LANGSFORD WILLING | BURROWS

**JACARANDA**
**MATHS QUEST**

# MATHEMATICAL METHODS 12
## FOR QUEENSLAND

**UNITS 3 & 4** | SECOND EDITION

$y = \cos x$
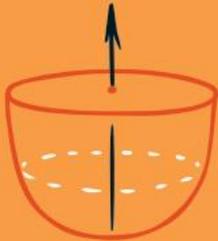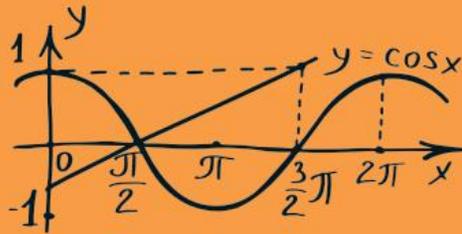
$\int e^x dx = e^x + c$
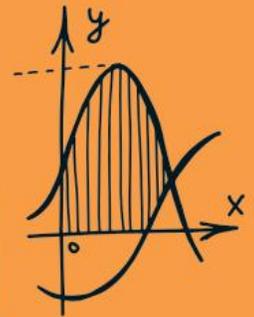
$\tan 60° = \sqrt{3}$

$\dfrac{d}{dx}\cos(x) = -\sin(x)$

$\dfrac{d}{dx}\ln(x) = \dfrac{1}{x}$

$\dfrac{\sin x}{\cos x}$

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# MATHS QUEST

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### UNITS 3 & 4 | SECOND EDITION

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# MATHS QUEST

# MATHEMATICAL METHODS 12
## FOR QUEENSLAND
### UNITS 3 & 4 | SECOND EDITION

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The publisher of this series acknowledges and pays its respects to Aboriginal Peoples and Torres Strait Islander Peoples as the traditional custodians of the land on which this resource was produced.

This suite of resources may include references to (including names, images, footage or voices of) people of Aboriginal and/or Torres Strait Islander heritage who are deceased. These images and references have been included to help Australian students from all cultural backgrounds develop a better understanding of Aboriginal and Torres Strait Islander Peoples' history, culture and lived experience.

It is strongly recommended that teachers examine resources on topics related to Aboriginal and/or Torres Strait Islander Cultures and Peoples to assess their suitability for their own specific class and school context. It is also recommended that teachers know and follow the guidelines laid down by the relevant educational authorities and local Elders or community advisors regarding content about all First Nations Peoples.

All activities in this resource have been written with the safety of both teacher and student in mind. Some, however, involve physical activity or the use of equipment or tools. **All due care should be taken when performing such activities**. To the maximum extent permitted by law, the authors and publisher disclaim all responsibility and liability for any injury or loss that may be sustained when completing activities described in this resource.

The publisher acknowledges ongoing discussions related to gender-based population data. At the time of publishing, there was insufficient data available to allow for the meaningful analysis of trends and patterns to broaden our discussion of demographics beyond male and female gender identification.

# Contents

**online only**

**Problem-solving and modelling task guide**

---

**UNIT 3** FURTHER CALCULUS AND INTRODUCTION TO STATISTICS     1

**on line only**

PRACTICE ASSESSMENT 1
**Problem-solving and modelling task**

PRACTICE ASSESSMENT 2
**Unit 3 Examination**

**online only**

PRACTICE ASSESSMENT 3
**Unit 4 Examination**

PRACTICE ASSESSMENT 4
**Units 3 & 4 Examination**

Glossary ........................................................................................................................... **online only**

# Learning with learnON
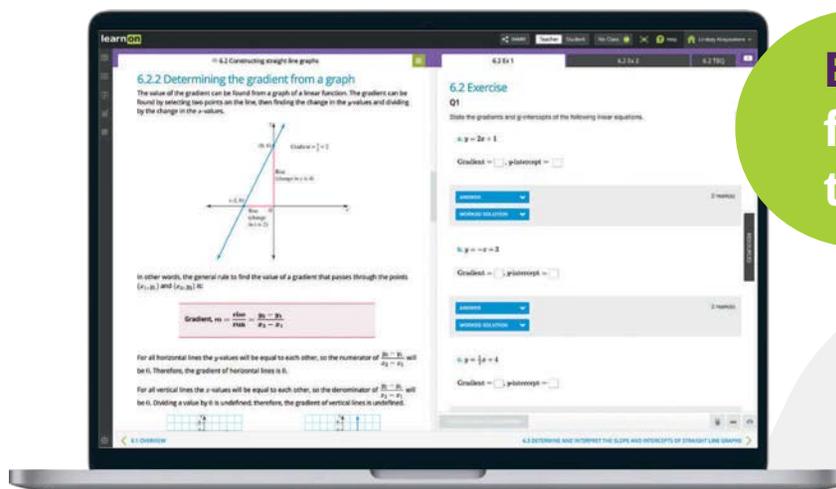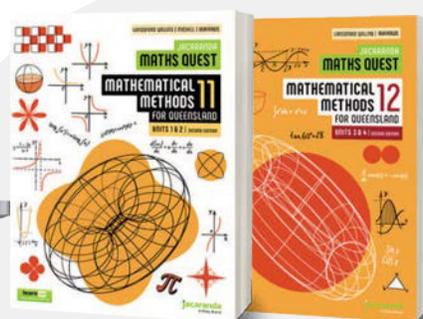
## JACARANDA MATHS QUEST
# MATHEMATICAL METHODS 12 UNITS 3 AND 4 FOR QUEENSLAND SECOND EDITION

## Developed by expert teachers for students

Tried, tested and trusted. The completely revised and updated second edition of *Jacaranda Maths Quest 12 Mathematical Methods Units 3 & 4 for Queensland* continues to focus on helping teachers achieve learning success for every student — ensuring no student is left behind and no student is held back.

**Because both what and how students learn matter**

### Learning is personal

Whether students need a challenge or a helping hand, you'll find what you need to create engaging lessons.
Whether in class or at home, students can get unstuck and progress! Scaffolded lessons, with detailed worked examples, including both TI and Casio calculator support. Differentiated question sets are all supported by detailed worked solutions. And brand-new exam-style questions and Past QCAA exam questions to support in-depth skill acquisition.

### Learning is effortful

Learning happens when students push themselves. With learnON, Australia's most powerful online learning platform, students can challenge themselves, build confidence and ultimately achieve success.

### Learning is rewarding

Through real-time results data, students can track and monitor their own progress and easily identify areas of strength and weakness.

And for teachers, Learning Analytics provide valuable insights to support student growth and drive informed intervention strategies.

# Learn online with Australia's most

- Trusted, syllabus-aligned theory
- Engaging, rich multimedia
- All the teacher support resources you need
- Deep insights into progress
- Immediate feedback for students
- Create custom assignments in just a few clicks.

Practical teaching advice and ideas for each lesson provided in teachON

Each lesson linked to content points from the QCAA Mathematical Methods 2025 General senior syllabus

Reading content and rich media including embedded interactivities and calculator support.

learn**on**

1.3 Wages  |  1.3 teachON  |  1.3 Ex 1

## LESSON 1.3
## Wages

### SYLLABUS LINKS

- Calculate weekly, fortnightly or monthly wages from an annual salary.

*Source:* QCE General Mathematics Syllabus extracts © QCAA; reproduced by permission.

Employees may be paid for their work in a variety of ways. Most receive either a wage or a salary.

Ways of paying employees

**Salary**
A fixed amount of money paid per year (annually), usually paid fortnightly or monthly regardless of the number of hours worked.

**Wages**
A fixed amount of money per hour worked.

It is usually based on an annual amount divided into weekly, fortnightly or monthly payments. There is no extra pay for hours worked outside the normal work period. There are benefits such as sick and annual leaves.

The weekly wage is the hourly rate multiplied by the hours worked. Hours worked outside the normal work period are paid at a higher rate. There are no additional payments such as sick or annual leaves.

Examples of salaried jobs include: architect, company director, data analyst, teacher, doctor, accountant, federal or state government minister.

Examples of waged jobs include: waiter, kitchen hand, bar attendant, receptionist, technician, retail assistant, fruit picker or packer.

## 1.3 Exercise

SELECT YOUR

ALL  |  SIM

1, 2,

### Simple fam

**Q1**

Nga is paid a sa

$ ☐

ANSWER

WORKED SOLUTI

**Q2**

Roger is paid a

$ ☐

ANSWER

WORKED SOLUTI

# powerful learning tool, learnON

Differentiated question sets

Practice exam questions

Teacher and student views

Textbook questions

Fully worked solutions

Digital documents

Interactivities

Exam question booklet

Enhanced teaching support resources

Interactive questions with immediate feedback including answers and fully worked solutions

SHARE  Teacher  Student  No Class  Help  Teacher diacriTech

1.3 Ex 2  1.3 TBQ  1

RESOURCES

Topic PDF  1

Solutions  1

Digital documents  7

Interactivities  3

Exam question booklet  1

TEACHER Digital documents  5

PATHWAY

PLE FAMILIAR  COMPLEX FAMILIAR  COMPLEX UNFAMILIAR

uestions:  Questions:  Questions:
3, 4, 5, 6, 7, 8  9, 10, 11  12, 13, 14

liar

ary of $67 860 per annum. Calculate Nga's fortnightly pay.

1 mark(s)

alary of $86 840 per annum. Calculate Roger's weekly pay.

1 mark(s)

# Online, these new editions are the complete package

**Trusted Jacaranda theory, plus tools to support teaching and make learning more engaging, personalised and visible.**

Learning matrix to monitor student's confidence level throughout topics.

Each topic is linked to content points from the QCAA Mathematical Methods 2025 General senior syllabus.
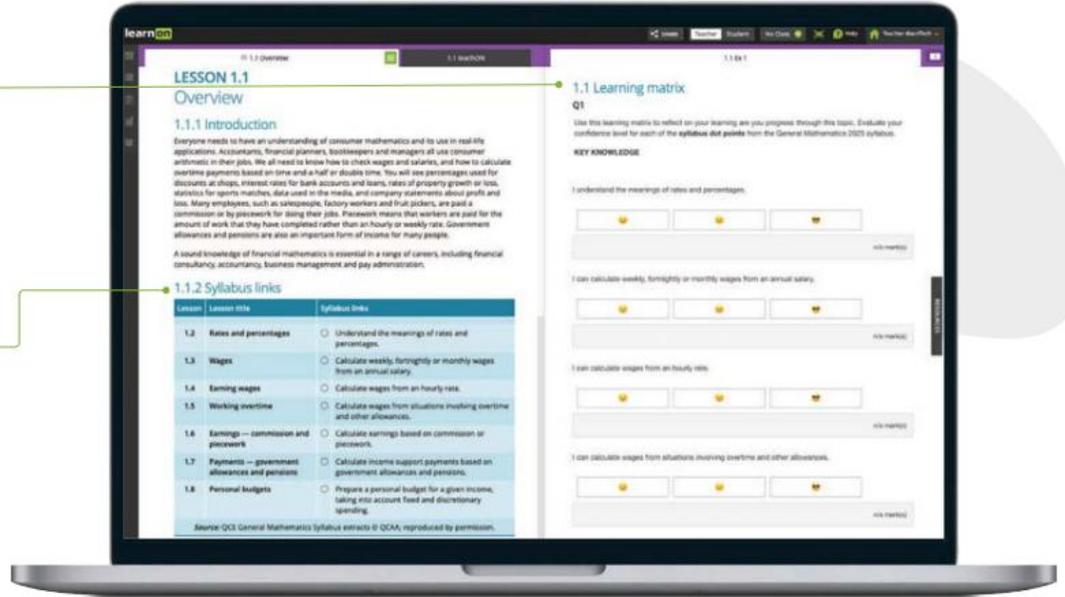
onResources link to targeted digital resources including fully worked solutions, interactivities and exam question booklets.

Interactive glossary terms help develop and support mathematical literacy.

Tables, flow charts and images break down content, allowing students to understand complex concepts.
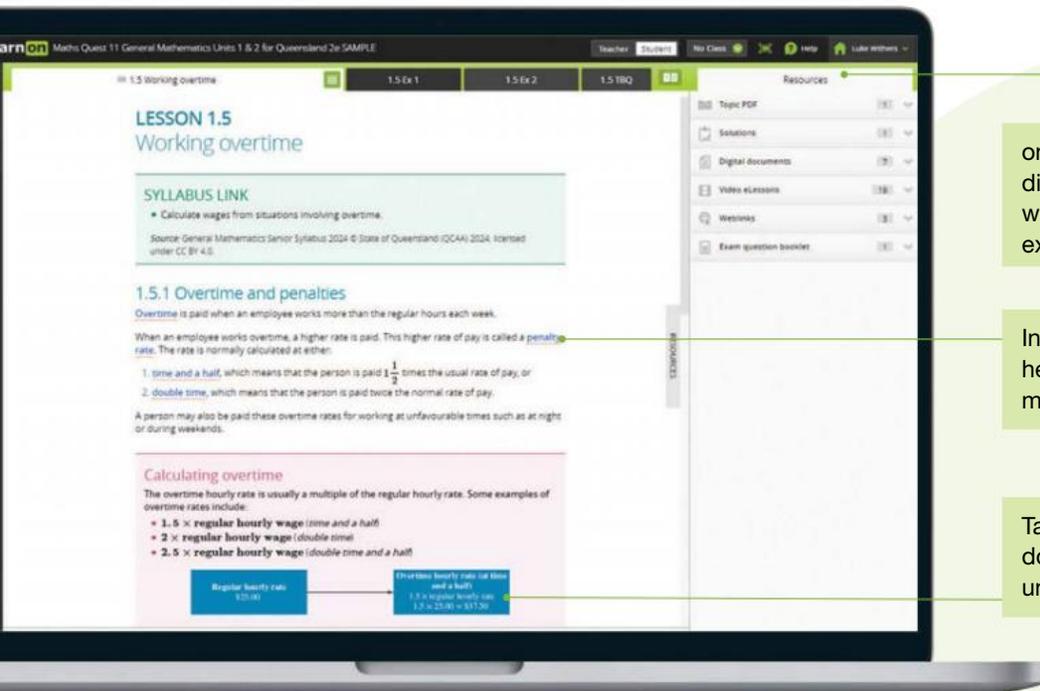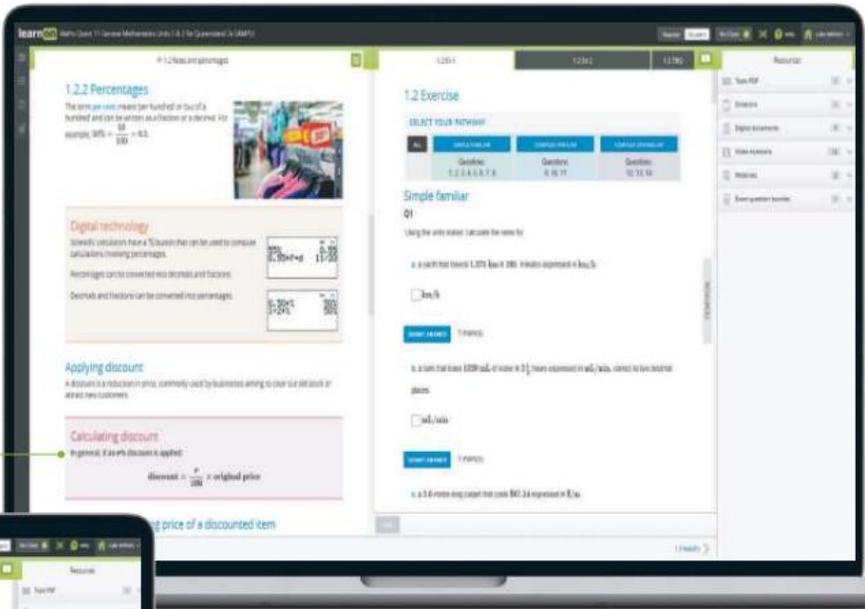
Pink highlight boxes summarise key information.
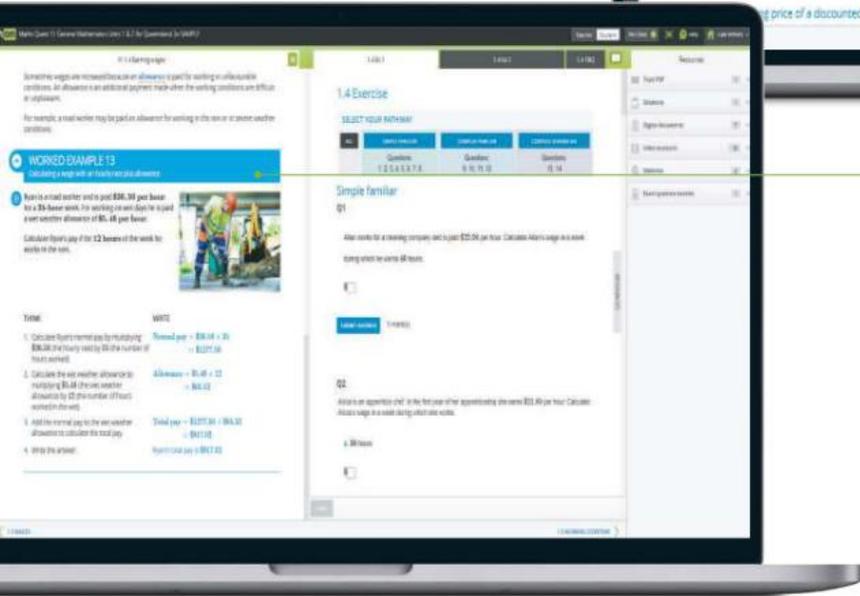
Worked examples break down the process of answering questions using a think/write format, including calculator support.
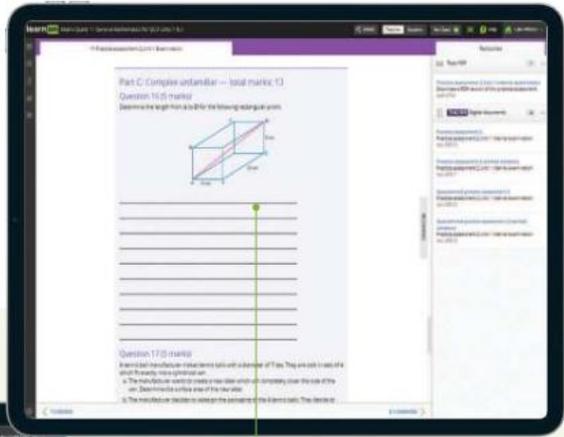
Each unit has practice assessments (Problem-solving and modelling task or examination) with sample responses to help build skills.
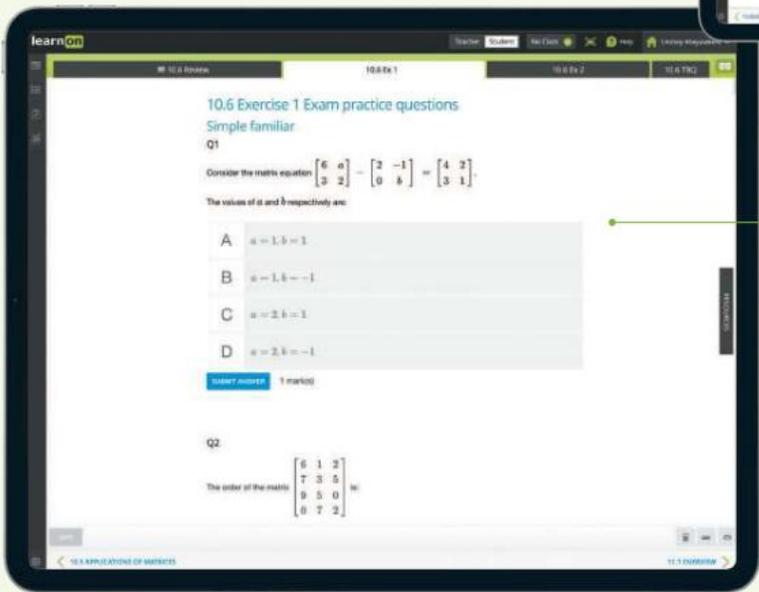
- Online and offline question sets contain practice questions and exam-style questions with exemplary responses and marking guides.
- Every question has immediate, corrective feedback to help students to overcome misconceptions as they occur and to study independently — in class and at home.

# The new jacTUTOR

## An AI tutor to build confidence and get unstuck

With jacTUTOR, every student can have the help of a personal tutor when they get stuck, in class or at home. This purpose-built tool sits safely within your favourite Jacaranda resource, so guidance will always be specific to that lesson, question and curriculum.

### A personal tutor for every student

Now every student can get the help they need, when they need it. jacTUTOR uses AI to create a fair and level playing field for all students.

### Get guidance, not the answer

jacTUTOR doesn't just give the answer away. Students are given prompts to help understand what they should be asking to get to a correct answer.

### Combat anxiety

Afraid to raise a hand or ask more questions? With jacTUTOR, students will find a safe space to ask questions, get clarification and try again.

### Consistency over confusion

There are so many ways to learn. To avoid confusion, jacTUTOR is built on the same trusted method used in the Jacaranda text.

### A safe space

To keep students safe, any concerning or inappropriate comments are automatically flagged and sent to their teacher.

# Teaching with learnON



learn on
one resource

Plan · Teach · Learn · Assess · Analyse

CLASS SET-UP · INTEGRATIONS · ONGOING SUPPORT

**Enhanced teacher support resources, including:**
- work programs and syllabus grids
- teaching advice and additional activities
- quarantined topic tests (with solutions)
- Quarantined PSMTs and examinations
  Custom exam-builder with question differentiation (SF/CF/CU) question filters

## Customise and assign

A testmaker enables you to create custom tests from the complete bank of thousands of questions (including past QCAA exam questions in year 12).
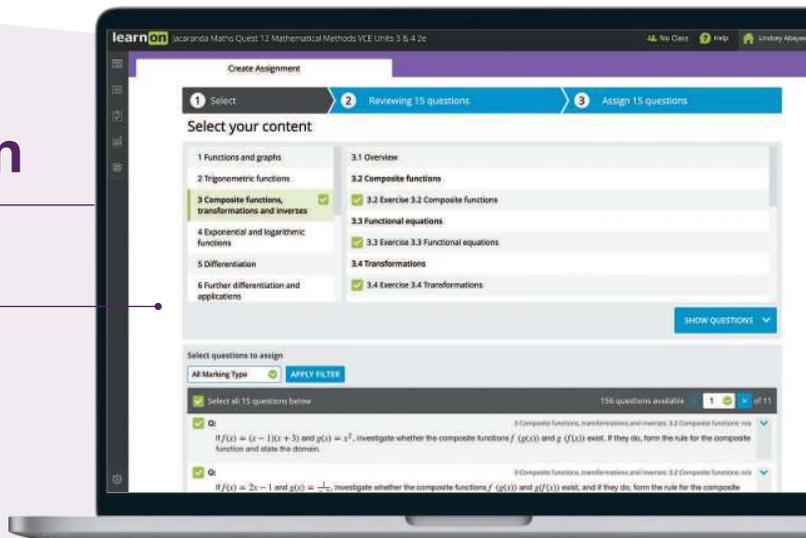


## Reports and results

Data analytics and instant reports provide data-driven insights into performance across the entire course.

Show students (and their parents or carers) their own assessment data in fine detail. You can filter their results to identify areas of strength and weakness.
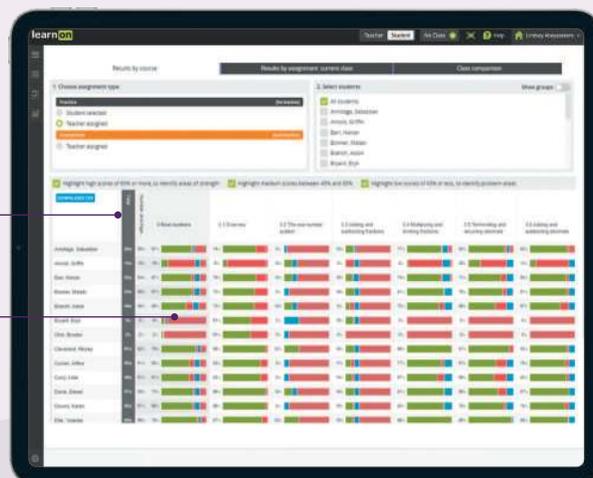
# Acknowledgements

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**The full list of acknowledgements can be found here:** www.jacaranda.com.au/acknowledgements/#2025

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# UNIT 3 Further calculus and introduction to statistics

# 1

# Calculus of exponential and logarithmic functions

## LESSON SEQUENCE

**Fully worked solutions for this chapter are available online.**

### EXAM PREPARATION
Access exam-style questions in every lesson, available online and access past QCAA exam questions in every review.

**on** Resources

| | | |
|---|---|---|
| **Solutions** | Solutions — Chapter 1 | |
| **Exam questions** | Exam question booklet — Chapter 1 | |
| **Digital documents** | Learning matrix — Chapter 1 | |
| | Chapter summary — Chapter 1 | |

# LESSON
## 1.1 Overview

### 1.1.1 Introduction

Calculus fundamentals include rates of change and limits. Exponential functions model many natural processes, such as investment growth, bacterial population increase and radioactive decay. The logarithmic function, the inverse of the exponential function, has practical applications such as data compression, earthquake magnitude measurement (Richter scale), artifact dating and determining radioactive decay in nuclear medicine.

Exponential and logarithmic functions are widely used in science, engineering and commerce.

### 1.1.2 Syllabus links

| Lesson | Lesson title | Syllabus links |
|--------|--------------|----------------|
| 1.2 | Limits | ○ Estimate the limit of $\dfrac{a^h - 1}{h}$ as $h \to 0$, using technology, for various values of $a > 0$. <br> ○ Recognise that $e$ is the unique number $a$ for which the above limit is $1$. |
| 1.3 | The exponential function | ○ Recognise and determine the qualitative features of the graph of $y = e^x$, including asymptote and intercept. |
| 1.4 | Differentiation of exponential functions | ○ Use the rules $\dfrac{d}{dx}e^x = e^x$ and $\dfrac{d}{dx}e^{f(x)} = f'(x)e^{f(x)}$. |
| 1.5 | The logarithmic function | ○ Recognise and determine the qualitative features of the graph of $y = \ln(x) = \log_e(x)$, including asymptote and intercept. <br> ○ Recognise and use the inverse relationship of the functions $y = e^x$ and $y = \ln(x)$. |
| 1.6 | Solving equations | ○ Solve equations involving exponential and logarithmic functions with base $e$, with and without technology. |
| 1.7 | Differentiation of logarithmic functions | ○ Use the rules $\dfrac{d}{dx}\ln(x) = \dfrac{1}{x}$ and $\dfrac{d}{dx}\ln\big(f(x)\big) = \dfrac{f'(x)}{f(x)}$. |
| 1.8 | Modelling and solving with exponential and logarithmic functions | ○ Model and solve problems that involve derivatives of exponential and logarithmic functions with and without technology. |

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# LESSON
## 1.2 Limits

## 1.2.1 Review of limits

The limit of a function, $y = f(x)$, is the value that the function approaches as $x$ approaches a given value.

Consider the limit of $f(x) = x + 1$ as approaches 1, using a spreadsheet.

| $x$ | $x + 1$ |
| --- | --- |
| 0.9 | 1.9 |
| 0.99 | 1.99 |
| 0.999 | 1.999 |
| 0.9999 | 1.9999 |

| $x$ | $x + 1$ |
| --- | --- |
| 1.1 | 2.1 |
| 1.01 | 2.01 |
| 1.001 | 2.001 |
| 1.0001 | 2.0001 |

As $x$ approaches 1 from the left-hand side, or below, the function approaches 2.

As $x$ approaches 1 from the right-hand side, or above, the function approaches 2.

As both are equal, the limit exists and is written as:

$$\lim_{x \to 1}(x + 1) = 2$$

Since the function $f(x) = x + 1$ is continuous, the limit can be found by direct substitution.

Consider a different function, $g(x) = \dfrac{x^2 - 1}{x - 1}$.

This function is undefined at $x = 1$.

However, $g(x)$ can be simplified to $g(x) = \dfrac{(x - 1)(x + 1)}{(x - 1)}$.

$$g(x) = (x + 1), x \neq 1$$

As shown above, since $\lim_{x \to 1}(x + 1) = 2$, then

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2 \text{ or } \lim_{x \to 1} g(x) = 2.$$



The graph of $y = g(x)$ is in fact a linear function with a point discontinuity at $(1, 2)$, as shown.
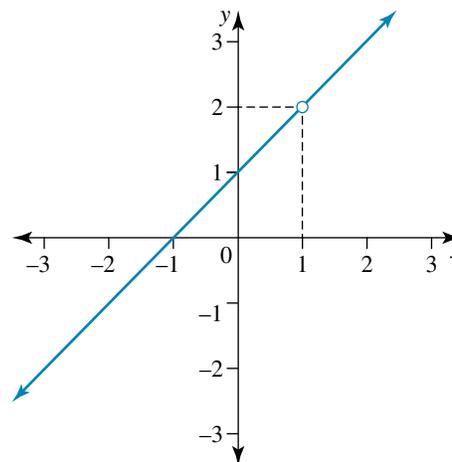
## WORKED EXAMPLE 1 Determining limits

**Determine the following limits.**

a. $\lim_{h \to 4}(3h - 5)$

b. $\lim_{x \to 0} \dfrac{x^2 + 5x + 6}{x + 2}$

c. $\lim_{x \to -2} \dfrac{x^2 + 5x + 6}{x + 2}$

**THINK**

**WRITE**

a. 1. Substitute $h = 4$ as the function is defined.

a. $\lim_{h \to 4}(3h - 5)$
$= 3 \times 4 - 5$
$= 7$

2. Answer the question.

$\lim_{h \to 4}(3h - 5) = 7$

b. 1. Substitute $x = 0$ as the function exists for this value.

b. $\lim_{x \to 0} \dfrac{x^2 + 5x + 6}{x + 2}$
$= \dfrac{6}{2}$
$= 3$

2. Answer the question.

$\lim_{x \to 0} \dfrac{x^2 + 5x + 6}{x + 2} = 3$

c. 1. The function is undefined at $x = -2$. Factorise the numerator and simplify.

c. $\lim_{x \to -2} \dfrac{x^2 + 5x + 6}{x + 2}$
$= \lim_{x \to -2} \dfrac{(x + 2)(x + 3)}{(x + 2)}$
$= \lim_{x \to -2}(x + 3)$

2. Substitute $x = -2$.

$= -2 + 3$
$= 1$

3. Answer the question.

$\lim_{x \to -2} \dfrac{x^2 + 5x + 6}{x + 2} = 1$

## 1.2.2 The derivative as a limit

In Unit 2, you were introduced to differentiation, the process of finding the rate of change of a function at any point.

Differentiation from first principles involves finding a limit as $h$ approaches 0.

For the function $y = f(x)$:

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

**Determine the derivative of $f(x) = 3x^2 - 4x + 7$ from first principles.**

| THINK | WRITE |
|---|---|
| 1. State the function. | $f(x) = 3x^2 - 4x + 7$ |
| 2. The derivative is equal to $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ | $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ |
| 3. Substitute for $f(x)$. | $f'(x) = \lim\limits_{h \to 0} \dfrac{[3(x+h)^2 - 4(x+h) + 7] - [3x^2 - 4x + 7]}{h}$ |
| 4. Expand and simplify. | $= \lim\limits_{h \to 0} \dfrac{3x^2 + 6xh + 3h^2 - 4x - 4h + 7 - 3x^2 + 4x - 7}{h}$ |
| | $= \lim\limits_{h \to 0} \dfrac{6xh + 3h^2 - 4h}{h}$ |
| 5. Factorise and simplify. | $= \lim\limits_{h \to 0} \dfrac{h(6x + 3h - 4)}{h}$ |
| | $= \lim\limits_{h \to 0} (6x + 3h - 4)$ |
| 6. Evaluate the limit as $h$ approaches 0. | $= 6x + 3 \times 0 - 4$ |
| | $= 6x - 4$ |
| 7. Answer the question. | $f'(x) = 6x - 4$ |

## 1.2.3 Estimating a limit

Technology, such as a spreadsheet, can be used to estimate the limit of a given expression.

Consider the limit of $\dfrac{a^h - 1}{h}$ as $h \to 0$ for various values of $a > 0$.

For $a = 2$:

| $h$ | $\dfrac{(2^h - 1)}{h}$ |
|---|---|
| 1 | 1 |
| 0.5 | 0.828 427 12 |
| 0.1 | 0.717 734 63 |
| 0.01 | 0.695 555 01 |
| 0.001 | 0.693 387 46 |
| 0.000 1 | 0.693 171 2 |
| 0.000 01 | 0.693 149 58 |

The limit is approaching 0.6931.

For $a = 3$:

| $h$ | $\dfrac{(3^h - 1)}{h}$ |
|---|---|
| 1 | 2 |
| 0.5 | 1.464 101 615 |
| 0.1 | 1.161 231 74 |
| 0.01 | 1.104 669 194 |
| 0.001 | 1.099 215 984 |
| 0.000 1 | 1.098 672 638 |
| 0.000 01 | 1.098 618 323 |

The limit is approaching 1.0986.

Can we find a value of $a$ where the fraction $\dfrac{a^h - 1}{h}$ has a limiting value of 1 as $h \to 0$?

The value of $a$ would lie in the interval $2 < a < 3$.

$$\text{If } \frac{a^h - 1}{h} = 1, h \neq 0, \text{then } a^h - 1 = h \implies a^h = 1 + h$$

$$a = (1 + h)^{\frac{1}{h}}$$

Consider the value of $a$ as $h \to 0$.

| $h$ | $(1 + h)^{\left(\frac{1}{h}\right)}$ |
|---|---|
| 1 | 2 |
| 0.5 | 2.25 |
| 0.1 | 2.593 742 46 |
| 0.01 | 2.704 813 829 |
| 0.001 | 2.716 923 932 |
| 0.000 1 | 2.718 145 927 |
| 0.000 01 | 2.718 268 237 |
| 0.000 001 | 2.718 280 469 |
| 0.000 000 1 | 2.718 281 693 |
| 0.000 000 01 | 2.718 281 815 |
| 0.000 000 001 | 2.718 281 827 |
| 1E − 10 | 2.718 281 828 |

The expression $(1 + h)^{\frac{1}{h}}$ is approaching 2.718 282 05… as $h \to 0$.

Euler's number, $e \approx 2.718\ 282$, is the value of $a$ that gives the value of $\lim\limits_{h \to 0} \dfrac{a^h - 1}{h}$ to be 1.

Like $\pi$, $e$ is an irrational number. An answer given in terms of $e$ is an exact answer.

Scientific and graphics calculators have an $e^x$ function that is treated in the same way as any other function (not to be confused with the key for the letter $e$).

## WORKED EXAMPLE 3 Evaluating powers of $e$

Evaluate the following, giving your answers to 4 decimal places.
a. $e^{1.5}$
b. $e^{-0.5} + 4$

| THINK | WRITE |
|---|---|
| a. 1. Locate the $e^x$ function on your calculator and enter 1.5. | a. $e^x$ <br> $e^{1.5}$ |
| 2. Press Enter. | $e^{1.5} = 4.481\,689\,07\ldots$ |
| 3. Write your answer correct to 4 decimal places. | $e^{1.5} \cong 4.4817$ to 4 decimal places |
| b. 1. Locate the $e^x$ function on your calculator and enter $-0.5$. | b. $e^x$ <br> $e^{-0.5}$ |
| 2. Add 4 to the answer. | $e^{-0.5} + 4$ |
| 3. Press Enter. | $e^{-0.5} + 4 = 4.606\,530\,6\ldots$ |
| 4. Write your answer correct to 4 decimal places. | $e^{-0.5} + 4 \cong 4.6065$ to 4 decimal places |

# Exercise 1.2 Limits

**learn on**

| 1.2 Exercise | 1.2 Exam questions on |
|---|---|

| Simple familiar | Complex familiar | Complex unfamiliar |
|---|---|---|
| 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 | 11, 12 | 13, 14 |

These questions are even better in jacPLUS!
- Receive immediate feedback
- Access sample responses
- Track results and progress

**Find all this and MORE in jacPLUS** ▶

### Simple familiar

1. **WE1a** Evaluate the following limits.
   a. $\lim_{h \to 3}(5h + 4)$
   b. $\lim_{h \to -2}(4 - 6h)$
   c. $\lim_{h \to 0}(6h^2 - 3h + 2)$

2. **WE1b** Evaluate the following limits.
   a. $\lim_{x \to 0} \dfrac{2x^2 + 7x + 3}{x - 1}$
   b. $\lim_{x \to 2} \dfrac{x^2 + 4x}{x + 2}$
   c. $\lim_{x \to -3} \dfrac{x^2 + 4x}{x + 1}$

3. **WE1c** Evaluate the following limits.
   a. $\lim_{h \to -3} \dfrac{h^2 - h - 12}{h + 3}$
   b. $\lim_{h \to 0} \dfrac{h^2 + 4h}{h}$
   c. $\lim_{x \to 3} \dfrac{x^2 - x - 6}{3 - x}$

4. Evaluate the following limits.
   a. $\lim_{h \to 0}(4x^2 + 5xh - h^2)$
   b. $\lim_{h \to 0} \dfrac{3x^2h + 4h^2}{h}$

5. **WE2** For the function $f(x) = x^2 - 6x$, calculate the derivative from first principles.

6. Use first principles to differentiate the function $f(x) = 5 + 3x - 2x^2$.

**7.** Estimate to 5 decimal places, using technology, the limit of $\dfrac{a^h - 1}{h}$ as $h \to 0$, where:

    **a.** $a = 2.5$             **b.** $a = 2.6$            **c.** $a = 2.7$

    **d.** $a = 2.8$             **e.** $a = 2.9$            **f.** $a = 2.718\,28$

**8.** **WE3** Evaluate the following, giving your answers to 4 decimal places.

    **a.** $e^2$                  **b.** $e^3$                  **c.** $e^{\frac{1}{2}}$

**9.** Evaluate the following, giving your answers to 3 decimal places.

    **a.** $2e^{-1}$               **b.** $\sqrt[3]{e}$               **c.** $\dfrac{4+e}{e}$

**10. a.** Use first principles to differentiate the function $y = 8x - x^2$.
    **b.** Calculate the gradient of the tangent to the curve $y = 8x - x^2$ at the point where $x = 2$.
    **c.** Hence, determine the equation of the tangent to the curve at $x = 2$.

### Complex familiar

**11. a.** Simplify the fraction $\dfrac{1}{x+h} - \dfrac{1}{x}$.

    **b.** Use first principles to determine the derivative of the function $f(x) = \dfrac{1}{x}$, $x \neq 0$.

**12. a.** Show that $\dfrac{1}{(x+h-2)} - \dfrac{1}{(x-2)} = \dfrac{-h}{(x-2)(x+h-2)}$.

    **b.** Evaluate $\lim\limits_{h \to 0} \dfrac{-1}{(x-2)(x+h-2)}$.

    **c.** Use first principles to determine the gradient function for $y = \dfrac{1}{(x-2)}$, $x \neq 2$.

    **d.** Determine the $x$-value(s) on the curve where the tangent is parallel to $9x + y - 7 = 0$.

### Complex unfamiliar

**13.** Determine $\lim\limits_{h \to 0} \dfrac{\sqrt{e+h} - \sqrt{e}}{h}$. Justify your procedures and decisions.

**14.** Using technology, determine the limit of $\left(1 + \dfrac{1}{n}\right)^n$ as $n \to \infty$. Comment on your result.

**Fully worked solutions for this chapter are available online.**

# LESSON
## 1.3 The exponential function

### 1.3.1 Review of exponential functions, $y = a^x$ where $a \in \mathbb{R}^+ \backslash \{1\}$

The graph of $f(x) = a^x$, $a > 1$ has the following features:
- The $y$-intercept is $(0, 1)$.
- The key points are $(1, a)$ and $\left(-1, \dfrac{1}{a}\right)$.
- The maximal domain is $x \in \mathbb{R}$.
- The range is $y \in \mathbb{R}^+$ or $y > 0$.
- As $x \to \infty$, $f(x) \to \infty$.
- As $x \to -\infty$, $f(x) \to 0$.
- The horizontal asymptote is $y = 0$ (the $x$-axis).
- It is a one-to-one function.
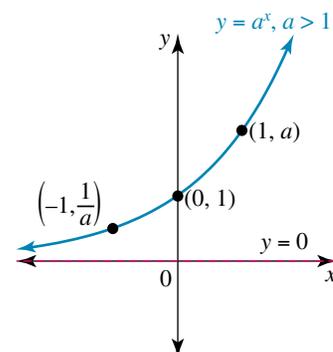
The graph of $f(x) = a^{-x}$, $a > 1$ is a reflection of $f(x) = a^x$ for $a > 1$ over the $y$-axis.

The graph of $f(x) = a^{-x}$, $a > 1$ has the following features:
- The $y$-intercept is $(0, 1)$.
- The key points are $(-1, a)$ and $\left(1, \dfrac{1}{a}\right)$.
- The maximal domain is $x \in \mathbb{R}$.
- The range is $y \in \mathbb{R}^+$ or $y > 0$.
- As $x \to \infty$, $f(x) \to 0$.
- As $x \to -\infty$, $f(x) \to \infty$.
- The horizontal asymptote is $y = 0$ (the $x$-axis).
- It is a one-to-one function.

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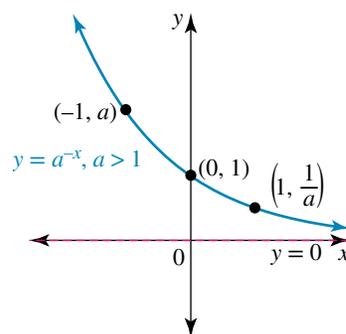
### WORKED EXAMPLE 4 Sketching exponential functions of the form $y = a^x$

**Sketch the following exponential functions, showing all important features.**
**a.** $f(x) = 2^x$        **b.** $f(x) = 2^{-x}$        **c.** $f(x) = -2^x$

**THINK**

**a. 1.** To sketch the graph of $f(x) = 2^x$, first determine the $y$-intercept, which occurs when $x = 0$.

**WRITE**

**a.** $f(x) = 2^x$
$f(0) = 2^0$
$f(0) = 1$
$y$-intercept $(0, 1)$

**2.** To help determine the shape of the curve, it is useful to know any two other points on the graph. For example, determine the coordinates of the points at $x = \pm 1$.

$f(1) = 2^1$  and  $f(-1) = 2^{-1}$

$f(1) = 2$        $f(-1) = \dfrac{1}{2}$

$(1, 2)$ and $\left(-1, \dfrac{1}{2}\right)$

**3.** State the equation of the horizontal asymptote.

$y = 0$

**4.** Sketch the graph.



**b. 1.** To sketch the graph of $f(x) = 2^{-x}$, first determine the $y$-intercept, which occurs when $x = 0$.

**b.** $f(x) = 2^{-x}$

$f(0) = 2^{-0}$

$f(0) = 1$

$y$-intercept $(0, 1)$

**2.** To help determine the shape of the curve, it is useful to know any two other points on the graph. For example, determine the coordinates of the points at $x = \pm 1$.

$f(1) = 2^{-1}$  and  $f(-1) = 2^{-1}$

$f(1) = \dfrac{1}{2}$        $f(-1) = 2$

$\left(1, \dfrac{1}{2}\right)$ and $(-1, 2)$

**3.** State the equation of the horizontal asymptote.

$y = 0$

**4.** Sketch the graph.
*Note:* This is a reflection of the curve in part **a** over the $y$-axis.



**c. 1.** To sketch the graph of $f(x) = -2^x$, first determine the $y$-intercept, which occurs when $x = 0$.

**c.** $f(x) = -2^x$

$f(0) = -2^0$

$f(0) = -1$

$y$-intercept $(0, -1)$

**2.** To help determine the shape of the curve, it is useful to know any two other points on the graph. For example, determine the coordinates of the points at $x = \pm 1$.

$f(1) = -2^1$  and  $f(-1) = -2^{-1}$

$f(1) = -2$        $f(-1) = -\dfrac{1}{2}$

$(1, -2)$ and $\left(-1, -\dfrac{1}{2}\right)$

**3.** State the equation of the horizontal asymptote.

$y = 0$

**4.** Sketch the graph.
   *Note:* This is a reflection of the curve in part **a** over the *x*-axis.

The graph shows $y = 0$, points $(0, -1)$, $\left(-1, -\frac{1}{2}\right)$, $(1, -2)$, and curve $f(x) = -2^x$.

## 1.3.2 The exponential functions, $y = e^x$

Remember that $e \approx 2.718\ 282$.

As $2 < e < 3$, the graph of $y = e^x$ lies between $y = 2^x$ and $y = 3^x$, and has much the same shape.

The graph of $y = e^x$ has the following key features:
- The *y*-intercept is $(0, 1)$.
- The key points are $(1, e)$ and $\left(-1, \dfrac{1}{e}\right)$.
- The horizontal asymptote is $y = 0$.
- The domain is $x \in \mathbb{R}$.
- The range is $y > 0$ or $y \in \mathbb{R}^+$.
- As $x \to \infty, y \to \infty$.
- As $x \to -\infty, y \to 0^+$.

The graph $y = e^x$ showing $(1, e)$, $(0, 1)$, and asymptote $y = 0$.
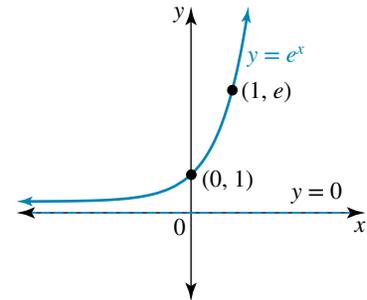
The graph shows an 'exponential **growth**' shape. Mathematical models of such phenomena, for example, population growth, usually involve the exponential function $y = e^x$. Exponential **decay** models usually involve the function $y = e^{-x}$.

The graph of $y = e^{-x}$ has the following key features:
- The *y*-intercept is $(0, 1)$.
- The key points are $(-1, e)$ and $\left(1, \dfrac{1}{e}\right)$.
- The horizontal asymptote is $y = 0$.
- The domain is $x \in \mathbb{R}$.
- The range is $y > 0$ or $y \in \mathbb{R}^+$.
- As $x \to \infty, y \to 0^+$.
- As $x \to -\infty, y \to \infty$.
- The graph is a reflection of $y = e^x$ in the *y*-axis.

The graph $y = e^{-x}$ showing $(-1, e)$, $(0, 1)$, and asymptote $y = 0$.
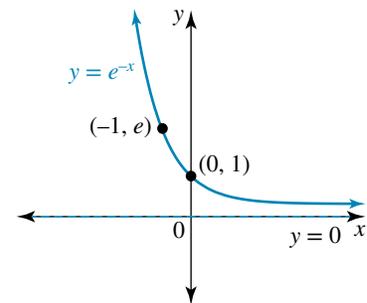
The graph can be transformed in the same way as other functions previously studied.

The graph of $y = -e^x$ is the graph of $y = e^x$ reflected in the *x*-axis.

## Sketching $y = e^x + k$

The graph of $y = e^x + k$ has the following key features:

- There is a vertical translation by $k$ units:
    - if $k > 0$, translated upwards
    - if $k < 0$, translated downwards.
- The horizontal asymptote is $y = k$.
- The $y$-intercept is $(0, 1 + k)$.
- The domain is $x \in \mathbb{R}$.
- The range is $y > k$ or $y \in (k, \infty)$.
- As $x \to \infty, y \to \infty$.
- As $x \to -\infty, y \to k$.
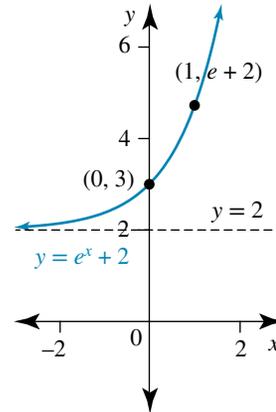
The graph shown is $y = e^x + 2$, where $k = 2$.

To sketch the graph of an exponential function, use the following steps.

- Identify the equation of the asymptote.
- Identify the shape, either growth or decay.
- Determine the $y$-intercept.
- Determine the $x$-intercept if there is one.
- Obtain the coordinates of at least one other point on the curve.

---

### WORKED EXAMPLE 5 Sketching exponential functions

**Sketch the function $f(x) = 2e^x$, showing all important features.**

**THINK**

**a. 1.** To sketch the graph of $f(x) = 2e^x$, first determine the $y$-intercept, which occurs when $x = 0$.

**2.** To help determine the shape of the curve, it is useful to know any two other points on the graph. For example, determine the coordinates of the points at $x = \pm 1$.

**3.** State the equation of the horizontal asymptote.

**4.** Sketch the function.

**WRITE**

**a.** $f(x) = 2e^x$
$f(0) = 2e^0$
$f(0) = 2 \times 1$
$f(0) = 2$
$y\text{-intercept } (0, 2)$

$f(1) = 2e^1 \quad$ and $\quad f(-1) = 2e^{-1}$
$f(1) = 2e \qquad\qquad f(-1) = \dfrac{2}{e}$
$(1, 2e) \approx (1, 5.44)$
$\left(-1, \dfrac{2}{e}\right) \approx (-1, 0.74)$

$y = 0$
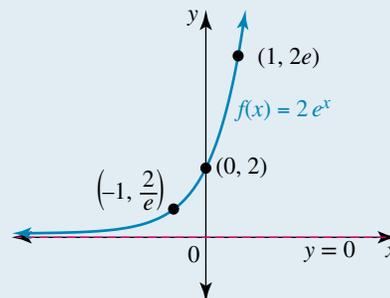
## WORKED EXAMPLE 6 Sketching translated exponential functions

**Sketch the function $f(x) = 2 + e^{-x}$, showing all important features.**

| THINK | WRITE |
|---|---|
| **a. 1.** To sketch the graph of $f(x) = 2 + e^{-x}$, first determine the $y$-intercept, which occurs when $x = 0$. | **a.** $f(x) = 2 + e^{-x}$<br>$f(0) = 2 + e^{-0}$<br>$f(0) = 2 + 1$<br>$f(0) = 3$<br>$y$-intercept $(0, 3)$ |
| **2.** To help determine the shape of the curve, it is useful to know any two other points on the graph. For example, determine the coordinates of the points at $x = \pm 1$. | $f(1) = 2 + e^{-1}$  and  $f(-1) = 2 + e^{-1}$<br>$f(-1) = 2 + e$<br>$(1, 2 + e^{-1}) \approx (1, 2.37)$<br>$(-1, 2 + e) \approx (-1, 4.72)$ |
| **3.** State the equation of the horizontal asymptote. | $y = 2$ |
| **4.** Sketch the function. | |
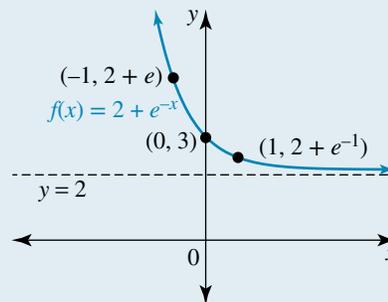


To determine the $x$-intercept if there is one, an exponential equation may need to be solved.

Recall that in Unit 2, indicial equations were solved. The same method is applied when solving indicial equations in base $e$. The following three examples illustrate the similarities when solving simple exponential, or indicial, equations. Scientific calculators include the functions $e^x$ and $\ln(x)$.

| | | |
|---|---|---|
| $2^x = 5$ | $e^x = 5$ | $3^x = 5$ |
| Take the $\log_2$ of both sides: | Take the $\log_e$ of both sides: | Take the $\log_3$ of both sides: |
| $\log_2(2^x) = \log_2 5$ | $\log_e(e^x) = \log_e 5$ | $\log_3(3^x) = \log_3 5$ |
| Simplify: | Simplify: | Simplify: |
| $x = \log_2 5$ | $x = \ln(5)$ | $x = \log_3 5$ |
| $x \approx 2.3219$ (to 4 d.p.) | $x \approx 1.6094$ (to 4 d.p.) | $x \approx 1.4650$ (to 4 d.p.) |

## WORKED EXAMPLE 7 Determining $x$-intercepts and sketching exponential functions

**Consider the function $f(x) = 2 - e^{-x}$.**
**a. Determine the coordinates of any axis intercepts.**
**b. Sketch the function $f(x) = 2 - e^{-x}$, showing all important features.**

| THINK | WRITE |
|---|---|
| **a. 1.** For the $y$-intercept, $x = 0$.<br>• Substitute $x = 0$.<br>• Evaluate. | **a.** $f(x) = 2 - e^{-x}$<br>$f(0) = 2 - e^0$<br>$f(0) = 2 - 1 = 1$ |

**2.** For the $x$-intercept, $y = 0$.

- Substitute $y = 0$.
- Rearrange the equation.
- Take the log (base $e$) of both sides.
- Use log laws to simplify.

$$2 - e^{-x} = 0$$
$$2 = e^{-x}$$
$$\log_e 2 = \log_e e^{-x}$$
$$-x \log_e e = \log_e 2$$
$$x = -\log_e 2$$

**3.** State the coordinates of the axis intercepts. (*Hint:* An approximation may be useful.)

Axis intercepts:
$(-\log_e 2, 0)$ and $(0, 1)$
Approximately:
$(-0.693, 0)$ and $(0, 1)$

**b. 1.** To sketch the graph of $f(x) = 2 - e^{-x}$, use the $x$- and $y$-intercepts found in part **a**.

**b.** Axis intercepts:
$(-\log_e 2, 0)$ and $(0, 1)$
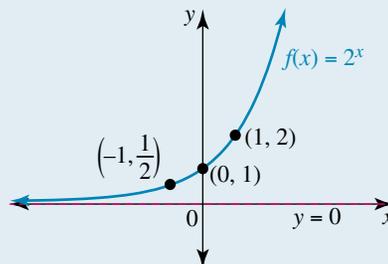Approximately:
$(-0.693, 0)$ and $(0, 1)$

**2.** To help determine the shape of the curve, it is useful to know any two other points on the graph. For example, determine the coordinates of the points at $x = \pm 1$.

$$f(1) = 2 - e^{-1} \quad \text{and} \quad f(-1) = 2 - e^{1}$$
$$f(-1) = 2 - e$$
$$(1, 2 - e^{-1}) \approx (1, 1.632)$$
$$(-1, 2 - e) \approx (-1, -0.718)$$

**3.** State the equation of the horizontal asymptote.

$y = 2$

**4.** Sketch the function.



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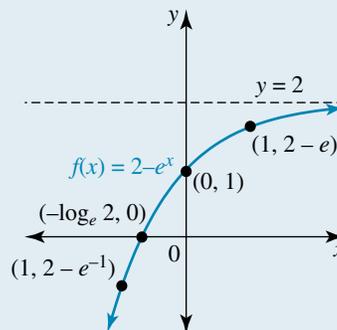
## WORKED EXAMPLE 8 Sketching transformed exponential graphs, stating the domain and range

**a.** Sketch the graph of the function $y = -\dfrac{1}{4}e^{x+1}$.

**b.** State the domain and range of the function.

**THINK**

**WRITE**

**a. 1.** State the equation of the asymptote.

**a.** $y = -\dfrac{1}{4}e^{x+1}$

The asymptote is $y = 0$.

**2.** Calculate the $y$-intercept.

$$y = -\frac{1}{4}e^{0+1}$$
$$= -\frac{1}{4}e$$

$y$-intercept: let $x = 0$.

The $y$-intercept is $\left(0, -\dfrac{e}{4}\right)$.

**3.** Calculate any $x$-intercepts.

There are no $x$-intercepts as the $x$-axis is an asymptote.

**4.** Locate another point if necessary and sketch the graph.

Let $x = -1$.
$$y = -\frac{1}{4}e^0$$
$$= -\frac{1}{4}e$$

Another point on the graph is $\left(-1, -\frac{1}{4}\right)$.
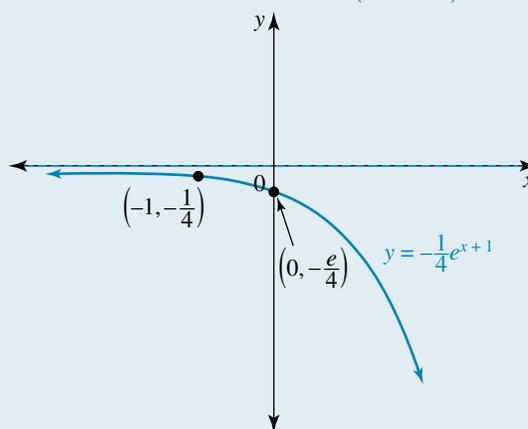


**b.** State the domain and range.

**b.** The domain is $\mathbb{R}$. The range is $\mathbb{R}^-$ or $y < 0$ or $y \in (-\infty, 0)$.

## Exercise 1.3 The exponential function

**learn on**

| 1.3 Exercise | 1.3 Exam questions **on** |
|---|---|

| Simple familiar | Complex familiar | Complex unfamiliar |
|---|---|---|
| 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 | 13, 14, 15, 16, 17 | 18, 19, 20 |

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### Simple familiar

**1.** **WE4** Sketch the following exponential functions, showing all important features.

    **a.** $f(x) = 4^x$            **b.** $f(x) = 4^{-x}$            **c.** $f(x) = -4^x$

**2.** On the same set of axes, sketch the graphs of $y = 10^x$ and $y = 10^{-x}$.

**3.** **WE5** Sketch the function $f(x) = 4e^x$, showing all important features.

**4.** Sketch the function $f(x) = -5e^x$, showing all important features.

5. **WE6** Sketch the function $f(x) = e^{-x} + 3$, showing all important features.

6. Sketch the function $f(x) = e^{2x} + 3$, showing all important features.

7. **WE7** Consider the function $f(x) = e^{2x} - 3$.
   a. Determine the coordinates of any axis intercepts for this function.
   b. Sketch the function $f(x) = e^{2x} - 3$, showing all important features.

8. Consider the function $f(x) = 4 - 2e^{-x}$.
   a. Determine the coordinates of any axis intercepts for this function.
   b. Sketch the function $f(x) = 4 - 2e^{-x}$, showing all important features.

9. Sketch the function $f(x) = 4e^{\frac{x}{2}}$, showing all important features.

10. Consider the function $y = 3e^{-\frac{x}{2}} - 6$.
    a. Determine the coordinates of any axis intercepts for the function.
    b. Sketch the function, showing all important features.

11. **WE8** Sketch the following graphs and state the domain and range of each graph.
    a. $y = -2e^x - 3$
    b. $y = 4e^{-3x} - 4$
    c. $y = 5e^{x-2}$

12. For each of the following functions, sketch the graph, state the range and identify the exact position of any intercepts the graph makes with the coordinate axes.
    a. $y = e^x - 3$
    b. $y = -2e^{2x} - 1$
    c. $y = \dfrac{1}{2}e^{-4x} + 3$

### Complex familiar

13. For each of the following functions, discuss the behaviour of the function as $x \to \pm\infty$. Sketch the graph, state the range and identify the exact position of any intercepts the graph makes with the coordinate axes.
    a. $y = 4 - e^{2x}$
    b. $y = 4e^{2x-6} + 2$

14. a. Sketch the graph of $y = 2e^{1-3x} - 4$, labelling any intercepts with the coordinate axes with their exact coordinates.
    b. Sketch the graph of $y = 3 \times 2^x - 24$ and state its domain and range.

15. The diagram shows the graph of $y = ae^x + b$. Determine the values of $a$ and $b$.
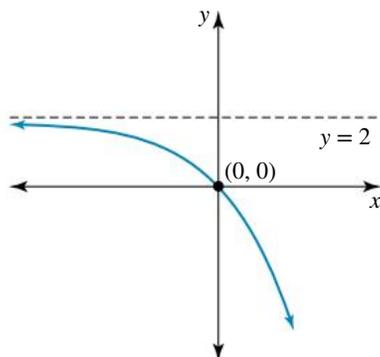
**16.** The graph shown is of the function $f(x) = ae^x + b$. Determine the values of $a$ and $b$, and write the function as a mapping.
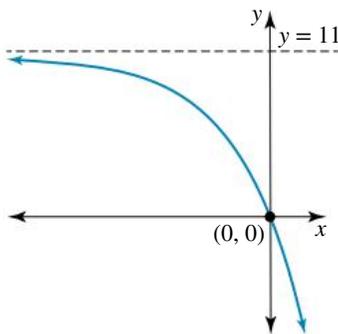


**17.** The graph shown has an equation of the form $y = Ae^{nx} + k$. Determine its equation.
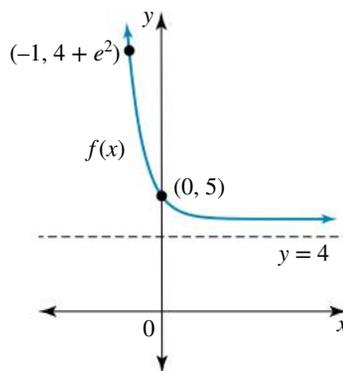


**Complex unfamiliar**

**18.** Consider the function $f(x) = 2e^{-x} + 1$.
Determine the values of $x$ if $f(x) < 3$ and discuss why $f(x) < 0$ has no real solutions.

**19.** Consider the functions $f(x) = 4 - e^x$, $x \geq 0$ and $g(x) = 4 - e^x$.
Discuss why these two functions do not have the same range. State the range of each function.

**20.** The graph of $y = Ae^{x-2} + B$ contains the point $(2, 10)$, and as $x \to -\infty$, $y \to -2$.

The graph passes through the point $\left(a, 2\left(\dfrac{6}{e} - 1\right)\right)$. Determine the value of $a$.

Justify your procedures.

**Fully worked solutions for this chapter are available online.**

# LESSON
## 1.4 Differentiation of exponential functions

### 1.4.1 The derivative of the exponential function

The derivative of the exponential function can be found using first principles.

If $f(x) = e^x$, then

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} \\
&= \lim_{h \to 0} \frac{e^x e^h - e^x}{h} \\
&= \lim_{h \to o} \frac{e^x \left( e^h - 1 \right)}{h} \\
&= e^x \lim_{h \to 0} \frac{e^h - 1}{h}
\end{aligned}
$$

In Lesson 1.2, it was shown that $\lim_{h \to 0} \dfrac{a^h - 1}{h} = 1$ for $a = e$; that is, $\lim_{h \to 0} \dfrac{e^h - 1}{h} = 1$.

Substituting into $f'(x) = e^x \lim_{h \to 0} \dfrac{e^h - 1}{h}$:

$$ f'(x) = e^x $$

---

**Derivative of $e^x$**

**If $y = e^x$, then**

$$ \frac{dy}{dx} = e^x $$

**Alternatively,**

$$ \frac{d}{dx}e^x = e^x $$

**The derivative of the exponential function is itself.**

*Note:* **This rule only applies to exponential functions of base $e$.**

---

The derivative of $y = e^{f(x)}$ can be found using the **chain rule**, studied in Unit 2.

If $y = e^{f(x)}$, let $u = f(x)$.

Then $y = e^u$.

$\dfrac{dy}{du} = e^u$ and $\dfrac{du}{dx} = f'(x)$

The chain rule states: $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

Substitute: $\dfrac{dy}{dx} = e^u \times f'(x)$

Replace $u$ as $f(x)$: $\dfrac{dy}{dx} = f'(x)e^{f(x)}$

---

### Derivative of $e^{f(x)}$

If $y = e^{f(x)}$, then

$$\frac{dy}{dx} = f'(x)e^{f(x)}$$

**Alternatively,**

$$\frac{d}{dx}e^{f(x)} = f'(x)e^{f(x)}$$

$$\frac{d}{dx}e^u = e^u \times \frac{du}{dx}$$

---

## WORKED EXAMPLE 9 Determine the derivative using the chain rule

Using the chain rule, differentiate $y = e^{-5x}$ with respect to $x$.

| THINK | WRITE |
|---|---|
| **1.** Write the equation. | $y = e^{-5x}$ |
| **2.** Substitute $u = -5x$. | $y = e^u$ and $u = -5x$ |
| **3.** Determine $\dfrac{dy}{du}$ and $\dfrac{du}{dx}$. | $\dfrac{dy}{du} = e^u$ and $\dfrac{du}{dx} = -5$ |
| **4.** Use the chain rule to find $\dfrac{dy}{dx}$. | $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$ <br> $\dfrac{dy}{dx} = e^u \times (-5)$ |
| **5.** State the derivative in terms of $x$. | $\dfrac{dy}{dx} = -5e^{-5x}$ |
| Alternatively, recognise and apply the formula $\dfrac{d\left(e^{f(x)}\right)}{dx} = f'(x)e^{f(x)}$ where $f(x) = -5x$. | $y = e^{-5x}$ <br> $\dfrac{dy}{dx} = -5e^x$ |

## WORKED EXAMPLE 10 Determining the derivative of an exponential function

**Determine the derivative of $y = e^{2x+1}$.**

| THINK | WRITE |
|---|---|
| **1.** Write the equation. | $y = e^{2x+1}$ |
| **2.** Recognise and apply the formula $\dfrac{d\left(e^{f(x)}\right)}{dx} = f'(x)e^{f(x)}$ where $f(x) = 2x + 1$. | $\dfrac{dy}{dx} = 2e^{2x+1}$ |

## WORKED EXAMPLE 11 Differentiating an exponential function 1

**Differentiate the following.**

**a.** $f(x) = e^x(e^x - 3)$

**b.** $f(x) = \dfrac{e^{3x} - 2e^{-x}}{e^x}$

| THINK | WRITE |
|---|---|
| **a. 1.** Write the equation. | **a.** $f(x) = e^x(e^x - 3)$ |
| **2.** Write the equation in expanded form. | $f(x) = e^{2x} - 3e^x$ |
| **3.** Differentiate each term. | $f'(x) = 2e^{2x} - 3e^x$ |
| **b. 1.** Write the equation. | **b.** $f(x) = \dfrac{e^{3x} - 2e^{-x}}{e^x}$ |
| **2.** Separate the terms and simplify. | $f(x) = \dfrac{e^{3x}}{e^x} - \dfrac{2e^{-x}}{e^x}$ |
| | $f(x) = e^{2x} - 2e^{-2x}$ |
| **3.** Differentiate. | $f'(x) = 2e^{2x} + 4e^{-2x}$ |
| | $f'(x) = 2e^{2x} + \dfrac{4}{e^{2x}}$ |

## WORKED EXAMPLE 12 Differentiating an exponential function 2

**Determine the derivative of the function $y = e^{\left(x^3 - x\right)}$.**

| THINK | WRITE |
|---|---|
| **1.** Write the equation. | $y = e^{(x^3 - x)}$ |
| **2.** Substitute $u = x^3 - x$. | $y = e^u$ and $u = x^3 - x$ |
| **3.** Determine $\dfrac{dy}{du}$ and $\dfrac{du}{dx}$. | $\dfrac{dy}{du} = e^u$ and $\dfrac{du}{dx} = 3x^2 - 1$ |
| **4.** Use the chain rule to find $\dfrac{dy}{dx}$. | $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$ |
| | $\dfrac{dy}{dx} = e^u \times (3x^2 - 1)$ |

**5.** State the derivative in terms of $x$.

$\dfrac{dy}{dx} = (3x^2 - 1)e^{(x^3 - x)}$

Alternatively:

**1.** Write the equation.

$y = e^{(x^3 - x)}$

**2.** Recognise and apply the formula

$\dfrac{d\left(e^{f(x)}\right)}{dx} = f'(x)e^{f(x)}$ where $f(x) = x^3 - x$.

$\dfrac{dy}{dx} = (3x^2 - 1)\,e^{(x^3 - x)}$

---

## Exercise 1.4 Differentiation of the exponential function   learn**on**

| **1.4 Exercise** | **1.4 Exam questions**  on |
|---|---|

| Simple familiar | Complex familiar | Complex unfamiliar |
|---|---|---|
| 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 | 13, 14, 15, 16 | 17, 18, 19, 20 |

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### Simple familiar

**1.** **WE9** Differentiate the following.

**a.** $y = e^{10x}$

**b.** $y = e^{\frac{1}{3}x}$

**c.** $y = e^{\frac{x}{4}}$

**d.** $y = e^{-x}$

**e.** $y = 3\,e^{2 - \frac{x}{3}}$

**f.** $y = 4e^{-5x}$

**2.** **WE10** Differentiate the following.

**a.** $y = e^{6x - 2}$

**b.** $y = e^{8 - 6x}$

**c.** $y = 2e^{5x + 3}$

**d.** $y = 4e^{7 - 2x}$

**e.** $y = -3e^{8x + 1}$

**f.** $y = -2e^{6 - 5x}$

**3.** Differentiate the following.

**a.** $y = 10e^{6 - 9x}$

**b.** $y = -5e^{3x + 4}$

**c.** $y = 6e^{-7x}$

**d.** $y = 2e^{\frac{x}{2} + 1}$

**e.** $y = 3e^{2 - \frac{x}{3}}$

**f.** $y = -4e^{\frac{x}{4} + 5}$

**4.** **MC** The derivative of $y = e^{3x + 2}$ is equal to:

**A.** $3e^{3x + 2}$

**B.** $(3x + 2)e^{3x + 2}$

**C.** $3e^{3x}$

**D.** $3xe^{3x + 2}$

**5.** **WE11a** Differentiate the following.

**a.** $f(x) = 2(e^x + 1)$

**b.** $f(x) = 3e^{2x}(e^x + 1)$

**c.** $f(x) = 5(e^{-4x} + 2x)$

**d.** $f(x) = (e^x + 2)(e^{-x} + 3)$

6. **WE11b** Differentiate the following.

   **a.** $f(x) = \dfrac{3e^{3x} + e^{-6x}}{e^x}$

   **b.** $f(x) = \dfrac{4e^{7x} - 2e^{-x}}{e^{-2x}}$

7. **WE12** Determine the derivatives of the following.

   **a.** $y = e^{x^2 + 3x}$       **b.** $y = e^{x^2 - 3x + 1}$       **c.** $y = e^{x^2 - 2x}$       **d.** $f(x) = e^{2 - 5x}$

8. Use the formula $\dfrac{d\left(e^{f(x)}\right)}{dx} = f'(x)\, e^{f(x)}$ to differentiate the following functions.

   **a.** $f(x) = e^{6 - 3x + x^2}$    **b.** $g(x) = e^{x^3 + 3x - 2}$    **c.** $h(x) = 3e^{4x^2 - 7x}$    **d.** $y = -5e^{1 - 2x - 3x^2}$

9. **MC** The derivative of $6e^{x^3 - 5x}$ is equal to:

   **A.** $6(3x^2 - 5)e^{x^3 - 5x}$

   **B.** $(3x^2 - 5)e^{x^3 - 5x}$

   **C.** $6(x^3 - 5x)e^{x^3 - 5x}$

   **D.** $6(3x^2 - 5)e^{3x^2 - 5}$

10. If $f(x) = 5e^{9 - 4x}$, determine the exact value of $f'(2)$.

11. If $g(x) = 2e^{x^2 - 3x + 2}$, determine the exact value of $g'(0)$.

12. Calculate the exact value of $h'(-1)$ if $h(x) = -5e^{x^2 + 3x}$.

**Complex familiar**

13. Determine the equation of the tangent to the curve $y = \dfrac{4}{e^{2x}}$ at $x = \dfrac{1}{2}$ and evaluate the $y$-intercept correct to 3 decimal places.

14. Show that the equation of the tangent to the curve with equation $y = e^{-2x}$ at the point where $x = -\dfrac{1}{2}$ passes through the origin.

15. Determine the equation of the tangent to the curve $y = e^{x^2 + 3x - 4}$ at the point where $x = 1$.

16. Determine the equations of the tangent and the line perpendicular to the curve $y = e^{-3x} - 2$ at the point where $x = 0$.

**Complex unfamiliar**

17. For the function $f(x) = e^{-2x + 3} - 2e$, determine $\{x: f'(x) = -2\}$.

18. If $f(x) = \dfrac{e^{3x} + 2}{e^x}$, determine $\{x: f'(x) = 0\}$.

19. The curve with the rule $A = A_0 e^{-0.69t}$ passes through the point $(0, 2)$. Determine $\dfrac{dA}{dt}$ when $t = 0$.

20. For the function with the rule $f(x) = Ae^x + Be^{-3x}$, where $A$ and $B$ are non-zero real constants, show that $f'(x) = 0$ when $e^{4x} = \dfrac{3B}{A}$.

**Fully worked solutions for this chapter are available online.**

# LESSON
## 1.5 The logarithmic function

### 1.5.1 Defining the natural logarithm

Recall from Unit 2 that the term logarithm is another name for the exponent, index or power, including expressions in $e$.

Consider the following indicial equations.

$$\text{Exponent} \qquad\qquad\qquad \text{Exponent}$$
$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$
$$\text{Base number} \to 10^2 = 100 \qquad\qquad \text{Base number} \to e^p = q$$

Written as logarithms they become:

$$\log_{10}(100) = 2 \leftarrow \text{Exponent} \qquad \log_e q = (p) \leftarrow \text{Exponent}$$
$$\uparrow \qquad\qquad\qquad\qquad\qquad \uparrow$$
$$\text{Base number} \qquad\qquad\qquad \text{Base number}$$

The logarithm in base $e$ is known as the natural logarithm or the Napierian logarithm.

The natural logarithm is written as $\log_e(x)$ or $\ln(x)$.

The logarithmic laws apply in the normal way when $e$ is the base.

**The logarithm laws**

1. $\log_a(m) + \log_a(n) = \log_a(mn)$
2. $\log_a(m) - \log_a(n) = \log_a\left(\dfrac{m}{n}\right)$
3. $\log_a(m^n) = n \log_a(m)$
4. $\log_a(1) = 0$
5. $\log_a(a) = 1$
6. $\log_a(0) = \text{undefined}$
7. $\log_a(x)$ is defined for $x > 0$ and $a \in \mathbb{R}^+ \backslash \{1\}$.
8. $a^{\log_a(m)} = m$

## WORKED EXAMPLE 13 Simplifying expressions using the index and logarithm laws

**Simplify the following.**

a. $\log_4(64) + \log_4(16) - \log_4(256)$

b. $2\log_3(7) - 2\log_3(21)$

c. $\dfrac{\log_3(16)}{\log_3(64)}$

**THINK**

**a. 1.** Express all the numbers in base 4 and, where possible, apply the log law $\log_a(m^n) = n\,\log_a(m)$.

**2.** Apply $\log_a(a) = 1$ and simplify.

**b. 1.** Apply the law $n\log_a(m) = \log_a(m^n)$.

**2.** Apply the law $\log_a(m) - \log_a(n) = \log_a\left(\dfrac{m}{n}\right)$ and simplify.

**3.** Convert $\dfrac{1}{3}$ to $3^{-1}$ and apply $\log_a(a) = 1$.

**c. 1.** Apply the law $n\log_a(m) = \log_a(m^n)$.
*Note:* The 16 and 64 cannot be cancelled, as when they are with the log function, they represent single numbers. Therefore, the 16 and 64 cannot be separated from their logarithm components.

**2.** Cancel the logs as they are identical.

**WRITE**

**a.** $\log_4(64) + \log_4(16) - \log_4(256)$
$= \log_4(4^3) + \log_4(4^2) - \log_4(4^4)$
$= 3\log_4(4) + 2\log_4(4) - 4\log_4(4)$

$= 3 \times 1 + 2 \times 1 - 4 \times 1$
$= 1$

**b.** $2\log_3(7) - 2\log_3(21)$
$= \log_3(7^2) - \log_3(21^2)$

$= \log_3\left(\dfrac{7^2}{21^2}\right)$

$= \log_3\left(\dfrac{7}{21}\right)^2$

$= 2\log_3\left(\dfrac{1}{3}\right)$

$= 2\log_3(3^{-1})$
$= -2\log_3(3)$
$= -2$

**c.** $\dfrac{\log_3(16)}{\log_3(64)}$

$= \dfrac{\log_3(2^4)}{\log_3(2^6)}$

$= \dfrac{4\log_3(2)}{6\log_3(2)}$

$= \dfrac{4}{6}$

$= \dfrac{2}{3}$

## WORKED EXAMPLE 14 Solving simple equations with logarithms

**Solve the following for $x$, giving your answers correct to 3 decimal places where appropriate.**

**a.** $\log_e(x) = 3$

**b.** $\ln(x) + \ln(2) = \ln(6)$

| THINK | WRITE |
|---|---|
| **a. 1.** Rewrite the equation as an exponential. | **a.** $\log_e(x) = 3$ <br> $e^3 = x$ |
| **2.** Evaluate. | $x = 20.085\,536\,9$ |
| **3.** Answer the question. | $x = 20.086$ (to 3 decimal places) |
| **b. 1.** Apply log laws to rewrite the equation. | **b.** $\ln(x) + \ln(2) = \ln(6)$ <br> $\ln(2x) = \ln(6)$ <br> $(2x) = 6$ |
| **2.** Answer the question. | $x = 3$ |

## 1.5.2 The inverse relationship of $y = e^x$ and $y = \log_e(x)$

The logarithmic function can also be thought of as the inverse of the exponential function.

For the inverse of a function to exist, the function needs to be a one-to-one function. This means that every $x$-value is mapped or paired to a unique $y$-value. The exponential function, $y = e^x$, satisfies this condition.

Inverse relationships are the reflection over the straight line $y = x$. Alternately, $y = x$ is a line of symmetry between a function and its inverse. Special notation is used for inverse functions.

For a function, $f(x)$, its inverse function is written as $f^{-1}(x)$.

To obtain the inverse of the exponential function $y = e^x$:
- interchange the $x$ and $y$ variables, so the function becomes $x = e^y$
- make $y$ the subject: $y = \log_e(x)$ or $y = \ln(x)$.

The graph illustrates the relationship between $y = e^x$ and $y = \log_e(x)$.
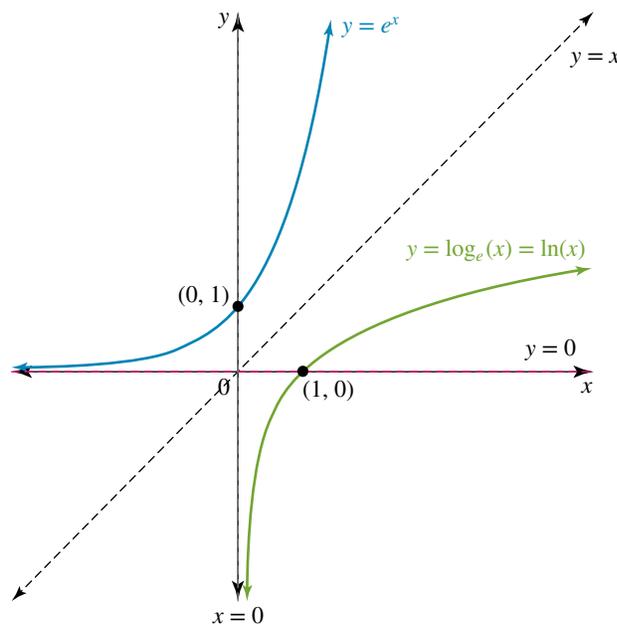
Note also that since the $x$ and $y$ variables are interchanged:
- the domain of the function, $f$, becomes the range of the inverse function, $f^{-1}$
- the range of the function, $f$, becomes the domain of the inverse function, $f^{-1}$

The table below illustrates this between the inverse functions $y = e^x$ and $y = \log_e(x)$.

| Function | Type of function | Domain | Range |
|---|---|---|---|
| $y = e^x$ | One-to-one | $x \in \mathbb{R}$ | $y \in (0, \infty)$ or $y \in \mathbb{R}^+$ |
| $y = \ln(x) = \log_e(x)$ | One-to-one | $x \in (0, \infty)$ or $x \in \mathbb{R}^+$ | $y \in \mathbb{R}$ |

## 1.5.3 The logarithmic function, $y = \ln(x) = \log_e(x)$

The graph of $y = \ln(x)$ has the following key features:
- The $x$-intercept is $(1, 0)$.
- A key point is $(e, 1)$.
- The vertical asymptote is $x = 0$.
- The domain is $x \in (0, \infty)$ or $x \in \mathbb{R}^+$ or $x > 0$.
- The range is $y \in \mathbb{R}$.
- As $x \to \infty, y \to \infty$.
- As $x \to 0, y \to -\infty$.
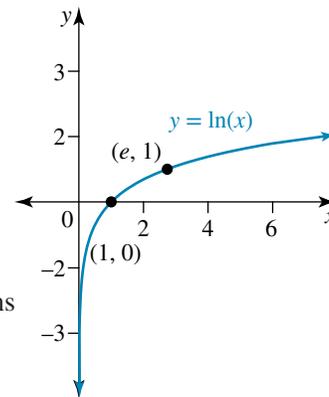
The graph of $y = \ln(x)$ can be transformed in the same way as other functions previously studied.

For example:

$y = -\ln(x)$ is the graph of $y = \ln(x)$ reflected in the $x$-axis with the domain $x > 0$.
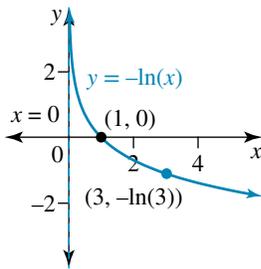
$y = \ln(-x)$ is the graph of $y = \ln(x)$ reflected in the $y$-axis with the domain $x < 0$.
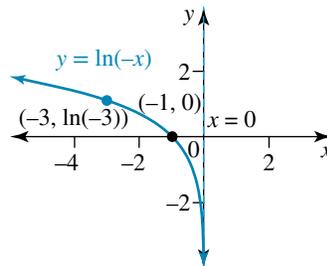
### Sketching $y = \ln(x - h)$

The graph of $y = \ln(x - h)$ has the following key features:
- A vertical asymptote with equation $x = h$
- A horizontal translation of $h$ units:
  - if $h > 0$, translated to the right
  - if $h < 0$, translated to the left
- The $x$-intercept is $(1 + h, 0)$.
- The domain is $x \in (h, \infty)$ or $x > h$.
- The range is $y \in \mathbb{R}$.
- As $x \to \infty, y \to \infty$.
- As $x \to h, y \to -\infty$.
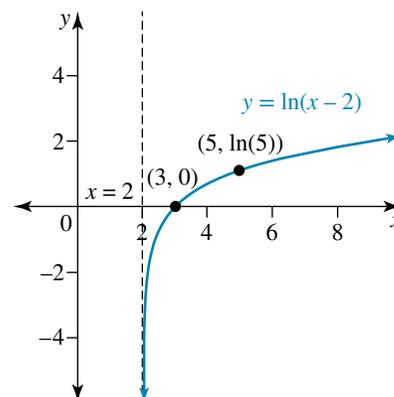
To sketch the graph of a logarithmic function, use the following steps:
- Identify the equation of the asymptote.
- Determine the domain and range.
- Determine the $x$-intercept.
- Determine the $y$-intercept if there is one.
- Obtain the coordinates of another point on the curve.

## WORKED EXAMPLE 15 Sketching logarithmic functions

**Sketch the graphs of the following, showing all important characteristics. State the domain and range in each case.**

a. $y = \log_e(x - 2)$

b. $y = \log_e(x + 1) + 2$

c. $y = \dfrac{1}{4}\log_e(2x)$

d. $y = -\log_e(-x)$

**THINK**

a. 1. The basic graph of $y = \log_e(x)$ has been translated 2 units to the right, so $x = 2$ is the vertical asymptote.

2. Locate the $x$-intercept, when $y = 0$.

3. Determine another point through which the graph passes.

4. Sketch the graph.

b. 1. The basic graph of $y = \log_e(x)$ has been translated up 2 units and 1 unit to the left, so $x = -1$ is the vertical asymptote.

2. Locate the $x$-intercept, when $y = 0$.

**WRITE**

a. $y = \log_e(x - 2)$
The domain is $(2, \infty)$. The range is $\mathbb{R}$.

$x$-intercept, $y = 0$: $\log_e(x - 2) = 0$
$$e^0 = x - 2$$
$$1 = x - 2$$
$$x = 3$$

When $x = 4$, $y = \log_e(2)$.
The point is $(4, \log_e(2))$.



b. $y = \log_e(x + 1) + 2$
The domain is $(-1, \infty)$. The range is $\mathbb{R}$.

The graph cuts the $x$-axis where $y = 0$.
$$\log_e(x + 1) + 2 = 0$$
$$\log_e(x + 1) = -2$$
$$e^{-2} = x + 1$$
$$x = e^{-2} - 1$$

**3.** Locate the $y$-intercept, when $x = 0$.

The graph cuts the $y$-axis where $x = 0$.
$y = \log_e(1) + 2$
$\quad = 2$

**4.** Sketch the graph.



$y = \log_e(x + 1) + 2$

$(0, 2)$

$(e^{-2} - 1, 0)$

$x = -1$
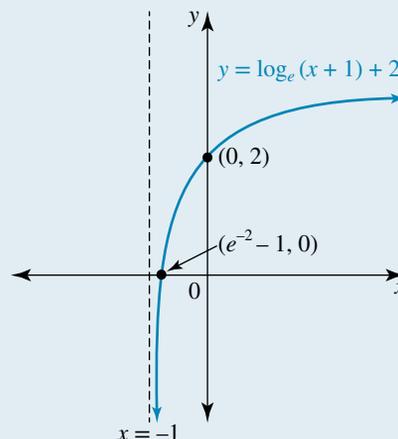
**c. 1.** The basic graph of $y = \log_e(x)$ has been dilated by factor $\dfrac{1}{4}$ from the $x$-axis and by factor $\dfrac{1}{2}$ from the $y$-axis. The vertical asymptote remains $x = 0$.

**c.** $y = \dfrac{1}{4}\log_e(2x)$

The domain is $(0, \infty)$. The range is $\mathbb{R}$.

**2.** Locate the $x$-intercept, when $y = 0$.

$x$-intercept, $y = 0$: $\dfrac{1}{4}\log_e(2x) = 0$

$$\log_e(2x) = 0$$
$$e^0 = 2x$$
$$1 = 2x$$
$$x = \dfrac{1}{2}$$

**3.** Determine another point through which the graph passes.

When $x = 1$, $y = \log_e(2)$.
The point is $(1, \log_e(2))$.

**4.** Sketch the graph.



$y = \dfrac{1}{4}\log_e(2x)$

$(1, \log_e(2))$

$\left(\dfrac{1}{2}, 0\right)$

$x = 0$
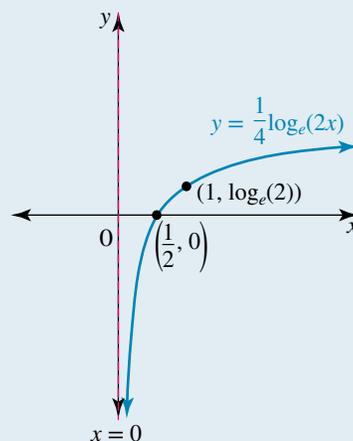
**d. 1.** The basic graph of $y = \log_e(x)$ has been reflected in both axes. The vertical asymptote remains $x = 0$.

**2.** Locate the $x$-intercept when $y = 0$.

**3.** Determine another point through which the graph passes.

**4.** Sketch the graph.

**d.** $y = -\log_e(-x)$

The domain is $(-\infty, 0)$. The range is $\mathbb{R}$.

$x$-intercept, $y = 0$: $-\log_e(-x) = 0$
$$\log_e(-x) = 0$$
$$e^0 = -x$$
$$x = -1$$

When $x = -2$, $y = -\log_e(2)$.
The point is $(-2, -\log_e(2))$.



$y = -\log_e(-x)$

$(-1, 0)$

$(-2, -\log_e(2))$

$x = 0$

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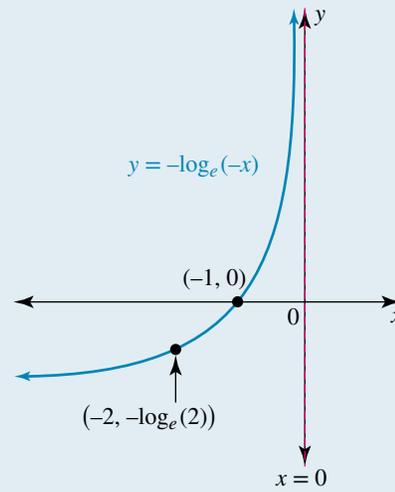
# Exercise 1.5 The logarithmic function

| 1.5 Exercise | 1.5 Exam questions **on** |
|---|---|

| Simple familiar | Complex familiar | Complex unfamiliar |
|---|---|---|
| 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 | 13, 14, 15, 16 | 17, 18, 19, 20 |

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**Simple familiar**

**1.** **WE13** Simplify the following.

**a.** $\log_7(49) + \log_2(32) - \log_5(125)$

**b.** $5 \log_{11}(6) - 5 \log_{11}(66)$

**c.** $\dfrac{\log_4 25}{\log_4 625}$

**2.** Simplify the following.

**a.** $7 \log_4(x) - 9 \log_4(x) + 2 \log_4(x)$

**b.** $\log_7(2x - 1) + \log_7(2x - 1)^2$

**c.** $\log_{10}(x - 1)^3 - 2 \log_{10}(x - 1)$

3. **WE14** Solve for $x$ in each of the following. Give exact answers when appropriate; otherwise, give answers correct to 3 decimal places.

   a. $\log_e(x) = 1$ 
   
   b. $\log_e(x) = 2$ 
   
   c. $\log_e(x) = -2$

   d. $\log_e(x) = -1$ 
   
   e. $\log_e(x) = 0.3$ 
   
   f. $\log_e(x) = -0.69$

4. Solve the following for $x$. Give exact answers when appropriate; otherwise, give answers correct to 3 decimal places.

   a. $\log_e(2x) = 2$ 
   
   b. $\log_e(3x) = 1$ 
   
   c. $\log_e(x^3) = 3$

5. **MC** Simplifying $3 \log_e(5) + 2 \log_e(2) - \log_e(20)$ gives:

   A. $\log_e\left(\dfrac{19}{20}\right)$ 
   
   B. $\log_e(109)$ 
   
   C. $\log_e(480)$ 
   
   D. $2 \log_e(5)$

6. **WE15** Sketch the graphs of the following functions, showing all important characteristics. State the domain and range for each graph.

   a. $y = \log_e(x + 4)$ 
   
   b. $y = \log_e(x) + 2$

   c. $y = 4 \log_e(x)$ 
   
   d. $y = -\log_e(x - 4)$

7. Sketch the following graphs, clearly showing any axis intercepts and asymptotes.

   a. $y = \log_e(x) + 3$ 
   
   b. $y = \log_e(x) - 5$ 
   
   c. $y = \log_e(x) + 0.5$

8. Sketch the following graphs, clearly showing any axis intercepts and asymptotes.

   a. $y = \log_e(x - 4)$ 
   
   b. $y = \log_e(x + 2)$ 
   
   c. $y = \log_e(x + 0.5)$

9. Sketch the following graphs, clearly showing any axis intercepts and asymptotes.

   a. $y = \dfrac{1}{4} \log_e(x)$ 
   
   b. $y = 3 \log_e(x)$ 
   
   c. $y = 6 \log_e(x)$

10. Sketch the following graphs, clearly showing any axis intercepts and asymptotes.

    a. $y = \log_e(3x)$ 
    
    b. $y = \log_e\left(\dfrac{x}{4}\right)$ 
    
    c. $y = \log_e(4x)$

11. Consider the function $f(x) = e^x + 2$.

    a. State the domain and range of the function, $f$.
    b. Define the inverse function, $f^{-1}$.
    c. State the domain and range of the inverse function, $f^{-1}$.
    d. On the same set of axes, sketch the graphs of $f$ and $f^{-1}$.

12. Consider the function $f(x) = e^{-x}$.

    a. State the domain and range of the function, $f$.
    b. Define the inverse function, $f^{-1}$.
    c. State the domain and range of the inverse function, $f^{-1}$.
    d. On the same set of axes, sketch the graphs of $f$ and $f^{-1}$.
    e. Use technology to determine, correct to 3 decimal places, the point of intersection of the graphs of $f$ and $f^{-1}$.

**Complex familiar**

13. Sketch the following graphs, clearly showing any axis intercepts and asymptotes.

    a. $y = 1 - 2 \log_e(x - 1)$ 
    
    b. $y = \log_e(2x + 4)$ 
    
    c. $y = \dfrac{1}{2} \log_e\left(\dfrac{x}{4}\right) + 1$

**14.** The graph of a logarithmic function of the form $y = a\log_e(x - h) + k$ is shown. Determine the values of $a$, $h$ and $k$.
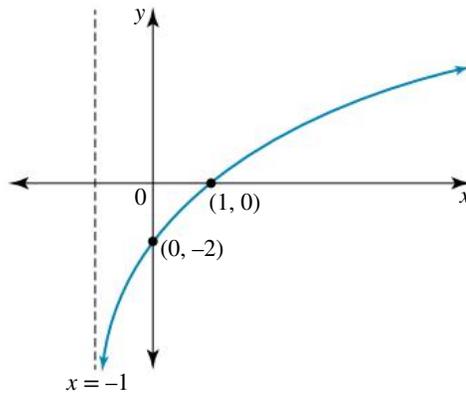


**15.** Consider the function defined by the rule $f(x) = \ln(3 - x)$.

   **a.** State the domain and range of the function $f$ and explain why the domain is restricted.
   **b.** Calculate, in exact form, the coordinates of any axis intercepts of $f$.
   **c.** Determine the equation of the inverse function, $f^{-1}$, and state the domain and range of the inverse function.
   **d.** State, in exact form, the coordinates of any axis intercepts of the inverse function, $f^{-1}$.
   **e.** On the same set of axes, sketch the graphs of $f$ and $f^{-1}$, showing all relevant features.
   **f.** Using technology, determine the coordinates of the point(s) of intersection of $f$ and $f^{-1}$.

**16.** Consider the function defined by the rule $f(x) = \log_e(2x - 1)$.

   **a.** State the domain and range of the function $f$ and explain why the domain is restricted.
   **b.** Calculate, in exact form, the coordinates of any axis intercepts of $f$.
   **c.** Determine the equation of the inverse function, $f^{-1}$, and state the domain and range of the inverse function.
   **d.** State, in exact form, the coordinates of any axis intercepts of the inverse function, $f^{-1}$.
   **e.** On the same set of axes, sketch the graphs of $f$ and $f^{-1}$, showing all relevant features.
   **f.** Explain why the two graphs do not intersect.

**Complex unfamiliar**

**17.** The table below gives values for $x$ and $y$ that relate to the equation $y = a\log_e(bx)$.

| $x$ | 1 | 2 | 3 |
|---|---|---|---|
| $y$ | $-3\log_e(2)$ | 0 | $m$ |

Determine the exact value of $m$, identifying all mathematical procedures used.

**18.** The equation $y = a\log_e(bx)$ relates $x$ to $y$. The table below shows values for $x$ and $y$.

| $x$ | 1 | 2 | 3 |
|---|---|---|---|
| $y$ | $\log_e(2)$ | 0 | $w$ |

Determine the value of $w$ correct to 4 decimal places. Justify your answer.

**19.** The graph of $y = m \log_2(nx)$ passes through the points $(-2, 3)$ and $\left(-\dfrac{1}{2}, \dfrac{1}{2}\right)$. Show that the values of $m$ and $n$ are $1.25$ and $-2^{\frac{7}{5}}$ respectively. Justify the procedures you used.

**20.** The graph of the function $f: (-5, \infty) \to \mathbb{R}, f(x) = \log_e(x+5) + 1$ is shown.
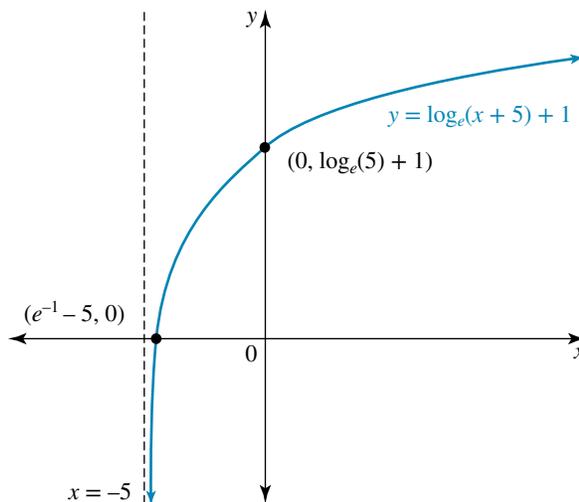


Sketch the graphs of the function, $f$ and the inverse function, $f^{-1}$. Determine the coordinates of the point(s) of intersection of the graphs, correct to 3 decimal places.

**Fully worked solutions for this chapter are available online.**

# LESSON
## 1.6 Solving equations

### 1.6.1 Solving indicial equations in $e$

When we solve an equation such as $3^x = 81$, the technique is to convert both sides of the equation to the same base. For example, $3^x = 3^4$, therefore, $x = 4$.

When we solve an equation such as $x^3 = 125$, we write both sides of the equation with the same index. For example, $x^3 = 5^3$, therefore, $x = 5$.

Many equations, such as $3^x = 4$, cannot be solved using the index laws. To remove the $x$ from the power, or index, we use logarithms. In Unit 2, we used logarithms to a given base.

We can also solve using natural logarithms. This is illustrated in the following example.

$$3^x = 4 \qquad\qquad\qquad 3^x = 4$$

Take $\log_3$ of both sides: Take natural logs of both sides:

$$\log_3(3^x) = \log_3(4) \qquad\qquad \ln(3^x) = \ln(4)$$

Simplify using laws of logarithms: Simplify using laws of logarithms:

$$x\log_3(3) = \log_3(4) \qquad\qquad x\ln(3) = \ln(4)$$

$$x = \log_3(4) \qquad\qquad x = \frac{\ln(4)}{\ln(3)}$$

Evaluate correct to 5 decimal places: Evaluate correct to 5 decimal places:

$$x \approx 1.261\,86 \qquad\qquad x \approx 1.261\,86$$

Determining solutions to indicial equations with $e$:
- The laws of indices apply in the same way if $e$ is the base.
- Equations involving $e$ are solved using the same methods as any equation involving indices.
- Solving may require the use of logarithms with base $e$, the natural logarithms.
- The laws of logarithms apply, using the notation $\log_e(x)$ or $\ln(x)$ as found on your calculator.
- Since $e^x > 0$, not all equations have real solutions. For example, $e^x = -1$ has no real solution.

### WORKED EXAMPLE 16 Solving an equation in $e$

Solve $6e^x = 15 + e^x$ for $x$, giving your answer:
a. as an exact number
b. correct to 3 decimal places.

| THINK | WRITE |
|---|---|
| a. 1. Write the equation. | a. $\quad 6e^x = 15 + e^x$ |
| 2. Collect like terms. | $5e^x = 15$ |
| 3. Find $e^x$. | $e^x = 3$ |
| 4. Take the log of both sides. (*Note:* $\log_e(x) = \ln(x)$) | $\ln(e^x) = \ln(3)$ |
| 5. State the solution as an exact number. | $x = \ln(3)$ |
| b. 1. Use your calculator to determine the approximation. | b. $x = 1.098\,61$ |
| 2. State the solution correct to 3 decimal places. | $x = 1.099$ |

### WORKED EXAMPLE 17 Solving quadratic equations in $e$

Solve $e^x - 3e^{-x} = 2$ for $x$, giving your answer(s) correct to 2 decimal places.

| THINK | WRITE |
|---|---|
| 1. Write the equation. | $e^x - 3e^{-x} = 2$ |
| 2. Rewrite without negative indices. | $e^x - \dfrac{3}{e^x} = 2$ |

▶

| | |
|---|---|
| **3.** Multiply each term by $e^x$. | $(e^x)^2 - 3 = 2e^x$ |
| **4.** Recognise a quadratic equation in $e^x$. | $(e^x)^2 - 2e^x - 3 = 0$ |
| **5.** Let $a = e^x$. | $a^2 - 2a - 3 = 0$ |
| **6.** Factorise the quadratic expression. | $(a-3)(a+1) = 0$ |
| **7.** Solve for $a$. | $a = 3$ or $a = -1$ |
| **8.** Substitute $e^x$ for $a$. | $e^x = 3$ or $e^x = -1$ |
| **9.** Solve for $x$. | $\ln(e^x) = \ln(3),\ e^x \neq -1$ |
| | $x = \ln(3)$ |
| **10.** State the solution correct to 2 decimal places. | $x = 1.10$ (to 2 decimal places) |

## 1.6.2 Solving equations in $\ln(x)$

Determining solutions to equations involving $\ln(x)$:
- The laws of logarithms apply to $\log_e(x)$ or $\ln(x)$.
- Solving may require the use of the laws of indices.
- Equations in logarithms follow the normal rules for solving equations, such as combining like terms.
- Since $\ln(x)$ is only defined for $x > 0$, not all equations will have real solutions.

### WORKED EXAMPLE 18 Solving equations in $\ln(x)$ using the index laws

a. Solve $\log_e(x) = 0.5$, giving your answer correct to 2 decimal places.
b. Solve $2\ln(x) - \ln(3) = 4$ for $x$, giving your answer correct to 2 decimal places.

**THINK**

**WRITE**

a. **1.** Rewrite as an exponential equation.

$\log_e(x) = 0.5$
$x = e^{0.5}$

  **2.** Evaluate.

$x \approx 1.648\,721\,270\,7$

  **3.** Answer the question.

$x = 1.65$ (to 2 decimal places)

b. **1.** Apply log laws to combine the left-hand side.

$2\ln(x) - \ln(3) = 4$
$\ln(x^2) - \ln(3) = 4$
$\ln\left(\dfrac{x^2}{3}\right) = 4$

  **2.** Rewrite as an exponential equation.

$\dfrac{x^2}{3} = e^4$
$x^2 = 3e^4$

  **3.** Solve for $x$. Remember: $\log(a)$ is only defined for $a > 0$.

$x = \pm\sqrt{3e^4}$
$x = \sqrt{3e^4}$
$x = 12.798\,220\,58$

  **4.** Evaluate.

  **5.** Answer the question.

$x = 12.80$ (to 2 decimal places)

## WORKED EXAMPLE 19 Solving an equation in ln(x) using the logarithm laws

**Solve the following equation for $x$:**

$$\ln(2x + 1) + \ln(2x - 1) = 2\ln(3)$$

| THINK | WRITE |
|---|---|
| 1. Use log laws to combine terms:<br>LHS: $\log(m) + \log(n) = \log(mn)$<br>RHS: $n\log(m) = \log(mn)$ | $\ln(2x + 1) + \ln(2x - 1) = 3\ln(2)$<br>$\ln(2x + 1)(2x - 1) = \ln(2^3)$ |
| 2. Since both are in terms of ln, equate the two sides. | $(2x + 1)(2x - 1) = 2^3$ |
| 3. Expand and simplify. | $4x^2 - 1 = 8$<br>$4x^2 = 9$<br>$x = \pm\dfrac{3}{2}$ |
| 4. Remember $\log(a)$ is only defined if $a > 0$.<br>Hence, $x > \dfrac{1}{2}$ for $\ln(2x - 1)$ to be defined. | $x > \dfrac{1}{2}$ |
| 5. State the answer. | $\therefore x = \dfrac{3}{2}$ |

## Exercise 1.6 Solving equations

| **1.6 Exercise** | **1.6 Exam questions** on |
|---|---|

| Simple familiar | Complex familiar | Complex unfamiliar |
|---|---|---|
| 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 | 13, 14, 15, 16 | 17, 18, 19, 20 |

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- Access sample responses
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### Simple familiar

1. **WE16** Solve $3e^x + 8 = 5e^x$ for $x$, giving your answer:
   a. as an exact number
   b. correct to 3 decimal places.

2. Solve for $x$, giving your answer correct to 3 decimal places.
   a. $e^x = 5$
   b. $e^x = \dfrac{1}{2}$
   c. $e^x = 2.6$
   d. $e^{-x} = 6$
   e. $3 = 2e^x$
   f. $3e^{-x} - 10 = 0$

3. Solve for $x$ in each of the following, giving your answer in exact form.
   a. $(e^x - 1)(e^x - 2) = 0$
   b. $(e^x - 1)(e^x + 3) = 0$
   c. $(e^{-x} - 1)(e^{2x} - 4) = 0$
   d. $(3e^{-x} - 2)(2e^x - 1) = 0$
   e. $(2e^x + 1)(e^x - 4) = 0$
   f. $(3e^x - 2)(e^x + 4) = 0$

4. Solve the following equations for $x$.

   a. $e^{x-2} - 2 = 7$    b. $e^{\frac{x}{4}} + 1 = 3$    c. $e^{2x} = 3e^x$    d. $e^{x^2} + 2 = 4$

5. **WE17** Solve $e^x - 15e^{-x} = 2$ for $x$, giving your answer(s) correct to 2 decimal places.

6. Solve for $x$ in each of the following, giving your answers in exact form.

   a. $5e^x - 12e^{-x} - 11 = 0$    c. $2e^x = 9 + 5e^{-x}$
   b. $3e^x + 6e^{-x} = 11$    d. $e^x = 25e^{-x}$

7. Solve the following equations for $x$.

   a. $e^{2x} = e^x + 12$    b. $e^x = 12 - 32e^{-x}$    c. $e^{2x} - 4 = 2e^x$    d. $e^x - 12 = -\dfrac{5}{e^x}$

8. **WE18a** Solve for $x$ in each of the following. Give exact answers when appropriate; otherwise, give answers correct to 3 decimal places.

   a. $\log_e(x) = 1$    b. $\log_e(x) = 2$    c. $\log_e(x) = -2$

   d. $\log_e(x) = -1$    e. $\log_e(x) = 0.3$    f. $\log_e(x) = -0.69$

9. Solve the following for $x$. Give exact answers when appropriate; otherwise, give answers correct to 3 decimal places.

   a. $\log_e(2x) = 2$    b. $\log_e(3x) = 1$    c. $\log_e(x^3) = 3$

10. Solve the following for $x$. Give exact answers when appropriate; otherwise, give answers correct to 3 decimal places.

    a. $\log_e(x - 1) = -1$    b. $\log_e(2x + 1) = -2$    c. $\log_e(-x) = 0.36$

    d. $\log_e(-x) = 0.72$    e. $\log_e(1 - x) = -0.54$    f. $\log_e(2 + x) = -0.83$

11. **WE18b** Solve the following for $x$, giving your answer in exact form and correct to 2 decimal places.

$$2\ln(x) + 6 = \ln(9)$$

12. **WE19** Solve the following for $x$. Give exact answers when appropriate; otherwise, give answers correct to 3 decimal places.

    a. $\log_e(x) + \log_e(5) = 8$    b. $2\ln(x) - \ln(5) = 9$

    c. $1 + \ln(x) = \ln(6)$    d. $2 - \log_e(x) = \log_e 10$

### Complex familiar

13. Solve the following for $x$, giving exact answers.

    a. $\log_e(x) + \log_e(5) - \log_e(10) = \log_e(3)$    b. $\log_e(x) + \log_e(3) - \log_e(9) = \log_e(4)$

    c. $2\log_e(3) + \log_e(x) - \log_e(2) = \log_e(3)$    d. $3\log_e(2) + \log_e(x) - \log_e(4) = \log_e(5)$

    e. $\log_e(6) + \log_e(2) - \log_e(x) = \log_e(4)$    f. $\log_e(4) + \log_e(3) - \log_e(x) = \log_e(2)$

14. Solve the following for $x$, giving exact answers.

    a. $\log_e(x) + \log_e(x + 1) = \log_e(2)$    b. $\log_e(x) + \log_e(2x - 1) = \log_e(3)$

    c. $\log_e(x - 1) + \log_e(x + 2) = \log_e(4)$    d. $\log_e(x + 1) + \log_e(2x - 1) = \log_e(5)$

15. If $\log_e y = \log_e(x) + \log_e(a)$, determine an equation relating $x$ and $y$ that does not involve logarithms.

16. Determine the value of $x$ if $2\log_e(x) - \log_e(3x) = a$.

**17.** Discuss whether the following equation can be expressed with $y$ as the subject.

$$2\log_e(x) + 1 = \log_e(y)$$

**18.** Determine the value(s) of $x$ for the following equation and discuss the reasonableness of your solution.

$$2e^{mx} = 5 + 4e^{-mx}, \text{ where } m \in \mathbb{R} \setminus \{0\}$$

**19.** If $y = ae^{-kx}$, $y = 3.033$ when $x = 2$ and $y = 1.1157$ when $x = 6$, determine the values of the constants $a$ and $k$. Give answers correct to 2 decimal places, and verify your solutions

**20.** The compound interest formula $A = Pe^{rt}$ is an indicial equation, where $A$ is the amount of interest, $P$ is the principal, $r$ is the annual interest rate and $t$ is the number of years. If a principal amount of money, $P$, is invested for 5 years, the interest earned is \$12 840.25, but if this same amount is invested for 7 years, the interest earned is \$14 190.66. Determine the integer rate of interest and the principal amount of money invested, to the nearest dollar.

**Fully worked solutions for this chapter are available online.**

# LESSON
# **1.7** Differentiation of logarithmic functions

## SYLLABUS LINKS

- Use the rules $\dfrac{d}{dx}\ln(x) = \dfrac{1}{x}$ and $\dfrac{d}{dx}\ln\big(f(x)\big) = \dfrac{f'(x)}{f(x)}$

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## 1.7.1 Proof for the derivative of $y = \ln(x)$

The proof for the derivative of $y = \log_e(x)$ relies on its link to its inverse function $y = e^x$ along with the chain rule.

Let $y = \log_e(x)$.

Then $x = e^y$.

Differentiate both sides with respect to $x$:

$$\frac{d}{dx}(x) = \frac{d}{dx}(e^y)$$

$$1 = e^y \times \frac{dy}{dx} \text{ (using the formula } \frac{d}{dx}(e^{f(x)}) = f'(x) \times e^{f(x)})$$

$$\frac{dy}{dx} = \frac{1}{e^y}$$

Substitute $x = e^y$: $\dfrac{dy}{dx} = \dfrac{1}{x}, x > 0$ (*Note:* The log function $y = \log_e(x)$ has a restricted domain, $x > 0$.)

## 1.7.2 The derivative of $y = \ln f(x)$

Let $y = \log_e f(x)$. Then $f(x) = e^y$.

Differentiate both sides with respect to $x$:

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}(e^y)$$

$f'(x) = e^y \times \dfrac{dy}{dx}$ (using the formula $\dfrac{d}{dx}\left(e^{f(x)}\right) = f'(x) \times e^{f(x)}$)

$$\frac{dy}{dx} = \frac{f'(x)}{e^y}$$

Substitute $f(x) = e^y$: $\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$

*Note:* The above rules are only applicable for logarithmic functions of base $e$.

For example, if $y = \log_e(4x)$, $\dfrac{dy}{dx} = \dfrac{4}{4x} = \dfrac{1}{x}$.

---

### The derivative of the logarithmic function

$$\frac{d}{dx}\ln(x) = \frac{1}{x} \text{ for } x > 0$$

$$\frac{d}{dx}\ln(f(x)) = \frac{f'(x)}{f(x)}$$

---

### WORKED EXAMPLE 20 Differentiating logarithmic functions 1

**Differentiate the following functions by first simplifying using the log laws.**
**a.** $y = 3\ln(2x)$
**b.** $y = 3\ln\left(\sqrt{x}\right)$

**THINK**

**WRITE**

**a. 1.** Simplify the function by applying the log laws.

**a.** $y = 3\ln(2x)$
$y = 3(\ln(2) + \ln(x))$
$y = 3\ln(2) + 3\ln(x)$

**2.** Differentiate the function.

$\dfrac{dy}{dx} = 0 + 3 \times \dfrac{1}{x}$

**3.** Simplify.

$\dfrac{dy}{dx} = \dfrac{3}{x}$

**b. 1.** Rewrite the function using fractional indices.

**b.** $y = 3\ln\left(\sqrt{x}\right)$

$y = 3\ln\left(x^{\frac{1}{2}}\right)$

**2.** Simplify by applying the log laws.

$$y = 3 \times \frac{1}{2} \times \ln(x)$$

$$y = \frac{3}{2} \ln(x)$$

**3.** Differentiate the function.

$$\frac{dy}{dx} = \frac{3}{2} \times \frac{1}{x}$$

**4.** Simplify.

$$\frac{dy}{dx} = \frac{3}{2x}$$

---

## WORKED EXAMPLE 21 Differentiating logarithmic functions 2

**Differentiate the following using the formula** $\dfrac{d}{dx}(\ln(f(x))) = \dfrac{f'(x)}{f'(x)}.$

**a.** $y = \ln(2x + 1)$  
**b.** $y = 4\ln(-3x), x < 0$

**c.** $y = 5\ln(x^2)$  
**d.** $y = \log_e(3x^2 - 13x + 15)$

| THINK | WRITE |
|---|---|
| **a. 1.** Let $y = \ln(2x + 1) = \ln(f(x))$. State $f(x)$ and $f'(x)$. | **a.** $f(x) = 2x + 1$ <br> $f'(x) = 2$ |
| **2.** Substitute into the formula $\dfrac{d}{dx}(\ln(f(x))) = \dfrac{f'(x)}{f(x)}$. | $\dfrac{dy}{dx} = \dfrac{2}{2x+1}$ |
| **b. 1.** Let $y = 4\ln(-3x) = 4\ln f(x), x < 0$. State $f(x)$ and $f'(x)$. | **b.** $f(x) = -3x$ <br> $f'(x) = -3$ |
| **2.** Substitute into the formula $\dfrac{d}{dx}(\ln(f(x))) = \dfrac{f'(x)}{f(x)}$. | $\dfrac{dy}{dx} = 4 \times \dfrac{-3}{-3x}$ |
| **3.** Simplify. <br> *Note:* $\dfrac{d}{dx}(\log_e(kx)) = \dfrac{1}{x}$ may be useful. | $\dfrac{dy}{dx} = \dfrac{4}{x}, x < 0$ |
| **c. 1.** Let $y = 5\ln(x^2) = 5\ln(f(x))$. State $f(x)$ and $f'(x)$. | **c.** $f(x) = x^2$ <br> $f'(x) = 2x$ |
| **2.** Substitute into the formula $\dfrac{d}{dx}(\ln(f(x))) = \dfrac{f'(x)}{f(x)}$. | $\dfrac{dy}{dx} = 5 \times \dfrac{2x}{x^2}$ |
| **3.** Simplify. <br> *Note:* This function could have been rewritten as $y = 10\ln(x)$ using log laws. | $\dfrac{dy}{dx} = \dfrac{10}{x}$ |
| **d. 1.** Let $y = \log_e(3x^2 - 13x + 15) = \log_e(f(x))$. State $f(x)$ and $f'(x)$. | **d.** $f(x) = 3x^2 - 13x + 15$ <br> $f'(x) = 6x - 13$ |
| **2.** Substitute into the formula. | $\dfrac{dy}{dx} = \dfrac{6x - 13}{3x^2 - 13x + 15}$ |

# Exercise 1.7 Differentiation of the logarithmic functions

| 1.7 Exercise | 1.7 Exam questions **on** |
|---|---|

| Simple familiar | Complex familiar | Complex unfamiliar |
|---|---|---|
| 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 | 13, 14, 15, 16 | 17, 18, 19, 20 |

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## Simple familiar

1. **WE20** Differentiate each of the following with respect to $x$.

   a. $y = \ln(10x)$
   b. $y = \ln(5x)$
   c. $y = \ln(-x)$
   d. $y = \log_e(-6x)$
   e. $y = 3\log_e(4x)$
   f. $y = -6\log_e 9x$

2. Differentiate each of the following with respect to $x$.

   a. $y = \ln\left(\dfrac{x}{2}\right)$
   b. $y = \ln\left(\dfrac{x}{3}\right)$
   c. $y = 4\ln\left(\dfrac{x}{5}\right)$
   d. $y = -5\ln\left(-\dfrac{2x}{3}\right),\ x < 0$

3. **MC** The derivative of $\log_e(8x)$ is:

   A. $\dfrac{1}{8}x$
   B. $\dfrac{8}{x}$
   C. $\dfrac{1}{x}$
   D. $\log_e 8$

4. **WE21** Differentiate each of the following with respect to $x$.

   a. $y = \ln(2x + 5)$
   b. $y = \ln(6x + 1)$
   c. $y = \ln(3x - 4)$
   d. $y = \log_e(8x - 1)$
   e. $y = \log_e(3 - 5x)$
   f. $y = \log_e(2 - x)$

5. Differentiate the following with respect to $x$.

   a. $y = 6\ln(5x + 2)$
   b. $y = 8\ln(4x - 2)$
   c. $y = -4\ln(12x + 5)$
   d. $y = -7\ln(8 - 9x)$

6. Differentiate the following with respect to $x$.

   a. $y = \log_e(3x^4)$
   b. $y = \log_e(x^2 + 3)$
   c. $y = \log_e(x^2 + 4x)$
   d. $y = \ln(x^2 - 3x + 2)$
   e. $y = \ln(x^3 + 2x^2 - 7x)$
   f. $y = \ln(x^2 - 2x^3 + x^4)$

7. **MC** The derivative of $f(x) = \ln(x^2 - 5x + 2)$ is:

   A. $\dfrac{1}{x^2 - 5x + 2}$
   B. $2x - 5$
   C. $\dfrac{1}{x(2x - 5)}$
   D. $\dfrac{2x - 5}{x^2 - 5x + 2}$

8. **MC** The derivative of $\log_e(3x - 2)$ is:

   A. $\dfrac{1}{3x - 2}$
   B. $\dfrac{1}{3x}$
   C. $\dfrac{1}{3(3x - 2)}$
   D. $\dfrac{3}{3x - 2}$

9. **MC** The derivative of $2\log_e(x^2 + x)$ is:

   A. $\dfrac{2(2x + 1)}{x^2 + x}$
   B. $\dfrac{2x + 1}{x^2 + x}$
   C. $\dfrac{2x}{x^2 + x}$
   D. $\dfrac{4x}{x^2 + x}$

10. Calculate the gradient of the tangent to the following functions at the specified point.

    a. $y = 2\ln(x),\ x = 5$

    b. $y = \dfrac{1}{3}\ln(4x + 1),\ x = 2$

    c. $y = \ln(x^2 + 3),\ x = 3$

11. Find the equation of the tangent to each of the given curves at the specified point.

a. $y = \log_e(2x - 2)$ at $\left(\dfrac{3}{2}, 0\right)$

b. $y = 3\log_e(x)$ at $(e, 3)$

c. $y = \dfrac{1}{2}\log_e x^2$ at $(e, 1)$

12. a. Calculate the gradient of the curve $y = 3\log_e(x - 5)$ at the point where $x = 6$.
   b. Hence, find the equations of the tangent to the curve and the line perpendicular to the curve at the point where $x = 6$.

**Complex familiar**

13. **MC** If $y = \log_e \sqrt{x^2 - 6x + 9}$, then the derivative is:

A. $\dfrac{2x - 6}{\sqrt{x^2 - 6x + 9}}$

B. $\dfrac{1}{x - 3}$

C. $\dfrac{x - 3}{\sqrt{x^2 - 6x + 9}}$

D. $\dfrac{1}{(x - 3)^2}$

14. Simplify each of the following using the log laws and differentiate with respect to $x$.

a. $y = \ln\left(\sqrt{2x + 1}\right)$

b. $y = \ln\left(\sqrt{3 - 4x}\right)$

c. $y = \ln\left(\sqrt{x^2 + 2}\right)$

d. $y = \ln(x + 3)^{\frac{1}{4}}$

15. Simplify each of the following using the log laws and differentiate with respect to $x$.

a. $f(x) = \ln\left(\dfrac{1}{x + 3}\right)$

b. $f(x) = \ln(3x - 2)^4$

c. $f(x) = \ln(5x + 8)^{-2}$

d. $f(x) = \ln\left(\dfrac{2}{4 + 3x}\right)$

16. The graph of the function defined by the rule $y = 2\log_e(2x)$ is shown. Determine the equation of the tangent at $\left(\dfrac{e}{2}, e\right)$.
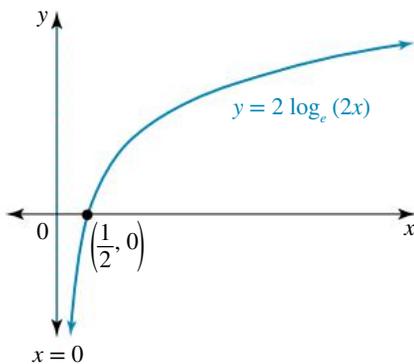


**Complex unfamiliar**

17. Obtain the equation of the tangent to the curve $y = 4\log_e(3x - 1)$ at the point where the tangent is parallel to the line $6x - y + 2 = 0$.

18. The line $y = x$ is a tangent to the curve $y = \log_e(x - 1) + b$, where $b$ is a constant. Determine the value of $b$.

19. The equation of a line perpendicular to the curve $y = \log_e(2(x - 1))$ has the equation $y = -2x + k$, where $k$ is a constant. Calculate the value of $k$, correct to 1 decimal place.

20. The tangent to the curve $y = \log_e(2x - 1)$ at $x = n$ intersects the $x$-axis at $x = 0.3521$. Calculate the value of the integer constant $n$.

**Fully worked solutions for this chapter are available online.**

# LESSON
## **1.8** Modelling and solving with exponential and logarithmic functions

## 1.8.1 Applications of exponential functions

Exponential functions are commonly used to model a number of real-life situations — population grown and decay, cellular or bacterial growth and decay, radioactive decay, Newton's law of cooling, and investments.

The following general equation can be used to represent exponential growth and decay.

---

**Exponential growth and decay**

$$P = P_0 e^{kt}$$

where $P_0$ is the initial amount and $k$ is a constant.

For growth: $k > 0$

For decay: $k < 0$

---

In modelling that involves exponential growth and decay, the rate of change of the function often implies the change with respect to time, $\dfrac{dy}{dt}$.

