in values.	$= \frac{0}{3.5}$
		$= 0$
3	State the solution with units.	Therefore, the average velocity for the trip is 0 km/h.

AROC stands for average rate of change.

For the graph of a function, the **average rate of change** is calculated using $\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$, where (x_1, y_1) and (x_2, y_2) are points on the graph.

You are calculating the gradient of a **secant**, a straight line that passes through the 2 points.



WORKED EXAMPLE 3 Average rate of change 2

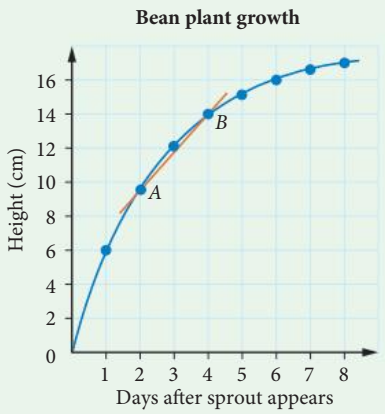
The graph shows the average height of bean plants measured as they grow. **Use** the graph to calculate the average rate of change for

a the first two days

b the next two days

c days 5 and 6.

Evaluate how the rate of change changes over time.



Steps	Working
a 1 State the formula.	$\text{ARoC} = \frac{h_2 - h_1}{d_2 - d_1}$
2 Substitute in values.	$= \frac{9.5 - 0}{2 - 0}$
3 State the solution with units.	$= 4.75$ cm/day
b 1 Substitute in values.	$\text{ARoC} = \frac{14 - 9.5}{4 - 2}$
2 State the solution with units.	$= 2.25$ cm/day
c 1 Substitute in values.	$\text{ARoC} = \frac{16 - 15}{6 - 5}$
2 State the solution with units.	$= 1$ cm/day
3 Comment on change in the average rate of change over time.	Although the plants are still increasing in height, the rate of change is slowing down. Therefore, the overall plant height is increasing at a decreasing rate.

EXERCISE 7.1 Average rates of change

ANSWERS p. 390

Mastery

1 WORKED EXAMPLE 1 This table shows how the length of a spring changes as different masses are hung on it.

Mass (g)	30	50	80	120	140	150
Length (cm)	15	16.8	18	19.8	22.2	23.4

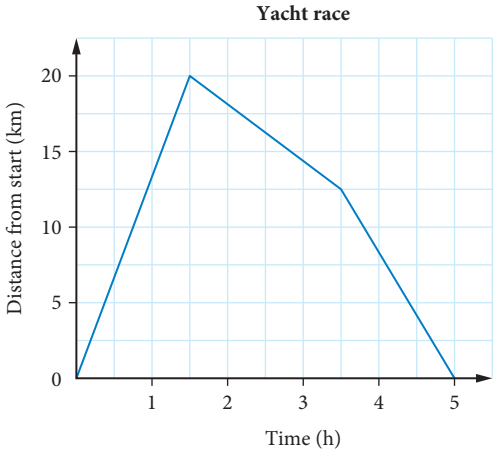
a Draw a graph to show the information.

b Comment on the shape of the graph.

c What is the average rate of change of length with respect to mass?

d What is the length for a mass of 0 g, and what does it represent?

2 WORKED EXAMPLE 2 The graph shows the distance of a yacht from the starting line in a race with 3 legs (sections).



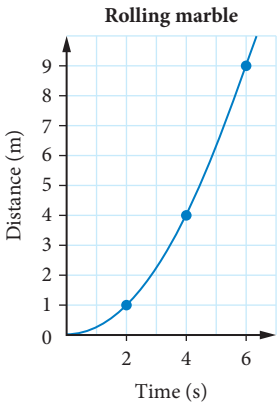
a What is the fastest speed shown on the graph?

b On which leg would the spinnaker (sail) be most likely to be used (the fastest leg, with the wind behind)?

c Which leg is the beat to windward (the slowest leg, with the most tacking needed)?

d What is the average speed of the yacht shown on the graph?

3 WORKED EXAMPLE 3 The distance a marble rolls down a long plank at an angle of 5° is shown on the graph.



Find the average rate of change of distance (speed) from

a 0 s to 2 s **b** 2 s to 4 s

c 4 s to 6 s **d** 0 s to 6 s

4 Express each statement as a rate.

a It costs $36.75 for 15 kg of potatoes.

b A wheel turns 180 times in 4 minutes.

c A car travels 3000 m in 100 s.

d An acceleration of 50 m/s^2 is produced by a force of 8 N (newtons).

e A$800 is worth the same as US$640.

5 A GPS shows the distance and time that it should take to drive a car between Bundaberg and Mackay via Gladstone and Rockhampton. The distance is 334 km and the time given is 3 hours 40 minutes. What average driving speed has been assumed?

6 A car travelling at 30 m/s increases its speed to 50 m/s in 5 s. What is the average rate of change of speed?

7 Use a numerical method to determine whether the quantities in the table are related by a fixed rate.

m	10	20	50	100	250
n	1.5	3	7.5	15	37.5

8 Use a numerical method to determine whether the quantities in the table are related by a fixed rate.

r	0.125	0.5	1.25	1.875	2.25
D	0.5	2.0	5.0	7.5	9.0

9 An air-conditioning system in a car has a small leak. The temperature in the car, with the air-conditioning on full, gradually increases, as shown in the table. During this time, the external temperature remains at about 28°C.

Time	11 am	11:20 am	11:40 am	12 pm	12:20 pm	12:40 pm	1 pm	1:20 pm
Temp (°C)	18.0	19.8	21.3	22.5	23.5	24.3	25.0	25.5

a Draw a graph.

b Comment on the rate of change of the temperature in the car.

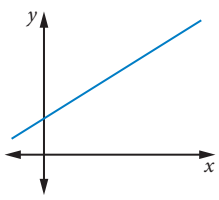
c What is the average rate of change of temperature during this time?

d Will the air-conditioning be of any use at 3 pm?

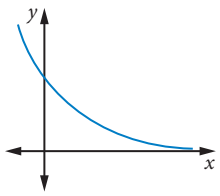
e Will it be of any use the next day?

10 Each graph models a rate-of-change situation. Describe each in terms of the change occurring and the rate of change.

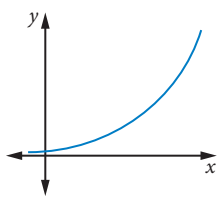
a



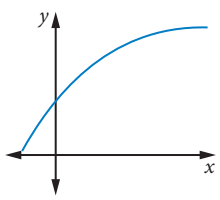
b



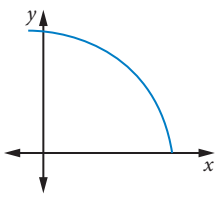
c



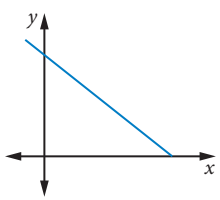
d



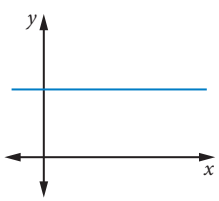
e



f



g



11 The function $f(x) = \ln(x)$ is shown.

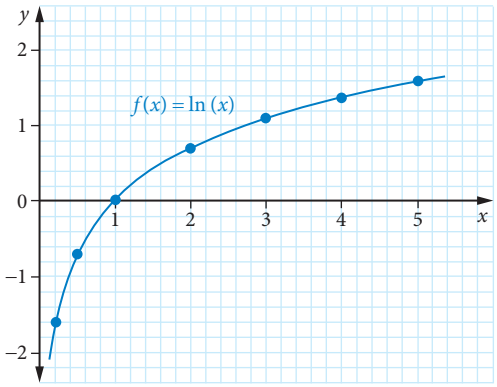
Use the graph to find the average rate of change of $\ln(x)$ from

a $x = 0.5$ to $x = 3$

b $x = 1$ to $x = 4$

c $x = 1$ to $x = 5$

d $x = 0.2$ to $x = 3$



Exam practice

12 (3 marks) The following information was obtained by changing the amount of watering of young tomato plants and observing their growth over 3 weeks. Use a graph to determine whether growth is related to water by a constant rate.

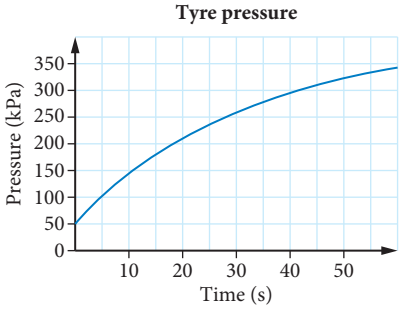
Water per day (mL)	20	50	100	200	300
Growth (cm)	5	20	35	60	65

13 (3 marks) A group of students made a simple balance with a pan on one side of the fulcrum and a sliding weight on the other. They put different masses in the pan and moved the sliding weight to make it balance. The distance between the sliding weight and the fulcrum was measured. Their results are given in the table. Determine whether these measurements are consistent with a constant rate relationship.

Mass in pan (g)	50	100	200	250	400
Distance (cm)	5	12	27	29	53

14 (4 marks) The pressure in a bicycle tyre as it is being inflated is shown in the graph.

Describe how the rate of change of pressure changes as time increases. Use average rates of change to support your claim.



15 (4 marks) CF A plane travels from Sydney to Brisbane (about 600 km) in 1 hour. It takes 1 hour to get from the Sydney central business district (CBD) to the airport (about 20 km) and about 15 minutes to get from Brisbane airport to the Brisbane CBD (about 15 km). Check-in at the airport takes about 40 minutes, and it takes about 20 minutes to disembark and collect luggage.

Calculate the difference between the average speed of the plane and the average speed for the entire journey from Sydney CBD to Brisbane CBD.

7.2 Instantaneous rates of change

Instantaneous rate of change

A **tangent** is a straight line that touches the graph of a function at one point. The **instantaneous rate of change** of a function at a point on its graph is the gradient (**slope**) of the tangent at that point.

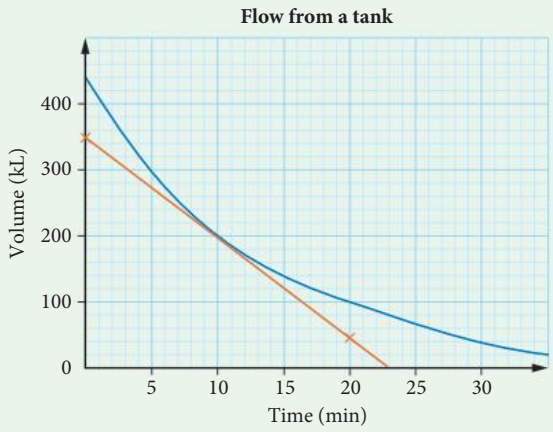


Worksheet
Instantaneous rates of change

WORKED EXAMPLE 4 Instantaneous rate of change

A valve on a tank of water is opened to drain its contents.

Use the tangent to the graph shown here to find the flow rate 10 minutes after the valve is opened.



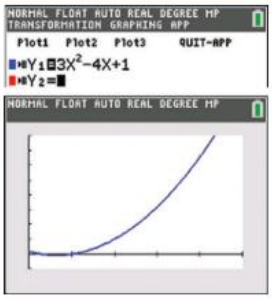
Steps	Working
1 State the formula for gradient.	$m = \frac{y_2 - y_1}{x_2 - x_1}$
2 State the coordinates of points that sit on the tangent line, marked with a cross on the diagram	(20, 50) and (0, 350)
3 Substitute points into the gradient formula and solve.	$m = \frac{50 - 350}{20 - 0}$ $= \frac{-300}{20}$ $= -15$
4 Interpret solution.	Water is draining from the tank at a flow rate of 15 kL/min.

TI-Nspire
Rates of change and the derivative

TECHNOLOGY Instantaneous rates of change

Determine the instantaneous rate of change of the function $y = 3x^2 - 4x + 1$ at $x = 3$.

TI-84 Plus CE

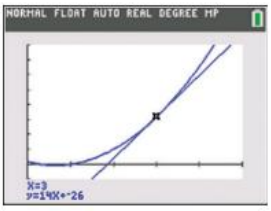


1 Press 2nd y= ([stat plot]) and sure all Plots are off.

2 Press y= and enter the function in Y1.

3 Set the window appropriately ($0 \le x \le 5$, $-5 \le y \le 40$). Use (–) for negative.

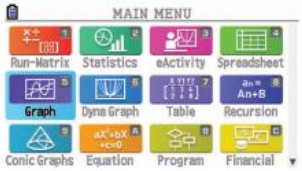
4 Press graph.



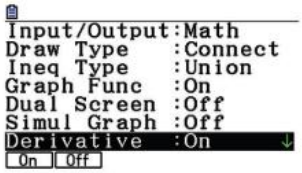
5 Press 2nd prgm ([draw]), and choose 5: Tangent(and 3 to draw the tangent at $x = 3$.

The tangent line is $y = 14x - 26$. The gradient, 14, is the instantaneous rate of change at $x = 3$.

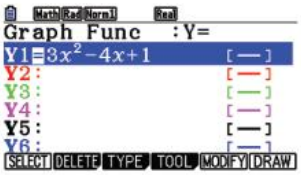
Casio fx-CG50 AU



1 From the **MAIN MENU**, press **5** for **Graph**.

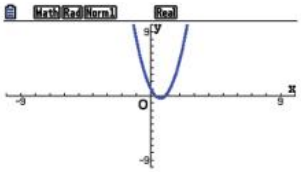


2 Press SHIFT then MENU to open the SET UP screen. Scroll down to Derivative and press F1 (On) to ensure it is turned On. Press EXIT to return to the Graph screen.

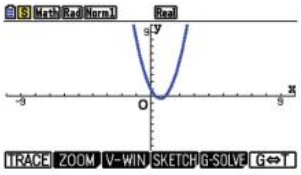


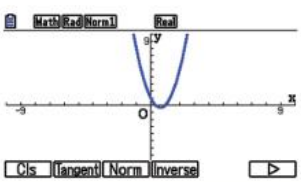
3 Type in the RHS of the function into Y1, then EXE. Press the up arrow to highlight the function.

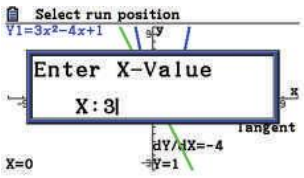
4 Press F6 (DRAW) to graph the function.
(Press SHIFT then F3 **V-Window** to alter the View Window, if required.)



5 To sketch the tangent, press SHIFT then F4 (SKETCH), then F2 (Tangent).

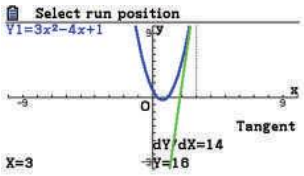






6 Press X, θ, T. The screen above will appear with the value.

7 Type the x value in and press EXE.



The dY/dX value represents the IRoC at the x value entered.

EXERCISE 7.2 Instantaneous rates of change

ANSWERS p. 391

Recap

1 A car trip from Warwick to Brisbane takes 2 h 20 min and covers 160 km. The average speed for the trip is

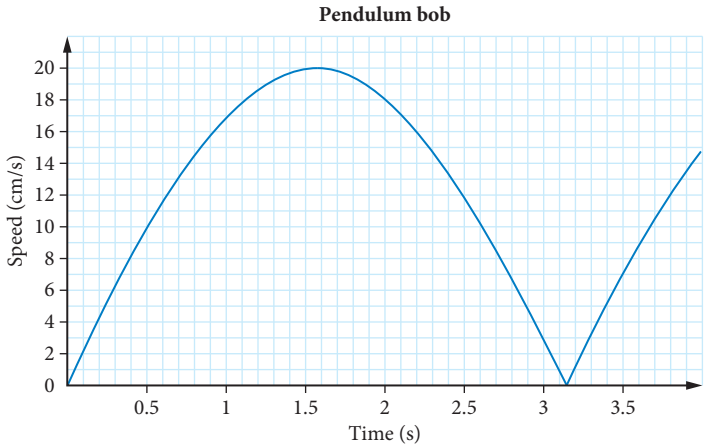
A 49.5 km/h **B** 52.5 km/h **C** 68.57 km/h **D** 72.73 km/h

2 The temperature of a water on a stove top is measured. 35 seconds after the water started heating up, the temperature was 50°C, while 75 seconds after the water started heating, the temperature was 88°C. The average rate of temperature change is

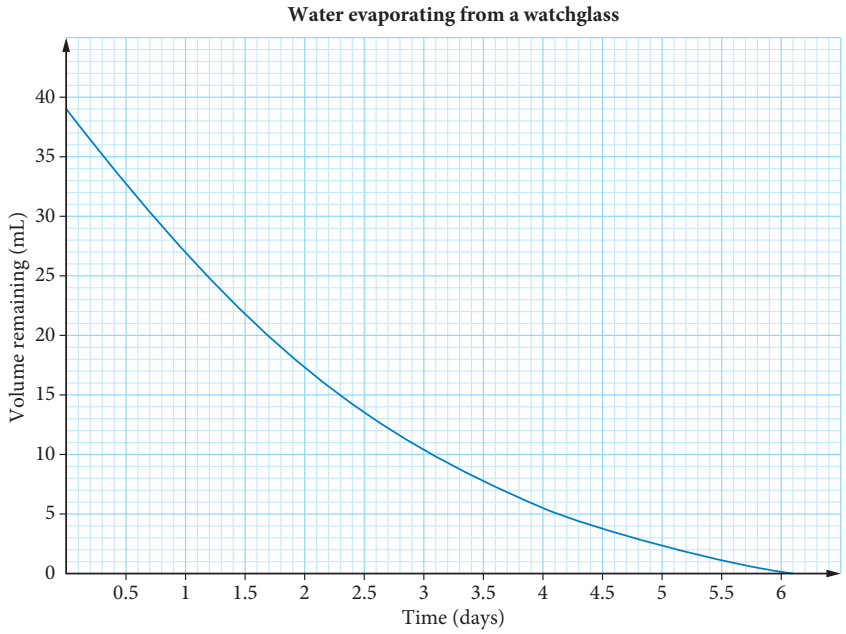
A 0.95° C/min **B** 1.05° C/min **C** 57° C/min **D** 63.16° C/min

Mastery

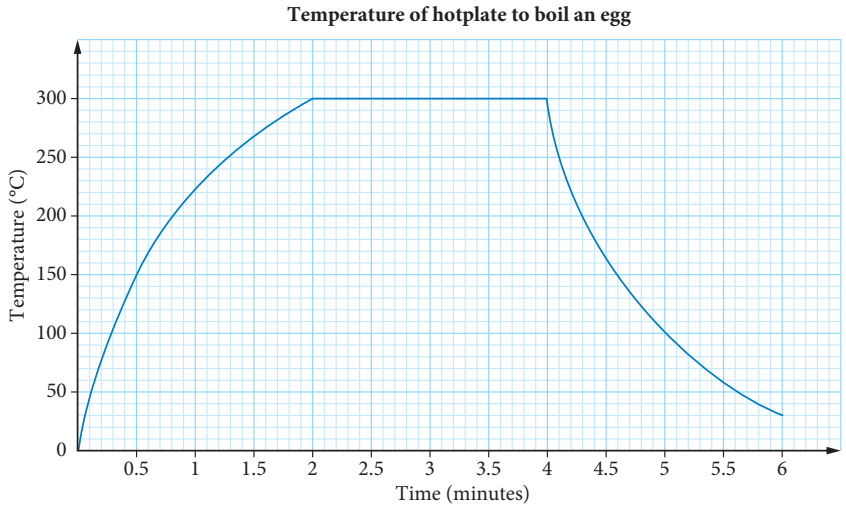
3 WORKED EXAMPLE 4 This graph shows the speed of the bob of a long pendulum. Use tangents to find the rate of change of speed of the bob at 1 s, 2 s and 3 s.



4 The amount of water left in a watchglass is shown in the graph. Use tangents to find the rate of evaporation at 1 day and 4 days.



5 The temperature of a hot plate used to boil an egg is shown in the graph. Use tangents to find the rate of change of temperature at 1 minute, 3 minutes and 5 minutes.



Exam practice

6 TECH-FREE (3 marks)

a What is the slope of the graph $f(x) = 3 - 2x$?

b What is the slope of the tangent at $x = 5$?

c What is the slope of the tangent at other points?

7 TECH-FREE (3 marks) Plot the graph of $f(x) = 2x^2 + x - 4$ for $-2 \leq x \leq 3$. Use a tangent to find the gradient at

a $x = -1$ **b** $x = 1$ **c** $x = 2$

8 TECH-FREE (3 marks) Plot the graph of $f(x) = 5 - 3x - x^2$ for $-3 \leq x \leq 4$. Use a tangent to find the gradient at

a $x = -2$ **b** $x = 0$ **c** $x = 3$

9 (3 marks) Use a graphics calculator to draw the graph of $f(x) = x^3 - 6$ for $-2 \leq x \leq 2$. Use a tangent to find the gradient at

a $x = -1$ **b** $x = 0$ **c** $x = 1$

10 (3 marks) Use a graphics calculator to draw the graph of $f(x) = (x + 4)(x - 3)$ for $-5 \leq x \leq 4$. Use a tangent to find the gradient at

a $x = -3$ **b** $x = -1$ **c** $x = 1$

11 TECH-FREE (5 marks) CF What is the rate of change of $y = x^2 - 2x + 3$ at $x = 4$?

12 TECH-FREE (7 marks) CF The distance travelled by a drag-racing car in the first seconds after taking off is given by the equation $d = 5t - t^3 + 8t^2$, where d is in metres and t is in seconds. Use a graph to find the velocity at 2 seconds and the time taken to travel 100 m. What is the velocity at 100 m?

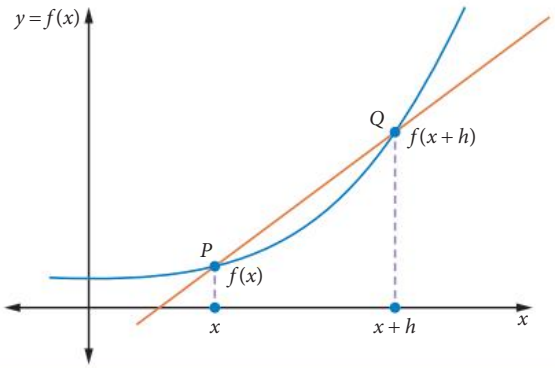
7.3 Difference quotients

Consider 2 points on the graph of $y = f(x)$ that are close together, $P(x, f(x))$ and $Q(x + h, f(x + h))$, where the difference in x-coordinates, h, is small.

Now consider the gradient of the secant (line) that goes through these 2 points. This gradient is called the **difference quotient** and is a measure of the average rate of change between the points.

The difference quotient between the points P and Q is found using the gradient formula:

$$\frac{\Delta y}{\Delta x} = \frac{f(x + h) - f(x)}{(x + h) - x} = \frac{f(x + h) - f(x)}{h}$$



Difference quotient

The difference quotient is an average rate of change of the form

$$\frac{\Delta y}{\Delta x} = \frac{f(x + h) - f(x)}{h}$$

WORKED EXAMPLE 5 Difference quotients

Calculate the difference quotient for the function $f(x) = 2x + 1$.

Steps	Working
1 Substitute $f(x)$ and $f(x + h)$ into the difference quotient.	$\frac{\Delta y}{\Delta x} = \frac{f(x + h) - f(x)}{h}$ $= \frac{2(x + h) + 1 - (2x + 1)}{h}$
2 Expand brackets.	$= \frac{2x + 2h + 1 - 2x - 1}{h}$
3 Simplify by collecting like terms	$= \frac{2h}{h}$
4 Cancel the h in the numerator with the h in the denominator.	$= 2$

This makes sense because the gradient of the line $y = 2x + 1$ is 2.

Exam hack

Avoid these common student errors when determining difference quotients:

- $f(x + h) \neq f(x) + h$
- $(x + h)^2 \neq x^2 + h^2$ Use FOIL to expand the brackets and obtain $(x + h)^2 = x^2 + 2xh + h^2$.
- $\frac{3h - 2}{h} \neq \frac{3 - 2}{1}$ h must be a common factor of the entire numerator to cancel.

WORKED EXAMPLE 6 Difference quotients

Calculate the difference quotient for the function $f(x) = x^2 + 2x + 1$.

Steps	Working
1 Substitute $f(x)$ and $f(x + h)$ into the difference quotient.	$\frac{\Delta y}{\Delta x} = \frac{f(x + h) - f(x)}{h}$ $= \frac{\left((x + h)^2 + 2(x + h) + 1\right) - \left(x^2 + 2x + 1\right)}{h}$
2 Expand brackets.	$= \frac{\left(x^2 + 2hx + h^2 + 2x + 2h + 1\right) - \left(x^2 + 2x + 1\right)}{h}$ $= \frac{x^2 + 2hx + h^2 + 2x + 2h + 1 - x^2 - 2x - 1}{h}$
3 Simplify.	$= \frac{2hx + h^2 + 2h}{h}$
4 Take h out as a common factor in the numerator and cancel it with the h in the denominator.	$= \frac{h(2x + h + 2)}{h}$ $= 2x + h + 2$

You can use difference quotients to see what happens as the 2 points get closer together, and the secant becomes closer to a tangent. This is done by reducing the value of h to values closer to 0.

WORKED EXAMPLE 7 Analysing difference quotients as h gets smaller

Using the difference quotient from the above example, **determine** the values of the quotient when

a $h = 1$ **b** $h = 0.1$

c $h = 0.01$ **d** $h = 0.001$

Describe what appears to be happening.

Steps	Working
a Substitute $h = 1$ into the difference quotient.	$\frac{\Delta y}{\Delta x} = 2x + h + 2$ $= 2x + 1 + 2$ $= 2x + 3$
b Substitute $h = 0.1$ into the difference quotient.	$\frac{\Delta y}{\Delta x} = 2x + h + 2$ $= 2x + 0.1 + 2$ $= 2x + 2.1$
c Substitute $h = 0.01$ into the difference quotient.	$\frac{\Delta y}{\Delta x} = 2x + h + 2$ $= 2x + 0.01 + 2$ $= 2x + 2.01$

d Substitute $h = 0.001$ into the difference quotient.

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= 2x + h + 2\\ &= 2x + 0.001 + 2\\ &= 2x + 2.001\end{aligned}$$

Look for patterns in the solutions to each part.

In each of the difference quotients, the $2x$ term remains the same. The constant term starts out high but appears to be approaching a value of 2 as the h value gets smaller.

EXERCISE 7.3 Difference quotients

ANSWERS p. 392

Recap

1 The instantaneous rate of change of $y = x^2$ at $x = 2$ is

A -4 **B** 2 **C** 4 **D** $2x$

2 The instantaneous rate of change of $y = 5x - 6$ at $x = 1$ is

A -6 **B** 1 **C** 1.2 **D** 5

Mastery

3 WORKED EXAMPLE 5 Given $f(x) = 4x + 1$, simplify each expression.

a $f(x + h)$ **b** $f(x + h) - f(x)$ **c** $\dfrac{f(x + h) - f(x)}{h}$

4 Given $f(x) = 7 - 3x$, simplify each expression.

a $f(x + h)$ **b** $f(x + h) - f(x)$ **c** $\dfrac{f(x + h) - f(x)}{h}$

5 WORKED EXAMPLE 6 Given $f(x) = x^2 - 5x$, simplify each expression.

a $f(x + h)$ **b** $f(x + h) - f(x)$ **c** $\dfrac{f(x + h) - f(x)}{h}$

6 Calculate the difference quotient for each function.

a $2x^2 - 4x + 3$ **b** $4x^2 + 3x - 2$ **c** $8 - \frac{1}{2}x^2$

7 WORKED EXAMPLE 7 Use the answer to question **6c** to determine the value of the quotient when

a $h = 1$ **b** $h = 0.1$ **c** $h = 0.01$ **d** $h = 0.001$

Describe what is happening to the value as $h \to 0$.

8 Use the answer to Question **6a** to determine the value of the quotient when:

a $h = 1$ **b** $h = 0.1$ **c** $h = 0.01$ **d** $h = 0.001$

Describe what is happening to the value as $h \to 0$.

9 Calculate the difference quotient for each expression.

a $2x + 7$ **b** $8 - 5x$ **c** $6x - 4$

10 Given $f(x) = 3x^2$, simplify each expression.

a $f(x + h)$ **b** $f(x + h) - f(x)$ **c** $\dfrac{f(x + h) - f(x)}{h}$

11 Given $f(x) = x^2 - 5x$, simplify each expression.

a $f(x + h)$ **b** $f(x + h) - f(x)$ **c** $\dfrac{f(x + h) - f(x)}{h}$

Exam practice TECH-FREE

12 (3 marks) Calculate the difference quotient for each expression.

a $6x^2 - 2$ **b** $4x - 4x^2$ **c** $\frac{x^2}{3} + \frac{1}{4}x$

13 (3 marks) Calculate the difference quotient for $f(x) = x^3 - x - 9$.

14 (3 marks) CF Given $f(x) = \frac{1}{x}$, simplify each expression.

a $f(x + h)$ **b** $f(x + h) - f(x)$ **c** $\frac{f(x + h) - f(x)}{h}$

15 (3 marks) CF Calculate the difference quotient for each expression.

a $\frac{1}{x + 1}$ **b** $\frac{3}{x - 2}$ **c** $\frac{1}{3 - 2x}$

16 (3 marks) CF Calculate the difference quotient for $f(x) = \frac{x - 3}{5 - x}$.

17 (3 marks) CF Calculate the difference quotient for $f(x) = \frac{(x - 8)^2}{x}$.

18 (3 marks) CU Show that the difference quotient for $f(x) = \sqrt{x^2 - 1}$ is given by $\frac{2x + h}{\sqrt{(x + h)^2 - 1} + \sqrt{x^2 - 1}}$.

Worksheet
Limits

7.4 Limits

The limit of a function

The **limit of a function** $f(x)$ is the value that the function approaches as the value of x approaches a value $x = c$.

A limit is shown by $\lim\limits_{x \to c} f(x)$, where $\to$ represents the phrase 'as it approaches'.

You can approach a limit from the right or left. You can show this as:

- $\lim\limits_{x \to c^-} f(x)$ means the limit of $f(x)$ as you approach c from the negative direction (the left)
- $\lim\limits_{x \to c^+} f(x)$ means the limit of $f(x)$ as you approach c from the positive direction (the right)
- $\lim\limits_{x \to c} f(x)$ means from either direction so $\lim\limits_{x \to c^-} f(x) = \lim\limits_{x \to c^+} f(x)$.

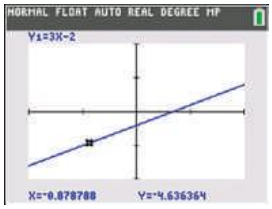
WORKED EXAMPLE 8 The limit of a linear function

Determine $\lim\limits_{x \to 0} f(x)$ for $f(x) = 3x - 2$.

Steps	Working
1 Substitute values of x that get closer to $x = 0$ from the right.	x: 1, 0.1, 0.01, 0.001 $f(x)$: 1, −1.7, −1.97, −1.997 As $x \to 0^+$, $f(x) \to -2$.
2 Substitute values of x that get closer to $x = 0$ from the left.	x: −1, −0.1, −0.01, −0.001 $f(x)$: −5, −2.3, −2.03, −2.003 As $x \to 0^-$, $f(x) \to -2$.
3 As x approaches 0 from both sides, $f(x)$ approaches −2, so this is the limit.	$\lim\limits_{x \to 0} f(x) = -2$ ← Notice that the limit is the same as the value of $f(0)$ but this isn't always the case.

This worked example can also be completed using technology.

TI-84 Plus CE



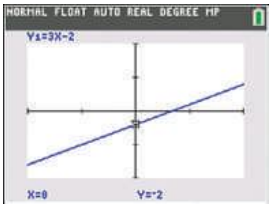
1 Press **y=** and enter the function in Y1.

2 Set the **window** appropriately ($-2 \leq x \leq 2$, $-10 \leq y \leq 10$). Use (–) for negative.

3 Press **graph**.

4 Press **trace**.

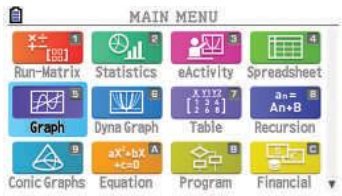
5 Press the left-hand cursor key repeatedly so the cross is below zero.



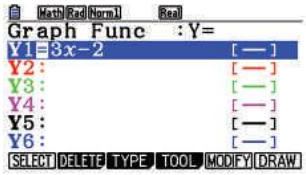
6 Pressing the right-hand cursor key moves the cross along the graph. The values of x and y are displayed.

As $x \to 0$, $y \to -2$, so $\lim_{x \to 0} f(x) = -2$.

Casio fx-CG50 AU



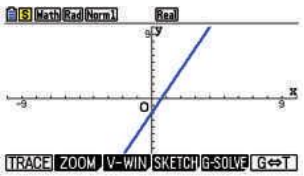
1 From the **MAIN MENU**, press **5** for **Graph**.

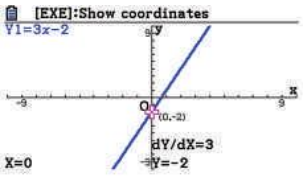


2 Type in the RHS of the function into Y1, then **EXE**. Press the up arrow to highlight the function.

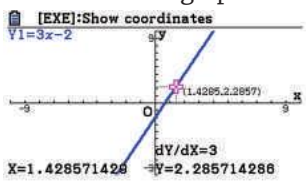
3 Press **F6** (DRAW) to graph the function.

(Press **SHIFT** then **F3** **V-Window** to alter the View Window, if required.)





4 Press **SHIFT** then **F1** (TRACE) to place the tracer onto the graph.



5 Pressing the right arrow repeatedly will increase the x value, while pressing the left arrow repeatedly will decrease the x value.

(Although specific x values from the table might not be possible, the limit can still be read as the points move towards given x value.)

TI-Nspire
Rates of change and the derivative

WORKED EXAMPLE 9 The limit of a function at a discontinuity

For the function $f(x) = \dfrac{x}{\sqrt{x+1}-1}$

a **justify** that a discontinuity occurs at $x = 0$

b **use** a table of values to find $\lim\limits_{x \to 0} f(x)$.

Steps	Working
a Substitute $x = 0$ into $f(x)$ and simplify.	$f(0) = \dfrac{0}{\sqrt{0+1}-1}$ $= \dfrac{0}{\sqrt{1}-1}$ $= \dfrac{0}{1-1}$ $= \dfrac{0}{0}$ which is undefined.
b **1** Substitute values of x that get closer to $x = 0$ from the right.	x: 1, 0.1, 0.01, 0.001 $f(x)$: 2.4142, 2.0488, 2.0050, 2.0005 As $x \to 0^+$, $f(x) \to 2$.
2 Substitute values of x that get closer to $x = 0$ from the left.	x: −1, −0.1, −0.01, −0.001 $f(x)$: 1, 1.9487, 1.9950, 1.9995 As $x \to 0^-$, $f(x) \to 2$.
3 State the solution.	$\lim\limits_{x \to 0} f(x) = 2$

Even though $f(0)$ doesn't exist, the limit of $f(x)$ as $x \to 0$ does exist. What does the graph of $y = f(x)$ look like?

You can work out a limit when the function is in its simplest form.

WORKED EXAMPLE 10 The limit of a simplified function

Determine $\lim\limits_{x \to 1}\left(\dfrac{x^2 + 3x - 4}{x - 1}\right)$.

Steps	Working
1 Factorise numerator and cancel with denominator.	$\lim\limits_{x \to 1}\left(\dfrac{(x-1)(x+4)}{x-1}\right)$
2 Work out the limit.	$\lim\limits_{x \to 1}(x+4) = 1 + 4$ $= 5$

Even though $f(1)$ doesn't exist, the limit of $f(x)$ as $x \to 1$ does exist. What does the graph of $y = f(x)$ look like?

A limit does not exist when the values:

- approached from the right and from the left are different
- become arbitrarily large ($\to \pm\infty$).

WORKED EXAMPLE 11 A non-existent limit

Determine $\lim_{x \to \frac{\pi}{2}} \tan(x)$.

Steps

1 Substitute values of x that get closer to $x = \frac{\pi}{2}$ from the left.

> What does the graph of $y = \tan(x)$ look like at $x = \frac{\pi}{2}$?

2 State the solution.

Working

x	$\frac{0.9\pi}{2}$	$\frac{0.99\pi}{2}$	$\frac{0.999\pi}{2}$
$\tan(x)$	6.314	63.6567	636.6192

As $x \to \frac{\pi}{2}^-$, $\tan(x) \to \infty$.

Since $\tan(x) \to \infty$, the limit does not exist at $x = \frac{\pi}{2}$.

> There is no need to work out the limit from the other direction as $\tan(x) \to -\infty$ from the left.

EXERCISE 7.4 Limits

ANSWERS p. 392

Recap

1 The difference quotient for the function $f(x) = 3x + 4$ is

A 3 **B** $3x$ **C** $3 + h$ **D** $4 + h$

2 The difference quotient for the function $f(x) = 2x^2 + 7x - 4$ is

A $2(x + h)^2 + 7(x + h) - 4$ **B** $4xh + 2h^2 + 7h$

C $4x + 7 + 2h$ **D** $h(4x + 2h + 7)$

Mastery

3 WORKED EXAMPLE 8 Complete each table of values and use the result to estimate the stated limit.

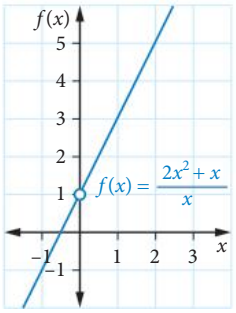
a $\lim_{x \to 4} (4x - 7)$

x	3.9	3.99	3.999	4	4.001	4.01	4.1
$f(x)$				?			

b $\lim_{x \to 1} (2x^2 + x - 5)$

x	0.9	0.99	0.999	1	1.001	1.01	1.1
$f(x)$				?			

4 WORKED EXAMPLE 9 The graph of $f(x) = \frac{2x^2 + x}{x}$ is shown.



a Find the value of $f(x)$ when $x = 0$.

b Use a table of values to find $\lim_{x \to 0} f(x)$.

> What is the other way of working out this limit?

5 WORKED EXAMPLE 10 Copy and complete this table of values to estimate $\lim_{x \to 2}\left(\frac{x+4}{x^2+2x-8}\right)$, then evaluate the limit by first simplifying the function.

x	1.9	1.99	1.999	2	2.001	2.01	2.1
$f(x)$				?			

6 WORKED EXAMPLE 11 Determine each limit (if it exists).

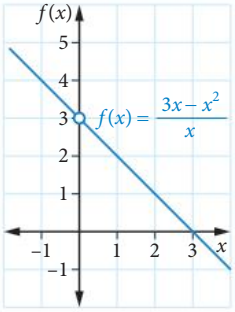
a $\lim_{x \to 1}\left(\frac{4}{x-1}\right)$

b $\lim_{x \to 5}\left(\frac{x^2-16}{x-5}\right)$

7 The graph of $f(x) = \frac{3x - x^2}{x}$ is shown.

a Find the value of $f(x)$ when $x = 0$.

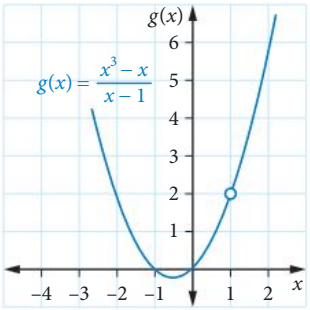
b Use a table of values to find $\lim_{x \to 0} f(x)$.



8 The graph of $g(x) = \frac{x^3 - x}{x - 1}$ is shown.

a Find the value of $g(x)$ when $x = 1$.

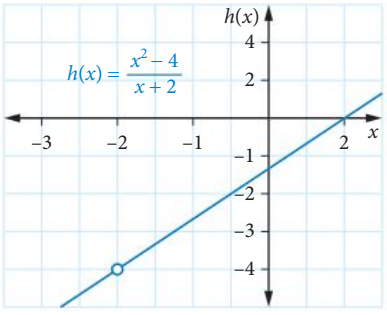
b Use a table of values to find $\lim_{x \to 1} g(x)$.



9 The graph of $h(x) = \frac{x^2 - 4}{x + 2}$ is shown.

a Find the value of $h(x)$ when $x = -2$.

b Use a table of values to find $\lim_{x \to -2} h(x)$.



10 Use a table to find each limit if it exists.

a $\lim_{x \to 3}(2x + 1)$

b $\lim_{x \to 2}(4 - 3x)$

c $\lim_{x \to 5}(x^2 - 4)$

d $\lim_{x \to 2}\left(\frac{x^2-4}{x-2}\right)$

e $\lim_{x \to 3}\left(\frac{5}{x-1}\right)$

9780170484909

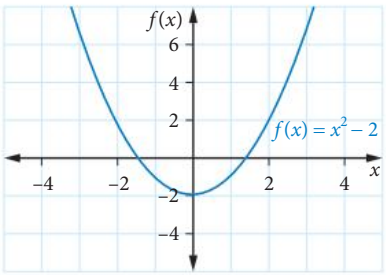
11 Use a table of values to calculate each limit.

a $\lim_{x \to 1}\left(\frac{x^2 - 1}{x - 1}\right)$ **b** $\lim_{x \to 4}\left(\frac{16 - x^2}{x - 4}\right)$ **c** $\lim_{x \to -1}\left(\frac{x^2 - 2x - 3}{x + 1}\right)$

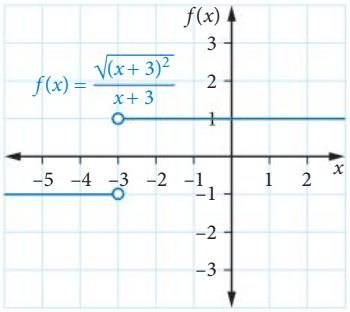
d $\lim_{x \to -3}\left(\frac{x^2 + 7x + 12}{x + 3}\right)$ **e** $\lim_{x \to 0}\left(\frac{\sqrt{x + 3} - \sqrt{3}}{x}\right)$

12 For each function, find the limit if it exists. If the limit does not exist, explain why.

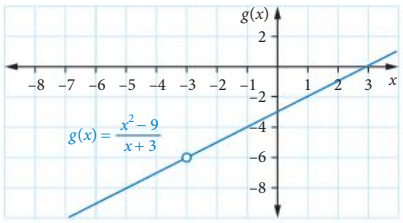
a $\lim_{x \to 0} f(x)$



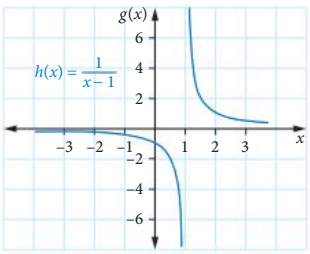
b $\lim_{x \to -3} f(x)$



c $\lim_{x \to -3} g(x)$



d $\lim_{x \to 1} h(x)$



13 For each function, find the limit if it exists. If the limit does not exist, explain why.

a $f(x) = \frac{x - 1}{x + 1}$, $\lim_{x \to -1} f(x)$ **b** $f(x) = \frac{3x}{x^2 + 1}$, $\lim_{x \to 1} f(x)$

c $g(x) = \frac{\sqrt{x + 5} - 4}{x - 2}$, $\lim_{x \to 2} g(x)$

14 Use a graphics calculator to find each limit if possible.

a $\lim_{x \to 4}(7x - 3)$ **b** $\lim_{x \to 2}\left(\frac{4}{x - 1}\right)$

c $\lim_{x \to 5}\left(\frac{x^2 - 25}{x - 5}\right)$ **d** $\lim_{x \to -2}\left(\frac{x^2 + x - 2}{x + 2}\right)$

15 Use an algebraic method to calculate each limit (if it exists).
Use a graphics calculator to confirm your results graphically.

a $\lim_{x \to -3}\left(\frac{x^2 + x - 6}{x + 3}\right)$ **b** $\lim_{x \to -1}\left(\frac{x^2 - x - 2}{x + 1}\right)$

c $\lim_{x \to -5}\left(\frac{x^2 + 6x + 5}{x + 5}\right)$ **d** $\lim_{x \to 1}\left(\frac{1 - 4x + 3x^2}{1 - x}\right)$

e $\lim_{x \to 0}\left(\frac{2x^2 + 7x - 4}{x - 4}\right)$

f $\lim_{x \to -1}\left(\frac{x^3 - 8}{x - 2}\right)$

g $\lim_{x \to 2}\left(\frac{x^2 - 2x - 3}{x^2 - 8x + 15}\right)$

h $\lim_{x \to 1}\left(\frac{x - 1}{x^3 - x^2 + x - 1}\right)$

Exam practice

16 (8 marks) CF Sam takes a square piece of cardboard with side length 12 cm and creates an open box by cutting equal squares x cm by x cm from the corners and turning up the sides.

a Draw and label a diagram that represents the box. *[1 mark]*

b Verify that the volume of the box is given by $V = 4x(6 - x)^2$. *[2 marks]*

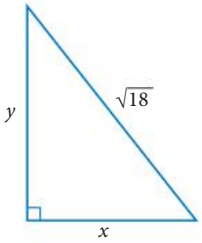
c Draw a graph of V for $0 \leq x \leq 6$. *[2 marks]*

d At what value of x does the box appear to have maximum volume? *[1 mark]*

e Use a table of values to estimate the limit of V at the value of x determined in part **d**. *[2 marks]*

17 (6 marks) CF Elya wants to build a garden bed in the shape of a right-angled triangle. The length of the hypotenuse must be $\sqrt{18}$ m and she wants the area of the triangle to be as large as possible.

Use an equation for the area of the triangle (A) and limits to find the value of x for which the area of the triangle is greatest.



Video
Differentiation from first principles

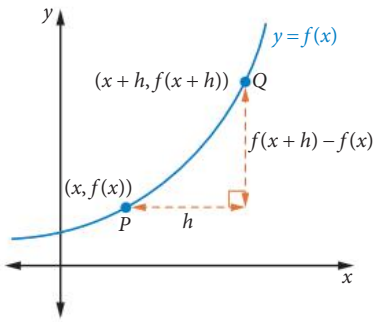
Worksheets
Finding derivatives from first principles

Differentiation from first principles

7.5 The derivative function

In the diagram, as $h \to 0$, Q approaches P and the secant through P and Q becomes closer to being a tangent. The **difference quotient** then changes from being an average rate of change from P to Q to the **gradient of the tangent** at P.

This creates a new function called $f'(x)$ that gives the gradient of the tangent to the graph of $y = f(x)$ for any value of x.
This function $f'(x)$ is 'derived' from the original function $f(x)$.



The derivative of a function

The **derivative**, or **gradient function**, of $f(x)$ is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

where $f'(x)$ is the notation representing the derivative of $f(x)$.

Other notations include $\frac{dy}{dx}$, y', $\frac{d}{dx}f(x)$ and $\frac{df}{dx}$.

The first d in $\frac{d}{dx}$ is read as 'the derivative of', while the second d in $\frac{d}{dx}$ is read as 'with respect to'.

Therefore, $\frac{dy}{dx}$ is read as the 'derivative of y with respect to x'.

The derivative at the point x is both the **instantaneous rate of change** and the **gradient** of the curve (or the gradient of the tangent to the curve) at x.

Finding the derivative of a function is called **differentiating** or **deriving** a function.
Finding the derivative using the limit formula is called **differentiation from first principles**.

WORKED EXAMPLE 12 The derivative from first principles 1

Find the derivative of $f(x) = 4x^2 + 6x - 1$ from first principles.

Steps	Working
1 Substitute $f(x)$ and $f(x + h)$ into the difference quotient.	$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$ $= \lim_{h \to 0} \frac{\left(4(x + h)^2 + 6(x + h) - 1\right) - (4x^2 + 6x - 1)}{h}$
2 Simplify.	$= \lim_{h \to 0} \frac{\left(4(x^2 + 2hx + h^2) + 6x + 6h - 1\right) - \left(4x^2 + 6x - 1\right)}{h}$ $= \lim_{h \to 0} \frac{\left(4x^2 + 8hx + 4h^2 + 6x + 6h - 1\right) - \left(4x^2 + 6x - 1\right)}{h}$ $= \lim_{h \to 0} \frac{8hx + 4h^2 + 6h}{h}$ $= \lim_{h \to 0} (8x + 4h + 6)$
3 Simplify the limit.	$= 8x + 6$
4 Write the answer.	$f'(x) = 8x + 6$

If differentiation from first principles is done correctly, you will be able to simplify like terms in the numerator and divide by the h.

This function $f'(x)$ gives the gradient of the tangent to the parabola $y = 4x^2 + 6x - 1$ for any value of x on the curve.

Notice that the derivative of the quadratic function above is a linear function.

WORKED EXAMPLE 13 The derivative from first principles 2

Find the gradient function of $f(x) = x^3$.

Steps	Working
1 Use the definition of the derivative.	$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$ $= \lim_{h \to 0} \frac{(x + h)^3 - x^3}{h}$
2 Use the binomial expansion.	$= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$
3 Simplify.	$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$ $= \lim_{h \to 0} 3x^2 + 3xh + h^2$ $= 3x^2$
4 Write the answer.	The gradient function of $f(x) = x^3$ is $f'(x) = 3x^2$.

EXERCISE 7.5 The derivative function

ANSWERS p. 393

Recap

1 $\lim\limits_{x \to 1} (5 - 7x) =$

A -2 **B** 4 **C** 5 **D** limit doesn't exist

2 $\lim\limits_{x \to 2} \left(\dfrac{x^2 - x - 6}{x^2 - 4} \right) =$

A -4 **B** 1 **C** 1.5 **D** limit doesn't exist

Mastery

3 WORKED EXAMPLE 12 Find the derivative of each function.

a $f(x) = 3x - 7$ **b** $f(x) = 8 - x$ **c** $g(x) = x^2 - 2$ **d** $h(x) = x^2 + 5x$

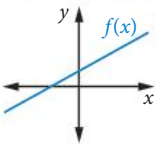
4 WORKED EXAMPLE 13 Find the gradient function of each function.

a $f(x) = 5x$ **b** $g(x) = 7 - 2x$ **c** $h(x) = 3 + x^2$ **d** $f(x) = 3x^2 - 4x + 7$

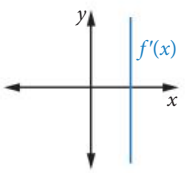
Exam practice TECH-FREE

5 The graph of $f(x)$ is shown.

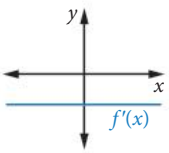
Which graph below shows a possible gradient function for $f(x)$?



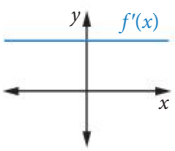
A



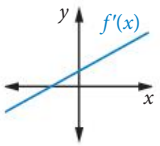
B



C

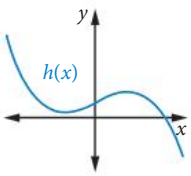


D

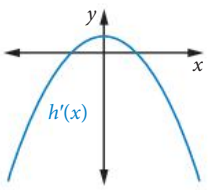


6 The graph of $h(x)$ is shown.

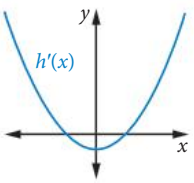
Which graph below shows a possible gradient function for $h(x)$?



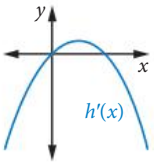
A



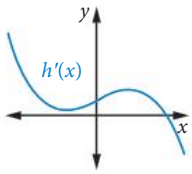
B



C



D



7 (3 marks) Find the derivative of $g(x) = (2x + 5)^2$ using first principles.

8 (5 marks) Find the derivative of $f(x) = 2x^3 + 7$ using first principles.

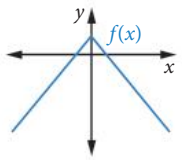
9 (9 marks) Find the derivatives of each function from first principles.

a $9x + 2$ **b** $x - 7$ **c** $3x^2$

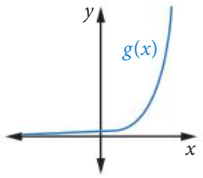
d 9 **e** $10x^2 + 3x + 1$ **f** $1 - 5x - x^2$

g $3x - 4x^2 + 2$ **h** $7x^3$ **i** x^4

10 (3 marks) CF This graph represents a function, $f(x)$.
Draw a possible corresponding gradient function.



11 (3 marks) CF This graph represents a function, $g(x)$.
Draw a possible corresponding gradient function.



12 (5 marks) CF Use first principles to show that $\frac{d}{dx}(cx^2) = 2cx$, where c is a constant.

7.6 The derivative of x^n

From your work on differentiating from first principles in the previous exercise, you may have noticed a pattern in the derivatives of powers of x, as shown in this table.

$f(x)$	$f'(x)$	
x^2	$2x$	
x^3	$3x^2$	(from Worked example 13)
x^4	$4x^3$	(from Question 9)
x^5	$5x^4$	
x^6	$6x^5$	
$9x + 2$	9	(from Question 9)
$7x^3$	$21x^2$	(from Question 9)
$4x^2 + 6x - 1$	$8x + 6$	(from Worked example 12)

The derivative of x^n

If $f(x) = x^n$, then $f'(x) = n \times x^{n-1}$.

If $f(x) = ax^n$, then $f'(x) = an \times x^{n-1}$.

This is called the **power rule** of differentiation.

WORKED EXAMPLE 14 The derivative of power functions

Find the derivative of

a $f(x) = x^9$ **b** $y = 11x^6$

Steps	Working
a Decrease the power by 1. Multiply by the original power.	$f'(x) = 9 \times x^{9-1}$ $= 9x^8$
b Decrease the power by 1. Multiply the coefficient by the original power.	$\frac{dy}{dx} = 6 \times 11x^{6-1}$ $= 66x^5$

The derivatives of linear and constant terms

For $f(x) = ax$, $f'(x) = a$. ← This makes sense because the gradient of $y = ax$ is always a.

For $f(x) = a$, $f'(x) = 0$. ← This makes sense because the gradient of $y = a$ is always 0.

You can prove both of the above using first principles or the derivative of ax^1 and ax^0.

You can use the derivative of $f(x)$ to find the gradient at a specific value of x.

$f'(c)$ means the value of the derivative at $x = c$.

WORKED EXAMPLE 15 The derivative at a given point

Determine the value of the derivative of $f(x) = 2x^5$ at $x = 2$.

Steps	Working
1 Determine derivative.	$f'(x) = 5 \times 2x^4$ $= 10x^4$
2 Substitute $x = 2$.	$f'(2) = 10 \times 2^4$ $= 160$

EXERCISE 7.6 The derivative of x^n

ANSWERS p. 393

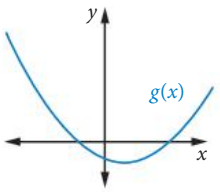
Recap

1 Using first principles, the derivative of the function $f(x) = 6x^4$ is

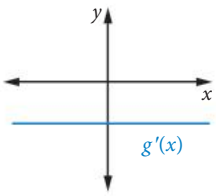
A $24x^4$ **B** $6x^3$ **C** $24x^3$ **D** $24x$

2 The graph of $g(x)$ is shown.

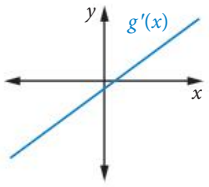
The graph that shows a possible gradient function for $g(x)$ is



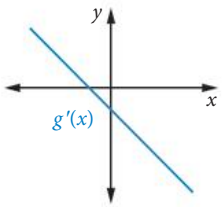
A



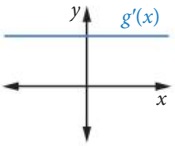
B



C



D



Mastery

3 WORKED EXAMPLE 14 Use the power rule for derivatives to differentiate each function.

a $f(x) = x^2$ **b** $y = x^3$ **c** $f(x) = x^5$ **d** $g(x) = x^8$

e $y = x^{20}$ **f** $f(x) = x^{17}$ **g** $h(x) = x^{12}$ **h** $y = x^{22}$

4 WORKED EXAMPLE 15 Determine the value of the derivative at $x = 2$ for each function in Question **3**.

Exam practice

5 TECH-FREE (4 marks) Determine the derivative of each function.

a $f(x) = 5x^3$ **b** $f(x) = 6x^4$ **c** $f(x) = 3x^5$ **d** $f(x) = 2x^6$

7.7 The derivative of a polynomial

You differentiate a polynomial by differentiating each power term individually.

WORKED EXAMPLE 16 Derivative of a polynomial function

Determine the derivative of $f(x) = 2x^5 - 6x^3 + x^2 + 10x - 3$.

Steps	Working
Differentiate each term of the polynomial separately, then simplify.	$f'(x) = 5 \times 2x^{5-1} - 3 \times 6x^{3-1} + 2x^{2-1} + 10x^{1-1}$ $f'(x) = 10x^4 - 18x^2 + 2x + 10$

Each term of a polynomial can be considered its own function and therefore, each function is differentiated individually. This leads to the following relationships:

- $\frac{d}{dx}\left[f(x) + g(x)\right] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$
- $\frac{d}{dx}\left[f(x) - g(x)\right] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$
- $\frac{d}{dx}\left[k \times f(x)\right] = k \times \frac{d}{dx}f(x)$

WORKED EXAMPLE 17 Applications of derivatives

The displacement of a particle is given by the equation $x(t) = t^3 + 2t^2 - 5t - 5$, where t is in seconds and x is in metres. Given that velocity is the derivative function of displacement and the particle starts its motion at $t = 0$, **determine**

a the velocity at 2 seconds **b** the time(s) when the particle is at rest.

Steps	Working
a 1 Determine the velocity function as the derivative of displacement.	$v(t) = x'(t) = 3t^2 + 4t - 5$
2 Substitute $t = 2$ into $v(t)$.	$v(2) = 3 \times 2^2 + 4 \times 2 - 5$ $= 12 + 8 - 5$ $= 15$
3 Interpret the solution.	Therefore, the velocity at 2 seconds is 15 m/s.
b 1 Solve $v = 0$.	Particle at rest when $v(t) = 0$. $0 = 3t^2 + 4t - 5$ $t = -2.12, 0.79$
2 Apply domain restriction and interpret the solution.	Since $t \geq 0$, $t = 0.79$ Therefore, the particle is at rest at approx. 0.79 seconds.

Velocity is the rate of change of **displacement**. You can use variables other than x, y and $f(x)$.

Worksheets
Gradient functions
Derivatives of polynomials
Derivatives of linear products
Derivatives of a sum of terms
Basic differentiation

Video
Displacement and velocity

EXERCISE 7.7 The derivative of a polynomial

ANSWERS p. 393

Recap

1 The derivative of $f(x) = 4x^3$ is

A $4x^2$ **B** $3x^2$ **C** $12x^3$ **D** $12x^2$

2 The value of $f'(3)$ for $f(x) = 5x - 2x^2$ is

A -7 **B** -3 **C** -1 **D** 3

Mastery

3 WORKED EXAMPLE 16 Find the derivative of each function.

a $6x^4 - 5x^3 + 2x^2 + 8x - 9$ **b** $9x^3 + 10x^4 - 3x^5 + 18$

c $x^3 + x^2 + x + 1$ **d** $4x^9 - 8x^{10} + 20$

e $15x^{12} + 12x^{15} - 36x^5$ **f** $5 - x^7 + 7x^9 + x^{10}$

g $5x^8 + 12x^9 - 6x^{10} + x^{12}$ **h** $11 - 3x + x^2 + 8x^3 - x^4$

i $5x^4 - 17x + 28 + 16x^3$ **j** $25x^7 - 8x^6 + 14x^5 + 2x - 2$

4 WORKED EXAMPLE 17 If $f(x) = 2x^4 - 5x^3 - 7x + 9$, find

a $f'(1)$ **b** $f'(-1)$ **c** $f'(5)$ **d** $f'(-3)$

Exam practice TECH-FREE

5 (10 marks) Determine the derivative of each function.

a $\frac{d}{dx}(3x^4 - 6x^2 + 18)$ **b** $\frac{d}{du}(3u^4 - 6u^2 + 18)$

c $\frac{d}{du}(5m^6 + 8m^5 - 24m + 8)$ **d** $\frac{d}{du}(9y + 6y^5 - 4y^3 + 2y^7)$

e $\frac{d}{dz}(3z^2 - 9z^3 + 38z - 5)$ **f** $\frac{d}{dt}(5t^7 + 9t^4 - 16t)$

g $\frac{d}{dr}(21 - 13r - 6r^2 + 9r^3)$ **h** $\frac{d}{da}(-6a^4 + 2a^2 - 10)$

i $\frac{d}{dp}(8p - 3p^4 - 6p^7 - 5p^9)$ **j** $\frac{d}{dt}(8 + 4t^2 - 2t^3 + 11t^4)$

6 (12 marks) Find the derivative of each function by first expanding the expression.

a $(7 + 2x)(3 + 5x)$ **b** $(4 - 3x)(9 - 2x)$ **c** $(8 + 3x)(6 - 7x)$

d $(4x + 7)(8 - 3x)$ **e** $(9x + 2)(2x - 11)$ **f** $(8 - x)(7x - 2)$

7 (12 marks) Find the derivative of each function by first expanding the expression.

a $(7 + x)^2$ **b** $(x - 4)^2$ **c** $(3 + 2x)^2$

d $(4 + 3x)^2$ **e** $(4x - 5)^2$ **f** $(8 + 3x)^2$

8 (12 marks) Find the derivative of each function.

a $(x - 3x^2)(1 + x^3)$ **b** $(2x^2 - x^3)(5x + 3x^2)$ **c** $(4x^2 + 2x^3)(5x^3 - 3x^4)$

d $(x - 7x^2)(2 + 3x^2 + x^3)$ **e** $(2x^2 + 3x^3)(3 - 4x^3 - 6x^4)$ **f** $(8x + 3x^2 - 4x^3)(2x^2 + 7x^3)$

9 (8 marks) Find the derivative of each function by first simplifying the expression.

a $\frac{7x^5 - 3x^2}{x}, x \neq 0$ **b** $\frac{12x^7 + 8x^5 - 4x^3}{x^2}, x \neq 0$

c $\frac{3x^2 + 2x - 5}{x - 1}, x \neq 1$ **d** $\frac{8x^2 - 2x - 21}{2x + 3}, x \neq -1\frac{1}{2}$

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10 (3 marks) Using $f(x) = 3x - 7$ and $g(x) = 4x^3 + 5x^2 - 9$ determine if

$$\frac{d}{dx}[f(x) \times g(x)] = \frac{d}{dx}f(x) \times \frac{d}{dx}g(x)$$

11 (4 marks) CF $y = 2x^2 - 3x$ and $z = 3 - 7x^2 + 2x^3$.

Given that y' means $y'(x)$ and z' means $z'(x)$, compare the results for $y' \times z'$ with the results for $(y \times z)'$.

12 (8 marks) CF

a Differentiate each function and factorise the answer. *[6 marks]*

i $(x + 1)^2$ **ii** $(x + 1)^3$ **iii** $(2x + 1)^2$

iv $(2x + 1)^3$ **v** $(3x + 1)^2$ **vi** $(3x + 1)^3$

b Using the results from part **a**, write a general formula for the derivative of $(ax + 1)^n$, where a and n are constants. *[2 marks]*

EXAM QUESTION ANALYSIS

Exam-style question (4 marks)

The total surface area of a cylinder is given by $A(r) = 2\pi rh + 2\pi r^2$. If the radius is changing but the height is always 2, determine the function for the rate of change of the total surface area with respect to the radius r and determine the rate of change when $r = 2$.



Video Exam question analysis: Rates of change and the derivative

Reading the question

- Two variables to consider:
  - r is the radius of the cylinder and is changing, and
  - h is the height of the cylinder and is constant.

Thinking about the question

- Rate of change means derivative is needed.
- Derivative can't be done with two variables.
- Since r is changing, that is the variable you derive with respect to.
- h is considered part of the coefficient and is not derived (differentiated).

Worked solution (✓ = 1 mark)

$A(r) = 2\pi rh + 2\pi r^2$

$= 2\pi r(2) + 2\pi r^2$ as $h = 2$

$= \mathbf{4\pi r + 2\pi r^2}$ ✓ Simplification of function to a single variable

$A'(r) = 4\pi + 2\pi \times 2r$ ✓ Deriving with respect to r

$= \mathbf{4\pi + 4\pi r}$ ✓ Simplified solution

When $r = 2$,

$A'(2) = 4\pi + 4\pi(2)$

$= 4\pi + 8\pi$

$= \mathbf{12\pi}$ ✓ Substitution and solution

7 Chapter summary

Average rates of change

- **Average rate of change** $= \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$.
- It is the gradient of the secant passing through two points on a function.

Difference quotients

$$\frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h}$$

Limits

- The **limit of a function** $f(x)$ is a value that the function approaches as x approaches a value $x = c$.

The derivative function

- **Derivative** or **gradient function** of $f(x)$ is $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.
- Other notations include: $\frac{dy}{dx}$, y', $\frac{d}{dx} f(x)$ and $\frac{df}{dx}$.
- $\frac{dy}{dx}$ is read as the 'derivative of y with respect to x'
- The derivative at point x is the instantaneous rate of change at x.
- It is the gradient of the **tangent** at a point on a function; the value of the derivative of the function at that point.

The derivative of x^n

- If $f(x) = x^n$, then $f'(x) = n \times x^{n-1}$
- If $f(x) = ax^n$, then $f'(x) = an \times x^{n-1}$.

The derivative of a polynomial

- Deriving a polynomial function is done by deriving each power term individually.

Cumulative examination 1

Technology-free

Perusal time: 2 minutes Working time: 30 minutes

TECH-FREE *Calculators are **not** permitted.*

Section 1 (3 marks): 3 multiple choice questions

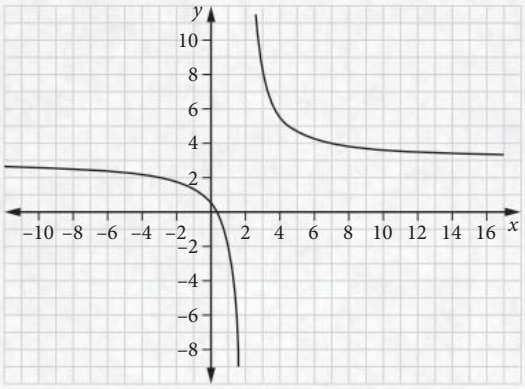
Section 2 (15 marks): 4 short response questions

Total: 18 marks

Section 1 3 multiple choice questions

3 marks

1 This graph has the equation



A $y = \dfrac{5}{x+2} + 3$

B $y = \dfrac{5}{x-2} + 3$

C $y = \dfrac{5}{x-2} - 3$

D $y = \dfrac{5}{x+2} - 3$

2 A bag contains 5 green, 3 black and 2 yellow marbles. A marble is selected and its colour noted. The marble is not replaced back into the bag. The probability of selecting 2 black marbles is

A $\dfrac{1}{18}$ **B** $\dfrac{3}{50}$ **C** $\dfrac{1}{15}$ **D** $\dfrac{9}{100}$

3 The derivative of the function $f(x) = x^3 - 6x^2 + 10$ is

A $3x^2 - 12x$ **B** $3x^2 - 2x$ **C** $3x^2 + 12x$ **D** $3x^3 - 12x^2$

Section 2 4 short response questions

15 marks

4 (4 marks) Simplify the expression $\dfrac{1}{3+\sqrt{2}} + \dfrac{3}{2-\sqrt{2}}$ to one fraction with a rational denominator.

5 (4 marks) Determine the x-intercepts of the parabola with equation $y = x^2 + 7x - 3$.

6 (3 marks) Find $\lim\limits_{x \to \frac{1}{2}} \dfrac{2x^2 + 5x - 3}{2x - 1}$.

7 (4 marks) Use first principles to determine the derivative of the function $f(x) = -3x^2 - 5x + 2$.

Worksheet
Maths Methods Year 11 formula sheet

Worksheet
Maths Methods Year 11 formula sheet

Cumulative examination 2

Technology-active

Perusal time: 2 minutes Working time: 30 minutes

Calculators are ***permitted***.

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

Section 1 (3 marks): 3 multiple choice questions

Section 2 (15 marks): 4 short response questions

Total: 18 marks

Section 1 3 multiple choice questions

3 marks

1 The leading coefficient of the polynomial $f(x) = 3x^2 - 6x^3 + 2x - 9$ is

A –6 **B** 2 **C** 3 **D** 6

2 A flight from Brisbane to Singapore takes 7 h 40 min and covers 6150 km. The average speed of the flight is

A 796.30 km/h **B** 802.17 km/h **C** 831.08 km/h **D** 849.13 km/h

3 Given $f(x) = x^5 - 2x^3 + 6x^2 - 4$, $f'(3)$ is

A –922 **B** 3 **C** 239 **D** 387

Section 2 4 short response questions

15 marks

4 (4 marks) A boater looks at a tide chart and sees that at 6:15 am, there is a high tide of 6 metres at the boat ramp. A low tide of 1.5 metres is set to occur at 12:30 pm. The boater plans to leave after 7 am and needs a minimum of 2 metres of depth to operate safely.

a Determine whether it is safe for the boater to leave the boat ramp at 7 am. Justify your response. *[3 marks]*

b By what time must the boater return to the boat ramp? *[1 mark]*

5 (4 marks) A die is rolled 3 times and its number noted. Determine the probability that each roll of the die is greater than or equal to 5.

6 (3 marks) Given the function $f(x) = 4x^4 - 2x^2 + 1$, determine the value of x when the derivative value is 6.

7 (4 marks) A turning point of a function occurs when the gradient of the function is equal to 0. Determine the coordinates of both turning points of the function $f(x) = 0.1x^3 - 0.225x^2 - 0.75x + 2$.

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CHAPTER

8 EXPONENTIAL FUNCTIONS



Syllabus coverage
Nelson MindTap chapter resources
Terminology
8.1 Index laws and fractional indices
8.2 Scientific notation
8.3 Exponential functions
8.4 Graphing exponential functions
8.5 Translating exponential functions
8.6 Exponential equations
8.7 Modelling with exponential functions
Exam question analysis
Chapter summary
Cumulative examination 1
Cumulative examination 2

Prior learning
Exponential functions

Syllabus coverage

UNIT 2, TOPIC 1: EXPONENTIAL FUNCTIONS

Indices and index laws

- Use indices (including negative and fractional indices) and the index laws.
- Convert radicals to and from fractional indices.
- Understand and use scientific notation.

Introduction to exponential functions

- Recognise and determine the qualitative features of the graph of $y = r^x$ (where $r > 0$), including asymptote and intercept.
- Recognise and determine the effect of the parameters h, k and r on the graph of $y = r^{(x-h)} + k$ (where $r > 0$), with and without technology.
- Sketch the graphs of exponential functions, with and without technology.
- Solve equations involving exponential equations, with and without technology.
- Model and solve problems that involve exponential functions, with and without technology.

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Videos (7):
- **8.1** Fractional indices 1 • Fractional indices 2
- **8.2** Scientific notation • Significant figures
- **8.3** Exponential growth 1
- **8.6** Exponential equations 1
- **Exam question analysis** Exponential functions

Prior learning (1):
- **8.1** Exponential functions

Worksheets (12):
- **8.1** Review of index laws • Indices • Fractional indices and radicals
- **8.2** Scientific notation 1 • Scientific notation 2 • Scientific notation and significant figures • Scientific notation problems
- **8.4** Graphing exponentials
- **8.5** Translating exponential graphs
- **8.6** Solving exponential equations
- **8.7** Using exponential models
- **Cumulative exams** Maths Methods Year 11 formula sheet

Puzzle (1):
- **8.1** Indices squaresaw (Advanced)

TI-Nspire CX II instructions (1):
- **8.6, 8.7** Exponential functions

Nelson MindTap

To access resources above, visit **cengage.com.au/nelsonmindtap**

Terminology

base	decay factor	decay function	exponent
exponential function	growth factor	growth function	horizontal translation
index	index laws	index term	initial value
order	radical	scientific notation	significant figures
standard notation	vertical translation		

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8.1 Index laws and fractional indices

The index laws

For any **base** number a and any positive integer n, a^n is the product of n factors of a. a^n is called a **power** of a and n is called the **index** (plural: indices) or **exponent**.

- $a^m \times a^n = a^{m+n}$
- $a^m \div a^n = a^{m-n}$
- $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$
- $\left(a^m\right)^n = a^{mn}$

You know that $\frac{5^2}{5^2} = \frac{25}{25} = 1$.

Also, $\frac{5^2}{5^2} = 5^{2-2} = 5^0$.

It makes sense to say $5^0 = 1$.

In the same way, $5^{2-6} = \frac{5^2}{5^6} = \frac{5^2}{5^2 \times 5^4} = \frac{1}{5^4}$.

It makes sense to say $5^{-4} = \frac{1}{5^4}$.

$\left(5^{\frac{1}{3}}\right)^3 = 5^1 = 5$, so $5^{\frac{1}{3}} = \sqrt[3]{5}$.

More index laws

- $a^0 = 1, a \neq 0$
- $a^{-m} = \frac{1}{a^m}, a \neq 0$
- $a^{\frac{p}{q}} = \sqrt[q]{a^p} = \left(\sqrt[q]{a}\right)^p$, where $a > 0$, p is an integer and q is a positive integer.

WORKED EXAMPLE 1 Simplifying index expressions

Simplify each expression and express with positive indices.

a $\frac{4h^{-6} \times 3h^2}{h^{-3} \times 6h^5}$ **b** $\frac{2^{m-n} \times 3^{m-n} \times 6^n}{12^m}$

Steps	Working
a 1 Write the expression.	$\frac{4h^{-6} \times 3h^2}{h^{-3} \times 6h^5}$
2 Simplify the numerical part. Apply index laws.	$= \frac{12}{6} \times h^{(-6+2-(-3)-5)}$
3 Evaluate and simplify.	$= 2h^{-6}$
4 Write with a positive index.	$= \frac{2}{h^6}$

Videos
Fractional indices 1
Fractional indices 2

Worksheets
Review of index laws
Indices
Fractional indices and radicals

Puzzle
Indices squaresaw (Advanced)

b 1	Write the expression.	$\dfrac{2^{m-n} \times 3^{m-n} \times 6^n}{12^m}$
2	Express bases as products of prime factors.	$= \dfrac{2^{m-n} \times 3^{m-n} \times (2 \times 3)^n}{(2^2 \times 3)^m}$
3	Apply index laws.	$= \dfrac{2^{m-n} \times 3^{m-n} \times 2^n \times 3^n}{2^{2m} \times 3^m}$
		$= 2^{(m-n+n-2m)} \times 3^{(m-n+n-m)}$
4	Simplify.	$= 2^{-m} \times 3^0$
5	Write with a positive index.	$= \dfrac{1}{2^m}$

WORKED EXAMPLE 2 Simplifying index expressions

Simplify each expression and express with positive indices.

a $x^3 \times \sqrt{x}$ **b** $\sqrt[6]{8a^4} \times \sqrt{8} \times a^{-\frac{8}{3}}$ **c** $9^{2\frac{1}{2}}$

Steps		Working
a 1	Write the expression.	$x^3 \times \sqrt{x}$
2	Apply the definition of a fractional index.	$= x^3 \times x^{\frac{1}{2}}$
3	Apply index laws.	$= x^{3+\frac{1}{2}}$
4	Evaluate exponents.	$= x^{\frac{7}{2}}$
b 1	Write the expression.	$\sqrt[6]{8a^4} \times \sqrt{8} \times a^{-\frac{8}{3}}$
2	Express bases as products of prime factors.	$= \sqrt[6]{2^3 a^4} \times \sqrt{2^3} \times a^{-\frac{8}{3}}$
3	Apply the definition of a fractional index.	$= (2^3 a^4)^{\frac{1}{6}} \times (2^3)^{\frac{1}{2}} \times a^{-\frac{8}{3}}$
4	Apply index laws.	$= 2^{\frac{3}{6}} a^{\frac{4}{6}} \times 2^{\frac{3}{2}} \times a^{-\frac{8}{3}}$
		$= 2^{\frac{1}{2}+\frac{3}{2}} \times a^{\frac{2}{3}+\left(-\frac{8}{3}\right)}$
5	Evaluate exponents and simplify.	$= 2^2 a^{-2}$
6	Write with a positive exponent.	$= \dfrac{4}{a^2}$
c 1	Write the expression.	$9^{2\frac{1}{2}}$
2	Write the exponent as an improper fraction.	$= 9^{\frac{5}{2}}$
3	Write using the radical sign.	$= \left(\sqrt[2]{9}\right)^5$
4	Find the square root.	$= 3^5$
5	Find the power.	$= 243$

EXERCISE 8.1 Index laws and fractional indices

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Round answers to 4 decimal places where appropriate.

Mastery

1 WORKED EXAMPLE 1 Simplify each expression and express with positive indices.

a $r^{-6} \times r^4$

b $4y^{-5} \times 3y^{-4}$

c $\dfrac{12p^6 \times 6p^{-2}}{8p^{-1} \times 3p}$

d $\dfrac{4j^5 \times 5j^{-4}}{10j^7}$

e $\dfrac{2m^3 \times m^{-10}}{12m^6}$

f $\dfrac{5k^{-3}}{2k^4} \div \dfrac{10k^5}{k^{-2}}$

g $3g^{-3}h^4 \times 5g^4h^{-6}$

h $5k^3m^{-6} \times (k^{-3}m^{-4})^{-5}$

i $\dfrac{8v^{-4}w^2 \times 4v^{10}w^{-8}}{2vw^{-9} \times v^{-6}w^3}$

j $\dfrac{12p^{-4}q^{-3} \times (2pq^4)^2}{(3p^{-1}q^4)^{-3} \times (18p^4)^2}$

k $\left(\dfrac{e^2f^{-4}}{e^{-5}f}\right)^3 \times \dfrac{4e^{-3}f^4}{3e^6f^6}$

l $\dfrac{c^7d^8}{d^{-5}} \div \dfrac{(8c^{-2}d^3)^{-1}}{4^{-2}c^{-2}d^3}$

2 WORKED EXAMPLE 2 Simplify each expression and express with positive indices.

a $a^4 \times \sqrt{a}$

b $b^2 \times \sqrt{b^3}$

c $3n^{-3} \times 2\sqrt{n^5}$

d $6g^5 \times 2\sqrt[3]{g^4}$

e $4\sqrt[3]{w^2} \times 3\sqrt{p^5w^6}$

f $5\sqrt[3]{m^3n^{-2}} \times 2\sqrt{m^5n^6}$

g $7\sqrt[4]{s^2u^6} \times \sqrt{9s^{-8}u^3}$

h $8\sqrt[3]{p^{12}q^{-8}} \times 3\sqrt[4]{p^{-10}q^9}$

3 Evaluate each expression.

a $16^{\frac{3}{4}}$

b 9^0

c $36^{\frac{1}{2}}$

d $27^{\frac{2}{3}}$

e $49^{\frac{3}{2}}$

f $36^{\frac{6}{4}}$

g $8^{-\frac{2}{3}}$

h $\left(\dfrac{3}{4}\right)^{-2}$

4 Simplify each expression and express with positive indices.

a a^{-2}

b 4^{-3}

c g^{-7}

d 5^{-4}

e y^{-6}

f $(cd^2)^{-4}$

g $4h^{-4}$

h $(4h)^{-4}$

5 Simplify each expression and express without denominators.

a $\dfrac{1}{y^4}$

b $\dfrac{1}{4^3}$

c $\dfrac{1}{h^9}$

d $\dfrac{t^4}{k^6}$

e $\dfrac{y^2}{p^6}$

f $\dfrac{4j^8}{i^6}$

g $\dfrac{3y^{-2}}{m^5}$

h $\dfrac{2m^{-1}}{z^4}$

6 Simplify each expression and express as roots (using radical signs).

a $6^{\frac{1}{2}}$ **b** $z^{\frac{1}{4}}$ **c** $216^{\frac{1}{3}}$ **d** $y^{\frac{3}{5}}$

e $49^{\frac{3}{2}}$ **f** $v^{\frac{7}{3}}$ **g** $8^{2\frac{1}{3}}$ **h** $16^{1\frac{1}{4}}$

7 Simplify each expression and express as powers.

a $\sqrt{k}$ **b** $\sqrt[4]{3}$ **c** $\sqrt{5}$ **d** $\sqrt[5]{4}$

e $\sqrt[3]{k}$ **f** $\sqrt{c^3}$ **g** $\left(\sqrt[3]{64}\right)^4$ **h** $\sqrt[4]{e^3}$

8 Evaluate each expression.

a $8^{\frac{1}{3}}$ **b** 36^0 **c** 17^0 **d** $81^{\frac{1}{2}}$

e $64^{\frac{2}{3}}$ **f** $32^{\frac{6}{5}}$ **g** $\left(\frac{1}{16}\right)^{-\frac{1}{4}}$ **h** 18^0

9 Simplify each expression and express with positive indices.

a $\dfrac{2^{n+2} \times 2^{n-3}}{2^{2n-5}}$ **b** $\dfrac{5^{2k} \times 5^{k-1}}{5^{2k-2}}$ **c** $\dfrac{3^{j+1} \times 2^{j+3}}{6^j}$ **d** $\dfrac{6^{2n-m} \times 9^{n+m}}{36^{n-m}}$

Exam practice TECH-FREE

10 $\left(\frac{1}{9}\right)^{-\frac{1}{2}} =$

A 81 **B** $\frac{1}{81}$ **C** 3 **D** 9

11 $\left(2\frac{1}{4}\right)^2 =$

A $4\frac{1}{16}$ **B** $5\frac{1}{16}$ **C** $6\frac{1}{4}$ **D** $9\frac{1}{4}$

12 (4 marks) CU Express $\dfrac{15^{4t-5} \times 21^{2t+1}}{35^{2t-2} \times 45^{2t-3}}$ as a product of powers of prime numbers.

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8.2 Scientific notation

Very large and very small numbers are difficult to read, write and compare.

You can use **scientific notation** (or **standard notation**) to write very large and very small numbers.

Scientific notation

Numbers in scientific notation use a decimal from 1 to 10 multiplied by a power of 10.

$$690\,000 = 6.9 \times 10^5$$

$$0.000\,25 = 2.5 \times 10^{-4}$$

Numbers are written in scientific notation by putting the decimal point after the first non-zero digit.

Count how many places you need to move the decimal point to work out the power of 10.



Videos
Scientific notation
Significant figures

Worksheets
Scientific notation 1
Scientific notation 2
Scientific notation and significant figures
Scientific notation problems

WORKED EXAMPLE 3 Converting numbers to scientific notation

Write each number in scientific notation.

a 61 000 **b** 764 000 000 000

c 0.0008 **d** 0.000 000 472

Steps	Working
a Put the decimal point after the 6.	6.1 0 0 0
You have to move the decimal point *forward* 4 places.	
The power of 10 is 4.	$= 6.1 \times 10^4$
b Write the decimal point after the 7.	7.6 4 0 0 0 0 0 0 0 0 0
You have to move the decimal point *forward* 11 places.	
The power is 11.	$= 7.64 \times 10^{11}$
c Write the decimal place after the 8.	0.0 0 0 8
You have to move the decimal point *back* 4 places.	
The power is −4.	$= 8 \times 10^{-4}$
d Write the decimal place after the 4.	0.0 0 0 0 0 0 0 4 7 2
You have to move the decimal point *back* 7 places.	
The power is −7.	$= 4.72 \times 10^{-7}$

WORKED EXAMPLE 4 Converting scientific notation to ordinary numbers

Write as ordinary numbers.

a 6.25×10^4 **b** 9.1×10^{-2}

Steps	Working
a Move the decimal point 4 places to the right.	6.25×10^4
It's positive so it makes a big number.	$= 62\,500$
b Move the decimal point 2 places to the left.	9.1×10^{-2}
It's negative so it makes a small number.	$= 0.091$

You can convert to and from scientific notation using the powers of 10.

$6.25 \times 10^4 = 6.25 \times 10\,000 = 62\,500$ (the 6 is in the ten thousands place)

$9.1 \times 10^{-2} = 9.1 \times \frac{1}{100} = 0.091$ (the 9 is in the hundredths place)

Significant figures

Scientific notation can be used to find the number of **significant figures** of a number. It is the number of digits that *must* be written for the decimal when writing the number in scientific notation.

The number of significant figures is a measure of accuracy.

WORKED EXAMPLE 5 Determining significant figures

How many significant figures are in these numbers?

a 57 800 **b** 0.0065 **c** 3.001 25

d 620 035 **e** 3 **f** 3.200

Steps	Working
a There are 3 digits in the decimal part.	$57\,800 = 5.78 \times 10^4$ There are 3 significant figures.
b There are 2 digits in the decimal part.	$0.0065 = 6.5 \times 10^{-3}$ There are 2 significant figures.
c There are 6 digits in the decimal part.	$3.001\,25 = 3.001\,25 \times 10^0$ There are 6 significant figures.
d There are 6 digits.	$620\,035 = 6.200\,35 \times 10^5$ There are 6 significant figures. Zeros are significant if they are *between* non-zero digits.
e There is only 1 digit.	3 There is 1 significant figure.
f The zeros are at the end of a decimal number are assumed to be significant.	3.200 There are 4 significant figures. However, 3.2 has 2 significant figures.

Exam hack

Zeros at the beginning of a decimal are not significant.
Zeros in the middle of a decimal are significant.

EXERCISE 8.2 Scientific notation

ANSWERS p. 394

Recap

1 $\frac{2x^5}{x^3} =$

A $2x^8$ **B** $2x^{\frac{5}{3}}$ **C** $2x^{15}$ **D** $2x^2$

2 $\sqrt{\left(4x^7\right)^0} =$

A $2x^{\frac{7}{2}}$ **B** 1 **C** 2 **D** 0

Mastery

3 WORKED EXAMPLE 3 Write each number in scientific notation.

a	6300	**b**	142 000 000	**c**	0.0075	**d**	0.000 000 938
e	500 000	**f**	0.0006	**g**	210	**h**	0.79
i	31 090 000	**j**	0.040 62	**k**	9 623 000	**l**	0.000 000 002

4 WORKED EXAMPLE 4 Write each number as an ordinary number.

a	4.3×10^{5}	**b**	2.1×10^{-3}	**c**	9.84×10^{6}	**d**	3.19×10^{-1}
e	6×10^{-9}	**f**	2×10^{4}	**g**	4.2×10^{1}	**h**	5.8×10^{-2}
i	1.6×10^{0}	**j**	9.5×10^{-5}	**k**	4.64×10^{2}	**l**	2.1×10^{3}

5 WORKED EXAMPLE 5 State the number of significant figures in each number.

a	24 700	**b**	0.000 35	**c**	700	**d**	0.000 005
e	10	**f**	9.65	**g**	3.047	**h**	105 030 000
i	510 000	**j**	0.050 63	**k**	0.000 087 5	**l**	1.8

6 Evaluate each expression and write the answer in scientific notation.

a	$4.6 \times 10^{-8} \times 3.2 \times 10^{9}$	**b**	$(6.1 \times 10^{9}) \div (8.34 \times 10^{-2})$
c	$5.64 \times 10^{7} \times 3.9 \times 10^{11}$	**d**	$(8 \times 10^{6}) \times (2 \times 10^{-9}) \div (4 \times 10^{3})$
e	$7.1 \times 10^{5} \div (1.6 \times 10^{-9}) \times 2.3 \times 10^{-4}$	**f**	$4.15 \times 10^{9} + 3.84 \times 10^{7}$
g	$8.63 \times 10^{-8} - 1.96 \times 10^{-12}$	**h**	$4.6 \times 10^{11} \times 3.1 \times 10^{-5} + 5.8 \times 10^{7}$

7 Evaluate each expression and write the answer in scientific notation.

a	$\dfrac{4861 \times 2 \times 10^{6}}{3.58 \times 10^{-5}}$	**b**	$\dfrac{3.1 \times 10^{6} \times 8.2 \times 10^{12}}{5.8 \times 10^{11} \times 0.067}$	**c**	$\dfrac{0.5 \times 1.8 \times 10^{-3}}{2.4 \times 10^{-6} \times 3.2 \times 10^{11}}$
d	$\dfrac{9.85 \times 10^{9} + 9.84 \times 10^{9}}{9.85 \times 10^{-9} - 9.84 \times 10^{-9}}$	**e**	$\dfrac{4.71 \times 10^{6} + 2.95 \times 10^{7}}{3.5 \times 10^{9} \times \pi}$	**f**	$\dfrac{\pi \times 5.3 \times 10^{-6}}{2 \times 10^{9} \times 3.604 \times 10^{18}}$

8 Round each number to the number of significant figures shown in brackets.

a	42 (1 sf)	**b**	475 (2 sf)	**c**	247.6 (3 sf)
d	0.000 006 03 (2 sf)	**e**	24 641 (1 sf)	**f**	3.6084 (2 sf)
g	63 480 (3 sf)	**h**	18 946 724 965 (4 sf)	**i**	36.000 000 478 (3 sf)

Exam practice

9 (4 marks each) CF The distance of planets from the Sun is shown in the table.

An astronomical unit (AU) is defined as exactly 149 597 870.7 km, the average distance between the Earth and the Sun. If the planets were in a straight line from the Sun, determine the distance (in AU correct to 2 decimal places) between

a Earth and Saturn

b Mercury and Neptune.

Planet	Distance from the Sun (km)
Mercury	5.79×10^{7}
Venus	1.082×10^{8}
Earth	1.496×10^{8}
Mars	2.279×10^{8}
Jupiter	7.786×10^{8}
Saturn	1.4335×10^{9}
Uranus	2.8725×10^{9}
Neptune	4.4951×10^{9}

8.3 Exponential functions

An **exponential function** has the variable in the exponent or power.

Exponential functions

The general exponential function has the form $y = A \times r^x$, where $r > 0$.
When $r > 1$, it is a **growth function**.
When $0 < r < 1$, it is a **decay function**.
r is the **growth** or **decay factor**.
A is the **initial value**, since $y = A \times r^0 = A$.

r for rate.

Video
Exponential growth 1

WORKED EXAMPLE 6 Exponential growth

A bacterial colony has about 2000 bacteria. The number of bacteria triples every day.

Assuming that growth continues in this way

a how many bacteria are in the colony after 60 hours

b how many bacteria are in the colony after 5 days?

Steps	Working
a 1 Write the known data.	The initial value is 2000 and the growth factor is 3.
2 Write the function and change 60 h to days.	$n(x) = 2000 \times 3^x$, where x is in days 60 h = 2.5 days
3 Find the required value.	$n(2.5) = 2000 \times 3^{2.5} = 31\,176.91\ldots$
4 Round the answer appropriately.	There are about 31 000 bacteria after 60 hours.
b 1 Find the required value.	$n(5) = 2000 \times 3^5 = 486\,000$
2 Write the answer.	There are about 486 000 bacteria after 5 days.

An exponential growth function becomes very large very quickly.

An exponential decay function becomes very small very quickly.

WORKED EXAMPLE 7 Exponential decay

The amount of air left in a leaking spacecraft is decreasing by 30% every minute.

When the leak first occurred, the pressure was 105 kPa. What will the pressure be after 6 minutes?

Steps	Working
1 Write the known data.	The initial value is 105 and the decay factor is 0.7.
2 Write the function.	$P(t) = 105 \times 0.7^t$
3 Find the required value.	$P(6) = 105 \times 0.7^6 = 12.353\ldots$
4 Round the answer appropriately.	After 6 minutes the pressure is about 12.4 kPa.

Exam hack

A percentage increase of 10% means the growth factor r = (100 + 10)%, or 1.1.
A percentage decrease of 10% means the decay factor r = (100 – 10)%, or 0.9.

In the spacecraft example, the air pressure would be about 0.002 kPa after half an hour. There would be so little air left, it would be virtually undetectable.

Theoretically, exponential decay never reaches zero, but it does become extremely small.

EXERCISE 8.3 Exponential functions

ANSWERS p. 395

Recap

1 213 000 000 written in scientific notation is

A 213×10^6 **B** 2.13×10^8 **C** 2.13×10^9 **D** 2.13×10^{-8}

2 $4.1 \times 10^5 \times 7.1 \times 10^4$ is equal to

A 2.911×10^9 **B** 2.911×10^{10} **C** 2.911×10^{20} **D** 2.911×10^{21}

Mastery

3 WORKED EXAMPLE 6 An algal bloom covers 3 m^2. The area doubles in size every week. What area will it cover after 10 weeks?

4 WORKED EXAMPLE 7 A large capacitor of 0.2 coulombs charge discharges to 0.1 C in 1 ms (0.001 s). The decay of the charge is exponential.

a What will the charge be after 20 ms?

b 1 ampere (1 A) of current is 1 C/s. What was the average current in the first 0.01 s?

c When will the charge be undetectable (less than 10^{-12} C)?

Exam practice

5 (4 marks) $f(x) = 0.3 \times 4.1^x$ and $g(x) = 8 \times 0.93^x$. Find the values of $f(x)$ and $g(x)$ at $x = -2$, 0, 2 and 4, correct to 3 decimal places, and comment on the nature of the functions.

6 (3 marks) An algal bloom covers about 3 m^2. The area increases by about 5% each day. What area will it cover after 20 days?

7 (3 marks) The value of a car depreciates by about 15% per year. What is a car bought for $24 900 worth after 5 years?

8 (3 marks) The atmospheric pressure decreases by about 12% for every 1000 m increase in altitude. The pressure at sea level is approximately 101 kPa. People need a minimum pressure of about 35 kPa to breathe. What is the highest altitude at which you could breathe?

9 (3 marks) There were about 606 000 crimes reported to police in Queensland in 2023. The number of reported crimes has grown by about 3.3% a year over the last 10 years. How many crimes would you expect to be reported in 2030?

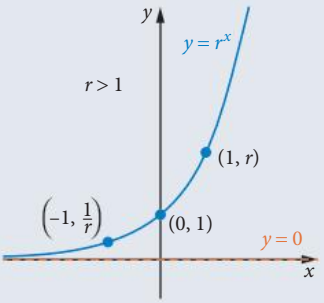
Worksheet
Graphing exponentials

8.4 Graphing exponential functions

Graphs of exponential functions

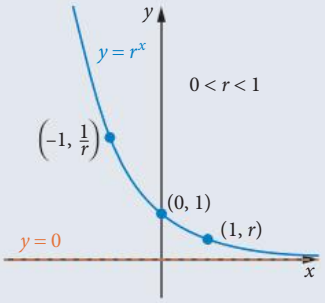
For an exponential function $y = r^x$ where $r > 1$:

- the graph is always increasing and passes through the points $\left(-1, \frac{1}{r}\right)$, $(0, 1)$ and $(1, r)$
- the y-intercept is 1
- as $x \to \infty, y \to \infty$
- as $x \to -\infty, y \to 0$, a horizontal asymptote exists at $y = 0$.



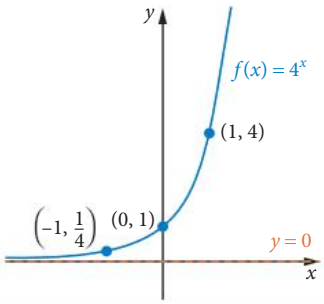
For an exponential function $y = r^x$ where $0 < r < 1$:

- the graph is always decreasing and passes through the points $\left(-1, \frac{1}{r}\right)$, $(0, 1)$ and $(1, r)$
- the y-intercept is 1
- as $x \to -\infty, y \to \infty$
- as $x \to \infty, y \to 0$; a horizontal asymptote exists at $y = 0$.



WORKED EXAMPLE 8 Graph of exponential growth 1

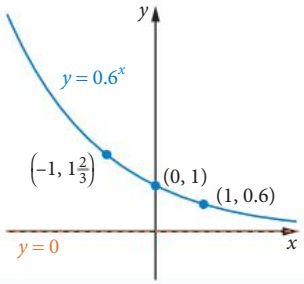
Sketch the graph of $f(x) = 4^x$.

Steps	Working
1 State the general shape: $r = 4 > 1$.	The function is increasing with asymptote $y = 0$.
2 Find important points.	$f(0) = 1, f(1) = 4, f(-1) = \frac{1}{4}$: $\left(-1, \frac{1}{4}\right)$, $(0, 1)$ and $(1, 4)$ are on the graph.
3 State the behaviour as $x \to \pm\infty$.	As $x \to -\infty, f(x) \to 0$. As $x \to \infty, f(x) \to \infty$.
4 Sketch the graph. Use the important points to get a smooth curve.	

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WORKED EXAMPLE 9 Graph of exponential decay

Sketch the graph of $y = 0.6^x$.

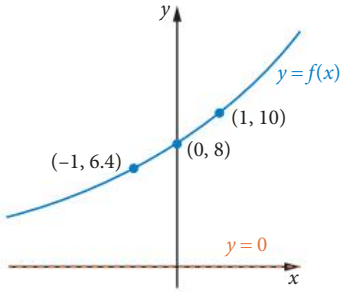
Steps	Working
1 State the general shape: $r = 0.6 < 1$.	The function is decreasing with asymptote $y = 0$.
2 Find important points.	$\left(-1, 1\frac{2}{3}\right)$, $(0, 1)$ and $(1, 0.6)$ are on the graph.
3 State the behaviour as $x \to \pm\infty$.	As $x \to -\infty$, $y \to \infty$. As $x \to \infty$, $y \to 0$.
4 Sketch the graph. Use the important points to get a smooth curve.	

Most applications do not have an initial value of 1. In these cases you can use an initial value A, so the function is $f(x) = Ar^x$.

WORKED EXAMPLE 10 Graph of exponential growth 2

An exponential growth function $f(x)$ has an initial value of 8. At time $t = 1$, the value is 10.

Draw a graph of the function.

Steps	Working
1 Write the equation for exponential growth.	$f(x) = Ar^x$
2 Write the initial value of $f(x)$.	The initial value $A = 8$.
3 Write the revised equation.	$f(x) = 8r^x$
4 To find r, substitute $f(1) = 10$.	$10 = 8r^1$ $r = \frac{10}{8}$ $= 1.25$
5 Write the function $f(x)$.	$f(x) = 8 \times 1.25^x$
6 State the general shape.	The function is increasing with asymptote $y = 0$.
7 Find important points of $f(x)$.	$(-1, 6.4)$, $(0, 8)$ and $(1, 10)$ are on the graph of $f(x)$.
8 State the behaviour as $x \to \pm\infty$.	As $x \to -\infty$, $f(x) \to 0$. As $x \to \infty$, $f(x) \to \infty$.
9 Sketch the graph of $f(x)$.	

EXERCISE 8.4 Graphing exponential functions

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Recap

1 The price of a bag of oranges is currently $5. The price increases 3% each year. If this trend continues, the price of the bag of oranges is 10 years' time would be

A $0 **B** $6.72 **C** $11.59 **D** $295 245

2 A car depreciates (decreases) in value over time. If a car was bought in 2016 for $24 990 and depreciates 10% each year, determine its value in 2025.

A $2499 **B** $8713.47 **C** $9681.64 **D** $10 757.38

Mastery

3 WORKED EXAMPLE 8 Sketch the graph of $f(x) = 6^x$.

4 WORKED EXAMPLE 9 Sketch the graph of $y = 0.7^x$.

5 WORKED EXAMPLE 10 $f(x)$ is an exponential function with an initial value of 5. After 1 unit of time, it has a value of 10. Sketch the function.

6 Sketch the graph of $y = 2.5^x$.

7 Sketch the graph of $f(x) = 2 \times 3^x$.

8 Sketch the graph of $y = 0.5 \times 4^x$.

9 Sketch the graph of $f(x) = 3 \times 0.6^x$.

10 Sketch the graph of $y = 5 \times 1.5^x$.

11 Sketch the graph of $f(x) = -3^x$.

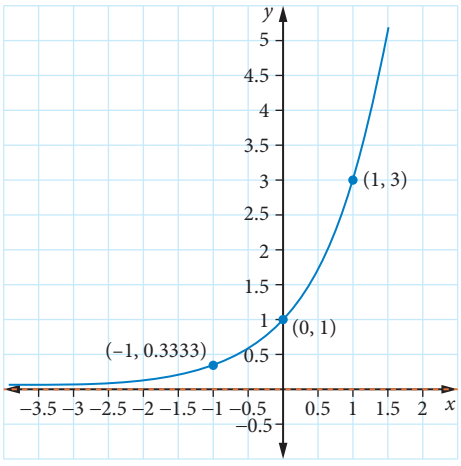
12 Sketch the graph of $y = -4 \times 0.8^x$.

13 Sketch the graph of $y = -0.5 \times 2.5^x$.

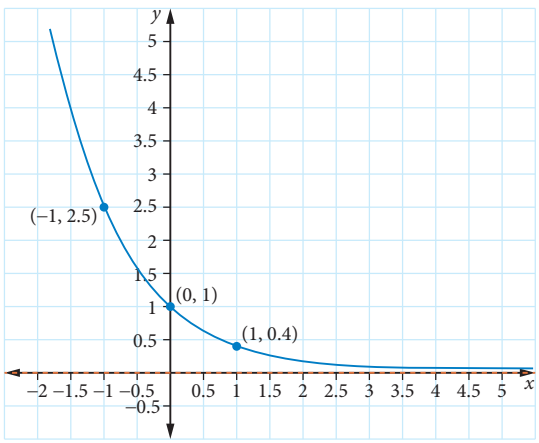
Exam practice

14 TECH-FREE (2 marks each) CF Determine the equation of each exponential graph.

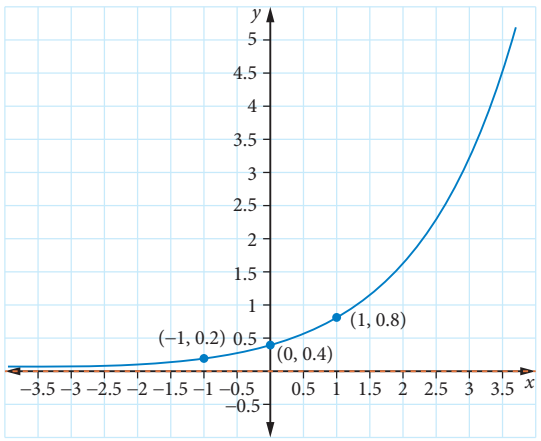
a



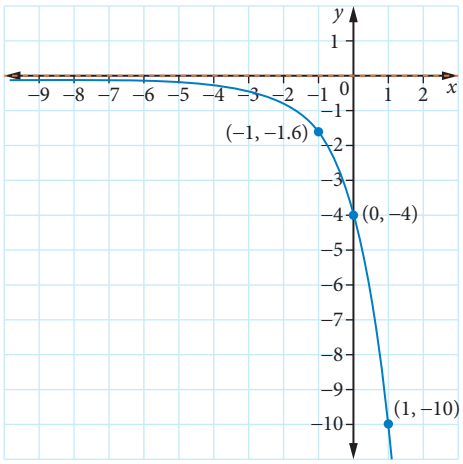
b



c



d



8.5

8.5 Translating exponential functions

Worksheet
Translating exponential graphs

Horizontal translations

$y = r^{x-h}$ is translated horizontally by h units compared to $y = r^x$.

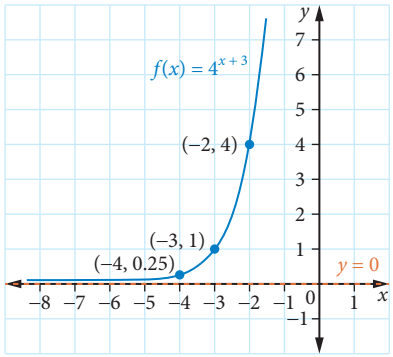
- For $h > 0$, the graph translates h units to the right.
- For $h < 0$, the graph translates h units to the left.

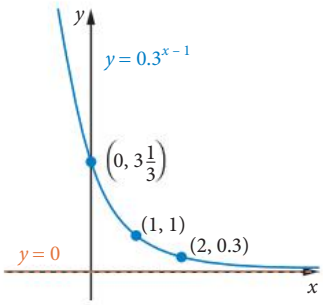
You can think of h as making the y values occur later or earlier, shifting them horizontally. The y-intercept is also an important point.

WORKED EXAMPLE 11 Horizontal translations

Sketch the graph of each exponential function.

a $f(x) = 4^{x+3}$ **b** $y = 0.3^{x-1}$

Steps	Working
a 1 State the general shape.	The function is increasing with asymptote $y = 0$.
2 $h = -3$: Find the points with $x = -3, -4, -2$ and 0.	It is the graph of $y = 4^x$ translated 3 units left. $f(-3) = 1, f(-4) = \frac{1}{4}, f(-2) = 4, f(0) = 64$
3 State the behaviour as $x \to \pm\infty$.	As $x \to -\infty, f(x) \to 0$. As $x \to \infty, f(x) \to \infty$.
4 Sketch the graph. Join the important points with a smooth curve.	

b **1**	State the general shape.	The function is decreasing with asymptote $y = 0$.
2	$h = 1$: Find important points at $x = 1$, 0 and 2.	It is the graph of $y = 0.3^x$ translated 1 unit right. $f(1) = 1, f(0) = 3\frac{1}{3}, f(2) = 0.3$
3	State the behaviour as $x \to \pm\infty$.	As $x \to -\infty, y \to \infty$. As $x \to \infty, y \to 0$.
4	Sketch the graph.	

Vertical translations

$y = r^x + k$ is translated vertically by k units compared to $y = r^x$.

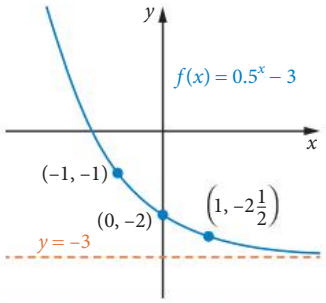
- For $k > 0$, the graph translates k units up.
- For $k < 0$, the graph translates k units down.
- The horizontal asymptote moves to $y = k$.

Draw a dashed line for your asymptote. Write the equation of the asymptote on the line.

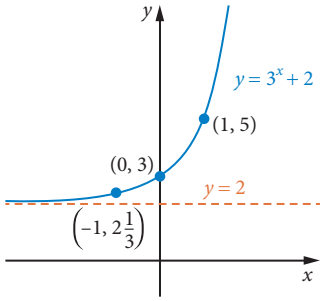
WORKED EXAMPLE 12 Vertical translations

Sketch the graph of each exponential function.

a $f(x) = 0.5^x - 3$ **b** $y = 3^x + 2$

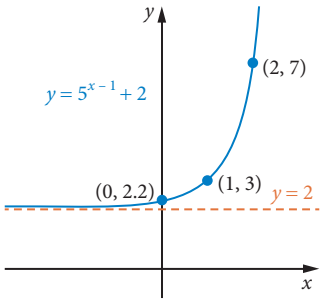
Steps		Working
a **1**	State the general shape.	The function is decreasing with asymptote $y = -3$.
2	Find important points at $x = 0$, -1 and 1.	It is the graph of $y = 0.5^x$ translated 3 units down. $f(0) = -2, f(-1) = -1, f(1) = -2\frac{1}{2}$
3	State the behaviour as $x \to \pm\infty$.	As $x \to -\infty, f(x) \to \infty$. As $x \to \infty, f(x) \to -3$.
4	Sketch the graph.	

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b 1 State the general shape.	The function is increasing with asymptote $y = 2$.
2 Find important points at $x = 0, -1$ and 1.	It is the graph of $y = 3^x$ translated 2 units up. $f(0) = 3, f(-1) = 2\frac{1}{3}, f(1) = 5$
3 State the behaviour as $x \to \pm\infty$.	As $x \to -\infty, y \to 2$. As $x \to \infty, y \to \infty$.
4 Sketch the graph.	

WORKED EXAMPLE 13 Combined translations

Sketch the graph of $y = 5^{x-1} + 2$.

Steps	Working
1 State the general shape. $k = 2$	The function is increasing with asymptote $y = 2$.
2 $h = 1$: Find important points at $x = 1, 0$ and 1.	It is the graph of $y = 5^x$ translated 1 unit right and 2 units up. $f(1) = 3, f(0) = 2.2, f(2) = 7$
3 State the behaviour as $x \to \pm\infty$.	As $x \to -\infty, y \to 2$. As $x \to \infty, y \to \infty$.
4 Sketch the graph.	

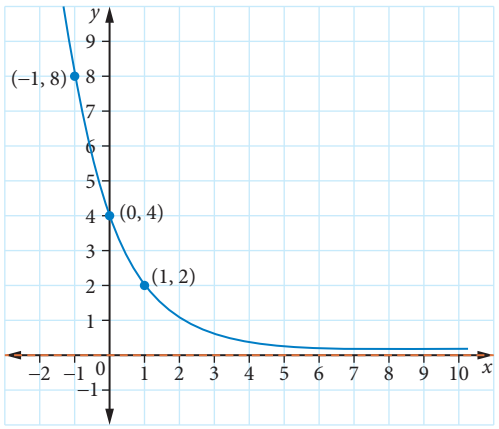
EXERCISE 8.5 Translating exponential functions

ANSWERS p. 396

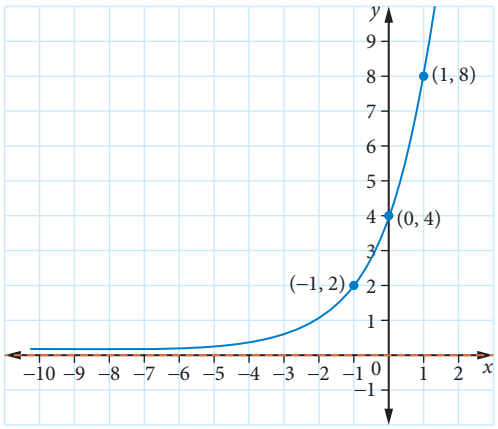
Recap

1 The graph of $y = 4 \times 2^x$ is

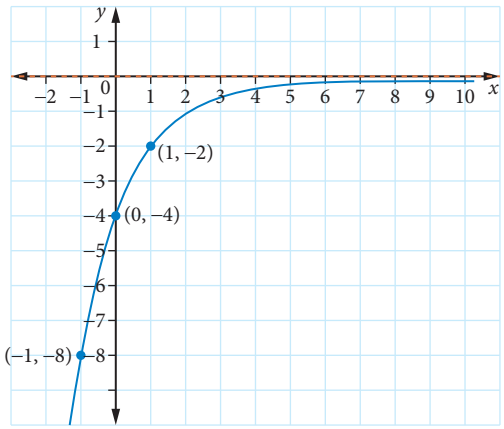
A



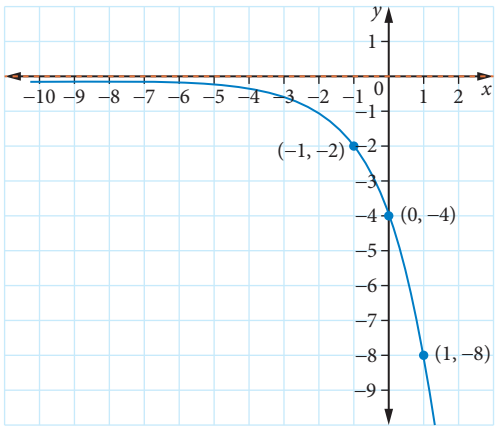
B



C

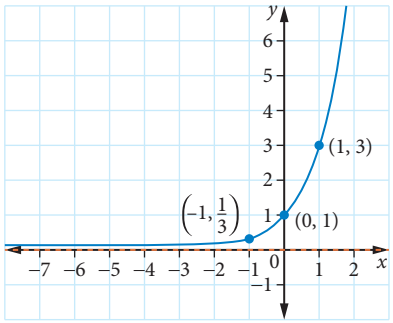


D

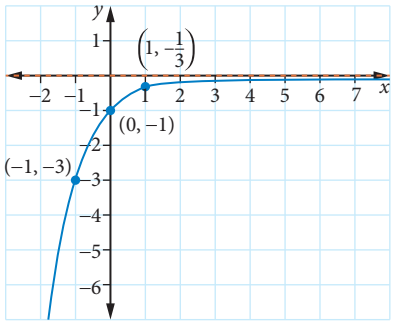


2 The graph of $y = 3^{-x}$ is

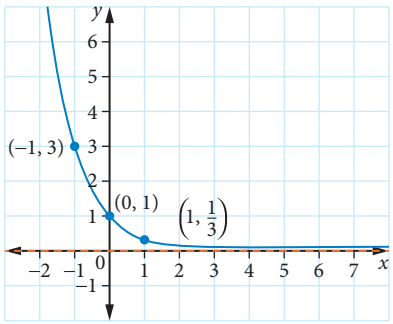
A



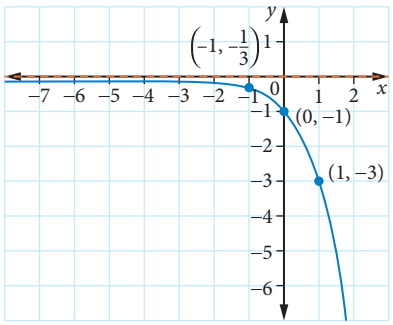
B



C



D



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Mastery

3 WORKED EXAMPLE 11 Sketch the graph of $y = 0.2^{x-2}$.

4 WORKED EXAMPLE 12 Sketch the graph of $f(x) = 2^x + 3$.

5 WORKED EXAMPLE 13 Sketch the graph of $y = 0.7^{x-2} + 3$.

6 Sketch the graph of $f(x) = 3^{x+3}$.

7 Sketch the graph of $y = 0.5^{x-1}$.

8 Sketch the graph of $y = 0.8^x - 2$.

9 Sketch the graph of $f(x) = 5^x + 1$.

10 Sketch the graph of $f(x) = 6^{x+2} - 1$.

11 Sketch the graph of $y = 0.9^{x-1} - 3$.

12 Sketch the graph of $y = -0.6^{x-3} + 2$.

13 Sketch the graph of $f(x) = 2 \times 3^{x+1} + 4$.

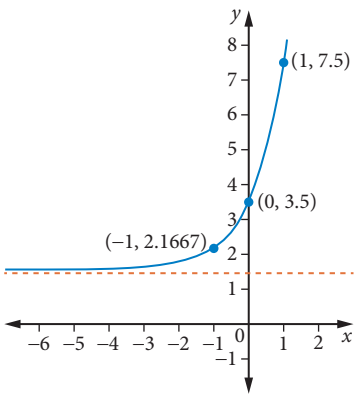
14 Sketch the graph of $y = -0.4 \times 0.7^{2x-4} - 1$.

15 Sketch the graph of $f(x) = 0.5 \times 2^{-x+2} - 4$.

Exam practice

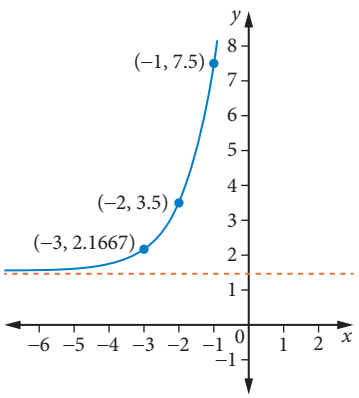
16 (6 marks) CF Determine the equation of each exponential graph.

a



[3 marks]

b



[3 marks]

Video
Exponential equations 1

Worksheet
Solving exponential equations

8.6 Exponential equations

WORKED EXAMPLE 14 Exponential equations

Solve each exponential equation.

a $3^{2x-5} = 27$ **b** $16^x - 5 \times 4^x = 24$

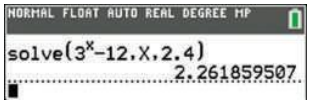
Steps	Working
a 1 Write the powers to the same base: $27 = 3^3$.	$3^{2x-5} = 3^3$
2 Equate indices and solve.	$2x - 5 = 3$ $2x = 8$ $x = 4$
3 Check in the original equation.	LHS $= 3^{2 \times 4 - 5}$ $= 3^3$ $= 27$ $=$ RHS ✓OK
4 Write the answer.	The solution is $x = 4$.
b 1 Write in powers to the same base: $16 = 4^2$ to form a quadratic equation.	$(4^2)^x - 5 \times 4^x - 24 = 0$ $4^{2x} - 5 \times 4^x - 24 = 0$ $(4^x)^2 - 5 \times 4^x - 24 = 0$
2 Substitute for 4^x to make equation clearer.	$y^2 - 5y - 24 = 0$ for $y = 4^x$
3 Solve the quadratic equation.	$(y - 8)(y + 3) = 0$ $y = 8$ or $y = -3$
4 Express in terms of 4^x.	$4^x = 8$ or $4^x = -3$
5 Discard the false solution.	$4^x = -3$ is not possible, so $4^x = 8$.
6 Write in powers to the same base and equate indices to solve.	$2^{2x} = 2^3$ $2x = 3$ $x = \frac{3}{2}$
7 Check in the original equation.	LHS $= 16^{\frac{3}{2}} - 5 \times 4^{\frac{3}{2}}$ $= \left(\sqrt{16}\right)^3 - 5 \times \left(\sqrt{4}\right)^3$ $= 4^3 - 5 \times 2^3$ $= 64 - 5 \times 8$ $= 24$ $=$ RHS ✓OK
8 Write the answer.	The solution is $x = \frac{3}{2}$.

You can use technology for exponential equations.

TECHNOLOGY Exponential equations

Use a graphics calculator to solve $3^x = 12$, correct to 3 decimal places.

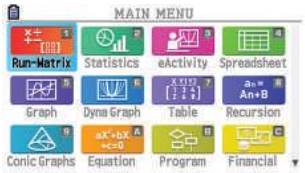
TI-84 Plus CE



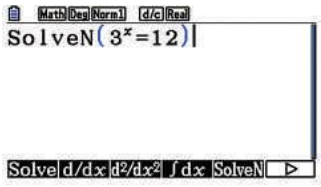
Write as *expression* = 0: $3^x - 12 = 0$.

1 Use 2nd 0 ([catalog]) to select **solve(**.
2 Enter the equation in the form *expression, variable, guess*).
3 Press ENTER.

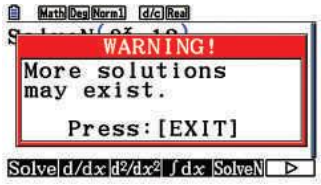
Casio fx-CG50 AU



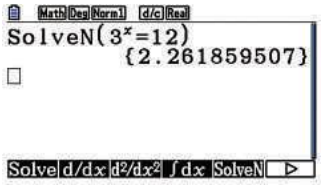
1 From the **MAIN MENU**, press **1** to select **Run-Matrix**.
2 Press OPTN, then press F4 (CALC) and then press F5 (SolveN).



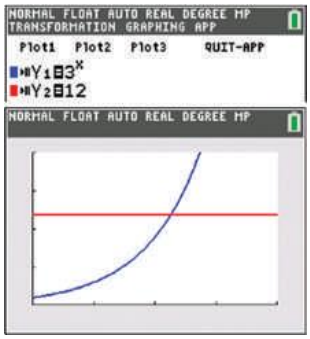
3 Type the equation into the calculator, remembering to close the brackets at the end of the equation.
4 Press EXE.



5 Press EXIT if this warning appears.
6 The value will appear on the screen.

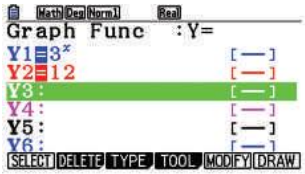


Alternatively, solve graphically



1 Press y=, enter 3^x in Y1 and 12 in Y2.
2 Set the window appropriately ($0 \le x \le 4$, $0 \le y \le 20$).

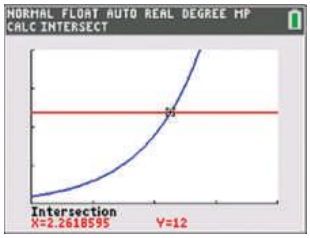
1 From the **MAIN MENU**, press **5** to select **Graph**.



2 Insert each side of the equation as a separate Y function.
3 Press SHIFT then F3 (V-WIN) to ensure domain and range are set appropriately.

TI-Nspire
Exponential functions

3 Press graph.

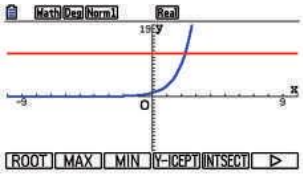


4 Press 2nd trace ([calc]) and choose **5: intersect**.

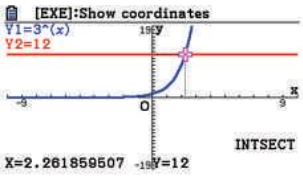
5 Press ENTER twice to select the functions and guess, say, 2.

$x \approx 2.262$

4 Press F6 (DRAW) to sketch the graphs.



5 Press F5 (G-Solv) and then press F5 (INTSECT) to determine the intersection point of the graphs.



6 Coordinates of the intersection point will appear at the bottom of the screen.

$x \approx 2.262$

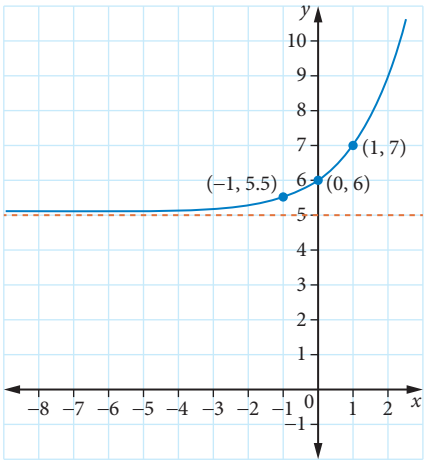
EXERCISE 8.6 Exponential equations

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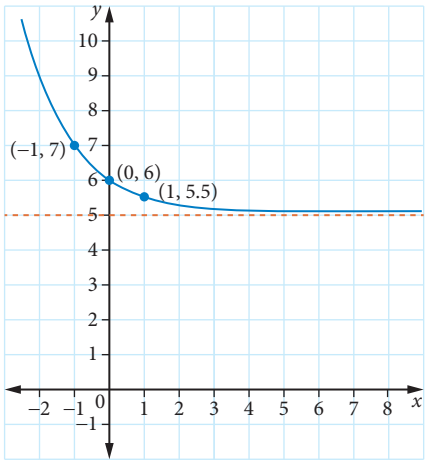
Recap

1 The sketch of $y = -2^x + 5$ is

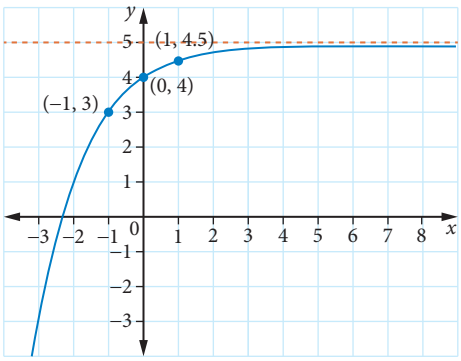
A



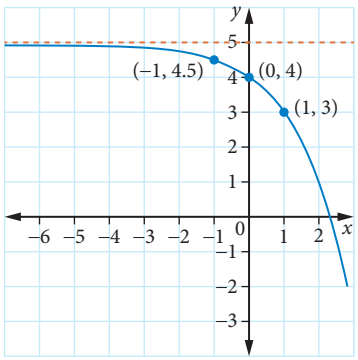
B



C

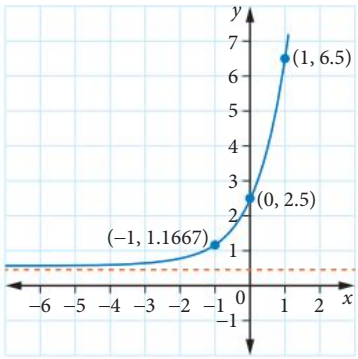


D



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2 A possible equation for this graph is



A $y = 3 \times 2^x + 0.5$

B $y = -2 \times 3^x + 0.5$

C $y = 2 \times 3^x + 0.5$

D $y = 2 \times 3^{-x} + 0.5$

Mastery

3 WORKED EXAMPLE 14 TECH-FREE Solve each exponential equation.

a $2^{x+1} = 32$ **b** $9^{x-5} = 243$ **c** $4^{3x+1} = 32$

d $5^{3-2x} - 125 = 0$ **e** $16^{x-2} - 8 = 0$ **f** $49 - 343^{3x+3} = 0$

g $64 - 16^{5-2x} = 0$ **h** $25^{1-x} - 125 = 0$ **i** $64^{3x+1} = 32^{5x-4}$

Exam practice

4 TECH-FREE (5 marks) Solve each exponential equation.

a $9^x - 90 \times 3^x + 729 = 0$ **b** $4 \times 4^x = 3 \times 2^{x+3} + 64$

5 (6 marks) Solve each exponential equation using a graphics calculator.

a $50^{2x-3} - 28 = 0$ **b** $5^x - 4^x - 3 = 0$ **c** $10^{3x} - 150 = 2^{4x}$

6 TECH-FREE (6 marks) CF Solve the equation $8^x + 7 \times 2^{x+1} = 7 \times 4^x + 8$.

7 TECH-FREE (6 marks) CF Solve the equation $3^{3x} - 12 \times 9^x + 63 \times 3^x = 4 \times 3^3$.

8.7 Modelling with exponential functions

There are many real-life applications of exponential functions.

Worksheet Using exponential models

WORKED EXAMPLE 15 Using an exponential model

The temperature T of a hot object as it cools is modelled by $(T - T_E) = (T_0 - T_E) \times 0.9^t$, where T_E is the temperature of the environment, T_0 is the initial temperature and t is the time in minutes.

The initial temperature is 88°C and the temperature of the surroundings is 26°C.

a Find the temperature after 10 minutes.

b Find the time taken to cool to 40°C.

c **Verify** and **evaluate** the model.

Steps	Working
a 1 Substitute values into the equation.	$T - 26 = (88 - 26) \times 0.9^t$
2 Rearrange to make T the subject.	$T = 26 + 62 \times 0.9^t$
3 Substitute $t = 10$.	$= 26 + 62 \times 0.9^{10}$ $= 47.618\ldots$
4 Write the answer.	After 10 minutes, the temperature is about 48°C.
b 1 Substitute $T = 40$ and solve for t.	$T = 26 + 62(0.9^t)$ $40 = 26 + 62(0.9^t)$ $14 = 62(0.9^t)$ Solve using a graphics calculator. $t = 14.12\ldots$ minutes
2 Write the answer.	It takes a bit over 14 minutes to cool to 40°C.
c 1 Verify the model against the data.	$T(0) = 26 + 62 \times 0.9^0 = 26 + 62 = 88°C$ As $t \to \infty$, $0.9^t \to 0$, so $T(t) \to 26°C$ so the model fulfils the conditions.
2 Evaluate the model.	The model allows calculation of temperatures at any time, so is a good predictor of future conditions.

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You can use your graphics calculator to fit data to a model. The method is called **regression analysis**.

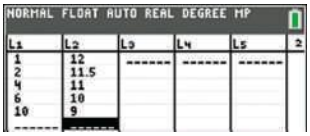
WORKED EXAMPLE 16 Exponential models from data

The amplitude of a pendulum is found to change with time as shown in the table.

Try an exponential model for the amplitude and predict the amplitude after 20 minutes.
Evaluate the reasonableness of your model.

Time in minutes	1	2	4	6	10
Amplitude in cm	12	11.5	11	10	9

TI-84 Plus CE



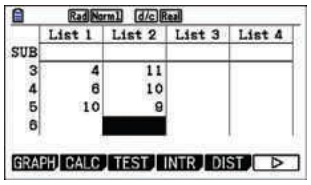
1 Press stat, then **1** to choose **1: Edit**.
If there is already data in the lists, you can press DEL repeatedly to remove it, or just overtype it. Put the times in L1, move across to L2 and put in the amplitudes. Do not leave any old data at the bottom.



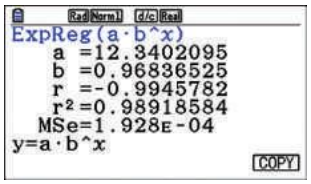
2 Press 2nd $y=$ (stat plot) and **1:Plot1**.
3 Turn Plot1 On, move down, make the Type a scatterplot, XList: L1 (2nd 1) and YList: L2.
4 Press stat again, move across to CALC and choose **0: ExpReg**.
5 Make sure that XList is L1, YList is L2 and FreqList is clear.
6 Move down and press ENTER to calculate.

CASIO fx-CG50 AU

1 From the **MAIN MENU**, press **2** to select **Statistics** and enter the data in List 1 and List 2.



2 Press F2 (CALC), then press F3 (REG), then F6 to scroll across, then F2 (EXP) and then press F2 (ab^x) to find the function.



Steps	Working
1 Write the function.	The function is $A(t) \approx 12.34 \times (0.9684)^t$.
2 Find the value after 20 minutes.	$A(20) \approx 12.34 \times (0.9684)^{20} \approx 6.492$
3 Write the answer.	The amplitude is given by $12.34 \times (0.9684)^t$, where t is in minutes. It is about 6.5 cm after 20 minutes.
4 Evaluate the model.	The value of r^2 is very close to 1, so the exponential model works well for the 10 minutes shown. It may not be as good a model over a longer time.

TI-Nspire
Exponential functions

EXERCISE 8.7 Modelling with exponential functions

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Recap

1 The solution to $27 = 9^{x-1}$ is

A 2.5 **B** 3 **C** 3.5 **D** 4

2 The solution to $4^x - 3 \times 2^x - 4 = 0$ is

A -1 **B** 0 **C** 2 **D** 4

Mastery

3 WORKED EXAMPLE 15 A full 30 000 L cylindrical rainwater tank has sprung a leak in the seam at the bottom of the tank. As the water leaks out, the pressure of water decreases. This means that the tank leaks more slowly over time. The level of water in the tank is given by $L = 295 \times 0.97^t$ cm, where t is the time in hours from the start of the leak. The leak was discovered 3 hours after it started.

a What is the diameter of the tank?

b How much water leaked out before the leak was discovered?

c How long will it take for 5000 L to leak out?

4 WORKED EXAMPLE 16 The voltage across a capacitor as it discharges through a resistor was measured by students. The values are shown below.

Time (t seconds)	4	8	12	20	30	40
Voltage (kV)	49	40	33	22	13	8

a Use an exponential model to find the original voltage. The capacitor is considered to be fully discharged when the voltage drops below 20 V. How many seconds does this take?

b If the original voltage was 3 times as high, how many seconds would the capacitor take to drop to a voltage of 5 kV?

Evaluate your model and answers.

Exam practice

5 (4 marks)

a The concentration of paracetamol in an adult's blood is about $25 \times (0.65)^t$ µg/mL t hours after taking a standard adult dose of 1 gram. What is the level after 2 hours? *[2 marks]*

b The therapeutic level is considered to be a minimum of 5 µg/mL, but high levels are extremely toxic. How long after taking a dose does the effect wear off? *[2 marks]*

6 (4 marks) The population of rabbits in an area increases by an average of 50% every month. How long would it take 2 rabbits to overrun an area at a population of 3000 rabbits?

7 (5 marks) A sample of radioactive material loses 4% of its radioactivity per annum. Calculate the half-life of the material. The half-life is the time for half the material to decay.

8 (5 marks) A sample of carbon from an ancient midden has only 40% of the original amount of carbon-14 present. The carbon-14 is known to have decayed at a rate of 1.2% per century from the time the midden was used. How old is the midden?

9 (5 marks) CF The size of a bacteria colony on an agar plate in an incubator increases by 13.5% every hour. Construct a function to model the size and use it to find the size of a colony after a day, given that it begins with a size of 0.1 mm^2. Are there any limitations on your model?

10 (6 marks) CF The temperature in the core of a nuclear reactor is about 64°C. In a disaster such as that at Fukushima, Japan in 2011, the temperature rise can cause a dangerous explosion. The table shows temperatures recorded in a nuclear core after a problem was encountered.

Time (min)	0	5	10	20	30
Temperature (°C)	64	69	73	89	101

If the temperature exceeds 1500°C, the core begins to melt down.

Use an exponential model to find how long the operators have got to avert a disaster.

11 (6 marks) CF Iodine-131 decays to xenon, so the mass of a sample decreases as the xenon escapes into the atmosphere. The mass of some iodine-131 after being made is shown below.

Time (days)	0	1	2	3	4	5	6	7
Mass (mg)	87	77	71	67	59	56	51	46

Use an exponential model for the amount of iodine-131 remaining to find the half-life of iodine-131 and the time for it to decay to 10 mg. Verify and your model and evaluate its limitations.

12 (6 marks) CU The difference in temperature between a hot pie and its surroundings decreases by 4% every minute. A hot pie bought at a Broncos–Raiders game starts at a temperature of 72°C, while the temperature at the ground is 15°C. It takes 3 minutes to walk back to the stand, where you start eating the pie.

Find the temperature when you start eating, the temperature 10 minutes after buying the pie and the time taken for its temperature to drop below 40°C. Evaluate the usefulness of your model.

Video
Exam question analysis: Exponential functions

EXAM QUESTION ANALYSIS

Exam-style question (7 marks)

A beekeeper is starting a new colony of bees. The population B of bees, in thousands, is given by

$$B(t) = 4 \times 4^t$$

where t is the number of years since the establishment of the colony.

a Determine the initial population of the bee colony. *[1 mark]*

b Determine the increase in the population of the bee colony in the first 6 months. *[2 marks]*

After three years, the beekeeper notices that the number of bees is beginning to decline. The declining population, b, in thousands, has the form $b(t) = A \times r^t$, where t is the number of years since the start of the decline.

c Determine the values of A and r if one year after the start of the decline the bee population is 100 000. *[4 marks]*

Reading the question

- Determine initial population.
- Determine change in population over time.
- Calculate parameters A and r in the declining population equation.

Thinking about the question

- Population is in thousands.
- t is in years, part **a** asks for 6 months; change to years.
- 'Increase in population' means the change in population.
- There is a change in the time for $t = 0$ when the equations change.
- Decline means r should be negative.

Worked solution (✓ = 1 mark)

a $B(0) = 4 \times 4^0 = 4$

Therefore, initial population is **4000 bees**. ✓ Correct value, inclusive of thousands.

b 6 months = 0.5 year

$B(0.5) = 4 \times 4^{0.5}$

$= 8$

8000 bees after 6 months.

Increase in population = 8000 – 4000

= 4000 ✓ Relationship to find increase

The population has increased by about **4000 bees** in the first 6 months.

✓ Solved value, inclusive of thousands

c $B(3) = \mathbf{4 \times 4^3 = 256}$ ✓ Relationship between populations

$A = \mathbf{256}$ ✓ Solve for A value

$b(t) = 256r^t$

Substitute $b(1) = 100$

$\mathbf{100 = 256 \times r^1}$ ✓ Substitution into $b(t)$

$r = \dfrac{100}{256} \approx \mathbf{0.390\,625}$ ✓ Solve for r value

8 Chapter summary

Index laws and fractional indices

- $a^m \times a^n = a^{m+n}$
- $a^m \div a^n = a^{m-n}$
- $(a^m)^n = a^{mn}$
- $a^0 = 1,\ a \neq 0$
- $a^{-m} = \frac{1}{a^m},\ a \neq 0$
- $(ab)^m = a^m \times b^m$
- $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$
- $a^{\frac{p}{q}} = \sqrt[q]{a^p} = \left(\sqrt[q]{a}\right)^p$, where a is positive, p is an integer and q is a positive integer

Scientific notation

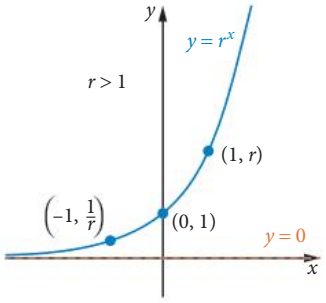
- Numbers in **scientific notation** have a **decimal** from 1 to 10, multiplied by a power of 10. Large numbers have positive indices, small numbers have negative indices.
- The number of **significant figures** of a number is the number of digits that *must* be written for the decimal when writing the number in scientific notation.

Exponential functions

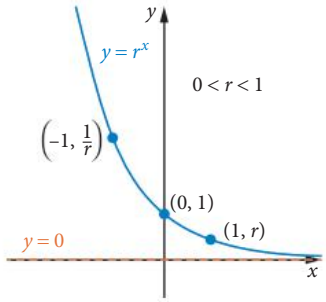
- The general **exponential function** has the form $y = A \times r^x$, where $r > 0$.
- When $r > 1$, it is a **growth function**.
- When $0 < r < 1$, it is a **decay function**.
- r is the **growth** or **decay factor**.
- A is the **initial value**.

Graphing exponential functions $y = r^x$

- For an exponential function $y = r^x$ where $r > 1$



  - the graph passes through the points $\left(-1, \frac{1}{r}\right)$, $(0, 1)$ and $(1, r)$
  - the y-intercept is 1
  - as $x \to \infty,\ y \to \infty$. As $x \to -\infty,\ y \to 0$
  - a horizontal asymptote exists at $y = 0$.
- For an exponential function $y = r^x$ where $0 < r < 1$,



  - the graph passes through the points $\left(-1, \frac{1}{r}\right)$, $(0, 1)$ and $(1, r)$
  - the y-intercept is 1
  - as $x \to \infty,\ y \to 0$
  - as $x \to -\infty,\ y \to \infty$
  - a horizontal asymptote exists at $y = 0$.
- Applications with an initial value of $A \neq 1$ can be modelled as $f(x) = Ar^x$.

Translating exponential functions $y = r^{x-h} + k$

- $y = r^{x-h}$ is translated horizontally h units compared to $y = r^x$.
  - For $h > 0$, the graph translates h units to the right.
  - For $h < 0$, the graph translates h units to the left.
- $y = r^x + k$ is translated vertically k units compared to $y = r^x$.
  - For $k > 0$, the graph translates k units up.
  - For $k < 0$, the graph translates k units down.
  - The horizontal asymptote moves to $y = k$.

Exponential equations

- Solve equations with related bases by making the bases the same.
- Use technology for exponential equations with different bases.

Modelling with exponential functions

Using the initial value A, use $A(t) = A \times k^t$.

Regression analysis using a graphics calculator is used to fit an exponential model to data.

Worksheet
Maths Methods Year 11 formula sheet

Cumulative examination 1

Technology-free

Perusal time: 2 minutes Working time: 30 minutes

TECH-FREE *Calculators are **not** permitted.*

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

Section 1 (3 marks): 3 multiple choice questions

Section 2 (15 marks): 4 short response questions

Total: 18 marks

Section 1 3 multiple choice questions

3 marks

1 The exact value of $\cos\left(\frac{2\pi}{3}\right)$ is

A $\frac{1}{2}$ **B** $\frac{\sqrt{3}}{2}$ **C** $-\frac{\sqrt{3}}{2}$ **D** $-\frac{1}{2}$

2 A standard deck of 52 cards has 13 hearts. The probability of not drawing a heart from a single draw from the deck is

A $\frac{1}{52}$ **B** $\frac{1}{13}$ **C** $\frac{13}{52}$ **D** $\frac{3}{4}$

3 In the exponential function $y = -6 \times 4^{x-1} + 3$, the 3 is the

A horizontal translation **B** initial value

C growth factor **D** vertical translation

Section 2 4 short response questions

15 marks

4 (3 marks) Describe the transformations that the graph $y = 5^x$ undergoes to become $y = -2 \times 5^x + 3$.

5 (4 marks) Solve $2^{3x-2} = 128$.

6 (4 marks) Use differentiation by first principles to determine the derivative of $f(x) = -5x^2 + 2x - 6$.

7 (4 marks) Determine all values of x such that $1 - \tan\left(\frac{2\pi x}{3}\right) = 2$ for $0 \leq x \leq 2\pi$.

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Cumulative examination 2

Technology-active

Perusal time: 2 minutes Working time: 30 minutes

*Calculators are **permitted**.*

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

Section 1 (3 marks): 3 multiple choice questions

Section 2 (15 marks): 4 short response questions

Total: 18 marks

Section 1 3 multiple choice questions

3 marks

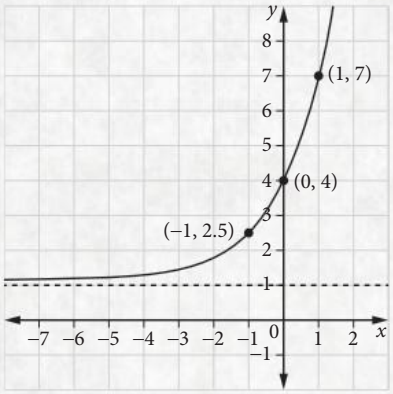
Worksheet
Maths Methods Year 11 formula sheet

1 The gradient of the function $y = 4x^2 - 5x + 2$ at $x = 2$ is

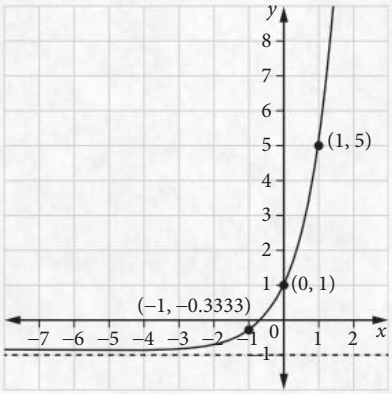
A -1 **B** 6 **C** 11 **D** 27

2 The graph of the exponential function $y = 3 \times 2^x - 1$ is

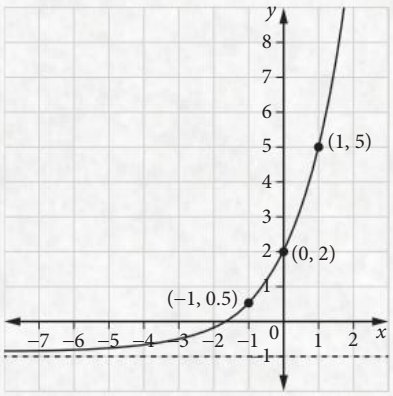
A



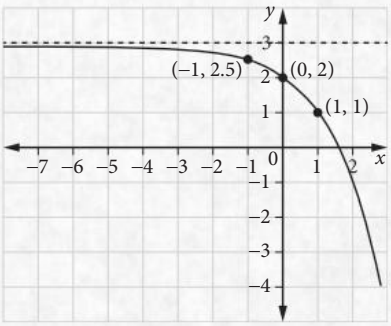
B



C



D



3 Solve $5^x = 20$

A $x = 1.86$ **B** $x = 2$ **C** $x = 4$ **D** $x = 100$

Section 2 4 short response questions

15 marks

4 (3 marks) Sketch the graph of $y = \sqrt{x - 1} + 5$, labelling all key points.