It may be necessary to restrict the domain of the exponential function to suit the context of the problem. For example, time ($t$) cannot be negative.

---

### WORKED EXAMPLE 22 Applying exponential growth

The number of bacteria on a culture plate, $N$, can be defined by the rule

$$N(t) = 2000e^{0.3t}, t \geq 0$$

where $t$ is the time, in seconds, the culture has been growing.
a. Determine how many bacteria were initially present.
b. Calculate how many bacteria, to the nearest whole number, are present after 10 seconds.
c. Determine the rate at which the bacteria population is increasing after 10 seconds. Give your answer correct to the nearest whole number.

| THINK | WRITE |
|---|---|
| a. 1. The initial time is when $t = 0$. Substitute $t = 0$ and evaluate. | a. $N(t) = 2000e^{0.3t}$ <br> $N(0) = 2000e^{0.3(0)}$ <br> $= 2000$ |

---

**2.** Write the answer.  Initially there were 2000 bacteria present.

**b. 1.** After 10 seconds, $t = 10$. Substitute $t = 10$.

**b.** $N(t) = 2000e^{0.3t}$

$N(10) = 2000e^{0.3(10)}$

$= 2000e^3$

$= 40\,171.074$

**2.** Write the answer.  After 10 seconds there were 40 171 bacteria present.

**c. 1.** Differentiate $\left( \dfrac{d(e^{f(x)})}{dx} = f'(x)e^{f(x)} \right)$ to find the rate of change with respect to time.

**c.** $N(t) = 2000e^{0.3t}$

$\dfrac{dN}{dt} = 2000 \times 0.3e^{0.3t}$

$\dfrac{dN}{dt} = 600e^{0.3t}$

**2.** After 10 seconds, $t = 10$. Substitute $t = 10$ and evaluate.

$\dfrac{dN}{dt} = 600e^{0.3(10)}$

$= 600e^3$

$= 12\,051.322$

**3.** Write the answer with correct units.  After 10 seconds the bacteria are increasing at a rate of 12 051 bacteria/second.

---

**TI | THINK**  **WRITE**  **CASIO | THINK**  **WRITE**

**a. 1.** On a Calculator page, press MENU, then select: 2: Add Graphs. Complete the entry line in the $f1(x) = 2000e^{0.3x}$

**a. 1.** On a Main Menu screen, select: Graph. Complete the entry line in the Y1 tab as: $2000e^{0.3x}$ *Note:* The independent variable $t$ has been replaced with $x$.

**2.** Sketch the graph by pressing the ENTER button.

**2.** Sketch the graph by pressing either the DRAW or EXE button.

**3.** To calculate the initial value, select: Menu 8: Geometry 1: Points & Lines 2: Point On.

**3.** To calculate the initial value, select: G-Solv (SHIFT) Y-CAL. Complete the entry line in X: as 0.
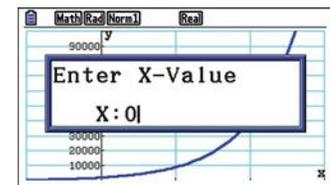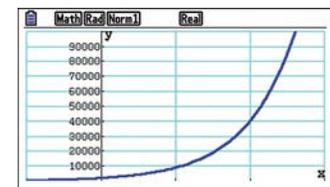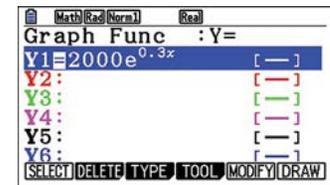
**4.** Move the cursor and select the curve representing $f1(x)$. Select the point of $x = 0$.

**4.** The answer appears on the screen. In this case, when $x = 0, y = 2000$.



**5.** The answer appears on the screen. In this case, when $x = 0, y = 2000$.



**5.** To calculate the number of bacteria after 10 seconds, select: G-Solv (SHIFT F5) Y-CAL. Complete the entry line in X: as 10.
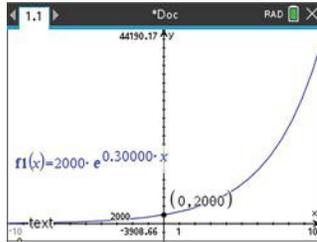


**b. 1.** Complete the entry line in the textbox as 10 for the $x$-value. The answer appears on the screen. In this case, when $x = 10$, $y = 40\ 171.074$.
*Note:* You can also calculate an $x$-value by entering a desired $y$-value.



**b. 1.** The answer appears on the screen. In this case, when $x = 10$, $y = 40\ 171.074$.
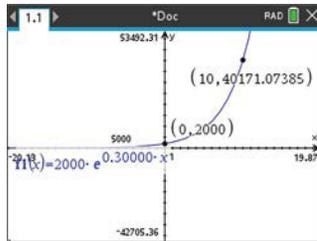


**c. 1.** On a Calculator page, press MENU, then select:
4: Calculus
1: Numerical Derivative at a Point.



**c. 1.** On a Run Matrix screen, select: Math $\dfrac{d}{dx}$.
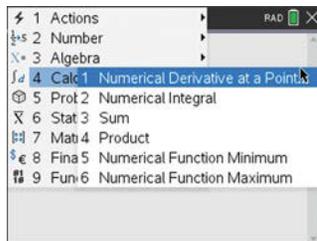


**2.** Complete the value entry line as: 10 Press the OK button.



**2.** Complete the entry line as:
$\dfrac{d}{dx}(2000e^{0.3x})$
Enter 10 in the $x$-value box. Press the EXE button to complete the calculation.
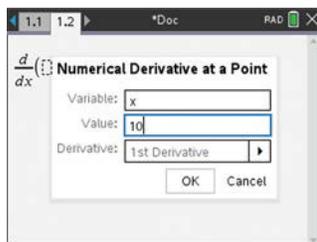


**3.** Complete the entry line as: Press the ENTER button to complete the calculation.
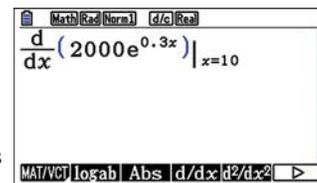


**3.** The answer appears on the screen. In this case, when $x = 10$,
$\dfrac{dy}{dx} = 12\ 051.322.$
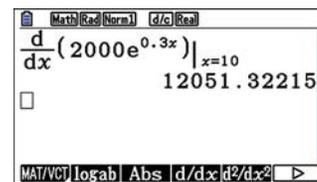
## WORKED EXAMPLE 23 Applying exponential decay

The mass, $M$ g, of a radioactive substance is initially 20 g. The substance has been stored for 30 years already. The mass in any year is given by

$$M(t) = M_0 e^{-0.00152t}$$

where $M_0$ is a constant and $t$ is the time in years.
a. Determine the value of $M_0$.
b. Calculate the amount of the substance remaining after a further 30 years. Give your answer correct to 2 decimal places.
c. Determine the rate of decay at this time. Give your answer correct to 2 decimal places.

| THINK | WRITE |
|---|---|
| a. 1. The mass, $M$ g, of a radioactive substance is initially 20 g. Substitute $t = 0, M(0) = 20$. | a. $M(t) = M_0 e^{-0.00152t}$ <br> $M(0) = M_0 e^{-0.00152(0)} = 20$ <br> $M_0 e^0 = 20$ <br> $\Rightarrow M_0 = 20$ |
| 2. Write the answer. | $M_0 = 20$ |
| b. 1. Rewrite the equation with $M_0 = 20$. | b. $M(t) = 20e^{-0.00152t}$ |
| 2. The mass, $M$ g, of a radioactive substance is initially 20 g. After 30 years ($t = 30$), a further 30 years is when $t = 60$. Substitute $t = 60$ and evaluate. | $M(60) = 20e^{-0.00152(60)}$ <br> $= 18.2567$ |
| 3. Write the answer. | After a further 30 years, the mass is 18.26 g. |
| c. 1. Differentiate to find the rate of change with respect to time. | c. $M(t) = 20e^{-0.001\,52t}$ <br> $M'(t) = 20 \times (-0.00152)\, e^{-0.00152t}$ <br> $M'(t) = -0.0304e^{-0.00152t}$ |
| 2. Evaluate the rate of change at $t = 60$ by substitution. | $M'(60) = -0.0304e^{-0.00152(60)}$ <br> $= -0.027\,75$ |
| 3. Answer the question with correct units. | The rate of decay after 60 years is 0.03 g/year. *Note:* The question asked for the rate of decay, so the negative sign is not included in the final answers. The negative indicates a rate that is decreasing. |

## WORKED EXAMPLE 24 Applying exponential decay with horizontal asymptote

The population of foxes on the outskirts of a city is starting to increase.
Data collected suggest that a model for the number of foxes is given by $N(t) = 480 - 320e^{-0.3t}$, $t \geq 0$
where $N$ is the number of foxes $t$ years after the observations began.

a. Determine how many foxes were present initially at the start of the observations.
b. Determine by how many the population of foxes had grown at the end of the first year of observations.
c. Determine after how many months the model predicts the number of foxes would be double the initial population.
d. Sketch the graph of $N$ versus $t$.
e. Explain why this model does not predict that the population of foxes will grow to 600.

**THINK**

**WRITE**

a. 1. Calculate the initial number, that is when $t = 0$.

a. $N(t) = 480 - 320e^{-0.3t}$ When $t = 0$,
$N(0) = 480 - 320e^0$
$= 480 - 320$
$= 160$

2. Answer the question.

There were 160 foxes present initially.

b. 1. Calculate the number of foxes, $N$, after 1 year, $t = 1$.

b. When $t = 1$, $N(1) = 480 - 320e^{-0.3}$
$\approx 242.94$
After the first year 243 foxes were present.

2. Express the change over the first year in context.

Over the first year the population grew from 160 to 243, an increase of 83 foxes.

c. 1. Calculate the required value of $t$.
*Note:* An algebraic method requiring logarithms has been used here.

c. Let $N = 2 \times 160 = 320$.
$320 = 480 - 320e^{-0.3t}$
$320e^{-0.3t} = 160$
$e^{-0.3t} = \dfrac{1}{2}$
$-0.3t = \log_e\left(\dfrac{1}{2}\right)$
$t = -\dfrac{1}{0.3}\log_e\left(\dfrac{1}{2}\right)$
$t \approx 2.31$

2. Answer the question.

$0.31 \times 12 \approx 4$
The population doubles after 2 years and 4 months.

**d.** Sketch the graph.

**d.** $N(t) = 480 - 320e^{-0.3t}$
The horizontal asymptote is $N = 480$.
The $y$-intercept is $(0, 160)$.
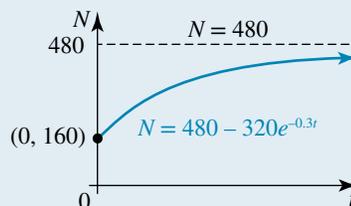


**e.** Give an explanation for the claim that this model will not predict the population of foxes to grow to 600.

**e.** The presence of an asymptote on the graph shows that as $t \to \infty$, $N \to 480$ Hence, $N$ can never reach 600. The population will never exceed 480 according to this model.

## 1.8.2 Applications of logarithmic functions

Functions involving the logarithmic function, along with its inverse, the exponential function, can be used to model real-life situations.

The derivatives of logarithmic functions are used to determine:
- the gradient of a logarithmic curve at a given point
- the rate of change of the function with respect to time, $\dfrac{dy}{dt}$.

As with the exponential function, it may be necessary to restrict the domain of the logarithmic function to suit the context of the problem. For example, time ($t$) cannot be negative.

### WORKED EXAMPLE 25 Applying logarithmic functions to equations of tangents

The graph of the function $f(x) = 0.5\ln(x - 1)$ is shown.
**a.** State the domain and range of $f$.
**b.** Calculate the value of the constant a given that $(a, 0)$ is the $x$-axis intercept.
**c.** Determine the equation of the tangent at $(a, 0)$.
**d.** Determine the equation of the line perpendicular to the curve at $(a, 0)$.
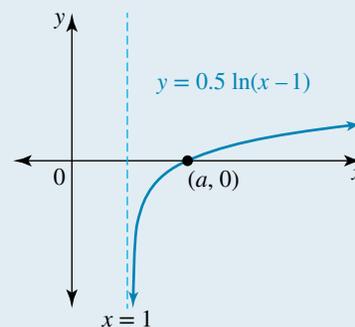


**THINK**

**a.** State the domain and range of the function.

**b. 1.** To find the $x$-intercept, let $f(x) = 0$.

**2.** Solve $0.5\log_e(x - 1) = 0$ for $x$.

**WRITE**

**a.** Domain $= (1, \infty)$
Range $= \mathbb{R}$

**b.** $0.5\ln(x - 1) = 0$

$\ln(x - 1) = 0$
$e^0 = x - 1$
$1 = x - 1$
$x = 2$

**3.** Answer the question.

$(a, 0) \equiv (2, 0)$
$\therefore a = 2$

**c. 1.** Determine the derivative of the function.

**c.** $f(x) = 0.5 \ln(x-1)$

$f'(x) = \dfrac{1}{2} \times \dfrac{1}{x-1}$

$\quad\;\, = \dfrac{1}{2(x-1)}$

**2.** Substitute $x = 2$ into the derivative to find the gradient at this point.

$f'(2) = \dfrac{1}{2(2-1)}$

$\quad\;\;\, = \dfrac{1}{2}$

The gradient at $x = 2$ is $\dfrac{1}{2}$.

**3.** State the general equation for a tangent.

The equation of the tangent is
$y - y_1 = m_T(x - x_1)$.

**4.** State the known information.

The gradient of the tangent at
$(x_1, y_1) = (2, 0)$ is $m_T = \dfrac{1}{2}$.

**5.** Substitute the values into the general equation.

$y - 0 = \dfrac{1}{2}(x - 2)$

**6.** Simplify.

$y = \dfrac{1}{2}x - 1$

**d. 1.** A perpendicular line has a gradient of $m_p = -\dfrac{1}{m_T}$.

**d.** $m_p = -\dfrac{1}{\frac{1}{2}}$

$m_p = -2$

**2.** State the general equation for a line perpendicular to the tangent.

$y - y_1 = m_p(x - x_1)$

**3.** State the known information.

$m_p = -2, \; (x_1, y_1) = (2, 0)$

**4.** Substitute the values into the general equation.

$y - 0 = -2(x - 2)$

**5.** Simplify.

$y = -2x + 4$

# Exercise 1.8 Modelling and solving with exponential and logarithmic functions

| Simple familiar | Complex familiar | Complex unfamiliar |
|---|---|---|
| 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 | 13, 14, 15, 16 | 17, 18, 19, 20 |

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## Simple familiar

**1.** **WE22** The number of bacteria on a culture plate, $N$, can be defined by the rule

$$N(t) = 500e^{0.46t}, t \geq 0$$

where $t$ is the time, in hours, that the culture has been growing.

**a.** Determine how many bacteria were initially present.
**b.** Calculate how many bacteria, to the nearest whole number, are present after 5 hours.
**c.** Determine the rate at which the population of bacteria is increasing after 5 hours. Give your answer correct to the nearest whole number.

**2.** The bilby is an endangered species that can be found in the Kimberley in Western Australia as well as some parts of South Australia, the Northern Territory and Queensland. The gestation time for a bilby is 2–3 weeks and when they are born, they are only about 11 mm in length. The growth of a typical bilby can be modelled by the rule

$$L = L_0 e^{0.599t}$$

where $L_0$ is its length in millimetres at birth and $L$ is the length of the bilby in millimetres $t$ months after its birth.

**a.** Determine the value of $L_0$.
**b.** Determine the rate of change of length of the bilby at time $t$ months.
**c.** Determine the rate at which the bilby is growing when it is 3 months old. Give your answer correct to 3 decimal places.

**3.** **WE23** The mass, $M$ grams, of a particular radioactive substance can be modelled by the exponential function

$$M(t) = M_0 e^{-0.005t}$$

where $M_0$ is a constant and $t$ is the time in days, Initially the substance weighed 50 grams.

**a.** Determine the value of $M_0$.
**b.** Calculate the amount of the substance remaining after 10 days. Give your answer correct to 2 decimal places.
**c.** Determine the rate of decay at this time. Give your answer correct to 2 decimal places.

**4.** Changing $\delta$-gluconolactone into gluconic acid can be modelled by the equation $y = y_0 e^{-0.6t}$ where $y$ is the number of grams of $\delta$-gluconolactone present $t$ hours after the process has begun. Suppose 200 grams of $\delta$-gluconolactone is to be changed into gluconic acid.

  **a.** Determine the value of $y_0$.

  **b.** Determine how many grams of $\delta$-gluconolactone will be present after 1 hour. Give your answer correct to the nearest gram.

  **c.** Determine how long it will take to reduce the amount of $\delta$-gluconolactone to 50 grams. Give your answer correct to the nearest quarter of an hour.

  **d.** Determine the rate of change in the $\delta$-gluconolactone after 2 hours. Give your answer to 1 decimal place.

**5.** The decay of radon-222 gas is given by the equation $y = y_0 e^{-0.18t}$, where $y$ is the amount of radon remaining after $t$ days. If initially there was 10 grams of radon-222 gas, determine:

  **a.** the value of $y_0$

  **b.** the amount of gas, to the nearest integer, remaining after 2 days

  **c.** how many days it will take for the mass to reach half its original mass

  **d.** the rate of decay after 5 days, correct to 2 decimal places.

**6.** An amount of \$1000 is invested in a building society where the 5% p.a. interest is compounded continuously. The amount in the account after $t$ years can be modelled by the equation $A = A_0 \times e^{rt}$, where $A_0$ is the initial value of the investment and $r$ is the continuous interest rate expressed as a decimal.

  **a.** State the value of $A_0$ and $r$.

  **b.** Calculate the amount in the account, correct to the nearest cent, after:

    **i.** 1 year           **ii.** 5 years           **iii.** 10 years.

  **c.** Determine the rate at which the investment was increasing, correct to the nearest cent per year, after:

    **i.** 1 year           **ii.** 5 years           **iii.** 10 years.

  **d.** Estimate how long, to the nearest year, it would take for the investment to double in value.

**7.** **WE25** The graph of the function $f\colon (2, \infty) \to \mathbb{R}, f(x) = 2\ln(x - 2)$ is shown.

  **a.** State the domain and range of $f$.

  **b.** Find the value of the constant $a$, given that $(a, 0)$ is the $x$-axis intercept.

  **c.** Find the equations of the tangent at $(a, 0)$.

  **d.** Find the equation of the line perpendicular to the curve at $(a, 0)$.

**8.** **a.** Calculate the gradient of the curve $y = 3\log_e(x - 5)$ at the point where $x = 6$.

  **b.** Hence, find the equations of the tangent to the curve and the line perpendicular to the curve at the point where $x = 6$.
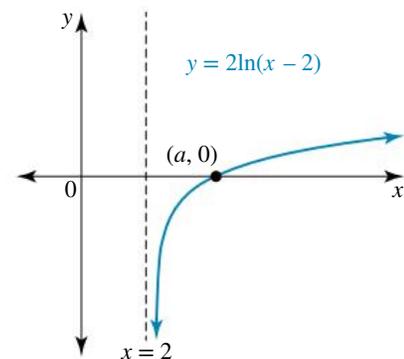
**9.** A body that is at a higher temperature than its surroundings cools according to Newton's Law of Cooling, which states that

$$T = T_0 e^{-zt}$$

where $T_0$ is the original *excess* of temperature, $T$ is the excess of temperature in degrees Celsius after $t$ minutes and $z$ is a constant.

  **a.** The original temperature of the body was 95 °C and the temperature of the surroundings was 20 °C. Find the value of $T_0$.

  **b.** Determine the rate at which the temperature is decreasing after a quarter of an hour if it is known that $z = 0.034$. Give your answer correct to 3 decimal places.

**10.** The number of people living in Boomerville at any time, $t$ years, after the first settlers arrived can be modelled by the equation

$$P = P_0 e^{kt}$$

Initially, on 1 January 1850, Boomerville had a population of 500 people. By 1 January 1860, the population had grown to 675.

**a.** Determine the value of $P_0$.
**b.** Calculate the value of $k$ correct to 2 decimal places.
**c.** Using this value of $k$ to 2 decimal places, determine the population on 1 January 1900. Give your answer to the nearest 10 people.
**d.** Determine the rate at which the population was increasing on 1 January 1900. Give your answer to the nearest whole number.

**11.** The mass, $m$ kg, of a radioactive isotope remaining in a sample $t$ hours after observations began is given by the rule

$$m(t) = ae^{-kt}$$

Initially there is 2 kg of the isotope. After 3 hours the mass of the isotope has decreased to 1.1 kg.

**a.** Determine the values of $a$ and $k$. Give your answers correct to 1 decimal place where necessary.
**b.** Using these values, find the rate of change of the isotope as a function of $t$.
**c.** Calculate the rate of decay of the isotope after 6 hours. Give your answer to 2 decimal places.
**d.** Determine the half-life of this isotope, that is, the time it takes for the isotope to reduce to half its original mass. Give your answer correct to 1 decimal place.

**12.** An unstable gas decomposes in such a way that the amount present, $A$ units, at time $t$ minutes is given by the equation
$$A = A_0 e^{-kt}$$

where $k$ and $A_0$ are constants. It was known that initially there were 120 units of unstable gas.

**a.** Find the value of $A_0$.
**b.** After 2 minutes there were 90 units of the gas left. Find the value of $k$.
**c.** Determine the rate at which the gas is decomposing when $t = 5$. Give your answer correct to 3 decimal places.
**d.** Determine whether there will ever be no gas left. Explain your answer.

### Complex familiar

**13.** **WE24** The population of possums in an inner city suburb is starting to increase. Observations of the numbers present suggest a model for the number of possums in the suburb given by

$$P(t) = 83 - 65e^{-0.2t}, t \geq 0$$

where $P$ is the number of possums observed and $t$ is the time in months since observations began.

**a.** Determine how many possums were present at the start of the observations.
**b.** Determine by how many the population of possums had grown at the end of the first month of observations. Give your answer to the nearest whole number.
**c.** Determine when the model predicts the number of possums would be twice the initial population.
**d.** Sketch the graph of $P$ versus $t$.
**e.** Explain why this model does not predict the population of possums will grow to 100.

**14.** Manoj pours himself a mug of coffee but gets distracted by a phone call before he can drink the coffee. The temperature of the cooling mug of coffee is given by

$$T = 20 + 75e^{-0.062t}$$

where $T$ is the temperature of the coffee $t$ minutes after it was initially poured into the mug.

   **a.** Determine the initial temperature of the coffee when it was first poured.

   **b.** Sketch the graph of temperature, $T\,°C$ against time, $t$ minutes.

   **c.** Determine the temperature that the coffee will cool to if left unattended.

   **d.** Determine how long it takes for the coffee to reach a temperature of 65 °C.
Give your answer correct to 2 decimal places.

   **e.** Determine the rate of change in the temperature of the coffee after 10 minutes, correct to 1 decimal place. Explain why the rate of change is negative.

**15.** The population of Australia since 1950 can be modelled by the rule

$$P = P_0 e^{0.016t}$$

where $P_0$ is the population in millions at the beginning of 1950 and $P$ is the population in millions $t$ years after 1950. It is known that there were 8.2 million people in Australia at the beginning of 1950.

   **a.** Calculate the population in millions at the beginning of 2015, correct to 1 decimal place.

   **b.** Determine the year and month in which the population reached 20 million.

   **c.** Determine the rate of change of population at the turn of the century, namely the year 2000, correct to 2 decimal places.

   **d.** Determine the year in which the rate of increase of the population was predicted to exceed 400 000 people per year.

**16.** The pressure of the atmosphere, $P$ cm of mercury, decreases with the height, $h$ km above sea level, according to the law

$$P = P_0 e^{-kh}$$

where $P_0$ is the pressure of the atmosphere at sea level and $k$ is a constant. At 500 m above sea level, the pressure is 66.7 cm of mercury, and at 1500 m above sea level, the pressure is 52.3 cm of mercury.

   **a.** Determine the values of $P_0$ and $k$, correct to 2 decimal places.

   **b.** Find the rate at which the pressure is falling when the height above sea level is 5 km. Give your answer correct to 2 decimal places.

### Complex unfamiliar

**17.** The number of rats, $N$, in a derelict house $t$ months after it was last occupied is given by
$N = 25 + 95 \ln(t + 1)$.
Determine how long, correct to1 decimal place, it would take for the number of rats to double, and determine the rate of change in the number of rats after 4 months.

**18.** The graph shown is that of the function

$$f: \mathbb{R} \to \mathbb{R}, f(x) = e^{-2x} + ze^{-x} + 2$$

where $z$ is a constant.
Determine the equation of the tangent to the curve, $f$, at the point $(n, 0)$.
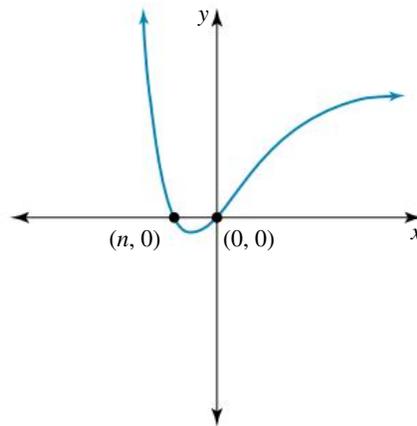


**19.** An entrance to a local suburban park has a series of posts connected with heavy chains as shown.
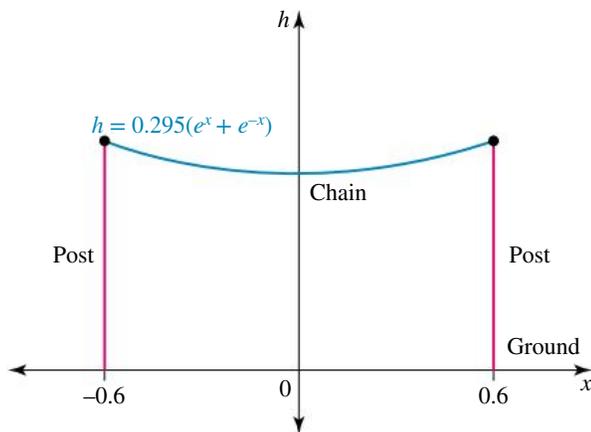


The chain between any two posts can be modelled by the curve defined by

$$h = 0.295(e^x + e^{-x}), -0.6 \le x \le 0.6$$

where $h$ metres is the height of the chain above the ground and $x$ metres is the horizontal distance from the midpoint of the posts. The $x$-axis represents the ground. The posts are positioned at $x = -0.6$ and $x = 0.6$. Calculate the angle the chain makes with the post positioned on the right-hand side of the structure (i.e. at $x = 0.6$). Give your answer correct to 1 decimal place.

**20.** Newton's Rule of Cooling states that the rate of change of the temperature of a particle is proportional to the difference between the temperature of the particle and the constant temperature of the surrounding medium. The temperature, $T\,°C$, of a particle when placed in a medium with a constant temperature of $A\,°C$ can be modelled by the equation

$$T = T_0 e^{-kt} + A$$

where $t$ is time in minutes and $T_0$ is a constant.
A metal ball has been heated to a temperature of $200\,°C$ and is placed into a room that is maintained at a constant temperature of $30\,°C$. After 5 minutes, the temperature of the ball has dropped to $150\,°C$.
Determine how long it will take for the temperature of the ball to reach $40\,°C$, giving your answer correct to 2 decimal places.
Verify, with reasons, that the metal ball would never reach $10\,°C$ if left in the room.

**Fully worked solutions for this chapter are available online.**

# LESSON
## 1.9 Review

## 1.9.1 Summary

## 1.9 Exercise

**learn**on

| 1.9 Exercise | 1.9 QCAA questions on |
|---|---|

| Simple familiar | Complex familiar | Complex unfamiliar |
|---|---|---|
| 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 | 13, 14, 15, 16 | 17, 18, 19, 20 |

### Simple familiar

1. Solve the following for $x$, giving your answers to 3 decimal places.
   a. $e^{x+1} = 6$
   b. $2e^{4-x} - 5 = 0$
   c. $e^{-2x} = 8$
   d. $4 - e^{x-2} = 0$

2. **MC** If $\log_e(2x) = a$, then $x$ is equal to:
   A. $2e^a$
   B. $2a^e$
   C. $\dfrac{e^a}{2}$
   D. $e^{2a}$

3. **MC** A solution for $x$ in $\log_e(x-3) + \log_e(x-2) = \log_e(12)$ is:
   A. 6 only
   B. 6 and $-1$
   C. $-1$ only
   D. 3 and 2

4. Solve the following for $x$, giving your answers in exact form.
   a. $e^{2x} - 2e^x = 0$
   b. $(e^x + 1)(e^x - 3) = 0$
   c. $e^{2x} + 2e^x = 8$
   d. $2e^{2x} - 9e^x + 4 = 0$

5. Sketch the following graphs and state the domain and range of each graph.
   a. $y = 2e^x + 1$
   b. $y = 3 - 3e^{-\frac{x}{2}}$
   c. $y = -\dfrac{1}{4}e^{x+1}$

6. Sketch the graphs of the following functions, showing all important features. For each graph, state the domain, the range and the equations of any asymptotes, and describe the transformations that have been applied to $y = \log_e(x)$ to achieve the function.
   a. $y = \ln(x+4)$
   b. $y = \ln(x-4)$
   c. $y = \ln(x) + 4$
   d. $y = 4 - \ln(x)$

7. Sketch the following graphs and state the domain and range of each graph.

   a. $y = -2e^x - 3$        b. $y = 4e^{-3x} - 4$        c. $y = 5e^{x-2}$

8. Differentiate the following functions with respect to $x$ and state any restrictions on $x$.

   a. $y = \dfrac{1}{2}\ln(x^2 - 2x + 7)$    b. $y = \ln\left(\dfrac{x+2}{x-3}\right)$      c. $y = \log_e(x+2)^2$

9. Differentiate the following with respect to $x$.

   a. $y = \ln\left(\dfrac{2x+1}{x-5}\right)$    b. $y = \log_e\left(\dfrac{7}{x-3}\right)$      c. $y = \ln(9x^2 - 6x + 7)$

10. Determine the derivative of each of the following functions.

    a. $y = e^{-\frac{1}{3}x}$                   b. $y = 3x^4 - e^{-2x^2}$

    c. $y = \dfrac{4e^x - e^{-x} + 2}{3e^{3x}}$        d. $y = \left(e^{2x} - 3\right)^2$

11. Consider the function defined by the rule $f(x) = \dfrac{1}{2}e^{3x} + e^{-x}$. Determine the gradient of the curve when $x = 0$.

12. **MC** The gradient of the line perpendicular to the curve $y = \log_e(2x)$ at $x = 4$ is equal to:

    A. $-4$       B. $\dfrac{1}{4}$       C. $\dfrac{1}{2}$       D. $-\dfrac{1}{2}$

### Complex familiar

13. The graph of $y = Ae^{-x^2}$, where $A$ is a constant, is shown. Answer the following questions correct to 2 decimal places where appropriate.

    a. Determine the value of $A$.
    b. Find $\dfrac{dy}{dx}$.
    c. Determine the gradient of the tangent to the curve at the point where:

       i. $x = -0.5$        ii. $x = 1$
    d. Show that the equation of the tangent at the point where $x = 1$ is given by $10x + ey - 15 = 0$.

    
    (0, 5)
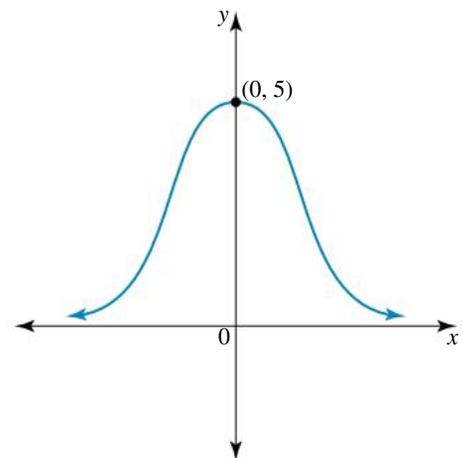
14. If $y = \ln(e^x + e^{-x})$, show that $\dfrac{dy}{dx} = \dfrac{(e^{2x} - 1)}{(e^{2x} + 1)}$.

15. The population of a certain town was 250 000 at the beginning of the year 2000 and 400 000 at the beginning of the year 2010. It was found that the relationship between the population, $P$ thousands, and time, $t$ years, could be modelled by the relationship $P(t) = A\,e^{kt}$ where $A$ and $k$ are constants.

    a. Determine the values of $A$ and $k$, giving your answers correct to 3 decimal places where necessary.
    b. Calculate how many people lived in the town at the beginning of the year 2015. Give your answer to the nearest thousand.
    c. The local council has determined that the population should not exceed 750 000. Identify the year in which this is likely to occur.

**16.** Let $h$ be the graph of the function $h : D \rightarrow \mathbb{R}$, $h(x) = 2 \log_e(1 - 3x)$, where $D$ is the largest possible domain over which $h$ is defined.

    **a.** Determine $D$.

    **b.** Calculate the exact coordinates of the intercepts of the graph with the $x$- and $y$-axes.

    **c.** Use calculus to show that the rate of change of $h$ with respect to $x$ is always negative.

    **d.** **i.** Determine the rule for $h^{-1}$.

       **ii.** State the domain and range of $h^{-1}$.

    **e.** Using technology, or otherwise, sketch, on one set of axes, the graphs of $h$ and $h^{-1}$. Show any asymptotes with their equations.

### Complex unfamiliar

**17.** Microbiologists have been working with a certain type of bacteria that continues to thrive providing it has a favourable growth medium. For a particular experiment, they started with 500 bacteria and observed that the population doubles every 8 hours. The relationship between the number of bacteria, $P$, and the time, $t$ hours since the bacteria started multiplying, is given by

$$P(t) = P_0 \, e^{kt}$$

where $P_0$ and $k$ are constants.

Show that $k = \dfrac{1}{8} \log_e(2)$ and determine when the rate of increase in the colony size would be double the rate after 8 hours.

**18.** The number of people with the flu virus, $N$, in a particular town 5 days after a vaccine is introduced is

$$N = 3000 - 500 \, \log_e(8t + 1)$$

Determine, to the nearest person, the number of people infected after 5 days, and the rate of change of the number of people in the town infected with flu at this time.

**19.** When money is invested in a bank at a constant rate of $r\%$ with continuously compounding interest, the accumulated amount $\$A$ at a time $t$ years after the start of the investment is modelled by the equation

$$A = A_0 e^{rt}$$

where $A_0$ is the amount to be invested.
You invested $\$10\,000$ with this bank at a rate of $4.5\%$. If the rate did not change, estimate the time it would take for the investment to triple in value. Give your answer to the nearest month.

**20.** The graph of the function $y = \log_e(x^2 + 1)$ is shown.
Points A and B lie on the curve with $x$-values of 2 and $-2$ respectively. The tangents at A and B intersect the $x$-axis at P and Q respectively. Show that the point of intersection, T, of the tangents lies on the $y$-axis and that the length of PQ is less than 0.1 units.
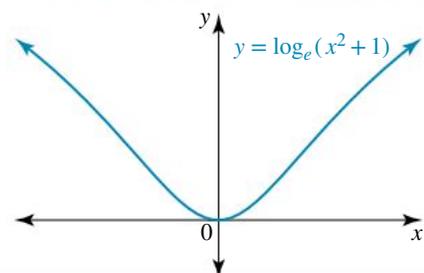
# 1.9 Past QCAA exam questions

**Question 1 (1 mark)**

*Source:* QCAA 2023 Mathematical Methods, Paper 1, Section 1, Q2; © State of Queensland (QCAA) 2024.

If $f(x) = e^{6-2x}$, determine the value of $f'(2)$.

**A.** $e^2$      **B.** $2e^2$      **C.** $-e^2$      **D.** $-2e^2$

**Question 2 (1 mark)**

*Source:* QCAA 2023 Mathematical Methods, Paper 2, Section 1, Q8; © State of Queensland (QCAA) 2024.

The number of koalas in a conservation park is modelled by $N = 15\ln(7t + 1), t \geq 1$, where $t$ represents the time (years) since the park opened.

There were $20$ koalas in the park when it opened. Determine the approximate rate of change in the number of koalas when $t = 3$.

**A.** $46$      **B.** $26$      **C.** $25$      **D.** $5$

**Question 3 (1 mark)**

*Source:* QCAA 2022 Mathematical Methods, Paper 1, Section 1, Q6; © State of Queensland (QCAA) 2024.

Which graph represents the function $f(x) = -3 - \ln(x + 3)$?

**A.**
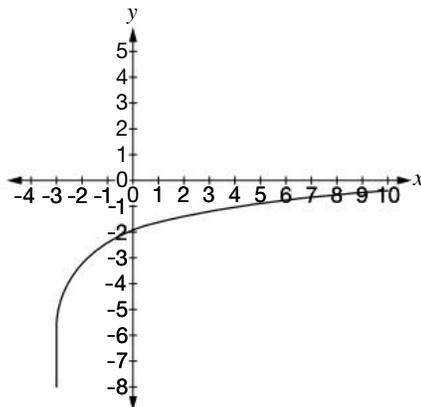


**B.**
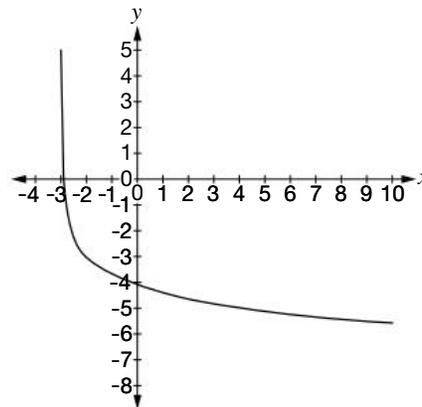


**C.**
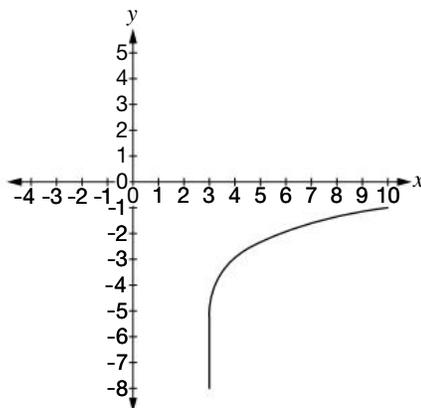


**D.**
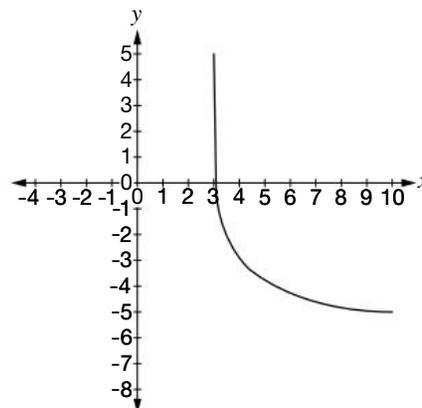
**Question 4 (5 marks)**

*Source:* QCAA 2023 Mathematical Methods, Paper 1, Section 2, Q16; © State of Queensland (QCAA) 2024.

Solve for $x$ in the equation $4 + 7e^{-2x} = 3e^{2x}$.

**Question 5 (4 marks)**

*Source:* QCAA 2023 Mathematical Methods, Paper 2, Section 2, Q15; © State of Queensland (QCAA) 2024.

Determine the derivative of $f(x) = \ln x^2 + \ln(x-5)^3$. Express the derivative as a single fraction in its simplest and factorised form.

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# Answers

## Chapter 1 Calculus of exponential and logarithmic functions

### 1.2 Limits

#### 1.2 Exercise

**1. a.** 19      **b.** 16      **c.** 2

**2. a.** $-3$      **b.** 3      **c.** $\dfrac{3}{2}$

**3. a.** $-7$      **b.** 4      **c.** $-5$

**4. a.** $4x^2$      **b.** $3x^2$

**5.** $f'(x) = 2x - 6$

**6.** $f'(x) = 3 - 4x$

**7. a.** 0.916 29      **b.** 0.955 51      **c.** 0.993 25
     **d.** 1.029 62      **e.** 1.064 71      **f.** 1.000 00

**8. a.** 7.3891      **b.** 20.0855      **c.** 1.6487

**9. a.** 0.736      **b.** 1.396      **c.** 2.472

**10. a.** $\dfrac{dy}{dx} = 8 - 2x$      **b.** 4      **c.** $y = 4x + 4$

**11. a.** $\dfrac{-h}{(x+h)x}$      **b.** $\dfrac{dy}{dx} = \dfrac{-1}{x^2}$

**12. a.** Sample responses can be found in the worked solutions in the online resources.

     **b.** $\dfrac{-1}{(x-2)^2}$

     **c.** $\dfrac{dy}{dx} = \dfrac{-1}{(x-2)^2}$

     **d.** $x = \dfrac{5}{3}, \dfrac{7}{3}$

**13.** $\dfrac{1}{2\sqrt{e}}$

**14.** $e$

### 1.3 The exponential function

#### 1.3 Exercise

**1. a.**



**b.**



**c.**



**2.**



**3.**

**4.**



Graph showing $y = -5e^x$, with $y = 0$ asymptote, points $(-1, -1.84)$, $(0, -5)$, $(1, -13.59)$

**5.**



Graph showing $y = 3 + e^{-x}$, with $y = 3$ asymptote, points $(-1, 5.718)$, $(0, 4)$, $(1, 3.368)$

**6.**



Graph showing $y = 3 + e^{2x}$, with $y = 3$ asymptote, points $(-1, 3.135)$, $(0, 4)$, $(1, 10.389)$
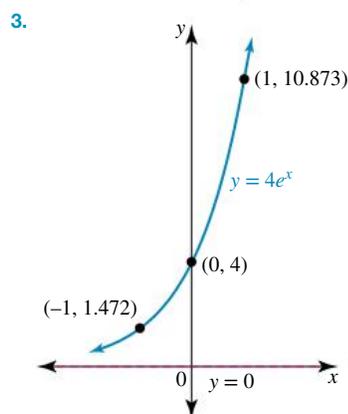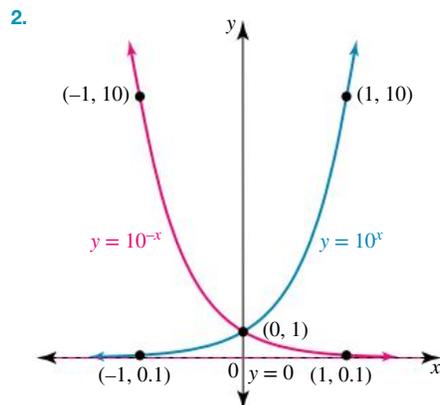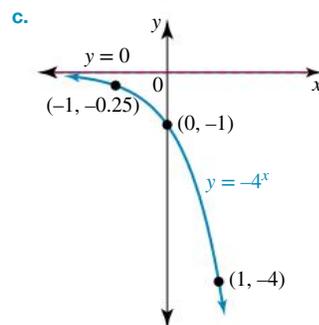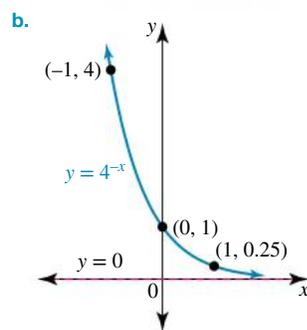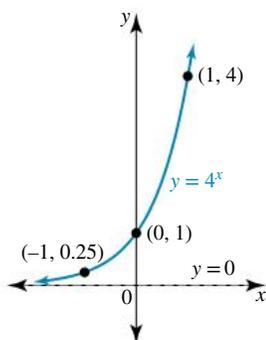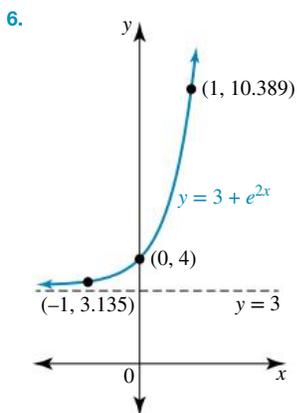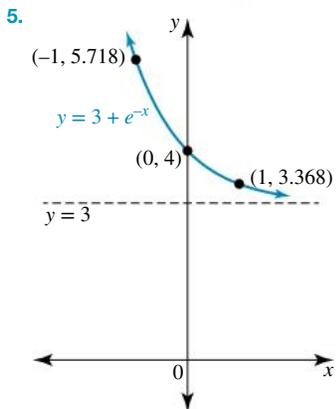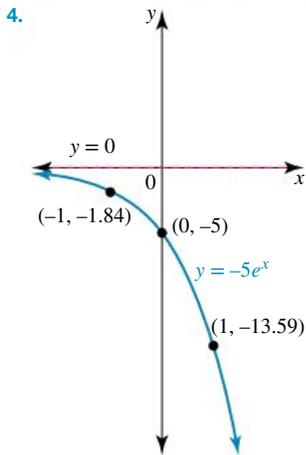
**7. a.** Intercepts: $\left(\dfrac{1}{2}\ln(3),\ 0\right)$ and $(0, -2)$

**b.**



Graph showing $y = e^{2x} - 3$, with $y = -3$ asymptote, points $(0.549, 0)$, $(0, -2)$

**8. a.** Intercepts: $(-\ln(2), 0)$ and $(0, 2)$

**b.**



Graph showing $f(x) = 2 - e^x$, with $y = 2$ asymptote, points $(1, 2-e)$, $(0, 1)$, $(-\log_e 2, 0)$, $(1, 2 - e^{-1})$

**9.**



Graph showing $y = 4e^{\frac{x}{2}}$, with $y = 0$ asymptote, points $(1, 6.595)$, $(0, 4)$, $(-1, 2.426)$
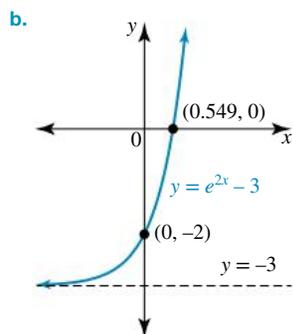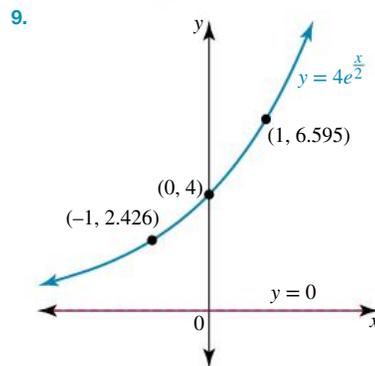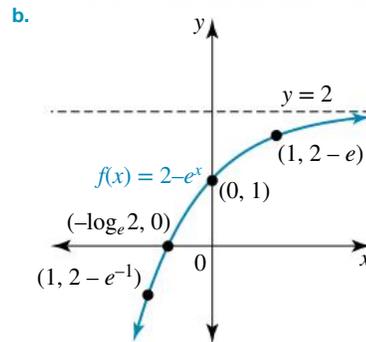
**10. a.** Intercepts: $(-2\ln(2),\ 0)$ and $(0, -3)$
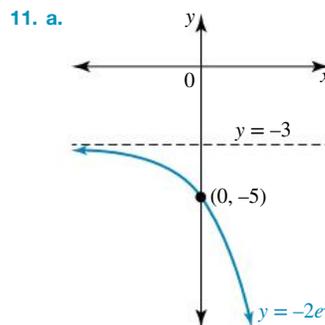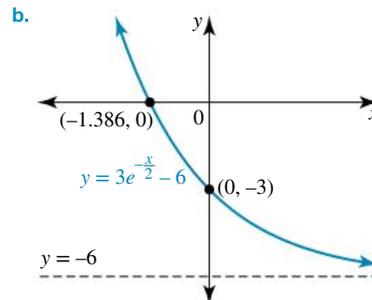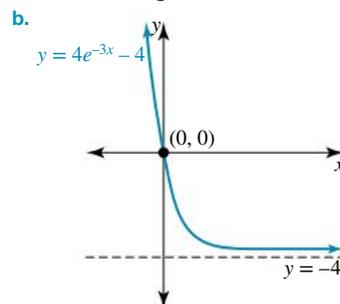
**b.**



Graph showing $y = 3e^{-\frac{x}{2}} - 6$, with $y = -6$ asymptote, points $(-1.386, 0)$, $(0, -3)$

**11. a.**



Graph showing $y = -2e^x - 3$, with $y = -3$ asymptote, point $(0, -5)$

Domain $\mathbb{R}$, range $(-\infty, -3)$

**b.**



Graph showing $y = 4e^{-3x} - 4$, with $y = -4$ asymptote, point $(0, 0)$

Domain $\mathbb{R}$, range $(-4, \infty)$

**c.**



$y = 5e^{x-2}$
$(2, 5)$
$(0, 5e^{-2})$  $y = 0$

Domain $\mathbb{R}$, range $\mathbb{R}^+$

**12. a.**



$y = e^x - 3$
$(0, -2)$  $(\log_e(3), 0)$
$y = -3$

Range $(-3, \infty)$

**b.**



$y = -1$
$(0, -3)$  $y = -2e^{2x} - 1$

Range $(-\infty, -1)$

**c.**



$y = \frac{1}{2}e^{-4x} + 3$
$(0, 3.5)$
$y = 3$

Range $(3, \infty)$

**13. a.**



$y = 4$
$(0, 3)$
$(\log_e(2), 0)$
$y = 4 - e^{2x}$

Range $(\infty, 4)$

**b.**



$y = 4e^{2x-6} + 2$
$(3, 6)$
$y = 2$
$(0, 4e^{-6} + 2)$

Range $(2, \infty)$

**14. a.**



$(0, 2e - 4)$
$\left(\frac{1}{3}(1 - \log_e(2)), 0\right)$
$y = 2e^{1-3x} - 4$
$y = -4$

**b.**



$y = 3 \times 2^x - 24$
$(3, 0)$
$(0, -21)$
$y = -24$
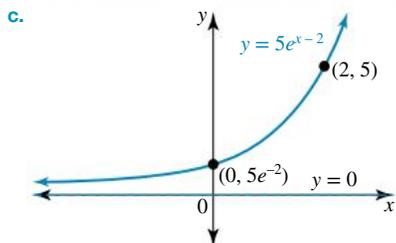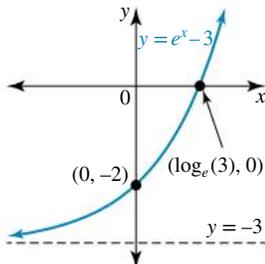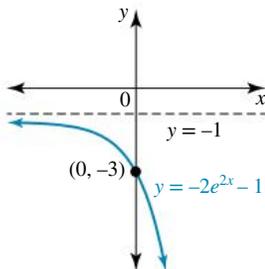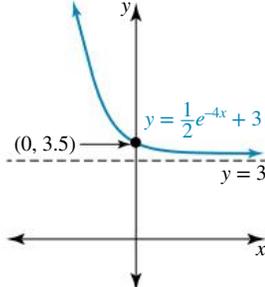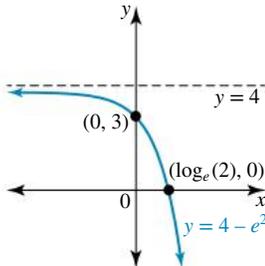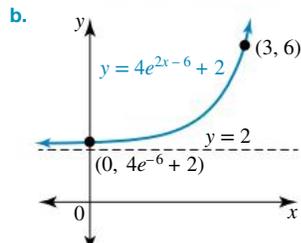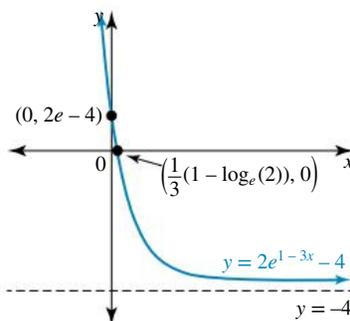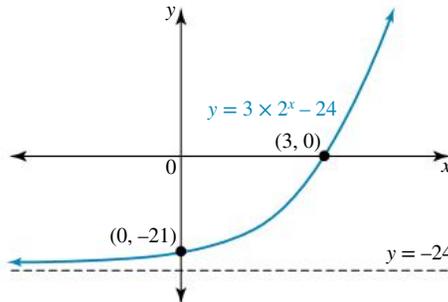
Domain $\mathbb{R}$, range $(-24, \infty)$

**15.** $a = -2, b = 2$

**16.** $a = -11, b = 11, f: \mathbb{R} \to \mathbb{R}, f(x) = -11e^x + 11$

**17.** $a = 1, n = -2, y = e^{-2x} + 4$

**18.** $x > 0$; sample responses can be found in the worked solutions in the online resources.

**19.** Range of $f(x)$: $y \leq 3$ or $y \in (-\infty, 3]$
Range of $g(x)$: $y < 4$ or $y \in (-\infty, 4)$

**20.** $a = 1$

## 1.4 Differentiation of the exponential function

### 1.4 Exercise

**1. a.** $10e^{10x}$  **b.** $\frac{1}{3}e^{\frac{x}{3}}$  **c.** $\frac{1}{4}e^{\frac{x}{4}}$

**d.** $-e^{-x}$  **e.** $6e^{3x}$  **f.** $-20e^{-5x}$

**2. a.** $6e^{6x-2}$  **b.** $-6e^{8-6x}$  **c.** $10e^{5x+3}$
**d.** $-8e^{7-2x}$  **e.** $-24e^{8x+1}$  **f.** $10e^{6-5x}$

**3. a.** $-90e^{6-9x}$  **b.** $-15e^{3x+4}$  **c.** $-42e^{-7x}$
**d.** $e^{\frac{x}{2}+1}$  **e.** $-e^{2-\frac{x}{3}}$  **f.** $-e^{\frac{x}{4}+5}$

**4.** A

**5. a.** $2e^x$  **b.** $3e^{2x}(3e^x + 2)$
**c.** $-10(2e^{-4x} - 1)$  **d.** $3e^x - 2e^{-x}$

**6. a.** $6e^{2x} - 7e^{-7x}$  **b.** $36e^{9x} - 2e^x$

**7. a.** $(2x + 3)e^{x^2+3x}$  **b.** $(2x - 3)e^{x^2-3x+1}$
**c.** $2(x - 1)e^{x^2-2x}$  **d.** $-5e^{2-5x}$

**8. a.** $(2x-3)e^{6-3x+x^2}$ **b.** $3(x^2+1)e^{x^3+3x-2}$

  **c.** $3(8x-7)e^{4x^2-7x}$ **d.** $10(1+3x)e^{1-2x-3x^2}$

**9.** A

**10.** $-20e$

**11.** $-6e^2$

**12.** $-5e^{-2}$

**13.** $y = -\dfrac{8x}{e} - \dfrac{8}{e}$; $y$-intercept $(0, -2.943)$

**14.** $y = -2e^x$

**15.** $y = 5x - 4$

**16.** Tangent: $y = -3x - 1$; perpendicular: $y = \dfrac{1}{3}x - 1$

**17.** $x = \dfrac{3}{2}$

**18.** $x = 0$

**19.** $-1.38$

**20.** Sample responses can be found in the worked solutions in the online resources.

## 1.5 The logarithmic function

### 1.5 Exercise

**1. a.** 4 **b.** $-5$ **c.** $\dfrac{1}{2}$

**2. a.** 0 **b.** $3\log_7(2x-1)$ **c.** $\log_{10}(x-1)$

**3. a.** $x = e$ **b.** $x = e^2$ **c.** $x = 0.135$

  **d.** $x = 0.368$ **e.** $x = 1.350$ **f.** $x = 0.502$

**4. a.** $x = 3.695$ **b.** $x = 0.906$ **c.** $x = e$

**5.** D

**6. a.** Domain $= (-4, \infty)$, range $= \mathbb{R}$
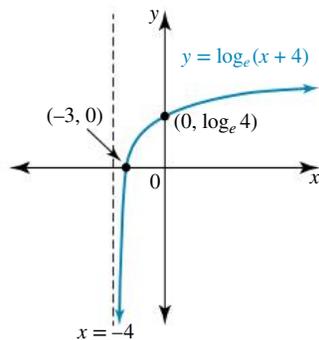


  **b.** Domain $= (0, \infty)$, range $= \mathbb{R}$
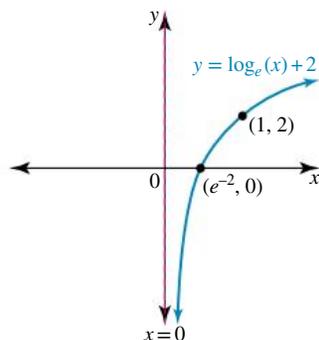


  **c.** Domain $= (0, \infty)$, range $= \mathbb{R}$



  **d.** Domain $= (4, \infty)$, range $= \mathbb{R}$



**7. a.**



  **b.**

**c.**



$y = \log_e(x) + 0.5$

$(1, 0.5)$

$\left(e^{-\frac{1}{2}}, 0\right)$

$x = 0$

**8. a.**



$y = \log_e(x - 4)$

$(10, \log_e 6)$

$(5, 0)$

$x = 4$

**b.**



$y = \log_e(x + 2)$

$(0, \log_e 2)$

$(-1, 0)$

$x = -2$

**c.**



$y = \log_e(x + 0.5)$

$(0.5, 0)$

$(0, -\log_e 2)$

$x = -0.5$

**9. a.**



$\left(2, \frac{1}{4}\log_e 2\right)$   $y = \frac{1}{4}\log_e(x)$

$(1, 0)$

$x = 0$

**b.**



$y = 3\log_e(x)$

$(2, 3\log_e 2)$

$(1, 0)$

$x = 0$

**c.**



$(2, 6\log_e 2)$   $y = 6\log_e(x)$

$(1, 0)$

$x = 0$

**10. a.**



$y = \log_e(3x)$

$(1, \log_e 3)$

$\left(\frac{1}{3}, 0\right)$

$x = 0$

**b.**



$y = \log_e\left(\frac{x}{4}\right)$

$(4, 0)$

$(1, -\log_e 4)$

$x = 0$

**c.**


$y = \log_e(4x)$, $(1, \log_e 4)$, $(0.25, 0)$, $x = 0$

**c.**


$y = \frac{1}{2}\log_e\left(\frac{x}{4}\right) + 1$, $(4e^{-2}, 0)$, $(4, 1)$, $x = 0$
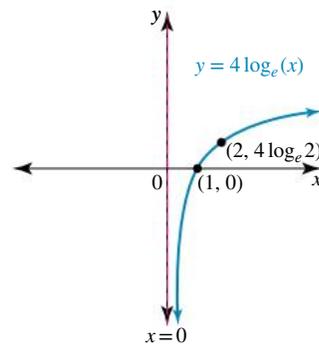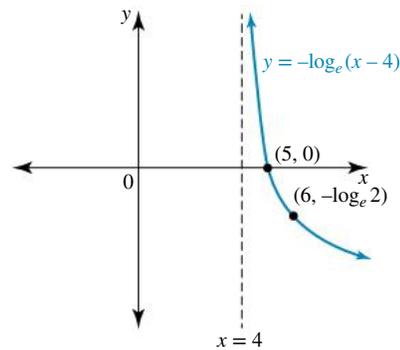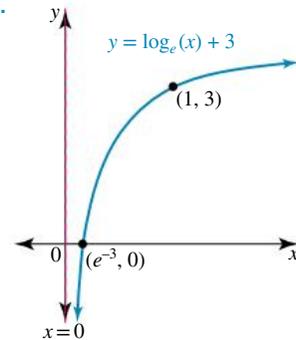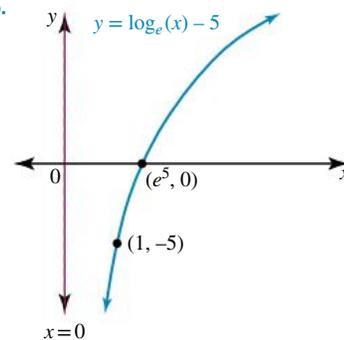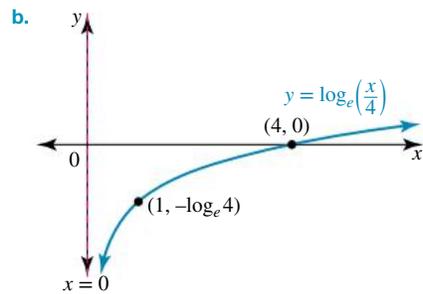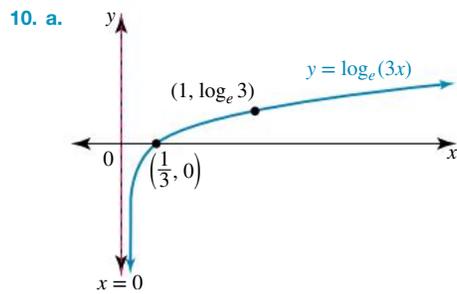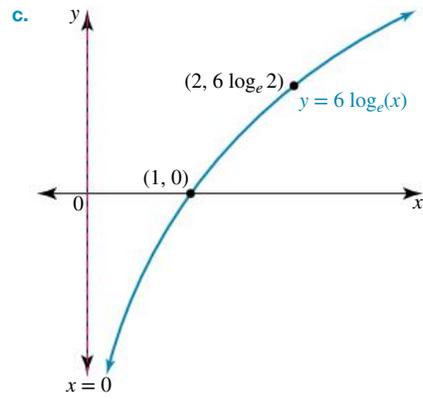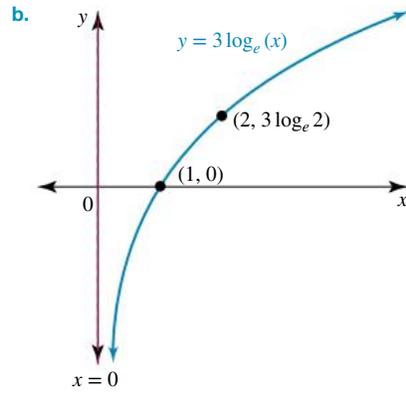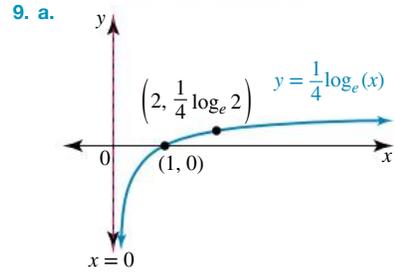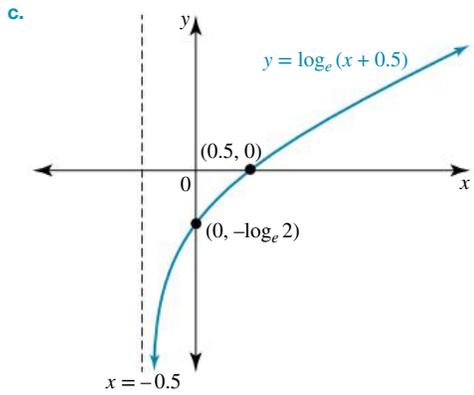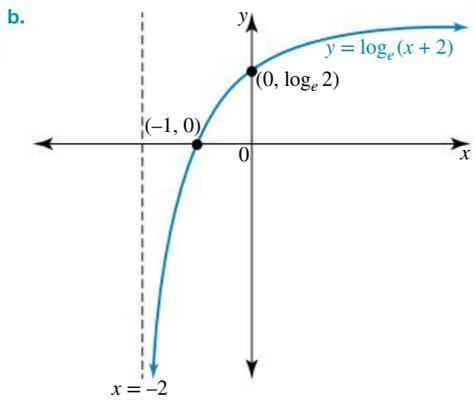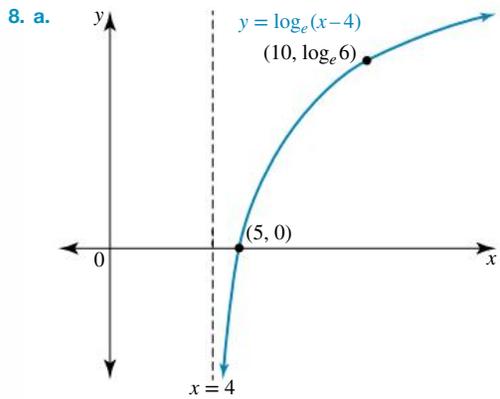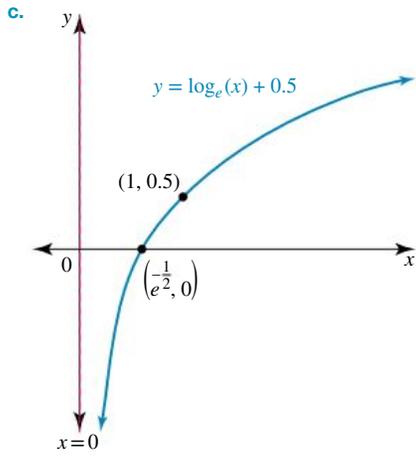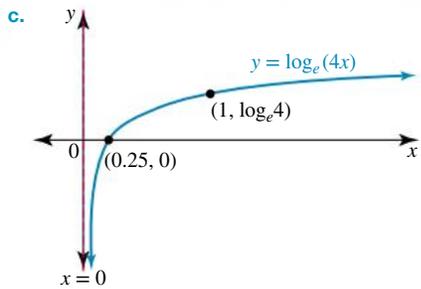
**11. a.** Domain: $x \in \mathbb{R}$ Range: $y > 2$ or $y \in (2, \infty)$

**b.** $f^{-1}(x) = \ln(x - 2)$ or $y = \ln(x - 2)$

**c.** Domain: $x > 2$ or $x \in (2, \infty)$ Range: $y \in \mathbb{R}$


$y = e^x + 2$, $(0, 3)$, $y = x$, $(0, 0)$, $(3, 0)$, $y = \ln(x - 2)$
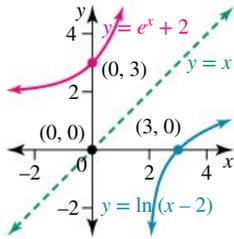
**12. a.** Domain: $x \in \mathbb{R}$ Range: $y > 0$ or $y \in (0, \infty)$

**b.** $f^{-1}(x) = -\ln(x)$ or $y = -\ln(x)$

**c.** Domain: $x > 0$ or $x \in (0, \infty)$ Range: $y \in \mathbb{R}$

**d.**


$(0, 1)$, $y = x$, $x = 0$, $(0, 0)$, $y = e^{-x}$, $(1, 0)$, $y = 0$, $y = -\ln(x)$

**e.** $(0.567, 0.567)$

**13. a.**


$y = 1 - 2\log_e(x - 1)$, $(2, 1)$, $(e^{0.5} + 1, 0)$, $x = 1$

**b.**


$y = \log_e(2x + 4)$, $(0, \log_e 4)$, $(-1.5, 0)$, $x = -2$
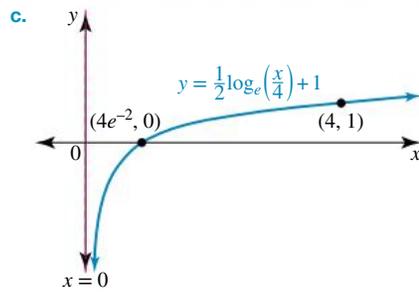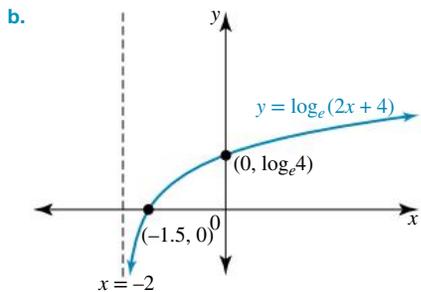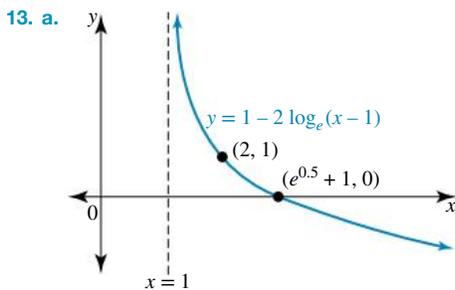
**14.** $a = \dfrac{2}{\log_e(2)}$, $h = -1$, $k = -2$

**15. a.** Domain: $x < 3$; range: $y \in \mathbb{R}$

**b.** $(2, 0)$, $(0, \log_e 3)$

**c.** $f^{-1}(x) = 3 - e^x$; domain of $f^{-1}(x)$: $x \in \mathbb{R}$; range of $f^{-1}(x)$: $y < 3$

**d.** $(0, 2)$ and $\left(\log_e 3, 0\right)$

**e.**


$y = x$, $y = 3 - e^x$, $(0, 2)$, $y = 3$, $y = \ln(3 - x)$, $(0, 1.099)$, $(0.792, 0.792)$, $(2, 0)$, $(1.099, 0)$, $x = 3$
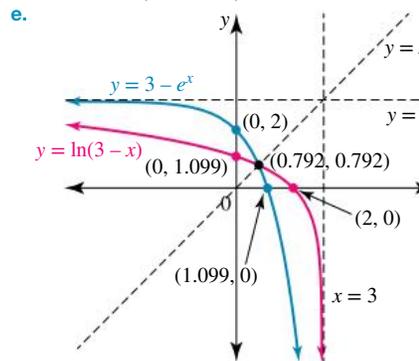
**f.** $(0.792, 0.792)$, $(-17.086, 2.999\ 999\ 96)$ and $(2.999\ 999\ 96, -17.086)$

**16. a.** Domain: $x > \dfrac{1}{2}$; range: $y \in \mathbb{R}$

**b.** $(1, 0)$

**c.** $f^{-1}(x) = \dfrac{1}{2}(e^x + 1)$; domain of $f^{-1}(x)$: $x \in \mathbb{R}$; range of $f^{-1}(x)$: $y > \dfrac{1}{2}$

**d.** $(0, 1)$

**e.**


$x = \frac{1}{2}$, $y = x$, $y = \frac{1}{2}(e^x + 1)$, $(0, 1)$, $y = \frac{1}{2}$, $(1, 0)$, $y = \ln(2x - 1)$
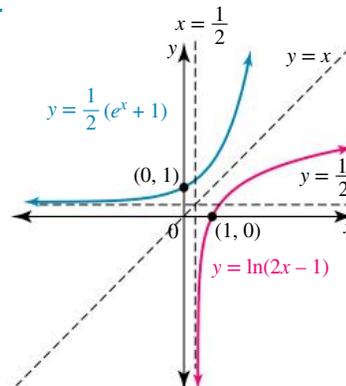
**f.** Inverse functions are reflections in the line $y = x$. The functions $y = \log_e(x)$ and $y = e^x$ lie on either side of $y = x$, so they do not intersect. The functions $f(x) = \log_e(2x - 1)$ and $f^{-1}(x) = \frac{1}{2}(e^x + 1)$ have been translated further away from the line of symmetry, so there is no point of intersection.

**17.** $m = 3 \ln\left(\frac{3}{2}\right)$

**18.** $-0.4055$

**19.** $(-2, 3) \Rightarrow 3 = m \log_2(-2n)$ [1]

$$\left(-\frac{1}{2}, \frac{1}{2}\right) \Rightarrow \frac{1}{2} = m \log_2\left(-\frac{n}{2}\right) \quad [2]$$

**20.** $(-4.998, -4.998), (3.091, 3.091)$

## 1.6 Solving equations

### 1.6 Exercise

**1. a.** $x = \ln(4)$  **b.** $x = 1.386$

**2. a.** $x \approx 1.609$  **b.** $x \approx -0.693$  **c.** $x \approx 0.956$
  **d.** $x \approx -1.792$  **e.** $x \approx 0.405$  **f.** $x \approx -1.204$

**3. a.** $x = 0, \ln(2)$  **b.** $x = 0$

  **c.** $x = 0, \ln(2)$  **d.** $x = \ln\left(\frac{3}{2}\right), -\ln(2)$

  **e.** $x = \ln(4)$  **f.** $x = \ln\left(\frac{2}{3}\right)$

**4. a.** $2\log_e(3) + 2$  **b.** $4\log_e(2)$

  **c.** $\log_e(3)$  **d.** $-\sqrt{\log_e(2)}, \sqrt{\log_e(2)}$

**5.** $x = 1.61$

**6. a.** $x = \ln(3)$  **b.** $x = \ln(3), \ln\left(\frac{2}{3}\right)$

  **c.** $x = \ln(5)$  **d.** $x = \ln(5)$

**7. a.** $2\log_e(2)$
  **b.** $2\log_e(2), \ 3\log_e(2)$

  **c.** $\log_e\left(\sqrt{5} + 1\right)$

  **d.** $\log_e\left(6 - \sqrt{31}\right), \ \log_e\left(\sqrt{31} + 6\right)$

**8. a.** $x = e$  **b.** $x = e^2$  **c.** $x = 0.135$
  **d.** $x = 0.368$  **e.** $x = 1.350$  **f.** $x = 0.502$

**9. a.** $x = 3.695$  **b.** $x = 0.906$  **c.** $x = e$

**10. a.** $x = 1.368$  **b.** $x = -0.432$  **c.** $x = -1.433$
  **d.** $x = -2.054$  **e.** $x = 0.417$  **f.** $x = -1.564$

**11.** $x = 3e^{-3}$ or $x = \frac{3}{e^3}$; $x \approx 0.15$

**12. a.** $x = 596.192$  **b.** $x = 201.284$

  **c.** $x = \frac{6}{e}$  **d.** $x = \frac{e^2}{10}$

**13. a.** $x = 6$  **b.** $x = 12$

  **c.** $x = \frac{2}{3}$  **d.** $x = \frac{5}{2}$ or $x = 2.5$

  **e.** $x = 3$  **f.** $x = 6$

**14. a.** $x = 1$  **b.** $x = \frac{3}{2}$ or $x = 1.5$

  **c.** $x = 2$  **d.** $x = \frac{3}{2}$ or $x = 1.5$

**15.** $y = ax$

**16.** $3e^a$

**17.** $y = ex^2$

**18.** $x = \frac{1}{m} \log_e\left(\frac{\sqrt{57} + 5}{4}\right)$, $m \in \mathbb{R} \setminus \{0\}$

**19.** $a = 5, k = 0.25$

**20.** $P = \$10\,000, r = 5\%$

## 1.7 Differentiation of the logarithmic functions

### 1.7 Exercise

**1. a.** $\frac{dy}{dx} = \frac{1}{x}$  **b.** $\frac{dy}{dx} = \frac{1}{x}$  **c.** $\frac{dy}{dx} = \frac{1}{x}$
  **d.** $\frac{dy}{dx} = \frac{1}{x}$  **e.** $\frac{dy}{dx} = \frac{3}{x}$  **f.** $\frac{dy}{dx} = \frac{-6}{x}$

**2. a.** $\frac{dy}{dx} = \frac{1}{x}$  **b.** $\frac{dy}{dx} = \frac{1}{x}$
  **c.** $\frac{dy}{dx} = \frac{4}{x}$  **d.** $\frac{dy}{dx} = \frac{-5}{x}$

**3.** C

**4. a.** $\frac{dy}{dx} = \frac{2}{2x + 5}$  **b.** $\frac{dy}{dx} = \frac{6}{6x + 1}$
  **c.** $\frac{dy}{dx} = \frac{3}{3x - 4}$  **d.** $\frac{dy}{dx} = \frac{8}{8x - 1}$
  **e.** $\frac{dy}{dx} = \frac{5}{5x - 3}$  **f.** $\frac{dy}{dx} = \frac{-1}{2 - x}$ or $\frac{1}{x - 2}$

**5. a.** $\frac{dy}{dx} = \frac{30}{5x + 2}$  **b.** $\frac{dy}{dx} = \frac{16}{2x - 1}$
  **c.** $\frac{dy}{dx} = \frac{-48}{12x + 5}$  **d.** $\frac{dy}{dx} = \frac{63}{8 - 9x}$

**6. a.** $\frac{dy}{dx} = \frac{4}{x}$  **b.** $\frac{dy}{dx} = \frac{2x}{(x^2 + 3)}$
  **c.** $\frac{dy}{dx} = \frac{2(x + 2)}{x(x + 4)}$  **d.** $\frac{dy}{dx} = \frac{2x - 3}{x^2 - 3x + 2}$
  **e.** $\frac{dy}{dx} = \frac{3x^2 + 4x - 7}{x(x^2 + 2x - 7)}$  **f.** $\frac{dy}{dx} = \frac{2(2x - 1)}{x(x - 1)}$

**7.** D

**8.** D

**9.** A

**10. a.** $\frac{dy}{dx} = \frac{2}{5}$  **b.** $\frac{dy}{dx} = \frac{4}{27}$  **c.** $\frac{dy}{dx} = \frac{1}{2}$

**11. a.** $y = 2x - 3$

  **b.** $y = \frac{3}{e}x$ or $3x - ey = 0$

  **c.** $y = \frac{1}{e}x$ or $x - ey = 0$

**12. a.** $\frac{dy}{dx} = 3$

  **b.** Tangent: $y = 3x - 18$; perpendicular: $x + 3y - 6 = 0$

**13.** B

**14. a.** $\dfrac{dy}{dx} = \dfrac{1}{2x+1}$ **b.** $\dfrac{dy}{dx} = \dfrac{-2}{3-4x}$

**c.** $\dfrac{dy}{dx} = \dfrac{x}{x^2+2}$ **d.** $\dfrac{dy}{dx} = \dfrac{1}{4(x+3)}$

**15. a.** $f'(x) = \dfrac{-1}{x+3}$ **b.** $f'(x) = \dfrac{12}{3x-2}$

**c.** $f'(x) = \dfrac{-10}{5x+8}$ **d.** $f'(x) = \dfrac{-3}{4+3x}$

**16.** $y = \dfrac{4}{e}x + e - 2$

**17.** $y = 6x + 4\log_e(2) - 6$

**18.** 2

**19.** $k = 7.4$

**20.** $n = 2$

## 1.8 Modelling and solving with exponential and logarithmic functions

### 1.8 Exercise

**1. a.** 500 bacteria **b.** 4987 bacteria **c.** 2294 bacteria/hour

**2. a.** 11 **b.** $\dfrac{dL}{dt} = 6.589e^{0.599t}$

**c.** 39.742 mm/month

**3. a.** 50 **b.** 47.56 grams **c.** 0.24 grams/day

**4. a.** 200 **b.** 110 grams

**c.** $2\dfrac{1}{4}$ hours **d.** $-36.1$ grams/hour

**5. a.** 10 **b.** 7 grams
**c.** 4 days **d.** 0.73 grams/day

**6. a.** $A_0 = 1000$; $r = 0.05$

**b. i.** \$1051.27 **ii.** \$1284.03 **iii.** \$1648.72

**c. i.** \$52.56/year **ii.** \$64.20/year **iii.** \$82.44/year

**d.** 14 years

**7. a.** Domain: $\{x : x \in (2, \infty)\}$; range: $\{y : y \in \mathbb{R}\}$

**b.** $a = 3$

**c.** $y = 2x - 6$

**d.** $y = -\dfrac{1}{2}x + \dfrac{3}{2}$

**8. a.** $\dfrac{dy}{dx} = 3$

**b.** Tangent: $y = 3x - 18$; perpendicular: $x + 3y - 6 = 0$

**9. a.** 75 **b.** 1.531 °C/min

**10. a.** 500 **b.** 0.03
**c.** 2240 **d.** 67 people/year

**11. a.** $a = 2$; $k = 0.2$ **b.** $\dfrac{dm}{dt} = -0.4e^{-0.2t}$

**c.** 0.12 kg/h **d.** 3.5 h

**12. a.** 120

**b.** $\dfrac{1}{2}\ln\left(\dfrac{4}{3}\right)$

**c.** 8.408 units/min

**d.** As $t \to \infty$, $A \to 0$. Technically the graph approaches the line $A = 0$ (asymptotic behaviour). However, the value of $A$ would be so small that in effect, after a long period of time, there is no gas left.

**13. a.** 18 possums
**b.** 12 possums
**c.** 1.62 months
**d.**
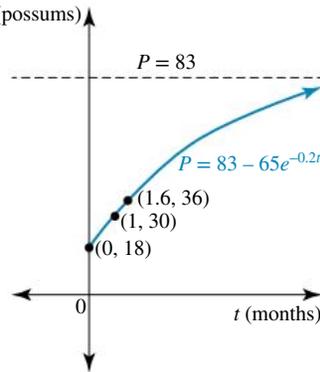


**e.** The presence of the asymptote at $P = 83$ shows that as $t \to \infty$, $P \to 83$. The population can never exceed 83, so the population cannot grow to 100 possums.

**14. a.** 95 °C
**b.**
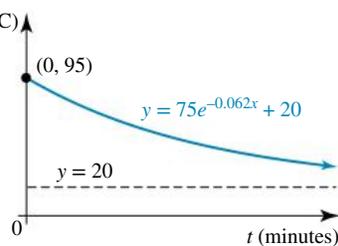


**c.** Approximately 20 °C
**d.** 8.24 min
**e.** After 10 minutes, the coffee is cooling at a rate of 2.5 °C/minute. The temperature is decreasing, so the rate of change will be negative.

**15. a.** 23.2 million **b.** September 2005
**c.** 0.29 million/year **d.** 2019

**16. a.** 75.32 cm; $k = 0.24$
**b.** 5.45 cm of mercury/km

**17.** 0.3 months; 19 rats/month

**18.** $y = -2x - 2\ln(2)$

**19.** 69.4°

**20.** 40.65 minutes
From the graph, or equation, the temperature is always greater than the room temperature of 30° C.

## 1.9 Review

### 1.9 Exercise

**1. a.** 0.792 **b.** 3.084 **c.** $-1.040$ **d.** 3.386

**2.** C

**3.** A

**4. a.** $\ln(2)$ **b.** $\ln(3)$
**c.** $\ln(2)$ **d.** $-\ln(2)$, $\ln(4)$

**5. a.** Domain: $x \in \mathbb{R}$, range: $y \in (1, \infty)$



$y = 2e^x + 1$
$(0, 3)$
$y = 1$

**b.** Domain: $x \in \mathbb{R}$, range: $y \in (-\infty, 3)$
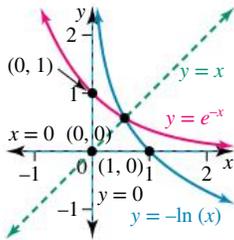


$y = 3$
$y = 3 - 3e^{-\frac{x}{2}}$
$(0, 0)$

**c.** Domain: $x \in \mathbb{R}$, range: $y \in (-\infty, 0)$



$y = 0$
$\left(-1, -\frac{1}{4}\right)$
$y = -\frac{1}{4}e^{x+1}$
$\left(0, -\frac{e}{4}\right)$
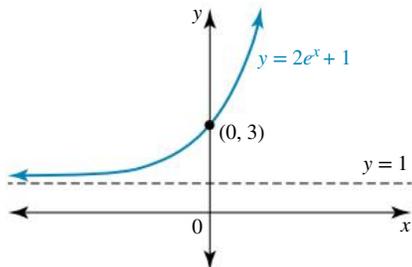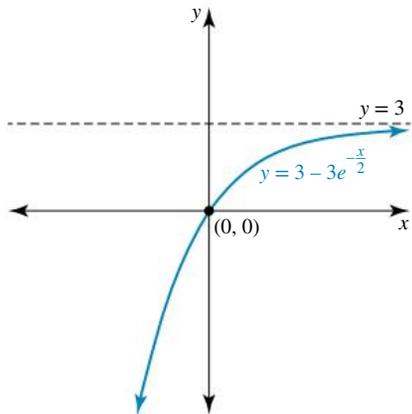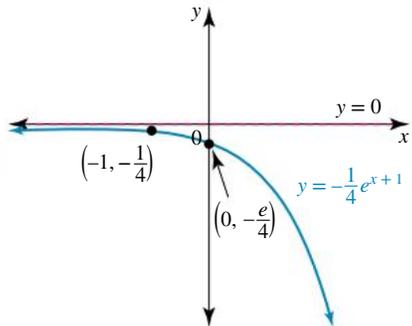
**6. a.** $y = \ln(x + 4)$
Domain: $x > -4$
Range: $y \in \mathbb{R}$
Asymptote: $x = -4$
Transformation: horizontal translation of 4 units to the left (in the negative direction)



$y = \ln(x + 4)$
$(0, 1.386)$
$(-3, 0)$
$x = -4$
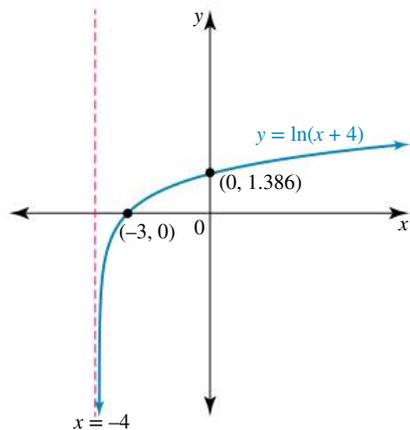
**b.** $y = \ln(x - 4)$
Domain: $x > 4$
Range: $y \in \mathbb{R}$
Asymptote: $x = 4$
Transformation: horizontal translation of 4 units to the right (in the positive direction)



$y = \ln(x - 4)$
$(6, \ln(2))$
$(5, 0)$
$x = 4$
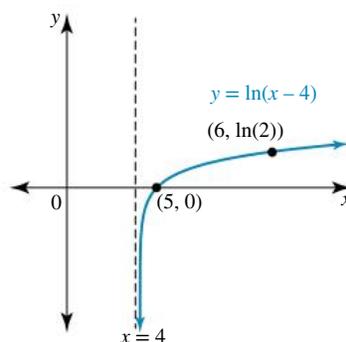
**c.** $y = \ln(x) + 4$
Domain: $x > 0$
Range: $y \in \mathbb{R}$
Asymptote: $x = 0$
Transformation: vertical translation of 4 units upwards (in the positive direction)



$y = \ln(x) + 4$
$(1, 4)$
$(0.018, 0)$
$x = 0$
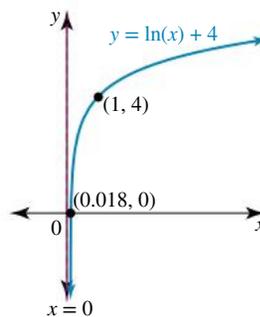
**d.** $y = 4 - \ln(x)$
Domain: $x > 0$
Range: $y \in \mathbb{R}$
Asymptote: $x = 0$
Transformation: vertical translation of 4 units upwards (in the positive direction) and reflection in the $x$-axis



$(1, 4)$
$y = 4 - \ln(x)$
$(e^4, 0)$
$x = 0$
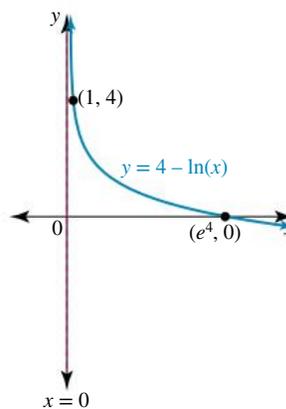
**7. a.** Domain: $x \in \mathbb{R}$, range: $y \in (-\infty, -3)$
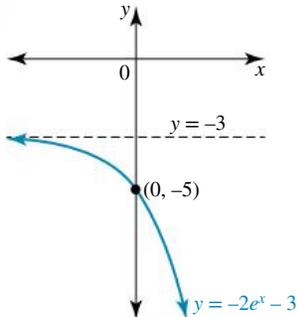


**b.** Domain: $x \in \mathbb{R}$, range: $y \in (-4, \infty)$
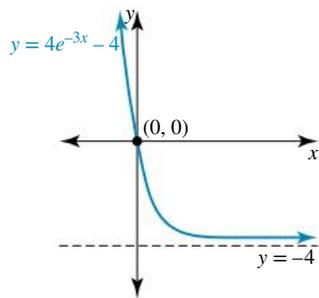


**c.** Domain: $x \in \mathbb{R}$, range: $y \in (0, \infty)$
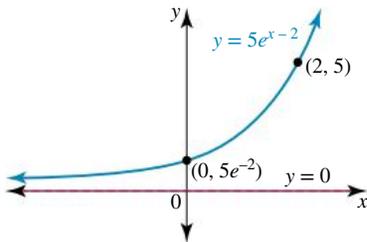


**8. a.** $\dfrac{dy}{dx} = \dfrac{x-1}{x^2 - 2x + 7}$; no restrictions on $x$. Domain: $x \in \mathbb{R}$

**b.** $\dfrac{dy}{dx} = \dfrac{-5}{(x+2)(x-3)}$; $x < -2$ or $x > 3$

**c.** $\dfrac{dy}{dx} = \dfrac{2}{x+2}$; $x \in \mathbb{R}, x \neq -2$

**9. a.** $\dfrac{dy}{dx} = \dfrac{-11}{(2x+1)(x-5)}$

**b.** $\dfrac{dy}{dx} = \dfrac{-1}{x-3}$

**c.** $\dfrac{dy}{dx} = \dfrac{6(3x-1)}{9x^2 - 6x + 7}$

**10. a.** $-\dfrac{1}{3}e^{-\frac{1}{3}x}$

**b.** $12x^3 + 4xe^{-2x^2}$

**c.** $-\dfrac{8}{3}e^{-2x} + \dfrac{4}{3}e^{-4x} - 2e^{-3x}$

**d.** $4e^{4x} - 12e^{2x}$

**11.** $\dfrac{1}{2}$

**12.** A

**13. a.** $A = 5$

**b.** $\dfrac{dy}{dx} = -10xe^{-x^2}$

**c. i.** 3.89

**ii.** $-3.68$

**d.** Sample responses can be found in the worked solutions in the online resources.

**14.** Sample responses can be found in the worked solutions in the online resources.

**15. a.** $A = 250; k = 0.047$

**b.** 506 000

**c.** 2023

**16. a.** $D = \left\{ x : x \in \left( -\infty, \dfrac{1}{3} \right) \right\}$

**b.** $(0, 0)$

**c.** Sample responses can be found in the worked solutions in the online resources.

**d. i.** $f^{-1}(x) = \dfrac{1}{3} - \dfrac{1}{3}e^{\frac{x}{2}}$

**ii.** Domain: $x \in \mathbb{R}$; range: $y < \dfrac{1}{3}$

**e.**
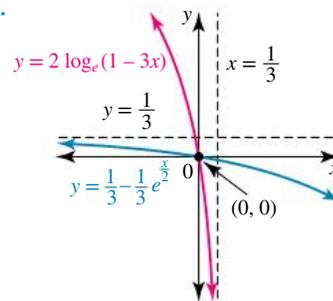


**17.** After 16 hours

**18.** 1143 people; decrease of 98 people/day

**19.** 24 years and 5 months

**20.** Sample responses can be found in the worked solutions in the online resources.

## 1.9 Past QCAA exam questions

**1.** D

**2.** D

**3.** B

**4.** $x = \dfrac{1}{2} \ln \left( \dfrac{7}{3} \right)$

**5.** $\dfrac{5(x-2)}{x(x-5)}$

# 2 Calculus of trigonometric functions

## LESSON SEQUENCE

**Fully worked solutions for this chapter are available online.**

## EXAM PREPARATION

Access exam-style questions in every lesson, available online and access past QCAA exam questions in every review.

**on** Resources

| | |
|---|---|
| **Solutions** | Solutions — Chapter 2 |
| **Exam questions** | Exam questions booklet — Chapter 2 |
| **Digital documents** | Learning matrix — Chapter 2 |
| | Chapter summary — Chapter 2 |

# LESSON
## 2.1 Overview

### 2.1.1 Introduction

Early studies of triangles can be traced back to Egyptian and Babylonian mathematicians around 4000 BC. They understood the ratio of sides of similar triangles but not angles. Babylonian astronomers kept detailed records of the rising and setting of stars as well as the motion of planets. The development of trigonometry as we know it today began with Isaac Newton, reaching its current form when Leonhard Euler published his analysis of trigonometric functions in 1748.

In the past, the principal application of trigonometry for many cultures was in astronomy. Today, trigonometric functions are used to model many physical phenomena that are cyclical or periodic in nature. Examples include the motion of a pendulum and electrical currents, which are both periodic. GPS and cell phones rely on formulas involving sine and cosine; TV and radios transmit images and sounds modelled on sine and cosine functions.

### 2.1.2 Syllabus links

| Lesson | Lesson title | Syllabus links |
|--------|--------------|----------------|
| 2.2 | Review of the unit circle | [Review from Unit 1, Topic 4]<br>○ Understand the unit circle definition of $\cos(\theta)$, $\sin(\theta)$ and $\tan(\theta)$ and periodicity using radians.<br>○ Understand and use the exact values of $\cos(\theta)$, $\sin(\theta)$ and $\tan(\theta)$ at integer multiples of $\dfrac{\pi}{6}$ and $\dfrac{\pi}{4}$. |
| 2.3 | Review of the trigonometric functions | [Review from Unit 1, Topic 4]<br>○ Sketch the graphs of $y = \sin(x)$, $y = \cos(x)$ on extended domains.<br>○ Recognise and determine the effect of the parameters $a, b\ h$ and $k$ on the graphs of $y = a\sin(b(x-h)) + k$, $y = a\cos(b(x-h)) + k$, with and without technology.<br>○ Sketch the graphs of $y = a\sin(b(x-h)) + k$, $y = a\cos(b(x-h)) + k$, with and without technology. |
| 2.4 | Differentiation of trigonometric functions | ○ Use the rules $\dfrac{d}{dx}\sin(x) = \cos(x)$ and $\dfrac{d}{dx}\sin(f(x)) = f'(x)\cos(f(x))$<br><br>○ Use the rules $\dfrac{d}{dx}\cos(x) = -\sin(x)$ and $\dfrac{d}{dx}\cos(f(x)) = -f'(x)\sin(f(x))$ |
| 2.5 | Modelling and solving with trigonometric functions | ○ Model and solve problems that involve derivatives of trigonometric functions, with and without technology. |

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# LESSON
## 2.2 Review of the unit circle

### 2.2.1 The unit circle and radians

The radius of the unit circle is 1 unit. Remember that angles, including those in the unit circle, can be measured in degrees, minutes and seconds or in radians. Recall that 1 degree is equal to 60 minutes and that 1 minute is equal to 60 seconds. An angle of 1 radian, written $1^c$, is equal to the angle formed at the centre of the unit circle by an arc of length 1 unit.
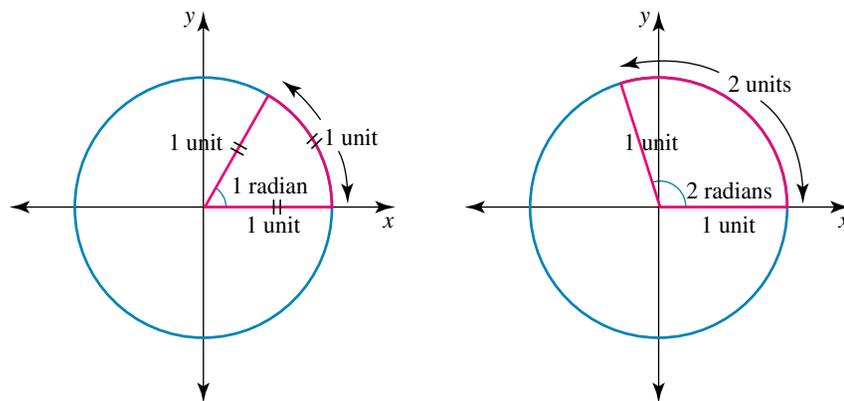


The circumference of the unit circle is $2\pi$, so $360° = 2\pi$ radians.

---

**The relationship between radian and degree measure**

$$\pi \text{ radians} = 180°$$

$$1^c = \left(\frac{180}{\pi}\right) \text{ degrees} \quad \text{or} \quad 1° = \left(\frac{\pi}{180}\right) \text{ radians}$$

---

Angles in the unit circle are measured from the positive $x$-axis, with positive angles formed when moving anticlockwise around the circle and negative angles formed when moving clockwise around the circle.

The radian symbol is often omitted when the angle is written. An angle without a symbol is therefore assumed to be in radians.

The figure below shows the relationship between radian and degree measure visually.



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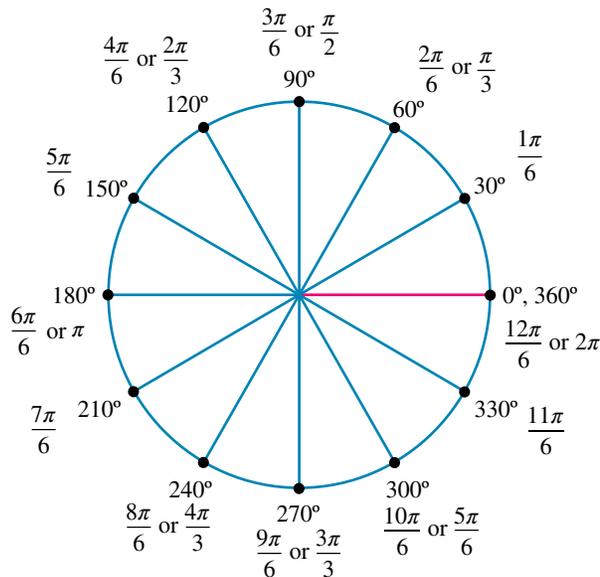
## WORKED EXAMPLE 1 Converting degrees and radians

**a.** Convert the following angles to radians, correct to 2 decimal places where necessary.
  **i.** $40°$                                      **ii.** $125.5°$

**b.** Convert the following angles to degrees, correct to the nearest minute where necessary.

  **i.** $\left(\dfrac{7\pi}{12}\right)$            **ii.** $2.5$

**THINK**

**a. i. 1.** Substitute $1° = \dfrac{\pi}{180}$.

  **2.** Simplify.

  **ii. 1.** Substitute $1° = \dfrac{\pi}{180}$.

  **2.** Simplify.

**b. i. 1.** Substitute $\pi = 180°$.

  **2.** Simplify.

  **ii. 1.** Substitute $1^c = \dfrac{180}{\pi}$ degrees.

  **2.** Simplify.

  **3.** Convert to degrees and minutes.

**WRITE**

**a. i.** $40°$

  $= 40 \times \dfrac{\pi}{180}$

  $= \dfrac{2\pi}{9}$

  **ii.** $125.5°$

  $= 125.5 \times \dfrac{\pi}{180}$

  $= 2.190\,388$

  $= 2.19$

**b. i.** $\dfrac{7\pi}{12} = \dfrac{7 \times 180}{12}$

  $= 105°$

  **ii.** $2.5$

  $= 2.5 \times \dfrac{180}{\pi}$

  $= 143.239\,44\,...$

  $= 143°14'$

## 2.2.2 The unit circle and the sine, cosine and tangent ratios

The point P lies on the unit circle. It can be found by rotating the point $(1, 0)$ through an angle of $\theta$, giving the point $P(\theta)$, or by using the right-angled triangle, with Cartesian coordinates $P(x, y)$.

From the right-angled triangle ONP:

$$\cos(\theta) = \frac{x}{1} = x$$

$$\sin(\theta) = \frac{y}{1} = y$$

$$\tan(\theta) = \frac{y}{x} = \frac{\sin(\theta)}{\cos(\theta)}$$

$x^2 + y^2 = 1$, giving the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$.
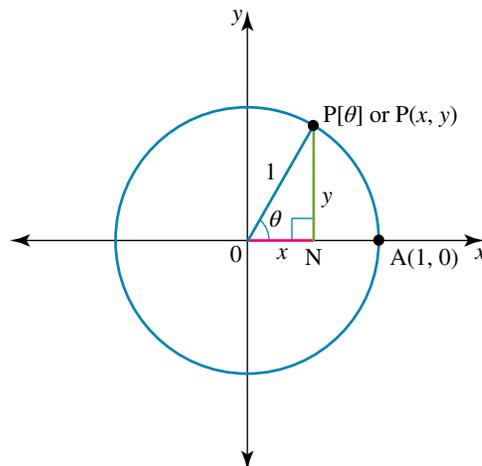
**The tangent ratio**

$$\mathbf{P}(\theta) = (x, y) = (\cos(\theta), \ \sin(\theta))$$

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

**The Pythagorean identity**

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

## 2.2.3 Symmetry and the unit circle

Using the symmetry of the unit circle, the following relationships for angles in other quadrants can be determined.

In the second quadrant, the point $P(\pi - \theta)$ is equivalent to the point $(-x, y)$, giving:

$$\cos(\pi - \theta) = -x = -\cos(\theta)$$

$$\sin(\pi - \theta) = y = \sin(\theta)$$

$$\tan(\pi - \theta) = \frac{y}{-x} = \frac{\sin(\theta)}{-\cos(\theta)} = -\tan(\theta)$$

Similar results can be found for the third and fourth quadrants and are given in the table below.
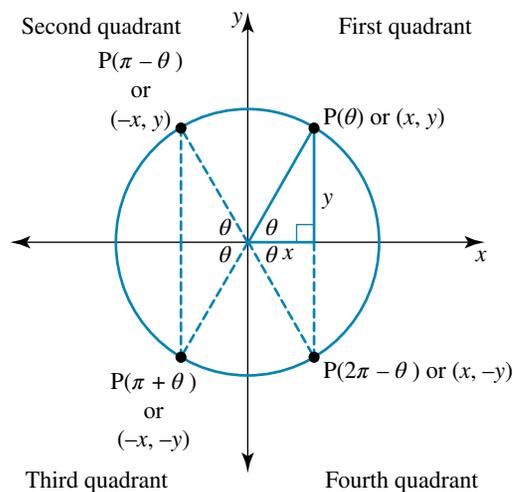
## Symmetry properties of the trigonometric functions

| 2nd quadrant | 1st quadrant |
|---|---|
| $\sin(\pi - \theta) = \sin(\theta)$ <br> $\cos(\pi - \theta) = -\cos(\theta)$ <br> $\tan(\pi - \theta) = -\tan(\theta)$ | $\sin(\theta)$ <br> $\cos(\theta)$ <br> $\tan(\theta)$ |
| **S** <br> **Sin positive** | **A** <br> **All positive** |
| **T** <br> **Tan positive** | **C** <br> **Cos positive** |
| $\sin(\pi + \theta) = -\sin(\theta)$ <br> $\cos(\pi + \theta) = -\cos(\theta)$ <br> $\tan(\pi + \theta) = \tan(\theta)$ | $\sin(2\pi - \theta) = -\sin(\theta)$ <br> $\cos(2\pi - \theta) = \cos(\theta)$ <br> $\tan(2\pi - \theta) = -\tan(\theta)$ |
| 3rd quadrant | 4th quadrant |

Or simply:

This can be remembered using mnemonics such as:

| | |
|---|---|
| **A**dd | **A**ll |
| **S**ugar | **S**tations |
| **T**o | **T**o |
| **C**offee | **C**entral |
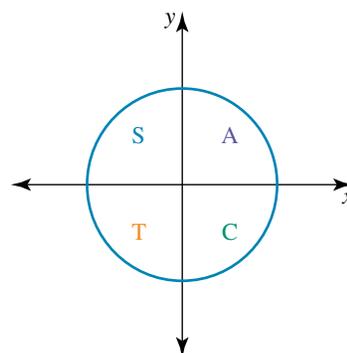
Angles measured in the clockwise direction are called negative angles.

$$\cos(-\theta) = x = \cos(\theta)$$
$$\sin(-\theta) = -y = -\sin(\theta)$$
$$\tan(-\theta) = \frac{-y}{x} = \frac{-\sin(\theta)}{\cos(\theta)} = -\tan(\theta)$$

### Negative angles

$$\cos(-\theta) = \cos(\theta)$$
$$\sin(-\theta) = -\sin(\theta)$$
$$\tan(-\theta) = -\tan(\theta)$$

*Note:* These relationships are true no matter which quadrant the negative angle is in.

## WORKED EXAMPLE 2 Identifying symmetry properties

If $\sin(\alpha) = \dfrac{3}{5}$ and $\alpha$ is in the first quadrant, determine the exact values of the following.

a. $\cos(\alpha)$        b. $\tan(\alpha)$        c. $\sin(\pi + \alpha)$

d. $\cos(2\pi - \alpha)$        e. $\tan(\pi - \alpha)$        f. $\cos(-\alpha)$

**THINK**

**WRITE**

a. 1. Draw a right-angled triangle where the opposite is 3 and the hypotenuse is 5, showing $\sin(\alpha) = \dfrac{3}{5}$ with the adjacent side of 4 (found using Pythagoras' theorem).

a.



$$a^2 + b^2 = c^2$$
$$3^2 + b^2 = 5^2$$
$$b^2 = 16$$
$$b = 4$$

2. State the answer.
*Note:* An alternative method is to use the Pythagorean identity, as shown.

$$\cos(\alpha) = \frac{4}{5}$$
$$\sin^2(\alpha) + \cos^2(\alpha) = 1$$
$$\cos^2(\alpha) = 1 - \sin^2(\alpha)$$
$$\cos^2(\alpha) = 1 - \left(\frac{3}{5}\right)^2$$
$$\cos^2(\alpha) = 1 - \frac{9}{25}$$
$$\cos^2(\alpha) = \frac{16}{25}$$
$$\cos(\alpha) = \pm\frac{4}{5}$$

But $\alpha$ is in the first quadrant, so $\cos(\alpha) = \dfrac{4}{5}$.

b. Use the right-angled triangle to calculate $\tan(\alpha)$.

b. $\tan(\alpha) = \dfrac{3}{4}$

c. 1. $\pi + \alpha$ is in the third quadrant, where sine is negative.

c. $\sin(\pi + \alpha) = -\sin(\alpha)$

2. Substitute.

$\sin(\pi + \alpha) = -\dfrac{3}{5}$

d. 1. $(2\pi - \alpha)$ is in the fourth quadrant, where cosine is positive.

d. $\cos(2\pi - \alpha) = \cos(\alpha)$

2. Substitute.

$\cos(2\pi - \alpha) = \dfrac{4}{5}$

e. 1. $(\pi - \alpha)$ is in the second quadrant, where tangent is negative.

e. $\tan(\pi - \alpha) = -\tan(\alpha)$

2. Substitute.

$\tan(\pi - \alpha) = -\dfrac{3}{4}$

| | |
|---|---|
| **f. 1.** $(-\alpha)$ is in the fourth quadrant, where cosine is positive. | **f.** $\cos(-\alpha) = \cos(\alpha)$ |
| **2.** Substitute. | $\cos(-\alpha) = \dfrac{4}{5}$ |

## 2.2.4 Special values and the unit circle

Using the axis intercepts of the unit circle, the values of sine, cosine and tangent for the angles $0$, $\dfrac{\pi}{2}$, $\pi$, $\dfrac{3\pi}{2}$ and $2\pi$ can be determined.

In summary:

| Angle ($\theta$) | Point P on the unit circle | $\sin(\theta) = y$ value | $\cos(\theta) = x$ value | $\tan(\theta) = \dfrac{y}{x}$ value |
|---|---|---|---|---|
| $0$ | $(1, 0)$ | $0$ | $1$ | $0$ |
| $\dfrac{\pi}{2}$ | $(0, 1)$ | $1$ | $0$ | Undefined |
| $\pi$ | $(-1, 0)$ | $0$ | $-1$ | $0$ |
| $\dfrac{3\pi}{2}$ | $(0, -1)$ | $-1$ | $0$ | Undefined |
| $2\pi$ | $1, 0$ | $0$ | $1$ | $0$ |

## 2.2.5 Exact values

Exact values for $30°$, $45°$ and $60°$ or $\dfrac{\pi}{6}$, $\dfrac{\pi}{4}$ and $\dfrac{\pi}{3}$ can be obtained using an equilateral triangle and a right-angled isosceles triangle.
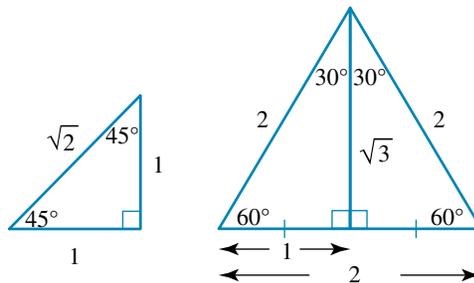


The table below provides a summary of these angles and their ratios.

| Angle ($\theta$) | $\sin(\theta)$ | $\cos(\theta)$ | $\tan(\theta)$ |
|---|---|---|---|
| $30°$ or $\dfrac{\pi}{6}$ | $\dfrac{1}{2}$ | $\dfrac{\sqrt{3}}{2}$ | $\dfrac{1}{\sqrt{3}}$ |
| $45°$ or $\dfrac{\pi}{4}$ | $\dfrac{1}{\sqrt{2}}$ | $\dfrac{1}{\sqrt{2}}$ | $1$ |
| $60°$ or $\dfrac{\pi}{3}$ | $\dfrac{\sqrt{3}}{2}$ | $\dfrac{1}{2}$ | $\sqrt{3}$ |

## WORKED EXAMPLE 3 Determining exact values

**Give exact values for each of the following trigonometric expressions.**

**a.** $\cos\left(\dfrac{2\pi}{3}\right)$　　**b.** $\tan\left(\dfrac{7\pi}{4}\right)$　　**c.** $\cos\left(-\dfrac{\pi}{6}\right)$　　**d.** $\sin\left(\dfrac{11\pi}{3}\right)$　　**e.** $\sin\left(\dfrac{7\pi}{3}\right)$

**THINK**

**a. 1.** Rewrite the angle in terms of $\pi$ and calculate the corresponding angle in the 1st quadrant.

**2.** The angle is in the 2nd quadrant, so cosine is negative.

**3.** Write the answer.

**b. 1.** Rewrite the angle in terms of $2\pi$ and calculate the corresponding angle in the 1st quadrant.

**2.** The angle is in the 4th quadrant, so tangent is negative.

**3.** Write the answer.

**c. 1.** Rewrite the negative angle as $\cos(-\theta) = \cos(\theta)$.

**2.** Write the answer.

**d. 1.** Rewrite the angle in terms of a multiple of $2\pi$.

**2.** Subtract the extra multiple of $2\pi$ so the angle is within 1 revolution of the unit circle.

**3.** The angle is in the 4th quadrant, so sine is negative.

**4.** Write the answer.

**e. 1.** Rewrite the angle in terms of a multiple of $2\pi$.

**2.** Subtract the 2 revolutions of the unit circle.

**3.** The angle corresponds to the point $(0, -1)$, and sine is the $y$-value.

**4.** Write the answer.

**WRITE**

**a.** $\cos\left(\dfrac{2\pi}{3}\right) = \cos\left(\pi - \dfrac{\pi}{3}\right)$

$= -\cos\left(\dfrac{\pi}{3}\right)$

$= -\dfrac{1}{2}$

**b.** $\tan\left(\dfrac{7\pi}{4}\right) = \tan\left(2\pi - \dfrac{\pi}{4}\right)$

$= -\tan\left(\dfrac{\pi}{4}\right)$

$= -1$

**c.** $\cos\left(-\dfrac{\pi}{6}\right) = \cos\left(\dfrac{\pi}{6}\right)$

$= \dfrac{\sqrt{3}}{2}$

**d.** $\sin\left(\dfrac{11\pi}{3}\right) = \sin\left(4\pi - \dfrac{\pi}{3}\right)$

$= \sin\left(2\pi - \dfrac{\pi}{3}\right)$

$= -\sin\left(\dfrac{\pi}{3}\right)$

$= -\dfrac{\sqrt{3}}{2}$

**e.** $\sin\left(\dfrac{7\pi}{2}\right) = \sin\left(4\pi - \dfrac{\pi}{2}\right)$

$= \sin\left(-\dfrac{\pi}{2}\right)$

$= -1$

## 2.2.6 Solving trigonometric equations

Trigonometric equations involve working with the special angles that have exact values as well as angles that can be analysed using technology.

To solve a basic trigonometric equation, follow these steps.
- Identify the quadrants in which solutions lie from the sign.
- Obtain the base angle or first quadrant value.
- Use the base angle to generate the values for the quadrants required from their symmetrical forms.

For example, if $\sin(x) = a, 0 \leq x \leq 2\pi$, remember:
- if $a > 0, x$ must lie in quadrants 1 and 2 where sine is positive
- if $a < 0, x$ must lie in quadrants 3 and 4 where sine is negative.

### WORKED EXAMPLE 4 Solving trigonometric equations

**Solve the following equations.**

**a.** $\sqrt{2}\cos(x) + 1 = 0, 0 \leq x \leq 2\pi$

**b.** $2\sin(x) = -1.5, 0 \leq x \leq 720°$, **correct to 2 decimal places**

**c.** $\tan(\theta) - 1 = 0, -\pi \leq \theta \leq \pi$

**THINK**

**a. 1.** Express the equation with the trigonometric function as the subject.

**2.** Identify the quadrants in which the solutions lie.

**3.** Use knowledge of exact values to state the first quadrant base.

**4.** Generate the solutions using the appropriate quadrant forms.

**5.** Calculate the solutions from their quadrant forms.

**b. 1.** Express the equation with the trigonometric function as the subject.

**2.** Identify the quadrants in which the solutions lie.

**3.** Calculate the base using technology, as an exact value is not possible.

**WRITE**

**a.** $\sqrt{2}\cos(x) + 1 = 0$
$$\sqrt{2}\cos(x) = -1$$
$$\cos(x) = -\frac{1}{\sqrt{2}}$$

Cosine is negative in quadrants 2 and 3.

The base is $\frac{\pi}{4}$, since $\cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$.

$$x = \pi - \frac{\pi}{4}, \pi + \frac{\pi}{4}$$

$$x = \frac{3\pi}{4}, \frac{5\pi}{4}$$

**b.** $2\sin(x) = -1.5$
$$\sin(x) = -0.75$$

Sine is negative in quadrants 3 and 4.
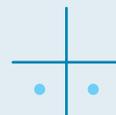
The base is $\sin^{-1}(0.75) = 48.59°$.

**4.** Generate the solutions using the appropriate quadrant forms. As $x \in$ [0°, 720°], there will be 4 positive solutions from 2 anticlockwise rotations.

$x = 180° + 48.59°, 360° - 48.59°, 540° + 48.59°, 720° - 48.59°$

**5.** Calculate the solutions from their quadrant forms. Alternatively, the solve function on technology can be used to determine the solutions (but remember to define the domain).

$x = 228.59°, 311.41°, 588.59°, 671.41°$

**c. 1.** Express the equation with the trigonometric function as the subject.

**c.** $\tan(\theta) - 1 = 0$
$\tan(\theta) = 1$

**2.** Identify the quadrants in which the solutions lie.

Tangent is positive in quadrants 1 and 3.



**3.** Use knowledge of exact values to state the first quadrant base.

The base is $\dfrac{\pi}{4}$, since $\tan\left(\dfrac{\pi}{4}\right) = 1$.

**4.** Generate the solutions using the appropriate quadrant forms. As the domain is $x \in [-\pi, \pi]$, there will be 1 positive solution and 1 negative solution.

$x = \dfrac{\pi}{4}, -\pi + \dfrac{\pi}{4}$

**5.** Calculate the solutions from their quadrant forms.

$x = \dfrac{\pi}{4}, \dfrac{-3\pi}{4}$

## Changing the domain

Equations such as $\sin(2x) = 1, 0 \le x \le 2\pi$ can be expressed in the basic form by the substitution $\theta = 2x$. However, the accompanying domain must be changed to be the domain for $\theta$. This requires the end points of the domain for $x$ to be multiplied by 2. Hence, $0 \le x \le 2\pi \Rightarrow 2 \times 0 \le 2x \le 2 \times 2\pi$ gives the domain requirement for $\theta$ as $0 \le \theta \le 4\pi$.

This allows the equation to be written as $\sin(\theta) = 1, 0 \le \theta \le 4\pi$.

---

### WORKED EXAMPLE 5 Solving by changing the domain

**Solve the following equations for $x$.**
**a.** $2\sin(2x) - 1 = 0, 0 \le x \le 2\pi$
**b.** $2\cos(2x - \pi) - 1 = 0, -\pi \le x \le \pi$

**THINK**

**a. 1.** Change the domain to be that for the given multiple of the variable.

**WRITE**

**a.** $2\sin(2x) - 1 = 0, 0 \le x \le 2\pi$

Multiply each value by 2:

$2\sin(2x) - 1 = 0, 0 \le 2x \le 4\pi$

**2.** Express the equation with the trigonometric function as the subject.

$$2\sin(2x) - 1 = 0$$
$$2\sin(2x) = 1$$
$$\sin(2x) = \frac{1}{2}$$

**3.** Solve the equation for $2x$. As $2x \in [0, 4\pi]$, each of the 2 revolutions will generate 2 solutions, giving a total of 4 values for $2x$.

Sine is positive in quadrants 1 and 2.

The base is $\frac{\pi}{6}$.
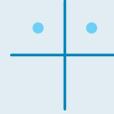
$$2x = \frac{\pi}{6}, \pi - \frac{\pi}{6}, 2\pi + \frac{\pi}{6}, 3\pi - \frac{\pi}{6}$$
$$2x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$$

**4.** Calculate the solutions for $x$.
*Note:* Dividing by 2 at the very end brings the solutions back within the domain originally specified, namely $0 \le x \le 2\pi$.

$$x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$$

**b. 1.** Change the domain to that for the given multiple of the variable.

**b.** $2\cos(2x - \pi) - 1 = 0, \ -\pi \le x \le \pi$

Multiply each value by 2:

$2\cos(2x - \pi) - 1 = 0, \ -2\pi \le 2x \le 2\pi$

Subtract $\pi$ from each value:

$2\cos(2x - \pi) - 1 = 0, \ -3\pi \le 2x - \pi \le \pi$

**2.** Express the equation with the trigonometric function as the subject.

$$2\cos(2x - \pi) - 1 = 0$$
$$2\cos(2x - \pi) = 1$$
$$\cos(2x - \pi) = \frac{1}{2}$$

**3.** Solve the equation for $(2x - \pi)$. The domain of $[-3\pi, \pi]$ involves 2 complete rotations of the unit circle, so there will be 4 solutions, 3 of which will be negative and 1 of which will be positive.

Cosine is positive in quadrants 1 and 4.

The base is $\frac{\pi}{3}$.
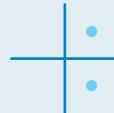
$$2x - \pi = \frac{\pi}{3}, -\frac{\pi}{3}, -2\pi + \frac{\pi}{3}, -2\pi - \frac{\pi}{3}$$
$$2x - \pi = \frac{\pi}{3}, -\frac{\pi}{3}, -\frac{5\pi}{3}, -\frac{7\pi}{3}$$

**4.** Calculate the solutions for $x$.

$$2x = \frac{\pi}{3} + \pi, -\frac{\pi}{3} + \pi, -\frac{5\pi}{3} + \pi, -\frac{7\pi}{3} + \pi$$
$$= \frac{4\pi}{3}, \frac{2\pi}{3}, -\frac{2\pi}{3}, -\frac{4\pi}{3}$$
$$x = \frac{2\pi}{3}, \frac{\pi}{3}, -\frac{\pi}{3}, -\frac{2\pi}{3}$$

## Further types of trigonometric equations

Trigonometric equations may require algebraic techniques or the use of relationships between the functions before they can be reduced to the basic form $f(x) = a$, where $f$ is either sin, cos or tan.

- Equations of the form $\sin(x) = a \cos(x)$ can be converted to $\tan(x) = a$ by dividing both sides of the equation by $\cos(x)$.
- Equations of the form $\sin^2(x) = a$ can be converted to $\sin(x) = \pm\sqrt{a}$ by taking the square roots of both sides of the equation.
- Equations of the form $\sin^2(x) + b \sin(x) + c = 0$ can be converted to standard quadratic equations by using the substitution $A = \sin(x)$.

Because $-1 \leq \sin(x) \leq 1$ and $-1 \leq \cos(x) \leq 1$, neither $\sin(x)$ nor $\cos(x)$ can have values greater than 1 or less than $-1$. This may have implications requiring the rejection of some steps when working with sine or cosine trigonometric equations. As $\tan(x) \in \mathbb{R}$, there is no restriction on the values the tangent function can take.

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### WORKED EXAMPLE 6 Solving further trigonometric equations

**Solve the following equations.**
**a.** $\sin(2x) = \cos(2x)$, $0 \leq x \leq 2\pi$
**b.** $2\sin^2(\theta) + 3\sin(\theta) - 2 = 0$, $0 \leq x \leq 2\pi$
**c.** $\cos^2(2\alpha) - 1 = 0$, $-\pi \leq \alpha \leq \pi$

| THINK | WRITE |
|---|---|
| **a. 1.** Change the domain to that for the given multiple of the variable. | **a.** $0 \leq x \leq 2\pi$ <br> Multiply by 2: <br> $0 \leq 2x \leq 4\pi$ |
| **2.** Reduce the equation to one trigonometric function by dividing by $\cos(2x)$. | $\sin(2x) = \cos(2x)$ <br> $\dfrac{\sin(2x)}{\cos(2x)} = \dfrac{\cos(2x)}{\cos(2x)}$ providing $\cos(2x) \neq 0$ <br> $\tan(2x) = 1$ |
| **3.** Solve the equation for $2x$. | Tangent is positive in quadrants 1 and 3. <br><br>  <br><br> The base is $\dfrac{\pi}{4}$. <br> $2x = \dfrac{\pi}{4}, \pi + \dfrac{\pi}{4}, 2\pi + \dfrac{\pi}{4}, 3\pi + \dfrac{\pi}{4}$ <br> $= \dfrac{\pi}{4}, \dfrac{5\pi}{4}, \dfrac{9\pi}{4}, \dfrac{13\pi}{4}$ |
| **4.** Calculate the solutions for $x$. Note that the answers are within the prescribed domain of $0 \leq x \leq 2\pi$. | $x = \dfrac{\pi}{8}, \dfrac{5\pi}{8}, \dfrac{9\pi}{8}, \dfrac{13\pi}{8}$ |
| **b. 1.** Use substitution to form a quadratic equation. | **b.** $2\sin^2(\theta) + 3\sin(\theta) - 2 = 0$ <br> Let $A = \sin(\theta)$. <br> $2A^2 + 3A - 2 = 0$ |
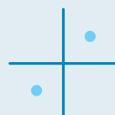
**2.** Solve the quadratic equation.