5 (4 marks) Two 6-sided die are rolled and the two values are added together.
Determine the probability that the sum is greater than 9.

6 (4 marks) Sketch the graph of $y = -6 \times 4^{x-1} + 3$.

7 (4 marks) A population of koalas was counted in northern Brisbane. The population at the time was 400. The population is decreasing by 13% per year. If this trend continues, determine in how many years there will be no koalas left in northern Brisbane.

CHAPTER

9 LOGARITHMS

Syllabus coverage
Nelson MindTap chapter resources
Terminology
9.1 Logarithms
9.2 Logarithm laws
9.3 The change of base theorem
9.4 Applying logarithms to exponential equations
Exam question analysis
Chapter summary
Cumulative examination 1
Cumulative examination 2

Prior learning
Logarithms

Syllabus coverage

UNIT 2, TOPIC 2: LOGARITHMS AND LOGARITHMIC FUNCTIONS

Logarithms and logarithmic laws

- Define logarithms as indices, where $a^x = b$ is equivalent to $x = \log_a(b)$, and convert between both forms.
- Use logarithmic laws and definitions
  - $\log_a(x) + \log_a(y) = \log_a(xy)$
  - $\log_a(x) - \log_a(y) = \log_a\left(\frac{x}{y}\right)$
  - $\log_a(x^n) = n\log_a(x)$
  - $\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$
  - $\log_a(a) = 1$
  - $\log_a(1) = 0$
- Solve equations involving indices using logarithms, with and without technology.

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Videos (4):
9.1 Logarithms
9.2 Logarithm laws
9.4 Exponential equations 2
Exam question analysis Logarithms

Prior learning (1):
9.1 Logarithms

Worksheets (6):
9.1 Logarithms
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9.4 Solving exponential equations
- Exponential functions assignment
- Exponential functions extension

Cumulative exams Maths Methods Year 11 formula sheet

Puzzles (6):
9.1 Logarithms 1 • Logarithms 2
9.2 Logarithm laws
9.4 Logarithms - Solving equations 1
- Logarithms - Solving equations 2
- Exponential equations

Nelson MindTap

To access resources above, visit
cengage.com.au/nelsonmindtap

Terminology

base	change of base theorem	common logarithm	exponential equation
general logarithm	index (indices)	logarithm	logarithm laws
power			

9.1 Logarithms

John Napier invented **logarithms** in 1614. They were used for multiplication, division and trigonometry until about 1970, when modern calculators became available.

General and common logarithms

The **logarithm of x to the base a** is the power m such that $a^m = x$, where $a, x > 0$ and $a \neq 1$.

$m = \log_a(x)$

The **common logarithm** of x has the base 10. It is written as $\log(x)$.

$m = \log(x)$

It means $10^m = x$.

The index statement $a^m = x$ means the same as the logarithm statement $\log_a(x) = m$.



Video Logarithms

Worksheet Logarithms

Puzzles Logarithms 1

Logarithms 2

WORKED EXAMPLE 1 Index and logarithm form

Write in index form.

a $\log_3(243) = 5$

b $\log_4\left(\frac{1}{16}\right) = -2$

Write in logarithmic form.

c $5^3 = 125$

d $2^{-5} = \frac{1}{32}$

Steps	Working
a 1 Write the equivalence statement.	$m = \log_a(x) \Leftrightarrow a^m = x$
2 Substitute $a = 3$, $x = 243$ and $m = 5$.	$\log_3(243) = 5 \Leftrightarrow 3^5 = 243$
3 State the result.	The index form of $\log_3(243) = 5$ is $3^5 = 243$.
b 1 Write the equivalence statement.	$m = \log_a(x) \Leftrightarrow a^m = x$
2 Substitute $a = 4$, $x = \frac{1}{16}$ and $m = -2$.	$\log_4\left(\frac{1}{16}\right) = -2 \Leftrightarrow 4^{-2} = \frac{1}{16}$
3 State the result.	The index form of $\log_4\left(\frac{1}{16}\right) = -2$ is $4^{-2} = \frac{1}{16}$.
c 1 Write the equivalence statement.	$a^m = x \Leftrightarrow \log_a(x) = m$
2 Substitute $a = 5$, $x = 25$ and $m = 3$.	$5^3 = 125 \Leftrightarrow \log_5(125) = 3$
3 State the result.	The log form of $5^3 = 125$ is $\log_5(125) = 3$.
d 1 Write the equivalence statement.	$a^m = x \Leftrightarrow \log_a(x) = m$
2 Substitute $a = 2$, $x = \frac{1}{32}$ and $m = -5$.	$2^{-5} = \frac{1}{32} \Leftrightarrow \log_2\left(\frac{1}{32}\right) = -5$
3 State the result.	The log form of $2^{-5} = \frac{1}{32}$ is $\log_2\left(\frac{1}{32}\right) = -5$.

When you are asked to find the logarithm of a number to a particular base, think 'What power of the base gives the number?'.

WORKED EXAMPLE 2 General logarithms

Evaluate each logarithm.

a $\log_5(25)$ **b** $\log_3\left(\frac{1}{81}\right)$ **c** $\log_{\frac{1}{2}}\left(\frac{1}{16}\right)$

d $\log_{15}(1)$ **e** $\log_{23}(23)$ **f** $\log(100\,000)$

Steps	Working
a 1 Write in index form and solve.	$5^m = 25 \Rightarrow m = 2$
2 State the answer.	$\log_5(25) = 2$
b 1 Write in index form and solve.	$3^m = \frac{1}{81} \Rightarrow m = -4$
2 Write the answer.	$\log_3\left(\frac{1}{81}\right) = -4$
c 1 Write in index form and solve.	$\left(\frac{1}{2}\right)^m = \frac{1}{16} \Rightarrow m = 4$
2 Write the answer.	$\log_{\frac{1}{2}}\left(\frac{1}{16}\right) = 4$
d 1 Write in index form and solve.	$15^m = 1 \Rightarrow m = 0$
2 Write the answer.	$\log_{15}(1) = 0$
e 1 Write in index form and solve.	$23 = 23^m \Rightarrow m = 1$
2 Write the answer.	$\log_{23}(23) = 1$
f 1 Write in index form and solve.	$10^m = 100\,000 \Rightarrow m = 5$
2 Write the answer.	$\log(100\,000) = 5$

EXERCISE 9.1 Logarithms

ANSWERS p. 398

Mastery

1 WORKED EXAMPLE 1 Write each equation in index form.

a $\log_5(125) = 3$ **b** $\log_2(32) = 5$ **c** $\log_4(64) = 3$ **d** $\log_7(7) = 1$

e $\log_3(2187) = 7$ **f** $\log_9(1) = 0$ **g** $\log_5(625) = 4$ **h** $\log_{13}(169) = 2$

2 Write each equation in index form.

a $\log_{16}(4) = \frac{1}{2}$ **b** $\log_8\left(\frac{1}{2}\right) = -\frac{1}{3}$ **c** $\log_3\left(\frac{1}{9}\right) = -2$ **d** $\log_4\left(\frac{1}{16}\right) = -2$

e $\log_{\frac{1}{2}}(8) = -3$ **f** $\log_{\frac{1}{4}}(2) = -\frac{1}{2}$ **g** $\log_{32}(4) = \frac{2}{5}$ **h** $\log_5\left(\frac{1}{\sqrt[3]{5}}\right) = -\frac{1}{3}$

3 Write each equation in logarithmic form.

a $8^2 = 64$ **b** $3^4 = 81$ **c** $2^6 = 64$

d $12^0 = 1$ **e** $15^1 = 15$ **f** $6^3 = 216$

4 Write each equation in logarithmic form.

a $36^{\frac{1}{2}} = 6$ **b** $27^{-\frac{1}{3}} = \frac{1}{3}$ **c** $4^{-2} = \frac{1}{16}$

d $\left(\frac{1}{2}\right)^{-4} = 16$ **e** $\left(\frac{1}{9}\right)^{-\frac{1}{2}} = 3$ **f** $9^{\frac{3}{2}} = 27$

5 WORKED EXAMPLE 2 Evaluate each logarithm.

a $\log_2(16)$ **b** $\log_3(81)$ **c** $\log_5(125)$ **d** $\log_6(1296)$

e $\log_{17}(1)$ **f** $\log_{13}(13)$ **g** $\log_{10}(1000)$ **h** $\log_4(4096)$

6 Write the value of

a $\log_3(1)$ **b** $\log_5(5)$ **c** $\log_4(1)$ **d** $\log_7(1)$

e $\log_{12}(12)$ **f** $\log_{10}(1)$ **g** $\log_{10}(10)$ **h** $\log_9(9)$

7 Write the value of

a $\log_8(0)$ **b** $\log_4(-4)$ **c** $\log_a(0), a > 0$ **d** $\log_b(-x), x > 0$

8 Evaluate each logarithm.

a $\log_2\left(\frac{1}{32}\right)$ **b** $\log_3\left(\frac{1}{9}\right)$ **c** $\log_5\left(\frac{1}{625}\right)$ **d** $\log_4\left(\frac{1}{64}\right)$

e $\log_{\frac{1}{2}}\left(\frac{1}{8}\right)$ **f** $\log_{\frac{1}{2}}(64)$ **g** $\log_{\frac{1}{3}}\left(\frac{1}{81}\right)$ **h** $\log_{\frac{1}{5}}(125)$

9 Write the value of

a $\log_3(27)$ **b** $\log_5(625)$ **c** $\log_8(2)$ **d** $\log_2(16)$

e $\log_8(4)$ **f** $\log_{10}(100)$ **g** $\log_{100}(10)$ **h** $\log_4(0.25)$

i $\log_7(1)$ **j** $\log_2(\sqrt{8})$ **k** $\log_8(\sqrt{2})$ **l** $\log_4(0.5)$

Exam practice TECH-FREE

10 $\log_4(64) =$

A 3 **B** 4 **C** 5 **D** 6

11 Given $\log_3(a) = 0$, $a =$

A 0 **B** 1 **C** 2 **D** 3

12 $\log_x(x^2) =$

A x **B** 0 **C** 1 **D** 2

13 Given $p = 4^{2x}$, $x =$

A $\log_4(2p)$ **B** $2\log_4(p)$ **C** $\frac{1}{2}\log_4(p)$ **D** $\frac{1}{2}\log_2(p)$

14 Given $6y = \log_{10}(x^2)$, $x =$

A 10^{6y} **B** $\sqrt{5^y}$ **C** $\sqrt{5^{3y}}$ **D** $\sqrt{10^{3y}}$

Video
Logarithm laws

Worksheet
Logarithms review

Puzzle
Logarithm laws

9.2 Logarithm laws

$5^2 \times 5^4 = 5^{2+4} = 5^6$. As powers, $5^2 = 25$, $5^4 = 625$ and $5^6 = 15\,625$.

The equivalents as logs are: $\log_5(25) = 2$, $\log_5(625) = 4$ and $\log_5(15\,625) = 6$.

$2 + 4 = 6$, and $25 \times 625 = 15\,625$, so $\log_5(25) + \log_5(625) = \log_5(15\,625)$.

Thus, $\log_5(25) + \log_5(625) = \log_5(25 \times 625)$.

In general, the index law $a^n \times a^m = a^{n+m}$ translates to the logarithm law $\log_a(x) + \log_a(y) = \log_a(xy)$.

The other index laws translate in similar ways to the logarithm laws.

Logarithm laws

$$\log_a(xy) = \log_a(x) + \log_a(y)$$

$$\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$$

$$\log_a(x^n) = n\log_a(x)$$

WORKED EXAMPLE 3 Logarithm laws

Evaluate each logarithm without using a calculator.

a $\log_3(\sqrt{3})$ **b** $\log_4(28) + \log_4(5) - \log_4(35)$ **c** $\dfrac{\log_7(8)}{\log_7(32)}$

Steps	Working
a 1 Write $\sqrt{3}$ as a power.	$\log_3(\sqrt{3}) = \log_3\left(3^{\frac{1}{2}}\right)$
2 Use the logarithm laws.	$= \frac{1}{2}\log_3(3)$
3 $3^1 = 3$, so $\log_3(3) = 1$.	$= \frac{1}{2}$
b 1 Write the expression.	$\log_4(28) + \log_4(5) - \log_4(35)$
2 Use the logarithm laws.	$= \log_4\left(\frac{28 \times 5}{35}\right)$
3 Simplify.	$= \log_4(4)$
4 $4^1 = 4$, so $\log_4(4) = 1$.	$= 1$
c 1 Write the expression.	$\frac{\log_7(8)}{\log_7(32)}$
2 Express the terms using powers of the same base.	$= \frac{\log_7(2^3)}{\log_7(2^5)}$
3 Use the logarithm laws.	$= \frac{3\log_7(2)}{5\log_7(2)}$
4 Simplify.	$= \frac{3}{5}$

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WORKED EXAMPLE 4 Common logarithms

Given $\log_{10}(7.93) \approx 0.899$ and $10^{0.3557} \approx 2.268$, **evaluate**

a $\log_{10}(79.3)$ **b** $\log_{10}(0.000\,793)$

c $10^{2.3557}$ **d** $10^{-1.6443}$

Steps	Working
a 1 Write in scientific notation.	$\log_{10}(79.3) = \log_{10}(7.93 \times 10^1)$
2 Use the log laws.	$= \log_{10}(7.93) + \log_{10}(10^1)$
3 Substitute the values and simplify.	$\approx 0.899 + 1$
	$= 1.899$
b 1 Write in scientific notation.	$\log_{10}(0.000\,793) = \log_{10}(7.93 \times 10^{-4})$
2 Use the log laws.	$= \log_{10}(7.93) + \log_{10}(10^{-4})$
3 Substitute the values and simplify.	$= 0.899 + (-4)$
	$= -3.101$
c 1 Isolate the part you know.	$10^{2.3557} = 10^{0.3557} \times 10^2$
2 Evaluate each part.	$\approx 2.268 \times 100$
3 Write the answer.	$= 226.8$
d 1 Write -1.6443 in terms of 0.3557.	$-1.6443 = 0.3557 + (-2)$
2 Write as an expression with powers.	$10^{-1.6443} = 10^{0.3557} \times 10^{-2}$
3 Evaluate.	$\approx 2.268 \times 10^{-2}$
	$= 0.022\,68$

You can use the log laws to simplify algebraic expressions.

WORKED EXAMPLE 5 Simplifying and expanding with logarithm laws

a Write as a single logarithm: $2\log_3(x) + 3\log_3(x) - 4\log_3(y)$.

b Expand $\log_5\left(\frac{x^4}{\sqrt{y}z^5}\right)$ using the properties and laws of logarithms.

Steps	Working
a 1 Write the expression.	$2\log_3(x) + 3\log_3(x) - 4\log_3(y)$
2 Use the logarithm laws.	$= \log_3(x^2) + \log_3(x^3) - \log_3(y^4)$
3 Use the logarithm laws.	$= \log_3\left(\frac{x^2 \times x^3}{y^4}\right)$
4 Simplify.	$= \log_3\left(\frac{x^5}{y^4}\right)$
b 1 Write the expression.	$\log_5\left(\frac{x^4}{\sqrt{y}z^5}\right)$
2 Express all terms as powers.	$= \log_5\left(\frac{x^4}{y^{\frac{1}{2}}z^5}\right)$
3 Use the logarithm laws.	$= \log_5(x^4) - \log_5\left(y^{\frac{1}{2}}\right) - \log_5(z^5)$
4 Use the logarithm laws.	$= 4\log_5(x) - \frac{1}{2}\log_5(y) - 5\log_5(z)$

EXERCISE 9.2 Logarithm laws

ANSWERS p. 398

Recap

1 $\log_3\left(\frac{1}{81}\right) =$

A -27 **B** -4 **C** 4 **D** 27

2 $\log_a(a^3) =$

A 3 **B** 2 **C** a **D** a^2

Mastery

3 WORKED EXAMPLE 3 TECH-FREE Evaluate each expression.

a $\log_7\left(\sqrt{7}\right)$ **b** $\log_5(10) + \log_5(4) - \log_5(5)$ **c** $\frac{\log_6(16)}{\log_6(4)}$

d $\log_2(16) + \log_2(8) - \log_2(4)$ **e** $\log_3(27) - \log_3(9) + 3$ **f** $\frac{\log_3(27)}{\log_3\left(\frac{1}{3}\right)}$

g $\log_8\left(\frac{1}{4}\right) + \log_8(2) + 1$ **h** $\frac{\log_2(0.25)}{\log_2(0.5)}$

4 Simplify each expression.

a $\log_2(32)$ **b** $\log_{10}(0.001)$

c $\log_2\left(\frac{1}{256}\right)$ **d** $\log_2(32) - \log_3(27)$

e $\log_{10}(200) - \log_{10}(100)$ **f** $\log_2(16) + \log_2(32) + \log_2(64)$

g $8\log_2(2) - 2\log_2(8)$ **h** $\log_3\left(\frac{1}{9}\right) + \log_3(81) - \log_3(27)$

5 WORKED EXAMPLE 4 Given $10^{0.6235} \approx 4.202$, find the value of

a $10^{3.6235}$ **b** $10^{1.6235}$ **c** $10^{-2.3875}$

6 Given $10^{0.1234} \approx 1.329$, find the value of

a $10^{-0.8766}$ **b** $10^{4.1234}$ **c** $10^{-3.8366}$

7 Given $\log_{10}(5.7) \approx 0.7559$, find

a $\log_{10}(57)$ **b** $\log_{10}(5700)$ **c** $\log_{10}(0.57)$

8 Given $\log_{10}(2.84) \approx 0.4533$, find

a $\log_{10}(28.4)$ **b** $\log_{10}(0.284)$ **c** $\log_{10}(0.002\,84)$

9 Given $\log_{10}(9.735) \approx 0.9883$, find

a $\log_{10}(97\,350)$ **b** $\log_{10}(0.9735)$ **c** $\log_{10}(0.000\,9735)$

10 WORKED EXAMPLE 5 Write as a single logarithm.

a $3\log_2(x) + \log_2(x) - 2\log_2(x)$ **b** $3\log_5(x) + 4\log_5(x) - 6\log_5(x)$

c $2\log_4(x^2) + 4\log_4(x) - 3\log_4(x^3)$ **d** $4\log_{10}(x) - 5\log_{10}(x) + \log_{10}(y)$

e $\log_7(x+1)^2 + 3\log_7(x+1)$ **f** $4\log_3(x) + 2\log_3(y) - 3\log_3(y)$

g $4\log_8(x) - \log_8(x) + \log_8(y^2)$ **h** $2\log_4(x-2) - \log_4(x-2)^3$

11 Expand using the properties and laws of logarithms.

a $\log_2\left(\frac{xy}{z}\right)$ **b** $\log_4\left(\frac{4x}{y}\right)$ **c** $\log_7\left(\frac{6}{xy}\right)$ **d** $\log_5\left(\frac{5x}{y^2z}\right)$

e $\log_{10}\left(\frac{x^3\sqrt{y}}{z^5}\right)$ **f** $\log_9\left(\frac{x^4\sqrt[3]{y}}{\sqrt{z}}\right)$ **g** $\log_5\left(\sqrt{\frac{x^3y}{z^3}}\right)$ **h** $\log_8\left(\sqrt[3]{\frac{x^4y^2}{z^2}}\right)$

12 $\log_7(5) = 0.827$. Calculate each logarithm correct to 3 decimal places.

a $\log_7(\sqrt[3]{5})$ **b** $\log_7(25)$ **c** $\log_7(245)$

Exam practice TECH-FREE

13 $\log_3(18) - \log_3(6) =$

A 1 **B** 2 **C** 3 **D** 4

14 Given $\log_{10}(3.56) = 0.5514$, $\log_{10}(0.0356) =$

A −2.5514 **B** −2.4486 **C** −1.5596 **D** −1.4486

15 $3\log_2(x^2) + 1 - 2\log_2(x) =$

A $\log_2(2x^4)$ **B** $\log_2(2x)$ **C** $2\log_2(x^2)$ **D** $\log_2(2x^2)$

16 (2 marks) Given that $a^b = 36$, find $\log_6 a$ in terms of b.

17 (2 marks) If $\log_a(5) - 2\log_a(x) = 0$, find x.

9.3 The change of base theorem

There is a formula for writing a logarithm in terms of a logarithm of another base.

For example, you can see that $\log_4(8) = \frac{3}{2}$ because $4^{\frac{3}{2}} = \sqrt{4^3} = 2^3 = 8$.

But you can also write both 8 and 4 as a logarithm to base 2: $\log_2(8) = 3$ and $\log_2(4) = 2$.

So, $\log_4(8) = \frac{\log_2(8)}{\log_2(4)}$.

Change of base theorem

For $a, b, x > 0$ and $a, b \neq 0$

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$$

You can prove the **change of base theorem** by writing $x = a^y$, taking logarithms to the base b and using $y = \log_a(x)$.

You can use this formula to find logarithms to any base using common (base 10) logarithms, or you can use the general log template on your graphics calculator.

WORKED EXAMPLE 6 Logarithms with any base

Evaluate $\log_8(29)$, correct to 4 decimal places.

Steps	Working
1 Write the expression.	$\log_8(29)$
2 Change the base to 10.	$= \frac{\log_{10}(29)}{\log_{10}(8)}$
3 Evaluate on your calculator.	$= 1.619\,326\ldots$
4 Round to the accuracy needed.	≈ 1.6193

Use the log button for $\log_{10}$.

You can use the change of base theorem to solve exponential equations.

WORKED EXAMPLE 7 Solving exponential equations using change of base 1

Solve $7^x = 12$, correct to 4 significant figures.

Solution

1 Write the equation.	$7^x = 12$
2 Rewrite as a logarithm.	$x = \log_7(12)$
3 Use the change of base theorem.	$= \dfrac{\log_{10}(12)}{\log_{10}(7)}$
4 Use your calculator.	$= 1.276\,989\ldots$
5 Round to 4 significant figures.	≈ 1.277

Exam hack

Check your answer by calculating $7^{1.277}$. Is it close to 12?

WORKED EXAMPLE 8 Solving exponential equations using change of base 2

Solve $25^{2x-1} = 64$, correct to 4 decimal places.

Steps	**Working**
1 Write the equation.	$25^{2x-1} = 64$
2 Rewrite as a logarithm.	$2x - 1 = \log_{25}(64)$
3 Use the change of base theorem.	$= \dfrac{\log_{10}(64)}{\log_{10}(25)}$
4 Use your calculator.	$= 1.292\,029\ldots$
5 Solve the equation.	$2x - 1 = 1.292\,029\ldots$
	$2x = 2.292\,029\ldots$
	$x = 1.146\,014\ldots$
6 Round appropriately.	$x \approx 1.1460$

Exam hack

Show the essential steps for this question such as steps 2, 5 and 6. No working = 1 mark only.

EXERCISE 9.3 The change of base theorem

ANSWERS p. 398

Recap

1 $\log_3(45) + \log_3(21) - 2\log_3(7) =$

A $\log_3(15)$ **B** $\log_3(7)$ **C** $\log_3\left(\dfrac{135}{7}\right)$ **D** $\log_3\left(\dfrac{1}{7}\right)$

2 If $2\log_{10}(x^3) + \log_{10}\left(\dfrac{1}{x^4}\right) + 2 = \log_{10}(25)$, then $x =$

A $\dfrac{1}{5}$ **B** $\dfrac{1}{4}$ **C** $\dfrac{1}{2}$ **D** 5

Mastery

3 WORKED EXAMPLE 6 Use the change of base theorem to express each logarithm in terms of common logarithms.

a $\log_5(12)$ **b** $\log_4(20)$ **c** $\log_2(50)$

d $\log_9(300)$ **e** $\log_{11}(0.5)$ **f** $\log_7(15)$

4 Use the change of base theorem to evaluate each logarithm, correct to 4 significant figures.

a $\log_2(10)$ **b** $\log_5(15)$ **c** $\log_3(12)$

d $\log_8(24)$ **e** $\log_{17}(4.8)$ **f** $\log_{21}(17)$

5 WORKED EXAMPLE 7 Solve each exponential equation, correct to 4 decimal places.

a $3^x = 50$ **b** $5^x = 10$ **c** $8^x = 100$

d $7^x = 3$ **e** $0.5^x = 10$ **f** $1.8^x = 20$

6 Find x, correct to 4 decimal places.

a $10^x = 20$ **b** $20^x = 8$ **c** $15^x = 10$

d $0.1^x = 15$ **e** $0.8^x = 21$ **f** $0.05^x = 17$

7 WORKED EXAMPLE 8 Solve each equation, correct to 4 significant figures.

a $5^{x+1} = 40$ **b** $9^{2x+3} = 72$ **c** $7^{3x-1} = 59$

d $8^{2-x} = 17$ **e** $11^{3-2x} = 41$ **f** $20^{5x-4} = 36$

Exam practice

8 TECH-FREE $\log_{16}(36) =$

A $\log_2(\sqrt{3}) + \frac{1}{2}$ **B** $\log_2(\sqrt{3}) - \frac{1}{2}$ **C** $\log_2(3) + 2$ **D** $\log_2(3) + \frac{1}{2}$

9 TECH-FREE $\dfrac{\log_{10}(64)}{\log_{10}(4)} =$

A 3 **B** 4 **C** 10 **D** 16

10 TECH-FREE Solve $4^x = 400$.

A $x = \log_4(25) + 1$ **B** $x = \log_4(25) + 2$ **C** $x = 10$ **D** $x = 100$

11 (2 marks) Solve $6^{3x-4} = 22$, correct to 4 significant figures.

12 TECH-FREE (3 marks) CF The sample space $S = \{A, B, C, D\}$.
$P(A) = \log_a(2)$, $P(B) = \log_a(3)$, $P(C) = \log_a(4)$, and $P(D) = \log_a(5)$.
Find the value of a.

Video
Exponential equations 2

Worksheets
Solving exponential equations

Exponential functions assignment

Exponential functions extension

Puzzles
Logarithms - Solving equations 1

Logarithms - Solving equations 2

Exponential equations

9.4 Applying logarithms to exponential equations

You can use logarithms to solve exponential equations.

WORKED EXAMPLE 9 Exponential equations 1

Solve $9^x = 3^{x+1} + 4$, correct to 4 significant figures.

Steps	Working
1 Write the equation.	$9^x = 3^{x+1} + 4$
2 Write in base 3.	$(3^2)^x = 3^{x+1} + 4$
3 Apply the index laws.	$(3^x)^2 = 3^x \times 3^1 + 4$
4 Choose a variable for 3^x to write as a quadratic equation.	Let $a = 3^x$. $a^2 = 3a + 4$ $a^2 - 3a - 4 = 0$
5 Factorise to solve.	$(a - 4)(a + 1) = 0$ $a = -1$ or 4
6 Substitute 3^x for a.	$3^x = -1$ or $3^x = 4$
7 But $3^x > 0$.	$3^x = 4$ only
8 Take logarithms of both sides.	$\log_{10}(3^x) = \log_{10}(4)$
9 Use a logarithm law and solve for x.	$x \log_{10}(3) = \log_{10}(4)$ $x = \dfrac{\log_{10}(4)}{\log_{10}(3)}$
10 Use a calculator to evaluate.	$x = 1.261\,859\ldots$
11 Round off and state the result.	$x \approx 1.262$

Notice that this is really the change of base formula.

Exam hack

Remember that a question worth more than 1 mark must have the essential working shown.

If there are several bases, isolate the variable by taking logarithms of both sides.

WORKED EXAMPLE 10 Exponential equations 2

Solve $12^x = 4^{x-3}$, correct to 4 decimal places.

Steps	Working
1 Write the equation.	$12^x = 4^{x-3}$
2 Take logarithms of both sides.	$\log_{10}(12^x) = \log_{10}(4^{x-3})$
3 Use a logarithm law.	$x\log_{10}(12) = (x - 3)\log_{10}(4)$
4 Expand to make x the subject.	$x\log_{10}(12) = x\log_{10}(4) - 3\log_{10}(4)$
5 Group like terms.	$x\log_{10}(12) - x\log_{10}(4) = -3\log_{10}(4)$
6 Factorise and use another logarithm law.	$x(\log_{10}(12) - \log_{10}(4)) = -3\log_{10}(4)$ $x\log_{10}\left(\dfrac{12}{4}\right) = -3\log_{10}(4)$ $x\log_{10}(3) = -3\log_{10}(4)$ $x = \dfrac{-3\log_{10}(4)}{\log_{10}(3)}$
7 Evaluate.	$\approx -3.785\,578\ldots$
8 Round to the required accuracy.	$x \approx -3.7856$

EXERCISE 9.4 Applying logarithms to exponential equations

ANSWERS p. 398

Recap

1 Express $\log_6(21)$ in logarithms to the base 3.

2 $\dfrac{\log_2(45)}{\log_2(5)} =$

A 9 **B** $1 + 2\log_5(3)$ **C** $1 + 2\log_3(5)$ **D** $\log_2(9)$

Mastery

3 WORKED EXAMPLE 9 Solve each exponential equation, expressing answers correct to 4 significant figures where necessary.

a $4^{x+2} = 64$ **b** $3^{x-1} = 81$ **c** $4^x = 3^{x+1}$

d $6^x = 2^{x-3}$ **e** $9^x = 3^{x+3} - 5$ **f** $16^x = 4^{x+2} + 1$

4 WORKED EXAMPLE 10 Solve each equation, correct to 4 significant figures.

a $4^x = 7^{x-1}$ **b** $8^x = 5^{x+1}$ **c** $7^x = 25^{x-2}$

d $15^{x+1} = 6^{2x-3}$ **e** $24^{x+1} = 9^{2x-5}$ **f** $13^{2x} = 7^{5x-1}$

5 Solve each equation, expressing answers correct to 4 significant figures where necessary.

a $7^{2x} - 3 \times 7^x + 1 = 0$ **b** $10^{x+3} = 6^{2x}$

c $6^x = 2^{x-3}$ **d** $5^{x+2} = 4$

e $4^{x-3} = \dfrac{1}{16}$ **f** $8^{x+4} = 4^{3x-1}$

Exam practice

6 TECH-FREE If $\log_3(2x - 1) = 4$, then $x =$

A 30.5 **B** 32.5 **C** 40 **D** 41

7 If $3^{4x+3} = 4$, $x =$

A −0.43 **B** 1.26 **C** 1.33 **D** 2.25

8 TECH-FREE (2 marks) Solve $7^{2x-3} = 49$.

9 (3 marks) Solve $5^{2x} - 3 \times 5^x = 4$, correct to 3 decimal places.

Video
Exam question analysis: Logarithms

EXAM QUESTION ANALYSIS

Exam-style question (6 marks)

Potassium–Argon dating is used in archaeology to estimate the age of rocks.
When molten rock crystallises, the radioactive decay of potassium into argon begins.
Any argon present in the rock comes from exponential decay of the potassium.

The ratio of potassium to argon is used to find the age of the rock.

The time taken for half the original potassium to decay is 1.31 billion years.
This is called the **half-life** of the potassium.

The amount of potassium in the rock can be modelled using the equation $A = A_0k^t$, where t is the time in millions of years, A is the amount of potassium, A_0 is the amount at $t = 0$ and k is a constant.

How old is a rock in which the ratio of potassium to argon is 6 : 1?

Give your answer correct to the nearest million years.

Reading the question

- It is an exponential decay problem involving the equation $A = A_0k^t$.
- t is not measured in years, but *millions* of years.
- Half-life of potassium is given.
- Age of rock with potassium : argon ratio of 6 : 1 is required.

Thinking about the question

- Need to find the values of A_0 and k in the formula.
- A_0 is the amount of potassium at the start.
- The value of A_0 isn't given in the problem but the half-life and 6 : 1 ratio is.

Worked solution

$A = A_0k^t$

When t = 1.31 billion years

$= 1.31 \times 10^9$ years

$= \mathbf{1.31 \times 10^3}$ (t is measured in millions) ✓

$A = \frac{1}{2}A_0$ (half-life)

Substitute into formula to find k:

$\frac{1}{2}\mathbf{A_0} = \mathbf{A_0k^{1.31 \times 10^3}}$ ✓

$\frac{1}{2} = k^{1.31 \times 10^3}$

Solve for k using a graphics calculator:

$k = \mathbf{0.99947102\ldots}$ ✓

So, $A = A_0(0.99947102\ldots)^t$.

At present, potassium : argon = 6 : 1, so potassium : original potassium = 6 : 7

Need to find t when $A = \frac{6}{7}A_0$.

$\frac{6}{7}A_0 = A_0(0.999\,471\,02\ldots)^t$ ✓

$\frac{6}{7} = (0.999\,471\,02\ldots)^t$

Solve for t using a graphics calculator:

$t = 291.334\,07\ldots$ ✓

The rock is about 291 million years old. ✓

9 Chapter summary

Logarithms

- The **logarithm** of x to the base a is the power m such that $a^m = x$, where $a, x > 0$ and $a \neq 1$.

  $m = \log_a(x) \Leftrightarrow a^m = x$
- The **common logarithm** of x has the base 10. It is written as $\log(x)$.

  $m = \log(x) \Leftrightarrow 10^m = x$

Logarithm laws

- $\log_a(xy) = \log_a(x) + \log_a(y)$
- $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$
- $\log_a(x^n) = n\log_a(x)$

Change of base theorem

- For $a, b, x > 0$ and $a, b \neq 0$

  $\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$
- Logarithm laws and the change of base theorem can be used to solve exponential equations.

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Cumulative examination 1

Technology-free

Perusal time: 2 minutes Working time: 30 minutes

TECH-FREE *Calculators are **not** permitted.*

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

Section 1 (3 marks): 3 multiple choice questions

Section 2 (15 marks): 5 short response questions

Total: 18 marks

Section 1 3 multiple choice questions

3 marks

Worksheet
Maths Methods Year 11 formula sheet

1 $\log_3\left(\frac{1}{9}\right) =$

A -3 **B** -2 **C** 2 **D** 3

2 $3\log_2(x) + \log_2(3x) =$

A $3\log_2(3x)$ **B** $4\log_2(3x)$

C $4\log_2(x) + \log_2(3)$ **D** $3\log_2(3x^2)$

3 The circle $x^2 + y^2 - 8x + 12y = 12$ has

A centre $(-4, 6)$ and radius $\sqrt{12}$ **B** centre $(4, -6)$ and radius $\sqrt{12}$

C centre $(4, -6)$ and radius 8 **D** centre $(-4, 6)$ and radius 8

Section 2 5 short response questions

15 marks

4 (2 marks) A straight line with gradient −2 passes through the point (3, 1).
What is the vertical distance from the x-axis to the point on the line at $x = -3$?

5 (3 marks) Solve $(2x - 3)(x + 6) = 8$.

6 (2 marks) Solve $2\sin(x) + \sqrt{3} = 0$ for $0 \leq x < 2\pi$.

7 (4 marks) The loudness of a sound decreases with the distance from the source of the sound. A small chainsaw is rated at 105 decibels (dB). The standard distance for measurement of a tool's loudness is 1 metre. The operator's ears are at a distance of about 500 mm from the source of the sound.
The formula for the loudness at a distance $L(r_2)$ for a loudness of $L(r_1)$ at the distance r_1 is

$$L(r_2) = L(r_1) + 20\log_{10}\left(\frac{r_2}{r_1}\right)$$

a Write an expression for the loudness of this chainsaw at the operator's ears. *[1 mark]*

b Noises more than 85 dB over a period of 8 hours causes permanent hearing loss.
How far away from the chainsaw must you be for the noise to be 85 dB? *[3 marks]*

8 (4 marks) An old water pipeline 25 km long must have leaks repaired roughly once a week. This involves 20 km of the pipeline being replaced by new pipe. The new pipe has a lower probability of leaking than the old pipe. In fact, 1 km of the old pipe is 10 times as likely to leak as 1 km of the new pipe. Any leak decreases the pressure in the pipe, so the likelihood of two leaks on the same day is small. Given that there is a leak, what is the probability that it is in the new section? Leave your answer as a fraction.

Cumulative examination 2

Technology-active

Perusal time: 2 minutes Working time: 30 minutes

*Calculators are **permitted**.*

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

Section 1 (3 marks): 3 multiple choice questions

Section 2 (15 marks): 5 short response questions

Total: 18 marks



Worksheet
Maths Methods Year 11 formula sheet

Section 1 3 multiple choice questions

3 marks

1 The solution of $5^{2x-3} = 8$ is

A -1.113 **B** -0.854 **C** 1.887 **D** 2.146

2 The asymptotes of $f(x) = \dfrac{3}{x+2} - 4$ are

A $x = 3$ and $y = 4$ **B** $x = -2$ and $y = -4$ **C** $x = 2$ and $y = -4$ **D** $x = -2$ and $y = 4$

3 $\sqrt{27} + \sqrt{18} - \sqrt{12} + 3\sqrt{2} =$

A $5\sqrt{3}$ **B** $\sqrt{3} + 6\sqrt{2}$ **C** $3\sqrt{3} + 4\sqrt{2}$ **D** $6\sqrt{3} + 4\sqrt{2}$

Section 2 5 short response questions

15 marks

4 (2 marks) Solve $3^{2x} = 3^x + 2$, correct to 3 decimal places.

5 (3 marks) Sketch the graph of $3\cos\left(2\left(x - \dfrac{\pi}{6}\right)\right) + 1$ for the domain $[0, \pi]$, showing important points.

6 (2 marks) Write $\log_4(x) + 2\log_2(x)$ as a single logarithm.

7 (4 marks) $f(x) = (x+1)(x-1)(x-3)$

a Find the derivative of $f(x)$. *[1 mark]*

b Sketch $f(x)$ and its derivative on the same set of axes for an appropriate domain. *[2 marks]*

c State a relationship between the graphs of a function and its derivative. *[1 mark]*

8 (4 marks) The rate of reaction of granular zinc and hydrochloric acid is given by

$$R = k\,[H^+]^2$$

where k is a constant and $[H^+]$ is the concentration of hydrogen ions. k is related to the surface area of the zinc granules, the temperature and the heat of reaction.

The pH of the acid is given by $pH = -\log_{10}([H^+])$.

What change of pH will halve the rate of reaction?

Give an exact answer and an approximation to 2 decimal places.

CHAPTER

10 APPLICATIONS OF DERIVATIVES

Syllabus coverage
Nelson MindTap chapter resources
Terminology
10.1 Instantaneous rates of change
10.2 Tangents and normals to a curve
10.3 Displacement, velocity and acceleration
10.4 Stationary points
10.5 Local and global maximum and minimum points
10.6 Curve sketching
Exam question analysis
Chapter summary
Cumulative examination 1
Cumulative examination 2

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Syllabus coverage

UNIT 2, TOPIC 4: APPLICATIONS OF DIFFERENTIAL CALCULUS

Graphical applications of derivatives

- Determine instantaneous rates of change.
- Determine the equation of a tangent and a normal of the graph of $y = f(x)$.
- Construct and interpret displacement–time graphs, with velocity as the slope of the tangent.
- Recognise that velocity is the instantaneous rate of change of displacement with respect to time.
- Use the first derivative of a function to determine and identify the nature of stationary points.
- Sketch curves associated with power functions and polynomials up to degree 4; find stationary points and local and global maxima and minima with and without technology; and examine behaviour as $x \to \infty$ and $x \to -\infty$.

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Prior learning
Applications of derivatives

Videos (3):
10.2 Equation of a tangent
10.4 Stationary points
Exam question analysis Applications of derivatives

Prior learning (1):
10.1 Applications of derivatives

Worksheets (15):
10.1 Properties of derivatives assignment • Properties of derivatives problems
10.2 Slopes of curves • Tangents to a curve
10.3 Straight-line motion 1 • Rates of change 2 • Motion in a straight line • Graphs of rates of change • Instantaneous rates of change
10.4 Stationary points • The sign of the derivative
10.5 Greatest and least values
10.6 Sketching curves • Gradient functions
Cumulative exams Maths Methods Year 11 formula sheet

TI-Nspire CX II instructions (1):
10.2, 10.4 Applications of derivatives

Spreadsheet (1):
10.2 Tangent tracer

Nelson MindTap

To access resources above, visit
cengage.com.au/nelsonmindtap

Terminology

acceleration	derivative	displacement	global maximum
global minimum	instantaneous rate of change	local maximum	local minimum
marginal cost	maximum	minimum	normal
point of horizontal inflection	position	stationary points	tangent
turning point	velocity		

Worksheets
Properties of derivatives assignment

Properties of derivatives problems

10.1 Instantaneous rates of change

In Chapter 7, *Rates of change and the derivative*, you saw that the **instantaneous rate of change** of a function is its **derivative**.

WORKED EXAMPLE 1 Instantaneous rate of change

Determine the rate of change of $f(x) = 3x^3 + 7x^2 - 4x + 1$ at $x = 1$.

Steps	Working
1 Differentiate $f(x)$.	$f(x) = 3x^3 + 7x^2 - 4x + 1$ $f'(x) = 9x^2 + 14x - 4$
2 Substitute $x = 1$ into $f'(x)$.	$f'(1) = 9 \times 1^2 + 14 \times 1 - 4$ $= 9 + 14 - 4$ $= 19$
3 State the result.	Therefore, the rate of change of $f(x)$ at $x = 1$ is 19.

Businesses analyse their production and sales processes to ensure they are getting the maximum profit. You should know these terms:

- **revenue** $R(x)$: total amount made from sales of items (also known as units).
- **cost** $C(x)$: total amount spent to produce saleable items (includes production costs, advertising costs, wages of staff, etc.).
- **profit** $P(x)$: total amount of money the business receives from sales after costs have been paid.
- $P(x) = R(x) - C(x)$.

Businesses and economists study the change in cost of producing 1 additional unit. This is known as the **marginal cost** of production. Marginal cost approximates the slope of the tangent of the cost equation $C(x)$. Marginal cost $\approx C'(x)$.

WORKED EXAMPLE 2 Marginal cost

The total production cost per week for producing x widgets is given by

$$C(x) = 50 + 5x + 0.2x^2, \text{ for } 0 \leq x \leq 200$$

where cost is in $ and x is the number of units sold.

a **Calculate** the average rate of change in cost as x increases from 2 to 7 units.

b **Determine** the marginal cost for 5 items.

c **Determine** the marginal cost for 15 items.

d **Determine** the number of widgets that need to be sold for the marginal cost to be $61.

Steps	Working
a 1 State the formula for the average rate of change.	$\text{AROC} = \frac{\Delta C(x)}{\Delta x}$
2 Calculate the average rate of change.	$= \frac{C(7) - C(2)}{7 - 2}$ $= \frac{94.8 - 60.8}{5}$
3 Interpret the solution.	$= 6.8$ Therefore, the average rate of change of cost from $x = 2$ to $x = 7$ is $6.80/unit.

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b 1	Differentiate $C(x)$.	$C(x) = 50 + 5x + 0.2x^2$ $C'(x) = 5 + 0.4x$
2	Substitute $x = 5$.	$C'(5) = 5 + 0.4 \times 5$ $= 7$
3	Interpret the solution.	Therefore, the marginal cost at $x = 5$ is \$7/unit.
c 1	Substitute $x = 15$ into $C'(x)$.	$C'(15) = 5 + 0.4 \times 15$ $= 11$
2	Interpret the solution.	Therefore, the marginal cost at $x = 15$ is \$11/unit.
d 1	State the marginal cost.	$C'(x) = 61$
2	Substitute for $C'(x)$.	$5 + 0.4x = 61$
3	Solve for x.	$x = \frac{56}{0.4}$ $x = 140$
4	Interpret the solution.	Therefore, 140 units would need to be made to reach the marginal cost of \$61/unit.

You can use the derivative to work out rates of change with respect to time in physical contexts.

WORKED EXAMPLE 3 Flow rate

The volume of water flowing out of a pipe is given by the function

$$V = 500 + 640t - 4t^2$$

where V is the volume of water in litres and t is time in hours.

a **Determine** the flow rate when $t = 2$.

b **Calculate** how long it will take the pipe to stop flowing.

Steps		Working
a 1	Differentiate V.	$V = 500 + 640t - 4t^2$ $V'(t) = 640 - 8t$
2	Substitute $t = 2$.	$V'(2) = 640 - 8 \times 2$ $= 624$
3	Interpret the solution.	Therefore, the flow rate after 2 hours is 624 L/h.
b 1	Interpret question.	'Stop flowing' means flow rate is 0.
2	Substitute for $V'(t)$.	$V'(t) = 0$ $640 - 8t = 0$
3	Solve for t.	$8t = 640$ $t = 80$
4	Interpret the solution.	Therefore, the pipe will stop flowing after 80 hours.

EXERCISE 10.1 Instantaneous rates of change

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Mastery

1 WORKED EXAMPLE 1 Determine the rate of change of $y = 7x^3 + 2x^2 - 12x + 5$ at

a $x = 1$ **b** $x = -2$ **c** $x = 3$ **d** $x = 0$ **e** $x = 5$

2 Determine the rate of change of $P(x) = 2x^2 + 4x^3 - 18x$ at

a $x = -2$ **b** $x = 3$ **c** $x = -1$ **d** $x = 0$ **e** $x = -4$

3 Consider $y = t^2 - 2t^3 + 5t - 6$.

a Find the average rate of change from $t = 2$ to $t = 8$.

b Find the instantaneous rate of change at

i $t = 2$ **ii** $t = 8$ **iii** $t = 5$.

c Compare the average rate of change and the instantaneous rate of change.

4 WORKED EXAMPLE 2 The cost function for the production of ovens is given by

$$C(x) = 0.04x^3 - 10x^2 + 1200x + 2000$$

where the cost $C(x)$ of producing x ovens is given in dollars.

a Find the cost of producing 60 ovens.

b Find the average cost of producing each oven if 60 ovens are made.

c Use the derivative to find the marginal cost of producing an extra oven at 60 ovens.

d Compare the average and marginal costs at

i 60 ovens **ii** 100 ovens **iii** 130 ovens.

5 The total revenue function for a particular item is

$$R(x) = 3x$$

where the revenue is in dollars and x is the number of items sold.

The total cost function is

$$C(x) = 300 + 2x - 0.2x^2$$

a Calculate the instantaneous rate of change of profit if

i 3 items are produced and sold **ii** 10 items are produced and sold.

b What can you conclude from the answer to part **a** above?

6 WORKED EXAMPLE 3 A gas-filled balloon has a small leak. Its volume is given by

$$V(t) = 400 - 5t - \frac{t^2}{50}$$

where V is in cm^3 and t is in seconds.

a Find the rate of change of volume after 20 seconds.

b Find the rate of change of volume after 30 seconds.

c When does the volume function cease to be valid?

7 A block of ice in the shape of a cube is melting uniformly.

a Find the rate of change of the surface area with respect to its side length, x cm.

b At what rate is the surface area of the block changing when the side length is 4 cm?

Exam practice

8 TECH-FREE (3 marks) Find the instantaneous rate of change of each function.

a $A = 3t^3 + 2t + 5$ *[1 mark]*

b $k(t) = 6t^2 - 5t - 8$ *[1 mark]*

c $P = 3m^4 - 2m + 6$ *[1 mark]*

9 (3 marks) The profit function for the monthly sales of cars by a dealership is

$$P(x) = 900x - 2x^2 - 100$$

where x is the number of cars sold.

a Find the instantaneous rate of change of profit when

i 150 cars are sold *[1 mark]*

ii 250 cars are sold. *[1 mark]*

b Explain the meaning of the results obtained in part **a.** *[1 mark]*

10 (4 marks) CF An ant-lion makes a hole in the sand in the shape of an inverted cone and sits in the bottom of the hole. The slope of the sides is 45°. When an ant falls into the trap, the ant-lion gets its reward but has to re-dig the hole.

After an ant falls in, the hole collapses so it is only 1.5 cm deep. Find the rate of change of volume of the hole at a depth of d cm and the amount of sand the ant-lion must remove to restore it to a depth of 1.7 cm.

10.2 Tangents and normals to a curve

Video Equation of a tangent

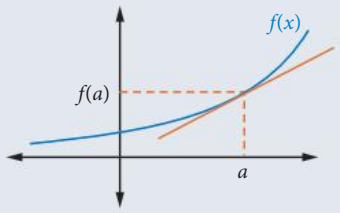
Worksheets Slopes of curves

Tangents to a curve

Spreadsheet Tangent tracer

Gradient of a tangent

A **tangent** is a straight line that intersects a function at only one point.
The gradient of a tangent at $x = a$ is $f'(a)$.



WORKED EXAMPLE 4 Equation of a tangent

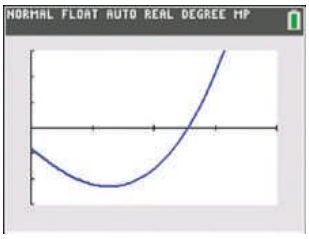
Determine the equation of the tangent to $y = x^3 + 2x^2 - 10x - 4$ at $x = 3$.

Steps	Working
1 Differentiate y.	$y = x^3 + 2x^2 - 10x - 4$ $\frac{dy}{dx} = 3x^2 + 4x - 10$
2 Determine the gradient of the tangent at $x = 3$.	When $x = 3$, $\frac{dy}{dx} = 3 \times 3^2 + 4 \times 3 - 10$ $= 27 + 12 - 10$ $= 29$
3 Find the coordinates of the point where the tangent touches the curve.	$y = 3^3 + 2 \times 3^2 - 10 \times 3 - 4$ $= 27 + 18 - 30 - 4$ $= 11$ The point $(3, 11)$ lies on the tangent.
4 Substitute the point and gradient into the straight line equation.	$y - 11 = 29(x - 3)$ $y = 29x - 87 + 11$ $y = 29x - 76$

TECHNOLOGY Equation of a tangent

Determine the equation of the tangent to $y = x^3 + 2x^2 - 10x - 4$ at $x = 3$.

TI-84 Plus CE

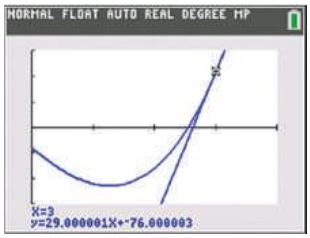


1 Press 2nd y= ([stat plot]) and make sure all Plots are off.

2 Press y= and enter the function in Y1.

3 Set the window appropriately ($0 \le x \le 4$, $-15 \le y \le 15$). Use (–) for negative.

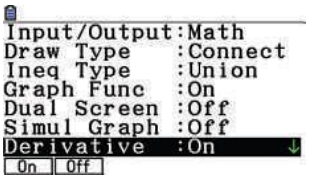
4 Press graph.



5 Press 2nd prgm ([draw]), and choose 5: Tangent(and 3 ENTER to draw the tangent at $x = 3$.

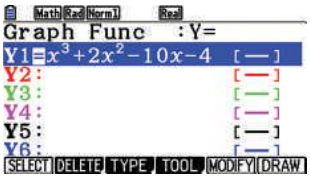
The equation of the tangent is $29x - 76$.

Casio fx-CG50 AU

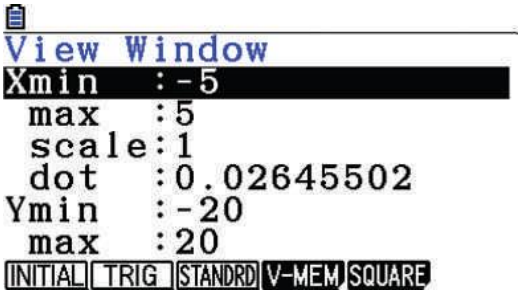


1 From the **MAIN MENU** screen, press **5** to select **Graph**.

2 Press SHIFT then MENU to open the SET UP menu. Scroll down to Derivative and press F1 to ensure it is turned on. Press EXIT to return to the Graph screen.



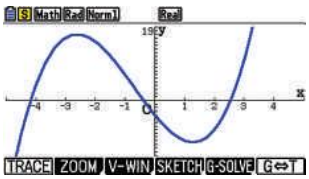
3 Type the RHS of the function into the space after **Y1:**.

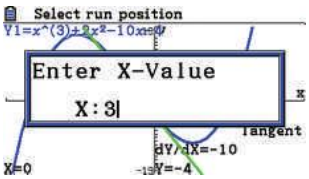


4 Press SHIFT then F3 to open the View Window.

5 Set appropriate Xmin, Xmax, Ymin and Ymax values for the question you are solving (setting the scale for the x- and y- axes is optional).

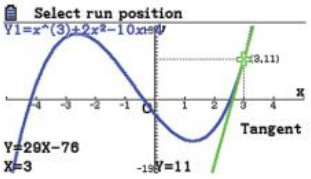
6 Press EXE again after saving the last value to return to the Graph screen, then F6 to draw the graph.





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7 Press SHIFT then F4 to open the Sketch tabs, press F2 to draw a tangent, then enter the x value where you want the tangent drawn. Press EXE to confirm the x value. Press EXE again to draw the tangent.



8 Equation for the tangent appears in the bottom-left corner of the screen, above the coordinates of the point on the tangent and function.

Gradient of a normal

A **normal** is a line that is perpendicular to a tangent, at the point where the tangent meets the graph. If m_T and m_N are the gradients of the tangent and normal respectively, then

$$m_N \times m_T = -1$$

or $$m_N = -\frac{1}{m_T}$$

WORKED EXAMPLE 5 Equation of a normal

Determine the equation of the normal to $y = x^3 + 2x^2 - 10x - 4$ at $x = 3$.

Steps	Working
1 Differentiate y.	$y = x^3 + 2x^2 - 10x - 4$ $\frac{dy}{dx} = 3x^2 + 4x - 10$
2 Find the gradient of the tangent at $x = 3$.	When $x = 3$, $\frac{dy}{dx} = 3 \times 3^2 + 4 \times 3 - 10$ $= 29$
3 Find the gradient of the normal.	$m_T \times m_N = -1$ $m_N = -\frac{1}{29}$
4 Find the coordinates of the point where the normal touches the curve.	$y = 3^3 + 2 \times 3^2 - 10 \times 3 - 4$ $= 11$ The point $(3, 11)$ lies on the normal.
5 Substitute the point and gradient into the point–gradient equation.	$y - 11 = -\frac{1}{29}(x - 3)$ $y = -\frac{1}{29}x + \frac{3}{29} + 11$ $y = -\frac{1}{29}x + \frac{322}{29}$

WORKED EXAMPLE 6 Problems with tangents and normals

The tangent to $f(x) = 12 + 4x - x^2$ at $x = 3$ crosses the x-axis at A.
The normal to the same function at $x = 3$ crosses the x-axis at B.

Calculate the distance between A and B.

Steps	Working
1 Differentiate $f(x)$.	$f(x) = 12 + 4x - x^2$ $f'(x) = 4 - 2x$
2 Determine the gradient of the tangent at $x = 3$.	$f'(3) = 4 - 2 \times 3$ $= -2$
3 Find the coordinates of the point where the tangent and normal touch the curve.	$f(3) = 12 + 4 \times 3 - 3^2$ $= 15$ $(3, 15)$ is on the tangent and normal.
4 Substitute the point and gradient of the tangent into the point–gradient equation.	$y - 15 = -2(x - 3)$ $y = -2x + 6 + 15$ $y = -2x + 21$
5 Determine the x-intercept of the tangent.	$0 = -2x + 21$ $x = 10.5$ The coordinates of point A are $(10.5, 0)$.
6 Determine the gradient of the normal.	$m_T \times m_N = -1$ $m_N = -\frac{1}{-2}$ $= \frac{1}{2}$
7 Substitute the point and gradient into the point–gradient equation.	$y - 15 = \frac{1}{2}(x - 3)$ $y = \frac{1}{2}x - \frac{3}{2} + 15$ $y = \frac{1}{2}x + \frac{27}{2}$
8 Determine the x-intercept.	$0 = \frac{1}{2}x + \frac{27}{2}$ $x = -27$ Therefore, the coordinates of point B are $(-27, 0)$.
9 Find the distance between A and B.	$d = A - B$ $= 10.5 - (-27)$ $= 37.5$ A and B are 37.5 units apart.

EXERCISE 10.2 Tangents and normals to a curve

ANSWERS p. 399

Recap

1 The instantaneous rate of change of the function $f(x) = 2x^3 - 2x + 5$ at $x = 1$ is

A 0 **B** 1 **C** 4 **D** 6

2 The flow rate of a pipe whose equation is $V = \pi t^2 - 3t$ at $t = 3$ is

A 3 **B** 3.28 **C** 15.85 **D** 19.27

Mastery

3 WORKED EXAMPLE 4 Find the equation of the tangent to each function at the x value indicated.

a $f(x) = x^2 - 1$ at $x = 2$

b $y = x^2 + 4x + 1$ at $x = -1$

c $f(x) = x^3 - x^2 + 2x - 2$ at $x = -1$

d $y = x^3 - 2x + 3$ at $x = 1$

e $f(x) = x^3 - 2x^2$ at $x = 1$

f $f(x) = x^3 - 2x^2 + 4$ at $x = 2$

4 WORKED EXAMPLE 5 Find the equation of the normal to each tangent in Question **3**.

5 WORKED EXAMPLE 6 The tangent to the curve $y = 2x^2 + 4x - 4$ at $A(0, -4)$ cuts the x-axis at point B. Find

a the gradient of AB

b the length AB (correct to 2 decimal places)

c the acute angle AB makes with the x-axis (correct to one decimal place).

Exam practice

6 TECH-FREE The slope of the tangent to $f(x) = 5x^2 - 3x + 2$ at $x = 3$ is

A −21 **B** 21 **C** 27 **D** 33

7 TECH-FREE The slope of the normal to $y = x^3 + 2x - 8$ at $x = -2$ is

A $-\frac{1}{14}$ **B** $\frac{1}{14}$ **C** −14 **D** 14

8 TECH-FREE (4 marks) A curve has the equation $y = x^3 - 3x^2 - 2x$. Find the equation of the tangents to this curve with gradient 7.

9 TECH-FREE (2 marks) Given $y = x^3 - 6x^2 + 11x - 6$, find all points on the function where the gradient of the tangent is:

a −1 **b** 26

10 (5 marks) The tangent to the curve $y = 2x^2 + 4x - 1$ drawn at P, where $x = -2$, cuts the y-axis at point Q. Find the length PQ (correct to 2 decimal places).

11 (4 marks) CF Find the equation of the tangent to $y = x^2 - 6x - 4$ that is parallel to the line $4x - y + 7 = 0$.

12 (6 marks) CF Find the equations of the tangent to the curve $f(x) = 15 - x - x^2$ at the points where the curve crosses the x-axis.

13 (6 marks) CU A quadratic function passes through the point (2, 15). The slope of the normal is $-\frac{1}{4}$ at $x = 1$ and the slope of the tangent is −8 at $x = -1$. Find the equation of the function.

Worksheets
Straight-line motion 1

Rates of change 2

Motion in a straight line

Graphs of rates of change

Instantaneous rates of change

10.3 Displacement, velocity and acceleration

Displacement, velocity and acceleration

- **Displacement** is the change in position from a start point to an end point ignoring the path taken.
- **Velocity** is the rate of change of displacement of a moving object.
- **Acceleration** is the rate of change of velocity of a moving object.

Displacement, velocity and acceleration have both magnitude and direction. They can be positive or negative.

- $v = \frac{dx}{dt}$
- $a = \frac{dv}{dt}$

WORKED EXAMPLE 7 Velocity and acceleration

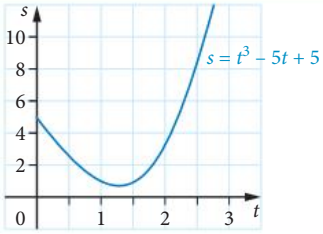
The displacement of an object from a fixed point is given by

$$s = t^3 - 5t + 5$$

a **Sketch** the displacement–time graph.

b **Describe** what is happening to the velocity.

c **Calculate** the velocity and acceleration of the object at $t = 3$.

Steps	Working
a Use graphics calculator or graphing software to create the graph.	
b Describe the gradient of the displacement–time graph.	Velocity starts off negative at $t = 0$. It is 0 at $t \approx 1.25$. After this point, the velocity is positive and increasing.
c **1** Differentiate s.	$v = \frac{ds}{dt} = 3t^2 - 5$
2 Substitute in $t = 3$.	$v(3) = 3 \times 3^2 - 5$ $= 22$
3 Differentiate v.	$a = \frac{dv}{dt} = 6t$
4 Substitute in $t = 3$.	$a(3) = 6 \times 3$ $= 18$
5 Interpret solutions.	At $t = 3$, the velocity of the object is 22 m/s, and its acceleration is 18 m/s^2.

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Exam hack

An object is considered to be 'at rest' or stationary when it has a velocity of 0. An acceleration of 0 means that the velocity is constant.

WORKED EXAMPLE 8 Find when an object is at rest

The displacement of a particle is defined by the equation

$$x(t) = 2t^2 - 4t + 1$$

where t is time in seconds and x is position in metres.

a **Determine** the velocity at $t = 3$.

b **Calculate** when the particle will be at rest.

c **Describe** what is happening to the velocity before and after it is at rest.

Steps	Working
a 1 Differentiate $x(t)$.	$v(t) = x'(t) = 4t - 4$
2 Substitute $t = 3$.	$v(3) = 4 \times 3 - 4$
3 Interpret the solution.	$= 8$ At $t = 3$, the velocity of the particle is 8 m/s.
b 1 Interpret the question.	Particle is at rest when $v(t) = 0$.
2 Substitute for $v(t)$.	$4t - 4 = 0$ $4t = 4$ $t = 1$ Therefore, the particle is at rest at 1 second.
c 1 Solve for velocity prior to coming to rest.	$v(0.5) = 4 \times 0.4 - 4$ $= -2$
2 Describe the velocity of the particle.	The velocity of the particle is negative at $t = 0$, is 0 at $t = 1$ and is positive after that.

EXERCISE 10.3 Displacement, velocity and acceleration

ANSWERS p. 399

Recap

1 The equation of the tangent to $f(x) = -x^2 + 6x - 1$ at $x = 0$ is

A $y = -\frac{x}{6} - 1$ **B** $y = -1$ **C** $y = 6x - 1$ **D** $y = -0.176x + 0.03$

2 The equation of the normal to $f(x) = -x^2 + 6x - 1$ at $x = 0$ is

A $y = -\frac{x}{6} - 1$ **B** $y = -1$ **C** $y = 6x - 1$ **D** $y = -0.176x + 0.03$

Mastery

3 WORKED EXAMPLE 7 The displacement of an object from a fixed point is given by

$$s(t) = 4t^2 + 3t - 17$$

where t is in seconds and s is in metres.

a Sketch a displacement–time graph.

b Describe the velocity.

c Find an equation for the velocity of the object at t seconds.

d Find an equation for the acceleration of the object at t seconds.

e Find the velocity and acceleration of the object at 4 seconds.

4 The position of a particle is given by

$$s = \frac{t^3}{2} - 8t$$

where t is in seconds and s is in cm.

a Show that the particle is initially at the origin.

b Find any other times that the particle is at the origin.

c Sketch a displacement–time graph.

d Describe the velocity.

e Find the velocity of the particle after 3 seconds.

f Find the acceleration of the particle after 4 seconds.

g When is the velocity at $5\frac{1}{2}$ cm/s?

5 WORKED EXAMPLE 8 The position of a body is given by

$$x = 3t^2 - 14t + 8$$

where t is in seconds and x is in metres.

a Find the initial velocity of the body.

b Show that the acceleration is constant.

c What is the position of the body after 5 seconds?

d When does the body come to rest?

e What is the position of the body when it comes to rest?

6 The position of a particle is given by

$$s = 4t^2 - 14t + 5$$

where s is in metres and t is in seconds.

a Where is the particle initially?

b When is the object at rest?

c What is the object's displacement after 1 second?

d What is the total distance travelled in the first 7 seconds?

7 The position of an object is given by

$$s = 8t - t^2$$

where s is in cm and t is in seconds.

a When is the particle at the origin?

b When is the object at rest?

c What is the object's displacement after 6 seconds?

d Sketch a displacement–time graph.

e Describe the velocity.

f What is the total distance travelled in the first 6 seconds?

8 The position function of a point, P, over time t seconds is given by

$$s(t) = t^3 - 12t^2 + 36t - 20$$

a When is the point at rest?

b What is the total distance travelled in the first 8 seconds?

9 The position function of a particle is given by

$$x(t) = t^3 - 6t^2 + 9t$$

where x is in metres and t is in seconds.

a What is the velocity when $t = 4$?

b What is the velocity after 2 seconds?

c When is the object at rest?

d When is the particle moving forwards (positive direction)?

e When is the particle moving backwards (negative direction)?

f What is the total distance travelled in the first 7 seconds?

10 The height (in metres) of a ball thrown vertically into the air is given by

$$h(t) = 17 + 5t - 5t^2$$

where t is in seconds.

a Find the initial displacement and velocity of the ball.

b Calculate the ball's acceleration.

c When will the ball be at the origin?

d Calculate the ball's maximum displacement from the origin.

e Sketch the graph showing the ball's motion up until it reaches the origin. Describe its velocity.

11 The position of an object is given by

$$s(t) = 20t - 5t^3$$

where $-1 < t < 3$, s is in metres and t is in seconds.

a When is the object's velocity zero?

b When is the object's acceleration zero?

c When is the velocity negative?

d When is the acceleration positive?

e Sketch the graphs of $s(t)$, $v(t)$ and $a(t)$.

Exam practice

12 TECH-FREE (6 marks) In a video game, a point is programmed to move across the screen. The position of the point is given by $x(t) = 9t - \frac{3t^3}{4}$, where x is in cm and t is in seconds. When the point reaches the edge of the screen ($x = 0$ or 15), the t is reset to 0 and the point begins moving again according to $x(t)$.

a Where and when does the point come to rest? *[2 marks]*

b What is the acceleration when the point comes to rest? *[2 marks]*

c When does the point first reach the edge of the screen after $t = 0$? *[2 marks]*

13 TECH-FREE (6 marks) CF The position of a particle moving along a horizontal line is $x(t) = t^3 - 23t^2 + 120t$ for $0 < t < 15$.

Find when the particle changes direction and when it is slowing down and speeding up.

14 (7 marks) CU The position of ball A, falling vertically from rest from the top of a tower, is given by $h_A = 63 - 4.9t^2$.

At the same instant as ball A begins to fall, another ball, B, is projected vertically upwards from the bottom of the tower. The position of B is given by $h_B = 21t - 4.9t^2$.

In both cases, h is in metres and t is in seconds. Determine the velocity of each ball when the balls collide.

10.4 Stationary points

Stationary points

A function is flat at a **stationary point**; it is neither increasing nor decreasing.
Stationary points occur when $f'(x) = 0$.

- A stationary point can be a **local minimum turning point**, where the function decreases to the stationary point, then increases afterwards.
- It can be a **local maximum turning point**, where the function increases to the stationary point, then decreases afterwards.
- A stationary point can be a **point of horizontal inflection**, where the function increases (or decreases) to the stationary point, then continues to increase (or decrease) afterwards.

The properties of minimum points, maximum points and points of inflection are shown in this table.

	$f'(x)$ before the stationary point	$f'(x)$ at the stationary point	$f'(x)$ after the stationary point	Type of point	Change in $f'(x)$ as x increases
$f(x)$	+	0	−	Maximum	Decreasing
$f(x)$	−	0	+	Minimum	Increasing
$f(x)$	+	0	+	Inflection on a rising curve	Increasing to zero, then increasing again
$f(x)$	−	0	−	Inflection on a falling curve	Decreasing to zero, then decreasing again

To determine the type of stationary point, find the gradient on each side.

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WORKED EXAMPLE 9 Identifying stationary points

Determine the location and nature of the stationary point of $y = x^2 - 4x + 7$.

Steps	Working
1 Differentiate y.	$y = x^2 - 4x + 7$ $\frac{dy}{dx} = 2x - 4$
2 Function is stationary when $\frac{dy}{dx} = 0$.	Stationary points occur when $\frac{dy}{dx} = 2x - 4 = 0$.
3 Solve for x.	$2x = 4$ $x = 2$
4 Determine the y-coordinate of the stationary point.	$y = 2^2 - 4 \times 2 + 7$ $= 3$ Therefore, the function is stationary at $(2, 3)$.
5 Find the gradient either side of the stationary point.	

x	0	2	3
$f'(x)$	-4	0	$+2$
Shape	╲	—	╱

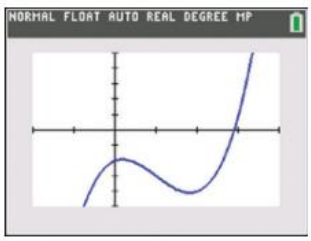
You can choose any points to the left of the stationary point or right of the stationary point. Choose the easy ones.

6 Answer the question. The function has a local minimum at $(2, 3)$.

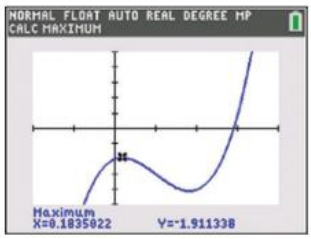
TECHNOLOGY Identifying stationary points

Determine the location and nature of the stationary points of $y = x^3 - 3x^2 + x - 2$.

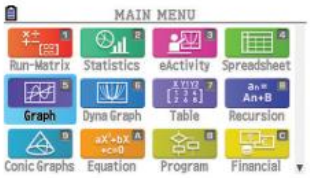
TI-84 Plus CE

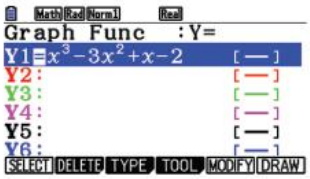


1 Press **y=** and enter the function in Y1.
2 Press **graph**.
3 Set the **window** appropriately.
Use (–) for negative.
4 Press **graph** again.

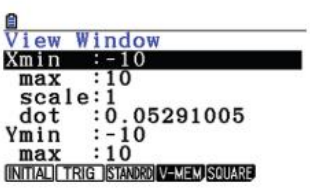


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1 From the **MAIN MENU** screen, press **5** to select **Graph**.
2 Type the RHS of the function into the space after **Y1:**.



3 Press **SHIFT** then **F3** to enter the View-Window screen.

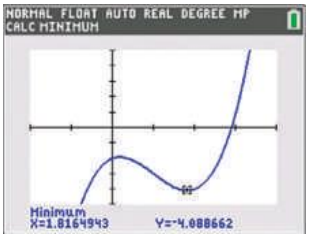
TI-Nspire Applications of derivatives

5 Press 2nd trace ([calc]) and choose 4: maximum.

6 Move the cross to set the left bound and press ENTER.

7 Move the cross to set the right bound and press ENTER.

8 Make a guess of the x value.

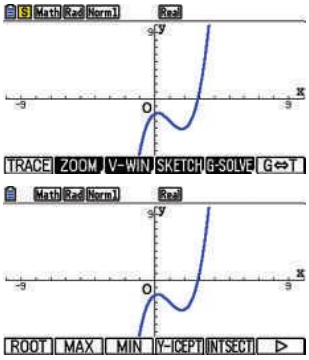


9 Press 2nd trace ([calc]) again to find the maximum.

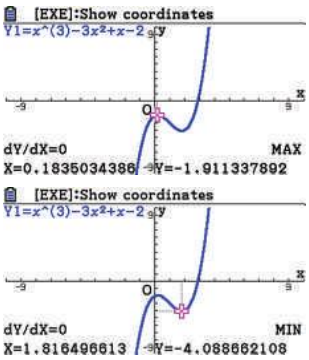
There is a maximum at about (0.184, –1.911) and a minimum at about (1.816, –4.089).