$(2A - 1)(A + 2) = 0$

$A = \dfrac{1}{2}$ or $A = -2$

But $A = \sin(\theta)$.

$\sin(\theta) = \dfrac{1}{2}$ or $\sin(\theta) = -2$

**3.** Solve each trigonometric equation separately.

$\sin(\theta) = \dfrac{1}{2}$
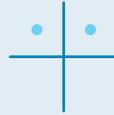


Sine is positive in quadrants 1 and 2.

The base is $\dfrac{\pi}{6}$.

$\theta = \dfrac{\pi}{6}, \pi - \dfrac{\pi}{6}$

$\theta = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$

$\sin(\theta) = -2$

There is no solution as $-1 \le \sin(\theta) \le 1$.

**4.** Write the answer.

$\theta = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$

**c. 1.** Change the domain to that for the given multiple of the variable.

**c.** $-\pi \le \alpha \le \pi$

Multiply by 2:

$-2\pi \le 2\alpha \le 2\pi$

**2.** Use substitution to form a quadratic equation and factorise by applying the difference of perfect squares method.

$\cos^2(2\alpha) - 1 = 0$

Let $A = \cos(2\alpha)$.

$A^2 - 1 = 0$

**3.** Solve the quadratic equation.

$(A - 1)(A + 1) = 0$

$A = 1, -1$

But $A = \cos(2\alpha)$.

$\therefore \cos(2\alpha) = 1$ or $\cos(2\alpha) = -1$

**4.** Solve each trigonometric equation separately.

$\cos(2\alpha) = 1$

$2\alpha = -2\pi, 0, 2\pi$

$\alpha = -\pi, 0, \pi$

$\cos(2\alpha) = -1$

$2\alpha = -\pi, \pi$

$\alpha = -\dfrac{\pi}{2}, \dfrac{\pi}{2}$

**5.** Write the answers in numerical order.

$\therefore \alpha = -\pi, -\dfrac{\pi}{2}, 0, \dfrac{\pi}{2}, \pi$

| **2.2 Exercise** | **2.2 Exam questions** on | These questions are even better in jacPLUS! |

| Simple familiar | Complex familiar | Complex unfamiliar |
|---|---|---|
| 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 | 13, 14, 15, 16 | 17, 18, 19, 20 |

These questions are even better in jacPLUS!
- Receive immediate feedback
- Access sample responses
- Track results and progress

**Find all this and MORE in jacPLUS** ▶

## Simple familiar

1. **WE1a** Convert the following angles to radians, giving your answers in exact form where possible.
   a. $15°$
   b. $120°$
   c. $130°$
   d. $63.9°$
   e. $78.82°$
   f. $310°$

2. **WE1b** Change the following angles to degrees, giving your answers to 2 decimal places when necessary. Note that radians are denoted by a superscript c.
   a. $5^c$
   b. $4.8^c$
   c. $2.56^c$
   d. $\dfrac{3\pi^c}{10}$
   e. $\dfrac{5\pi^c}{6}$
   f. $\dfrac{5\pi^c}{4}$

3. **WE2** Evaluate the following, given that $\sin(\alpha) = \dfrac{5}{13}$ and $\alpha$ lies in the first quadrant.
   a. $\cos(\alpha)$
   b. $\tan(\alpha)$
   c. $\sin(\pi - \alpha)$
   d. $\cos(\pi + \alpha)$
   e. $\cos(2\pi - \alpha)$
   f. $\tan(-\alpha)$

4. **WE3** Calculate the exact values of each of the following.
   a. $\tan\left(\dfrac{3\pi}{4}\right)$
   b. $\cos\left(\dfrac{5\pi}{6}\right)$
   c. $\sin\left(-\dfrac{\pi}{4}\right)$
   d. $\cos\left(\dfrac{7\pi}{3}\right)$
   e. $\tan\left(-\dfrac{\pi}{3}\right)$
   f. $\sin\left(\dfrac{11\pi}{6}\right)$

5. For the given triangle, determine the values of:
   a. $\sin(\theta)$
   b. $\tan(\theta)$
   c. $\cos(\theta)$
   d. $\sin(90° - \theta)$
   e. $\cos(90° - \theta)$
   f. $\tan(90° - \theta)$

   (Right triangle with hypotenuse 13, opposite side 5, adjacent side 12, angle $\theta$)

6. Consider $\sin(x) = \dfrac{5}{6}$.
   a. Show that $\sin^2(x) + \cos^2(x) = 1$.
   b. Show that $1 + \tan^2(x) = \dfrac{1}{\cos^2(x)}$.
   c. Explain why you didn't need to consider the quadrant in which $x$ was lying for your answers to parts **a** and **b**.

7. **WE4** Solve the following equations.
   a. $2\cos(\theta) + \sqrt{3} = 0$ for $0 \le \theta \le 2\pi$
   b. $\tan(x) + \sqrt{3} = 0$ for $0° \le x \le 720°$
   c. $2\cos(\theta) = 1$ for $-\pi \le \theta \le \pi$

8. a. Solve the equation $2\sin(\theta) + 1 = 0$, $0° \le \theta \le 360°$.
   b. Solve $\sin(x) = 1$, $-2\pi \le x \le 2\pi$.
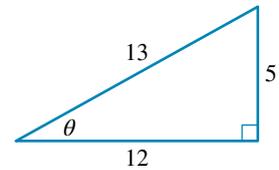
9. **WE5** Solve the following equations.
   a. $2\cos(3\theta) - \sqrt{2} = 0$ for $0 \le \theta \le 2\pi$
   b. $2\sin(2x + \pi) + \sqrt{3} = 0$ for $-\pi \le x \le \pi$

10. Solve $2\cos\left(3\theta - \dfrac{\pi}{2}\right) + \sqrt{3} = 0$, $0 \le \theta \le 2\pi$.

11. **WE6** Solve the equation $\cos^2(\theta) - \sin(\theta)\cos(\theta) = 0$ for $0 \le \theta \le 2\pi$.

12. Solve $\{\theta: 2\cos^2(\theta) + 3\cos(\theta) = -1, 0 \le \theta \le 2\pi\}$.

13. Calculate the exact values of the following.

   a. $\cos\left(\dfrac{7\pi}{6}\right) + \cos\left(\dfrac{2\pi}{3}\right)$
   b. $2\sin\left(\dfrac{7\pi}{4}\right) + 4\sin\left(\dfrac{5\pi}{6}\right)$
   c. $\sqrt{3}\tan\left(\dfrac{5\pi}{4}\right) - \tan\left(\dfrac{5\pi}{3}\right)$

14. Determine the exact values of the following.

   a. $\sin^2\left(\dfrac{8\pi}{3}\right) + \sin\left(\dfrac{9\pi}{4}\right)$
   b. $2\cos^2\left(-\dfrac{5\pi}{4}\right) - 1$
   c. $\dfrac{\tan\left(\frac{17\pi}{4}\right)\cos(-7\pi)}{\sin\left(-\frac{11\pi}{6}\right)}$

15. Solve the following for $x$ given that $-\pi \le x \le \pi$.

   a. $2\sin\left(2x + \dfrac{\pi}{4}\right) = \sqrt{2}$
   b. $2\cos(x + \pi) = \sqrt{3}$
   c. $\tan(x - \pi) = -1$

16. Solve the following for $\alpha$ where $-\pi \le \alpha \le \pi$.

   a. $\sin(\alpha) - \cos^2(\alpha)\sin(\alpha) = 0$
   b. $\sin(2\alpha) = \sqrt{3}\cos(2\alpha)$
   c. $\sin^2(\alpha) = \cos^2(\alpha)$
   d. $4\cos^2(\alpha) - 1 = 0$

17. The weight on a spring moves in such a way that its speed,
    $v$ cm/s, is given by the rule $v = 12 + 3\sin\left(\dfrac{\pi t}{3}\right)$.

    Determine the initial speed of the weight and discuss when the
    spring has the same speed. Justify your answer.

18. A particle moves in a straight line so that its distance, $x$ metres, from a point O is given by the equation
    $x = 3 + 4\sin(2t)$, where $t$ is the time in seconds after the particle begins to move.

    Determine the time, correct to 2 decimal places, when the particle first reaches the point O, and discuss,
    justifying your response, why the particle is never more than 4 metres from its initial position.

**19.** The height, $h(t)$ metres, that the water reaches up the side of the bank of the Brisbane river is determined by the rule

$$h(t) = 0.5 \cos\left(\frac{\pi t}{12}\right) + 1.0$$

where $t$ is the number of hours after 6 am. Determine the height of the water up the side of the bank at 10 pm, and discuss, with reasons, when the height of the water will again reach this level.

**20.** Sketch the graphs of $y = \sin(x)$ and $y = \cos(2x)$ for $0 \le x \le 2\pi$ and determine the values of $x$ where $\sin(x) = \cos(2x)$ (for $0 \le x \le 2\pi$), giving your answers correct to 2 decimal places. Discuss why technology was useful in solving this equation.

**Fully worked solutions for this chapter are available online.**

# LESSON
## 2.3 Review of the trigonometric functions

### SYLLABUS LINKS

[Review from Unit 1, Topic 4]

- Sketch the graphs of $y = \sin(x)$, $y = \cos(x)$ on extended domains.
- Recognise and determine the effect of the parameters $a, b$ $h$ and $k$ on the graphs of $y = a\sin(b(x - h)) + k$, $y = a\cos(b(x - h)) + k$, with and without technology.
- Sketch the graphs of $y = a\sin(b(x - h)) + k$, $y = a\cos(b(x - h)) + k$, with and without technology.

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## 2.3.1 Graphs of $y = \sin(x)$ and $y = \cos(x)$

Trigonometric functions, or circular functions, are periodic functions such as $= \sin(x)$ and $y = \cos(x)$.

The graph of the sine function has a wave shape that repeats itself every $2\pi$ units. Its **period** is $2\pi$ as shown in its graph.

The graph oscillates about the line $y = 0$ (the $x$-axis), rising and falling by up to 1 unit. This gives the graph its range of $[-1, 1]$ with a mean, or equilibrium, position of $y = 0$ and an **amplitude** of 1.
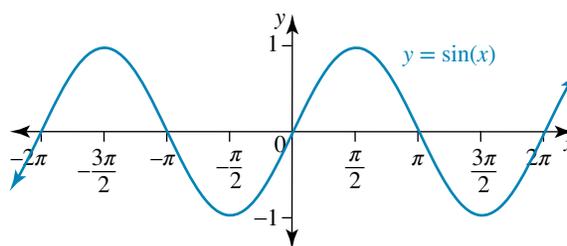
The graph of the cosine function has the same wave shape with period $2\pi$ oscillating about the mean position of $y = 0$ and an amplitude of 1.



Although the domain of both the sine and cosine functions is $\mathbb{R}$, they are usually sketched for a given restricted domain.
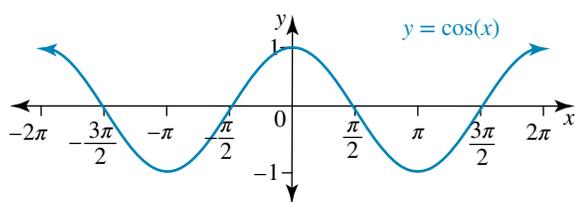
The two graphs of $y = \sin(x)$ and $y = \cos(x)$ are 'out of phase' by $\dfrac{\pi}{2}$; that is, $\cos\left(x - \dfrac{\pi}{2}\right) = \sin(x)$. In other words, a horizontal shift of the cosine graph by $\dfrac{\pi}{2}$ units to the right gives the sine graph. Likewise, a horizontal shift of the sine graph by $\dfrac{\pi}{2}$ units to the left gives the cosine graph; $\sin\left(x + \dfrac{\pi}{2}\right) = \cos(x)$.

> **Key features of the sine and cosine functions**
>
> **The graphs of $y = \sin(x)$ and $y = \cos(x)$ have:**
> - **period $2\pi$**
> - **amplitude 1**
> - **mean position $y = 0$**
> - **domain $\mathbb{R}$**
> - **range $[-1, 1]$.**

## 2.3.2 Graphs of $y = a\sin(bx) + k$ and $y = a\cos(bx) + k$

The **value of $a$** affects the amplitude of the sine and cosine functions. The amplitude measures the distance — the rise and fall — from the equilibrium or mean position.
- If $a > 0$, the amplitude is $a$.
- If $a < 0$, the amplitude is $|a|$ and the graph is inverted or reflected in the $x$-axis.

For example:

$y = 2\sin(x)$

Amplitude $= 2$

*Note:* The graph shows a dilation of 2 from the $x$-axis.
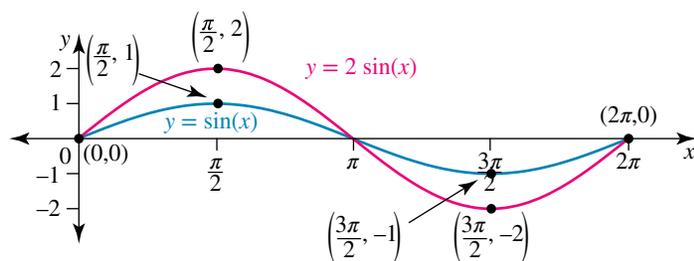


$y = -\dfrac{1}{2}\cos(x)$

Amplitude $= \dfrac{1}{2}$

Reflection in the $x$-axis

*Note:* The graph shows a dilation of $\dfrac{1}{2}$ from the $x$-axis.
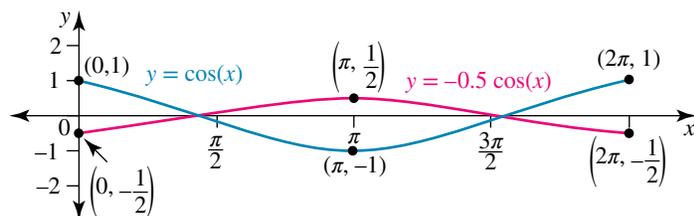
The **value of *b*** affects the period of the sine and cosine functions.

- The period is $\dfrac{2\pi}{b}$.
- If $b < 0$, the function is reflected over the *y*-axis.

For example:

$y = \cos(2x)$

$\text{Period} = \dfrac{2\pi}{2} = \pi$

*Note:* The graph shows a dilation of $\dfrac{1}{2}$ from the *y*-axis.
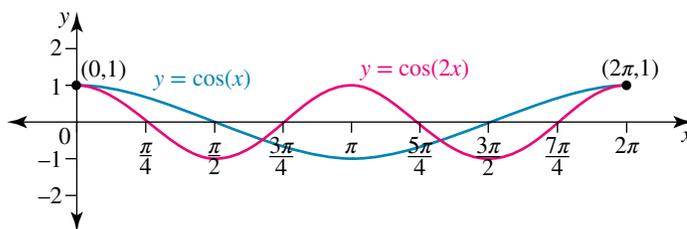


$y = \sin\left(\dfrac{x}{2}\right)$

$\text{Period} = 2\pi \div \dfrac{1}{2} = 4\pi$

*Note:* The graph shows a dilation of 2 from the *y*-axis.
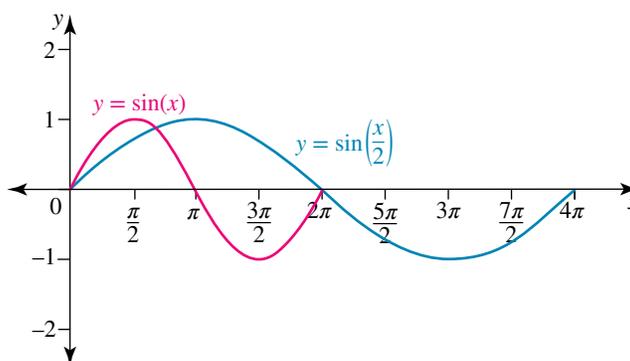


The **value of *k*** affects the equilibrium or mean position about which the sine and cosine functions oscillate, as it represents a vertical translation of the function.

- The graph oscillates about the line $y = k$.
- The range of the function changes.

For example:

$y = \sin(x) - 2$

Line of oscillation: $y = -2$

*Note:* This graph shows a vertical translation down by 2 units.
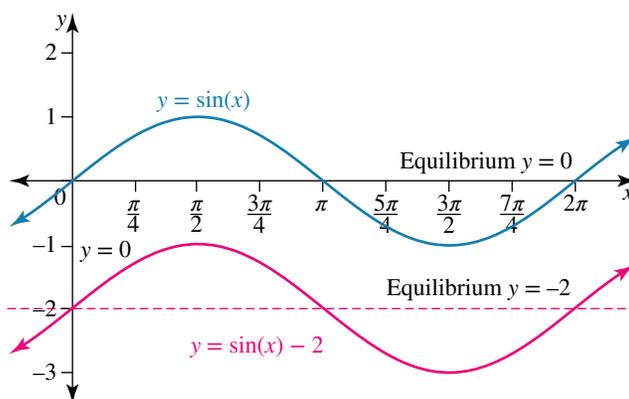


**Features of $y = a\,\sin(bx) + k$ and $y = a\,\cos(bx) + k$**

The graphs of $y = a\,\sin(bx) + k$ and $y = a\,\cos(bx) + k$ have:

- **period $\dfrac{2\pi}{b}$**
- **amplitude $|a|$**
- **mean position $y = k$**
- **domain $\mathbb{R}$**
- **range $[k - a,\ k + a]$.**

## WORKED EXAMPLE 7 Sketching trigonometric graphs

**Sketch the graph of $y = 3\sin(2x) + 4, 0 \le x \le 2\pi$.**

| THINK | WRITE |
|---|---|
| **1.** State the period and amplitude of the graph. | $y = 3\sin(2x) + 4, 0 \le x \le 2\pi$ <br> The period is $\dfrac{2\pi}{2} = \pi$. <br> The amplitude is 3. |
| **2.** State the mean position and the range. | The mean position is $y = 4$. <br> The range of the graph is $[4 - 3, 4 + 3] = [1, 7]$. |
| **3.** Construct appropriate scales on the axes and sketch the graph. | |

| TI \| THINK | WRITE | CASIO \| THINK | WRITE |
|---|---|---|---|
| **1.** On a Graphs page, set the Graphing Angle to Radian. Press MENU, then select: 4: Window/Zoom 1: Window Settings … Complete the fields as: XMin: $-\pi$ XMax: $3\pi$ XScale: $\pi/4$ YMin: 0 YMax: 8 YScale: 1 then select OK. <br> *Note:* The calculator will only give decimal approximations for intercepts, minimums and maximums, so it is important to have the *x*-axis scale as a multiple of $\pi$ so that important points can be easily read from the graph. | | **1.** Put the Calculator in Radian mode. On a Graph screen, press SHIFT then F3 to open the V_WIN. Complete the fields as: Xmin: $-\pi$ max: $3\pi$ scale: $\pi/4$ Ymin: 0 max: 8 scale: 1 then press EXIT. <br> *Note:* The calculator will only give decimal approximations for intercepts, minimums and maximums, so it is important to have the *x*-axis scale as a multiple of $\pi$ so that important points can be easily read from the graph. | |
| **2.** Complete the entry line for function 1 as: $f1(x) = 3\sin(2x) + 4 \mid 0 \le x \le 2\pi$ then press ENTER. | | **2.** Complete the entry line for *y*1 as: $y1 = 3\sin(2x) + 4, [0, 2\pi]$ then press EXE. Select DRAW by pressing F6. | |
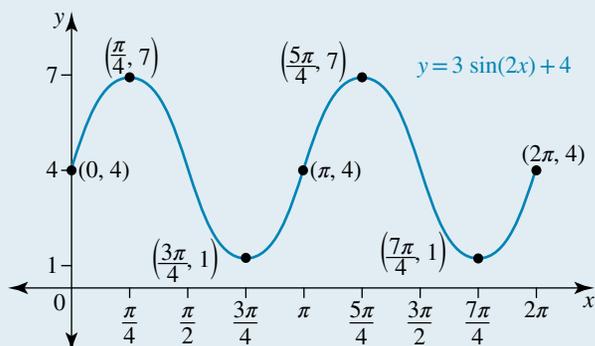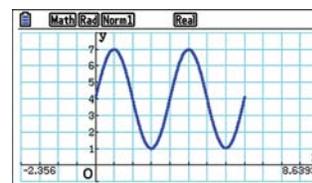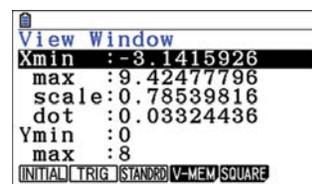
**The diagram shows the graph of a cosine function. State its mean position, amplitude and period, and give a possible equation for the function.**
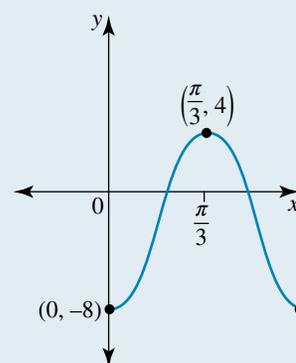


**THINK**

1. Deduce the mean position.

2. State the amplitude.

3. State the period.

4. Determine a possible equation for the given graph.

**WRITE**

The minimum value is $-8$ and the maximum value is 4, so the mean position is $y = \dfrac{-8 + 4}{2} = -2$.

The amplitude is the distance from the mean position to either its maximum or minimum. The amplitude is $A = \dfrac{4 - 8}{2} = 6$.

At $x = \dfrac{\pi}{3}$, the graph is halfway through its cycle, so its period is $\dfrac{2\pi}{3}$.

Let the equation be $y = A\cos(Bx) + D$.
The graph is an inverted cosine shape, so $A = -6$.
The period is $\dfrac{2\pi}{B}$.
$\dfrac{2\pi}{B} = \dfrac{2\pi}{3}$
$B = 3$
The mean position is $y = -2$, so $D = -2$.
The equation could be $y = -6\cos(3x) - 2$.

## 2.3.3 Graphs of $y = a\sin(b(x - h)) + k$ and $y = a\cos(b(x - h)) + k$

Horizontal translations do not affect the period, amplitude or mean position of the graphs of sine or cosine functions. The presence of a horizontal translation of $h$ units is recognised from the equation in the form given as $y = a\sin(b(x - h)) + k$ in exactly the same way it is for any other type of function. The graph will have the same shape as $y = a\sin(bx) + k$, but it will be translated to the right or to the left, depending on whether $h$ is positive or negative, respectively. Translations affect the position of the maximum and minimum points and any $x$-intercepts and $y$-intercept.

The function $y = a\cos(b(x - h)) + k$ follows the same patterns as those for $y = a\sin(b(x - h)) + k$.

When sketching trigonometric functions, the following steps may be useful.
- Identify the period, amplitude, line of oscillation (or mean position) and range.
- Sketch the graph without any horizontal translation.
- Determine the coordinates of the end points for the given domain.
- Sketch the graph with the horizontal translation.
- Apply the vertical translation to the graph.
- Determine the coordinates of the $x$-intercept and $y$-intercept if included.

## WORKED EXAMPLE 9 Sketching trigonometric graphs with translations

**a.** Sketch the graph of the function $f$: $\left[0, \dfrac{3\pi}{2}\right] \to \mathbb{R}$, $f(x) = 4 \cos\left(2x + \dfrac{\pi}{3}\right)$.

**b.** Hence, sketch the graph of the function $g$: $\left[0, \dfrac{3\pi}{2}\right] \to \mathbb{R}$, $g(x) = 6 - 4 \cos\left(2x + \dfrac{\pi}{3}\right)$.

**THINK**

**a. 1.** State the period, amplitude, mean position and horizontal translation by rewriting the function in the form $y = A \cos(B(x + C)) + D$.
*Note:* It is a common error not to factorise to determine $B$.

**2.** Sketch the graph without the horizontal translation:
$y = 4\cos(2x)$.

**WRITE**

**a.** $f$: $\left[0, \dfrac{3\pi}{2}\right] \to \mathbb{R}$, $f(x) = 4\cos\left(2x + \dfrac{\pi}{3}\right)$

$f(x) = 4\cos\left(2\left(x + \dfrac{\pi}{6}\right)\right)$

The period is $\dfrac{2\pi}{B}$; in this case, $\dfrac{2\pi}{2} = \pi$.

The amplitude is 4.
The mean position is $y = 0$.
The horizontal translation is $\dfrac{\pi}{6}$ to the left.
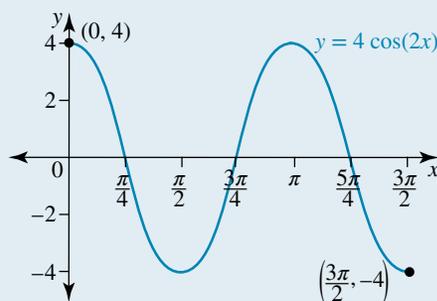
**3.** Calculate the coordinates of the end points of the domain of the given function.

$$f(0) = 4 \cos\left(\frac{\pi}{3}\right)$$
$$= 4 \times \frac{1}{2}$$
$$= 2$$

$$f\left(\frac{3\pi}{2}\right) = 4 \cos\left(3\pi + \frac{\pi}{3}\right)$$
$$= 4 \times \frac{-1}{2}$$
$$= -2$$

The end points of the graph are $(0, 2)$ and $\left(\frac{3\pi}{2}, -2\right)$.

**4.** Calculate or deduce the positions of the $x$-intercepts.

Each $x$-intercept on $y = 4\cos(2x)$ is translated $\frac{\pi}{6}$ units to the left. Alternatively, let $y = 0$

$$4\cos\left(2x + \frac{\pi}{3}\right) = 0$$

$$\cos\left(2x + \frac{\pi}{3}\right) = 0, \frac{\pi}{3} \le 2x + \frac{\pi}{3} \le 3\pi + \frac{\pi}{3}$$

$$2x + \frac{\pi}{3} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$$

$$2x = \frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}$$

$$x = \frac{\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}$$

**5.** Apply the horizontal translation to key points on the graph already sketched and hence sketch the function over its given domain.
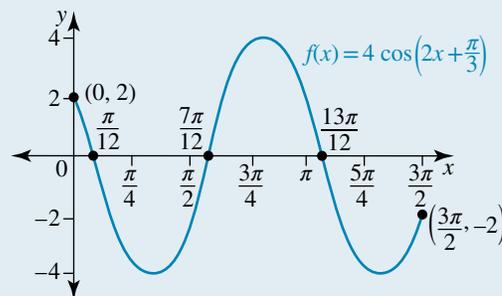


**b. 1.** State the period, amplitude, line of oscillation and range.
*Note:* $g(x)$ is related to $f(x)$ from part **a**.
The curve $f(x)$ has been reflected in the $x$-axis and translated vertically upwards by 6.

**b.** $g: \left[0, \frac{3\pi}{2}\right] \to \mathbb{R}, g(x) = 6 - 4\cos\left(2x + \frac{\pi}{3}\right)$

$$g(x) = -4\cos\left(2\left(x + \frac{\pi}{6}\right)\right) + 6$$

Period $= \pi$
Amplitude $= 4$
Line of oscillation: $y = 6$
Range $= [2, 10]$

**2.** State the translations.

Horizontal translation $\frac{\pi}{6}$ to the left. Vertical translation upwards by 6 units.

**3.** Calculate the end points for the given domain.

*Note:* Since $g(x)$ is a reflection and vertical translation of $f(x)$, the end points could be obtained easily using the end points of $f(x)$.

$$g(0) = -4\cos\left(\frac{\pi}{3}\right) + 6$$

$$g(0) = -4 \times \frac{1}{2} + 6$$

$$g(0) = 4$$

$$g\left(\frac{3\pi}{2}\right) = -4\cos\left(2 \times \frac{3\pi}{2} + \frac{\pi}{3}\right) + 6$$

$$g\left(\frac{3\pi}{2}\right) = -4\cos\left(\frac{10\pi}{3}\right) + 6$$

$$g\left(\frac{3\pi}{2}\right) = -4 \times \frac{-1}{2} + 6$$

$$g\left(\frac{3\pi}{2}\right) = 8$$

The end points are $(0, 4)$ and $\left(\frac{3\pi}{2}, 8\right)$.

**4.** Using the transformations of $f(x)$, the function $g(x)$ will not have any $x$-intercepts.

There are no $x$-intercepts since $\cos\left(2x + \frac{\pi}{3}\right) \neq \frac{3}{2}$.

**5.** Sketch the function $y = g(x)$ using the graph of $f(x)$ along with the extra information and transformations.
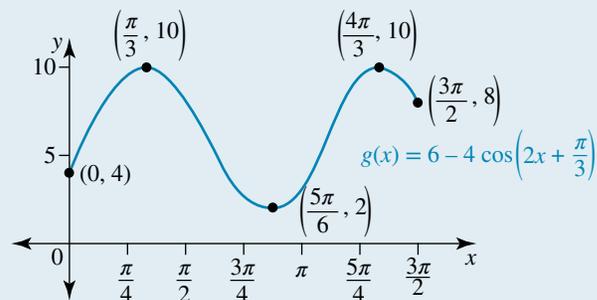


# Exercise 2.3 Review of the trigonometric functions

**learn on**

| **2.3 Exercise** | **2.3 Exam questions** on |
|---|---|

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| **Simple familiar** | **Complex familiar** | **Complex unfamiliar** |
|---|---|---|
| 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 | 13, 14, 15, 16 | 17, 18, 19, 20 |

### Simple familiar

**1.** Sketch the following graphs for $0 \leq x \leq 2\pi$.

   **a.** $y = \sin(x)$                    **b.** $y = \sin(2x)$                       **c.** $y = 2\sin(x)$

   **d.** $y = \sin(x) + 2$            **e.** $y = 2 - \sin(x)$

2. Sketch the following graphs for $-2\pi \leq x \leq 2\pi$.

   a. $y = \cos(x)$     b. $y = \cos\left(\dfrac{x}{2}\right)$     c. $y = 3\cos(x)$     d. $y = 3 - \cos(x)$     e. $y = \cos(x) - 3$

3. **WE7** Sketch the graph of $y = 2\cos(4x) - 3, 0 \leq x \leq 2\pi$.

4. Sketch the graph of $y = 2 - 4\sin(3x), \ 0 \leq x \leq 2\pi$.

5. Sketch the graph of $y = -7\cos(4x)$ for $0 \leq x \leq \pi$, stating any axis intercepts.

6. For $-\pi \leq x \leq 2\pi$, sketch the function $y = \dfrac{1}{2}\cos(2x) + 3$.

7. Sketch the graph of $f\colon\ [0, 2\pi] \to \mathbb{R}, \ f(x) = 1 - 2\sin\left(\dfrac{3x}{2}\right)$, locating any intercepts with the coordinate axes.

8. **WE8** The diagram shows the graph of a sine function. State its mean position, amplitude and period, and give a possible equation for the function.
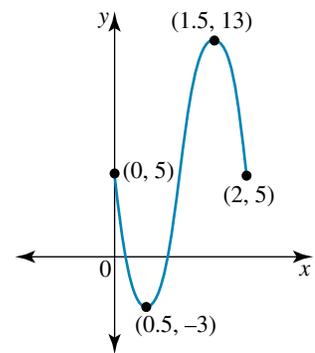


9. The diagram shows the graph of a cosine function. State its line of oscillation, amplitude and period, and give a possible equation for the function.
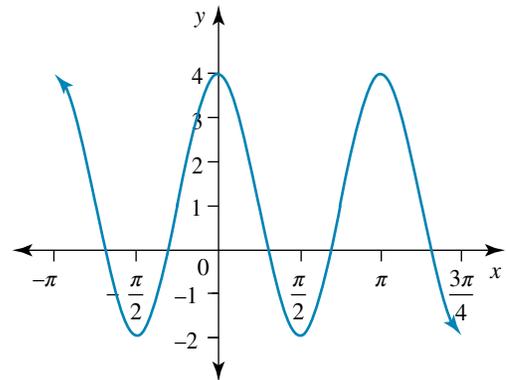


10. a. **WE9** Sketch the graph of the function $f\colon\ \left[0, \dfrac{3\pi}{2}\right] \to \mathbb{R}, \ f(x) = -6\sin\left(3x - \dfrac{3\pi}{4}\right)$.

    b. Hence or otherwise, sketch the function $g\colon\ \left[0, \dfrac{3\pi}{2}\right] \to \mathbb{R}, \ g(x) = 7 - 6\sin\left(3x - \dfrac{3\pi}{4}\right)$, showing all important features.

11. a. Sketch the function $y = 2\sin\left(x + \dfrac{\pi}{4}\right), \ 0 \leq x \leq 2\pi$.

    b. Hence or otherwise, sketch the function $y = 2\sin\left(x + \dfrac{\pi}{4}\right) - 1, 0 \leq x \leq 2\pi$, showing all important features.

12. State the maximum value of the function $f(x) = 2 - 3\cos\left(x + \dfrac{\pi}{12}\right)$ and give the first positive value of $x$ for when this maximum occurs.

**13. a.** Sketch the function $f$: $\left[-\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right] \to \mathbb{R}$, $f(x) = 4\cos\left(3x - \dfrac{\pi}{2}\right)$, showing all axis intercepts.

   **b.** Hence or otherwise, sketch the function $g$: $\left[-\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right] \to \mathbb{R}$, $g(x) = 4 - 4\cos\left(3x - \dfrac{\pi}{2}\right)$.

   **c.** Explain how the functions in parts **a** and **b** are related.

**14. a.** Sketch the graph of $y = 2\cos(3x)$ for one complete cycle.

   **b.** Sketch the graph that would result from the function $y = 2\cos(3x)$ being translated $\dfrac{\pi}{3}$ units to the right and 3 units vertically up.

   **c.** State an equation for the graph formed in part **b**.

**15. a.** Sketch $y = \sin(x), 0 \le x \le 4\pi$
   **b.** Hence, sketch $y = \sin^2(x), 0 \le x \le 4\pi$. Check your graph using technology.

**16. a.** Solve the equation $2\sin(2x) + \sqrt{3} = 0$ for $x \in [0, 2\pi]$.
   **b.** Sketch the graph of $y = \sin(2x)$ for $x \in [0, 2\pi]$.

   **c.** Hence, determine $\left\{x:\ \sin 2x < -\dfrac{\sqrt{3}}{2}, 0 \le x \le 2\pi\right\}$.

**17.** James is in a boat out at sea fishing. The weather makes a change for the worse and the water becomes very choppy. The depth of water above the sea bed can be modelled by the function with equation

$$d = 1.5\ \sin\left(\dfrac{\pi t}{12}\right) + 12.5$$

where $d$ is the depth of water in metres and $t$ is the time in hours since the change of weather began. James has heard on the radio that the cycle of weather should have passed within 12 hours, and when the height of water above the sea bed is at a minimum after that, it will be safe to return to shore. If the weather change occurred at 9:30 am, determine when James will be able to return to shore.

**18.** A very young girl is learning to skip. The graph showing this skipping for one cycle is given.
The general equation for this graph is given by

$$h = a\cos(n(t - 0.5)) + c$$

where $h$ is the height in millimetres of the girl's feet above the ground and $t$ is the time in seconds the girl has been skipping.
Determine how soon after the beginning of the turn the girl's feet are 40 mm off the ground.
Give your answer correct to 1 decimal place.
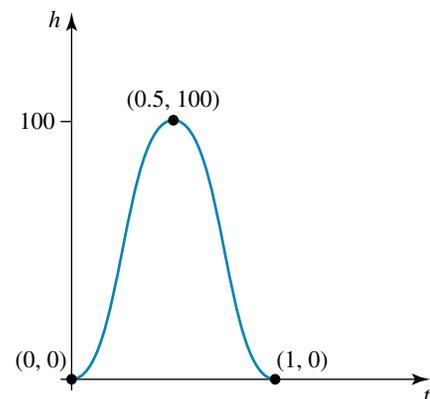
**19.** A young girl and boy are lifted onto a seesaw in a playground. At this time the seesaw is horizontal with respect to the ground.

Initially the girl's end of the seesaw rises. Her height above the ground, $h$ metres, $t$ seconds after the seesaw starts to move is modelled by

$$h(t) = a \sin(nt) + k$$

The greatest height above the ground that the girl reaches is 1.7 metres, and the least distance above the ground that she reaches is 0.7 metres. It takes 2 seconds for her to seesaw between these heights.

Sketch the graph showing the height of the girl above the ground during the first 6 seconds and state its equation.

**20.** The water level in a harbour, $h$ metres below a level jetty, at time $t$ hours after 7 am is given by

$$h = 3 - 2.5 \sin\left(\frac{1}{2}(t-1)\right)$$

Sketch the graph of $h$ versus $t$ for $0 \le t \le 24$ and determine the times at which the low and high tides first occur. Give your answers correct to the nearest minute.

**Fully worked solutions for this chapter are available online.**

# LESSON
## 2.4 Differentiation of trigonometric functions

**SYLLABUS LINKS**

- Use the rules $\dfrac{d}{dx} \sin(x) = \cos(x)$ and $\dfrac{d}{dx} \sin(f(x)) = f'(x) \cos(f(x))$
- Use the rules $\dfrac{d}{dx} \cos(x) = -\sin(x)$ and $\dfrac{d}{dx} \cos(f(x)) = -f'(x) \sin(f(x))$

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### 2.4.1 Investigating the differential of $y = \sin(x)$ using technology

The derivative of a function gives the slope of the curve at any point.

By considering the slope of the tangent at points such as $x = 0, \dfrac{\pi}{2}, \pi, \dfrac{3\pi}{2}$ and $2\pi$, the general shape of the derived function can be observed. Only one cycle needs to be considered.

Use the 'Analyze graph' menu on your calculator to calculate $\dfrac{dy}{dx}$ at any point on the curve $y = \sin(x)$. Any number of points may be considered.

The table below summarises the gradients of $y = \sin(x)$ from $x = 0$ to $x = 2\pi$, giving values to 1 decimal place where necessary.

| $x$ radians | $\dfrac{dy}{dx}$, gradient at the point |
|:---:|:---:|
| $0$ | $1$ |
| $\dfrac{\pi}{4}$ | $0.7$ |
| $\dfrac{\pi}{2}$ | $0$ |
| $\dfrac{3\pi}{4}$ | $-0.7$ |
| $\pi$ | $-1$ |
| $\dfrac{5\pi}{4}$ | $-0.7$ |
| $\dfrac{3\pi}{2}$ | $0$ |
| $\dfrac{7\pi}{4}$ | $0.7$ |
| $2\pi$ | $1$ |

Consider also the sign of the gradient of the sine curve for various intervals.

| $x$ intervals | Sign of gradient |
|:---:|:---:|
| $0 < x < \dfrac{\pi}{2}$ | Positive |
| $\dfrac{\pi}{2} < x < \dfrac{3\pi}{2}$ | Negative |
| $\dfrac{3\pi}{2} < x < 2\pi$ | Positive |

This information is illustrated in the diagram. It can be seen that the derived function has the shape of a cosine graph.

*Note:* Adding more points gives a better shape.
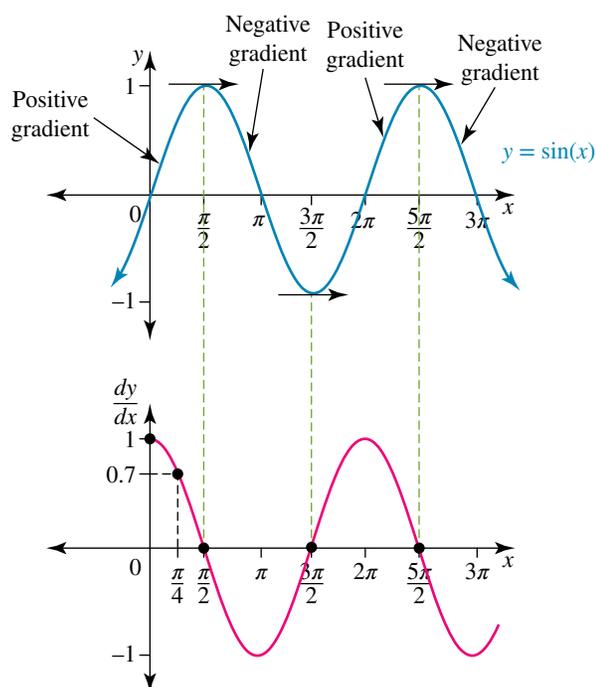
## 2.4.2 Investigating the derivative of $y = \sin(x)$ using first principles

The derivative of $y = \sin(x)$ can be investigated using differentiation from first principles.

Consider $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \sin(x)$ where $x$ is an angle measurement in radians.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f(x) = \sin(x)$$

$$f(x+h) = \sin(x+h)$$

$$\frac{f(x+h) - f(x)}{h} = \frac{\sin(x+h) - \sin(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

To evaluate this limit, we must look at the unit circle.

$$\angle NOM = x, \ \angle QOM = x + h$$

$$\angle PQO = \frac{\pi}{2} - (x+h)$$

$$\angle RQS = \frac{\pi}{2} - \left(\frac{\pi}{2} - (x+h)\right)$$

$$= x + h$$



By definition,

$$\sin(x) = MN$$

$$\sin(x+h) = PQ$$

$$\sin(x+h) - \sin(x) = PQ - MN = QR$$

$$\frac{\sin(x+h) - \sin(x)}{h} = \frac{QR}{h}$$
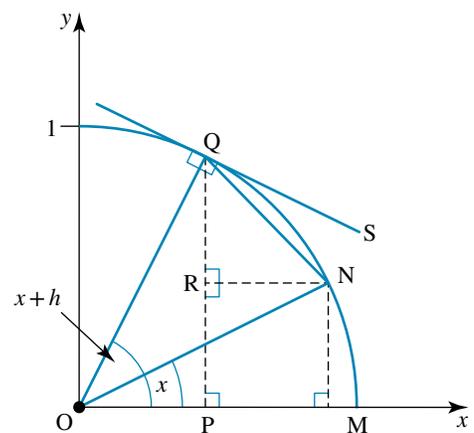
From the diagram, it can be seen that $\angle RQS = x + h$ and the arc QN has length $h$.

As $h \to 0$, $\angle RQS$ approaches $\angle RQN$, which approaches $x$. Furthermore, the arc QN approaches the chord QN.

Consequently, $\dfrac{QR}{h} \to \dfrac{QR}{QN}$, but by definition, $\dfrac{QR}{QN} = \cos(x)$.

Hence,

$$f'(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$= \lim_{h \to 0} \frac{QR}{h}$$

$$= \cos(x)$$

The derivative of cosine can also be investigated geometrically or graphically, using the same methods as shown for the sine function. To differentiate trigonometric functions, the angles need to be in radian measure.

## 2.4.3 Differentiation of the sine and cosine functions

The derivatives for sine and cosine are summarised below.

---

### Derivatives of sin(x) and cos(x)

$$\frac{d}{dx}\sin(x) = \cos(x)$$

$$\frac{d}{dx}\cos(x) = -\sin(x)$$

---

Using the chain rule, introduced in Unit 2 and studied further in the next chapter, the following results can also be obtained.

---

### Derivatives of sin(f(x)) and cos(f(x))

$$\frac{d}{dx}\sin(f(x)) = f'(x)\cos(f(x))$$

$$\frac{d}{dx}\cos(f(x)) = -f'(x)\sin(f(x))$$

---

These rules only apply if the angle $x$ is measured in radians.

---

### Relationship between degrees and radians

$$\pi \text{ radians} = 180°$$

---

### WORKED EXAMPLE 10 Determining the derivative of trigonometric functions

**Determine the derivative of each of the following functions.**
**a.** $y = 7\sin(5x)$
**b.** $y = \cos(x^2 + 2x - 3)$

| THINK | WRITE |
|---|---|
| **a. 1.** State $f(x)$ and $f'(x)$. | **a.** $f(x) = 5x$<br>$f'(x) = 5$ |
| **2.** Substitute into the formula $\dfrac{d}{dx}\sin(f(x)) = f'(x)\cos(f(x))$. | $\dfrac{dy}{dx} = 7\cos(5x) \times 5$ |
| **3.** Simplify and answer the question. | $\dfrac{dy}{dx} = 35\cos(5x)$ |
| **b. 1.** State $f(x)$ and $f'(x)$. | **b.** $f(x) = x^2 + 2x - 3$<br>$f'(x) = 2x + 2$ |
| **2.** Substitute into the formula $\dfrac{d}{dx}\cos(f(x)) = -f'(x)\sin(f(x))$. | $\dfrac{dy}{dx} = -\sin(x^2 + 2x - 3) \times (x + 2)$<br>$\dfrac{dy}{dx} = -(2x + 2)\sin(x^2 + 2x - 3)$ |
| **3.** Simplify and answer the question. | $\dfrac{dy}{dx} = -2(x + 1)\sin(x^2 + 2x - 3)$ |

## WORKED EXAMPLE 11 Determining a derivative of a trigonometric function in degrees

Determine the derivative of the function $y = \sin(6x°)$.

**THINK**

1. The function cannot be differentiated as the angle is not measured in radians. Convert the angle to radian measure.

2. Differentiate by applying the formula $\frac{d}{dx}\sin(f(x)) = f'(x)\cos(f(x))$.

**WRITE**

$$\sin(6x°) = \sin\left(6 \times \frac{\pi}{180}x\right)$$

$$= \sin\left(\frac{\pi}{30}x\right)$$

$$y = \sin\left(\frac{\pi}{30}x\right)$$

$$\frac{dy}{dx} = \frac{\pi}{30}\cos\left(\frac{\pi}{30}x\right)$$

## WORKED EXAMPLE 12 Determining equation of tangents

Determine the equation of the tangent to the curve $y = \sin(3x) + 1$ at the point where $x = \frac{\pi}{3}$.

**THINK**

1. First determine the coordinates of the point; that is, determine the $y$-value when $x = \frac{\pi}{3}$.

2. Determine the derivative of the function.

3. Determine the gradient at the point where $x = \frac{\pi}{3}$.

4. Substitute the appropriate values into the rule $y - y_1 = m(x - x_1)$ to determine the equation of the tangent.

5. Simplify and answer the question.

**WRITE**

When $x = \frac{\pi}{3}$,

$$y = \sin\left(3 \times \frac{\pi}{3}\right) + 1$$

$$= \sin(\pi) + 1$$
$$= 0 + 1$$
$$= 1$$

The point is $\left(\frac{\pi}{3}, 1\right)$.

$$\frac{dy}{dx} = 3\cos(3x)$$

$$x = \frac{\pi}{3}, \frac{dy}{dx} = 3\cos\left(3 \times \frac{\pi}{3}\right)$$
$$= 3\cos(\pi)$$
$$= 3(-1)$$
$$= -3$$

$$m = -3, (x_1, y_1) = \left(\frac{\pi}{3}, 1\right)$$

$$y - y_1 = m(x - x_1)$$

$$y - 1 = -3\left(x - \frac{\pi}{3}\right)$$

$$y = -3x + \pi + 1$$

The equation of the tangent is $y = 1 + \pi - 3x$.

# Exercise 2.4 Differentiation of the trigonometric functions

**2.4 Exercise** | **2.4 Exam questions** on

| Simple familiar | Complex familiar | Complex unfamiliar |
|---|---|---|
| 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 | 13, 14, 15, 16 | 17, 18, 19, 20 |

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## Simple familiar

1. **WE10a** Determine the derivative of each of the following functions.
   a. $y = \sin(8x)$
   b. $y = \sin(-6x)$
   c. $y = \sin(x)$
   d. $y = \sin\left(\dfrac{x}{3}\right)$
   e. $y = \sin\left(-\dfrac{x}{2}\right)$
   f. $y = \sin\left(\dfrac{2x}{3}\right)$

2. Differentiate each of the following.
   a. $y = \cos(3x)$
   b. $y = \cos(-2x)$
   c. $y = \cos\left(\dfrac{x}{3}\right)$
   d. $y = \cos(21x)$
   e. $y = \cos(-7x)$
   f. $y = \cos\left(\dfrac{\pi x}{4}\right)$

3. Differentiate each of the following.
   a. $y = \sin(2x + 3)$
   b. $y = \sin(6 - 7x)$
   c. $y = \sin(5x - 4)$
   d. $y = \sin\left(\dfrac{3x + 2}{4}\right)$
   e. $y = \sin\left(\dfrac{8 - 7x}{3}\right)$
   f. $y = 5\pi \sin(2\pi x)$

4. Differentiate each of the following.
   a. $y = \cos(8 - x)$
   b. $y = \cos(6 - 5x)$
   c. $y = \cos\left(\dfrac{2x + 3}{3}\right)$
   d. $y = \cos\left(\dfrac{4x - 1}{5}\right)$
   e. $y = 4\pi(\cos 10\pi x)$
   f. $y = -6\cos(-2x)$

5. **WE10b** Determine the derivative of each of the following.
   a. $y = \cos(x^2 - 4x + 3)$
   b. $y = \sin(10 - 5x + x^2)$
   c. $y = \sin(e^x)$
   d. $y = \cos(x^2 + 7x)$
   e. $y = \cos(4x - x^2)$
   f. $y = \sin(x^2 + 3x)$

6. **WE11** Determine the derivative of the function $y = 9\cos(10x°)$.

7. For each of the following functions, determine $\dfrac{dy}{dx}$.
   a. $y = 2\cos(3x)$
   b. $y = \cos(x°)$
   c. $y = 3\cos\left(\dfrac{\pi}{2} - x\right)$
   d. $y = -4\sin\left(\dfrac{x}{3}\right)$
   e. $y = \sin(12x°)$
   f. $y = 2\sin\left(\dfrac{\pi}{2} + 3x\right)$

8. If $f(x) = 4\sin(2x)$, determine $f'\left(\dfrac{\pi}{3}\right)$.

9. If $f(x) = -2\cos(4x)$, determine $f'\left(\dfrac{2\pi}{3}\right)$.

10. **WE12** Determine the equation of the tangent to the curve $y = -\cos(x)$ at the point where $x = \dfrac{\pi}{2}$.

11. Determine the equation of the tangent to the curve with equation $y = 3\cos(x)$ at the point where $x = \dfrac{\pi}{6}$.

12. Determine the equation of the tangent to the curve $y = 4\sin(2x)$ at the point where $x = 45°$.

### Complex familiar

13. Determine the exact value of $g'(-30°)$ for $g(x) = 6 - \cos(5x)$.

14. Determine the point on the curve with equation $y = -2\sin\left(\dfrac{x}{2}\right), x \in [0, 2\pi]$ where the gradient is equal to $\dfrac{1}{2}$.

15. Consider the function $f\colon [0, 2\pi] \to \mathbb{R},\ f(x) = \sin(x) - \cos(x)$. Calculate:
    a. $f(0)$
    b. $\{x\colon f(x) = 0\}$
    c. $f'(x)$
    d. $\{x\colon f'(x) = 0\}$

16. Consider the function $f\colon [-\pi, \pi] \to \mathbb{R},\ f(x) = \sqrt{3}\cos(x) + \sin(x)$. Calculate:
    a. $f(0)$
    b. $\{x\colon f(x) = 0\}$
    c. $f'(x)$
    d. $\{x\colon f'(x) = 0\}$

### Complex unfamiliar

17. Determine the $x$-values over the domain $x \in [-\pi, \pi]$ for which the gradients of the functions $f(x) = \sin(2x)$ and $f(x) = \cos(2x)$ are equal.

18. For the function $f(x) = x - \sin(2x), -\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$, determine the point(s) where the gradient is 0. Give your answer correct to 3 decimal places.

19. For the function $f(x) = 2x + \cos(3x), 0 \le x \le \dfrac{\pi}{2}$, determine the point(s) where the gradient is 0. Give your answer correct to 3 decimal places.

20. Determine the angle that the curve with equation $y = \sin(2x)$ makes with the positive direction of the $x$-axis the first time it intersects the $x$-axis when $x > 0$. Give your answer correct to 1 decimal place.

**Fully worked solutions for this chapter are available online.**

# LESSON
## 2.5 Modelling and solving with trigonometric functions

### 2.5.1 Applications of trigonometric functions

The trigonometric functions are used to model many practical situations that include periodic behaviour or cyclic phenomena. Examples of these include the rise and fall of the tides, temperature fluctuations, and the heights of moving objects about a fixed point.

In solving practical problems:
- read the question carefully
- state clearly the model being considered
- sketch clear graphs if required
- note any restrictions, particularly on the domain, that apply to the problem
- differentiate the function to determine the rate of change or slope of the curve
- consider the units of measurement
- answer the question.

---

**WORKED EXAMPLE 13 Modelling tidal heights and problem solving**

**While out in his trawler, John North, a fisherman, notes that the height of the tide in the harbour can be found by using the equation**

$$h = 5 + 2\cos\left(\frac{\pi}{6}t\right)$$

**where $h$ metres is the height of the tide and $t$ is the number of hours after midnight.**
**a.** **Determine the height of the high tide and when it occurs in the first 24 hours.**
**b.** **Determine the difference in height between high and low tides.**
**c.** **Sketch the graph of $h$ for $0 \leq t \leq 24$.**
**d.** **John North knows that his trawler needs a depth of 6 metres to enter the harbour. Determine when he is able to bring his boat back into the harbour.**
**e.** **Determine the rate of change of the height of the tide when John North first brings his boat back into the harbour. Give your answer to 2 decimal places.**

| THINK | WRITE |
|---|---|
| **a. 1.** Write the given equation. | **a.** $h = 5 + 2\cos\left(\frac{\pi}{6}t\right)$ |

**2.** For high tide, calculate the maximum value of $h$.

For maximum $h$,

$$\cos\left(\frac{\pi}{6}t\right) = 1$$

$$\text{So } h = 5 + 2 \times 1$$

$$= 7$$

**3.** Determine when high tide occurs.

$$\frac{\pi}{6}t = 0, 2\pi, 4\pi, \ldots$$

$$t = 0, 12, 24, \ldots$$

A high tide of height 7 m occurs at midnight, noon the next day, and midnight the next night.

**b. 1.** Calculate the minimum value of $h$.

**b.** For minimum $h$,

$$\cos\left(\frac{\pi}{6}t\right) = -1$$

$$\text{So } h = 5 + 2 \times -1$$

$$= 3$$

**2.** Calculate the difference between high and low tides.

The difference between high and low tides is $7 - 3 = 4$ metres.

**c.** Use the information from parts **a** and **b** to sketch the graph.

**c.**
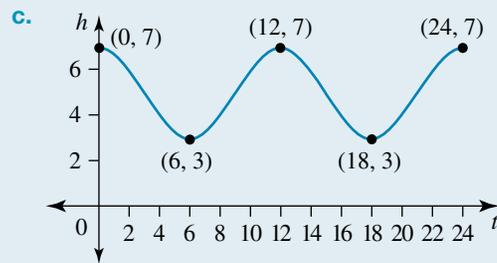


**d. 1.** Determine $t$ using the equation when $h = 6$.

**d.** When $h = 6$,

$$5 + 2\cos\left(\frac{\pi}{6}t\right) = 6$$

$$2\cos\left(\frac{\pi}{6}t\right) = 1$$

$$\cos\left(\frac{\pi}{6}t\right) = \frac{1}{2}$$

$$\frac{\pi}{6}t = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, \ldots$$

$$t = 2, 10, 14, 22 \ldots$$

**2.** Write the answer in words.

From the graph we can see that John North can bring his boat back into the harbour before 2 am, between 10 am and 2 pm, and between 10 pm and 2 am the next morning.

**e. 1.** Differentiate the function.

**e.** $h = 5 + 2\cos\left(\frac{\pi}{6}t\right)$

$$\frac{dh}{dt} = -2\sin\left(\frac{\pi}{6}t\right) \times \frac{\pi}{6}$$

$$= -\frac{\pi}{3}\sin\left(\frac{\pi}{6}t\right)$$

2. From parts **c** and **d**, state when the boat first enters the harbour.

From $h = 6$, $t = 10$:

3. Determine the rate of change of depth at $t = 10$ and simplify.

$$\frac{dh}{dt} = -\frac{\pi}{3} \sin\left(\frac{\pi}{6} \times 10\right)$$

$$= -\frac{\pi}{3} \sin\left(\frac{5\pi}{3}\right)$$

$$= -\frac{\pi}{3} \times \frac{-\sqrt{3}}{2}$$

$$\frac{dh}{dt} = 0.91 \text{ m/h}$$

4. Evaluate and state the answer with correct units.

The water level is increasing at a rate of 0.91 m/h when the boat enters the harbour.

---

## WORKED EXAMPLE 14 Modelling temperature and problem solving

**The temperature on a particular day can be modelled by the function**

$$T(t) = -3 \cos\left(\frac{\pi t}{9}\right) + 18, 0 \le t \le 18$$

**where $t$ is the time in hours after 5:00 am and $T$ is the temperature in degrees Celsius. For the remaining 6 hours of the 24-hour period, the temperature remains constant.**

**a. Calculate the temperature at 8:00 am.**

**b. Determine the time(s) of the day at which the temperature is 20 °C. Give your answer correct to the nearest minute.**

**c. Determine $\dfrac{dT}{dt}$.**

**d. Determine the rate of change of temperature at the time(s) found in part b, correct to 2 decimal places.**

**THINK**

**a.** At 8:00 am, $t = 3$. Substitute this value into the equation.

**b. 1.** Substitute $T = 20$ into the equation.

**2.** Solve the equation for $0 \le t \le 18$.

**3.** Interpret your answers and convert the $t$-values to times of the day.

**WRITE**

**a.** $T(3) = -3 \cos\left(\frac{3\pi}{9}\right) + 18$

$$= -3 \cos\left(\frac{\pi}{3}\right) + 18$$

$$= -3 \times \frac{1}{2} + 18$$

$$= -1.5 + 18$$

$$= 16.5 \text{ °C}$$

**b.** $20 = -3 \cos\left(\frac{\pi t}{9}\right) + 18$

$$\cos\left(\frac{\pi t}{9}\right) = \frac{-2}{3}$$

$\dfrac{2}{3}$ suggests 0.841 069, and cosine is negative in quadrants 2 and 3.

**4.** Write the answer.
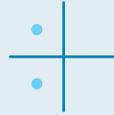


$$\frac{\pi t}{9} = \pi - 0.841\,069, \pi + 0.841\,069$$
$t = 6.5905, 11.4095$ after 5 am
The temperature is 20 °C at 11:35 am and 4:25 pm.

**c.** Determine $\dfrac{dT}{dt}$.

**c.** $\dfrac{dT}{dt} = -3 \times \dfrac{\pi}{9}\left(-\sin\left(\dfrac{\pi t}{9}\right)\right)$

$\qquad = \dfrac{\pi}{3}\sin\left(\dfrac{\pi t}{9}\right)$

**d. 1.** Substitute $t = 6.6$ (11:36 am) and $t = 11.4$ (4:24 pm) into $\dfrac{dT}{dt}$.

**d.** When $t = 6.6$ (11:36 am),

$\dfrac{dT}{dt} = \dfrac{\pi}{3}\sin\left(\dfrac{6.6 \times \pi}{9}\right)$

$\qquad = 0.78$

When $t = 11.4$ (4:24 pm),

$\dfrac{dT}{dt} = \dfrac{\pi}{3}\sin\left(\dfrac{11.4 \times \pi}{9}\right)$

$\qquad = -0.78$

**2.** Write the answer.

At 11:36 am the temperature is increasing at a rate of 0.78 °C/h.
At 4:24 pm the temperature is decreasing at a rate of 0.78 °C/h.

---

**TI | THINK**    **WRITE**    **CASIO | THINK**    **WRITE**

**a. 1.** On a Graphs page, Complete the entry line:
$f1(x) = -3\cos\left(\dfrac{\pi x}{9}\right) + 18$
*Note:* The independent variable $t$ has been replaced with $x$.



**a. 1.** On a Main Menu screen, select Graph. Complete the entry line in the Y1 tab as:
$-3\cos\left(\dfrac{\pi x}{9}\right) + 18$
*Note:* The independent variable $t$ has been replaced with $x$.
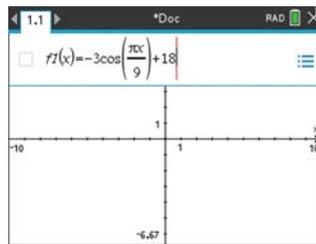


**2.** Sketch the graph by pressing the ENTER button.
*Note:* You may need to zoom out to see the curve.
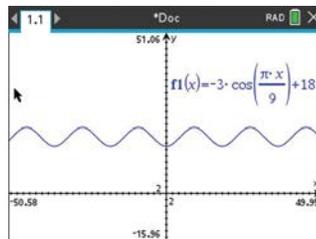


**2.** Sketch the graph by pressing either the DRAW or EXE button.
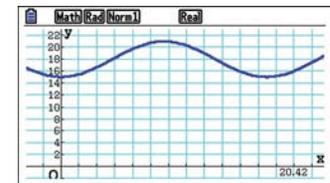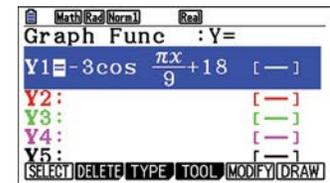
**3.** To calculate the *x*-value(s) for when $y = 20$, select: Menu 8: Geometry 1: Points & Lines 2: Point On.
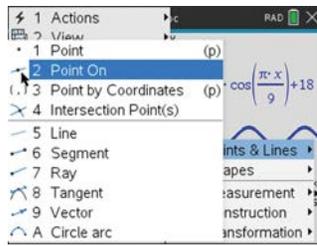
**4.** Move the cursor and select the curve representing
$f1(x) = -3\cos\left(\dfrac{\pi x}{9}\right) + 18$
Press the ESC (escape) button. This allows you to move the textbox indicating the coordinates of the point P(*x*, *y*). Complete the entry line in the textbox as 20 for the *y*-value. Press the ENTER button to perform the calculation.
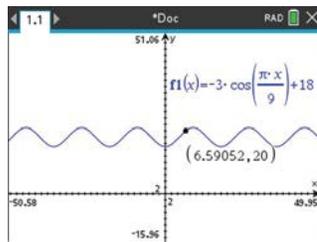
**5.** Determine the next solution by moving the cursor point on the line to a position that is close to the desired solution. Complete the entry line in the textbox as 20 for the *y*-value. Press the ENTER button to perform the calculation. The second solution exists at $x = 11.4$.
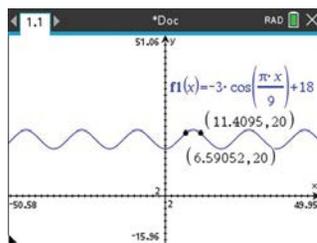
**3.** To calculate the *x*-value(s) for when $y = 20$, select: G-Solv (SHIFT F5) X-CAL. Complete the entry line in Y: as 20. Press the EXE button to perform the calculation.
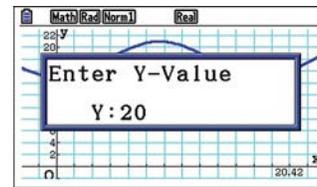
**4.** The answer appears on the screen. The first solution exists at $x = 6.6$. *Note:* The family of solutions can be calculated by pressing the directional cursor button either left or right.
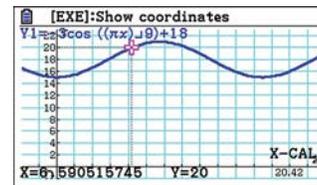
**5.** Determine the next solution by pressing the directional cursor button to the right. The second solution exists at $x = 11.4$.
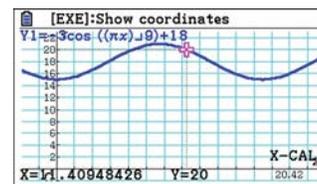
# Exercise 2.5 Modelling and solving with trigonometric functions

| 2.5 Exercise | 2.5 Exam questions  on |
|---|---|

| Simple familiar | Complex familiar | Complex unfamiliar |
|---|---|---|
| 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 | 13, 14, 15, 16 | 17, 18, 19, 20 |

## Simple familiar

1. **WE13** Fred Greenseas and John North are competing to catch the most fish. Fred Greenseas decides to fish in an inlet several kilometres east of the place where John North fishes. There is a sandbar at the entrance to the inlet, and the depth of water in metres on the sandbar is modelled by the function $d(t) = 6 + 2.5 \sin\left(\dfrac{\pi t}{6}\right)$, where $t$ is the number of hours after 12 noon.

   a. Determine the greatest depth of the water on the sandbar and the time it first occurs.
   b. Determine how many hours pass before there is once again the maximum depth of water on the sandbar.
   c. Determine the least amount of water on the sandbar.
   d. Sketch the graph of $d$ for $0 \le t \le 24$.
   e. Fred Greenseas needs a depth of at least 7.25 metres to cross the sandbar. Determine between what hours he is able to enter and leave the inlet.
   f. Determine the rate of change of the water level when Fred Greenseas first crosses the sandbar. Give your answer correct to 2 decimal places.

2. A student wanting to catch fish to sell at a local market on Sunday has discovered that more fish are in the water at the end of the pier when the depth of water is greater than 8.5 metres. The depth of the water (in metres) is given by $d = 7 + 3 \sin\left(\dfrac{\pi}{6} t\right)$, where $t$ hours is the number of hours after midnight on Friday.

   a. Determine the maximum and minimum depth of the water at the end of the pier.
   b. Sketch a graph of $d$ against $t$ from midnight of Friday until midday on Sunday, and determine the initial rate of change of the depth of the water.
   c. Determine when the water first reaches maximum depth.
   d. Determine when the student should be on the pier in order to catch the most fish.
   e. If the student can fish for only 2 hours at a time, determine when she should fish in order to sell the freshest fish at the market from 10:00 am on Sunday morning.

**3.** The mean daily maximum temperature in Tarabon, an experimental town in a glass dome, is modelled by the function $T(m) = 18 + 7 \cos\left(\dfrac{\pi}{6}m\right)$, where $T$ is in degrees Celsius and $m$ is the number of months after 1 January 2026.

   **a.** Calculate the mean daily maximum temperature in March 2026 and August 2026.

   **b.** Determine the rate of change of the mean daily maximum temperatures in August 2026.
Give your answer to 1 decimal place.

   **c.** Determine the highest mean daily maximum temperature in Tarabon and the months in which it occurs.

   **d.** Calculate the mean daily maximum temperature in February 2027.

   **e.** If the pattern continued, determine how many months would pass before the mean daily maximum temperature would be the same again as it was in February 2027.

**4.** The height above the ground of the middle of a skipping rope as it is being turned in a child's game is found by using the equation $h = a \sin(nt) + c$, where $t$ is the number of seconds after the rope has begun to turn. During the game, the maximum height the rope reaches is 1.8 metres, and it takes 2 seconds for the rope to complete a full turn.

   **a.** Determine the values of $a$, $n$ and $c$, and hence write the equation of $h$ in terms of $t$.

   **b.** Sketch the graph of $h$ against $t$ for $0 \le t \le 5$.

   **c.** Determine when the rope will first be exactly 25 cm above the ground, and determine the rate at which it is changing at that instant. Give your answers correct to 1 decimal place.

**5.** The population of a colony of frogs rises and falls according to the breeding season. The population can be modelled by the equation $P(t) = 100 \sin \dfrac{\pi t}{2} + 500$, where $t$ is the number of months since the beginning of the year.

   **a.** Determine the population at the beginning of the year.

   **b.** Sketch the graph to represent this population of frogs for the year.

   **c.** Determine the first time at which the population is greatest.

   **d.** Determine the rate of change of the population of frogs with respect to the number of months of the year when the population reaches its mean or equilibrium population of frogs.

**6.** **WE14** A mass oscillates up and down at the end of a metal spring. The length of the spring, $L$ cm, after time $t$ seconds is modelled by the function $L(t) = 2 \sin(\pi t) + 10$ for $t \ge 0$.

   **a.** Determine the length of the spring when the mass is not oscillating, that is, when it is at the mean position, P.

   **b.** Determine $\dfrac{dL}{dt}$.

   **c.** Calculate the exact value of $\dfrac{dL}{dt}$ after 1 second.

7. The area, $A$ cm$^2$, of the triangle shown is given by $A = 18\sin(\theta)$, with the area changing for various values of $\theta$.
Determine the rate of change of the area when the triangle becomes an equilateral triangle.
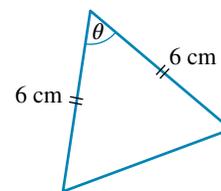
8. The temperature in °C on a particular winter's day in an inland town can be modelled by the function

$$T = 2\sin\left(\frac{\pi t}{9}\right) + 12, 0 \le t \le 24$$

where $t$ is the time in hours after 8:00 am.

a. Calculate the temperature, correct to the nearest degree, at 12 noon.

b. Determine $\dfrac{dT}{dt}$.

c. Determine the rate of change of the temperature at midnight. Give your answer correct to 3 decimal places.

9. A section of a rollercoaster track at a local fun park is shown.
The track can be described by the rule

$$h = 4\cos\left(\frac{\pi d}{25}\right) + 5, 0 \le d \le 25$$

where $h$ is the height in metres above ground level and $d$ is the horizontal distance in metres from the top of the descent. Note that the $d$-axis represents the ground.
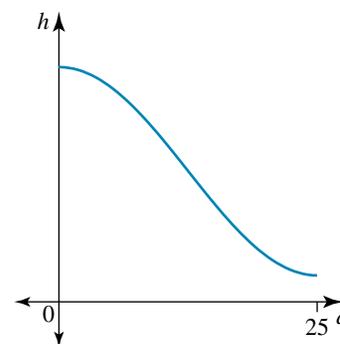
a. Determine how high a rollercoaster car is above the ground at the beginning of its descent.

b. Determine $\dfrac{dh}{dd}$.

c. Determine the gradient of the rollercoaster track, correct to 3 decimal places, when:

   i. $d = 5$ m

   ii. $d = 15$ m.

10. Between 6 am and 6 pm on a given day the height, $H$ metres, of the tide in a harbour is given by

$$H(t) = 1.5 + 0.5\sin\left(\frac{\pi t}{6}\right), 0 \le t \le 12$$

a. Determine the period of the function.

b. Determine the value of $H$ at low tide and the times low tide occurs.

c. Determine $\dfrac{dH}{dt}$.

d. Calculate the exact value of $\dfrac{dH}{dt}$ at 7:30 am.

e. Determine the second time during the given time interval that $\dfrac{dH}{dt}$ equals the value found in part d.

**11.** A young girl is learning to skip. The graph showing this skipping for one cycle is given.
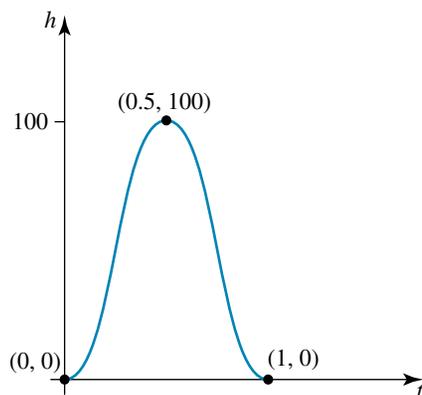


The general equation for this graph is given by $h = a\cos(nt) + c$, where $h$ is the height in millimetres of the girl's feet above the ground and $t$ is the time in seconds the girl has been skipping.

**a.** Calculate the values of the constants $a$, $n$ and $c$, and hence restate the equation for one cycle of the skipping.

**b.** Determine $\dfrac{dh}{dt}$.

**c.** Calculate the value of $\dfrac{dh}{dt}$ when $t = 0.25$ seconds.

**12.** The height, $h$ metres, above ground level of a chair on a rotating Ferris wheel is modelled by the function

$$h = 5 - 3.5 \cos\left(\frac{\pi t}{30}\right)$$

where $t$ is measured in seconds.

**a.** People can only enter a chair when it is at its lowest position, at the bottom of the rotation. They enter the chair from a platform. Determine how high is the platform is above ground level.

**b.** Determine the highest point reached by the chair.

**c.** Determine how long 1 rotation of the wheel takes.

**d.** During a rotation, determine for how long a chair is higher than 7 m off the ground. Give your answer to 1 decimal place.

**e.** Determine $\dfrac{dh}{dt}$.

**f.** Calculate the first two times, correct to 2 decimal places, when a chair is descending at a rate of 0.2 m/s.

**13.** A section of a water slide at a local aquatic complex is shown.
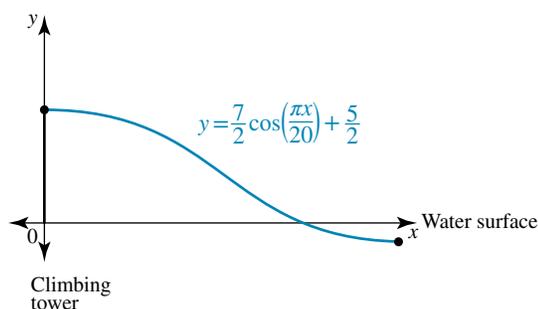


The water slide can be defined by the rule

$$y = \frac{7}{2} \cos\left(\frac{\pi x}{20}\right) + \frac{5}{2}, 0 \le x \le 20$$

where $y$ is the height in metres of the water slide above the water surface and $x$ is the horizontal distance in metres between the start of the slide and the end of the slide. (*Note:* The $x$-axis represents the water surface.)

**a.** Determine how high a person must climb in order to reach the top of the water slide.

**b.** Determine $\dfrac{dy}{dx}$.

**c.** Calculate the exact gradient of the water slide:

   **i.** when $x = 5$          **ii.** when $x = 10$.

**d. i.** Determine how far from the climbing tower, to the nearest whole metre, the slide comes into contact with the water surface.

   **ii.** Determine the obtuse angle that the slide makes with the water surface at this point. Give your answer correct to 2 decimal places.

**14.** Given that $f\colon \left[0, \dfrac{\pi}{2}\right] \to \mathbb{R}, \ f(x) = 2\sin(4x) + 1$, determine:

**a.** the values of $x$ for which $f(x) = 0.5$, giving your answer correct to 3 decimal places

**b.** the coordinates where the gradient of the function is zero

**c.** the value of $f'(x)$ when $x = \dfrac{\pi}{4}$

**d.** the interval over which the gradient is positive.

**15.** The diagram shows a garden bed bordered by wooden sleepers. BDC is a triangular herb garden and ABDE is a rectangular garden for vegetables.

**a.** Determine BD and CD in terms of $a$ and $\theta$, where $a$ is a constant, $\theta$ is $\angle$BCD as shown and $0 < \theta < \dfrac{\pi}{2}$.

**b.** Calculate the total length, $L$ metres, of sleepers required to surround the garden bed. (This should include BD as well as the sleepers defining the perimeter.)

**c.** Determine $\dfrac{dL}{d\theta}$ in terms of $\theta$ and $a$.

**d.** Let $a = 2$ and use CAS to sketch $\dfrac{dL}{d\theta}$ for $0 < \theta < \dfrac{dL}{d\theta}$. Hence, determine when $\dfrac{dL}{d\theta} = 0$, correct to 1 decimal place.
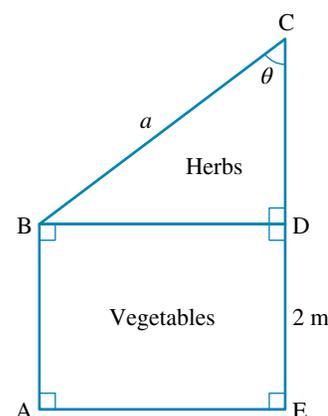
**16.** The depth of water in an inlet has been monitored over a 24-hour period from 4 am on Monday. It was observed that the depth, $D$ metres, was modelled by the function

$$D(t) = 2.5 + 0.5 \sin\left(\frac{\pi t}{3}\right), 0 \le t \le 24$$

where $t$ is time in hours.

a. Determine the depth of the water at 4 am on Monday.
b. Calculate the depth of the water at midday on Monday, correct to 2 decimal places.
c. Determine the maximum depth of the water during the 24-hour period and the time this first occurred.
d. Sketch the function, $D(t)$.
e. Calculate the rate of change of the depth of water, $D'(t)$, at any time, and sketch the function $y = D'(t)$.
f. Determine when during the 24-hour period the flow of water into the inlet was the greatest.

### Complex unfamiliar

**17.** A wire frame is shaped as shown in the diagram. The diagonals shown are 100 cm long, and each diagonal makes an angle of $\theta$ with the horizontal.
Show that the length of wire required to form the shape is given by

$$L = 300 \cos(\theta) + 400 \sin(\theta) + 200$$

where $L$ is the total length of wire in centimetres and $\theta$ is the angle shown in radians. Also, determine the length of wire required, correct to 2 decimal places, when $\dfrac{dL}{d\theta} = 0$.
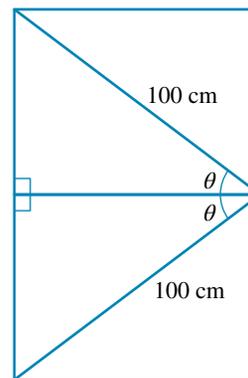
**18.** At a skateboard park, a new skateboard ramp has been constructed. A cross-section of the ramp is shown.
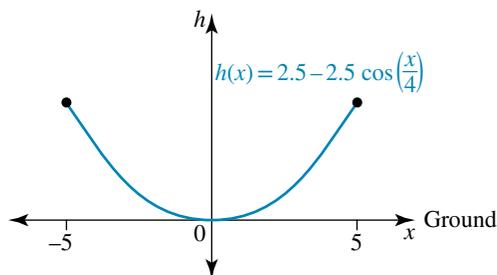


The equation that approximately defines this curve is given by

$$h(x) = 2.5 - 2.5 \cos\left(\frac{x}{4}\right), -5 \le x \le 5$$

where $h$ is the height in metres above the ground level and $x$ is the horizontal distance in metres from the lowest point of the ramp to each end of the ramp.

Determine where the gradient of the ramp is equal to 0.58 m/m, giving your answer correct to 3 decimal places.

**19.** An industrial process is known to cause the production of two separate toxic gases that are released into the atmosphere. At a factory where this industrial process occurs, the technicians work a 12-hour day from 6:00 am until 6:00 pm.

The emission of the toxic gas X can be modelled by the rule

$$x(t) = 1.5 \sin\left(\frac{\pi t}{3}\right) + 1.5, 0 \leq t \leq 12$$

and the emission of the toxic gas Y can be modelled by the rule

$$y(t) = 2.0 - 2.0 \cos\left(\frac{\pi t}{3}\right), 0 \leq t \leq 12$$

Determine the time at which the rates of change of the emissions are the same for the first time. Give your answer to the nearest minute.

**20.** The diagram represents the cross-section of a waterfall feature in an Australian native garden. It consists of an undulating surface of corrugated plastic with vertical posts at each end. The relationship that defines this surface can be expressed by

$$h(x) = 10 \cos\left(\frac{7x}{2}\right) - 5x + 90, 0 \leq x \leq 4.5$$

where $h$ centimetres represents the vertical height of the water feature and $x$ metres is the horizontal distance from the left-hand post. The posts supporting the undulating surface over which the water falls are situated at the points $x = 0$ and $x = 4.5$, as shown.
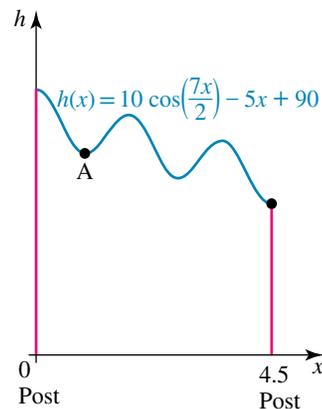


Determine the coordinates of point A, the first point in the interval $[0, 4.5]$ where the gradient of the undulating surface is zero, and the slope of the undulating surface at $x = 0.4$. Give your answers correct to 1 decimal place.

**Fully worked solutions for this chapter are available online.**

# LESSON
## 2.6 Review

### 2.6.1 Summary

Hey students! Now that it's time to revise this chapter, go online to:

**Access the chapter summary**

**Review your results**

**Practise exam questions**

**Find all this and MORE in jacPLUS**

### 2.6 Exercise

**learn on**

| 2.6 Exercise | 2.6 QCAA questions **on** |
| --- | --- |

| Simple familiar | Complex familiar | Complex unfamiliar |
| --- | --- | --- |
| 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 | 13, 14, 15, 16 | 17, 18, 19, 20 |

These questions are even better in jacPLUS!
- Receive immediate feedback
- Access sample responses
- Track results and progress

**Find all this and MORE in jacPLUS**

**Simple familiar**

1. Convert the following angles to degrees, giving your answers to the nearest minute where necessary.
   a. $\dfrac{3\pi}{4}$
   b. $\dfrac{13\pi}{12}$
   c. 2.1
   d. 1.76

2. Convert the following angles to radians, giving your answers correct to 2 decimal places where necessary.
   a. $35°$
   b. $280°$
   c. $128.5°$
   d. $230°48'$

3. Evaluate the following, given that $\cos(\theta) = \dfrac{2}{3}$ and $0 \le \theta \le \dfrac{\pi}{2}$.
   a. $\cos(\pi - \theta)$
   b. $\sin(\pi - \theta)$
   c. $\tan(\pi + \theta)$
   d. $\sin(3\pi + \theta)$
   e. $\tan(\pi - \theta)$
   f. $\cos(-\theta)$

4. State the exact values of the following.
   a. $\sin(120°)$
   b. $\cos(135°)$
   c. $\tan(330°)$
   d. $\cos(225°)$
   e. $\sin(210°)$
   f. $\tan(150°)$

5. State the exact values of the following.
   a. $\sin\left(\dfrac{3\pi}{4}\right)$
   b. $\cos\left(\dfrac{5\pi}{6}\right)$
   c. $\tan\left(\dfrac{2\pi}{3}\right)$
   d. $\cos\left(\dfrac{4\pi}{3}\right)$
   e. $\cos\left(\dfrac{9\pi}{4}\right)$
   f. $\sin\left(\dfrac{13\pi}{6}\right)$

6. State the exact values of the following.

   **a.** $\tan\left(-\dfrac{\pi}{4}\right)$
   
   **b.** $\cos\left(-\dfrac{3\pi}{4}\right)$
   
   **c.** $\sin\left(-\dfrac{2\pi}{3}\right)$

   **d.** $\tan\left(-\dfrac{5\pi}{6}\right)$
   
   **e.** $\sin\left(-\dfrac{7\pi}{6}\right)$
   
   **f.** $\cos\left(-\dfrac{5\pi}{4}\right)$

7. Determine all solutions to the following equations in the domain $0 \le \theta \le 2\pi$.

   **a.** $\cos(\theta) = 0$
   
   **b.** $\sin(\theta) = -\dfrac{1}{\sqrt{2}}$
   
   **c.** $\cos(\theta) = \dfrac{1}{\sqrt{2}}$

   **d.** $\sin(\theta) = -1$
   
   **e.** $\cos(\theta) = -\dfrac{\sqrt{3}}{2}$

8. Determine all the values of $\theta$ between $0°$ and $360°$ for which:

   **a.** $\sin(\theta) = 1$
   
   **b.** $\cos(\theta) = \dfrac{1}{2}$
   
   **c.** $\sin(\theta) = \dfrac{\sqrt{3}}{2}$

   **d.** $\cos(\theta) = -1$
   
   **e.** $\sin(\theta) = \dfrac{1}{\sqrt{2}}$

9. Determine all the solutions to the following equations in the domain $0 \le x \le 2\pi$.

   **a.** $2\sin(x) = 1$
   
   **b.** $3\cos(x) = 0$
   
   **c.** $2\sin(x) = -\sqrt{3}$

   **d.** $\sqrt{2}\cos(x) = 1$
   
   **e.** $\sqrt{3}\tan(x) + 1 = 0$

10. Solve the following equations for $-\pi \le x \le \pi$, giving your answers in exact form.

   **a.** $4\sin(x) + 2 = 6$
   
   **b.** $3\cos(x) - 3 = 0$
   
   **c.** $2\sin(3x) - 5 = -4$

   **d.** $\sqrt{2}\cos(3x) + 2 = 3$
   
   **e.** $2\cos(2x) + \sqrt{3} = 0$

11. Determine all values between 0 and $2\pi$ for the following equations. Give exact answers for parts **a–d**; give answers correct to 4 decimal places for parts **e** and **f**.

   **a.** $\sin(x) = \cos(x)$
   
   **b.** $\sin(2x) = \cos(2x)$
   
   **c.** $\sin(2x) = \sqrt{3}\cos(2x)$

   **d.** $\sqrt{3}\sin(3x) = \cos(3x)$
   
   **e.** $\sin(3x) + 2\cos(3x) = 0$
   
   **f.** $\sin(x) + 3\cos(x) = 0$

12. Solve the following for $\alpha$ where $0 \le \alpha \le 2\pi$. Give answers in exact form.

   **a.** $\sin^2(2\alpha) + \sin(2\alpha) - 2 = 0$

   **b.** $2\cos^2(3\alpha) + \cos(3\alpha) - 1 = 0$

   **c.** $2\sin^2\left(\alpha - \dfrac{\pi}{2}\right) = \sin\left(\alpha - \dfrac{\pi}{2}\right)$

### Complex familiar

13. Sketch the graph of the function $y = 2\sin(2x)$ for $-\pi \le x \le \pi$. State the amplitude, period and range.

14. Sketch the graphs of the following for $0 \le x \le 2\pi$. State the period, amplitude and range, and the coordinates of the end points. Calculate any axis intercepts, giving your answers in exact form where applicable.

   **a.** $y = 2\sin(2x + \pi)$
   
   **b.** $y = 3\cos(3x + \pi)$

   **c.** $y = 2\sin\left(x - \dfrac{\pi}{4}\right) - 1$
   
   **d.** $y = \cos\left(\dfrac{1}{2}(x - \pi)\right) + 1$

**15.** Determine $\dfrac{dy}{dx}$ for the following functions.

**a.** $y = 4\cos(8x^2 - 3x + 2)$  
**b.** $y = 4 - 3\sin(2x^2 + x - 5)$  
**c.** $y = 6\sin(3x^2 - 2) + 5\cos(3x^2 - 2)$  
**d.** $y = \cos(x^2 + 2x + 1)$  
**e.** $y = 2\sin(4 - 3x^2) - 6x + 7$  
**f.** $y = \sin(-x) - \cos(-2x)$

**16.** Determine the equation of the tangent and the line perpendicular to the tangent to the curve $y = 3\cos(x)$ at the point where $x = \pi$.

### Complex unfamiliar

**17.** During a particular day in a Mediterranean city, the temperature inside an office building between 10:00 am and 7:30 pm fluctuates so that $t$ hours after 10 am the temperature, $T\,°C$, is given by

$$T = 19 + 6\sin\left(\dfrac{\pi t}{6}\right).$$

When the temperature reaches 24 °C, an air conditioner in the boardroom is switched on. It is switched off when the temperature in the rest of the building falls below 24 °C.
Determine when the air conditioner is on in the boardroom.

**18.** The profile of water waves produced by a wave machine in a scientific laboratory is modelled by the trigonometric function defined by
$f(x) = a\,\sin(x) + b\,\cos(x)$.
Given that the graph of the wave profile passes through the points $(0, 7)$ and $\left(\dfrac{\pi}{2}, 3\right)$, determine the maximum and minimum swells for the wave profile, correct to 1 decimal place. Justify the procedures used to determine these maximum and minimum swells.
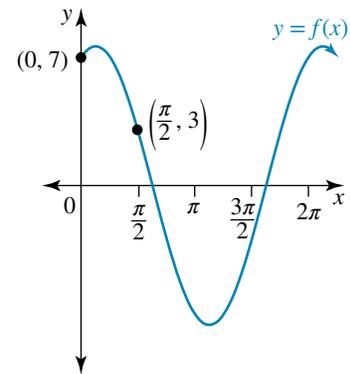
**19.** The diagram shows a property in the country that includes a farmhouse that is rented out for holidays.
F represents the farmhouse, MP is a road and TF is a track from the farmhouse to the road. MP is modelled by the equation

$$y = 2\sin(2x) + \dfrac{5}{2}$$

T is the point $\left(\dfrac{2\pi}{3}, t\right)$. Given that the track, TF, is perpendicular to the road at point T, determine the length of the track from the farmhouse to the road. Express your answer in the form $\left(a\sqrt{5} + b\sqrt{15}\right)$, where $a$ and $b$ are real constants.
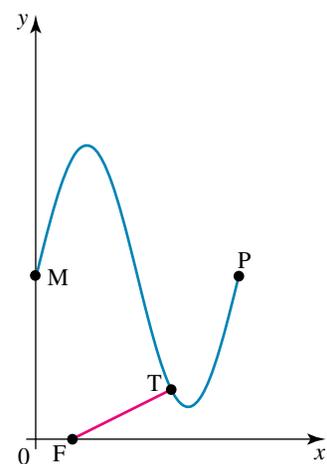
**20.** During one day in October, the temperature, $T\,°C$, is given by $T = 19 - 3\sin\left(\dfrac{\pi}{12}t\right)$, where $t$ is the time in hours after midnight.
Determine the fastest rate the temperature is rising, correct to 3 decimal places, and when this rate occurs.

# 2.6 Past QCAA exam questions

### Question 1 (1 mark)

*Source:* QCAA 2023 Mathematical Methods, Paper 2, Section 1, Q1; © State of Queensland (QCAA) 2024.

If $f(x) = \sin(3x)$, determine the value of $f'\left(\dfrac{\pi}{8}\right)$.

**A.** 2.772 **B.** 1.148 **C.** 0.929 **D.** 0.383

### Question 2 (1 mark)

*Source:* QCAA 2022 Mathematical Methods, Paper 2, Section 1, Q1; © State of Queensland (QCAA) 2024.

The position (in cm) of a particle is given by $x = \cos(4t)$, where $t$ is time (in seconds).

The velocity of the particle when $t = 5$ is

**A.** $1.6323 \, \text{cm s}^{-1}$ **B.** $0.4081 \, \text{cm s}^{-1}$ **C.** $-0.9129 \, \text{cm s}^{-1}$ **D.** $-3.6518 \, \text{cm s}^{-1}$

### Question 3 (1 mark)

*Source:* QCAA 2021 Mathematical Methods, Paper 2, Section 1, Q8; © State of Queensland (QCAA) 2024.

The displacement (in metres) of a particle is given by $s(t) = -3\cos(t) + 2\sin(t)$, where $t$ is in seconds.

The instantaneous velocity of the particle at time $t = \dfrac{\pi}{2}$ seconds is

**A.** $-3 \, \text{m s}^{-1}$ **B.** $-2 \, \text{m s}^{-1}$ **C.** $2 \, \text{m s}^{-1}$ **D.** $3 \, \text{m s}^{-1}$

### Question 4 (4 marks)

*Source:* QCAA 2023 Mathematical Methods, Paper 2, Section 2, Q12; © State of Queensland (QCAA) 2024.

The graph shows the water level under a bridge over a 12-hour period.
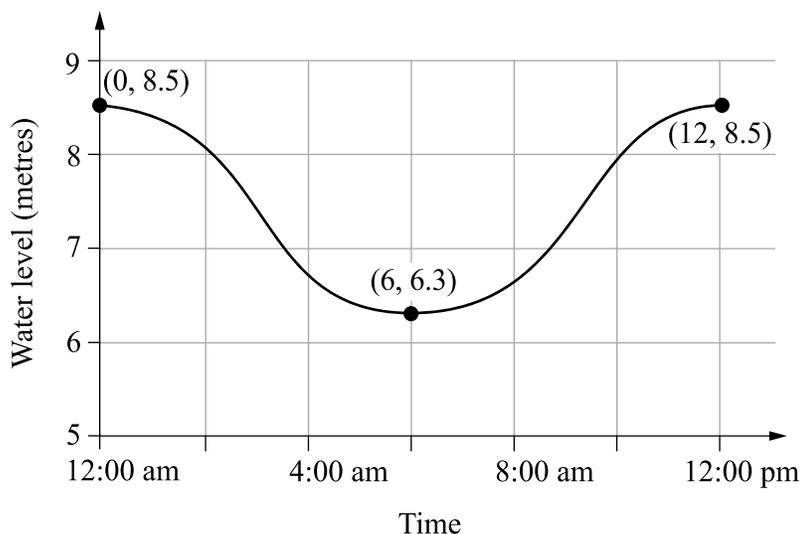


**a.** Determine the equation of the cosine function that models the water level as a function of time after 12:00 am. **[1 mark]**

**b.** How long in the 12-hour period shown is the rate of change of water level more than 0.55 metres per hour? **[3 marks]**

**Question 5 (4 marks)**

*Source:* QCAA 2022 Mathematical Methods, Paper 2, Section 2, Q13; © State of Queensland (QCAA) 2024.

A sandy beach has a fence on one side and ocean on the other. The width of the beach is the distance (in metres) from the fence to the water's edge. The width, $w(t)$, at a certain point is given by

$$w(t) = a + b \sin\left(\frac{\pi}{6}t - \frac{\pi}{3}\right), 0 \leq t \leq 24$$

where $t$ is time (in hours) since 6 am. The width of the beach is $8$ metres at 8 am and $3$ metres at 5 pm.

**a.** Determine $a$ and $b$. **[2 marks]**

**b.** Determine the rate of change of the width of the beach at 8 am and the first time after this when this rate of change is repeated. **[2 marks]**

**Fully worked solutions for this chapter are available online.**

# Answers

## Chapter 2 Calculus of trigonometric functions

### 2.2 Review of the unit circle

#### 2.2 Exercise

**1. a.** $\dfrac{\pi}{12}$ **b.** $\dfrac{2\pi}{3}$ **c.** $\dfrac{13\pi}{18}$

**d.** 1.12 **e.** 1.38 **f.** $\dfrac{31\pi}{18}$

**2. a.** 286.48° **b.** 275.02° **c.** 146.68°

**d.** 54° **e.** 150° **f.** 225°

**3. a.** $\dfrac{12}{13}$ **b.** $\dfrac{5}{12}$

**c.** $\dfrac{5}{13}$ **d.** $-\dfrac{12}{13}$

**e.** $\dfrac{12}{13}$ **f.** $-\dfrac{5}{12}$

**4. a.** $-1$ **b.** $-\dfrac{\sqrt{3}}{2}$ **c.** $-\dfrac{1}{\sqrt{2}}$

**d.** $\dfrac{1}{2}$ **e.** $-\sqrt{3}$ **f.** $-\dfrac{1}{2}$

**5. a.** $\dfrac{5}{13}$ **b.** $\dfrac{5}{12}$ **c.** $\dfrac{12}{13}$

**d.** $\dfrac{12}{13}$ **e.** $\dfrac{5}{13}$ **f.** $\dfrac{12}{5}$

**6. a.** Sample responses can be found in the worked solutions in the online resources.

**b.** Sample responses can be found in the worked solutions in the online resources.

**c.** Since the ratios are squared, there is no need to consider the quadrant for the angle.

**7. a.** $\dfrac{5\pi}{6}, \dfrac{7\pi}{6}$

**b.** 120°, 300°, 480°, 660°

**c.** $\dfrac{-\pi}{3}, \dfrac{\pi}{3}$

**8. a.** 210°, 330° **b.** $\dfrac{-3\pi}{2}, \dfrac{\pi}{2}$

**9. a.** $\dfrac{\pi}{12}, \dfrac{7\pi}{12}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{17\pi}{12}, \dfrac{23\pi}{12}$

**b.** $\dfrac{-5\pi}{6}, \dfrac{-2\pi}{3}, \dfrac{\pi}{6}, \dfrac{\pi}{3}$

**10.** $\dfrac{4\pi}{9}, \dfrac{5\pi}{9}, \dfrac{10\pi}{9}, \dfrac{11\pi}{9}, \dfrac{16\pi}{9}, \dfrac{17\pi}{9}$

**11.** $\dfrac{\pi}{4}, \dfrac{\pi}{2}, \dfrac{5\pi}{4}, \dfrac{3\pi}{2}$

**12.** $\dfrac{2\pi}{3}, \pi, \dfrac{4\pi}{3}$

**13. a.** $\dfrac{-\left(1+\sqrt{3}\right)}{2}$ **b.** $2-\sqrt{2}$ **c.** $2\sqrt{3}$

**14. a.** $\dfrac{3+2\sqrt{2}}{4}$ **b.** 0 **c.** $-2$

**15. a.** $-\pi, \dfrac{-3\pi}{4}, 0, \dfrac{\pi}{4}, \pi$

**b.** $\dfrac{-5\pi}{6}, \dfrac{5\pi}{6}$

**c.** $\dfrac{-\pi}{4}, \dfrac{3\pi}{4}$

**16. a.** $-\pi, 0, \pi$

**b.** $\dfrac{-5\pi}{6}, \dfrac{-\pi}{3}, \dfrac{\pi}{6}, \dfrac{2\pi}{3}$

**c.** $\dfrac{-3\pi}{4}, \dfrac{-\pi}{4}, \dfrac{\pi}{4}, \dfrac{3\pi}{4}$

**d.** $\dfrac{-2\pi}{3}, \dfrac{-\pi}{3}, \dfrac{\pi}{3}, \dfrac{2\pi}{3}$

**17.** 12 cm/s; the spring has the same speed every 3 seconds from the start.

**18.** 1.99 seconds; sample responses for the discussion can be found in the worked solutions in the online resources.

**19.** 0.75 metres; 2 pm the next day. See the worked solutions for reasons.

**20.**
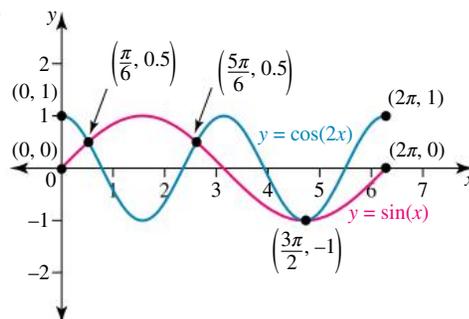


$x \approx 0.52, 2.62, 4.71$

### 2.3 Review of the trigonometric functions

#### 2.3 Exercise

**1. a.**



**b.**

**c.**

$y = 2\sin(x)$

$\left(\dfrac{\pi}{2}, 2\right)$; $(0, 0)$; $(\pi, 0)$; $(2\pi, 0)$; $\left(\dfrac{3\pi}{2}, -2\right)$

x-axis: $\dfrac{\pi}{4}$, $\dfrac{\pi}{2}$, $\dfrac{3\pi}{4}$, $\pi$, $\dfrac{5\pi}{4}$, $\dfrac{3\pi}{2}$, $\dfrac{7\pi}{4}$, $2\pi$

**c.**

$y = 3\cos(x)$

$(-2\pi, 3)$; $(0, 3)$; $\left(\dfrac{3\pi}{2}, 0\right)$; $(2\pi, 3)$; $\left(-\dfrac{3\pi}{2}, 0\right)$; $\left(-\dfrac{\pi}{2}, 0\right)$; $\left(\dfrac{\pi}{2}, 0\right)$; $(-\pi, -3)$; $(\pi, -3)$

**d.**

$y = \sin(x) + 2$; $y = 2$

$\left(\dfrac{\pi}{2}, 3\right)$; $(\pi, 2)$; $(0, 2)$; $(2\pi, 2)$; $\left(\dfrac{3\pi}{2}, 1\right)$

**d.**

$y = 3 - \cos(x)$; $y = 3$

$\left(-\dfrac{3\pi}{2}, 3\right)$; $(-\pi, 4)$; $\left(\dfrac{\pi}{2}, 3\right)$; $(\pi, 4)$; $\left(\dfrac{3\pi}{2}, 3\right)$; $(-2\pi, 2)$; $\left(-\dfrac{\pi}{2}, 3\right)$; $(0, 2)$; $(2\pi, 2)$

**e.**

$y = 2 - \sin(x)$; $y = 2$

$\left(\dfrac{3\pi}{2}, 3\right)$; $(0, 2)$; $(\pi, 2)$; $(2\pi, 2)$; $\left(\dfrac{\pi}{2}, 1\right)$

**e.**

$y = \cos(x) - 3$; $y = -3$

$(-2\pi, -2)$; $(-\pi, -4)$; $\left(\dfrac{\pi}{2}, -3\right)$; $(2\pi, -2)$; $\left(-\dfrac{3\pi}{2}, -3\right)$; $\left(-\dfrac{\pi}{2}, -3\right)$; $(0, -2)$; $(\pi, -4)$; $\left(\dfrac{3\pi}{2}, -3\right)$

**2. a.**

$y = \cos(x)$

$(-2\pi, 1)$; $\left(-\dfrac{\pi}{2}, 0\right)$; $(0, 1)$; $\left(\dfrac{3\pi}{2}, 0\right)$; $\left(-\dfrac{3\pi}{2}, 0\right)$; $\left(\dfrac{\pi}{2}, 0\right)$; $(2\pi, 1)$; $(-\pi, -1)$; $(\pi, -1)$

**3.**

$(0, -1)$; $(2\pi, -1)$; $y = -3$; $y = 2\cos(4x) - 3$

x-axis: $\dfrac{\pi}{4}$, $\dfrac{\pi}{2}$, $\dfrac{3\pi}{4}$, $\pi$, $\dfrac{5\pi}{4}$, $\dfrac{3\pi}{2}$, $\dfrac{7\pi}{4}$, $2\pi$

**b.**

$y = \cos\left(\dfrac{x}{2}\right)$

$(0, 1)$; $(-\pi, 0)$; $(\pi, 0)$; $(-2\pi, -1)$; $(2\pi, -1)$
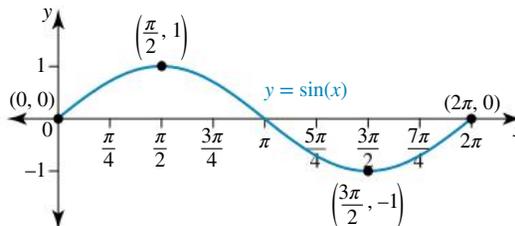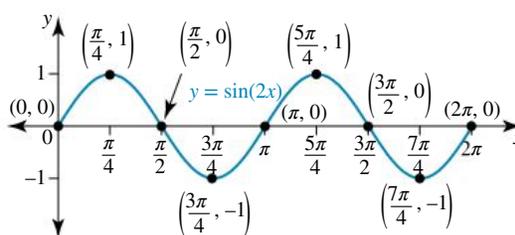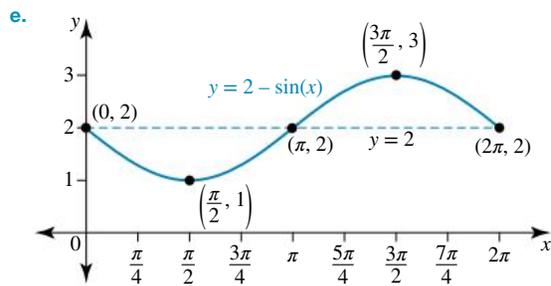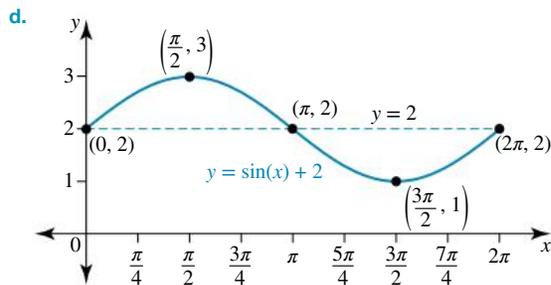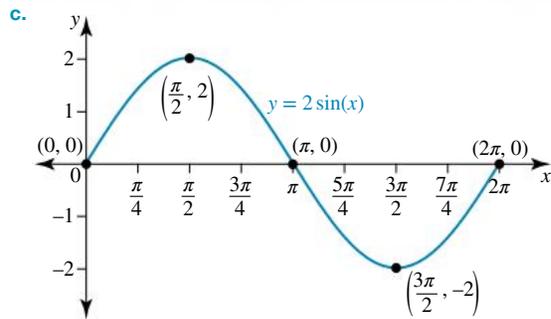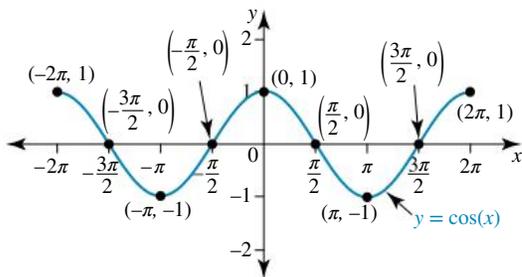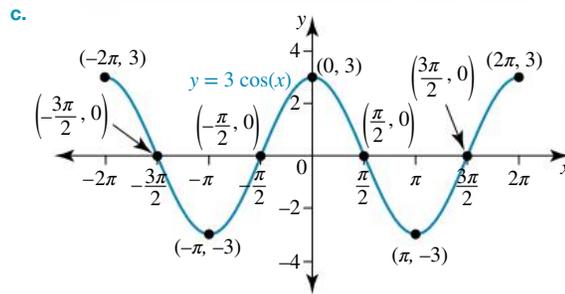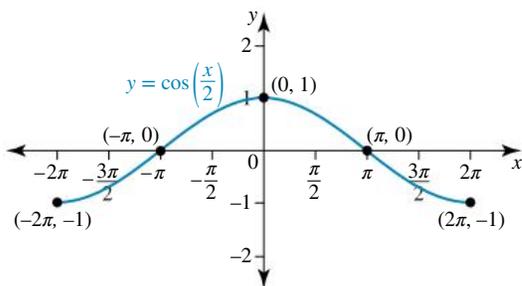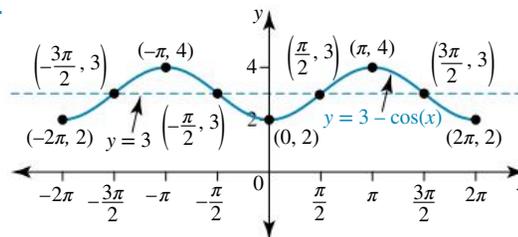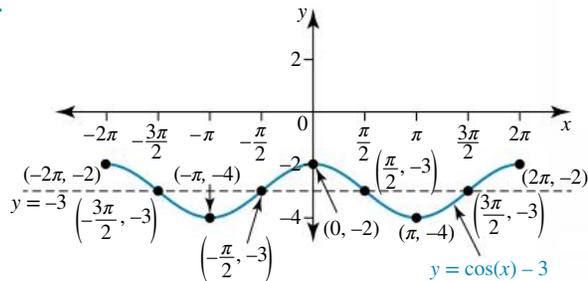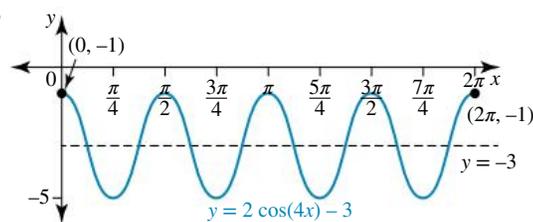
**4.** See the figure at the bottom of the page*

**5.**



$\left(\frac{\pi}{4}, 7\right)$  $\left(\frac{3\pi}{4}, 7\right)$

$y = -7\cos(4x)$

$\left(\frac{\pi}{8}, 0\right)$  $\left(\frac{3\pi}{8}, 0\right)$  $\left(\frac{5\pi}{8}, 0\right)$  $\left(\frac{7\pi}{8}, 0\right)$

$(0, -7)$  $\left(\frac{\pi}{2}, -7\right)$  $(\pi, -7)$

**6.**



$(-\pi, 3.5)$  $(0, 3.5)$  $y = \frac{1}{2}\cos(2x) + 3$  $(\pi, 3.5)$  $(2\pi, 3.5)$

$\left(-\frac{\pi}{2}, 2.5\right)$  $\left(\frac{\pi}{2}, 2.5\right)$  $\left(\frac{3\pi}{2}, 2.5\right)$  $y = 3$

**7.**



$f(x) = 1 - 2\sin\left(\frac{3x}{2}\right)$

$y = 1$  $(2\pi, 1)$

$(0, 1)$
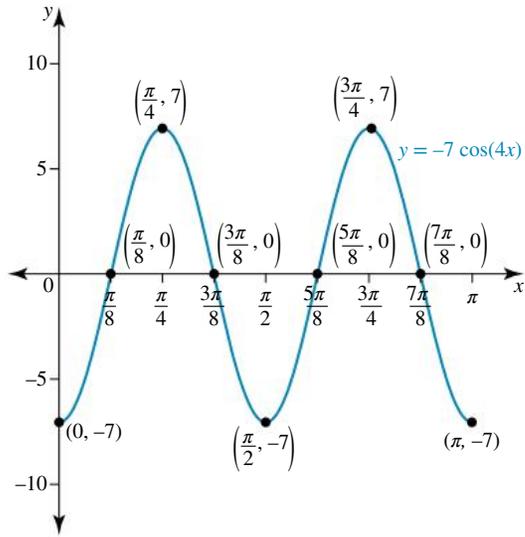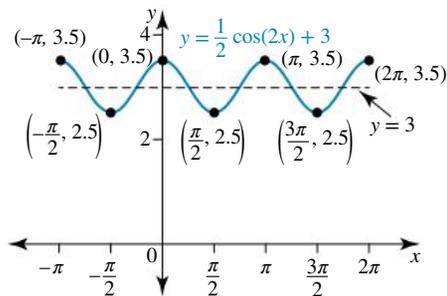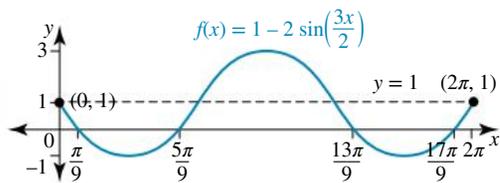
**8.** Mean position: $y = 5$; amplitude: 8; period: 2;
$y = -8\sin(\pi x) + 5$

**9.** Line of oscillation (mean position): $y = 1$; amplitude: 3;
period: $\pi$; $y = 3\cos(2x) + 1$

**10. a.**



$(0, 3\sqrt{2})$  $\left(\frac{3\pi}{2}, 3\sqrt{2}\right)$

$f(x) = -6\sin\left(3x - \frac{3\pi}{4}\right)$

**b.**



$y = 7 - 6\sin\left(3x - \frac{3\pi}{4}\right)$

$\left(\frac{\pi}{12}, 13\right)$  $\left(\frac{3\pi}{4}, 13\right)$  $\left(\frac{17\pi}{12}, 13\right)$

$(0, 11.243)$  $\left(\frac{3\pi}{2}, 11.243\right)$

$\left(\frac{5\pi}{12}, 1\right)$  $\left(\frac{13\pi}{12}, 1\right)$  $y = 7$

**11. a.**



$\left(\frac{\pi}{4}, 2\right)$

$y = 2\sin\left(x + \frac{\pi}{4}\right)$  $(2\pi, \sqrt{2})$

$(0, \sqrt{2})$

$\left(\frac{3\pi}{4}, 0\right)$  $\left(\frac{7\pi}{4}, 0\right)$

$\left(\frac{5\pi}{4}, -2\right)$

*4.



$\left(\frac{\pi}{2}, 6\right)$  $\left(\frac{7\pi}{6}, 6\right)$  $\left(\frac{11\pi}{6}, 6\right)$

$y = 2 - 4\sin(3x)$  $\left(\frac{5\pi}{3}, 2\right)$

$\left(\frac{\pi}{3}, 2\right)$  $\left(\frac{2\pi}{3}, 2\right)$  $\left(\frac{4\pi}{3}, 2\right)$  $y = 2$

$y = 2$  $(2\pi, 2)$

$(0, 2)$  $\left(\frac{\pi}{18}, 0\right)$  $\left(\frac{5\pi}{18}, 0\right)$  $\left(\frac{17\pi}{18}, 0\right)$  $\left(\frac{29\pi}{18}, 0\right)$

$\left(\frac{\pi}{6}, -2\right)$  $\left(\frac{5\pi}{6}, -2\right)$  $\left(\frac{3\pi}{2}, -2\right)$

$\left(\frac{13\pi}{18}, 0\right)$  $\left(\frac{25\pi}{18}, 0\right)$

**b.**



Graph of $y = 2\sin\left(x + \dfrac{\pi}{4}\right) - 1$ with points $(0, 0.414)$, $\left(\dfrac{\pi}{4}, 1\right)$, $\left(\dfrac{7\pi}{12}, 0\right)$, $\left(\dfrac{23\pi}{12}, 0\right)$, $(2\pi, 0.414)$, $\left(\dfrac{5\pi}{4}, -3\right)$; $y = -1$.

**12.** Maximum: 5; $x = \dfrac{11\pi}{12}$

**13. a.**



$y = 4\cos\left(3x - \dfrac{3\pi}{2}\right)$ with points $\left(-\dfrac{\pi}{2}, 4\right)$, $\left(-\dfrac{\pi}{3}, 0\right)$, $(0, 0)$, $\left(\dfrac{\pi}{3}, 0\right)$, $\left(\dfrac{2\pi}{3}, 0\right)$, $(\pi, 0)$, $\left(\dfrac{3\pi}{2}, 4\right)$, $\left(\dfrac{4\pi}{3}, 0\right)$.

**b.**



$y = 4 - 4\cos\left(3x - \dfrac{\pi}{2}\right)$ with points $\left(-\dfrac{\pi}{2}, 0\right)$, $(0, 4)$, $\left(\dfrac{\pi}{6}, 0\right)$, $\left(\dfrac{5\pi}{6}, 0\right)$, $\left(\dfrac{3\pi}{2}, 0\right)$; $y = 4$.
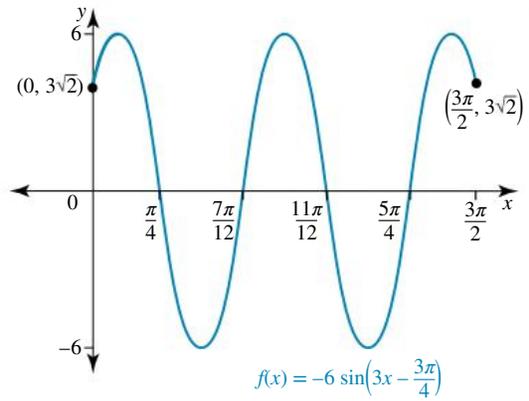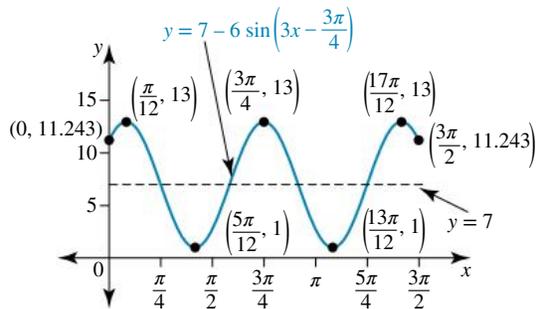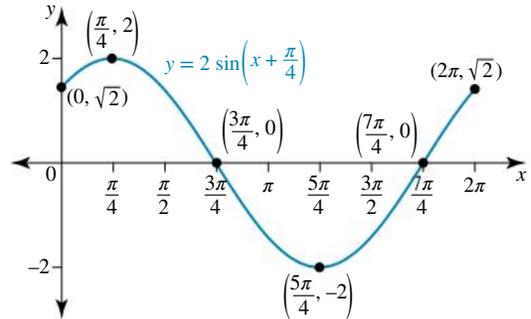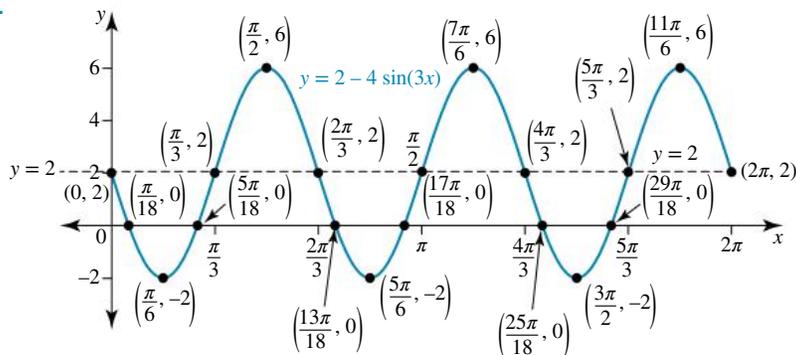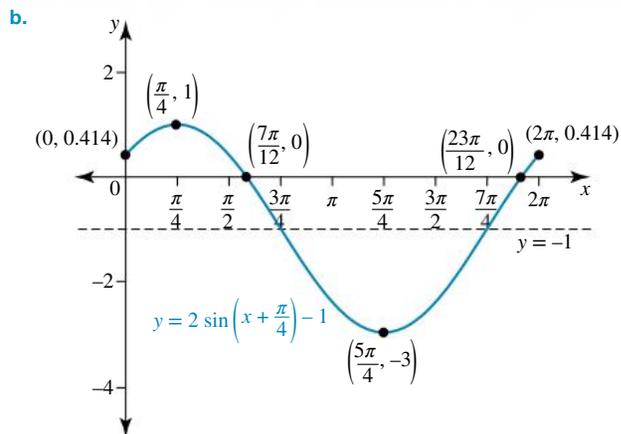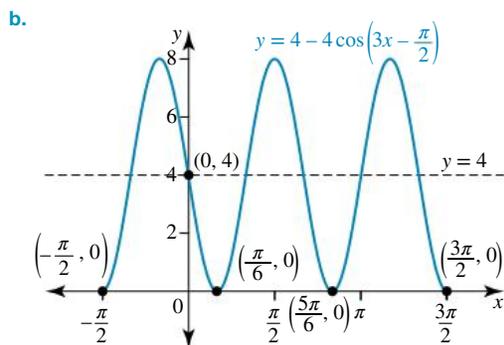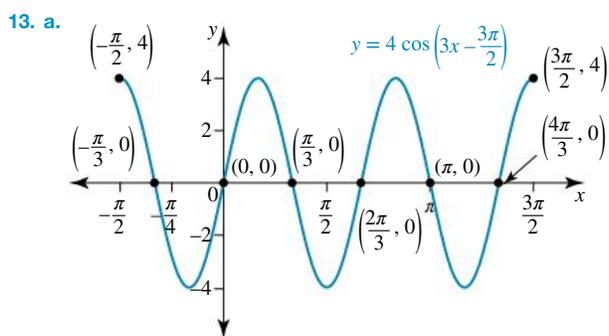
**c.** The curve $g(x)$ is $f(x)$ reflected in the $x$-axis (or inverted) and translated vertically up by 4 unit, oscillating around $y = 4$. Neither the period nor the amplitude have changed.

**14. a.**



$y = 2\cos(3x)$ with points $(0, 2)$, $\left(\dfrac{\pi}{6}, 0\right)$, $\left(\dfrac{\pi}{2}, 0\right)$, $\left(\dfrac{2\pi}{3}, 2\right)$, $\left(\dfrac{\pi}{3}, -2\right)$.

**b.**



$y = 2\cos(3x - \pi) + 3$ with points $\left(\dfrac{\pi}{3}, 5\right)$, $(\pi, 5)$, $(0, 1)$, $\left(\dfrac{2\pi}{3}, 1\right)$; $y = 3$.

**c.** $y = 2\cos(3x - \pi) + 3$

**15. a.**



$y = \sin(x)$ with points $\left(\dfrac{\pi}{2}, 1\right)$, $\left(\dfrac{5\pi}{2}, 1\right)$, $(0, 0)$, $(\pi, 0)$, $(2\pi, 0)$, $(3\pi, 0)$, $(4\pi, 0)$, $\left(\dfrac{3\pi}{2}, -1\right)$, $\left(\dfrac{7\pi}{2}, -1\right)$.

**b.**



$y = (\sin(x))^2$ with points $\left(\dfrac{\pi}{2}, 1\right)$, $\left(\dfrac{3\pi}{2}, 1\right)$, $\left(\dfrac{5\pi}{2}, 1\right)$, $\left(\dfrac{7\pi}{2}, 1\right)$, $(0, 0)$, $(\pi, 0)$, $(2\pi, 0)$, $(3\pi, 0)$, $(4\pi, 0)$.