4 Set appropriate Xmin, Xmax, Ymin and Ymax values for the question you are solving (setting the scale for the x- and y-axes is optional).

5 Press EXE again after saving the last value to return to the Graph screen, then **menu** to draw the graph.



6 Press SHIFT to open the soft keys on the screen, then press F5 (G-Solv).



7 Press F2 to find any maximum stationary points and F3 to find any minimum stationary points.

(Note: if there are multiples of the same type of points, use the arrows to scroll from one to the next and back.)

WORKED EXAMPLE 10 Stationary points of a cubic function

Determine the location and nature of all stationary points of the function $f(x) = x^3 + x^2 - 5x - 4$.

Steps	Working
1 Differentiate $f(x)$.	$f(x) = x^3 + x^2 - 5x - 4$ $f'(x) = 3x^2 + 2x - 5$
2 Function is stationary when $f'(x) = 0$.	Stationary points occur when
3 Solve for x.	$3x^2 + 2x - 5 = 0$ $x = -\frac{5}{3}, 1$
4 Find the y-coordinate of each stationary point.	$f\left(-\frac{5}{3}\right) = \left(-\frac{5}{3}\right)^3 + \left(-\frac{5}{3}\right)^2 - 5\left(-\frac{5}{3}\right) - 4$ $= \frac{67}{27}$ $f(1) = 1^3 + 1^2 - 5 \times 1 - 4$ $= -7$ The stationary points are $\left(-\frac{5}{3}, \frac{67}{27}\right)$ and $(1, -7)$.
5 Find the gradient either side of the stationary points.	
6 Interpret the solution.	The stationary points are a local maximum at $\left(-\frac{5}{3}, \frac{67}{27}\right)$ and a local minimum at $(1, -7)$.

x	-2	$-\frac{5}{3}$	0	1	2
$f'(x)$	Positive	0	Negative	0	Positive
Shape	/	—	\	—	/

Video
Stationary points

WORKED EXAMPLE 11 Stationary points of a quartic function

Determine the location and nature of all stationary points of the function $f(x) = x^4 - 7x^3 + 15x^2 - 13x + 4$.

Steps	Working
1 Differentiate $f(x)$.	$f(x) = x^4 - 7x^3 + 15x^2 - 13x + 4$ $f'(x) = 4x^3 - 21x^2 + 30x - 13$
2 Function is stationary when $f'(x) = 0$.	Stationary points occur when $f'(x) = 4x^3 - 21x^2 + 30x - 13 = 0$
3 Solve for x.	$x = 1, \frac{13}{4}$
4 Determine the y-coordinate of each stationary point.	$f(1) = 1^4 - 7 \times 1^3 + 15 \times 1^2 - 13 \times 1 + 4$ $= 0$ $f\left(\frac{13}{4}\right) = \left(\frac{13}{4}\right)^4 - 7 \times \left(\frac{13}{4}\right)^3 + 15 \times \left(\frac{13}{4}\right)^2 - 13 \times \left(\frac{13}{4}\right) + 4$ $= -8.54$ Therefore, the function has stationary points at $(1, 0)$ and $(3.25, -8.54)$.

5 Assess the nature of the stationary points by looking at the gradient of the function either side of the stationary points.

x	0	1	2	3.25	4
$f'(x)$	Negative	0	Negative	0	Positive
Shape	╲	—	╲	—	╱

Therefore, the function has a point of inflection on a falling curve at $(1, 0)$ and a minimum turning point at $(3.25, -8.54)$.

Exam hack

In a Technology-active exam, it is quicker to use your calculator to determine the location of stationary points.

In a Technology-free exam, factorise the derivative to find stationary points.

EXERCISE 10.4 Stationary points

ANSWERS p. 400

Recap

1 A particle has a displacement equation of $s(t) = 0.3t^2 + t - 5$. The velocity at $t = 2$ is

A -1.8 **B** 1.6 **C** 2.2 **D** 5

2 A particle has a displacement equation of $s(t) = 0.3t^2 + t - 5$. The particle has velocity 2 at

A $t = 6.08$ **B** $t = 1.67$ **C** $t = 2.74$ **D** $t = 3.33$

Mastery

3 WORKED EXAMPLE 9 Identify the location and nature of any stationary points of each function.

a $f(x) = 2x^2 - 8x + 3$ **b** $f(x) = 6 + 4x - x^2$

c $f(x) = x^3 - x^2 - 8x + 10$ **d** $f(x) = 2x^3 + 3x^2 - 12x$

e $f(x) = x^3 + 3x - 5$ **f** $f(x) = x^3 + 3x^2 - 9x$

4 WORKED EXAMPLE 10 Identify the location and nature of the stationary points of $f(x) = x^3 - 3x + 1$.

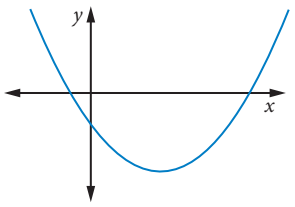
5 Identify the location and nature of the stationary points of $f(x) = x^3 - 9x^2 + 24x$.

6 WORKED EXAMPLE 11 Identify the location and nature of the stationary points of $f(x) = 3x^4 - 4x^3$.

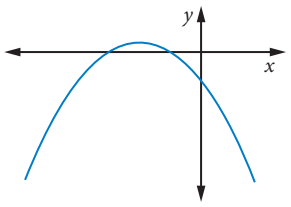
7 Identify the location and nature of the stationary points of $f(x) = x^3 + 6x^2 + 12x + 10$.

8 Copy each graph, then label the parts of the curve where the gradient is positive '+', stationary '0' or negative '–'.

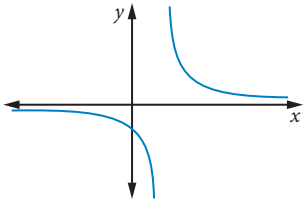
a



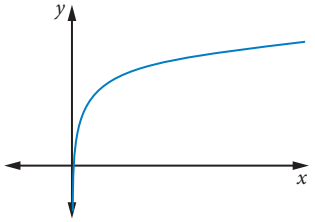
b



c

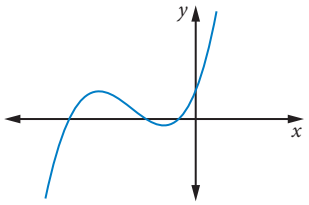


d

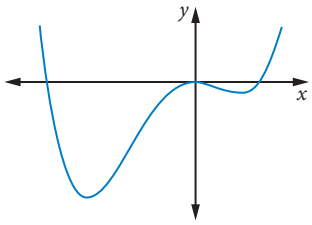


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e

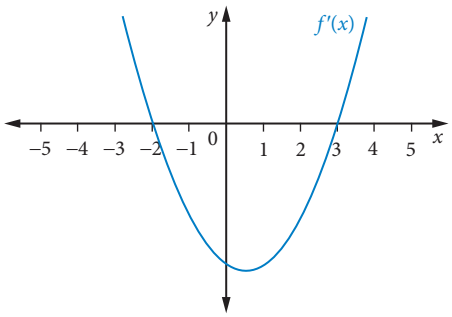


f

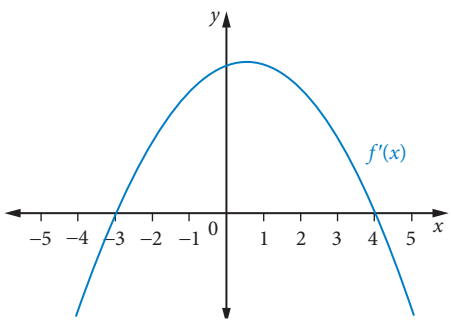


9 The graphs of the derivatives of a number of functions are shown below. In each case, state the value of x for which the function has a stationary point and identify the nature of the stationary point.

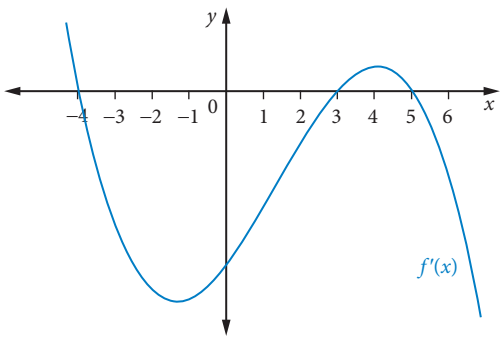
a



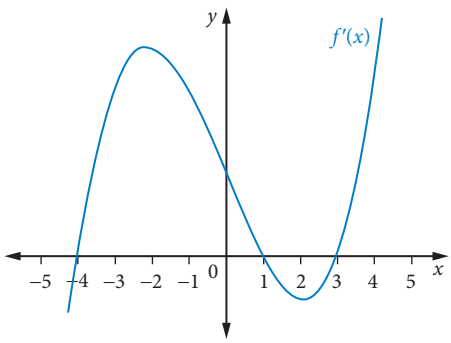
b



c



d



10 For the function $f(x) = 3x^2 - 12x + 5$, find

a the point at which the function is stationary

b all x values for which the function is increasing

c all x values for which the function is decreasing.

11 Find the values of x for which each function is increasing and decreasing.

a $f(x) = x^3 + 6x^2 + 8$ **b** $f(x) = -x^3 - 2x^2 + 7x + 5$ **c** $f(x) = x^4 - 3x^3 + x^2$

12 Given the function $f(x) = 5x^2 + 4x - 8$, show that the function is

a decreasing when $x = -3$ **b** stationary when $x = -\frac{2}{5}$ **c** increasing when $x = 2$.

13 Given the function $f(x) = x^3 + 5x^2 + x$, show that the function is

a increasing when $x = -4$ **b** decreasing when $x = -2$ **c** increasing when $x = 2$.

14 Given the function $f(x) = x^3 + 3x^2 - 24x$, show that the function is

a increasing when $x = -5$ **b** decreasing when $x = -1$ **c** increasing when $x = 3$.

Exam practice TECH-FREE

15 (4 marks) Identify the location and nature of the stationary points of $f(x) = 2 + 3x - x^3$.

16 (4 marks) Identify the location and nature of the stationary points of $f(x) = 12x^2 - x^3 - 48x + 69$.

17 (5 marks) CF $f(x) = x^2 + kx + 7$ has a stationary point at $x = 2$. Find the value of k and identify the location and nature of the stationary point.

18 (6 marks) CF $f(x) = x^3 + ax^2 + bx + 2$ has stationary points at $x = 1$ and $x = 3$. Find the values of a and b and identify the location and nature of the stationary points.

19 (7 marks) CU $f(x) = x^4 + ax^2 + b$ has a stationary point at $(-1, 2)$. Identify all the stationary points.

Worksheet
Greatest and least values

10.5 Local and global maximum and minimum points

A local maximum or local minimum is not necessarily the greatest or least value of a function.

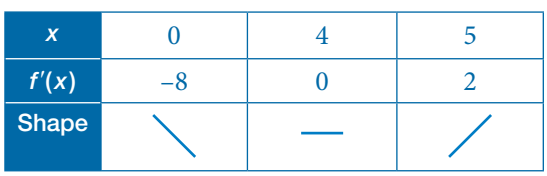
The greatest and least overall values of a function are called the **global maximum** and **global minimum** respectively.

For a function with a restricted domain, the global maximum or minimum point can only be in one of 2 places:

- a minimum or maximum stationary point, or
- at the start point or end point of the stated domain.

WORKED EXAMPLE 12 Maximum and minimum of a quadratic function

Determine all local and global maxima and minima of $f(x) = x^2 - 8x + 18$ for $2 \leq x \leq 5$.

Steps	Working
1 Find values at the end points.	$f(2) = 2^2 - 8 \times 2 + 18$ $= 6$ $f(5) = 5^2 - 8 \times 5 + 18$ $= 3$ The end points are $(2, 6)$ and $(5, 3)$.
2 Find when $f'(x) = 0$ for stationary points.	$f'(x) = 2x - 8 = 0$ $2x = 8$ $x = 4$
3 Find the y value.	$f(4) = 4^2 - 8 \times 4 + 18$ $= 2$ The function has a stationary point at $(4, 2)$.
4 Determine the nature of the stationary point.	 The function has a local minimum at $(4, 2)$.
5 Compare the end points and the local minimum.	The function has values $(2, 6)$ and $(5, 3)$ at the end points and a local minimum of $(4, 2)$.
6 Write the answer.	Therefore, the function has a local and global minimum at $(4, 2)$ and a global maximum at $(2, 6)$.

EXERCISE 10.5 Local and global maximum and minimum points

ANSWERS p. 401

Recap

1 At most, how many stationary points can a quartic function have?

A 1 **B** 2 **C** 3 **D** 4

2 What is the nature of the stationary point of the function $y = (x - 2)^3$?

A a point of inflection **B** a maximum **C** there is no stationary point **D** a minimum

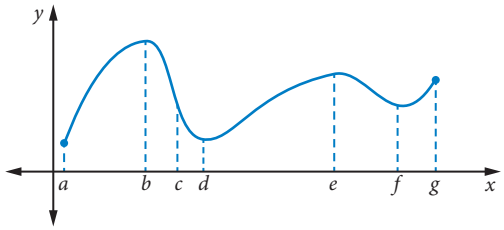
Mastery

3 WORKED EXAMPLE 12 Find the global maximum of the function $f(x) = x^2 + 6x + 5$ for $-6 \leq x \leq 1$.

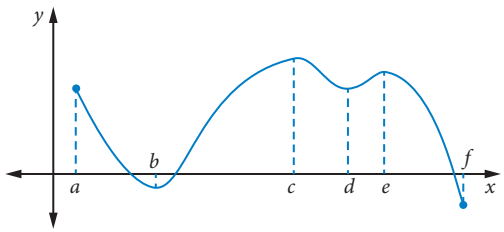
4 Find the global maximum and minimum of the function $f(x) = x^3 - 12x$ for $-3 \leq x \leq 5$.

Exam practice

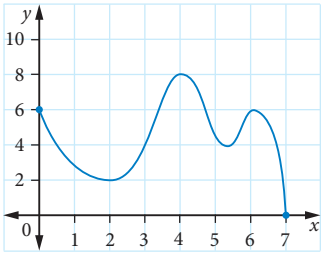
5 TECH-FREE (4 marks) At each x value a, b, c, d, e, f and g, state whether the point on the graph is a local maximum, local minimum, global maximum, global minimum, or neither a maximum nor a minimum.



6 TECH-FREE (4 marks) At each x value a, b, c, d, e, and f, state whether the point on the graph is a local maximum, local minimum, global maximum, global minimum, or neither a maximum nor a minimum.

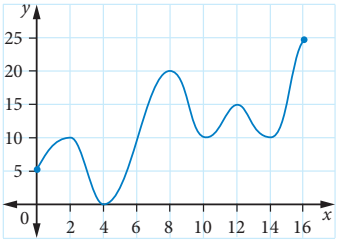


7 TECH-FREE (4 marks) The graph of a function is shown.



State the global and local maximum and minimum values of the function.

8 TECH-FREE (4 marks) For the graph shown, state the global and local maximum and minimum values of the function.



9 TECH-FREE (4 marks) Find the greatest and least values of $f(x) = x^2 - 2x + 3$ for $-2 \leq x \leq 2$.

10 TECH-FREE (4 marks) Find the global minimum of the function $f(x) = x^2 + 6x + 5$ for $-6 \leq x \leq 1$.

11 TECH-FREE (4 marks) Find the global minimum of the function $f(x) = -3x^2 - 10x + 8$ for $-5 \leq x \leq 2$.

12 TECH-FREE (6 marks) Find the global minimum and maximum of the function $f(x) = x^3 - 5x^2 + 6x + 4$ for $-3 \leq x \leq 1$.

13 TECH-FREE (6 marks) Find the global minimum and maximum of the function $f(x) = x^3 - 3x^2 + 1$ for $\frac{1}{2} \leq x \leq 4$.

14 (3 marks) Use a graphics calculator to find the global minimum and maximum of the function $f(x) = x^4 + x^3 - 16x^2 - 4x + 48$ for $-3 \leq x \leq 3$.

15 (3 marks) Use a graphics calculator to find the global minimum and maximum of the function $f(x) = 3x^5 - 5x^3 + 5$ for $-1 \leq x \leq 1\frac{1}{2}$.

16 (3 marks) Use a graphics calculator to find the global minimum and maximum of the function $f(x) = x^4 - 2x^3 - 5$ for $-2 \leq x \leq 3$.

17 (3 marks) Use a graphics calculator to find the global minimum and maximum of the function $f(x) = x^5 - 5x^3$ for $-2 \leq x \leq 2$.

18 (5 marks) CF The position of a particle moving along a horizontal line is $s(t) = -t^3 + 18t^2 - 81t$ for $0 < t < 12$, where s is in metres and t is in seconds. Find the greatest velocity reached by the particle and the time at which this occurs.

19 (5 marks) CF Find the points on the graph of the function $f(x) = 1 - 2x + 3x^2 - x^3$ between $x = 0$ and $x = 3$ where the gradient is greatest and least.

20 (5 marks) CF Find the points on the graph of the function $f(x) = -x^3 + \frac{15}{2}x^2 - 12x - 2$ for $0 \leq x \leq 6$ where the gradient is the greatest and least.

Worksheets
Sketching curves
Gradient functions

10.6 Curve sketching

Sketching functions

Find as many of the following features of the graph as you can:

- behaviour at the extremes: when the domain is not restricted, use the leading (dominant) term to assess the direction $f(x)$ will approach as $x \to \pm\infty$
- endpoint values for a restricted domain
- zeros (x-intercepts) if possible
- sign: between and beyond zeros
- y-intercept: $f(0)$
- derivative: find stationary points and where the function is increasing or decreasing.

WORKED EXAMPLE 13 Sketching a factored cubic

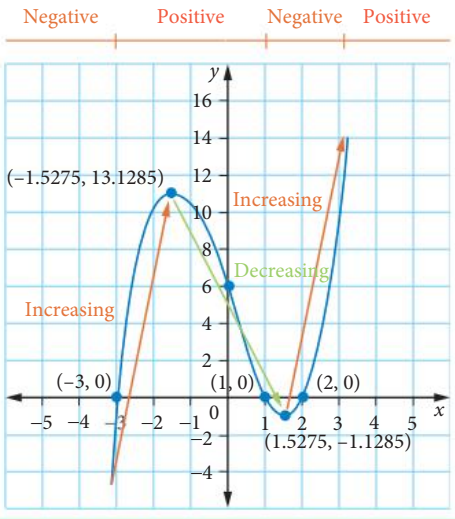
Sketch the function $f(x) = (x - 1)(x - 2)(x + 3)$.

Steps	Working
1 Interpret the function's general shape.	Function is a cubic with a positive leading coefficient. Therefore, sketch should have standard cubic shape.
2 Assess behaviour at extreme points.	As $x \to +\infty, f(x) \to +\infty$. As $x \to -\infty, f(x) \to -\infty$.
3 Solve for x-intercepts.	Using the null factor law: $x - 1 = 0, x - 2 = 0$ or $x + 3 = 0$ $x = -3, 1, 2$
4 Use a table for signs.	

x	-4	-3	0	1	1.5	2	3
$f(x)$	-30	0	6	0	$-\frac{9}{8}$	0	12
Sign	$-$		$+$		$-$	$+$	

5 Determine y-intercept.	$f(0) = 6$
6 Differentiate $f(x)$.	$f(x) = x^3 - 7x + 6$ $f'(x) = 3x^2 - 7$
7 Find stationary points and determine their nature.	$0 = 3x^2 - 7$ $x = \pm\sqrt{\frac{7}{3}}$ $x \approx \pm 1.5275$ Use a graphics calculator to determine the associated y values. Stationary points occur at $(-1.5275, 13.1285)$ and $(1.5275, -1.1285)$.

x	-2	-1.5275	0	1.5275	2
$f'(x)$	$+$	0	$-$	0	$+$
Shape	/	—	\	—	/

8 Determine the nature of these stationary points.	There is a local maximum at $(-1.5275, 13.1285)$ and a local minimum at $(1.5275, -1.1285)$.
9 Use the table for increasing and/or decreasing intervals.	$f(x)$ is increasing before $x = -1.5275$ and after $x = 1.5275$. $f(x)$ is decreasing between $x = -1.5275$ and $x = 1.5275$.
10 Use all the information to sketch the graph.	

Exam hack

When you see a curve sketching question in a Technology-active exam, the first thing you should do is graph the function on your graphics calculator. You still need to show working for the important features.

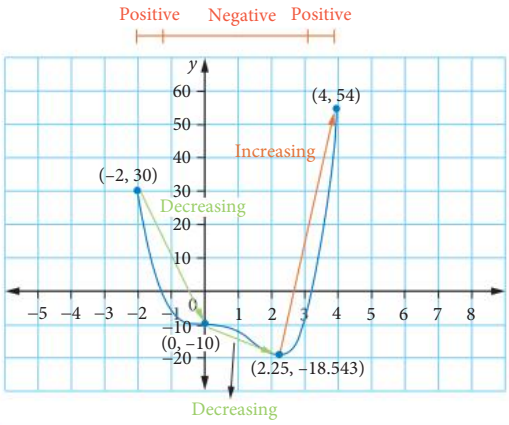
WORKED EXAMPLE 14 Sketching a polynomial function

Sketch the function $y = x^4 - 3x^3 - 10$ for $-2 \leq x \leq 4$.

Steps	Working
1 Interpret the function's general shape.	Function is a quartic with positive leading coefficient. Therefore, sketch should have standard quartic shape.
2 Determine end point values.	For $x = -2$, $y = (-2)^4 - 3 \times (-2)^3 - 10$ $= 30$ For $x = 4$, $y = 4^4 - 3 \times 4^3 - 10$ $= 54$ Therefore, the function will start at $(-2, 30)$ and end at $(4, 54)$.
3 Try to solve for x-intercepts.	$x^4 - 3x^3 - 10 = 0$ is not easily solved.
4 Create a table to assess sign of the function.	
5 Determine y-intercept.	$f(0) = -10$
6 Differentiate $f(x)$ and solve $f'(x) = 0$ for stationary points.	$y = x^4 - 3x^3 - 10$ $f'(x) = 4x^3 - 9x^2 = 0$ $= x^2(4x - 9) = 0$
7 Find the y values.	$x^2 = 0$ or $4x - 9 = 0$ $x = 0$ or $x = 2.25$ When $x = 2.25$, $y = 2.25^4 - 3 \times 2.25^3 - 10$ ≈ -18.543
8 List the stationary points.	The stationary points are $(0, -10)$, $(2.25, -18.543)$.
9 Determine the nature of these stationary points.	
10 Describe the stationary points.	There is a point of inflection on a falling curve at $(0, -10)$ and a local minimum at $(2.25, -18.543)$.
11 State the increasing and/or decreasing intervals.	$f(x)$ is decreasing before $x = 2.25$, except at $x = 0$. $f(x)$ is increasing after $x = 2.25$.
12 Use all information to create the sketch.	

x	-2	0	4
$f(x)$	30	-10	54
Sign	+	−	+

x	-1	0	1	2.25	3
$f'(x)$	−	0	−	0	+
Shape	╲	—	╲	—	╱



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EXERCISE 10.6 Curve sketching

ANSWERS p. 402

Recap

1 The global maximum of the function $f(x) = -x^3 + 2x^2 - x + 1$ for $0 \leq x \leq 3$ is

A both $(0, 1)$ and $(1, 1)$ **B** $(1, 1)$ only **C** $(0, 1)$ only **D** $(0.33, 0.85)$

2 The global minimum of the function $f(x) = -x^3 + 2x^2 - x + 1$ for $0 \leq x \leq 3$ is

A $(0.33, 0.85)$ **B** $(3, -11)$ **C** $(1.75, 0)$ **D** $(0, 1)$

Mastery

3 WORKED EXAMPLE 13 Sketch the graph of each function showing all key features.

a $g(x) = 10x - x^2 - 21$ **b** $h(x) = x^2 - x - 8$

c $q(x) = x^3 - x^2 - x - 1$ **d** $f(x) = 3x - x^3 - 18$

4 WORKED EXAMPLE 14 Sketch the graph of each function, showing all key features.

a $f(x) = x^3 - 3x^2 + 3x - 1$ **b** $g(x) = x^4 - 2x^3 + 6$

c $f(x) = 3x^5 - 5x^3 + 5$ **d** $h(x) = (2x + 1)^2(x + 3)$

5 Use the graph of $g(x)$ to estimate the

a coordinates of stationary points

b y value at $x = -4$

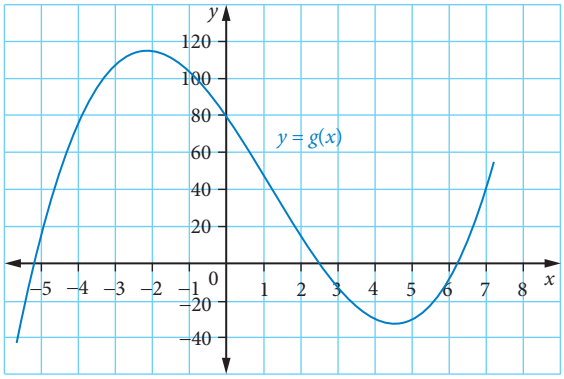
c y value at $x = 1$

d intervals where the function is decreasing

e intervals where the function is increasing

f rate of change at $x = -4$

g rate of change at $x = 1$.



6 Use the graph of $p(x)$ to estimate the

a coordinates of stationary points

b y value at $x = 1$

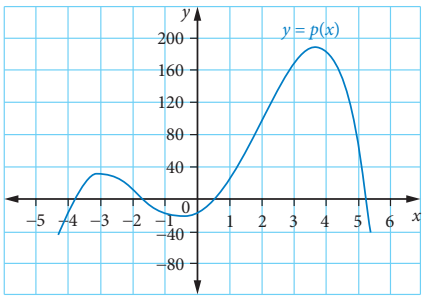
c y value at $x = 2$

d intervals where the rate of change of the function is positive

e intervals where the rate of change of the function is negative

f rate of change at $x = 2$

g rate of change at $x = -4$.



7 Sketch the graph of each function showing all key features.

a $h(x) = 2x^2 + x - 4$ **b** $f(x) = x^3 + 3x^2 - 24x$

c $f(x) = 2x^6 - 6x^4$ **d** $f(x) = 7 + 2x^2 - x^4$

8 Sketch the graph of each function between $-5 \leq x \leq 5$, showing all key features.

a $y = 2 + 5x - x^3$ **b** $y = 2x^3 - 6x^2 + 6x - 5$ **c** $f(x) = x^3 - 9x^2$

d $y = (x - 1)^3(x + 2)^2$ **e** $f(x) = x^5 - 4x^3$ **f** $y = (x^2 - 1)(x^2 - 4)$

Exam practice

9 TECH-FREE (6 marks) CU A cubic polynomial function $f(x) = ax^3 + bx^2 + cx + d$ can have either 0, 1 or 2 stationary points. Determine the x value of any stationary points and the conditions for which only 1 stationary point would occur.

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Video
Exam question analysis: Applications of derivatives

EXAM QUESTION ANALYSIS

© QCAA 2018 Unit 1 and 2 Sample Assessment Instrument 1Q9 MODIFIED TECH-FREE (6 marks)

a If $x^3 + 4x - 5$ can be factorised to $(x - 1)(x^2 + bx + c)$ where b and c are constants, find the values of b and c. *[2 marks]*

b A curve has the function $y = x^4 + 8x^2 - 20x + 8$.
Show that the only stationary point for this function is a minimum.
Use mathematical reasoning to justify your response. *[4 marks]*

Reading the question

- The question involves the stationary points of a function.
- The question asks to prove that a statement is correct.
- The question asks to provide reasons or evidence to support the statement.

Thinking about the question

- 'Only stationary point' means one stationary point. Therefore, when $\frac{dy}{dx} = 0$, there is only one solution.
- Minimum stationary point means the function decreases before the stationary point and then increases after the stationary point.
- 'Justify' in part **b** means to provide the algebraic solution.

Worked solution (✓ = 1 mark)

a $x^3 + 4x - 5 = (x - 1)(x^2 + bx + c)$
$= x^3 + bx^2 + cx - x^2 - bx - c$
$= x^3 + (b - 1)x^2 + (c - b)x - c$

$\therefore b - 1 = 0, c - b = 4, c = 5$

$\therefore b = 1$, ✓ $c = 5$ ✓

b $y = x^4 + 8x^2 - 20x + 8$

$\frac{dy}{dx} = 4x^3 + 16x - 20$

$0 = 4(x^3 + 4x - 5)$

$\mathbf{0 = x^3 + 4x - 5}$ ✓ Determining derivative and letting equal 0

From part (a),

$x^3 + 4x - 5 = (x - 1)(x^2 + x + 5)$

So, $x^3 + 4x - 5 = 0$ becomes

$(x - 1)(x^2 + x + 5) = 0$

Using the null factor law,

$x - 1 = 0$ or $x^2 + x + 5 = 0$

$x = 1$ $\Delta = b^2 - 4ac$

$\Delta = 1^2 - 4 \times 1 \times 5$

$= -19$

< 0

Therefore, there are no real solutions to the quadratic term. ✓ Justification of no roots from quadratic factor

Hence, the only stationary point occurs when $x = 1$. ✓ Statement of location of only stationary point

Finally, determine the nature of the stationary point.

x	0	1	2
$f'(x)$	Negative	0	Positive
Shape	╲	—	╱

Minimum point at $x = 1$.

Therefore, there is only one stationary point for $y = x^4 + 8x^2 - 20x + 8$, which is a minimum at $x = 1$. ✓ Statement proving original statement

10 Chapter summary

Instantaneous rates of change

- **Marginal cost**: the change in $C(x)$ of producing 1 additional unit. Therefore, a normal marginal cost $= C'(x)$.
- Physical applications including area, volume, temperature and pressure.

Tangents and normals to a curve

- A tangent is a straight line that intersects a function at only one point.
- The gradient of a tangent at $x = a$ is $f'(a)$.
- A normal is a line that is perpendicular to a tangent, at the point where the tangent meets the graph.
- $m_T \times m_N = -1$

Displacement, velocity and acceleration

- **Displacement** is the change in position from a start point to an end point ignoring the path taken. It is represented by the variable x, though you may see the variable s also used.
- **Velocity** is the rate of change of displacement of a moving object, so $v = \frac{dx}{dt}$.
- **Acceleration** is the rate of change of velocity of a moving object, so $a = \frac{dv}{dt}$.

Stationary points

- A **local minimum**, where the function decreases to the stationary point, then increases afterwards.
- A **local maximum**, where the function increases to the stationary point, then decreases afterwards.
- Collectively, local minimums and local maximums are called turning points.
- A **point of horizontal inflection**, where the function increases (or decreases) to the stationary point, then continues to increase (or decrease) afterwards.

	$f'(x)$ before the stationary point	$f'(x)$ at the stationary point	$f'(x)$ after the stationary point	Type of point	Change in $f'(x)$ as x increases
$f(x)$	+	0	−	Maximum	Decreasing
$f(x)$	−	0	+	Minimum	Increasing
$f(x)$	+	0	+	Inflection on a rising curve	Increasing to zero, then increasing again
$f(x)$	−	0	−	Inflection on a falling curve	Decreasing to zero, then decreasing again

Local and global maximum and minimum points

- Local maximum is used to describe a maximum point. Local minimum is used to describe a minimum point.
- The greatest and least overall values of a function are called the **global maximum** and **global minimum** respectively.
- For a function with a restricted domain, a global maximum or minimum point is either a minimum or maximum stationary point, or the start point or end point of the stated domain.

Curve sketching

Aim to find these features of the graph of $y = f(x)$:

- behaviour at the extremes: when the domain is not restricted, use the leading (dominant) term to assess the direction $f(x)$ will approach as $x \to \pm\infty$
- end point values for a restricted domain
- zeros (x-intercept(s)) if possible
- sign: between and beyond zeros
- y-intercept: $f(0)$
- derivative: find stationary points and where the function is increasing or decreasing.

Cumulative examination 1

Technology-free

Perusal time: 2 minutes Working time: 30 minutes

TECH-FREE *Calculators are **not** permitted.*

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

Section 1 (3 marks): 3 multiple choice questions

Section 2 (15 marks): 4 short response questions

Total: 18 marks



Worksheet
Maths Methods Year 11 formula sheet

Section 1 3 multiple choice questions

3 marks

1 The derivative of $7x^3 + 6x^2$ is

A $21x^2 + 12x$ **B** $21x^3 + 12x$ **C** $3x^2 + 2x$ **D** $7x^2 + 6x$

2 $\dfrac{5\log(2x)}{3\log(x)} =$

A $\log(32x^2)$ **B** $\log(32x^5 - x^3)$ **C** 1 **D** $\log(7x)$

3 A stationary point where the gradient function is decreasing as x increases is also known as a

A minimum turning point **B** rising point of inflection

C maximum turning point **D** falling point of inflection

Section 2 4 short response questions

15 marks

4 (3 marks) Describe the transformations to transform $y = 7^x$ into $y = 14^{x+2} - 1$.

5 (3 marks) Use first principles to differentiate $6x^2 - x + 1$.

6 (4 marks) Find the length of the tangent to $f(x) = x^3 - 5x^2 + x + 26$ at $x = 3$ cut off by the axes.

7 (5 marks) Determine the intersection point of the normals to the function $f(x) = 2 - 2x^2$ at the zeros of $f(x)$.

Worksheet
Maths Methods Year 11 formula sheet

Cumulative examination 2

Technology-active

Perusal time: 2 minutes Working time: 30 minutes

*Calculators are **permitted**.*

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

Section 1 (3 marks): 3 multiple choice questions

Section 2 (15 marks): 4 short response questions

Total: 18 marks

Section 1 3 multiple choice questions

3 marks

1 The value of x that satisfies the equation $3 \times 2^x = 5$ is

A -2.322 **B** 0.737 **C** 1 **D** 2.322

2 The global maximum of the function $f(x) = 2x^3 - 5x^2 + 3$ in the domain $0 \leq x \leq 2$ is at

A $(1, 0)$ **B** $(0, 3)$ **C** $(1.67, -1.63)$ **D** $(2, -1)$

3 The equation of the tangent to the function $f(x) = x^2 - 3x + 2$ at $x = 0.5$ is

A $y = -2x + 1.75$ **B** $y = 0.75x + 0.5$

C $y = 0.5x + 0.5$ **D** $y = -2x + 0.75$

Section 2 4 short response questions

15 marks

4 (4 marks) Simplify $\dfrac{\sqrt{5}+\sqrt{2}}{\sqrt{5}-\sqrt{2}}$.

5 (4 marks) Find the stationary points of $f(x) = 3x^3 - 5x^2 - 3x + 4$, correct to 2 decimal places.

6 (3 marks) Solve $5^{3x} - 3 \times 5^{2x} - 5x = 0$, correct to 3 decimal places.

7 (4 marks) Sketch the graph of $y = 5x - 3x^2 + 6 - x^3$, identifying all key features.

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CHAPTER

11 LOGARITHMIC FUNCTIONS

Syllabus coverage
Nelson MindTap chapter resources
Terminology
11.1 Logarithmic functions
11.2 Graphing logarithmic functions
11.3 Translating logarithmic functions
11.4 Logarithmic equations
11.5 Modelling with logarithmic functions
Exam question analysis
Chapter summary
Cumulative examination 1
Cumulative examination 2

Prior learning
Logarithmic functions

Syllabus coverage

UNIT 2, TOPIC 2: LOGARITHMS AND LOGARITHMIC FUNCTIONS

Logarithmic functions

- Recognise and determine the qualitative features of the graph of $y = \log_a(x)$ (where $a > 1$), including asymptote and intercept.
- Recognise and determine the effect of the parameters a, h and k on the graph of $y = \log_a(x - h) + k$ (where $a > 1$), with and without technology.
- Sketch graphs of logarithmic functions, with and without technology.
- Solve equations involving logarithmic functions with and without technology.
- Model and solve problems that involve logarithmic functions, e.g. decibels in acoustics and the Richter scale for earthquake magnitude, with and without technology.

Mathematical Methods 2025 v1.1 General senior syllabus p. 21,
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Video (1):
Exam question analysis Logarithmic functions

Prior learning (1):
11.1 Logarithmic functions

Worksheets (5):
11.2 Plotting log functions
11.4 Logarithmic and exponential equations
11.5 Logarithmic functions assignment
• Logarithmic functions extension
Cumulative exams Maths Methods Year 11 formula sheet

Puzzles (5):
11.2 Logarithmic graphs match-up
• Exponential and log graphs match-up
11.4 Logarithms – Solving equations 1
• Logarithms – Solving equations 2
Chapter summary Logarithmic functions word puzzle

TI-Nspire CX II instructions (1):
11.2, 11.4 Logarithmic functions

Nelson MindTap

To access resources above, visit
cengage.com.au/nelsonmindtap

Terminology

half-life	logarithmic equation	logarithmic function
logarithmic scale	Richter scale	

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11.1 Logarithmic functions

Logarithmic functions

A logarithmic function has the form $f(x) = \log_a(x)$, where $a, x > 0, a \neq 1$.
Its domain is $x > 0$ and its range is all real numbers.

Remember that $\log_a(x) = m$ and $x = a^m$ are equivalent statements.

WORKED EXAMPLE 1 Logarithmic functions

$f(x) = \log_4(x)$.

Determine $f\left(\frac{1}{16}\right), f\left(\frac{1}{4}\right), f(1), f(4), f(8)$ and $f(16)$.

Steps	Working
1 Write as exponentials.	For what values of m does $4^m = \frac{1}{16}, \frac{1}{4}$, 1, 4, 8 and 16?
2 Write the indices needed.	$4^{-2} = \frac{1}{16}, 4^{-1} = \frac{1}{4}, 4^0 = 1, 4^1 = 4, 4^{\frac{3}{2}} = 8$ and $4^2 = 16$
3 Write the answers.	$f(x) = -2, -1, 0, 1, 1.5$ and 2 respectively.

EXERCISE 11.1 Logarithmic functions

ANSWERS p. 404

Mastery

1 WORKED EXAMPLE 1 $f(x) = \log_3(x)$.

Determine

a $f(9)$ **b** $f(243)$ **c** $f\left(\frac{1}{9}\right)$

d $f\left(\frac{1}{3}\right)$ **e** $f\left(\frac{1}{729}\right)$ **f** $f(27)$

2 Evaluate $h(x) = \log_2(x)$ at $x =$

a 0.125 **b** 32 **c** 256

d $\frac{1}{16}$ **e** $\frac{1}{32}$ **f** 128

3 $f(x) = \log_4(x)$.

Determine $f(0.03125), f(0.125), f(0.25), f(4), f(16), f(256)$

4 $f(x) = \log_{16}(x)$.

Determine $f(0.03125), f(0.125), f(0.25), f(4), f(16), f(256)$

Exam practice

5 TECH-FREE (3 marks) $f(x) = \log_{27}(x)$. What are the values of $f(3), f(729)$ and $f\left(\frac{1}{81}\right)$?

6 (3 marks) $f(x) = \log_{\frac{1}{6}}(x)$. What are the values of $f\left(\frac{1}{36}\right), f(6)$ and $f(\sqrt{216})$?

Worksheet
Plotting log functions

Puzzles
Logarithmic graphs match-up

Exponential and log graphs match-up

11.2 Graphing logarithmic functions

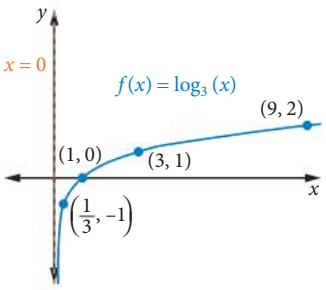
You already know that $\log_3(1) = 0$, $\log_3(3) = 1$ and $\log_3\left(\frac{1}{3}\right) = -1$.

When $x > 1$, $\log_3(x)$ is positive, when $x < 1$, $\log_3(x)$ is negative, and as $x \to 0$, $\log_3(x) \to -\infty$.

As x increases, $\log_3(x)$ increases, and as x decreases, $\log_3(x)$ decreases. You can use this knowledge to sketch the graph of $y = \log_3(x)$.

WORKED EXAMPLE 2 Graphing logarithmic functions 1

Sketch the graph of $f(x) = \log_3(x)$, labelling important features.

Steps	Working
1 Write the important values.	$\log_3\left(\frac{1}{3}\right) = -1$, $\log_3(1) = 0$, $\log_3(3) = 1$
2 List important points on the graph.	$\left(\frac{1}{3}, -1\right)$, $(1, 0)$, $(3, 1)$ are on the graph.
3 State the behaviour as x gets close to 0.	As $x \to 0$, $\log_3(x) \to -\infty$. There is a vertical asymptote at $x = 0$.
4 State the behaviour as x becomes large.	As $x \to \infty$, $\log_3(x) \to \infty$.
5 State another convenient point.	$f(9) = 2$
6 Put the information on the graph and join with a smooth curve. Label the asymptote and graph.	

You can sketch the graph of $f(x) = 2\log_3(x)$ as a dilation from the x-axis of $f(x) = \log_3(x)$.

You just need to multiply the y values of $f(x) = \log_3(x)$ by 2.

WORKED EXAMPLE 3 Graphing logarithmic functions 2

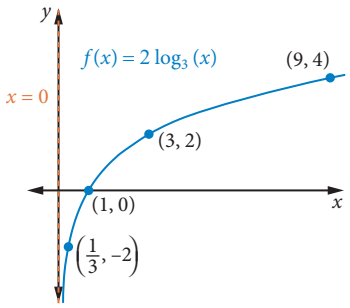
Sketch the graph of $f(x) = 2\log_3(x)$, labelling important features.

Steps	Working
1 Write the important values.	$\log_3\left(\frac{1}{3}\right) = 2 \times -1$, $\log_3(1) = 2 \times 0$, $\log_3(3) = 2 \times 1$
2 List important points on the graph.	$\left(\frac{1}{3}, -2\right)$, $(1, 0)$, $(3, 2)$ are on the graph.
3 State the behaviour as x gets close to 0.	As $x \to 0$, $\log_3(x) \to -\infty$. There is a vertical asymptote at $x = 0$.
4 State the behaviour as x becomes large.	As $x \to \infty$, $\log_3(x) \to \infty$.
5 State another convenient point.	$f(9) = 2 \times 2 = 4$

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6 Put the information on the graph and join with a smooth curve.

Label the asymptote and graph.



Graphs of logarithmic functions

The graph of a logarithmic function of the form $y = \log_a(x)$ where $a > 1$ has

- a vertical asymptote at $x = 0$
- a zero at $(1, 0)$
- important points $(a, 1)$ and $\left(\frac{1}{a}, -1\right)$
- as $x \to \infty, f(x) \to \infty$
- as $x \to 0, f(x) \to -\infty$.

You can use your graphics calculator to draw logarithmic graphs.

TECHNOLOGY Logarithmic graphs

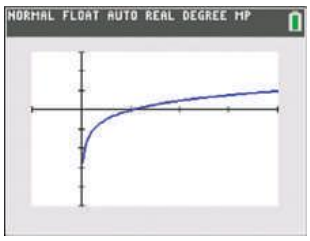
Draw the graph of $y = \log_2(x)$ on your graphics calculator.

TI-Nspire
Logarithmic functions

TI-84 Plus CE

1 Select y= and use catalog to select **logBASE(**. Fill in the required information.

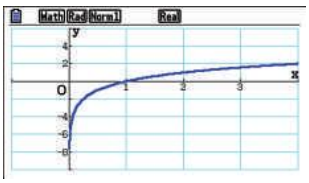
2 Adjust the window settings as required and draw the graph.



Casio fx-CG50 AU

1 From the **MAIN MENU**, press **5** to select **Graph** and use CATALOG (scroll down to logab) to enter $Y1 = \log_2(x)$.

2 Press F6 (DRAW) to graph. Set the V-Window as required.



EXERCISE 11.2 Graphing logarithmic functions

ANSWERS p. 404

Recap

1 Evaluate $\log_5(125)$.

A -3 **B** -1.5 **C** 0.5 **D** 3

2 Evaluate $\log_{25}(125)$.

Mastery

3 WORKED EXAMPLE 2 Sketch the graph of each function, labelling important features.

a $f(x) = \log_2(x)$ **b** $f(x) = \log_4(x)$ **c** $f(x) = \log_5(x)$

d $f(x) = \log_6(x)$ **e** $f(x) = \log_{2.5}(x)$ **f** $f(x) = \log_{1.5}(x)$

4 WORKED EXAMPLE 3 Sketch the graph of each function, labelling important features.

a $f(x) = 3\log_3(x)$ **b** $f(x) = 4\log_2(x)$ **c** $f(x) = 2.5\log_4(x)$

d $f(x) = 0.8\log_5(x)$ **e** $f(x) = 0.5\log_2(x)$ **f** $f(x) = 0.4\log_6(x)$

5 Sketch the graph of each function, labelling important features.

a $f(x) = -\log_2(x)$ **b** $f(x) = -\log_3(x)$ **c** $f(x) = -\log_5(x)$

Exam practice TECH-FREE

6 (3 marks) Sketch the graph of $f(x) = -\log_8(x)$, labelling important features.

7 (3 marks) Sketch the graph of $p(x) = 3\log_5(x)$, labelling important features.

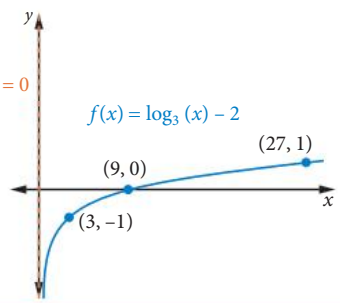
8 (4 marks) CF Use the graphs of $f(x) = 3 - (x-2)^2$ and $g(x) = 2\log_3(x)$ to solve $3 - (x-2)^2 = 2\log_3(x)$.

11.3 Translating logarithmic functions

You can use the graph of $y = \log_a(x)$ to sketch the graph of $y = \log_a(x) + k$.

WORKED EXAMPLE 4 Graphing $\log_a(x) + k$

Sketch the graph of $f(x) = \log_3(x) - 2$, labelling important features.

Steps	Working
1 State the important values of $\log_3(x)$.	$\log_3\left(\frac{1}{3}\right) = -1, \log_3(1) = 0, \log_3(3) = 1$
2 Subtract 2 to get values of $f(x)$.	$f\left(\frac{1}{3}\right) = -3, f(1) = -2, f(3) = -1$
3 State the behaviour as x gets close to 0.	As $x \to 0$, $\log_3(x) \to -\infty$. There is a vertical asymptote at $x = 0$.
4 State the behaviour as x becomes large.	As $x \to \infty$, $\log_3(x) \to \infty$.
5 Find the zero if possible.	$f(x) = 0$ when $\log_3(x) - 2 = 0$ $\log_3(x) = 2$ $x = 3^2 = 9$ $(9, 0)$
6 Find another point where x is a power of 3.	$f(27) = \log_3(27) - 2$ $= 3 - 2$ $= 1$ $(27, 1)$
7 Put the points on the graph. Join them in a smooth curve. Label the asymptote and the graph.	

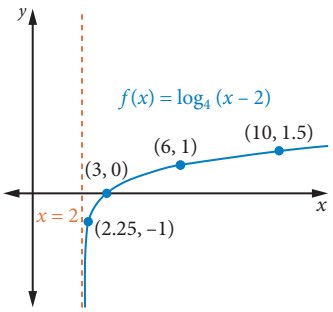
You can use the graph of $y = \log_a(x)$ to sketch the graph of $y = \log_a(x - h)$.

The graph of $y = \log_a(x - 5)$ is the graph of $y = \log_a(x)$ shifted 5 units to the right.

The vertical asymptote is shifted to $x = h$.

WORKED EXAMPLE 5 Graphing $\log_a(x - h)$

Sketch the graph of $f(x) = \log_4(x - 2)$, labelling important features.

Steps	Working
1 State the important values of $\log_4(x)$.	$\log_4(0.25) = -1, \log_4(1) = 0, \log_4(4) = 1$
2 Add 2 to the above x values to write corresponding values of $f(x)$.	$f(2.25) = -1, f(3) = 0, f(6) = 1$
3 State the behaviour as x gets close to 2.	As $x \to 2$, $\log_4(x - 2) \to -\infty$. There is a vertical asymptote at $x = 2$.
4 State the behaviour as x becomes large.	As $x \to \infty$, $\log_3(x) \to \infty$.
5 Find another point where $x - 2$ is a power of 4, such as 8.	$f(10 - 2) = \log_4(8) = \frac{3}{2} = 1.5$ $(10, 1.5)$
6 Put the points and asymptote on the graph. Join them in a smooth curve. Label the asymptote and the graph.	

Functions of the form $f(x) = \log_a(x - h) + k$ combine the features of Worked examples 4 and 5.

Translations of $y = \log_a(x)$

The graph of $y = \log_a(x - h) + k$ is the graph of $y = \log_a(x)$ translated:

- to the right by h units for $h > 0$ (the vertical asymptote shifted to $x = h$)
- to the left by h units for $h < 0$
- up by k units for $k > 0$
- down by k units for $k < 0$.

WORKED EXAMPLE 6 Graphing $\log_a(x - h) + k$

Sketch the graph of $f(x) = \log_2(x + 4) - 3$, labelling important features.

Steps	Working
1 State important points of $y = \log_2(x)$.	$\log_2(0.5) = -1, \log_2(1) = 0, \log_2(2) = 1$
2 How is $f(x)$ different to $y = \log_2(x)$?	$\log_2(x + 4) - 3$ is shifted 4 units left and 3 units down.
3 Subtract 4 from the x values and 3 from the y-coordinates of the important points.	$f(x) = \log_2(x + 4) - 3$ passes through $(-3.5, -4), (-3, -3)$ and $(-2, -2)$.

4 Find the zero.

There is a y-intercept because the curve has been shifted left.

$\log_2(x+4) - 3 = 0$

$x + 4 = 2^3$

$x + 4 = 8$

$x = 8 - 4 = 4$. The zero is $(4, 0)$.

5 Find the y-intercept.

$f(0) = \log_2(4) - 3 = -1$. The y-intercept is $(0, -1)$.

6 State the asymptote.

The vertical asymptote is at $x = -4$.

7 State the behaviour as x becomes large.

As $x \to \infty, f(x) \to \infty$.

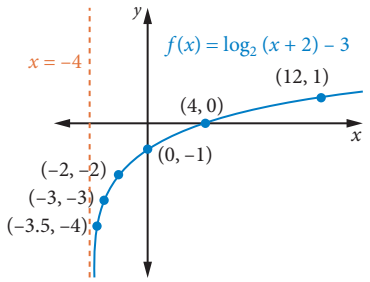
8 Find another point where $x + 4$ is a power of 2.

$f(12) = \log_2(12 + 4) - 3 = 1$, so $(12, 1)$.

9 Put the points and asymptote on the graph.

Join them in a smooth curve.

Label the asymptote and the graph.



EXERCISE 11.3 Translating logarithmic functions

ANSWERS p. 405

Recap

1 Sketch the graph of $f(x) = \log_7(x)$.

2 Sketch the graph of $f(x) = \log_5(x)$.

Mastery

3 WORKED EXAMPLE 4 Sketch the graph of each function, labelling important features.

a $y = \log_2(x) + 3$ **b** $y = \log_5(x) - 1$ **c** $y = \log_3(x) + 2$

4 Find the equation of the function produced by a translation of

a $y = \log_4(x)$, 5 units up

b $y = \log_{10}(x)$, 3 units down

c $y = \log_7(x)$, 2 units up

d $y = \log_6(x)$, 7 units down

5 WORKED EXAMPLE 5 Sketch the graph of each function, labelling important features.

a $y = \log_3(x+2)$ **b** $y = \log_5(x+1)$ **c** $y = \log_4(x-4)$

6 Find the equation of the function produced by a translation of

a $y = \log_5(x)$, 5 units to the left

b $y = \log_{0.4}(x)$, 3 units to the right

c $y = \log_{18}(x)$, 2 units to the left

d $y = \log_3(x)$, 7 units to the right

7 WORKED EXAMPLE 6 Sketch the graph of each function, labelling important features.

a $y = \log_2(x+2) - 1$ **b** $y = \log_3(x-1) + 2$ **c** $y = \log_5(x+2) + 1$

Exam practice

8 TECH-FREE (3 marks) Sketch the graph of $f(x) = \log_4(x + 3) + 2$, labelling important features.

9 TECH-FREE (3 marks) Sketch the graph of $f(x) = \log_5(x - 2) - 4$, labelling important features.

10 TECH-FREE (4 marks) CF Sketch the graph of $y = -\log_5(x + 1.5) + 2.5$.

11 (4 marks) CU Find solution(s) of the simultaneous equations $y = \log_4(x - 1) + 3$ and $y = (x - 2)(x - 6)$. Give answers correct to 2 decimal places.

11.4 Logarithmic equations

You can solve many equations with logarithms using the laws of logarithms.

Remember that if $\log_a(x) = \log_a(y)$, then $x = y$.

WORKED EXAMPLE 7 Logarithmic equations using laws

Solve each equation.

a $\log_7(5x - 1) = 2\log_7(\sqrt{x})$

b $\log_2(x) + \log_2(x - 2) = \log_2(6 - x)$

c $\log(4) - \log(5 - x) = \log(x)$

Steps	Working
a 1 Write the problem.	$\log_7(5x - 1) = 2\log_7(\sqrt{x})$
2 Simplify the RHS using logarithm laws.	$\log_7(5x - 1) = \log_7(\sqrt{x})^2$ $\log_7(5x - 1) = \log_7(x)$
3 Use equivalence.	$5x - 1 = x$
4 Solve for x.	$4x = 1$ $x = \frac{1}{4}$
5 Check the solution, starting with the more complex side.	$\text{LHS} = \log_7\left(5 \times \frac{1}{4} - 1\right)$ $= \log_7\left(\frac{1}{4}\right)$ $\text{RHS} = 2\log_7\left(\sqrt{\frac{1}{4}}\right)$ $= \log_7\left(\sqrt{\frac{1}{4}}\right)^2$ $= \log_7\left(\frac{1}{4}\right)$ $= \text{LHS}$
6 State the solution.	$x = \frac{1}{4}$
b 1 Write the problem.	$\log_2(x) + \log_2(x - 2) = \log_2(6 - x)$
2 Simplify the LHS using logarithm laws.	$\log_2[x(x - 2)] = \log_2(6 - x)$
3 Use equivalence.	$x(x - 2) = 6 - x$

Worksheet
Logarithmic and exponential equations

Puzzles
Logarithms - Solving equations 1
Logarithms - Solving equations 2

4	Expand the brackets and solve for x.	$x^2 - 2x = 6 - x$ $x^2 - x - 6 = 0$ $(x - 3)(x + 2) = 0$ $x = -2$ or 3
5	Check the LHS for $x = -2$.	$\text{LHS} = \log_2(-2) + \log_2(-2 - 2)$
6	State conclusion for $x = -2$.	$\log_2(-2)$ and $\log_2(-4)$ do not exist. $x = -2$ is not a solution.
7	Check the LHS for $x = 3$.	$\text{LHS} = \log_2(3) + \log_2(3 - 2)$ $= \log_2(3) + \log_2(1)$ $= \log_2(3) + 0$ $= \log_2(3)$
8	Check the RHS for $x = 3$.	$\text{RHS} = \log_2(6 - 3)$ $= \log_2(3)$ $= \text{LHS}$
9	State the solution.	LHS = RHS, so the solution is $x = 3$.
c 1	Write the problem.	$\log(4) - \log(5 - x) = \log(x)$
2	Simplify the LHS using logarithm laws.	$\log\left(\frac{4}{5 - x}\right) = \log(x)$
3	Use equivalence.	$\frac{4}{5 - x} = x$
4	Simplify and solve for x.	$4 = x(5 - x)$ $4 = 5x - x^2$ $x^2 - 5x + 4 = 0$ $(x - 4)(x - 1) = 0$ $x = 1$ or 4
5	Check LHS for $x = 1$.	$\text{LHS} = \log(4) - \log(5 - 1)$ $= \log(4) - \log(4)$ $= 0$
6	Check RHS.	$\text{RHS} = \log(1)$ $= 0$ $= \text{LHS}$
7	State conclusion for $x = 1$.	$x = 1$ is a solution.
8	Check LHS for $x = 4$.	$\text{LHS} = \log(4) - \log(5 - 4)$ $= \log(4) - \log(1)$ $= \log(4)$
9	Check RHS.	$\text{RHS} = \log(4)$ $= \text{LHS}$
10	State conclusion for $x = 4$.	$x = 4$ is a solution.
11	State the solution.	$x = 1$ or 4

You may need to convert an equation with logarithms to an exponential equation.

WORKED EXAMPLE 8 Logarithmic equations and exponentials

Solve each equation.

a $\log_2(5x+12)=6$

b $\log_3(x)+\log_3(x-6)=3$

c $\log_5(x^2-2x)=1+\log_5(2-x)$

Steps	Working
a 1 Write the problem.	$\log_2(5x+12)=6$
2 Write in exponential form and solve for x.	$2^6=5x+12$ $64=5x+12$ $5x=52$ $x=\frac{52}{5}$
3 Check the solution.	For $x=\frac{52}{5}$ $\text{LHS}=\log_2\left(5\times\frac{52}{5}+12\right)$ $=\log_2(64)$ $=6$ $\text{RHS}=6$
4 State the solution.	LHS = RHS, so $x=\frac{52}{5}$ is the solution.
b 1 Write the equation.	$\log_3(x)+\log_3(x-6)=3$
2 Simplify the RHS.	$\log_3[x(x-6)]=3$
3 Write in exponential form.	$x(x-6)=3^3$
4 Simplify and solve for x.	$x^2-6x=27$ $x^2-6x-27=0$ $(x+3)(x-9)=0$ $x=-3$ or 9
5 Check LHS for $x=-3$.	$\text{LHS}=\log_3(-3)+\log_3(-3-6)$ But $\log_3(-3)$ and $\log_3(-9)$ do not exist.
6 State the result for $x=-3$.	$x=-3$ is not a solution.
7 Check LHS for $x=9$.	$\text{LHS}=\log_3(9)+\log_3(9-6)$ $=\log_3(9)+\log_3(3)$ $=2+1$ $=3$ $=\text{RHS}$
8 State the solution.	$x=9$

c **1** Write the equation. $\log_5(x^2 - 2x) = 1 + \log_5(2 - x)$

2 Combine the logarithms and simplify.
$\log_5(x^2 - 2x) - \log_5(2 - x) = 1$

$$\log_5\left(\frac{x^2 - 2x}{2 - x}\right) = 1$$

3 Write in exponential form.
$$\frac{x^2 - 2x}{2 - x} = 5^1 = 5$$

4 Solve for x.
$$x^2 - 2x = 5(2 - x)$$
$$x^2 - 2x = 10 - 5x$$
$$x^2 + 3x - 10 = 0$$
$$(x - 2)(x + 5) = 0$$
$$x = -5 \text{ or } 2$$

5 Check LHS for $x = -5$.
$\text{LHS} = \log_5[(-5)^2 - 2 \times (-5)]$
$= \log_5(35)$

6 Check RHS for $x = -5$.
$\text{RHS} = 1 + \log_5[2 - (-5)]$
$= \log_5(5) + \log_5(7)$
$= \log_5(5 \times 7)$
$= \log_5(35)$
$= \text{LHS}$

7 State the conclusion for $x = -5$. $x = -5$ is a solution.

8 Write LHS for $x = 2$.
$\text{LHS} = \log_5[(2)^2 - 2 \times 2]$
$= \log_5(0)$
But $\log_5(0)$ does not exist.

9 State the solution. The solution is $x = -5$.

You can use a graphics calculator to solve logarithmic equations.

TECHNOLOGY Logarithmic equations on a graphics calculator

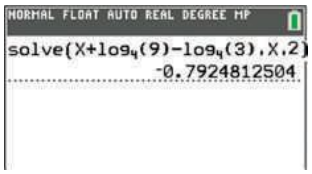
Solve $x + \log_4(9) = \log_4(3)$ using a graphics calculator.

TI-84 Plus CE

1 Use catalog to select **solve(**.

2 Enter the equation in the form *expression*, where *expression* is RHS – LHS. (Use catalog to choose logBASE).

3 Complete with , X, 2), where X is the variable and 2 is a guess of the solution.



Write the answer.

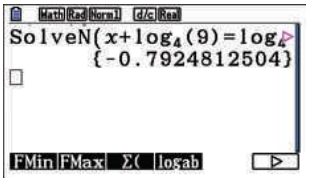
$x \approx -0.792$

Casio fx-CG50 AU

1 From the **MAIN MENU**, choose **Run-Matrix**.

2 Press OPTN, then F4 (CALC), and F5 (SolveN).

3 Enter the equation and the variable to solve for. Press F6 (▷) and F4 (logab) for log.



Write the answer.

$x \approx -0.792$

You can also solve logarithmic equations by finding zeros of a graph on a graphics calculator.

TI-Nspire Logarithmic functions

EXERCISE 11.4 Logarithmic equations

ANSWERS p. 406

Recap

1 Sketch the graph of $y = \log_3(x - 3) - 2$, labelling important features.

2 Sketch the graph of $y = \log_4(x + 1) + 3$, labelling important features.

Mastery

3 WORKED EXAMPLE 7 Solve each equation.

a $\log_5(7x + 3) = \log_5(5x + 9)$

b $\log_8(x) - \log_8(x + 6) = \log_8(5x + 12)$

c $\log(x - 2) - \log(2x - 3) = \log(2)$

d $\log_3[(x - 2)(x + 3)] = \log_3(14)$

e $\log_6(2x + 1) = \log_6(x + 2) - \log_6(3)$

f $\log_7(x + 4) + \log_7(x - 2) = \log_7(4x)$

g $\log(3x + 5) = \log(7x - 12)$

h $\log_2(3x + 2) = \log_2(x - 2)$

i $\log_6(2x) + \log_6(4) = \log_6(x + 12) - \log_6(2)$

4 WORKED EXAMPLE 8 Solve each equation.

a $\log_2(5x + 7) = 5$

b $\log(4x - 1) = 3$

c $\log_3(5x - 11) = 2$

d $\log_5(4x + 11) = 2$

e $\log_3(9x + 2) = 4$

f $\log_3(4x - 9) = 3$

g $\log_3(3x + 11) = 4$

h $\log_7(6x - 5) = 2$

i $\log_3(x^2 - 6x) = 3$

5 Solve each equation.

a $\log_2(x+5) - \log_2(2x-1) = 5$

b $2\log_3(x) - 1 = \log_3(2x-3)$

c $\log_4(4x-2) + \log_4(3x+1) = 5$

d $\log_2(x+1) - \log_2(x-4) = 3$

e $\log_2(x) + \log_2(2-2x) = -1$

f $\log_3(x) + \log_3(2x+1) = 1$

g $\log_6(x) + \log_6(x+5) = 2$

h $\log_2(2x+12) - \log_2(3x) = 2$

6 Solve each equation.

a $\log_6(x) + \log_6(x-9) = 2$

b $\log_4(x) + \log_4(x-12) = 3$

c $\log_6(x) + \log_6(x-1) = 1$

d $\log_6(x^2-2x) = \log_6(5x-12)$

e $\log_7(6x) - \log_7(4-x) = \log_7(3)$

f $\log(x) + \log(x+3) = \log(5x)$

g $\log_5(x-1) + \log_5(x+3) = \log_5(x^2-3x+12)$

h $[\log_3(x)]^2 = 3 - 2\log_3(x)$

i $\log_5(2x) + \log_5(x+2) = \log_5(6)$

Exam practice

7 TECH-FREE (3 marks) Solve $\log_8(2x-1) + \log_8(2x+1) = 3\log_8(3)$.

8 TECH-FREE (3 marks) Solve $\log_3(x) - \log_3(x-1) = 2$.

9 TECH-FREE (3 marks) Solve $\log_4(2x-3) - \log_4(x+2) = 1$.

10 TECH-FREE (3 marks) Solve $\log_8(2) + \log_8(4x^2) = 2$.

11 (3 marks) CF Solve $2^{\log_2(x-5)} + 2(x-5) - 12 = 0$.

12 (3 marks) CF Find a value of x that satisfies $\log_7(4x+31) = \log_{16}\left(2^6 + 72 \times \frac{1}{2} + \sqrt{24\,336}\right)$.

11.5 Modelling with logarithmic functions

The loudest sounds you can hear have 10 000 000 000 times the energy of the quietest sounds you can hear. Scientists use **logarithmic scales** for measurements that change by factors like this.

The **Richter scale** for earthquakes, the pH scale for acidity and the decibel scale for sound are logarithmic scales.

Remember that $\log(x)$ means common logarithms (with base 10).

WORKED EXAMPLE 9 The Richter scale

Earthquake magnitude, M, is often calculated using $M(x) = \log\left(\frac{x}{x_0}\right)$, where x is the amplitude of the earthquake wave on a seismograph and x_0 is the amplitude of the smallest detectable earthquake (the reference intensity or standard wave).

a An earthquake has intensity $x = 40\,000x_0$. **Calculate** its magnitude (correct to 1 decimal place).

b An earthquake has magnitude 5.7. **Determine** how many times the standard intensity it is.

Steps	Working
a 1 Write the formula.	$M(x) = \log\left(\frac{x}{x_0}\right)$
2 Substitute values.	$= \log\left(\frac{40\,000x_0}{x_0}\right)$
3 Simplify and evaluate.	$= \log(40\,000)$ $= 4.6020\ldots$
4 State the result to 1 decimal place.	The magnitude is about 4.6.
b 1 Write the formula.	$M(x) = \log\left(\frac{x}{x_0}\right)$
2 Substitute known values.	$5.7 = \log\left(\frac{x}{x_0}\right)$
3 Write in index form.	$10^{5.7} = \frac{x}{x_0}$
4 Make x the subject.	$x = 10^{5.7}x_0$ $= 501\,187.23\ldots x_0$
5 State the result.	The earthquake is about 500 000 times more intense than the standard wave.

Exam hack

Round the answer so it has accuracy appropriate to the information given.

Worksheets
Logarithmic functions assignment
Logarithmic functions extension

WORKED EXAMPLE 10 Sound loudness

Loudness, in decibels (dB), is related to the intensity, I, of sound by the equation

$$L = 10\log\left(\frac{I}{I_0}\right)$$

where I_0 is 10^{-12} watts per square metre (W/m^2). It is the lowest sound that can be heard by the human ear.

a The bark of a large dog has a loudness of 85 dB. **Calculate** its intensity in W/m^2.

b The intensity of a sound is increased by a factor of 4.

Calculate the change in the loudness, correct to 2 decimal places.

Steps	Working
a 1 Write the formula.	$L = 10\log\left(\frac{I}{I_0}\right)$
2 Substitute known values.	$85 = 10\log\left(\frac{I}{10^{-12}}\right)$
3 Simplify.	$8.5 = \log(10^{12}\, I)$
4 Write in index form to make I the subject.	$10^{8.5} = 10^{12}\, I$
5 Solve.	$I = 10^{8.5\,-\,12}$
6 State the result.	$I = 10^{-3.5}$ W/m^2
b 1 Choose symbols for the sounds.	Let L_1 and I_1 be the loudness and intensity of the original sound and L_2 and I_2 be the loudness and intensity of the louder sound.
2 Write an equation for the intensity.	$I_2 = 4I_1$
3 Write the formula for loudness L_2.	$L_2 = 10\log\left(\frac{I_2}{10^{-12}}\right)$
4 Substitute the intensity of I_2.	$= 10\log\left(\frac{4I_1}{10^{-12}}\right)$
5 Use the log laws.	$= 10\left(\log(4) + \log\left(\frac{I_1}{10^{-12}}\right)\right)$
6 Simplify.	$= 10\log\left(\frac{I_1}{10^{-12}}\right) + 10\log(4)$
7 The 1st term is the loudness formula for L_1.	$= L_1 + 10\log(4)$
8 Evaluate log(4) and simplify.	$= L_1 + 10 \times 0.6020\ldots$ $= L_1 + 6.020\ldots$
9 Write the result.	The loudness increases by about 6.02 dB.

Exam hack

Don't simplify more than is needed for the result.

WORKED EXAMPLE 11 Using logarithms in exponential growth

The compound interest formula is $A = P(1 + i)^n$, where A is the value (total amount) after n years of an initial amount P invested at the interest rate i expressed as a decimal.

Determine how long it would take \$8000 to accumulate to \$10 000 at 6% annual interest.

Steps	Working
1 Write the known values.	$P = 8000$, $A = 10\,000$, $i = 0.06$
2 Write the formula and substitute values.	$A = P(1 + i)^n$ $10\,000 = 8000(1 + 0.06)^n$
3 Simplify and isolate the power.	$10\,000 = 8000 \times 1.06^n$ $1.06^n = \frac{10\,000}{8000}$
4 Take logarithms of both sides.	$\log(1.06^n) = \log\left(\frac{10}{8}\right)$
5 Use logarithm laws and solve for n.	$n \log(1.06) = \log(1.25)$ $n = \frac{\log(1.25)}{\log(1.06)}$
6 Calculate the answer.	$= 3.829\ldots$
7 Write the answer.	As the interest is annual, it would take 4 years to accumulate to \$10 000.

Exam hack

Make sure your answers reflect the information in the question.

EXERCISE 11.5 Modelling with logarithmic functions

ANSWERS p. 407

Recap

1 Solve each equation.

a $\log_2(5x) = \log_2(2x + 9)$

b $\log_5(4m - 5) = \log_5(2m - 1)$

c $\log_3(16 + 2k) = \log_3(k^2 - 4k)$

d $\log_7(2p) + \log_7(p + 2) = \log_7(6)$

2 Solve each equation.

a $\log_2(5w + 6) = 4$

b $\log_3(v) + \log_3(v - 6) = 3$

c $\log_2(m + 1) - \log_2(m - 4) = 3$

d $\log_{12}(x) + \log_{12}(x - 1) = 1$

Mastery

3 WORKED EXAMPLE 9 An earthquake is measured to have an intensity 758 times as great as a standard wave. What is the magnitude of this earthquake, correct to one decimal place?

4 WORKED EXAMPLE 10 A drill press has a noise rating of 50 decibels and a compressor has a noise rating of 62 decibels. How many times more intense is the compressor noise than the drill press noise?

5 WORKED EXAMPLE 11 How long will it take for money to double in value if it is invested at 10% annual interest?

6 An earthquake in San Francisco measured 8.3 on the Richter scale. An earthquake recorded in Chile was found to be 4 times the strength of the San Franciscan earthquake. What did the Chilean earthquake measure on the Richter scale?

7 An earthquake in Ecuador measured 8.3 on the Richter scale, and an earthquake in New Zealand measured 7.1 on the Richter scale. How much stronger was the earthquake in Ecuador?