**16. a.** $\dfrac{2\pi}{3}, \dfrac{5\pi}{6}, \dfrac{5\pi}{3}, \dfrac{11\pi}{6}$

**b.**



$y = \sin(2x)$ with points $(0, 0)$, $(2\pi, 0)$.
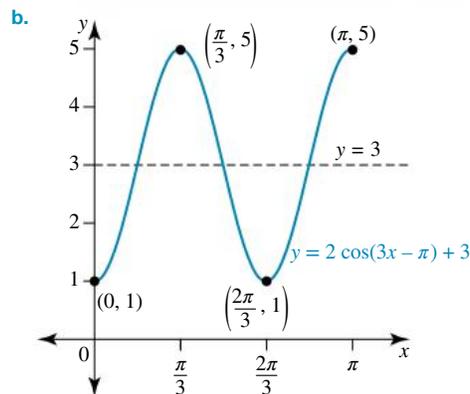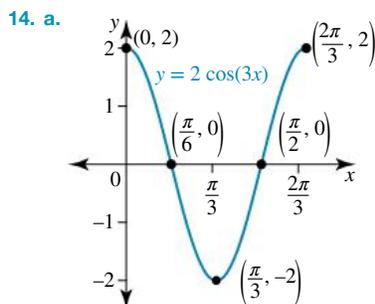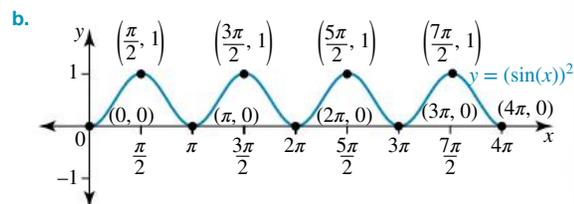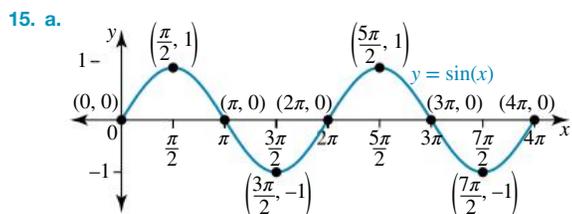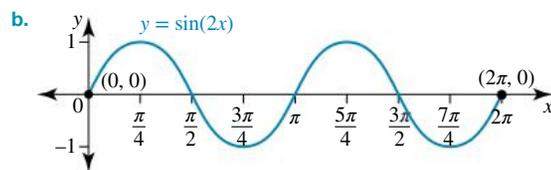
**c.** $\left\{x: \dfrac{2\pi}{3} < x < \dfrac{5\pi}{6}\right\} \cup \left\{x: \dfrac{5\pi}{3} < x < \dfrac{11\pi}{6}\right\}$

**17.** 3:30 am the following day



$d = 1.5\sin\left(\dfrac{\pi t}{12}\right) + 12.5$ with points $(0, 12.5)$, $(6, 14)$, $(18, 11)$, $(24, 12.5)$.
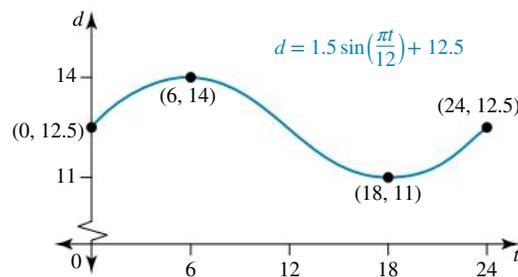
**18.** 0.2 seconds

**19.**
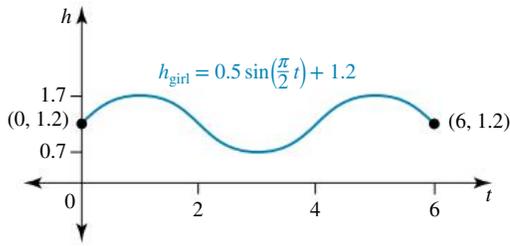


$$h(t) = 0.5 \sin\left(\frac{\pi}{2}t\right) + 1.2$$

**20.**
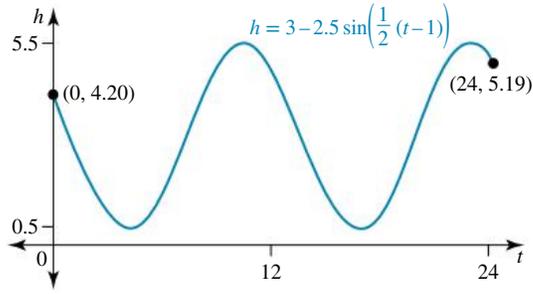


Low tide at 11:08 am
High tide at 5:25 pm

## 2.4 Differentiation of trigonometric functions

### 2.4 Exercise

**1. a.** $8\cos(8x)$      **b.** $-6\cos(-6x)$

   **c.** $\cos(x)$      **d.** $\dfrac{1}{3}\cos\left(\dfrac{x}{3}\right)$

   **e.** $-\dfrac{1}{2}\cos\left(-\dfrac{x}{2}\right)$      **f.** $\dfrac{2}{3}\cos\left(\dfrac{2x}{3}\right)$

**2. a.** $-3\sin(3x)$      **b.** $2\sin(-2x)$

   **c.** $-\dfrac{1}{3}\sin\left(\dfrac{x}{3}\right)$      **d.** $-21\sin(21x)$

   **e.** $7\sin(-7x)$      **f.** $-\dfrac{\pi}{4}\sin\left(\dfrac{\pi x}{4}\right)$

**3. a.** $2\cos(2x+3)$      **b.** $-7\cos(6-7x)$

   **c.** $5\cos(5x-4)$      **d.** $\dfrac{3}{4}\cos\left(\dfrac{3x+2}{4}\right)$

   **e.** $-\dfrac{7}{3}\cos\left(\dfrac{8-7x}{3}\right)$      **f.** $10\pi^2\cos(2\pi x)$

**4. a.** $\sin(8-x)$      **b.** $5\sin(6-5x)$

   **c.** $-\dfrac{2}{3}\sin\left(\dfrac{2x+3}{3}\right)$      **d.** $-\dfrac{4}{5}\sin\left(\dfrac{4x-1}{5}\right)$

   **e.** $-40\pi^2\sin(10\pi x)$      **f.** $-12\sin(-2x)$

**5. a.** $2(2-x)\sin(x^2-4x+3)$

   **b.** $(2x-5)\cos(10-5x+x^2)$

   **c.** $e^x\cos(e^x)$

   **d.** $-(2x+7)\sin(x^2+7x)$

   **e.** $2(x-2)\sin(4x-x^2)$

   **f.** $(2x+3)\cos(x^2+3x)$

**6.** $\dfrac{-\pi}{2}\sin\left(\dfrac{\pi}{18}x\right)$

**7. a.** $\dfrac{dy}{dx} = -6\sin(3x)$

   **b.** $\dfrac{dy}{dx} = -\dfrac{\pi}{180}\sin\left(\dfrac{\pi x}{180}\right)$

   **c.** $\dfrac{dy}{dx} = 3\sin\left(\dfrac{\pi}{2} - x\right)$

   **d.** $\dfrac{dy}{dx} = -\dfrac{4}{3}\cos\left(\dfrac{x}{3}\right)$

   **e.** $\dfrac{dy}{dx} = \dfrac{\pi}{15}\cos\left(\dfrac{\pi x}{15}\right)$

   **f.** $\dfrac{dy}{dx} = 6\cos\left(\dfrac{\pi}{2} + 3x\right)$

**8.** $-4$

**9.** $4\sqrt{3}$

**10.** $y = x - \dfrac{\pi}{2}$

**11.** $y = -\dfrac{3}{2}x + \dfrac{\pi}{4} + \dfrac{3\sqrt{3}}{2}$

**12.** $y = 4$

**13.** $-\dfrac{5}{2}$

**14.** $\left(\dfrac{4\pi}{3}, -\sqrt{3}\right)$

**15. a.** $f(0) = -1$

   **b.** $\dfrac{\pi}{4}, \dfrac{5\pi}{4}$

   **c.** $f'(x) = \cos(x) + \sin(x)$

   **d.** $\dfrac{3\pi}{4}, \dfrac{7\pi}{4}$

**16. a.** $f(0) = \sqrt{3}$

   **b.** $\dfrac{-\pi}{3}, \dfrac{2\pi}{3}$

   **c.** $f'(x) = -\sqrt{3}\sin(x) + \cos(x)$

   **d.** $\dfrac{-5\pi}{6}, \dfrac{\pi}{6}$

**17.** $\dfrac{-5\pi}{8}, \dfrac{-\pi}{8}, \dfrac{3\pi}{8}, \dfrac{7\pi}{8}$

**18.** $(-0.524, 0.342)$ and $(0.524, -0.342)$

**19.** $(0.243, 1.232)$ and $(0.804, 0.863)$

**20.** $116.6°$

## 2.5 Modelling and solving with trigonometric functions

### 2.5 Exercise

**1. a.** 8.5 m at 3 pm

   **b.** 12 hours

   **c.** 3.5 m

**d.**
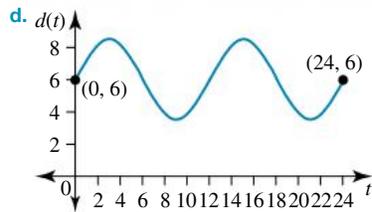


**e.** Between 1 pm and 5 pm, and again between 1 am and 5 am the next day

**f.** 1.13 m/h

**2. a.** 10 m; 4 m

**b.**



$\dfrac{\pi}{2}$ m/h
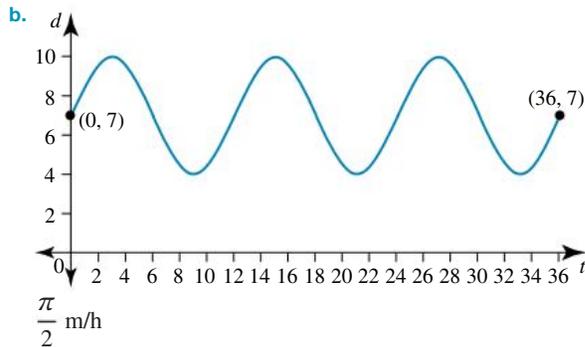
**c.** 3 am

**d.** Between 1 am and 5 am on Saturday, between 1 pm and 5 pm on Saturday, and between 1 am and 5 am on Sunday

**e.** Between 3 am and 5 am on Sunday morning

**3. a.** 18 °C; 14.5 °C

**b.** 3.2 degrees/month

**c.** January and December

**d.** 21.5 °C

**e.** October

**4. a.** $a = 0.9; n = \pi; c = 0.9$
$h = 0.9\sin(\pi t) + 0.9$

**b.**
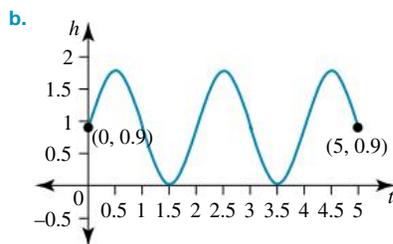


**c.** 1.3 s; $-2.0$ m/s

**5. a.** 500 frogs

**b.**
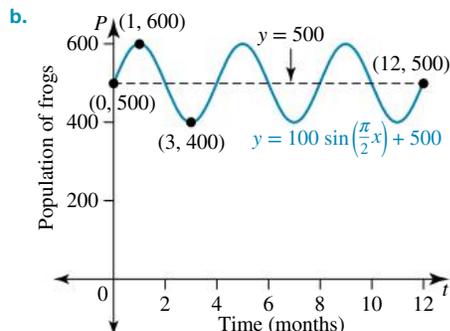


**c.** After 1 month

**d.** Increasing or decreasing at $50\pi$ frogs/month

**6. a.** 10 cm

**b.** $\dfrac{dL}{dt} = 2\pi\,\cos(\pi t)$

**c.** $-2\pi$ cm/s

**7.** 9 cm$^2$/radian

**8. a.** 14 °C

**b.** $\dfrac{dT}{dt} = \dfrac{2\pi}{9}\cos\left(\dfrac{\pi}{9}t\right)$

**c.** 0.535° C/h

**9. a.** 9 m

**b.** $\dfrac{dh}{dd} = -\dfrac{4\pi}{25}\sin\left(\dfrac{\pi}{25}d\right)$

**c. i.** $-0.295$ m/m

**ii.** $-0.478$ m/m

**10. a.** 12 h

**b.** 1 m at 3 pm

**c.** $\dfrac{dH}{dt} = \dfrac{\pi}{12}\cos\left(\dfrac{\pi t}{6}\right)$

**d.** $\dfrac{\pi\sqrt{2}}{24}$

**e.** 4:30 pm

**11. a.** $h = -50\cos(2\pi t) + 50$

**b.** $\dfrac{dh}{dt} = 100\pi\,\sin(2\pi t)$

**c.** $100\pi$ mm/s

**12. a.** 1.5 m

**b.** 8.5 m

**c.** 60 s

**d.** 18.4 s

**e.** $\dfrac{dh}{dt} = \dfrac{7\pi}{60}\sin\left(\dfrac{\pi t}{30}\right)$

**f.** 35.51 s; 54.49 s

**13. a.** 6 m

**b.** $\dfrac{dy}{dx} = -\dfrac{7\pi}{40}\sin\left(\dfrac{\pi t}{20}\right)$

**c. i.** $-\dfrac{7\sqrt{2}\pi}{80}$ m/m

**ii.** $-\dfrac{7\pi}{40}$ m/m

**d. i.** 15 m

**ii.** 158.96°

**14. a.** 0.849, 1.508

**b.** $\left(\dfrac{\pi}{8}, 3\right), \left(\dfrac{3\pi}{8}, -1\right)$

**c.** $-8$

**d.** $\left[0, \dfrac{\pi}{8}\right) \cup \left(\dfrac{3\pi}{8}, \dfrac{\pi}{2}\right]$

**15. a.** $BD = a\sin(\theta), CD = a\cos(\theta)$

**b.** $L = a + 2a\sin(\theta) + a\cos(\theta) + 4$

**c.** $2a\cos(\theta) - a\sin(\theta)$

**d.** $\theta = 1.1^c$

**16. a.** 2.5 m

**b.** 2.93 m

**c.** 3 m at 5:30 am

**d.**
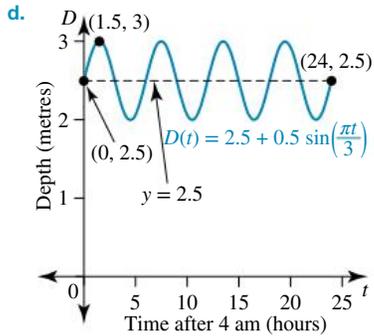


**e.** $D'(t) = \dfrac{\pi}{6}\cos\left(\dfrac{\pi}{3}t\right)$
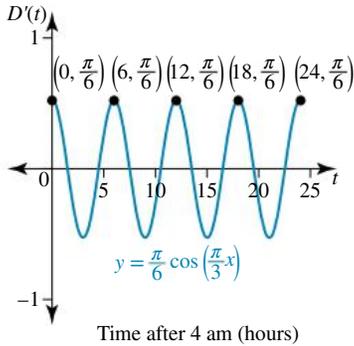


**f.** 4 am, 10 am, 4 pm, 10 pm and 4 am

**17.** Sample responses can be found in the worked solutions in the online resources; the maximum length is 700 cm.

**18.** $x = 4.756$ m

**19.** 6:37 am

**20.** $(0.9, 75.4), -39.5$

## 2.6 Review

## 2.6 Exercise

**1. a.** 135°   **b.** 195°

**c.** 120°19′   **d.** 100°50′

**2. a.** $\dfrac{7\pi}{36}$   **b.** $\dfrac{14\pi}{9}$

**c.** 2.24   **d.** 4.03

**3. a.** $-\dfrac{2}{3}$   **b.** $\dfrac{\sqrt{5}}{3}$   **c.** $\dfrac{\sqrt{5}}{2}$

**d.** $-\dfrac{\sqrt{5}}{3}$   **e.** $-\dfrac{\sqrt{5}}{2}$   **f.** $\dfrac{2}{3}$

**4. a.** $\dfrac{\sqrt{3}}{2}$   **b.** $-\dfrac{1}{\sqrt{2}} = -\dfrac{\sqrt{2}}{2}$

**c.** $-\dfrac{1}{\sqrt{3}} = -\dfrac{\sqrt{3}}{3}$   **d.** $-\dfrac{1}{\sqrt{2}} = -\dfrac{\sqrt{2}}{2}$

**e.** $\dfrac{-1}{2}$   **f.** $-\dfrac{1}{\sqrt{3}} = -\dfrac{\sqrt{3}}{3}$

**5. a.** $\dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$   **b.** $\dfrac{-\sqrt{3}}{2}$

**c.** $-\sqrt{3}$   **d.** $\dfrac{-1}{2}$

**e.** $\dfrac{\sqrt{2}}{2}$   **f.** $\dfrac{1}{2}$

**6. a.** $-1$   **b.** $-\dfrac{1}{\sqrt{2}} = -\dfrac{\sqrt{2}}{2}$

**c.** $\dfrac{-\sqrt{3}}{2}$   **d.** $\dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$

**e.** $\dfrac{1}{2}$   **f.** $-\dfrac{1}{\sqrt{2}} = -\dfrac{\sqrt{2}}{2}$

**7. a.** $\dfrac{\pi}{2}, \dfrac{3\pi}{2}$   **b.** $\dfrac{5\pi}{4}, \dfrac{7\pi}{4}$   **c.** $\dfrac{\pi}{4}, \dfrac{7\pi}{4}$

**d.** $\dfrac{3\pi}{2}$   **e.** $\dfrac{5\pi}{6}, \dfrac{7\pi}{6}$

**8. a.** 90°   **b.** 60°, 300°   **c.** 60°, 120°

**d.** 180°   **e.** 45°, 135°

**9. a.** $\dfrac{\pi}{6}, \dfrac{5\pi}{6}$   **b.** $\dfrac{\pi}{2}, \dfrac{3\pi}{2}$   **c.** $\dfrac{4\pi}{3}, \dfrac{5\pi}{3}$

**d.** $\dfrac{\pi}{4}, \dfrac{7\pi}{4}$   **e.** $\dfrac{5\pi}{6}, \dfrac{11\pi}{6}$

**10. a.** $\dfrac{\pi}{2}$

**b.** 0

**c.** $-\dfrac{11\pi}{18}, -\dfrac{7\pi}{18}, \dfrac{\pi}{18}, \dfrac{5\pi}{18}, \dfrac{13\pi}{18}, \dfrac{17\pi}{18}$

**d.** $\dfrac{-3\pi}{4}, \dfrac{-7\pi}{12}, \dfrac{-\pi}{12}, \dfrac{\pi}{12}, \dfrac{7\pi}{12}, \dfrac{3\pi}{4}$

**e.** $\dfrac{-7\pi}{12}, \dfrac{-5\pi}{12}, \dfrac{5\pi}{12}, \dfrac{7\pi}{12}$

**11. a.** $\dfrac{\pi}{4}, \dfrac{5\pi}{4}$

**b.** $\dfrac{\pi}{8}, \dfrac{5\pi}{8}, \dfrac{9\pi}{8}, \dfrac{13\pi}{8}$

**c.** $\dfrac{\pi}{6}, \dfrac{2\pi}{3}, \dfrac{7\pi}{6}, \dfrac{5\pi}{3}$

**d.** $\dfrac{\pi}{18}, \dfrac{7\pi}{18}, \dfrac{13\pi}{18}, \dfrac{19\pi}{18}, \dfrac{25\pi}{18}, \dfrac{31\pi}{18}$

**e.** 0.6781, 1.7253, 2.7725, 3.8197, 4.8669, 5.9141

**f.** 1.8925, 5.0341

**12. a.** $\dfrac{\pi}{4}, \dfrac{5\pi}{4}$

**b.** $\dfrac{\pi}{9}, \dfrac{\pi}{3}, \dfrac{5\pi}{9}, \dfrac{7\pi}{9}, \pi, \dfrac{11\pi}{9}, \dfrac{13\pi}{9}, \dfrac{5\pi}{3}, \dfrac{17\pi}{9}$

**c.** $\dfrac{\pi}{2}, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}, \dfrac{3\pi}{2}$

**13.** $y = 2 \sin 2x$, $-\pi \le x \le \pi$; amplitude = 2; period = $\pi$; range $-2 \le y \le 2$
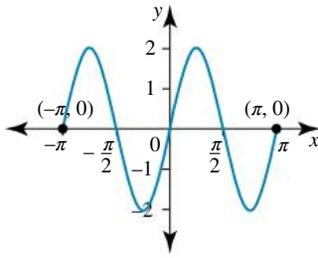


**14. a.** $y = 2 \sin(2x + \pi)$ for $0 \le x \le 2\pi$; period: $\pi$; amplitude: 2; range: $[-2, 2]$; end points: $(0, 0)$ and $(2\pi, 0)$

For $x$-intercepts: $x = 0, \dfrac{\pi}{2}, \pi, \dfrac{3\pi}{2}, 2\pi$
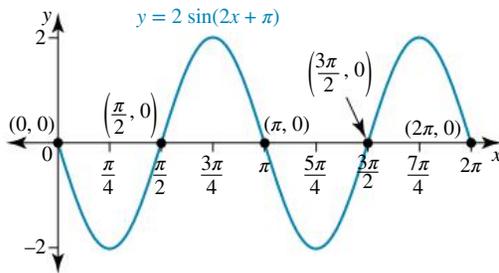


**b.** $y = 3 \cos(3x + \pi)$; period: $\dfrac{2\pi}{3}$; amplitude: 3; range: $[-3, 3]$; end points: $(0, -3)$ and $(2\pi, -3)$

For $x$-intercepts: $x = \dfrac{\pi}{6}, \dfrac{\pi}{2}, \dfrac{5\pi}{6}, \dfrac{7\pi}{6}, \dfrac{3\pi}{2}, \dfrac{11\pi}{6}$
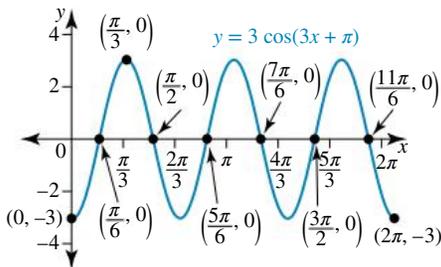


**c.** $y = 2 \sin\left(x - \dfrac{\pi}{4}\right) - 1$; period: $2\pi$; amplitude: 2; range: $[-3, 1]$; end points: $\left(0, -\sqrt{2} - 1\right)$ and $\left(2\pi, -\sqrt{2} - 1\right)$

For $x$-intercepts: $x = \dfrac{5\pi}{12}, \dfrac{13\pi}{12}$
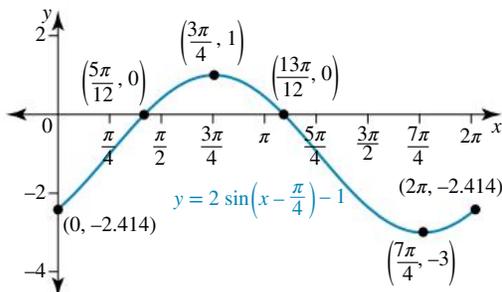


**d.** $y = \cos\left(\dfrac{1}{2}(x - \pi)\right) + 1$; period: $4\pi$; amplitude: 1; range: $[0, 2]$; end points: $(0, 1)$ and $(2\pi, 1)$

For $x$-intercepts: there are no solutions for these restricted $x$-values.
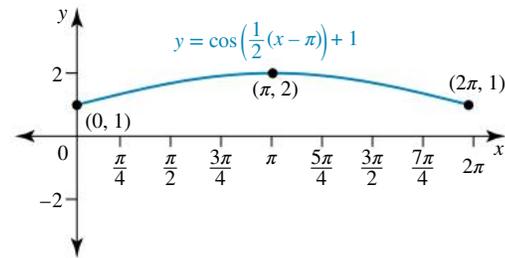


**15. a.** $-4(16x - 3) \sin(8x^2 - 3x + 2)$

**b.** $-3(4x + 1) \cos(2x^2 + x - 5)$

**c.** $36x \cos(3x^2 - 2) - 30x \sin(3x^2 - 2)$

**d.** $-2(x + 1) \sin(x^2 + 2x + 1)$

**e.** $-12x \cos(4 - 3x^2) - 6$

**f.** $-\cos(-x) - 2 \sin(-2x)$

**16.** Tangent: $y = -3$; perpendicular: $x = \pi$

**17.** The air conditioner is on from 11:53 am until 2:07 pm.

**18.** 7.6 and $-7.6$ respectively

**19.** $\dfrac{5}{2}\sqrt{5} - \sqrt{15}$ units

**20.** 0.785 degrees/hour at midday

### 2.6 Past QCAA exam questions

**1.** B

**2.** D

**3.** D

**4. a.** $1.1 \cos\left(\dfrac{\pi t}{6}\right) + 7.4$

**b.** 1.151 hours

**5. a.** $a = 8$
  $b = 5$

**b.** $t = 14$

# 3 Differentiation rules

## LESSON SEQUENCE

Fully worked solutions of this chapter are available online.

## EXAM PREPARATION

Access exam-style questions in every lesson, available online and access past QCAA exam questions in every review.

**on** Resources

| | |
|---|---|
| **Solutions** | Solutions — Chapter 3 |
| **Exam questions** | Exam question booklet — Chapter 3 |
| **Digital documents** | Learning matrix — Chapter 3<br>Chapter summary — Chapter 3 |

# LESSON
## 3.1 Overview

### 3.1.1 Introduction

Differential calculus is a branch of mathematical analysis concerned with determining how a change in one variable will affect another related variable. Calculus is the study of change, the slopes of curves and the rate of change between two variables. It is generally thought that Sir Isaac Newton (1643–1727, on the right), in England, and Gottfried Leibniz (1646–1716, on the left), in Germany, independently discovered calculus in the mid-seventeenth century. Both of them were building on earlier studies of motion and areas: Newton was investigating the laws of motion and gravity as well as geometry, whereas Leibniz was focused on understanding tangents to curves. Although Leibniz was the first to publish his results, controversy remained between the two as to who invented the notation which is still used today.

Although early study of differential calculus involved ratios and geometry, during the eighteenth century it became more algebraic in nature. Today, calculus is used in many different areas. In economics and commerce, examples of rates of change include marginal costs, the increase or decrease in production costs if another unit is produced, and predictions on the stock market. In science, the rate of growth of bacteria or the rate of decay of a substance can be expressed as a differential equation. In engineering, optimisation — determining the value of one variable that would either maximise or minimise a related variable — is used extensively along with graphing curves.

### 3.1.2 Syllabus links

| Lesson | Lesson title | Syllabus links |
|--------|--------------|----------------|
| 3.2 | The chain rule | ○ Use the chain rule to determine the derivative of composite functions involving exponential, logarithmic and trigonometric functions, expressing derivatives in simplest and factorised form. |
| 3.3 | The product rule | ○ Use the product rule to determine the derivative of exponential, logarithmic and trigonometric functions, expressing derivatives in simplest and factorised form. |
| 3.4 | The quotient rule | ○ Use the quotient rule to determine the derivative of exponential, logarithmic and trigonometric functions, expressing derivatives in simplest and factorised form. |
| 3.5 | Modelling and solving using the differentiation rules | ○ Solve problems that involve combinations of the chain rule, product rule and quotient rule to differentiate exponential, logarithmic and trigonometric functions. |

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# LESSON
## 3.2 The chain rule

### 3.2.1 Composite functions

A **composite function**, also known as a function of a function, consists of two or more functions nested within each other.

Consider $g(x) = x^4$ and $h(x) = 2x + 1$. If $f(x) = g(h(x))$, then:

$$f(x) = g(2x + 1)$$
$$= (2x + 1)^4$$

The function $f(x)$ can be differentiated if it is expanded.

$$f(x) = 16x^4 + 32x^3 + 24x^2 + 8x + 1$$
$$f'(x) = 64x^3 + 96x^2 + 48x + 8$$
$$= 8(8x^3 + 12x^2 + 6x + 1)$$
$$= 8(2x + 1)^3$$

The chain rule allows us to reach this same outcome without having to expand. The chain rule also allows us to differentiate composite functions that we cannot expand.

It is worth noting, however, that $g(h(x))$ is not necessarily equal to $h(g(x))$.

For example, in this instance, $g(h(x)) = (2x + 1)^4$ and $h(g(x)) = h(x^4) = 2x^4 + 1$.

### 3.2.2 The chain rule

> **The chain rule**
>
> If $y = f(g(x))$,
>
> $$\frac{dy}{dx} = f'(g(x)) \times g'(x)$$
>
> Alternatively, if $y = f(u)$ and $u = g(x)$,
>
> $$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

# 3.2.3 Proof of the chain rule

The proof of the chain rule is as follows.

If $f(x) = m(n(x))$,

then $f(x + h) = m(n(x + h))$.

Therefore, $\dfrac{f(x + h) - f(x)}{h} = \dfrac{m(n(x + h)) - m(n(x))}{h}$.

Multiply the numerator and the denominator by $n(x + h) - n(x)$, as it is expected that at some stage $n'(x)$ will appear somewhere in the rule.

$$\frac{f(x + h) - f(x)}{h} = \frac{n(x + h) - n(x)}{h} \times \frac{m(n(x + h)) - m(n(x))}{n(x + h) - n(x)}$$

$$f'(x) = \lim_{h \to 0} \left[ \frac{m(n(x + h)) - m(n(x))}{n(x + h) - n(x)} \times \frac{n(x + h) - n(x)}{h} \right]$$

$$= \lim_{h \to 0} \left[ \frac{m(n(x + h)) - m(n(x))}{n(x + h) - n(x)} \right] \times \lim_{h \to 0} \left[ \frac{n(x + h) - n(x)}{h} \right]$$

By definition, $n'(x) = \lim\limits_{h \to 0} \dfrac{n(x + h) - n(x)}{h}$. Also, if we let $n(x) = A$ and $n(x + h) = A + B$, then $n(x + h) - n(x)$ $= A + B - A$, so that

$$\frac{m(n(x + h)) - m(n(x))}{n(x + h) - n(x)} = \frac{m(A + B) - n(A)}{B}.$$

Also, as $h \to 0$, $B \to 0$.

Consequently, $\lim\limits_{B \to 0} \dfrac{m(A + B) - m(A)}{B} = m'(A)$.

Therefore, $\lim\limits_{h \to 0} \left[ \dfrac{m(n(x + h)) - m(n(x))}{n(x + h) - n(x)} \right] = m'(n(x))$.

Bringing this all together, we can see the following.

If $f(x) = m(n(x))$,

$f'(x) = m'(n(x)) \times n'(x)$.

Using Leibniz notation, this becomes

$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$, where $y = f(u)$ and $u$ is a function of $x$.

Consider again $y = f(x) = (2x + 1)^4$. The chain rule can be used to determine the derivative of this function.

Let $u = 2x + 1$. $\qquad \therefore \dfrac{du}{dx} = 2$

Also let $y = u^4$. $\qquad \therefore \dfrac{dy}{du} = 4u^3$.

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

By the chain rule, $= 4u^3 \times 2$

$$= 8u^3$$

Since $u = 2x + 1$, $\dfrac{dy}{dx} = 8(2x + 1)^3$.

---

## WORKED EXAMPLE 1 Using the chain rule to determine the derivative of a composite function 1

**a.** Use the chain rule to determine the derivative of $y = (x^2 + 3x + 5)^7$.

**b.** Determine $\dfrac{dy}{dx}$ for the function $y = (4x - 7)^{\frac{2}{3}}$.

**THINK**

**WRITE**

**a. 1.** Write the function to be derived.

**a.** $y = (x^2 + 3x + 5)^7$

**2.** Let $u$ equal the inner function and rewrite.

$y = u^7$ and $u = x^2 + 3x + 5$

**3.** Differentiate to determine $\dfrac{dy}{du}$ and $\dfrac{du}{dx}$.

$\dfrac{dy}{du} = 7u^6$ and $\dfrac{du}{dx} = 2x + 3$

**4.** Apply the chain rule.

$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

$\dfrac{dy}{dx} = 7u^6 \times (2x + 3)$

**5.** Substitute for $u$ and simplify.

$\dfrac{dy}{dx} = 7(2x + 3)(x^2 + 3x + 5)^6$

**b. 1.** Write the function to be derived.

**b.** $y = (4x - 7)^{\frac{2}{3}}$

**2.** Let $u$ equal the inner function and rewrite.

$y = u^{\frac{2}{3}}$ and $u = 4x - 7$

**3.** Differentiate to determine $\dfrac{dy}{du}$ and $\dfrac{du}{dx}$.

$\dfrac{dy}{du} = \dfrac{2}{3}u^{\frac{-1}{3}}$ and $\dfrac{du}{dx} = 4$

**4.** Apply the chain rule.

$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

$\dfrac{dy}{dx} = \dfrac{2}{3}u^{\frac{-1}{3}} \times 4$

**5.** Substitute for $u$ and simplify.

$\dfrac{dy}{dx} = \dfrac{8}{3\sqrt[3]{4x - 7}}$

In complex functions, the chain rule may need to be applied more than once. This is demonstrated in the following worked example.

---

**WORKED EXAMPLE 2 Using the chain rule to determine the derivative of a composite function 2**

**Determine the derivatives of the following functions.**

**a.** $y = \sqrt{\log_e(6x + 5)}$

**b.** $y = \ln(x^4 + 4x^2)$

| THINK | WRITE |
|---|---|
| **a. 1.** Write the function to be derived, expressing the square root as a power. | **a.** $y = \sqrt{\log_e(6x + 5)}$. <br><br> $y = \left(\log_e(6x + 5)\right)^{\frac{1}{2}}$ |
| **2.** Let $u$ equal the inner function and rewrite. | $y = u^{\frac{1}{2}}$ and $u = \log_e(6x + 5)$ |
| **3.** Differentiate to determine $\dfrac{dy}{du}$ and $\dfrac{du}{dx}$. | $\dfrac{dy}{du} = \dfrac{1}{2}u^{\frac{-1}{2}}$ and $\dfrac{du}{dx} = \dfrac{6}{(6x + 5)}$ |
| **4.** Apply the chain rule. | $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$ <br><br> $\dfrac{dy}{dx} = \dfrac{1}{2}u^{\frac{-1}{2}} \times \dfrac{6}{(6x + 5)}$ |
| **5.** Substitute for $u$ and simplify. | $\dfrac{dy}{dx} = \dfrac{1}{2} \times \dfrac{1}{\sqrt{\log_e(6x + 5)}} \times \dfrac{6}{(6x + 5)}$ <br><br> $\dfrac{dy}{dx} = \dfrac{3}{(6x + 5)\sqrt{\log_e(6x + 5)}}$ |
| **b. 1.** Write the function to be derived. | **b.** $y = \ln(x^4 + 4x^2)$ |
| **2.** Let $u$ equal the inner function and rewrite. | $y = \ln(u)$ and $u = x^4 + 4x^2$ |
| **3.** Differentiate to determine $\dfrac{dy}{du}$ and $\dfrac{du}{dx}$. | $\dfrac{dy}{du} = \dfrac{1}{u}$ and $\dfrac{du}{dx} = 4x^3 + 8x$ |
| **4.** Apply the chain rule. | $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$ <br><br> $\dfrac{dy}{dx} = \dfrac{1}{u} \times (4x^3 + 8x)$ |
| **5.** Substitute for $u$ and simplify, expressing your answer in simplest and factorised form. | $\dfrac{dy}{dx} = \dfrac{1}{(x^4 + 4x^2)} \times (4x^3 + 8x)$ <br><br> $= \dfrac{(4x^3 + 8x)}{(x^4 + 4x^2)}$ <br><br> $= \dfrac{4x(x^2 + 2)}{x^2(x^2 + 4)}$ <br><br> $\dfrac{dy}{dx} = \dfrac{4(x^2 + 2)}{x(x^2 + 4)}$ |

a. **Determine the derivative of $y = \cos^2(e^{2x})$.**
b. **Evaluate the derivative when $x = 0$, giving your answer correct to 4 decimal places.**

**THINK**

**WRITE**

a. 1. Write the function to be derived.

2. Let $u$ equal the inner function.

3. Use the chain rule to differentiate this inner function.

4. Consider the outer function.

5. Differentiate.

6. Apply the chain rule.

7. Substitute for $u$ and simplify.

b. 1. Substitute $x = 0$ into $\dfrac{dy}{dx}$.

2. Answer the question.

a. $y = \cos^2(e^{2x})$

$y = [\cos(e^{2x})]^2$

$y = u^2$ and $u = \cos(e^{2x})$

$\dfrac{du}{dx} = -\sin(e^{2x}) \times 2e^{2x}$

$\dfrac{du}{dx} = -2e^{2x}\sin(e^{2x})$

$y = u^2$

$\dfrac{dy}{du} = 2u$

$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

$\dfrac{dy}{dx} = 2u(-2e^{2x}\sin(e^{2x}))$

$\dfrac{dy}{dx} = -4e^{2x}\cos(e^{2x})\sin(e^{2x})$

b. $\dfrac{dy}{dx} = -4e^{0}\cos(e^{0})\sin(e^{0})$

$\dfrac{dy}{dx} = -4\cos(1)\sin(1)$

$\dfrac{dy}{dx} = -1.818\,594\,85$

$\dfrac{dy}{dx} = -1.8186$ to 4 decimal places.

For the function with the rule $y = (x-1)^{\frac{2}{3}}$, determine:

a. $\dfrac{dy}{dx}$

b. the equations of the tangents at $(2, 1)$ and $(0, 1)$.

**THINK**

**WRITE**

a. 1. Write the function to be derived.

a. $y = (x-1)^{\frac{2}{3}}$

2. Apply the chain rule to determine the derivative. Multiply the derivative of the outer function with the derivative of the inner function. Write the answer in surd form.

$\dfrac{dy}{dx} = \dfrac{2}{3}(x-1)^{-\frac{1}{3}} \times (1)$

$= \dfrac{2}{3\sqrt[3]{(x-1)}}$

b. 1. Determine the gradient at $x = 2$.

b. When $x = 2$:

$\dfrac{dy}{dx} = \dfrac{2}{3\sqrt[3]{2-1}}$

$= \dfrac{2}{3}$

2. Determine the equation of the tangent at $x = 2, y = 1$.

If $m_T = \dfrac{2}{3}$, $(x_1, y_1) = (2, 1)$.

$y - y_1 = m(x - x_1)$

$y - 1 = \dfrac{2}{3}(x - 2)$

$y - 1 = \dfrac{2}{3}x - \dfrac{4}{3}$

$y = \dfrac{2}{3}x - \dfrac{1}{3}$

or $2x - 3y = 1$

3. Determine the gradient at $x = 0$.

When $x = 0$:

$\dfrac{dy}{dx} = \dfrac{2}{3\sqrt[3]{0-1}}$

$= -\dfrac{2}{3}$

4. Determine the equation of the tangent at $x = 0, y = 1$.

If $m_T = -\dfrac{2}{3}$, $(x_1, y_1) = (0, 1)$.

$y - y_1 = m(x - x_1)$

$y - 1 = -\dfrac{2}{3}(x - 0)$

$y - 1 = -\dfrac{2}{3}x$

$y = -\dfrac{2}{3}x + 1$

or $2x + 3y = 3$

# Exercise 3.2 The chain rule

learn**on**

**3.2 Exercise**   |   **3.2 Exam questions** **on**

| Simple familiar | Complex familiar | Complex unfamiliar |
|---|---|---|
| 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 | 13, 14, 15, 16 | 17, 18, 19, 20 |

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## Simple familiar

**1.** **WE1a** Differentiate each of the following functions.

**a.** $y = (5x - 4)^3$

**b.** $y = \sqrt{3x + 1}$

**c.** $y = \dfrac{1}{(2x + 3)^4}$

**d.** $y = \dfrac{1}{7 - 4x}$

**e.** $y = (5x + 3)^{-6}$

**f.** $y = (4 - 3x)^{\frac{4}{3}}$

**2.** **WE1b** Determine $\dfrac{dy}{dx}$ for each of the following functions.

**a.** $f(x) = (4 - 3x)^5$

**b.** $y = \sqrt{3x^2 - 4}$

**c.** $f(x) = (x^2 - 4x)^{\frac{1}{3}}$

**d.** $g(x) = (2x^3 + x)^{-2}$

**e.** $g(x) = \left(x - \dfrac{1}{x}\right)^6$

**f.** $y = (x^2 - 3x)^{-1}$

**3.** **WE2** Determine the derivative of each of the following.

**a.** $\sqrt{\log_e(3 - 2x)}$

**b.** $\dfrac{1}{2}\log_e\left(\sqrt{x - 2}\right)$

**c.** $y = \ln\left(x^3 + 3x^2\right)$

**d.** $y = \ln\left(e^{3x} + 6\right)$

**4.** **WE3** Use the chain rule to determine the derivatives of the following.

**a.** $y = \sin^2(x)$

**b.** $y = e^{\cos(3x)}$

**5.** If $y = \sin^3(x)$, determine the exact value of $\dfrac{dy}{dx}$ when $x = \dfrac{\pi}{3}$.

**6.** Determine the derivatives of the following functions.

**a.** $g(x) = 3(x^2 + 1)^{-1}$

**b.** $g(x) = e^{\cos(x)}$

**c.** $g(x) = \sqrt{(x + 1)^2 + 2}$

**d.** $g(x) = \dfrac{1}{\sin^2(x)}$

**e.** $f(x) = \sqrt{x^2 - 4x + 5}$

**7.** **WE4** For the function $y = \dfrac{1}{(2x - 1)^2}$, determine:

**a.** $\dfrac{dy}{dx}$

**b.** the equation of the tangent to the curve at the point where $x = 1$.

**8.** Let $f(x) = (x - 1)^3$ and $g(x) = e^x$.

**a.** Write the rule for $f(g(x))$.

**b.** Determine $h'(x)$ where $h(x) = f(g(x))$.

**c.** Determine the equation of the tangent of $y = h(x)$ at the origin.

**9.** **MC** If $y = e^{3\cos(5x)}$, then $\dfrac{dy}{dx}$ is:

**A.** $15\sin(5x)e^{3\cos(5x)}$

**B.** $-15\sin(5x)e^{3\cos(5x)}$

**C.** $e^{-15\sin(5x)}$

**D.** $-15\cos(5x)e^{-3\sin(5x)}$

**CHAPTER 3** Differentiation rules **137**

**10.** **MC** If $y = \sin^2(5x)$, then $\dfrac{dy}{dx}$ is:

**A.** $2\sin(5x)$       **B.** $-2\sin(5x)\cos(5x)$       **C.** $-10\sin(5x)\cos(5x)$       **D.** $10\sin(5x)\cos(5x)$

**11.** Simplify each of the following functions and use the chain rule to determine $g'(x)$.

**a.** $g(x) = \dfrac{\sqrt{6x-5}}{6x-5}$

**b.** $g(x) = \dfrac{(x^2+2)^3}{\sqrt{x^2+2}}$

**12.** For each of the following functions, use the chain rule to determine $f'(x)$.

**a.** $f(x) = 3\cos(x^2-1)$       **b.** $f(x) = 5e^{3x^2-1}$       **c.** $f(x) = \left(x^3 - \dfrac{2}{x^2}\right)^{-2}$

**d.** $f(x) = \dfrac{\sqrt{2-x}}{2-x}$       **e.** $f(x) = \cos^3(2x+1)$

### Complex familiar

**13.** If $f(x) = e^{\sin^2(x)}$, determine $f'\left(\dfrac{\pi}{4}\right)$.

**14.** Differentiate the following functions and hence determine the gradients at the given $x$-values.

**a.** $f(x) = (2-x)^{-2}$; determine $f'\left(\dfrac{1}{2}\right)$.       **b.** $f(x) = e^{2x^2}$; determine $f'(-1)$.

**c.** $f(x) = \sqrt[3]{(3x^2-2)^4}$; determine $f'(1)$.       **d.** $f(x) = (\cos(3x)-1)^5$; determine $f'\left(\dfrac{\pi}{2}\right)$.

**15.** If $f(x) = \sin^2(2x)$, determine the points where $f'(x) = 0$ for $x \in [0, \pi]$.

**16.** For the functions $f(x) = 2\sin(x)$ and $h(x) = e^x$:

    **a.** state the rule for:

        **i.** $m(x) = f(h(x))$       **ii.** $n(x) = h(f(x))$

    **b.** determine when $m'(x) = n'(x)$ over the interval $x \in [0, 3]$, correct to 3 decimal places.

### Complex unfamiliar

**17.** The function $g(x)$ is defined as $g(x) = f(h(x))$ where $f(x) = \sqrt[3]{x^2}$ and $h(x) = 2x-1$.
Determine the coordinates of the point of intersection of the tangents to the function $g(x)$ at the points $(1, 1)$ and $(0, 1)$.

**18.** The function $f$ is defined by $f\colon [0, \infty) \to \mathbb{R}$, $f(x) = (2\log_e(x))^2$, and the function $g$ is defined by $g\colon [0, \infty) \to \mathbb{R}$, $g(x) = 2\log_e(x)$.
The curves intersect at two points, $x = 1$ and $x = a$, $a > 1$. Determine the value of $a$ and show that $f'(a) = 2g'(a)$.

**19.** The function $f$ has a rule $f(x) = \sqrt{x^2-1}$ and the function $g$ has the rule $g(x) = x+3$.
If $h(x) = f(g(x))$, determine $h'(x)$, expressing your answer in the form $\dfrac{x+a}{\sqrt{(x+b)(x+c)}}$ where $a$, $b$, $c$ are integers.

**20.** The line perpendicular to the graph $y = g(f(x))$ where $f(x) = \dfrac{1}{x}$ and $g(x) = x - \dfrac{1}{x^2}$ is given by $y = -x + a$, where $a$ is a real constant. Calculate the possible value(s) of $a$.

**Fully worked solutions for this chapter are available online.**

# LESSON
## 3.3 The product rule

### 3.3.1 Differentiation using the product rule

There are many functions that have rules that are the product of two simpler functions, such as $x \sin(x)$ or $e^x (2x + 1)$. In order to differentiate such functions, we need to apply the **product rule**.

> **The product rule**
>
> If $\qquad\qquad y = uv$
>
> then $\qquad \dfrac{d}{dx}(uv) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

### 3.3.2 Proof of the product rule

Let $f(x) = u(x)v(x)$

so $f(x + h) = u(x + h)v(x + h)$.

$$\frac{f(x+h) - f(x)}{h} = \frac{u(x+h)v(x+h) - u(x)v(x)}{h}$$

Add and subtract $u(x)v(x + h)$, as it is expected that at some stage $v'(x)$ will appear somewhere in the rule.

$$\frac{f(x+h) - f(x)}{h} = \frac{u(x+h)\,v(x+h) - u(x)v(x+h) + u(x)v(x+h) - u(x)v(x)}{h}$$

$$= \frac{[u(x+h) - u(x)]\,v(x+h) + u(x)\,[v(x+h) - v(x)]}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{[u(x+h) - u(x)]\,v(x+h) + u(x)\,[v(x+h) - v(x)]}{h}$$

$$= \lim_{h \to 0} \left[ \frac{u(x+h) - u(x)}{h} \times v(x+h) \right] + \lim_{h \to 0} \left[ \frac{v(x+h) - v(x)}{h} \times u(x) \right]$$

$$= \lim_{h \to 0} \frac{u(x+h) - u(x)}{h} \times \lim_{h \to 0} v(x+h) + \lim_{h \to 0} \frac{v(x+h) - v(x)}{h} \times \lim_{h \to 0} u(x)$$

$$= u'(x)v(x) + v'(x)u(x)$$

$$= u(x)v'(x) + v(x)u'(x)$$

Using Leibnitz notation, if $y = uv$,

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

## WORKED EXAMPLE 5 Using the product rule

**Differentiate the following functions.**
**a.** $f(x) = x\sin(x)$
**b.** $y = e^{2x}(3x + 1)$

| THINK | WRITE |
|---|---|
| **a. 1.** Define $u$ and $v$ as functions of $x$. | **a.** $f(x) = x\sin(x)$ $u = x$ and $v = \sin(x)$ |
| **2.** Differentiate with respect to $x$. | $\frac{du}{dx} = 1$ and $\frac{dv}{dx} = \cos(x)$ |
| **3.** Apply the product rule and simplify. | $\frac{dy}{dx} = u \times \frac{dv}{dx} + v \times \frac{du}{dx}$ $\frac{dy}{dx} = x \times \cos(x) + \sin(x) \times 1$ $\frac{dy}{dx} = x\cos(x) + \sin(x)$ |
| **b. 1.** Define $u$ and $v$ as functions of $x$. | **b.** $y = e^{2x}(3x + 1)$ $u = e^{2x}$ and $v = (3x + 1)$ |
| **2.** Differentiate with respect to $x$. | $\frac{du}{dx} = 2e^{2x}$ and $\frac{dv}{dx} = 3$ |
| **3.** Apply the product rule and simplify. | $\frac{dy}{dx} = u \times \frac{dv}{dx} + v \times \frac{du}{dx}$ $\frac{dy}{dx} = e^{2x} \times 3 + (3x + 1) \times 2e^{2x}$ $\frac{dy}{dx} = 3e^{2x} + 6xe^{2x} + 2e^{2x}$ $\frac{dy}{dx} = e^{2x}(6x + 5)$ |

## WORKED EXAMPLE 6 Using the product rule

**The graph of $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 e^x$ is shown. Using calculus, determine the coordinates where $f'(x) = 0$.**
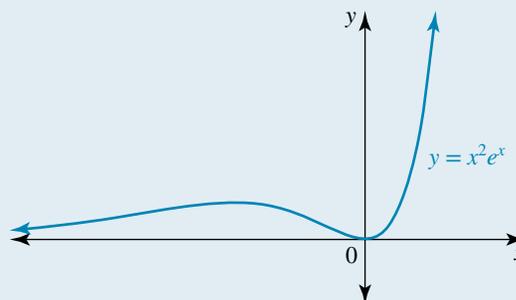


$y = x^2 e^x$

| THINK | WRITE |
|---|---|
| **1.** Define $u$ and $v$ as functions of $x$. | $f(x) = x^2 e^x$ <br> Let $u(x) = x^2$ and $v(x) = e^x$. |
| **2.** Differentiate $u$ and $v$ with respect to $x$. | $u'(x) = 2x$ <br> $v'(x) = e^x$ |
| **3.** Apply the product rule to determine $f'(x)$. | $f'(x) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$ <br> $= x^2 \times e^x + e^x \times 2x$ <br> $= x^2 e^x + 2xe^x$ |
| **4.** Solve $f'(x) = 0$. | $x^2 e^x + 2xe^x = 0$ <br> $e^x x(x+2) = 0$ <br> $e^x > 0$ for all values of $x$. <br> Either $x = 0$ or $x + 2 = 0$. <br> $\therefore x = 0,\ -2$ |
| **5.** Substitute the $x$-values to determine the corresponding $y$-values. | When $x = -2$, <br> $y = (-2)^2 e^{-2}$ <br> $= 4e^{-2}$ <br> When $x = 0$, <br> $y = (0)^2 e^0$ <br> $= 0$ |
| **6.** Write the answer. | The coordinates where the gradient is zero are $(0, 0)$ and $(-2, 4e^{-2})$. |

## WORKED EXAMPLE 7 Determining equations of tangents using the product rule

Given that $y = e^{2x}(x+1)^2$, determine $\dfrac{dy}{dx}$ and hence calculate the equation of the tangent to the curve at the point $(0, 1)$.

| THINK | WRITE |
|---|---|
| **1.** Define $u$ and $v$ as functions of $x$. | $y = e^{2x}(x+1)^2$ <br> Let $u = e^{2x}$ and $v = (x+1)^2$. |
| **2.** Differentiate $u$ and $v$ with respect to $x$. | $\dfrac{du}{dx} = 2e^{2x}$ $\quad$ $\dfrac{dv}{dx} = 2(x+1)$ |
| **3.** Apply the product rule to determine $\dfrac{dy}{dx}$ and simplify. | $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$ <br> $\dfrac{dy}{dx} = e^{2x} \times 2(x+1) + (x+1)^2 \times 2e^{2x}$ <br> $= 2e^{2x}(x+1) + 2e^{2x}(x+1)^2$ <br> $= 2e^{2x}(x+1)(1+x+1)$ <br> $= 2e^{2x}(x+1)(x+2)$ |

**4.** Evaluate $\dfrac{dy}{dx}$ when $x = 0$.

When $x = 0$, then
$$\dfrac{dy}{dx} = 2e^0(0+1)(0+2)$$
$$= 4$$

**5.** Determine the equation of the tangent.

If $m = 4$ and $(x_1, y_1) = (0, 1)$,
$$y - y_1 = m(x - x_1)$$
$$y - 1 = 4(x - 0)$$
$$y - 1 = 4x$$
$$y = 4x + 1$$

## Exercise 3.3 The product rule

**learn**`on`

| 3.3 Exercise | 3.3 Exam questions `on` |
|---|---|

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| Simple familiar | Complex familiar | Complex unfamiliar |
|---|---|---|
| 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 | 13, 14, 15, 16 | 17, 18, 19, 20 |

### Simple familiar

**1.** **WE5** For each of the following functions, determine the derivative function.

**a.** $f(x) = \sin(3x)\cos(3x)$     **b.** $f(x) = x^2 e^{3x}$     **c.** $f(x) = (x^2 + 3x - 5)e^{5x}$

**2.** Determine $\dfrac{dy}{dx}$ for the following functions.

**a.** $y = x^2(x+1)^5$     **b.** $y = x^3(2x-1)^4$     **c.** $y = (4x+1)^3(3x-2)^5$

**3.** Determine the derivative functions of the following.

**a.** $y = x\ln(x)$     **b.** $y = 2x^2\ln(3x)$     **c.** $y = (6x^2 + 9x)\log_e(2x+3)$

**4.** Differentiate the following.

**a.** $x^2 e^{5x}$     **b.** $x^{-2}(2x+1)^3$     **c.** $x\cos(x)$     **d.** $2\sqrt{x}(4-x)$

**5.** Differentiate the following.

**a.** $3x^{-2}e^{x^2}$     **b.** $e^{2x}\sqrt{4x^2-1}$     **c.** $x^2\sin^3(2x)$     **d.** $(x-1)^4(3-x)^{-2}$

**6.** **WE6** Given $f(x) = 2x^2(1-x)^3$, use calculus to determine the coordinates where $f'(x) = 0$.

**7.** Given the function $f(x) = (x+1)\sin(x)$, determine $f'(x)$ and hence determine the gradient of the function when $x = 0$.

**8.** **WE7** Given that $y = (x^2+1)e^{3x}$, determine $\dfrac{dy}{dx}$ and hence calculate the equation of the tangent to the curve at $x = 0$.

**9.** If $f(x) = x(x^2+x)^4$, determine $f'(1)$.

**10.** **MC** If $f(x) = (x-a)^3 g(x)$, the derivative of $f(x)$ is equal to:

    **A.** $3(x-a)g'(x)$                              **B.** $3(x-a)^2 g'(x)$

    **C.** $3g'(x)$                                   **D.** $3(x-a)^2 g(x) + (x-a)^3 g'(x)$

**11.** **MC** The derivative of $12p(1-p)^8$ with respect to $p$ is equal to:

    **A.** $96p(1-p)^8$     **B.** $-96p(1-p)^8$     **C.** $12(1-p)^7(1-9p)$     **D.** $12(1-p)^8(1-9p)$

**12.** **MC** Let $y = 2x^3 \sin(x)$. The derivative $\dfrac{dy}{dx}$ when $x = \dfrac{\pi}{2}$ is:

    **A.** $\dfrac{3\pi^2}{2}$             **B.** $4\pi^2$             **C.** $\dfrac{3\pi}{2}$             **D.** $\dfrac{\pi^2}{2}$

### Complex familiar

**13.** If $f(x) = 2x^4 \cos(2x)$, determine $f'\left(\dfrac{\pi}{2}\right)$.

**14.** If $f(x) = \sqrt{x}\,\sin^2(2x^2)$, determine $f'(\sqrt{\pi})$.

**15.** For the function with the rule $y = xe^x$, determine the equation of the line perpendicular to the tangent to the curve at the point where $x = 1$.

**16.** The graph of $f: \mathbb{R}^+ \to \mathbb{R}$, $f(x) = e^{-\frac{x}{2}} \sin(x)$ is shown.

    **a.** Calculate the values of $x$ when $f(x) = 0$ for $x \in [0,\ 3\pi]$.

    **b.** Use calculus to determine the values of $x$ when $f'(x) = 0$ for $x \in [0,\ 3\pi]$. Give your answers correct to 2 decimal places.



### Complex unfamiliar

**17.** The graph of $y = e^{-x^2}(1-x)$ is shown.

Determine the coordinates of the point of intersection of the tangent where the curve intersects the $x$-axis with the line perpendicular to the curve where the curve crosses the $y$-axis. Give your answer correct to 2 decimal places.



**18.** The graph of the function $\left( f: \mathbb{R} \to \mathbb{R},\ f(x) = 3x^3 e^{-2x} \right)$ is shown.

Determine the exact coordinates where $f'(x) = 0$ and the equation of the tangent to the curve at $x = 1$.

**19.** If $f(x) = g(x) \sin(2x)$ and $f'\left(\dfrac{\pi}{2}\right) = -3\pi$, calculate the constant $a$ if $g(x) = ax^2$. Verify your solution.

**20.** The gradient of the function $g(x)$ at the point $(2a,\ 6)$ is 3.
Another function, $f(x)$, is defined as $f(x) = (x - a)^2\, g(x)$.
Determine, with reasons for your decisions, the equation of the tangent to $f(x)$ at the point where $x = 2a$.

**Fully worked solutions for this chapter are available online.**

# LESSON
## 3.4 The quotient rule

### SYLLABUS LINKS

- Use the quotient rule to determine the derivative of exponential, logarithmic and trigonometric functions, expressing derivatives in simplest and factorised form.

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### 3.4.1 Differentiation using the quotient rule

When one function is divided by a second function, such as $f(x) = \dfrac{x}{x^2 + 1}$ or $y = \dfrac{e^x}{\cos(x)}$, we have the quotient of the two functions. To differentiate such functions, the quotient rule is applied.

> **The quotient rule**
>
> If
> $$y = \frac{u}{v}$$
>
> then
> $$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

### 3.4.2 Proof of the quotient rule

This rule can be proven as follows by using the product rule.

If $f(x) = \dfrac{u(x)}{v(x)}$, then $f(x) = u(x) \times [v(x)]^{-1}$.

$$f'(x) = u(x) \times -1 \times [v(x)]^{-2} \times v'(x) + [v(x)]^{-1} \times u'(x)$$

$$= -\frac{u(x)v'(x)}{[v(x)]^2} + \frac{u'(x)}{[v(x)]}$$

$$= \frac{u'(x)v(x)}{[v(x)]^2} - \frac{u(x)v'(x)}{[v(x)]^2}$$

$$= \frac{v(x)u'(x) - u(x)v'(x)}{[v(x)]^2}$$

In Leibnitz notation, the quotient rule states that if $y = \dfrac{u}{v}$, then $\dfrac{dy}{dx} = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$.

For example, we can differentiate $y = \dfrac{x}{x^2 - 1}$ as follows.

Let $u = x$ so that $\dfrac{du}{dx} = 1$ and let $v = x^2 - 1$ so that $\dfrac{dv}{dx} = 2x$.

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(x^2 - 1)(1) - x(2x)}{(x^2 - 1)^2}$$

$$= \frac{x^2 - 1 - 2x^2}{(x^2 - 1)^2}$$

$$= \frac{-x^2 - 1}{(x^2 - 1)^2}$$

$$= -\frac{x^2 + 1}{(x^2 - 1)^2}$$

Note that although the numerator has been factorised and simplified, it is more common not to expand the denominator.

Always check that the quotient rule is the best method to use to differentiate the function.
For example, $y = \dfrac{x - 2}{\sqrt{x - 2}}$ can be broken down to $y = \sqrt{x - 2}$; therefore, the chain rule should be used.

Also, $y = \dfrac{5x^2 - 2x}{\sqrt{x}}$ can be split into separate fractions, and each term can be differentiated using the basic differentiation rule. Before applying the quotient rule, always check if the function can be simplified first.

---

### WORKED EXAMPLE 8 Using the quotient rule to determine the derivative of a composite function 1

**Determine the derivatives of the following functions.**

**a.** $f(x) = \dfrac{x}{x^2 + 1}$

**b.** $y = \dfrac{e^x}{\cos(x)}$

| THINK | WRITE |
|---|---|
| **a. 1.** Define $u$ and $v$ as functions of $x$. | **a.** $f(x) = \dfrac{x}{x^2 + 1}$ <br> $u = x$ and $v = x^2 + 1$ |
| **2.** Differentiate $u$ and $v$ with respect to $x$. | $\dfrac{du}{dx} = 1$ and $\dfrac{dv}{dx} = 2x$ |
| **3.** Apply the quotient rule and simplify. | $\dfrac{dy}{dx} = \dfrac{v \times \frac{du}{dx} - u \times \frac{dv}{dx}}{v^2}$ <br><br> $\dfrac{dy}{dx} = \dfrac{(x^2 + 1) \times 1 - x \times 2x}{(x^2 + 1)^2}$ <br><br> $\dfrac{dy}{dx} = \dfrac{1 - x^2}{(x^2 + 1)^2}$ |

**b. 1.** Define $u$ and $v$ as functions of $x$.

**b.** $y = \dfrac{e^x}{\cos(x)}$

$u = e^x$ and $v = \cos(x)$

**2.** Differentiate $u$ and $v$ with respect to $x$.

$\dfrac{du}{dx} = e^x$ and $\dfrac{dv}{dx} = -\sin(x)$

**3.** Apply the quotient rule and simplify.

$\dfrac{dy}{dx} = \dfrac{v \times \frac{du}{dx} - u \times \frac{dv}{dx}}{v^2}$

$\dfrac{dy}{dx} = \dfrac{\cos(x) \times e^x - e^x \times (-\sin(x))}{(\cos(x))^2}$

**4.** Factorise the numerator.

$\dfrac{dy}{dx} = \dfrac{e^x(\cos(x) + \sin(x))}{\cos^2(x)}$

---

## WORKED EXAMPLE 9 Using the quotient rule with a function in $t$

**Determine the derivative of $y = \dfrac{\sin(2t)}{t^2}$ with respect to $t$.**

| THINK | WRITE |
|---|---|
| **1.** Define $u$ and $v$ as functions of $t$. | $y = \dfrac{\sin(2t)}{t^2}$<br><br>Let $u = \sin(2t)$ and $v = t^2$. |
| **2.** Differentiate $u$ and $v$ with respect to $t$. | $\dfrac{du}{dt} = 2\cos(2t)$<br><br>$\dfrac{dv}{dt} = 2t$ |
| **3.** Apply the quotient rule to determine $\dfrac{dy}{dt}$ and simplify. | $\dfrac{dy}{dt} = \dfrac{v\frac{du}{dt} - u\frac{dv}{dt}}{v^2}$<br><br>$= \dfrac{t^2 2\cos(2t) - \sin(2t) \times 2t}{\left(t^2\right)^2}$<br><br>$= \dfrac{2t(t\cos(2t) - \sin(2t))}{t^4}$<br><br>$= \dfrac{2(t\cos(2t) - \sin(2t))}{t^3}$ |

Determine the derivative of $f(x) = \dfrac{\cos(3x)}{2e^x - x}$ and hence determine the gradient at the point where $x = 0$.

| THINK | WRITE |
|---|---|
| 1. Define $u$ and $v$ as functions of $x$. | $f(x) = \dfrac{\cos(3x)}{2e^x - x}$ <br> Let $u(x) = \cos(3x)$ and $v(x) = 2e^x - x$. |
| 2. Differentiate $u$ and $v$ with respect to $x$. | $\dfrac{du}{dx} = -3\sin(3x)$ <br> $\dfrac{dv}{dx} = 2e^x - 1$ |
| 3. Apply the quotient rule to determine $\dfrac{dy}{dx}$ and simplify. | $f'(x) = \dfrac{v(x)\frac{du}{dx} - u(x)\frac{dv}{dx}}{v^2}$ <br> $= \dfrac{(2e^x - x) \times -3\sin(3x) - \cos(3x) \times (2e^x - 1)}{(2e^x - x)^2}$ <br> $= \dfrac{-3(2e^x - x)\sin(3x) - (2e^x - 1)\cos(3x)}{(2e^x - x)^2}$ |
| 4. Evaluate $f'(0)$. | $f'(0) = \dfrac{-3(2e^0 - 0)\sin(0) - (2e^0 - 1)\cos(0)}{(2e^0 - 0)^2}$ <br> $= \dfrac{0 - 1}{4}$ <br> $= -\dfrac{1}{4}$ |

## Exercise 3.4 The quotient rule

| 3.4 Exercise | 3.4 Exam questions on |
|---|---|

| Simple familiar | Complex familiar | Complex unfamiliar |
|---|---|---|
| 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 | 13, 14, 15, 16 | 17, 18, 19, 20 |

### Simple familiar

1. **WE8** If $y = \dfrac{x + 3}{x + 7}$ is expressed as $y = \dfrac{u}{v}$, determine:

   a. $u$ and $v$
   b. $\dfrac{du}{dx}$ and $\dfrac{dv}{dx}$
   c. $\dfrac{dy}{dx}$

2. If $f(x) = \dfrac{x^2 + 2x}{5 - x}$ is expressed as $f(x) = \dfrac{u}{v}$, determine:

   a. $u$ and $v$
   b. $\dfrac{du}{dx}$ and $\dfrac{dv}{dx}$
   c. $f'(x)$

3. Determine the derivative of each of the following.

   a. $y = \dfrac{2x}{x^2 - 4}$

   b. $y = \dfrac{x^2 + 7x + 6}{3x + 2}$

   c. $f(x) = \dfrac{4x - 7}{10 - 3x}$

4. **MC** If $h(x) = \dfrac{8 - 3x^2}{x}$, then $h'(x)$ equals:

   A. $\dfrac{8 - 9x^2}{x^2}$

   B. $\dfrac{-3x^2 + 8}{x^2}$

   C. $\dfrac{-3x^2 - 8}{x^2}$

   D. $\dfrac{-3x^2 + 8}{x}$

5. **WE9** Use the quotient rule to determine the derivatives of:

   a. $\dfrac{e^{2x}}{e^x + 1}$

   b. $\dfrac{\cos(3t)}{t^3}$

6. Determine the derivative of $\dfrac{x + 1}{x^2 - 1}$.

7. **WE10** If $y = \dfrac{\sin(x)}{e^{2x}}$, determine the gradient of the function at the point where $x = 0$.

8. If $y = \dfrac{5x}{x^2 + 4}$, calculate the gradient of the function at the point where $x = 1$.

9. Differentiate the following.

   a. $y = \dfrac{\sin^2(x^2)}{x}$

   b. $y = \dfrac{3x - 1}{2x^2 - 3}$

   c. $\dfrac{e^x}{\cos(2x + 1)}$

   d. $\dfrac{e^{-x}}{x - 1}$

10. Differentiate the following.

    a. $\dfrac{\sin(x)}{\sqrt{x}}$

    b. $f(x) = \dfrac{(5 - x)^2}{\sqrt{5 - x}}$

    c. $f(x) = \dfrac{x - 4x^2}{2\sqrt{x}}$

    d. $y = \dfrac{3\sqrt{x}}{x + 2}$

11. Determine the derivative of each of the following.

    a. $\dfrac{\log_e(x^2)}{2x - 1}$

    b. $\dfrac{2 \log_e(2x)}{e^{2x} + 1}$

12. Determine the equation of the tangent to the curve $y = \dfrac{e^x}{x^2 + 2}$ when $x = 0$.

---

**Complex familiar**

13. Calculate the gradient of the tangent to the curve with equation $y = \dfrac{2x}{(3x + 1)^{\frac{3}{2}}}$ at the point where $x = 1$.

14. Show that $\dfrac{d}{dx}\left(\dfrac{1 + \cos(x)}{1 - \cos(x)}\right) = -\dfrac{-2\sin(x)}{(\cos(x) - 1)^2}$.

15. Consider the curve defined by the rule $y = \dfrac{2x - 1}{3x^2 + 1}$.

    a. Determine the rule for the gradient.
    b. Determine the value(s) of $x$ for which the gradient is equal to 0.875. Give your answers correct to 4 decimal places.

**16.** Let $f\colon [-2,\, 2] \to \mathbb{R},\, f(x) = \dfrac{\sin(2x-3)}{e^x}$. The graph of this function is shown.
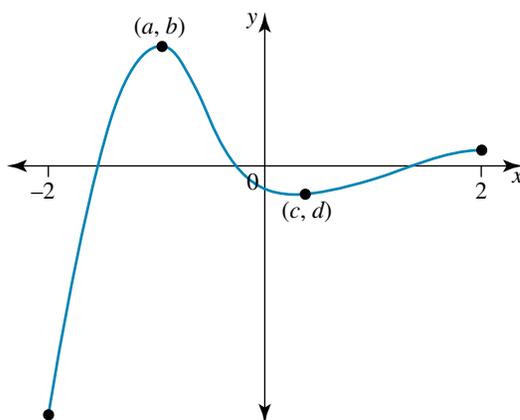


**a.** The stationary points occur at $(a, b)$ and $(c, d)$. Calculate the values of $a$, $b$, $c$ and $d$, giving your answers correct to 3 decimal places.

**b.** Calculate the gradient of the tangent to the curve at the point where $x = 1$, correct to 3 decimal places.

**Complex unfamiliar**

**17.** If $\dfrac{d}{dx}\left(\dfrac{e^{-3x}}{e^{2x}+1}\right) = \dfrac{e^{-x}\left(a+be^{-2x}\right)}{\left(e^{2x}+1\right)^2}$, determine the exact values of $a$ and $b$. Justify any decisions you made.

**18.** Consider the curve with the rule $y = \dfrac{x-5}{x^2+5x-14}$. Determine the exact length of the interval joining the axis intercepts of the tangent to the curve where $x = 1$.

**19.** Determine the exact area of the triangle between the axis intercepts and the tangent to the curve $y = \tan(x)$ where $x = \dfrac{\pi}{4}$.

**20.** Given that $f(x) = \dfrac{\sqrt{2x-1}}{\sqrt{2x+1}}$, calculate $m$ such that $f'(m) = \dfrac{2}{5\sqrt{15}}$. Explain your reasoning and the method used to determine the value of $m$.

**Fully worked solutions for this chapter are available online.**

# LESSON
## 3.5 Modelling and solving using the differentiation rules

## 3.5.1 Modelling and solving with derivatives

Differentiation can be applied to many different situations. These include:
- determining tangents to curves at specific points
- curve sketching
- rates of change, investigating how a change in one variable affects another related variable
- kinematics, the study of motion
- determining maximum or minimum values within given constraints — optimisation.

Previously, problems involving these situations have been restricted to polynomial functions and exponential, logarithmic and trigonometric functions. Now it is possible to solve problems involving combinations of these functions using the chain rule, product rule and quotient rule.

This section introduces and recalls some of these concepts. Chapter 4 covers these situations in more detail, with the introduction of the second derivative.

In problem solving, the various rules of differentiation may need to be used first before the problem can be solved. In some cases, the domain of the function may be restricted to meet the given constraints of the problem or situation.

## 3.5.2 Tangents and curve sketching

### Tangents

The derivative of a function gives the gradient of the tangent to the curve at any point.
- If $\dfrac{dy}{dx} > 0$, the function is increasing.
- If $\dfrac{dy}{dx} < 0,$ the function is decreasing.
- If $\dfrac{dy}{dx} = 0$, the function has a stationary point.

### Stationary points

**Stationary points**

**There are three types of stationary points:**
- **local minimum turning point**
- **local maximum turning point**
- **stationary point of inflection**

**For a curve $y = f(x)$, at a stationary point, $f'(x) = 0.$**

The word 'local' means the point is a maximum or minimum turning point in a particular locality or neighbourhood.

The nature of the stationary point is determined by examining the slope of the tangent immediately before and after the stationary point. This is summarised in the following table.

| | Minimum turning point | Maximum turning point | Stationary point of inflection |
|---|---|---|---|
| Stationary point |  |  |   or |
| Slope of tangent |  |  |   or |

---
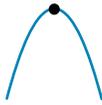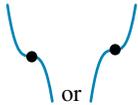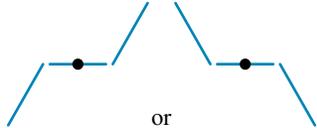
## WORKED EXAMPLE 11 Determining the tangent to a curve

Given that $y = e^{2x}(x+1)^2$, evaluate $\dfrac{dy}{dx}$ and hence determine the equation of the tangent to the curve at the point $(0, 1)$.

| THINK | WRITE |
|---|---|
| 1. Define $u$ and $v$ as functions of $x$. | $y = e^{2x}(x+1)^2$ <br> Let $u = e^{2x}$ and $v = (x+1)^2$. |
| 2. Differentiate $u$ and $v$ with respect to $x$. | $\dfrac{du}{dx} = 2e^{2x}$ <br> $\dfrac{dv}{dx} = 2(x+1)$ |
| 3. Apply the product rule to determine $\dfrac{dy}{dx}$ and simplify. | $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$ <br> $\dfrac{dy}{dx} = e^{2x} \times 2(x+1) + (x+1)^2 \times 2e^{2x}$ <br> $\quad = 2e^{2x}(x+1) + 2e^{2x}(x+1)^2$ <br> $\quad = 2e^{2x}(x+1)(1+x+1)$ <br> $\quad = 2e^{2x}(x+1)(x+2)$ |
| 4. Evaluate $\dfrac{dy}{dx}$ when $x = 0$. | When $x = 0$, then $\dfrac{dy}{dx} = 2e^0(0+1)(0+2)$ <br> $\quad = 4$ |
| 5. Determine the equation of the tangent. | If $m = 4$ and $(x_1, y_1) = (0, 1)$, $y - y_1 = m(x + x_1)$ <br> $y - 1 = 4(x - 0)$ <br> $y - 1 = 4x$ <br> $y = 4x + 1$ |

## Curve sketching

To sketch the graph of any function $y = f(x)$, apply the following steps:

- Obtain the $y$-intercept by evaluating $f(0)$.
- Obtain the $x$-intercepts by solving, if possible, $f(x) = 0$.
- Differentiate $y = f(x)$ and solve $f'(x) = 0$ to determine the stationary points.
- Identify the nature of any stationary point and determine its coordinates.
- Calculate the coordinates of the end points of the domain where appropriate.
- Consider the values of $f(x)$ as $x \to \pm\infty$.
- Determine the equations of any asymptotes.
- Identify any other key features of the graph where appropriate.

---

### WORKED EXAMPLE 12 Sketching curves

Consider the function $f(x) = e^x(x-2)^3$.
a. **Calculate $f'(x)$ and hence determine the coordinates of the stationary points.**
b. **By investigating the sign of $f'(x)$, state the nature of these stationary points.**
c. **Investigate the values of $f(x)$ as $x \to \pm\infty$. State the equations of any asymptotes.**
d. **Calculate any axis intercepts.**
e. **Sketch the curve of $y = f(x)$, showing all important features.**
f. **State the domain and range of the function.**

| THINK | WRITE |
|---|---|
| a. 1. Define $u$ and $v$ as functions of $x$. | a. $f(x) = e^x(x-2)^3$   $u = e^x$ and $v = (x-2)^3$ |
| 2. Differentiate with respect to $x$. | $\dfrac{du}{dx} = e^x$ and $\dfrac{dv}{dx} = 3(x-2)^2$ |
| 3. Apply the product rule and simplify by factorising. | $\dfrac{dy}{dx} = u \times \dfrac{dv}{dx} + v \times \dfrac{du}{dx}$ <br> $\dfrac{dy}{dx} = e^x \times 3(x-2)^2 + (x-2)^3 \times e^x$ <br> $\dfrac{dy}{dx} = e^x(x-2)^2[3 + (x-2)]$ <br> $f'(x) = e^x(x-2)^2(x+1)$ |
| 4. Stationary points exist when $\dfrac{dy}{dx} = 0$. | $e^x(x-2)^2(x+1) = 0$ <br> $e^x = 0$ is undefined, so $x = 2$ or $x = -1$ |
| 5. Determine $y$-values for $x = 2$, $x = -1$. | The stationary points are $\left(-1, \dfrac{-27}{e}\right)$ and $(2, 0)$. |
| b. 1. Construct a table of values for $f'(x)$ for suitable values of $x$. | b. $f'(x) = e^x(x-2)^2(x+1)$ |

| $x$ | $-2$ | $-1$ | $0$ | $2$ | $3$ |
|---|---|---|---|---|---|
| $f'(x)$ | $-16e^{-2}$ | $0$ | $4$ | $0$ | $4e^3$ |
| | \ | — | / | — | / |

| | |
|---|---|
| b. 2. State the nature of the stationary points by considering the direction of the tangents. | $\left(-1, \dfrac{-27}{e}\right)$ is a minimum stationary point; $(2, 0)$ is a horizontal point of inflexion (or stationary point of inflexion) |

**c. 1.** Consider the behaviour of $f(x)$ as $x \to \infty$ (or as $x$ becomes very large).

**c.** $f(x) = e^x(x-2)^3$
$e^x \to \infty; \ (x-2)^3 \to \infty$
$\therefore$ as $x \to \infty$,
$f(x) \to \infty$

**2.** Consider the behaviour of $f(x)$ as $x \to -\infty$ (or as $x$ becomes very small).

$e^x \to 0; \ (x-2)^3 \to -\infty$
$\therefore$ as $x \to -\infty$, $f(x) \to 0$ from the negative side.

**3.** State equations of asymptotes.
*Note:* $y = 0$ is only an asymptote for small values of $x$.

$y = 0$ is an asymptote.

**d. 1.** For $x$-intercepts, $y = 0$.

**d.** $e^x(x-2)^3 = 0$
$\therefore x = 2$ or $(2, \ 0)$

**2.** For $y$-intercepts, $x = 0$.

$f(0) = e^0(-2)^3$

**3.** State the axis intercepts.

$\therefore y = -8$ or $(0, -8)$
The axis intercepts are $(2, 0)$ and $(0, -8)$.

**e. 1.** Draw axes and plot the axis intercepts and stationary points, noting their nature.

**e.**
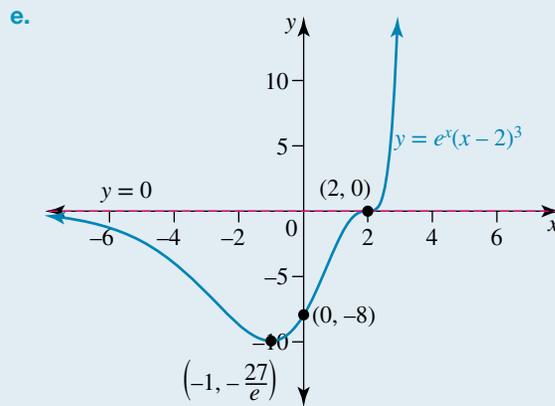
**2.** Remember the $x$-axis is an asymptote on the left, as $x \to -\infty$.



**f. 1.** State the domain.

**f.** $f(x) = e^x(x-2)^3$
$f(x)$ is defined for all values of $x$.
The domain is $x \in \mathbb{R}$.

**2.** State the range.

The minimum $y$-value is $\dfrac{-27}{e}$.

The range is $y \geq \dfrac{-27}{e}$ or $y \in \left[ \dfrac{-27}{e}, \infty \right)$.

---

**TI | THINK**   **WRITE**

**a. 1.** On a Graphs page complete the entry line
$f1(x) = e^x(x-2)^3$
and press ENTER.
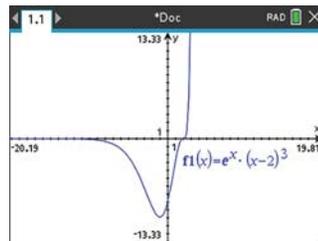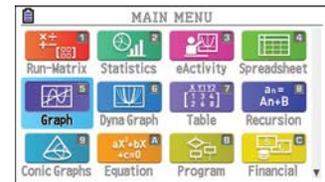


**CASIO | THINK**   **WRITE**

**a. 1.** On a Main Menu screen, select: Graph.

**2.** On a Calculator page, press MENU, then select:
6: Analyze Graph
2: Minimum.
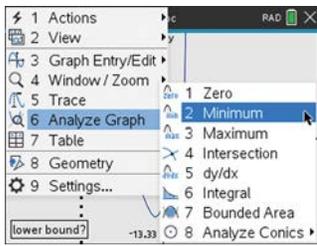


**3.** Inspect the graph and set lower and upper bounds to either side of the minimum value.
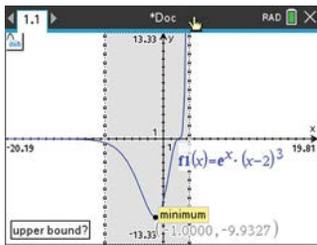


**4.** The answer appears on the screen.
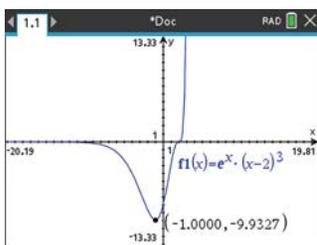


**2.** Complete the function entry line in the Y1 tab as:
$e^x (x-2)^3$
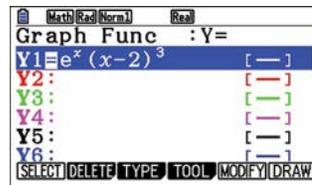


**3.** Press the DRAW button to sketch the graph.
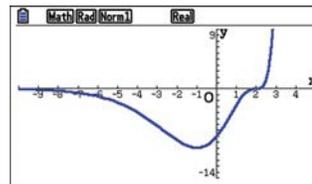


**4.** Determine the minimum value by selecting:
SHIFT F5 (G-Solv) MIN.
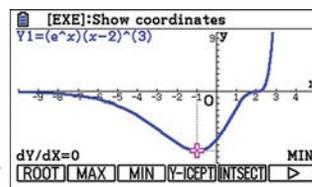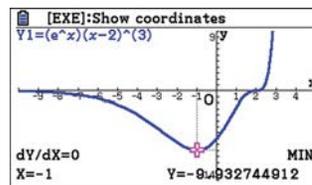


**5.** The answer appears on the screen.



**d. 1.** On a Calculator page, press MENU, then select:
6: Analyze Graph
2: Minimum.
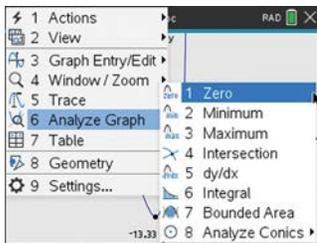


**2.** Inspect the graph and set lower and upper bounds to either side of the x-intercept.
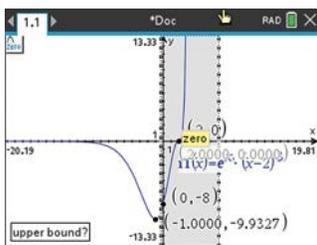


**3.** The answer appears on the screen.
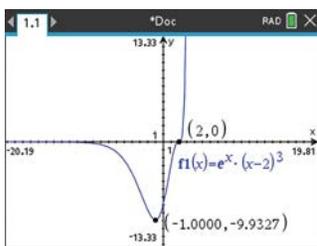


**d. 1.** Determine the x-intercept by selecting:
SHIFT F5 (G-Solv) ROOT.
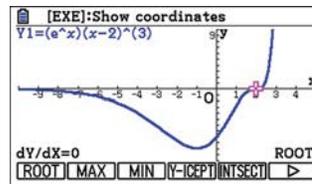


**2.** The answer appears on the screen.
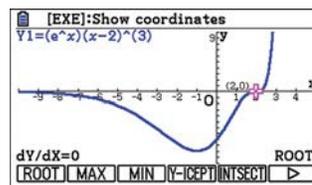


**3.** Determine the y-intercept by selecting:
SHIFT F5 (G-Solv) ROOT.
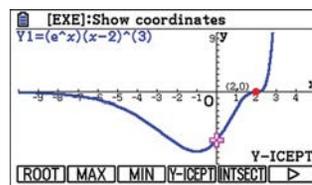
**4.** On a Calculator page, select: Menu 8: Geometry 1: Points & Lines 2: Point On.

**4.** The answer appears on the screen.
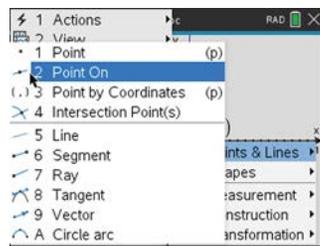
**5.** Move the cursor and select the curve representing $f1(x)\,e^x\,(x-2)^3$. Press the ESC (escape) button. Complete the entry line in the textbox as 0 for the $x$-value. Press ENTER to perform the calculation. The answer appears on the screen.
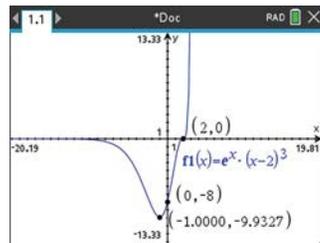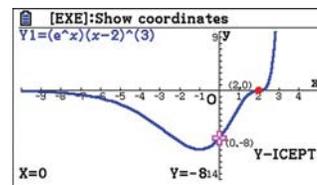
## 3.5.3 Rates of change and kinematics

The **instantaneous rate of change**, or simply the rate of change, of the function $y = f(x)$ is given by the derivative, $\dfrac{dy}{dx}$ or $f'(x)$.

For example:

- $\dfrac{dV}{dt}$ could be the rate of change of volume with respect to time, with possible units being litres/minute

- $\dfrac{dV}{dh}$ could be the rate of change of volume with respect to height, with possible units being litres/centimetre

To calculate these rates, $V$ would need to be expressed as a function of one independent variable, either time or height.

**Kinematics,** the study of the motion of a particle moving in a straight line, involves determining rates of change of displacement and velocity with respect to time.

- **Displacement**, $x$, gives the position of a particle by specifying both its distance and direction from a fixed point, the origin. Common units are centimetres, metres and kilometres.
  - If $x > 0$, the particle is to the right of the origin.
  - If $x < 0$, the particle is to the left of the origin.
  - If $x = 0$, the particle is at the origin.

- **Velocity**, $v$, measures the rate of change of displacement with respect to time, so $v = \dfrac{dx}{dt}$.

  Common units are centimetres per second, metres per second and kilometres per hour.
  - If $v > 0$, the particle is moving to the right.
  - If $v < 0$, the particle is moving to the left.
  - If $v = 0$, the particle is stationary, or instantaneously at rest.
- The term '**initially**' means at the start, or when $t = 0$.

The number of mosquitoes, $N$, around a dam on a certain night can
be modelled by the equation

$$N = \frac{400}{2t+1} + 100t + 1000$$

where $t$ equals hours after sunset.
Determine:
a. the initial number of mosquitoes
b. the rate of change at any time, $t$
c. the rate of change when $t = 4$ hours.

| THINK | WRITE |
|---|---|
| a. 1. Write the rule. | a. $N = \dfrac{400}{2t+1} + 100t + 1000$ |
| 2. Calculate $N$ when $t = 0$. | $N = \dfrac{400}{1} + 0 + 1000$ <br> $N = 1400$ |
| 3. Answer the question. | Initially, there were 1400 mosquitoes. |
| b. 1. Differentiate $N$ with respect to $t$. | b. $N = 400(2t+1)^{-1} + 100t + 1000$ <br> $\dfrac{dN}{dt} = 400 \times -1(2t+1)^{-2} \times 2 + 100$ |
| 2. Simplify. | $\dfrac{dN}{dt} = \dfrac{-800}{(2t+1)^2} + 100$ |
| c. 1. Calculate $\dfrac{dN}{dt}$ when $t = 4$. | c. $\dfrac{dN}{dt} = \dfrac{-800}{(2 \times 4 + 1)^2} + 100$ <br> $= \dfrac{-800}{(9)^2} + 100$ <br> $= \dfrac{7300}{81}$ <br> $\approx 90.1$ |
| 2. Answer the question. | After 4 h, the rate of change is approximately 90.1 mosquitoes per hour. |

## WORKED EXAMPLE 14 Solving problems involving kinematics

The displacement, $x$ metres, of a particle after $t$ seconds is given by the equation $x = 4\sin(2t) + 3$.
a. Derive an expression for the velocity, $v$ m/s, of the particle.
b. Determine the time at which the particle is first at rest and its position at this time.

**THINK**

a. 1. State the displacement function.

    2. Differentiate with respect to $t$.

    3. State the expression for velocity.

b. 1. At rest, $v = 0$.

    2. Solve for $t$.

    3. The first time the particle is at rest is the lowest value of $t$.

    4. Substitute to determine $x$ when $t = \dfrac{\pi}{4}$.

    5. Answer the question.

**WRITE**

a. $x = 4\sin(2t) + 3$

$v = \dfrac{dx}{dt}$

$v = 4\cos(2t) \times 2$

$v = 8\cos(2t)$

b. $8\cos(2t) = 0$

$\cos(2t) = 0$

$2t = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dots$

$t = \dfrac{\pi}{4}, \dfrac{2\pi}{4}, \dots$

$t = \dfrac{\pi}{4}$ seconds

$x = 4\sin\left(2 \times \dfrac{\pi}{4}\right) + 3$

$= 4\sin\left(\dfrac{\pi}{2}\right) + 3$

$= 4 + 3$

$= 7$

The particle is first at rest after $\dfrac{\pi}{4}$ seconds and it is 7 metres to the right of the origin.

## 3.5.4 Maximum and minimum problems

In many practical situations, it is necessary to determine the maximum or minimum value of the function that describes it. For example, if you were running your own business, you would always want to minimise the production costs while maximising the profits.

To solve optimisation, or maximum and minimum, problems, apply the following steps.
- Draw diagrams when necessary.
- Determine the variables and any connection between them.
- State or determine the function to be optimised.
- Differentiate the function and determine the stationary points and their nature.
- Reject any unrealistic solution.
- Answer the question.

## WORKED EXAMPLE 15 Solving problems involving maximum profit

The profit, $P$, per item that a store makes by selling $n$ items of a certain type each day is
$P = 40\sqrt{n + 25} - 200 - 2n$.
a. Determine the number of items that need to be sold to maximise the profit on each item.
b. Calculate:
  i. the maximum profit per item
  ii. the total profit per day made by selling this number of items.

**THINK**

**a. 1.** Rewrite with powers.

**2.** Differentiate with respect to $n$.

**3.** Simplify.

**4.** Solve $\dfrac{dP}{dn} = 0$ for $n$.

**5.** Draw a sign diagram to justify your answer.

**6.** State the answer.

**b. i. 1.** Calculate $P(75)$.

**2.** State the answer.

**ii. 1.** Calculate the total profit for selling 75 items per day.

**2.** State the answer.

**WRITE**

**a.** $P = 40\sqrt{n + 25} - 200 - 2n$

$P = 40(n + 25)^{\frac{1}{2}} - 200 - 2n$

$\dfrac{dP}{dn} = 40 \times \dfrac{1}{2} \times (n + 25)^{\frac{-1}{2}} \times 1 - 2$

$\dfrac{dP}{dn} = \dfrac{20}{\sqrt{n + 25}} - 2$

$\dfrac{20}{\sqrt{n + 25}} - 2 = 0$

$\dfrac{20}{\sqrt{n + 25}} = 2$

$20 = 2\sqrt{n + 25}$

$\sqrt{n + 25} = 10$

$n + 25 = 100$

$n = 75$

| $n$ | 70 | 75 | 80 |
|---|---|---|---|
| $\dfrac{dp}{dn}$ | $\approx 0.052$ | 0 | $\approx -0.048$ |
| **Slope** | / | — | \ |

The maximum profit per item is obtained when 75 items are sold each day.

**b.** $P = 40\sqrt{n + 25} - 200 - 2n$
$P = 40\sqrt{75 + 25} - 200 - 2 \times 75$
$P = 40 \times 10 - 200 - 150$
$P = 50$

The maximum profit per item is $50.

Total profit $= \$50 \times 75$

The maximum total profit per day for selling 75 items is $3750.

## WORKED EXAMPLE 16 Solving problems involving minimimum distance

Determine the minimum distance from the curve $y = 2x^2$ to the point $(4, 0)$, correct to 2 decimal places. You do not need to justify your answer.
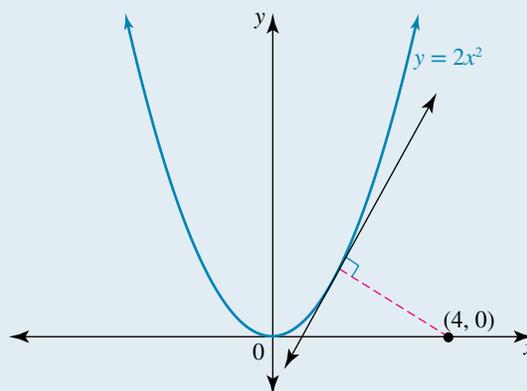


**THINK**

1. Let $P$ be the point on the curve such that the distance from $P$ to the point $(4, 0)$ is a minimum.

2. Write the formula for the distance between the two points.

3. Express the distance between the two points as a function of $x$ only.

4. Differentiate $d(x)$ using the chain rule.

5. Solve $d'(x) = 0$ using technology.

6. Evaluate $d(0.741)$.

7. Write the answer.

**WRITE**

1. $P = (x, y)$

2. $d(x) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

   $= \sqrt{(x - 4)^2 + (y - 0)^2}$

   $= \sqrt{(x - 4)^2 + y^2}$

3. $y = 2x^2$

   $\therefore d(x) = \sqrt{(x - 4)^2 + (2x^2)^2}$

   $= (x^2 - 8x + 16 + 4x^4)^{\frac{1}{2}}$

4. $d'(x) = \frac{1}{2} \times (4x^4 + x^2 - 8x + 16)^{-\frac{1}{2}} \times (16x^3 + 2x - 8)$

   $= \frac{16x^3 + 2x - 8}{2\sqrt{4x^4 + x^2 - 8x + 16}}$

   $= \frac{8x^3 + x - 4}{\sqrt{4x^4 + x^2 - 8x + 16}}$

5. $0 = \frac{8x^3 + x - 4}{\sqrt{4x^4 + x^2 - 8x + 16}}$

   $0 = 8x^3 + x - 4$

   $x = 0.741$

6. $d(0.741) = \sqrt{4(0.741)^4 + (0.741)^2 - 8(0.741) + 16}$

   $= 3.439$

7. The minimum distance is $3.44$ units (to 2 decimal places).

A new window is to be made to allow more light into a room. The window will have the shape of a rectangle surmounted by a semicircle. The frame of the window will be made from aluminium measuring 336 cm.

a. Show that the area, $A$ cm$^2$, of the window is $A = 336x - \dfrac{1}{2}(4 + \pi)x^2$, where $x$ is the radius of the semicircle in centimetres.

b. Hence, determine the width of the window for which the area is greatest. Give your answer to the nearest centimetres.

c. Structural limitations mean that the width of the window should not exceed 84 cm. Determine what the dimensions of the window of maximum area should now be. Give your answer to the nearest centimetres.

| THINK | WRITE |
|---|---|
| a. 1. Draw a diagram to illustrate the window where radius of the semicircle is $x$ cm and the height of the rectangle is $h$ cm. | a.<br> |
| 2. Use the perimeter to form an expression connecting the two variables, $x$ and $h$. | $p = \pi x + 2x + 2h$<br>$336 = \pi x + 2x + 2h$ |
| 3. Express $h$ in terms of $x$. | $2h = 336 - \pi x - 2x$<br>$h = \dfrac{1}{2}(336 - \pi x - 2x)$ |
| 4. Express the area as a function of $x$ by substituting for $h$. | $A = 2xh + \dfrac{1}{2}\pi x^2$<br>$A = 2x \times \dfrac{1}{2}(336 - \pi x - 2x) + \dfrac{1}{2}\pi x^2$<br>$A = 336x - \pi x^2 - 2x^2 + \dfrac{1}{2}\pi x^2$<br>$A = 336x - 2x^2 - \dfrac{1}{2}\pi x^2$ |
| 5. Express the area in the required form. | $A = 336x - \dfrac{1}{2}(4 + \pi)x^2$ |

**b. 1.** Differentiate.

**b.** $\dfrac{dA}{dx} = 336 - (4 + \pi)x$

**2.** Determine the stationary point and its nature.

$336 - (4 + \pi)x = 0$

$$x = \dfrac{336}{4 + \pi}$$

$x \approx 47.05$

| $x$ | 40 | $\dfrac{336}{4 + \pi}$ | 50 |
|---|---|---|---|
| $\dfrac{dA}{dx}$ | positive | 0 | negative |
| **slope** | / | — | \ |

The maximum area occurs when $x = \dfrac{336}{4 + \pi}$ cm or

$x \approx 47$ cm (to the nearest cm).

**c. 1.** State the restrictions for the window.

**c.** The width of the window is not to exceed 84 cm.

$2x \le 84$

$x \le 42$

**2.** State the restricted domain for $A(x)$.

Restricted domain of area: $x \in (0, 42]$

**3.** Sketch the graph of $A(x)$.
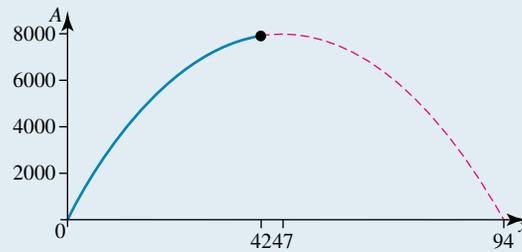


**4.** Determine, for the restricted domain, the value of $x$ for the greatest area.

The maximum area occurs when $x = 42$ cm.

**5.** Calculate the height, $h$.

$h = \dfrac{1}{2}(336 - \pi \times 42 - 2 \times 42)$

$h = 60.0266$

**6.** Calculate the dimensions.

Width $= 2x = 84$  Total height $= h + x = 60 + 42 = 102$

**7.** State the answer.

With the restrictions, the area of the window will be greatest if the width is 84 cm and the total height is 102 cm.

# Exercise 3.5 Modelling and solving using the differentiation rules

| 3.5 Exercise | 3.5 Exam questions **on** |
|---|---|

| Simple familiar | Complex familiar | Complex unfamiliar |
|---|---|---|
| 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 | 13, 14, 15, 16 | 17, 18, 19, 20 |

### Simple familiar

1. **WE11** For the function $y = \sqrt{3x^2 + 2x}$, determine:
   a. $\dfrac{dy}{dx}$
   b. the equation of the tangent to the curve at $x = 2$.

2. Use the chain rule to determine the derivative of $y = \dfrac{1}{(2x-1)^2}$ and hence determine the equation of the tangent to the curve at the point where $x = 1$.

3. a. Evaluate $f'(-1)$ if $f(x) = \dfrac{3}{\sqrt{5-4x}}$.
   b. Hence, determine the equation of the tangent to the curve $y = f(x)$ at the point where $x = -1$.

4. For the function with the rule $y = xe^x$, determine the equations of the tangent and the line perpendicular to the curve at the point where $x = 1$.

5. **WE12** Consider the function $f(x) = \dfrac{x}{x^2 + 1}$.
   a. Calculate $f'(x)$ and hence determine the coordinates of the stationary points.
   b. By investigating the sign of $f'(x)$, state the nature of these stationary points.
   c. Investigate the values of $f(x)$ as $x \to \pm \infty$. State the equations of any asymptotes.
   d. Calculate any axis intercepts.
   e. Sketch the curve of $y = f(x)$, showing all important features.
   f. State the domain and range of the function.

6. Consider the function $f(x) = \ln(x^2 + 1)$.
   a. Calculate $f'(x)$ and hence determine the coordinates of the stationary point.
   b. By investigating the sign of $f'(x)$, state the nature of the stationary point.
   c. Investigate the values of $f(x)$ as $x = \pm 1, \pm 2, \pm 3$. Explain why, in this logarithmic function, $x$-values can be negative.
   d. Sketch the curve of $y = f(x)$, showing all important features.
   e. State the domain and range of the function.

7. **WE13** A colony of viruses can be modelled by the rule

$$N(t) = \frac{2t}{(t+0.5)^2} + 0.5$$

where $N$ hundred thousand is the number of viruses on a nutrient plate $t$ hours after they started multiplying.
   a. Determine how many viruses were present initially.
   b. Determine $N'(t)$.
   c. Determine the maximum number of viruses and when this maximum will occur.
   d. Calculate the rate at which the virus numbers were changing after 10 hours.

**8.** **WE14** The displacement, $x$ metres, of a particle after $t$ seconds is given by the equation $x = 2\cos(4t) - 5$.

   **a.** Derive an expression for the velocity, $v$ m/s, of the particle.
   **b.** The particle is initially at rest. Determine the next time the particle is at rest and its position at this time.

**9.** The displacement, $x$ metres from the origin, of a particle moving in a straight line after $t$ hours is given by the equation $x(t) = 6 - 4\sin\left(\dfrac{\pi}{6}t\right)$ for $0 \le t \le 24$.

   **a.** State the period and amplitude for the function.
   **b.** Determine the initial position of the particle.
   **c.** Derive an expression for velocity, $v$ m/h.
   **d.** Determine the position of the particle when it is first at rest.
   **e.** Sketch the function, $x(t)$. What observations can you make from the graph?

**10.** **WE15** The profit per item, $P$, that a store makes by selling $n$ items of a certain type each day is given by $P = 80\sqrt{n+8} - 15 - 5n$.

   **a.** Determine the number of items that need to be sold to maximise the profit on each item.
   **b.** Calculate:

      **i.** the maximum profit per item
      **ii.** the total profit per day made by selling this number of items.

**11.** **WE16** Determine the minimum distance from the line $y = 2\sqrt{x}$ to the point $(5, 0)$.

**12.** **WE17** A rectangle with its base on the $x$-axis is inscribed in the semicircle $y = \sqrt{4 - x^2}$.

   **a.** Show that the area, $A$, of the rectangle is $A = 2x\sqrt{4 - x^2}$.
   **b.** Hence, determine the dimensions of the largest rectangle that can be inscribed in this semicircle.
   **c.** State the maximum area of the rectangle.
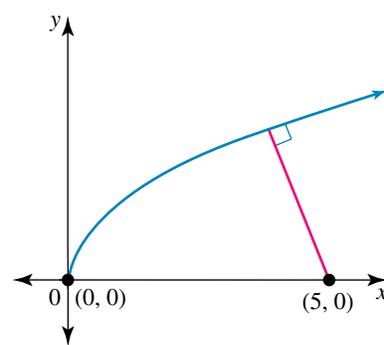


### Complex familiar

**13.** A population of butterflies in an enclosure at a zoo is modelled by

$$N = 220 - \frac{150}{t+1}, \ t \ge 0$$

where $N$ is the number of butterflies $t$ years after observations of the butterflies commenced.

   **a.** How long did it take for the butterfly population to reach 190 butterflies, and at what rate was the population growing at that time?
   **b.** Determine the time at which the growth rate was 12 butterflies per year. Give your answer correct to 2 decimal places.
   **c.** Determine the growth rate after 10 years. Give your answer correct to 2 decimal places.
   **d.** Sketch the graphs of population versus time and rate of growth versus time, and explain what happens to each as $t \to \infty$.

**14.** The length of a snake, $L$ cm, at any time $t$ weeks after it is born is modelled as:

$L = 12 + 6t + 2\sin\dfrac{\pi t}{4}$, $0 \leq t \leq 20$.

Calculate:

**a.** the length at:

   **i.** birth

   **ii.** 20 weeks

**b.** $R$, the rate of growth, at any time, $t$

**c.** the maximum and minimum growth rate. Give your answers correct to 3 decimal places where necessary.

**15.** The population of cheetahs, $P$, in a national park in Africa since 1 January 2010 can be modelled as $N = 100te^{-\frac{t}{12}} + 500$ where $t$ is the number of years.

**a.** Determine when this model predicts that the maximum population will be reached.

**b.** Calculate the maximum population of cheetahs that will be reached.

**c.** Calculate how many cheetahs there will be on 1 January in:

   **i.** 2034.

   **ii.** 2094.

**16.** The graph of the function $f: \mathbb{R}^+ \to \mathbb{R}$, $f(x) = \dfrac{1}{10x} + \log_e(x)$ is shown.
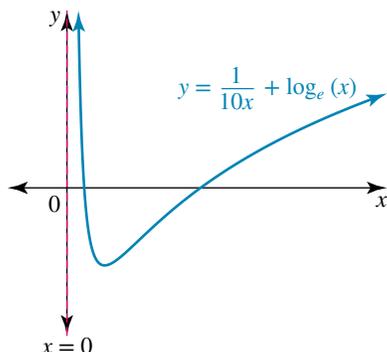


Use calculus to determine the coordinates of the minimum turning point.

**Complex unfamiliar**

**17.** The amount of money in a savings account $t$ years after the account was opened on 1 January 2025 is given by the equation

$$A(t) = 1000 - 12te^{\frac{4-t^3}{8}} \text{ for } t \in [0, 6].$$

Determine the least amount of money in the account and when this amount was reached. Give the year and month.

**18.** Determine the minimum distance from the point $(1, 1)$ to the straight line $y = x - 4$.
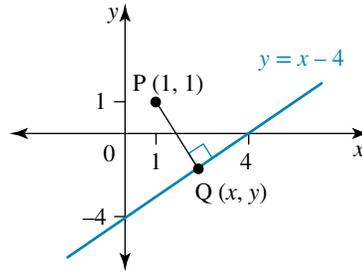


**19.** The owner of an apartment wants to create a stained glass feature in the shape of a rectangle surmounted by an isosceles triangle of height equal to half its base. This will be adjacent to a door opening on to a balcony. The owner has 150 cm of plastic edging to place around the perimeter of the figure, and wants to determine the dimensions of the figure with the greatest area.

Determine the width and the height of the figure for which the area is greatest. Give your answers correct to 1 decimal place.
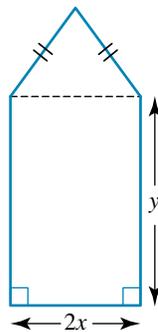


**20.** A country town has decided to construct a new road. The $x$-axis is also the position of the railway line that connects Sydney with Brisbane. The road can be approximated by the equation $y = (4x^2 - 5x)e^x$.
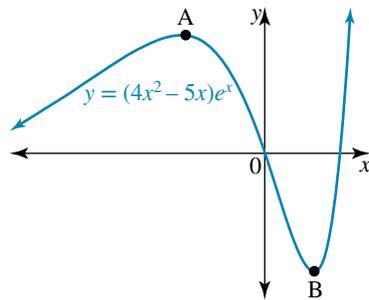


The country town has two general stores, located at A and B. Determine which store is closest to the railway line. Justify your decision.

**Fully worked solutions for this chapter are available online.**

# LESSON
## 3.6 Review

### 3.6.1 Summary

Hey students! Now that it's time to revise this chapter, go online to:

- Access the chapter summary
- Review your results
- Practise exam questions

**Find all this and MORE in jacPLUS**

### 3.6 Exercise

| 3.6 Exercise | 3.6 QCAA questions on |
|---|---|

These questions are even better in jacPLUS!
- Receive immediate feedback
- Access sample responses
- Track results and progress

**Find all this and MORE in jacPLUS**

| Simple familiar | Complex familiar | Complex unfamiliar |
|---|---|---|
| 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 | 13, 14, 15, 16 | 17, 18, 19, 20 |

**Simple familiar**

1. Determine $\dfrac{dy}{dx}$ for each of the following functions.

   **a.** $y = 3\left(2x^2 + 5x\right)^5$

   **b.** $y = \left(4x - 3x^2\right)^{-2}$

   **c.** $y = \left(x + \dfrac{1}{x}\right)^6$

   **d.** $y = 4(5 - 6x)^{-4}$

2. Use the product rule to differentiate each of the following.

   **a.** $y = x^2 \sin(x)$

   **b.** $y = 3x \sin(x)$

   **c.** $y = x^5 \cos(3x + 1)$

   **d.** $y = \sin(x) \cos(x)$

   **e.** $y = 8 \sin(5x) \, \log_e(5x)$

   **f.** $y = 5 \cos(2x) \sin(x)$

3. Differentiate each of the following, expressing your answer in simplest form.

   **a.** $y = \sin\left(\dfrac{4x}{3}\right) \cos(x)$

   **b.** $y = 2x^{-3} \sin(2x + 3)$

   **c.** $y = 4e^{-5x} \sin(2 - x)$

   **d.** $y = \dfrac{1}{\sqrt{x}} \cos(6x)$

   **e.** $y = \sin x \, \log_e(x)$

   **f.** $y = \pi x \cos(2\pi x)$

4. Determine the derivative of each of the following.

   **a.** $y = \dfrac{\sin(x)}{x}$

   **b.** $y = \dfrac{\sin(4x)}{\cos(2x)}$

   **c.** $y = \dfrac{\cos(x)}{x}$

   **d.** $y = \dfrac{\cos(x)}{e^x}$

   **e.** $y = \dfrac{\sin\left(\sqrt{x}\right)}{x}$

   **f.** $y = \dfrac{2 \cos(3 - 2x)}{x^2}$

5. **MC** The derivative of $f(x) = x^2 \sin(2x)$ is:

A. $f'(x) = 4x \cos(2x)$
B. $f'(x) = 2x \sin(2x) + x^2 \cos(2x)$
C. $f'(x) = 2x \sin(2x) + 2x^2 \cos(2x)$
D. $f'(x) = 2x \sin(x) + 2x^2 \cos(x)$

6. **MC** The derivative of $f(x) = \dfrac{\sin(4x)}{4x + 1}$ is:

A. $f'(x) = \dfrac{4(4x + 1) \cos(x) - 4 \sin(4x)}{(4x + 1)^2}$

B. $f'(x) = \dfrac{(4x - 1) \cos(4x) - 4 \sin(4x)}{(4x + 1)^2}$

C. $f'(x) = \dfrac{4(4x - 1) \cos(4x) - 4 \sin(4x)}{(4x + 1)^2}$

D. $f'(x) = \dfrac{4 \sin(4x) - 4(4x - 1) \cos(4x)}{(4x + 1)^2}$

7. Given that $y = (x^2 + 1)e^{3x}$, determine the equation of the tangent to the curve at $x = 0$.

8. A curve is represented by the equation $y = ax \cos(3x)$ where $a$ is a constant.

   a. If $\dfrac{dy}{dx} = -5$ when $x = \pi$, determine the value of $a$.

   b. Determine the equation of the line perpendicular to the curve at $x = \dfrac{\pi}{3}$.

9. The function $f: \mathbb{R} \to \mathbb{R}, f(x) = 6 \ln(x^2 - 4x + 8)$ has one stationary point.

   a. Use calculus to determine the coordinates of this stationary point.
   b. Determine the nature of this stationary point.
   c. Sketch the function, $f$.

10. Sketch the following functions by determining their stationary points and any axis intercepts. State the range of each function.

   a. $y = x^4 - 4x^3$

   b. $y = \dfrac{4}{x^2 + 1}$

11. The volume of water, $V$ litres, in a bath $t$ minutes after the plug is removed is given by $V = 0.4(8 - t)^3$, $0 \le t \le 8$.

   a. Determine the rate at which the water is leaving the bath after 3 minutes.
   b. Determine when the rate of water leaving the bath is the greatest.

12. A particle moves in a straight line so that its displacement a point, $O$, at any time, $t$, is $x = \sqrt{3t^2 + 4}$. Determine:

   a. the velocity as a function of time
   b. the velocity when $t = 2$.

**13.** Metal box guttering has to be formed on a common wall between two adjacent town houses. The cross-section of the box guttering is shown.

For the most efficient elimination of rain water, this box guttering needs to have a maximum cross-sectional area within the given dimensions.

**a.** Determine an expression for $h$, the height of the trapezium, in terms of the angle $x$ in radians, as shown.

**b.** Determine an expression for $b$, the base length of the trapezium, in terms of $x$.

**c.** Show that the cross-sectional area of the box guttering, $A$ cm$^2$, is given by $A = 200 \sin(x)(2 \cos(x) + 1)$.

**d.** Determine, correct to 3 decimal places, the value of $x$ that gives maximum cross-sectional area, and determine this maximum area correct to the nearest cm$^2$.
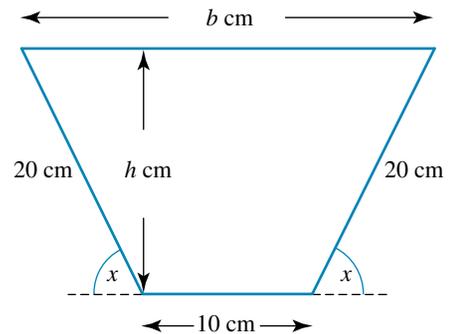


**14.** A bushwalker can walk at a rate of 5 km/h through clear land and 3 km/h through bushland. She has to get from point A to point B following a route indicated on the diagram. $\left( Note: \text{Time} = \dfrac{\text{distance}}{\text{speed}} \right)$
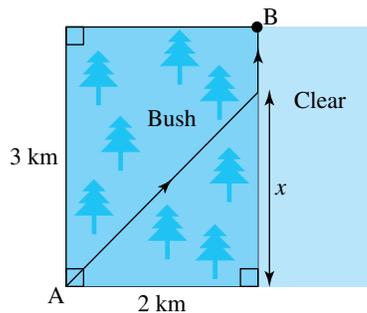


**a.** Determine the distance walked in terms of $x$:

**i.** through the bush

**ii.** through clear land.

**b.** If the total time taken is $T$ hours, express $T$ as a function of $x$.

**c.** Derive $\dfrac{dT}{dx}$.

**d.** Hence, determine the value of $x$ so that the route is covered in a minimum time.

**e.** Calculate the minimum time to complete this route. Give your answer in hours and minutes.

**15.** The line perpendicular to the graph $y = g(f(x))$, where $f(x) = \dfrac{1}{x}$ and $g(x) = x - \dfrac{1}{x^2}$, is given by $y = -x + a$, where $a$ is a real constant. Calculate the possible value(s) of $a$.

**16.** Consider the functions $f(x) = 2 \sin(x)$ and $h(x) = e^x$.

**a.** State the rule for:

**i.** $m(x) = f(h(x))$

**ii.** $n(x) = h(f(x))$

**b.** Determine when $m'(x) = n'(x)$ over the interval $x \in [0, 3]$, correct to 3 decimal places.

**17.** A veterinarian has administered a painkiller by injection to a sick horse.

The concentration of painkiller in the blood, $y$ mg/L, can be defined by the rule

$$y = \frac{3t}{\left(4 + t^2\right)}$$

where $t$ is the number of hours since the medication was administered.

Determine the maximum concentration of painkiller in the blood and the time at which the rate of change of painkiller in the blood is decreasing at a rate of 0.06 mg/L/h.

**18.** The graph of the function $f: \mathbb{R} \to \mathbb{R}$ $f(x) = x^4 e^{-3x}$ is shown. The derivative $f'(x)$ may be written in the form $f'(x) = e^{-3x}(mx^4 + nx^3)$, where $m$ and $n$ are real constants.
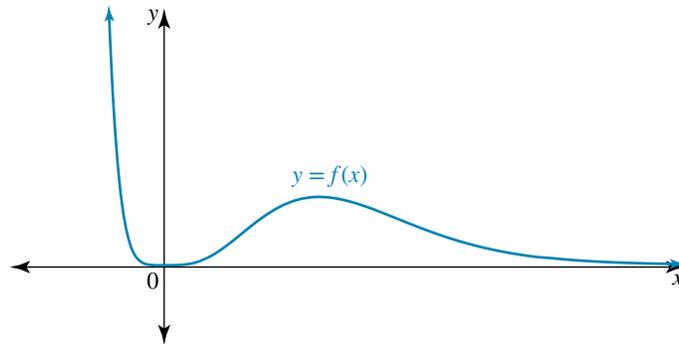


Determine the exact values of $m$ and $n$, and determine the coordinates of the stationary points.

**19.** Let $f: [-2, \ 2] \to \mathbb{R}$, $f(x) = \dfrac{\sin(2x - 3)}{e^x}$. The graph of this function is shown.
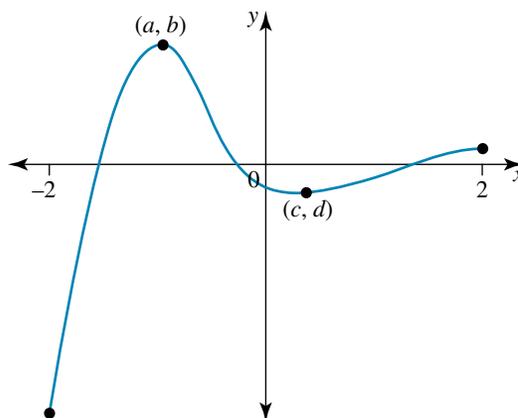


The stationary points occur at $(a, b)$ and $(c, d)$. Determine the values of $a$, $b$, $c$ and $d$, and the range of the function. Give your answers correct to 3 decimal places.

**20.** The population of rabbits on a particular island $t$ weeks after a virus is introduced is modelled by $P = 1200\,e^{0.1t}$, where $P$ is the number of rabbits.

After 15 weeks the virus has become ineffective, and the population of rabbits starts to increase again according to the model

$$P = P_0 + 10\,(t-15)\log_e(2t-29)$$

where $t$ is the number of weeks since the virus was first introduced.

Determine the population of rabbits, the rate of change of the population after 30 weeks, and how many weeks it takes the population to return to its original size.

Give your answers correct to the nearest integer.

## 3.6 Past QCAA exam questions

### Question 1 (1 mark)

*Source:* QCAA 2023 Mathematical Methods, Paper 2, Section 1, Q9; © State of Queensland (QCAA) 2024.

If $f(x) = e^{3x}(x+1)^2$ and $f'(x) = ae^{3x}(x+1)$, determine the expression for $a$.

**A.** $3x+5$

**B.** $3x+3$

**C.** $5x+5$

**D.** $5x+3$

### Question 2 (1 mark)

*Source:* QCAA 2020 Mathematical Methods, Paper 1, Section 1, Q5; © State of Queensland (QCAA) 2024.

The equation of the tangent to the curve $f(t) = te^t$ at $t = 1$ is

**A.** $y = et$

**B.** $y = 2et - e$

**C.** $y = et - e^2 + 1$

**D.** $y = 2et - 2e^2 + 1$

### Question 3 (3 marks)

*Source:* QCAA 2020 Mathematical Methods, Paper 1, Section 2, Q11 © State of Queensland (QCAA) 2024.

Determine the derivative of each of the following with respect to $x$.

**a.** $y = \dfrac{1}{\sin(x)}$ [1 mark]

**b.** $y = x^2 \times e^{-x}$ [2 marks]

Express your answer in factorised form.

**Question 4 (5 marks)**

*Source:* QCAA 2021 Mathematical Methods, Paper 1, Section 2, Q11; © State of Queensland (QCAA) 2024.

Determine the derivative with respect to $x$ of the following functions.

**a.** $y = (e^x + 1)^3$ [2 marks]

**b.** $y = \dfrac{\sin(x)}{x^2}$ (Give your answer in simplest form.) [3 marks]

**Question 5 (7 marks)**

*Source:* QCAA 2021 Mathematical Methods, Paper 1, Section 2, Q20; © State of Queensland (QCAA) 2024.

The population of rabbits $(P)$ on an island, in hundreds, is given by $P(t) = t^2 \ln(3t) + 6, t > 0$, where $t$ is time in years.

Determine the intervals of time when the population is increasing and the intervals when it is decreasing.

**Fully worked solutions for this chapter are available online.**

# Answers

## Chapter 3 Differentiation rules

### 3.2 The chain rule

#### 3.2 Exercise

**1. a.** $\dfrac{dy}{dx} = 15(5x - 4)^2$ **b.** $\dfrac{dy}{dx} = \dfrac{3}{2\sqrt{3x+1}}$

**c.** $\dfrac{dy}{dx} = \dfrac{-8}{(2x+3)^5}$ **d.** $\dfrac{dy}{dx} = \dfrac{4}{(7-4x)^2}$

**e.** $\dfrac{dy}{dx} = \dfrac{-30}{(5x+3)^7}$ **f.** $\dfrac{dy}{dx} = -4\sqrt[3]{4-3x}$

**2. a.** $\dfrac{dy}{dx} = -15(4-3x)^4$

**b.** $\dfrac{dy}{dx} = \dfrac{3x}{\sqrt{3x^2-4}}$

**c.** $\dfrac{dy}{dx} = \dfrac{2}{3}(x-2)(x^2-4x)^{-\frac{2}{3}}$

**d.** $\dfrac{dy}{dx} = -2(6x^2+1)(2x^3+x)^{-3}$

**e.** $\dfrac{dy}{dx} = 6\left(1 + \dfrac{1}{x^2}\right)\left(x - \dfrac{1}{x}\right)^5$

**f.** $\dfrac{dy}{dx} = -(2x-3)(x^2-3x)^{-2}$

**3. a.** $\dfrac{1}{(2x-3)\sqrt{\log_e(3-2x)}}$ **b.** $\dfrac{1}{4(x-2)}$

**c.** $\dfrac{3(x+2)}{x(x+3)}$ **d.** $\dfrac{3e^{3x}}{e^{3x}+6}$

**4. a.** $\dfrac{dy}{dx} = 2\ \cos(x)\sin(x)$ **b.** $\dfrac{dy}{dx} = -3\sin(3x)e^{\cos(3x)}$

**5.** $\dfrac{9}{8}$

**6. a.** $-\dfrac{6x}{(x^2+1)^2}$ **b.** $-\sin(x)e^{\cos(x)}$

**c.** $\dfrac{x+1}{\sqrt{x^2+2x+3}}$ **d.** $\dfrac{-2\cos(x)}{\sin^3(x)}$

**e.** $\dfrac{x-2}{\sqrt{x^2-4x+5}}$

**7. a.** $\dfrac{dy}{dx} = -\dfrac{4}{(2x-1)^3}$ **b.** $y = -4x + 5$

**8. a.** $f(g(x)) = (e^x - 1)^3$ **b.** $h'(x) = 3e^x\,(e^x - 1)^2$
**c.** $y = 0$

**9.** B

**10.** D

**11. a.** $g'(x) = \dfrac{-3}{(6x-5)^{\frac{3}{2}}}$ **b.** $g'(x) = 5x(x^2+2)^{\frac{3}{2}}$

**12. a.** $-6x\sin(x^2-1)$

**b.** $30xe^{3x^2-1}$

**c.** $-\dfrac{6x^5 + 8}{x^3\left(x^3 - \frac{2}{x^2}\right)^3}$

**d.** $\dfrac{1}{2(2-x)^{\frac{3}{2}}}$

**e.** $-6\sin(2x+1)\cos^2(2x+1)$

**13.** $\sqrt{e}$

**14. a.** $\dfrac{16}{27}$ **b.** $-4e^2$ **c.** $8$ **d.** $15$

**15.** $(0,\ 0),\ \left(\dfrac{\pi}{4},\ 1\right),\ \left(\dfrac{\pi}{2},\ 0\right),\ \left(\dfrac{3\pi}{4},\ 1\right),\ (\pi,\ 0)$

**16. a. i.** $f(h(x)) = 2\ \sin(e^x)$
**ii.** $h(f(x)) = e^{2\ \sin(x)}$

**b.** $1.555,\ 2.105,\ 2.372$

**17.** $\left(\dfrac{1}{2},\ \dfrac{1}{3}\right)$

**18.** $a = \sqrt{e}$
Sample responses can be found in the worked solutions in the online resources.

**19.** $\dfrac{x+3}{\sqrt{(x+2)(x+4)}}$

**20.** $a = -3$

### 3.3 The product rule

#### 3.3 Exercise

**1. a.** $f'(x) = 3\ \cos^2(3x) - 3\ \sin^2(3x)$
**b.** $f'(x) = 3x^2 e^{3x} + 2xe^{3x}$
**c.** $f'(x) = (5x^2 + 17x - 22)e^{5x}$

**2. a.** $\dfrac{dy}{dx} = x(x+1)^4(7x+2)$

**b.** $\dfrac{dy}{dx} = x^2(2x-1)^3(14x-3)$

**c.** $\dfrac{dy}{dx} = 3(4x+1)^2(3x-2)^4(32x-3)$

**3. a.** $\dfrac{dy}{dx} = 1 + \ln(x)$

**b.** $\dfrac{dy}{dx} = 2x + 4x\ln(3x)$

**c.** $\dfrac{dy}{dx} = 6x + 3(4x+3)\log_e(2x+3)$

**4. a.** $\dfrac{dy}{dx} = 5x^2 e^{5x} + 2xe^{5x}$

**b.** $\dfrac{dy}{dx} = \dfrac{2(x-1)(2x+1)^2}{x^3}$

**c.** $\dfrac{dy}{dx} = -x\sin(x) + \cos(x)$

**d.** $\dfrac{dy}{dx} = \dfrac{4-3x}{\sqrt{x}}$

**5. a.** $\dfrac{dy}{dx} = \dfrac{6e^{2x}\left(x^2-1\right)}{x^3}$

**b.** $\dfrac{dy}{dx} = \dfrac{2e^{2x}(4x^2+2x-1)}{\sqrt{4x^2-1}}$

**c.** $\dfrac{dy}{dx} = 2x \sin^2(2x)[3x \cos(2x) + \sin(2x)]$

**d.** $\dfrac{dy}{dx} = \dfrac{2(x-5)(x-1)^3}{(x-3)^3}$

**6.** $(0,\ 0)\,,(1,\ 0)\,,\left(\dfrac{2}{5},\ \dfrac{216}{3125}\right)$

**7.** $f'(x) = \sin(x) + (x+1)\ \cos(x)$; gradient $= 1$

**8.** $\dfrac{dy}{dx} = 2xe^{3x} + 3\left(x^2+1\right)e^{3x}$; $\dfrac{dy}{dx} = 3.$
The equation of the tangent is $= 3x + 1$.