8 A jazz band uses 2 different amplifiers. One is a 25 W/m^2 amplifier, and the other is a 500 W/m^2 model. How many decibels louder is the 500 W/m^2 model (at the same distance)? Give your answer correct to 2 decimal places.

9 A person's eardrum can be ruptured if exposed to a noise that has a loudness of 160 dB. Calculate the intensity (W/m^2) of sound that would cause your eardrum to be ruptured.

10 The pH of a solution is $pH = -\log[H^+]$, where $[H^+]$ is the concentration of hydrogen ions in moles/litre (mol/L). Find the hydrogen ion concentration for each solution, correct to 3 decimal places.

a lemon juice, pH 1.8

b battery acid, pH 0

c vinegar, pH 3

d baking soda, pH 8

e household ammonia, pH 11

11 The apparent magnitudes, m_1 and m_2, of 2 objects in the sky are related by

$$m_1 - m_2 = -2.5\log_{10}\left(\frac{I_1}{I_2}\right)$$

where I_1 and I_2 are their brightness or intensity of the objects respectively.

a The planet Mercury has an apparent magnitude of –2 and a brightness of 6.31. The supernova SN1006 had an apparent magnitude of –7. Calculate the brightness of the supernova.

b The star Proxima Centauri has an apparent magnitude of 11. Calculate the brightness of the Proxima Centauri.

c The Sun has an apparent magnitude of –27 and the full moon has an apparent magnitude of –13. How many times brighter is the Sun than the full moon?

12 The decay of a radioactive substance is modelled by $Q(t) = Q_0 \times 2.5^{-nt}$, where Q kg is the amount of the substance present at time t years and Q_0 and n are positive constants.

a Show that the constant Q_0 represents the initial amount of the substance.

b If the half-life of the radioactive substance is 1000 years, calculate n to 2 significant figures.

c If there was 45 kg of the radioactive substance initially, how many kilograms would decay in 10 years?

13 The population of the world can be modelled using the formula

$$P = 6.9(1.011)^t$$

where t is the number of years after 2011 and P is the world population in billions.

Estimate

a the population in the year 2045 to the nearest 0.1 billion

b by what year the population will be double what it was in 2011.

14 A particular bacterial colony doubles its population every 15 hours. The experiment begins with 100 bacterial cells, and the number of cells is modelled by the formula $N = 100\left(2^{\frac{t}{15}}\right)$, where t is the number of hours since the experiment started.

After how many hours would you expect to have 300 bacterial cells?

Exam practice

The following information is used for Questions 15 to 17.

A logarithmic scale for photo-power is $P = \log\left(\frac{X}{X_0}\right)$, where X is the number of cells of the plant that perform photosynthesis. The reference number is $X_0 = 10^5$ cells.

15 TECH-FREE The photo-power of a single-celled algal cell is

A 10 **B** -5 **C** 0 **D** 5

16 TECH-FREE The number of photosynthetic cells of a plant with photo-power 4 is

A 10^1 **B** 10^4 **C** 10^5 **D** 10^9

17 TECH-FREE Two plants have photo-power 4 and 9. How much larger is the second plant than the first in terms of the number of photosynthetic cells?

A 4 times **B** 9 times **C** 10 000 times **D** 90 000 times

18 (4 marks) CU The Kelly Kiloton Index (*KKI*) for earthquakes uses the energy of TNT explosions. The table below shows the amount of TNT, in tonnes, with the same energy as earthquakes on the Richter scale.

Richter scale measure, R	2	3	4
Tonnes of TNT, *KKI*	1	32	1024

The 1957 Alaskan earthquake measured 9.1 on the Richter scale.

Develop a formula for *KKI* in terms of R. Use your formula to find the TNT equivalent of the Alaskan earthquake.

19 (4 marks) CU Carbon-14 in living animals and plants has a known ratio to normal carbon-12. Carbon-12 does not decay. Carbon-14 does decay with a half-life of 5750 years. The decay is modelled by $N = N_0 \times 2.72^{-kt}$, where N is the number of radioactive atoms present after t years. N_0 and k are positive constants.

After death, the amount of carbon present in an organism decreases as the carbon-14 decays.

Find the age of a skeleton that has lost 70% of its carbon-14 atoms. Give your answer correct to the nearest year.

Video
Exam question analysis: Logarithmic functions

EXAM QUESTION ANALYSIS

© QCAA 2021 2Q15 Paper 2 Question 15 (5 marks)

A new internet search engine gives a ranking R to each website based on the function

$$R = \log_{10}(50h^2)$$

where h is the number of hits (visits) the website has received.

If a website currently has 100 hits, determine how many more hits they need to increase their ranking by 1.

Reading the question

- You have the ranking function, R
- h = number of hits
- Currently have 100 hits.
- You need to increase hits to increase the ranking by 1.

Thinking about the question

- It has a log function.
- The current ranking depends on the number of hits.
- New ranking needs to be 1 higher than the current ranking.

Worked solution (✓ = 1 mark)

Website currently has 100 hits ($h = 100$)

$$R = \log_{10}(50 \times 100^2)$$

$$= \mathbf{5.6989\ldots} \quad ✓$$

If ranking is increased by 1, then $R = 5.6989\ldots + 1 = 6.6989\ldots$

Substitute R into $R = \log_{10}(50h^2)$ to find new h:

$$\mathbf{6.6989\ldots = \log_{10}(50h^2)} \quad ✓$$

Solve for h using a graphics calculator:

$$\boldsymbol{h = 316.2277\ldots} \quad ✓$$

Increase in hits = $316.2277\ldots - 100$

$$= 216.2277\ldots$$

$$\approx \mathbf{217} \quad ✓$$

$\therefore$ the website requires an additional **217 hits** to increase their ranking by 1. ✓ The solution showed logical organisation of communication steps.

11 Chapter summary

Logarithmic functions

- A general logarithmic function has the form $f(x) = \log_a(x)$, where $a, x > 0, a \neq 1$.
- Its domain is $x > 0$ and its range is all real numbers.

Graphing logarithmic function

- The graph of a logarithmic function of the form $y = \log_a(x)$, where $a > 1$, has
  - a vertical asymptote at $x = 0$
  - a zero at $(1, 0)$
  - important points $(a, 1)$ and $\left(\frac{1}{a}, -1\right)$
  - as $x \to \infty, f(x) \to \infty$
  - as $x \to 0, f(x) \to -\infty$.

Translating logarithmic functions

- Compared to the graph of $y = \log_a(x)$, $y = \log_a(x - h) + k$ is translated:
  - to the right by h units for $h > 0$ (with the vertical asymptote shifted to $x = h$)
  - to the left by h units for $h < 0$
  - up by k units for $k > 0$
  - down by k units for $k < 0$.

Logarithmic equations

- Use the laws of logarithms to solve logarithmic equations.
- $\log_a(x) = \log_a(y) \Rightarrow x = y$
- Sometimes you may need to convert an equation with logarithms to an exponential equation.

Modelling with logarithmic functions

- Scientists use **logarithmic scales** for measurements that change by large factors.
- The **Richter scale** for earthquakes, the loudness scale and the pH scale for acidity are all logarithmic scales.

Puzzle
Logarithmic functions word puzzle

Worksheet
Maths Methods Year 11 formula sheet

Cumulative examination 1

Technology-free

Perusal time: 2 minutes Working time: 30 minutes

TECH-FREE *Calculators are **not** permitted.*

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

Section 1 (3 marks): 3 multiple choice questions

Section 2 (15 marks): 4 short response questions

Total: 18 marks

Section 1 3 multiple choice questions

3 marks

1 Evaluate $\log_8(32)$.

A $\frac{5}{83}$ **B** 1.5 **C** $1\frac{2}{3}$ **D** 4

2 The asymptote of $f(x) = \log_5(x - 5) + 3$ is

A $x = -5$ **B** $x = -3$ **C** $x = 3$ **D** $x = 5$

3 Evaluate $\cos\left(\frac{5\pi}{3}\right)$.

A $-\frac{\sqrt{3}}{2}$ **B** $-\frac{1}{2}$ **C** $\frac{1}{2}$ **D** $\frac{\sqrt{3}}{2}$

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Section 2 4 short response questions

15 marks

4 (3 marks) Sketch the graph of $f(x) = \log_4(x - 3) + 2$, labelling all important features.

5 (5 marks) $g(x) = 2x^3 + x^2 - 4x + 8$.

a Find the values of x where the stationary points of $g(x)$ occur. *[3 marks]*

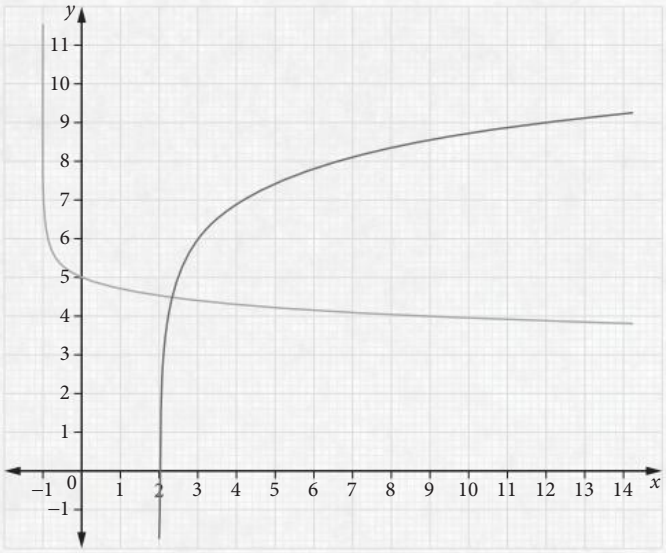
b Find the greatest value of $g(x)$ on the interval $[-2, 2]$. *[2 marks]*

6 (4 marks) © QCAA 2020 1Q18 MODIFIED

The function $f(x)$ has the form given by $f(x) = 3\log(x - a) + b$.

The function $g(x)$ has the form given by $g(x) = -\log(x + c) + d$.

A section of the graphs of the two functions is shown.



Determine the values of a, b, c and d.

7 (3 marks) The temperature, $f(t)$ (in °C), of a warehouse must be kept in a specific range and is modelled using the equation

$$f(t) = 10 + 3\cos(0.3t)$$

where t is the time in hours from midday and $0 \leq t < 24$.

State the minimum temperature and the approximate time(s) of the minimum(s).

You may assume $\pi \approx 3.14$.

Worksheet
Maths Methods Year 11 formula sheet

Cumulative examination 2

Technology-free

Perusal time: 2 minutes Working time: 30 minutes

*Calculators are **permitted**.*

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

Section 1 (3 marks): 3 multiple choice questions

Section 2 (15 marks): 4 short response questions

Total: 18 marks

Section 1 3 multiple choice questions

3 marks

1 Solve $5^x = 28$ correct to one decimal place.

A $x = 0.5$ **B** $x = 1.4$ **C** $x = 2.1$ **D** $x = 5.6$

2 Solve $x^2 - 9x + 12 = 0$ correct to one decimal place.

A $x = -1.6, 7.4$ **B** $x = 1.6, 7.4$ **C** $x = 3.3, 14.7$ **D** $x = 5.7, 3.3$

3 The number of combinations of 5 players chosen from 24 players is

A 1771 **B** 12 560 **C** 42 504 **D** 85 008

Section 2 4 short response questions

15 marks

4 (4 marks) Solve each equation, correct to 2 decimal places if necessary.

a $\log_5(x + 2) + \log_5(4) = \log_5(x - 4)$ *[1 mark]*

b $\log_6(x - 1) + \log_6(x - 3) = 2$ *[3 marks]*

5 (4 marks) Find the derivative of $x^4 - 2x$ from first principles.

6 (3 marks) CF 'Crunchy Fried Crocodile Burgers' uses chicken, fish or crocodile meat.

The sauces are peri-peri, garlic aioli and tartare sauce.

The ingredients are tomato, lettuce, cucumber and kimchi.

There are 2 burger options.

Burger 1: One variety of meat, one sauce and 2 ingredients

Burger 2: 2 portions of meat, 2 sauces and 3 ingredients

70% of customers choose Burger 1, and 25% of Burger 1 customers have peri-peri sauce.

56% of customers who choose Burger 2 have peri-peri sauce.

One customer does not have peri-peri. What is the probability they chose Burger 2?
Give your answer correct to 2 decimal places.

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7 (4 marks) **CU** The Palermo Technical Impact Hazard Scale (P) rates the potential hazard of a Near-Earth Object (NEO). A positive value indicates some concern.

$$P = \log_{10}\left(\frac{p_i}{8.71 \times 10^{-6}}\right)$$

where p_i represents the impact probability.

For the NEO asteroid Apophis, p_i is estimated by the senior astronomer, Julie, as 1 in 100 000.

John says he thinks the P value should be 2 units higher than Julie's estimate. He says this is because the chance of impact is 1000 times greater than her estimate.

Determine Julie's P value and evaluate the reasonableness of John's opinion.

CHAPTER

12 DIFFERENTIATION RULES



Syllabus coverage
Terminology
Nelson MindTap chapter resources
12.1 The chain rule
12.2 The product rule
12.3 The quotient rule
12.4 Combining the rules
Exam question analysis
Chapter summary
Cumulative examination 1
Cumulative examination 2

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Syllabus coverage

UNIT 2, TOPIC 5: FURTHER DIFFERENTIATION

Differentiation rules

- Use the chain rule, if $y = f(u)$ and $u = g(x)$ then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$, to determine the derivative of composite functions involving power and polynomial functions.
- Use the product rule, $\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$, to determine the derivative of products of functions involving power and polynomial functions.
- Use the quotient rule, $\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$, to determine the derivative of quotients of functions involving power and polynomial functions.
- Solve problems that involve combinations of the chain rule, product rule and quotient rule to differentiate functions involving power and polynomial functions, expressing derivatives in simplest and factorised form.

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Prior learning
Differentiation rules

Videos (3):
12.1 The chain rule
12.3 The quotient rule 1
Exam question analysis Differentiation rules

Prior learning (1):
12.1 Differentiation rules

Worksheets (6):
12.1 The chain rule
12.2 The product rule
12.3 The quotient rule
• Mixed differentiation problems
12.4 Rules of differentiation
Cumulative exams Maths Methods Year 11 formula sheet

Puzzle (1):
Chapter summary Derivatives find-a-word

TI-Nspire CX II instructions (1):
12.1 Differentiation rules

Nelson MindTap
To access resources above, visit
cengage.com.au/nelsonmindtap

Terminology

chain rule	derivative	differentiate	gradient
instantaneous rate of change	product rule	quotient rule	

Video
The chain rule

Worksheet
The chain rule

12.1 The chain rule

The **derivative** of a function is both the gradient and the instantaneous rate of change of that function. You can use the **chain rule** to derive (**differentiate**) more complex functions.

The chain rule

If $y = f(u)$ and $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.
or
If $h(x) = f(g(x))$, then $h'(x) = f'(g(x))\,g'(x)$.

WORKED EXAMPLE 1 The chain rule 1

Use the chain rule to differentiate the function $y = (x^2 + 6x - 2)^5$.

Steps	Working
1 Let u or $g(x)$ be the 'inside' function.	Let $u = x^2 + 6x - 2$.
2 Write the original function in terms of u or $g(x)$.	$y = u^5$
3 Differentiate each of the equations.	$\frac{du}{dx} = 2x + 6$ $\quad$ $\frac{dy}{du} = 5u^4$
4 Apply the chain rule.	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
	$= 5u^4 \times (2x + 6)$
5 Substitute back into u.	$= 5 \times (2x + 6) \times (x^2 + 6x - 2)^4$
6 Factorise the $2x + 6$ and simplify.	$= 10(x + 3)(x^2 + 6x - 2)^4$

Exam hack

When an exam question says, "You do not need to simplify" (or something similar), do not simplify the derivative. You will waste valuable exam time or might make a mistake and lose marks.

WORKED EXAMPLE 2 The chain rule 2

Use the chain rule to differentiate the function $h(x) = (3x - 2)^{10}$.

Steps	Working
1 Let u or $g(x)$ be the 'inside' function.	$g(x) = 3x - 2$
2 Write the original function in terms of u or $g(x)$.	$h(x) = f(g(x)) = (g(x))^{10}$
Differentiate each.	$g'(x) = 3$
	$f'(g(x)) = 10(g(x))^9$
3 Apply the chain rule.	$h'(x) = f'(g(x))g'(x)$
	$= 10(g(x))^9 \times 3$
4 Substitute back into $g(x)$.	$= 30(3x - 2)^9$

WORKED EXAMPLE 3 Finding the gradient

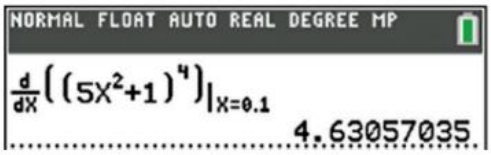
Determine the gradient of the function $y = (5x^2 + 1)^4$ at $x = 0.1$.

Steps	Working
1 Identify the 'inside' function.	$u = 5x^2 + 1$
2 Write the original function in terms of u.	$y = u^4$
Differentiate each.	$\frac{du}{dx} = 10x$ $\quad$ $\frac{dy}{du} = 4u^3$
3 Apply the chain rule.	$\frac{dy}{dx} = 4u^3 \times 10x$
4 Simplify and substitute back into u.	$\frac{dy}{dx} = 40x(5x^2 + 1)^3$
5 Substitute $x = 0.1$ into the derivative to determine the gradient at that value.	$y'(0.1) = 40 \times 0.1 \times (5 \times (0.1)^2 + 1)^3$ $= 4 \times (1.05)^3$ $= 4.6305$
6 Write the answer.	The gradient is 4.6305.

TECHNOLOGY The gradient of a function at a given point

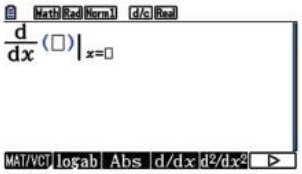
Determine the gradient of the function $y = (5x^2 + 1)^4$ at $x = 0.1$.

TI-84 Plus CE



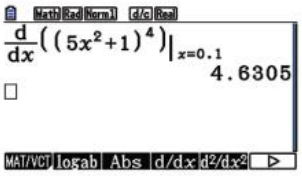
1 Use 2nd 0 (catalog) to select **nderiv(**, complete the variable, function and value. Press ENTER.

Casio fx-CG50 AU



1 From the **MAIN MENU** screen, press **1** for **Run-Matrix**.

2 Press F4 (MATH), F4 (d/dx).

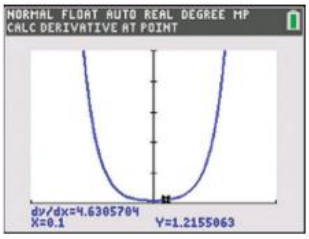


3 Type the RHS of the function into the box inside the brackets after $\frac{d}{dx}$ and the x value into the box after $x =$.

4 Press EXE for the solution, which matches the solution solved for algebraically.

TI-Nspire Differentiation rules

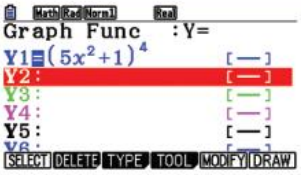
Alternatively, the gradient at a point can be solved graphically.



1 Press y= and enter the function in Y1.

2 Set the window to $-1 \leq x \leq 1$, $0 \leq y \leq 50$ and press graph.

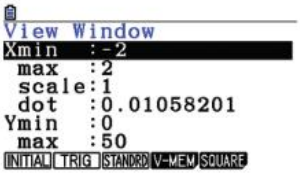
3 Press 2nd trace () and choose **6**: dy/dx. Type 0.1 and press ENTER.



1 From the **MAIN MENU** screen, press 5 for **Graph**.

2 Press SHIFT then MENU to open the SET UP menu. Scroll down to Derivative and press F1 to ensure it is turned On. Press EXIT to return to the Graph screen.

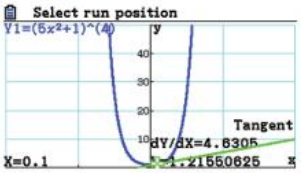
3 Type the RHS of the function into the space after Y1:



4 Press SHIFT then F3 to open the View Window.

5 Set appropriate Xmin, Xmax, Ymin and Ymax values for the question you are solving (setting the scale for the x- and y- axes is optional).

6 Press EXE again after saving the last value to return to the Graph screen, then F6 to draw the graph.



7 Press SHIFT then F4 to open the Sketch tabs, press F2 to draw a tangent, then enter the x value where you want the tangent drawn. Press EXE to draw the tangent.

8 dY/dX value appears on the screen, which matches the solution solved algebraically.

The gradient at $x = 0.1$ is about 4.63.

EXERCISE 12.1 The chain rule

ANSWERS p. 407

Mastery

1 WORKED EXAMPLES 1, 2 Use the chain rule to differentiate each expression.

a $7(3t - 6)^9$

b $(4m^2 + 1)^5$

c $8(3v - 6)^3$

d $-9(x^2 + 4x + 6)^7$

e $4(3x + 2)^4 + 5(3x + 2)^2 - 9$

f $6(3 - 4x)^4 - 5(3 - 4x)^2 + 3 - 4x$

g $5(3p + 7)^6 + 4(3p + 7)^3 - 6p - 14$

h $(3 - 2x)^3 - 5(3 - 2x)^2 + 5x - 8$

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2 WORKED EXAMPLE 3 Determine the gradient of each function at the stated value.

a $y = (x - 4)^3$ at $x = -1$ **b** $y = 5t + 0.5(t - 9)^2$ at $t = 20$

Exam practice

3 TECH-FREE The derivative of $4(5 - 2x)^3$ is

A $12(5 - 2x)^2$ **B** $60(5 - 2x)^2$ **C** $-24(5 - 2x)^2$ **D** $-16(5 - 2x)^2$

4 TECH-FREE The gradient of the function $y = -3(3x - 1)^4$ at $x = 0$ is

A -12 **B** 9 **C** 12 **D** 36

5 TECH-FREE (4 marks) Differentiate $5(2x - 3)^5 + 2(2x - 3)^2$.

6 © QCAA 2020S 2Q13 (2 marks) Determine $\frac{dM}{dt}$ if $M(t) = \frac{1}{\sqrt{16 + 3t^2}}$.

7 TECH-FREE (5 marks) CF The displacement of a particle (in metres) is given by the equation $x(t) = -5t + 0.5(t - 3)^2$, where t is in seconds. Determine the displacement of the particle when it has a velocity of 0.

8 (4 marks) CF The displacement of a particle (in metres) is given by the equation $x(t) = (4 - 3t)^3 + (t - 2)^2 + 5t - 1$, where t is in seconds. Determine the displacement and velocity of the particle when the acceleration of the particle is 0.

9 (4 marks) CF Find the point where the tangent of the curve $y = (2x - 3)^3$ at $x = \frac{7}{4}$ intersects the y-axis.

10 (6 marks) CU Determine the coordinates of all points of the curve $y = 2(x^2 - 3)^4$ where the gradient of the tangent is equal to 0.

12.2 The product rule



Worksheet
The product rule

Is the derivative of $(3x + 2)(2x - 5)$ equal to the derivative of $(3x + 2)$ times the derivative of $(2x - 5)$?
For $f(x) = 3x + 2$ and $g(x) = 2x - 5$, $f'(x) = 3$ and $g'(x) = 2$.
Let $h(x) = f(x) \times g(x) = (3x + 2)(2x - 5) = 6x^2 - 11x - 10$.
Thus, $h'(x) = 12x - 11$, but $f'(x) \times g'(x) = 3 \times 2$.
So you can't find the derivative of a product by multiplying the derivatives of each part.
You have to use the **product rule**.

The product rule

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

or

If $h(x) = f(x)g(x)$, then $h'(x) = f(x)g'(x) + f'(x)g(x)$.

WORKED EXAMPLE 4 The product rule

Use the product rule to differentiate $y = (x^3 - 2x^2 + 3)(x^3 + x^2 + 4x - 1)$.

Steps	Working
1 Choose u and v (or $f(x)$ and $g(x)$).	Let $u = x^3 - 2x^2 + 3$ and $v = x^3 + x^2 + 4x - 1$.
2 Differentiate each part.	$\frac{du}{dx} = 3x^2 - 4x$ $\quad$ $\frac{dv}{dx} = 3x^2 + 2x + 4$
3 Apply the product rule.	$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
4 Write the answer.	$= (x^3 - 2x^2 + 3)(3x^2 + 2x + 4) + (x^3 + x^2 + 4x - 1)(3x^2 - 4x)$

WORKED EXAMPLE 5 Justifying the product rule

Use the product rule to **determine** a simplified derivative of the function $h(x) = (3x - 2)(4x + 1)$.
Verify the solution by first expanding $h(x)$ and then differentiating.

Steps	Working
1 Choose u and v (or $f(x)$ and $g(x)$).	$f(x) = 3x - 2$ $\quad$ $g(x) = 4x + 1$
2 Differentiate each part.	$f'(x) = 3$ $\quad$ $g'(x) = 4$
3 Apply the product rule.	$h'(x) = f(x)g'(x) + g(x)f'(x)$ $= (3x - 2) \times 4 + (4x + 1) \times 3$ $= 12x - 8 + 12x + 3$ $= 24x - 5$
4 Expand $h(x)$.	$h(x) = 12x^2 + 3x - 8x - 2$ $= 12x^2 - 5x - 2$
5 Differentiate $h(x)$.	$h'(x) = 24x - 5$ The solution is verified.

A shorter notation for the product rule

If $y = u \times v$, then $\frac{dy}{dx} = vu' + uv'$.

This is not in the QCAA Formula book, but it is a commonly used notation.

WORKED EXAMPLE 6 Applying the product rule to find a gradient

Use the product rule to **determine** the gradient of the function $y = (2x^2 - 4)(x^2 + 4x - 4)$ at $x = 1$.

Steps	Working
1 Choose u and v.	$u = 2x^2 - 4$ $\quad$ $v = x^2 + 4x - 4$
2 Differentiate each part.	$u' = 4x$ $\quad$ $v' = 2x + 4$
3 Apply the product rule.	$\frac{dy}{dx} = vu' + uv'$ $\frac{dy}{dx} = (x^2 + 4x - 4)(4x) + (2x^2 - 4)(2x + 4)$
4 Substitute $x = 1$ into the derivative.	$y'(1) = (1^2 + 4 \times 1 - 4)(4 \times 1) + (2 \times 1^2 - 4)(2 \times 1 + 4)$ $= (1)(4) + (-2)(6)$ $= -8$
5 Write the answer.	The gradient of $y = (2x^2 - 4)(x^2 + 4x - 4)$ at $x = 1$ is -8.

EXERCISE 12.2 The product rule

ANSWERS p. 407

Recap

1 Given the displacement equation $x(t) = 6\left(\frac{t}{2} - 5\right)^3$, the acceleration equation is

A $a(t) = 36\left(\frac{t}{2} - 5\right)$ **B** $a(t) = \frac{27\left(\frac{t}{2} - 5\right)}{2}$ **C** $a(t) = 9\left(\frac{t}{2} - 5\right)^2$ **D** $a(t) = 9\left(\frac{t}{2} - 5\right)$

2 The gradient of the tangent to the function $y = \frac{-2}{(2x + 3)^2}$ at $x = 0$ is

A does not exist **B** -0.5926 **C** -0.2222 **D** 0.2963

Mastery

3 WORKED EXAMPLES 4, 5 Use the product rule to differentiate each expression.

a $x^4(3x + 2)$ **b** $(5x - 7)(6x + 1)$ **c** $2x^2(5 + 2x)$

d $3x^3(2x^2 - 1)$ **e** $6x(4x^3 + 3)$ **f** $(4x - 3)(2 + 5x)$

4 Find the derivative of each expression.

a $(z^5 - z^3 + z - 1)(9 + 3z^2 - z^4)$

b $(3p + 4)(p^2 - 2p - 4)$

c $(5v^3 - 3v^2 + v - 9)(7v^4 - 2v^3 - v^2 + 8v + 6)$

d $(4b^5 + 3b^2 - 6b + 2)(b^4 + 3b^2 - 6b^3 + 21)$

e $(3u - 8u^2 - 6u^3 + 8)(7u^3 - 4u^2 + 9u - 6)$

5 WORKED EXAMPLE 6 Determine the gradient of each function at the specified x values.

a $y = x^3(1 - x)$ at $x = 9$ **b** $y = (1 + x^4)(x^2 - 1)$ at $x = 2$

Exam practice TECH-FREE

6 (6 marks) Differentiate the expression $(3x^5 - 3x^2 + 8)(2x^2 + 5x + 2) - (2x^4 + 5x^3)(x^5 - 6)$.

7 (3 marks) CF Given that $f(x) = g(x)h(x)$, $g(5) = 10$, $h(5) = -2$, $g'(5) = 8$ and $h'(5) = 1$, find $f'(5)$.

8 (4 marks) CF Find any points on the curve $y = x^2(2x - 5)$ where the gradient of the tangent is 4.

9 (6 marks) CU Determine the derivative of $(3t^2 - 2t + 5)(t^3 + 4t^2 + t - 6)(5t^4 + t^3 - 7t + 8)$.

10 (8 marks) CU Given that the derivative of $y = (a - 2x)(x^2 + 7x - b)$ is $\frac{dy}{dx} = -6x^2 - 20x + 32$, determine the values of a and b.



Worksheets
The quotient rule

Mixed differentiation problems

12.3 The quotient rule

You use the **quotient rule** to differentiate when a function is divided by another function.

The quotient rule

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

or

$$\text{If } h(x) = \frac{f(x)}{g(x)}, \text{ then } h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{\left(g(x)\right)^2}.$$

The positions of the functions in the quotient rule is important. For the original function, u or $f(x)$ must be the numerator, while v or $g(x)$ must be the denominator. Then when applying the quotient rule, remember to differentiate in alphabetical order for the numerator: u' or $f'(x)$ first, minus v' or $g'(x)$ second.

WORKED EXAMPLE 7 Quotient rule 1

Differentiate $y = \dfrac{x^2 - 6x + 3}{2x + 11}$.

Steps	Working
1 Choose u and v (or $f(x)$ and $g(x)$).	$u = x^2 - 6x + 3$ $\quad v = 2x + 11$
2 Differentiate each part.	$\dfrac{du}{dx} = 2x - 6$ $\quad \dfrac{dv}{dx} = 2$
3 Apply the quotient rule, differentiating u first on top.	$\dfrac{dy}{dx} = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ $= \dfrac{(2x + 11)(2x - 6) - 2(x^2 - 6x + 3)}{(2x + 11)^2}$

Exam hack

When the functions are not linear, don't simplify unless you are asked to.

WORKED EXAMPLE 8 Quotient rule 2

Differentiate $h(x) = \dfrac{3x - 2}{4x + 1}$.

Steps	Working
1 Choose u and v (or $f(x)$ and $g(x)$).	$f(x) = 3x - 2$ $\quad g(x) = 4x + 1$
2 Differentiate each part.	$f'(x) = 3$ $\quad g'(x) = 4$
3 Apply the quotient rule, differentiating $f(x)$ first on top.	$h'(x) = \dfrac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$ $= \dfrac{3(4x + 1) - (3x - 2)4}{(4x + 1)^2}$
4 Expand and simplify the numerator.	$= \dfrac{12x + 3 - 12x + 8}{(4x + 1)^2}$ $= \dfrac{11}{(4x + 1)^2}$

An alternate notation for the quotient rule

If $y = \frac{u}{v}$, then $\frac{dy}{dx} = \frac{vu' - uv'}{v^2}$.

WORKED EXAMPLE 9 Applying the quotient rule to find a gradient

Determine the gradient of $y = \frac{2x^2 - 4}{x^2 + 4x - 4}$ at $x = 1$.

Steps	Working
1 Choose u and v.	$u = 2x^2 - 4$ $\quad v = x^2 + 4x - 4$
2 Differentiate each part.	$u' = 4x$ $\quad v' = 2x + 4$
3 Apply the quotient rule.	$\frac{dy}{dx} = \frac{vu' - uv'}{v^2}$ $= \frac{(x^2 + 4x - 4)(4x) - (2x^2 - 4)(2x + 4)}{(x^2 + 4x - 4)^2}$
4 Substitute $x = 1$ into the derivative.	$y'(1) = \frac{(1^2 + 4 \times 1 - 4)(4 \times 1) - (2 \times 1^2 - 4)(2 \times 1 + 4)}{(1^2 + 4 \times 1 - 4)^2}$ $= \frac{(1)(4) - (-2)(6)}{1^2}$ $= 16$
5 Write the answer.	The gradient of $y = \frac{2x^2 - 4}{x^2 + 4x - 4}$ at $x = 1$ is 16.

Video
The quotient rule 1

EXERCISE 12.3 The quotient rule

ANSWERS p. 408

Recap

1 The derivative of $(x - 2)(x^2 + 3x - 5)$ is

A $(x^2 + 3x - 5) + (x - 2)(2x + 3)$ **B** $(x - 2)(x^2 + 3x - 5) + (2x + 3)$

C $(x^2 + 3x - 5)^2$ **D** $1 + (2x + 3)^2$

2 The gradient of $y = (x^3 - 6x^2 + 2)(x^2 + x + 1)$ at $x = 5$ is

A -4 **B** 212 **C** 346 **D** 1490

Mastery

3 WORKED EXAMPLE 7 Find the derivative of each expression.

a $\frac{5q + 9}{3 - 5q}$ **b** $\frac{8m - 7}{2m - 5}$ **c** $\frac{3x + 10}{2x - 5}$

d $\frac{t + 5}{t - 5}$ **e** $\frac{3(2t - 5)}{4(5t + 8)}$ **f** $\frac{12 - 5x}{5(2x - 9)}$

4 WORKED EXAMPLE 8 Find the derivative of each expression.

a $\frac{1}{2x^2 - 3}$ **b** $\frac{5}{2x + 3x^2}$ **c** $\frac{x^2}{x^3 - 1}$

d $\frac{3x^2}{4 - x^2}$ **e** $\frac{x^2 - 2}{x^2 + 1}$ **f** $\frac{3x^2 - x}{2x^3 + 3x}$

5 WORKED EXAMPLE 9 Determine the gradient of each expression in Question **4** at $x = 3$.

Exam practice

6 © SCSA WA 2021 1Q1a **TECH-FREE** (3 marks) Differentiate $\frac{3x+1}{x^3}$ and simplify your answer.

7 © SCSA WA 2022 1Q1a **TECH-FREE** (3 marks) Consider the derivative function $f'(x) = \frac{4x}{x^2+3}$. Determine the rate of change of $f'(x)$ when $x = 1$.

8 (5 marks) **CF** Given that $f(x) = \frac{g(x)}{x^2-3}$, $g(1) = 4$ and $g'(1) = 3$, find $f'(1)$.

9 (5 marks) **CF** Find any values of x on the curve $y = \frac{x^2-1}{x+2}$ at which the gradient of the tangent is $\frac{11}{16}$.

Worksheet
Rules of differentiation

12.4 Combining the rules

You can combine the chain, product and quotient rules to work out complicated derivatives.

WORKED EXAMPLE 10 Combining differentiation rules

Differentiate the function $y = (4x-1)^3(6x^2+7x-2)^2$.

Steps	Working
1 Determine the differentiation rules to be used.	The function is the product of a cubic and a quadratic, each with functions requiring the chain rule.
2 Define u and v.	$u = (4x-1)^3$ $\quad v = (6x^2+7x-2)^2$
3 Differentiate u and v separately using the chain rule.	$\frac{du}{dx} = 12(4x-1)^2$ $\frac{dv}{dx} = 2(6x^2+7x-2)(12x+7)$
4 Apply the product rule.	$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$ $= (4x-1)^3 \times 2(6x^2+7x-2)(12x+7)$ $+ (6x^2+7x-2)^2 \times 12(4x-1)^2$ $= 2(12x+7)(4x-1)^3(6x^2+7x-2)$ $+ 12(4x-1)^2(6x^2+7x-2)^2$

EXERCISE 12.4 Combining the rules

ANSWERS p. 408

Recap

1 The derivative of $(3x-2)(x^2-5x+2)$ is

A $3(3x-2)+(2x-5)(x^2-5x+2)$
B $3(x^2-5x+2)-(2x-5)(3x-2)$
C $3(2x-5)$
D $3(x^2-5x+2)+(2x-5)(3x-2)$

2 The derivative of $\frac{3x-2}{x^2-5x+2}$ is

A $\frac{(2x-5)(3x-2)-3(x^2-5x+2)}{(3x-2)^2}$
B $3(x^2-5x+2)-(2x-5)(3x-2)$
C $\frac{3(x^2-5x+2)-(2x-5)(3x-2)}{(x^2-5x+2)^2}$
D $\frac{(2x-5)(3x-2)-3(x^2-5x+2)}{(x^2-5x+2)^2}$

Mastery

3 WORKED EXAMPLE 10 Differentiate each expression with respect to x.

a $(x + 3)(x + 5)^3$

b $(3x - 1)^4(x - 5)^5$

4 Differentiate each expression.

a $5(3x - 2)^4(x + 8)^3$

b $(4t + 5)^3(2t - 7)^2$

c $6(m - 6)(4m + 3)^8$

d $(2k + 9)^4(3k - 5)^7$

e $4(3g - 4)^5(5g - 4)^6$

f $-6(x + 7)^5(x - 6)^4$

g $-3(6 - 5r)^3(2r + 9)^{11}$

h $-5(8 - 6h)^7(-4 - 3h)^6$

i $(5t + 8)(2t - 3)$

j $-(3p + 10)^7(8p + 1)^8$

Exam practice TECH-FREE

5 (18 marks) Differentiate each expression.

a $\dfrac{(3x - 4)^2}{3(2x + 3)^3}$ *[4 marks]*

b $\dfrac{15c^7}{5(9 - 4c)^2}$ *[4 marks]*

c $\dfrac{1}{(t - 6)^3} - \dfrac{4}{(2t + 7)^2} + \dfrac{2(3t - 5)}{7(2t + 9)^4}$ *[6 marks]*

d $\dfrac{4p^5 - 5p^4 + 2p^3 - p^2 - 4p + 2}{(5p - 1)^6}$ *[4 marks]*

6 (5 marks) If $f(x) = \dfrac{2x^2 + 3x - 7}{(3x - 2)^3}$, find $f'(-2)$.

7 (4 marks) Find any points on the curve $y = 2(x^2 - 3)^4$ where the gradient of the tangent is equal to 0.

8 (6 marks) CF If $\dfrac{d}{dx}(3x(ax + b)^2) = 36x^2 + 12abx + 3$, determine the values of a and b.

9 (4 marks) CF Given the function $f(x) = \dfrac{ax^2 + b}{ax + b}$, show that $f'(x) = \dfrac{a(ax^2 + 2bx - b)}{(ax + b)^2}$.

Video
Exam question analysis: Differentiation rules

EXAM QUESTION ANALYSIS

Exam-style question **TECH-FREE** (6 marks)

The displacement in metres, $x(t)$, of a power boat t seconds after it is launched is given by

$$x(t) = \frac{5t(t-6)^2}{6},\ t \geq 0$$

Determine how many seconds it will take for the boat to reach an acceleration of $0\ \text{m/s}^2$ and the velocity at this time.

Reading the question

- Two values to determine:
  - t when $a(t) = 0$, and
  - $v(t)$ at this time.
- Units for time is seconds, displacement is metres, therefore velocity units are metres per second and acceleration are metres per second per second.

Thinking about the question

- Recall the relationship between displacement $x(t)$, velocity $v(t)$ and acceleration $a(t)$:
  - $v(t) = \frac{d}{dt}x(t)$
  - $a(t) = \frac{d}{dt}v(t)$
- Substitute the acceleration of 0 for $a(t)$ then solve to determine t.
- Substitute this t value into $v(t)$ to determine velocity and interpret.

Worked solution (✓ = 1 mark)

$$x(t) = \frac{5t(t-6)^2}{6}$$

Use the product rule to find velocity:

$$u = \frac{5t}{6} \text{ and } v = (t-6)^2$$

$\mathbf{u' = \frac{5}{6}}$ **and** $\mathbf{v' = 2(t-6)^1 \times 1 = 2(t-6)}$ **by the chain rule** ✓ starts applying a differentiation rule

Velocity $v(t) = u \times v' + v \times u'$

$$= \frac{5t}{6} \times 2(t-6) + (t-6)^2 \times \frac{5}{6}$$

$$= \frac{5}{6}\left[2t(t-6) + (t-6)^2\right]$$

$$= \frac{5}{6}\left[\mathbf{2t^2 - 12t + (t-6)^2}\right]$$ ✓ determines expression for velocity

Differentiate again to find acceleration:

$$\text{Acceleration } a(t) = \frac{5}{6}\left[4t - 12 + 2(t-6)^1 \times 1\right]$$

$$= \frac{5}{6}\left[4t - 12 + 2t - 12\right]$$

$$= \frac{5}{6}(6t - 24)$$ ✓ determines expression for acceleration

Find when $a(t) = 0$:

$$0 = \frac{5}{6}(6t - 24)$$

$$6t - 24 = 0$$

$$t = \frac{24}{6}$$

$$= \mathbf{4\ s}$$ ✓ equates acceleration to 0 and determines t

Substitute into velocity formula:

$$v(4) = \frac{5}{6}\left[2(4)^2 - 12 \times 4 + (4 - 6)^2\right]$$

$$= \mathbf{-10\ m/s}$$ ✓ evaluates $v(4)$

Therefore, the boat reaches an acceleration of $0\ \text{m/s}^2$ at **4 seconds** and has a velocity of **10 m/s** in the negative direction. ✓ statement and interpretation of solutions

12 Chapter summary

Puzzle
Derivatives find-a-word

The chain rule

- The **chain rule** is used to differentiate functions that have functions inside them.

If $y = f(u)$ and $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

or

if $h(x) = f(g(x))$, then $h'(x) = f'(g(x))g'(x)$.

The product rule

- The **product rule** is used to differentiate when two functions are multiplied together.

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

or

if $y = u \times v$, then $\frac{dy}{dx} = vu' + uv'$.

or

if $h(x) = f(x)g(x)$, then

$h'(x) = f(x)g'(x) + f'(x)g(x)$.

The quotient rule

- The **quotient rule** is used to differentiate when a function is divided by another function.

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

or

if $y = \frac{u}{v}$, then $\frac{dy}{dx} = \frac{vu' - uv'}{v^2}$.

or

If $h(x) = \frac{f(x)}{g(x)}$, then

$$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}.$$

Worksheet
Maths Methods Year 11 formula sheet

Cumulative examination 1

Technology-free

Perusal time: 2 minutes Working time: 30 minutes

TECH-FREE *Calculators are **not** permitted.*

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

Section 1 (3 marks): 3 multiple choice questions

Section 2 (15 marks): 4 short response questions

Total: 18 marks

Section 1 3 multiple choice questions

3 marks

1 A circle with a radius of 5 and a centre at $(2, -1)$ has the equation

A $(x-2)^2 + (y+1)^2 = 25$ **B** $(x-1)^2 + (y+2)^2 = 25$

C $(x+2)^2 + (y-1)^2 = 25$ **D** $(x+1)^2 + (y-2)^2 = 25$

2 The derivative of $y = (2x+3)^4$ is

A $4(2x+3)^3$ **B** $8(2x+3)^4$ **C** $8(2x+3)^5$ **D** $8(2x+3)^3$

3 The derivative of $(3x+1)(x-2)^5$ is

A $3(3x+1) + 5(x-2)^9$ **B** $(3x+1)(x-2)^5 + 15(x-2)^4$

C $3(x-2)^5 + 5(3x+1)(x-2)^4$ **D** $15(3x+1)(x-2)^4$

Section 2 4 short response questions

15 marks

4 (3 marks) Differentiate $\dfrac{x-2}{x+2}$. Write your answer in simplest form.

5 (4 marks) Solve $2x^2 - 10x - 12 = 0$.

6 (4 marks) Solve the equation $2\sin\left(\dfrac{2\pi x}{3}\right) = -1$ for $0 \leq x \leq \pi$.

7 (4 marks) The displacement of a particle is given by the function $x(t) = (2t-3)^3 + kt$. When acceleration $a(t) = 0$, velocity $v(t) = 6$. Determine k.

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Cumulative examination 2

Technology-active

Perusal time: 2 minutes Working time: 30 minutes

*Calculators are **permitted**.*

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

Section 1 (3 marks): 3 multiple choice questions

Section 2 (15 marks): 4 short response questions

Total: 18 marks



Worksheet
Maths Methods Year 11 formula sheet

Section 1 3 multiple choice questions

3 marks

1 $(4.15 \times 10^5) \times (3.66 \times 10^4)$ simplifies to

A 7.81×10^9 **B** 15.19×10^9 **C** 1.52×10^{10} **D** 1.52×10^{20}

2 Given that $y = -3(2x - 5)^4$, the value of $y'(2)$ is

A -24 **B** -8 **C** 12 **D** 24

3 The stationary point(s) of the function $y = x^3 + x^2 - 10x + 3$ are

A $(-3.82, 0)$, $(0.31, 0)$ and $(2.51, 0)$ **B** $(0, 3)$

C $(-2.19, 19.19)$ and $(1.52, -6.38)$ **D** $(-0.33, 6.41)$

Section 2 4 short response questions

15 marks

4 (3 marks) Determine the gradient of the tangent to $y = (2x + 5)^2(3x - 1)$ at $x = 10$.

5 (4 marks) A multiple choice quiz has 4 questions on it, each with 5 possible options. Calculate the percentage chance that by simply guessing the correct answer for each question, you will get all 4 questions correct.

6 (4 marks) The equation to calculate the magnitude of an earthquake is

$$M = \log_{10}\left(\frac{I}{S}\right)$$

where I is the intensity of the earthquake and S is the intensity of a 'standard earthquake'.

An earthquake in Chile in 2012 had a magnitude of 7.2. In the Los Santos region of Colombia, earthquakes of magnitude 3.5 are a regular occurrence. Determine how many times more intense the Chilean earthquake was compared to the Colombian earthquakes.

7 (4 marks) The gradient of a normal line to $f(x) = \dfrac{4kx - 2}{(x + 1)^2}$ at $x = 4$ is -2.

Determine the x-intercept of the normal line.

Answers

CHAPTER 1

EXERCISE 1.1

1 **a** $15h + 30$ **b** $-6mn + 12n$
c $-10q^2 + 26q$ **d** $-21g^2h + 49h^2$
e $20t - 8tx + 28ty$ **f** $6hj^2 - 15h^2j + 3hj$
g $5y^2 - 16y + 18$ **h** $-6y^4 + 2xy^4 + 4by^4 + 12y^4z^2$

2 **a** $2x^2 + 4x + 5xy + 4y + 3y^2$
b $6x^2 - 17xy + 10x + 12y^2 - 15y$
c $6a^2 + 8ab + 9a - 4b - 6$
d $6c^2 - cd - 3c - d^2 - d$
e $-2v^2 + 2vw - 2cv + 4w^2 + 4cw$
f $-3b^3 - 12ab^2 + 15b^2 - 9a^2b + 15ab$
g $9r^3 - 15r^2 - 2r + 8$
h $6t^3 - 5t^2g - 7tg^2 + 4g^3$
i $m^3 + 7m^2f - 5mf^2 - 75f^3$
j $25k^3 - 95k^2a + 64ka^2 - 12a^3$
k $15v^3 - 85v^2 - 40v + 560$
l $42b^4 - 231b^3 + 210b^2 + 168b$

3 **a** $42m^2 + m - 56$ **b** $5a^2 - 8ab + 3b^2$
c $10p^2 - 31pg + 15g^2$ **d** $21t^2 + 16tj + 3j^2$
e $35u^2 - 9uw - 18w^2$ **f** $64x^2 + 32xy - 45y^2$

4 **a** $a^2 - 2ab + b^2$ **b** $k^2 - 2km + m^2$
c $x^2 + 2xy + y^2$ **d** $16h^2 + 24hm + 9m^2$
e $4p^2 - 20pj + 25j^2$ **f** $49a^2 + 50ab + 25b^2$
g $a^2 - b^2$ **h** $x^2 - y^2$
i $g^2 - 9h^2$ **j** $y^2 - 36$
k $9q^2 - 16$ **l** $16r^2 - 25s^2$

5 **a** $x^2 + 2xy + y^2 - 1$ **b** $a^2 + 4ab + 4b^2 - 16$
c $9t^2 - s^2 - 4s - 4$ **d** $16w^2 - z^2 - 10z - 25$

6 **a** $x^3 + 3ax^2 + 3a^2x + a^3$
b $x^3 - 15x^2 + 75x - 125$
c $27x^3 - 54x^2 + 36x - 8$
d $27v^3 - 108v^2w + 144vw^2 - 64w^3$
e $8m^3 + 36m^2n + 54mn^2 + 27n^3$
f $x^6 - 9x^4y^2 + 27x^2y^4 - 27y^6$

7 **a** $a^3 + b^3$ **b** $m^3 + n^3$ **c** $a^3 - b^3$
d $x^3 - y^3$ **e** $p^3 + 125$ **f** $8 + 27k^3$

8 C **9** B

10 **a** $k^2 + 9k + 14m^2$ **b** $12x^2 - 11xy - 15y^2$
c $4e^2 + 12de + 9e^2$ **d** $25f^2 - 16g^2$

11 **a** $27p^3 + 108p^2r + 144pr^2 + 64r^3$
b $16x^4 - 24x^2y^2 - 9y^4$

EXERCISE 1.2

1 **a** $(3 + 5)^2 = 64$, $3^2 + 5^2 = 34$, they are not equal.
b $(3 + 5)^2 = 3^2 + 2 \times 3 \times 5 + 5^2$
$= 3^2 + 30 + 5^2$
$= 9 + 30 + 25$
$= 64$

2 **a** $a^2 + 2ab + b^2$ **b** $a^2 - b^2$

3 C

4 **a** $7\sqrt{3}$ **b** $3\sqrt{2}$ **c** $3\sqrt{3}$
d $3\sqrt{7}$ **e** $5\sqrt{2}$ **f** $2\sqrt{5}$
g $4\sqrt{5}$ **h** $6\sqrt{2}$ **i** $5\sqrt{3}$

5 **a** $2\sqrt{7}$ **b** $6\sqrt{5}$ **c** $2\sqrt{3}$
d $3\sqrt{11}$ **e** $2\sqrt{2}$ **f** $3\sqrt{5}$
g $5\sqrt{5}$ **h** $6\sqrt{3}$ **i** $4\sqrt{2}$

6 **a** $9\sqrt{7}$ **b** $9\sqrt{5}$ **c** $5\sqrt{11}$
d $5\sqrt{7}$ **e** $4\sqrt{7}$ **f** $9\sqrt{2}$

7 B **8** C

EXERCISE 1.3

1 **a** $5\sqrt{11}$ **b** 24 **c** $13\sqrt{2}$

2 C

3 **a** $9\sqrt{7}$ **b** $\sqrt{7}$ **c** $5\sqrt{5}$
d $6\sqrt{5}$ **e** $20\sqrt{2}$ **f** $14\sqrt{2}$

4 **a** $10 + \sqrt{3}$ **b** $4 - 2\sqrt{5}$ **c** $3\sqrt{11}$

5 **a** $17\sqrt{2}$ **b** $5\sqrt{7}$ **c** $2\sqrt{3}$

6 **a** $4\sqrt{6}$ **b** $11\sqrt{5}$ **c** $-\sqrt{7}$
d $3\sqrt{7}$ **e** $\sqrt{3}$ **f** $6\sqrt{11}$

7 C **8** D

EXERCISE 1.4

1 **a** $3\sqrt{2}$ **b** $\sqrt{3}$ **c** $\sqrt{7}$

2 **a** $11\sqrt{2}$ **b** $10\sqrt{2}$ **c** $2\sqrt{3}$

3 **a** 40 **b** 42 **c** 30
d 297 **e** 18 **f** 54

4 **a** 120 **b** 84 **c** 90
d $54\sqrt{22}$ **e** 90 **f** 108

5 **a** 8 **b** $11\sqrt{3}$ **c** $5\sqrt{3}$

6 **a** $\frac{13}{10}$ **b** $\frac{7}{3}$ **c** $\frac{2}{3}$

7 B **8** C **9** A

10 **a** 2 **b** 12 **c** $10\sqrt{30}$
d $\frac{2}{45}$ **e** 12 **f** 2

EXERCISE 1.5

1 **a** $4\sqrt{7}$ **b** $2\sqrt{2}$ **c** 147

2 **a** $\frac{3\sqrt{2}}{2}$ **b** $\frac{\sqrt{2}}{3}$ **c** $\frac{\sqrt{3}}{2}$

3 **a** $5\sqrt{3} - \sqrt{6}$ **b** $\sqrt{7} + 1$ **c** $15 + 2\sqrt{5}$
d $3\sqrt{2} + 5\sqrt{3} - \sqrt{6} - 5$ **e** $15 + 7\sqrt{3}$
f $71 + 3\sqrt{3}$

4 **a** $43 + 30\sqrt{2}$ **b** $10 - 2\sqrt{21}$ **c** $540 + 14\sqrt{11}$
d $8\sqrt{3} + 19$ **e** 74 **f** 5

5 B **6** D **7** B

EXERCISE 1.6

1 **a** $9\sqrt{7}$ **b** $4\sqrt{5}$ **c** 48

2 **a** $3\sqrt{2} + 6$ **b** $3\sqrt{3}$ **c** 36

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3 **a** $\frac{\sqrt{15}}{5}$ **b** $\frac{\sqrt{21}}{3}$ **c** $\frac{\sqrt{10}}{2}$

4 **a** $\frac{4\sqrt{5}}{35}$ **b** $\frac{2\sqrt{7}}{21}$ **c** $\frac{1}{2}$

5 **a** $\frac{16+2\sqrt{7}}{57}$ **b** $\frac{28-7\sqrt{5}}{11}$ **c** $\frac{18+2\sqrt{15}}{11}$

d $4+\sqrt{15}$ **e** $5+\sqrt{35}$

f $\frac{3\sqrt{2}+\sqrt{6}-\sqrt{3}-1}{5}$

6 **a** $\frac{\sqrt{10}}{5}$ **b** $\frac{\sqrt{15}}{3}$ **c** $\frac{\sqrt{14}}{2}$

d $\frac{\sqrt{21}}{7}$ **e** $\frac{2\sqrt{7}}{7}$ **f** $\frac{\sqrt{5}}{3}$

g $\frac{\sqrt{3}}{5}$ **h** $\frac{4\sqrt{2}}{7}$ **i** $\frac{5-5\sqrt{5}}{-4}$

j $\frac{10(2+\sqrt{7})}{-3}$ **k** $\frac{9(3-\sqrt{11})}{-2}$ **l** $\frac{2(4+\sqrt{13})}{3}$

7 A **8** C **9** B

10 $2\sqrt{2}-\sqrt{5}$

EXERCISE 1.7

1 **a** $7\sqrt{7}$ **b** $6\sqrt{3}$ **c** 42

2 **a** $6\sqrt{15}$ **b** $\frac{-2\sqrt{15}}{5}$ **c** $\frac{7\sqrt{3}}{9}$

3

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1
1 11 55 165 330 462 462 330 165 55 11 1
1 12 66 220 495 792 924 792 495 220 66 12 1

4 **a** 35 **b** 56 **c** 126 **d** 15

e 210 **f** 84 **g** 8 **h** 35

i 120 **j** 36

5 **a** 1287 **b** 1820 **c** 1330 **d** 5005

e 3432 **f** 8568 **g** 7315 **h** 1716

i 77 520 **j** 27 405

6 **a** 4 **b** 2 **c** 10

d 6 **e** 12 **f** 16

7 **a** $t^8 - 8t^7v + 28t^6v^2 - 56t^5v^3 + 70t^4v^4 - 56t^3v^5 + 28t^2v^6 - 8tv^7 + v^8$

b $n^{12} - 12n^{11}t + 66n^{10}t^2 - 220n^9t^3 + 495n^8t^4 - 792n^7t^5 + 924n^6t^6 - 792n^5t^7 + 495n^4t^8 - 220n^3t^9 + 66n^2t^{10} - 12nt^{11} + t^{12}$

c $a^{15} + 15a^{14}q + 105a^{13}q^2 + 455a^{12}q^3 + 1365a^{11}q^4 + 3003a^{10}q^5 + 5005a^9q^6 + 6435a^8q^7 + 6435a^7q^8 + 5005a^6q^9 + 3003a^5q^{10} + 1365a^4q^{11} + 455a^3q^{12} + 105a^2q^{13} + 15aq^{14} + q^{15}$

d $r^{14} - 14r^{13}u + 91r^{12}u^2 - 364r^{11}u^3 + 1001r^{10}u^4 - 2002r^9u^5 + 3003r^8u^6 - 3432r^7u^7 + 3003r^6u^8 - 2002r^5u^9 + 1001r^4u^{10} - 364r^3u^{11} + 91r^2u^{12} - 14ru^{13} + u^{14}$

e $q^{11} + 11q^{10}g + 55q^9g^2 + 165q^8g^3 + 330q^7g^4 + 462q^6g^5 + 462q^5g^6 + 330q^4g^7 + 165q^3g^8 + 55q^2g^9 + 11qg^{10} + g^{11}$

f $n^5 + 20n^4c + 160n^3c^2 + 640n^2c^3 + 1280nc^4 + 1024c^5$

g $y^6 - 30y^5a + 375y^4a^2 - 2500y^3a^3 + 9375y^2a^4 - 18\,750ya^5 + 15\,625a^6$

h $p^7 - 21p^6q + 189p^5q^2 - 945p^4q^3 + 2835p^3q^4 - 5103p^2q^5 + 5103pq^6 - 2187q^7$

i $m^8 + 16m^7k + 112m^6k^2 + 448m^5k^3 + 1120m^4k^4 + 1792m^3k^5 + 1792m^2k^6 + 1024mk^7 + 256k^8$

j $k^9 - 27k^8e + 324k^7e^2 - 2268k^6e^3 + 10\,206k^5e^4 - 30\,618k^4e^5 + 61\,236k^3e^6 - 78\,732k^2e^7 + 59\,049ke^8 - 19\,683e^9$

k $81v^4 + 216v^3n + 216v^2n^2 + 96vn^3 + 16n^4$

l $81n^4 - 432n^3m + 864n^2m^2 - 768nm^3 + 256m^4$

m $243m^5 - 1620m^4g + 4320m^3g^2 - 5760m^2g^3 + 3840mg^4 - 1024g^5$

n $128m^7 + 1344m^6c + 6048m^5c^2 + 15\,120m^4c^3 + 22\,680m^3c^4 + 20\,412m^2c^5 + 10\,206mc^6 + 2187c^7$

o $2187x^7 - 10\,206x^6y + 20\,412x^5y^2 - 22\,680x^4y^3 + 15\,120x^3y^4 - 6048x^2y^5 + 1344xy^6 - 128y^7$

8 **a** $q^9 - 9q^8x + 36q^7x^2$

b $n^{20} + 20n^{19}y + 190n^{18}y^2$

c $4096g^{12} - 24\,576g^{11}v + 67\,584g^{10}v^2$

d $e^{20} - 60e^{19}m + 1710e^{18}m^2$

e $16\,384m^7 + 143\,360m^6u + 537\,600m^5u^2$

9 $3^{12} \times 2^4 \times {}^{16}C_4\, x^{12}y^4 = 15\,475\,561\,920x^{12}y^4$

10 $2^8 \times 5^4 \times {}^{12}C_4\, m^8n^4 = 79\,200\,000m^8n^4$

11 495 **12** 125 970 **13** B

14 D **15** B **16** 3 838 380

17 924 **18** 5005 **19** 792

20 2 598 960 **21** 1 or 8 **22** $\frac{1}{56}$

CUMULATIVE EXAMINATION 1

1 D **2** A **3** D

4 **a** $y^2 + 16y + 64$ **b** $4c^2 - 25d^2$

c $8g^2 - 14gh - 15h^2$

5 **a** $-5\sqrt{3}$ **b** 144 **c** 2

6 $\frac{16\sqrt{3}-3\sqrt{2}}{12}$ **7** $(20+4\sqrt{5})^2 = 480 + 160\sqrt{5}$ cm^2

CUMULATIVE EXAMINATION 2

1 C **2** D **3** C

4 **a** $a^6 + 6a^5b^1 + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6a^1b^5 + b^6$

b Proof: see worked solutions

5 **a** $\frac{10!}{5! \times 5!}$ or $\frac{10 \times 9 \times 8 \times 7 \times 6}{5 \times 4 \times 3 \times 2 \times 1}$

b 252

6 2.317×10^{13} and -3.454×10^{13} **7** n

CHAPTER 2

EXERCISE 2.1

1 **a** $2(3a+4)$ **b** $-7(2x+y)$ **c** $m(m+4)$

d $2x(x-2)$ **e** $ab(a+1)$ **f** $12ab(3b+4a)$

2 **a** $(7b-4)^2$ **b** $(x+4y)^2$ **c** $(2b-5c)^2$

3 **a** $(n-7)(n+7)$ **b** $(10+9g)(10-9g)$
c $(9v-11w)(9v+11w)$

4 **a** $(2+x)(a+b)$ **b** $(3a^2-17)(x+y)$
c $-(3a+5)(a+2)$ **d** $(a+b)(c-1)$
e $(5i-2j)(g-h)$ **f** $(k-2n)(m^2+h^2)$

5 **a** $(v-2)(v+8)$ **b** $(x+1)(x-16)$
c $(y+10)(y-2)$ **d** $(m+3)(m+4)$
e $(p-3)(p-8)$

6 **a** $(2z-3)(z-4)$ **b** $(3p-2)(2p+3)$

7 **a** $(3k-5)(k-2)$ **b** $(2n-7)(n+5)$

8 **a** $(3x-5)(x-2)$ **b** $(x+8)(x-2)$
c $(7x-6)(x+1)$ **d** $(2x-3)(x-4)$
e $(4x+3)(x-4)$ **f** $(4x+3)(2x+3)$
g $(5x+1)(x+7)$ **h** $(3x+7)(x-7)$
i $(7x+4)(x-8)$

9 D **10** C **11** A **12** D

13 $(2d+1)(d-5)$

14 $(k-8)(k-1)$

15 $-2(s-11)(s-1)$

16 $-x(3x+2z)$

17 $(x-2)^2$

EXERCISE 2.2

1 B

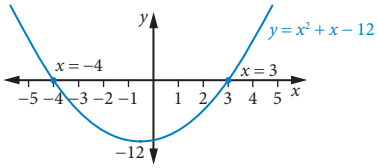
2 $(ax+3)(x^2-8)$

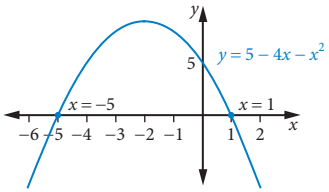
3 **a** $x=0$ or $x=12$ **b** $x=5$ or $x=-2$
c $x=0$ or $x=8$ **d** $x=-7$ or $x=-2$
e $x=5$ or $x=3$ **f** $x=5$ or $x=-1$
g $x=3$ **h** $x=-\frac{1}{2}$ or $x=-2$
i $x=-\frac{1}{3}$ or $x=-3$

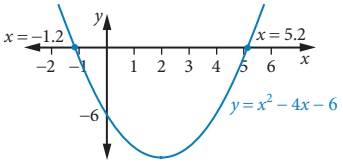
4 **a** $x=\frac{4}{3}$ or $x=-3$ **b** $x=-1$ or $x=-4$
c $x=-1$ or $x=8$ **d** $x=\frac{3}{8}$ or $x=\frac{1}{2}$
e $m=-\frac{5}{6}$ or $m=1$ **f** $x=-7$ or $x=7$
g $x=1$ or $x=7$ **h** $x=1\frac{1}{8}$ or $x=3$
i $x=-\frac{1}{2}$ or $x=8$

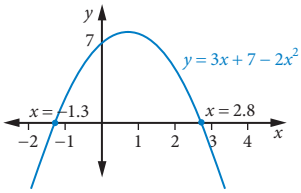
5 **a** $k=-5$ or $k=1$ **b** $v=-7$ or $v=5$
c $j=1$ or $j=8$ **d** $x=-11$ or $x=12$
e $m=-1\frac{1}{3}$ or $m=\frac{2}{3}$
f $b=-2-\sqrt{10}$ or $b=-2+\sqrt{10}$
g $a=\frac{-1-\sqrt{5}}{2}$ or $a=\frac{-1+\sqrt{5}}{2}$
h $z=\frac{3-\sqrt{41}}{4}$ or $z=\frac{3+\sqrt{41}}{4}$
i $v=\frac{9-\sqrt{69}}{2}$ or $v=\frac{9+\sqrt{69}}{2}$
j $x=\frac{15-\sqrt{65}}{2}$ or $x=\frac{15+\sqrt{65}}{2}$
k $x=-2$ or $x=1\frac{1}{2}$
l $c=-2$ or $c=1\frac{2}{3}$
m $m=\frac{3-\sqrt{7}}{2}$ or $m=\frac{3+\sqrt{7}}{2}$
n $r=\frac{6-\sqrt{15}}{7}$ or $r=\frac{6+\sqrt{15}}{7}$
o $h=\frac{7-\sqrt{79}}{10}$ or $h=\frac{7+\sqrt{79}}{10}$

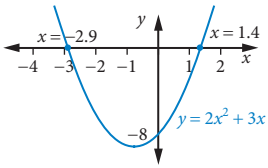
6 **a** $x=-3.5$ and $x=0.7$ **b** $x=-1.4$ and $x=4.8$
c $x=-1.5$ and $x=3.2$ **d** $x=-3.6$ and $x=2.7$

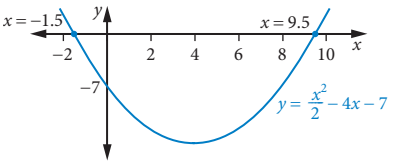
7 **a**


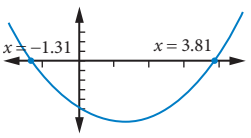
b


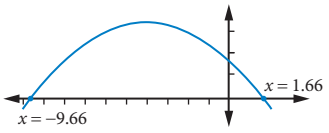
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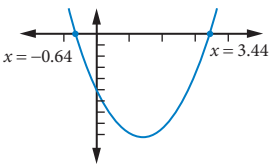
d


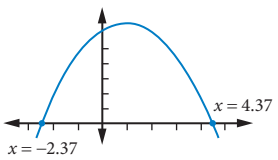
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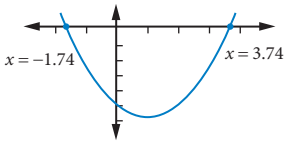
f


8 **a**


b


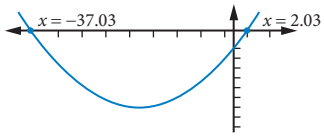
c


d


e


9780170484909

f



9 C **10** B **11** D **12** A

13 $x = \frac{1}{3}, x = -3.5$ **14** $x = 1.4, x = -1$

15 $x = -3 \pm 2\sqrt{2}$

16 equations: $l^2 - 0.9l = 16.2$ or $w^2 + 0.9w = 16.2$
dimensions: 3.6 m × 4.5 m

17 equation: $l^2 + 12l = 160$
length: 8 m

EXERCISE 2.3

1 $x = -\frac{3}{2}, x = 9$

2 $x \approx -10.69$ and $x \approx -1.31$

3 **a** $x = -9$ or $x = -5$ **b** $x = -14$ or $x = 21$
c $f = -2\frac{1}{3}$ or $f = 2$
d $x = -2 \pm \sqrt{11} \approx -5.32$ or 1.32
e $x = -1.79$ or $x = 2.79$ **f** $t = -1.68$ or $t = 10.68$
g $x = -7$ or $x = 20$ **h** $d = -2.64$ or $d = 1.14$

4 **a** $x = -7.36$ or $x = 1.36$ **b** $x = -1\frac{1}{3}$ or $x = \frac{1}{2}$
c $x = -2.12$ or $x = 6.12$ **d** $x = 3.413$ or $x = 2.733$
e $x = 2.54$ or $x = 9.46$ **f** $x = -2$ or $x = -1\frac{1}{2}$
g $x = -28.79$ or $x = -0.21$ **h** $x = 0.11$ or $x = 1.76$

5 **a** 2 roots **b** 1 root **c** no roots
d no roots **e** 1 root **f** 2 roots

6 C **7** A **8** A

9 no real solutions

10 $\Delta = 122.25$ **11** $x \approx 5.23, x \approx -4.43$

12 $(g - 2)^2 \geq 0$ for all values of g. The discriminant is a perfect square, therefore the solutions will be rational.

13 4 km/h and 5 km/h **14** $20

15 40 ohms and 60 ohms

EXERCISE 2.4

1 C **2** D

3 **a** no **b** yes **c** no

4 **a** 4 **b** 1

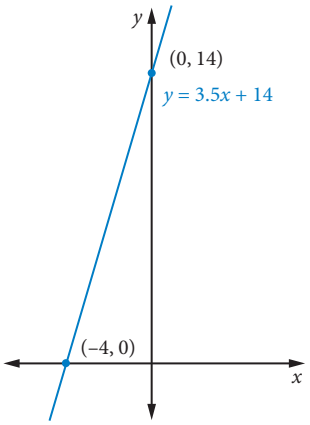
5 **a** $m = -3, c = -4$ **b** $m = 2, c = -5$
c $m = 1, c = 0$ **d** $m = -\frac{9}{8}, c = 9$
e $m = 0, c = 3$ **f** $m = 1.5, c = -7.5$

6 **a** $5x - y - 1 = 0$ **b** $y = 14$
c $9x + 7y - 197 = 0$

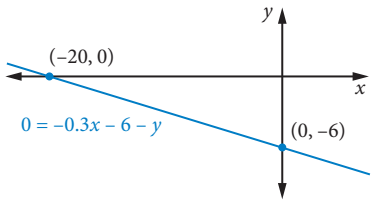
7 **a** $y = 6x - 1$ **b** $y = -x + 2$
c $y = 4$ **d** $y = -0.5x - 6.5$
e $x = 6$ **f** $y = \frac{2}{3}x + 2$

8 **a** $y = -\frac{3}{4}x - 3$ **b** $y = 2$ **c** $x = -1$
d $y = -\frac{1}{3}x + 2$ **e** $y = -7$ **f** $y = 2x + 12.5$

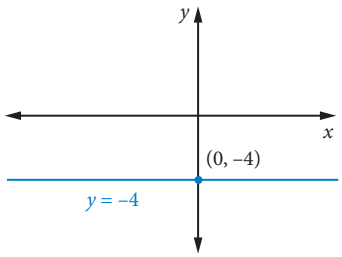
9 **a**



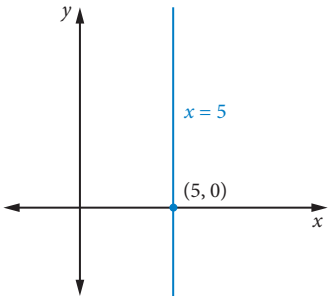
b



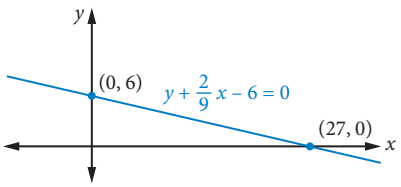
c



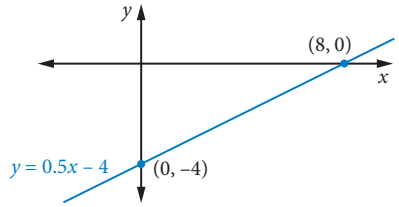
d



e



f

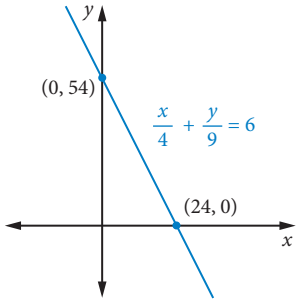


10 C **11** D **12** A **13** A

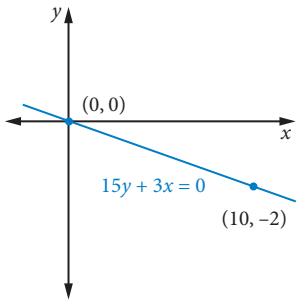
14 $y = \frac{2}{5}x + 4.2$ *or* $2x - 5y + 21 = 0$

ANSWERS

15

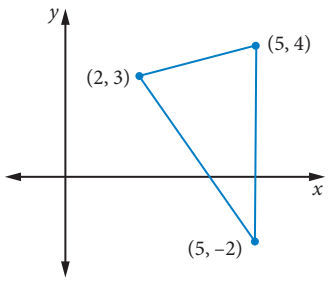


16



17 **a** L_1 and L_5 **b** L_3 and L_6

18



b $x - 5 = 0$, $x - 3y + 7 = 0$, $5x + 3y - 19 = 0$

c $(4, 2)$ inside, $(3, -1)$ outside

19 $m_{AB}m_{AC} = \frac{5}{2} \times -\frac{2}{5} = -1$. As the lines are perpendicular, the triangle is right-angled.