**9.** 112

**10.** D

**11.** C

**12.** A

**13.** $f'\left(\dfrac{\pi}{2}\right) = -\pi^3$

**14.** 0

**15.** Perpendicular line $y = -\dfrac{x}{2e} + \left(\dfrac{1}{2e} + e\right)$

**16. a.** $x = 0,\ \pi,\ 2\pi,\ 3\pi$　　**b.** $x = 1.11,\ 4.25,\ 7.39$

**17.** $(-0.46,\ 0.54)$

**18.** $(0,\ 0)$ and $\left(\dfrac{3}{2},\ \dfrac{81}{8e^3}\right)$; $y = \dfrac{3}{e^2}x$

**19.** $\dfrac{6}{\pi}$

**20.** $y = 3a\,(a+4)\,x - 6a^2(a+3)$

## 3.4 The quotient rule

### 3.4 Exercise

**1. a.** $u = x + 3;\ v = x + 7$　　**b.** $\dfrac{du}{dx} = 1;\ \dfrac{dv}{dx} = 1$

**c.** $\dfrac{dy}{dx} = \dfrac{4}{(x+7)^2}$

**2. a.** $u = x^2 + 2x;\ v = 5 - x$　**b.** $\dfrac{du}{dx} = 2x+2;\ \dfrac{dv}{dx} = -1$

**c.** $f'(x) = \dfrac{10 + 10x - x^2}{(5-x)^2}$

**3. a.** $\dfrac{dy}{dx} = \dfrac{-2(x^2+4)}{(x^2-4)^2}$　**b.** $\dfrac{dy}{dx} = \dfrac{3x^2 + 4x - 4}{(3x+2)^2}$

**c.** $f'(x) = \dfrac{19}{(10-3x)^2}$

**4.** C

**5. a.** $\dfrac{dy}{dx} = \dfrac{e^{2x}(2 + e^x)}{(e^x + 1)^2}$

**b.** $\dfrac{dy}{dt} = \dfrac{-3(t\sin(3t) + \cos(3t))}{t^4}$

**6.** $\dfrac{-1}{(x-1)^2}$

**7.** 1

**8.** $\dfrac{3}{5}$

**9. a.** $\dfrac{dy}{dx} = \dfrac{4x^2 \sin(x^2)\cos(x^2) - \sin^2(x^2)}{x^2}$

**b.** $\dfrac{dy}{dx} = \dfrac{-6x^2 4x - 9}{(2x^2 - 3)^3}$

**c.** $\dfrac{dy}{dx} = \dfrac{e^x \cos(2x+1) + 2e^x \sin(2x+1)}{\cos^2(2x+1)}$

**d.** $\dfrac{dy}{dx} = -\dfrac{xe^{-x}}{(x-1)^2}$

**10. a.** $\dfrac{dy}{dx} = \dfrac{2x\cos(x) - \sin(x)}{2x\sqrt{x}}$

**b.** $\dfrac{dy}{dx} = \dfrac{-3}{2}\sqrt{5-x}$

**c.** $f'(x) = \dfrac{1}{4\sqrt{x}} - 3\sqrt{x}$

**d.** $\dfrac{dy}{dx} = \dfrac{6 - 3x}{2\sqrt{x}(x+2)^2}$

**11. a.** $\dfrac{2\,(2x - 1) - 2x\log_e\left(x^2\right)}{x(2x-1)^2}$

**b.** $\dfrac{2e^{2x} + 2 - 4e^{2x}x\log_e(2x)}{x\left(e^{2x}+1\right)^2}$

**12.** $y = \dfrac{1}{2}x + \dfrac{1}{2}$

**13.** $\dfrac{-1}{32}$

**14.** Sample responses can be found in the worked solutions in the online resources.

**15. a.** $\dfrac{-6x^2 + 6x + 2}{(3x^2 + 1)^2}$

**b.** $x = -0.1466, 0.5746$

**16. a.** $a = -1.088, b = 2.655, c = 0.483, d = -0.552$

**b.** 0.707

**17.** $a = -5, b = -3$

**18.** $\dfrac{3\sqrt{281}}{80}$ units

**19.** $\dfrac{(\pi - 2)^2}{16}$ units$^2$

**20.** $m = 2$

## 3.5 Modelling and solving using the differentiation rules

### 3.5 Exercise

**1. a.** $\dfrac{dy}{dx} = \dfrac{(3x + 1)}{\sqrt{3x^2 + 2x}}$

**b.** $y = \dfrac{7}{4}x + \dfrac{1}{2}$ or $7x - 4y + 2 = 0$

**2.** $y = -4x + 5$

**3. a.** $\dfrac{2}{9}$

**b.** $y = \dfrac{2}{9}x + \dfrac{11}{9}$ or $2x - 9y + 11 = 0$

**4.** Tangent: $y = 2e\,x - e$; normal: $y = -\dfrac{1}{2e}x + \left(\dfrac{1+2e^2}{2e}\right)$

**5. a.** $f'(x) = \dfrac{1-x^2}{(x^2+1)^2}$

Stationary points: $\left(-1, -\dfrac{1}{2}\right)$ and $\left(1, \dfrac{1}{2}\right)$

**b.** Local minimum stationary point at $\left(-1, -\dfrac{1}{2}\right)$

Local maximum stationary point at $\left(1, \dfrac{1}{2}\right)$

**c.** As $x \to \infty$, $\dfrac{x}{x^2+1} \to 0$ (positive side).

As $x \to -\infty$, $\dfrac{x}{x^2+1} \to 0$ (negative side). Equation of asymptote: $y = 0$

**d.** Intercept at $(0, 0)$

**e.**
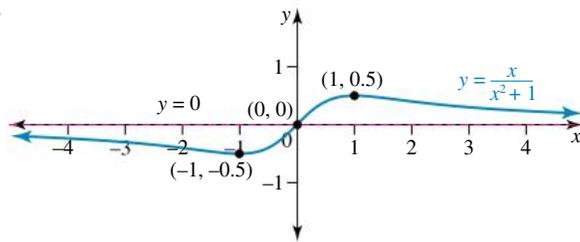


**f.** Domain: $x \in \mathbb{R}$; range $-\dfrac{1}{2} \le y \le \dfrac{1}{2}$

**6. a.** $f'(x) = \dfrac{2x}{x^2+1}$; stationary point: $(0, 0)$

**b.** Local minimum turning point

**c.** For all $x$-values, $x^2 \ge 0$ and $x^2 + 1 \ge 1$. Hence, this logarithmic function is defined and is greater than or equal to zero, even when $x$-values are negative.

**d.**
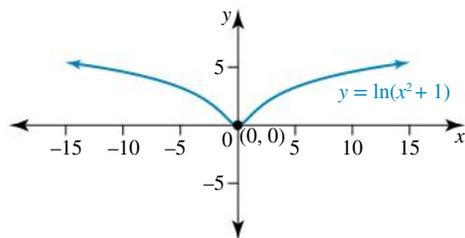


**e.** Domain: $x \in \mathbb{R}$; range: $y \ge 0$

**7. a.** 50 000 viruses

**b.** $\dfrac{0.5 - 2t^2}{(t+0.5)^4}$

**c.** 150 000 viruses after half an hour

**d.** $-1641$ viruses/hour

**8. a.** $v = -8\sin(4t)$      **b.** $\dfrac{\pi}{4}$ s; $-7$ m

**9. a.** 12 h; 4

**b.** 6 m to the right of the origin

**c.** $v = -\dfrac{2\pi}{3}\cos\left(\dfrac{\pi}{6}t\right)$

**d.** 2 m to the right of the origin

**e.**
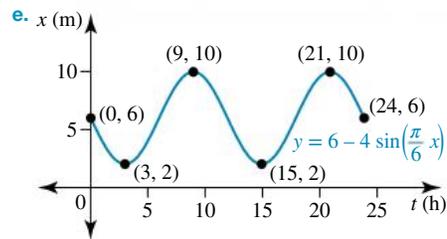


The particle is at rest at the turning points of the curve where the displacement is 2 m and 10 m. The particle oscillates between these two positions.

**10. a.** 56 items

**b. i.** $345          **ii.** $19 320

**11.** 4 units

**12. a.** Sample responses can be found in the worked solutions in the online resources.

**b.** Base: $2\sqrt{2}$ units; height: $\sqrt{2}$ units

**c.** 4 square units

**13. a.** 4 years; 6 butterflies/year

**b.** 2.54 years

**c.** 1.24 butterflies/year

**d.**
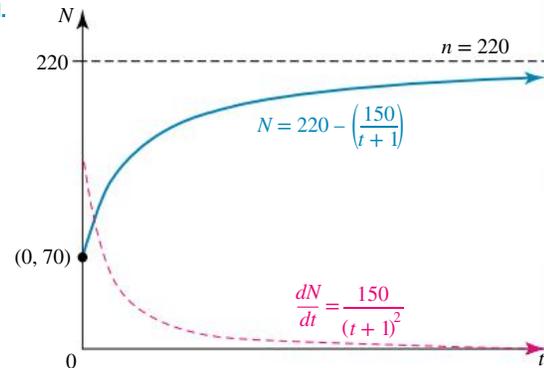


As $t \to \infty$, $N \to 220$ and $\dfrac{dN}{dt} \to 0$.

**14. a. i.** 12 cm          **ii.** 132 cm

**b.** $\dfrac{dL}{dt} = 6 + \dfrac{\pi}{2}\cos\left(\dfrac{\pi t}{4}\right)$

**c.** Maximum rate of growth $= 6 + \dfrac{\pi}{2} \approx 7.571$ cm/week;

minimum rate of growth $= 6 - \dfrac{\pi}{2} \approx 4.429$ cm/week

**15. a.** 12 years, 1 January 2022

**b.** 941 cheetahs

**c. i.** 824 cheetahs          **ii.** 507 cheetahs

**16.** $\left(0.1, 1 - \log_e(10)\right)$

**17.** $980.34, May 2026

**18.** $2\sqrt{2}$ units

**19.** Width: 39.2 cm height: 47.3 cm

**20.** General store B, by approximately 0.48 units

## 3.6 Review

### 3.6 Exercise

**1. a.** $15(4x+5)(2x^2+5x)^4$

**b.** $-4(2-3x)(4x-3x^2)^{-3}$

**c.** $6\left(x+\dfrac{1}{x}\right)^5\left(1-\dfrac{1}{x^2}\right)$

**d.** $96(5-6x)^{-5}$

**2. a.** $x^2\cos(x)+2x\sin(x)$

**b.** $3x\cos(x)+3\sin(x)$

**c.** $5x^4\cos(3x+1)-3x^5\sin(3x+1)$

**d.** $\cos^2(x)-\sin^2(x)$

**e.** $\dfrac{8}{x}\sin(5x)+40\cos(5x)\log_e(5x)$

**f.** $5\cos(x)\cos(2x)-10\sin(x)\sin(2x)$

**3. a.** $\dfrac{4}{3}\cos(x)\cos\left(\dfrac{4x}{3}\right)-\sin(x)\sin\left(\dfrac{4x}{3}\right)$

**b.** $4x^{-3}\cos(2x+3)-6x^{-4}\sin(2x+3)$

**c.** $-4e^{-5x}\cos(2-x)-20e^{-5x}\sin(2-x)$

**d.** $\dfrac{-6\sin(6x)}{\sqrt{x}}-\dfrac{\cos(6x)}{2\sqrt{x^3}}$

**e.** $\dfrac{1}{x}\sin(x)+\cos(x)\log_e(x)$

**f.** $\cos(2\pi x)-2\pi^2 x\sin(2\pi x)$

**4. a.** $\dfrac{x\cos(x)-\sin(x)}{x^2}$

**b.** $\dfrac{4\cos(2x)\cos(4x)+2\sin(2x)\sin(4x)}{\cos^2(2x)}$

**c.** $\dfrac{-x\sin(x)-\cos(x)}{x^2}$

**d.** $\dfrac{-(\sin(x)+\cos(x))}{e^x}$

**e.** $\dfrac{\sqrt{x}\cos(\sqrt{x})-2\sin(\sqrt{x})}{2x^2}$

**f.** $\dfrac{4x\sin(3-2x)-4\cos(3-2x)}{x^3}$

**5.** C

**6.** A

**7.** $y=3x+1$

**8. a.** 5

**b.** $y=\dfrac{1}{5}x-\dfrac{26\pi}{15}$

**9. a.** $\left(2,12\log_e(2)\right)$

**b.** Local minimum

**c.**



Graph of $y=6\ln(x^2-4x+8)$ with minimum at $(2, 8.318)$.

**10. a.**



Graph of $y=x^4-4x^3$ with points $(0,0)$, $(4,0)$ and $(3,-27)$.
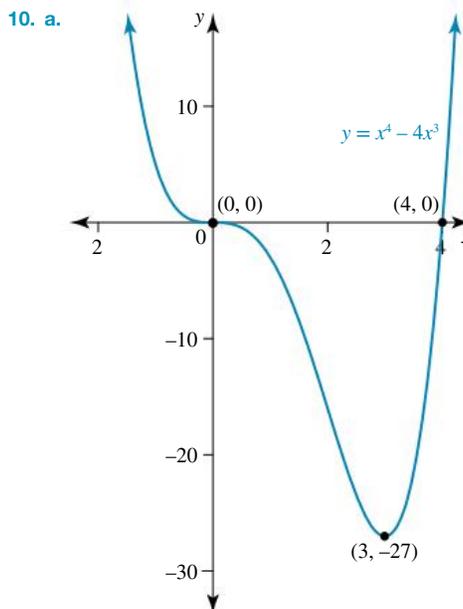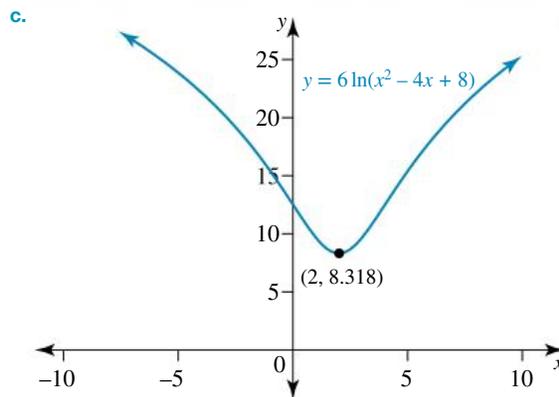
Stationary points: $(0,0)$ and $(3,-27)$
Axis intercepts: $(0,0)$ and $(4,0)$
Range: $y\geq-27$

**b.**



Graph of $y=\dfrac{4}{x^2+1}$ with point $(0,4)$ and asymptote $y=0$.
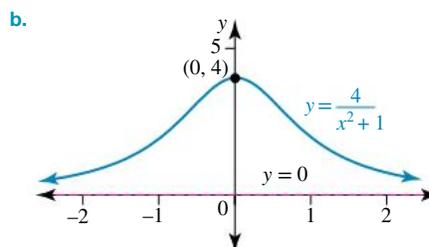
Stationary point: $(0,4)$
Axis intercept: $(0,4)$
Range: $0<y\leq4$

**11. a.** 30 L/ min

**b.** at $t=0$, at the beginning

**12. a.** $v=\dfrac{3t}{\sqrt{3t^2+4}}$    **b.** 1.5

**13. a.** $h=20\sin(x)$

**b.** $b=10+40\cos(x)$

**c.** Sample responses can be found in the worked solutions in the online resources.

**d.** For a maximum, the angle $x$ is $0.936^c$ and the maximum area is 352 cm$^2$.

**14. a. i.** $\sqrt{4+x^2}$ km      **ii.** $(3-x)$ km

**b.** $T = \dfrac{3}{5} - \dfrac{x}{5} + \dfrac{1}{3}\sqrt{4+x^2}$

**c.** $\dfrac{dT}{dx} = \dfrac{x}{3\sqrt{4+x^2}} - \dfrac{1}{5}$

**d.** 1.5 km

**e.** 1 h 8 min

**15.** $a = -3$

**16. a. i.** $m(x) = 2\sin(e^x)$      **ii.** $n(x) = e^{2\sin(x)}$

**b.** $x = 1.555,\ 2.105,\ 2.372, 2.660, 2.844$

**17.** 0.75 mg/L; and 6 hours

**18.** $m = -3,\ n = 4;\ (0,0),\ \left(\dfrac{4}{3},\ \dfrac{256}{81e^4}\right)$

**19.** $a = -1.088,\ b = 2.655,\ c = 0.483,\ d = -0.552$;
range $= \left[-4.855,\ 2.655\right]$

**20.** 782 rabbits; 44 rabbits/week; 39 weeks

## 3.6 Past QCAA exam questions

**1.** A

**2.** B

**3. a.** $\dfrac{dy}{dx} = \dfrac{-\cos(x)}{\sin^2(x)}$      **b.** $\dfrac{dy}{dx} = xe^{-x}(2-x)$

**4. a.** $\dfrac{dy}{dx} = 3e^x(e^x + 1)^2$

**b.** $\dfrac{dy}{dx} = \dfrac{x\cos(x) - 2\sin(x)}{x^3}$

**5.** The population decreases for $0 < t < \dfrac{1}{3\sqrt{e}}$ but increases for

$t > \dfrac{1}{3\sqrt{e}}$.

# 4 The second derivative and applications of differentiation

## LESSON SEQUENCE

**Fully worked solutions for this chapter are available online.**

## EXAM PREPARATION

Access exam-style questions in every lesson, available online and access past QCAA exam questions in every review.

**on Resources**

| | |
|---|---|
| **Solutions** | Solutions — Chapter 4 |
| **Exam questions** | Exam question booklet — Chapter 4 |
| **Digital documents** | Learning matrix — Chapter 4 |
| | Chapter summary — Chapter 4 |

# LESSON
## 4.1 Overview

### 4.1.1 Introduction

This is the final chapter in the area of differential calculus, the study of how a change in one variable will affect a related variable. You have learned about differentiation and its applications to curve sketching and determining equations of tangents to curves, as well as practical applications. In coming chapters, you will also study antidifferentiation, or integration, along with applications such as determining areas under curves.

In this chapter we will explore the second derivative — the rate of change of the derivative. You will discover that using the second derivative can help in discussing the behaviour of functions. For example, it allows us to answer questions such as: what is the concavity of the function? Is the critical point the largest value of the function? The second derivative also has many practical applications. In problems of optimisation, the second derivative assists in determining the nature of the critical points. In kinematics, the second derivative of the displacement of a particle with respect to time is the acceleration of the function. Scientists, economists and engineers are all concerned with determining critical values.

### 4.1.2 Syllabus links

| Lesson | Lesson title | Syllabus links |
|--------|-------------|----------------|
| 4.2 | Second derivatives | ○ Understand the concept of the second derivative as the rate of change of the first derivative function. <br> ○ Recognise acceleration as the second derivative of displacement position with respect to time. |
| 4.3 | Concavity and points of inflection | ○ Understand the concepts of concavity and points of inflection and their relationship with the second derivative. |
| 4.4 | Curve sketching | ○ Understand and use the second derivative test for finding local maxima and minima. <br> ○ Sketch the graph of a function using first and second derivatives to locate stationary points and points of inflection. |
| 4.5 | Modelling and solving optimisation problems | ○ Model and solve optimisation problems from a wide variety of fields using first and second derivatives, where the function to be optimised is either given or to be developed. |

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# LESSON
## 4.2 Second derivatives

## 4.2.1 The second derivative

The second derivative is, in fact, the 'derivative of the derivative' of a function.

If $y = f(x)$ is the equation of the curve, then:

- the first derivative, $\dfrac{dy}{dx} = f'(x)$, is the rate of change of $y$ with respect to $x$, or in other words, the gradient of the curve;
- the rate of change of the derivative with respect to $x$ is the **second derivative**.

### Notation for the second derivative

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = f''(x)$$

When we differentiate to determine either the first or second derivatives, we may need to use the rules for differentiation, such as the chain, product or quotient rules.

---

### WORKED EXAMPLE 1 Determining the second derivative

**Determine** $\dfrac{d^2y}{dx^2}$ **if** $y = x^4 + 2x^3 - 4x^2 + 5.$

| THINK | WRITE |
|---|---|
| 1. Write the equation. | $y = x^4 + 2x^3 - 4x^2 + 5$ |
| 2. Differentiate to determine the first derivative and simplify. | $\dfrac{dy}{dx} = 4 \times x^3 + 2 \times 3x^2 - 4 \times 2x$ <br> $\dfrac{dy}{dx} = 4x^3 + 6x^2 - 8x$ |
| 3. Differentiate to determine the second derivative and simplify. | $\dfrac{d^2y}{dx^2} = 4 \times 3x^2 + 6 \times 2x - 8$ <br> $\dfrac{d^2y}{dx^2} = 12x^2 + 12x - 8$ |

## WORKED EXAMPLE 2 Calculating the second derivative at a point

If $f(x) = \dfrac{4\sqrt{x^5}}{3x^2}$, calculate $f''(9)$.

**THINK**

**WRITE**

1. Express the function in simplified form using index laws.

$$f(x) = \dfrac{4\sqrt{x^5}}{3x^2}$$
$$= \dfrac{4x^{\frac{5}{2}}}{3x^2}$$
$$= \dfrac{4}{3}x^{\frac{5}{2}-2}$$
$$= \dfrac{4}{3}x^{\frac{1}{2}}$$

2. Determine the first derivative, using the basic laws for differentiation, and simplify.

$$f'(x) = \dfrac{4}{3} \times \dfrac{1}{2}x^{-\frac{1}{2}}$$
$$= \dfrac{2}{3}x^{-\frac{1}{2}}$$

3. Determine the second derivative, using the basic laws for differentiation, by differentiating the first derivative again; then simplify.

$$f''(x) = \dfrac{2}{3} \times \dfrac{-1}{2}x^{-\frac{3}{2}}$$
$$= \dfrac{-1}{3\sqrt{x^3}}$$

4. Substitute in the value for $x$.

$$f''(9) = \dfrac{-1}{3\sqrt{9^3}}$$
$$= \dfrac{-1}{3 \times 27}$$

5. State the final result.

$$f''(9) = -\dfrac{1}{81}$$

## WORKED EXAMPLE 3 Using the product rule to determine the second derivative

Determine the second derivative, $\dfrac{d^2y}{dx^2}$, of $y = x^2 \ln(3x + 5)$.

**THINK**

**WRITE**

1. Write the equation.

$$y = x^2 \ln(3x + 5)$$

2. Use the product rule and simplify. If $y = uv$, then $y' = uv' + vu'$.

$$\dfrac{dy}{dx} = \ln(3x + 5) \times 2x + x^2 \times \dfrac{1}{(3x + 5)} \times 3$$
$$= 2x \ln(3x + 5) + \dfrac{3x^2}{(3x + 5)}$$

3. Differentiate with respect to $x$ again, using the product rule for the first term and the quotient rule for the second term of $\dfrac{dy}{dx}$.

$$\frac{d^2y}{dx^2} = \left\{ \ln(3x+5) \times 2 + 2x \times \frac{1}{(3x+5)} \times 3 \right\}$$
$$+ \frac{(3x+5) \times 6x - 3x^2(3)}{(3x+5)^2}$$

4. Simplify.

$$\frac{d^2y}{dx^2} = 2\,\ln(3x+5) + \frac{6x}{(3x+5)} + \frac{(9x^2+30x)}{(3x+5)^2}$$

*Note:* Sometimes you may be able to combine terms with common denominators, but this is not necessary unless you are asked to express your answer in a particular form.

## 4.2.2 Acceleration

From previous chapters, you are familiar with the relationships between displacement and velocity as functions of time. **Acceleration** measures the rate of change of velocity with respect to time.

Common units for acceleration include m/s$^2$.

If $x = f(t)$ is the displacement of a particle from the origin at time $t$, then:
- velocity, the rate of change of displacement with respect to time, is

$$v = \frac{dx}{dt} = f'(t)$$

- acceleration, the rate of change of velocity with respect to time, is

$$a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = f''(t)$$

Position, velocity and acceleration are linked by calculus. Differentiation enables us to obtain the velocity function from the position function, and to obtain the acceleration function from the velocity function. This is illustrated in the flowchart below.

**Relationship between $x$, $v$ and $a$**

$$x \xrightarrow{\text{derive}} v \xrightarrow{\text{derive}} a$$
$$x \xrightarrow{\quad} \frac{dx}{dt} \xrightarrow{\quad} \frac{dv}{dt}$$

## WORKED EXAMPLE 4 Determining acceleration and the second derivative

An object travels in a straight line so that its displacement from the origin, $x$ m/s, at time $t$ seconds is given by $x(t) = t e^{-t}$, $t \in [0, 5]$.

**a.** Determine an expression for:
   **i.** velocity as a function of time
   **ii.** acceleration as a function of time.
**b.** Calculate the initial speed of the object.
**c.** Determine when the object is at rest and identify its position at that time.
**d.** Determine when the acceleration is positive.

| THINK | WRITE |
|---|---|
| **a. i. 1.** Differentiate displacement with respect to time to determine velocity. Use the product rule. | **a.** $x(t) = t e^{-t}$ <br> $v(t) = \dfrac{dx}{dt}$ <br> $= e^{-t} \times 1 + t \times \left(e^{-t} \times -1\right)$ |
| **2.** Simplify. | $= e^{-t} - t e^{-t}$ <br> $v(t) = e^{-t}(1 - t)$ |
| **ii. 1.** Differentiate velocity with respect to time to determine acceleration. Use the product rule. | $a(t) = \dfrac{dv}{dt} = \dfrac{d^2x}{dt^2}$ <br> $= (1 - t) \times \left(e^{-t} \times -1\right) + e^{-t} \times (-1)$ |
| **2.** Simplify. | $= -e^{-t} + t e^{-t} - e^{-t}$ <br> $a(t) = e^{-t}(t - 2)$ |
| **b.** Substitute $t = 0$ into velocity equation. | **b.** $v(0) = e^{0}(1 - 0)$ <br> $v = 1 \text{ m/s}$ |
| **c. 1.** At rest, $v = 0$. | **c.** $v(t) = e^{-t}(1 - t)$ <br> $e^{-t}(1 - t) = 0$ |
| **2.** Solve for $t$. | $t = 1 \text{ s}$ |
| **3.** For position, calculate displacement when $t = 1$. | $x(t) = t e^{-t}$ <br> $x(1) = e^{-1} = \dfrac{1}{e} \approx 0.368$ |
| **4.** Answer the question. | The object is at rest after 1 second and its position is $\dfrac{1}{e}$ metres or approximately 0.368 metres to the right of the origin. |
| **d. 1.** State the expression for acceleration. | **d.** $a(t) = e^{-t}(t - 2)$ |
| **2.** Solve for $a(t) > 0$. | $e^{-t}(t - 2) > 0$ <br> $t > 2$ |
| **3.** Answer the question and note the restricted domain for time. | Acceleration is positive when $t \in (2, 5]$ seconds. |

# Exercise 4.2 Second derivatives

| **4.2 Exercise** | **4.2 Exam questions** on |
|---|---|

| Simple familiar | Complex familiar | Complex unfamiliar |
|---|---|---|
| 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 | 13, 14, 15, 16 | 17, 18, 19, 20 |

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## Simple familiar

1. **WE1** Determine $\dfrac{d^2y}{dx^2}$ for each of the following functions.

   **a.** $y = x^4 - 5x^3 + x^2 - 9$     **b.** $y = x^3 - 4x^2$     **c.** $y = 4 - x^2$

   **d.** $y = x^2(8 - x)$     **e.** $y = (2x - 1)^4$

2. Determine the second derivatives of the following.

   **a.** $x\sqrt{x}$     **b.** $\dfrac{1}{x^2}$     **c.** $4\,e^{2x+3}$

   **d.** $\cos\left(\dfrac{2x}{5}\right)$     **e.** $3\,\sin(4x - \pi)$

3. Determine $f''(x)$ if $f(x)$ is given by:

   **a.** $x\ln(x)$     **b.** $e^{3x^2}$     **c.** $\ln(x + 1)$

4. **WE2** If $f(x) = \dfrac{8\sqrt{x^3}}{3x}$, calculate $f''(4)$.

5. If $f(x) = 8\,\cos\left(\dfrac{x}{2}\right)$, calculate $f''\left(\dfrac{\pi}{3}\right)$.

6. **a.** If $f(x) = \dfrac{4x^2}{3\sqrt{x}}$, calculate $f''(4)$.

   **b.** If $f(x) = \dfrac{2}{3x - 5}$, calculate $f''(1)$.

7. **a.** If $f(x) = 4\,\log_e(2x - 3)$, calculate $f''(3)$.

   **b.** If $f(x) = e^{x^2}$, calculate $f''(1)$.

8. **WE3** Determine the second derivative, $\dfrac{d^2y}{dx^2}$, of $y = x^3\,\log_e(2x^2 + 5)$.

9. Determine $\dfrac{d^2y}{dx^2}$ for $y = \dfrac{x^4}{e^{3x}}$.

10. Determine $\dfrac{d^2y}{dx^2}$ if:

    **a.** $y = \log_e(x^2 + 4x + 13)$     **b.** $e^{3x}\cos(4x)$

11. The velocity of an object that is initially 3 m left of O is given by $v(t) = 3t^2 - 2t - 5$ m/s. Determine:

    **a.** the displacement from O at any time $t$     **b.** the acceleration at any time $t$

    **c.** when the object is at rest     **d.** the distance travelled in the first second

    **e.** the acceleration when the velocity is 0.

**12.** An object travelling in a straight line has its displacement (in metres) after $t$ seconds given by $x(t) = 2\cos(3t-1) + 3$.

    **a.** Determine the maximum and minimum displacement.

    **b.** Determine when the velocity is first equal to 0.

    **c.** Determine the time interval between the object's first and next instance of being at rest.

    **d.** Determine an expression for the acceleration.

**Complex familiar**

**13.** **WE4** An object travels in a straight line so that its displacement from the origin, $x$ m/s, at time $t$ seconds is given by $x(t) = 8te^{-\frac{t}{2}}$, $t \in [0, 6]$.

    **a.** Determine an expression for:

        **i.** velocity as a function of time

        **ii.** acceleration as a function of time.

    **b.** Calculate the initial speed of the object.

    **c.** Determine when the object is at rest and identify its position at that time.

    **d.** Determine when the acceleration is positive.

**14.** The displacement, $x$ metres, of a particle at any time, $t$ seconds, is given by the equation $x = 6\sin\left(\dfrac{\pi}{4}(2t-1)\right)$.

    **a.** Determine where the particle is initially.

    **b.** Calculate when the particle is first at rest.

    **c.** Calculate the acceleration of the particle at 3.5 seconds.

**15.** Consider the function $f(x) = e^{\sin(x)}$.

    **a.** Show that the gradient of the function at $x = \pi$ is $-1$.

    **b.** Determine $f''(\pi)$.

**16.** Consider the function $y = e^x \sin(x)$.

    **a.** Show that the function has a stationary point at $x = \dfrac{3\pi}{4}$.

    **b.** Evaluate $\dfrac{d^2y}{dx^2}$ when $x = \dfrac{3\pi}{4}$. Give your answer correct to 2 decimal places.

**Complex unfamiliar**

**17.** Determine the values of $a$ and $b$ if it is known that, for all $x$, the second derivative of the function $f(x) = 2\sin(3x) + 4\cos(2x)$ is given by $f''(x) = a\sin(3x) + b\cos(2x)$.

**18.** The position, $x$ metres, of a particle at a time $t$ seconds is given by $x = 6e^{-\frac{t}{2}}$.

    If its acceleration is $a$, show that $a = \dfrac{x}{4}$.

**19.** The position, $x$ metres, of a particle from an origin O at a time $t$ seconds is given by $x = 4\cos\left(\dfrac{t}{2}\right)$.

    If its acceleration is $a$, show that $x = -4a$ and determine the time when it comes to rest.

**20.** Given that $y = x\sqrt{a + bx}$, verify that $y\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 = a + 3bx$.

**Fully worked solutions for this chapter are available online.**

# LESSON
## 4.3 Concavity and points of inflection

### 4.3.1 Concavity

The shape of continuous functions is often described in terms of its **concavity**. Continuous functions are said to be either **concave up** (sometimes referred to as convex) or **concave down**.

A function is concave up when the gradient of the function is increasing, so the rate of change of the gradient is positive.

A function is concave down when the gradient of the function is decreasing, so the rate of change of the gradient is negative.

Remember, the rate of change of the gradient with respect to $x$ is the second derivative.

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = f''(x)$$

Concavity may be summarised as follows.

| Shape | First derivative | Second derivative |
|---|---|---|
| Concave up (or convex) | $\dfrac{dy}{dx}$ is increasing. | $\dfrac{d^2y}{dx^2} > 0$ |
| Concave down | $\dfrac{dy}{dx}$ is decreasing. | $\dfrac{d^2y}{dx^2} < 0$ |
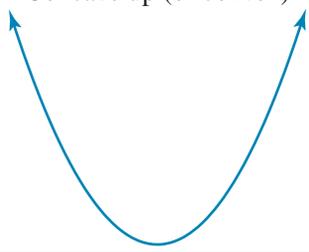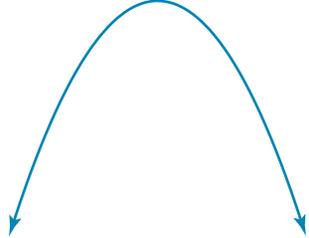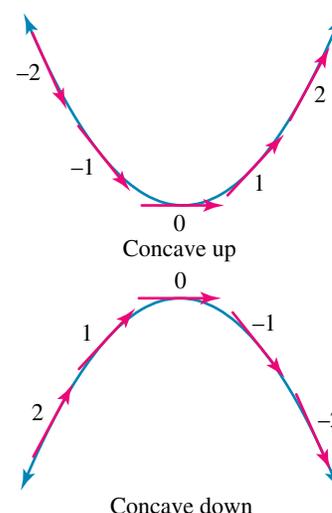
## 4.3.2 Points of inflection

On a continuous curve, the point where the concavity changes is called a **point of inflection**. At this point, the curve changes from being concave up to concave down or vice versa.

The rate of change of the gradient is 0 at a point of inflection, as it has either stopped increasing and started to decrease or vice versa.

For a point of inflection, the following must be true:

$\dfrac{d^2y}{dx^2} = 0$ **AND** the sign changes either side to show the change in concavity.

A point of inflection occurs when there is a change in concavity, giving two possible shapes.
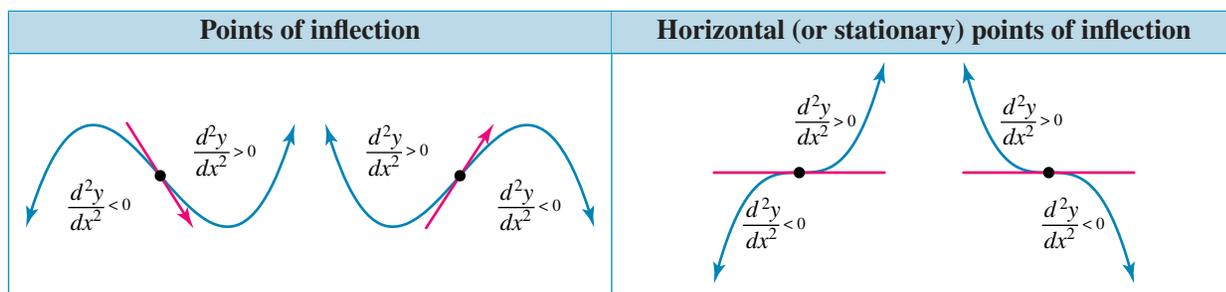


The tangent to the curve at the point of inflection crosses the curve at this point. (In contrast, the tangent to the curve at a maximum or minimum does not cross the curve at that point.)

In summary, concavity and points of inflection depend on the second derivative $\dfrac{d^2y}{dx^2}$ or $f''(x)$.

The analysis of the second derivative may be used to determine the nature of stationary points on a curve.

---

### WORKED EXAMPLE 5 Describing the shape of a curve

a. **Describe the shape of the curve $y = x^3 - 3x^2$ at the point where:**
   i. $x = 2$
   ii. $x = -2$
b. **Determine the coordinates of the point of inflection.**

| THINK | WRITE |
|---|---|
| a. Differentiate twice to determine the second derivative. | a. $y = x^3 - 3x^2$ <br> $\dfrac{dy}{dx} = 3x^2 - 6x$ <br> $\dfrac{d^2y}{dx^2} = 6x - 6$ |
|   i. Substitute $x = 2$. | i. When $x = 2$: $\dfrac{d^2y}{dx^2} = 12 - 6$ <br> $= 6$ which is positive <br> The curve is concave up at $x = 2$. |
|   ii. Substitute $x = -2$. | ii. When $x = -2$: <br> $\dfrac{d^2y}{dx^2} = -12 - 6$ <br> $= -18$ which is negative <br> The curve is concave down at $x = -2$. |
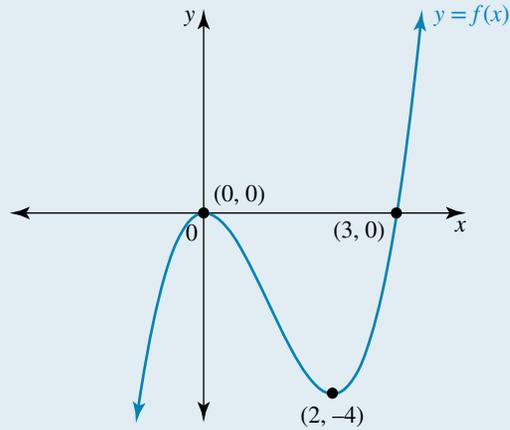
**b. 1.** Solve $\dfrac{d^2y}{dx^2} = 0$.

**b.** $\dfrac{d^2y}{dx^2} = 6x - 6$

$6x - 6 = 0$

$x = 1$

**2.** Check either side for change in concavity.

| $x$ | $1^-$ | $1$ | $1^+$ |
|---|---|---|---|
| $\dfrac{d^2y}{dx^2}$ | $<0$ | $=0$ | $>0$ |

The concavity has changed either side of $x = 1$, so the point of inflection is $(1, 2)$.

---

## WORKED EXAMPLE 6 Determining the concavity of a function

**Consider the function $f(x) = 12x^2 - x^3$.**
**a. Determine where the function is:**
  **i. concave up**
  **ii. concave down.**
**b. Hence, state the coordinates of the point of inflection.**

| THINK | WRITE |
|---|---|
| **a. i. 1.** For concavity, determine the second derivative. | **a.** $f(x) = 12x^2 - x^3$ <br> $f'(x) = 24x - 3x^2$ <br> $f''(x) = 24 - 6x$ |
| **2.** The function is concave up when $f''(x) > 0$. Solve for $x$. | $24 - 6x > 0$ <br> $24 > 6x$ <br> $x < 4$ <br> The function is concave up for $x < 4$. |
| **ii.** The function is concave down when $f''(x) < 0$. Solve for $x$. | $24 - 6x < 0$ <br> $24 < 6x$ <br> $x > 4$ <br> The function is concave down for $x > 4$. |

**b.** The point of inflection is where concavity changes. State the point at $x = 4$.

**b.** At $x = 4$:
$$f(4) = 12 \times 16 - 64$$
$$= 128$$
The point of inflection is $(4, 128)$.

---

## WORKED EXAMPLE 7 Investigating concavity

By investigating the concavity of the curve $y = x^4$, explain why the curve does not have a point of inflection. A sketch of the curve may be useful.

| THINK | WRITE |
|---|---|
| **1.** For concavity, determine the second derivative. | $y = x^4$ <br><br> $\dfrac{dy}{dx} = 4x^3$ <br><br> $\dfrac{d^2y}{dx^2} = 12x^2$ |
| **2.** The point of inflection exists when $\dfrac{d^2y}{dx^2} = 0$ and changes sign. | $12x^2 = 0$ <br> $x = 0$ |

| $x$ | $0^-$ | $0$ | $0^+$ |
|---|---|---|---|
| $\dfrac{d^2y}{dx^2}$ | $> 0$ | $= 0$ | $> 0$ |

| | |
|---|---|
| **3.** State a reason for your decision. | The second derivative does not change sign either side of $x = 0$. Therefore, $x = 0$ is not a point of inflection. <br> $\dfrac{d^2y}{dx^2} \geq 0$ for all $x$, so the curve is always concave up. |
| **4.** Sketch the curve. | A sketch of $y = x^4$ is shown, demonstrating that it is always concave up. |
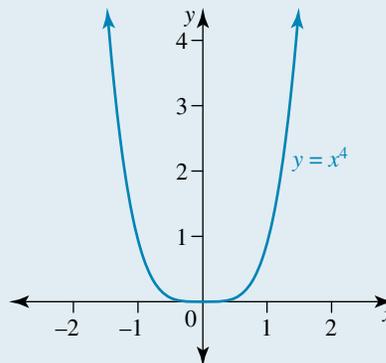
# Exercise 4.3 Concavity and points of inflection

learnon

## 4.3 Exercise

| Simple familiar | Complex familiar | Complex unfamiliar |
|---|---|---|
| 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 | 13, 14, 15, 16 | 17, 18, 19, 20 |

## 4.3 Exam questions  on

These questions are even better in jacPLUS!
- Receive immediate feedback
- Access sample responses
- Track results and progress

**Find all this and MORE in jacPLUS** ▶

### Simple familiar

1. **WE5** **a.** Describe the shape of the curve $y = x^3 - 9x^2 + 8$ at the point where:
   **i.** $x = 4$
   **ii.** $x = -4$
   **b.** Determine the coordinates of the point of inflection.

2. **a.** Describe the shape of the curve $y = x^3 + 6x^2$ at the point where:
   **i.** $x = -3$
   **ii.** $x = 3$
   **b.** Determine the coordinates of the point of inflection.

3. **a.** Describe the shape of the curve $y = 4x^2 - x^3$ at the point where:
   **i.** $x = 0$
   **ii.** $x = 1$
   **b.** Determine the coordinates of the point of inflection.

4. **WE6** Consider the function $f(x) = x^3 + 9x^2$.
   **a.** Determine where the function is:
   **i.** concave up
   **ii.** concave down.
   **b.** Hence, state the coordinates of the point of inflection.

5. **a.** For the function $y = x^3 + 2x^2 - 3x + 1$, determine where the function is:
   **i.** concave up
   **ii.** concave down.
   **b.** Hence, state the coordinates of the point of inflection.

6. **WE7** By investigating the concavity of the curve $y = 6 - x^4$, explain why the curve does not have a point of inflection. A sketch of the curve may be useful.

7. By investigating the concavity of the curve $y = 2x^6 - 4$, explain why the curve does not have a point of inflection. A sketch of the curve may be useful.

8. Determine the coordinates of the point(s) of inflection for the following functions.
   **a.** $y = x^3 - 3x^2 - 9x + 5$
   **b.** $y = -x^3 + 9x^2 - 15x - 20$

CHAPTER 4 The second derivative and applications of differentiation **189**

9. Consider the function $f(x) = x^4 + 4x^3 - 16x + 3$.

   a. Determine where the function is:

      i. concave up
      ii. concave down.

   b. State, with reasons, the coordinates of the point(s) of inflection of the function.

10. Determine the point(s) of inflection of the function $f(x) = \dfrac{1}{2}x^2 - 3x^4$. Hence, state where the function is concave down.

11. Determine the coordinates, with reasons, of the point(s) of inflection of the graph of $y = x^4 - 4x^3$.

12. Determine the value of $k$ if the function $f(x) = 2x^3 - kx^2 + 3x$ has a point of inflection when $x = 3$.

13. Consider the function $f(x) = x^4 + kx^3$.

    a. Determine the value for $k$ if the function $f$ has a point of inflection at $x = 1$.
    b. Hence, determine the interval where the function is concave up.

14. Show that the function $f(x) = x \log_e(x)$, $x > 0$ is always concave up.

15. Consider the function $y = xe^x$.

    a. Determine where the function is:

       i. concave up
       ii. concave down.

    b. State the coordinates of the point(s) of inflection.

16. a. Sketch the graph of $y = 2 \sin(x) + 3$, $x \in [0, 2\pi]$.
    b. Determine where the function is:

       i. concave up
       ii. concave down.

    c. Hence, state the coordinates of the point(s) of inflection.

17. Consider the curves $y = (2x - 3)^3 + 4$ and $y = (2x - 3)^4 + 4$.
    Discuss the similarities and differences between the curves in relation to any point of inflection.

18. The function $f(x) = x^3 + bx^2 + cx + d$ has a stationary point of inflection at $(1, -2)$.
    Determine the values of $b$, $c$ and $d$.

19. A cubic polynomial $y = ax^3 + bx^2 + cx$ has a point of inflection at $x = -2$. The tangent at the point of inflection has the equation $y = 21x + 8$. Determine the values of $a$, $b$ and $c$.

20. Show that the graph of $y = x^3 - 2ax^2 + a^2x$, where $a \in R \backslash \{0\}$, crosses the x-axis at $(a, 0)$ and $(0, 0)$, has an

    inflection point at $\left( \dfrac{2a}{3}, \dfrac{2a^3}{27} \right)$, and that the equation of the tangent to the curve at the point of inflection is

    given by $y = \dfrac{8a^3}{27} - \dfrac{a^2x}{3}$.

**Fully worked solutions for this chapter are available online.**

# LESSON
## 4.4 Curve sketching

### 4.4.1 Reviewing polynomial shapes

You will recall the general shapes of quadratic and cubic functions. These are summarised in the following table. Remember, these graphs may also be inverted.

| Quadratics (degree 2) | Cubics (degree 3) |
|---|---|
|  |  |
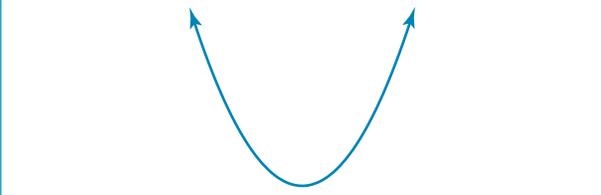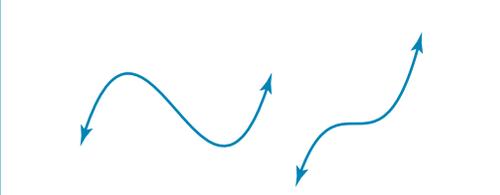
Knowledge about the first and second derivatives of a function will allow us to make important observations about a wider variety of functions and help us sketch the graph more accurately.

### 4.4.2 Stationary points, points of inflection and derivatives

A stationary point on a curve is defined as a point where the gradient is zero; that is, where $\dfrac{dy}{dx} = f'(x) = 0$.

Recall the three types of stationary points:
- local maximum turning points
- local minimum turning points
- horizontal (or stationary) points of inflection.

Previously, you have determined the nature of the stationary points by calculating the slope of the tangent, $\dfrac{dy}{dx}$ or $f'(x)$, on either side of the stationary point. At times, this method is still useful.

The nature of the stationary point can also be determined by considering concavity.
- For a maximum turning point, the curve is concave down.
- For a minimum turning point, the curve is concave up.
- For a horizontal (or stationary) point of inflection:
  - the first derivative, $\dfrac{dy}{dx}$ or $f'(x)$, equals zero
  - the second derivative, $\dfrac{d^2y}{dx^2}$ or $f''(x)$, equals zero, and changes sign either side of the point.

Non-stationary points of inflection occur when the second derivative equals zero and changes sign either side of the point but does not have the first derivative equal to zero.

Stationary points and points of inflection are summarised in the following table.

| Shape | First derivative | Second derivative |
|---|---|---|
| Maximum turning point  | $\dfrac{dy}{dx} = 0$ | $\dfrac{d^2y}{dx^2} < 0$ |
| Minimum turning point  | $\dfrac{dy}{dx} = 0$ | $\dfrac{d^2y}{dx^2} > 0$ |
| Stationary point of inflection (tangent is horizontal)  | $\dfrac{dy}{dx} = 0$ | $\dfrac{d^2y}{dx^2} = 0$ and changes sign either side  |
| Non-stationary point of inflection (non-horizontal tangent)  | $\dfrac{dy}{dx} \neq 0$ | $\dfrac{d^2y}{dx^2} = 0$ and changes sign either side  |
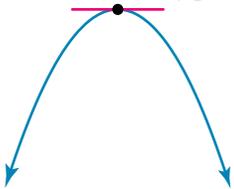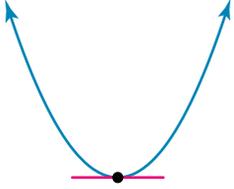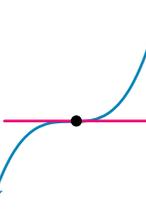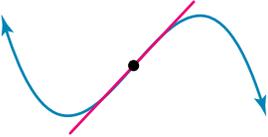
## 4.4.3 Curve sketching

Using the first and second derivatives of a function allows curves to be sketched with greater accuracy.

The **first derivative** of a function gives the gradient at any point. This is summarised in the following table.

| First derivative | | |
|---|---|---|
| $\dfrac{dy}{dx} > 0$ | $\dfrac{dy}{dx} < 0$ | $\dfrac{dy}{dx} = 0$ |
| **Increasing function** | **Decreasing function** | **Stationary point** |
|  |  |  |
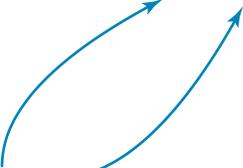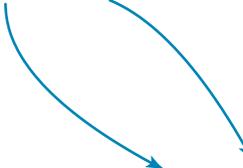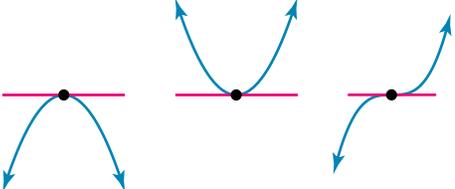
The **second derivative** of a function gives its concavity. This is summarised in the following table.

| Second derivative | | |
|---|---|---|
| $\dfrac{d^2y}{dx^2} > 0$ | $\dfrac{d^2y}{dx^2} < 0$ | $\dfrac{d^2y}{dx^2} = 0$ and changes sign |
| Concave up | Concave down | Point of inflection |
|  |  |  |
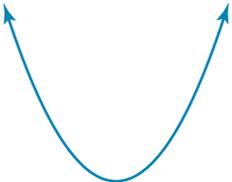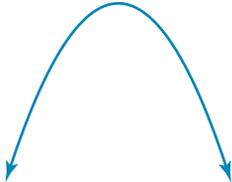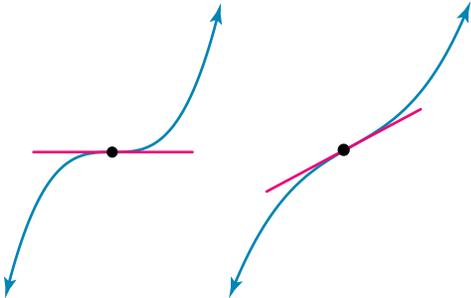
The relationships between the function $y = x^4 - 4x^2 + 10$ and its derivatives are shown.

At the point $(0, 10)$:

$$\frac{dy}{dx} = 0$$

$\dfrac{d^2y}{dx^2}$ changes from positive to negative

The point is a stationary point of inflection.

At the point $(3, -17)$:

$$\frac{dy}{dx} = 0$$

$$\frac{d^2y}{dx^2} > 0$$

The point is a minimum turning point.

At the point $(2, -6)$:

$\dfrac{d^2y}{dx^2} = 0$ and changes from negative to positive

The point is a point of inflection.

When sketching the graph of a function, $f(x)$, consider the following:
- Determine the $y$-intercept by evaluating $f(0)$.
- Determine the $x$-intercept(s) by solving $f(x) = 0$, if possible.
- Determine the coordinates of the stationary point(s) and their nature.
- Determine the coordinates of any point(s) of inflection.
- Consider restrictions on the domain.
- Calculate the coordinates of the end points of the domain, where appropriate.
- Identify vertical and horizontal asymptotes, where appropriate.
- Consider the direction of $f(x)$ as $x \to \pm\infty$.
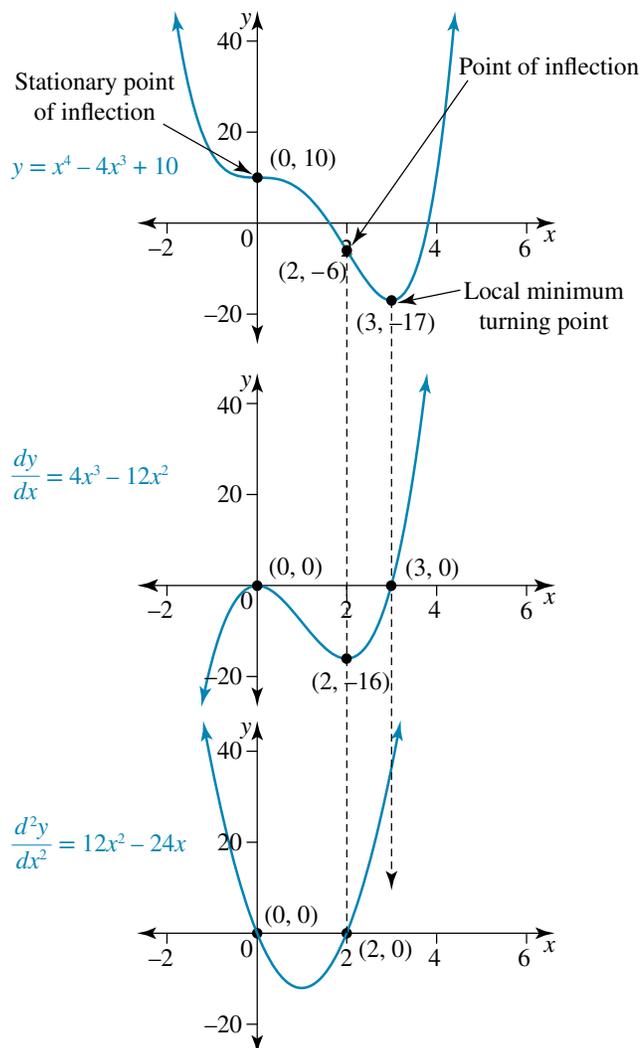
## WORKED EXAMPLE 8 Sketching a cubic function

**Sketch the graph of the function $f : R \to R, f(x) = x^3 + 6x^2 + 9x$ by determining the coordinates of all axis intercepts as well as any stationary points and their nature. Include on your sketch the coordinates of any point(s) of inflection.**

| THINK | WRITE |
|---|---|
| **1.** State the function and differentiate to determine the first and second derivatives. | $f(x) = x^3 + 6x^2 + 9x$ <br> $f'(x) = 3x^2 + 12x + 9$ <br> $f''(x) = 6x + 12$ |
| **2.** For $x$-axis intercepts: <br> factorise the function <br> solve for $f(x) = 0$. | $f(x) = x^3 + 6x^2 + 9x$ <br> $\quad = x(x^2 + 6x + 9)$ <br> $x(x+3)(x+3) = 0$ <br> $\quad x(x+3)^2 = 0$ <br> The $x$-intercepts are $(0, 0)$ and $(-3, 0)$. |
| **3.** For stationary points, $f'(x) = 0$. | $f'(x) = 3x^2 + 12x + 9$ <br> $\quad = 3(x^2 + 4x + 3)$ <br> $3(x+3)(x+1) = 0$ <br> $x = -3$ or $x = -1$ |
| **4.** Determine the nature of the stationary points using the second derivative and determine the corresponding values of $f(x)$. | When $x = -3$: <br> $f''(-3) = -18 + 12 = -6 < 0$, so concave down <br> $f(-3) = 0$ <br> The point $(-3, 0)$ is a maximum turning point. <br> When $x = -1$: <br> $f''(-1) = -6 + 12 = 6 > 0$, so concave up <br> $f(-1) = -1 + 6 - 9 = -4$ <br> The point $(-1, -4)$ is a minimum turning point. |
| **5.** For points of inflection, $f''(x) = 0$ and changes sign either side of that value. | $f''(x) = 6x + 12$ <br> $6x + 12 = 0$ <br> $\quad x = -2$ <br><br> Check for change of sign either side of $x = -2$. <br><br> <table><tr><td>$x$</td><td>$-2^-$</td><td>$-2$</td><td>$-2^+$</td></tr><tr><td>$\dfrac{d^2y}{dx^2}$</td><td>$< 0$</td><td>$0$</td><td>$> 0$</td></tr></table> |
| **6.** Determine $f(x)$ for the point and make a statement. | $f(-2) = -8 + 24 - 18 = -2$ <br> The second derivative has changed sign, so there is a point of inflection at $(-2, -2)$. |

**7.** Sketch the curve, showing all important features.



The graph shows $f(x) = x^3 + 6x^2 + 9x$ with labelled points $(-3, 0)$, $(0, 0)$, $(-2, -2)$ and $(-1, -4)$.

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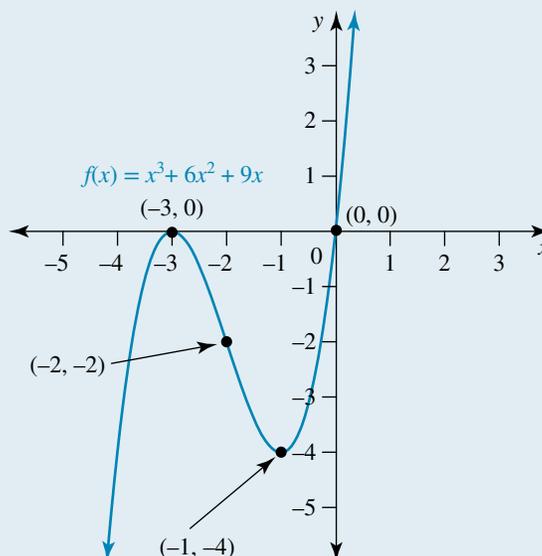
## WORKED EXAMPLE 9 Sketching a quartic function with a horizontal point of inflection

**Consider the equation $y = x^4 - 4x^3$.**

**a. Sketch the graph, stating the coordinates of all axis intercepts, any stationary points and their nature, and any point(s) of inflection.**

**b. Determine the values of $x$ where the function is:**
  **i. decreasing**
  **ii. concave up.**

| THINK | WRITE |
|---|---|
| **a. 1.** State the function and differentiate to determine the first and second derivatives. | **a.** $y = x^4 - 4x^3$ <br> $\dfrac{dy}{dx} = 4x^3 - 12x^2$ <br><br> $\dfrac{d^2y}{dx^2} = 12x^2 - 24x$ |
| **2.** For $x$-axis intercepts: factorise the function solve for $y = 0$. | $y = x^4 - 4x^3$ <br> $x^3(x - 4) = 0$ <br> The $x$-intercepts are $(0, 0)$ and $(4, 0)$. |
| **3.** For stationary points, solve for $\dfrac{dy}{dx} = 0$. | $\dfrac{dy}{dx} = 4x^3 - 12x^2$ <br> $4x^2(x - 3) = 0$ <br> $x = 0, 3$ |

4. Determine their nature using the second derivative and determine the corresponding $y$-value. *Remember:* If the second derivative is 0, you need to check either side by taking a value of $x$ close to the point. (For example, either side of $x = 0$, take $x = \pm 0.1$.)

When $x = 0$:
$$\frac{d^2y}{dx^2} = 12 \times 0 - 24 \times 0 = 0$$
Possible point of inflection; check for change of sign either side of $x = 0$.

| $x$ | $0^-$ | $0$ | $0^+$ |
|---|---|---|---|
| $\dfrac{d^2y}{dx^2}$ | $> 0$ | $0$ | $< 0$ |

The second derivative has changed sign from concave up to concave down, so there is a horizontal (or stationary) point of inflection at $(0, 0)$.

When $x = 3$:
$$\frac{d^2y}{dx^2} = 12 \times 9 - 24 \times 3 = 36 > 0, \text{ so concave up}$$
$$y = 3^4 - 4 \times 3^3 = -27$$
The point $(3, -27)$ is a minimum turning point.

5. For points of inflection:
$\dfrac{d^2y}{dx^2} = 0$ and changes sign.

$$\frac{d^2y}{dx^2} = 12x^2 - 24x$$
$$12x(x - 2) = 0$$
$$x = 0, 2$$

| $x$ | $2^-$ | $2$ | $2^+$ |
|---|---|---|---|
| $\dfrac{d^2y}{dx^2}$ | $< 0$ | $0$ | $> 0$ |

6. Determine the $y$-value for the point

$$y = 2^4 - 4 \times 2^3 = -16$$
The concavity has changed, so there is a point of inflection at $(2, -16)$.

7. Sketch the curve, showing all of the critical points.


$y = x^4 - 4x^3$
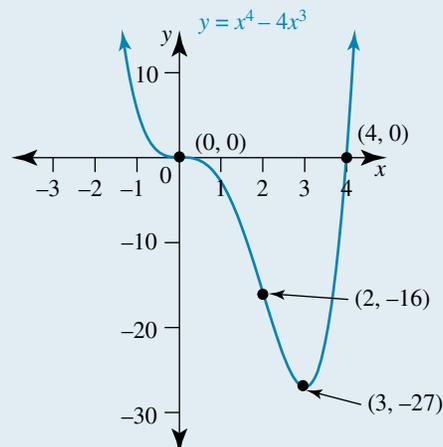
**b i.** For a decreasing function, $\dfrac{dy}{dx} < 0$.

Read from the graph. Remember the gradient cannot equal 0.

**b. i.** The function is decreasing when $x < 0$ or $0 < x < 3$.

**ii.** For concave up, $\dfrac{d^2y}{dx^2} > 0$.

Read from the graph, noting the points of inflection already found.

**ii.** There are points of inflection at $(0, 0)$ and $(2, -16)$.
The function is concave up when $x < 0$ or $x > 2$.

---

## WORKED EXAMPLE 10 Sketching a function

**Sketch the graph of $f(x) = 2xe^x, x \le 1$, showing the important features, including stationary points and points of inflection. Give your answers correct to 3 decimal places where necessary.**

| THINK | WRITE |
|---|---|

**1.** State the function and differentiate using the appropriate rules to determine the first and second derivatives.

$$f(x) = 2xe^x$$
$$f'(x) = 2x \times e^x \times 1 + e^x \times 2$$
$$= 2xe^x + 2e^x$$

$$f''(x) = (2x \times e^x \times 1 + e^x \times 2) + 2e^x \times 1$$
$$= 2xe^x + 2e^x + 2e^x$$
$$= 2xe^x + 4e^x$$

**2.** For $x$-axis intercepts, solve for $f(x) = 0$.

$$f(x) = 2xe^x$$
$$2xe^x = 0$$
$$x = 0$$
The $x$-intercept is $(0, 0)$.

**3.** For stationary points, solve for $f'(x) = 0$.

$$f'(x) = 2xe^x + 2e^x$$
$$2e^x(x + 1) = 0$$
$$x = -1$$

**4.** Determine the nature of the stationary points using the second derivative and determine the corresponding value of $f(x)$.

When $x = -1$:
$f''(-1) = -2e^{-1} + 4e^{-1} = 2e^{-1} > 0$, so concave up
$$f(-1) = -2e^{-1} = -\dfrac{2}{e}$$
The point $\left(-1, -\dfrac{2}{e}\right) \approx (-1, -0.736)$ is a minimum turning point.

**5.** For points of inflection, $f''(x) = 0$ and changes sign on either side.

$$f''(x) = 2xe^x + 4e^x$$
$$2e^x(x + 2) = 0$$
$$x = -2$$

**6.** Determine $f(x)$ for the point and make a statement.

| $x$ | $-2^-$ | $-2$ | $-2^+$ |
|---|---|---|---|
| $\dfrac{d^2y}{dx^2}$ | $<0$ | $0$ | $>0$ |

$f(-2) = -4\,e^{-2} = -\dfrac{4}{e^2}$

The concavity has changed, so the point
$\left(-2, -\dfrac{4}{e^2}\right) \approx (-2, -0.541)$ is a point of inflection.

**7.** Determine the coordinates of the end point of the restricted domain.

$f(1) = 2e$
The end point is $(1, 2e) \approx (1, 5.437)$ correct to 3 decimal places.

**8.** Consider the behaviour of the graph to the left as $x$ becomes very small.

As $x \to -\infty$, $e^x \to 0$
$\therefore$ as $x \to -\infty, x\,e^x \to 0$
The function approaches the $x$-axis, which will be a horizontal asymptote on the left side of the graph.

**9.** Sketch the curve, showing all important features.
Include a closed circle at the end point.
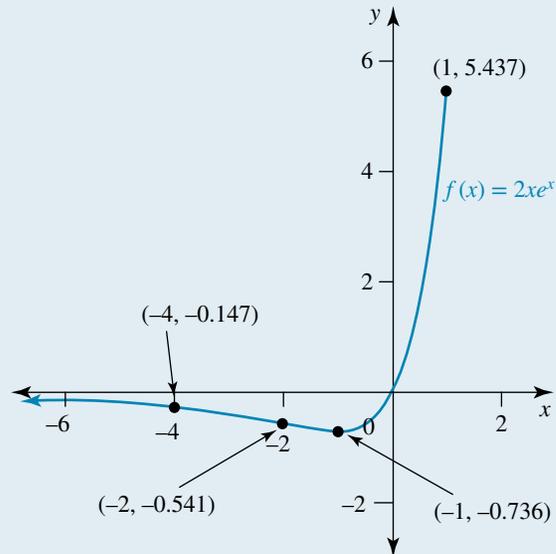
### Simple familiar

1. **WE8** Sketch the graph of the function $f : R \to R, f(x) = x^3 - 4x^2 + 4x$ by determining the coordinates of all axis intercepts as well as any stationary points and their nature. Include on your sketch the coordinates of any point(s) of inflection.

2. Sketch the graphs of each of the following by determining the coordinates of all axis intercepts and any stationary points, and establishing their nature. Also determine the coordinates of the points of inflection.
   a. $y = x^3 - 27x$
   b. $y = 9x - x^3$

3. Sketch the graphs of each of the following by determining the coordinates of all axis intercepts and any stationary points, and establishing their nature. Also determine the coordinates of the points of inflection. State where the functions are:
   i. increasing
   ii. concave up.
   a. $y = x^3 + 12x^2 + 36x$
   b. $y = -x^3 + 10x^2 - 25x$

4. Sketch the graphs of each of the following by determining the coordinates of all axis intercepts and any stationary points, and establishing their nature. Also determine the coordinates of the points of inflection.
   a. $y = x^3 - 3x^2 - 9x - 5 = (x - 5)(x + 1)^2$
   b. $y = -x^3 + 9x^2 - 15x - 25 = -(x - 5)^2(x + 1)$

5. **WE9** Consider the equation $y = 8x^3 - x^4$.
   a. Sketch the graph, stating the coordinates of the axis intercepts, any stationary points and their nature, and any point(s) of inflection.
   b. Determine the values of $x$ where the function is:
      i. decreasing
      ii. concave up.

6. a. Sketch the graph of the function $f(x) = x^4 - 8x^2 - 9$, stating the coordinates of the axis intercepts, any stationary points and their nature, and any point(s) of inflection.
   b. State the values of $x$ where $f(x)$ is both increasing and concave up.

7. a. Sketch the graph of the function $f(x) = (x - 1)^3 + 8$, showing all important features.
   b. State the values of $x$ where $f(x)$ is both increasing and concave down.

8. Sketch the graphs of each of the following by determining the coordinates of all axis intercepts and any stationary points, and establishing their nature. Also determine the coordinates of the points of inflection. The functions have been given in both factorised and expanded form for ease of calculations.
   a. $y = x^4 + 4x^3 - 16x - 16 = (x - 2)(x + 2)^3$
   b. $y = x^4 - 6x^2 + 8x - 3 = (x - 1)^3(x + 3)$

9. Sketch the graph of $y = x^4 - 10x^2 + 9$ by determining the coordinates of all axis intercepts and stationary points and establishing their nature. Determine the coordinates of the point(s) of inflection.

10. Sketch the graph of $y = x^4 - 24x^2 + 80$ by determining the coordinates of all axis intercepts and stationary points, and establishing their nature. Calculate the coordinates of the point of inflection.

11. Sketch the graph of $y = 6x^2 - x^4$ by determining the coordinates of all axis intercepts, stationary points and points of inflection, and establishing their nature.

12. A cubic polynomial, $y = x^3 + bx^2 + cx + d$, crosses the $y$-axis at $y = 5$ and has a point of inflection at $(1, -21)$. Determine the equation of the tangent at the point of inflection.

### Complex familiar

13. **WE10** Sketch the graph of $f(x) = 3xe^{-x}$, showing the important features, including stationary points and points of inflection. Give your answers correct to 3 decimal places where necessary.

14. Sketch the graph of $f(x) = 4 - 10xe^x$, $-4 \leq x \leq 0$, showing the important features, including stationary points and points of inflection. Give your answers correct to 3 decimal places where necessary.

15. The function $f(x) = x^3 + bx^2 + cx + d$ crosses the $x$-axis at $x = 3$ and has a point of inflection at $(2, -4)$. Calculate the values of $b$, $c$ and $d$.

16. Consider the function $f(x) = \dfrac{1}{2} \log_e(x^2 + 1)$.

   a. State the domain of $f(x)$.
   b. Determine the coordinates and nature of the stationary point.
   c. By considering the second derivative, show that the function has two points of inflection. State the coordinates of these points.
   d. Sketch the graph of $f(x)$ for a suitable domain.

### Complex unfamiliar

17. Consider the function $f(x) = \dfrac{10 \log_e(x)}{x}$.

   Sketch the graph of $f(x)$ for a suitable domain and discuss the behaviour of the function as $x$ increases.

18. Sketch the function $y = \dfrac{1}{x^2 + 1}$, showing all important features.

19. Sketch the function $y = \dfrac{x}{x^2 + 1}$, showing all important features.

20. Sketch the function $y = \dfrac{x^2 + 1}{x}$, showing all important features.

**Fully worked solutions for this chapter are available online.**

# LESSON
## 4.5 Modelling and solving optimisation problems

### 4.5.1 Optimisation

In many practical situations, it is necessary to determine the maximum or minimum value of a function. Optimisation was discussed in Chapter 3.

Checking the nature of the stationary point(s) using the second derivative may now be easier and quicker instead of using a sign diagram of the first derivative.

When solving optimisation problems, the following steps may be useful.
- Draw a diagram if possible and label it with as few variables as possible.
- Determine a connection between the variables from the information given.
  These may include:
  – Pythagoras' theorem
  – trigonometry
  – similar triangles
  – formulas for area and volume
  – standard formulas.
- Determine an expression for the quantity to be optimised in terms of one variable.
- Determine if the domain of the function is restricted.
- Differentiate the expression to determine the stationary point(s).
- Reject any unrealistic solutions.
- Determine the nature of the stationary point(s) by either substituting into the second derivative or using a sign diagram of the first derivative.
- Check whether the answer is the absolute maximum or minimum by evaluating the end points of the domain if it is restricted.
- Sketch the graph of the function to check for realistic values.
- Answer the actual problem.

### WORKED EXAMPLE 11 Determining the minimum sum of squares

**The sum of two positive numbers is 10. Determine the numbers if the sum of their squares is a minimum.**

| THINK | WRITE |
|---|---|
| 1. Define the two positive numbers. | Let the two positive numbers be $x$ and $y$. |
| 2. Write an expression for the sum, $S$, of the squares of the two numbers. | $S = x^2 + y^2$     [1] |
| 3. State the relationship between the variables. | $x + y = 10$     [2] |
| 4. Express $y$ in terms of $x$ using [2]. | $y = 10 - x$ |
| 5. Express $S$ in terms of one variable, $x$. | $S = x^2 + (10 - x)^2$ |
| 6. Simplify the expression. | $S = x^2 + 100 - 20x + x^2$ <br> $= 2x^2 - 20x + 100$ ▶ |

**7.** Differentiate to obtain $\dfrac{dS}{dx}$.

$\dfrac{dS}{dx} = 4x - 20$

**8.** For the maximum or minimum, solve $\dfrac{dS}{dx} = 0$.

$4x - 20 = 0$

$x = 5$

**9.** Determine the second derivative to verify the minimum.

$\dfrac{d^2S}{dx^2} = 4 > 0$

$S(x)$ is concave up for all $x$, so $S$ is minimum at $x = 5$.

**10.** State the value of $y$.

When $x = 5$, $y = 10 - 5 = 5$.

**11.** Answer the question.

The two numbers that add to 10 and have the minimum sum of their squares are 5 and 5.

---

## WORKED EXAMPLE 12 Determining maximum volume of a container

A cuboid container with a base length twice its width is to be made with $48\,\text{m}^2$ of metal.

**a.** Determine that the height, $h$ m, is $h = \dfrac{8}{x} - \dfrac{2x}{3}$, where $x$ m is the width of the base.

**b.** Express the volume, $V\,\text{m}^3$, in terms of $x$.

**c.** Determine the dimensions of the container with maximum volume.

**d.** Hence, calculate the maximum volume of the container.

**THINK**

**WRITE**

**a. 1.** Draw a diagram of the cuboid.

**a.**
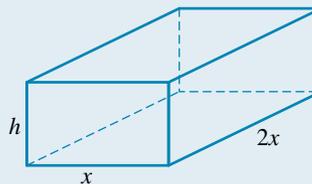


**2.** Let $x =$ width of base and hence express length in terms of $x$.

Let $x =$ width and $h =$ height, so length $= 2x$.

**3.** Calculate the total surface area (TSA) of the cuboid in terms of $x$ and $h$.

$\text{TSA} = 2\,[2x(x) + 2x\,(h) + x\,(h)]$

$\qquad = 2(2x^2 + 3xh)$

$\qquad = 4x^2 + 6xh$

**4.** Express $h$ in terms of $x$.

As $\text{TSA} = 48\,\text{m}^2$

$4x^2 + 6xh = 48$

$6xh = 48 - 4x^2$

$h = \dfrac{48 - 4x^2}{6x}$

$\quad = \dfrac{48}{6x} - \dfrac{4x^2}{6x}$

$h = \dfrac{8}{x} - \dfrac{2x}{3}$

**b. 1.** Determine the volume, $V$, in terms of $x$ and $h$.    **b.** $V = x(2x)h$

**2.** Express the volume in terms of $x$ by substituting for $h$.

$$V(x) = 2x^2 \left( \frac{8}{x} - \frac{2x}{3} \right)$$
$$= 16x - \frac{4x^3}{3}$$

**c. 1.** Differentiate to obtain $\dfrac{dV}{dx}$.

**c.**
$$\frac{dV}{dx} = 16 - \frac{4}{3} \times 3x^2$$
$$\frac{dV}{dx} = 16 - 4x^2$$

**2.** For maximum/minimum values, solve $\dfrac{dV}{dx} = 0$.

$$16 - 4x^2 = 0$$
$$x^2 = 4$$
$$x = \pm 2$$
Reject $x = -2$, as width cannot be negative.

**3.** Determine the second derivative to verify the maximum.

$$\frac{d^2V}{dx^2} = -8x$$
When $x = 2$:
$$\frac{d^2V}{dx^2} = -16 < 0$$
$V(x)$ is concave down, so there is a maximum turning point at $x = 2$.

**4.** Calculate the dimensions by substituting $x = 2$ into the expression for $h$.

$$h = \frac{8}{x} - \frac{2x}{3}$$
$$= \frac{8}{2} - \frac{4}{3}$$
$$= \frac{8}{3}$$

**5.** State the dimensions of the cuboid.

$x = 2$, $2x = 4$, $h = \dfrac{8}{3}$
The dimensions of the cuboid with maximum volume are $2$ m by $4$ m by $\dfrac{8}{3}$ m.

**d. 1.** Calculate the volume, $V = lbh$. Alternatively, substitute into the expression for volume.

**d.** $V = 2 \times 4 \times \dfrac{8}{3} = \dfrac{64}{3}$ m$^3$ or
$$V(x) = 16x - \frac{4}{3}x^3$$
$$V(2) = 16 \times 2 - \frac{4}{3} \times 2^3$$
$$= 32 - \frac{32}{3}$$
$$= \frac{64}{3}$$

**2.** State the answer.

The maximum volume is $\dfrac{64}{3}$ m$^3$.

| **Simple familiar** | **Complex familiar** | **Complex unfamiliar** |
| --- | --- | --- |
| 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 | 13, 14, 15, 16 | 17, 18, 19, 20 |

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### Simple familiar

1. **WE11** The sum of two positive numbers is 32. Determine the numbers if their product is a maximum.

2. The sum of two positive numbers is 8. Determine the numbers if the sum of the cube of one and the square of the other is a minimum.

3. **WE12** The total surface area of a closed cylinder is 200 cm$^2$. The base radius is $r$ cm and the height is $h$ cm.
   a. Express $h$ in terms of $r$.
   b. Prove that the volume, $V$ cm$^3$, is $V = 100r - \pi r^3$.
   c. Prove that for maximum volume the height must equal the diameter of the base.
   d. Calculate, to the nearest integer, the minimum volume if $2 \le r \le 4$.

4. An open rectangular storage bin is to have a volume of 12 metres$^3$. The cost of the materials for its sides is $10 per square metre, and the material for the reinforced base costs $25 per square metre. If the dimensions of the base are $x$ and $y$ metres and the bin has a height of 1.5 metres, determine, with justification, the cost, to the nearest dollar, of the cheapest bin that can be formed under these conditions.

5. The owner of an apartment wants to create a stained glass feature in the shape of a rectangle surmounted by an isosceles triangle of height equal to half its base. This will be adjacent to a door opening onto a balcony.
   The owner has 150 cm of plastic edging to place around the perimeter of the figure and wants to determine the dimensions of the figure with the greatest area.

   a. Show that the area, $A$ in cm$^2$, of the stained glass figure is $A = 150x - \left(2\sqrt{2} + 1\right)x^2$.
   b. Determine, correct to 1 decimal place, the width and the height of the figure for which the area is greatest.
   c. Due to structural limitations, the width of the figure should not exceed 30 cm. Determine the required dimensions of the stained glass figure of maximum area.
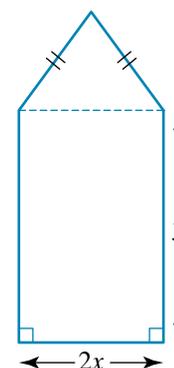
6. A pen for holding farm animals has dimensions $l \times w$ metres. This pen is to be partitioned so that there are four spaces of equal area as shown.
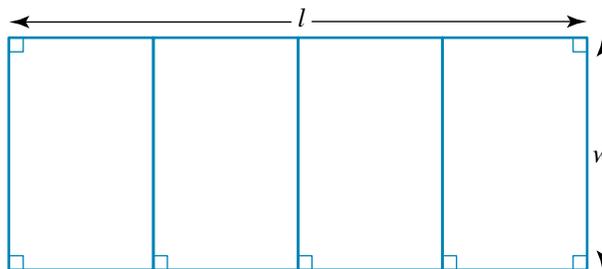
   The farmer has 550 metres of fencing material to construct this pen.
   a. Calculate the required length and width in order to maximise the area of the pen.
   b. Calculate the maximum area.

7. A piece of wire of length 40 cm is cut into two pieces. Each of these two pieces is then bent into the shape of a square. Determine the minimum value of the sum of the areas of the two squares.

8. A rectangular box with an open top is to be constructed from a rectangular sheet of cardboard measuring 16 cm by 10 cm. The box will be made by cutting equal squares of side length $x$ cm out of the four corners and folding the flaps up.
   a. Express the volume as a function of $x$.
   b. Determine the dimensions of the box with greatest volume and give this maximum volume.

9. A piece of wire 240 cm long is used to reinforce the edges of a rectangular box where the length of the box is three times the width.
   a. Determine the maximum volume of the box and state its dimensions.
   b. If the length of the box needs to be restricted to 12 cm or less, determine the maximum volume of the box and state its dimensions.

10. A manufacturer makes open steel boxes with square bases. The volume of each steel box is required to be 2 m³. The steel costs $50 per square metre.
    a. If the length of the square base is $x$ m, show that the total cost, $C$, of the steel is

$$C = 50x^2 + \frac{400}{x}$$

    b. Determine the dimensions of the box if the manufacturer wishes to minimise costs. Give your answers correct to 2 decimal places.

11. The amount of a drug, $D$ milligrams, in the bloodstream at a time $t$ hours after it is administered is given by $D(t) = 30te^{-\frac{t}{3}}$.
    a. Calculate the instantaneous rate of change of the amount of the drug after 1.5 hours, correct to 1 decimal place.
    b. Determine the time when the amount of drug is a maximum and calculate the maximum amount of the drug in the body, correct to 2 decimal places.
    c. Determine for how long the amount of the drug in the body is more than 10 milligrams, write your answer in hours correct to 2 decimal places.

**12.** A right circular cone is inscribed in a sphere of radius 7 cm. In the diagram shown, O is the centre of the sphere, C is the centre of the circular base of the cone, and V is the vertex of the cone.
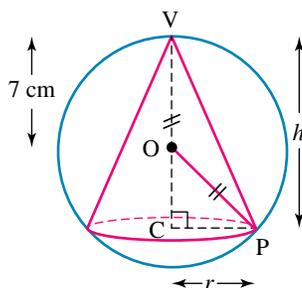


The formula for the volume of a cone of height $h$ cm and base radius $r$ cm is $V_{cone} = \dfrac{1}{3}\pi r^2 h$.

**a.** Show that the volume, $V\,\text{cm}^3$, of the cone satisfies the relationship $V = \dfrac{1}{3}\pi(14h^2 - h^3)$.

**b.** Hence, obtain the exact values of $r$ and $h$ for which the volume is greatest, justifying your answer.

**Complex familiar**

**13.** A section of a rose garden is enclosed by edging to form the shape of a sector ABC of radius $r$ metres and arc length $l$ metres. The perimeter of this section of the garden is 8 metres.
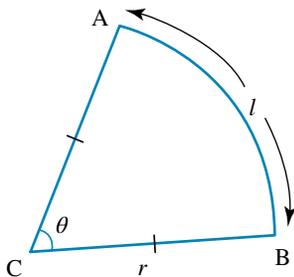


**a.** If $\theta$ is the angle in radian measure subtended by the arc at $C$, express $\theta$ in terms of $r$.

**b.** The formula for the area of a sector is $A_{sector} = \dfrac{1}{2}r^2\theta$. Express the area of a sector in terms of $r$.

**c.** Calculate the value of $\theta$ when the area is greatest.

**14.** A colony of blue wrens, also known as superb fairy wrens, survives in a national park because the wooded areas have rich undergrowth and a plentiful supply of insects, the wrens' main food source. Breeding begins in spring and continues until late summer.

The population of the colony any time $t$ months after 1 September can be modelled by the function

$$P(t) = 200te^{-\frac{t}{4}} + 400,\ 0 \le t \le 12$$

where $P$ is the number of birds in the colony. Determine:

**a.** the initial population of the birds

**b.** when the largest number of birds is reached

**c.** the maximum number of birds, to the nearest bird.

**15.** A rower is in a boat 4 km from the nearest point, O, on a straight beach. His destination is 8 km along the beach from O. Calculate the optimal point on the beach where he should row to reach his destination in the least possible time, given that he rows at 5 km/h and walks at 8 km/h. Give your answer correct to 1 decimal place.
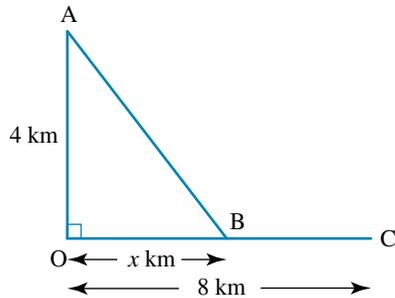


**16.** The amount of money in a savings account $t$ years after the account was opened on 1 January 2016 is given by the equation

$$A(t) = 1000 - 12te^{\frac{4-t^3}{8}} \text{ for } t \in [0, 6]$$

  **a.** Calculate how much money was in the account when the account was first opened.
  **b.** Determine the lowest amount of money in the account.
  **c.** Determine when the account contained its lowest amount. Give the year and month.
  **d.** Calculate how much money was in the account at the end of the 6 years.

**Complex unfamiliar**

**17.** A cylinder has a surface area of $220\pi$ cm$^2$. Determine the height and radius of each end of the cylinder so that the volume of the cylinder is maximised, and determine the maximum volume for the cylinder. Give answers correct to 2 decimal places.

**18.** A liquid is being stored in a cylindrical container. The volume of the container is 10 litres. The metal used to make the cylinders is rare. Determine the dimensions of the container if the costs are to be as cheap as possible, given that the container will be full of liquid. Express your answers correct to 2 decimal places.

**19.** A cone is 10 cm high and has a base radius of 8 cm. Determine the radius and height of a cylinder that is inscribed in the cone such that the volume of the cylinder is a maximum. Determine the maximum volume of the cylinder, correct to the nearest cubic centimetre.

**20.** An ornamental fish pond has the shape shown.
A plastic cover is being made for the pond for the winter months. If the surface area of the pond, $A$, is a constant, show that the perimeter of the pond is a minimum when both $R$ and $l$ are equal to $\sqrt{\dfrac{2A}{\pi + 4}}$.
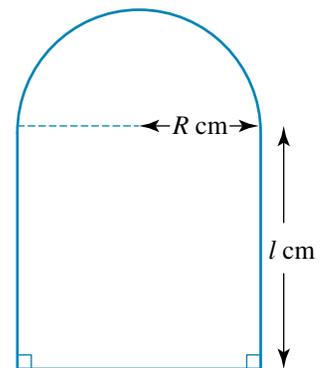


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# LESSON
## 4.6 Review

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| Simple familiar | Complex familiar | Complex unfamiliar |
|---|---|---|
| 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 | 13, 14, 15, 16 | 17, 18, 19, 20 |

**Simple familiar**

1. **MC** The graph of $y = (x+2)^3$ has:
   - **A.** 1 turning point
   - **B.** 2 turning points
   - **C.** 1 point of inflection
   - **D.** 0 stationary points

2. **MC** The graph of $x^3 + 2x^2 + x - 2$ has
   - **A.** 2 points of inflection
   - **B.** 1 turning point and 1 point of inflection
   - **C.** 3 turning points
   - **D.** 2 turning points

3. **MC** The graph of $g(x)$ has the following properties:
   - $g'(x) = 0$ if $x = -3$, 1 and 4
   - $g'(x) < 0$ if $x < -3$ and $1 < x < 4$
   - $g'(x) > 0$ for all other $x$.

   Identify the diagram that shows the graph of $g(x)$.

   **A.**
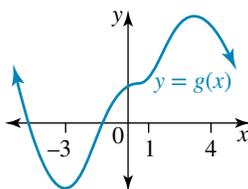
   **B.**
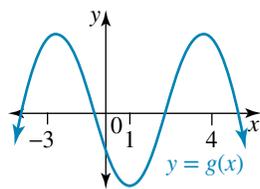
   **C.**
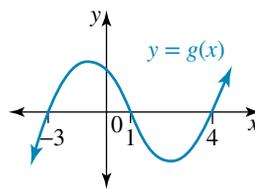
   **D.**
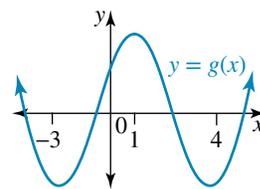
4. **MC** The graph of $f'(x)$ shown indicates that the graph of $f(x)$ has:
   A. a turning point at $x=2$ and $x=-4$
   B. a turning point at $x=2$ and a point of inflection at $x=-4$
   C. a turning point at $x=-4$ and a point of inflection at $x=2$
   D. 2 points of inflection at $x=-4$ and $x=2$
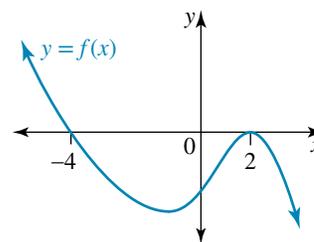


5. A particle moves in a straight line so that at time $t$ seconds its displacement, $x$ metres, from a fixed origin O is given by $x(t)=t^3-6t^2+9t$, $t\geq0$.
   a. Determine how far the particle is from O after 2 seconds.
   b. Calculate the velocity of the particle after 2 seconds.
   c. Determine after how many seconds the particle reaches the origin again, and calculate its velocity at that time.
   d. Determine the particle's acceleration when it reaches the origin again.

*Questions 6 to 9 relate to the isosceles triangle shown, which has a perimeter of 40 cm.*
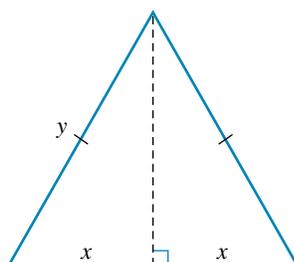


6. **MC** The value of $y$ in terms of $x$ is:
   A. $40-2x$      B. $20-x$      C. $40-x$      D. $20-2x$

7. **MC** The height of the triangle in terms of $x$ is:
   A. $\sqrt{400-40x}$    B. $20-\sqrt{40x}$    C. $\sqrt{400-40x+2x^2}$    D. $\sqrt{400-40x-x^2}$

8. **MC** The area in terms of $x$ is:
   A. $x\sqrt{400-40x+x^2}$   B. $2x\sqrt{400-40x+x^2}$   C. $x\sqrt{400-40x}$   D. $2x\sqrt{400-40x}$

9. **MC** The maximum area of the triangle is obtained if $x$ equals:
   A. $6\dfrac{2}{3}$ cm      B. $10$ cm      C. $5$ cm      D. $10\dfrac{2}{5}$ cm

10. A playground is being constructed by the local council. The shape of the playground is shown. All measurements are in metres. The perimeter of the playground is known to be 96 metres.
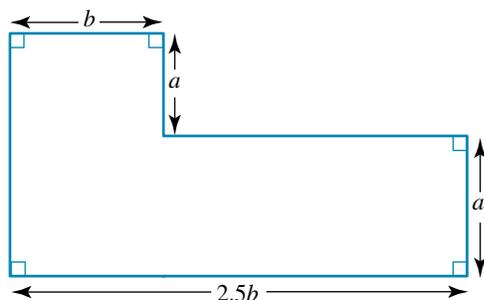


   a. Determine the values of $a$ and $b$ that give a maximum area for the playground.
   b. Determine the maximum area.

**11.** At a time $t$ seconds a bullet is fired vertically upwards, its height $h$ metres above the surface of the Earth is given by $h = 10 + 49t - 4.9t^2$.

  **a.** Determine from what height it was projected.
  **b.** Calculate its initial speed of projection.
  **c.** Determine how long before it reaches its maximum height.
  **d.** Calculate the maximum height reached.
  **e.** Determine how long before it reaches ground level, in seconds to 1 decimal place.
  **f.** Calculate its speed when it strikes the ground, in m/s to 2 decimal places.
  **g.** Show that its acceleration is always constant.

**12.** The position of a particle, $x$ metres, from the origin at time $t$ seconds is given by
$x(t) = \dfrac{1}{4}e^{2t} - 4t^2 - 3t + 10$.

  **a.** Determine the initial position of the particle.
  **b.** Calculate, correct to 2 decimal places, the velocity of the particle after 2 seconds.
  **c.** Derive an expression for the acceleration at any time.
  **d.** Determine when the acceleration of the particle is negative.

### Complex familiar

**13.** The displacement of an object, $x$ metres, from a fixed point at any time, $t$ hours, is given by
$x(t) = 2 \cos\left(\dfrac{\pi t}{12}\right) + 10$.

  **a.** Determine the initial position of the particle.
  **b.** Calculate the velocity of the particle after 3 hours.
  **c.** Show that the particle was initially at rest and determine when the particle is again at rest.
  **d.** Calculate the position and acceleration of the particle at this time.

**14.** Sketch the graphs of each of the following by determining the coordinates of all axis intercepts and any stationary points, and establishing their nature. Also determine the coordinates of the point(s) of inflection.

  **a.** $y = x^3 - x^2 - 16x + 16$
  **b.** $y = (2 - x)(x + 1)(x + 6)$

**15.** A manufacturing company is required to produce cylindrical cans (for tuna) of volume $50\,\text{cm}^3$. The tin used to produce the cans costs 40 cents per $100\,\text{cm}^2$.

  **a.** Determine the area of tin required, $A$, in terms of the radius, $r$.
  **b.** Calculate the radius of the can (to the nearest tenth) for minimum area.
  **c.** Hence, calculate the minimum area (to the nearest tenth).
  **d.** Determine the cost of the tin required to produce $10\,000$ such cans. Give your answer to the nearest \$20.

**16.** Water is being poured into a vase. The volume, $V\,\text{mL}$, of water in the vase after $t$ seconds is given by $V = \dfrac{2}{3}t^2(15 - t)$, $0 \le t \le 10$.

  **a.** Calculate the volume after 10 seconds.
  **b.** Determine the rate at which the water is flowing into the vase at $t$ seconds.
  **c.** Determine the rate of flow after 3 seconds.
  **d.** Determine when the rate of flow is the greatest, and determine the rate of flow at this time.

**17.** The amount of antibiotic drugs, $A(t)$ milligrams, in the body of a patient after time $t$ hours, where $t \geq 0$, is given by $A(t) = 500te^{-\frac{t}{4}}$.

Determine the maximum amount of the drug, in milligrams correct to 2 decimal places, in the patient's body and for how long, to the nearest minute, the amount of the drug in the body is over 350 milligrams.

**18.** A cylinder of cheese is to be removed from a spherical piece of cheese of radius 8 cm.
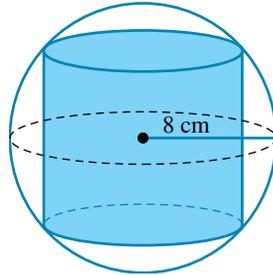


Determine the maximum volume of the cylinder of cheese.
Express your answer to the nearest unit.

**19.** Consider the function $y = \dfrac{18}{x^2 - 9}$.

Discuss the behaviour of the function as $x \to \pm\infty$ and sketch the graph of the function, showing clearly all important features, including the equation(s) of any asymptotes.

**20.** Consider the function $y = (x - 2)^n + 1$, where $n$ is an integer greater than or equal to 2. By discussing various values of $n$, draw conclusions about the shape of functions that can be expressed in this form.

## 4.6 Past QCAA exam questions

**Question 1 (6 marks)**

*Source:* QCAA 2023 Mathematical Methods, Paper 1, Section 2, Q17; © State of Queensland (QCAA) 2024.

A chemical is added to the water in a swimming pool at 10:00 am to prevent algae. The amount of chemical absorbed into the water over time $t$ (hours) is represented by

$$A = 10t^2 - 4t^3, \, 0 \leq t \leq 1\frac{2}{3}$$

Determine the time of day when the rate of absorption of the chemical is at its maximum. Use calculus techniques to verify that your time corresponds to a maximum rate.

**Question 2 (1 mark)**

*Source:* QCAA 2023 Mathematical Methods, Paper 2, Section 1, Q4; © State of Queensland (QCAA) 2024.

The displacement (m) of a moving particle is given by $d = e^{0.5t} - 1$ where $t$ is time (s).

The acceleration $(ms^{-2})$ of the particle when $t = 4$ is

**A.** 7.3891 **B.** 6.3891 **C.** 3.6945 **D.** 1.8473

**Question 3 (4 marks)**

*Source:* QCAA 2022 Mathematical Methods, Paper 1, Section 2, Q15; © State of Queensland (QCAA) 2024.

The derivative of a function is given by $f'(x) = e^x(x - 4)$.

Determine the interval on which the graph of $f(x)$ is both decreasing and concave up.

**Question 4 (3 marks)**

*Source:* QCAA 2022 Mathematical Methods, Paper 1, Section 2, Q16; © State of Queensland (QCAA) 2024.

A section of the graphs of the first and second derivatives of a function are shown.

Sketch a possible graph of the function on the same axes over the domain $0 \leq x \leq 2\pi$.

Explain all reasoning used to produce the sketch.
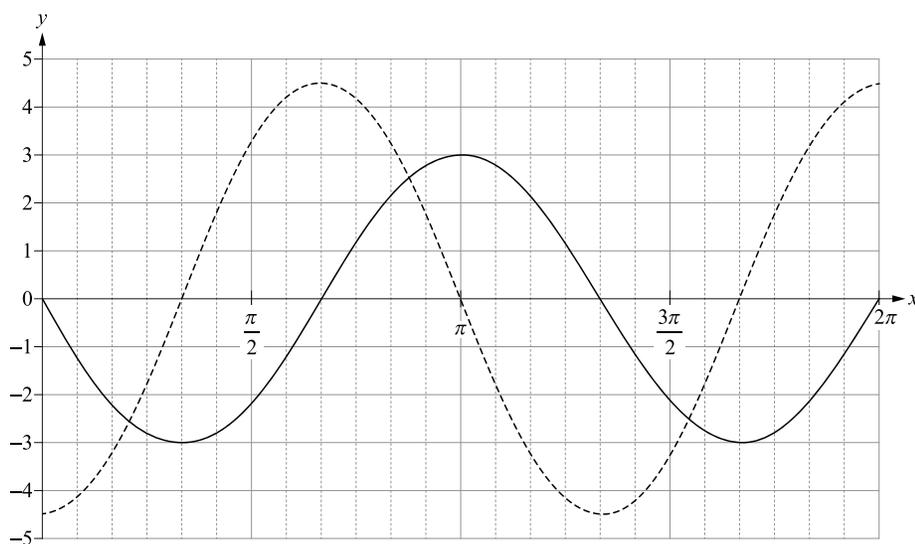


**Question 5 (1 mark)**

*Source:* QCAA 2022 Mathematical Methods, Paper 2, Section 1, Q2; © State of Queensland (QCAA) 2024.

Identify the correct features of the function $f(x) = xe^x$.

**A.** $f'(-1) = 0, f''(-1) < 0$        **B.** $f'(-1) = 0, f''(-1) > 0$

**C.** $f'(-1) < 0, f''(-1) < 0$        **D.** $f'(-1) < 0, f''(-1) > 0$

**Fully worked solutions for this chapter are available online.**

# Answers

## Chapter 4 The second derivative and applications of differentiation

### 4.2 Second derivatives

**4.2 Exercise**

1. a. $12x^2 - 30x + 2$
   b. $6x - 8$
   c. $-2$
   d. $16 - 6x$
   e. $48(2x - 1)^2$

2. a. $\dfrac{3}{4\sqrt{x}}$
   b. $\dfrac{6}{x^4}$
   c. $16\,e^{2x+3}$
   d. $-\dfrac{4}{25}\cos\left(\dfrac{2x}{5}\right)$
   e. $-48\sin(4x - \pi)$

3. a. $\dfrac{1}{x}$   b. $6e^{3x^2}(6x^2 + 1)$
   c. $\dfrac{-1}{(x+1)^2}$

4. $\dfrac{-1}{12}$

5. $-\sqrt{3}$

6. a. $\dfrac{1}{2}$   b. $\dfrac{-9}{2}$

7. a. $\dfrac{-16}{9}$   b. $6e$

8. $\dfrac{20x^3(2x^2 + 7)}{(2x^2 + 5)^2} + 6x\ln(2x^2 + 5)$

9. $3x^2(3x^2 - 8x + 4)e^{-3x}$

10. a. $\dfrac{-2(x^2 + 4x - 5)}{(x^2 + 4x + 13)^2}$
    b. $-e^{3x}(7\cos(4x) + 24\sin(4x))$

11. a. $x(t) = t^3 - t^2 - 5t - 3$   b. $a(t) = 6t - 2$
    c. $1\dfrac{2}{3}$ s   d. $5$ m
    e. $8$ m/s$^2$

12. a. Maximum: 5 m; minimum: 1 m
    b. $\dfrac{1}{3}$ s
    c. $\dfrac{\pi}{3}$ s
    d. $a(t) = -18\cos(3t - 1)$

13. a. i. $v(t) = 4e^{-\frac{t}{2}}(2 - t)$
    ii. $a(t) = 2e^{-\frac{t}{2}}(t - 4)$

   b. 8 m/s
   c. 2 s; $\dfrac{16}{e}$ m $\approx 5.886$ m from the origin
   d. $t \in (4, 6]$

14. a. Initially, the particle has a position of $-3\sqrt{2}$ m, or $3\sqrt{2}$ m to the left of the origin.
    b. The particle is first at rest after 1.5 s.
    c. The acceleration of the particle at 3.5 s is $\dfrac{3\pi^2}{2}$ m/s$^2$.

15. a. Sample responses can be found in the worked solutions in the online resources.
    b. 1

16. a. Sample responses can be found in the worked solutions in the online resources.
    b. $-14.92$ (correct to 2 decimal places)

17. $a = -18,\ b = -16$

18. Sample responses can be found in the worked solutions in the online resources.

19. Sample responses can be found in the worked solutions in the online resources.

20. Sample responses can be found in the worked solutions in the online resources.

### 4.3 Concavity and points of inflection

**4.3 Exercise**

1. a. i. Concave up
      ii. Concave down
   b. $(3, -46)$

2. a. i. Concave down
      ii. Concave up
   b. $(-2, 16)$

3. a. i. Concave up
      ii. Concave up
   b. $\left(\dfrac{4}{3}, 4\dfrac{20}{27}\right)$

4. a. i. $x > -3$
      ii. $x < -3$
   b. $(-3, 54)$

5. a. i. $x > -\dfrac{2}{3}$
      ii. $x < -\dfrac{2}{3}$
   b. $\left(-\dfrac{2}{3}, 3\dfrac{16}{27}\right)$

6. The second derivative does not change sign either side of $x = 0$. In fact, $\dfrac{d^2y}{dx^2} \le 0$ for all $x$, so the curve is always concave down.

7. The second derivative does not change sign either side of $x = 0$. In fact, $\dfrac{d^2y}{dx^2} \ge 0$ for all $x$, so the curve is always concave up.

8. a. $(1, -6)$   b. $(3, -11)$

**9. a. i.** $x < -2$ or $x > 0$

**ii.** $-2 < x < 0$

**b.** $(-2, 19), (0, 3)$

**10. a.** $\left(-\dfrac{1}{6}, \dfrac{5}{432}\right), \left(\dfrac{1}{6}, \dfrac{5}{432}\right)$

**b.** $x < -\dfrac{1}{6}$ or $x > \dfrac{1}{6}$

**11.** Horizontal point of inflection and $(0, 0)$ and point of inflection at $(2, -16)$

**12.** 18

**13. a.** $-2$  **b.** $x < 0$ or $x > 1$

**14.** Sample responses can be found in the worked solutions in the online resources.

**15. a. i.** $x > -2$

**ii.** $x < -2$

**b.** $\left(-2, -\dfrac{2}{e^2}\right)$

**16. a.**



$y = 2\sin(x) + 3$
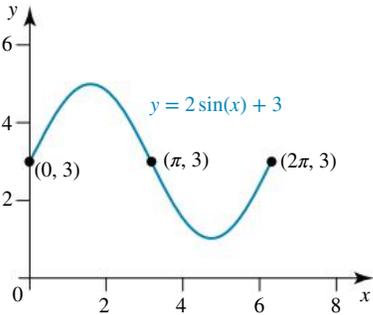
**b. i.** $x \in (\pi, 2\pi)$

**ii.** $x \in (0, \pi)$

**c.** $(\pi, 3)$

**17.** Sample responses can be found in the worked solutions in the online resources.

**18.** $b = -3, c = 3, d = -3$

**19.** $a = -1, b = -6, c = 9$

**20.** Sample responses can be found in the worked solutions in the online resources.

## 4.4 Curve sketching

### 4.4 Exercise

**1.**



$f(x) = x^3 - 4x^2 + 4x$

**2. a.**



$y = x^3 - 27x$

**b.**



$y = 9x - x^3$

**3. a.**



$y = x^3 + 12x^2 + 36x$

**i.** $x < -6$ or $x > -2$

**ii.** $x > -4$

**b.**



$y = -x^3 + 10x^3 - 25x$

**i.** $\dfrac{5}{3} < x < 5$

**ii.** $x < \dfrac{10}{3}$

**4. a.**



$(-1, 0)$   $(5, 0)$
$(0, -5)$
$(1, -16)$
$y = x^3 - 3x^2 - 9x - 5$
$(3, -32)$

**b.**



$y = -x^3 + 9x^2 - 15x - 25$
$(-1, 0)$   $(5, 0)$
$(3, -16)$
$(0, -25)$
$(1, -32)$

**5. a.**



$(6, 432)$
$(4, 256)$
$y = 8x^3 - x^4$
$(0, 0)$   $(8, 0)$

**b. i.** $x > 6$

**ii.** $0 < x < 4$

**6. a.**



$(-3, 0)$   $(3, 0)$
$(0, -9)$   $f(x) = x^4 - 8x^2 - 9$
$(-1.155, -17.889)$   $(1.155, -17.889)$
$(-2, -25)$   $(2, -25)$

**b.** $-2 < x < -\dfrac{2\sqrt{3}}{3}$ or $x > 2$

**7. a.**



$f(x) = (x - 1)^3 + 8$
$(0, 7)$   $(1, 8)$
$(-1, 0)$

**b.** $x < 1$

**8. a.**



$(2, 0)$
$(-2, 0)$
$y = (x - 2)(x + 2)^3$
$(0, -16)$
$(1, -27)$

**b.**



$y = (x - 1)^3(x + 3)$
$(-3, 0)$
$(1, 0)$
$(0, -3)$
$(-1, -16)$
$(-2, -27)$
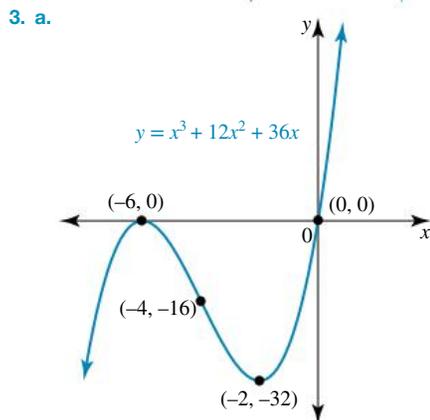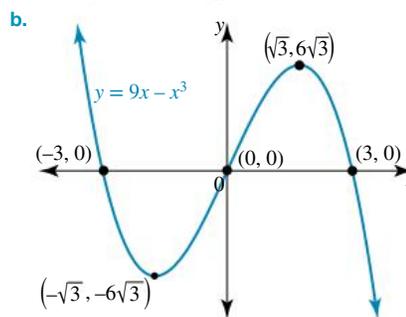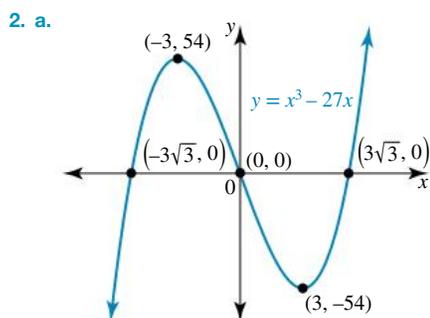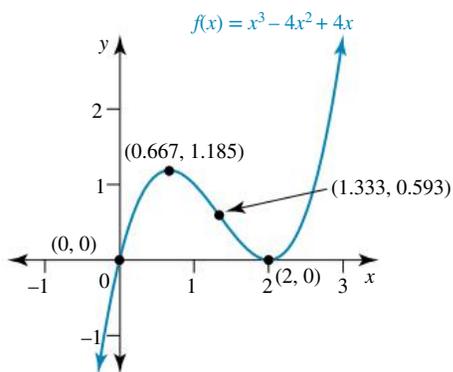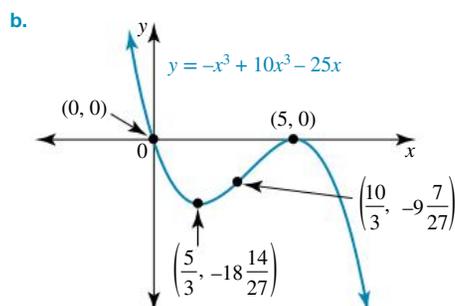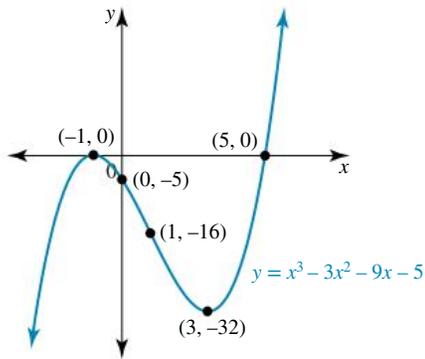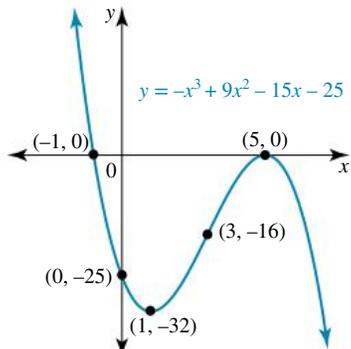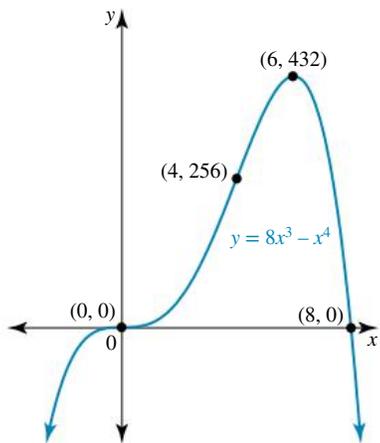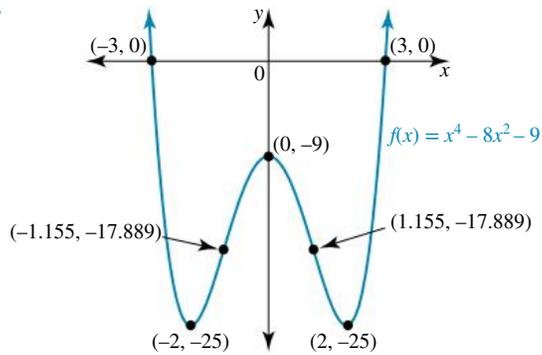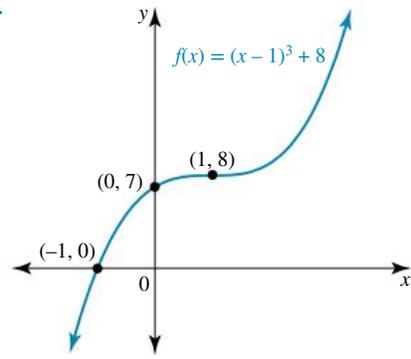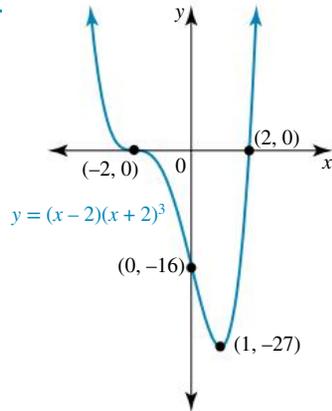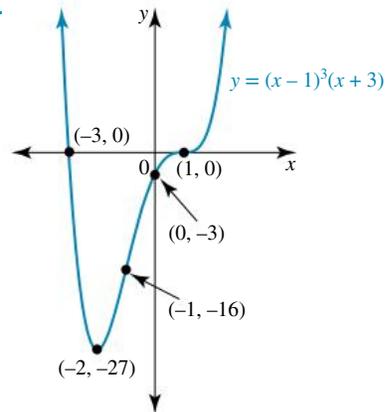
**9.** $\left(\dfrac{-\sqrt{15}}{3}, \dfrac{-44}{9}\right)$ and $\left(\dfrac{\sqrt{15}}{3}, \dfrac{-44}{9}\right)$

$y = x^4 - 10x^2 + 9$
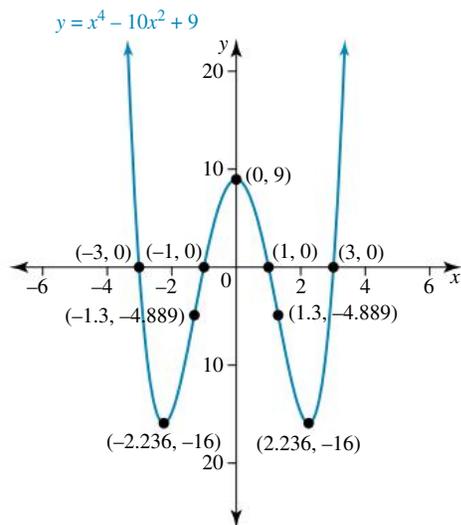


**10.** $\left(\pm 2\sqrt{5}, 0\right)$, $(\pm 2, 0)$, $(0, 80)$ local max.,
$\left(\pm 2\sqrt{3}, -64\right)$ min., $(\pm 2, 0)$ inflection

$y = x^4 - 24x^2 + 80$



**11.**



$y = 6x^2 - x^4$

**12.** $y = -27x + 6$

**13.**



$f(x) = 3xe^{-x}$

**14.**



$f(x) = 4 - 10xe^x$
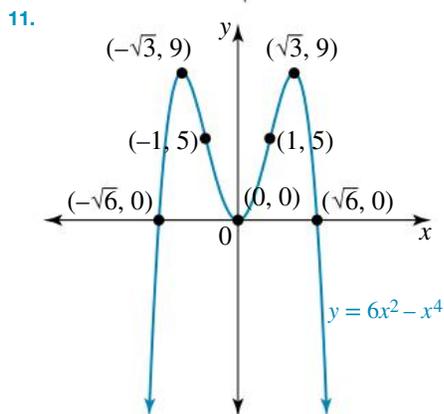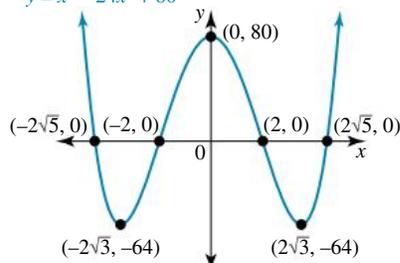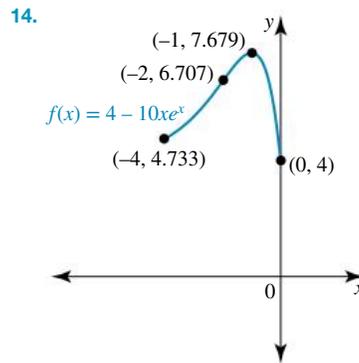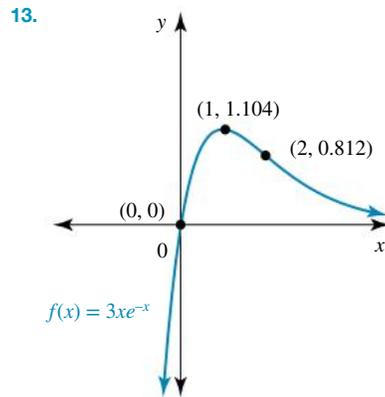
**15.** $b = -6$, $c = 15$, $d = -18$

**16. a.** $x \in R$

**b.** $(0, 0)$; relative minimum

**c.** $\left(-1, \dfrac{1}{2}\ln(2)\right)$, $\left(1, \dfrac{1}{2}\ln(2)\right)$ or $(-1, 0.347)$, $(1, 0.347)$

**d.**



$f(x) = \dfrac{1}{2}\log_e(x^2+1)$

**17.**



$f(x) = \dfrac{10\log_e(x)}{x}$
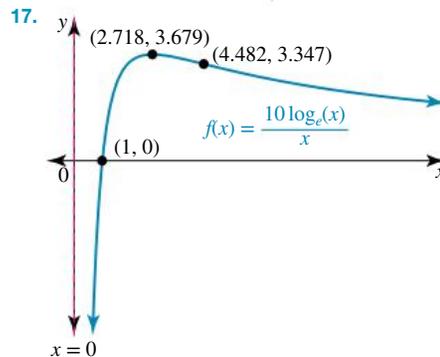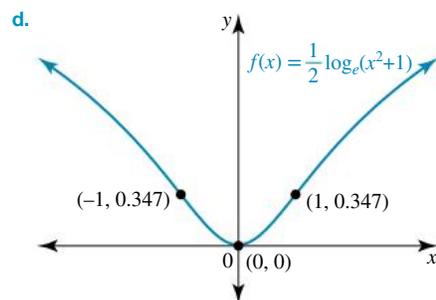
Sample responses can be found in the worked solutions in the online resources.