20 \$115

EXERCISE 2.5

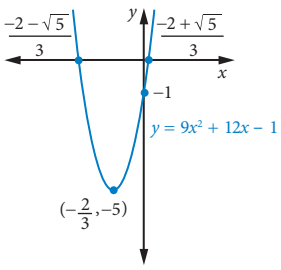
1 D **2** $1.5x + 7$

3 **a** The graph is reflected in the x-axis, dilated by a factor of 4 from the x-axis and moved upwards 2 units.

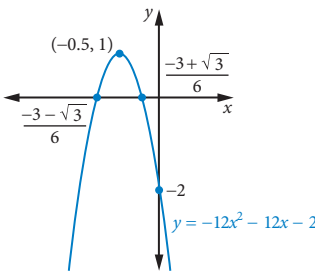
b The graph is dilated by a factor of $\frac{1}{2}$ towards the x-axis, moved to the left 3 units.

c The graph is reflected in the x-axis, dilated by a factor of $\frac{2}{5}$ towards the x-axis. It is moved to the right 4 units and down 7 units.

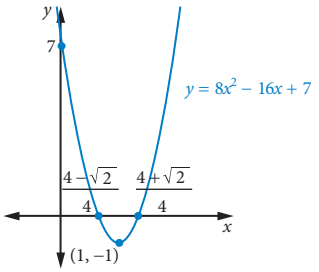
4 **a**



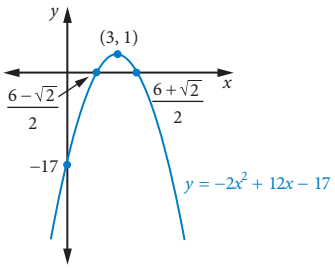
b



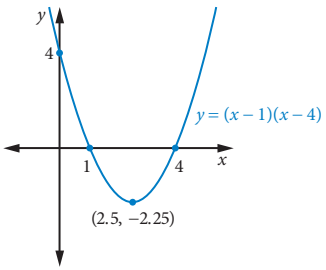
c



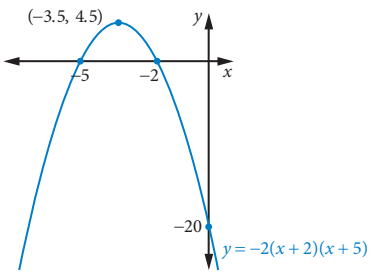
d



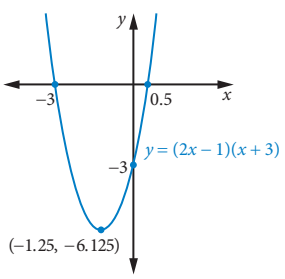
5 **a**



b

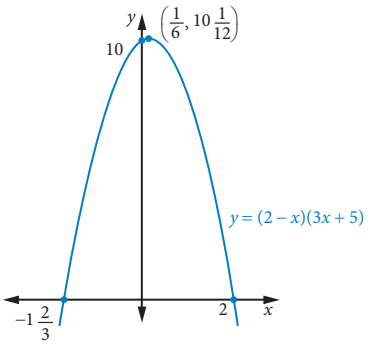


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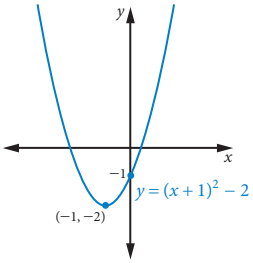


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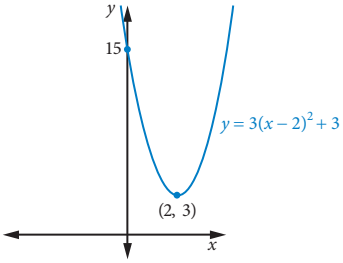
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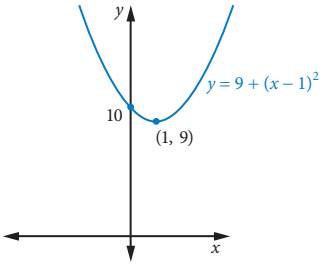
6 a



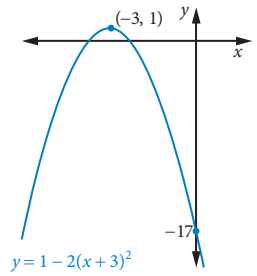
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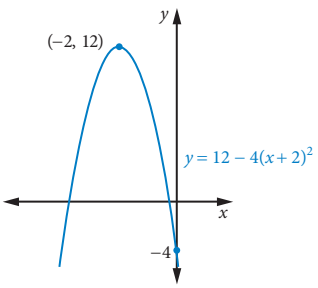
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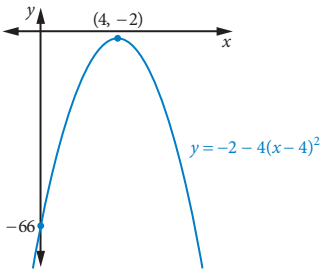
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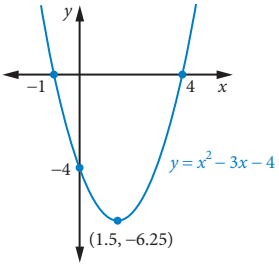
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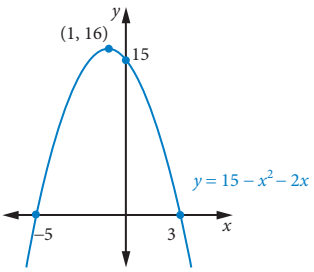
f



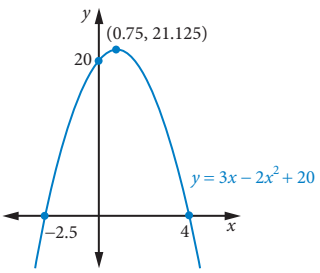
7 a



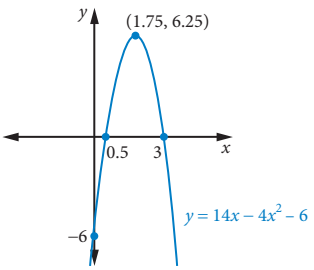
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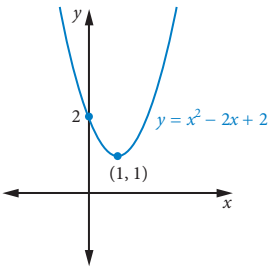
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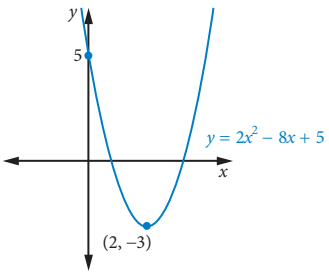
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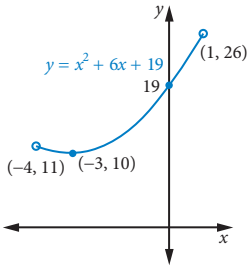
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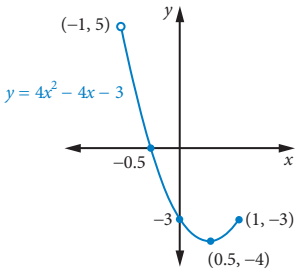
f



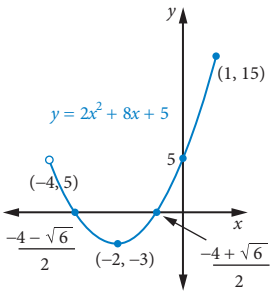
8 a range: $10 \leq y < 26$



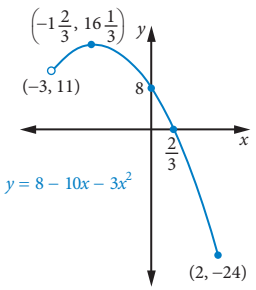
b range: $-4 \leq y < 5$



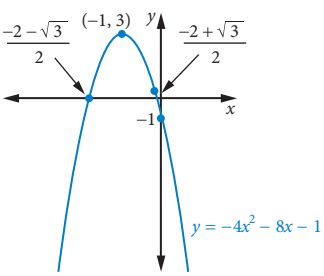
c range: $-3 \leq y \leq 15$



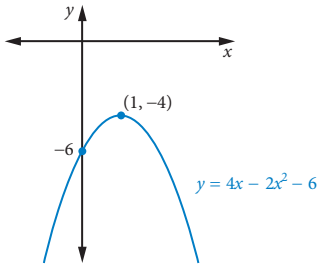
d range: $-24 \leq y \leq 16\frac{1}{3}$



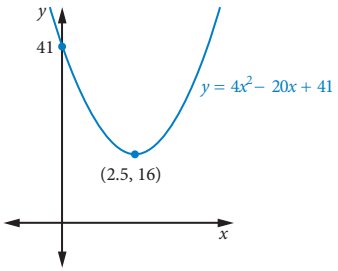
e range: $y \leq 3$



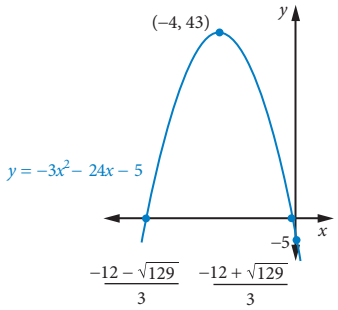
f range: $y \leq -4$



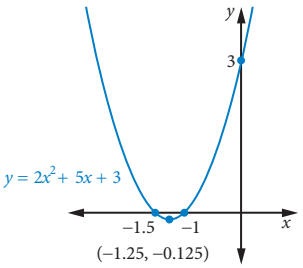
g range: $y \geq 16$



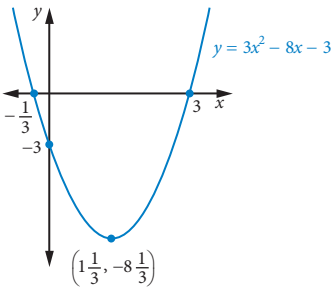
h range: $y \leq 43$



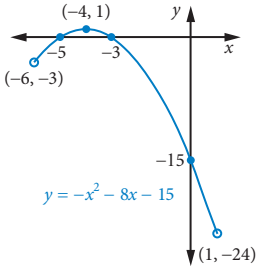
i range: $y \geq -0.125$



j range: $y \geq -8\frac{1}{3}$

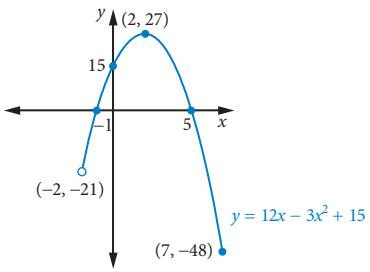


k range: $-24 < y \leq 1$



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l range: $-48 \leq y \leq 27$



9 a y-intercept: 2; x-intercepts: –5.65, –0.35; minimum, (–3, –7)

b y-intercept: 9; x-intercepts: –3.71, 1.21; maximum, (–1.25, 12.125)

c y-intercept: –1; x-intercepts: 0.04, 5.71; maximum, (2.875, 32.0625)

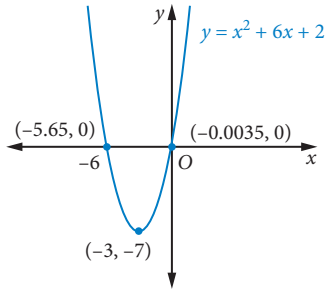
d y-intercept: 7; x-intercepts: none; minimum, (1.25, 3.875)

10 a D **b** F **c** A
d B **e** C **f** E

11 D **12** D

13 (–1, 19); minimum

14



15 Proof: see worked solution

16 a $y = -x^2 + 4x - 6$ **b** $y = 1.5x^2 - 1.5x - 3$
c $y = -0.8x^2 + 0.8x + 4.8$

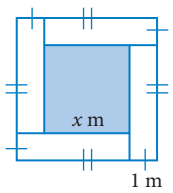
17 25

EXERCISE 2.6

1 no real solutions **2** D **3** 17 and 18

4 a $j + 11 = 2\left(\frac{j}{3} + 11\right)$ **b** 33

5 a



b Botanical garden $4(x + 1) \times 0.1 = 5$

c 11.5 m by 11.5 m

6 a $\$22x^3 - 110x^2 + 210x$ per panel

b 2.5 m **c** \$7250

7 a $15s + 25(2s - 6) = 15 \times 2s + 25(s + 6)$ where s is the weight of the originally lighter pack

b 30 kg and 60 kg

8 $y = 0.25x^2 - 4.5x + 50$

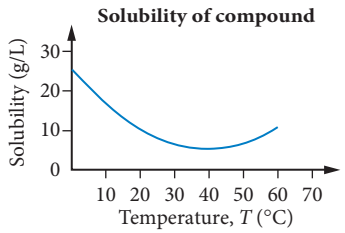
9 $y = \frac{85x - 5x^2 - 30}{6}$

10 $5\sqrt{5}$ **11** $\frac{8\sqrt{2}}{3}$

12 For $n < 50\,000$, $850n - 0.016\,995n^2 + 21\,000\,000$; for $n \leq 50\,000$: $0.25n + 21\,000\,000$; 12 000 000; no, because the profit keeps increasing.

13 $R(T) = 80 + \frac{13T}{1200} - \frac{T^2}{360\,000}$, 89 Ω. The data matches the model, but the model is unlikely to work when the wire gets close to melting. The resistance will certainly not become negative.

14 40°C



15 maximum height: (2.5, 33.25); h-intercept: 2; t-intercepts: –0.08, 5.08. It takes 2.5 s to reach maximum height and 5.08 s to reach the ground. The model matches the data given.

16 1 km horizontally and 75 m vertically up (with maximum gradient). The function matches the data. It only really matters that the entry and exit points of the tunnel match the model, as long as the rest of the escarpment is above it.

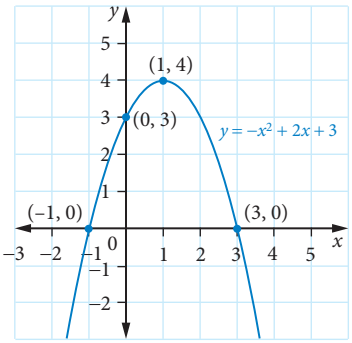
CUMULATIVE EXAMINATION 1

1 B **2** C **3** A

4 $y = 3(x + 2)(x^2 + 1)$ **5** $y = 1.1875x + 0.25$

6 $\sqrt{6} + 3\sqrt{2}$ **7** LHS = RHS = 0

8 range: $(-\infty, 4]$



CUMULATIVE EXAMINATION 2

1 D **2** A **3** B

4 $x = 1 \pm \sqrt{6}$

5 $x = \frac{3 + \sqrt{69}}{6} \approx 1.884$ or $\frac{3 - \sqrt{69}}{6} \approx -0.884$

6 350

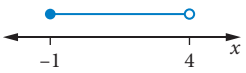
7 $(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$
$(3x + 1)^4 = 81x^4 + 108x^3 + 54x^2 + 12x + 1$

8 equations: $6h^2 + 32h = 88$ or $\frac{2l^2}{3} + \frac{32l}{3} = 88$
dimensions: 2 cm × 4 cm × 6 cm

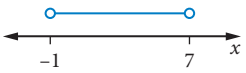
CHAPTER 3

EXERCISE 3.1

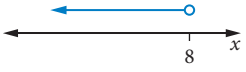
1 a



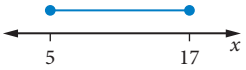
b



c



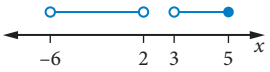
d



e



f



2 a $\{x: -1 \leq x < 4\}$

b $\{x: -1 < x < 7\}$

c $\{x: x < 8\}$

d $\{x: 5 \leq x \leq 17\}$

e $\{x: 1 \leq x < 4\} \cup \{x: x \geq 6\}$

f $\{x: -6 < x < 2\} \cup \{x: 3 < x \leq 6\}$

3 a $[-1, \infty)$ **b** $[-5, 5)$ **c** $[-1, \infty)$

d $\left(\frac{\pi}{2}, \pi\right)$ **e** $\{0, \infty)$ **f** $(\infty, 3) \cup [4, \infty)$

4 a domain = {4, 5, 6, 7}, range = {1, 3, 4, 8}

b domain = R, range = R

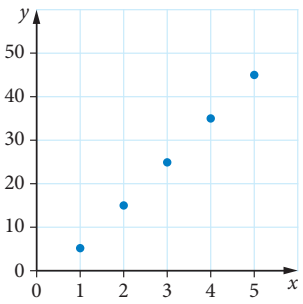
c domain = $x \geq -1$, range = R

d domain = $-2.5 \leq x \leq 2.5$, range = $0 \leq y \leq 2.5$

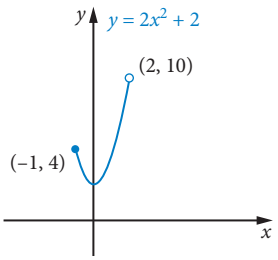
e domain = $1 \leq x < 5$, range = $-2 < y \leq 2$

f domain = $x \leq 1$, range = R

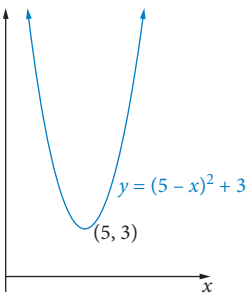
5 a



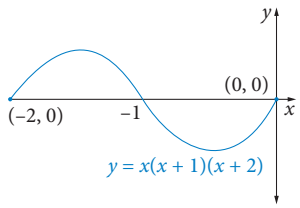
b



c



d



6 a 1 : 1 **b** n : 1 **c** 1 : n

d n : n **e** 1 : 1 **f** 1 : 1

7 B **8** B **9** B **10** A

11 D **12** C **13** $[-1.5, 1)$

EXERCISE 3.2

1 a {(4, −6), (5, 7), (8, 7), (9, −6)}

b

x	4	5	8	9
y	6	7	7	6

c domain = {4, 5, 8, 9}, range = {−6, 7}

d discrete

2 $(-1, 5]$

3 a function **b** function

c not a function **d** not a function

e function **f** function

g function **h** not a function

i function **j** function

4 a n : 1; function **b** n : n; not a function

c n : 1; function **d** n : n; not a function

e 1 : n not a function **f** n : 1; function

5 a domain: real numbers, range: real numbers

b domain: $x \leq 6$, range: $y \leq 0$

c domain: real numbers, range: $y \geq 2$

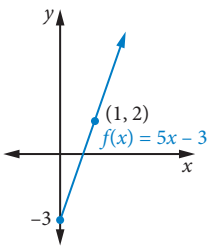
d domain: $x \neq 0$ or $(-\infty, 0) \cup (0, \infty)$,
range: $y \neq 0$ or $(-\infty, 0) \cup (0, \infty)$

e domain: real numbers, range: $y \leq 4$

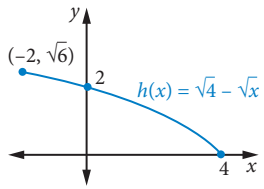
f domain: $x > -1$ or $(-1, \infty)$, range: $(-4, \infty]$

6 range: $[-36.125, \infty)$

7 a

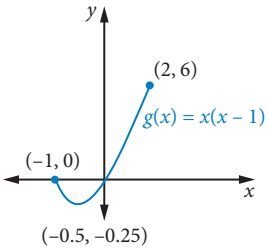


b

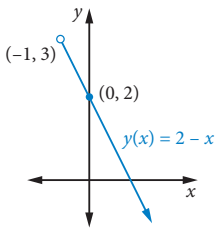


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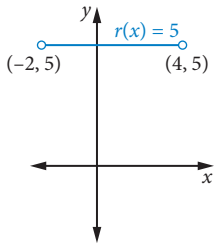
c



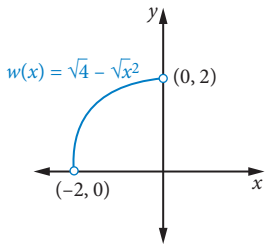
d



e



f



8 B **9** B **10** C **11** A

12 **a** function **b** function
c function **d** function

13 domain: $x \neq -12$, range: $y \neq 0$

14 $-3 \leq g(x) < 4$

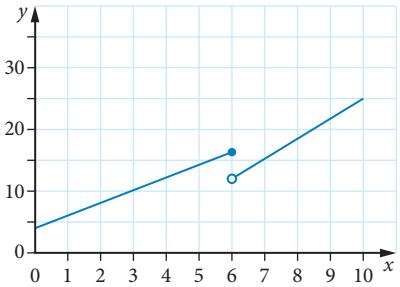
EXERCISE 3.3

1 C

2 **a** not a function **b** function
c not a function **d** function
e function **f** not a function
g function **h** not a function
i function **j** function

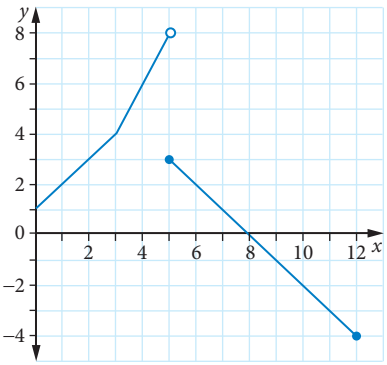
3 **a** $f(5) = 14, f(6) = 16, f(7) = 16$

b



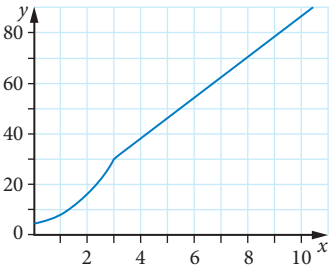
4 **a** $f(2) = 3, f(3) = 4, f(4) = 6, f(5) = 3, f(7) = 1$

b



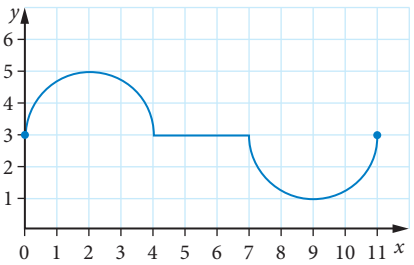
5 **a** $f(1) = 7, f(3) = 31, f(5) = 47$

b

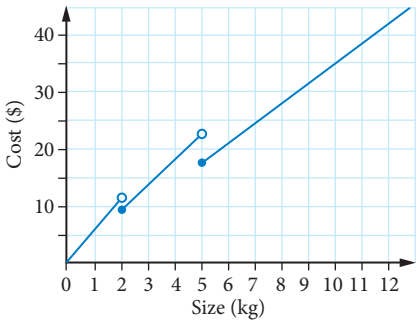


6 **a** $f(2) = 5, f(6) = 3, f(8) = 3 - \sqrt{3} \approx 1.27$

b



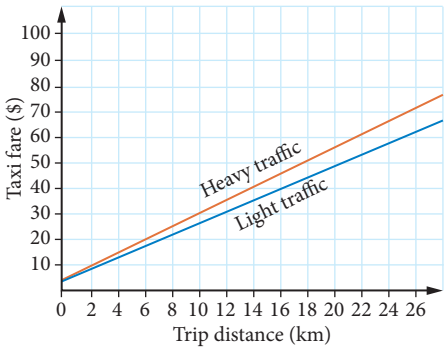
7 $P(x) = \begin{cases} 5.5x, & 0 < x < 2 \\ 4.5x, & 2 \leq x < 5 \\ 3.5x, & x \geq 5 \end{cases}$



8 **a** $C = 4.4 + 2.17d + \frac{60d}{45} \times 0.05 \times 0.81$

b $C = 4.4 + 2.17d + \frac{60d}{30} \times 0.25 \times 0.81$

c



ANSWERS

9 C **10** B

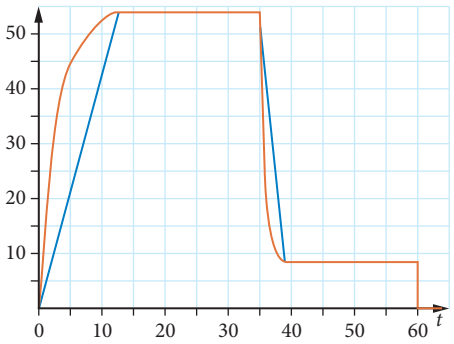
11 a $$v = \begin{cases} 54 \times \dfrac{t}{12}, & t \leq 12 \\ 54, & 12 \leq t < 35 \\ 54 - 46\left(\dfrac{t-35}{3.5}\right), & 35 \leq t < 38.5 \\ 8, & 38.5 \leq t < 60 \\ 0, & 60 \leq t \end{cases}$$

b $$v = \begin{cases} 56(1 - 0.75^t), & t \leq 12 \\ 54, & 12 \leq t < 35 \\ 8 + 46 \times 0.25^{t-35}, & 35 \leq t < 38.5 \\ 8, & 38.5 \leq t < 60 \\ 0, & 60 \leq t \end{cases}$$

c First: 13.5 m/s, 22.5 m/s, 45 m/s
Second: 32.4 m/s, 42.7 m/s, 52.8 m/s
The second increases faster.

d First: 40.9 m/s, 27.7 m/s, 14.6 m/s, 8 m/s
Second: 19.5 m/s, 10.9 m/s, 8.7 m/s, 8 m/s
The second decreases faster.

e



The first function curves up to 54 m/s after jumping and curves down to 8 m/s after the parachute is opened.

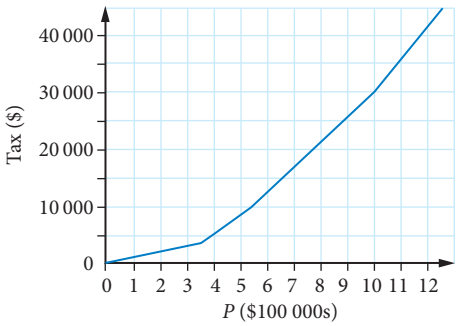
12 a

$$T(P) = \begin{cases} 0.01P, & 0 < P < \$350\,000 \\ 3500 + 0.035(P - 350\,000), & \$350\,000 \leq P < \$540\,000 \\ 10\,150 + 0.045(P - 540\,000), & \$540\,000 \leq P < \$1\,000\,000 \\ 30\,850 + 0.0575(P - 1\,000\,000), & \$1\,000\,000 \leq P \end{cases}$$

where the price P is rounded up to the nearest \$100

b \$3000, \$12 850, \$26 350

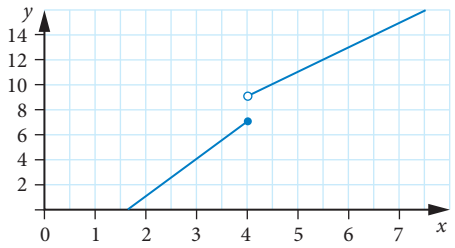
c



EXERCISE 3.4

1 1, 7, 13

2



3 a no **b** yes **c** no **d** yes
e no **f** yes **g** no **h** yes

4 a $-x^4$, 4th degree **b** $3x^7$, 7th degree
c $5x$, 1st degree **d** t^9, 9th degree
e 5, 0th degree **f** $-y^{10}$, 10th degree

5 a 6th degree **b** $\frac{1}{4}$
c 7 **d** -2

6 a $x^2 + 5x + 6$ **b** $a^3 + 6a^2 - 7a$
c $3m^5 - 12m^4 + 12m^3$ **d** $6x^3 - 13x^2 + 4x + 3$
e $-8x^3 + 36x^2 - 54x + 27$ **f** $y^4 - 4y^2$

7 a 194 **b** -4 **c** 44 **d** 19.66

8 Proof: See worked solutions.

9 33 544

10 a $b = 6$ **b** $c = 2\frac{2}{3}$
c $a = 3$ **d** $a = -2$ and $b = 1$
e $d = -6$ **f** $a = -1$

11 a $f(x) = -2$ **b** $x = 0$ or $x = -1$
c $f(x) = a^3 + 8a^2 + 21a + 18$
d $f(-2)\, f(x + h) = -2x^3 - 6x^2h - 6xh^2 - 2h^3 - 4x^2 - 8xh - 4h^2 - 2x - 2h$

12 D **13** D **14** C **15** A

16 a $a = -1$ **b** $b = 9$ **c** $c = -4$
d $a = -1.5, b \neq 4$ **e** $a = 1.5$

17 $n = -1$ **18** $b = 11$

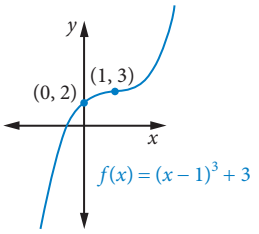
19 $A = 4$, $B = -2$ and $C = -7$.

EXERCISE 3.5

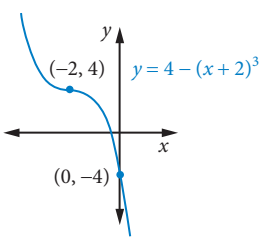
1 C

2 a 7 **b** -2 **c** 1 **d** $\frac{1}{5}$

3 a

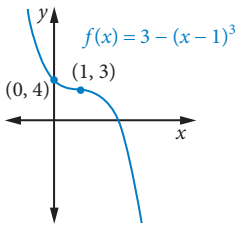


b

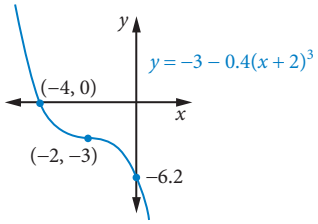


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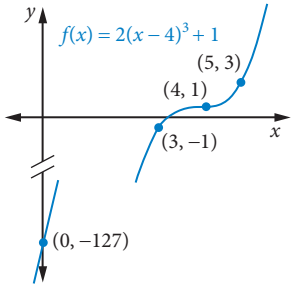
c



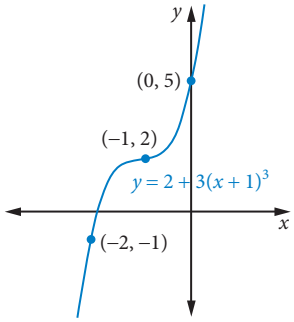
d



e

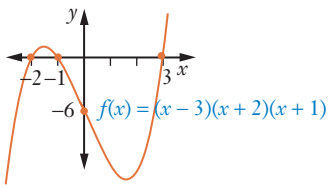


f

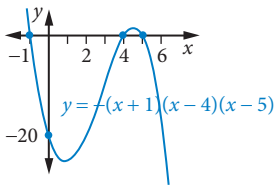


4 **a** range: $-11 \leq y \leq 24$ **b** range: $-62 < y \leq 10$
c range: $-67 < y \leq 24$ **d** range: $-124 \leq y \leq 28$
e range: $-11.8 < y < 49$ **f** range: $-123 \leq y \leq 7$

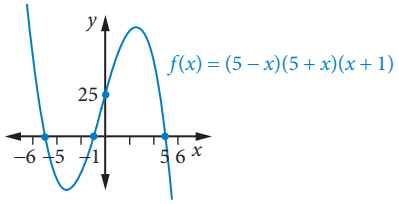
5 **a**



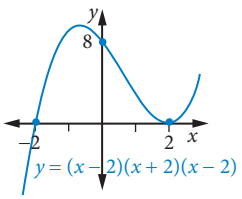
b



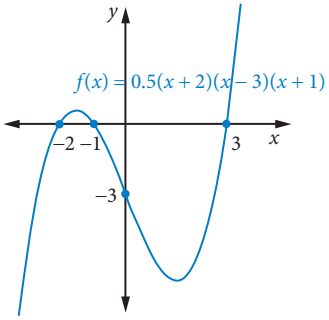
c



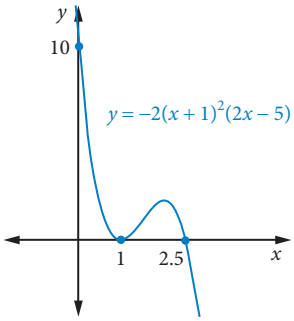
d



e

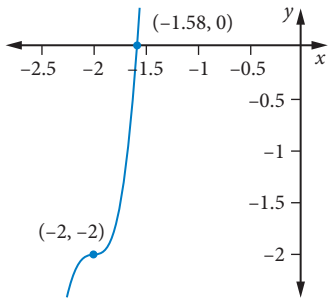


f



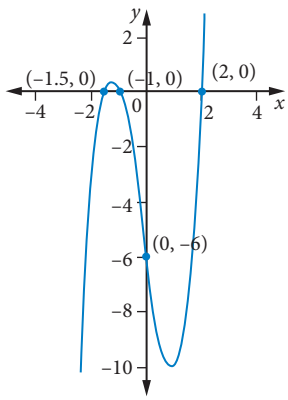
6 **a** C **b** F **c** B
d A **e** E **f** D

7



EXERCISE 3.6

1



2 It is translated 1 unit to the right and up 3 units.

3 **a** $x = 0$ and $x = \pm 5$ **b** $x = 0$ and $x = \pm 3$

c $x = 0$, $x = 1$ and $x = 2$ **d** $x = 0$ and $x = 2$

e $x = 0$, $x = 2$, $x = 3$ **f** $x = 0$, $x = 2$, $x = 3$

4 **a** $x = 0$ **b** $x = \frac{1}{2}$

c $x = 2$ **d** $x = -1$

5 **a** $x = -2$, $x = 3$, $x = -9$ **b** $x = 2$, $x = -3$, $x = 4$

c $x = 2$, $x = -2$, $x = 5$ **d** $x = 1$, $x = \frac{2}{3}$, $x = 6$

6 **a** $x = -3$, $x = -4$, $x = 9$ **b** $x = 4$, $x = -5$, $x = 2$

c $x = 6$, $x = 2.5$, $x = -\frac{4}{3}$ **d** $x = -5$, $x = -3$, $x = 7$

7 **a** $y = (x + 1)(x - 1)(x - 3)$

b $y = -(x + 1)^2(x - 2)$

8 D **9** D **10** $(-2.44, 0)$

11 $f(x) = 0.5(x - 2)(x + 2)(x - 3)$

12 $f(x) = -0.5(x + 3)^3 + 2$

13 $f(x) = -x(x - 3)(x + 1)$

EXERCISE 3.7

1 D

2 domain: R, range: R

3 **a** $x \approx -2.05$

b $x \approx -2.39$, $x \approx 0.81$ or $x \approx 2.58$

c $x \approx -2.66$, $x \approx -1.21$ or $x \approx 1.87$

d $x \approx -1.78$, $x \approx 1.29$ or $x \approx 3.49$

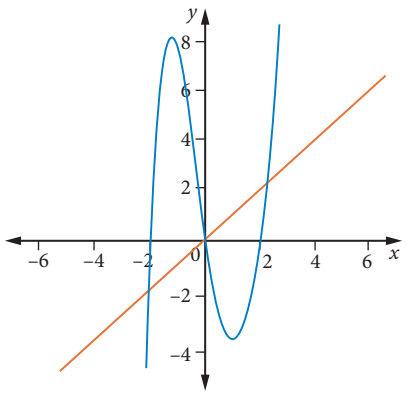
4 **a** $x = 0$, $x = \frac{1 - \sqrt{17}}{2}$ or $x = \frac{1 + \sqrt{17}}{2}$

b $x = -2$, $x = 1$ or $x = 1.5$

c $x = -3$, $x = -2 + \sqrt{6}$, $x = -2 - \sqrt{6}$

5 no: **a**, **b**, **e**; yes: **c**, **d**

6



$(-\sqrt{5}, -\sqrt{5})$, $(0, 0)$, $(\sqrt{5}, \sqrt{5})$

7 **a** 36.5, 91, 103.5 g/L

b 107.73 g/L. More nutrient, more algae.

c The mass cannot be negative, so the domain must be $0 \leq D \leq 40.84$ or smaller.

8 85.3%, 93.3%

9 Maximum volume is about 94 750 cm^3 with dimensions $28.29 \times 71.71 \times 46.71$ cm. Small changes reduce the volume, so the method seems to work well.

CUMULATIVE EXAMINATION 1

1 B **2** C **3** D

4 $n < -2$ or $n > -2$; that is, all real n, $n \neq 2$.

5 Reflection in the x-axis, dilation by a factor of 0.5 from the y-axis, translation 2 up and 2 to the right.

6 $(x - 1)^2 = \frac{1}{2}(y - 2)$

7

$$T(P) = \begin{cases} 0, & 0 < P < \$5000 \\ 0.01(P - 5000), & \$5000 \leq P < \$75\,000 \\ 1050 + 0.0355(P - 75\,000), & \$75\,000 \leq P < \$540\,000 \\ 17\,325 + 0.045(P - 540\,000), & \$540\,000 \leq P < \$1\,000\,000 \\ 38\,025 + 0.0575(P - 1\,000\,000), & \$1\,000\,000 \leq P \end{cases}$$



The model works correctly for the boundary conditions of each section. The function is an effective calculation method.

CUMULATIVE EXAMINATION 2

1 C **2** C **3** C

4 **a** $x = 0, -\sqrt{5}, \sqrt{5}$

b $\left(\frac{\sqrt{15}}{3}, \frac{-10\sqrt{15}}{9}\right), \left(\frac{-\sqrt{15}}{3}, \frac{10\sqrt{15}}{9}\right)$

5 time for descent ≈ 2.6 s; maximum height = 28 m

6 distance ≈ 40.00

7 Squares should be of length 10.74; maximum volume is 40 145.92 cm^3.

CHAPTER 4

EXERCISE 4.1

1 **a** $\frac{\pi}{3}$ **b** $\frac{\pi}{4}$ **c** $\frac{\pi}{2}$

d $\frac{3\pi}{4}$ **e** $\frac{2\pi}{3}$ **f** $\frac{5\pi}{9}$

g $\frac{7\pi}{3}$ **h** $\frac{2\pi}{5}$ **i** $\frac{8\pi}{5}$

2 **a** 0.9774 **b** 2.4435 **c** 4.8171

d 4.6740 **e** 9.8859

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3 **a** 120° **b** 150° **c** 315°
d 98.2° **e** 396°

4 **a** 57.3° **b** 126.1° **c** 45.8°
d 189.1° **e** 118.0°

5 **a** $\frac{2\pi}{3}$ **b** $\frac{-\pi}{6}$ **c** $\frac{\pi}{3}$

6 **a** $\frac{\sqrt{2}}{4}$ **b** $\frac{2\sqrt{3}+1}{2}$ **c** 1
d -3 **e** $-(2\sqrt{2}+3)$ **f** $\frac{2\sqrt{3}+7}{4}$

7 **a** -0.90 **b** 0.98 **c** 0.31 **d** 0.25

8 **a** 0.26 **b** 0.40 **c** -1.97 **d** 1.00
e 0.98

9 Proof: see worked solution

10 Proof: see worked solution

11 B **12** D **13** A

14 $x = 35°47'$

EXERCISE 4.2

1 **a** $\frac{3\pi}{2}$ **b** $\frac{5\pi}{4}$ **c** $\frac{11\pi}{6}$ **d** $\frac{8\pi}{5}$

2 **a** 0.8203 **b** 3.8528

3 **a** $\frac{5\pi}{3}$ cm **b** $\frac{85\pi}{6}$ mm **c** $\frac{9\pi}{10}$ m
d 8π cm **e** 6π mm **f** 23.78 cm

4 **a** $\frac{150}{\pi}$ mm **b** $\frac{63}{2\pi}$ cm **c** $\frac{240}{\pi}$ cm
d $\frac{58.8}{\pi}$ m **e** $\frac{559}{6\pi}$ cm **f** 1.68 m

5 22.7 cm

6 **a** 471.24 mm^2 **b** 73.89 m^2 **c** 1.35 m^2

7 **a** 7.14 cm **b** 4.76 m
c 2.55 cm **d** 1.25 cm

8 radius = 9.08 cm, area = 151 cm^2

9 **a** 1.30 (or 74.3°) **b** 23.3 cm

10 **a** 1.047 m **b** 1.014 m

11 28.3 cm **12** 91.7 m **13** 2737 cm^2

14 47.7% **15** B **16** B **17** D

EXERCISE 4.3

1 $\frac{55\pi}{12}$ or 14.40 cm

2 **a** $\frac{\sqrt{2}}{4}$ **b** 1

3 **a** $-\cos(\alpha)$ **b** $-\sin(x)$ **c** $\tan(\theta)$
d $\tan(\beta)$ **e** $-\tan(\alpha)$ **f** $-\sin(\alpha)$

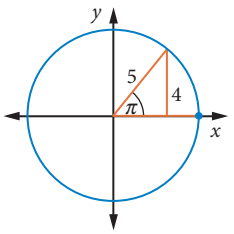
4 **a** $-\frac{1}{2}$ **b** $-\frac{\sqrt{3}}{2}$ **c** -1
d $\frac{1}{2}$ **e** $\frac{1}{\sqrt{3}}$ **f** $\frac{\sqrt{3}}{2}$

5 **a** 1 **b** -1 **c** 0
d 0 **e** 0 **f** 0

6 **a** $\frac{1}{2}$ **b** $-\frac{\sqrt{3}}{2}$ **c** 0 **d** 0

7 **a** $\frac{1}{2}$ **b** $\frac{1}{2}$ **c** $\sqrt{3}$ **d** $\frac{\sqrt{3}}{2}$

8 **a**



b 3

c $\cos(\theta) = \frac{3}{5}$, $\tan(\theta) = \frac{4}{3}$

9 **a** $\cos(\theta) = \frac{\sqrt{2}}{2}$, $\tan(\theta) = -1$

b $\sin(\theta) = \frac{\sqrt{2}}{2}$, $\tan(\theta) = 1$

c $\sin(\theta) = -\frac{7}{25}$, $\tan(\theta) = \frac{7}{24}$

d $\cos(\theta) = -\frac{5}{13}$, $\tan(\theta) = -\frac{12}{5}$

10 **a** $-1 - \frac{-\sqrt{2}}{2} = \frac{\sqrt{2}}{2} - 1 = \frac{\sqrt{2}-2}{2}$

b $\frac{-\sqrt{3}}{2} + \frac{1}{2} = \frac{1-\sqrt{3}}{2}$

c $-\frac{1}{2} + \frac{-\sqrt{3}}{2} = -\frac{1+\sqrt{3}}{2}$

d $\frac{1}{4}$ **e** 1

11 **a** $4\tan(x)$ **b** 0
c $-6\sin(5y)$ **d** $\cos(x)$

12 C **13** B **14** A

15 **a** $\cos(\theta) = -\frac{12}{13}$ and $\tan(\theta) = -\frac{5}{12}$

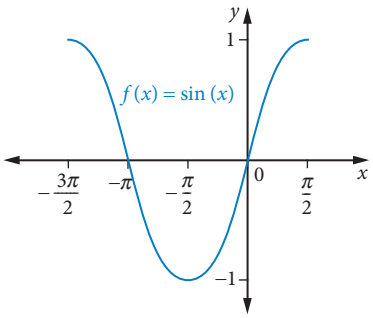
b $\sin(\theta) = -\frac{4}{5}$ and $\tan(\theta) = -\frac{4}{3}$

EXERCISE 4.4

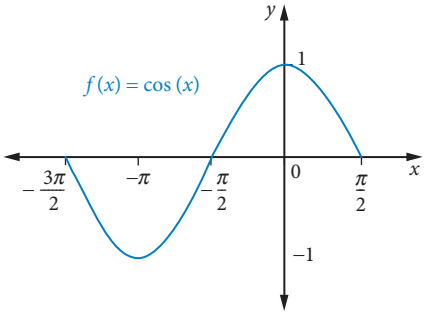
1 **a** $\frac{\sqrt{3}}{2}$ **b** $-\frac{\sqrt{3}}{2}$ **c** $\frac{1}{2}$
d -1 **e** $-\frac{\sqrt{3}}{2}$

2 **a** $-1 - \frac{\sqrt{2}}{2}$ **b** $-\sqrt{3}$ **c** 0 **d** $\frac{3}{4}$

3 **a**

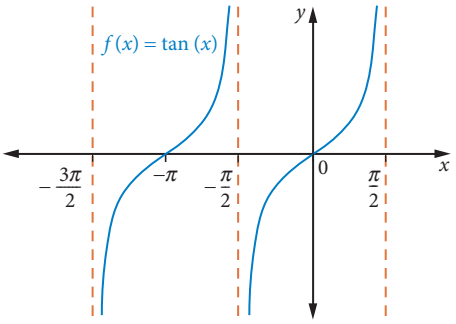


b

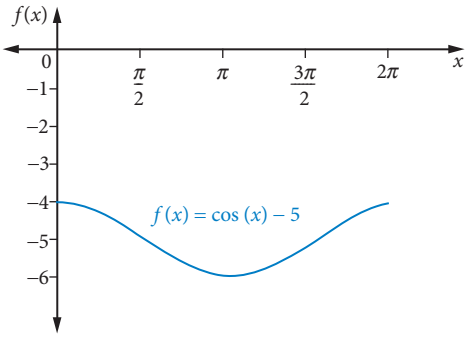


ANSWERS

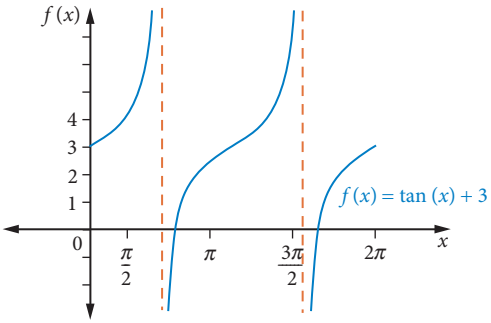
c



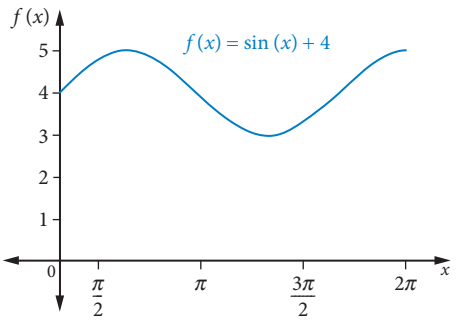
4 a



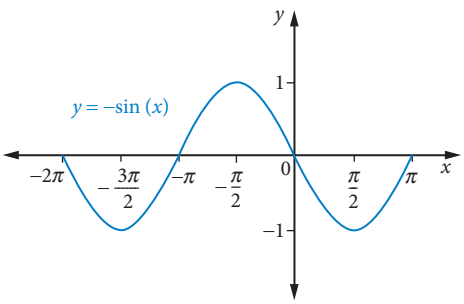
b



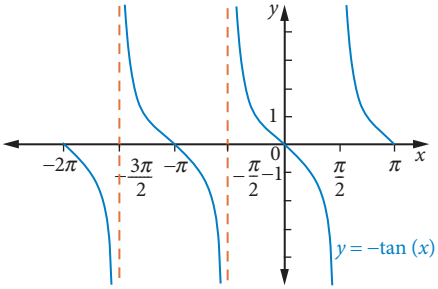
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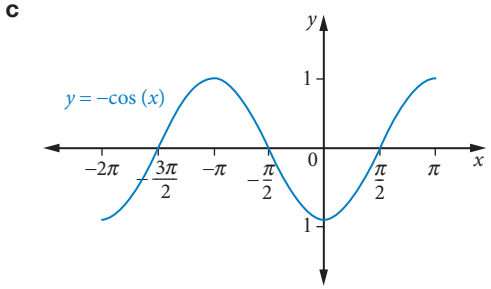
5 a



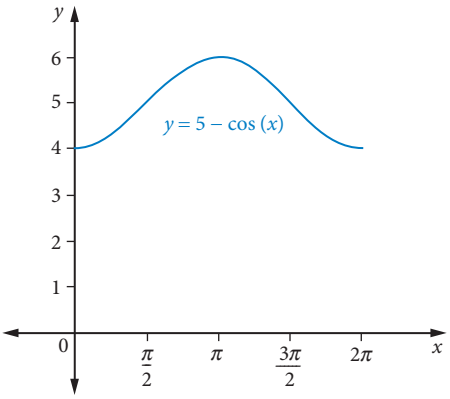
b



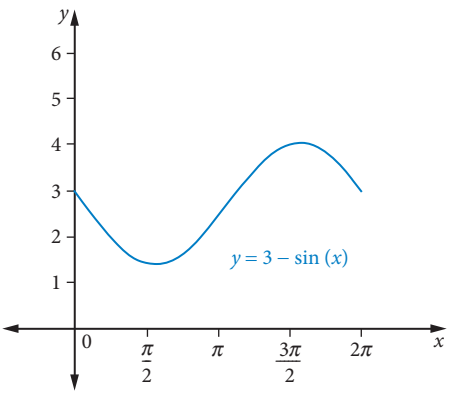
c



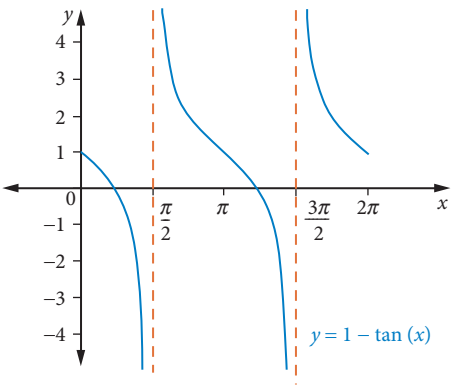
6 a



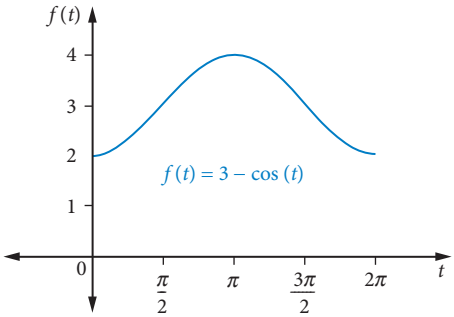
b



c



7



9780170484909

a $\frac{6+\sqrt{3}}{2}$ **b** $\frac{\pi}{3}$ and $\frac{5\pi}{3}$

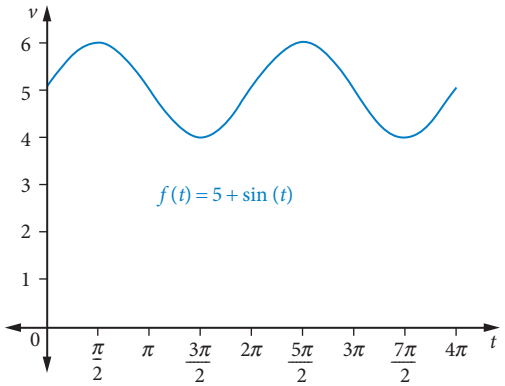
c 4 **d** 1.37 and 4.91

8 a **i** 4.5 m/s **ii** 5 m

iii 6 m/s **iv** $\frac{10+\sqrt{3}}{2}$ m/s

v $\frac{10+\sqrt{2}}{2}$ m/s **vi** $\frac{10-\sqrt{3}}{2}$ m/s

b



c 0.78 s, 2.36 s, 7.06 s, 8.65 s

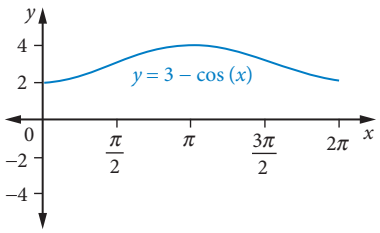
9 A **10** B **11** −1 and 9

12 amplitude 2, range $0 \leq y \leq 4$

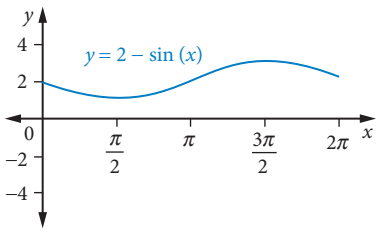
13 a $\frac{1}{50}$ s **b** $240\sqrt{2} \approx 340$ V **c** $240\sqrt{2}$ V

EXERCISE 4.5

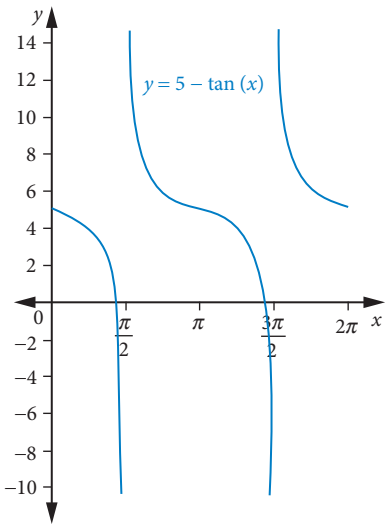
1 a



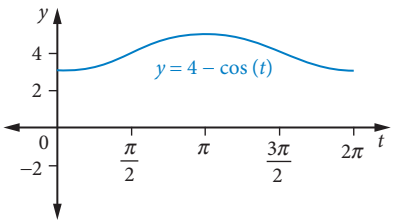
b



c



2 a



b $4+\frac{\sqrt{3}}{2}$

3 a amplitude = 30, period = 16

b amplitude = 10 m, period ≈ 63 s

4 a 3 **b** 7 **c** 2

d 2.8 **e** 5 **f** 8

g 10 **h** 15

5 a maximum = 1, minimum = −1

b maximum = 3, minimum = −3

c maximum = 4, minimum = −4

d maximum = −3, minimum = −5

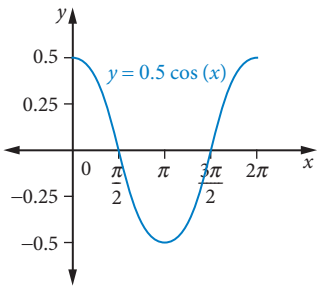
e maximum = 4, minimum = 2

f maximum = 7, minimum = 3

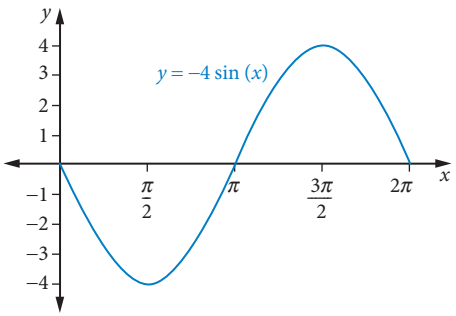
6 a 2π **b** π **c** 2π

d $\frac{\pi}{5}$ **e** 4π **f** 6π

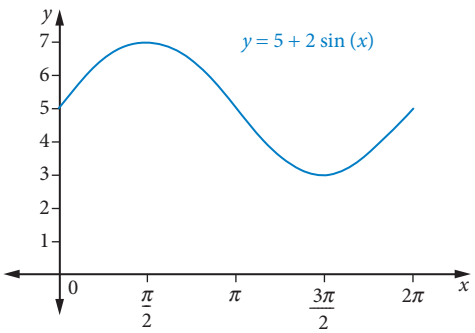
7 a



b

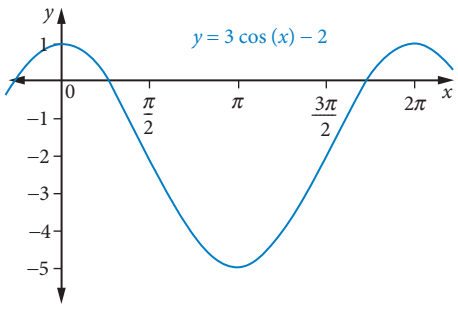


8 a

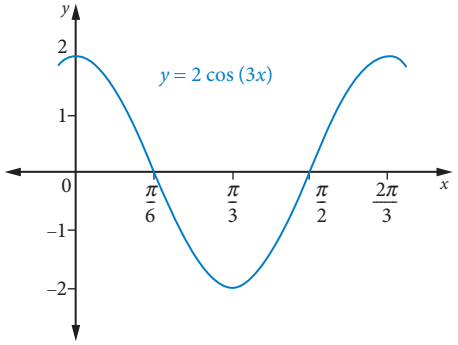


ANSWERS

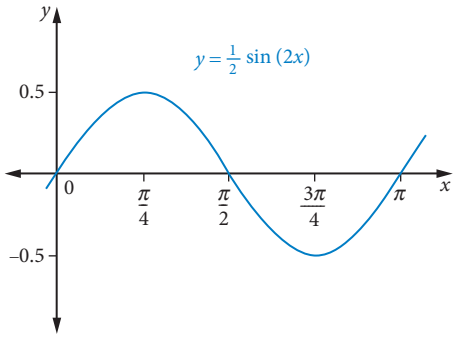
b



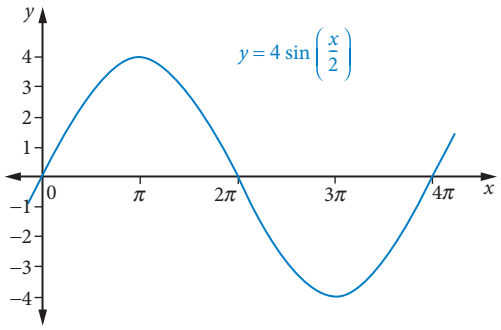
9 a amplitude = 2, period = $\frac{2\pi}{3}$



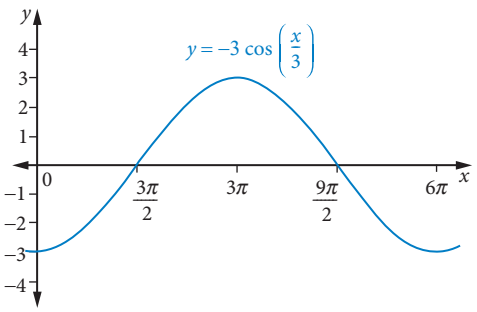
b amplitude = 0.5, period = π



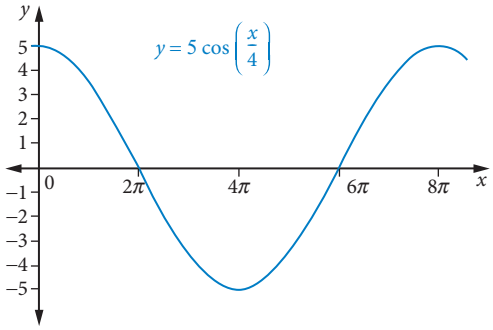
c amplitude = 4, period = 4π



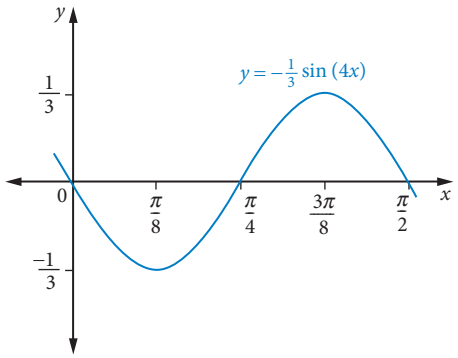
d amplitude = 3, period = 6π



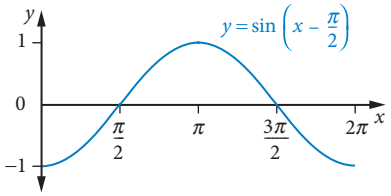
e amplitude = 5, period = 8π



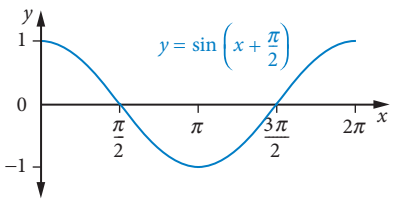
f amplitude = $\frac{1}{3}$, period = $\frac{\pi}{2}$



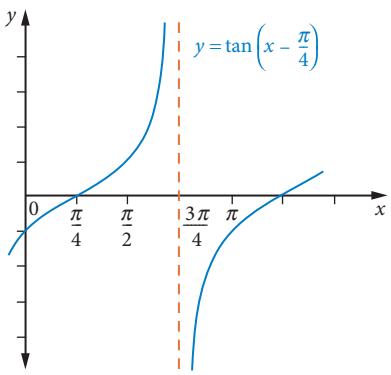
10 a



b

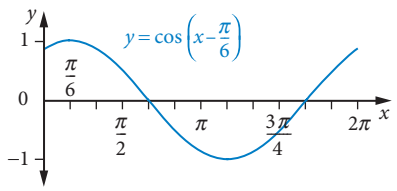


c

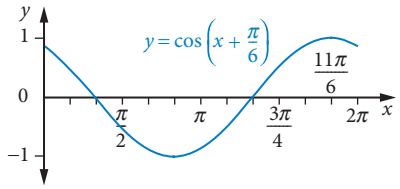


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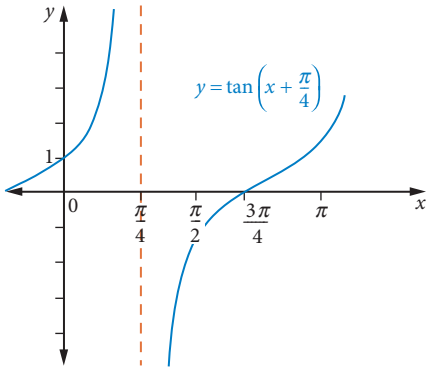
d



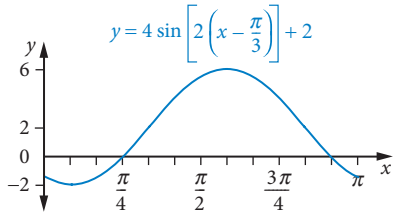
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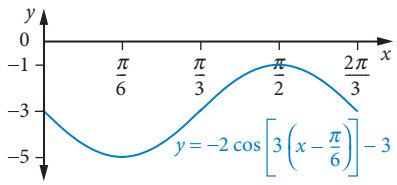
f



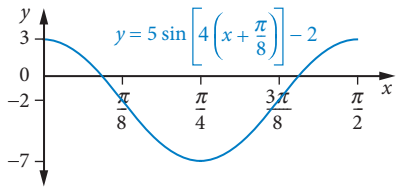
11 a



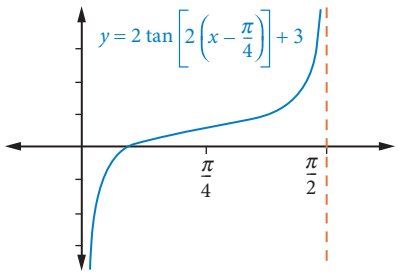
b



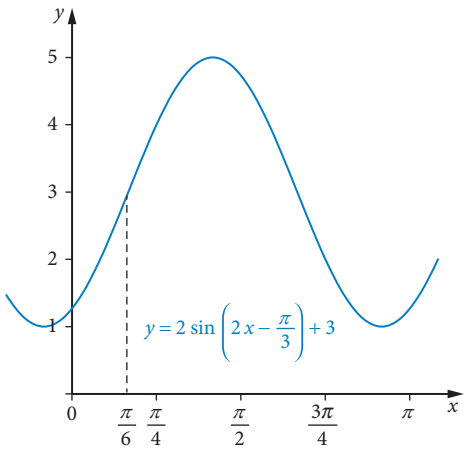
c



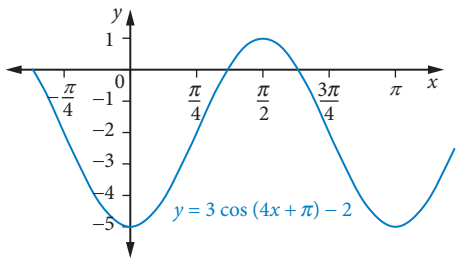
d



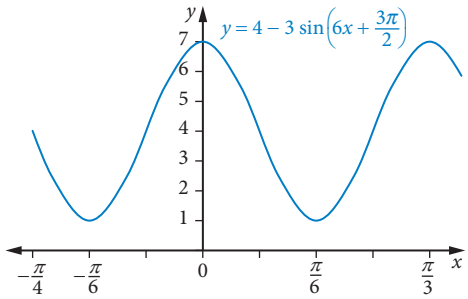
12 a



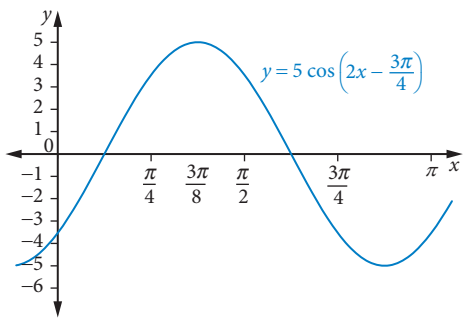
b



c

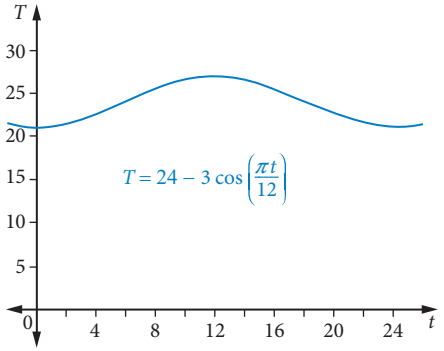


d



13 a amplitude = 0.8 m **b** period = 12 h

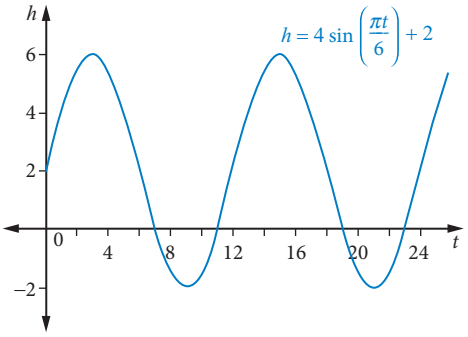
14 a



b 12 midday **c** 12 midnight

d about 8:48 am and 3:12 pm

15 a



b 6 m **c** 0.96 m

d between 12:29 am and 5:31 am and between 12:29 pm and 5:31 pm

16 $y = 3\cos\left(x + \frac{5\pi}{6}\right)$

17 $a = 2, b = 1, c = -\frac{\pi}{3}$ and $d = -1, y = 2\sin\left(x + \frac{\pi}{3}\right) - 1$

18 B **19** D **20** B

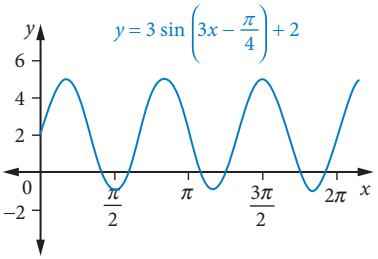
21 $x = 0.16, 2.46, 3.30, 5.60$

22 The maximum is at $t = 0$ and every 28 days thereafter. It is at 75% at $4\frac{2}{3}$ and $23\frac{1}{3}$ days after the full moon.

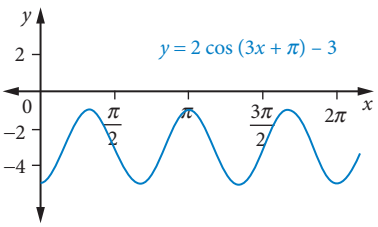
EXERCISE 4.6

1 $y = 2\cos(x - \pi) + 1$

2 a



b



3 a $x = -\pi, \pi$ **b** $x = \frac{\pi}{2}$

c $x = -\frac{7\pi}{4}, -\frac{3\pi}{4}$ **d** $x = 0, 2\pi$

e $x = -\pi, \pi$ **f** $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$

g $x = -\frac{\pi}{4}, \frac{3\pi}{4}$ **h** $x = \frac{\pi}{6}, \frac{5\pi}{6}$

4 a $\theta = 45°, 225°$ **b** $\theta = 30°, 150°$

c $\theta = 150°, 210°$ **d** $\theta = 225°, 315°$

e $\theta = 90°, 270°$ **f** $\theta = 60°, 240°$

g $\theta = 50°, 130°$ **h** $\theta = 115°, 245°$

5 a $x = 0.3398, 2.8018$ **b** $x = 1.8235$

c $x = 0.3805$ **d** $x = 0.9818$

e $x = 0.4429, 2.6987$ **f** $x = 1.7197$

6 a $x = \frac{\pi}{12}, \frac{5\pi}{12}$ **b** $x = \frac{\pi}{2}$ **c** $x = \frac{\pi}{4}$

7 a $\beta = 60°, 120°, 240°, 300°$

b $\beta = 90°, 270°$

c $\beta = 30°, 150°, 210°, 330°$

d $\beta = 45°, 135°, 225°, 315°$

8 a $\theta = 180°, 210°$ **b** $\theta = 270°, 300°$

c $\theta = 0, \frac{\pi}{4}, \pi, \frac{5\pi}{4}, 2\pi$ **d** $\theta = -\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}$

9 a $\alpha = \frac{\pi}{2}$ **b** $\alpha = \frac{\pi}{3}, \frac{5\pi}{3}$

c $\alpha = 2.3005, 3.9827$

d $\alpha = 0.3398, 2.8018, 3.6652, 5.7596$

10 a 0.34 m **b** 0.37 m

c 0.38 m when $t = \frac{\pi}{8}$

11 $x = 0.47, -1.51$

12 D **13** $x = \frac{2\pi}{3}, \frac{4\pi}{3}$

14 $x = -\frac{2\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{6}, \frac{5\pi}{6}$

15 $x = 100°, 160°$

16 18.9°C, midnight, 2 am, 10 am, 8 pm, 10 pm

EXERCISE 4.7

1 a 135°, 315° **b** 210°, 330°

c 30°, 330° **d** 45°, 135°

2 a $\frac{\pi}{6}, \frac{5\pi}{6}$ **b** 1.8235 **c** 0.3805

3 a yes **b** 8:00 pm **c** about 6:00 am

d about 24 h **e** about 10.5

4 a February

b period = 12 months, amplitude = 9%

5 a 0.75 s **b** time between heart beats

c 2 h 5 min

6 a $y = 5\cos\left(\frac{\pi x}{3}\right)$

b $y = 3\sin\left(\frac{\pi x}{4}\right)$

c $y = 8\cos\left(\frac{3\pi x}{4}\right)$

d $y = 6\sin\left(\frac{2\pi x}{3}\right)$

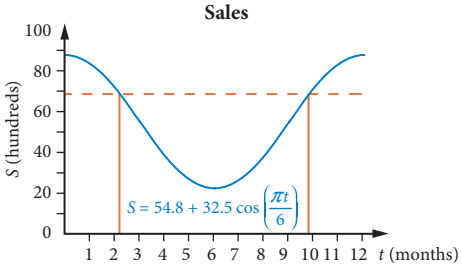
7 a maximum = 12 m, minimum = 2 m

b 120 **c** $a = 5, k = 4\pi, d = 7$

d $33\frac{1}{3}\%$

e It will only work for a limited time.

8 a

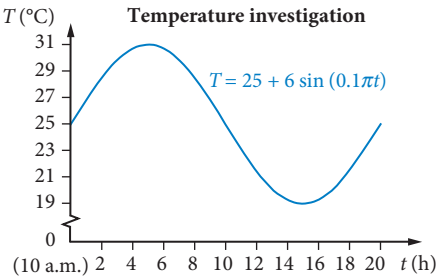


b Jan, Feb, Oct, Nov and Dec

c No, conditions change over time.

9780170484909

9 a



b 31°C, 19°C

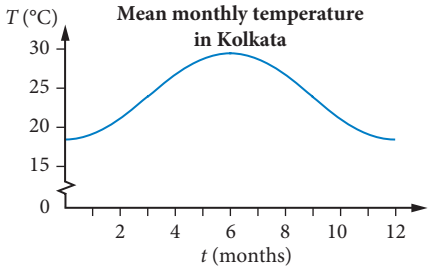
c 3 pm, 1 am

d **i** 11:05 am, 6:55 pm, …

ii 11:08 pm, 2:52 am, …

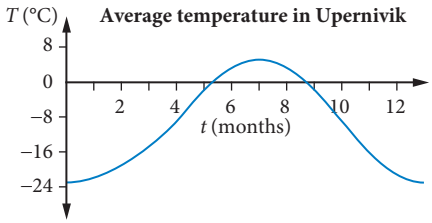
e Because the period was not 24 hours.

10 $T = 23.875 - 5.575\cos\left(\frac{\pi t}{6}\right)$



In the absence of climate change, averages over a long period will be good predictors of monthly trends, but not daily weather. General warming of the Earth makes conditions more extreme because there is more energy in the atmospheric system.

11 $T = -14\cos\left(\frac{\pi}{6}(t-1)\right) - 9$



The ice in Greenland is melting, so the model will not be useful for very long.

12 about 0.352 s, 0.648 s and 1.352 s

13 $d = 4 + 0.9\sin\left(\frac{6\pi t}{37}\right)$

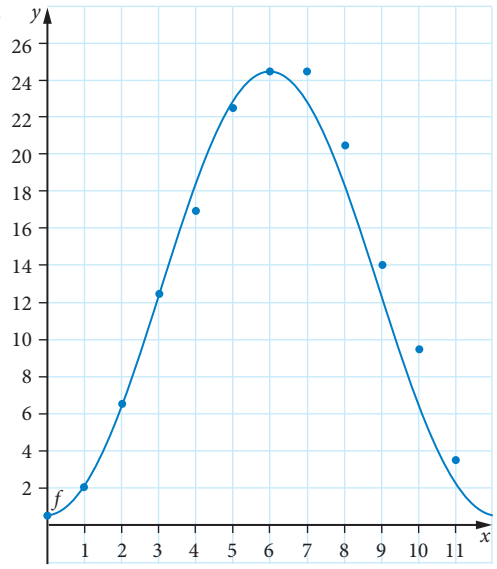
The boat with a draught of 3.5 m can enter for 8.45 hours, then cannot for 3.88 hours. The boat with a draught of 4.5 m can enter for 3.85 hours, then cannot for 8.48 hours. As long as the bottom does not change its nature, the model is useful.

14 a $\frac{1}{200}$ s **b** $y = 12\sin(400\pi t)$

c Particular notes do not vary, so it is very useful.

15 A **16** B **17** C

18



The period is 12 months, and the amplitude is 12°C.

The mean temperature is about 12.5°C.

$f(t) = 12.5 - 12\cos\left(\frac{\pi}{6}x\right)$

where $t = 0$ in January and t is in months.

19 a 230 W **b** 2408 W **c** 0.005 s

20 a 12.8 m/s

b A taller person should get greater distance.

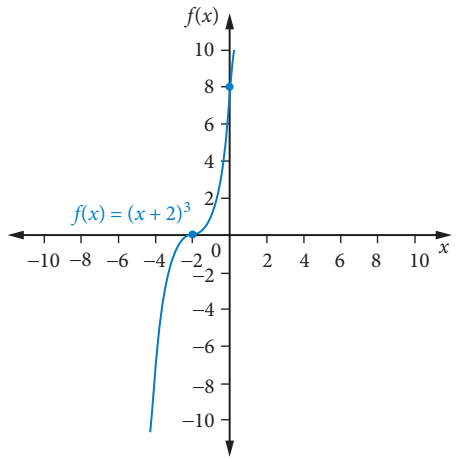
CUMULATIVE EXAMINATION 1

1 B **2** C **3** C

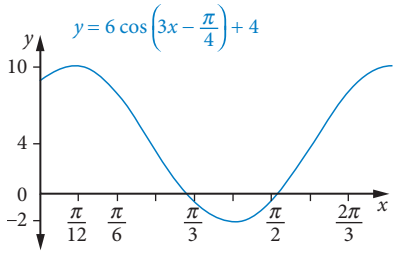
4 amplitude = 3, period = π, range: $-2 \le y \le 4$

5 $x = \frac{3+\sqrt{29}}{5}$ or $\frac{3-\sqrt{29}}{5} \approx 1.677$ or -0.477

6



7

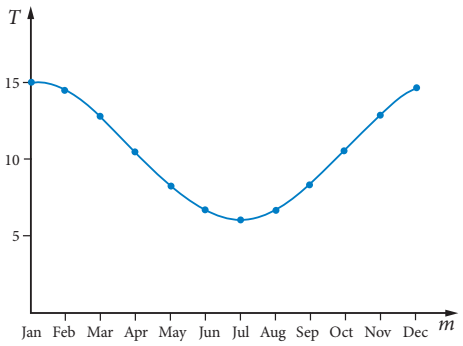


CUMULATIVE EXAMINATION 2

1 C **2** D **3** B

4 height = 8 cm, width = 13 cm, length = 21.125 cm

5 a



b 4.5°C

6 about .69

7 at 5:40 am and 6:20 am

CHAPTER 5

EXERCISE 5.1

1 a $A = \{14, 16, 18, 20, 22, 24, 26, 28, 30, 32\}$

b $B = \{12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25\}$

c $C = \{9, 11, 13, 15, 17, 19, 21, 23, 25\}$

d $D = \{\{\}, \{1\}, \{3\}, \{5\}, \{7\}, \{1, 3\}, \{1, 5\}, \{1, 7\}, \{3, 5\}, \{3, 7\}, \{5, 7\}, \{1, 3, 5\}, \{1, 3, 7\}, \{1, 5, 7\}, \{3, 5, 7\}, \{1, 3, 5, 7\}\}$

e F = {Jan, Feb, Mar, Apr, May, Jun, Jul, Aug, Sep, Oct, Nov, Dec}

2 a 10 **b** 14 **c** 9
d 16 **e** 12

3 a 9 **b** 9 **c** 4
d 15 **e** 0

4 a {letters of the alphabet} **b** {b}
c {m, p, r, t} **d** {a, e, i, o, u}
e {a, b, c, m, p, r} **f** {} **g** 21
h A and C, A and D, B and C, B and D
i {}, {p}, {r}, {t}, {p, r}, {p, t}, {r, t}, {p, r, t}
j {}, {m}, {p}, {r}, {m, p}, {m, r}, {p, r}

5 a {0, 1, 2, 3, 4, 5, 6, 7, 8, 9} **b** {2, 3, 4, 6, 7, 8, 9}
c {0, 9} **d** {0, 1, 2, 3, 4, 6, 7, 8, 9}
e {0, 1, 2, 3, 4, 6, 7, 8, 9} **f** {0, 9}
g $V = \{0, 3, 6, 9\}$ **h** M and P
i {}, {2}, {4}, {6}, {8}, {2, 4}, {2, 6}, {2, 8}, {4, 6), {4, 8}, {6, 8}, {2, 4, 6}, {4, 6, 8}, {2, 6, 8}, {2, 4, 8}, {2, 4, 6, 8}
j {}, {5}, {6}, {7}, {8}, {5, 6}, {5, 7}, {5, 8}, {6, 7), {6, 8}, {7, 8}, {5, 6, 7}, {6, 7, 8}, {5, 7, 8}, {5, 6, 8}

6 a {HH, HT, TH, TT} **b** A = {HH, HT}
c B = {HH, TH} **d** {HH}
e {HH, HT, TH} **f** C = {HT, TH, TT}
g D = {HT, TH} **h** E = {TT}
i F = {HH} **j** {HH}

7 a {Blue, Red, Green, Pink, Orange}
b {Blue, Red}
c {Green, Pink, Orange}
d {}
e {Blue, Red, Green, Pink, Orange}
f 2, 3, 0 and 5 respectively

8 $n(A) = 37$, $n(B) = 57$, $n(C) = 12$, $n(D) = 22$, $n(E) = 11$, $n(F) = 3$, $n(G) = 3$, $n(H) = 2$

9 D

10 {2, 3, 4, 6, 8, 9, 10, 12, 14, 15}

11 {6, 12, 15}

12 {Brisbane, Sydney, Canberra, Melbourne, Hobart, Adelaide, Perth, Darwin}

13 {R1R2, R1G1, R1G2, R1G3, R2G1, R2G2, R3G3, G1G2, G1G3, G2G3}

14 a {WB, WP, WY, WR, WR, BW, BP, BY, BR, BR, PW, PB, PY, PR, PR, YW, YB, YP, YR, YR, RW, RB, RP, RY, RR, RW, RB, RP, RY, RR}
b {BW, BP, BY, BR, BR, WB, PB, YB, RB, RB}
c {WR, WR, BR, BR, PR, PR, YR, YR, RW, RB, RP, RY, RR, RW, RB, RP, RY, RR}
d {WB, WY, BW, BP, BY, BR, BR, PB, PY, YW, YB, YP, YR, YR, RB, RY, RB, RY}
e {BY, YB}
f {}

EXERCISE 5.2

1 a $A = \{7, 10, 13, 16, 19\}$, $B = \{23, 29, 31, 37\}$

b $|A| = 5$, $|B| = 4$

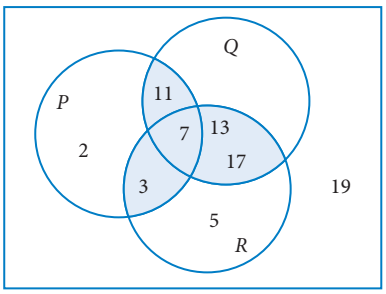
2 a {5, 13, 17, 19}

b {7, 13, 17}

c {}, {7}, {11}, {13}, {17}, {7, 11}, {7, 13}, {7, 17}, {11, 13}, {11, 17}, {13, 17}, {7, 11, 13}, {7, 11, 17}, {7, 13, 17}, {11, 13, 17}

d {3, 5, 7, 11, 13, 17}

e



3 a $\frac{1}{12}$ **b** $\frac{7}{24}$ **c** $\frac{1}{2}$

4 a {R, R, R, R, R, G, G, G, G, G, G, G, G, B, B, B, B, B, B, B}
b 0.35 **c** 0 **d** 0.4 **e** green

5 a $\frac{2}{5} = 0.4$ **b** $\frac{3}{5} = 0.6$ **c** white
d $\frac{1}{3}$ **e** $\frac{2}{3}$

6 a {V, V, V, V, V, V, H, H, H, H, H}
b $\frac{6}{11}$ **c** $\frac{5}{11}$ **d** $\frac{6}{11}$ **e** 1

7 a $\frac{1}{21}, \frac{2}{21}, \frac{1}{7}, \frac{4}{21}, \frac{5}{21}, \frac{2}{7}$ **b** $\frac{4}{7}$
c $\frac{10}{21}$ **d** $\frac{3}{7}$

8 **a** $\frac{184}{365}$ or $\frac{92}{183}$ (leap year) **b** $\frac{2}{7}$

c about $\frac{1}{12}$ **d** $\frac{1}{6}$ **e** $\frac{1}{7}$

9 B

10 $P(A) = 0.12$, $P(B) = 0.06$, $P(C) = 0.03$

11 **a** $\frac{2}{7}$ **b** $\frac{1}{21}$ **c** $\frac{2}{21}$

d $\frac{2}{21}$ **e** 0 **f** $\frac{2}{3}$

12 $P(\text{2 blue pens}) = \frac{1}{5}$; $P(\text{different coloured pens}) = \frac{11}{15}$

13 $\frac{2}{3}$

EXERCISE 5.3

1 $\frac{2}{23}$ **2** $\frac{1}{3}$

3 **a** $\frac{5}{33}$ **b** $\frac{7}{22}$ **c** $\frac{35}{66}$ **d** $\frac{1}{22}$

4 $\frac{58}{135}$

5 **a** $\frac{1}{3}$ **b** $\frac{3}{11}$

6 $P(H|O) = 0.517$, $P(H|\overline{O}) = 0.132$, so for obese women the risk is nearly 4 times greater.

7 **a** $\frac{1}{1024}$ **b** $\frac{15}{1024}$ **c** $\frac{45}{516}$ **d** $\frac{243}{1024}$

8 $\frac{3}{11}$ **9** B **10** $\frac{14}{95}$ **11** $\frac{3}{38}$

12 0.46 **13** $\frac{1}{2728}$

14 $\frac{9}{16}$ **15** 0.025%

EXERCISE 5.4

1 $\frac{1}{15}$ **2** $\frac{1}{64}$

3 **a** $\frac{1}{45}$ **b** $\frac{2}{45}$ **c** $\frac{14}{45}$

4 **a** $\frac{4}{9}$ **b** $\frac{1}{9}$ **c** $\frac{4}{9}$

d $\frac{1}{2}$ **e** $\frac{5}{18}$ **f** $\frac{7}{8}$

5 **a** $\frac{11}{40}$ **b** $\frac{3}{8}$

6 $\frac{1}{16}$

7 **a** $\frac{299}{600}$ **b** $\frac{1}{1800}$

8 **a** $\frac{1}{5}$ **b** $\frac{4}{5}$ **c** $\frac{1}{125}$ **d** $\frac{64}{125}$

e $\frac{16}{125}$ **f** $\frac{12}{125}$ **g** $\frac{48}{125}$

9 0.525

10 **a** $\frac{1}{12}$ **b** $\frac{1}{2}$ **c** $\frac{1}{2}$

d $\frac{1}{4}$ **e** $\frac{1}{6}$

11 **a** 0.26 **b** $\frac{4}{17}$ **c** no

12 A **13** A **14** B

15 **a** 0.384 **b** 0.992 **c** 0.02 **d** 0.98

16 about 0.493

17 0.000 504 = 0.0504%

Assumption: the 2 events are independent.

EXERCISE 5.5

1 $\frac{1}{16}$

2 **a** 0.097 **b** 0.027 **c** 0.009 409

3 **a** 0.075 **b** 0.025

c 0.275 **d** 38

4 **a** 0.06, 0.34, 0.26, 0.29, 0.02, 0.034

b $\frac{43}{150}$ **c** 7 **d** 8

5 **a** 61 **b** 51 **c** 69

6 **a** 48 **b** 22 **c** 0.458

d 0.104 **e** 23

7 **a** 0.0135 **b** 0.0016 **c** 0.0025

d 0.000 02 **e** 2

8 $\frac{2}{7} \approx 29\%$

9 About 0.092, assuming the lights are independent.

10 About 0.14, discounting couples and assuming that the people in the retirement home are typical.

11 About 10, assuming the conditions are similar.

12 About 38, assuming both schools have a similar mix of teachers.

CUMULATIVE EXAMINATION 1

1 C **2** B **3** A

4 30 cm^2

5 $1 - \frac{3 \times 38 \times 37}{4 \times 51 \times 50} \left(= \frac{703}{1700} \right)$

6 about 354

7 $f(x) = x^3 + x^2 - 0.5x - 3$

CUMULATIVE EXAMINATION 2

1 C **2** A **3** D **4** 0.1

5 The function $f(x) = \begin{cases} -0.3x + 5, & 0 < x < 2 \\ -0.35x + 4.7, & 2 \leq x < 4 \\ -0.1x + 3.4, & 4 \leq x < 10 \end{cases}$

Domain is $0 < x < 10$.

6 $x = 0.524, 2.618, 3.990, 5.435$

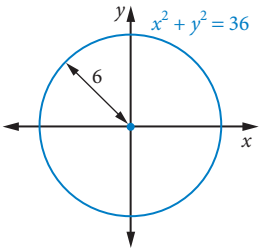
7 $P(HD|GF) \approx 0.659$ and $P(HD|\overline{GF}) \approx 0.524$, so there is about a 14% higher risk for those with the genetic factor. Genetic factors may be important predictors of heart trouble, but obesity is definitely related.

ANSWERS

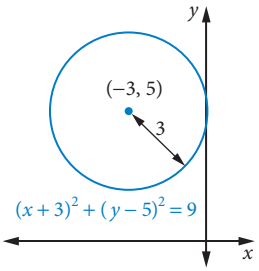
CHAPTER 6

EXERCISE 6.1

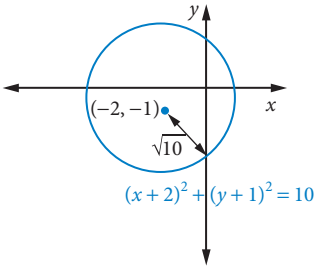
1 a circle with centre (0, 0) and radius 6



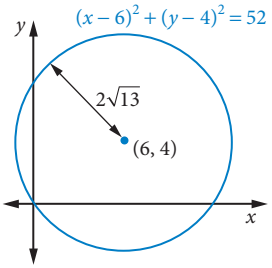
b circle with centre (−3, 5) and radius 3



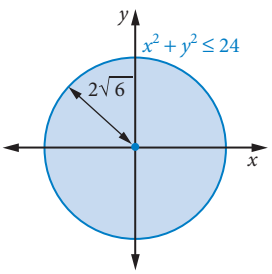
c circle with centre (−2, −1) and radius $\sqrt{10}$



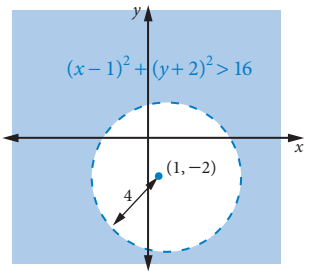
d circle with centre (6, 4) and radius $2\sqrt{13}$



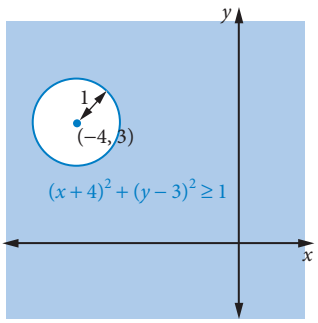
2 a region inside circle, centre (0, 0), radius $2\sqrt{6}$



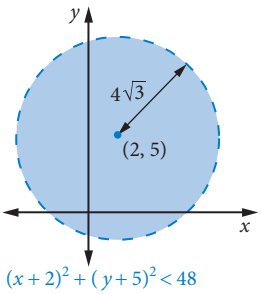
b region outside circle, centre (1, −2) and radius 4



c region outside circle, centre (−4, 3) and radius 1



d region inside circle, centre (2, 5) and radius $4\sqrt{3}$



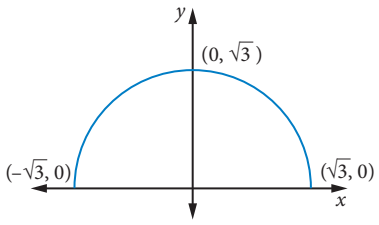
3 a (3, −4), 7 **b** (1, 3), 6
c (2.5, −3.5), 5 **d** (−2, −4), 3
e (−4, 2), 9

4 a $x^2 + y^2 = 16$ **b** $x^2 + y^2 = 1$
c $x^2 + y^2 = 6$ **d** $(x - 1)^2 + y^2 = 4$
e $(x + 3)^2 + (y + 3)^2 = 9$
f $y = -\sqrt{25 - x^2}$

5 a domain: [−4, 4], range: [−4, 4]
b domain: [−1, 1], range: [−1, 1]
c domain: $\left[-\sqrt{6}, \sqrt{6}\right]$, range: $\left[-\sqrt{6}, \sqrt{6}\right]$
d domain: [−1, 3], range: [−2, 2]
e domain: [−6, 0], range: [−6, 0]
f domain: [−5, 5], range: [−5, 0]

6 D **7** B **8** A **9** C

10 domain: $\left[-\sqrt{3}, \sqrt{3}\right]$, range: $[0, \sqrt{3}]$



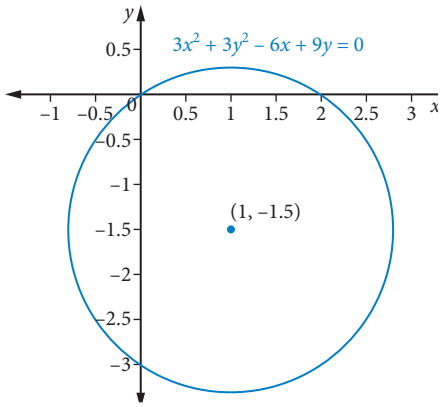
11 centre = (3, −1), radius = 2

12 No, it is not a function. It is 2 semicircles that form a circle and fails the vertical line text.

EXERCISE 6.2

1 $(x-11)^2+(y-13)^2=25$

2 radius $=\dfrac{\sqrt{13}}{2}$

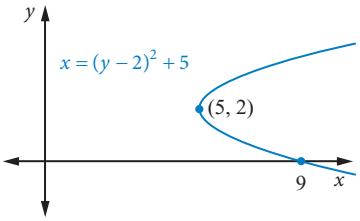


3 A quadratic relation fails the vertical line test.

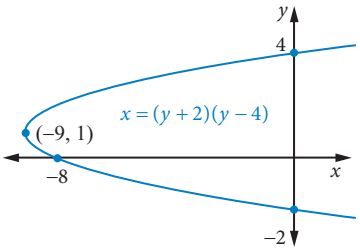
4 **a** vertex: $(-2, 3)$, domain: $[-2, \infty)$, range: R

b vertex: $(-8, -1)$, domain: $[-8, \infty)$, range: R

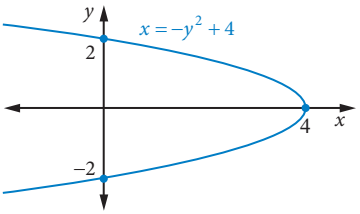
5 **a**



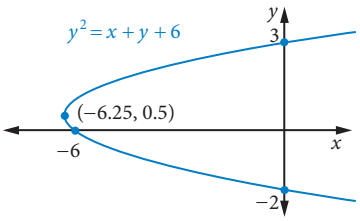
b



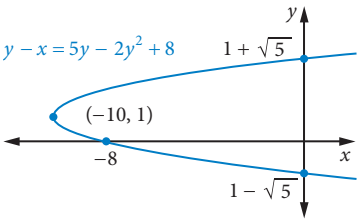
c



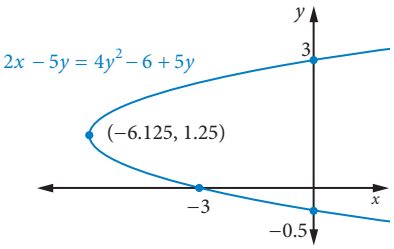
d



e



f

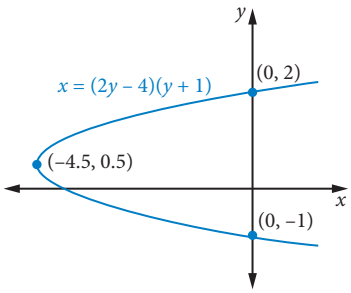


6 $x = 2y - y^2 - 8$

7 C **8** B **9** A **10** D

11 $(y+5)^2 = 2(x-3)$

12

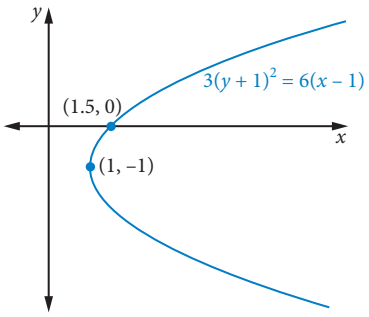


13 $x = y^2 - 16y + 24$

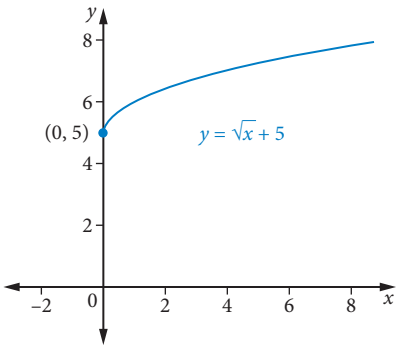
EXERCISE 6.3

1 A

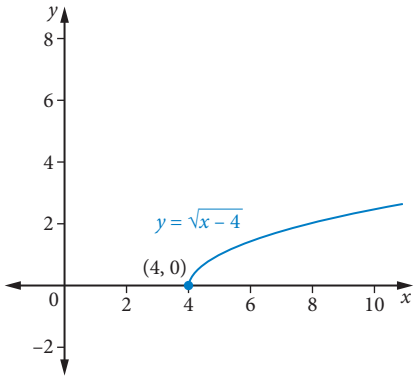
2



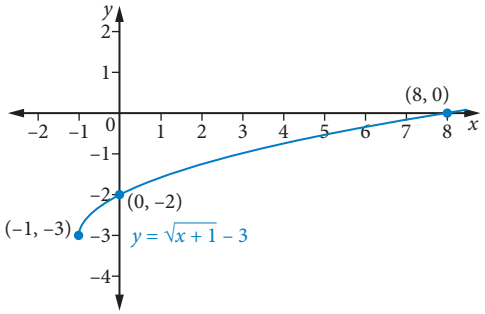
3 **a** domain: $[0, \infty)$, range: $[5, \infty)$



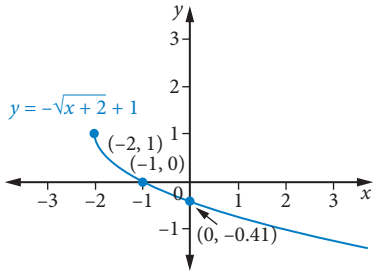
b domain: $[4, \infty)$, range: $[0, \infty)$



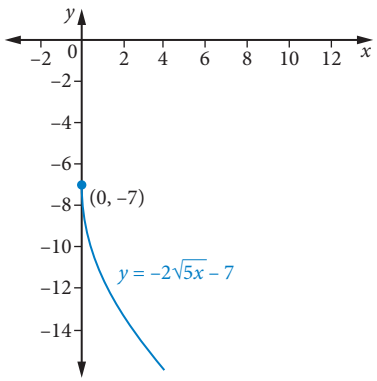
c domain: $[-1, \infty)$, range: $[-3, \infty)$



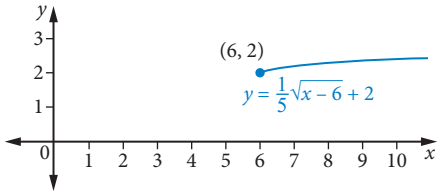
d domain: $[-2, \infty)$, range: $(-\infty, @1]$



e domain: $[0, \infty)$, range: $(-\infty, -7]$



f domain: $[6, \infty)$, range: $[2, \infty)$



4 **a** $y = \sqrt{x + 3} - 2$ **b** $y = \sqrt{x + 4} + 1$

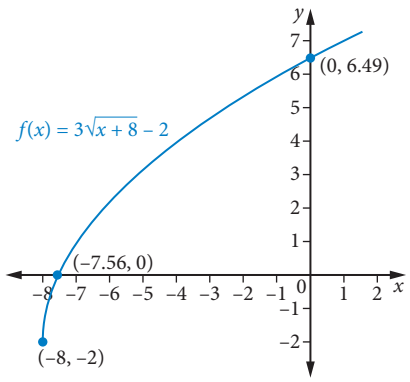
c $y = -2\sqrt{x} + 3$ **d** $y = \sqrt{x + 4} - 2$

5 The graph of $y = \sqrt{x}$ is

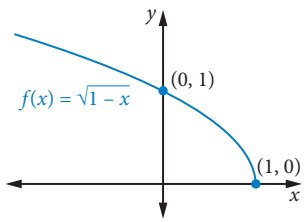
- reflected in the x-axis
- dilated by a factor of $\frac{1}{4}$ and become narrower
- horizontally translated right 9 units
- vertically translated up 5 units.

6 C **7** B **8** B

9



10

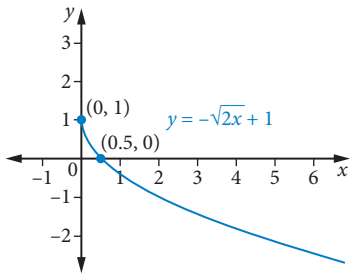


The graph of $f(x)$ is reflected in the y-axis and translated right 1 unit.

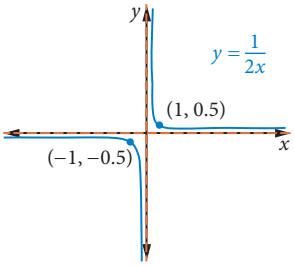
EXERCISE 6.4

1 C

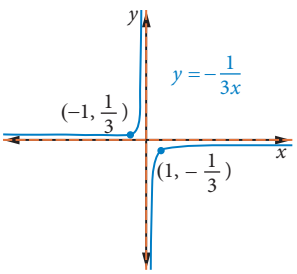
2



3 **a**

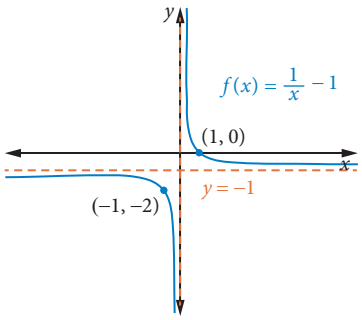


b

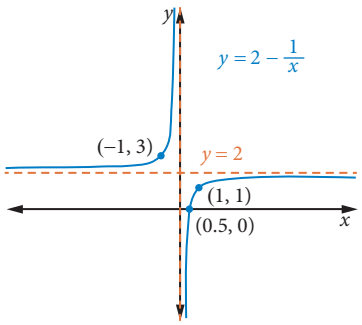


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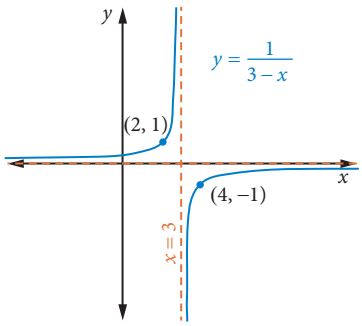
c



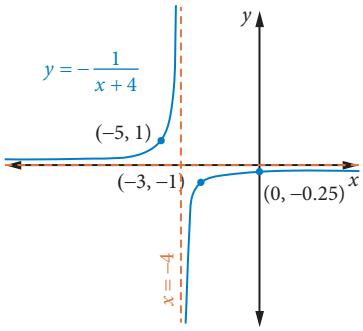
d



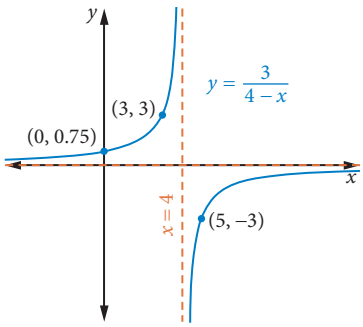
e



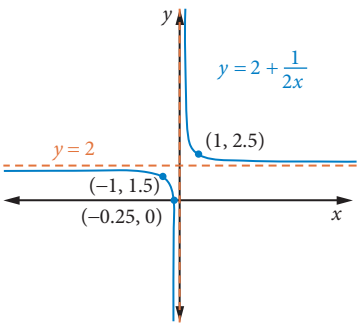
f



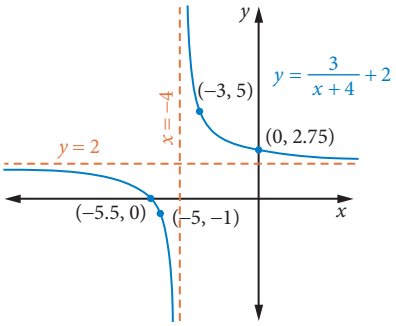
4 a



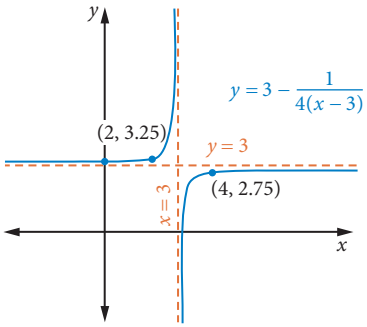
b



c



d



5 a Moved up 4 units and dilated by a factor of 4 (stretched).

b Moved 1 unit to the right and dilated by a factor of 2 (stretched).

c Moved to the left 5 units and down 7 units, reflected in the y-axis and dilated by a factor of 6 (stretched).

d Moved to the right 0.4 units and up $\frac{2}{3}$ units, reflected in the y-axis and dilated by a factor of 0.5 (compressed).

6 $y = \dfrac{-1}{x + 2} - 2$

7 domain: $R\backslash\{-2\}$, range: $R\backslash\{-2\}$,

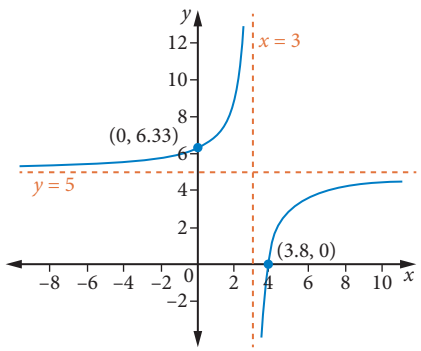
8 $f(x) = 2 + \dfrac{5}{x - 3}$; the equation matches the data.

9 B **10** A **11** C **12** C

13 $f(x) = 4 - \dfrac{4}{x + 2}$; the equation matches the data.

14 Moved to the left 4 units and up 2.6 units, reflected in the y-axis and dilated by a factor of 0.9 (compressed).

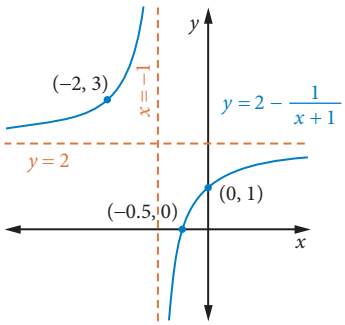
15



EXERCISE 6.5

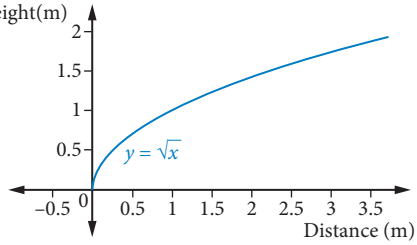
1 B

2



3 $y^2 = 5.625x$

4 a



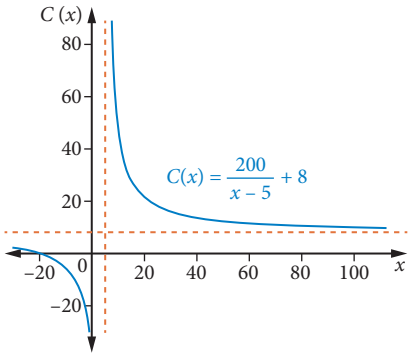
b domain: $0 \le x \le 7$

5 a The initial population is 11 birds.

b $P = \frac{10}{t+1} + 1$, domain: $[0, \infty)$, range: $(1, 11]$

6 a $C(100) \approx \$10.11$

b



c The domain is restricted to $x > 5$, range: $(8, \infty)$.

d As the number of items produced increases, the average cost decreases.

e No there is no exact minimum value, the average cost will approach \$8.

7 a $r_1 = 3$ cm, $r_2 = 18$ cm **b** 3 cm/s

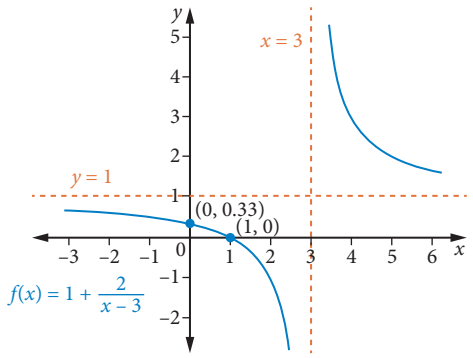
8 $(x-3)^2 + (y-4)^2 = 25$; additional area: 30.54 m^2

CUMULATIVE EXAMINATION 1

1 B **2** D **3** B **4** C

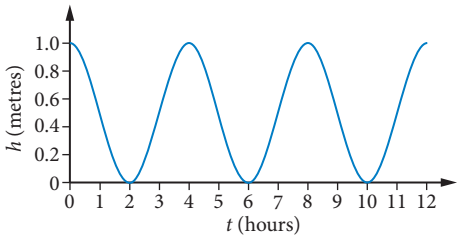
5 $y = -\frac{1}{2}\sqrt{x+1} + 4$

6



7 $\frac{-\sqrt{3}}{6}$

8 a



b 4 am, 8 am and 12 pm

CUMULATIVE EXAMINATION 2

1 B **2** A **3** D **4** C

5 The graph of y is translated right $\frac{1}{3}$ of a unit and up 3 units. It is dilated by a factor of $\frac{1}{2}$ and reflected in the x-axis.

6 a amplitude: 400, period: 6 months

b minimum = 800, maximum = 1600

c 1000 sloths

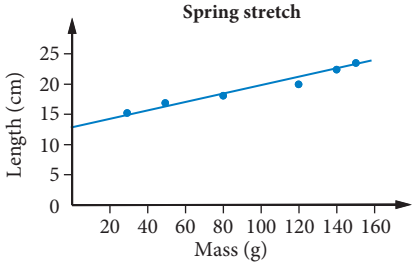
7 $\frac{6}{38}$

8 4.2 metres per month

CHAPTER 7

EXERCISE 7.1

1 a



b Length increases as mass increases. Relationship is approx. linear.

c 0.07 cm/g

d 13 cm. Original length of spring before any stretch.

2 a $\frac{20\text{ km}}{1.5\text{ h}} = 13.\dot{3}$ km/h **b** first leg

c second leg **d** $\frac{40\text{ km}}{5\text{ h}} = 8$ km/h

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3 **a** 0.5 m/s **b** 1.5 m/s
c 2.5 m/s **d** 1.5 m/s

4 **a** \$2.45/kg **b** 45 rev/min
c 30 m/s **d** 6.25 m/s^2/N
e AU\$1.25/US\$1 or AU\$1/US\$0.80

5 91.09 km/h

6 4 m/s^2

7

m	10	20	50	100	250
n	1.5	3	7.5	15	37.5
$\frac{\Delta n}{\Delta m}$		0.15	0.15	0.15	0.15

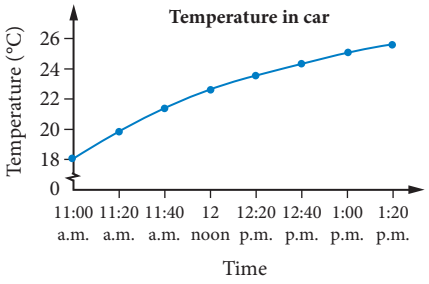
Therefore, quantities are related by a fixed rate of 0.15.

8

r	0.125	0.5	1.25	1.875	2.25
D	0.5	2.0	5.0	7.5	9.0
$\frac{\Delta D}{\Delta r}$		4	4	4	4

Therefore, quantities are related by a fixed rate of 4.

9 **a**

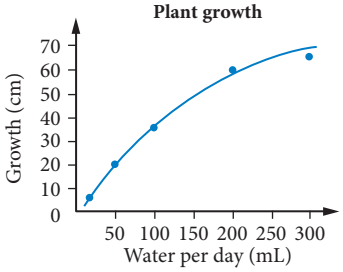


b Temperature is increasing but at a decreasing rate.
c 3.21°C/h
d Temperature rose 2.5°C from 12 pm–1 pm. Over the course of 2 h after 1 pm, temperature likely to exceed 28°C, therefore air-conditioning will not be of any use.
e no

10 **a** increasing at a constant rate
b decreasing at a decreasing rate
c increasing at an increasing rate
d increasing at a decreasing rate
e decreasing at an increasing rate
f decreasing at a constant rate
g no rate of change (constant)

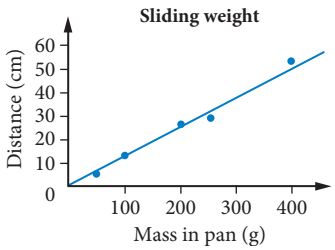
11 **a** 0.72 **b** 0.47 **c** 0.40 **d** 0.96

12



Plant growth does not increase constantly with respect to amount of water per day.

13



Distance appears to increase constantly with respect to mass in pan (not perfectly linear but could be explained by measurement error).

14 Pressure is increasing over time, but at a decreasing rate.
From 0–10 s, average rate of change is ~10 kPa/s, while from 40–50 s, average rate of change is ~2 kPa/s.

15 405 km/h

EXERCISE 7.2

1 C **2** A

Since these values are calculated using tangents, they may vary slightly from the stated answers.

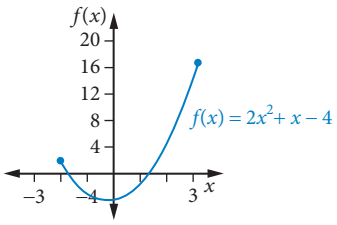
3 At $t = 1$, IRoC = 10 cm/s^2.
At $t = 2$, IRoC = –9 cm/s^2.
At $t = 3$, IRoC = –20 cm/s^2.

4 At $t = 1$, IRoC = –10.7 mL/day.
At $t = 4$, IRoC = –4 mL/day.

5 At $t = 1$, IRoC = 112.5°C/min.
At $t = 3$, IRoC = 0°C/min.
At $t = 5$, IRoC = –100°C/min.

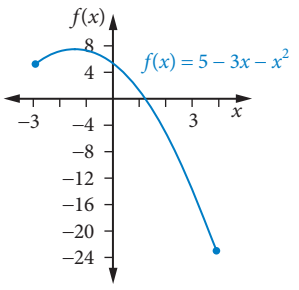
6 **a – c** At any point, slope is –2.

7



a –3 **b** 5 **c** 9

8



a 1 **b** –3 **c** –9

9 **a** 3 **b** 0 **c** 3

10 **a** –5 **b** –1 **c** 3

11 6

12 $v(2) = 25$ m/s; $d = 100$ when $t = 5$; $v(5) = 10$ m/s

EXERCISE 7.3

1 C **2** D

3 **a** $f(x+h) = 4x + 4h + 1$

b $f(x+h) - f(x) = 4h$

c $\dfrac{f(x+h) - f(x)}{h} = 4$

4 **a** $f(x+h) = 7 - 3x - 3h$

b $f(x+h) - f(x) = -3h$

c $\dfrac{f(x+h) - f(x)}{h} = -3$

5 **a** $f(x+h) = x^2 + 2xh + h^2 - 5x - 5h$

b $f(x+h) - f(x) = 2xh + h^2 - 5h$

c $\dfrac{f(x+h) - f(x)}{h} = 2x + h - 5$

6 **a** $4x - 4 + 2h$ **b** $8x + 3 + 4h$ **c** $-x - \dfrac{1}{2}h$

7 **a–d**

h	1	0.1	0.01	0.001
$\dfrac{f(x+h) - f(x)}{h}$	$-x - 0.5$	$-x - 0.05$	$-x - 0.005$	$-x - 0.0005$

x term remains the same, constant term approaches 0.

8 **a–d**

h	1	0.1	0.01	0.001
$\dfrac{f(x+h) - f(x)}{h}$	$4x - 6$	$4x - 4.2$	$4x - 4.02$	$4x - 4.002$

x term remains the same, constant term approaches -4.

9 **a** 2 **b** -5 **c** 6

10 **a** $3x^2 + 6xh + 3h^2$ **b** $6xh + 3h^2$ **c** $6x + 3h$

11 **a** $x^2 + 2xh + h^2 - 5x - 5h$ **b** $2xh + h^2 - 5h$

c $2x + h - 5$

12 **a** $12x + 6h$ **b** $4 - 8x - 4h$

c $\dfrac{2}{3}x + \dfrac{1}{4} + \dfrac{h}{3}$

13 $3x^2 + 3xh + h^2 - 1$

14 **a** $\dfrac{1}{x+h}$ **b** $\dfrac{-h}{x(x+h)}$ **c** $\dfrac{-1}{x(x+h)}$

15 **a** $\dfrac{-1}{(x+1)(x+h+1)}$ **b** $\dfrac{-3}{(x-2)(x+h-2)}$

c $\dfrac{2}{(3-2x)(3-2x-2h)}$

16 $\dfrac{2}{(5-x)(5-x-h)}$

17 $\dfrac{x^2 + hx - 64}{x(x+h)}$

18 Proof: see worked solution (HINT: eliminate the surds in the numerator by using the difference of 2 squares)

EXERCISE 7.4

1 A **2** C

3 **a**

x	3.9	3.99	3.999	4	4.001	4.01	4.1
$f(x)$	8.6	8.96	8.996	?	9.004	9.04	9.4

Therefore, $\lim\limits_{x \to 4}(4x - 7)$ is 9.

b

x	0.9	0.99	0.999	1	1.001	1.01	1.1
$f(x)$	−2.48	−2.0498	−2.0050	?	−1.9950	−1.9498	−1.48

Therefore, $\lim\limits_{x \to 1}(2x^2 + x - 5)$ is -2.

4 **a** undefined

b

x	−0.1	−0.01	−0.001	0	0.001	0.01	0.1
$f(x)$	0.8	0.98	0.998	?	1.002	1.02	1.2

Therefore, $\lim\limits_{x \to 0}\left(\dfrac{2x^2 + x}{x}\right)$ is 1.

5

x	1.9	1.99	1.999	2	2.001	2.01	2.1
$f(x)$	−10	−100	−1000	?	1000	100	10

Therefore, $\lim\limits_{x \to 2}\left(\dfrac{x+4}{x^2 + 2x - 8}\right)$ does not exist.

6 **a** limit does not exist

b limit does not exist

7 **a** undefined

b

x	−0.1	−0.01	−0.001	0	0.001	0.01	0.1
$f(x)$	3.1	3.01	3.001	?	2.999	2.99	2.9

Therefore, $\lim\limits_{x \to 0}\left(\dfrac{3x - x^2}{x}\right)$ is 3.

8 **a** undefined

b

x	0.9	0.99	0.999	1	1.001	1.01	1.1
$g(x)$	1.71	1.9701	1.9970	?	2.0030	2.0301	2.31

Therefore, $\lim\limits_{x \to 1}\left(\dfrac{x^3 - x}{x - 1}\right)$ is 2.

9 **a** undefined

b

x	−2.1	−2.01	−2.001	−2	−1.999	−1.99	−1.9
$h(x)$	−4.1	−4.01	−4.001	?	−3.999	−3.99	−3.9

Therefore, $\lim\limits_{x \to -2}\left(\dfrac{x^2 - 4}{x + 2}\right)$ is -4.

10 **a** 7 **b** -2 **c** 21

d 4 **e** 2.5

11 **a** 2 **b** -8 **c** -4

d 1 **e** 0.2887

12 **a** -2

b undefined; limit approaches different values at $x = -3$

c -6

d undefined; limit approaches ∞ as x approaches 1 from above, while it approaches $-\infty$ as x approaches 1 from below

13 **a** undefined; limit approaches $-\infty$ as x approaches 1 from the right, while it approaches ∞ as x approaches 1 from the left

b 1.5

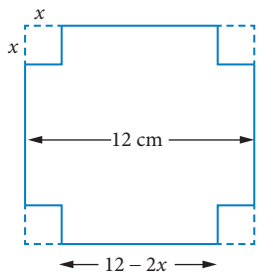
c undefined; limit approaches $-\infty$ as x approaches 2 from the right, while it approaches ∞ as x approaches 2 from the left

14 **a** 25 **b** 4 **c** 10 **d** -3

15 **a** -5 **b** -3 **c** -4 **d** -2

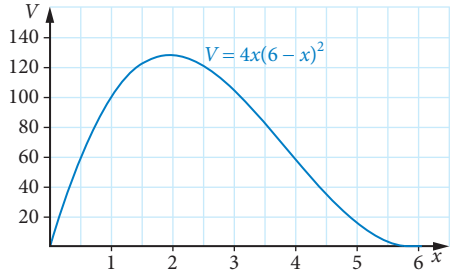
e 1 **f** 3 **g** -1 **h** $\frac{1}{2}$

16 **a**



b Proof: see worked solution

c



d $x = 2$

e

x	1.9	1.99	1.999	2	2.001	2.01	2.1
V	127.756	127.999	127.999	?	127.999	127.998	127.764

Therefore, the limit of V at $x = 2$ is 128.

17 $A = \frac{1}{2}x\sqrt{18 - x^2}$

From the graph, the greatest area appears to occur at $x = 3$. Confirm with table.

x	2.9	2.99	2.999	3	3.001	3.01	3.1
A	4.4903	4.4999	4.499 99	?	4.499 99	4.4999	4.4897

Therefore, the greatest area (4.5 m^2) occurs when $x = 3$.

EXERCISE 7.5

1 A **2** D

3 **a** $\frac{df}{dx} = 3$ **b** $\frac{df}{dx} = -1$

c $\frac{dg}{dx} = 2x$ **d** $\frac{dh}{dx} = 2x + 5$

4 **a** $f'(x) = 5$ **b** $g'(x) = -2$

c $h'(x) = 2x$ **d** $f(x) = 6x - 4$

5 C **6** A

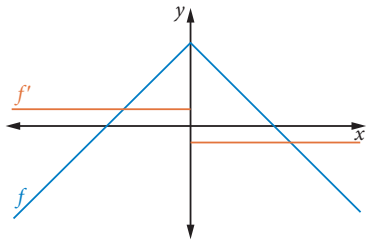
7 $g'(x) = 8x + 20$ **8** $f'(x) = 6x^2$

9 **a** 9 **b** 1 **c** $6x$

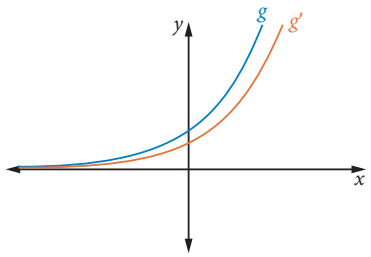
d 0 **e** $20x + 3$ **f** $-5 - 2x$

g $3 - 8x$ **h** $21x^2$ **i** $4x^3$

10



11



12 Proof: see worked solution

EXERCISE 7.6

1 C **2** B

3 **a** $f'(x) = 2x$ **b** $\frac{dy}{dx} = 3x^2$ **c** $f'(x) = 5x^4$

d $g'(x) = 8x^7$ **e** $\frac{dy}{dx} = 20x^{19}$ **f** $f'(x) = 17x^{16}$

g $h'(x) = 12x^{11}$ **h** $\frac{dy}{dx} = 22x^{21}$

4 **a** $f'(2) = 4$ **b** $y'(2) = 12$

c $f'(2) = 80$ **d** $g'(2) = 1024$

e $y'(2) = 10\,485\,760$ **f** $f'(2) = 1\,114\,112$

g $h'(2) = 24\,576$ **h** $y'(2) = 46\,137\,344$

5 **a** $f'(x) = 15x^2$ **b** $f'(x) = 24x^3$

c $f'(x) = 15x^4$ **d** $f'(x) = 12x^5$

EXERCISE 7.7

1 D **2** A

3 **a** $24x^3 - 15x^2 + 4x + 8$

b $27x^2 + 40x^3 - 15x^4$

c $3x^2 + 2x + 1$

d $36x^8 - 80x^9$

e $180x^{11} + 180x^{14} - 180x^4$

f $-7x^6 + 63x^8 + 10x^9$

g $40x^7 + 108x^8 - 60x^9 + 12x^{11}$

h $-3 + 2x + 24x^2 - 4x^3$

i $20x^3 - 17 + 48x^2$

j $175x^6 - 48x^5 + 70x^4 + 2$

4 **a** -14 **b** -30 **c** 618 **d** -358

5 **a** $12x^3 - 12x$ **b** $12u^3 - 12u$
c $30m^5 + 40m^4 - 24$
d $9 + 30y^4 - 12y^2 + 14y^6$
e $6z - 27z^2 + 38$ **f** $35t^6 + 36t^3 - 16$
g $-13 - 12r + 27r^2$ **h** $-24a^3 + 4a$
i $8 - 12p^3 - 42p^6 - 45p^8$ **j** $8t - 6t^2 + 44t^3$

6 **a** $20x + 41$ **b** $12x - 35$
c $-42x - 38$ **d** $-24x + 11$
e $36x - 95$ **f** $-14x + 58$

7 **a** $2x + 14$ **b** $2x - 8$
c $8x + 12$ **d** $18x + 24$
e $32x - 40$ **f** $18x + 48$

8 **a** $-15x^4 + 4x^3 - 6x + 1$
b $-15x^4 + 4x^3 + 30x^2$
c $-42x^6 - 12x^5 + 100x^4$
d $-35x^4 - 80x^3 + 9x^2 - 28x + 2$
e $-126x^6 - 144x^5 - 40x^4 + 27x^2 + 12x$
f $-168x^5 + 65x^4 + 248x^3 + 48x^2$

9 **a** $28x^3 - 3$ **b** $60x^4 + 24x^2 - 4$
c 3 **d** 4

10 LHS $= 48x^3 - 39x^2 - 70x - 27$
RHS $= 36x^2 + 30x$
As LHS $\neq$ RHS, statement is false.

11 $y' \times z' = 24x^3 - 74x^2 + 42x$
$(y \times z)' = 20x^4 - 80x^3 + 63x^2 + 12x - 9$
They are not the same.

12 **a** **i** $2(x+1)$ **ii** $3(x+1)^2$ **iii** $4(2x+1)$
iv $6(2x+1)^2$ **v** $6(3x+1)$ **vi** $9(3x+1)^2$
b $\frac{d}{dx}(ax+1)^n = an(ax+1)^{n-1}$

CUMULATIVE EXAMINATION 1

1 B **2** C **3** A
4 $\frac{48 + 19\sqrt{2}}{14}$ **5** $x = \frac{-7 \pm \sqrt{61}}{2}$
6 3.5 **7** $f'(x) = -6x - 5$

CUMULATIVE EXAMINATION 2

1 A **2** B **3** D
4 **a** Depth is greater than 2 m between approx. 1:21 am and 11:09 am, therefore it is safe for the boat to leave at 7 am.
b From graph on calculator, depth is decreasing through 2 m at 11:09 am. This is the time the boater must return to the boat ramp by.
5 $\frac{1}{27}$
6 0.8358
7 $(-1, 2.425)$ and $(2.5, 0.28125)$

CHAPTER 8

EXERCISE 8.1

1 **a** $\frac{1}{r^2}$ **b** $\frac{12}{y^9}$ **c** $3p^4$ **d** $\frac{2}{j^6}$
e $\frac{1}{6m^{13}}$ **f** $\frac{1}{4k^{14}}$ **g** $\frac{15g}{h^2}$ **h** $5k^{18}m^{14}$
i $16v^{11}$ **j** $\frac{4q^{17}}{p^{13}}$ **k** $\frac{4e^{12}}{3f^{17}}$ **l** $\frac{c^3d^{19}}{2}$

2 **a** $a^{\frac{9}{2}}$ **b** $b^{\frac{7}{2}}$ **c** $\frac{6}{n^{\frac{1}{2}}}$ **d** $12g^{\frac{19}{3}}$
e $12p^{\frac{5}{2}}w^{\frac{11}{3}}$ **f** $10m^{\frac{7}{2}}n^{\frac{7}{3}}$ **g** $\frac{21u^3}{s^{\frac{7}{2}}}$ **h** $\frac{24p^{\frac{3}{2}}}{q^{\frac{5}{12}}}$

3 **a** 8 **b** 1 **c** 6 **d** 9
e 343 **f** 216 **g** $\frac{1}{4}$ **h** $\frac{16}{9}$

4 **a** $\frac{1}{a^2}$ **b** $\frac{1}{4^3}$ or $\frac{1}{64}$ **c** $\frac{1}{g^7}$
d $\frac{1}{5^4}$ or $\frac{1}{625}$ **e** $\frac{1}{y^6}$ **f** $\frac{1}{c^4d^8}$
g $\frac{4}{h^4}$ **h** $\frac{1}{4^4h^4}$ or $\frac{1}{256h^4}$

5 **a** y^{-4} **b** 4^{-3} **c** h^{-9} **d** t^4k^{-6}
e y^2p^{-6} **f** $4j^8i^{-6}$ **g** $3y^{-2}m^{-5}$ **h** $2m^{-1}z^{-4}$

6 **a** $\sqrt{6}$ **b** $\sqrt[4]{z}$
c $\sqrt[3]{216}$ **d** $\sqrt[5]{y^3}$ or $\left(\sqrt[5]{y}\right)^3$
e $\sqrt{49^3}$ or $\left(\sqrt{49}\right)^3$ **f** $\sqrt[3]{v^7}$ or $\left(\sqrt[3]{v}\right)^7$
g $\sqrt[3]{8^7}$ or $\left(\sqrt[3]{8}\right)^7$ **h** $\sqrt[4]{16^5}$ or $\left(\sqrt[4]{16}\right)^5$

7 **a** $k^{\frac{1}{2}}$ **b** $3^{\frac{1}{4}}$ **c** $5^{\frac{1}{2}}$ **d** $4^{\frac{1}{5}}$
e $k^{\frac{1}{3}}$ **f** $c^{\frac{3}{2}}$ **g** $64^{\frac{4}{3}}$ **h** $e^{\frac{3}{4}}$

8 **a** 2 **b** 1 **c** 1 **d** 9
e 16 **f** 64 **g** 2 **h** 1

9 **a** 2^4 **b** 5^{k+1} **c** 3×2^3
d $3^{2n+3m} \times 2^m$

10 C **11** B **12** $3^{2t+2} \times 7^3$

EXERCISE 8.2

1 D **2** B
3 **a** 6.3×10^3 **b** 1.42×10^8 **c** 7.5×10^{-3}
d 9.38×10^{-7} **e** 5×10^5 **f** 6×10^{-4}
g 2.1×10^2 **h** 7.9×10^{-1} **i** 3.109×10^7
j 4.062×10^{-2} **k** 9.623×10^6 **l** 2×10^{-9}

4 **a** 430 000 **b** 0.0021 **c** 9 840 000
d 0.319 **e** 0.000 000 006 **f** 20 000
g 42 **h** 0.058 **i** 1.6
j 0.000 095 **k** 464 **l** 2100

5 **a** 3 **b** 2 **c** 1 **d** 1
e 1 **f** 3 **g** 4 **h** 5
i 2 **j** 4 **k** 3 **l** 2

6 **a** 1.472×10^{2} **b** 7.314×10^{10}
c 2.1996×10^{19} **d** 4×10^{-6}
e $1.0\,20\,625 \times 10^{11}$ **f** 4.1884×10^{9}
g 8.6104×10^{-8} **h** 7.226×10^{7}

7 **a** 2.716×10^{14} **b** 6.541×10^{8}
c 1.171875×10^{-9} **d** 1.969×10^{21}
e 9.774×10^{-3} **f** 2.310×10^{-33}

8 **a** 40 **b** 480 **c** 248
d 0.000 0060 **e** 20 000 **f** 3.6
g 63 500 **h** 18 950 000 000 **i** 36.0

9 **a** 8.58 AU **b** 29.66 AU

EXERCISE 8.3

1 B **2** B **3** $3072\,m^2$

4 **a** About 1.9×10^{-7} C
b 100 A
c 38 ms (0.038 s)

5

x	−2	0	2	4
$f(x)$	0.018	0.3	5.043	84.773
$g(x)$	9.250	8	6.919	5.984

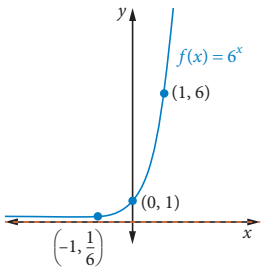
$f(x)$ is increasing at an increasing rate, while $g(x)$ is decreasing at a decreasing rate.

6 $7.96\,m^2$ **7** \$11 048.26
8 8290 m **9** 760 632 crimes

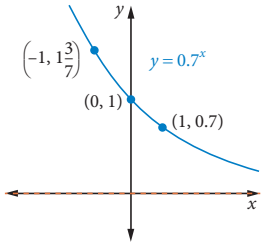
EXERCISE 8.4

1 B **2** C

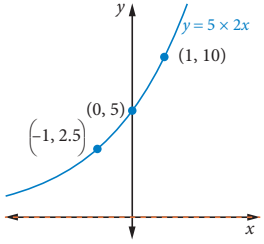
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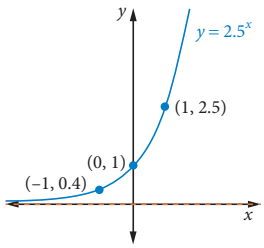
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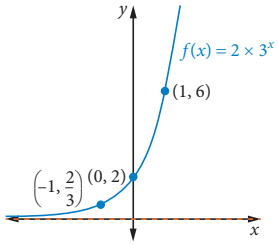
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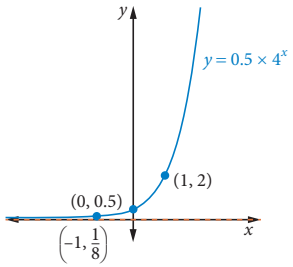
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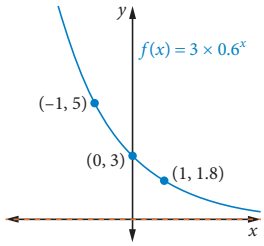
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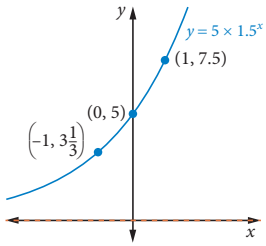
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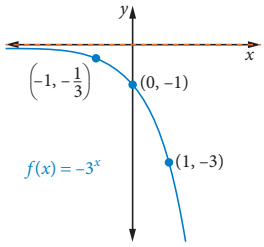
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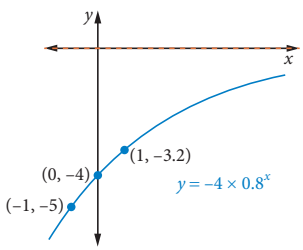
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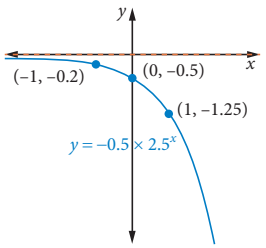


12



ANSWERS

13



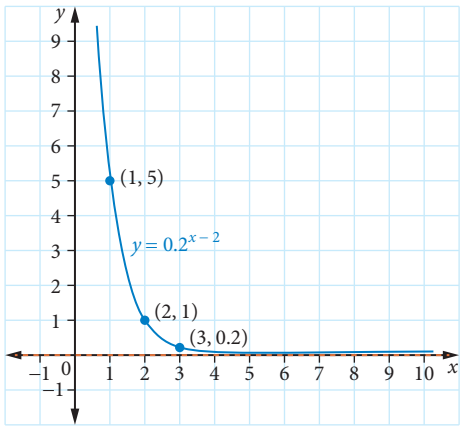
14 **a** $y = 3^x$ **b** $y = 0.4^x$

c $y = 0.4 \times 2^x$ **d** $y = -4 \times 2.5^x$

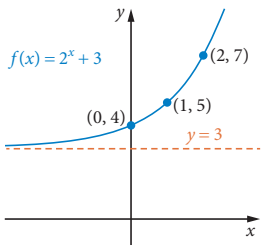
EXERCISE 8.5

1 B **2** C

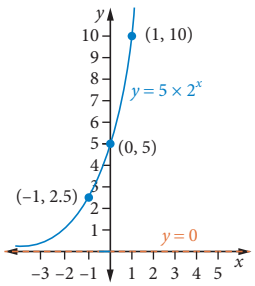
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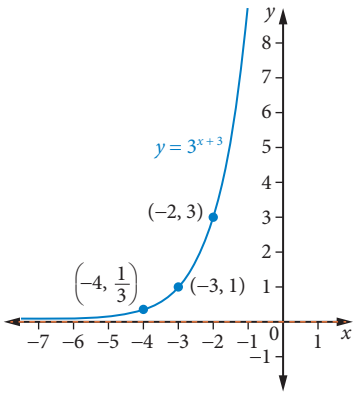
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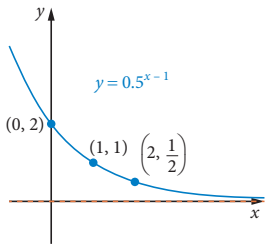
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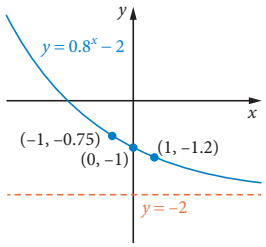
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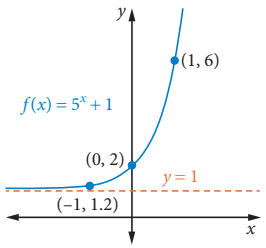
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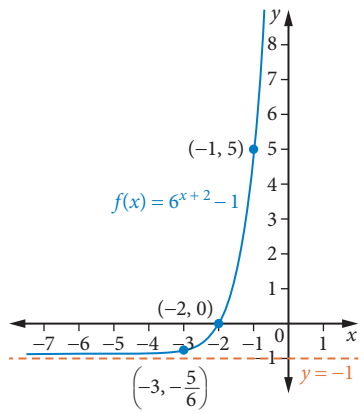
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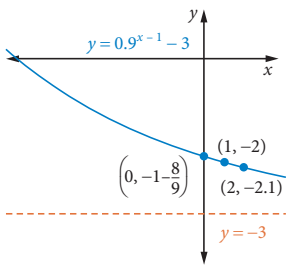
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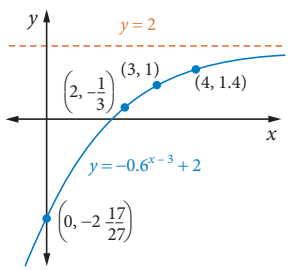
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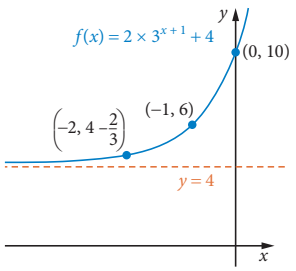


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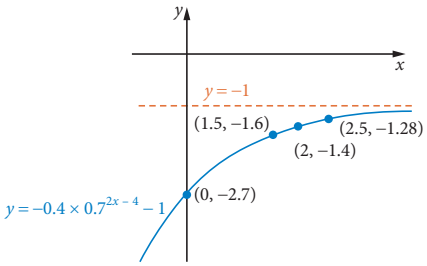


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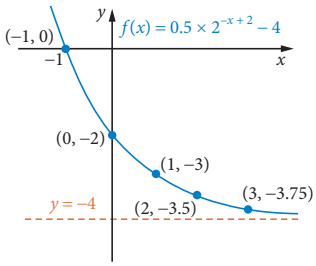
13



14



15



16 a $y = 2 \times 3^x + 1.5$ **b** $y = 2 \times 3^{x+2} + 1.5$

EXERCISE 8.6

1 D **2** C

3 a $x = 4$ **b** $x = 7.5$ **c** $x = 0.5$

d $x = 0$ **e** $x = \frac{11}{4}$ **f** $x = -\frac{7}{9}$

g $x = \frac{7}{4}$ **h** $x = -0.5$ **i** $x = \frac{26}{7}$

4 a $x = 2$ or $x = 4$ **b** $x = 3$

5 a $x \approx 1.9259$ **b** $x \approx 1.4732$ **c** $x \approx 0.7325$

6 $x = 0$, $x = 1$ or $x = 2$ **7** $x = 1$

EXERCISE 8.7

1 A **2** C

3 a $d = 3.60$ m **b** 2626 L **c** 5.98 hours

4 a $V = 60.1284 \times 0.9507^t$. Original voltage is 60.1284 kV. Capacitor fully discharged after $t = 159$ s.

b When original voltage is tripled, capacitor falls below 5 kV after 71 s.
r^2 (0.9999) is very high, confident in interpolating and extrapolating data from this model.

5 a Concentration after 2 hours is 10.5625 µg/mL.

b Dose wears off after 4 hours.

6 Area overrun after approx. 18 months.

7 17 years

8 7590 years

9 $C(t) = 0.1 \times 1.135^t$. Size of the colony after 24 hours is 2.1 mm^2. Over time, nutrients will become scarce, causing bacteria to die and the model to no longer be representative of the scenario.

10 $T = 63.7 \times 1.016^t$; 199 minutes

11 $M = 85.56 \times 0.9161^t$; the half-life is 8 days and the time until 10 mg is left is 24.5 days. Substitute values to verify the model using the given data. The extrapolation is for only a short time, so the model is probably OK for this period.

12 It was about 65°C when you started eating, about 53°C after 10 minutes and it takes about 20 minutes for its temperature to fall below 40°C. The temperature at the ground will change over time, meaning the model will become less useful over time. Other assumptions can be made, meaning the model is a general indicator of the temperature.

CUMULATIVE EXAMINATION 1

1 D **2** D **3** D

4 +3 is a vertical translation (up 3). Multiply by −2 is a reflection over the x-axis and dilation of factor of 2.