**18.**
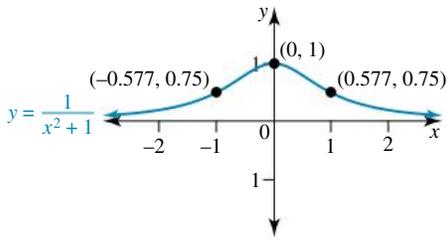


**19.**
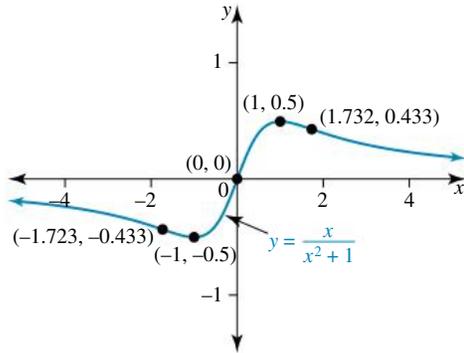


**20.**
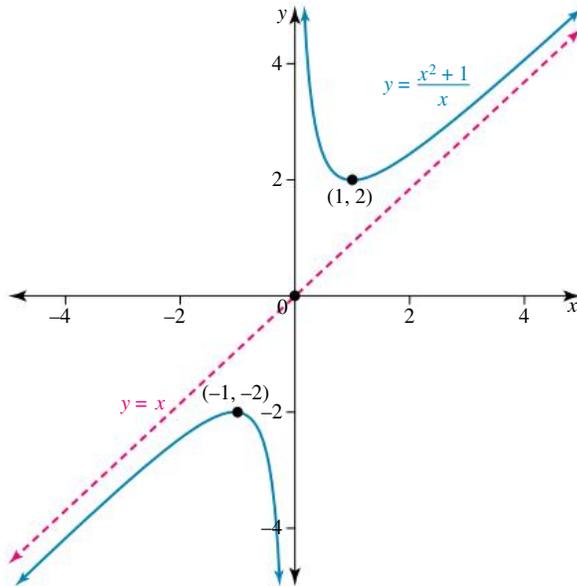


## 4.5 Modelling and solving optimisation problems

### 4.5 Exercise

**1.** 16, 16

**2.** 2 is cubed and 6 is squared.

**3.** **a.** $h = \dfrac{100 - \pi r^2}{\pi r}$

**b, c.** Sample responses can be found in the worked solutions in the online resources.

**d.** 175 cm$^3$

**4.** $370

**5.** **a.** Sample responses can be found in the worked solutions in the online resources.

**b.** Width $= 39.2$ cm and height $= 47.3$ cm

**c.** Width $= 30$ cm and height $= 53.8$ cm

**6.** **a.** $l = 137.5$ m, $w = 55$ m

**b.** $A_{\text{max}} = 7562.5$ m$^2$

**7.** 50 cm$^2$

**8.** **a.** $V = 4x^3 - 52x^2 + 160x$

**b.** Length: 12 cm; width: 6 cm; height: 2 cm; volume 144 cm$^3$

**9.** **a.** 6000 cm$^3$, dimensions: 30 cm $\times$ 10 cm $\times$ 20 cm

**b.** 2112 cm$^3$, dimensions: 12 cm $\times$ 4 cm $\times$ 44 cm

**10.** **a.** Sample responses can be found in the worked solutions in the online resources.

**b.** Square base of length 1.59 m with a height of 0.79 m (2 d.p.)

**11.** **a.** 9.1 mg/hour

**b.** 3 hours, 33.11 mg

**c.** 9.91 hours

**12.** **a.** Sample responses can be found in the worked solutions in the online resources.

**b.** $r = \dfrac{14\sqrt{2}}{3}, h = \dfrac{28}{3}$

**13.** **a.** $\theta = \dfrac{8 - 2r}{r}$    **b.** $A = 4r - r^2$    **c.** 2 radians

**14.** **a.** 400 birds

**b.** At the end of December

**c.** 694 birds

**15.** 3.2 km to the right of point O

**16.** **a.** $1000

**b.** $A_{\text{least}} = \$980.34$

**c.** May 2017

**d.** $1000

**17.** Radius: 6.06 cm; height: 12.11 cm; volume: 1395.04 cm$^3$

**18.** Radius 9.27 cm, height 18.53 cm (to 2 d.p.)

**19.** Radius: $5\dfrac{1}{3}$ cm; height: $3\dfrac{1}{3}$ cm; volume: 298 cm$^3$

**20.** Sample responses can be found in the worked solutions in the online resources.

## 4.6 Review

### 4.6 Exercise

**1.** C

**2.** D

**3.** D

**4.** C

**5.** **a.** 2 m    **b.** $-3$ m/s    **c.** 3 s; 0 m/s

**d.** 6 m/s$^2$

**6.** B

**7.** A

**8.** C

**9.** A

**10.** **a.** $a = 12$ m, $b = 9.6$ m

**b.** 403.2 m$^2$

**11.** **a.** 10 m    **b.** 49 m/s    **c.** 5 s

**d.** 132.5 m    **e.** 10.2 s    **f.** 50.96 m/s

**g.** $a = -9.8$ m/s$^2$

**12. a.** 10.25 m from the origin

**b.** 8.30 m/s

**c.** $a(t) = e^{2t} - 8$

**d.** $0 \le t < \ln\left(\sqrt{8}\right)$ or $t \in \left[0, \ln\left(2, \sqrt{2}\right)\right)$

**13. a.** 12 m from the fixed point

**b.** $\left(-\dfrac{\pi\sqrt{2}}{12}\right)$ m/h $\approx -0.370$ m/h

**c.** After 12 hours

**d.** 8 m from the fixed point; $\left(\dfrac{\pi^2}{72}\right)$ m/h$^2 \approx 0.137$ m/h$^2$

**14. a.**



$y = x^3 - x^2 - 16x + 16$

**b.**



$y = x^3 - 5x^2 + 8x + 12$
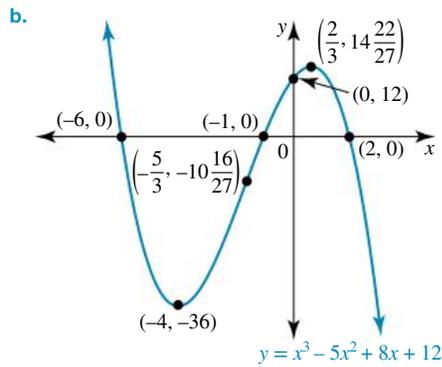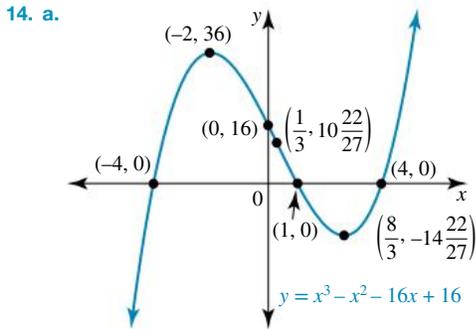
**15. a.** $A = 2\pi r^2 + \dfrac{100}{r}$     **b.** 2.0 cm

**c.** 75.1 cm$^2$     **d.** \$3000

**16. a.** $333\dfrac{1}{3}$ mL

**b.** $\dfrac{dV}{dt} = 20t - 2t^2$

**c.** 42 mL/s

**d.** 5 s; 50 mL/s

**17.** 735.76 mg; 10 hours and 10 minutes.

**18.** 1238 cm$^3$
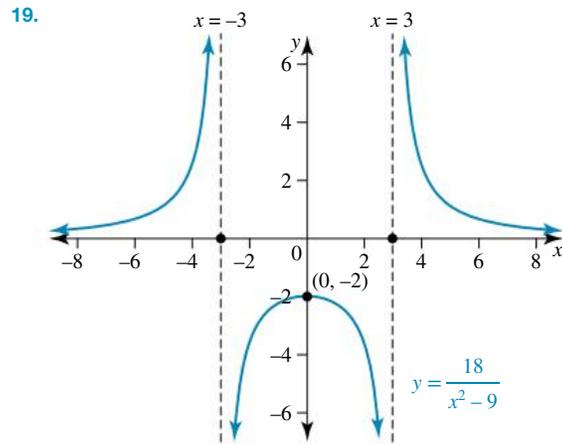
**19.**



Sample responses can be found in the worked solutions in the online resources.

**20.** Sample responses can be found in the worked solutions in the online resources.

## 4.6 Past QCAA exam questions

**1.** Sample responses can be found in the worked solutions in the online resources.

**2.** D

**3.** $3 < x < 4$

**4.** $f(x) = \displaystyle\int -3\sin\left(\dfrac{3x}{2}\right)dx = 2\cos\left(\dfrac{3x}{2}\right)$

See the image at the bottom of the page.*

**5.** B

*4.

# 5 Anti-differentiation

## LESSON SEQUENCE

**Fully worked solutions of this chapter are available online.**

## EXAM PREPARATION

Access exam-style questions in every lesson, available online and access past QCAA exam questions in every review.

**on** Resources

| | | |
|---|---|---|
| **Solutions** | Solutions — Chapter 5 | |
| **Exam questions** | Exam question booklet — Chapter 5 | |
| **Digital documents** | Learning matrix — Chapter 5 | |
| | Chapter summary — Chapter 5 | |

# LESSON
## 5.1 Overview

### 5.1.1 Introduction

In the previous chapter, you studied differential calculus along with some of its applications. Calculus involves the study of change, how a change in one variable will affect another related variable. Anti-differentiation, also known as integration, is the reverse process of differentiation. It, too, involves the study of change. If the rate of change between two variables, the derivative, is known, the relationship connecting the variables may be found using anti-differentiation.

### 5.1.2 Syllabus links

| Lesson | Lesson title | Syllabus links |
|---|---|---|
| 5.2 | **Anti-differentiation of rational functions** | ○ Recognise anti-differentiation as the reverse of differentiation. <br> ○ Use the notation $\int f(x)\,dx$ for anti-derivatives or indefinite integrals. <br> ○ Use the formula $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + c$ for $n \neq -1$ <br> ○ Understand and use the formulas $\int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$ and $\int k\,f(x)\,dx = k \int f(x)\,dx$ |
| 5.3 | **Anti-differentiation of exponential functions** | ○ Use the formula $\int e^x\,dx = e^x + c$ |
| 5.4 | **Anti-differentiation of logarithmic functions** | ○ Use the formula $\int \dfrac{1}{x}\,dx = \ln(x) + c$, for $x > 0$ |
| 5.5 | **Anti-differentiation of trigonometric functions** | ○ Use the formulas $\int \sin(x)\,dx = -\cos(x) + c$ and $\int \cos(x)\,dx = \sin(x) + c$ |
| 5.6 | **Further integration** | ○ Determine indefinite integrals of the form $\int f(ax+b)\,dx$. <br> ○ Determine $f(x)$ given $f'(x)$ and an initial condition $f(a) = b$. |
| 5.7 | **Modelling and solving using indefinite integrals** | ○ Determine displacement given velocity and the initial value of displacement. <br> ○ Determine displacement given acceleration and initial values of displacement and velocity. <br> ○ Model and solve problems that involve indefinite integrals, with and without technology. |

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# LESSON
## 5.2 Anti-differentiation of rational functions

### 5.2.1 Introduction

**Anti-differentiation**, also known as **integration**, is the reverse of differentiation. It allows us to determine $f(x)$ when we are given $f'(x)$.

Consider the following polynomial functions.

$$f(x) = x^2 + 7 \qquad g(x) = x^2 - 5 \qquad h(x) = x^2 + 2$$
$$f'(x) = 2x \qquad\quad g'(x) = 2x \qquad\quad h'(x) = 2x$$

The derivatives of the three functions all equal $2x$, so the **anti-derivative** of $2x$ could be either $x^2 + 7$, $x^2 - 5$, $x^2 + 2$. These three functions differ by a constant.

In general, if $f'(x) = 2x$, then $f(x) = x^2 + c$, where $c$ is a constant. This gives a family of curves that fit the criteria for the function $f$, that is $f'(x) = 2x$. Some of these curves are shown.

To determine a specific answer for $f(x)$ given $f'(x) = 2x$, additional information is required, such as a point through which the curve passes. This is discussed later in this chapter.
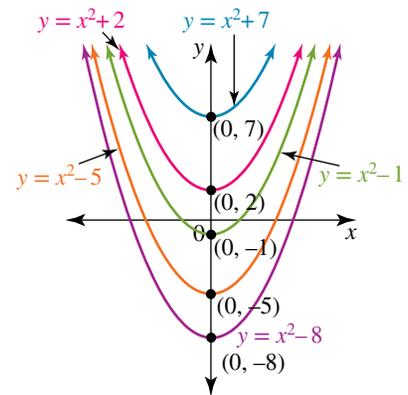


### 5.2.2 Notation for anti-derivatives

An example of this notation is $\int 2x\,dx = x^2 + c.$

This equation indicates that the anti-derivative, or **indefinite integral**, of $2x$ with respect to $x$ is equal to $x^2 + c$.

The indefinite integral of $f(x)$ is $\int f(x)\,dx.$

This is read as 'the integral of the function $f(x)$ with respect to the variable $x$'.

The $dx$ indicates that the variable is $x$.

Another example is $\int g(t)\,dt$, which reads as 'the integral of the function $g(t)$ with respect to the variable $t$'.

## 5.2.3 The anti-derivative of $x^n$, $n \neq -1$

Consider the following.

$$f(x) = x^3 \qquad\qquad f(x) = x^4 \qquad\qquad f(x) = x^5 \qquad\qquad f(x) = x^6$$
$$f'(x) = 3x^2 \qquad\qquad f'(x) = 4x^3 \qquad\qquad f'(x) = 5x^4 \qquad\qquad f'(x) = 6x^5$$

$$\therefore \int 3x^2 \, dx = x^3 + c \qquad \therefore \int 4x^3 \, dx = x^4 + c \qquad \therefore \int 5x^4 \, dx = x^5 + c \qquad \therefore \int 6x^5 \, dx = x^6 + c$$

So: \qquad\qquad\qquad So: \qquad\qquad\qquad So: \qquad\qquad\qquad So:

$$\int x^2 \, dx = \frac{1}{3} x^3 + c \qquad \int x^3 \, dx = \frac{1}{4} x^4 + c \qquad \int x^4 \, dx = \frac{1}{5} x^5 + c \qquad \int x^5 \, dx = \frac{1}{6} x^6 + c$$

This shows the general formula for integration is as follows.

> ### The general formula for integration
>
> $$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + c, \; n \neq -1$$
>
> **where $c$ is a constant.**

This formula is true for all real values of $n$ except for $n = -1$.

Consider the following.

$$f(x) = \sqrt{x} \qquad\qquad\qquad\qquad f(x) = \frac{1}{x^2}$$
$$= x^{\frac{1}{2}} \qquad\qquad\qquad\qquad = x^{-2}$$
$$f'(x) = \frac{1}{2} x^{-\frac{1}{2}} \qquad\qquad\qquad f'(x) = -2x^{-3}$$

$$\therefore \int \frac{1}{2} x^{-\frac{1}{2}} \, dx = \sqrt{x} + c \qquad\qquad \therefore \int -2x^{-3} \, dx = x^{-2} + c$$

So: \qquad\qquad\qquad\qquad\qquad\qquad\qquad So:

$$\int x^{-\frac{1}{2}} \, dx = 2\sqrt{x} + c \qquad\qquad\qquad \int x^{-3} \, dx = \frac{-1}{2} x^{-2} + c$$

---

### WORKED EXAMPLE 1 Anti-derivative of $x^n$

**Determine:**

**a.** $\displaystyle\int x^4 \, dx$ \qquad\qquad **b.** $\displaystyle\int \sqrt[3]{x} \, dx$ \qquad\qquad **c.** $\displaystyle\int \frac{1}{x^3} \, dx$

**THINK** \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad **WRITE**

**a. 1.** Apply the rule $\displaystyle\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$ \qquad **a.** $\displaystyle\int x^4 \, dx = \frac{x^{4+1}}{5} + c$

**2.** Simplify the answer. \qquad\qquad\qquad\qquad\qquad\qquad $\displaystyle = \frac{x^5}{5} + c$

**b. 1.** Write the integral in the form $\int x^n\,dx$

**b.** $\int \sqrt[3]{x}\,dx = \int x^{\frac{1}{3}}\,dx$

**2.** Apply the rule $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + c$

$$= \dfrac{x^{\frac{1}{3}+1}}{\frac{4}{3}} + c$$

**3.** Simplify the answer.

$$= \dfrac{3x^{\frac{4}{3}}}{4} + c$$

**c. 1.** Write the integral in the form $\int x^n\,dx$

**c.** $\int \dfrac{1}{x^3}\,dx = \int x^{-3}\,dx$

**2** Apply the rule $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + c$

$$= \dfrac{x^{-3+1}}{-2} + c$$

$$= -\dfrac{x^{-2}}{2} + c$$

**3.** Simplify the answer.

$$= -\dfrac{1}{2x^2} + c$$

## 5.2.4 Properties of anti-differentiation

Integration, being the reverse of differentiation, has the corresponding properties.

**Properties of integration**

$$\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

$$\int k\,f(x)\,dx = k \int f(x)\,dx$$

$$\int k\,dx = k \int 1\,dx = kx + c$$

**where $k$ and $c$ are constants.**

That is, we can anti-differentiate the separate components of an expression. For example:

$$\int \left(4x^3 + 6x^2 - 9x + 7\right) dx = \dfrac{4x^{3+1}}{4} + \dfrac{6x^{2+1}}{3} - \dfrac{9x^{1+1}}{2} + 7x + c$$

$$= x^4 + 2x^3 - \dfrac{9}{2}x^2 + 7x + c$$

To check your anti-derivative is correct, it is always good to differentiate your answer to see if the derivative matches the original expression.

## WORKED EXAMPLE 2 Applying anti-derivative properties

**Determine the following.**

**a.** $\displaystyle\int 6x^2\,dx$

**b.** $\displaystyle\int \left(10x^4 - 5x^2 + 7\right)dx$

| THINK | WRITE |
|---|---|
| **a. 1.** Apply the formula. | **a.** $\displaystyle\int 6x^2\,dx$<br><br>$= 6 \times \dfrac{x^3}{3} + c$ |
| **2.** Simplify. | $= 2x^3 + c$ |
| **b. 1.** Integrate each term separately by applying the formula. | **b.** $\displaystyle\int (10x^4 - 5x^2 + 7)\,dx$<br><br>$= 10 \times \dfrac{1}{5}x^5 - 5 \times \dfrac{1}{3}x^3 + 7x + c$ |
| **2.** Simplify. | $= 2x^5 - \dfrac{5}{3}x^3 + 7x + c$ |

## WORKED EXAMPLE 3 Determining $y$ given $\dfrac{dy}{dx}$

**Determine $y$ in terms of $x$ if $\dfrac{dy}{dx} = 2\sqrt{x} + \dfrac{3}{x^3} - 4.$**

| THINK | WRITE |
|---|---|
| **1.** Express as powers of $x$. | $\dfrac{dy}{dx} = 2\sqrt{x} + \dfrac{3}{x^3} - 4$<br><br>$\dfrac{dy}{dx} = 2x^{\frac{1}{2}} + 3x^{-3} - 4$ |
| **2.** Integrate each term separately by applying the formula. | $y = \displaystyle\int (2x^{\frac{1}{2}} + 3x^{-3} - 4)\,dx$<br><br>$= 2 \times \dfrac{x^{\frac{3}{2}}}{\frac{3}{2}} + 3 \times \dfrac{x^{-2}}{-2} - 4x + c$ |
| **3.** Simplify. | $y = \dfrac{4}{3}x^{\frac{3}{2}} - \dfrac{3}{2x^2} - 4x + c$ |

## WORKED EXAMPLE 4 Simplifying expressions and anti-differentiation

**Determine:**

a. an anti-derivative of $(2x - 3)(4 - x)$

b. $\int \left( \dfrac{x^4 - 2x^3 + 5}{x^3} \right) dx$

**THINK**

**WRITE**

a. 1. Expand the expression and simplify.

a. $\int (2x - 3)(4 - x)\, dx$

$= \int (8x - 2x^2 - 12 + 3x)\, dx$

$= \int (-2x^2 + 11x - 12)\, dx$

2. Integrate each term separately by applying the formula.

$= -2 \times \dfrac{x^3}{3} + 11 \times \dfrac{x^2}{2} - 12x + c$

3. Simplify.

$= -\dfrac{2}{3} x^3 + \dfrac{11}{2} x^2 - 12x + c$

b. 1. Expand the expression and simplify.

b. $\int \left( \dfrac{x^4 - 2x^3 + 5}{x^3} \right) dx$

$= \int (x^4 - 2x^3 + 5) \times x^{-3}\, dx$

$= \int (x - 2 + 5x^{-3})\, dx$

2. Integrate each term separately by applying the formula.

$= \dfrac{x^2}{2} - 2x + 5 \times \dfrac{x^{-2}}{-2} + c$

3. Simplify.

$= \dfrac{x^2}{2} - 2x - \dfrac{5}{2x^2} + c$

# Exercise 5.2 Anti-differentiation of rational functions

| 5.2 Exercise | 5.2 Exam questions on |
|---|---|

| Simple familiar | Complex familiar | Complex unfamiliar |
|---|---|---|
| 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 | 13, 14, 15, 16 | 17, 18, 19, 20 |

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## Simple familiar

1. **WE1** Determine

   a. $\displaystyle\int x^6 \, dx$
   b. $\displaystyle\int \sqrt[4]{x} \, dx$
   c. $\displaystyle\int \frac{1}{x^2} \, dx$
   d. $\displaystyle\int \frac{1}{\sqrt[3]{x^2}} \, dx$

2. **WE2** Determine the following.

   a. $\displaystyle\int x^7 \, dx$
   b. $\displaystyle\int (8x^3 + 4x) \, dx$
   c. $\displaystyle\int (3x^2 + 5x - 8) \, dx$
   d. $\displaystyle\int (2x^3 + 3x^2 - 6x - 9) \, dx$

3. Determine the following indefinite integrals.

   a. $\displaystyle\int (2x + 5) \, dx$
   b. $\displaystyle\int \left(3x^2 + 4x - 10\right) dx$
   c. $\displaystyle\int \left(10x^4 + 6x^3 + 2\right) dx$
   d. $\displaystyle\int \left(x^3 + 12 - x^2\right) dx$

4. **WE3** Determine $y$ in terms of $x$ if:

   a. $\dfrac{dy}{dx} = 4\sqrt{x} - \dfrac{1}{x^2}$
   b. $\dfrac{dy}{dx} = 6\sqrt{x} + \dfrac{3}{\sqrt{x}} + 8$

5. Integrate the following, expressing your answers with positive powers.

   a. $\dfrac{x^4}{5}$
   b. $\dfrac{x^{-4}}{3}$
   c. $\sqrt{x}$
   d. $x^{\frac{2}{3}}$

6. Determine the anti-derivatives of the following, expressing your answers with positive powers.

   a. $\dfrac{5}{x^3}$
   b. $\dfrac{-10}{x^6}$
   c. $\dfrac{8}{\sqrt{x}}$
   d. $\dfrac{-6}{x\sqrt{x}}$

7. **WE4a** Determine an anti-derivative of:

   a. $(x + 3)(x - 7)$
   b. $5\left(x^2 + 2x - 1\right)$
   c. $\left(x^2 + 4\right)(x - 7)$
   d. $x(x - 1)(x + 4)$

8. **WE4b** Determine:

   a. $\displaystyle\int \frac{x^2 + x^4}{x} \, dx$
   b. $\displaystyle\int \frac{x^2 + 2x - 1}{\sqrt{x}} \, dx$
   c. $\displaystyle\int \frac{10 - x + 2x^4}{x^3} \, dx$

9. Given that $f'(x) = x^2 - \dfrac{1}{x^2}$, determine the rule for $f$.

**10.** Determine:

$$\int 7x^2 - \frac{2}{5x^3}\, dx$$

**11.** Determine:

$$\int (4x^3 - 7x^2 + 2x - 1)\, dx$$

**12.** Determine:

$$\int \left(2\sqrt{x}\right)^3 dx$$

**13.** Determine:

a. $f(x)$ if $f'(x) = \frac{3}{2}x - 4x^2 + 2x^3$

b. an anti-derivative of $\dfrac{3}{\sqrt{x}} - 4x^3 + \dfrac{2}{5x^3}$

**14.** Determine:

a. $\displaystyle\int x(x-3)(2x+5)\, dx$

b. $\displaystyle\int \frac{3x^3 - x}{2\sqrt{x}}\, dx$

**15.** Calculate:

a. $\displaystyle\int \left(\frac{2}{\sqrt{x}} + \frac{3}{x^2} - \frac{1}{2x^3}\right) dx$

b. $\displaystyle\int (x+1)(2x^2 - 3x + 4)\, dx$

**16.** Determine an anti-derivative for each of the following functions.

a. $(2x+3)(3x-2)$

b. $\dfrac{x^3 + x^2 + 1}{x^2}$

c. $2\sqrt{x} - \dfrac{4}{\sqrt{x}}$

d. $\left(x^3 - \dfrac{2}{x^3}\right)^2$

**17.** The gradient function for a particular curve is given by $\dfrac{dy}{dx} = x^3 - 3\sqrt{x}$. Determine the general rule for the function, $y$.

**18.** Determine the general equation of the curve whose gradient at any point is given by $\dfrac{x^3 + 3x^2 - 3}{x^2}$.

**19.** Determine the general equation of the curve whose gradient at any point on the curve is given by $\sqrt{x} + \dfrac{1}{\sqrt{x}}$.

**20.** If $y = \sqrt{x^2 + 1}$, determine $\dfrac{dy}{dx}$ and hence find the anti-derivative of $\dfrac{5x}{\sqrt{x^2 + 1}}$.

**Fully worked solutions for this chapter are available online.**

# LESSON
## 5.3 Anti-differentiation of exponential functions

### 5.3.1 Anti-differentiation of exponential functions

As you have learned in Chapter 1:

$$-y = e^x \quad \text{and} \quad y = e^{ax},$$

$$\frac{dy}{dx} = e^x \quad \text{and} \quad \frac{dy}{dx} = ae^{ax}.$$

Therefore, it follows that:

$$\int e^x \, dx = e^x + c \text{ and } \int e^{ax} \, dx = \frac{1}{a} e^{ax} + c$$

where $c$ is a constant.

---

**Anti-differentiation of exponential functions**

$$\int e^x \, dx = e^x + c$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax} + c$$

where $a$ and $c$ are constants.

---

Generally:

---

**Anti-differentiation of exponential functions including constants**

$$\int ke^{ax} \, dx = k \int e^{ax} \, dx$$

$$= k \times \frac{1}{a} e^{ax} + c$$

$$= \frac{k}{a} e^{ax} + c$$

where $a$, $c$ and $k$ are constants.

---

## WORKED EXAMPLE 5 Anti-derivative of $e^x$

**Determine the following.**

**a.** $\displaystyle\int 8e^x\,dx$

**b.** $\displaystyle\int 8e^{2x}\,dx$

| THINK | WRITE |
|---|---|
| **a.** Apply the formula. | **a.** $\displaystyle\int 8e^x\,dx$ <br><br> $= 8\displaystyle\int e^x\,dx$ <br><br> $= 8e^x + c$ |
| **b. 1.** Apply the formula. | **b.** $\displaystyle\int 8e^{2x}\,dx$ <br><br> $= 8\displaystyle\int e^{2x}\,dx$ <br><br> $= 8 \times \dfrac{1}{2}\,e^{2x} + c$ |
| **2.** Simplify. | $= 4e^{2x} + c$ |

## WORKED EXAMPLE 6 Further anti-differentiation

**Determine $y$ if it is known that $\dfrac{dy}{dx} = (e^x + e^{-x})^2.$**

| THINK | WRITE |
|---|---|
| **1.** Expand the brackets. | $\dfrac{dy}{dx} = (e^x + e^{-x})^2$ <br><br> $\dfrac{dy}{dx} = (e^x + e^{-x})(e^x + e^{-x})$ <br><br> $\dfrac{dy}{dx} = (e^{2x} + e^0 + e^0 + e^{-2x})$ |
| **2.** Simplify. | $\dfrac{dy}{dx} = (e^{2x} + 2 + e^{-2x})$ |
| **3.** Integrate each term separately by applying the formula. | $y = \displaystyle\int (e^{2x} + 2 + e^{-2x})\,dx$ <br><br> $y = \dfrac{1}{2}\,e^{2x} + 2x + \dfrac{1}{-2}\,e^{-2x} + c$ |
| **4.** Simplify. | $y = \dfrac{1}{2}\,e^{2x} + 2x - \dfrac{1}{2}\,e^{-2x} + c$ |

# Exercise 5.3 Anti-differentiation of exponential functions

**5.3 Exercise** | **5.3 Exam questions** on | These questions are even better in jacPLUS!

| Simple familiar | Complex familiar | Complex unfamiliar |
|---|---|---|
| 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 | 13, 14, 15, 16 | 17, 18, 19, 20 |

- Receive immediate feedback
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## Simple familiar

1. **WE5** Determine the following.

   a. $\displaystyle\int e^{2x}\, dx$
   b. $\displaystyle\int e^{4x}\, dx$
   c. $\displaystyle\int e^{-x}\, dx$

2. Determine the following.

   a. $\displaystyle\int e^{-3x}\, dx$
   b. $\displaystyle\int 5e^{5x}\, dx$
   c. $\displaystyle\int 7e^{4x}\, dx$

3. Determine the anti-derivatives of the following.

   a. $e^{\frac{x}{3}}$
   b. $0.1e^{\frac{x}{4}}$
   c. $3e^{\frac{x}{2}}$

4. Determine the anti-derivatives of the following.

   a. $3e^{\frac{-x}{3}}$
   b. $e^x + e^{-x}$
   c. $\dfrac{e^x - e^{-x}}{2}$

5. **WE6** Determine $y$ if it is known that $\dfrac{dy}{dx} = (e^x - e^{-x})^2$.

6. Determine $y$ if it is known that $\dfrac{dy}{dx} = \left(1 + e^{2x}\right)^2$.

7. If $\dfrac{dy}{dx} = \left(e^{3x} + 6\right)^2$, determine $y$ as a function of $x$.

8. Determine:

   a. $\displaystyle\int (x^4 - e^{-4x})\, dx$
   b. $\displaystyle\int \left(\frac{1}{2}\, e^{2x} - \frac{2}{3}\, e^{-\frac{x}{2}}\right) dx$

9. Determine:

   a. $\displaystyle\int \frac{e^{2x} + 3e^{-5x}}{2e^x}\, dx$
   b. $\displaystyle\int (e^x - e^{2x})^2\, dx$

10. Determine the indefinite integral of $\left(e^{\frac{x}{2}} - \dfrac{1}{e^x}\right)^2$.

11. **MC** If $\displaystyle\int ae^{bx}\, dx = -2e^{2x} + c$, then

   A. $a = 4$ and $b = -2$
   B. $a = -2$ and $b = 2$
   C. $a = -4$ and $b = 2$
   D. $a = -4$ and $b = -2$

**12.** **MC** The anti-derivative of $e^{3x} + \dfrac{2}{e^{4x}}$ is:

A. $\dfrac{e^{3x}}{3} - \dfrac{8}{e^{4x}} + c$

B. $3e^{3x} + \dfrac{2}{e^{4x}} + c$

C. $3e^{3x} + \dfrac{e^{4x}}{2} + c$

D. $\dfrac{e^{3x}}{3} - \dfrac{1}{2e^{4x}} + c$

### Complex familiar

**13.** The gradient function of a curve is given by $f'(x) = 4e^{2x} + 8$. Determine the general rule for the function $f(x)$.

**14.** Determine the general rule for the function $y = f(x)$ if it is known that $\dfrac{dy}{dx} = e^{2x}\left(e^{2x} - e^{-2x}\right)$.

**15.** Determine the general equation of the curve whose gradient at any point is given by $6e^{3x} + 9x^2 - 2\sqrt{e^x}$.

**16.** Determine $\displaystyle\int \left(e^{2x} - e^{-3x}\right)^3 dx$.

### Complex unfamiliar

**17.** A curve has a gradient function $f'(x) = 4e^{-2x} + k$, where $k \in R$. The function has a stationary point when $x = 0$.
Determine the general rule for the function $f(x)$.

**18.** If it is known that $\displaystyle\int ae^{bx}\, dx = -3e^{3x} + c$, determine the exact values of the constants $a$ and $b$.

**19.** It is known that $\displaystyle\int (me^{nx} + px + q)\, dx = 5e^{2x} + 2x^2 - 3x + c$. Determine the exact values of the constants $m$, $n$, $p$ and $q$.

**20.** Determine the derivative of $e^{(x+1)^3}$ and hence $\displaystyle\int 9(x+1)^2 e^{(x+1)^3}\, dx$.

**Fully worked solutions for this chapter are available online.**

# LESSON
## 5.4 Anti-differentiation of logarithmic functions

### 5.4.1 Anti-differentiation of logarithmic functions

As you have learned in Chapter 1:

$$y = \ln(x) \qquad \text{and} \qquad y = \ln(ax + b),$$

$$\frac{dy}{dx} = \frac{1}{x} \qquad \text{and} \qquad \frac{dy}{dx} = \frac{a}{(ax + b)}.$$

Therefore, it follows that:

$$\int \frac{1}{x} dx = \ln(x) + c \text{ and } \int \frac{1}{(ax + b)} dx = \frac{1}{a} \ln(ax + b) + c$$

where $c$ is a constant.

---

**Anti-differentiation of logarithmic functions**

$$\int \frac{1}{x} dx = \ln(x) + c \text{ for } x > 0$$

$$\int \frac{1}{(ax + b)} dx = \frac{1}{a} \ln(ax + b) + c$$

where $a$, $b$ and $c$ are constants.

---

*Notes:*

1. This allows us to determine the anti-derivative of $x^n$, $n = -1$ as $\int x^{-1} dx = \int \dfrac{1}{x} dx = \ln(x)$, $x > 0$.

2. Remember that $\log_e(x)$ and $\ln(x)$ are equivalent expressions for logarithmic functions of base $e$.

**Determine:**

a. $\displaystyle\int \dfrac{3}{2x}\, dx$

b. $\displaystyle\int \dfrac{4}{2x+1}\, dx$

| THINK | WRITE |
|---|---|
| a. **1.** Move the factor of $\dfrac{3}{2}$ outside of the integral sign $\left(\text{using } \displaystyle\int k f(x)\, dx = k \int f(x)\, dx\right).$ | a. $\displaystyle\int \dfrac{3}{2x}\, dx$ $= \dfrac{3}{2}\displaystyle\int \dfrac{1}{x}\, dx$ |
| **2.** Apply the formula. | $= \dfrac{3}{2}\ln(x) + c$ |
| b. **1.** Move the factor of 4 outside of the integral sign. | b. $\displaystyle\int \dfrac{4}{2x+1}\, dx$ $= 4\displaystyle\int \dfrac{1}{(2x+1)}\, dx$ |
| **2.** Apply the formula. | $= 4 \times \dfrac{1}{2}\ln(2x+1) + c$ |
| **3.** Simplify. | $= 2\ln(2x+1) + c$ |

---

WORKED EXAMPLE 8 Further anti-differentiation

**Determine** $\displaystyle\int \dfrac{(2x+3)^2}{x}\, dx.$

| THINK | WRITE |
|---|---|
| **1.** Expand the numerator. | $\displaystyle\int \dfrac{(2x+3)^2}{x}\, dx$ $= \displaystyle\int \dfrac{4x^2 + 12x + 9}{x}\, dx$ |
| **2.** Express with separate fractions and simplify. | $= \displaystyle\int \left(\dfrac{4x^2}{x} + \dfrac{12x}{x} + \dfrac{9}{x}\right) dx$ $= \displaystyle\int \left(4x + 12 + \dfrac{9}{x}\right) dx$ |
| **3.** Integrate each term and simplify. | $= 4 \times \dfrac{x^2}{2} + 12x + 9\ln(x) + c$ $= 2x^2 + 12x + 9\ln(x) + c$ |

# Exercise 5.4 Anti-differentiation of logarithmic functions

learn on

| **5.4 Exercise** | **5.4 Exam questions** on |

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| **Simple familiar** | **Complex familiar** | **Complex unfamiliar** |
|---|---|---|
| 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 | 13, 14, 15, 16 | 17, 18, 19, 20 |

## Simple familiar

1. **WE7** Determine the following.

   a. $\displaystyle\int \frac{8}{x}\,dx$
   b. $\displaystyle\int \frac{6}{5x}\,dx$
   c. $\displaystyle\int \frac{7}{3x}\,dx$
   d. $\displaystyle\int \frac{4}{7x}\,dx$

2. Anti-differentiate the following.

   a. $\displaystyle\int \frac{3}{x+3}\,dx$
   b. $\displaystyle\int \frac{-2}{x+4}\,dx$
   c. $\displaystyle\int \frac{-6}{x+5}\,dx$
   d. $\displaystyle\int \frac{4}{3x+2}\,dx$

3. Determine the following.

   a. $\displaystyle\int \frac{8}{5x+6}\,dx$
   b. $\displaystyle\int \frac{3}{2x-5}\,dx$
   c. $\displaystyle\int \frac{-5}{3+2x}\,dx$
   d. $\displaystyle\int \frac{-2}{6+7x}\,dx$

4. Determine the following.

   a. $\displaystyle\int \frac{1}{5-x}\,dx$
   b. $\displaystyle\int \frac{3}{6-11x}\,dx$
   c. $\displaystyle\int \frac{-2}{4-3x}\,dx$
   d. $\displaystyle\int \frac{-8}{5-2x}\,dx$

5. **WE8** Determine $\displaystyle\int \frac{(2x+5)^2}{x}\,dx$.

6. Determine $\displaystyle\int \frac{(3x+2)^2}{x^2}\,dx$.

7. Anti-differentiate the following.

   a. $\dfrac{3-4x}{x}$
   b. $\dfrac{2x^2-3x+4}{x^2}$
   c. $\dfrac{(4-3x)^2}{2x}$
   d. $\dfrac{9+\sqrt{x}}{x}$

8. The gradient function of a curve is given by $f'(x)=x-\dfrac{4}{x}$. Determine the general rule for the function $f(x)$.

9. Determine the general equation of the curve whose gradient at any point is given by $2x+3-\dfrac{4}{5-x}$.

10. Determine the general rule for the function $y=f(x)$ if it is known that $\dfrac{dy}{dx}=x\left(1-\dfrac{1}{x}\right)^2$.

11. If $\displaystyle\int \frac{6}{4-2x}\,dx = a\log_e(b-x)+c$, determine the values of the constants $a$ and $b$.

234    Jacaranda Maths Quest 12 Mathematical Methods Units 3 & 4 for Queensland Second Edition

**12.** **MC** The anti-derivative of $x^2 - \dfrac{1}{x}$ is:

A. $2x - \dfrac{2}{x^2} + c$

B. $\dfrac{x^3}{3} - \ln(x) + c$

C. $3x^3 - \ln(x) + c$

D. $\dfrac{x^3}{3} - \dfrac{2}{x^2} + c$

### Complex familiar

**13. a.** Show that $\dfrac{2x-5}{x-3} = 2 + \dfrac{1}{x-3}$.

**b.** Hence, determine $\displaystyle\int \dfrac{2x-5}{x-3}\, dx$.

**14. a.** Show that $\dfrac{(x+2)^2}{x-2} = x + 6 + \dfrac{16}{x-2}$.

**b.** Hence, determine $\displaystyle\int \dfrac{(x+2)^2}{x-2}\, dx$.

**15.** Determine the values of $a$ and $b$ if $\displaystyle\int \dfrac{a}{bx+3}\, dx = 6\ln(2x+3) + c$, where $a, b, c \in R$.

**16.** A curve has a gradient function $f'(x) = \dfrac{k}{2x+3}$, where $k \in R$. It is known that the function has a gradient of 2 when $x = 1$.

**a.** Determine the value of $k$.

**b.** Hence, determine the general rule for the function $f(x)$.

### Complex unfamiliar

**17.** Differentiate $y = \left(\log_e(x)\right)^2$ and determine an anti-derivative for $\displaystyle\int \dfrac{4\log_e(x)}{x}\, dx$.

**18.** Differentiate $3x\log_e(x)$ and hence find an anti-derivative for $2\log_e(x)$.

**19.** If $f(x) = 2x\log_e(mx)$, state $f'(x)$ and hence determine $\displaystyle\int \log_e(mx)\, dx$, where $m$ is a constant.

**20.** If $\displaystyle\int \dfrac{d - 2x^b + x}{ax}\, dx = \dfrac{5}{4}\log_e(x) - \dfrac{x^2}{4} + \dfrac{x}{4} + c$ determine the values of the constants $a$, $b$ and $d$.

**Fully worked solutions for this chapter are available online.**

# LESSON
## 5.5 Anti-differentiation of trigonometric functions

### 5.5.1 Anti-differentiation of trigonometric functions

As you have learned in Chapter 2:

$$y = \sin(x) \qquad \text{and} \qquad y = \sin(ax + b),$$

$$\frac{dy}{dx} = \cos(x) \qquad\qquad \frac{dy}{dx} = a\cos(ax + b).$$

$$y = \cos(x) \qquad \text{and} \qquad y = \cos(ax + b),$$

$$\frac{dy}{dx} = -\sin(x) \qquad\qquad \frac{dy}{dx} = -a\sin(ax + b).$$

Therefore, it follows that:

$$\int \sin(x)\,dx = -\cos(x) + c \text{ and } \int \cos(x)\,dx = \sin(x) + c$$

where $c$ is a constant.

---

**Anti-derivative of sin($x$)**

$$\int \sin(x)\,dx = -\cos(x) + c$$

**and**

$$\int \sin(ax + b)\,dx = -\frac{1}{a}\cos(ax + b) + c$$

**where $a$, $b$ and $c$ are constants.**

---

**Anti-derivative of cos($x$)**

$$\int \cos(x)\,dx = \sin(x) + c$$

**and**

$$\int \cos(ax + b)\,dx = \frac{1}{a}\sin(ax + b) + c$$

**where $a$, $b$ and $c$ are constants.**

---

*Note:*
- The formulas apply for sine and cosine of linear functions only.

## WORKED EXAMPLE 9 Anti-derivative of sin(x) and cos(x)

**Anti-differentiate the following.**

**a.** $\sin(6x)$        **b.** $8\cos(4x)$        **c.** $3\sin\left(-\dfrac{x}{2}\right)$

| THINK | WRITE |
|---|---|
| **a.** Integrate by rule. | **a.** $\displaystyle\int \sin(6x)\,dx = -\dfrac{1}{6}\cos(6x) + c$ |
| **b. 1.** Integrate by rule. | **b.** $\displaystyle\int 8\cos(4x)\,dx = \dfrac{8}{4}\sin(4x) + c$ |
| **2.** Simplify the result. | $= 2\sin(4x) + c$ |
| **c. 1.** Integrate by rule. | **c.** $\displaystyle\int 3\sin\left(-\dfrac{x}{2}\right)\,dx = \dfrac{-3}{-\frac{1}{2}}\cos\left(-\dfrac{x}{2}\right) + c$ |
| **2.** Simplify the result. | $= 6\cos\left(-\dfrac{x}{2}\right) + c$ |

## WORKED EXAMPLE 10 Anti-differentiation of trigonometric functions

**Determine an anti-derivative of $f(x) = 2\sin(5x) + 3\cos\left(\dfrac{x}{3}\right)$.**

| THINK | WRITE |
|---|---|
| **1.** Separate the two terms. | $\displaystyle\int \left(2\sin(5x) + 3\cos\left(\dfrac{x}{3}\right)\right)\,dx$ |
| | $= \displaystyle\int 2\sin(5x)\,dx + \int 3\cos\left(\dfrac{x}{3}\right)\,dx$ |
| **2.** Move the factors outside of the integral sign. | $= 2\displaystyle\int \sin(5x)\,dx + 3\int \cos\left(\dfrac{x}{3}\right)\,dx$ |
| **3.** Apply the anti-differentiation rules for sin and cos. | $= -\dfrac{2}{5}\cos(5x) + 9\sin\left(\dfrac{x}{3}\right)$ |

## WORKED EXAMPLE 11 Determining the indefinite integral

**Determine the indefinite integral of $2e^{4x} - 5\sin(2x) + 4x$.**

| THINK | WRITE |
|---|---|
| **1.** Integrate each term separately. | $\displaystyle\int \left(2e^{4x} - 5\sin(2x) + 4x\right)\,dx$ |
| | $= 2\times\dfrac{1}{4}e^{4x} - 5\times\dfrac{-1}{2}\cos(2x) + 4\times\dfrac{x^2}{2} + c$ |
| **2.** Simplify. | $= \dfrac{1}{2}e^{4x} + \dfrac{5}{2}\cos(2x) + 2x^2 + c$ |

# Exercise 5.5 Anti-differentiation of trigonometric functions

**5.5 Exercise** | **5.5 Exam questions** on

| Simple familiar | Complex familiar | Complex unfamiliar |
|---|---|---|
| 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 | 13, 14, 15, 16 | 17, 18, 19, 20 |

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## Simple familiar

1. **WE9** Anti-differentiate the following.
   a. $\sin(3x)$
   b. $\sin(4x)$
   c. $\cos(7x)$
   d. $\dfrac{\cos(2x)}{3}$
   e. $\sin(-2x)$
   f. $\cos(-3x)$

2. Anti-differentiate the following.
   a. $\dfrac{4\sin(6x)}{3}$
   b. $8\sin(4x)$
   c. $-6\sin(3x)$
   d. $-2\cos(-x)$
   e. $\sin\left(\dfrac{x}{3}\right)$
   f. $\cos\left(\dfrac{x}{2}\right)$

3. Determine the indefinite integrals of the following.
   a. $3\sin\left(\dfrac{-x}{4}\right)$
   b. $-2\sin\left(\dfrac{x}{5}\right)$
   c. $4\cos\left(\dfrac{x}{4}\right)$

4. Determine the indefinite integrals of the following.
   a. $-6\cos\left(\dfrac{-x}{2}\right)$
   b. $4\sin\left(\dfrac{2x}{3}\right)$
   c. $6\cos\left(\dfrac{3x}{4}\right)$

5. Determine the indefinite integral of $\dfrac{1}{2}\cos(3x+4)-4\sin\left(\dfrac{x}{2}\right)$.

6. **WE10** Determine an anti-derivative of $\cos\left(\dfrac{2x}{3}\right)-\dfrac{1}{4}\sin(5-2x)$.

7. Determine $\displaystyle\int\left(\sin\left(\dfrac{x}{2}\right)-3\cos\left(\dfrac{x}{2}\right)\right)dx$.

8. If $f'(x)=7\cos(2x)-\sin(3x)$, determine a general rule for $f$.

9. **WE11** Determine the indefinite integral of:
   a. $e^{4x}+\sin(2x)+x^3$
   b. $3x^2-2\cos(2x)+6e^{3x}$

10. Determine:
    a. $\displaystyle\int(\sin(x)+\cos(x))\,dx$
    b. $\displaystyle\int(\sin(2x)-\cos(x))\,dx$
    c. $\displaystyle\int(\cos(4x)+\sin(2x))\,dx$
    d. $\displaystyle\int\left(\sin\left(\dfrac{x}{2}\right)-\cos(2x)\right)dx$

**11.** Determine:

a. $\displaystyle\int \left( 4\cos(4x) - \frac{1}{3}\sin(2x) \right) dx$

b. $\displaystyle\int (5x + 2\sin(x)) \, dx$

c. $\displaystyle\int \left( 3\sin\left(\frac{\pi x}{2}\right) + 2\cos\left(\frac{\pi x}{3}\right) \right) dx$

d. $\displaystyle\int \left( 3e^{6x} - 4\sin(8x) + 7 \right) dx$

**12.** Determine the indefinite integral of:

a. $e^{\frac{x}{3}} + \sin\left(\frac{x}{3}\right) + \frac{x}{3}$

b. $\cos(4x) + 3e^{-3x}$

**12.** Determine an anti-derivative of $\dfrac{1}{4x^2} + \sin\left(\dfrac{3\pi x}{2}\right)$.

**13.** The gradient of a tangent to a curve is given by $\dfrac{dy}{dx} = \cos(2x) - e^{-3x}$. Determine a possible general rule for the curve $y$.

**14.** A curve has a gradient function $f'(x) = k\sin(3x)$, where $k \in R$. It is known that the function has a gradient of 2 when $x = \dfrac{\pi}{2}$.

a. Determine the value of $k$.

b. Hence, determine the general rule for the function $f(x)$.

**15.** The gradient function of a curve is $f'(x) = 4\cos(2x) + k$ where $k \in R$. The gradient at the point when $x = \dfrac{5\pi}{6}$ is $-3$.

a. Determine the value of $k$.

b. Hence, determine the general rule for the function $f(x)$.

**17.** A curve has a gradient function $\dfrac{dy}{dx} = k\cos\left(2x + \dfrac{\pi}{3}\right)$, where $k \in R$. If $\dfrac{dy}{dx} = 5$ when $x = \dfrac{\pi}{2}$, determine the general rule for the function.

**18.** If it is known that $\displaystyle\int (3\sin(2x) + 8\cos(2x)) \, dx = p\sin(2x) + q\cos(2x)$, where $p, q \in R$, determine the values of $p$ and $q$.

**19.** If $f'(x) = a\sin(mx) - be^{nx}$ and $f(x) = \cos(2x) - 2e^{-2x} + 3$, calculate the value of the constants $a$, $b$, $m$ and $n$.

**20.** If $x(t) = 20 + \cos\left(\dfrac{\pi t}{4}\right)$ and $\dfrac{dy}{dt} = \dfrac{\pi}{20}x(t) - \pi$, determine a possible rule for $y$ in terms of $t$.

**Fully worked solutions for this chapter are available online.**

# LESSON
## 5.6 Further integration

### 5.6.1 Integration of functions of the form $f(ax+b)$

Consider the function:
$$f(x) = (ax+b)^{n+1}$$

Apply the chain rule:
$$f'(x) = (n+1)(ax+b)^n \times a$$
$$= a(n+1)(ax+b)^n$$

Hence:
$$\int a(n+1)(ax+b)^n\,dx = (ax+b)^{n+1}$$

So:
$$a(n+1)\int (ax+b)^n\,dx = (ax+b)^{n+1}$$

This gives us the general rule:

$$\int (ax+b)^n\,dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c, n \neq -1$$

---

**Integration of functions of the form $f(ax+b)$**

$$\int (ax+b)^n\,dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c, n \neq -1$$

$$\int e^{(ax+b)}\,dx = \frac{1}{a}e^{(ax+b)} + c$$

$$\int \frac{1}{(ax+b)}\,dx = \frac{1}{a}\ln(ax+b) + c$$

$$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + c$$

$$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + c$$

where $a$, $b$ and $c$ are constants.

---

The rules described above only apply if the expression inside the brackets is linear. If the expression is of any other kind, it may need to be expanded before integrating, or another method may need to be used, such as technology.

**Anti-differentiate each of the following.**

**a.** $(2x+3)^5$        **b.** $2\sqrt{5x+4}$        **c.** $\left(e^{2x-1}+3\right)^2$

| THINK | WRITE |
|---|---|
| **a. 1.** Recognise the linear function inside the brackets. | **a.** $\displaystyle\int (2x+3)^5\, dx$ |
| **2.** Integrate the expression. | $= \dfrac{1}{2\times 6}(2x+3)^6$ |
| **3.** Simplify. | $= \dfrac{1}{12}(2x+3)^6 + c$ |
| **b. 1.** Move the factor of 2 out of the integral and write the square root as a linear expression to a power of $\dfrac{1}{2}$. | **b.** $\displaystyle\int 2\sqrt{5x+4}\,dx$ <br><br> $= 2\displaystyle\int (5x+4)^{\frac{1}{2}}\, dx$ |
| **2.** Integrate. | $= 2\times\dfrac{1}{5}\times\dfrac{(5x+4)^{\frac{3}{2}}}{\frac{3}{2}}$ |
| **3.** Simplify. | $= \dfrac{4}{15}(5x+4)^{\frac{3}{2}} + c$ |
| **c. 1.** Expand the brackets. | **c.** $\displaystyle\int \left(e^{2x-1}+3\right)^2\, dx$ <br><br> $= \displaystyle\int \left(e^{4x-2}+6e^{2x-1}+9\right)\, dx$ |
| **2.** Integrate each term separately. | $= \dfrac{1}{4}\,e^{(4x-2)} + 6\times\dfrac{1}{2}\,e^{(2x-1)} + 9x$ |
| **3.** Simplify. | $= \dfrac{1}{4}\,e^{(4x-2)} + 3e^{(2x-1)} + 9x + c$ |

## 5.6.2 Initial conditions

Suppose we are asked to investigate $\dfrac{dy}{dx} = 2e^{2x}$. Because $y = e^{2x} + c$, this is a series of an infinite number of exponential functions. We call this a family of curves. The functions with $c$ values of 1, 0, $-2$ and $-4$, as shown in the diagram, are four of the possible functions for $y = e^{2x} + c$.

A specific function can only be found if we are given some additional information to allow us to evaluate the constant, $c$.
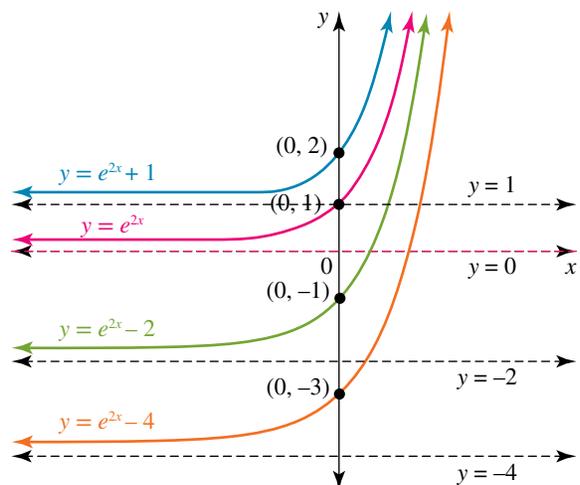
For example, we might be told that the curve passes through the origin. This lets us know that when $x = 0$, $y = 0$.

Hence,

$$y = e^{2x} + c$$
$$(0, 0) \Rightarrow 0 = e^{2(0)} + c$$
$$0 = 1 + c$$
$$c = -1$$

Therefore, $y = e^{2x} - 1$.

This additional information is referred to as an initial condition. The question could have been given as follows: 'If $f'(x) = 2e^{2x}$ find $f$ given that $f(0) = 0$.'

The graphs of $y = e^{2x} + c$ are sometimes referred to as a family of curves. Integrating the derivative function, together with the initial conditions, allows us to determine the original function.

---

## WORKED EXAMPLE 13 Sketching a family of curves

a. **Sketch a family of curves that have the derivative function $f'(x) = 2\cos(2x)$ for $0 \le x \le 2\pi$.**
b. **Determine the specific rule for this function if $f(\pi) = 2$.**

**THINK**

**WRITE**

a. 1. Apply the rule $\displaystyle\int \cos(ax)\, dx = \dfrac{1}{a}\sin(ax) + c$ to anti-differentiate the function.

a. $f(x) = \displaystyle\int 2\cos(2x)\, dx$

$\qquad = \sin(2x) + c$

2. Graph the function, first with $c = 0$. That is, sketch $f(x) = \sin(2x)$. Then translate this graph up or down to sketch the graphs with different $c$ values. Any $c$ values may be used.

$f(x) = \sin(2x) + c$

Amplitude $= 1$ and period $= \dfrac{2\pi}{2} = \pi$
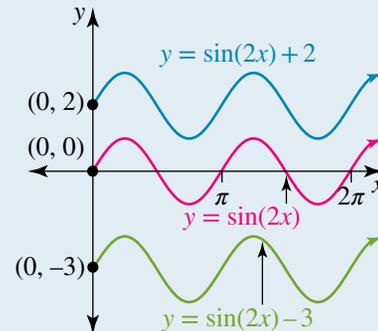


b. 1. Substitute the known point into the equation.

b. $\qquad\qquad f(x) = \sin(2x) + c$

$f(\pi) = 2 \Rightarrow 2 = \sin(2\pi) + c$

2. Simplify and determine the value for $c$.

$\qquad\qquad 2 = 0 + c$

$\qquad\qquad\quad c = 2$

3. State the rule for $f(x)$.

$\qquad\qquad f(x) = \sin(2x) + 2$

## WORKED EXAMPLE 14 Using initial conditions to find the function

Determine the equation of the curve that passes through the point (1, 0) if the gradient is given by $\dfrac{dy}{dx} = 3x^2 - 2x + 2.$

| THINK | WRITE |
|---|---|
| 1. Write the gradient rule and anti-differentiate to determine $y$. | $\dfrac{dy}{dx} = 3x^2 - 2x + 2$ <br><br> $y = \displaystyle\int (3x^2 - 2x + 2)\,dx$ <br><br> $= x^3 - x^2 + 2x + c$ |
| 2. Substitute the known point into the equation. | When $x = 1, y = 0$: <br> $0 = (1)^3 - (1)^2 + 2(1) + c$ <br> $0 = 1 - 1 + 2 + c$ <br> $c = -2$ |
| 3. State the rule for $y$. | $y = x^3 - x^2 + 2x - 2$ |

## 5.6.3 Integration by recognition

Recall that integration, or anti-differentiation, is the reverse of differentiation.

That is:
$$\frac{d}{dx}[f(x)] = g(x), \text{ then } \int g(x)\,dx = f(x) + c$$

Sometimes you may be required to find an anti-derivative of a very complex function. To complete this task, you will first be given a function to differentiate. The technique is then to recognise the patterns between the derivative you have found and the function you have been given to integrate.

## WORKED EXAMPLE 15 Integration by recognition using logarithmic functions

Differentiate $y = \ln(x^2 + 4)$ and hence determine $\displaystyle\int \frac{6x}{(x^2 + 4)}\,dx.$

| THINK | WRITE |
|---|---|
| 1. Differentiate using the chain rule. | $y = \ln\left(x^2 + 4\right)$ <br> Let $y = \ln(u)$ and $u = x^2 + 4.$ <br><br> $\dfrac{dy}{du} = \dfrac{1}{u}$ and $\dfrac{du}{dx} = 2x$ <br><br> $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$ <br><br> $= \dfrac{1}{u} \times 2x$ |

**2.** Simplify.

$$\frac{dy}{dx} = \frac{1}{(x^2+4)} \times 2x$$

$$= \frac{2x}{(x^2+4)}$$

**3.** Rewrite the result as an integral.

$$\therefore \int \frac{2x}{(x^2+4)}\, dx = \ln(x^2+4)$$

**4.** Adjust the left-hand side so that it matches the expression to be integrated.

$$3 \times \int \frac{2x}{(x^2+4)}\, dx = 3 \times \ln(x^2+4)$$

**5.** Write the answer.

$$\therefore \int \frac{6x}{(x^2+4)}\, dx = 3\ln\left(x^2+4\right) + c$$

---

### WORKED EXAMPLE 16 Integration by recognition using trigonometric functions

**Differentiate $x\cos(x)$ and hence determine an anti-derivative of $x\sin(x)$.**

| THINK | WRITE |
|---|---|
| **1.** Write the expression as a function. | Let $y = x\cos(x)$. |
| **2.** Differentiate using the product rule. | $\dfrac{dy}{dx} = x \times (-\sin(x)) + \cos(x) \times 1$ |
| **3.** Simplify. | $\dfrac{dy}{dx} = \cos(x) - x\sin(x)$ |
| **4.** Rewrite the result as an integral. | $\therefore \displaystyle\int (\cos(x) - x\sin(x))\, dx = x\cos(x)$ |
| **5.** Express as separate integrals. | $\displaystyle\int \cos(x)\, dx - \int x\sin(x)\, dx = x\,\cos(x)$ |
| **6.** Simplify by integrating. | $\sin(x) - \displaystyle\int x\sin(x)\, dx = x\cos(x)$ |
| **7.** Rearrange the equation to make the expression to be integrated the subject. | $\sin(x) - x\cos(x) = \displaystyle\int x\sin(x)\, dx$ |
| **8.** Write the answer. | $\therefore \displaystyle\int x\sin(x)\, dx = \sin(x) - x\cos(x) + c$ |

| Simple familiar | Complex familiar | Complex unfamiliar |
|---|---|---|
| 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 | 13, 14, 15, 16 | 17, 18, 19, 20 |

## Simple familiar

1. **WE12** Anti-differentiate each of the following.
   a. $(x+3)^2$
   b. $(x-5)^3$
   c. $2(2x+1)^4$
   d. $-2(3x-4)^5$
   e. $(6x+5)^4$
   f. $3(4x-1)^2$

2. Determine the anti-derivative for each of the following.
   a. $(4-x)^3$
   b. $(7-x)^4$
   c. $4(8-3x)^4$
   d. $-3(8-9x)^{10}$
   e. $(2x+3)^{-2}$
   f. $(6x+5)^{-3}$

3. Anti-differentiate:
   a. $(3x-5)^5$
   b. $\dfrac{1}{(2x-3)^{\frac{5}{2}}}$

4. Determine:
   a. $\displaystyle\int (2x+3)^4\, dx$
   b. $\displaystyle\int (1-2x)^{-5}\, dx$

5. Determine:
   a. $\displaystyle\int \left(e^{2x+1}-4\right)^2\, dx$
   b. $\displaystyle\int \left(2e^{3-x}+3e^{2-x}\right)^2\, dx$

6. **WE13** a. Sketch a family of curves related to the derivative function $f'(x)=3x^2$.

   b. Determine the rule for the function that belongs to this family of curves and passes through the point $(2, 16)$.

7. a. Sketch a family of curves related to the derivative function $f'(x)=-2\cos(2x)$.

   b. Determine the rule for the function that belongs to this family of curves and passes through the point $\left(\dfrac{\pi}{2}, 4\right)$.

8. a. Sketch a family of curves with the derivative function $f'(x)=3e^{-3x}$.
   b. Determine the rule for the function that belongs to this family of curves and passes through the point $(0, 1)$.

9. **WE14** Determine the equation of the function $f(x)$ given that:

   a. $f'(x)=4x+1$ and the curve passes through $(0, 2)$
   b. $f'(x)=5-2x$ and the curve passes through $(1, -1)$
   c. $f'(x)=x^{-2}+3$ and the curve passes through $(1, 4)$

10. Determine the equation of the function $f(x)$ given that:

    a. $f'(x)=x+\sqrt{x}$ and $f(4)=10$
    b. $f'(x)=x^{\frac{1}{3}}-3x^2+50$ and $f(8)=-100$
    c. $f'(x)=\dfrac{1}{\sqrt{x}}-2x$ and $f(1)=-5$

11. Determine the equation of the curve that passes through the point $(0, 3)$ if the gradient is given by $\frac{dy}{dx} = 2e^{2x} + e^{-x}$.

12. The gradient function of a particular curve is given by $f'(x) = \cos(2x) - \sin(2x)$. Determine the rule for this function if it is known that the curve passes through the point $(\pi, 2)$.

### Complex familiar

13. **WE15** Differentiate $\ln\left(3x^2 + 4\right)$ and hence determine an anti-derivative of $\frac{x}{(3x^2 + 4)}$.

14. Differentiate $x \ln(x)$ and hence determine an anti-derivative of $\ln(x)$.

15. **WE16** Differentiate $x \sin(x)$ and hence determine an anti-derivative of $x \cos(x)$.

16. Differentiate $y = 2xe^{3x}$ and hence determine an anti-derivative of $xe^{3x}$.

### Complex unfamiliar

17. Determine $f(x)$ if $f'(x) = x^3 - \frac{1}{x}$ and $f(1) = \frac{1}{4}$.

18. If $y = \ln(\cos(x))$, determine $\int \tan(x)\, dx$.

19. The graph of the gradient function $y = f'(x)$ is shown.



If $f(-2) = -5$, determine $f(4)$.

20. The graph of the gradient function $y = f'(x)$ is shown.



Determine the axis intercepts of the function $f(x)$ given $f(2) = -6$.
Give your answers to 2 decimal places where necessary.

**Fully worked solutions for this chapter are available online.**

# LESSON
## 5.7 Modelling and solving using indefinite integrals

## 5.7.1 Applications involving rates of change

Differentiation of a function gives the rate of change of the function, normally with respect to time.

Integration, or anti-differentiation, is the reverse operation and can be applied in many similar problems. Practical questions such as those involving rates of change may also be given in terms of the derivative function. Integrating the equation for the rate of change, along with initial conditions, allows us to determine the original function.

### WORKED EXAMPLE 17 Rates of change application

A young boy bought an ant farm. It is known that the ant population is changing at a rate defined by $\frac{dN}{dt} = 20e^{0.2t}, 0 \le t \le 20$, where $N$ is the number of ants in the colony and $t$ is the time in days since the ant farm has been set up.

a. Determine a rule relating $N$ to $t$ if initially there were 50 ants.
b. How many ants make up the colony after 8 days?

**THINK**

a. 1. Write the rate rule and anti-differentiate to determine the function for $N$.

2. Use the initial condition to determine the value of $c$.

3. State the equation for $N$.

**WRITE**

a. $\dfrac{dN}{dt} = 20e^{0.2t}$

$$N = \int (20e^{0.2t})\, dt$$

$$= \frac{20}{0.2} e^{0.2t} + c$$

$$= 100e^{0.2t} + c$$

When $t = 0$, $N = 50$:

$50 = 100e^{0.2(0)} + c$
$50 = 100 + c$
$c = -50$

$N = 100e^{0.2t} - 50$

**b. 1.** Substitute $t = 8$ into the population equation.

**b.** When $t = 8$:
$$N = 100e^{0.2(8)} - 50$$
$$= 100e^{1.6} - 50$$
$$= 445.3$$

**2.** Answer the question.
*Note:* It is reasonable to round down when counting elements from the natural world.

There are 445 ants after 8 days.

## 5.7.2 Linear motion

Kinematics, the study of the motion of a particle in a straight line, was introduced in Unit 2 where velocity is the rate of change of displacement with respect to time. Acceleration, as the second derivative of displacement with respect to time, was studied in the previous chapter.

### Displacement-velocity-acceleration relationships

- **Displacement, $x$**, gives the position of a particle, specifying both its distance and direction from the origin, $x = 0$.
- **Velocity, $v$**, measures the rate of change of displacement with respect to time, $t$, which means that $v = \dfrac{dx}{dt}$.

  It follows that displacement is the anti-derivative of velocity.
- **Acceleration, $a$**, measures the rate of change of velocity with respect to time, $t$, which means that

  $$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}.$$

  It follows that velocity is the anti-derivative of acceleration.

These relationships are summarised in the diagram.
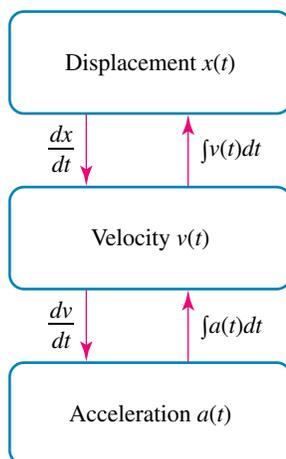


If a particle is slowing down, or decelerating, the equation for acceleration would be negative. That is, $a < 0$.

**In each of the following cases, express the position as a function of $t$ if initially the particle is at the origin.**

**a.** $v = t^3 - t$  **b.** $v = (2t - 3)^3$  **c.** $v = \dfrac{1}{(t-1)^2}$

**THINK**

**WRITE**

**a. 1.** Write the velocity equation and anti-differentiate to find the position function, $x$.

**a.** $v = \dfrac{dx}{dt} = t^3 - t$

$x = \displaystyle\int (t^3 - t)\, dt$

$x = \dfrac{1}{4} t^4 - \dfrac{1}{2} t^2 + c$

**2.** Substitute the initial condition into the formula for $x$ and determine $c$.

When $t = 0,\ x = 0$:

$0 = 0 + c$

$c = 0$

**3.** State the rule.

$x = \dfrac{1}{4} t^4 - \dfrac{1}{2} t^2$

**b. 1.** Write the velocity equation and anti-differentiate to find the position function, $x$.

**b.** $v = \dfrac{dx}{dt}$

$= (2t - 3)^3$

$x = \displaystyle\int (2t - 3)^3\, dt$

$= \dfrac{(2t-3)^4}{2\,(4)} + c$

$= \dfrac{1}{8}(2t - 3)^4 + c$

**2.** Substitute the initial condition into the formula for $x$ and determine $c$.

When $t = 0,\ x = 0$:

$0 = \dfrac{1}{8}(-3)^4 + c$

$= \dfrac{81}{8} + c$

$c = -\dfrac{81}{8}$

**3.** State the rule.

$x = \dfrac{1}{8}(2t - 3)^4 - \dfrac{81}{8}$

**c. 1.** Write the velocity equation and anti-differentiate to find the position function, $x$.

**c.** $v = \dfrac{dx}{dt}$

$= \dfrac{1}{(t-1)^2}$

$= (t - 1)^{-1}$

$x = \displaystyle\int (t - 1)^{-2}\, dt$

$= \dfrac{(t-1)^{-1}}{-1} + c$

$$= -(t-1)^{-1} + c$$
$$= -\frac{1}{(t-1)} + c$$

| | | |
|---|---|---|
| **2.** | Substitute the initial condition into the formula for $x$ and determine $c$. | When $t = 0$, $x = 0$:<br>$0 = -\dfrac{1}{(-1)} + c$<br>$0 = 1 + c$<br>$c = -1$ |
| **3.** | State the rule. | $x = -\dfrac{1}{(t-1)} - 1$ |

---

## WORKED EXAMPLE 19 Determining displacement from acceleration

The acceleration, $a$ m/s$^2$, of a particle moving in a straight line along the $x$-axis is given by

$$a = 27e^{-3t}$$

where $t$ is the time in seconds and $x$ is the displacement in metres.
a. Determine the equation for velocity of the particle if the particle is initially at rest.
b. Determine the equation for displacement if it is known that initially the particle was 3 metres to the left of the origin.

**THINK**

**WRITE**

| | | |
|---|---|---|
| **a. 1.** | Write the velocity equation as the anti-derivative of acceleration. | **a.** $a = 27e^{-3t}$<br>$v = \displaystyle\int 27e^{-3t}\, dt$ |
| **2.** | Anti-differentiate to determine velocity and simplify. | $= 27 \times \dfrac{1}{-3} e^{-3t} + c$<br>$= -9e^{-3t} + c$ |
| **3.** | Substitute initial conditions to determine $c$. | when $t = 0$, $v = 0$ as particle at rest.<br>$0 = -9 \times e^0 + c$<br>$c = 9$ |
| **4.** | State the equation for velocity. | $v = 9 - 9e^{-3t}$ |
| **b. 1.** | Write the velocity equation and anti-differentiate to determine the position equation, $x$. | **b** $v = \dfrac{dx}{dt}$<br>$= 9 - 9e^{-3t}$<br>$x = \displaystyle\int \left(9 - 9e^{-3t}\right) dt$<br>$= 9t + 3e^{-3t} + c$ |
| **2.** | Substitute the initial condition to determine $c$. Remember, left of the origin means the position is negative. | When $t = 0, x = -3$<br>$-3 = 9 \times 0 + 3e^0 + c$<br>$-3 = 3 + c$<br>$c = -6$ |
| **3.** | State the equation. | $x = 9t + 3e^{-3t} - 6$ |

# Exercise 5.7 Modelling and solving using indefinite integrals    **learn on**

| Simple familiar | Complex familiar | Complex unfamiliar |
|---|---|---|
| 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 | 13, 14, 15, 16 | 17, 18, 19, 20 |

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**Simple familiar**

1. **WE17** The rate of change of the length of a snake, $L$ cm, any time $t$ weeks after it is born is modelled as:

$$\frac{dL}{dt} = 6 + \frac{\pi}{2}\cos\left(\frac{\pi t}{4}\right), \ 0 \le t \le 20$$

   a. Determine an equation relating $L$ and $t$ if initially the snake was 12 cm in length.
   b. Determine the length of the snake after 20 weeks.

2. The rate of change of volume of a balloon as it is being blown up can be modelled by

$$\frac{dV}{dr} = \pi r^2$$

   where $V$ cm$^3$ is the volume of the balloon and $r$ cm is the radius of the balloon.

   a. Determine the rule for the volume of the balloon.
   b. Calculate the volume of the balloon when its radius is 4 cm.

3. A chemical factory has permission from the Environment Protection Authority to release particular toxic gases into the atmosphere for a period of 20 s no more than once every 3 h. This maintains safe levels of the gases in the atmosphere. This rate of emission is given by

$$\frac{dV}{dt} = 20t^2 - t^3 \ \text{cm}^3/\text{s}$$

   where $0 \le t \le 20$ and $V$ cm$^3$ is the total volume of toxic gases released over $t$ s. Calculate the total volume of toxic gases released during a 20-s release period.

4. It is known that the population of a certain species of bugs is changing at a rate defined by

$$\frac{dP}{dt} = 20e^{0.4t}, \ 0 \le t \le 10$$

   where $P$ is the number of bugs at any time $t$ days since the monitoring of the bugs commenced.

   a. Determine the relationship between $P$ and $t$ if initially the population consisted of 35 bugs.
   b. Calculate the number of bugs present after 6 days.

**5.** A newly established suburban area of Perth is growing at a rate modelled by the rule

$$\frac{dN}{dt} = 400 + 1000\sqrt{t}, 0 \le t \le 10$$

where $N$ is the number of families living in the suburb $t$ years after the suburb was established in 2015.

**a.** Determine a rule relating $N$ and $t$ if initially there were 40 families living in this suburb.

**b.** Calculate how many families will be living in the suburb 5 years after its establishment. Give your answer correct to the nearest number of families.

**6.** **WE18** In each of the following cases, determine the displacement as a function of $t$ if initially the particle is at the origin.

**a.** $v = (3t + 1)^{\frac{1}{2}}$

**b.** $v = \dfrac{1}{(t + 2)^2}$

**c.** $v = (2t + 1)^3$

**7.** A particle moves in a straight line so that its velocity, in metres per second, can be defined by the equation $v = 3t^2 + 6t, t \ge 0$. Determine:

**a.** the displacement of the particle, $x$ metres, as a function of $t$, if it is known that the particle was initially 2 metres to the left of the origin

**b.** the position of the particle after 5 seconds.

**8.** Determine the displacement of a particle that starts from the origin and has a velocity defined by:

**a.** $v = e^{(3t-1)}$

**b.** $v = -\sin(2t + 3)$

**9.** **WE19** A particle is moving along the $x$-axis and at time $t \ge 0$ seconds has an acceleration $a(t) = 24t - 6 \text{ m/s}^2$. If after one second its position is 4 metres and its initial velocity is 1 m/s, determine its position $x$ metres at a time $t$ seconds.

**10.** A particle moves back and forth along the $x$-axis and its acceleration is given by $a(t) = -2\sin\left(\dfrac{t}{3}\right) \text{ m/s}^2$ at time $t \ge 0$ seconds. If its initial velocity is 6 m/s and initially the particle starts at the origin, express the position $x$ metres in terms of $t$.

**11.** A particle attached to a spring moves up and down in a straight line so that at time $t$ seconds its velocity, $v$ m/s, is given by

$$v = 3\pi \sin\left(\frac{\pi t}{8}\right), t \ge 0$$

Initially the particle is stationary. Determine:

**a.** the rule relating the position of the particle, $x$ centimetres, to $t$

**b.** the maximum position of the particle

**c.** where the particle is, relative to the stationary position, after 4 seconds.

**12.** When a bus travels along a straight road in heavy traffic from one stop to another stop, the velocity at time $t$ s is given by

$$v = 0.25t\,(50 - t)$$

where $v$ is the velocity in m/s.

**a.** Calculate the greatest velocity reached by the bus.
**b.** Determine the rule for the position of the bus, $x$ metres, in terms of $t$.

**Complex familiar**

**13.** A population of sea lions on a distant island is growing according to the model

$$\frac{dP}{dt} = 30e^{0.3t},\ 0 \le t \le 10$$

where $P$ is the number of sea lions present after $t$ years.

**a.** If initially there were 50 sea lions on the island, determine the rule for the number of sea lions present, $P$, after $t$ years.
**b.** Determine the number of sea lions on the island after 10 years. Give your answer correct to the nearest whole sea lion.

**14.** The rate of change of the depth of water in a canal is modelled by the rule

$$\frac{dh}{dt} = \frac{\pi}{2} \cos\left(\frac{\pi t}{4}\right)$$

where $h$ is the height of the water in metres and $t$ is the number of hours since 6 am.

**a.** Determine an expression for $h$ in terms of $t$ if the water is 3 metres deep at 6 am.
**b.** Determine the maximum and minimum depths of the water.
**c.** For how many hours a day is the water level 4 metres or higher?

**15.** A particle moves back and forth along the $x$-axis and has an acceleration

$$a(t) = -2 \cos\left(\frac{t}{2}\right)\ \text{m/s}^2$$

at time $t \ge 0$ seconds. If initially the particle is at rest and starts from a point 4 metres from the origin, determine the position in metres of the particle after a time of $\dfrac{\pi}{2}$ seconds.

**16.** A car is moving along a straight road at a speed of 72 km/h when the driver brakes. The acceleration is given by

$$a(t) = 50 - 120t\,\text{m/s}^2$$

where $t$ is the time in seconds after the driver applies the brakes. Determine:

**a.** the time in seconds when the speed of the car has been reduced to 36 km/h
**b.** the distance in metres travelled in this time.

**17.** A particle, starting at the origin, moves in a straight line so that at time $t$ s its acceleration is given by

$$a(t) = \frac{-800}{(t+5)^3} \text{ m/s}^2$$

If the initial velocity of the particle is 16 m/s, determine how far it has travelled in the first 5 s.

**18.** A particle moves in a straight line. At time $t$ s, $t \geq 0$ its position from a fixed origin $O$ is $x$ m and its acceleration is $a$ m/s$^2$, is given by

$$a(t) = 4 - 4e^{-0.1t}$$

If initially it is moving away from the origin with a velocity of 12 m/s, determine how far in metres, correct to 3 decimal places, it has travelled in the first 5 s.

**19.** A car moves along a straight road. When travelling at 60 km/h the driver applies the brakes, its acceleration is given by

$$a(t) = -\frac{100t}{3} \text{ m/s}^2$$

where $t \geq 0$ is the time in seconds after the driver applies the brakes.
Determine the distance travelled in metres until the car comes to rest.



**20.** The rate of deflection from a horizontal position of a 3-m diving board when an 80-kg person is $x$ metres from its fixed end is given by

$$\frac{dy}{dx} = -0.03 (x+1)^2 + 0.03$$

where $y$ is the deflection in metres.
Determine the maximum deflection, correct to the nearest centimetre.

**Fully worked solutions for this chapter are available online.**

# LESSON
## 5.8 Review

### 5.8.1 Summary

## 5.8 Exercise

**learn on**

| 5.8 Exercise | 5.8 QCAA questions on |
|---|---|

| Simple familiar | Complex familiar | Complex unfamiliar |
|---|---|---|
| 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 | 13, 14, 15, 16 | 17, 18, 19, 20 |

### Simple familiar

1. Anti-differentiate each of the following.
   a. $3x^5$
   b. $5x^{-2}$
   c. $-2x^4$
   d. $2\sqrt{x}$
   e. $\dfrac{x^4}{5}$
   f. $(3x-8)^{-6}$

2. Determine the following.
   a. $\displaystyle\int \left(x^4 + 2x + \dfrac{1}{x}\right) dx$
   b. $\displaystyle\int (3x+1)^5 \, dx$
   c. $\displaystyle\int \dfrac{3x^2 + 2x - 1}{x^2} \, dx$
   d. $\displaystyle\int \dfrac{3}{2x+1} \, dx$
   e. $\displaystyle\int \dfrac{-5}{6-10x} \, dx$
   f. $\displaystyle\int 3(4x+1)^{-3} \, dx$

3. Determine the equation of the curve $f(x)$ given that:
   a. $f'(x) = (x+4)^3$ and the curve passes through $(-2, 5)$
   b. $f'(x) = 8(1-2x)^{-5}$ and $f(1) = 3$
   c. $f'(x) = (x+5)^{-1}$ and the curve passes through $(-4, 2)$
   d. $f'(x) = \dfrac{8}{7-2x}$ and $f(3) = 7$.

4. If a curve has a stationary point at $(1, 5)$ and a gradient of $8x + k$ where $k$ is a constant, determine:
   a. the value of $k$
   b. the value of $y$ when $x = -2$.

5. Determine an anti-derivative of:
   a. $(e^x - 3)^2$
   b. $(1 + e^{-x})^3$

6. Anti-differentiate the following.

a. $-2 \sin \left( \dfrac{5x}{2} \right)$

b. $-3 \cos \left( \dfrac{7x}{4} \right)$

c. $5 \sin (\pi x)$

d. $3 \cos \left( \dfrac{\pi x}{2} \right)$

e. $-2 \cos \left( \dfrac{\pi x}{3} \right)$

f. $- \sin \left( \dfrac{-4x}{\pi} \right)$

7. Integrate each of the following with respect to $x$.

a. $x^3 - \dfrac{1}{2x + 3} + e^{2x}$

b. $x^2 + 4 \cos(2x) - e^{-x}$

8. Integrate each of the following with respect to $x$.

a. $3 \sin \left( \dfrac{x}{2} \right) - 2 \cos \left( \dfrac{x}{3} \right) - e^{\frac{-x}{5}}$

b. $\sqrt{x} + 2x - 2 \sin \left( \dfrac{\pi x}{3} \right) + 5$

9. Determine the equation of the curve $f(x)$ given that:

a. $f'(x) = \cos(x)$ and $f \left( \dfrac{\pi}{2} \right) = 5$

b. $f'(x) = 4 \sin(2x)$ and $f(0) = -1$

c. $f'(x) = 3 \cos \left( \dfrac{x}{4} \right)$ and $f(\pi) = 9 \sqrt{2}$

d. $f'(x) = \cos \left( \dfrac{x}{4} \right) - \sin \left( \dfrac{x}{2} \right)$ and $f(2\pi) = -2$.

10. Determine $\dfrac{d}{dx} \left( \ln \left( x^2 + 3 \right) \right)$ and hence determine $\displaystyle\int \dfrac{12x}{\left( x^2 + 3 \right)} \, dx$.

11. Differentiate $\dfrac{\cos(x)}{\sin(x)}$ and hence determine an anti-derivative of $\dfrac{1}{\sin^2(x)}$.

12. a. Show that $\dfrac{6x - 5}{3 - 2x} = -3 + \dfrac{4}{3 - 2x}$.

b. Hence, determine $\displaystyle\int \dfrac{6x - 5}{3 - 2x} \, dx$.

### Complex familiar

13. A curve has a gradient function $f'(x) = 4 \cos(2x) + ke^x$, where $k$ is a constant, and a stationary point at $(0, -1)$. Calculate:

a. the value of $k$

b. the equation of the curve $f(x)$

c. $f \left( \dfrac{\pi}{6} \right)$ correct to 2 decimal places.

14. a. Sketch a family of curves related to the derivative function $f'(x) = -4e^{-2x}$.

b. Determine the rule for the function that belongs to this family of curves and passes through the origin.

15. Determine $f(x)$ for each of the following.

a. $f'(x) = 5 - 2x$ and $f(1) = 4$

b. $f'(x) = \sin \left( \dfrac{x}{2} \right)$ and $f(\pi) = 3$

c. $f'(x) = \dfrac{1}{(1 - x)^2}$ and $f(0) = 4$

16. **a.** Given that $y = e^{3x^2 - 2x + 1}$, determine $\dfrac{dy}{dx}$ and hence the anti-derivative of $(3x - 1) e^{3x^2 - 2x + 1}$.

    **b.** Given $y = x \sin(2x)$, determine $\dfrac{dy}{dx}$ and hence the anti-derivative of $x \cos(2x)$.

**Complex unfamiliar**

17. The acceleration of a particle is given by $a = 9 \cos(3t)$, where $x$ is the position in metres at time $t$ seconds.
    Initially the particle is at the origin with a velocity of 1 m/s.
    Determine where the particle is relative to the origin after 3 seconds. Give your answer correct to 3 decimal places.

18. An oil tanker off the coast of Queensland has sprung a leak, and the oil is threatening the marine life of the Great Barrier Reef. The oil leak has produced a circular oil slick that is spreading at a rate approximated by

$$\frac{dr}{dt} = \frac{30}{\sqrt{t+1}}, t \geq 0$$

    where $r$ metres is the radius of the oil slick and $t$ is the time in hours since the spill started.
    Three hours after the spill, the radius of the oil slick was 45 metres. The captain and crew of the tanker are working furiously to plug the hole to stop the oil leaking. They aim to have the hole plugged so that the radius of the circular slick does not exceed 75 metres. Determine how long they expect it will take them to plug the hole.

19. George went to his favourite coffee shop and ordered a large plunger of his favourite South American black coffee.
    The initial temperature of the coffee was 98 °C. George estimated the coffee was cooling at a rate of 8 °C per minute, but after 10 minutes its rate of cooling was 4 °C per minute. If $T$ °C is the temperature of the coffee $t$ minutes after the coffee was made, then a model for the rate of change of temperature is

$$\frac{dT}{dt} = me^{nt}$$

    where $m$ and $n$ are constants.
    Determine the temperature of the coffee 5 minutes after it was made. Give your answer correct to the nearest whole degree.

20. Over a 24-h period on a particular March day, starting at 12 am, the rate of change of the temperature for Brisbane was approximately

$$\frac{dT}{dt} = -\frac{5\pi}{12} \cos\left(\frac{\pi t}{12}\right)$$

    where $T$ is the temperature in °C and $t$ is the number of hours since midnight. The temperature at midnight was 20 °C.
    Determine the times during the day when the temperature was above 22.5 °C.

## 5.8 Past QCAA exam questions

**Question 1 (1 mark)**

*Source:* QCAA 2023 Mathematical Methods, Paper 1, Section 1, Q4; © State of Queensland (QCAA) 2024.

If the gradient of the function $f(x)$ is given by $\dfrac{20}{x^3}$, then $f(x)$ is equal to

**A.** $-\dfrac{60}{x^4} + c$

**B.** $-\dfrac{5}{x^4} + c$

**C.** $-\dfrac{10}{x^2} + c$

**D.** $-\dfrac{40}{x^2} + c$

**Question 2 (1 mark)**

*Source:* QCAA 2022 Mathematical Methods, Paper 2, Section 1, Q7; © State of Queensland (QCAA) 2024.

A marble moves in one direction in a straight line with velocity $v = 2\ln(t+1)$ (in metres per second) where $t$ is time (in seconds) since the marble passed through the origin.

Determine the distance from the origin the marble has rolled after 4 seconds.

**A.** $0.40\,\text{m}$

**B.** $1.60\,\text{m}$

**C.** $3.22\,\text{m}$

**D.** $8.09\,\text{m}$

**Question 3 (6 marks)**

*Source:* QCAA 2023 Mathematical Methods, Paper 1, Section 2, Q14; © State of Queensland (QCAA) 2024.

The rate of change in the number of bacteria in a science experiment is represented by $\dfrac{dP}{dt} = e^{2t}$, $t \geq 0$, where $t$ represents the time (hours) since starting the experiment and $P$ represents the number of bacteria present (thousands). Initially there are $60\,000$ bacteria present, i.e. $P(0) = 60$.

**a.** Determine the equation for $P(t)$. **[2 marks]**

**b.** Determine the change in the number of bacteria during the third hour. **[2 marks]**

**c.** Determine how long it will take for the number of bacteria present to double after starting the experiment. **[2 marks]**

**Question 4 (6 marks)**

*Source:* QCAA 2023 Mathematical Methods, Paper 2, Section 2, Q16; © State of Queensland (QCAA) 2024.

A particle is moving in a straight line. The velocity $\left(\text{ms}^{-1}\right)$ of the particle is given by $v(t) = \dfrac{20\sin(2t)}{6 - 5\cos(2t)}$, $t \geq 0$, where $t$ is time $(s)$ after moving from its initial position.

The initial position of the particle is $+6.0\,\text{m}$ from the origin.

**a.** Use calculus methods to determine an equation for the position of the particle from the origin at any time $t$. **[3 marks]**

**b.** Determine the position of the particle relative to the origin when it first reaches maximum velocity. **[3 marks]**

**Question 5 (6 marks)**

*Source:* QCAA 2023 Mathematical Methods, Paper 2, Section 2, Q19; © State of Queensland (QCAA) 2024.

Over a suitable domain, a hill has a cross-sectional area given by $\int h(x)\,dx = \dfrac{a}{b}\,e^{bx} + c$, where:

- $a, b$ and $c$ are constants, $b \neq 0$
- $h(x)$ represents vertical distance (m), $x$ represents horizontal distance (m).

It is known that $h(0) = 1.22$ and $h(40) = 25$.

Where the gradient of the hill is $0.86$ there is a tree stump. A second tree stump is located further up the hill. The difference in hill gradient between the two tree stumps is $0.44$.

A surveyor predicts that the vertical distance separating the two tree stumps is between $7.5\,\text{m}$ and $8.5\,\text{m}$. Evaluate the reasonableness of this prediction.

**Fully worked solutions for this chapter are available online.**

# Answers

## Chapter 5 Anti-differentiation

### 5.2 Anti-differentiation of rational functions

#### 5.2 Exercise

1. a. $\dfrac{x^7}{7} + c$  b. $\dfrac{4x^{\frac{5}{4}}}{5} + c$

   c. $\dfrac{-1}{x} + c$  d. $3\sqrt[3]{x} + c$

2. a. $\dfrac{x^8}{8} + c$  b. $2x^4 + 2x^2 + c$

   c. $x^3 + \dfrac{5}{2}x^2 - 8x + c$  d. $\dfrac{x^4}{2} + x^3 - 3x^2 - 9x + c$

3. a. $x^2 + 5x + c$  b. $x^3 + 2x^2 - 10x + c$

   c. $2x^5 + \dfrac{3}{2}x^4 + 2x + c$  d. $\dfrac{1}{4}x^4 + 12x - \dfrac{1}{3}x^3 + c$

4. a. $\dfrac{8}{3}x\sqrt{x} + \dfrac{1}{x} + c$

   b. $4x\sqrt{x} + 6\sqrt{x} + 8x + c$

5. a. $\dfrac{1}{25}x^5 + c$  b. $-\dfrac{1}{9x^3} + c$

   c. $\dfrac{2}{3}x^{\frac{3}{2}} + c$  d. $\dfrac{3}{5}x^{\frac{5}{3}} + c$

6. a. $-\dfrac{5}{2x^2} + c$  b. $\dfrac{2}{x^5} + c$

   c. $16\sqrt{x} + c$  d. $\dfrac{12}{\sqrt{x}} + c$

7. a. $\dfrac{1}{3}x^3 - 2x^2 - 21x + c$

   b. $\dfrac{5}{3}x^3 + 5x^2 - 5x + c$

   c. $\dfrac{1}{4}x^4 - \dfrac{7}{3}x^3 + 2x^2 - 28x + c$

   d. $\dfrac{1}{4}x^4 + x^3 - 2x^2 + c$

8. a. $\dfrac{1}{4}x^4 + \dfrac{1}{2}x^2 + c$

   b. $\dfrac{2}{5}x^{\frac{5}{2}} + \dfrac{4}{3}x^{\frac{3}{2}} - 2\sqrt{x} + c$

   c. $-5x^{-2} + x^{-1} + x^2 + c$

9. $f(x) = \dfrac{1}{3}x^3 + \dfrac{1}{x} + c$

10. $\dfrac{7}{3}x^3 + \dfrac{1}{5x^2} + c$

11. $x^4 - \dfrac{7}{3}x^3 + x^2 - x + c$

12. $\dfrac{16}{5}x^2\sqrt{x} + c$

13. a. $f(x) = \dfrac{3}{4}x^2 - \dfrac{4}{3}x^3 + \dfrac{1}{2}x^4 + c$

   b. $6\sqrt{x} - x^4 - \dfrac{1}{5x^2} + c$

14. a. $\dfrac{1}{2}x^4 - \dfrac{1}{3}x^3 - \dfrac{15}{2}x^2 + c$

   b. $\dfrac{3}{7}x^3\sqrt{x} - \dfrac{1}{3}x\sqrt{x} + c$

15. a. $4\sqrt{x} - \dfrac{3}{x} + \dfrac{1}{4x^2} + c$

   b. $\dfrac{1}{2}x^4 - \dfrac{1}{3}x^3 + \dfrac{1}{2}x^2 + 4x + c$

16. a. $2x^3 + \dfrac{5}{2}x^2 - 6x + c$  b. $\dfrac{1}{2}x^2 + x - \dfrac{1}{x} + c$

   c. $\dfrac{4}{3}x\sqrt{x} - 8\sqrt{x} + c$  d. $\dfrac{1}{7}x^7 - 4x - \dfrac{4}{5x^5} + c$

17. $y = \dfrac{1}{4}x^4 - 2x\sqrt{x} + c$

18. $y = \dfrac{1}{2}x^2 + 3x + \dfrac{3}{x} + c$

19. $y = \dfrac{2}{3}x\sqrt{x} + 2\sqrt{x} + c$

20. $\dfrac{dy}{dx} = \dfrac{x}{\sqrt{x^2+1}}$

   $\displaystyle\int \dfrac{5x}{\sqrt{x^2+1}} = 5\sqrt{x^2+1} + c$

### 5.3 Anti-differentiation of exponential functions

#### 5.3 Exercise

1. a. $\dfrac{1}{2}e^{2x} + c$  b. $\dfrac{1}{4}e^{4x} + c$  c. $-e^{-x} + c$

2. a. $-\dfrac{1}{3}e^{-3x} + c$  b. $e^{5x} + c$  c. $\dfrac{7}{4}e^{4x} + c$

3. a. $3e^{\frac{x}{3}} + c$

   b. $0.4e^{\frac{x}{4}} + c = \dfrac{2}{5}e^{\frac{x}{4}} + c$

   c. $6e^{\frac{x}{2}} + c$

4. a. $-9e^{\frac{-x}{3}} + c$  b. $e^x - e^{-x} + c$  c. $\dfrac{1}{2}e^x + \dfrac{1}{2}e^{-x} + c$

5. $y = \dfrac{1}{2}e^{2x} - 2x - \dfrac{1}{2}e^{-2x} + c$

6. $y = x + e^{2x} + \dfrac{1}{4}e^{4x} + c$

7. $y = \dfrac{1}{6}e^{6x} + 4e^{3x} + 36x + c$

8. a. $\dfrac{1}{5}x^5 + \dfrac{1}{4}e^{-4x} + c$

   b. $\dfrac{1}{4}e^{2x} + \dfrac{4}{3}e^{-\frac{x}{2}} + c$

9. a. $\dfrac{1}{2}e^x - \dfrac{1}{4}e^{-6x} + c$

   b. $\dfrac{1}{2}e^{2x} - \dfrac{2}{3}e^{3x} + \dfrac{1}{4}e^{4x} + c$

10. $e^x + 4e^{-\frac{x}{2}} - \dfrac{1}{2}e^{-2x} + c$

11. C

12. D

**13.** $f(x) = 2e^{2x} + 8x + c$

**14.** $y = \dfrac{1}{4}e^{4x} - x + c$

**15.** $y = 2e^{3x} + 3x^3 - 4e^{\frac{x}{2}} + c$

**16.** $\dfrac{1}{6}e^{6x} - 3e^x - \dfrac{3}{4}e^{-4x} + \dfrac{1}{9}e^{-9x} + c$

**17.** $f(x) = -2e^{-2x} - 4x + c$

**18.** $a = -9, b = 3$

**19.** $m = 10, n = 2, p = 4, q = -3$

**20.** $\dfrac{dy}{dx} = 3(x+1)^2 e^{(x+1)^3}$

$\displaystyle\int 9(x+1)^2 e^{(x+1)^3}\, dx = 3e^{(x+1)^3} + c$

## 5.4 Anti-differentiation of logarithmic functions

### 5.4 Exercise

**1. a.** $8\ln(x) + c$
    **b.** $\dfrac{6}{5}\ln(x) + c$

    **c.** $\dfrac{7}{3}\ln(x) + c$
    **d.** $\dfrac{4}{7}\ln(x) + c$

**2. a.** $3\ln(x+3) + c$
    **b.** $-2\ln(x+4) + c$

    **c.** $-6\ln(x+5) + c$
    **d.** $\dfrac{4}{3}\ln(3x+2) + c$

**3. a.** $\dfrac{8}{5}\ln(5x+6) + c$
    **b.** $\dfrac{3}{2}\ln(2x-5) + c$

    **c.** $-\dfrac{5}{2}\ln(3+2x) + c$
    **d.** $-\dfrac{2}{7}\ln(6+7x) + c$

**4. a.** $-\ln(5-x) + c$
    **b.** $-\dfrac{3}{11}\ln(6-11x) + c$

    **c.** $\dfrac{2}{3}\ln(4-3x) + c$
    **d.** $4\ln(5-2x) + c$

**5.** $2x^2 + 20x + 25\ln(x) + c$

**6.** $9x + 12\ln(x) - \dfrac{4}{x} + c$

**7. a.** $3\ln(x) - 4x + c$

    **b.** $2x - 3\ln(x) - \dfrac{4}{x} + c$

    **c.** $8\ln(x) - 12x + \dfrac{9}{4}x^2 + c$

    **d.** $9\ln(x) + 2\sqrt{x} + c$

**8.** $f(x) = \dfrac{1}{2}x^2 - 4\ln(x) + c$

**9.** $y = x^2 + 3x + 4\ln(5-x) + c$

**10.** $y = \dfrac{1}{2}x^2 - 2x + \ln(x) + c$

**11.** $a = -3, b = 2$

**12.** B

**13. a.** Sample responses can be found in the worked solutions in the online resources.

    **b.** $2x + \ln(x-3) + c$

**14. a.** Sample responses can be found in the worked solutions in the online resources.

    **b.** $\dfrac{1}{2}x^2 + 6x + 16\ln(x-2) + c$

**15.** $a = 12, b = 2$

**16. a.** 10

    **b.** $f(x) = 5\ln(2x+3) + c$

**17.** $2\left(\log_e(x)\right)^2$

**18.** $f'(x) = 3 + 3\log_e(x)$ and $\displaystyle\int 2\log_e(x)\, dx = 2x\log_e(x) - 2x$

**19.** $f'(x) = 2\log_e(mx) + 2$ and

$\displaystyle\int \log_e(mx)\, dx = x\log_e(mx) - x + c$

**20.** $a = 4, b = 2, d = 5$

## 5.5 Anti-differentiation of trigonometric functions

### 5.5 Exercise

**1. a.** $-\dfrac{1}{3}\cos(3x) + c$
    **b.** $-\dfrac{1}{4}\cos(4x) + c$

    **c.** $\dfrac{1}{7}\sin(7x) + c$
    **d.** $\dfrac{1}{6}\sin(2x) + c$

    **e.** $\dfrac{1}{2}\cos(-2x) + c$
    **f.** $-\dfrac{1}{3}\sin(-3x) + c$

**2. a.** $-\dfrac{2}{9}\cos(6x) + c$
    **b.** $-2\cos(4x) + c$

    **c.** $2\cos(3x) + c$
    **d.** $2\sin(-x) + c$

    **e.** $-3\cos\left(\dfrac{x}{3}\right) + c$
    **f.** $2\sin\left(\dfrac{x}{2}\right) + c$

**3. a.** $12\cos\left(\dfrac{-x}{4}\right) + c$
    **b.** $10\cos\left(\dfrac{x}{5}\right) + c$

    **c.** $16\sin\left(\dfrac{x}{4}\right) + c$

**4. a.** $12\sin\left(\dfrac{-x}{2}\right) + c$
    **b.** $-6\cos\left(\dfrac{2x}{3}\right) + c$

    **c.** $8\sin\left(\dfrac{3x}{4}\right) + c$

**5.** $\dfrac{1}{6}\sin(3x+4) + 8\cos\left(\dfrac{x}{2}\right) + c$

**6.** $\dfrac{3}{2}\sin\left(\dfrac{2x}{3}\right) - \dfrac{1}{8}\cos(5-2x) + c$

**7.** $f(x) = -2\cos\left(\dfrac{x}{2}\right) - 6\sin\left(\dfrac{x}{2}\right) + c$

**8.** $f(x) = \dfrac{7}{2}\sin(2x) + \dfrac{1}{3}\cos(3x) + c$

**9. a.** $\dfrac{1}{4}e^{4x} - \dfrac{1}{2}\cos(2x) + \dfrac{1}{4}x^4 + c$

    **b.** $x^3 - \sin(2x) + 2e^{3x} + c$

**10. a.** $\sin(x) - \cos(x) + c$

**b.** $-\dfrac{1}{2}\cos(2x) - \sin(x) + c$

**c.** $\dfrac{1}{4}\sin(4x) - \dfrac{1}{2}\cos(2x) + c$

**d.** $-2\cos\left(\dfrac{x}{2}\right) - \dfrac{1}{2}\sin(2x) + c$

**11. a.** $\sin(4x) + \dfrac{1}{6}\cos(2x) + c$

**b.** $\dfrac{5}{2}x^2 - 2\cos(2x) + c$

**c.** $\dfrac{6}{\pi}\left(\sin\left(\dfrac{\pi x}{3}\right) - \cos\left(\dfrac{\pi x}{2}\right)\right) + c$

**d.** $\dfrac{1}{2}e^{6x} + \dfrac{1}{2}\cos(8x) + 7x + c$

**12. a.** $3e^{\frac{x}{3}} - 3\cos\left(\dfrac{x}{3}\right) + \dfrac{1}{6}x^2 + c$

**b.** $\dfrac{1}{4}\sin(4x) - e^{-3x} + c$

**13.** $-\dfrac{1}{4x} - \dfrac{2}{3\pi}\cos\left(\dfrac{3\pi x}{2}\right) + c$

**14.** $y = \dfrac{1}{2}\sin(2x) + \dfrac{1}{3}e^{-3x} + c$

**15. a.** $-2$

**b.** $f(x) = \dfrac{2}{3}\cos(3x) + c$

**16. a.** $-5$

**b.** $f(x) = 2\sin(2x) - 5x + c$

**17.** $y = -5\sin\left(2x + \dfrac{\pi}{3}\right) + c$

**18.** $p = 4,\ q = -\dfrac{3}{2}$

**19.** $a = -2,\ b = -4,\ m = 2,\ n = -2,$

**20.** $\dfrac{dy}{dt} = \dfrac{\pi}{20}\cos\left(\dfrac{\pi t}{4}\right)\ y = \dfrac{1}{5}\sin\left(\dfrac{\pi t}{4}\right) + c$

## 5.6 Further integration

### 5.6 Exercise

**1. a.** $\dfrac{1}{3}(x+3)^3 + c$  **b.** $\dfrac{1}{4}(x-5)^4 + c$

**c.** $\dfrac{1}{5}(2x+1)^5 + c$  **d.** $-\dfrac{1}{9}(3x-4)^6 + c$

**e.** $\dfrac{1}{30}(6x+5)^5 + c$  **f.** $\dfrac{1}{4}(4x-1)^3 + c$

**2. a.** $-\dfrac{1}{4}(4-x)^4 + c$  **b.** $-\dfrac{1}{5}(7-x)^5 + c$

**c.** $-\dfrac{4}{15}(8-3x)^5 + c$  **d.** $\dfrac{1}{33}(8-9x)^{11} + c$

**e.** $-\dfrac{1}{2}(2x+3)^{-1} + c$  **f.** $-\dfrac{1}{12}(6x+5)^{-2} + c$

**3. a.** $\dfrac{1}{18}(3x-5)^6 + c$  **b.** $-\dfrac{1}{3(2x-3)^{\frac{3}{2}}} + c$
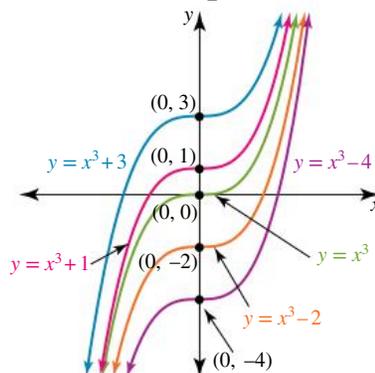
**4. a.** $\dfrac{1}{10}(2x+3)^5 + c$

**b.** $\dfrac{1}{8(1-2x)^4} + c$

**5. a.** $\dfrac{1}{4}e^{4x+2} - 4e^{2x+1} + 16x + c$
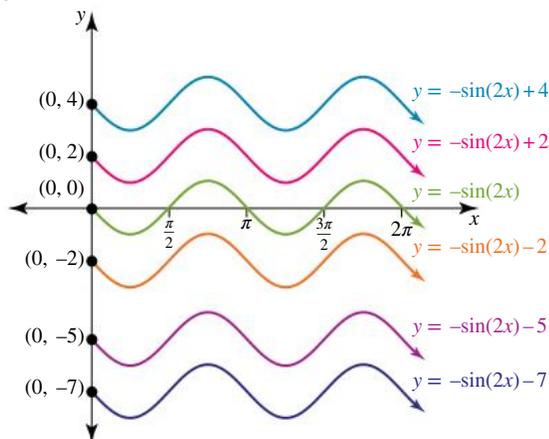
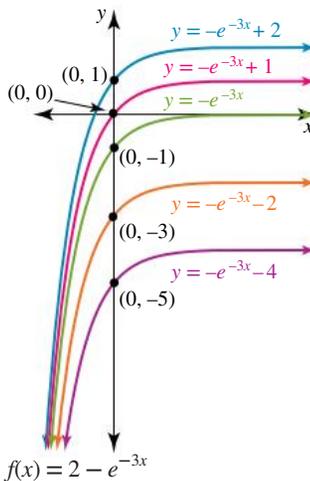**b.** $-2e^{6-2x} - 6e^{5-2x} - \dfrac{9}{2}e^{4-2x} + c$

**6. a.**



**b.** $f(x) = x^3 + 8$

**7. a.** $f(x) = -\sin(2x) + c$



**b.** $f(x) = 4 - \sin(2x)$

**8. a.** $f(x) = -e^{-3x} + c$

**b.**



$f(x) = 2 - e^{-3x}$

**9. a.** $f(x) = 2x^2 + x + 2$

   **b.** $f(x) = 5x - x^2 - 5$

   **c.** $f(x) = 3x + 2 - \dfrac{1}{x}$

**10. a.** $f(x) = \dfrac{1}{2}x^2 + \dfrac{2}{3}x^{\frac{3}{2}} - \dfrac{10}{3}$

   **b.** $f(x) = \dfrac{3}{4}x^{\frac{4}{3}} - x^3 + 50x$

   **c.** $f(x) = 2\sqrt{x} - x^2 - 6$

**11.** $y = e^{2x} - e^{-x} + 3$

**12.** $f(x) = \dfrac{1}{2}\sin(2x) + \dfrac{1}{2}\cos(2x) + \dfrac{3}{2}$

**13.** $\dfrac{6x}{(3x^2 + 4)}; \dfrac{1}{6}\ln(3x^2 + 4) + c$

**14.** $\ln(x) + 1; x\ln(x) - x + c$

**15.** $\sin(x) + x\cos(x); x\sin(x) + \cos(x) + c$

**16.** $2e^{3x} + 6xe^{3x}; \dfrac{1}{3}xe^{3x} - \dfrac{1}{9}e^{3x} + c$

**17.** $f(x) = \dfrac{1}{4}x^4 - \log_e(x), x > 0$

**18.** $-\ln(\cos(x)) + c$

**19.** $-5$

**20.** $x$-intercepts: $(-2.20, 0)$ and $(2.55, 0)$
   $y$-intercept: $(0, -8)$

## 5.7 Modelling and solving using indefinite integrals

### 5.7 Exercise

**1. a.** $L = 6t + 2\sin\left(\dfrac{\pi t}{4}\right) + 12$

   **b.** 132 cm

**2. a.** $V = \dfrac{\pi}{3}r^3$     **b.** $\dfrac{64\pi}{3}\text{cm}^3$

**3.** $V = 13333\dfrac{1}{3}\text{cm}^3$

**4. a.** $P = 50e^{0.4t} - 15$

   **b.** 536 bugs

**5. a.** $N = 400t + \dfrac{2000}{3}\sqrt{t^3} + 40$

   **b.** 9494 families

**6. a.** $x = \dfrac{2}{9}\sqrt{(3t + 1)^3} - \dfrac{2}{9}$

   **b.** $x = \dfrac{1}{2} - \dfrac{1}{(t + 2)}$

   **c.** $x = \dfrac{1}{8}(2t + 1)^4 - \dfrac{1}{8}$

**7. a.** $x = t^3 + 3t^2 - 2$

   **b.** 198 m

**8. a.** $x = \dfrac{1}{3}e^{(3t-1)} - \dfrac{1}{3e}$

   **b.** $x = \dfrac{1}{2}\cos(2t + 3) - \dfrac{1}{2}\cos(3)$

**9.** $x = 4t^3 - 3t^2 + t + 2$

**10.** $x = 18\sin\left(\dfrac{t}{3}\right)$

**11. a.** $x = 24 - 24\cos\left(\dfrac{\pi t}{8}\right)$

   **b.** Maximum position $= 48$ metres

   **c.** After 4 seconds the particle is 24 metres above the stationary position.

**12. a.** 156.25 m/s     **b.** $x = 6.25t^2 - \dfrac{1}{12}t^3$

**13. a.** $P = 100e^{0.3t} - 50$

   **b.** 1958 sea lions

**14. a.** $h = 2\sin\left(\dfrac{\pi t}{4}\right) + 3$

   **b.** Maximum depth: 5 m; minimum depth: 1 m

   **c.** 8 hours/day

**15.** $4\left(\sqrt{2} - 1\right)$ m

**16. a.** 1 s          **b.** 25 m

**17.** 40 m

**18.** 67.388 m

**19.** $11\dfrac{1}{9}$ m

**20.** 54 cm

## 5.8 Review

### 5.8 Exercise

**1. a.** $\dfrac{1}{2}x^6 + c$     **b.** $-\dfrac{5}{x} + c$     **c.** $-\dfrac{2}{5}x^5 + c$

   **d.** $\dfrac{4}{3}x^{\frac{3}{2}} + c$     **e.** $\dfrac{1}{25}x^5 + c$     **f.** $-\dfrac{1}{15(3x - 8)^5} + c$

**2. a.** $\dfrac{1}{5}x^5 + x^2 + \ln(x) + c$

   **b.** $\dfrac{1}{18}(3x + 1)^6 + c$

   **c.** $3x + 2\ln(x) + \dfrac{1}{x} + c$

   **d.** $\dfrac{3}{2}\ln(2x + 1) + c$

   **e.** $\dfrac{1}{2}\ln(6 - 10x) + c$

   **f.** $-\dfrac{3}{8}(4x + 1)^{-2} + c$

**3. a.** $f(x) = \dfrac{1}{4}(x + 4)^4 + 1$

   **b.** $f(x) = (1 - 2x)^{-4} + 2$

   **c.** $f(x) = \ln(x + 5) + 2$

   **d.** $f(x) = -4\ln(7 - 2x) + 7$

**4. a.** $-8$          **b.** 41

**5. a.** $\dfrac{1}{2}e^{2x} - 6e^x + 9x + c$

   **b.** $x - 3e^{-x} - \dfrac{3}{2}e^{-2x} - \dfrac{1}{3}e^{-3x} + c$

**6. a.** $\dfrac{4}{5}\cos\left(\dfrac{5x}{2}\right)+c$    **b.** $-\dfrac{12}{7}\sin\left(\dfrac{7x}{4}\right)+c$

**c.** $-\dfrac{5}{\pi}\cos(\pi x)+c$    **d.** $\dfrac{6}{\pi}\sin\left(\dfrac{\pi x}{2}\right)+c$

**e.** $-\dfrac{6}{\pi}\sin\left(\dfrac{\pi x}{3}\right)+c$    **f.** $-\dfrac{\pi}{4}\cos\left(\dfrac{-4x}{\pi}\right)+c$

**7. a.** $\dfrac{1}{4}x^4-\dfrac{1}{2}\ln(2x+3)+\dfrac{1}{2}e^{2x}+c$

**b.** $\dfrac{1}{3}x^3+2\sin(2x)+\ e^{-x}+c$

**8. a.** $-6\cos\left(\dfrac{x}{2}\right)-6\sin\left(\dfrac{x}{3}\right)+5e^{\frac{-x}{5}}+c$

**b.** $\dfrac{2}{3}x\sqrt{x}+x^2+\dfrac{6}{\pi}\cos\left(\dfrac{\pi x}{3}\right)+5x+c$

**9. a.** $f(x)=\sin(x)+4$

**b.** $f(x)=1-2\cos(2x)$

**c.** $f(x)=12\sin\left(\dfrac{x}{4}\right)+3\sqrt{2}$

**d.** $f(x)=4\sin\left(\dfrac{x}{4}\right)+2\cos\left(\dfrac{x}{2}\right)-4$

**10.** $\dfrac{2x}{(x^2+3)}$; $6\ln\left(x^2+3\right)+c$

**11.** $\dfrac{-1}{\sin^2(x)}$; $-\dfrac{\cos(x)}{\sin(x)}+c$

**12. a.** Sample responses can be found in the worked solutions in the online resources.
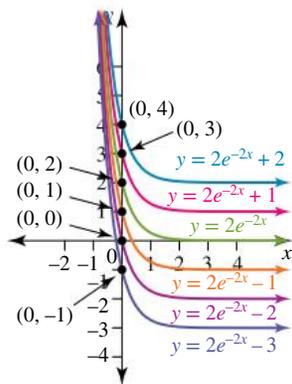
**b.** $-3x-2\ln(3-2x)+c$

**13. a.** $-4$

**b.** $f(x)=2\sin(2x)-4e^x+3$

**c.** $-2.02$

**14. a.**



(0, 4)
(0, 3)
$y=2e^{-2x}+2$
(0, 2)
(0, 1)
$y=2e^{-2x}+1$
(0, 0)
$y=2e^{-2x}$
$y=2e^{-2x}-1$
(0, −1)
$y=2e^{-2x}-2$
$y=2e^{-2x}-3$

**b.** $f(x)=2-e^{-3x}$

**15. a.** $f(x)=5x-x^2$

**b.** $f(x)=3-2\cos\left(\dfrac{x}{2}\right)$

**c.** $f(x)=\dfrac{1}{(1-x)}+3$

**16. a.** $\dfrac{dy}{dx}=2(3x-1)e^{3x^2-2x+1}$,

$\displaystyle\int(3x-1)e^{3x^2-2x+1}\,dx=\dfrac{1}{2}e^{3x^2-2x+1}$

**b.** $\dfrac{dy}{dx}=2x\cos(2x)+\sin(2x)$

$\displaystyle\int x\cos(2x)\,dx=\dfrac{x\sin(2x)}{2}+\dfrac{\cos(2x)}{4}$

**17.** 4.911 metres to the right of the origin.

**18.** 5 hours and 15 minutes

**19.** 64 °C

**20.** Between 2 pm and 10 pm.

## 5.8 Past QCAA exam questions

**1.** C

**2.** D

**3. a.** $P(t)=\dfrac{1}{2}\left(e^{2t}+119\right)$

**b.** $e^6-e^4$

**c.** $\dfrac{1}{2}\ln(121)$

**4. a.** $2\ln(6-5\cos(2t))+6$

**b.** 7.212 m

**5.** Sample responses can be found in the worked solutions in the online resources.

# 6 General discrete random variables

## LESSON SEQUENCE

Fully worked solutions of this chapter are available online.

## EXAM PREPARATION

Access exam-style questions in every lesson, available online and access past QCAA exam questions in every review.

### on Resources

| | |
|---|---|
| **Solutions** | Solutions — Chapter 6 |
| **Exam questions** | Exam question booklet — Chapter 6 |
| **Digital documents** | Learning matrix — Chapter 6 |
| | Chapter summary — Chapter 6 |

# LESSON
## 6.1 Overview

### 6.1.1 Introduction

When we toss a coin, the result can be either a Head or a Tail. When we throw a standard die, the result can only be one of the numbers from 1 to 6. When we select a card from a standard deck, there are only 52 possibilities.

The tossing of a coin, the selection of a card and the rolling of a die can all be regarded as experimental trials, the outcome of which cannot be predicted with absolute certainty ahead of time. The results of such trials are referred to as random variables, a term first formally coined by the Russian mathematician Andrei Kolmogorov in his 1933 text *Grundbegriffe der wahrscheinlichkeitsrechnung* ('Basic concepts of probability').

In this chapter we will only explore the probabilities of events with sample spaces that have a countable number of elements.

### 6.1.2 Syllabus links

| Lesson | Lesson title | Syllabus links |
| --- | --- | --- |
| 6.2 | Discrete random variables | ○ Understand the concepts of a discrete random variable and its associated probability function, and its use in modelling data. <br> ○ Use relative frequencies obtained from data to determine point estimates of probabilities associated with a discrete random variable. <br> ○ Recognise uniform discrete random variables and use them to model random phenomena with equally likely outcomes. <br> ○ Recognise non-uniform discrete random variables and use them to model random phenomena. |
| 6.3 | Expected values | ○ Determine and use the mean (expected value) of a discrete random variable as a measurement of centre, $E(X) = \mu = \sum p_i x_i$, where $p_i$ is the probability of outcome $x_i$ occurring. |
| 6.4 | Variance and standard deviation | ○ Determine and use the variance of a discrete random variable as a measure of spread, $\mathrm{Var}(X) = \sum p_i \left( x_i - \mu \right)^2$ where $p_i$ is the probability of outcome $x_i$ occurring, and $\mu$ is the mean. <br> ○ Determine and use the standard deviation of a discrete random variable, $\sqrt{\mathrm{Var}(X)}$, as a measure of spread. |
| 6.5 | Modelling and solving with discrete random variables | ○ Model and solve problems that involve discrete random variables and associated probabilities, with and without technology. |

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# LESSON
## 6.2 Discrete random variables

### 6.2.1 Discrete random variables

A **random variable** is one whose value cannot be predicted but is determined by the outcome of an experiment. For example, two dice are rolled simultaneously a number of times. The sum of the numbers appearing uppermost is recorded. The possible outcomes we could expect are {2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}. Since the outcome may vary each time the dice are rolled, the sum of the numbers appearing uppermost is a random variable.

Random variables are expressed as capital letters (for example $X, Y, Z$), and the values they can take on are represented by lowercase letters (for example $x, y, z$ respectively).

The above situation with dice illustrates an example of a **discrete random variable** since the possible outcomes were able to be *counted*. Discrete random variables generally deal with number or size.

A random variable that can take on any value is defined as a **continuous random variable**. Continuous random variables generally deal with quantities that can be *measured*, such as mass, height or time.

We will look at continuous random variables later in this course.

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**WORKED EXAMPLE 1 Identifying discrete random variables**

**Identify which of the following represent discrete random variables.**
**a. The number of goals scored at a football match**
**b. The height of students in a Maths Methods class**
**c. The shoe sizes worn by members of a volleyball team**
**d. The number of girls in a five-child family**
**e. The time taken to run a distance of 10 kilometres in minutes**

| THINK | WRITE |
|---|---|
| Determine whether the variable can be counted or needs to be measured. | |
| a. Goals can be counted. | a. Discrete |
| b. Height must be measured. | b. Continuous |
| c. The number of shoe sizes can be counted. | c. Discrete |
| d. The number of girls can be counted. | d. Discrete |
| e. Time must be measured. | e. Continuous |

## 6.2.2 Discrete probability distributions

When we are dealing with random variables, we often need to know the probabilities associated with them.

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### WORKED EXAMPLE 2 Drawing a probability distribution table

**Let $X$ represent the variable 'number of Tails' obtained in three tosses. Draw up a table that displays the values the discrete random variable can assume ($x$) and the corresponding probabilities.**

**THINK**

1. List all of the possible outcomes.

2. Draw up a table with two columns: one labelled 'Number of Tails', the other 'Probability'.

3. Enter the information into the table.

**WRITE**

HHH, HHT, HTH, HTT, THH, THT, TTH, TTT

| Number of Tails ($x$) | Probability ($p(x)$) |
|:---:|:---:|
| 0 | $\dfrac{1}{8}$ |
| 1 | $\dfrac{3}{8}$ |
| 2 | $\dfrac{3}{8}$ |
| 3 | $\dfrac{1}{8}$ |

---

The table above displays the **probability distribution** of the total number of Tails obtained in three tosses of a fair coin. Since the variable in this case is discrete, the table displays a discrete probability distribution.

In Worked Example 2, we used $X$ to denote the random variable and $x$ the value that the random variable could take. Thus, the **probability** can be denoted by $p(x)$ or $p(x = x)$, and the above table could be presented as shown below.

| $x$ | 0 | 1 | 2 | 3 |
|:---:|:---:|:---:|:---:|:---:|
| $p(x)$ | $\dfrac{1}{8}$ | $\dfrac{3}{8}$ | $\dfrac{3}{8}$ | $\dfrac{1}{8}$ |

Close inspection of this table shows important characteristics that satisfy all discrete probability distributions.

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### Key characteristics of a discrete probability distribution

1. Each probability lies in a restricted interval $0 \le p(x) \le 1$.
2. The probabilities of a particular experiment sum to 1; that is,

$$\sum p(x) = 1.$$

If these two characteristics are not satisfied, then there is no discrete probability distribution.

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## WORKED EXAMPLE 3 Drawing a probability distribution graph

**Draw a probability distribution graph of the outcomes in Worked example 2.**
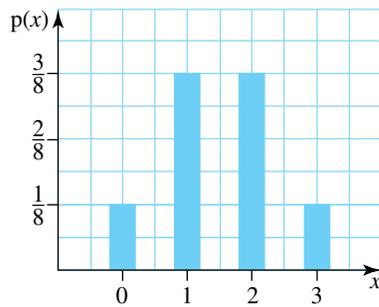
| THINK | WRITE |
|---|---|
| 1. Draw a set of axes in the first quadrant only. Label the horizontal axis $x$ and the vertical axis $p(x)$. | |
| 2. Mark graduations evenly along the horizontal and vertical axes, and label them with appropriate values. | |
| 3. Draw a straight line from each $x$-value to its corresponding probability. | |
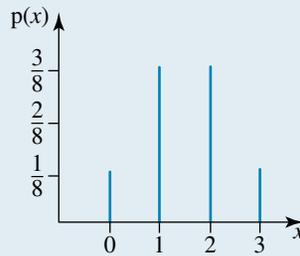
*Note:* The probability distribution graph may also be drawn as follows.

A column graph
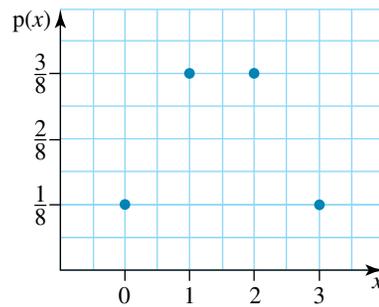
A dot graph

The tossing of an unbiased die 3 times to see how many sixes are obtained is an example of a **uniform distribution**, because all of the outcomes are equally likely. Another example is determining how many Heads are obtained when a single coin is tossed $n$ times. However, a **non-uniform distribution** exists when a biased coin is used, because all of the outcomes are not equally likely.

## WORKED EXAMPLE 4 Modelling a non-uniform probability distribution

**A motorist travels along a main road in Brisbane. In doing so they must travel through three intersections with traffic lights over a stretch of two kilometres. At each intersection the motorist will encounter either a red light or a green light (ignoring amber!). The probability that the motorist will have to stop because of a red light at any of the intersections is $\frac{2}{5}$.**

**Let $X$ be the number of red lights encountered by the motorist.**

**a. Use a tree diagram to produce a sample space for this situation.**
**b. Determine the probability of each outcome.**
**c. Determine the probability distribution for this random variable.**
**d. Test whether this probability distribution obeys the necessary properties for a discrete random variable distribution.**

**THINK**                    **WRITE**

**a. 1.** Set up a tree diagram to show the sample space.
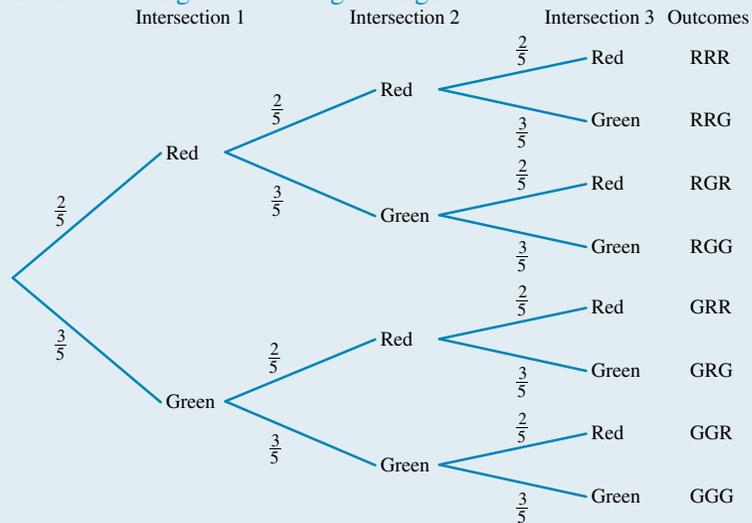
*Note:* $p(R) = \dfrac{2}{5}$,

$p(G) = \dfrac{3}{5}$.