5 $x = 3$

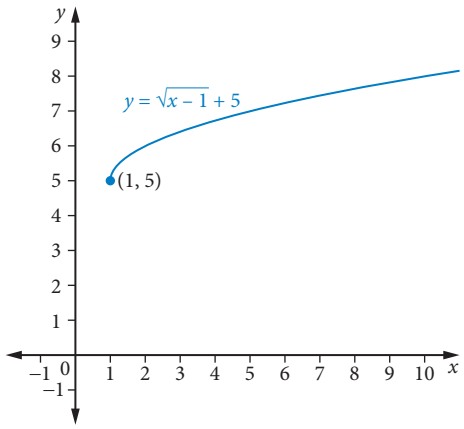
6 $f'(x) = -10x + 2$

7 $x = \frac{9}{8}, \frac{21}{8}$

CUMULATIVE EXAMINATION 2

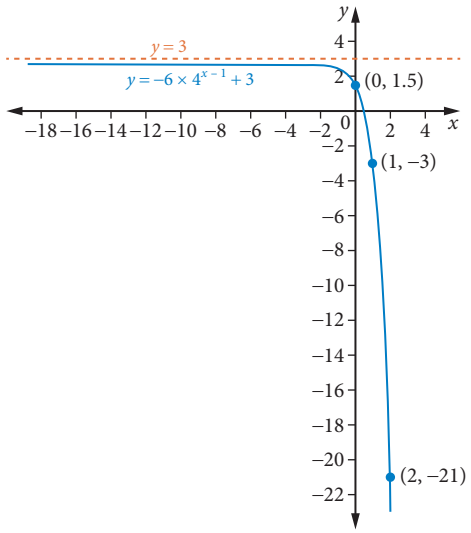
1 C **2** C **3** A

4



5 $\frac{1}{6}$

6



7 49 years

CHAPTER 9

EXERCISE 9.1

1 **a** $5^3 = 125$ **b** $2^5 = 32$ **c** $4^3 = 64$
d $7^1 = 7$ **e** $3^7 = 2187$ **f** $9^0 = 1$
g $5^4 = 625$ **h** $13^2 = 169$

2 **a** $16^{\frac{1}{2}} = 4$ **b** $8^{-\frac{1}{3}} = \frac{1}{2}$ **c** $3^{-2} = \frac{1}{9}$
d $4^{-2} = \frac{1}{16}$ **e** $\left(\frac{1}{2}\right)^{-3} = 8$ **f** $\left(\frac{1}{4}\right)^{-\frac{1}{2}} = 2$
g $32^{\frac{2}{5}} = 4$ **h** $5^{-\frac{1}{3}} = \frac{1}{\sqrt[3]{5}}$

3 **a** $\log_8(64) = 2$ **b** $\log_3(81) = 4$ **c** $\log_2(64) = 6$
d $\log_{12}(1) = 0$ **e** $\log_{15}(15) = 1$ **f** $\log_6(216) = 3$

4 **a** $\log_{36}(6) = \frac{1}{2}$ **b** $\log_{27}\left(\frac{1}{3}\right) = -\frac{1}{3}$ **c** $\log_4\left(\frac{1}{16}\right) = -2$
d $\log_{\frac{1}{2}}(16) = -4$ **e** $\log_{\frac{1}{9}}(3) = -\frac{1}{2}$ **f** $\log_9(27) = \frac{3}{2}$

5 **a** 4 **b** 4 **c** 3 **d** 4
e 0 **f** 1 **g** 3 **h** 6

6 **a** 0 **b** 1 **c** 0 **d** 0
e 1 **f** 0 **g** 1 **h** 1

7 all undefined

8 **a** −5 **b** −2 **c** −4 **d** −3
e 3 **f** −6 **g** 4 **h** −3

9 **a** 3 **b** 4 **c** $\frac{1}{3}$ **d** 4
e $\frac{2}{3}$ **f** 2 **g** $\frac{1}{2}$ **h** −1
i 0 **j** $\frac{3}{2}$ **k** $\frac{1}{6}$ **l** $-\frac{1}{2}$

10 A **11** B **12** D
13 C **14** A

EXERCISE 9.2

1 B **2** A

3 **a** $\frac{1}{2}$ **b** $\log_5(8)$ **c** 2 **d** 5
e 4 **f** −3 **g** $\frac{2}{3}$ **h** 2

4 **a** 5 **b** −3 **c** −8 **d** 2
e $\log_{10}(2)$ **f** 15 **g** 2 **h** −1

5 **a** 4202 **b** 42.02 **c** 004 202

6 **a** 0.1329 **b** 13 290 **c** 0.000 132 9

7 **a** 1.7559 **b** 3.7559 **c** −0.2441

8 **a** 1.4533 **b** −0.5467 **c** −2.5467

9 **a** 4.9883 **b** −0.0117 **c** −3.0117

10 **a** $\log_2(x^2)$ **b** $\log_5(x)$ **c** $\log_4\left(\frac{1}{x}\right)$
d $\log_{10}\left(\frac{y}{x}\right)$ **e** $\log_7(x+1)^5$ **f** $\log_3\left(\frac{x^4}{y}\right)$
g $\log_8(x^3y^2)$ **h** $\log_4\left(\frac{1}{x-2}\right)$

11 **a** $\log_2(x) + \log_2(y) - \log_2(z)$
b $1 + \log_4(x) - \log_4(y)$
c $\log_7(6) - \log_7(x) - \log_7(y)$
d $1 + \log_5(x) - 2\log_5(y) - \log_5(z)$
e $3\log_{10}(x) + \frac{1}{2}\log_{10}(y) - 5\log_{10}(z)$
f $4\log_9(x) + \frac{3}{2}\log_9(y) - \frac{1}{2}\log_9(z)$
g $\frac{1}{2}[3\log_5(x) + \log_5(y) - 3\log_5(z)]$
h $\frac{1}{3}[4\log_8(x) + 2\log_8(y) - 2\log_8(z)]$

12 **a** 0.276 **b** 1.654 **c** 2.827

13 A **14** D **15** A

16 $\frac{2}{b}$ **17** $x = \sqrt{5}$

EXERCISE 9.3

1 C **2** C

3 **a** $\frac{\log(12)}{\log(5)}$ **b** $\frac{\log(20)}{\log(4)}$ **c** $\frac{\log(50)}{\log(2)}$
d $\frac{\log(300)}{\log(9)}$ **e** $\frac{\log(0.5)}{\log(11)}$ **f** $\frac{\log(15)}{\log(7)}$

4 **a** 3.322 **b** 1.683 **c** 2.262
d 1.528 **e** 0.5537 **f** 0.9306

5 **a** $x = 3.5609$ **b** $x = 1.4307$ **c** $x = 2.2146$
d $x = 0.5646$ **e** $x = -3.3219$ **f** $x = 5.0966$

6 **a** $x = 1.3010$ **b** $x = 0.6941$ **c** $x = 0.8503$
d $x = -1.1761$ **e** $x = -13.6438$ **f** $x = -0.9457$

7 **a** $x = 1.292$ **b** $x = -0.5268$ **c** $x = 1.032$
d $x = 0.6375$ **e** $x = 0.7257$ **f** $x = 1.039$

8 A **9** A **10** B
11 1.908 **12** $a = 120$

EXERCISE 9.4

1 $\frac{\log_3(21)}{\log_3(6)}$ **2** B

3 **a** 1 **b** 5 **c** 3.819
d −1.893 **e** −1.529 or 2.994 **f** 2.003

4 **a** 3.477 **b** 3.424 **c** 5.057
d 9.233 **e** 11.64 **f** 0.4231

5 **a** 0.4946, −0.4946 **b** 5.393 **c** −1.893
d −1.139 **e** 1 **f** 4.667

6 D **7** A
8 2.5 **9** 0.861

CUMULATIVE EXAMINATION 1

1 B **2** C **3** C

4 13 **5** $x = 2$ **6** $x = \frac{4\pi}{3}, \frac{5\pi}{3}$

7 **a** $105 - 20\log_{10}(2)$ **b** 5 m

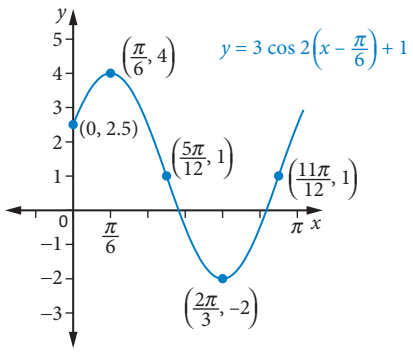
8 $\frac{2}{7}$

CUMULATIVE EXAMINATION 2

1 D **2** B

3 B **4** 0.631

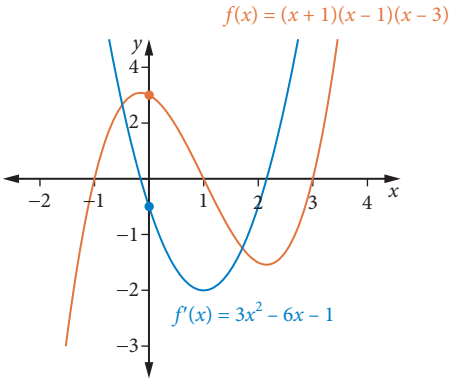
5



6 16

7 **a** $3x^2 - 6x - 1$

b



c Either: the zeroes of $f'(x)$ are at the maximum and minimum of $f(x)$; or, when $f'(x)$ is positive, $f(x)$ is increasing, and when it is negative, it is decreasing.

8 an increase of $\log_{10}(\sqrt{2}) \approx 0.15$

CHAPTER 10

EXERCISE 10.1

1 **a** 13 **b** 64 **c** 189
d −12 **e** 533

2 **a** 22 **b** 102 **c** −10
d −18 **e** 158

3 **a** −153
b **i** −15 **ii** −363 **iii** −135
c Given that $x = 5$ is midway between $x = 2$ and $x = 8$, its instantaneous rate of change should be similar to the average rate of change between $x = 2$ and $x = 8$. Values for gradients are similar.

4 **a** \$46 640 **b** \$777.33 **c** \$432/unit
d **i** Gradient is flattening out towards $x = 60$, so marginal cost is less than average cost.
ii Marginal cost: \$400/unit.
Average cost: \$620/unit. Gradient still flattening out towards $x = 100$, so marginal cost is less than average cost.
iii Marginal cost: \$628/unit.
Average cost: \$591/unit. Gradient has started to increase again towards $x = 130$, so marginal cost is more than average cost.

5 **a** **i** \$2.20/unit **ii** \$5/unit
b Although the number of items produced is still leading to a loss, the profit per unit is increasing.

6 **a** $-5.8\,\text{cm}^3/\text{s}$ **b** $-6.2\,\text{cm}^3/\text{s}$
c 63.75 s, when the balloon has a volume of 0.

7 **a** $\frac{\Delta SA}{\Delta t} = -12x$

b $-48\,\text{cm}^2/\text{cm}$

8 **a** $\frac{dA}{dt} = 9t^2 + 2$

b $k'(t) = 12t - 5$

c $\frac{dP}{dm} = 12m^3 - 2$

9 **a** **i** \$300/car
ii −\$100/car
b The dealership is making \$300/car when 150 cars are sold per month, but is losing \$100 per month when 250 cars are sold per month.

10 $\frac{\Delta V}{\Delta t} = \pi d^2$, $1.61\,\text{cm}^3$

EXERCISE 10.2

1 C **2** C

3 **a** $y = 4x - 5$ **b** $y = 2x$ **c** $y = 7x + 1$
d $y = x + 1$ **e** $y = -x$ **f** $y = 4x - 4$

4 **a** $y = -\frac{x}{4} + \frac{7}{2}$ **b** $y = -\frac{x}{2} - \frac{5}{2}$
c $y = -\frac{x}{7} - \frac{43}{7}$ **d** $y = -x + 3$
e $y = x - 2$ **f** $y = -\frac{x}{4} + \frac{9}{2}$

5 **a** $m = 4$ **b** $d = \sqrt{17}$ *or* 4.12 **c** 75.9°

6 C **7** A

8 $y = 7x + 5$ and $y = 7x - 27$

9 **a** (2, 0)
b (−1, −24), (5, 24)

10 $PQ = \sqrt{68}$ or 8.25 units

11 $y = 4x - 29$

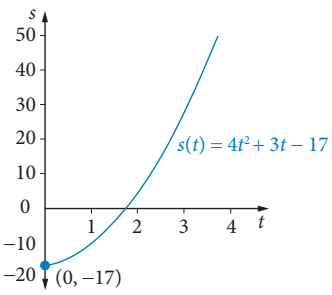
12 $y = 7.81x + 34.405$ and $y = -7.81x + 26.593$

13 $y = 3x^2 - 2x + 7$

EXERCISE 10.3

1 C **2** A

3 **a**



b The velocity starts out small and is increasing at an increasing rate.

c $v(t) = 8t + 3$

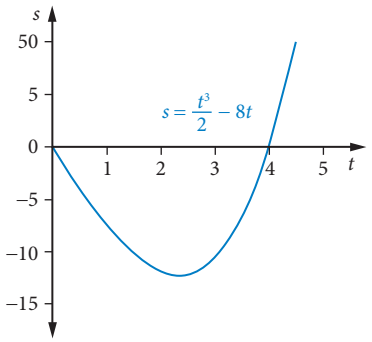
d $a(t) = 8$

e $v(4) = 35\,\text{m/s}$, $a(4) = 8\,\text{m/s}^2$

4 a Proof: see worked solution

b $t = 4$

c



d Velocity is negative after $t = 0$ and decreasing. After $t \approx 2.3$, velocity becomes positive and increases at an increasing rate.

e $v(3) = 5.5\,\text{cm/s}$

f $a(4) = 12\,\text{cm/s}^2$

g $t = 3\,\text{s}$

5 a $v(0) = -14\,\text{m/s}$ **b** $v(t) = 6t - 14$, $a(t) = 6$

c $x(5) = 13\,\text{m}$ **d** $t = \frac{7}{3}\,\text{s}$

e $x\left(\frac{7}{3}\right) = -\frac{25}{3}\,\text{m}$

6 a $s(0) = 5\,\text{m}$

b $v(t) = 8t - 14$, at rest when $t = \frac{7}{4}\,\text{s}$

c $s(1) = -5\,\text{m}$

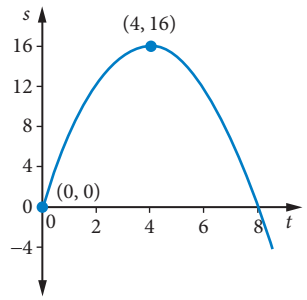
d 122.5 m

7 a $t = 0\,\text{s}$ and $t = 8\,\text{s}$

b $t = 4\,\text{s}$

c 12 cm

d



e Velocity is positive but decreasing from $t = 0$ until $t = 4$. After $t = 4$, velocity is negative and increasing.

f 20 cm

8 a $t = 2$ and $t = 6$

b 96 units

9 a $v(4) = 9\,\text{m/s}$ **b** $v(2) = -3\,\text{m/s}$

c $t = 1\,\text{s}$ and $t = 3\,\text{s}$ **d** $t < 1$ and $t > 3$

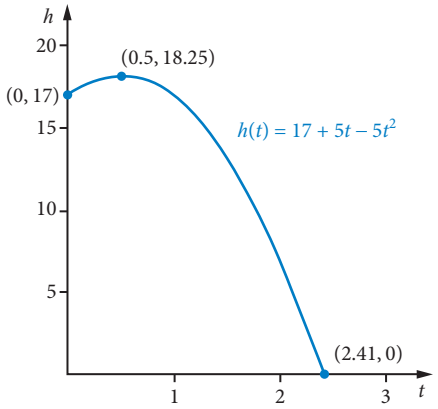
e $1 < t < 3$ **f** 120 m

10 a $h(0) = 17\,\text{m}$, $v(0) = 5\,\text{m/s}$

b $a(t) = -10\,\text{m/s}^2$

c $t = 2.41\,\text{s}$ **d** 18.25 m

e

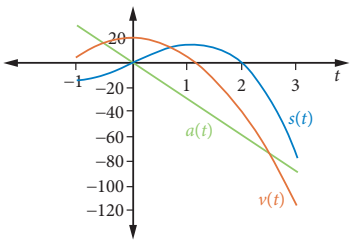


Velocity is positive but decreasing from $t = 0$ until $t = 0.5$. After this point, velocity is increasing in the negative direction.

11 a $t = 1.15\,\text{s}$ **b** $t = 0\,\text{s}$

c $t > 1.15\,\text{s}$ **d** $t < 0\,\text{s}$

e



12 a $t = 2\,\text{s}$, $x(2) = 12\,\text{cm}$

b $a(2) = -9\,\text{cm/s}^2$

c $t \approx 3.46\,\text{s}$

13 The direction changes at $t = 3\frac{1}{3}$ and $t = 12$. The particle is speeding up when $3\frac{1}{3} < t < 7\frac{2}{3}$ and $12 < t < 15$.

The particle is slowing down when $0 < t < 3\frac{1}{3}$ and $7\frac{2}{3} < t < 12$.

14 $v_A(3) = -29.4\,\text{m/s}$, while $v_B(3) = -8.4\,\text{m/s}$ at $t = 3$.

EXERCISE 10.4

1 C **2** B

3 a minimum turning point at $(2, -5)$

b maximum turning point at $(2, 10)$

c maximum turning point at $\left(-\frac{4}{3}, 16\frac{14}{27}\right)$ and minimum turning point at $(2, -2)$

d maximum turning point at $(-2, 20)$ and minimum turning point at $(1, -7)$

e no stationary points

f maximum turning point at $(-3, 27)$ and minimum turning point at $(1, -5)$

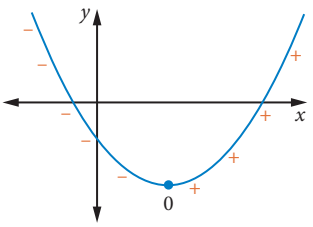
4 maximum turning point at $(-1, 3)$ and minimum turning point at $(1, -1)$

5 maximum turning point at $(2, 20)$ and minimum turning point at $(4, 16)$

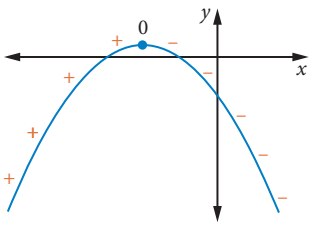
6 falling point of inflection at $(0, 0)$ and minimum turning point at $(1, -1)$

7 rising point of inflection at $(-2, 2)$

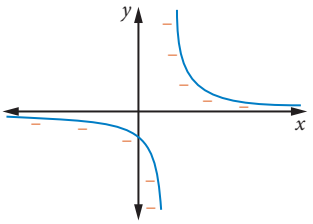
8 a



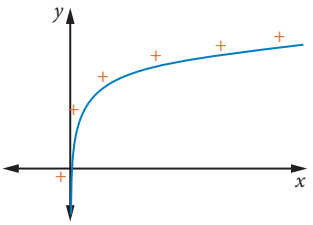
b



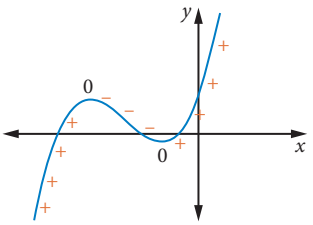
c



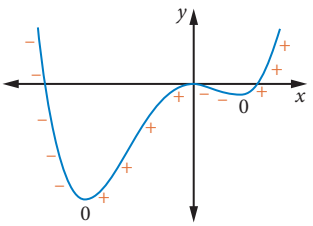
d



e



f



9 a local maximum at $x = -2$ and local minimum at $x = 3$

b local minimum at $x = -3$ and local maximum at $x = 4$

c local maximum at $x = -4$ and $x = 5$, local minimum at $x = 3$

d local minimum at $x = -4$ and $x = 3$, local maximum at $x = 1$

10 a local minimum at $(2, -7)$

b $x > 2$ **c** $x < 2$

11 a Increasing when $x < -4$ and $x > 0$, decreasing when $-4 < x < 0$.

b Increasing when $-2\frac{1}{3} < x < 1$, decreasing when $x < -2\frac{1}{3}$ and $x > 1$.

c Increasing when $0 < x < 0.25$ and $x > 2$, decreasing when $x < 0$ and $0.25 < x < 2$.

12 a gradient is negative at $x = -3$

b gradient is 0 at $x = -\frac{2}{5}$

c gradient is positive at $x = 2$

13 a gradient is positive at $x = -4$

b gradient is negative at $x = -2$

c gradient is positive at $x = 2$

14 a gradient is positive at $x = -5$

b gradient is negative at $x = -1$

c gradient is positive at $x = 3$

15 local minimum at $(-1, 0)$ local maximum at $(1, 4)$

16 falling point of inflection at $(4, 5)$

17 $k = -4$; local minimum at $(2, 3)$

18 $a = -6$ and $b = 9$; local maximum at $(1, 6)$ and local minimum at $(3, 2)$

19 $a = -2$ and $b = 3$; local minimums at $(-1, 2)$ and $(1, 2)$, local maximum at $(0, 3)$

EXERCISE 10.5

1 C **2** A **3** $(1, 12)$

4 global maximum at $(5, 65)$, global minimum at $(2, -16)$

5 *a* global minimum

b global and local maximum

c neither maximum nor minimum

d local minimum

e local maximum

f local minimum

g neither maximum nor minimum

6 *a* neither maximum nor minimum

b local minimum

c global and local maximum

d local minimum

e local maximum

f global minimum

7 global maximum: $(4, 8)$

global minimum: $(7, 0)$

local maximum: $(4, 8)$ and $(6, 6)$

local minimum $(2, 2)$ and $(5.25, 4)$

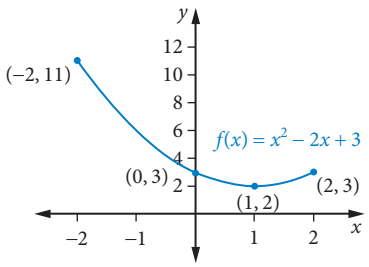
8 global maximum: $(16, 25)$

global minimum: $(4, 0)$

local maximum: $(2, 10)$, $(8, 20)$ and $(12, 15)$

local minimum $(4, 0)$, $(10, 10)$ and $(14, 10)$

9



Greatest value is 11. Least value is 2.

10 (−3, −4)

11 (2, −24)

12 global minimum at (−3, −86), global maximum at (0.78, 6.11)

13 global minimum at (2, −3), global maximum at (4, 17)

14 global minimum at (−3, −30), global maximum at (−0.12, 48.25)

15 global minimum at (1, 3), global maximum at (1.5, 10.91)

16 global minimum at (1.5, −6.69), global maximum at (−2, 27)

17 global minimum at (1.73, −10.39), global maximum at (−1.73, 10.39)

18 Greatest velocity is 27 m/s and occurs at $t = 6$.

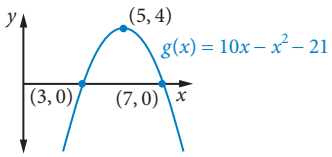
19 Greatest gradient is 1 and occurs at (1, 1).
Least gradient is −11 and occurs at (3, −5).

20 Greatest gradient is 6.75 and occurs at (2.5, −0.75).
Least gradient is −30 and occurs at (6, −20).

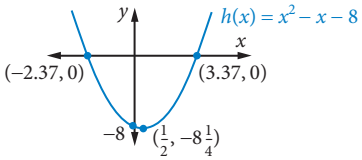
EXERCISE 10.6

1 A **2** B

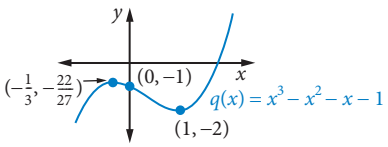
3 a



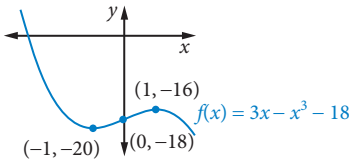
b



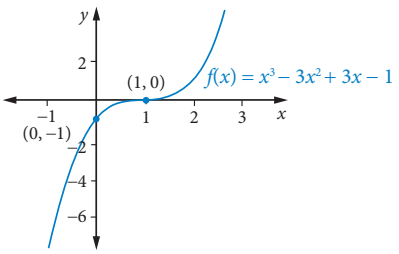
c



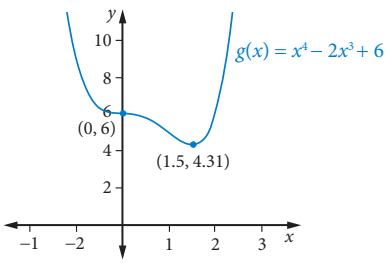
d



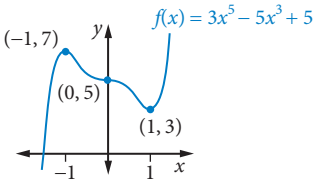
4 a



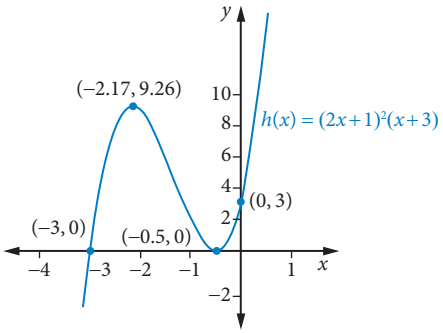
b



c



d



5 a local maximum at (−225, 115) and local minimum at (4.5, −30)

b (−4, 75) **c** (1, 50)

d decreasing when $-2.25 < x < 4.5$

e increasing when $x < -2.25$ and $x > 4.5$

f IRoC = 30

g IRoC = −33

(Note: since parts **f** and **g** are determined by drawing tangents, allow some tolerance in the solution.)

6 a local maximum at (−3, 30)and (3.5, 190), local minimum at (−0.5, −20)

b (1, 30) **c** (2, 100)

d rate of change is positive when $x < -3$ and $-0.5 < x < 3.5$

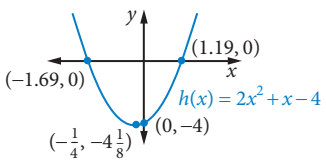
e rate of change is negative when $-3 < x < -0.5$ and $x > 3.5$

f IRoC = 66

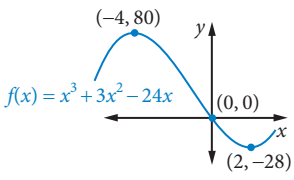
g IRoC = 75

(Note: since parts **f** and **g** are determined by drawing tangents, allow some tolerance in the solution.)

7 a

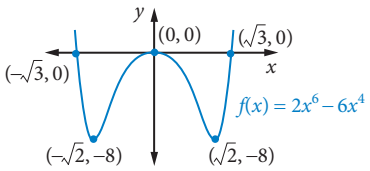


b

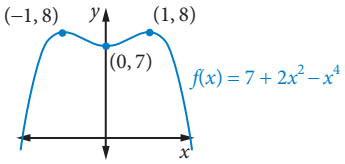


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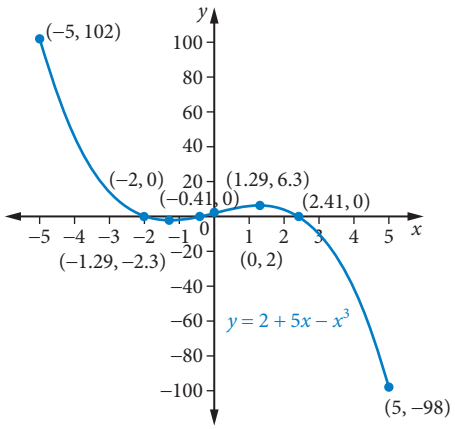
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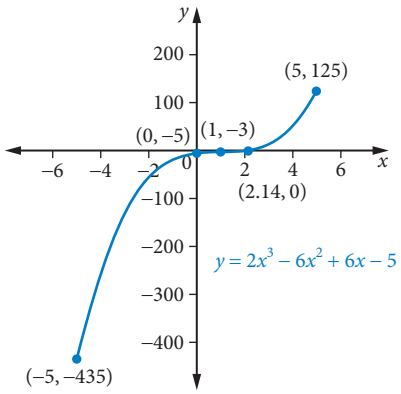
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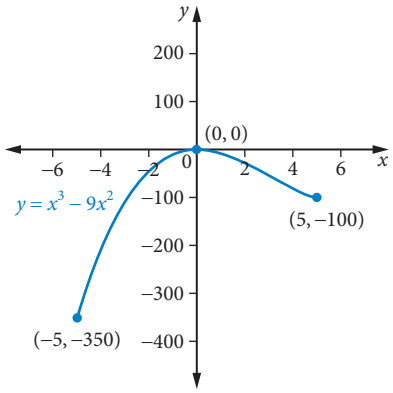
8 a



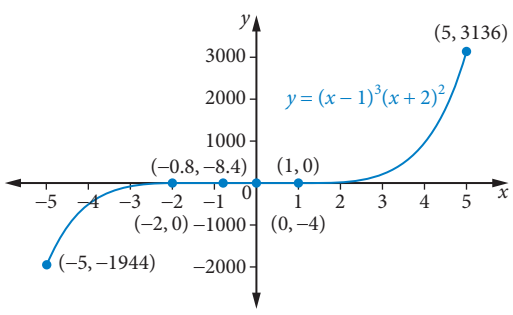
b



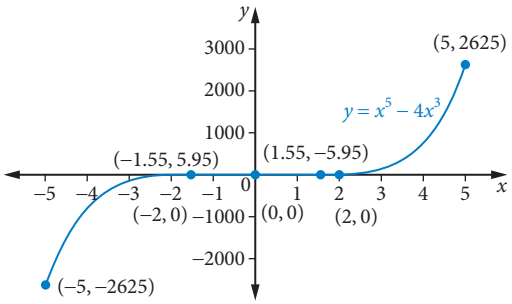
c



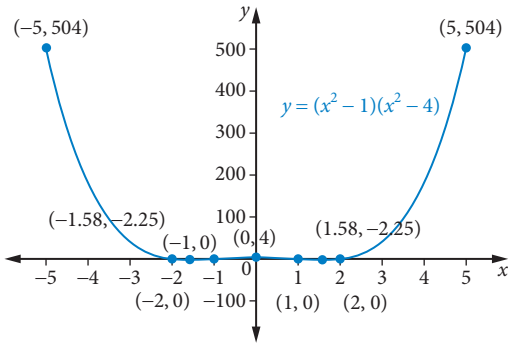
d



e



f



9 For two stationary points, they occur at $x = \dfrac{-b \pm \sqrt{b^2 - 3ac}}{3a}$. One stationary point occurs when $b^2 - 3ac = 0$, therefore it occurs at $x = -\dfrac{b}{3a}$.

CUMULATIVE EXAMINATION 1

1 A **2** B **3** D

4 $y = 7^x$ has been dilated by the factor 0.5 vertically and shifted 2 units to the left and 1 unit down.

5 $12x - 1$ **6** $\dfrac{17}{2}\sqrt{5}$ **7** (0, −0.25)

CUMULATIVE EXAMINATION 2

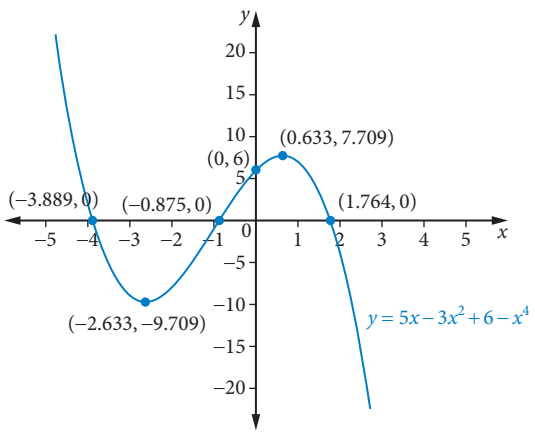
1 B **2** B **3** A

4 $\dfrac{7 + 2\sqrt{10}}{3}$

5 (−0.25, 4.39) and (1.36, −1.78)

6 $x = 0.742$

7



CHAPTER 11

EXERCISE 11.1

1 **a** 2 **b** 5 **c** −2
d −1 **e** −6 **f** 3

2 **a** −3 **b** 5 **c** 8
d −4 **e** −5 **f** 7

3 −2.5, −1.5, −1, 1, 2, 4

4 −1.25, −0.75, −0.5, 0.5, 1, 2

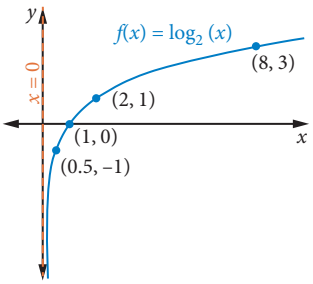
5 $\frac{1}{3}$, 2, $-\frac{4}{3}$

6 2, −1, $-\frac{3}{2}$

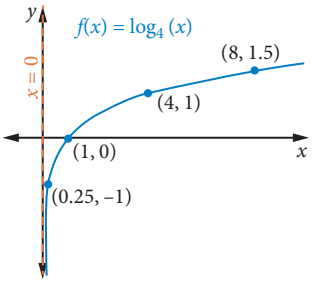
EXERCISE 11.2

1 D **2** 1.5

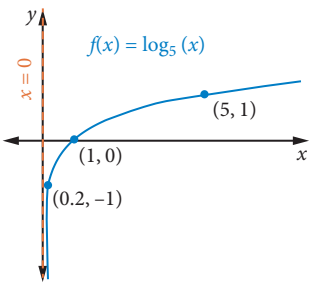
3 **a**



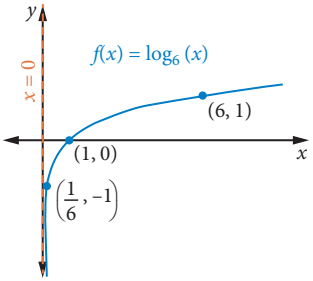
b



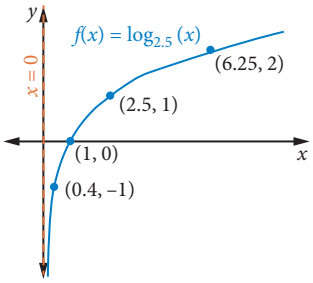
c



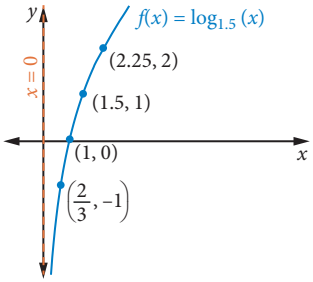
d



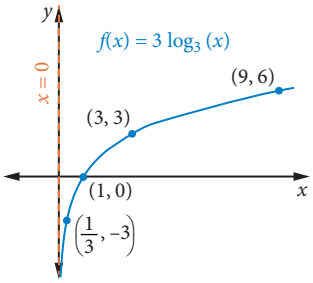
e



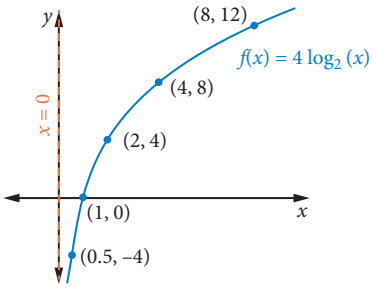
f



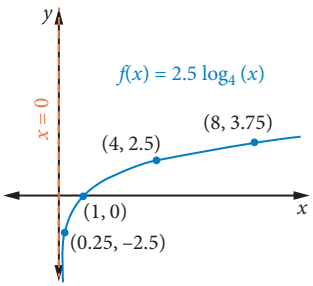
4 **a**



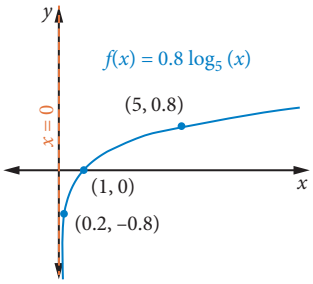
b



c

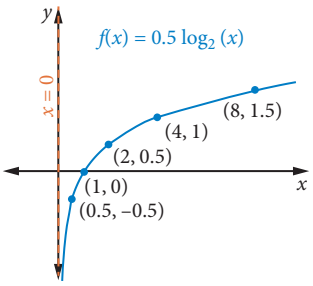


d

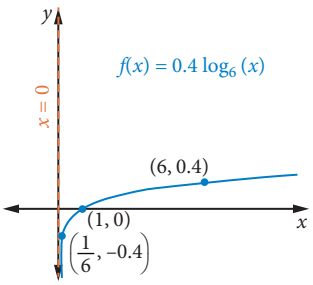


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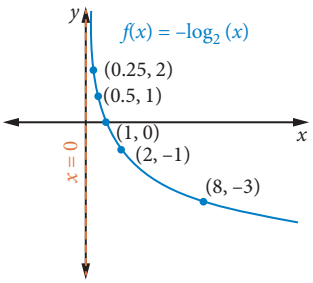
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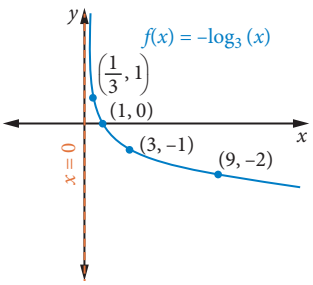
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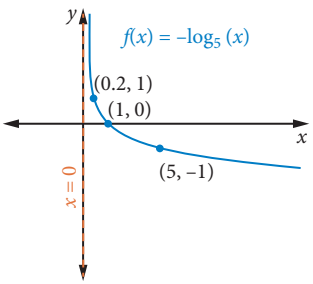
5 a



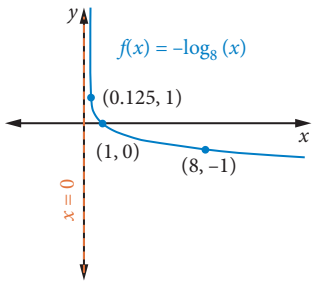
b



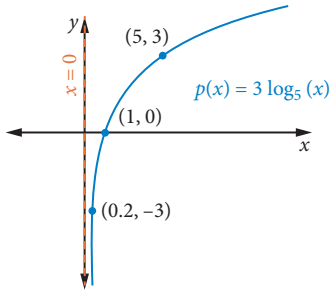
c



6



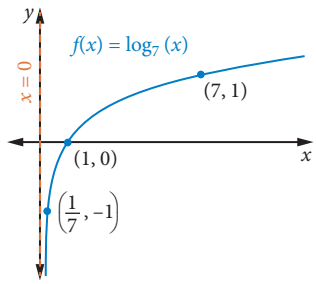
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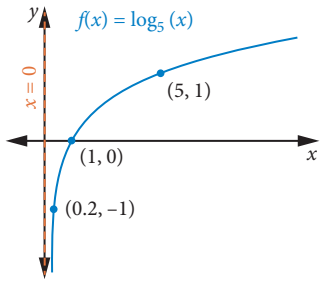
8 $x = 3$

EXERCISE 11.3

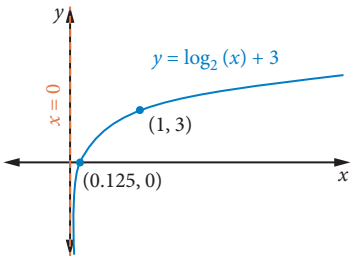
1



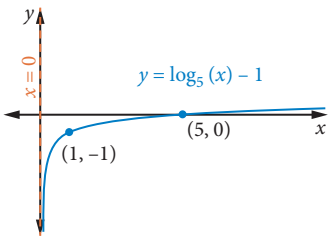
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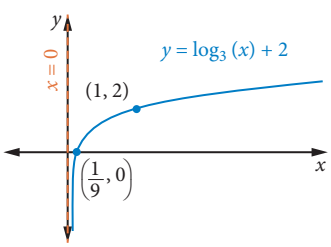
3 a



b



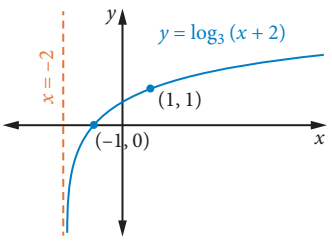
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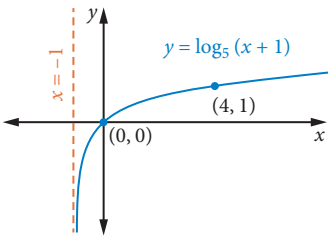
4 a $y = \log_4(x) + 5$ **b** $y = \log_{10}(x) - 3$

c $y = \log_7(x) + 2$ **d** $y = \log_6(x) - 7$

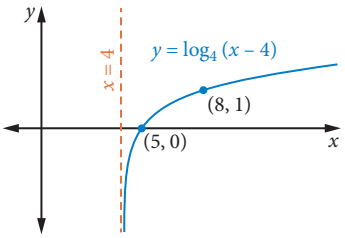
5 a



b



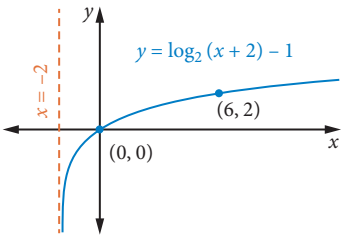
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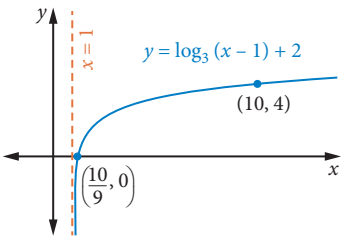
6 a $y = \log_5(x + 5)$ **b** $y = \log_{0.4}(x - 3)$

c $y = \log_{18}(x + 2)$ **d** $y = \log_3(x - 7)$

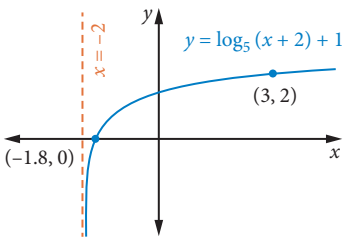
7 a



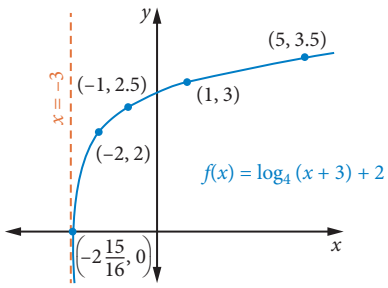
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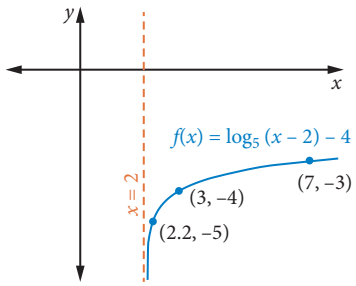
c



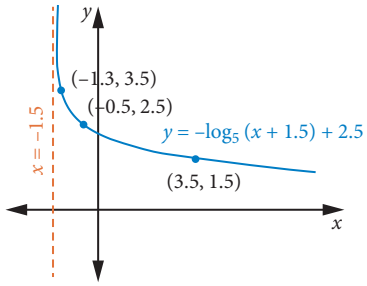
8



9



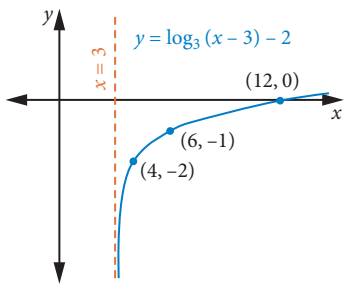
10



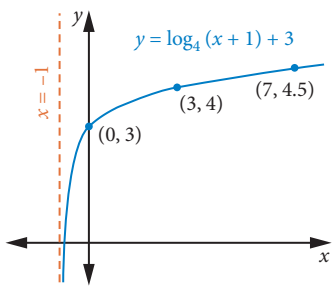
11 (1.46, 2.44) or (6.88, 4.28)

EXERCISE 11.4

1



2



3 a $x = 3$ **b** $x = 3$ **c** $x = \frac{4}{3}$

d $x = -5, 4$ **e** $x = -\frac{1}{5}$ **f** $x = -2, 4$

g $x = \frac{17}{4}$ **h** No solution **i** $x = \frac{4}{5}$

4 a $x = 5$ **b** $x = 250.25$ **c** $x = 4$

d $x = \frac{7}{2}$ **e** $x = \frac{79}{9}$ **f** $x = 9$

g $x = \frac{70}{3}$ **h** $x = 9$ **i** $x = -3$ or 9

9780170484909

5 **a** $x = \frac{37}{63}$ **b** $x = 3$ **c** No solution

d $x = \frac{33}{7}$ **e** $x = \frac{1}{2}$ **f** $x = 1$

g $x = 4$ **h** $x = \frac{6}{5}$

6 **a** $x = 12$ **b** $x = 16$ **c** $x = 3$

d $x = 3, 4$ **e** $x = \frac{4}{3}$ **f** $x = 2$

g $x = 3$ **h** $x = 3, \frac{1}{27}$ **i** $x = 1$

7 $x = \sqrt{7}$ **8** $x = \frac{9}{8}$

9 No solution **10** $x = \pm 2\sqrt{2}$

11 $x = 9$

EXERCISE 11.5

1 **a** $x = 3$ **b** $m = 2$

c $k = -2$ or 8 **d** $p = 1$

2 **a** $w = 2$ **b** $v = 9$ **c** $m = \frac{33}{7}$ **d** $x = 4$

3 2.9

4 $10^{1.2}$ or about 15.85

5 about 7 years (6.96)

6 8.9

7 16 times stronger (15.85)

8 13.01 dB

9 $10^4\,\text{W/m}^2$

10 **a** 0.016 mol/L **b** 1 mol/L **c** 10^{-3} mol/L

d 10^{-8} mol/L **e** 10^{-11} mol/L

11 **a** 631 **b** 3.98×10^{-5}

c about 400 000 times brighter (398 107)

12 **a** When $t = 0$, $Q(t) = Q_0$.

b $n \approx 0.000\,76$ **c** 0.3 kg

13 **a** 10.0 billion **b** 2075

14 24 hours

15 B **16** D **17** C

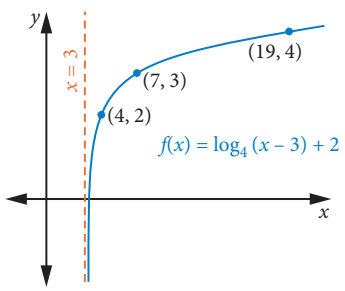
18 $KKI = 32^{R-2}$, 4.859×10^{10} tonnes

19 9985 years

CUMULATIVE EXAMINATION 1

1 C **2** D **3** C

4



5 **a** $(-1, 11), \left(\frac{2}{3}, 6\frac{10}{27}\right)$ **b** 20

6 $a = 2, b = 6, c = -1, d = 5$

7 The minimum is 7°C. It occurs at approximately 10:28 pm.

CUMULATIVE EXAMINATION 2

1 C **2** B **3** C

4 **a** no solution **b** $x \approx 8.08$

5 $4x^3 - 2$

6 P(Burger 2|no peri-peri) ≈ 0.20

7 P(Julie) ≈ 0.06, P(John) ≈ 3.06, using 1 in 100, so John's reasoning is incorrect.

CHAPTER 12

EXERCISE 12.1

1 **a** $189(3t - 6)^8$ **b** $40m(4m^2 + 1)^4$

c $72(3v - 6)^2$ **d** $-63(x^2 + 4x + 6)^6(2x + 4)$

e $48(3x + 2)^3 + 30(3x + 2)$

f $-96(3 - 4x)^3 + 40(3 - 4x) - 4$

g $90(3p + 7)^5 + 36(3p + 7)^2 - 6$

h $-6(3 - 2x)^2 + 20(3 - 2x) + 5$

2 **a** 75 **b** 16

3 C **4** D

5 $50(2x - 3)^4 + 8(2x - 3)$

6 $\frac{dM}{dt} = \frac{-3t}{\sqrt{(16 + 3t^2)^3}}$

7 $x(8) = -27.5$ m

8 $x(1.3457) = 6.1566$ m, $v(1.3457) = 3.6790$ m/s

9 $(0, -2.5)$

10 $(-\sqrt{3}, 0), (\sqrt{3}, 0)$ and $(0, 162)$

EXERCISE 12.2

1 D **2** D

3 **a** $4x^3(3x + 2) + 3x^4$

b $5(6x + 1) + 6(5x - 7)$

c $4x(5 + 2x) + 4x^2$

d $9x^2(2x^2 - 1) + 12x^4$

e $6(4x^3 + 3) + 72x^3$

f $4(2 + 5x) + 5(4x - 3)$

4 **a** $(9 + 3z^2 - z^4)(5z^4 - 3z^2 + 1) + (z^5 - z^3 + z - 1)(6z - 4z^3)$

b $3(p^2 - 2x - 4) + (3p + 4)(2p - 2)$

c $(7v^4 - 2v^3 - v^2 + 8v + 6)(15v^2 - 6v + 1) + (5v^3 - 3v^2 + v - 9)(28v^3 - 6v^2 - 2v + 8)$

d $(b^4 + 3b^2 - 6b^3 + 21)(20b^4 + 6b - 6) + (4b^5 + 3b^2 - 6b + 2)(4b^3 + 6b - 18b^2)$

e $(7u^3 - 4u^2 + 9u - 6)(3 - 16u - 18u^2) + (3u - 8u^2 - 6u^3 + 8)(21u^2 - 8u + 9)$

ANSWERS

5 **a** -2673 **b** 164

6 $(2x^2 + 5x + 2)(15x^4 - 6x) + (3x^5 - 3x^2 + 8)(4x + 5) - (x^5 - 6)(8x^3 + 15x^2) - (2x^4 + 5x^3)(5x^4)$

7 $f'(5) = -6$

8 $\left(-\frac{1}{3}, -\frac{17}{27}\right)$ and $(2, -4)$

9 $135t^8 + 424t^7 + 70t^6 - 126t^5 + 195t^4 - 212t^3 - 90t^2 - 238t + 346$
or $(6t - 2)(t^3 + 4t^2 + t - 6)(5t^4 + t^3 - 7t + 8) + (3t^2 - 2t + 5)(3t^2 + 8t + 1)(5t^4 + t^3 - 7t + 8) + (3t^2 - 2t + 5)(t^3 + 4t^2 + t - 6)(20t^3 + 3t^2 - 7)$

10 $a = 4, b = 2$

EXERCISE 12.3

1 A **2** B

3 **a** $\frac{5(3 - 5q) + 5(5q + 9)}{(3 - 5q)^2} = \frac{60}{(3 - 5q)^2}$

b $\frac{8(2m - 5) - 2(8m - 7)}{(2m - 5)^2} = \frac{-26}{(2m - 5)^2}$

c $\frac{3(2x - 5) - 2(3x + 10)}{(2x - 5)^2} = \frac{-35}{(2x - 5)^2}$

d $\frac{(t - 5) - (t + 5)}{(t - 5)^2} = \frac{-10}{(t - 5)^2}$

e $\frac{24(5t + 8) - 60(2t - 5)}{16(5t + 8)^2} = \frac{123}{4(5t + 8)^2}$

f $\frac{-25(2x - 9) - 10(12 - 5x)}{25(2x - 9)^2} = \frac{21}{5(2x - 9)^2}$

4 **a** $\frac{-4x}{(2x^2 - 3)^2}$

b $\frac{-5(2 + 6x)}{(2x + 3x^2)^2} = \frac{-10 - 30x}{(2x + 3x^2)^2}$

c $\frac{2x(x^3 - 1) - 3x^4}{(x^3 - 1)^2} = \frac{-2x - x^4}{(x^3 - 1)^2}$

d $\frac{6x(4 - x^2) + 6x^3}{(4 - x^2)^2} = \frac{24x}{(4 - x^2)^2}$

e $\frac{2x(x^2 + 1) - 2x(x^2 - 2)}{(x^2 + 1)^2} = \frac{6x}{(x^2 + 1)^2}$

f $\frac{(2x^3 + 3x)(6x - 1) - (3x^2 - x)(6x^2 + 3)}{(2x^3 + 3x)^2}$
$= \frac{-6x^4 + 4x^3 + 9x^2}{(2x^3 + 3x)^2}$

5 **a** $-\frac{4}{75}$ **b** $-\frac{100}{1089}$ **c** $-\frac{87}{676}$
d $\frac{72}{25}$ **e** $\frac{9}{50}$ **f** $-\frac{11}{147}$

6 $\frac{-3(2x + 1)}{x^4}$

7 $f''(1) = \frac{1}{2}$

8 $f'(1) = -\frac{7}{2}$

9 $x = \frac{-10 \pm 4\sqrt{5}}{5}$

EXERCISE 12.4

1 D **2** C

3 **a** $3(x + 3)(x + 5)^2 + (x + 5)^3$

b $12(3x - 1)^3(x - 5)^5 + 5(x - 5)^4(3x - 1)^4$

4 **a** $60(3x - 2)^3(x + 8)^3 + 15(x + 8)^2(3x - 2)^4$

b $12(4t + 5)^2(2t - 7)^2 + 4(2t - 7)(4t + 5)^3$

c $6(4m + 3)^8 + 192(m - 6)(4m + 3)^7$

d $8(2k + 9)^3(3k - 5)^7 + 21(2k + 9)^4(3k - 5)^6$

e $60(3g - 4)^4(5g - 4)^6 + 120(3g - 4)^5(5g - 4)^5$

f $-30(x + 7)^4(x - 6)^4 - 24(x - 6)^3(x + 7)^5$

g $45(6 - 5r)^2(2r + 9)^{11} - 66(6 - 5r)^3(2r + 9)^{10}$

h $210(8 - 6h)^6(-4 - 3h)^6 + 90(-4 - 3h)^5(8 - 6h)^7$

i $20t + 1$

j $-21(3p + 10)^6(8p + 1)^8 - 64(3p + 10)^7(8p + 1)^7$

5 **a** $\frac{2(3x - 4)(2x + 3) - 2(3x - 4)^2}{(2x + 3)^4}$

b $\frac{3c^6(63 - 20c)}{(9 - 4c)^3}$

c $-\frac{3}{(t - 6)^4} + \frac{16}{(2t + 7)^3} + \frac{6(2t + 9) - 16(3t - 5)}{7(2t + 9)^5}$

d $\frac{(20p^4 - 20p^3 + 6p^2 - 2p - 4)(5p - 1) - 30(4p^5 - 5p^4 + 2p^3 - p^2 - 4p + 2)}{(5p - 1)^7}$

6 $f'(-2) = \frac{85}{4096}$

7 $(-\sqrt{3}, 0)$, $(0, 162)$ and $(\sqrt{3}, 0)$

8 $a = \pm 2, b = \pm 1$

9 Proof: see worked solution

CUMULATIVE EXAMINATION 1

1 A **2** D **3** C

4 $\frac{4}{(x + 2)^2}$ **5** $x = -1$ or $x = 6$

6 $x = \frac{7}{4}$ and $x = \frac{11}{4}$ **7** $k = 6$

CUMULATIVE EXAMINATION 2

1 C **2** D **3** C

4 4775

5 $P(\text{all 4 correct}) = 0.16\%$

6 The Chilean earthquake was approx. 5012 times more intense than the earthquakes in Los Santos.

7 x-intercept = 2.4

Glossary and index

1 : 1 *See* **one-to-one**.

1 : *n* *See* **one-to-many**.

acceleration The rate of change or derivative of velocity with respect to time. (p. 211)

amplitude Distance from the average position to the extreme values of a trigonometric function. (p. 123)

and Mathematically, *A and B* means both *A* and *B* are true or both happen. (p. 158)

arc A portion of the circumference of a circle. (p. 108)

asymptote A function or relation characteristic: a horizontal line $y = a$ such that as $x \to \pm\infty$, $y \to a$; or a vertical line $x = a$ such that as $x \to a$, $y \to \pm\infty$. (p. 124)

average rate of change The rate of change of a function between 2 points on the function (x_1, y_1) and (x_2, y_2), given by $\frac{y_2 - y_1}{x_2 - x_1}$, which is also the gradient of the line passing through both points. (p. 212)

average speed The total distance travelled by an object divided by the elapsed time to travel that distance. (p. 211)

average velocity The displacement of an object divided by the total time for the displacement. (p. 211)

axis of symmetry A straight line such that a function or relation is symmetrical about the line. (p. 50)

base The value a in a term of the form a^n, the number or variable that has a power. (p. 243)

binomial An algebraic expression with 2 terms, such as $(x - 2)$. (p. 3)

binomial expansion Expansion of expressions of the type $(x + y)^n$ to produce $n + 1$ terms beginning with x^n. (p. 17)

binomial theorem The theorem that gives the expansion of $(x + y)^n$. (p. 19)

cardinal, cardinality The number of elements in a set. (p. 157)

certain An event that must occur; it has a probability of 1. (p. 163)

chain rule The rule for the derivative of a function of a function. (p. 350)
If $y = f(u)$ and $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.

change of base theorem The rule $\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$. (p. 281)

circle The graph of the relation $(x - h)^2 + (y - k)^2 = r^2$, where (h, k) is the centre and r is the radius. (p. 184)

coefficient A constant placed before and multiplied by a variable; for example, the 3 in $3x$. (p. 17)

combination A selection of a group of objects without regard to order. (p. 20)

common logarithm A base 10 logarithm written as $\log(x)$ or $\log_{10}(x)$ such that $y = \log(x) \Leftrightarrow 10^y = x$. (p. 275)

complement For a set or event A, the set of all elements excluding those in A; shown as $\bar{A}$ or A'. (p. 157)

concave down A curve with its inside below the curve, like $\cap$. (p. 50)

concave up A U-shaped curve with its inside above the curve. (p. 50)

conditional probability The probability of A given that B is true, shown as $P(A|B)$. (p. 166)

conjugate A binomial term with the opposite sign to the given term. The conjugate of $\sqrt{5} - 2$ is $\sqrt{5} + 2$. (p. 14)

constant A fixed number such as 2 or π. (p. 83)
See also **variable**.

continuous A function whose graph has no breaks or values at which it is undefined. (p. 71)

cosine (cos) For $0° \le \theta \le 90°$, the ratio of the adjacent side to the hypotenuse of the right-angled triangle containing θ; for any angle, the x-coordinate of the trigonometric point $P(x, y)$ with angle θ on a unit circle. (p. 109)

cost The price of an item, particularly before sale. (p. 294)
See also **marginal cost**.

cubic A polynomial of degree 3. (p. 87)

cycle One complete repetition of the pattern of a function. (p. 123)

decay factor The factor by which a quantity decreases in unit time. (p. 250)

decay function An exponential function $y = Ar^x$, where $0 < r < 1$; r is the decay factor. (p. 250)

degree (of a polynomial) The highest power of the variable in a polynomial. (p. 83)

dependent events Events A, B that are not independent; $P(A|B) \neq P(A)$. (p. 169) *See also* **conditional probability**, **independent events**.

dependent variable A variable whose value depends on that of another; the second variable of an ordered pair (x, y) of a function or relation. (p. 70) *See also* **independent variable**.

derivative Symbolised by $f'(x)$ or $\frac{dy}{dx}$, the instantaneous rate of change of a function with respect to one of its variables. (p. 230)

difference of 2 squares The rule $(x - y)(x + y) = x^2 - y^2$. (p. 4)

difference quotient $\frac{f(x + h) - f(x)}{h}$; calculates the slope of the secant through the points on the graph of $f(x)$ with x-coordinates x and $x + h$. (p. 221)

differentiate To find the derivative. (p. 350)

differentiation from first principles Differentiating a function from the basic definition and formula involving the **difference quotient** and the limit as $h \to 0$. (p. 231)

dilation A transformation that magnifies or reduces x or y values of a function or relation. (p. 50)

discontinuity A value at which a function is not defined. (p. 80)

discrete Values from a finite set of numbers. (p. 71)

discriminant Symbolised by Δ, the value under the square root symbol in the quadratic formula: $\Delta = b^2 - 4ac$. Its value determines whether the quadratic equation has 1, 2 or no solutions. (p. 41)

disjoint Sets or events with no elements in common. (p. 157)

displacement A change of position without reference to the route, measured from the starting point to the end point. (p. 211)

distributive law The identity showing how to expand brackets: $a(b + c) = ab + ac$. (p. 3)

domain The set of first (x) values of the ordered pairs (x, y) of a function or relation. (p. 70)

element A member of a set or event. (pp. 69, 157)

empty set (null set) A set with no elements. (p. 157)

event A collection of outcomes of a chance situation. (p. 158)

expected frequency The number of times an event is expected to occur in a chance situation. (p. 173)

exponent (index) The value n in a term of the form a^n. (p. 243) *See also* **power**.

exponential equation An equation in which the variable is part of an exponent (index), such as $2^x = 10$. (p. 260)

exponential function A function of the general form $y = Ar^x$, where $r > 0$. (p. 250)

factorial n ($n!$) The product $n \times (n - 1) \times (n - 2) \times \ldots \times 3 \times 2 \times 1$; with $0! = 1! = 1$. (p. 17)

favourable outcome An outcome or event whose frequency is required. (p. 173)

finite Having a definite number of elements. (p. 162)

flow rate The rate of change of volume of a fluid. (p. 211)

function A $1 : n$ relation; each x value is in at most one pair (x, y). (p. 73)

global maximum The greatest value of a function over its entire domain. (p. 312) *See also* **local maximum**.

global minimum The least value of a function over its entire domain. (p. 312) *See also* **local minimum**.

gradient The slope of a line, given by $m = \tan(\theta) = \dfrac{y_2 - y_1}{x_2 - x_1}$ for a straight line with inclination θ passing through (x_1, y_1) and (x_2, y_2). (p. 44)

gradient function The function obtained by finding the gradient at every point on the graph of a function; also called the **derivative**. (p. 230)

growth factor The factor by which a quantity increases in unit time. (p. 250)

growth function An exponential function $y = Ar^x$, where $r > 1$; r is the growth factor. (p. 250)

half-life The time taken for half a substance to disappear, especially in radioactive decay. (p. 266)

highest common factor The largest term that divides into every term of an expression. (p. 30)

horizontal asymptote An asymptote that is a horizontal line. (p. 194)

hybrid function *See* **piecewise function**.

hyperbola The shape of the function with the equation $y = \dfrac{1}{x}$. (p. 194)

hypotenuse The longest side of a right-angled triangle; the side opposite the right angle.

impossible An event that cannot occur; it has a probability of 0. (p. 163)

independent events Events A, B that do not affect each other's probabilities; $P(A|B) = P(A)$. (p. 169)

independent variable A variable whose variation does not depend on other variables; the first variable of an ordered pair. (p. 70) *See also* **dependent variable**.

index (plural: **indices**) *See* **exponent**.

initial value The coefficient A of an exponential function; the value at $x = 0$. (p. 250)

instantaneous rate of change The rate of change calculated at a particular instant, the gradient of the tangent (the derivative) at a point on the graph of a function. (p. 217)

intercept The point at which a line crosses one of the axes of a Cartesian plane. Also the value of the axis at that point. (p. 47)

intersection (sets) The set or event $A \cap B$ containing elements both A, B. (p. 157)

irrational number A real number that cannot be written as a simple fraction; one that is not rational. (p. 6) *See also* **rational number**.

interval notation A notation used to describe the domain and range of a function or solution using round and square brackets. (p. 69)

leading term The term in a polynomial which contains the highest power. (p. 83)

limit of a function The value that a function 'approaches' as the independent variable (input) approaches some value. (p. 224)

linear The relationship between variables is linear if the power of each is 1 and its graph is a straight line. (p. 44)

local maximum The greatest value of a function over an interval, particularly a short interval including the point with this value. (p. 306) *See also* **global maximum**.

local minimum The smallest value of a function over an interval, particularly a short interval including the point with this value. (p. 306) *See also* **global minimum**.

logarithm The exponent of a number expressed as a power to a particular base. (p. 275)

logarithmic scale A scale, such as the pH scale, with values that are logarithms of a variable. (p. 337)

many-to-many ($n : n$) A relation with some x values and some y values in more than one pair (x, y). (p. 71)

many-to-one ($n : 1$) A function or relation with more than one x value mapping onto the same y value; each y value in at most one pair (x, y). (p. 71) *See also* **one-to-one**.

mapping A visual representation of a relation with arrows between members of 2 sets. (p. 70)

marginal cost In economics, the cost of production of one more item. (p. 294)

monic polynomial A polynomial in which the coefficient of the leading term is 1. (pp. 32, 83)

mutually exclusive Sets or events with no elements in common. (p. 158)

$n:1$ *See* **many-to-one**.

$n:n$ *See* **many-to-many**.

normal A line perpendicular to another line or curve. (p. 299) *See also* **tangent**.

not Not A is the complement of A, the set or event with all the elements excluding those in A. (p. 158)

null factor law If the product of two or more terms is 0 then at least one of the terms must be equal to 0. If $xy = 0$, then $x = 0$ or $y = 0$ or $x = y = 0$. (p. 36)

null set *See* **empty set**.

one-to-many ($1:n$) A relation with each x value mapping onto more than one y value; e.g. $(4, 2)$, $(4, -2)$. (p. 71) *See also* **many-to-one**.

one-to-one ($1:1$) A function or relation with each x value having a unique y value (x, y). (p. 71)

or Mathematically, *A or B* means at least one of A and B are true or happen. (p. 158)

ordered pair A pair of numbers in brackets; (x, y). (p. 70)

outcome An individual result of a chance situation. (p. 158)

parabola The shape of the function $y = x^2$. (p. 35)

Pascal's triangle Triangular arrangement in rows of numbers of combinations of n objects. (p. 16)

perfect square A binomial expression of the form $(x + a)^2$. (pp. 4, 31)

period The interval of the independent variable (often time) needed to complete one cycle of a periodic function. (p. 123)

periodic function A function that repeats the same values after a constant interval called the period. (p. 123)

phase shift The measure of how far a periodic function is horizontally to the right of the usual position. (p. 129)

piecewise function A function with different rules for different parts of the domain. (p. 79)

point estimate An estimate of the probability of the value of a random variable being in a particular value. (p. 173)

point of horizontal inflection A point on a curve that changes between concave up and concave down and has a horizontal tangent; point where the derivative is zero but there is no change of sign of the derivative. (p. 306)

point of inflection Point where a curve changes between concavity upwards and downwards. (p. 87)

polynomial An expression of the form $P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2x^2 + a_1x + a_0$, where a_n, a_{n-1}, $a_{n-2} \ldots a_2$, a_1, a_0 are constants and $n \neq 0$. (p. 83)

power A repeated multiplication of the form $a^n = a \times a \times a \times \ldots$ to n terms, with base a and exponent (index) n. (p. 243)

power rule The rule that states $\frac{dy}{dx}(x^n) = nx^{n-1}$. (p. 233)

probability A number from 0 to 1 showing the mathematical chance that an event will occur. (p. 162)

product rule The rule for the derivative of the product of 2 functions. (p. 353)

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

proper subset A subset of A that excludes at least one element of A, shown as $B \subset A$. (p. 157)

quadratic A polynomial in which the leading term has a power of 2. $P(x) = ax^2 + bx + c$, where a, b and c are constants and $a \neq 0$, is a quadratic. (p. 31)

quadratic formula The formula for the solutions of a quadratic equation; given by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, for $ax^2 + bx + c = 0$. (p. 40)

quadratic function A function of the form $f(x) = ax^2 + bx + c$; a polynomial of degree 2, whose graph is a parabola. (p. 50)

quadratic relation A relation of the form $x = ay^2 + by + c$, whose graph is a 'sideways' parabola (p. 189)

quotient rule The rule for the derivative of the quotient of two functions. (p. 355)

$$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

radian The angle measure whose size is equal to the angle at the centre of a circle opposite an arc equal in length to the radius of the circle (p. 108)

random (at random) having no pattern; outcomes equally likely to occur; determined only by chance. (p. 162)

range The set of values of the dependent variable; the y values of the ordered pairs (x, y) of a function or relation. (p. 70)

rate The quotient of 2 quantities, with units separated by / in the order of division. (p. 209)

rate of change The rate of change of y with respect to x is the value $\frac{y_2 - y_1}{x_2 - x_1}$. (p. 209) *See also* **average rate of change, instantaneous rate of change.**

rationalise the denominator Make a denominator that contains a surd rational. (p. 13)

rational number A number that can be expressed as a fraction $\frac{a}{b}$ where a and b are integers with $b \neq 0$. (p. 6)

reference angle For a given angle in the unit circle, the related acute angle in the first quadrant. (p. 118)

reflection A transformation that reflects the graph of a given function across a line; $-f(x)$ across the x-axis (vertical) or $f(-x)$ across the y-axis (horizontal). (p. 126)

relation A set of ordered pairs (x, y); shown by a list, rule, mapping or part of a Cartesian plane. (p. 70)

relative frequency The proportion of occurrence of a data item, given by $\frac{\text{frequency}}{\text{total frequency}}$. (p. 173)

replacement (in probability) Replacing objects before a second or subsequent selection is made. (p. 162)

Richter scale A logarithmic scale of the strength of an earthquake. (p. 337)

root A solution to an equation of the form $f(x) = 0$. (p. 35)

sample space The collection of all possible outcomes in a chance situation. (p. 158)

scientific notation A number written in the form $b \times 10^n$ where $1 \leq b < 10$ and n is an integer; for example, 3.1×10^{-2}. (p. 247)

secant A straight line that passes through a curve. (p. 212)

sector Part of a circle that lies between two radii and the circumference. (p. 114)

set A well-defined collection of objects with a common attribute: you can tell whether or not an object is in the set. (pp. 69, 157)

significant figures The digits in a number that show how accurately the number has been written; for example, 38 000 has two significant figures. (p. 248)

simultaneous equations A set of equations in the same variables. (p. 97)

sine (sin) For $0° \leq \theta \leq 90°$: the ratio of the opposite side to the hypotenuse of the right-angled triangle containing θ; for all angles: the y-coordinate of the trigonometric point $P(x, y)$ with angle θ on a unit circle. (p. 109)

sinusoidal A function that has the general shape of the sine or cosine function. (p. 142)

slope (of a curve) The slope of the tangent line drawn at that point. (p. 217)

slope *See* **gradient**.

speed The rate of change of distance. (p. 211) *See also* **velocity**.

square root function The function with the form $f(x) = \sqrt{x - h} + k$. (p. 192)

standard triangles A right-angled triangle with a unit side, used to find exact trigonometric values of 30°, 45° or 60°. (p. 110)

stationary point A point where the gradient of a function is zero; a point where the derivative is zero. (p. 90)

subject (of a formula) The single variable on the left-hand side of a formula. (p 186)

subset All the elements of a subset B of A are also in A; shown as $B \subseteq A$. (p. 157)

surd A root that can't be simplified to a rational number. For example, $\sqrt{5}$ is a surd, whereas $\sqrt{9}$ can be simplified to 3, so it is not a surd. (p. 6)

tangent (ratio, **tan**) For $0° \leq \theta \leq 90°$: the ratio of the opposite side to the adjacent of the right-angled triangle containing θ; for all angles: the quotient of the y-coordinate and x-coordinate of the trigonometric point $P(x, y)$ with angle θ on a unit circle. (p. 109)

tangent (to a curve) A straight line drawn so that it just touches a curve. (pp. 217, 297) *See also* **normal**.

term Part of an algebraic expression separated by addition and/or subtraction. (p. 30)

translation A transformation that moves the graph of a given function. (p. 50)

trials The number of items in a data collection is called the number of trials. (p. 173)

trigonometric function A function of an angle expressed as a ratio of 2 of the sides of a right-angled triangle containing the angle, or by coordinates on a unit circle. (p. 123)

turning point Point where a curve changes between rising and falling; maximum or minimum point (p. 50)

union The set or event $A \cup B$ containing elements in at least one of A or B. (p. 157)

unit circle A circle of radius 1, particularly used for trigonometric ratios. (p. 117)

universal set The set U containing all possible elements of sets in a particular situation. (p. 157)

variable A value in an equation or formula, represented by a letter which can vary. (p. 17)

velocity The rate of change of displacement, the derivative of displacement. (p. 211) *See also* **speed**.

Venn diagram A diagram showing sets or events as circles in a rectangle representing the universal set or sample space. (p. 158)

vertex Turning point of a curve. (p. 189)

vertical asymptote An asymptote that is a vertical line. (p. 194)

vertical line test A function is intersected in only one point by any vertical line. (p. 74)

***y*-intercept** Point where the curve of a function or relation crosses the y-axis; the distance of this point from the origin. (p. 44)

zero An x value of a function $f(x)$ such that $f(x) = 0$; also the x-intercepts of the graphed function. (p. 50) *See also* **root**.

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