**a.** Let $R = $ a red light and $G = $ a green light.



Intersection 1    Intersection 2    Intersection 3  Outcomes

| | | | | | |
|---|---|---|---|---|---|
| | | | Red | Red | RRR |
| | | Red | | Green | RRG |
| | Red | | Green | Red | RGR |
| | | | | Green | RGG |
| | | | Red | Red | GRR |
| | Green | | | Green | GRG |
| | | | Green | Red | GGR |
| | | | | Green | GGG |

**2.** List the event or sample space.

$\xi = \{RRR, RRG, RGR, RGG, GRR, GRG, GGR, GGG\}$

**b.** Calculate the probability of each outcome.

**b.** $p(RRR) = \dfrac{2}{5} \times \dfrac{2}{5} \times \dfrac{2}{5} = \dfrac{8}{125}$

$p(RRG) = \dfrac{2}{5} \times \dfrac{2}{5} \times \dfrac{3}{5} = \dfrac{12}{125}$

$p(RGR) = \dfrac{2}{5} \times \dfrac{3}{5} \times \dfrac{2}{5} = \dfrac{12}{125}$

$p(RGG) = \dfrac{2}{5} \times \dfrac{3}{5} \times \dfrac{3}{5} = \dfrac{18}{125}$

$p(GRR) = \dfrac{3}{5} \times \dfrac{2}{5} \times \dfrac{2}{5} = \dfrac{12}{125}$

$p(GRG) = \dfrac{3}{5} \times \dfrac{2}{5} \times \dfrac{3}{5} = \dfrac{18}{125}$

$p(GGR) = \dfrac{3}{5} \times \dfrac{3}{5} \times \dfrac{2}{5} = \dfrac{18}{125}$

$p(GGG) = \dfrac{3}{5} \times \dfrac{3}{5} \times \dfrac{3}{5} = \dfrac{27}{125}$

**c. 1.** Set up the probability distribution by combining the outcomes related to each possible value of $x$.

**c.** $p(x = 0) = p(GGG) = \dfrac{27}{125}$

$p(x = 1) = p(RGG) + p(GRG) + p(GGR)$

$= \dfrac{18}{125} + \dfrac{18}{125} + \dfrac{18}{125}$

$= \dfrac{54}{125}$

$p(x = 2) = p(RRG) + p(RGR) + p(GRR)$

$= \dfrac{12}{125} + \dfrac{12}{125} + \dfrac{12}{125}$

$= \dfrac{36}{125}$

$p(x = 3) = p(RRR) = \dfrac{8}{125}$

**2.** Enter the combined results into a table.

$X =$ number of red lights

| $x$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $p(x)$ | $\dfrac{27}{125}$ | $\dfrac{54}{125}$ | $\dfrac{36}{125}$ | $\dfrac{8}{125}$ |

**d.** Test whether the two properties of a discrete random variable are obeyed.

**d.** Each $p(x)$ is such that $0 \le p(x) \le 1$

and $\sum p(x) = \dfrac{27}{125} + \dfrac{54}{125} + \dfrac{36}{125} + \dfrac{8}{125}$

$\qquad = \dfrac{125}{125}$

$\displaystyle\sum_{\text{all } x} p(x) = 1$

Therefore, both properties of a discrete random distribution are obeyed.

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## WORKED EXAMPLE 5 Identifying discrete probability distributions from tables

**a. State, giving reasons, whether each of the following represents a discrete probability distribution.**

**i.**

| $x$ | 0 | 2 | 4 | 6 |
|---|---|---|---|---|
| $p(x)$ | $-0.1$ | 0.3 | 0.4 | 0.2 |

**ii.**

| $x$ | $-3$ | $-1$ | 4 | 6 |
|---|---|---|---|---|
| $p(x)$ | 0.01 | 0.32 | 0.52 | 0.15 |

**iii.**

| $x$ | $-1$ | 0 | 1 | 2 |
|---|---|---|---|---|
| $p(x)$ | 0.2 | 0.1 | 0.2 | 0.3 |

**b. A random variable, $X$, has the following probability distribution.**

| $x$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $p(x)$ | $b$ | $2b$ | $0.5b$ | $0.5b$ | $b$ |

**Calculate the value of the constant $b$.**

**THINK**

**a. i. 1.** Check that each probability is a value from 0 to 1.

**2.** If this condition is satisfied, add the probabilities together to see if they add to 1.

**3.** Answer the question.

**ii. 1.** Check that each probability is a value from 0 to 1.

**2.** If this condition is satisfied, add the probabilities together to see if they add to 1.

**3.** Answer the question.

**WRITE**

**a. i.** Each probability does not meet the requirement $0 \le p(X = x) \le 1$, as $p(X = 0) = -0.1$.

As one of the probabilities is a negative value, there is no point checking the sum of the probabilities.

This is not a discrete probability distribution.

**ii.** Each probability does meet the requirement $0 \le p(X = x) \le 1$.

$\sum p(X = x) = 0.01 + 0.32 + 0.52 + 0.15$
$\qquad\qquad = 1$

Yes, this is a discrete probability function, as both of the conditions have been satisfied.

**iii. 1.** Check that each probability is a value from 0 to 1.

**2.** If this condition is satisfied, add the probabilities together to see if they add to 1.

**3.** Answer the question.

**iii.** Each probability does meet the requirement $0 \leq p(X = x) \leq 1$.

$$\sum p(X = x) = 0.2 + 0.1 + 0.2 + 0.3$$
$$= 0.8$$

As the sum of the probabilities is not equal to 1, this is not a discrete probability distribution.

**b. 1.** As we know this is a probability distribution, we can equate the probabilities to 1.

**2.** Simplify.

**3.** Solve for $b$.

**b.** $\sum p(X = x) = 1b + 2b + 0.5b + 0.5b + b = 1$

$$5b = 1$$

$$b = \frac{1}{5}$$

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## WORKED EXAMPLE 6 Solving for algebraic values in a discrete probability distribution

**Determine the value(s) of $k$ if the table represents a discrete probability distribution.**

| $x$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $p(x)$ | $2k$ | $5k - 0.3$ | $k$ | $3k$ | $k + 0.1$ |

**THINK**

**1.** If the table represents a discrete probability distribution, then the sum of all values of $p(x)$ will equal 1.

**2.** Add the values of $p(x)$ algebraically.

**3.** Simplify the equation by grouping like terms.

**4.** Solve for $k$.

**5.** Substitute the value of $k$ into each algebraic expression to confirm that $p(x) \geq 0$ as required for a discrete probability distribution.

**WRITE**

If it is a discrete probability distribution, $1 = \sum_{\text{all } x} p(x)$

$1 = 2k + (5k - 0.3) + k + 3k + (k + 0.1)$

$1 = 2k + 5k + k + 3k + k - 0.3 + 0.1$
$1 = 12k - 0.2$

$1 + 0.2 = 12k$
$1.2 = 12k$
$k = 0.1$

$p(X = 1) = 2k = 2(0.1) = 0.2$
$p(X = 2) = 5k - 0.3 = 5(0.1) - 0.3 = 0.5 - 0.3 = 0.2$
$p(X = 3) = k = 0.1$
$p(X = 4) = 3k = 3(0.1) = 0.3$
$p(X = 5) = k + 0.1 = 0.1 + 0.1 = 0.2$
In each case, $p(x) \geq 0$; therefore, $k = 0.1$.

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A discrete probability distribution may also be represented in the form of a function. In this case, it is referred to as a discrete probability function.

## WORKED EXAMPLE 7 Proving a function is a discrete probability function

1. Show that the function $p(x) = \dfrac{1}{42}(5x + 3)$, where $x = 0, 1, 2, 3$ is a discrete probability function.

2. Show that the function $p(x) = \dfrac{1}{100}x^2(6 - x)$, where $x = 2, 3, 4, 5$ is a discrete probability function.

| THINK | WRITE |
|---|---|
| **a. 1.** Substitute each of the $x$-values into the equation and obtain the corresponding probability. | **a.** When $x = 0$, $p(x) = \dfrac{3}{42}$ $= \dfrac{1}{14}$ |
| **2.** Simplify where possible. | When $x = 1$, $p(x) = \dfrac{8}{42}$ $= \dfrac{4}{21}$ When $x = 2$, $p(x) = \dfrac{13}{42}$ When $x = 3$, $p(x) = \dfrac{18}{42}$ $= \dfrac{3}{7}$ |
| **3.** Check whether each of the probabilities lies within the restricted interval $0 \leq p(x) \leq 1$. | All probabilities lie between 0 and 1. |
| **4.** Check whether the probabilities sum to 1. | $\dfrac{1}{14} + \dfrac{4}{21} + \dfrac{13}{42} + \dfrac{3}{7} = 1$ |
| **5.** Answer the question. | Yes, this is a probability function since both requirements have been met. |
| **b. 1.** Substitute each of the $x$-values into the equation and obtain the corresponding probability. | **b.** When $x = 2$, $p(x) = \dfrac{16}{100}$ $= \dfrac{4}{25}$ |
| **2.** Simplify where possible. | When $x = 3$, $p(x) = \dfrac{27}{100}$ When $x = 4$, $p(x) = \dfrac{32}{100}$ $= \dfrac{8}{25}$ When $x = 5$, $p(x) = \dfrac{25}{100}$ $= \dfrac{1}{4}$ |
| **3.** Check whether each of the probabilities lies within the restricted interval $0 \leq p(x) \leq 1$. | All probabilities lie between 0 and 1. |

**4.** Check whether the probabilities sum to 1.

$$\frac{4}{25} + \frac{27}{100} + \frac{8}{25} + \frac{1}{4} = 1$$

**5.** Answer the question.

Yes, this is a probability function since both requirements have been met.

## 6.2.3 Conditional probability for discrete variables

Consider the case of a probability experiment where we know part of the outcome. Suppose your friend Brett comes from a family of four children. What is the probability that there are three boys in Brett's family? Because you know Brett, you know that at least one of the four children is male.

Normally, the probability distribution of four children can be represented by the table shown below.

| $x$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $p(x)$ | 0.0625 | 0.25 | 0.375 | 0.25 | 0.0625 |

Therefore, $p(X = 3) = 0.25$ but we know that the number of males in the family is greater than 0. From the table, $p(X > 0) = 0.9375$. We can say that the probability that there are three males in the family, given that at least one is male is $\dfrac{0.25}{0.9375}$, is 0.266.

This is known as **conditional probability**.

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**Conditional probability rule**

$$p(x \mid X > n) = \frac{p(x = x \cap X > n)}{p(x > n)}$$

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### WORKED EXAMPLE 8 Calculating conditional probability

**Three balls are selected from a box containing 6 blue balls and 4 yellow balls. If the ball chosen after each selection is replaced before the next selection, determine:**
**a. the probability distribution for the following numbers of blue balls drawn**
   **i.  0 blue balls**
   **ii.  1 blue ball**
   **iii.  2 blue balls**
   **iv.  3 blue balls**
**b. the probability that 3 blue balls are chosen, given that at least 2 balls were blue.**

**THINK**

**a. i. 1.** Define the random variable.

   **2.** Assign values that $x$ can take.

   **3.** Determine the probability of each outcome.

**WRITE**

**a. i.**  Let $X =$ the number of blue balls.

   $x = 0, 1, 2, 3$

   $p(\text{blue}) = \dfrac{6}{10} \quad p(\text{yellow}) = \dfrac{4}{10}$

   $\quad\quad\quad = \dfrac{3}{5} \quad\quad\quad\quad = \dfrac{2}{5}$

**4.** Simplify where possible.

$p(X=0) \Rightarrow$ no blue, three yellow

$\qquad = p(YYY)$

$p(X=0) = \dfrac{2}{5} \times \dfrac{2}{5} \times \dfrac{2}{5}$

$\qquad = 0.064$

**ii.** $p(X=1) \Rightarrow$ one blue, two yellow

$\qquad = p(BYY) + p(YBY) + p(YYB)$

$p(X=1) = 3 \times \dfrac{3}{5} \times \dfrac{2}{5} \times \dfrac{2}{5}$

$\qquad = 0.288$

**iii.** $p(X=2) \Rightarrow$ two blue, one yellow

$\qquad = p(BBY) + p(BYB) + p(YBB)$

$p(X=2) = 3 \times \dfrac{3}{5} \times \dfrac{3}{5} \times \dfrac{2}{5}$

$\qquad = 0.432$

**iv.** $p(X=3) \Rightarrow$ three blue, no yellow

$\qquad = p(BBB)$

$p(X=3) = \dfrac{3}{5} \times \dfrac{3}{5} \times \dfrac{3}{5}$

$\qquad = 0.216$

**5.** Place all of the information in a table.

| $x$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $p(x)$ | 0.064 | 0.288 | 0.432 | 0.216 |

**6.** Check that the probabilities sum to 1.

$\sum p(x) = 0.064 + 0.288 + 0.432 + 0.216$

$\qquad = 1$

**b.**

**1.** Define the rule for conditional probability.

**b.** $p(X=3|X>1) = \dfrac{p(x=3 \cap X>1)}{p(x>1)}$

**2.** Determine each of the probabilities.

$p(X=3 \cap X>1) = p(x=3)$

$\qquad = 0.216$

$p(X>1) = 0.432 + 0.216$

$\qquad = 0.648$

**3.** Substitute values into the rule.

$p(X=3|X>1) = \dfrac{0.216}{0.648}$

**4.** Evaluate and simplify.

$= 0.3333 \left(\text{or } \dfrac{1}{3}\right)$

# Exercise 6.2 Discrete random variables

| **6.2 Exercise** | **6.2 Exam questions** on |
|---|---|

| Simple familiar | Complex familiar | Complex unfamiliar |
|---|---|---|
| 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 | 13, 14, 15, 16, 17 | 18, 19, 20 |

These questions are even better in jacPLUS!
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### Simple familiar

1. **WE1** Identify which of the following represent discrete random variables.

   a. The number of people at a tennis match
   b. The time taken to read this question
   c. The length of the left arms of students in your class
   d. The shoe sizes of twenty people
   e. The weights of babies at a maternity ward
   f. The number of grains in each of ten 250-gram packets of rice
   g. The height of jockeys competing in a certain race
   h. The number of books in Brisbane libraries

2. **WE2, 3** a. If $X$ represents the number of Heads obtained in two tosses of a coin, draw up a table that displays the values that the discrete random variable can assume and the corresponding probabilities.

   b. Draw a probability distribution graph of the outcomes in part a.

3. A fair coin is tossed three times and a note is taken of the number of Tails.

   a. List the possible outcomes.
   b. List the possible values of the random variable $X$, representing the number of Tails obtained in the three tosses.
   c. Draw a table showing the probability distribution of $X$.
   d. Determine $p(x \leq 2)$.

4. **WE4** A bag contains 3 red, 3 green and 4 yellow balls. A ball is withdrawn from the bag, its colour is noted, and then the ball is returned to the bag. This process is repeated on two more occasions. Let $Y$ be the number of green balls obtained.

   a. Use a tree diagram to produce the sample space for the experiment.
   b. Determine the probability of each outcome of $Y$.
   c. Determine the probability distribution for this random variable.
   d. Test whether this probability distribution obeys the necessary properties for a discrete random variable distribution.

5. An unbiased die is tossed twice. Let the random variable $X$ be the number of sixes obtained. Determine the probability distribution for this discrete random variable.

6. **WE5** State, giving reasons, whether each of the following represents a discrete probability distribution.

   a.

   | $y$ | 3 | 6 | 9 | 12 |
   |---|---|---|---|---|
   | $p(y)$ | 0.2 | 0.3 | 0.3 | 0.2 |

   b.

   | $y$ | $-2$ | $-1$ | 0 | 1 | 2 |
   |---|---|---|---|---|---|
   | $p(y)$ | 0.15 | 0.2 | 0.3 | 0.2 | 0.15 |

**7.** **WE5** **a.** Determine the value(s) of $k$ if the table represents a discrete probability distribution.

| $x$ | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| **p(x)** | $5k$ | $3k - 0.1$ | $2k$ | $k$ | $0.6 - 3k$ |

**b.**

| $y$ | $-6$ | $-3$ | 0 | 3 | 6 |
|---|---|---|---|---|---|
| **p(y)** | $0.5k$ | $1.5k$ | $2k$ | $1.5k$ | $0.5k$ |

**8.** Draw graphs for each of the following probability distributions.

**a.**

| $x$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| **p(x)** | 0.05 | 0.2 | 0.5 | 0.2 | 0.05 |

**b.**

| $x$ | 5 | 10 | 15 | 20 |
|---|---|---|---|---|
| **p(x)** | 0.5 | 0.3 | 0.15 | 0.05 |

**c.**

| $x$ | 2 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|
| **p(x)** | 0.1 | 0.2 | 0.4 | 0.2 | 0.1 |

**d.**

| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| **p(x)** | 0.1 | 0.2 | 0.3 | 0.4 |

**9.** Two fair dice are rolled simultaneously, and $X$, the sum of the two numbers appearing uppermost, is recorded.

   **a.** Draw up a table that displays the probability distribution of $X$.

   **b.** Calculate

      **i.** $p(X > 9)$

      **ii.** $p(X < 6)$

      **iii.** $p(4 \leq X < 6)$

      **iv.** $p(3 \leq X \leq 9)$

**10.** Two dice are weighted so that $p(2) = 0.2$, $p(1) = p(3) = p(5) = 0.1$ and $p(4) = p(6) = 0.25$. They are both rolled at the same time. Let $Z$ be the number of even numbers obtained.

   **a.** List the sample space.

   **b.** List the possible values of $Z$ and construct a probability distribution table.

   **c.** Determine $p(Z = 1)$.

**11.** **WE7** Show that the function $p(x) = \dfrac{1}{90}(8x + 2)$, where $x = 0, \ 1, \ 2, \ 3, \ 4$ is a probability function.

**12.** **WE8** Three balls are selected from a box containing 4 red balls and 5 blue balls. If the ball chosen after each selection is replaced before the next selection, determine

   **a.** the probability distribution for the following number of red balls drawn

      **i.** 0 red balls

      **ii.** 1 red ball

      **iii.** 2 red balls

      **iv.** 3 red balls

   **b.** the probability that three reds are chosen, given that at least one ball is red.

**13.** Show that the function $p(x) = \dfrac{1}{160}x^2(x+2)$, where $x = 1, 2, 3, 4$ is a probability function.

**14.** State, with reasons, whether the following are discrete probability distributions.

**a.** $p(x) = \dfrac{1}{7}(5-x), x \in \{1, 3, 4\}$

**b.** $p(x) = \dfrac{x^2 - x}{40}, x \in \{-1, 1, 2, 3, 4, 5\}$

**c.** $p(x) = \dfrac{1}{15}\sqrt{x}, x \in \{1, 4, 9, 16, 25\}$

**15.** Determine the value of $a$ if the following is a discrete probability function.

$$p(x) = \dfrac{1}{a}(15 - 3x), \ x \in \{1, 2, 3, 4, 5\}$$

**16.** A discrete random variable has the following probability distribution.

| $y$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $p(y)$ | $0.5k^2$ | $0.3 - 0.2k$ | $0.1$ | $0.5k^2$ | $0.3$ |

Determine the value(s) of $k$, correct to 4 decimal places, that meet the criteria for this to be a valid probability distribution function.

**17.** A mature British Blue female cat has just given birth to 4 kittens. Assume that there is an equally likely chance of a kitten being of either sex.

**a.** Use a tree diagram to list the sample space for the possible number of males and females in the litter.

**b.** Let $X$ be the number of females in the litter. Construct a probability distribution table for the gender of the kittens.

**c.** Determine the probability that 4 females will be born.

**d.** Determine the probability that at least 1 females will be born.

**e.** Determine the probability that at most 2 females will be born.

**18.** Matthew likes to collect differently shaped dice. Currently he has two tetrahedrons (4 sides), an icosahedron (20 sides), two dodecahedrons (12 sides) and an octahedron (8 sides) as well as two standard 6-sided cubes. He has decided to play a game of chance using the octahedral die (with sides numbered 1 to 8) and one dodecahedral die (with sides numbered 1 to 12). He tosses the dice simultaneously and notes the number showing uppermost on both dice.

This particular game of chance involves tossing the two dice simultaneously on three occasions.

The winner of the game must obtain two primes with each of the three tosses. Determine the probability of being a winner. Give your answer correct to 3 decimal places.

**19.** Diabetes is the name of a group of diseases that affect how the body uses blood glucose. If you have diabetes, it means that you have too much glucose in your blood. This can lead to serious health problems. Treatment for type 2 diabetes primarily involves monitoring your blood sugar level along with medications, insulin or both.

A new diabetes medication is to be trialled by 5 patients. From experiments that have been performed with mice, the success rate of the new medication is about 60%. Let $X$ denote the number of patients who improve their health with the new medication.

The new medication will be considered a success if 68% or more of the patients improve their health. Determine $p(X=3) + p(X=4) + p(X=5)$ and comment on the success of the new medication.

**20.** A game is played using a spinner that has been loaded so that it is more likely to land on the red side. In fact, $p\,(\text{red}) = \dfrac{2}{5}$, and $p\,(\text{blue}) = p\,(\text{green}) = p\,(\text{yellow}) = \dfrac{1}{5}$.

Each player pays $2 to play. The player spins the spinner a total of 3 times; however, once the spinner lands on the red side the game is over. If a player has a combination of any 3 colours, they win $1, but if the player has a combination of 3 colours that are all the same, they win $10. There are a total of 40 different outcomes for the game.

Suppose $X$ equals the amount of money won by playing the game, excluding the amount the person pays to play, so $X = \{0, 1, 10\}$. Construct the probability distribution. Give your answers correct to 4 decimal places.
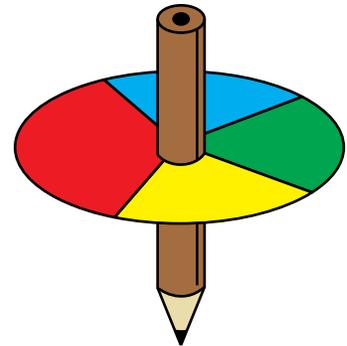
**Fully worked solutions for this chapter are available online.**

# LESSON
## 6.3 Expected values

### 6.3.1 Expected values

In past studies of statistics, the **mean** $(\bar{x})$ was defined as the average of a set of data or values. It was determined by the rule $\bar{x} = \dfrac{\sum xf}{\sum f}$, where $x$ represented the value a variable could assume and $f$ the frequency (i.e. the number of times the variable occurred).

When dealing with discrete random variables, the mean is called the **expected value** or expectation. Since the expected value signifies the average outcome of an experiment, it could be used to determine the feasibility of a situation.

Consider the following example. John tosses two coins. If two Heads are obtained, he wins $20. If one Head is obtained, he wins $10. If no Heads are obtained, he loses $25. John must consider his options and decide whether it is in his best interest to play. Determining the expected value (i.e. the average outcome) may help John in his decision-making process.

Allowing $X$ to represent the number of Heads obtained, the above information is summarised in the table below.

| Outcome | TT | TH or HT | HH |
|---|---|---|---|
| $x$ | 0 | 1 | 2 |
| $p(x)$ | $\dfrac{1}{4}$ | $\dfrac{1}{2}$ | $\dfrac{1}{4}$ |
| Win($\$$) | $-25$ | 10 | 20 |

The expectation or expected gain $= \dfrac{1}{4} \times -25 + \dfrac{1}{2} \times 10 + \dfrac{1}{4} \times 20$

$= -6.25 + 5 + 5$

$= \$3.75$

The average outcome or expected gain is $\$3.75$ per toss. This might seem appealing; however, if there is a charge of $\$5$ per game played, it would not be in John's best interest to participate because he would lose $\$1.25$ per game on average. The above game would not be considered **fair** since the cost to play does not equal the expected gain.

## The expected value (mean) of a discrete random variable

**The expected value of a discrete random variable, $X$, is denoted by $E(X)$.**

**The expected value of a discrete random variable is also called the mean, denoted by the symbol $\mu$ (mu).**

$$E(X) = \mu = p_1 x_1 + p_2 x_2 + p_3 x_3 + \ldots + p_n x_n$$

$$= \sum p_i x_i$$

**where $p_i$ is the probability of outcome $x_i$ occurring.**

*Note:* **The expected value will not always assume a discrete value.**

## WORKED EXAMPLE 9 Calculating the expected value

**Determine the expected value of a random variable that has the following probability distribution.**

| $x$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $p(x)$ | $\dfrac{2}{5}$ | $\dfrac{1}{10}$ | $\dfrac{3}{10}$ | $\dfrac{1}{10}$ | $\dfrac{1}{10}$ |

**THINK**

**1.** Write the rule for the expected value.

**2.** Substitute the values into the rule.

**3.** Evaluate.

**WRITE**

$E(X) = \sum p_i x_i$

$E(X) = 1 \times \dfrac{2}{5} + 2 \times \dfrac{1}{10} + 3 \times \dfrac{3}{10} + 4 \times \dfrac{1}{10} + 5 \times \dfrac{1}{10}$

$= \dfrac{2}{5} + \dfrac{2}{10} + \dfrac{9}{10} + \dfrac{4}{10} + \dfrac{5}{10}$

$= 2\dfrac{2}{5}$

| | TI \| THINK | WRITE | CASIO \| THINK | WRITE |
|---|---|---|---|---|

**TI | THINK**

1. On a Lists & Spreadsheet page, label the first column as $x$ and the second column as p. Enter the given $x$ values in the first column and their respective probabilities in the second column.

2. On a Calculator page, press MENU then select 4: Statistics 1: Stat Calculations 1: One-Variable Statistics … Select 1 for the Num of Lists then select OK. Complete the fields as X1 List: $x$ Frequency List: p then select OK.

3. The answer appears on the screen.

**WRITE**

The expected value is represented by the symbol $\bar{x}$ on the screen. The expected value is 2.4.

**CASIO | THINK**

1. On a Statistics screen, label List 1 as $x$ and List 2 as p. Enter the given $x$ values in the first column and their respective probabilities in the second column.

2. Select CALC by pressing F2, then select SET by pressing F6. Complete the fields as 1 Var XList: List 11 Var Freq: List 2 then press EXIT. Select 1-VAR by pressing F1.

3. The answer appears on the screen.

**WRITE**

The expected value is represented by the symbol $\bar{x}$ on the screen. The expected value is 2.4.

---

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## WORKED EXAMPLE 10 Determining an unknown probability in a distribution

Determine the unknown probability, $a$, and hence determine the expected value of a random variable that has the following probability distribution.

| $x$ | 2 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|
| p($x$) | 0.2 | 0.4 | $a$ | 0.1 | 0.1 |

**THINK**

1. Determine the unknown value of $a$ using the knowledge that the sum of the probabilities must total 1.

2. Write the rule for the expected value.

3. Substitute the values into the rule.

4. Evaluate.

**WRITE**

$$0.2 + 0.4 + a + 0.1 + 0.1 = 1$$
$$0.8 + a = 1$$
$$a = 1 - 0.8$$
$$= 0.2$$

$$E(X) = \sum p_i \, x_i$$

$$E(X) = 2 \times 0.2 + 4 \times 0.4 + 6 \times 0.2 + 8 \times 0.1 + 10 \times 0.1$$

$$= 0.4 + 1.6 + 1.2 + 0.8 + 1$$
$$= 5$$

## WORKED EXAMPLE 11 Determining multiple unknown probabilities in a distribution

Determine the values of $a$ and $b$ in the following probability distribution if $E(X) = 4.29$.

| $x$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|------|-----|-----|-----|-----|-----|-----|-----|
| $p(x)$ | 0.1 | 0.1 | $a$ | 0.3 | 0.2 | $b$ | 0.2 |

**THINK**

1. Write an equation for the unknown values of $a$ and $b$ using the knowledge that the sum of the probabilities must total 1. Call this equation [1].

2. Write the rule for the expected value.

3. Substitute the values into the rule.

4. Evaluate and call this equation [2].

5. Solve equations simultaneously. Multiply equation [1] by 3 and call it equation [3].

Subtract equation [3] from equation [2].
Solve for $b$.
Substitute $b = 0.03$ into equation [1].
Solve for $a$.

6. Answer the question.

**WRITE**

$0.1 + 0.1 + a + 0.3 + 0.2 + b + 0.2 = 1$
$$0.9 + a + b = 1$$
$$a + b = 1 - 0.9$$
$$a + b = 0.1 \qquad [1]$$

$E(X) = \sum p_i \, x_i$

$4.29 = 1 \times 0.1 + 2 \times 0.1 + 3 \times a + 4 \times 0.3 + 5 \times 0.2$
$\qquad + 6 \times b + 7 \times 0.2$
$\qquad = 0.1 + 0.2 + 3a + 1.2 + 1 + 6b + 1.4$

$4.29 - 3.9 = 3a + 6b$
$3a + 6b = 0.39 \qquad [2]$

$a + b = 0.1 \qquad\qquad [1]$
$3a + 6b = 0.39 \qquad\quad [2]$
$3 \times (a + b = 0.1)$
$3a + 3b = 0.3 \qquad\qquad [3]$

$3b = 0.09$
$b = 0.03$
$a + 0.03 = 0.1$
$[2]-[3]: \qquad a = 0.1 - 0.03$
$\qquad\qquad = 0.07$

$a = 0.07$ and
$b = 0.03$

Niki and Melanie devise a gambling game based on tossing three coins simultaneously. If three Heads or three Tails are obtained, the player wins $20. Otherwise the player loses $5. In order to make a profit they charge each person two dollars to play.

a. **Calculate the expected gain to the player.**
b. **Determine whether Niki and Melanie are expected to make a profit.**
c. **Explain whether this is a fair game.**

| THINK | WRITE |
|---|---|
| a. 1. Define the random variable. Place all of the information in a table. | a. Let $X =$ the number of Heads obtained. |

| $x$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $p(x)$ | $\dfrac{1}{8}$ | $\dfrac{3}{8}$ | $\dfrac{3}{8}$ | $\dfrac{1}{8}$ |
| Gain ($) | 20 | $-5$ | $-5$ | 20 |

| | |
|---|---|
| 2. Write the rule for the expected value. | $E(X) = \sum p_i\, x_i$ |
| 3. Substitute the values into the rule. | $= 20 \times \dfrac{1}{8} + -5 \times \dfrac{3}{8} + -5 \times \dfrac{3}{8} + 20 \times \dfrac{1}{8}$ |
| 4. Evaluate. | $= \dfrac{20}{8} - \dfrac{15}{8} - \dfrac{15}{8} + \dfrac{20}{8}$ |
| | $= \dfrac{10}{8}$ |
| | $= \$1.25$ |
| 5. Answer the question. | The player's expected gain per game is $1.25; however, as each game incurs a cost of $2, the player in fact loses 75c per game. |
| b. Answer question using results from part **a**. | b. The girls are expected to make a profit of 75c per game. |
| c. Answer question using results from part **a**. | c. No, this not a fair game, since the cost to play each game is more than the expected gain of each game. |

If $X$ is a random variable and $a$ is a constant, its expected value is defined by $E(aX) = aE(X)$. Furthermore, if $X$ is a random variable where $a$ and $b$ are constants, then the expected value of a linear function in the form $f(X) = aX + b$ is defined by:

$$E(aX + b) = aE(X) + b$$

If $a = 0$, then the expected value becomes:

$$E(0X + b) = 0E(X) + b$$
$$= b$$

These rules are called *expectation theorems* and are summarised below.

## Expectation theorems

$$E(aX) = aE(X) \qquad \text{where } X \text{ is a random variable and } a \text{ is a constant.}$$
$$E(aX + b) = aE(X) + b \qquad \text{where } X \text{ is a random variable } a \text{ and } b \text{ are constants.}$$
$$E(b) = b \qquad \text{where } b \text{ is a constant.}$$
$$E(X + Y) = E(X) + E(Y) \qquad \text{where } X \text{ and } Y \text{ are both random variables.}$$

These theorems make it easier to calculate the expected values.

## WORKED EXAMPLE 13 Expected values for linear functions

**A random variable has the following probability distribution.**

| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $p(x)$ | 0.25 | 0.26 | 0.14 | 0.35 |

**Evaluate:**
**a.** $E(X)$　　　　　　**b.** $E(3X)$　　　　　　**c.** $E(2X - 4)$　　　　　　**d.** $E(X^2)$

**THINK**

**WRITE**

**a. 1.** Write the rule for the expected value.

**a.** $E(X) = \sum p_i \, x_i$

**2.** Substitute the values into the rule.

$E(X) = 1 \times 0.25 + 2 \times 0.26 + 3 \times 0.14 + 4 \times 0.35$
$= 0.25 + 0.52 + 0.42 + 1.4$

**3.** Evaluate.

$= 2.59$

**b. 1.** Write the rule for the expected value.

**b.** $E(3X) = \sum 3p_i \, x_i$

**2.** Substitute the values into the rule.

$= (3 \times 1) \times 0.25 + (3 \times 2) \times 0.26 + (3 \times 3)$
$\times \ 250.14 + (3 \times 4) \times 0.35$

**3.** Evaluate.
*Notes:*
1. The probability remains the same.
2. Each $x$-value is multiplied by 3 is because of the new function, which is $3x$.

$= 3 \times 0.25 + 6 \times 0.26 + 9 \times 0.14 + 12 \times 0.35$
$= 0.75 + 1.56 + 1.26 + 4.2$
$= 7.77$

**c. 1.** Write the rule for the expected value.

**c.** $E(2X - 4) = \sum(2x_i - 4) \, p_i$

**2.** Substitute the values into the rule.

$= (2 \times 1 - 4) \times 0.25 + (2 \times 2 - 4) \times 0.26 +$
$(2 \times 3 - 4) \times 0.14 + (2 \times 4 - 4) \times 0.35$

**3.** Evaluate.
*Notes*:
The probability remains the same.
Each $x$-value is multiplied by 2 and then 4 is subtracted from the result, because of the new function, which is $2x - 4$.

$= -2 \times 0.25 + 0 \times 0.26 + 2 \times 0.14 + 4 \times 0.35$
$= -0.5 + 0 + 0.28 + 1.4$
$= 1.18$

**d. 1.** Write the rule for the expected value.    **d.** $E(X^2) = \sum x_i^2 p_i$

    **2.** Substitute the values into the rule.

$$= (1^2) \times 0.25 + (2^2) \times 0.26 +$$
$$(3^2) \times 0.14 + (4^2) \times 0.35$$

    **3.** Evaluate.

      *Notes:*

      The probability remains the same.
      Each $x$-value is squared because of
      the new function, which is $x^2$.

$$= 1 \times 0.25 + 4 \times 0.26 + 9 \times 0.14 + 16 \times 0.35$$
$$= 0.25 + 1.04 + 1.26 + 5.6$$
$$= 8.15$$

---

## WORKED EXAMPLE 14 Applying the expectation theorem

**Casey decides to apply for a job selling mobile phones. She receives a base salary of $200 per month and $15 for every mobile phone sold. The following table shows the probability of a particular number of mobile phones, $x$, being sold per month. Determine the expected salary Casey would receive each month.**

| $x$ | 50 | 100 | 150 | 200 | 250 |
|---|---|---|---|---|---|
| $p(x)$ | 0.48 | 0.32 | 0.1 | 0.06 | 0.04 |

**THINK**

**Method 1**

**1.** Define a random variable.

**2.** Write the rule for the expected salary.

**3.** Substitute the values into the rule.

**4.** Evaluate.

**5.** Answer the question.

**Method 2**

Using the expectation theorem:

**1.** Write the rule for the expected salary.

**2.** Substitute the values into the rule.

**WRITE**

Let $X =$ the number of mobile phones sold by Casey in a month.

$$E(15X + 200) = \sum (15x_i + 200)\, p_i$$

$$= (15 \times 50 + 200) \times 0.48 +$$
$$(15 \times 100 + 200) \times 0.32 +$$
$$(15 \times 150 + 200) \times 0.1 +$$
$$(15 \times 200 + 200) \times 0.06 +$$
$$(15 \times 250 + 200) \times 0.04$$

$$= 950 \times 0.48 + 1700 \times 0.32 +$$
$$2450 \times 0.1 + 3200 \times 0.06 +$$
$$3950 \times 0.04$$
$$= 456 + 544 + 245 + 192 + 158$$
$$= 1595$$

The expected salary Casey would receive each month would be $1595.

$$E(X) = \sum p_i x_i$$

$$= 50 \times 0.48 + 100 \times 0.32 + 150 \times 0.01 +$$
$$200 \times 0.06 + 250 \times 0.04$$

$$= 24 + 32 + 15 + 12 + 10$$

**3.** Evaluate.

$= 93$

**4.** Using the fact that $E(aX + b) = aE(X) + b$, calculate $E(15X + 200)$.

$E(15x + 200) = 15E(X) + 200$
$= 15 \times 93 + 200$
$= 1595$

*Note:* Using the expectation theorem is quicker because it is easier to evaluate $aE(X) + b$ than $E(aX + b)$.

# Exercise 6.3 Expected values

| 6.3 Exercise | 6.3 Exam questions **on** |
|---|---|

| Simple familiar | Complex familiar | Complex unfamiliar |
|---|---|---|
| 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 | 13, 14, 15, 16, 17 | 18, 19, 20 |

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**Simple familiar**

**1.** WE9 Determine the expected value of a random variable that has the following probability distribution.

| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $p(x)$ | $\dfrac{1}{8}$ | $\dfrac{1}{2}$ | $\dfrac{3}{16}$ | $\dfrac{3}{16}$ |

**2.** Determine the expected value of a random variable that has the following probability distribution.

| $x$ | −4 | −2 | 0 | 2 | 4 | 6 |
|---|---|---|---|---|---|---|
| $p(x)$ | 0.15 | 0.18 | 0.06 | 0.23 | 0.31 | 0.07 |

**3.** WE10 Determine the unknown probability, $a$, and hence determine the expected value of a random variable that has the following probability distribution.

| $x$ | 1 | 3 | 5 | 7 | 9 | 11 |
|---|---|---|---|---|---|---|
| $p(x)$ | 0.11 | 0.3 | 0.15 | 0.25 | $a$ | 0.1 |

**4.** Determine the unknown probability, $a$, and hence determine the expected value of a random variable that has the following probability distribution.

| $x$ | −2 | 1 | 4 | 7 | 10 | 13 |
|---|---|---|---|---|---|---|
| $p(x)$ | $\dfrac{5}{18}$ | $a$ | $\dfrac{1}{9}$ | $\dfrac{5}{18}$ | $\dfrac{1}{18}$ | $\dfrac{2}{9}$ |

**5.** Determine the value of $k$ and hence determine the expected value of a random variable with the following probability distribution.

| $x$ | 2 | 4 | 6 | 8 |
|---|---|---|---|---|
| $p(x)$ | $k$ | $2k$ | $3k$ | $4k$ |

6. Determine the unknown probability, $b$, and hence determine the expected value of a random variable that has the following probability distribution.

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $p(x)$ | $b$ | 0.2 | 0.02 | $3b$ | 0.1 | 0.08 |

7. Determine the value of $k$, and hence determine the expected value of a random variable that has the following probability distribution.

| $x$ | 4 | 8 | 12 | 16 | 20 |
|---|---|---|---|---|---|
| $p(x)$ | $6k$ | $2k$ | $k$ | $3k$ | $8k$ |

8. If $X$ represents the outcome of a fair die being rolled, determine:
   a. the probability distribution of each outcome
   b. $E(X)$.

9. Two fair dice are rolled simultaneously. If $X$ represents the sum of the two numbers appearing uppermost, determine:
   a. the probability distribution of each outcome
   b. $E(X)$.

10. If $X$ represents the number of Heads obtained when a fair coin is tossed twice, determine:
   a. the probability distribution of each outcome
   b. $E(X)$.

11. A fair coin is tossed 4 times. If $X$ represents the number of Tails obtained, determine:
   a. the probability distribution of each outcome
   b. $E(X)$.

12. **WE13** A random variable has the following probability distribution.

| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $p(x)$ | $\dfrac{2}{15}$ | $\dfrac{7}{15}$ | $\dfrac{1}{3}$ | $\dfrac{1}{15}$ |

Determine:
   a. $E(X)$
   b. $E(4X)$
   c. $E(2X+1)$
   d. $E(X^2)$.

**Complex familiar**

13. **WE11** Determine the values of $a$ and $b$ of the following distribution if $E(X) = 1.91$.

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $p(x)$ | 0.2 | 0.32 | $a$ | 0.18 | $b$ | 0.05 | 0.05 |

14. Determine the values of $a$ and $b$ of the following distribution if $E(X) = 2.41$.

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $p(x)$ | 0.2 | $a$ | 0.23 | 0.15 | $b$ | 0.12 |

**15.** **WE12** Lucas contemplates playing a new game that involves tossing three coins simultaneously. He will receive $15 if he obtains 3 Heads, $10 if he obtains 2 Heads and $5 if he obtains 1 Head. However, if he obtains no Heads he must pay $30. He must also pay $5 for each game he plays.

a. Calculate Lucas' expected gain.
b. Determine whether he should play the game. Explain why.
c. Explain whether this is this a fair game.

**16.** **WE14** Christian decides to apply for a job selling mobile phones. He receives a base salary of $180 per month and $12 for every mobile phone sold. The following table shows the probability of a particular number of mobile phones, $x$, being sold per month. Determine the expected salary Christian would receive each month.

| $x$ | 50 | 100 | 150 | 200 | 250 |
|---|---|---|---|---|---|
| $p(x)$ | 0.32 | 0.38 | 0.2 | 0.06 | 0.04 |

**17.** In a certain random experiment, the events $V$ and $W$ are independent events.
Let $X$ be the discrete random variable that defines the number of times events $V$ and $W$ occur.
$X = 0$ if neither $V$ nor $W$ occurs.
$X = 1$ if only one of $V$ and $W$ occurs.
$X = 2$ if both $V$ and $W$ occur.
Determine the value of E $(X)$ given that p $(V \cup W) = 0.7725$, P($V \cap W) = 0.2275$, and p $(V) < p(W)$.

**Complex unfamiliar**

**18.** $X$ is a discrete random variable with the following probability distribution.

| $x$ | −2 | 3 | 8 | 10 | 14 | $k$ |
|---|---|---|---|---|---|---|
| $p(x)$ | 0.1 | 0.08 | 0.07 | 0.27 | 0.16 | 0.32 |

Calculate the value of $k$ if the mean is 10.98.

**19.** Determine the mean of the discrete random variable, $Z$, for a probability function defined by
$p(z) = \dfrac{1}{38}(z^2 - 4), 2 \le z \le 5.$
Give your answer correct to 2 decimal places.

**20.** The probability distribution table for the discrete random variable, $Z$, is as follows.

| $z$ | 1 | 3 | 5 |
|---|---|---|---|
| $p(z)$ | $\dfrac{k^2}{7}$ | $\dfrac{5-2k}{7}$ | $\dfrac{8-3k}{7}$ |

Calculate the mean of $Z$ correct to 4 decimal places.

**Fully worked solutions for this chapter are available online.**

# LESSON
## 6.4 Variance and standard deviation

### 6.4.1 Calculating variance and standard deviation

The measure of spread of a random variable distribution tells us how the data is dispersed. The measure of spread is called the **variance**, and the square root of the variance gives the **standard deviation**. The variance is denoted by Var $(X)$ or $\sigma^2$ (sigma squared), and is defined as follows.

> **Variance**
> $$\mathbf{Var}(X) = \sum \mathbf{p}_i (x_i - \mu)^2$$

This may also be written as $\text{Var}(X) = \sigma^2 = \text{E}(X^2) - \left[\mu\right]^2$, where $\mu = \text{E}(X)$.

The derivation of this rule is as follows:

$$
\begin{aligned}
\text{Var}(X) &= \text{E}(X - \mu)^2 \\
&= \text{E}(X^2 - 2X\mu + \mu^2) \\
&= \text{E}(X^2) - \text{E}(2X\mu) + \text{E}(\mu^2) \\
&= \text{E}(X^2) - 2\mu\text{E}(X) + \mu^2
\end{aligned}
$$

Since $\text{E}(X) = \mu$,

$$
\begin{aligned}
&= \text{E}(X^2) - 2\mu^2 + \mu^2 \\
\text{Var}(x) &= \text{E}(X^2) - \mu^2 \\
\text{Var}(x) &= \text{E}(X^2) - [\text{E}(X)]^2
\end{aligned}
$$

The standard deviation of $X$ is the square root of the variance of $X$ and is denoted by $\text{SD}(X)$ or $\sigma$ (sigma).

> **Standard deviation**
> $$\mathbf{SD}(X) = \sigma = \sqrt{\mathbf{Var}(X)}$$

If the standard deviation is large, the spread of the data is large. If the standard deviation is small, the data is clumped together, close to the mean.

## WORKED EXAMPLE 15 Calculating variance and standard deviation

**A discrete random variable, $X$, has the following probability distribution.**

| $x$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $p(x)$ | 0.15 | 0.25 | 0.3 | 0.2 | 0.1 |

**Calculate the following to 4 decimal places.**
**a. $E(X)$**
**b. $Var(X)$**
**c. $SD(X)$**

**THINK**

**a. 1.** Write the rule to determine the expected value.

**2.** Substitute the appropriate values into the rule.

**3.** Simplify.

**b. 1.** Evaluate $E(X^2)$.

**2.** Write the rule for the variance.

**3.** Substitute in the appropriate values and evaluate.

**c. 1.** Write the rule for the standard deviation.

**2.** Substitute in the variance and evaluate.

**WRITE**

**a.** $E(X) = \sum_{\text{all } x} x_i\, p_i$

$E(X) = 1(0.15) + 2(0.25) + 3(0.3) + 4(0.2) + 5(0.1)$

$E(X) = 0.15 + 0.5 + 0.9 + 0.8 + 0.5$
$= 2.85$

**b.** $E(X^2) = \sum_{\text{all } x} x_i^2\, p_i$

$E(X^2) = 1^2(0.15) + 2^2(0.25) + 3^2(0.3) + 4^2(0.2) + 5^2(0.1)$

$= 0.15 + 1 + 2.7 + 3.2 + 2.5$

$= 9.55$

$Var(X) = E(X^2) - [E(X)]^2$

$Var(X) = 9.55 - (2.85)^2$
$= 9.55 - 8.1225$
$= 1.4275$

**c.** $SD(X) = \sqrt{Var(X)}$

$SD(X) = \sqrt{1.4275}$
$= 1.1948$

**TI | THINK**

**a. 1.** On a Lists & Spreadsheet page, label the first column as $x$ and the second column as p. Enter the given $x$-values in the first column and their respective probabilities in the second column.

**WRITE**

| | A x | B p | C | D |
|---|---|---|---|---|
| = | | | | |
| 1 | 1 | 0.15 | | |
| 2 | 2 | 0.25 | | |
| 3 | 3 | 0.3 | | |
| 4 | 4 | 0.2 | | |
| 5 | 5 | 0.1 | | |

B8

**CASIO | THINK**

**a. 1.** On a Statistics screen, label List 1 as $x$ and List 2 as p. Enter the given $x$-values in the first column and their respective probabilities in the second column.

**WRITE**

| | List 1 | List 2 | List 3 | List 4 |
|---|---|---|---|---|
| SUB | x | p | | |
| 1 | 1 | 0.15 | | |
| 2 | 2 | 0.25 | | |
| 3 | 3 | 0.3 | | |
| 4 | 4 | 0.2 | | |

0.15

GRAPH CALC TEST INTR DIST

| **TI | THINK** | **WRITE** | **CASIO | THINK** | **WRITE** |
|---|---|---|---|---|

**2.** On a Calculator page, press MENU then select:
6: Statistics
1: Stat Calculations
1: One-Variable Statistics …
Select 1 for the Num of Lists then select OK. Compete the fields as: X1 List: x Frequency List: p then select OK.

**2.** Select CALC by pressing F2, then select SET by pressing F6. Complete the fields as: 1Var XList: List 11 Var Freq: List 2 then press EXIT. Select 1-VAR by pressing F1.

**3.** The answer appears on the screen.

The expected value is represented by the symbol $\bar{x}$ on the screen. The expected value is 2.85.

**3.** The answer appears on the screen.

The expected value is represented by the symbol $\bar{x}$ on the screen. The expected value is 2.85.

**b. 1.** See the Calculator page from part **a**. Scroll down to find the variance.

**b. 1.** On the Run-Matrix screen, press OPTN and select STAT by pressing F5, then select Var by pressing F5. Select $\sigma^2$ by pressing F2. Complete the entry line as Variance $\sigma^2$ (List 1, List 2) then press EXE. *Note:* List can be found by pressing OPTN then selecting List then List again by pressing F1 twice.

**2.** The answer appears on the screen.

The variance is represented by the symbol $SSX := \Sigma(x - \bar{x})^2$ on the screen. The variance is 1.4275.

**2.** The answer appears on the screen.

The variance is 1.4275.

**c. 1.** See the Calculator page from part **a**.

**c. 1.** Return to the Statistics screen from part **a**.

**2.** The answer appears on the screen.

The standard deviation is represented by the symbol $\sigma x := \sigma_n x$ on the screen. The standard deviation is 1.194 78.

**2.** The answer appears on the screen.

The standard deviation is represented by the symbol $\sigma x$ on the screen. The standard deviation is 1.194 78.

## 6.4.2 Properties of the variance

The variance of a linear function has rules similar to those for the expectation of a linear function.

> ### Variance of a linear function
>
> $$\text{Var}(aX + b) = a^2\text{Var}(X)$$

This can be proved in the following manner.

$$
\begin{aligned}
\text{Var}(aX + b) &= \text{E}(aX + b)^2 - [\text{E}(aX + b)]^2 \\
&= \text{E}(a^2X^2 + 2abX + b^2) - [a\text{E}(X) + b]^2 \\
&= \text{E}(a^2X^2) + \text{E}(2abX) + \text{E}(b^2) - (a^2[\text{E}(X)]^2 + 2ab\text{E}(X) + b^2) \\
&= a^2\text{E}(X^2) + 2ab\text{E}(X) + b^2 - a^2[\text{E}(X)]^2 - 2ab\text{E}(X) - b^2 \\
&= a^2(\text{E}(X^2) - [\text{E}(X)]^2)
\end{aligned}
$$

But $\text{Var}(X) = \text{E}(X^2) - [\text{E}(X)]^2$, so:

$$\text{Var}(aX + b) = a^2\text{Var}(X)$$

### WORKED EXAMPLE 16 Variance of a linear function

A discrete probability function is defined by the rule $p(y) = \dfrac{1}{12}(10 - 3y), y \in \{1, 2, 3\}$.

**a.** Show that the sum of the probabilities is equal to 1.

**b.** Determine:

   **i.** $\text{E}(Y)$             **ii.** $\text{Var}(Y)$.

**c.** Determine:

   **i.** $\text{Var}(3Y - 1)$       **ii.** $\text{Var}(4 - 5Y)$.

| THINK | WRITE |
|---|---|
| **a. 1.** Evaluate the probabilities for the given values of $Y$. | **a.** $p(y) = \dfrac{1}{12}(10 - 3y)\ y \in \{1, 2, 3\}$ <br><br> $p(1) = \dfrac{1}{12}(10 - 3(1)) = \dfrac{7}{12}$ <br><br> $p(2) = \dfrac{1}{12}(10 - 3(2)) = \dfrac{4}{12} = \dfrac{1}{3}$ <br><br> $p(3) = \dfrac{1}{12}(10 - 3(3)) = \dfrac{1}{12}$ |
| **2.** Add the probabilities. | $p(Y = 1) + p(Y = 2) + p(Y = 3) = \dfrac{7}{12} + \dfrac{4}{12} + \dfrac{1}{12}$ <br><br> $= \dfrac{12}{12}$ <br><br> $= 1$ |

**b. i.** **1.** Write the rule to determine the expected value.

**b. i.** $E(Y) = \sum\limits_{\text{all } y} y_i p_i$

**2.** Substitute the appropriate values into the rule.

$E(Y) = 1\left(\dfrac{7}{12}\right) + 2\left(\dfrac{4}{12}\right) + 3\left(\dfrac{1}{12}\right)$

**3.** Simplify.

$= \dfrac{7}{12} + \dfrac{8}{12} + \dfrac{3}{12}$

$= \dfrac{18}{12}$

$= \dfrac{3}{2}$

**ii.** **1.** Evaluate $E(Y^2)$.

**ii.** $E(Y^2) = \sum y_i^2 p_i$

$= 1^2\left(\dfrac{7}{12}\right) + 2^2\left(\dfrac{4}{12}\right) + 3^2\left(\dfrac{1}{12}\right)$

$= \dfrac{7}{12} + \dfrac{16}{12} + \dfrac{9}{12}$

$= \dfrac{32}{12}$

$= \dfrac{8}{3}$

**2.** Write the rule for the variance.

$\text{Var}(Y) = E(Y^2) - [E(Y)]^2$

**3.** Substitute in the appropriate values and evaluate.

$\text{Var}(Y) = \dfrac{8}{3} - \left(\dfrac{3}{2}\right)^2$

$= \dfrac{8}{3} - \dfrac{9}{4}$

$= \dfrac{32 - 27}{12}$

$= \dfrac{5}{12}$

**c. i.** **1.** Apply the property of the variance: $\text{Var}(aY + b) = a^2\text{Var}(Y)$.

**c. i.** $\text{Var}(3Y - 1) = 3^2\text{Var}(Y)$

**2.** Substitute in the value of $\text{Var}(Y)$ and evaluate.

$\text{Var}(3Y - 1) = 9 \times \dfrac{5}{12}$

$= \dfrac{15}{4}$

**ii.** **1.** Apply the property of the variance: $\text{Var}(aY + b) = a^2\text{Var}(Y)$.

**ii.** $\text{Var}(4 - 5Y) = (-5)^2\text{Var}(Y)$

**2.** Substitute in the value of $\text{Var}(Y)$ and evaluate.

$\text{Var}(4 - 5Y) = 25 \times \dfrac{5}{12}$

$= \dfrac{125}{12}$

# Exercise 6.4 Variance and standard deviation

| 6.4 Exercise | 6.4 Exam questions **on** |
|---|---|

| Simple familiar | Complex familiar | Complex unfamiliar |
|---|---|---|
| 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 | 13, 14, 15, 16 | 17, 18, 19, 20 |

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### Simple familiar

**1.** **WE15** A discrete random variable, $X$, has the following probability distribution.

| $x$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $p(x)$ | 0.2 | 0.3 | 0.15 | 0.1 | 0.25 |

Calculate the following to 4 decimal places.

**a.** $E(X)$
**b.** $Var(X)$
**c.** $SD(X)$

**2.** A discrete random variable, $Y$, has the following probability distribution.

| $y$ | 1 | 4 | 7 | 10 | 13 |
|---|---|---|---|---|---|
| $p(y)$ | 0.15 | 0.2 | 0.3 | 0.2 | 0.15 |

Calculate the following to 4 decimal places.

**a.** $E(X)$
**b.** $Var(X)$
**c.** $SD(X)$

**3.** A discrete random variable, $X$, has the following probability distribution.

| $x$ | $-3$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $p(x)$ | $\dfrac{1}{9}$ | $\dfrac{1}{9}$ | $\dfrac{1}{9}$ | $\dfrac{2}{9}$ | $\dfrac{2}{9}$ | $\dfrac{1}{9}$ | $\dfrac{1}{9}$ |

Calculate the following, writing your answers as fractions.

**a.** $E(X)$
**b.** $Var(X)$
**c.** $SD(X)$

**4.** A discrete random variable, $Z$, has the following probability distribution.

| $z$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $p(z)$ | $\dfrac{1}{12}$ | $\dfrac{1}{4}$ | $\dfrac{1}{3}$ | $\dfrac{1}{6}$ | $\dfrac{1}{12}$ | $\dfrac{1}{12}$ |

Calculate the following, writing your answers as fractions.

**a.** $E(Z)$
**b.** $Var(Z)$
**c.** $SD(Z)$.

**5.** Recently the large supermarket chains have been waging a price war on bread. On a particular Tuesday, a standard loaf of bread was purchased from a number of outlets of different chains. The following table shows the probability distribution for the price of the bread, $X$.

| $x$ | $1 | $2 | $3 | $4 | $5 |
|---|---|---|---|---|---|
| $p(x)$ | 0.3 | 0.15 | 0.4 | 0.1 | 0.05 |

a. Calculate the expected cost of a loaf of bread on that given Tuesday.
b. Calculate the variance and the standard deviation of that loaf of bread, correct to 2 decimal places.

**6.** A discrete random variable, $X$, has the following probability distribution.

| $x$ | $-2$ | $0$ | $2$ | $4$ | $6$ |
|---|---|---|---|---|---|
| $p(x)$ | $k$ | $k$ | $2k$ | $3k$ | $3k$ |

a. Determine the value of the constant $k$.
b. Determine the expected value of $X$.
c. Determine the variance and the standard deviation of $X$, correct to 2 decimal places.

**7.** **WE16** A discrete probability function $Y$ is defined by the rule

$$p(y) = \frac{1}{27}(5y - 1), \; y \in \{1, 2, 3\}.$$

a. Show that the sum of the probabilities is equal to 1.
b. Determine:
  i. $E(Y)$
  ii. $\text{Var}(Y)$
c. Determine:
  i. $\text{Var}(3Y - 1)$
  ii. $\text{Var}(4 - 5Y)$

**8.** A discrete probability function is defined by $p(x) = \dfrac{x^2}{30}, \; x \in \{1, 2, 3, 4\}$.

Where appropriate, give your answers to the following to 2 decimal places.

a. Show that the sum of the probabilities is equal to 1.
b. Determine:
  i. $E(X)$
  ii. $\text{Var}(X)$
c. Determine:
  i. $\text{Var}(4X + 3)$
  ii. $\text{Var}(2 - 3X)$

**9.** Given that $E(X) = 4.5$, determine:
a. $E(2X - 1)$
b. $E(5 - X)$
c. $E(3X + 1)$

**10.** Given that $SD(X) = \sigma = 2.5$, determine:
a. $\text{Var}(6X)$
b. $\text{Var}(2X + 3)$
c. $\text{Var}(-X)$

**11.** A discrete probability function, $X$, is defined by the rule $p(x) = h(3-x)(x+1)$, $x = 0, 1, 2$.

   **a.** Show that the value of $h$ is $\dfrac{1}{10}$.

   **b.** Hence, determine the mean, variance and standard deviation of $X$. Where appropriate, give your answers to 4 decimal places.

**12.** A random variable, $Y$, has the following probability distribution.

| $y$ | $-1$ | $1$ | $3$ | $5$ | $7$ |
|---|---|---|---|---|---|
| $p(y)$ | $1-2c$ | $c^2$ | $c^2$ | $c^2$ | $1-2c$ |

   **a.** Determine the value of the constant $c$.

   **b.** Calculate $E(Y)$, the mean of $Y$.

   **c.** Calculate $\text{Var}(Y)$ and hence determine the standard deviation of $Y$, correct to 2 decimal places.

**13. a.** Determine the value of the constant $m$ if the discrete random variable $Z$ has the probability distribution shown and $E(Z) = 14.94$.

| $z$ | $-7$ | $m$ | $23$ | $31$ |
|---|---|---|---|---|
| $p(z)$ | $0.21$ | $0.34$ | $0.33$ | $0.12$ |

   **b.** Determine $\text{Var}(Z)$ and hence determine $\text{Var}(2(Z-1))$ and $\text{Var}(3-Z)$, correct to 2 decimal places.

**14.** A discrete probability function has the following distribution.

| $x$ | $1$ | $2$ | $3$ | $4$ | $5$ |
|---|---|---|---|---|---|
| $p(x)$ | $a$ | $0.2$ | $0.3$ | $b$ | $0.1$ |

The expected value of the function is 2.5.

   **a.** Calculate the values of the constants $a$ and $b$.

   **b.** Hence, calculate the variance and standard deviation of $X$. Where appropriate, give your answers to 4 decimal places.

**15.** Two octahedral dice (with faces numbered 1 to 8) are rolled simultaneously and the two numbers are recorded. Let $Z$ be the larger of the two numbers on the two dice.

   **a.** List the probability or event space and determine $n(\xi)$.

   **b.** State the probability distribution for $Z$.

   **c.** Calculate the expected value and standard deviation of $Z$, correct to 4 decimal places.

**16.** A discrete random variable, $X$, has the following probability distribution.

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ |
|---|---|---|---|---|---|---|---|
| $p(x)$ | $0.5k^2$ | $0.5k^2$ | $k+k^2$ | $4k$ | $2k$ | $2k+k^2$ | $7k^2$ |

   **a.** Determine the value of the constant $k$.

   **b.** Calculate the expected value of $X$.

   **c.** Calculate the standard deviation of $X$, correct to 4 decimal places.

**17.** For a discrete random variable, $Y$, the probability function is defined by the following.

$$p(y) = \begin{cases} ny, & y \in \{1, 2, 3, 4\} \\ n(7 - y), & y \in \{5, 6\} \end{cases}$$

Calculate the expected value, the variance and the standard deviation of $Y$, correct to 4 decimal places.

**18.** For a given discrete random variable, $X$, it is known that $E(X) = a$ and $Var(X) = 2a - 2$, where $a$ is a constant that is greater than zero.

If $E(X^2)$ is known to be 6, calculate $E(X)$ and $Var(X)$.

**19.** A dart competition at a local sports centre allows each player to throw one dart at the board, which has a radius of 20 centimetres. The board consists of five concentric circles, each with the same width.

The inner circle has a radius of 4 cm. The probability of landing on each band is determined by the area of that band available on the board.

The outer red band is called band E, the next white band is called band D and so on until you get to the inner red circle, which is band A.

The competition costs $1 to enter and the prizes are as follows:

If a dart hits band E, the player receives nothing.

If a dart hits band D, the player receives $1.

If a dart hits band C, the player receives $2.

If a dart hits band B, the player receives $5.

If a dart hits band A, the player receives $10.

Calculate the expected profit a player could make in dollars.

**20.** At a beginner's archery competition, each archer has two arrows to shoot at the target. A target is marked with ten evenly spaced concentric rings.

The following is a summary of the scoring for the beginner's competition.

Gold — 10 points

Red — 7 points

Blue — 5 points

Black — 3 points

White — 1 point

Let $X$ be the total score after a beginner shoots two arrows.

The probability of a beginner hitting each of the rings has been calculated as follows:

$p(\text{gold}) = 0.1, p(\text{red}) = 0.2, p(\text{blue}) = 0.3, p(\text{black}) = 0.2$ and $p(\text{white}) = 0.2$.

Calculate the expected score and the standard deviation for a beginner. Where appropriate, give your answers correct to 4 decimal places.

**Fully worked solutions for this chapter are available online.**

# LESSON
## 6.5 Modelling and solving with discrete random variables

The techniques used to determine expected values and the standard deviation can be applied to a wide variety of situations in which the outcomes are discrete random variables.

### WORKED EXAMPLE 17 Problem solving with discrete random variables

**The Bright Bit Coding Solutions company has 12 staff. The number of cups of coffee consumed by each member of staff over the course of an average day can be modelled by the probability distribution of $C$ as shown.**

| $c$ | 0 | 1 | 2 | 3 | 4 | 5 |
|------|------|------|------|------|------|------|
| $p(c)$ | 0.05 | 0.1 | 0.2 | 0.3 | 0.25 | 0.1 |

**A 500-g bag of coffee beans will make approximately 295 cups of coffee.**
**Determine how many bags of coffee beans should be purchased each month to meet the coffee consumption needs of the company. Assume that there are 22 working days in a month.**

| THINK | WRITE |
|-------|-------|
| 1. Write the rule to determine the expected value. | $E(C) = \sum_{\text{all } c} c_i \, p_i$ |
| 2. Substitute the appropriate values into the rule. | $E(C) = 0(0.05) + 1(0.1) + 2(0.2) + 3(0.3) + 4(0.25) + 5(0.1)$ |
| 3. Simplify and solve for $E(C)$ to determine the expected number of cups of coffee each employee will drink each day. | $E(C) = 0 + 0.1 + 0.4 + 0.9 + 1.0 + 0.5$ <br> $= 2.9$ cups of coffee |
| 4. Calculate the average number of coffees consumed by the whole staff each month. | Total each day $= 2.9 \times 12 = 34.8$ cups each day <br> Total each month $= 34.8 \times 22 = 765.6$ cups each month |
| 5. Calculate the number of bags of beans needed. | Number of bags $= \dfrac{765.6}{295} = 2.6$ bags |
| 6. State the number of whole bags needed. | The company will need to buy 3 bags of coffee beans each month. |

# Exercise 6.5 Modelling and solving with discrete random variables

**learn on**

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### Simple familiar

1. The probability distribution for the number of nails in a 500-g pack can be represented by the following.

| $n$ | 169 | 170 | 171 |
| --- | --- | --- | --- |
| $p(n)$ | 0.05 | 0.75 | 0.2 |

If a 500-g pack of nails costs $10.45, determine the average price per nail.

2. Gavin is an apprentice potter. On any random day, the number of pots he produces that crack in the kiln when they are fired has the following probability distribution.

| $k$ | 1 | 2 | 3 | 4 |
| --- | --- | --- | --- | --- |
| $p(k)$ | 0.48 | 0.36 | 0.12 | 0.04 |

a. Calculate the average number of cracked pots that Gavin produces each day.
b. Calculate the standard deviation for the distribution.
c. Determine the probability that Gavin will produce fewer than 3 cracked pots on a given day.



3. The probability distribution of the number of cream donuts sold in a bakery on any random day is as shown:

| $d$ | 2 | 3 | 4 | 5 | 6 |
| --- | --- | --- | --- | --- | --- |
| $p(d)$ | 0.05 | 0.25 | 0.4 | 0.25 | 0.05 |

Calculate the expected number of cream donuts that will be sold on any given day.

4. An appliance centre buys the Squeaky Kleen brand of dishwasher from the manufacturer for $400 and sells it to its customers for $750. They include a 2-year warranty where, if the dishwasher fails, they will replace it at no cost to the purchaser. Given that 20% of Squeaky Kleen dishwashers fail within the warranty period, evaluate the expected profit or loss over the warranty period.

**5.** A financial adviser for a large company has put forward a number of options to improve the company's profitability, $X$ (measured in hundreds of thousands of dollars). The decision to implement the options will be based on the cost of the options as well as their profitability. The company stands to make an extra profit of $1 million with a probability of 0.1, an extra profit of $750 000 with a probability of 0.3, an extra profit of 500 000 with a probability of 0.3, an extra profit of 250 000 with a probability of 0.2 and an extra profit of $100 000 with a probability of 0.1.

Determine:

**a.** $p(X \le \$500\,000)$

**b.** $P\left(X \ge \$2000 \,|\, X \le \$750\,000\right)$

**c.** the expected profit.

**6.** Jenni is the owner of a prize-winning female Corgi called Powderpuff Fluffybritches. She is hoping to breed her dog but knows that she will need to pay tax on any earnings above $27 500 per year. Consulting with other Corgi breeders, Jenni determines that the litter sizes of Corgis can be represented by the following probability distribution:

| $w$ | 2 | 3 | 4 | 5 | 6 | 7 |
|------|------|------|------|------|------|------|
| $p(w)$ | 0.02 | 0.14 | 0.26 | 0.40 | 0.16 | 0.02 |

Each puppy will sell for $3000, and Powderpuff can safely produce three litters in a two-year period. Determine Jenni's expected earnings each year from breeding her Corgi and evaluate whether she will be taxed on this income.

**7.** In a café, Marat is the barista 45% of the time while Neema is the cook 48% of the time. Anna, a customer who comes to the café every day, likes both Marat and Neema and puts money in the tip jar according to who is working.
If neither Marat nor Neema is working, she does not put anything in the tip jar. If only one of them is working, she will put $2 in the tip jar. If they are both working, she places $5 in the tip jar.
Assume that the events M and N are independent.

**a.** Calculate the probability that both Marat and Neema are working on a day that Anna comes in.

**b.** Calculate the probability that neither are working that day.

**c.** Construct the probability distribution for the amount of money that Anna will place in the tip jar each day.

**d.** Calculate the expected value and the standard distribution of this probability distribution. Round your answers to 2 decimal places

**8.** For any random variable, the probability distribution is such that approximately 95% of the distribution lies within two standard deviations of the mean (or expected value).
The number of Tails, $t$, when a coin is tossed six times has the following probability distribution:

| $t$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|------|------|------|------|------|------|------|------|
| $p(t)$ | 0.012 | 0.093 | 0.243 | 0.315 | 0.214 | 0.1 | 0.023 |

Determine if $P(\mu - 2\sigma \le t \le \mu + 2\sigma) \approx 0.95$.

**9.** The probability distribution for the number of defective diodes produced per 10 000 units by a machine in an electronics manufacturer is shown in the table.

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $p(x)$ | 0.09 | 0.14 | 0.32 | 0.20 | 0.05 | 0.10 | 0.10 |

It is proposed that the machine be replaced by a new model. The number of defective diodes produced per 12 000 units with the new model has the following probability distribution.

| $y$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $p(y)$ | 0.15 | 0.15 | 0.25 | 0.21 | 0.05 | 0.08 | 0.01 |

**a.** Determine the average number and the standard deviation of defective diodes produced by each machine per 10 000 units.

**b.** Evaluate which of the two machines produces the fewest defective diodes.

**10.** The game Zeno is played using a deck of cards, each of which is marked with a number between 0 and 5. The probability distribution of $Z$ is as follows,

| $z$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $p(z)$ | $m$ | $m$ | $m$ | $n$ | $n$ | $n$ |

where $m$ and $n$ are constants.

Given that $p(Z < 2) = 3\, p(Z > 4)$:

**a.** determine the values of $m$ and $n$

**b.** show that $E(Z) = \dfrac{11}{5}$

**c.** calculate the variance and standard deviation for $Z$, correct to 4 decimal places.

**11.** While doing her statistics homework, Caroline notices that she didn't write down two of the values for the probability table she had copied from the board during class.

| $x$ | 1 | 3 | 5 | 7 | 9 |
|---|---|---|---|---|---|
| $p(x)$ | 0.2 | 0.15 | | | 0.05 |

However, she did write down that the expected value of $x$ is equal to 4.6.

Determine the missing table values.

**12.** The number of customers, $X$, waiting in line at a bank just before closing time has a probability distribution as follows.

| $x$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $p(x)$ | $\dfrac{k^2}{4}$ | $\dfrac{5k-1}{12}$ | $\dfrac{3k-1}{12}$ | $\dfrac{4k-1}{12}$ |

**a.** Determine the value of the constant $k$.

**b.** Determine the expected number of customers waiting in line just before closing time.

**c.** Calculate the probability that the number of customers waiting in line just before closing time is no greater than $E(X)$.

**13.** Let $X$ be the number of dining suites sold by the dining suite department of a large furniture outlet on any given day. The probability function for this discrete random variable is as follows.

| $x$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $p(x)$ | 0.3 | 0.4 | 0.2 | 0.1 |

The dining suite department receives a profit of $350 for every dining setting sold. The daily running costs for the sales operation of the department are $120. The net profit per day is a function of the random variable such that $y(x) = 350x - 120$ dollars.

**a.** Set up a probability distribution table for the net profit, $\$Y$, per day.
**b.** Determine the expected daily profit for the dining suite department.

**14.** A loaded six-sided die and a biased coin are tossed simultaneously. The coin is biased such that the probability of obtaining a Head is three times the probability of obtaining a Tail. The loaded die has the following probabilities for each of the numbers 1 to 6.

$$p(1) = p(2) = p(5) = \frac{1}{12}$$

$$p(3) = p(4) = p(6) = \frac{1}{4}$$

When a player tosses the coin and die simultaneously, they receive the following outcomes.

| 10 points | 5 points | 1 points |
|---|---|---|
| 1T   2T   5T | 1H   2H   5H | All other results |

Let $X$ be the number of points scored from a simultaneous toss.

**a.** Construct a probability distribution table for the number of points scored.
**b.** Calculate the expected points received from a single toss, correct to 1 decimal place.
**c.** If 25 simultaneous tosses occurred, determine the expected score, correct to 1 decimal place.
**d.** Determine the minimum number of simultaneous tosses that would have to occur for the expected total to be a score of 100.

**Complex unfamiliar**

**15.** As part of the Australian Census, the number of residents living in each household in Australia is recorded. The results for households in the Greater Brisbane area in 2021 are shown below.

| Number of residents per household | Percentage of all households (Brisbane) |
|---|---|
| 1 | 23.6 |
| 2 | 33.5 |
| 3 | 16.7 |
| 4 | 16.0 |
| 5 | 6.5 |
| 6 or more | 3.6 |

A real estate company site claims that the 2021 census data is no longer representative of the suburb of Ashgrove due to the influx of young families with two or more children. Evaluate whether the real estate company's claim is supported, given that the Brisbane City Council presently records 5145 private dwellings in Ashgrove and a suburb population of 13 450.

**16.** An open-air cinema runs every summer evening in a city park from 1 November to 31 January except for Christmas Day and New Year's Eve. The cinema charges $10 per person. The number of people who turn up to the cinema depends upon the weather. The cinema does not operate if there is heavy rain. The probability distribution for the nightly attendance rate depending on the weather is shown below and is based upon the last ten years that the outdoor cinema has been running.

| Weather | Clear sky | Overcast | Showers | Heavy rain |
|---|---|---|---|---|
| $w$ | 300 | 270 | 190 | 0 |
| $p(w)$ | 0.60 | 0.18 | 0.12 | 0.10 |

The company who run the outdoor cinema are advised by the City Council that they can erect a large marquee in the park so they can run the cinema every night during the summer regardless of the weather. The largest marquee available allows room for 220 seats and will cost $30 000 per calendar month to rent. Evaluate whether the company will benefit from hiring a marquee every summer.

**17.** The television show *Steal or No Steal* features 26 cases with various amounts of money ranging from 50 cents to $200 000. The contestant chooses one case and then proceeds to open the other cases. At the end of each round, the banker makes an offer to end the game. The game ends when the contestant accepts the offer or when all the other 25 cases have been opened; in the latter event, the contestant receives the amount of money in the case they first chose. Suppose a contestant has five cases left and the amounts of $200 000, $100 000, $50 000, $15 000 and $1000 are still to be found.
Determine the expected amount that the banker should offer the contestant to end the game.

**18.** A bookstore sells both new and secondhand books. A particular new autobiography costs $65, a good-quality used autobiography costs $30 and a worn autobiography costs $12. A new cookbook costs $54, a good-quality used cookbook costs $25 and a worn cookbook costs $15. Let $X$ denote the total cost of buying two books (an autobiography and a cookbook). Assume that the purchases are independent of one another.
- The probability of buying a new autobiography is 0.4.
- The probability of buying a good-quality used autobiography is 0.3.
- The probability of buying a worn used autobiography is 0.3.
- The probability of buying a new cookbook is 0.4.
- The probability of buying a good-quality used cookbook is 0.25.
- The probability of buying a worn used cookbook is 0.35.
Calculate the expected cost of the two books.

**Fully worked solutions for this chapter are available online.**

# LESSON
## 6.6 Review

### 6.6.1 Summary

Hey students! Now that it's time to revise this topic, go online to:

| | Access the topic summary | | Review your results | | Watch teacher-led videos | | Practise exam questions |

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### 6.6 Exercise

**learn on**

| 6.6 Exercise | 6.6 QCAA questions on |

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| Simple familiar | Complex familiar | Complex unfamiliar |
| --- | --- | --- |
| 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 | 13, 14, 15, 16, 17 | 18, 19, 20 |

#### Simple familiar

1. **MC** Identify which of the following random variables is *not* discrete.

   A. The number of goals scored at a football match
   B. The number of T-shirts owned by a student
   C. The volume of soft drink consumed by a family over the period of a week
   D. The number of customers at a department store sale

2. **MC** Consider the discrete probability function with the following distribution.

   | $x$ | 2 | 4 | 6 | 8 | 10 |
   | --- | --- | --- | --- | --- | --- |
   | $p(x)$ | $2a$ | $3a$ | $4a$ | $5a$ | $6a$ |

   The value of the constant $a$ is:

   A. 20
   B. $\dfrac{1}{20}$
   C. $\dfrac{1}{2}$
   D. $\dfrac{1}{14}$

3. **MC** For a discrete random variable $X$ with a mean of 2.1 and a variance of 1.3, the values of $E(2X + 1)$ and $Var(2X + 1)$ are, respectively:

   A. 4.2 and 5.2
   B. 5.2 and 6.2
   C. 5.2 and 5.2
   D. 4.2 and 6.2

**4.** **MC** The random variable $Y$ has the following probability distribution.

| $y$ | $-2$ | 0 | 2 |
|---|---|---|---|
| $p(y)$ | $2p$ | $3p$ | $1-5p$ |

The mean of $Y$ is:

**A.** $2-11p$  **B.** $2-14p$  **C.** $1-3p$  **D.** $14p$

**5.** **MC** The probability distribution for the random variable $X$ is as follows.

| $x$ | $-1$ | 0 | 1 | 2 |
|---|---|---|---|---|
| $p(x)$ | $m$ | $m+n$ | $3m$ | $m-n$ |

If $E(X) = 0.4$, then $m$ and $n$ are equal to:

**A.** $m = \dfrac{1}{6}, \ n = \dfrac{1}{5}$

**B.** $m = \dfrac{1}{5}, \ n = \dfrac{1}{6}$

**C.** $m = \dfrac{1}{6}, \ n = \dfrac{2}{15}$

**D.** $m = \dfrac{2}{15}, \ n = \dfrac{1}{6}$

**6.** The probability distribution of $X$ is given by the formula $p(x) = \dfrac{x^2}{30}$, where $x = 1, \ 2, \ 3, \ 4$.

   **a.** Write the probability distribution of $X$ as a table.
   **b.** Calculate the expected value of $X$.

**7.** A biased coin is tossed twice. If the probability of obtaining a Head is $\dfrac{3}{5}$:

   **a.** Write the probability distribution table of the number of Heads in 2 tosses.
   **b.** Calculate the expected number of Heads.

**8.** Examine the following probability distribution table.

| $x$ | 4 | 9 | 16 | 25 | 36 |
|---|---|---|---|---|---|
| $p(x)$ | 0.16 | 0.21 | 0.35 | 0.08 | 0.2 |

Calculate the value of $p(X \geq 10)$.

**9.** A game is played where two dice are rolled and the sum of the two numbers showing uppermost is recorded. If players get a sum of 7, they win \$10. If they get a sum of 2 or 12, they win \$5. For any other sum, they must pay \$2.50. Explain whether this is a fair game.
Justify your response mathematically.

**10.** A discrete random variable, $Z$, has a probability distribution as follows.

| $z$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $p(z)$ | 0.1 | 0.25 | 0.35 | 0.25 | 0.05 |

Calculate:

   **a.** the expected value of $Z$
   **b.** the variance of $Z$
   **c.** the standard deviation of $Z$.

**11.** Maya constructed a spinner that will fall onto one of the numbers 1 to 5 with the following probability.

| Number | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| Probability | 0.1 | 0.3 | 0.3 | 0.2 | 0.1 |

Calculate the mean and the standard deviation of this distribution, correct to 2 decimal places.

**12.** A player rolls a fair die. If the player gets a 1 on the first roll, she rolls again and her score is the sum of the two results; otherwise, her score is the result of the first roll. The die cannot be thrown more than twice. Determine:

**a.** the probability distribution
**b.** the expected score
**c.** $p(X < \mu)$.

### Complex familiar

**13.** A biased coin is tossed four times. The probability of a Head from a toss is $a$ where $0 < a < 1$.

**a.** Determine, in terms of $a$, the probability of obtaining:

**i.** four Tails from four tosses
**ii.** one Head and three Tails from four tosses.

**b.** If the probability of obtaining four Heads is the same as the probability of obtaining one Head and three Tails, determine the value of $a$.

**14.** Alicia and Harry have devised a game where a biased spinner is spun. There are 4 colours on the wheel and the sectors are of varying sizes.

$p(\text{red}) = \dfrac{1}{20}$, $p(\text{blue}) = p(\text{green}) = 2 \times p(\text{red})$, and the other sector colour is yellow.

Players have to pay $2.00 to play. If the spinner lands on yellow, players receive nothing. If the spinner lands on green or blue, players get their money back. If the spinner lands on red, players win $5.

**a.** Determine the probability distribution for the amount of money a person can win.
**b.** Calculate the expected amount of money a player will win each game.

**15.** On any given day the number of text messages, $Y$, received by Garisht is a discrete random variable with a distribution as follows.

| y | 0 | 2 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|---|
| p(y) | 0.05 | 0.4 | 0.2 | 0.15 | 0.15 | 0.05 |

**a.** Determine the expected value of $Y$.
**b.** Determine the probability that Garisht receives no texts on four consecutive days.
**c.** Garisht received text messages on Thursday and Friday. Determine the probability that he received 10 text messages over these two days.

**16.** At Fast Eddy's Drive-In Theatre the cost is $10 per car, plus $3 per occupant. The variable $X$ represents the number of people in any car and is known to follow the probability distribution as shown below. Determine:

**a.** the expected cost per car
**b.** Fast Eddy's expected profit if 100 cars enter, and costs for wages, electricity, and so on are $500.

| x | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| p(x) | 0.4 | 0.2 | 0.3 | 0.1 |

**17.** A discrete random variable, $Z$, has a probability distribution as shown.

| $z$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $p(z)$ | $\dfrac{m}{5}$ | $\dfrac{1}{6}$ | $\dfrac{1}{6}$ | $\dfrac{1}{6}$ | $\dfrac{2m}{5}$ | $\dfrac{1}{10}(5-6m)$ |

This random variable describes the outcome of tossing a loaded die. The die is thrown twice. You may choose to use technology to answer questions **a–c**.

**a.** Prove that the chance of throwing a total of 11 is $\dfrac{10m - 12m^2}{25}$.

**b.** Determine the value of $m$ that makes this chance a maximum, and determine the maximum probability.

**c.** Using the value of $m$ from part **b**, determine:

    **i.** the expected value of $Z$ and the standard deviation of $Z$

    **ii.** $p(\mu - 2\sigma \le Z \le \mu + 2\sigma)$.

### Complex unfamiliar

**18.** A discrete random variable, $Z$, can only take the values 0, 1, 2, 3, 4, 5 and 6. The probability distribution for $Z$ is given by the following:

$p(Z=0) = p(Z=2) = p(Z=4) = p(Z=6) = m$
$p(Z=1) = p(Z=3) = p(Z=5) = n$
and $2p(0 < Z < 2) = p(3 < Z \le 6)$, where $m$ and $n$ are constants.
Determine the values of $m$ and $n$.

**19.** The number of passengers per car, $X$, entering Brisbane on a motorway on a workday morning is as follows.

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $p(x)$ | 0.37 | 0.22 | 0.21 | 0.1 | 0.05 | 0.05 |

The fees for cars at a toll booth on the motorway are as follows.
- Cars carrying no passengers: $2.50
- Cars carrying 1 or 2 passengers: $1.00
- Cars carrying more than 2 passengers: no fee

Determine the probability that, out of 10 cars selected at random, at least 8 cars have no passengers.

**20.** A random variable, $X$, represents the number of televisions serviced per week by a television serviceman. The probability distribution is as follows.

| $x$ | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $p(x)$ | 0.07 | 0.12 | 0.12 | 0.1 | 0.1 | 0.1 | 0.1 | 0.08 | 0.08 | 0.08 | 0.05 |

The serviceman is paid $20 for each television that he services plus a bonus depending on how many televisions he services a week. The bonuses are as follows:
- If less than 13 televisions are serviced, there is no bonus.
- If 13–16 televisions are serviced, he receives a bonus of $120.
- If more than 16 televisions are serviced, he receives a bonus of $250.

Determine the expected amount that the serviceman will be paid in a week.

## 6.6 Past QCAA exam questions

### Question 1 (4 marks)

*Source:* QCAA 2023 Mathematical Methods, Paper 1, Section 2, Q15; © State of Queensland (QCAA) 2024.

In a certain game, players throw one water balloon at a target. There is a one in four chance of hitting the target.

**a.** State the probabilities of all the possible outcomes for one throw at the target. **[2 marks]**

**b.** Let $H$ be the discrete random variable for one of the possible outcomes. Determine the mean and variance of the distribution of random variable $H$ when $20$ players throw a water balloon at the target. **[2 marks]**

### Question 2 (1 mark)

*Source:* Jacaranda practice exam question.

A random variable, $X$, has the following probability distribution.

| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $p(x)$ | $2b$ | $1.5b$ | $3b$ | $0.5b$ |

The value of $b$ is

**A.** 4 **B.** 1 **C.** 7 **D.** $\dfrac{1}{7}$

### Question 3 (1 mark)

*Source:* Jacaranda practice exam question.

Identify the expected value of a random variable with probability distribution shown below.

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $p(x)$ | $\dfrac{1}{32}$ | $\dfrac{3}{32}$ | $\dfrac{5}{16}$ | $\dfrac{5}{16}$ | $\dfrac{5}{32}$ | $\dfrac{3}{32}$ |

**A.** $\dfrac{11}{4}$ **B.** 1 **C.** $\dfrac{5}{2}$ **D.** $\dfrac{87}{32}$

### Question 4 (1 mark)

*Source:* Jacaranda practice exam question.

If $X$ represents the outcome of a fair die being rolled, then $\mathrm{E}(X)$ is equal to

**A.** $\dfrac{1}{6}$

**B.** 3

**C.** 3.5

**D.** 6

*Source:* Jacaranda practice exam question.

Given the probability distribution below, the standard deviation of $X$ is

| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $\mathbf{p}(x)$ | $\dfrac{1}{8}$ | $\dfrac{1}{2}$ | $\dfrac{3}{16}$ | $\dfrac{3}{16}$ |

**A.** 4.3750

**B.** 2.0917

**C.** 0.8711

**D.** 0.9333

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# Answers

## Chapter 6 General discrete random variables

### 6.2 Discrete random variables

#### 6.2 Exercise

**1. a.** Discrete    **b.** Continuous    **c.** Continuous
   **d.** Discrete    **e.** Continuous    **f.** Discrete
   **g.** Continuous    **h.** Discrete

**2. a.**

| $x$ | 0 | 1 | 2 |
|---|---|---|---|
| $p(x)$ | $\dfrac{1}{4}$ | $\dfrac{1}{2}$ | $\dfrac{1}{4}$ |

**b.**



**3. a.** HHH, HHT, HTH, HTT, THH, THT, TTH, TTT

**b.** $x = 0, 1, 2, 3$

**c.**

| $x$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $p(x)$ | $\dfrac{1}{8}$ | $\dfrac{3}{8}$ | $\dfrac{3}{8}$ | $\dfrac{1}{8}$ |

$\dfrac{7}{8}$

**4. a.**



$\xi = \{$RRR, RRG, RRY, RGR, RGG, RGY, RYR, RYG, RYY, GRR, GRG, GRY, GGR, GGG, GGY, GYR, GYG, GYY, YRR, YRG, YRY, YGR, YGG, YGY, YYR, YYG, YYY$\}$

**b.** $p(Y=3) = \dfrac{27}{1000}$; $p(Y=2) = \dfrac{189}{1000}$; $p(Y=1) = \dfrac{441}{1000}$; $p(Y=0) = \dfrac{343}{1000}$

**c.**

| $y$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $p(y)$ | $\dfrac{343}{1000}$ | $\dfrac{441}{1000}$ | $\dfrac{189}{1000}$ | $\dfrac{27}{1000}$ |

**d.** This is a discrete probability function.

**5.**

| $x$ | 0 | 1 | 2 |
|---|---|---|---|
| $p(x)$ | $\dfrac{25}{36}$ | $\dfrac{10}{36} = \dfrac{5}{18}$ | $\dfrac{1}{36}$ |

**6. i.** This is a discrete probability function.
   **ii.** This is a discrete probability function.

**7. a.** $k = \dfrac{1}{16}$      **b.** $k = \dfrac{1}{6}$

**8. a.**



**b.**



**c.**



**d.**



**9. a.**

| $x$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $p(x)$ | $\dfrac{1}{36}$ | $\dfrac{1}{18}$ | $\dfrac{1}{12}$ | $\dfrac{1}{9}$ | $\dfrac{5}{36}$ | $\dfrac{1}{6}$ | $\dfrac{5}{36}$ | $\dfrac{1}{9}$ | $\dfrac{1}{12}$ | $\dfrac{1}{18}$ | $\dfrac{1}{36}$ |

**b. i.** $\dfrac{1}{6}$    **ii.** $\dfrac{5}{18}$    **iii.** $\dfrac{7}{36}$    **iv.** $\dfrac{29}{36}$

**10. a.** $\xi = \{11, 12, 13, 14, 15, 16, 17, 18, 19, 110, 111, 112$
$21, 22, 23, 24, 25, 26, 27, 28, 29, 210, 211, 212$
$31, 32, 33, 34, 35, 36, 37, 38, 39, 310, 311, 312$
$41, 42, 43, 44, 45, 46, 47, 48, 49, 410, 411, 412$
$51, 52, 53, 54, 55, 56, 57, 58, 59, 510, 511, 512$
$61, 62, 63, 64, 65, 66, 67, 68, 69, 610, 611, 612$
$71, 72, 73, 74, 75, 76, 77, 78, 79, 710, 711, 712$
$81, 82, 83, 84, 85, 86, 87, 88, 89, 810, 811, 812\}$

**b.** $Z = [0, 1, 2]$

| $z$ | 0 | 1 | 2 |
|---|---|---|---|
| $p(z)$ | 0.09 | 0.42 | 0.49 |

**c.** 0.42

**11.** Sample responses can be found in the worked solutions in the online resources.

**12. a.   i.** 0.1715

   **ii.** 0.4115

   **iii.** 0.3292

   **iv.** 0.0878

   **b.** 0.1060

**13.** Sample responses can be found in the worked solutions in the online resources.

**14. a.** This is a discrete probability function.

   **b.** This is not a discrete probability function.

   **c.** This is a discrete probability function.

**15.** $a = 30$

**16.** $k = -0.4568$ or $0.6568$

**17. a.** F = female and M = male



$$\xi = \begin{cases} \text{FFFF, FFFM, FFMF, FFMM, FMFF, FMFM,} \\ \text{FMMF, FMMM, MFFF, MFFM, MFMF, MFMM,} \\ \text{MMFF, MMFM, MMMF, MMMM,} \end{cases}$$

**b.**

| $x$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $p(x)$ | $\dfrac{1}{16}$ | $\dfrac{4}{16} = \dfrac{1}{4}$ | $\dfrac{6}{16} = \dfrac{3}{8}$ | $\dfrac{4}{16} = \dfrac{1}{4}$ | $\dfrac{1}{16}$ |

**c.** $\dfrac{1}{16}$

**d.** $\dfrac{15}{16}$

**e.** $\dfrac{11}{16}$

**18.** 0.009

**19.** It is a success.

**20.**

| $x$ | \$0 | \$1 | \$10 |
|---|---|---|---|
| $p(x)$ | 0.7840 | 0.1920 | 0.0240 |

## 6.3 Expected values

### 6.3 Exercise

**1.** 2.4375

**2.** 1.16

**3.** $a = 0.09$ $E(X) = 5.42$

**4.** $a = \dfrac{1}{18}$; $E(Z) = 5\dfrac{1}{3}$

**5.** 6

**6.** $b = 0.15$; $E(X) = 2.39$

**7.** $k = 0.05$; $E(X) = 13$

**8. a.**

| $x$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $p(x)$ | $\dfrac{1}{6}$ | $\dfrac{1}{6}$ | $\dfrac{1}{6}$ | $\dfrac{1}{3}$ | $\dfrac{1}{6}$ | $\dfrac{1}{6}$ |

**b.** $3\dfrac{1}{2}$

**9. a.**

| $x$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $p(x)$ | $\dfrac{1}{36}$ | $\dfrac{1}{18}$ | $\dfrac{1}{12}$ | $\dfrac{1}{9}$ | $\dfrac{5}{36}$ | $\dfrac{1}{6}$ | $\dfrac{5}{36}$ | $\dfrac{1}{9}$ | $\dfrac{1}{12}$ | $\dfrac{1}{18}$ | $\dfrac{1}{36}$ |

**b.** 7

**10. a.**

| $x$ | 0 | 1 | 2 |
|---|---|---|---|
| $p(x)$ | $\dfrac{1}{4}$ | $\dfrac{1}{2}$ | $\dfrac{1}{4}$ |

**b.** 1

**11. a.**

| $x$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $p(x)$ | $\dfrac{1}{16}$ | $\dfrac{1}{4}$ | $\dfrac{3}{8}$ | $\dfrac{1}{4}$ | $\dfrac{1}{16}$ |

**b.** 2

**12. a.** $2\dfrac{1}{3}$   **b.** $9\dfrac{1}{3}$   **c.** $5\dfrac{2}{3}$   **d.** $6\dfrac{1}{15}$

**13.** $a = 0.15$, $b = 0.05$

**14.** $a = 0.1, b = 0.2$

**15. a.** \$3.75

   **b.** No, he shouldn't play the game because his loss per game is \$1.25.

   **c.** It is not a fair game because the expected gain is less than the initial cost of the game.

**16.** \$1452

**17.** $E(X) = 1$

**18.** $k = 17$

**19.** 4.42

**20.** 2.4286

## 6.4 Variance and standard deviation

### 6.4 Exercise

**1. a.** 2.9      **b.** 2.19      **c.** 1.48

**2. a.** $E(Y) = 7$

   **b.** $SD(Y) = 3.7947$

**3. a.** $E(X) = \dfrac{1}{9}$

   **b.** $SD(X) = \dfrac{260}{81}$

   **c.** $SD(X) = \dfrac{\sqrt{65}}{9}$

**4. a.** $E(Z) = \dfrac{19}{6}$

   **b.** $SD(Z) = \dfrac{65}{36}$

   **c.** $SD(X) = \dfrac{\sqrt{65}}{6}$

**5. a** $2.45

   **b** $Var(X) = \$1.35$, $SD(X) = \$1.16$

**6. a.** $k = \dfrac{1}{10}$

   **b.** 3.2

   **c.** $Var(X) = 6.56$, $SD(X) = 2.56$

**7. a.** Sample responses can be found in the worked solutions in the online resources.

   **b. i.** $\dfrac{64}{27}$

     **ii.** $\dfrac{386}{729}$

   **c. i.** $\dfrac{386}{81}$

     **ii.** $\dfrac{9650}{729}$

**8. a.**

| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $p(x)$ | $\dfrac{1}{30}$ | $\dfrac{4}{30} = \dfrac{2}{15}$ | $\dfrac{9}{30} = \dfrac{3}{10}$ | $\dfrac{16}{30} = \dfrac{8}{15}$ |

   **b. i.** 3.33

     **ii.** 0.69

   **c. i.** 11.02

     **ii.** 6.20

**9. a.** 8      **b.** 0.5      **c.** 14.5

**10. a.** 225      **b.** 25      **c.** 6.25

**11. a.** $h = \dfrac{1}{10}$

    **b.** $E(X) = 1$, $Var(X) = 0.6$, $SD(X) = 0.7746$

**12. a.** $c = \dfrac{1}{3}$

    **b.** $E(Y) = 3$

    **c.** $Var(Y) = 11.56$, $SD(Y) = 3.40$

**13. a.** $m = 15$

    **b.** $Var(Z) = 153.48$, $Var(2\,(Z-1)) = 613.91$, $Var(3 - Z) = 153.48$

**14. a.** $a = 0.3$; $b = 0.1$

    **b.** $Var(X) = 1.65$, $SD(X) = 1.2845$

**15. a.** E = {11, 12, 13, 14, 15, 16, 17, 18
21, 22, 23, 24, 25, 26, 27, 28
31, 32, 33, 34, 35, 36, 37, 38
41, 42, 43, 44, 45, 46, 47, 48
51, 52, 53, 54, 55, 56, 57, 58
61, 62, 63, 64, 65, 66, 67, 68
71, 72, 73, 74, 75, 76, 77, 78
81, 82, 83, 84, 85, 86, 87, 88}

    **b.**

| $z$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $p(z)$ | $\dfrac{1}{64}$ | $\dfrac{3}{64}$ | $\dfrac{5}{64}$ | $\dfrac{7}{64}$ | $\dfrac{9}{64}$ | $\dfrac{11}{64}$ | $\dfrac{13}{64}$ | $\dfrac{15}{64}$ |

    **c.** $E(Z) = 5.8125$, $SD(Z) = 1.8781$

**16. a.** $k = 0.1$

    **b.** $E(X) = 1.695$

    **c.** $SD(X) = 1.167$

**17. a.** $n = \dfrac{1}{13}$

    **b.** $Var(Y) = 1.7870$, $E(Y) = 3.5385$, $SD(Y) = 1.3368$

**18.** $E(X) = 2$, $Var(X) = 2$

**19.** $0.68

**20.** $E(X) = 9.4$, $SD(X) = 3.7974$

## 6.5 Modelling and solving with discrete random variables

### 6.5 Exercise

**1.** 6 cents/nail

**2. a.** 1.72

    **b.** 0.8461

    **c.** 0.84

**3.** 4 donuts

**4.** $250 per dishwasher

**5. a.** 0.6

    **b.** $\dfrac{8}{9}$

    **c.** $535 000

**6.** No as under the tax free threshold

**7. a.** 0.216

    **b.** 0.286

    **c.**

| $t$ | 0 | $2 | $5 |
|---|---|---|---|
| $p(t)$ | 0.286 | 0.498 | 0.216 |

    **d.** $E(T) = \$2.08$; $SD(T) = \$1.76$

**8.** Yes. Sample responses can be found in the worked solutions in the online resources.

**9. a.** Machine X has an expected number of 2.68 defective diodes with a standard deviation of 1.71 diodes per 10 000; machine Y has an expected number of 1.62 defective diodes with a standard deviation of 1.3 diodes per 10 000.

    **b.** Machine Y produces fewer defective diodes per 10 000 units.

**10. a.** $n = \dfrac{2}{15}$; $m = \dfrac{1}{5}$

   **b.** Sample responses can be found in the worked solutions in the online resources.

   **c.** $\text{Var}(Z) = 2.8267$; $\text{SD}(Z) = 1.6813$

**11.** $a = 0.35$; $b = 0.25$

**12. a.** $k = 1$

   **b.** 1.4

   **c.** $\dfrac{7}{12}$

**13. a.** Let $Y$ be the net profit per day.

| $Y$ | $120 | $230 | $580 | $930 |
|---|---|---|---|---|
| $p(y)$ | 0.3 | 0.4 | 0.2 | 0.1 |

   **b.** $265

**14. a.**

| $x$ | 1 | 5 | 10 |
|---|---|---|---|
| $p(x)$ | $\dfrac{12}{16} = \dfrac{3}{4}$ | $\dfrac{3}{16}$ | $\dfrac{1}{16}$ |

   **b.** 2.3

   **c.** 57.8

   **d.** 44

**15.** The claim is not supported. Sample responses can be found in the worked solutions in the online resources.

**16.** The company is better off remaining an open-air cinema.

**17.** $73 200

**18.** $71.70

## 6.6 Review

### 6.6 Exercise

**1.** C

**2.** B

**3.** C

**4.** B

**5.** C

**6. a.**

| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $p(x)$ | $\dfrac{1}{30}$ | $\dfrac{2}{15}$ | $\dfrac{3}{10}$ | $\dfrac{8}{15}$ |

   **b.** $3\dfrac{1}{3}$

**7. a.**

| $x$ | 0 | 1 | 2 |
|---|---|---|---|
| $p(x)$ | $\dfrac{4}{25}$ | $\dfrac{12}{25}$ | $\dfrac{9}{25}$ |

   **b.** 1.2

**8.** 0.63

**9.** Yes

**10. a.** 2.9

   **b.** 1.09

   **c.** 1.044

**11.** Mean $= 2.9$; SD $= 1.14$

**12. a.**

| $x$ | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|
| $p(x)$ | $\dfrac{7}{36}$ | $\dfrac{7}{36}$ | $\dfrac{7}{36}$ | $\dfrac{7}{36}$ | $\dfrac{7}{36}$ | $\dfrac{1}{36}$ |

   **b.** $4\dfrac{1}{12}$

   **c.** $\dfrac{7}{12}$

**13. a. i.** $(1-a)^4$

     **ii.** $4a(1-a)^3$

   **b.** $a = \dfrac{1}{5}$

**14. a.** $p(x=0) = \dfrac{3}{4}$; $p(x=2) = \dfrac{1}{5}$; $p(x=5) = \dfrac{1}{20}$

   **b.** $0.65

**15. a.** 4.2

   **b.** $\dfrac{1}{160\,000}$

   **c.** 0.1994

**16. a.** $19.30/car

   **b.** $1430

**17. a.** Sample responses can be found in the worked solutions in the online resources.

   **b.** $m = \dfrac{5}{12}$; maximum probability $= \dfrac{1}{12}$

   **c. i.** $E(Z) = 3.9167$, $SD(Z) = 1.6562$

     **ii.** 1

**18.** $m = \dfrac{1}{10}$; $n = \dfrac{1}{5}$

**19.** 0.007 14

**20.** $412.10

### 6.6 Past QCAA exam questions

**1. a.** $p(\text{hit}) = \dfrac{1}{4}$, $p(\text{miss}) = \dfrac{3}{4}$

   **b.** $E\left(H_{20}\right) = 5$

      $\text{Var}\left(H_{20}\right) = \dfrac{15}{4}$

**2.** D

**3.** A

**4.** C

**5.** D

# 7 Bernoulli and binomial distributions

## LESSON SEQUENCE

**Fully worked solutions of this chapter are available online.**

## EXAM PREPARATION

Access exam-style questions in every lesson, available online and access past QCAA exam questions in every review.

**on** Resources

| | |
|---|---|
| **Solutions** | Solutions — Chapter 7 |
| **Exam questions** | Exam question booklet — Chapter 7 |
| **Digital documents** | Learning matrix — Chapter 7<br>Chapter summary — Chapter 7 |

# LESSON
## 7.1 Overview

### 7.1.1 Introduction

The binomial distribution models the number of successes in a fixed number of independent trials, each with the same probability of success. It's widely used in scenarios like quality control and genetics. The Bernoulli distribution is a special case of the binomial distribution with only one trial. It describes a single experiment with two possible outcomes: success or failure.

### 7.1.2 Syllabus Links

| Lesson | Lesson title | Syllabus links |
|--------|--------------|----------------|
| 7.2 | **Bernoulli distributions** | ○ Use a Bernoulli random variable as a model for two-outcome situations. <br> ○ Identify contexts suitable for modelling by Bernoulli random variables. <br> ○ Recognise and determine the mean $p$ and variance $p(1-p)$ of the Bernoulli distribution with parameter $p$. <br> ○ Model and solve problems that involve Bernoulli random variables and associated probabilities, with and without technology. |
| 7.3 | **Binomial distributions** | ○ Understand the concepts of Bernoulli trials and the concept of a binomial random variable as the number of 'successes', $r$, in $n$ independent Bernoulli trials, with the same probability of success $p$ in each trial. <br> ○ Identify contexts suitable for modelling by binomial random variables. <br> ○ Determine and use the probabilities $P(X=r) = \binom{n}{r} p^r (1-p)^{n-r}$ associated with the binomial distribution with parameters $n$ and $p$. <br> ○ Use the language of probability, including *at most, at least, no more than, no less than, inclusive* and *between*. <br> ○ Model and solve problems that involve binomial distributions and associated probabilities with and without technology. |
| 7.4 | **The mean and variance of a binomial distribution** | ○ Calculate the mean $np$ and variance $np(1-p)$ of a binomial distribution using technology and algebraic methods. <br> ○ Use the language of probability, including *at most, at least, no more than, no less than, inclusive* and *between*. <br> ○ Model and solve problems that involve binomial distributions and associated probabilities with and without technology. |
| 7.5 | **Modelling and solving with Bernoulli and binomial distributions** | ○ Use the language of probability, including *at most, at least, no more than, no less than, inclusive* and *between*. <br> ○ Model and solve problems that involve binomial distributions and associated probabilities with and without technology. |

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# LESSON
## 7.2 Bernoulli distributions

## 7.2.1 Bernoulli trials

In probability theory, the **Bernoulli distribution** is a discrete probability distribution of the simplest kind. It is named after the Swiss mathematician Jacob Bernoulli (1654–1705). The term '**Bernoulli trial**' refers to a single event that has only 2 possible outcomes, a success or a failure, with each outcome having a fixed probability. The following are examples of Bernoulli trials.

- Will a coin land Heads up?
- Will a newborn child be a male or a female?
- Are a random person's eyes blue or not?
- Will a person vote for a particular candidate at the next local council elections or not?
- Will you pass or fail an examination?

### WORKED EXAMPLE 1 Identifying a Bernoulli trial

**Determine which of the following can be defined as a Bernoulli trial.**
**a. Interviewing a random person to see if they have had a flu vaccination this year**
**b. Rolling a die in an attempt to obtain an even number**
**c. Choosing a ball from a bag which contains 3 red balls, 5 blue balls and 4 yellow balls**

| THINK | WRITE |
|---|---|
| **a.** Check for the characteristics of a Bernoulli trial. | **a.** Yes, this is a Bernoulli trial, as there are 2 possible outcomes. A person either has or has not had a flu vaccination this year. |
| **b.** Check for the characteristics of a Bernoulli trial. | **b.** Yes, this is a Bernoulli trial, as there are 2 possible outcomes. The die will show either an odd number or an even number. |
| **c.** Check for the characteristics of a Bernoulli trial. | **c.** No, this is not a Bernoulli trial, as success has not been defined. |

### ✦ 7.2.2 Bernoulli random variables

int-6430

Bernoulli distributions are controlled by the probability of success, $p$. Given that there are only two possible outcomes for a single Bernoulli trial and that the sum of probabilities for that trial is 1, it can be seen that the probability of failure, $q$, is equal to $1 - p$.

This means that the probability distribution table for a single Bernoulli trial looks like this:

| $x$ | 0 | 1 |
|---|---|---|
| $P(X = x)$ | $1 - p$ | $p$ |

where $X$ represents the number of successes.

## 7.2.3 Mean of a Bernoulli distribution

As you will recall from your earlier studies of probability, the mean of a discrete random variable distribution is referred to as the **expected value**, represented by $E(X)$ or $\mu$. The expected value of a discrete random variable is the sum of each value of $X$ in the distribution multiplied by its probability:

$$E(X) = \sum_{\text{all } x} x \, P(x)$$

In the case of the probability distribution for a Bernoulli trial

$$
\begin{aligned}
E(X) &= \sum x \, P(x) \\
&= 0 \, (1 - p) + 1 \times p \\
&= p
\end{aligned}
$$

## 7.2.4 Variance and standard deviation of a Bernoulli distribution

The **variance** (written as $\text{Var}(X)$ or $\sigma^2$) and **standard deviation** (written $\text{SD}(X)$ or $\sigma$) of any distribution are measures of spread used to indicate the range over which an outcome deviates from the expected value.

Substituting the values from the Bernoulli probability distribution table into the equation for variance:

$$
\begin{aligned}
\text{Var}(X) &= E\left(X^2\right) - [E(X)]^2 \\
&= \left[0^2 \, (1 - p) + 1^2 p\right] - p^2 \\
&= p - p^2 \\
&= p(1 - p)
\end{aligned}
$$

As the standard deviation is calculated by taking the square root of the variance, we can determine an expression for the standard deviation of a Bernoulli distribution:

$$
\begin{aligned}
\text{SD}(X) &= \sqrt{\text{Var}(X)} \\
&= \sqrt{p \, (1 - p)}
\end{aligned}
$$

*Note:* When calculating $\text{Var}(X)$ and $\text{SD}(X)$, it is conventional to round to 4 decimal places unless otherwise indicated.

**WORKED EXAMPLE 2 Calculating mean, variance and standard deviation for a Bernoulli distribution**

**A new cream has been developed for the treatment of dermatitis. In laboratory trials the cream was found to be effective in 72% of the cases. Hang's doctor has prescribed the cream for her. Let $X$ be the effectiveness of the cream.**
**a. Construct a probability distribution table for $X$.**
**b. Determine $E(X)$.**
**c. Determine the variance and the standard deviation of $X$, correct to 4 decimal places.**

| THINK | WRITE |
|---|---|
| **a.** Construct a probability distribution table and clearly state the value of $p$. | **a.** $p = $ success with cream $= 0.72$ |

| $x$ | 0 | 1 |
|---|---|---|
| $P(x)$ | 0.28 | 0.72 |

| THINK | WRITE |
|---|---|
| **b. 1.** State the rule for the expected value. | **b.** $E(X) = \displaystyle\sum_{\text{all } x} xP(x)$ |
| **2.** Substitute the appropriate values and evaluate. | $E(X) = 0 \times 0.28 + 1 \times 0.72$ $= 0.72$ |
| **c. 1.** Determine $E(X^2)$. | **c.** $E(X^2) = 0^2 \times 0.28 + 1^2 \times 0.72$ $= 0.72$ |
| **2.** Calculate the variance. | $Var(X) = E(X^2) - [E(X)]^2$ $Var(X) = 0.72 - (0.72)^2$ $= 0.2016$ |
| **3.** Calculate the standard deviation. | $SD(X) = \sqrt{0.2016}$ $= 0.4490$ |

| TI | THINK | WRITE | CASIO | THINK | WRITE |
|---|---|---|---|
| **a. 1.** On a New Document page, select: 4: Add Lists & Spreadsheets. Define the list names of each column by completing the entry lines: list 1 list 2 | | **a. 1.** On a Main Menu screen, select: Statistics. Complete the entry line in List 1 as: 0 1 Complete the entry line in List 2 as: 0.28 0.72 | |

**2.** Complete the entries in list 1 as:
0
1
Complete the entries in list 2 as:
0.28
0.72
The probability table has been constructed.

**b. 1.** To complete the statistical calculations, select:
MENU
4: Statistics
1: Stat Calculations
1: One-Variable Statistics.

**2.** Set the number of lists to 1 and press the OK button.

**3.** Complete the entry lines as: X1 list: 'list1
Frequency List: 'list2
1$^{st}$ Result Column: c[]
Press the OK button.

**4.** The expected value E(X) can be read from the screen.

**c.** The variance can be read by scrolling down the screen.

**2.** Select List 2 as a frequency list by pressing the SET button.
Press the LIST button.
Complete the entry line as:
2
Press the EXE button twice.

**3.** The probability table has been constructed.

**b.** To complete the statistical calculations, press the 1−VAR button.
The expected value, E(X), can be read from the screen.

**c.** The standard deviation can be read from the screen.
*Note:* The variance can be calculated using SD(X) = $\sqrt{\mathrm{Var}(X)}$.

| **7.2 Exercise** | **7.2 Exam questions** on |
|---|---|

| Simple familiar | Complex familiar | Complex unfamiliar |
|---|---|---|
| 1, 2, 3, 4, 5, 6, 7, 8, 9 | 10, 11, 12, 13 | 14 |

**These questions are even better in jacPLUS!**
- Receive immediate feedback
- Access sample responses
- Track results and progress

**Find all this and MORE in jacPLUS** ▶

### Simple familiar

1. **WE1** Determine which of the following can be defined as a Bernoulli trial.

   a. Spinning a spinner with 3 coloured sections
   b. A golfer is at the tee of the first hole of a golf course. As she is an experienced golfer, the chance of her getting a hole in one is 0.15. Will she get a hole in one at this first hole?
   c. A card is drawn from a standard pack of 52 cards. What is the chance of drawing an ace?

2. Determine which of the following can be defined as a Bernoulli trial.

   a. A new drug for arthritis is said to have a success rate of 63%. Jing Jing has just been prescribed the drug to treat her arthritis, and her doctor is interested in whether her symptoms improve or not.
   b. Juanita has just given birth to a baby, and we are interested in the gender of the baby, in particular whether the baby is a girl.
   c. You are asked what your favourite colour is.
   d. A telemarketer rings random telephone numbers in an attempt to sell a magazine subscription and has a success rate of 58%. Will the next person he rings subscribe to the magazine?

3. State clearly why the following are not Bernoulli trials.

   a. A bag contains 12 balls, 5 of which are black, 3 of which are white and 4 of which are red. Paul has just drawn a ball from the bag without returning it. Now it is Alice's turn to draw a ball from the bag. Does she get a red one?
   b. A die is tossed and the outcome is recorded.
   c. A fairy penguin colony at Phillip Island in Victoria is being studied by an ecologist. Will the habitat be able to sustain the colony in the future?

4. **WE2** Caitlin is playing basketball for her local club. The chance that Caitlin scores a basket is 0.42. The ball has just been passed to her and she shoots. Let $X$ be the random variable that defines Caitlin sinking the ball. (Assume $X$ obeys the Bernoulli distribution.)

   a. Set up a probability distribution for this discrete random variable.
   b. Determine $E(X)$.
   c. Determine:
      i. $Var(X)$
      ii. $SD(X)$

**5.** A discrete random variable, $Z$, has a Bernoulli distribution as follows.

| $z$ | 0 | 1 |
|---|---|---|
| $\mathbf{P}(z)$ | 0.37 | 0.63 |

   **a.** Determine $E(Z)$.          **b.** Determine $Var(Z)$.          **c.** Determine $SD(Z)$.

**6.** Eli and Jacinta are about to play a game of chess. As Eli is a much more experienced chess player, the chance that he wins is 0.68. Let $Y$ be the discrete random variable that defines the fact the Eli wins.

   **a.** Construct a probability distribution table for $Y$.
   **b.** Evaluate:

   **i.** $E(Y)$          **ii.** $Var(Y)$          **iii.** $SD(Y)$

**7.** During the wet season, the probability that it rains on any given day in Cairns in northern Queensland is 0.89. I am going to Cairns tomorrow and it is the wet season. Let $X$ be the chance that it rains on any given day during the wet season.

   **a.** Construct a probability distribution table for $X$.
   **b.** Evaluate:
   **i.** $E(X)$                    **ii.** $Var(X)$                    **iii.** $SD(X)$

**8.** It has been found that when breast ultrasound is combined with a common mammogram, the rate in which breast cancer is detected in a group of women is 7.2 per 1000. Louise is due for her two-yearly mammography testing, which will involve an ultrasound combined with a mammogram. Let $Z$ be the discrete random variable that breast cancer is detected.

   **a.** Determine the probability that Louise has breast cancer detected at this next test.
   **b.** Construct a probability distribution table for $Z$.

**9.** A manufacturer of sweets reassures their customers that when they buy a box of their 'All Sorts' chocolates there is a 33% chance that the box will contain one or more toffees. Kasper bought a box of 'All Sorts' and selected one. Let $Y$ be the discrete random variable that Kasper chose a toffee.

   **a.** Construct a probability distribution table for $Y$.
   **b.** Determine $E(Y)$.

**Complex familiar**

**10.** $X$ is a discrete random variable that has a Bernoulli distribution. It is known that the variance for this distribution is 0.21.

   **a.** Determine the probability of success, $p$, where $p > 1 - p$.
   **b.** Determine $E(X)$.

**11.** $Y$ is a discrete random variable that has a Bernoulli distribution. It is known that the standard deviation for this distribution is 0.4936.

   **a.** Determine the variance of $Y$ correct to 4 decimal places.
   **b.** Determine the probability of success, $p$, if $p > 1 - p$.
   **c.** Determine $E(Y)$.

**12.** $Z$ is a discrete random variable that has a Bernoulli distribution. It is known that the variance of $Z$ is 0.1075.

   **a.** Determine the probability of success, correct to 4 decimal places, if P(success) > P(failure).

   **b.** Construct a probability distribution table for $Z$.

   **c.** Evaluate the expected value of $Z$.

**13.** $Y$ is a discrete random variable that has a Bernoulli distribution. It is known that the standard deviation of $Y$ is 0.3316.

   **a.** Calculate the variance correct to 2 decimal places.

   **b.** Calculate the probability of success correct to 4 decimal places if P(success) > P(failure).

**Complex unfamiliar**

**14.** Jaiden is told she can take one jellybean from a container of jellybeans without looking at the colour first. She really wants a red one but knows that there are only 5 red jellybeans in there. Let $R$ be the discrete random variable that she selects a red jellybean. If Jaiden is lucky and selects a red jellybean, Var($R$) becomes $\dfrac{19}{900}$ less than it was before the selection.

Determine the total number of jellybeans in the container before Jaiden made her selection.

**Fully worked solutions for this chapter are available online.**

# LESSON
## 7.3 Binomial distributions

### SYLLABUS LINKS

- Understand the concepts of Bernoulli trials and the concept of a binomial random variable as the number of 'successes', $r$, in $n$ independent Bernoulli trials, with the same probability of success $p$ in each trial.
- Identify contexts suitable for modelling by binomial random variables.
- Determine and use the probabilities $P(X = r) = \dbinom{n}{r} p^r (1 - p)^{n-r}$ associated with the binomial distribution with parameters $n$ and $p$.
- Use the language of probability, including *at most, at least, no more than, no less than, inclusive* and *between*.
- Model and solve problems that involve binomial distributions and associated probabilities with and without technology.

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A **binomial distribution** results from multiple Bernoulli trials with the following characteristics:

1. Trials are independent.
2. Each trial has two outcomes: success or failure.
3. The success probability, $p$, is consistent for each trial.

We can express a discrete random variable $X$ with a binomial distribution in the form $X \sim \text{Bi}\,(n, p)$, where $p$ is the success probability and $n$ is the number of trials.

Consider Cassandra, a Science student guessing the last 5 multiple choice questions on her exam. Each question has 5 choices, with only one correct answer. This situation represents a binomial distribution with five independent Bernoulli trials. Each trial has two possible outcomes: guessing correctly $\left(\text{success}, p = \dfrac{1}{5}\right)$ or incorrectly $\left(\text{failure}, 1 - p = \dfrac{4}{5}\right)$.

There are six possible results: 0 to 5 correct answers. Combinations don't matter because the order of correct answers is irrelevant.

As you will recall from the binomial theorem in Unit 1, the probability of getting $r$ successes out of $n$ trials can be calculated using the equation

$$P(X = r) = \binom{n}{r} p^r (1 - p)^{n-r}$$

where $\binom{n}{r} = {}^nC_r = \dfrac{n!}{(n-r)!\ r!}$.

If $X$ represents the number of questions answered correctly, then we may calculate the probabilities for the distribution as shown in the following table given that $n = 5$, $p = \dfrac{1}{5}$ and $1 - p = \dfrac{4}{5}$.

C represents a correctly guessed answer and I represents an incorrect answer.

| Number of correct answers, $r$ | Possible outcomes | Probability $P(x = r) = \binom{n}{r} p^r (1 - p)^{n-r}$ |
|---|---|---|
| 0 | IIIII | $P(X = 0) = \binom{5}{0} \left(\dfrac{1}{5}\right)^0 \left(\dfrac{4}{5}\right)^5 = \dfrac{1024}{3125} = 0.3277$ |
| 1 | CIIII, ICIII, IICII, IIICI, IIIIC | $P(X = 1) = \binom{5}{1} \left(\dfrac{1}{5}\right)^1 \left(\dfrac{4}{5}\right)^4 = \dfrac{1280}{3125} = 0.4096$ |
| 2 | CCIII, ICCII, IICCI, IIICC, CICII, ICICI, IICIC, CIICI, ICIIC, CIIIC | $P(X = 2) = \binom{5}{2} \left(\dfrac{1}{5}\right)^2 \left(\dfrac{4}{5}\right)^3 = \dfrac{640}{3125} = 0.2048$ |
| 3 | IICCC, ICICC, CIICC, ICCIC, CICIC, CCIIC, ICCCI, CICCI, CCICI, CCCII | $P(X = 3) = \binom{5}{3} \left(\dfrac{1}{5}\right)^3 \left(\dfrac{4}{5}\right)^2 = \dfrac{160}{3125} = 0.0512$ |
| 4 | ICCCC, CICCC, CCICC, CCCIC, CCCCI | $P(X = 4) = \binom{5}{4} \left(\dfrac{1}{5}\right)^4 \left(\dfrac{4}{5}\right)^1 = \dfrac{20}{3125} = 0.0064$ |
| 5 | CCCCC | $P(X = 5) = \binom{5}{5} \left(\dfrac{1}{5}\right)^5 \left(\dfrac{4}{5}\right)^0 = \dfrac{1}{3125} = 0.0003$ |

The results can then be written in the form of a probability distribution table:

| $r$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $P(X = r)$ | 0.3277 | 0.4096 | 0.2048 | 0.0512 | 0.0064 | 0.0003 |

From this table, we see Cassandra is most likely to guess 1 out of 5 questions correctly and least likely to guess all 5 correctly. If the order is specified, the binomial probability distribution rule cannot be used. We can calculate her chances of getting 3 answers right but not her chances of getting the first, fourth and fifth questions right using the binomial distribution.

---

**WORKED EXAMPLE 3 Calculating probability using binomial distribution**

**A new drug for hay fever is known to be successful in 40% of cases.**
**Ten hay-fever sufferers take part in the testing of the drug.**
**Determine the probability that:**
**a. 4 people are cured**
**b. no people are cured**
**c. at least 2 people are cured.**

| THINK | WRITE |
|---|---|
| **a. 1.** Check that all the characteristics have been satisfied for a binomial distribution. | **a.** This is a binomial distribution with $n$ independent trials and two outcomes, $p$ and $(1-p)$. |
| **2.** Write down the rule for the binomial probability distribution. | $P(X=r) = \binom{n}{r} p^r (1-p)^{n-r}$ |
| **3.** Define and assign values to variables. | $n = 10$<br>Let $X$ = number of people cured.<br>Therefore, $r = 4$.<br>$p = 0.4$<br>$(1-p) = 0.6$ |
| **4.** Substitute the values into the rule. | $P(X=4) = \binom{10}{4}(0.4)^4(0.6)^6$ |
| **5.** Evaluate. | $= 210 \times 0.0256 \times 0.046\,656$<br>$= 0.250\,822\,656$ |
| **6.** Round the answer off to 4 decimal places. | $= 0.2508$ |
| **7.** Answer the question. | The probability that 4 people are cured is 0.2508. |
| **b. 1.** Define and assign values to variables. | **b.** $n = 10$<br>Let $X$ = number of people cured.<br>Therefore, $r = 0$.<br>$p = 0.4$<br>$(1-p) = 0.6$ |
| **2.** Substitute the values into the rule. | $P(X=0) = \binom{10}{0}(0.4)^0(0.6)^{10}$ |
| **3.** Evaluate. | $= 1 \times 1 \times 0.006\,046\,617\,6 = 0.006\,046\,6176$ |
| **4.** Round off the answer to 4 decimal places. | $= 0.0060$ |
| **5.** Answer the question. | The probability that no people are cured is 0.0060. |

**c. 1.** The condition that at least 2 people are cured means that the probability will be the sum of probabilities for which $2 \leq r \leq 10$; that is, $P(X \geq 2) = \sum_{r=2}^{10} \binom{10}{r} p^r(1-p)^{10-r}$. This expression would require the summation of 9 terms. It would be easier in this case to determine the probability of the complementary event and subtract it from 1.

**c.** $P(X \geq 2) = 1 - P(X > 2)$
$P(X \geq 2) = 1 - [P(X = 0) + P(X = 1)]$

**2.** Define and assign values to variables.

From **b**, $P(X = 0) = 0.0060$.
For $P(X = 1)$: $n = 10$
Let $X =$ number of people cured.
Therefore, $r = 1$.
$p = 0.4$
$(1 - p) = 0.6$

**3.** Substitute the values into the expression.

$P(X \geq 2) = 1 - \left[ (0.0060) + \binom{10}{1} (0.4)^1 (0.6)^9 \right]$

**4.** Evaluate.

$= 1 - [(0.0060) + (10 \times 0.4 \times 0.010\,077\,696)]$
$= 1 - [0.0060 + 0.040\,310\,784]$
$= 0.953\,642\,598$

**5.** Round off the answer to 4 decimal places.

$= 0.9536$

**6.** Answer the question.

The probability that at least 2 people are cured is 0.9536.

---

**TI | THINK**  **WRITE**

**a. 1.** On a Calculator page, select:
MENU
6: Statistics
5: Distributions
A: Binomial Pdf…

**2.** Complete the entry lines as:
n: 10
p: 0.4
X Value: 4
then press OK.

**3.** The answer appears on the screen.

binomPdf(10,0.4,4)   0.250823

**CASIO | THINK**  **WRITE**

**a. 1.** On a Statistics screen, select:
DIST
BINOMIAL
Bpd.

**2.** Select Variable by pressing the $\boxed{\text{Var}}$ button.

**3.** Complete the entry lines as:
x: 4
Numtrial: 10
p: 0.4
then press EXE.

|  | The answer appears on the screen. | Rad Norm1 d/c Real<br>Binomial P.D<br>    p=0.25082265 |

**c. 1.** On a Calculator page, select:
MENU
6: Statistics
5: Distributions
B: Binomial Cdf…

| 1 Actions | | RAD ☐ ✕ |
|---|---|---|
| | | 0.250823 |
| 5 t Cdf… | | |
| 6 Inverse t… | | |
| 7 χ² Pdf… | | |
| 8 χ² Cdf… | Stat Calculations ▸ | |
| 9 Inverse χ³… | Stat Results | |
| A Binomial Pdf… | List Math ▸ | |
| **B Binomial Cdf…** | List Operations ▸ | |
| C Inverse Binomial… | Distributions ▸ | |
| D Inverse Binomial N… | Confidence Intervals ▸ | |
| ▼ | Stat Tests ▸ | |

**2.** Complete the entry lines as:
n: 10
p: 0.4
Lower Bound: 2
Upper Bound: 10
then press the OK button.

| ◀ 1.1 ▶ | *Doc | RAD ☐ ✕ |
|---|---|---|
| bin **Binomial Cdf** | | ⟩ |
| Num Trials, n: | 10 | ▶ |
| Prob Success, p: | 0.4 | ▶ |
| Lower Bound: | 2 | ▶ |
| Upper Bound: | 10 | ▶ |
| | OK | Cancel |

**3.** The answer appears on the screen.

| ◀ 1.1 ▶ | *Doc | RAD ☐ ✕ |
|---|---|---|
| binomPdf(10,0.4,4) | | 0.250823 |
| binomCdf(10,0.4,2,10) | | 0.953643 |
| ❘ | | |

**c. 1.** On a Statistics screen, select:
DIST
BINOMIAL
Bcd.

| Rad Norm1 d/c Real | | | |
|---|---|---|---|
| List 1 | List 2 | List 3 | List 4 |
| SUB | | | |
| 1 | | | |
| 2 | | | |
| 3 | | | |
| 4 | | | |
| Bpd | Bcd | InvB | |

**2.** Complete the entry lines as:
Lower: 2
Upper: 10
Numtrial:10
p: 0.4
then press the EXE button.

Rad Norm1 d/c Real
Binomial C.D
Data   :Variable
Lower  :2
Upper  :10
Numtrial:10
p    :0.4
Save Res:None  ↓

**3.** The answer appears on the screen.

Rad Norm1 d/c Real
Binomial C.D
    p=0.95364259

---

## WORKED EXAMPLE 4 Calculating conditional probability using binomial distribution

**It is known that 52% of the population participates in sport on a regular basis. Five random individuals are interviewed and asked whether they participate in sport on a regular basis. Let $X$ be the number of people who regularly participate in sport.**
**a. Construct a probability distribution table for $X$.**
**b. Determine the probability that 3 people or fewer play sport.**
**c. Determine the probability that at least one person plays sport, given that no more than 3 people play sport.**
**d. Determine the probability that the first person interviewed plays sport but the next 2 do not.**

**THINK**

**a. 1.** Write the rule for the probabilities of the binomial distribution.

**2.** Substitute $r = 0$ into the rule and simplify.

**3.** Substitute $r = 1$ into the rule and simplify.

**WRITE**

**a.** $X \sim \text{Bi}(5, 0.52)$
$P(X = r) = {}^nC_r(1 - p)^{n-r}p^r$

$P(X = 0) = {}^5C_0(0.48)^5$
$\qquad\qquad = 0.025\,48$

$P(X = 1) = {}^5C_1(0.48)^4(0.52)$
$\qquad\qquad = 0.138\,02$

▶

**4.** Substitute $r = 2$ into the rule and simplify.

$$P(X=2) = {}^5C_2(0.48)^3(0.52)^2$$
$$= 0.299\,04$$

**5.** Substitute $r = 3$ into the rule and simplify.

$$P(X=3) = {}^5C_3(0.48)^2(0.52)^3$$
$$= 0.323\,96$$

**6.** Substitute $r = 4$ into the rule and simplify.

$$P(X=4) = {}^5C_4(0.48)(0.52)^4$$
$$= 0.175\,48$$

**7.** Substitute $r = 5$ into the rule and simplify.

$$P(X=5) = {}^5C_5(0.52)^5$$
$$= 0.038\,02$$

**8.** Construct a probability distribution table and check that $\sum\limits_{\text{all } r} P(X=r) = 1$.

| $x$ | $P(X=r)$ |
|---|---|
| 0 | 0.025 48 |
| 1 | 0.138 02 |
| 2 | 0.299 04 |
| 3 | 0.323 96 |
| 4 | 0.175 48 |
| 5 | 0.038 02 |

$$\sum\limits_{\text{all } r} P(X=r) = 1$$

**b. 1.** Interpret the question and write the probability to be found.

**b.** $P(X \le 3)$

**2.** State the probabilities included in $P(X \le 3)$.

$$P(X \le 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$$
$$P(X \le 3) = 1 - (P(X=4) + P(X=5))$$

**3.** Substitute the appropriate probabilities and evaluate.

$$P(X \le 3) = 1 - (0.175\,48 + 0.038\,02)$$
$$P(X \le 3) = 0.7865$$

**c. 1.** State the rule for conditional probability.

**c.** $P(X \ge 1 | X \le 3) = \dfrac{P(X \ge 1 \cap X \le 3)}{P(X \le 3)}$

**2.** Evaluate $P(X \ge 1 \cap X \le 3)$.

$$P(X \ge 1 \cap X \le 3) = P(1 \le X \le 3)$$
$$= P(X=1) + P(X=2) + P(X=3)$$
$$= 0.138\,02 + 0.299\,04 + 0.323\,96$$
$$= 0.761\,02$$

**3.** Substitute the appropriate values into the rule.

$$P(X \ge 1 | X \le 3) = \dfrac{P(X \ge 1 \cap X \le 3)}{P(X \le 3)}$$
$$= \dfrac{0.761\,02}{0.7865}$$

**4.** Simplify.

$$P(X \ge 1 | X \le 3) = 0.9676$$

**d. 1.** Order has been specified for this question. Therefore, the binomial probability distribution rule cannot be used. The probabilities must be multiplied together in order.

**d.** $S = $ plays sport, $N = $ doesn't play sport
$$P(SNN) = P(S) \times P(N) \times P(N)$$

**2.** Substitute the appropriate values and evaluate.

$$P(SNN) = 0.52 \times 0.48 \times 0.48$$
$$= 0.1198$$

---

## WORKED EXAMPLE 5 Calculating $n$ given $p$ for a binomial distribution

The probability of an Olympic archer hitting the centre of a target 80 metres away is 0.7. Determine the smallest number of arrows he must shoot to ensure that the probability he hits the centre at least once is more than 0.9.

**THINK**

**1.** Write the rule for the probabilities of the binomial distribution.

**2.** The upper limit of successes is unknown, because $n$ is unknown. Therefore, $P(X \geq 1)$ cannot be found by adding up the probabilities. However, the required probability can be found by subtracting from 1 the only probability not included in $P(X \geq 1)$.

**3.** Substitute in the appropriate values and simplify.

**4.** Rearrange and take the log of both sides to determine the value of $n$.

**5.** Interpret the result and answer the question.

**WRITE**

$X \sim \text{Bi}(n, 0.7)$
$P(X \geq 1) > 0.9$

$P(X \geq 1) = 1 - P(X = 0)$

$P(X \geq 1) = 1 - P(X = 0)$
$1 - P(X = 0) > 0.9$
$1 - {}^n C_x (1-p)^{n-r} p^r > 0.9$
$1 - {}^n C_0 (0.3)^n (0.7)^0 > 0.9$
$1 - 1 \times (0.3)^n \times 1 > 0.9$
$1 - (0.3)^n > 0.9$

$1 - 0.9 > (0.3)^n$
$\log_{10}(0.1) > \log_{10}(0.3)^n$
$\log_{10}(0.1) > n \log_{10}(0.3)$
$n > \dfrac{\log_{10}(0.1)}{\log_{10}(0.3)}$
$n > 1.91249$

$n = 2$ (as $n$ must be an integer). The smallest number of arrows the archer needs to shoot in order to guarantee a probability of 0.9 of hitting the centre is 2.

# Exercise 7.3 Binomial distributions

**7.3 Exercise**    **7.3 Exam questions** on

| Simple familiar | Complex familiar | Complex unfamiliar |
|---|---|---|
| 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 | 13, 14, 15, 16 | 17, 18, 19, 20 |

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### Simple familiar

1. Identify which of the following constitute binomial probability distributions.
   a. Rolling a die 10 times and recording the number that comes up
   b. Rolling a die 10 times and recording the number of 3s that come up
   c. Spinning a spinner numbered 1 to 10 and recording the number that is obtained
   d. Tossing a coin 15 times and recording the number of Tails obtained
   e. Drawing a card from a fair deck, without replacement, and recording the number of picture cards
   f. Drawing a card from a fair deck, with replacement, and recording the number of black cards
   g. Selecting 3 marbles from a jar containing 3 yellow marbles and 2 black marbles, without replacement

2. Seven per cent of items made by a certain machine are defective. The items are packed and sold in boxes of 50. Determine the probability of 5 items being defective in a box.

3. Nadia has a 40% chance of getting a red light on the way to work. Determine the probability of Nadia getting a red light on 4 out of 5 days.

4. Peter has 4 chances to knock an empty can off a stand by throwing a ball. On each throw, the probability of success is $\frac{1}{5}$. Determine the probability that he will knock the empty can off the stand:
   a. once
   b. twice
   c. at least once.

5. **WE3** Fifty-five per cent of the local municipality support the local council. If 8 people are selected at random, determine the probability that:
   a. half support the council
   b. all 8 support the council
   c. 5 support the council
   d. 3 oppose the council.

6. The probability of Colin beating Maria at golf is 0.4. If they play once a week throughout the entire year and the outcome of each game is independent of any other, determine the probability that they will have won the same number of matches.

7. It is known that 5 out of every 8 people eat Superflakes for breakfast. Determine the probability that half of a random sample of 20 people surveyed eat Superflakes.

8. On a certain evening, during a ratings period, two television stations put their best shows on against each other. The ratings showed that 39% of people watched Channel 6, while only 30% of people watched Channel 8. The rest watched other channels. A random sample of 10 people were surveyed the next day. Determine the probability that exactly:
   a. 6 watched Channel 6
   b. 4 watched Channel 8.

9. **WE4** Jack is an enthusiastic darts player and on average is capable of achieving a bullseye 3 out of 7 times. Jack will compete in a five-round tournament. Let $Y$ be the discrete random variable that defines the number of bullseyes Jack achieves.

   a. Construct a probability distribution table for $Y$, giving your answers correct to 4 decimal places.
   b. Determine the probability that Jack will score at most 3 bullseyes.
   c. Determine the probability that Jack will score 1 or more bullseyes, given that he scored at most 3 bullseyes.
   d. Determine the probability that his first shot missed, his second shot was a bullseye and then his next 2 shots missed.

10. At a poultry farm, eggs are collected daily and classified as large or medium. Then they are packed into cartons containing 12 eggs of the same classification. From experience, the director of the poultry farm knows that 42% of all eggs produced at the farm are considered to be large. Ten eggs are randomly chosen from a conveyor belt on which the eggs are to be classified. Let $Z$ be the discrete random variable that gives the number of large eggs.

   a. Determine $P(Z=0)$, $P(Z=1)$ … $P(Z=9)$, $P(Z=10)$ for this binomial distribution.
   b. Construct a probability distribution table for $Z$.
   c. Determine $P(Z \geq 5 | Z \leq 8)$.

11. If $X$ has a binomial distribution so that $n = 15$ and $p = 0.62$, calculate:

   a. $P(X=10)$          b. $P(X \geq 10)$          c. $P(X < 4 | X \leq 8)$

12. The tram that stops outside Maia's house is late 20% of the time. If there are 12 times during the day that the tram stops outside Maia's house, calculate, correct to 4 decimal places:

   a. the probability that the tram is late 3 times
   b. the probability that the tram is late 3 times for at least 6 out of the next 14 days.

   **Complex familiar**

13. **WE 5** The probability of winning a prize in a particular competition is 0.2. Determine how many tickets someone would need to buy in order to guarantee them a probability of at least 0.85 of winning at least one prize.

14. Lizzie and Matt enjoy playing card games. The probability that Lizzie will beat Matt is 0.67. Determine how many games do they need to play so that the probability of Matt winning at least one game is more than 0.9.

15. A particular medication used by asthma sufferers has been found to be beneficial if used 3 times a day. In a trial of the medication it was found to be successful in 63% of the cases. Eight random asthma sufferers have had the medication prescribed for them.

   a. Construct a probability distribution table for the number of sufferers who have benefits from the medication, $X$.
   b. Determine the probability that no more than 7 people will benefit from the medication.
   c. Determine the probability that at least 3 people will benefit from the medication, given that no more than 7 will.
   d. Determine the probability that the first person won't benefit from the medication, but the next 5 will.

**16.** Lilly knows that the chance of her scoring a goal during a basketball game is 0.75. Determine the least number of shots that Lilly must attempt to ensure that the probability of her scoring at least 1 goal in a match is more than 0.95.

**17.** A competition is being run by Big Bun Burgers in which every purchase made by a customer gives them an entry into a prize draw.
The chance of winning each prize is given in the table.

| Prize | Chance of winning |
| --- | --- |
| Free Banger burger | 1 in 100 |
| Big Bun Burger t-shirt | 1 in 500 |
| Movie tickets | 1 in 2 000 |
| Gaming console | 1 in 10 000 |
| Sports car | 1 in 2 000 000 |

Determine the probability that you will win at least one prize if you make 10 purchases during the competition period.

**18.** In the game of Bucket Bean, throwers aim to get a bag filled with beans to land in a bucket set up several metres away. A Three Toss game is won if the player makes at least two successful throws out of three attempts. A Four Toss game is won if the player makes at least three successful throws out of four attempts. Khoo is twice as likely to win a Three Toss game than he is a Four Toss game. Calculate Khoo's probability of making a successful throw. (Round your answer to 4 decimal places.)

**19.** In a team ultramarathon, teams of four or six runners score points equal to the number of runners who complete the race. However, if more than half of the team does not complete the race, no points are scored by the team at all.
Assuming that each runner in a team has an equal probability $p$ of completing the race, determine the range of values that $p$ can take if a four-person team is more likely to score points than a six-person team.

**20.** In a ceramics factory, a new technique is being tested for making porcelain vases. A random sample of 40 vases is selected from a large batch of the new vases and each is inspected for flaws. If the sample has fewer than 2 flawed vases, the batch is accepted and moves on to the glazing section. If the sample contains more than 2 flawed vases, the batch is rejected and sent to be broken up into porcelain grit.
If the sample contains exactly 2 flawed vases, a different random sample of 20 vases is selected from the batch and checked for flaws. If this second sample contains no flawed vases, the batch will be accepted.
If it is known that 5% of the vases will be flawed using this new technique, calculate the probability that a batch will be rejected. Give your answer correct to 4 decimal places.

**Fully worked solutions for this chapter are available online.**

# LESSON
## 7.4 The mean and variance of a binomial distribution

### ✦ 7.4.1 Mean and variance

int-6432

When we are working with a binomial probability distribution, it is very useful to know the mean (or expected value), the variance and the standard deviation. These values are calculated the same way as for other probability distributions that you would have encountered in Year 11.

For a binomial distribution $X \sim \text{Bi}(n, p)$, the mean is calculated simply from the following equation:

$$\mu = \text{E}(X) = np$$

The variance of the binomial distribution $X \sim \text{Bi}(n, p)$ is calculated using the equation

$$\sigma^2 = \text{Var}(X) = np(1-p).$$

Therefore, as the standard deviation $\sigma = \text{SD}(X) = \sqrt{\text{Var}(X)}$, $\sigma = \text{SD}(X) = \sqrt{np(1-p)}$.

---

**Describing a binomial probability distribution**

For a binomial distribution $X \sim \text{Bi}(n, \ p)$,

$$\mu = \text{E}(X) = np$$

$$\sigma^2 = \text{Var}(X) = np(1-p)$$

$$\sigma = \text{SD}(X) = \sqrt{np(1-p)}$$

---

**WORKED EXAMPLE 6 Calculating mean, variance and standard deviation for a binomial distribution**

a. **A test consists of 20 multiple choice questions, each with 5 alternatives for the answer. A student has not studied for the test, so she chooses the answers at random. Let $X$ be the discrete random variable that describes the number of correct answers.**
   i. **Determine the expected number of correct questions answered.**
   ii. **Determine the standard deviation of the correct number of questions answered, correct to 4 decimal places.**

▶

**b.** **A binomial random variable, $Z$, has a mean of 8.4 and a variance of 3.696.**

   **i.** **Determine the probability of success, $p$.**

   **ii.** **Determine the number of trials, $n$.**

| THINK | WRITE |
|---|---|
| **a. i.** **1.** Write the rule for the expected value. | **a. i.** $E(X) = np$ |
|    **2.** Substitute the appropriate values and simplify. | $n = 20, p = \dfrac{1}{5}$<br>$E(X) = np$<br>$\quad = 20 \times \dfrac{1}{5}$<br>$\quad = 4$ |
|    **3.** Write the answer. | The expected number of questions correct is 4. |
| **ii.** **1.** Write the rule for the variance. | **ii.** $Var(X) = np(1-p)$ |
|    **2.** Substitute the appropriate values and evaluate. | $Var(X) = 20 \times \dfrac{1}{5} \times \dfrac{4}{5}$<br>$\quad = \dfrac{16}{5}$<br>$\quad = 3.2$ |
|    **3.** Write the rule for the standard deviation. | $SD(X) = \sqrt{Var(X)}$ |
|    **4.** Substitute the variance and evaluate. | $\sigma = \sqrt{\dfrac{16}{5}}$<br>$\quad = 1.7889$ |
| **b. i.** **1.** Write the rules for the variance and expected value. | **b. i.** $E(Z) = np$<br>$Var(Z) = np(1-p)$ |
|    **2.** Substitute the known information and label the two equations. | $8.4 = np \qquad\qquad\qquad [1]$<br>$Var(Z) = np(1-p) \quad [2]$<br>$3.696 = np(1-p)$ |
|    **3.** To cancel out the $n$, divide equation [2] by equation [1]. | $[2] \div [1]: \dfrac{np(1-p)}{np} = \dfrac{3.696}{8.4}$ |
|    **4.** Simplify. | $1 - p = 0.44$<br>$\quad p = 0.56$ |
|    **5.** Write the answer. | The probability of success is 0.56. |
| **ii.** **1.** Substitute $p = 0.56$ into $E(Z) = np$ and solve for $n$. | **ii.** $E(Z) = np$<br>$8.4 = n \times 0.56$<br>$\quad n = 15$ |
|    **2.** Write the answer. | There are 15 trials. |

# Exercise 7.4 The mean and variance of a binomial distribution learnon

**7.4 Exercise** | **7.4 Exam questions** on

| Simple familiar | Complex familiar | Complex unfamiliar |
|---|---|---|
| 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 | 13, 14 | 15, 16, 17, 18 |

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### Simple familiar

1. **WE6a** A fair die is tossed 25 times. Let $X$ be the discrete random variable that represents the number of ones achieved. Determine, correct to 4 decimal places:
   a. the expected number of ones achieved
   b. the standard deviation of the number of ones achieved.

2. **WE6b** A binomial random variable, $Z$, has a mean of 32.535 and a variance of 9.021 95.
   a. Determine the probability of success, $p$.
   b. Determine the number of trials, $n$.

3. A fair coin is tossed 10 times. Determine:
   a. the expected number of Heads
   b. the variance for the number of Heads
   c. the standard deviation for the number of Heads.

4. A card is selected at random from a standard playing pack of 52 and then replaced. This procedure is completed 20 times. Determine:
   a. the expected number of picture cards
   b. the variance for the number of picture cards
   c. the standard deviation for the number of picture cards.

5. Six out of every 10 cars manufactured are white. Twenty cars are randomly selected. Calculate:
   a. the expected number of white cars
   b. the variance for the number of white cars
   c. the standard deviation for the number of white cars.

6. A fair die is rolled 10 times. Determine:
   a. the expected number of 2s rolled
   b. the probability of obtaining more than the expected number of 2s.

7. Eighty per cent of rabbits that contract a certain disease will die. If a group of 120 rabbits contract the disease, determine how many would you expect to:
   a. die
   b. live.

8. A binomial random variable has a mean of 10 and a variance of 5. Determine:
   a. the probability of success, $p$
   b. the number of trials, $n$.

9. A binomial random variable has a mean of 12 and a variance of 3. Determine:
   a. the probability of success, $p$
   b. the number of trials, $n$.

CHAPTER 7 Bernoulli and binomial distributions **335**

**10.** For each of the following binomial random variables, calculate:

   **i.** the expected value

   **ii.** the variance.

     **a.** $X \sim \text{Bi}(45, 0.72)$

     **b.** $Y \sim \text{Bi}\left(100, \dfrac{1}{5}\right)$

     **c.** $Z \sim \text{Bi}\left(72, \dfrac{2}{9}\right)$

**11.** Four per cent of pens made at a certain factory do not work. If pens are sold in boxes of 25, determine the probability that a box contains more than the expected number of faulty pens.

**12.** A statistician estimates the probability that a spectator at a Brisbane Lions versus Collingwood AFL match barracks for Brisbane is $\dfrac{1}{2}$. At an AFL grand final between these two teams there are 100 000 spectators. Determine:

   **a.** the expected number of Brisbane supporters

   **b.** the variance of the number of Brisbane supporters

   **c.** the standard deviation of the number of Brisbane supporters.

### Complex familiar

**13.** Thirty children are given 5 different yoghurts to try. The yoghurts are marked A to E, and each child has to select his or her preferred yoghurt. Each child is equally likely to select any brand. The company running the tests manufactures yoghurt B.

Analyse whether the tests indicate that half of the children selected yoghurt B as their favourite, and justify your response with calculations.

**14.** A binomial experiment is completed 16 times and has an expected value of 10.16. Determine the standard deviation of the experimental results.

### Complex unfamiliar

**15.** A factory manufactures 2500 DVDs each day. Two per cent of these DVDs are faulty. Each day, the quality control department tests the DVDs and destroys all of the faulty DVDs.

If it costs $1.20 to manufacture a DVD and the manufacturer sells the non-faulty DVDs for $12 each, determine the expected profit made by the manufacturer each day.

**16.** A discrete random variable $X$ can be modelled by a binomial distribution. Given that $n = 30$ and $\text{Var}(X) = 3.15$, determine the value of $p$ for this distribution.

**17.** A farmer is testing the 420 cows in his dairy herd for cowpox. The first group are tested and the variance in the number of cows with cowpox is 4.224. When the remainder of the herd are tested, the variance in the number who are positive for cowpox is 40.128. Assuming that all of the cows have the same probability of testing positive for cowpox, determine the probability of a cow having cowpox and the number of cows in the first group tested.

**18.** In a binomial distribution, $\text{E}(X) = \dfrac{3}{4}$ and $\text{Var}(X) = \dfrac{5}{9}$. Determine the probability that $X = 2$.

**Fully worked solutions for this chapter are available online.**

# LESSON
## 7.5 Modelling and solving with Bernoulli and binomial distributions

The binomial distribution has important applications in medical research, quality control, simulation and genetics. In this section we will explore some of these areas.

### WORKED EXAMPLE 7 Determining the probability of diabetes in a population sample

**It has been found that 9% of the population have diabetes. A sample of 15 people were tested for diabetes. Let $X$ be the random variable that gives the number of people who have diabetes.**

**a. Determine $P(X \leq 5)$.**

**b. Determine $E(X)$ and $SD(X)$.**

| THINK | WRITE |
|---|---|
| a. 1. Define and assign values to variables. | a. $n = 15$<br>Let $X =$ number of people who have diabetes.<br>Therefore, $0 \leq r \leq 5$.<br>$p = 0.09$<br>$(1 - p) = 0.91$ |
| 2. Substitute the values into the rule. | $P(X \leq 5) = \sum_{r=0}^{5} \binom{15}{r} (0.09)^r (0.91)^{15-r}$ |
| 3. Evaluate by using the binomial expansion. | $= \binom{15}{0} (0.91)^{15} + \binom{15}{1} (0.09)^1 (0.91)^{14}$<br><br>$+ \binom{15}{2} (0.09)^2 (0.91)^{13} + \binom{15}{3} (0.09)^3 (0.91)^{12}$<br><br>$+ \binom{15}{4} (0.09)^4 (0.91)^{11} + \binom{15}{5} (0.09)^5 (0.91)^{10}$<br><br>$= 0.243\,008 + 0.360\,507 + 0.249\,582 + 0.106\,964$<br>$\quad + 0.031\,736 + 0.006\,905$<br>$= 0.998\,702$<br>$= 0.9987$ |

Alternatively, calculate the binomial cdf using the distribution function of your graphics calculator.

| 1.1 ▶ | *Doc | RAD ⊡ ✕ |
|---|---|---|
| binomCdf(15,0.09,0,5) | | 0.998701545398 |

4. Round the answer off to 4 decimal places.

    The probability that 5 or fewer people of the 15 selected have diabetes is 0.9987.

5. Answer the question.

b. 1. State the rule for the expected value.

    b. $E(X) = np$

2. Substitute the appropriate values and simplify.

$$E(X) = 15 \times 0.09$$
$$= 1.35$$

3. Determine the variance.

$$Var(X) = np(1-9)$$
$$= 15 \times 0.09 \times 0.91$$
$$= 1.2285$$

4. Determine the standard deviation.

$$SD(X) = \sqrt{Var(X)}$$
$$= \sqrt{1.2285}$$
$$= 1.1084$$
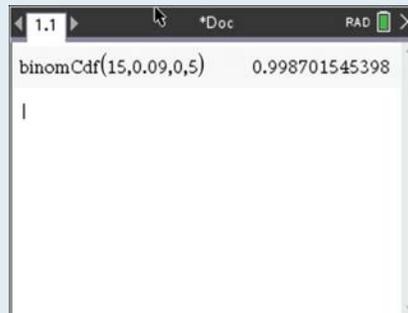
## Exercise 7.5 Modelling and solving with Bernoulli and binomial distributions

**learn** on

| 7.5 Exercise | 7.5 Exam questions   on |
|---|---|

| Simple familiar | Complex familiar | Complex unfamiliar |
|---|---|---|
| 1, 2, 3, 4, 5, 6 | 7, 8, 9, 10, 11, 12, 13 | 14, 15, 16 |

These questions are even better in jacPLUS!
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### Simple familiar

1. The most common type of colour blindness — congenital red-green colour blindness — affects 8% of males but only 0.5% of females in the human population. In a Year 12 class of 58 males and 56 females, determine how many more colour-blind males than colour-blind females would be expected. (Round your answer up to the nearest integer value.)

2. Approximately 7.5% of stars in the Milky Way galaxy are G-type stars like our sun.
If there are 1.7 billion stars within our section of the galaxy, calculate the expected number, variance and standard deviation of G-type stars in this region. Give your answer correct to 2 decimal places using scientific notation. (This assumes that the distribution of G-type stars is uniform throughout the Milky Way.)

3. **WE7** It is thought that about 30% of teenagers receive their spending money from part-time jobs. Ten random teenagers were interviewed about their spending money and how they obtained it. Let $Y$ be the random variable that defines the number of teenagers who obtain their spending money by having a part-time job.

   **a.** Determine $P(Y \geq 7)$.
   **b.** Determine $E(Y)$ and $SD(Y)$.

4. In Australia, it is estimated that 30% of the population over the age of 25 have hypertension. A statistician wishes to investigate this, so he arranges for 15 random adults over the age of 25 to be tested to see if they have high blood pressure. Let $X$ be the random variable that defines the number of adults over the age of 25 with hypertension.

   **a.** Determine $P(X \leq 5)$.
   **b.** Determine $E(X)$.
   **c.** Determine $SD(X)$.

5. Consider a woman with the genotype XX and a man with the genotype XY. Their offspring have an equal chance of inheriting one of these genotypes. Determine the probability that 6 of their 7 offspring have the genotype XY.
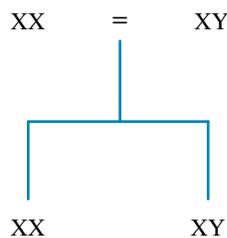


6. Silicon chips are tested at the completion of the fabrication process. Chips either pass or fail the inspection, and if they fail they are destroyed. The probability that a chip fails an inspection is 0.02. Determine the probability that in a manufacturing run of 250 chips, only 7 will fail the inspection.
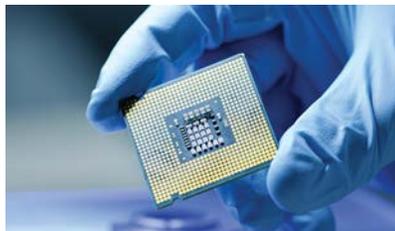


**Complex familiar**

7. The proportion of defective fuses made by a certain company is 0.02. A sample of 30 fuses is taken for quality control inspection.

   **a.** Determine the probability that there are no defective fuses in the sample.
   **b.** Determine the probability that there is only 1 defective fuse in the sample.
   **c.** Determine how many defective fuses you would expect in the sample.
   **d.** The hardware chain that sells the fuses will accept the latest batch for sale only if, upon inspection, there is at most 1 defective fuse in the sample of 30. Determine the probability that they accept the batch.
   **e.** Ten quality control inspections are conducted monthly for the hardware chain. Determine the probability that all of these inspections will result in acceptable batches.

8. Suppose that 85% of adults with allergies report systematic relief with a new medication that has just been released. The medication has just been given to 12 patients who suffer from allergies. Let $Z$ be the discrete random variable that defines the number of patients who get systematic relief from allergies with the new medication.

   a. Determine the probability that no more than 8 people get relief from allergies.
   b. Given that no more than 8 people get relief from allergies after taking the medication, determine the probability that at least 5 people do.
   c. Calculate:

       i. E($Z$)                                            ii. SD($Z$)

9. A manufacturer of electric kettles has a process of randomly testing the kettles as they leave the assembly line to see if they are defective. For every 50 kettles produced, 3 are selected and tested for any defects. Let $X$ be the binomial random variable that is the number of kettles that are defective, so that $X \sim \text{Bi}(3, p)$.

   a. Construct a probability distribution table for $X$, giving your probabilities in terms of $p$.
   b. Assuming $P(X = 0) = P(X = 1)$, determine the value of $p$ where $0 < p < 1$.
   c. Determine:

       i. $\mu$                                               ii. $\sigma$

10. The probability of a person in Australia suffering anaemia is 1.3%. A group of 100 different Australians of differing ages were tested for anaemia.

    a. Determine the probability that at least 5 of the 100 Australians suffer from anaemia. Give your answer correct to 4 decimal places.
    b. Determine the probability that 4 of the 100 Australians suffer from anaemia, given that less than 10 do. Give your answer correct to 4 decimal places.

11. Edie is completing a multiple choice test of 20 questions. Each question has 5 possible answers.

    a. If Edie randomly guesses every question, determine the probability, correct to 4 decimal places, that she correctly answers 10 or more questions.
    b. If Edie knows the answers to the first 4 questions but must randomly guess the answers to the other questions, determine the probability that she correctly answers a total of 10 or more questions. Give your answer correct to 4 decimal places.

12. Six footballers are chosen at random and asked to kick a football. The probability of a footballer being able to kick at least 50 m is 0.7.

    a. Determine the probability, correct to 4 decimal places, that:
        i. only the first three footballers chosen kick the ball at least 50 m
        ii. exactly three of the footballers chosen kick the ball at least 50 m
        iii. at least three of the footballers chosen kick the ball at least 50 m, given that the first footballer chosen kicks it at least 50 m.

    b. Determine the minimum number of footballers required to ensure that the probability that at least one of them can kick the ball 50 m is at least 0.95.

13. Lori is a goal shooter for her netball team. The probability of her scoring a goal is 0.85. In one particular game, Lori had 12 shots at goal. Determine the probability, correct to 4 decimal places, that:

    a. she scored at least 9 goals
    b. only her last 9 shots were goals
    c. she scored exactly 10 goals, given that her last 9 shots were goals.

**14.** The chance of winning a prize in the local raffle is 0.08. Determine the least number of tickets Siena needs to purchase so that the chance of both her and her sister each winning at least one prize is more than 0.8.

**15.** A regional community is trying to ensure that their local water supply has fluoride added to it, as a medical officer found that a large number of children aged between 8 and 12 have at least one filling in their teeth. In order to push their cause, the community representatives have asked a local dentist to check the teeth of ten 8–12-year-old children from the community.

Let $X$ be the binomial random variable that defines the number of 8–12-year-old children who have at least one filling in their teeth: $X \sim \text{Bi}(10, p)$. Determine the value of $p$, correct to 4 decimal places, if $P(X \le 8) = 0.9$.

**16.** Capture and recapture is a method to estimate populations by tagging organisms and later recapturing some to see how many are tagged. A biologist tags possums in a 1-km$^2$ bushland area. The next day, she finds 10 more possums than on the first day, with 5 tagged. She estimates that the number of possums tagged on day 1 was one-eighth of the total population. Calculate her estimate for the total possum population.

**Fully worked solutions for this chapter are available online.**

# LESSON
## 7.6 Review

### 7.6.1 Summary

Hey students! Now that it's time to revise this topic, go online to:

- Access the topic summary
- Review your results
- Watch teacher-led videos
- Practise exam questions

**Find all this and MORE in jacPLUS**

### 7.6 Exercise

**learnon**

| 7.6 Exercise | 7.6 QCAA questions **on** |
|---|---|

| Simple familiar | Complex familiar | Complex unfamiliar |
|---|---|---|
| 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 | 13, 14, 15, 16 | 17, 18, 19, 20 |

These questions are even better in jacPLUS!
- Receive immediate feedback
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#### Simple familiar

1. **MC** Identify which of the following does not represent a binomial distribution.
   - **A.** Rolling a die four times and recording the number of 2s
   - **B.** Tossing a coin 10 times and recording the number of Heads
   - **C.** Rolling two dice simultaneously 20 times and recording the outcomes
   - **D.** Drawing a card with replacement and recording the number of aces obtained

2. **MC** A Bernoulli random variable, $X$, has a probability of failure of 0.35. The expected value and variance of $X$ are respectively:
   - **A.** 0.35 and 0.2275
   - **B.** 0.35 and 0.65
   - **C.** 0.65 and 0.2275
   - **D.** 0.65 and 0.35

3. **MC** Suppose that $X$ is a binomial random variable with a mean of 12 and a standard deviation of 3. The probability of success, $p$, in any trial is:
   - **A.** 0.25
   - **B.** 0.35
   - **C.** 0.50
   - **D.** 0.75

4. **MC** The probability that the 7:35 am bus arrives on time is 0.45. Determine the probability that the bus is on time at least once in the next 5 days.

   A. $1 - (0.55)^5$
   B. $(0.55)^5$
   C. $(0.45)^5$
   D. $1 - (0.45)^5$

5. **MC** If $X$ is a binomial random variable with $n = 15$ and $p = \dfrac{1}{5}$, the mean and variance of $X$ are closest to:

   A. $\mu = 3,\ \sigma^2 = 12$
   B. $\mu = 12,\ \sigma^2 = 2.4$
   C. $\mu = 2.4,\ \sigma^2 = 12$
   D. $\mu = 3,\ \sigma^2 = 2.4$

6. One-quarter of all customers at a particular bookstore buy non-fiction books. If 5 customers purchase a book on a particular day, determine the probability that 3 of them purchased a non-fiction book.

7. One in every 100 new cars is returned with faulty steering. A survey is taken of 300 buyers. Determine the probability that:

   a. none have cars with faulty steering
   b. one has a car with faulty steering.

8. A binomial random variable has a mean of 10 and variance of 8. Determine:

   a. the probability of success, $p$
   b. the number of trials, $n$.

9. Six out of every 10 cars manufactured are white. If 20 cars are selected at random, determine how many you would expect to be white.

10. A random variable $X$ follows a Bernoulli sequence with a probability of success of 0.86. Determine the variance of $X$ correct to 2 decimal places.

11. A coin is biased such that the probability of a tail is 0.7. Determine the probability that at most 1 Tail will be observed when the coin is tossed 5 times.

12. A binomial experiment is completed 16 times and has an expected value of 10.16.

   a. Determine the probability of success, $p$.
   b. Determine the variance and the standard deviation.

**Complex familiar**

13. Five per cent of watches made at a certain factory are defective. Watches are sold to retailers in boxes of 20. Determine:

   a. the expected number of defective watches in each box
   b. the probability that a box contains more than the expected number of defective watches per box
   c. the probability of a 'bad batch', if a 'bad batch' entails more than a quarter of the box being defective.

**14.** Ten per cent of all Olympic athletes are tested for drugs at the conclusion of their event. One per cent of all athletes use performance-enhancing drugs. Of the 1000 Olympic wrestlers competing from all over the world, Australia sends 10. Determine:

   **a.** the expected number of Australian wrestlers who are tested for drugs

   **b.** the probability that half the Australian wrestlers are tested for drugs

   **c.** the probability that at least 2 Australian wrestlers are tested for drugs

   **d.** the expected number of drug users among all Olympic wrestlers.

**15.** One-fifth of Australia's population has a British background. Fifty Australians are randomly selected and questioned about their ancestry. Determine (correct to 4 decimal places) the probability that at least 96% of the selected people have a non-British background.

**16.** Speedy Saverio's Pizza House claims to cook and deliver 90% of pizzas within 15 minutes of the order being placed. If your pizza is not delivered within this time, it is free. On one busy Saturday night, Saverio has to make 150 deliveries.

   **a.** Determine how many deliveries are expected to be made within 15 minutes of placing the order.

   **b.** Determine the probability of receiving a free pizza on this night.

   **c.** If Saverio loses an average of $4 for every late delivery, determine how much he would expect to lose on late deliveries this night.

---

**Complex unfamiliar**

**17.** An experiment consists of 3 independent trials. Each trial results in a success or failure. The probability of success in a trial is $p$. Determine in terms of $p$ the probability of exactly 1 success given at least 1 success.

**18.** Keepers at a zoo are concerned that their herd of 10 giraffes are low in iron. In order to investigate this, they ask the zoo vet to take blood samples from all the giraffes to check the iron levels. Let $X$ be the binomial random variable that defines the number of giraffes that have low iron levels. For this distribution, $X \sim \text{Bi}(10, p)$. Determine the value of $p$, correct to 4 decimal places, if $P(X \le 8) = 0.9$.

**19.** While on holiday at the Gold Coast, Jordan and Bronte play a total of $n$ games of mini-golf. The probability that Jordan wins any game is 0.15. Determine how many games of mini-golf must they play if the probability of Jordan winning exactly 2 games is 0.2759.

**20.** A barrel contains 100 balls, some of which have a stripe painted on them. Five balls are randomly selected from the barrel with replacement after each ball has been withdrawn. Let $p$ be the proportion of striped balls in the barrel such that $0 < p < 1$. Using technology, determine the value of $p$ for which the probability that exactly 1 of the 5 balls chosen has a stripe will be greatest.

# 7.6 Past QCAA exam questions

## Question 1 (1 mark)

*Source:* QCAA 2023 Mathematical Methods, Paper 1, Section 1, Q3; © State of Queensland (QCAA) 2024.

A bag contains 10 buttons of the same shape and size in different colours: 5 blue, 3 green and 2 red.

If 3 buttons are randomly drawn from the bag, which probability can be calculated using the binomial distribution?

**A.** $P(3$ green$)$ with replacement
**B.** $P(3$ blue$)$ without replacement
**C.** $P(2$ green and 1 red$)$ with replacement
**D.** $P(2$ red and 1 blue$)$ without replacement

## Question 2 (1 mark)

*Source:* QCAA 2023 Mathematical Methods, Paper 2, Section 1, Q2; © State of Queensland (QCAA) 2024.

The probability of hitting a bullseye on a standard dartboard is 1 in 1250. What is the probability of hitting a bullseye exactly once in 10 attempts?

**A.** $\binom{9}{1}\left(\frac{1}{1250}\right)^1 \times \left(\frac{1249}{1250}\right)^9$

**B.** $\binom{9}{1}\left(\frac{1}{1250}\right)^9 \times \left(\frac{1249}{1250}\right)^1$

**C.** $\binom{10}{1}\left(\frac{1}{1250}\right)^1 \times \left(\frac{1249}{1250}\right)^9$

**D.** $\binom{10}{1}\left(\frac{1}{1250}\right)^9 \times \left(\frac{1249}{1250}\right)^1$

## Question 3 (1 mark)

*Source:* QCAA 2023 Mathematical Methods, Paper 2, Section 1, Q7; © State of Queensland (QCAA) 2024.

The distribution of a certain sample proportion has a mean of 0.70 and a standard deviation of 0.02.

Determine the sample size.
**A.** 525
**B.** 750
**C.** 1750
**D.** 2500

**Question 4 (4 marks)**

*Source:* QCAA 2023 Mathematical Methods, Paper 1, Section 2, Q11; © State of Queensland (QCAA) 2024.

Two random samples (A and B) were obtained using two different Bernoulli experiments. Each Bernoulli trial in the random samples was recorded as 1 (for success) or 0 (for failure). The results are shown.

| **A** | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 1 |
|-------|---|---|---|---|---|---|---|---|---|---|
| **B** | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 0 |

In sample A, for each trial the mean is $0.8$ and the variance is $0.16$.

**a.** Use the sample B results to determine the mean and variance for each trial in sample B. **[2 marks]**

**b.** Compare the variability about the means of samples A and B. **[2 marks]**

**Question 5 (5 marks)**

*Source:* QCAA 2023 Mathematical Methods, Paper 1, Section 2, Q13; © State of Queensland (QCAA) 2024.

At a certain airport, the departure of one in five international flights is delayed every day. The status of any flight is independent of other flights.

One international flight is selected at random each day for three days. Each selection is recorded as either 'delayed' or 'not delayed'.

**a.** State two conditions that make this context suitable for modelling using a binomial random variable. **[2 marks]**

**b.** Calculate the probability that at least two of the selected flights were delayed. **[3 marks]**

**Fully worked solutions for this chapter are available online.**

# Answers

## Chapter 7 Bernoulli and binomial distributions

### 7.2 Bernoulli distributions

#### 7.2 Exercise

**1. b** and **c** are Bernoulli trials.

**2. a**, **b** and **d** are Bernoulli trials.

**3. a.** The probability of success is not able to be calculated (as Paul has not replaced the ball he drew).

  **b.** There are more than two outcomes.

  **c.** The probability of success is not able to be calculated.

**4. a.**

| $x$ | 0 | 1 |
|---|---|---|
| $P(X=x)$ | 0.58 | 0.42 |

  **b.** 0.42

  **c. i.** 0.2436     **ii.** 0.4936

**5. a.** 0.63    **b.** 0.2331    **c.** 0.4828

**6. a.**

| $y$ | 0 | 1 |
|---|---|---|
| $P(Y=y)$ | 0.32 | 0.68 |

  **b. i.** 0.68    **ii.** 0.2176    **iii.** 0.4665

**7. a.**

| $x$ | 0 | 1 |
|---|---|---|
| $P(X=x)$ | 0.11 | 0.89 |

  **b. i.** $E(X) = 0.89$

   **ii.** $Var(X) = 0.0979$

   **iii.** $SD(X) = 0.3129$

**8. a.** 0.0072

  **b.**

| $z$ | 0 | 1 |
|---|---|---|
| $P(Z=z)$ | 0.9928 | 0.0072 |

**9. a.**

| $y$ | 0 | 1 |
|---|---|---|
| $P(Y=y)$ | 0.67 | 0.33 |

  **b.** 0.33

**10. a.** 0.7    **b.** 0.7

**11. a.** 0.2436    **b.** 0.58    **c.** 0.58

**12. a.** 0.8775

  **b.**

| $z$ | 0 | 1 |
|---|---|---|
| $P(Z=z)$ | 0.1225 | 0.8775 |

  **c.** 0.8775

**13. a.** 0.11    **b.** 0.8742

**14.** 25 jellybeans

### 7.3 Binomial distributions

#### 7.3 Exercise

**1. b**, **d** and **f** are binomial; **a**, **c**, **e** and **g** are not binomial.

**2.** 0.1359

**3.** 0.0768

**4. a.** $\dfrac{256}{625}$    **b.** $\dfrac{96}{625}$    **c.** $\dfrac{369}{625}$

**5. a.** 0.2627      **b.** 0.0084
  **c.** 0.2568      **d.** 0.2568

**6.** 0.0381

**7.** 0.0924

**8. a.** 0.1023    **b.** 0.2001

**9. a.** See the table at the bottom of the page.*

  **b.** 0.8891

  **c.** 0.9315

  **d.** 0.0800

**10. a**, **b.** See the table at the bottom of the page.*

  **c.** 0.4164

**11. a.** 0.1997    **b.** 0.4665    **c.** 0.0034

**12. a.** 0.2362    **b.** 0.0890

**13.** 9 tickets

**14.** 6 games

**15. a.** See the table at the bottom of the page.*

  **b.** 0.9752

  **c.** 0.9655

  **d.** 0.0367

**16.** 3 shots

**17.** 0.1191

**18.** 0.3924

**19.** $\left(0 < p < 0.6\right)$

**20.** 0.5014

### 7.4 The mean and variance of a binomial distribution

#### 7.4 Exercise

**1. a.** 4.1667

  **b.** $Var(X) \approx 3.472$; $SD(X) = 1.8634$

**2. a.** 0.7227    **b.** 45

**3. a.** 5    **b.** 2.5    **c.** 1.58

**4. a.** 4.62    **b.** 3.55    **c.** 1.88

**5. a.** 12    **b.** 4.8    **c.** 2.19

**6. a.** 1.67    **b.** 5155

**7. a.** 96    **b.** 24

**\* 10. a , b.**

| $z$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $P(Z=z)$ | 0.0043 | 0.0312 | 0.1017 | 0.1963 | 0.2488 | 0.2162 | 0.1304 | 0.0540 | 0.0147 | 0.0024 | 0.0002 |

**\* 15. a.**

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $P(X=x)$ | 0.0004 | 0.0048 | 0.0285 | 0.0971 | 0.2067 | 0.2815 | 0.2397 | 0.1166 | 0.0248 |

**8. a.** $\dfrac{1}{2}$  **b.** 20

**9. a.** $\dfrac{3}{4}$  **b.** 16

**10. a. i.** 32.4  **ii.** 9.072
  **b. i.** 20  **ii.** 16
  **c. i.** 16  **ii.** 12.4

**11.** 0.2642

**12. a.** 50 000  **b.** 25 000  **c.** 158.11

**13. a.** 6 children
  **b.** Yoghurt B is far more popular than expected.

**14.** 1.93

**15.** \$26 400/day

**16.** $p = 0.88$ or $p = 0.12$

**17.** 0.12

**18.** $\dfrac{96}{625}$

## 7.5 Modelling and solving with Bernoulli and binomial distributions

### 7.5 Exercise

**1.** 5 more

**2.** $E(X) = 1.23 \times 10^8$
  $\text{Var}(X) = 1.17 \times 10^8$
  $\text{SD}(X) = 1.09 \times 10^4$

**3. a.** 0.0106
  **b.** $E(Y) = 3$; $SD(Y) = 1.4491$

**4. a.** 0.7216  **b.** 4.5  **c.** 1.7748

**5.** 0.0547

**6.** 0.1051

**7. a.** 0.5455
  **b.** 0.3340
  **c.** 0.6
  **d.** 0.8795
  **e.** 0.2678

**8. a.** 0.0922
  **b.** 0.9992
  **c. i.** 10.2  **ii.** 1.2369

**9. a.**

| $x$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $P(X=x)$ | $(1-p)^3$ | $3(1-p)^2 p$ | $3(1-p)p^2$ | $p^3$ |

  **b.** $\dfrac{1}{4}$
  **c. i.** $\dfrac{3}{4}$
  **ii.** $\dfrac{3}{4}$

**10. a.** 0.0101  **b.** 0.0319

**11. a.** 0.0026  **b.** 0.0817

**12. a. i.** 0.0093  **ii.** 0.1852  **iii.** 0.9692
  **b.** 3 footballers

**13. a.** 0.9078  **b.** 0.0008  **c.** 0.0574

**14.** 37 tickets

**15.** 0.6632

**16.** 240 possums

## 7.6 Review

### 7.6 Exercise

**1.** C

**2.** C

**3.** A

**4.** A

**5.** D

**6.** 0.0879

**7. a.** 0.0490  **b.** 0.1486

**8. a.** 0.2  **b.** 50

**9.** 12

**10.** 0.12

**11.** 0.030 78

**12. a.** 0.635  **b.** 1.9257

**13. a.** 1  **b.** 0.2642  **c.** 0.0003

**14. a.** 1  **b.** 0.0015  **c.** 0.2639
  **d.** 10

**15.** 0.0013

**16. a.** 135
  **b.** 0.1
  **c.** Expected loss of \$60

**17.** $\dfrac{3p(1-p)^2}{1-(1-p)^3}$

**18.** 0.6632

**19.** 10

**20.** $p = 0.2$

### 7.6 Past QCAA exam questions

**1.** A

**2.** C

**3.** A

**4. a.** Mean $= 0.5$; variance $= 0.25$
  **b.** Sample B has a larger variance than sample A, which means that sample B has more variability about the mean.

**5. a.** There are only two outcomes for each selection ('delayed' or 'on time'); the probabilities of each outcome do not change for each trial.
  **b.** $\dfrac{13}{125}$

# UNIT 4 Further calculus, trigonometry and statistics

# 8 Further integration

## LESSON SEQUENCE

**Fully worked solutions of this chapter are available online.**

## EXAM PREPARATION

Access exam-style questions in every lesson, available online and access past QCAA exam questions in every review.

**on** Resources

| | |
|---|---|
| **Solutions** | Solutions — Chapter 8 |
| **Exam questions** | Exam question booklet — Chapter 8 |
| **Digital documents** | Learning matrix — Chapter 8 |
| | Chapter summary — Chapter 8 |

# LESSON
## 8.1 Overview

### 8.1.1 Introduction

In previous chapters, you explored various aspects of calculus, which examines how changes in one variable affect another. Historically, integration began with determining areas under curves. This chapter investigates methods for estimating these areas, leading to the fundamental theorem of calculus and its applications to the definite integral. Integration has many real-life applications. In economics, knowing marginal costs helps predict total costs. In science, the rate of change, like an oil spill, helps determine the affected area. Kinematics, or motion in a straight line, also uses differential and integral calculus. In probability, integrals calculate the likelihood of a value falling.

### 8.1.2 Syllabus links

| Lesson | Lesson title | Syllabus links |
|---|---|---|
| 8.2 | **Estimating the area under a curve** | ○ Use sums of the form $\sum_i f(x_i)\,\delta x_i$ to estimate the area under the curve $y = f(x)$. |
| 8.3 | **The fundamental theorem of calculus** | ○ Recognise the definite integral $\int_a^b f(x)\,dx$ as a limit of sums of the form $\sum_i f(x_i)\,\delta x_i$. <br> ○ Understand the fundamental theorem of calculus, $\int_a^b f(x)\,dx = F(b) - F(a)$, and use it to calculate definite integrals. |
| 8.4 | **Areas under a curve** | ○ Use the definite integral $\int_a^b f(x)\,dx$ to determine the area under the curve $y = f(x)$ between $x = a$ and $x = b$ if $f(x) > 0$ over this interval. <br> ○ Calculate the area enclosed by a curve and the $x$-axis over a given domain, with and without technology. |
| 8.5 | **Areas between curves** | ○ Calculate the area between curves, with and without technology. |
| 8.6 | **Modelling and solving with definite integrals** | ○ Use the trapezoidal rule, $\int_a^b f(x)\,dx \approx \frac{w}{2}\left[f(x_0) + 2\left(f(x_1) + f(x_2) + \ldots f(x_{n-1})\right) + f(x_n)\right]$, where $w = \frac{b-a}{n}$, to approximate an area and the value of a definite integral, with and without technology. <br> ○ Calculate total change by integrating instantaneous or marginal rates of change, with and without technology. <br> ○ Model and solve problems that involve definite integrals, including motion problems, with and without technology. |

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# LESSON
## 8.2 Estimating the area under a curve

### Introduction

There are several different ways to estimate or approximate the area between a curve and the $x$-axis.

This section considers three methods:
- the left end-point rectangle method
- the right end-point rectangle method
- the trapezoidal method.

### 8.2.1 The left end-point rectangle method

Consider the curve defined by the function $f(x) = x^2 + 2$ and the area between this curve and the $x$-axis from $x = 0$ to $x = 3$.
- Construct rectangles of width 1 unit.
- The height of the rectangle is given by the function on the left side, so the height of the first rectangle is $f(0)$, the height of the second rectangle is $f(1)$ and so on.
- Approximate area:

$$\begin{aligned} A &= 1 \times f(0) + 1 \times f(1) + 1 \times f(2) \\ &= 2 + 3 + 6 \\ &= 11 \text{ square units} \end{aligned}$$
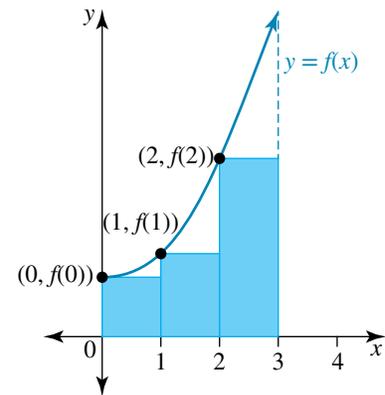


- Construct rectangles of width 0.5 units.
- The height is again given by the function on the left side of the rectangle.
- Approximate area:

$$\begin{aligned} A &= 0.5 \times f(0) + 0.5 \times f(0.5) + 0.5 \times f(1) + 0.5 \times f(1.5) \\ &\quad + 0.5 \times f(2) + 0.5 \times f(2.5) \\ &= 0.5 \times (f(0) + f(0.5) + f(1) + f(1.5) + f(2) + f(2.5)) \\ &= 0.5 \times (2 + 2.25 + 3 + 4.25 + 6 + 8.25) \\ &= 12.875 \text{ square units} \end{aligned}$$
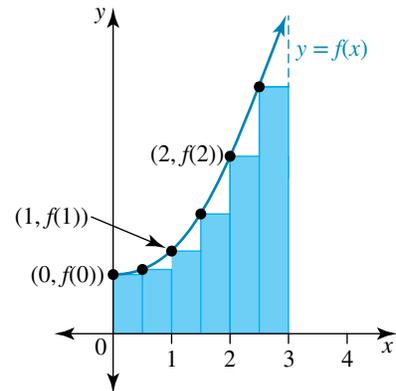
Observe from these examples that:
- the narrower the rectangles, the closer the approximation is to the actual area under the curve
- the approximate area is less than the actual area
- if the function was a decreasing function, for example $f(x) = 10 - x^2$, then the area using the left end-points would be greater than the actual area under the curve.
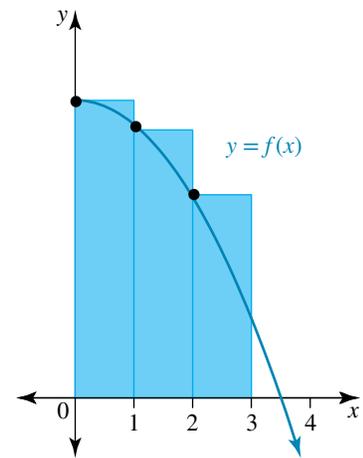


## 8.2.2 The right end-point rectangle method

Again, consider the curve defined by the function $f(x) = x^2 + 2$ and the area between this curve and the $x$-axis from $x = 0$ to $x = 3$.
- Construct rectangles of width 1 unit.
- The height of the rectangle is given by the function on the right side, so the height of the first rectangle is $f(1)$, the height of the second rectangle is $f(2)$ and so on.
- Approximate area:

$$A = 1 \times f(1) + 1 \times f(2) + 1 \times f(3)$$
$$= 3 + 6 + 11$$
$$= 20 \text{ square units}$$
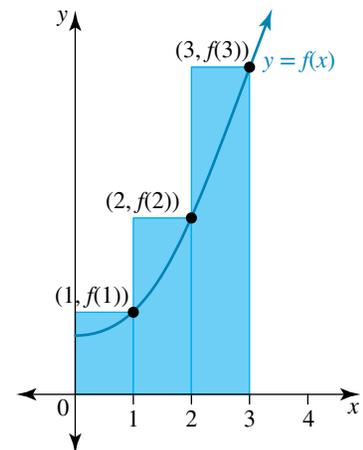


- Construct rectangles of width 0.5 units.
- The height is again given by the function on the right side of the rectangle.
- Approximate area:

$$A = 0.5 \times f(0.5) + 0.5 \times f(1) + 0.5 \times f(1.5) + 0.5 \times f(2)$$
$$\quad + 0.5 \times f(2.5) + 0.5 \times f(3)$$
$$= 0.5 \times (f(0.5) + f(1) + f(1.5) + f(2) + f(2.5) + f(3))$$
$$= 0.5 \times (2.25 + 3 + 4.25 + 6 + 8.25 + 11)$$
$$= 17.375 \text{ square units}$$

Observe from these examples that:
- the narrower the rectangles, the closer the approximation is to the actual area under the curve
- the approximate area is greater than the actual area
- if the function was a decreasing function, for example $f(x) = 10 - x^2$, then the area using the right end-points would be less than the actual area under the curve.
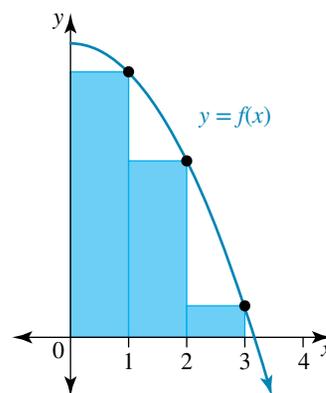


## 8.2.3 The trapezoidal method

There are several different ways to approximate or estimate the area between a curve and the $x$-axis.

Recall that for a trapezium, area $A = \dfrac{1}{2}(a + b) \times h$, where $a$ and $b$ are the lengths of the parallel sides and $h$ is the width of the trapezium.
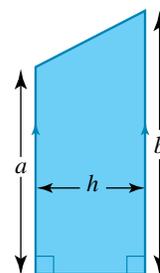


Once again, consider the curve defined by the function $f(x) = x^2 + 2$ and the area between this curve and the $x$-axis from $x = 0$ to $x = 3$.
- Construct trapeziums of width 1 unit.
- The lengths of the parallel sides are defined by $f(x)$.
- Area of $T_1 = \dfrac{1}{2}(f(0) + f(1)) \times 1$

$$= \dfrac{1}{2}(2 + 3)$$

$$= \dfrac{5}{2}$$

- Area of $T_2 = \dfrac{1}{2}(f(1) + f(2)) \times 1$

$$= \dfrac{1}{2}(3 + 6)$$

$$= \dfrac{9}{2}$$

- Area of $T_3 = \dfrac{1}{2}(f(2) + f(3)) \times 1$

$$= \dfrac{1}{2}(6 + 11)$$

$$= \dfrac{17}{2}$$

- Total area $= \dfrac{5}{2} + \dfrac{9}{2} + \dfrac{17}{2}$

$$= \dfrac{31}{2}$$

$$= 15.5 \text{ square units}$$
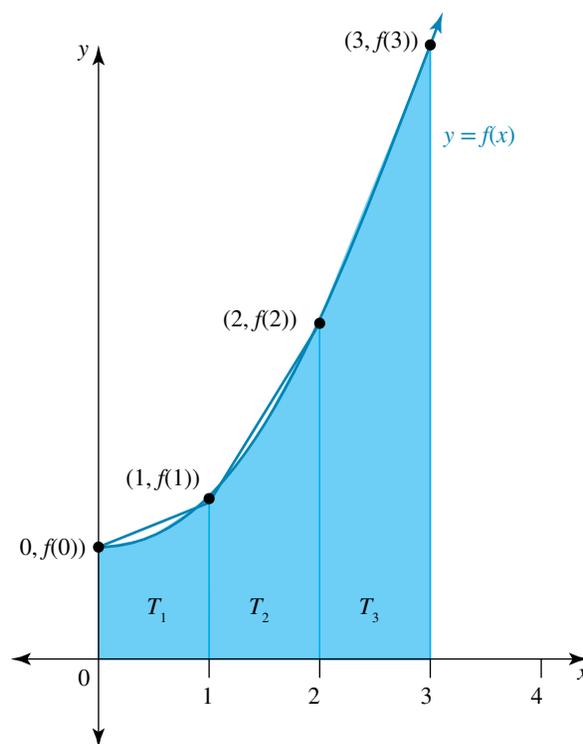


*Note:* When the widths of the rectangles and the trapezoids are the same, the average of the areas found using left end-point rectangles and right end-point rectangles gives the same result as the trapezoidal rule.

For example, using a width of 1 unit:

$$\text{Area using left end-points} = 11$$
$$\text{Area using right end-points} = 20$$
$$\text{Average of areas} = (11 + 20)/2$$
$$= 15.5 \text{ square units, the area of the trapezium.}$$

Observe from these three methods

- the narrower the width of the rectangles or trapeziums, the closer the approximation is to the actual area under the curve
- the trapezoidal method, or rule, gives a closer approximation to the actual area under the curve.

### WORKED EXAMPLE 1 Determining approximate areas using all the methods

Points on a function, $y = f(x)$, are shown in the diagram. An approximation of the area between the function $y = f(x)$ and the $x$-axis from $x = 1$ to $x = 7$ is to be found using the given information.
Determine the approximate area using:
a. the left end-point rectangle method
b. the right end-point rectangle method
c. the trapezoidal method.
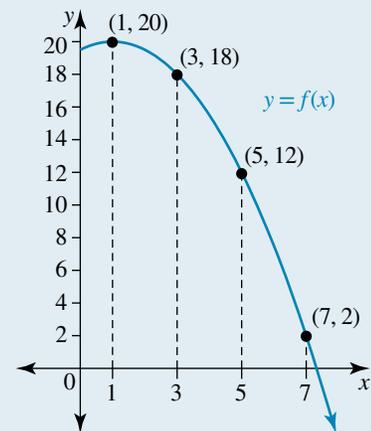


**THINK**

a. 1. Draw the left end-point rectangles on the graph. State the width and heights of the rectangles.

**WRITE**

a.
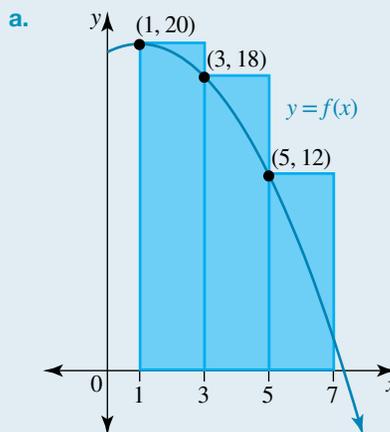


The width of each rectangle is 2 units.
The heights of the rectangles are 20, 18 and 12 units.

2. Calculate the approximate area by adding the areas of the rectangles.

$A = 2 \times 20 + 2 \times 18 + 2 \times 12$
$\quad = 100$
The area is approximately 100 square units.

**b. 1.** Draw the right end-point rectangles on the graph. State the widths and heights of the rectangles.

**b.**



The width of each rectangle is 2 units.

The heights of the rectangles are 18, 12 and 2 units.

**2.** Calculate the approximate area by adding the areas of the rectangles.

$A = 2 \times 18 + 2 \times 12 + 2 \times 2$
$\quad = 64$

The area is approximately 64 square units.

**c. 1.** Draw trapeziums on the graph. State the widths of the trapeziums.

**c.**
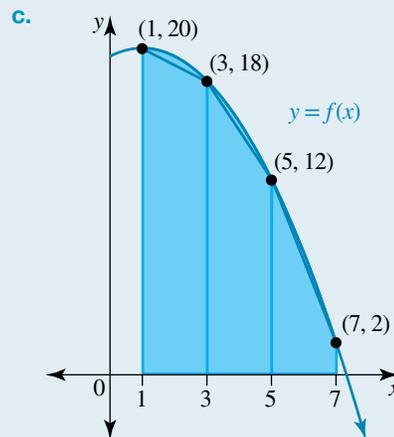


The width of each trapezium is 2 units.

**2.** Use the formula to calculate the area of each trapezium.

$T_1 = \dfrac{1}{2}(20 + 18) \times 2 = 38$

$T_2 = \dfrac{1}{2}(18 + 12) \times 2 = 30$

$T_3 = \dfrac{1}{2}(12 + 2) \times 2 = 14$

$\text{Area} = 38 + 30 + 14$
$\qquad\quad = 82$

**3.** Calculate the approximate area by adding the areas of the trapeziums.

The area is approximately 82 square units.

The graph of the function defined by the rule $f(x) = e^x$ is shown. Give your answers to the following correct to 2 decimal places.

a. Use the left end-point rectangle method with rectangles of width 0.5 units to calculate the area between the curve and the $x$-axis from $x = 0$ to $x = 2.5$.
b. Use the right end-point rectangle method with rectangles of width 0.5 units to calculate the area between the curve and the $x$-axis from $x = 0$ to $x = 2.5$.



**THINK**

a. 1. Draw the left end-point rectangles on the graph. State the widths and heights of the rectangles.

**WRITE**

a.
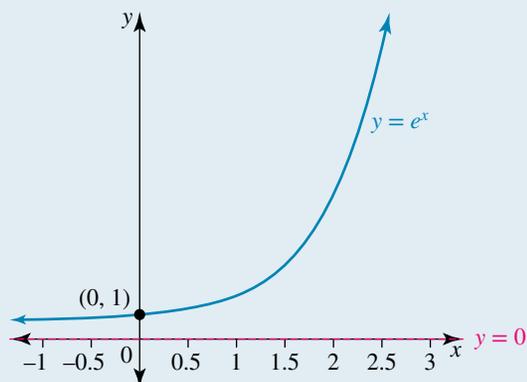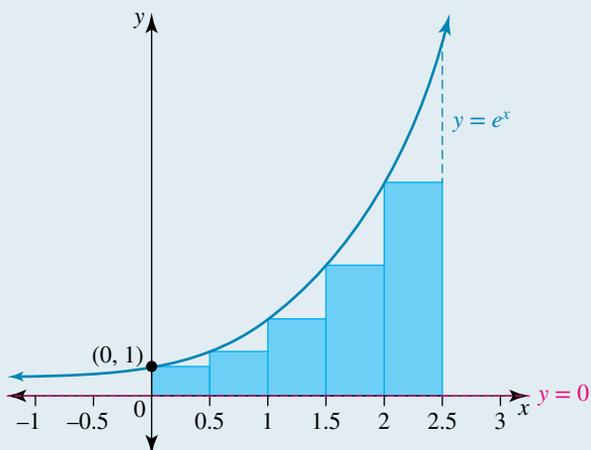


The width of each rectangle is 0.5 units. The heights of the rectangles are $f(0)$, $f(0.5)$, $f(1)$, $f(1.5)$ and $f(2)$.

2. Determine the approximate area by adding the areas of all the rectangles.

$A = 0.5f(0) + 0.5f(0) + 0.5f(1) + 0.5f(1.5) + 0.5f(2)$
$= 0.5\,[f(0) + f(0.5) + f(1) + f(1.5) + f(2)]$
$= 0.5\,[e^0 + e^{0.5} + e^1 + e^{1.5} + e^2]$
$= 0.5 \times 17.2377$
$\approx 8.62$

The area is approximately 8.62 square units.

**b. 1.** Draw the right end-point rectangles on the graph. State the widths and heights of the rectangles.

**b.**
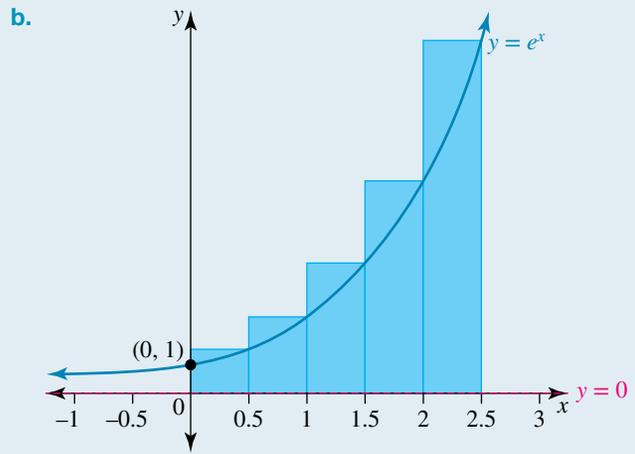


The width of each rectangle is 0.5 units.
The heights of the rectangles are $f(0.5), f(1), f(1.5), f(2)$ and $f(2.5)$.

**2.** Determine the approximate area by adding the areas of all the rectangles.

$A = 0.5f(0.5) + 0.5f(1) + 0.5f(1.5) + 0.5f(2) + 0.5f(2.5)$
$= 0.5 \left[ f(0.5) + f(1) + f(1.5) + f(2) + f(2.5) \right]$
$= 0.5 \left[ e^{0.5} + e^1 + e^{1.5} + e^2 + e^{2.5} \right]$
$= 0.5 \times 28.4202$
$\approx 14.21$
The area is approximately 0.5 square units.

## WORKED EXAMPLE 3 Estimating area using the trapezoidal rule

Determine an approximation for the area enclosed by the graph of $f(x) = 0.2x^2 + 3$, the $x$-axis and the lines $x = 1$ and $x = 3$ using the trapezoidal rule with interval widths of 0.5 units.

**THINK**

**1.** Sketch the graph of $f(x)$.
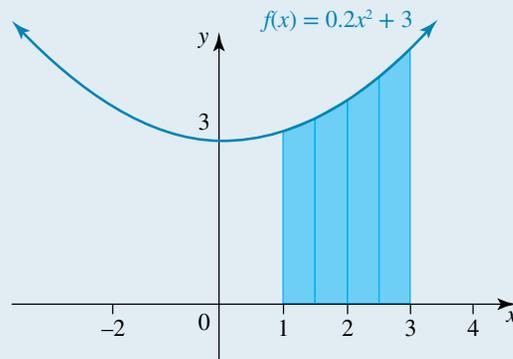
**2.** Draw trapeziums of width 0.5 units from $x = 1$ to $x = 3$.

**WRITE**

**3.** Evaluate the height of each vertical side of the trapeziums by substituting the appropriate $x$-value into $f(x)$.

$$f(1) = 0.2(1)^2 + 3 = 3.2$$
$$f(1.5) = 0.2(1.5)^2 + 3 = 3.45$$
$$f(2) = 0.2(2)^2 + 3 = 3.8$$
$$f(2.5) = 0.2(2.5)^2 + 3 = 4.25$$
$$f(3) = 0.2(3)^2 + 3 = 4.8$$

**4.** Calculate the area of each trapezium.

$$T_1 = \frac{1}{2}(3.2 + 3.45) \times 0.5 = 1.6625$$

$$T_2 = \frac{1}{2}(3.45 + 3.8) \times 0.5 = 1.8125$$

$$T_3 = \frac{1}{2}(3.8 + 4.25) \times 0.5 = 2.0125$$

$$T_4 = \frac{1}{2}(4.25 + 4.8) \times 0.5 = 2.2625$$

**5.** Calculate the approximate area by adding the areas of the trapeziums.

Total area of trapeziums
$$= 1.6625 + 1.8125 + 2.0125 + 2.2625$$
$$= 7.75$$
Therefore, the area under the curve is approximately 7.75 square units.

# Exercise 8.2 Estimating the area under a curve

**learn** on

| **8.2 Exercise** | **8.2 Exam questions** on |
|---|---|

| Simple familiar | Complex familiar | Complex unfamiliar |
|---|---|---|
| 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 | 11, 12, 13, 14, 15 | — |

### Simple familiar

**1.** **WE1** For the curve $f(x)$ shown, determine an approximation for the area between the curve and the $x$-axis from $x = 1$ to $x = 5$. Use lower rectangles with a width of 2 units.
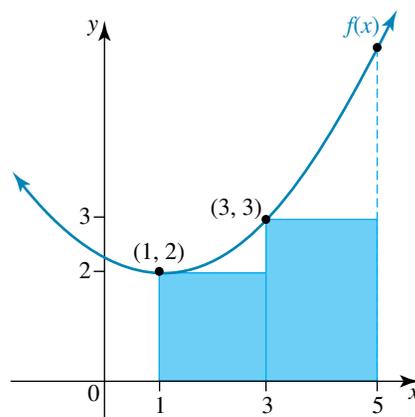
**2.** For each of the following curves, determine an approximation for the area between the curve and the $x$-axis from $x = 1$ to $x = 5$ by calculating the areas of the shaded rectangles.

**a.**



**b.**
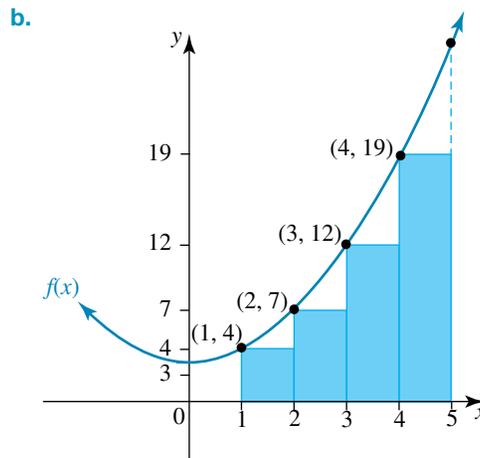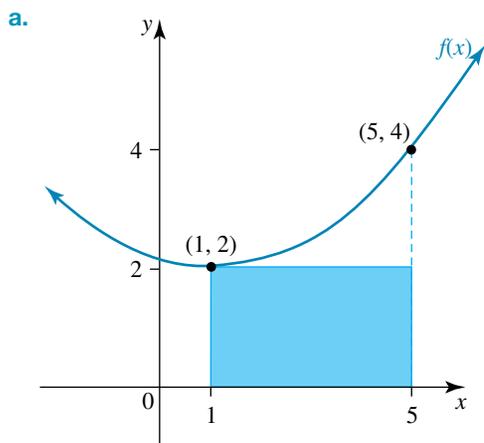


**3.** **WE2** The left end-point rectangle method and the right end-point rectangle method are shown for the calculation of the approximate area between the curve $f(x) = \dfrac{1}{x}$, $x > 0$, and the $x$-axis from $x = 0.5$ to $x = 2.5$.
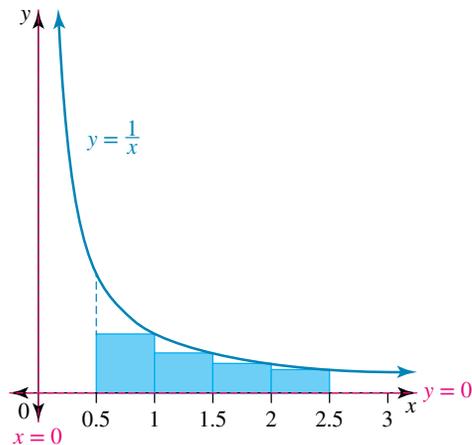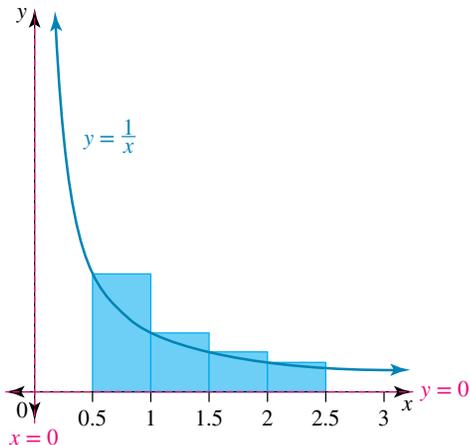


Calculate the approximate area under the curve:

**a.** using the left end-point rectangle rule

**b.** using the right end-point rectangle rule.

**4.** The graph of $f : [0, 4] \rightarrow \mathbb{R}, f(x) = -(x-1)^2 + 9$ is shown.
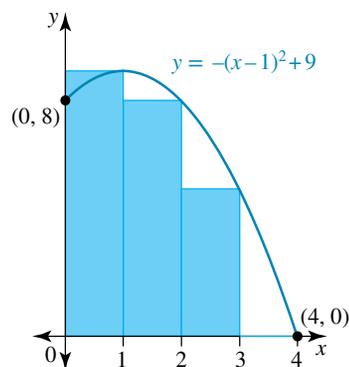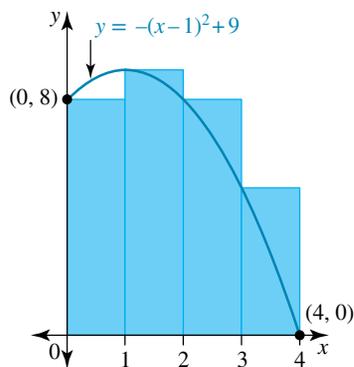


**a.** Use the left end-point rule with rectangles 1 unit wide to estimate the area between the curve and the $x$-axis from $x = 0$ to $x = 4$.

**b.** Use the right end-point rule with rectangles 1 unit wide to estimate the area between the curve and the $x$-axis from $x = 0$ to $x = 4$.

**5.** **WE3** Determine an approximation for the area enclosed by the graph of $f(x) = x^2$, the $x$-axis and the lines $x = 1$ and $x = 3$ with interval widths of 1 unit. Use the trapezoidal method.
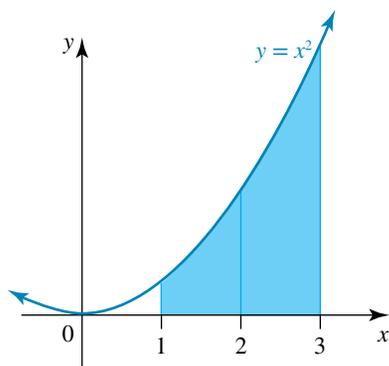


**6.** For each of the following curves, calculate an approximate area between the curve and the $x$-axis over the interval indicated by calculating the areas of the shaded trapeziums.

a.



b.
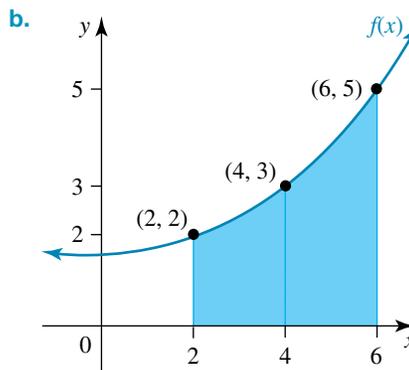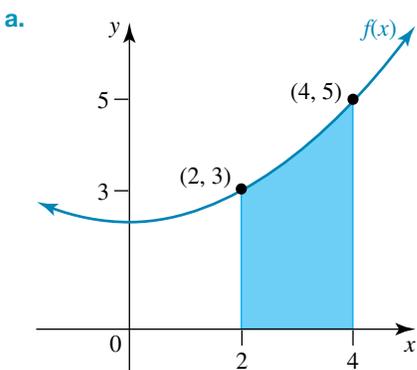


**7.** Using width intervals of 1 unit, calculate an approximation for the area between the graph of $f(x) = x^2 + 4$ and the $x$-axis from $x = 1$ to $x = 4$ using:
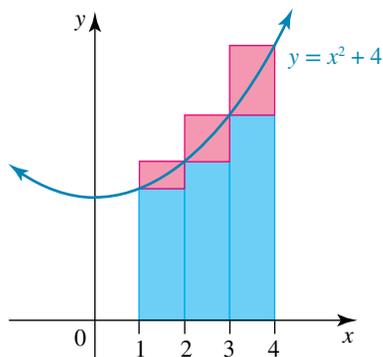


a. lower rectangles
b. upper rectangles
c. averaging of the lower and upper rectangle areas.

**8.** For each of the following figures, determine the approximate area between the curve and the $x$-axis over the interval indicated by calculating the area of the shaded rectangles. Give your answers in exact form.

**a.**



$y = e^x$

$x = -1$ to $x = 2$

**b.**



$y = \log_e(x)$

$x = 1$ to $x = 5$

**9.** For each of the following figures, determine the approximate area between the curve and the $x$-axis over the interval indicated by calculating the area of the shaded rectangles. Give your answers in exact form.

**a.** $f(x) = -x^2 - 4x$



$x = -3$ to $x = -1$

**b.** $x = 2$ to $x = 6$



$y = x^3 - 6x^2$
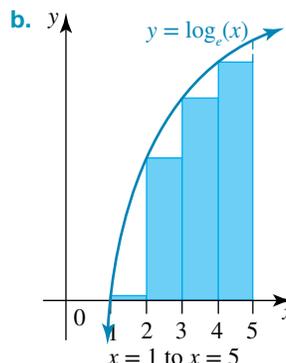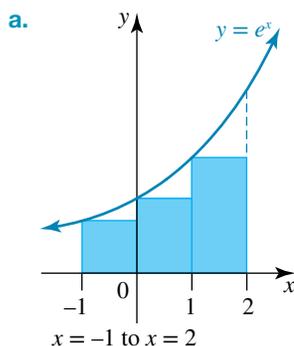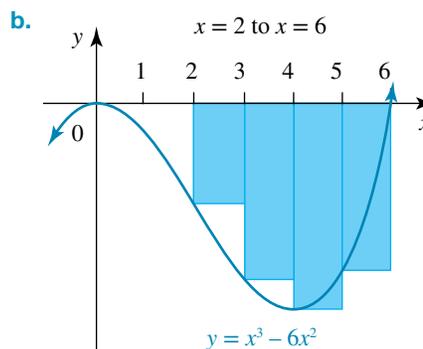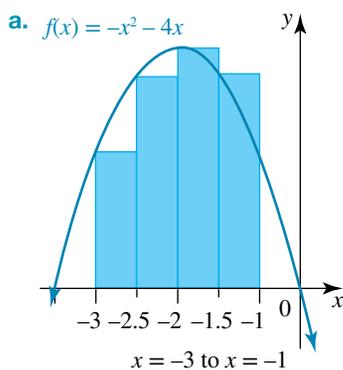
**10.** For each of the following figures, determine the approximate area between the curve and the $x$-axis over the interval indicated by calculating the shaded area using the trapezoidal rule. Give your answers in exact form.

**a.**



$y = 10 - x^2$

**b.**



$y = e^x$
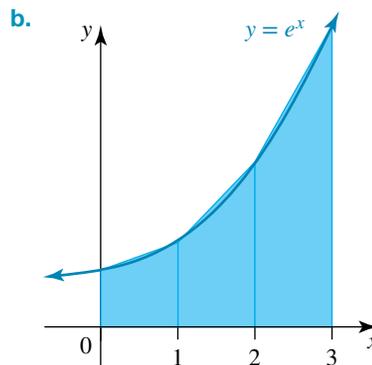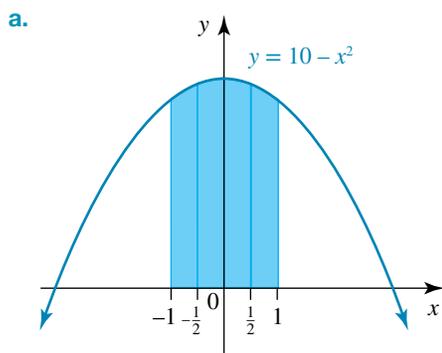
**Complex familiar**

**11.** Calculate approximations for the area between the graph of $f(x) = (x-1)^3$ and the $x$-axis between $x = 1$ and $x = 4$ (correct to 1 decimal place) using the trapezoidal rule and interval widths of:

    **a.** 1 unit                                           **b.** 0.5 units.

**12.** Calculate approximations for the area under the graph of $y = \dfrac{1}{x}$ between $x = 0.5$ and $x = 2.5$ (correct to 2 decimal places) using the trapezoidal rule and interval widths of:

    **a.** 1 unit                                           **b.** 0.5 units.

**13.** Consider the function defined by the rule $f: \mathbb{R} \to \mathbb{R}$,
$f(x) = -0.01x^3(x-5)(x+5)$, $x \geq 0$. The graph of the function is shown. Use the left end-point rule with rectangles 1 unit wide to approximate the area bound by the curve and the $x$-axis.
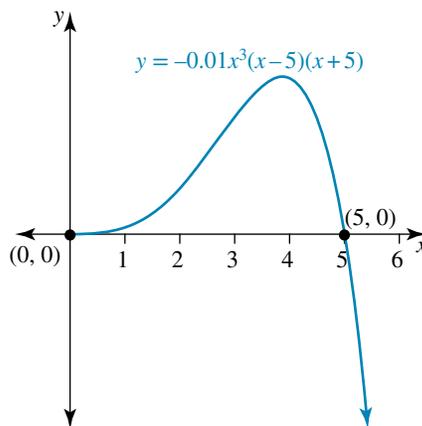


**14.** The graph of the function $f(x) = 4 - \dfrac{1}{4}x^2$ is shown.
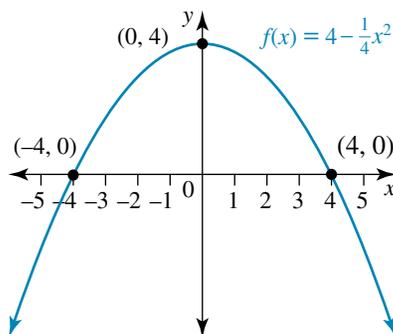


Estimate the area bound by the curve and the $x$-axis using the right end-point method with rectangles of width 1 unit.

**15.** The graph of $f(x) = \sqrt{x}(4-x)$ for $x \in [0, a]$ is shown.
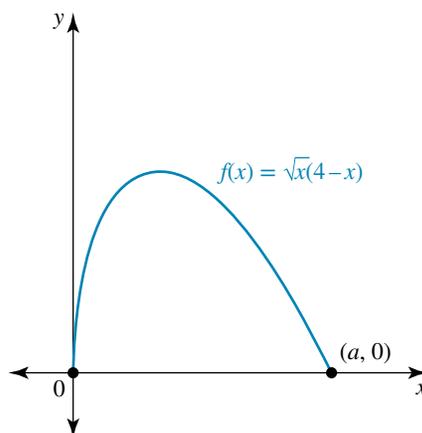


a. The graph intersects the $x$-axis at the point $(a, 0)$ as shown. Calculate the value of the constant $a$.
b. Use both the left end-point and the right end-point rules to determine the approximate area between the curve and the $x$-axis from $x = 0$ to $x = a$. Use a rectangle width of 1 and give your answers correct to 2 decimal places.

**Fully worked solutions for this chapter are available online.**

# LESSON
## 8.3 The fundamental theorem of calculus

### 8.3.1 The fundamental theorem of calculus

Consider the function $y = f(x)$, where $f(x) \geq 0$, that is continuous for $x \in [a, b]$.

Let $A(x)$ represent the area between the curve and the $x$-axis from $x = a$ to $x$.

Similarly, let $A(x + \delta x)$ represent the area between the curve and the $x$-axis from $x = a$ to $x + \delta x$.
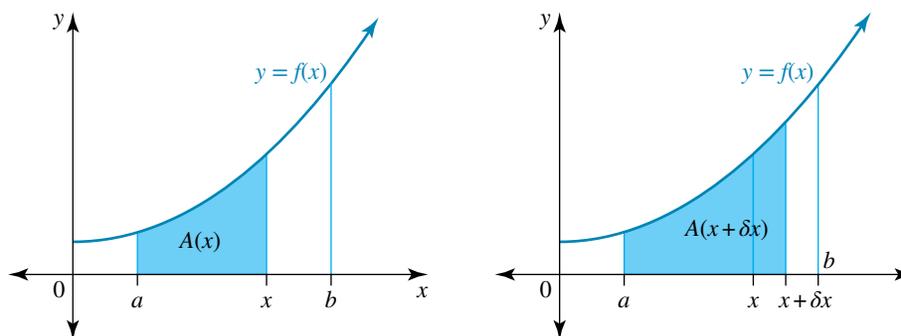


The difference between these areas is $A(x + \delta x) - A(x)$. From the previous lesson, we know that this area lies between the areas of the left end-point rectangle and the right end-point rectangle.
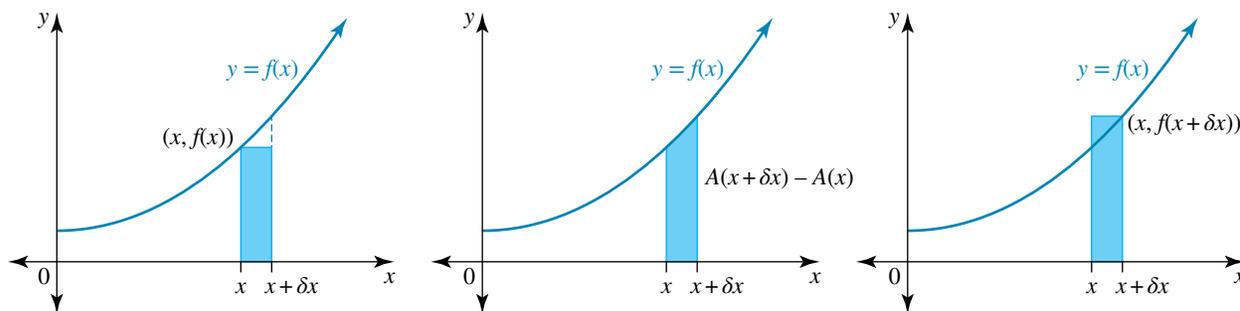


Left end-point rectangle method     Actual area under the curve     Right end-point rectangle method

Therefore, $f(x) \times \delta x \leq A(x + \delta x) - A(x) \leq f(x + \delta, x) \times \delta x$

$$f(x) \leq \frac{A(x + \delta x) - A(x)}{\delta x} \leq f(x + \delta x)$$

As the width of each rectangular strip becomes smaller, the areas become closer together.

This means as $\delta x \to 0$, $f(x + \delta x) \to f(x)$

or $\displaystyle\lim_{\delta x \to 0} \frac{A(x + \delta x) - A(x)}{\delta x} = f(x)$.

By definition of differentiation from first principles, $\displaystyle\lim_{\delta x \to 0} \frac{A(x + \delta x) - A(x)}{\delta x} = \frac{d}{dx}(A(x))$.

Thus, $\dfrac{d}{dx}(A(x)) = f(x)$.

Integrating both sides with respect to $x$ gives:

$$\int \frac{d}{dx}(A(x)) \, dx = \int f(x) \, dx$$

Or simply:

$$A(x) = \int f(x) \, dx$$

Let the antiderivative of $f(x)$ be $F(x)$.

Then:

$$A(x) = F(x) + c$$

Or:

$$\int f(x) \, dx = F(x) + c$$

When $x = a$:

$$\int f(x) \, dx = F(a) + c$$

But:

$$\int f(x) \, dx = 0, \text{ as the area defined is zero at } x = a.$$

$$\therefore c = -F(a)$$

This gives:

$$\int f(x) \, dx = F(x) - F(a)$$

Let $x = b$:

$$\int f(x) \, dx = F(b) - F(a)$$

That is, the area between the graph of $y = f(x)$, the $x$-axis and $x = a$ and $x = b$ is given by $F(b) - F(a)$, where $F(x)$ is the antiderivative of $f(x)$.

This is the **fundamental theorem of integral calculus**. It allows areas under graphs to be calculated exactly. The fundamental theorem can be stated as:

$$\text{Area} = \int_a^b f(x) \, dx$$
$$= \left[ F(x) \right]_a^b$$
$$= F(b) - F(a)$$

**The fundamental theorem of integral calculus**

$$\int_a^b f(x) \, dx = \left[ F(x) \right]_a^b$$
$$= F(b) - F(a)$$

where $F(x)$ is the antiderivative of $f(x)$.

## 8.3.2 Properties of the definite integral

$\int_a^b f(x)\,dx$ is called the **definite integral** as it is a real number and not a function. As found in the previous

lessons, the **indefinite integral**, $\int f(x)\,dx$, involves determining only an antiderivative of the function.

For the definite integral:
- there is no need to add $+c$ as the two $c$ constants in $F(a)$ and $F(b)$ cancel each other out
- the variables $a$ and $b$ are called the **terminals** of the definite integral and indicate the range of values of $x$ over which the integral is taken
- the function, $f(x)$, to be integrated is called the **integrand**.

Definite integrals have the following properties, assuming the functions, $f(x)$ and $g(x)$ are continuous functions for $x \in [a, b]$ or $a \le x \le b$.

---

### Properties of definite integrals

$$\int_a^a f(x)\,dx = 0$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

$$\int_a^b k f(x)\,dx = k \int_a^b f(x)\,dx$$

$$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \text{ provided } a < c < b$$

---

### WORKED EXAMPLE 4 Determining definite integrals

**Determine the following definite integrals.**

a. $\displaystyle\int_0^3 (3x^2 + 4x - 1)\,dx$

b. $\displaystyle\int_1^2 \frac{4}{(2x+1)^3}\,dx$

**THINK**

a. **1.** Antidifferentiate each term of the integrand and write in the form $[F(x)]_a^b$.

**2.** Substitute the values of $a$ and $b$ into $F(b) - F(a)$.

**3.** Evaluate the integral.

**WRITE**

a. $\displaystyle\int_0^3 (3x^2 + 4x - 1)\,dx$

$= \left[x^3 + 2x^2 - x\right]_0^3$

$= \left[3^3 + 2(3)^2 - 3\right] - \left[0^3 + 2(0)^2 - 0\right]$

$= 42 - 0$

$= 42$

**b. 1.** Express the integrand in simplest index form.

**b.** $\displaystyle\int_1^2 \frac{4}{(2x+1)^3}\,dx$

$$= \int_1^2 4\,(2x+1)^{-3}\,dx$$

**2.** Antidifferentiate by rule.

$$= \left[\frac{4\,(2x+1)^{-2}}{2\times-2}\right]_1^2$$

$$= \left[-(2x+1)^{-2}\right]_1^2$$

**3.** Express the integral with a positive index number.

$$= \left[\frac{-1}{(2x+1)^2}\right]_1^2$$

**4.** Substitute the values of $a$ and $b$ into $F(b) - F(a)$.

$$= \left[-\frac{1}{5^2}\right] - \left[\frac{-1}{3^2}\right]$$

**5.** Evaluate the integral.

$$= -\frac{1}{25} + \frac{1}{9}$$

$$= \frac{16}{225}$$

| TI | THINK | WRITE | CASIO | THINK | WRITE |
|---|---|---|---|---|---|
| **a. 1.** On a Calculator page, press MENU, then select: 4: Calculus 2: Numerical Integral. |  | **a. 1.** On a Run-Matrix page, select: SHIFT F5 $\displaystyle\int dx$. |  |
| **2.** Complete the entry line as: $\displaystyle\int_0^3 3x^2 + 4x - 1\,dx$ then press the ENTER button. |  | **2.** Complete the entry line as: $\displaystyle\int_0^3 3x^2 + 4x - 1\,dx$ then press the EXE button. |  |
| **3.** The answer appears on the screen. |  | **3.** The answer appears on the screen. |  |
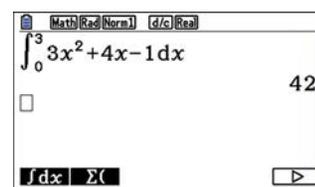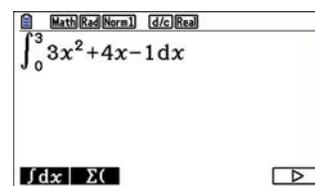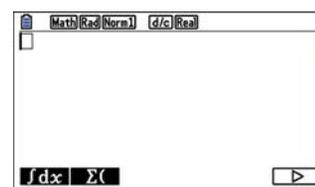
## WORKED EXAMPLE 5 Determining definite integrals

**Evaluate the following definite integrals.**

**a.** $\displaystyle\int_0^{\frac{\pi}{2}} \cos(x)\,dx$

**b.** $\displaystyle\int_0^2 \left(e^{-x}+2\right)dx$

| THINK | WRITE |
|---|---|
| **a. 1.** Antidifferentiate the given function and specify the end points for the calculation using square brackets. | **a.** $\displaystyle\int_0^{\frac{\pi}{2}} \cos(x)\,dx$ <br><br> $= \left[\sin(x)\right]_0^{\frac{\pi}{2}}$ |
| **2.** Substitute the upper and lower end points into the antiderivative and calculate the difference between the two values. | $= \sin\left(\dfrac{\pi}{2}\right) - \sin(0)$ <br> $= 1 - 0$ <br> $= 1$ |
| **b. 1.** Antidifferentiate the given function and specify the end points for the calculation using square brackets. | **b.** $\displaystyle\int_0^2 (e^{-x}+2)\,dx = \left[-e^{-x}+2x\right]_0^2$ |
| **2.** Substitute the upper and lower end points into the antiderivative and calculate the difference between the two values. | $= (-e^{-2}+2(2)) - (-e^0 + 2(0))$ <br> $= -\dfrac{1}{e^2} + 4 + 1$ <br> $= 5 - \dfrac{1}{e^2}$ |

Even if the function is unknown, we can use the properties of definite integrals to evaluate the values of related integrals. This is demonstrated in the following worked example.

## WORKED EXAMPLE 6 Applying definite integral properties

**a.** Given that $\displaystyle\int_1^3 f(x)\,dx = 8$, determine:

**i.** $\displaystyle\int_1^3 2f(x)\,dx$      **ii.** $\displaystyle\int_1^3 (f(x)+1)\,dx$      **iii.** $\displaystyle\int_3^1 f(x)\,dx$      **iv.** $\displaystyle\int_1^3 (f(x)-x)\,dx$

**b.** Calculate $k$ if $\displaystyle\int_1^k (x+2)\,dx = 0$.

| THINK | WRITE |
|---|---|
| **a. i.** Apply the definite integral property $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$. | **a. i.** $\displaystyle\int_a^b 2f(x)\,dx = 2\int_1^3 f(x)\,dx$ <br> $= 2 \times 8$ <br> $= 16$ |

**ii. 1.** Apply the definite integral property

$$\int_a^b (f(x) \pm g(x)) \, dx = \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx.$$

**ii.** $\int_1^3 (f(x) + 1) \, dx = \int_1^3 f(x) \, dx + \int_1^3 1 \, dx$

**2.** Integrate the second function and evaluate.

$$= 8 + \left[ x \right]_1^3$$
$$= 8 + (3 - 1)$$
$$= 10$$

**iii.** Apply the definite integral property

$$\int_a^b f(x) \, dx = -\int_b^a f(x) \, dx.$$

**iii.** $\int_3^1 f(x) \, dx = -\int_1^3 f(x) \, dx$

$$= -8$$

**iv. 1.** Apply the definite integral property

$$\int_a^b (f(x) \pm g(x)) \, dx = \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx.$$

**iv.** $\int_1^3 (f(x) - x) \, dx = \int_1^3 f(x) \, dx - \int_1^3 x \, dx$

**2.** Integrate the second function and evaluate.

$$= 8 - \left[ \frac{1}{2} x^2 \right]_1^3$$
$$= 8 - \left( \frac{1}{2} (3)^2 - \frac{1}{2} (1)^2 \right)$$
$$= 8 - \left( \frac{9}{2} - \frac{1}{2} \right)$$
$$= 8 - 4$$
$$= 4$$

**b. 1.** Antidifferentiate and substitute the values of 1 and $k$.

**b.** $0 = \int_1^k (x + 2) \, dx$

$$0 = \left[ \frac{1}{2} x^2 + 2x \right]_1^k$$

$$0 = \left( \frac{1}{2} k^2 + 2k \right) - \left( \frac{1}{2} (1)^2 + 2(1) \right)$$

**2.** Simplify and solve for $k$.

$$0 = \frac{1}{2} k^2 + 2k - \frac{5}{2}$$
$$0 = k^2 + 4k - 5$$
$$0 = (k + 5)(k - 1)$$
$$k = -5 \text{ or } k = 1$$

**3.** Write the answer.

$$k = 1, -5$$

# Exercise 8.3 The fundamental theorem of calculus

| **8.3 Exercise** | **8.3 Exam questions** on |
|---|---|

| Simple familiar | Complex familiar | Complex unfamiliar |
|---|---|---|
| 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 | 13, 14, 15, 16 | 17, 18, 19, 20 |

These questions are even better in jacPLUS!
- Receive immediate feedback
- Access sample responses
- Track results and progress

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## Simple familiar

**1.** **WE4** Determine the following definite integrals.

a. $\displaystyle\int_0^1 x^2 \, dx$

b. $\displaystyle\int_0^3 x^3 \, dx$

c. $\displaystyle\int_3^4 (x^2 - 2x) \, dx$

d. $\displaystyle\int_2^6 \frac{1}{x^2} \, dx$

e. $\displaystyle\int_0^2 \left(x^3 + 3x^2 - 2x\right) dx$

**2.** Evaluate the following definite integrals.

a. $\displaystyle\int_1^3 \frac{2x^3 + 5x^2}{x} \, dx$

b. $\displaystyle\int_1^5 \frac{3}{5x} \, dx$

c. $\displaystyle\int_0^1 \frac{-4}{(3x-4)^5} \, dx$

d. $\displaystyle\int_3^7 \frac{1}{\sqrt{2x-5}} \, dx$

e. $\displaystyle\int_{-2}^0 \frac{6}{\sqrt{8-3x}} \, dx$

**3.** **WE5** Evaluate the following definite integrals.

a. $\displaystyle\int_0^{\frac{\pi}{2}} \sin(x) \, dx$

b. $\displaystyle\int_{\frac{\pi}{2}}^{\pi} 3 \sin(4x) \, dx$

c. $\displaystyle\int_0^{\pi} 5 \sin\left(\frac{x}{4}\right) dx$

d. $\displaystyle\int_{\pi}^{2\pi} -2 \sin\left(\frac{x}{3}\right) dx$

e. $\displaystyle\int_{-\pi}^0 \cos(2x) \, dx$

**4.** Evaluate the exact values of the following definite integrals.

a. $\displaystyle\int_0^2 e^{4x} \, dx$

b. $\displaystyle\int_{-2}^0 e^{\frac{x}{3}} \, dx$

c. $\displaystyle\int_{-1}^1 -4e^{-2x} \, dx$

d. $\displaystyle\int_1^2 (3e^{6x} + 5x) \, dx$

e. $\displaystyle\int_1^4 \left(\frac{5}{x} + e^{\frac{x}{2}}\right) dx$

**5.** Evaluate the following.

a. $\displaystyle\int_0^3 (3x^2 - 2x + 3) \, dx$

b. $\displaystyle\int_1^2 \frac{2x^3 + 3x^2}{x} \, dx$

c. $\displaystyle\int_{-1}^1 \left(e^{2x} - e^{-2x}\right) dx$

d. $\displaystyle\int_{2\pi}^{4\pi} \sin\left(\frac{x}{3}\right) dx$

e. $\displaystyle\int_{-3}^{-1} \frac{2}{\sqrt{1-3x}} \, dx$

6. Evaluate the following.

a. $\displaystyle\int_{-3}^{2} (x+1)^3 \, dx$

b. $\displaystyle\int_{0}^{1} (e^x + e^{-x})^2 \, dx$

7. Determine:

a. $\displaystyle\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} 8\cos(4x) \, dx$

b. $\displaystyle\int_{\frac{-\pi}{3}}^{\frac{\pi}{2}} \left[\cos(2x) - \sin\left(\frac{x}{2}\right)\right] dx$

8. **WE6** Given that $\displaystyle\int_{2}^{5} m(x) \, dx = 7$ and $\displaystyle\int_{2}^{5} n(x) \, dx = 3$, determine:

a. $\displaystyle\int_{2}^{5} 3m(x) \, dx$

b. $\displaystyle\int_{2}^{5} (2m(x) - 1) \, dx$

c. $\displaystyle\int_{5}^{2} (m(x) + 3) \, dx$

d. $\displaystyle\int_{2}^{5} (2m(x) + n(x) - 3) \, dx$

9. Given that $\displaystyle\int_{0}^{5} f(x) \, dx = 7.5$ and $\displaystyle\int_{0}^{5} g(x) \, dx = 12.5$, determine:

a. $\displaystyle\int_{0}^{5} -2f(x) \, dx$

b. $\displaystyle\int_{5}^{0} g(x) \, dx$

c. $\displaystyle\int_{0}^{5} (3f(x) + 2) \, dx$

d. $\displaystyle\int_{0}^{5} (g(x) + f(x)) \, dx$

e. $\displaystyle\int_{0}^{5} (8g(x) - 10f(x)) \, dx$

f. $\displaystyle\int_{0}^{3} g(x) \, dx + \int_{3}^{5} g(x) \, dx$

10. If $\displaystyle\int_{0}^{k} 3x^2 \, dx = 8$, determine the value of $k$.

11. If $\displaystyle\int_{1}^{k} \frac{2}{x} \, dx = \log_e(9)$, determine the value of $k$.

12. **MC** If $\displaystyle\int_{1}^{h} \frac{3}{x^2} \, dx = -\frac{12}{5}$, the value of $h$ is:

A. $\dfrac{9}{5}$

B. $\dfrac{5}{81}$

C. 3

D. $\dfrac{5}{9}$

**13.** **WE6** Calculate the value of $k$ if $\displaystyle\int_{k}^{1} \left(4x^3 - 3x^2 + 1\right) dx = 0.$

**14.** Determine the value of $a$ if $\displaystyle\int_{0}^{a} e^{\frac{x}{2}} \, dx = 4.$

**15.** Determine the value of $a$ if $\displaystyle\int_{0}^{a} e^{-2x} \, dx = \frac{1}{2}\left(1 - \frac{1}{e^8}\right).$

**16.** If $\displaystyle\int_{1}^{k} (2x - 3) \, dx = 7 - 3\sqrt{5},$ determine $k$, given $k > 1.$

Complex unfamiliar

**17.** If $y = x\sin(x)$, determine the value of $\displaystyle\int_{-\pi}^{\frac{\pi}{2}} 2x\cos(x) \, dx.$

**18.** If $y = e^{x^3 - 3x^2}$, determine the value of $\displaystyle\int_{0}^{1} (x^2 - 2x)e^{x^3 - 3x^2} \, dx.$

**19.** Differentiate $\log_e (e^x + 1)^2$ and hence calculate $\displaystyle\int_{1}^{5} \frac{e^x}{e^x + 1} \, dx$, correct to 4 decimal places.

**20.** The graph of the function $f: \mathbb{R} \to \mathbb{R}, f(x) = x^3 - 8x^2 + 21x - 14$ is shown.
The graph cuts the $x$-axis at the point $(a, 0)$.
Determine $\displaystyle\int_{a}^{5} f(x) \, dx.$



**Fully worked solutions for this chapter are available online.**

# LESSON
## 8.4 Areas under a curve

### 8.4.1 Definite integrals and areas

The area between a continuous function, $y = f(x)$, and the $x$-axis between $x = a$ and $x = b$ is shown in the diagram below.

As we have already seen, this area can be approximated by dividing it into a series of thin vertical strips or rectangles of width $\delta x$. The approximate value of the area is the sum of the areas of all the rectangles, whether they are left end-point or right end-point rectangles.

As the number of strips increases, $\delta x \to 0$

From the fundamental theorem of calculus, the shaded area, $A$, can be expressed as the limiting sum of the rectangles or the definite integral.

$$A = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} f(x)\, \delta x = \int_a^b f(x)\, dx$$
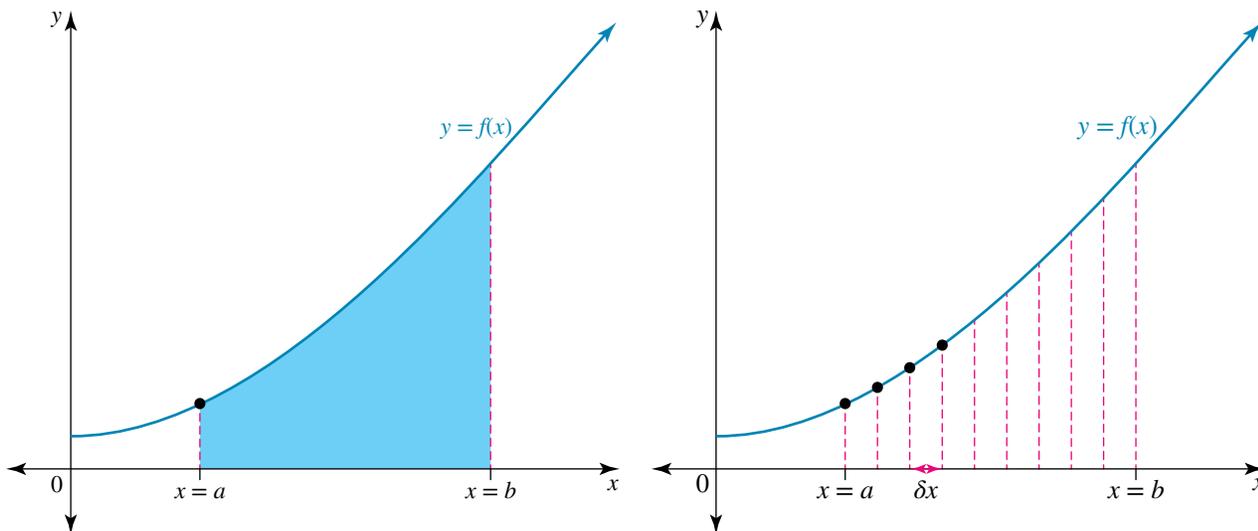


### Area under the curve

$$A = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} x\, \delta x = \int_a^b x\, dx$$

This statement allows us to calculate the area between a curve and the $x$-axis from $x = a$ to $x = b$.

## WORKED EXAMPLE 7 Determining areas under curves

**Determine the area bound by the curve defined by the rule $y = e^{-x} + 3$ and the $x$-axis from $x = 0$ to $x = 3$.**

**THINK**

1. Sketch the graph of the given function and shade the required area.

**WRITE**
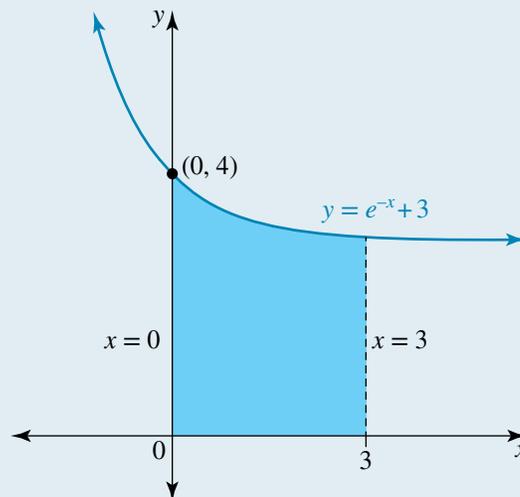


2. Write the integral needed to determine the area.

$$A = \int_0^3 (e^{-x} + 3)\, dx$$

3. Antidifferentiate the function and evaluate.

$$A = \left[ -e^{-x} + 3x \right]_0^3$$

$$= \left( -e^{-3} + 3(3) \right) - \left( -e^0 + 3(0) \right)$$
$$= -e^{-3} + 9 + 1$$
$$= -e^{-3} + 10$$

4. Write the answer.

The area is $-e^{-3} + 10$ square units.

---

**TI | THINK**

1. On a Graphs page, complete the entry line, $f1(x) = e^{-x} + 3$ then press ENTER.

**WRITE**
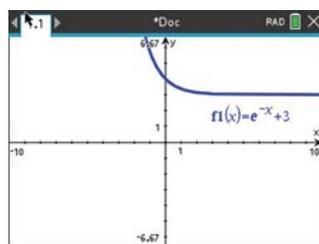


**CASIO | THINK**

1. On a Graph screen, complete the entry line $y1 = (e^{-x} + 3$ then press EXE. Press F6 to DRAW.

**WRITE**
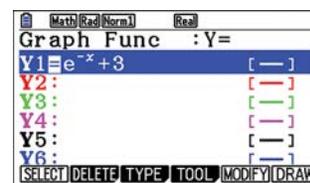
**2.** To determine the area bounded by the curve and the *x*-axis, press MENU, then select: 6 Analyse Graph 7 Bounded Area Select the curve and then the *x*-axis. Press 0 for the lower bound and press ENTER. Press 3 for the upper bounded and press ENTER.
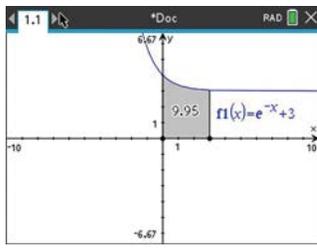


**3.** The answer appears on the screen.
*Note:* The answer will appear in decimal form.

The area is $-e^{-3} + 10$ square units.

**2.** To determine the area bounded by the curve and the *x*-axis, select G-Solv by pressing F5, then F6 for more options. Select $\int dx$ by pressing F3 then select $\int dx$ by pressing F1. Select $x = 0$ and press EXE. Press 3 for $x = 3$ and press EXE.
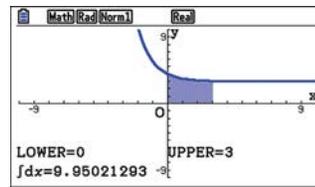


**3.** The answer appears on the screen.
*Note:* The answer will appear in decimal form.

The area is $-e^{-3} + 10$ square units.

## 8.4.2 Signed areas

When we calculate the area between the graph of a function, $y = f(x)$, and the *x*-axis from

$x = a$ to $x = b$ using the definite integral $\int_a^b f(x)\, dx$, the result can be either positive, negative or zero.

Consider the function defined by $y = x^3 - 4x$. The function has *x*-intercepts at $x = -2,\ 0,\ 2$.

For the area above the *x*-axis:

$$\int_{-2}^{0} \left(x^3 - 4x\right) dx = \left[\frac{x^4}{4} - 2x^2\right]_{-2}^{0}$$

$$= (0) - (4 - 8)$$

$$= 4$$



- The definite integral is positive as the area lies above the *x*-axis and $y = f(x) > 0$.
- The shaded area, $A_1$, is 4 square units.
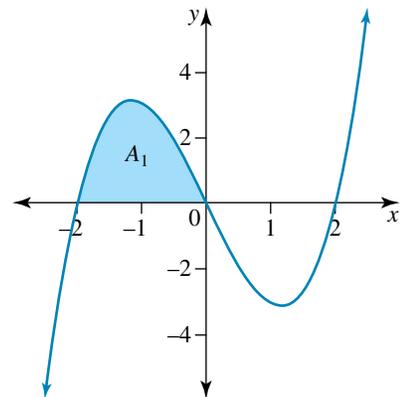
For the area below the *x*-axis:

$$\int_{0}^{2} \left(x^3 - 4x\right) dx = \left[\frac{x^4}{4} - 2x^2\right]_{0}^{2}$$

$$= (4 - 8) - (0)$$

$$= -4$$



- The definite integral is negative as the area lies below the *x*-axis and $y = f(x) < 0$.
- The shaded area, $A_2$, is 4 square units, the absolute value, or the negative, of the definite integral.
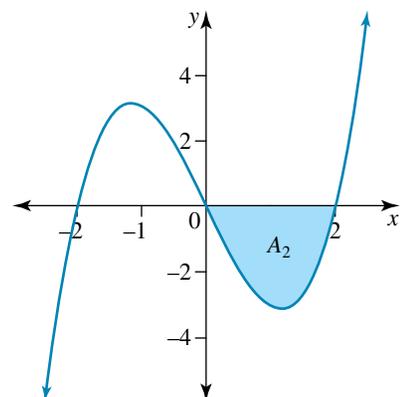
## Combining areas

For the area between the curve and the $x$-axis:

$$\int_{-2}^{2} (x^3 - 4x)\, dx = \left[\frac{x^4}{4} - 2x^2\right]_{-2}^{2}$$

$$= (4 - 8) - (4 - 8)$$

$$= 0$$

- This definite integral does not give the required area between the curve and the $x$-axis.
- This shows that it is imperative to have a 'picture' of the function to determine when $f(x) > 0$ and when $f(x) < 0$; otherwise, we are just evaluating the definite integral rather than finding the necessary area.
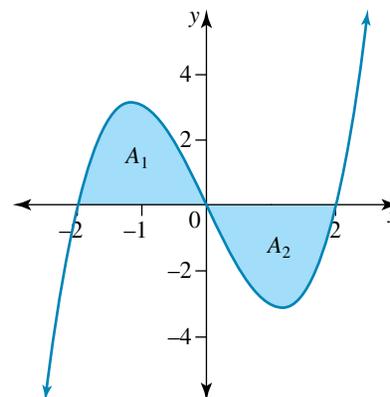
### Calculation of area under a curve

The total area between the function $y = f(x)$ and the $x$-axis from $x = a$ to $x = b$ is given by:

$$A_{total} = \int_{c}^{b} f(x)\, dx - \int_{a}^{c} f(x)\, dx$$

$$= A_2 - A_1$$

**or**

$$A_{total} = \int_{c}^{b} f(x)\, dx + \left|\int_{a}^{c} f(x)\, dx\right|$$

$$= A_2 + |A_1|$$
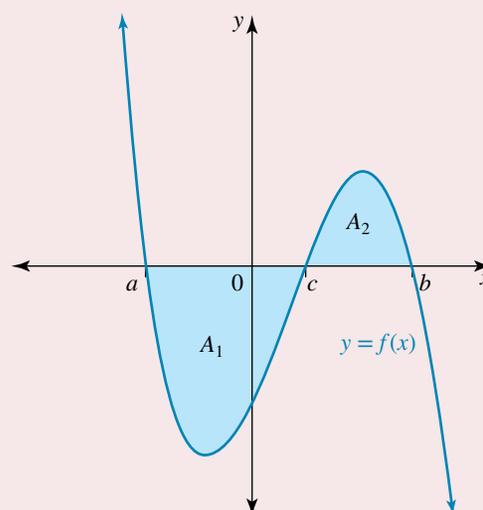
### WORKED EXAMPLE 8 Determining a shaded area

**Calculate the shaded area.**

$y = x^2 - 4x$
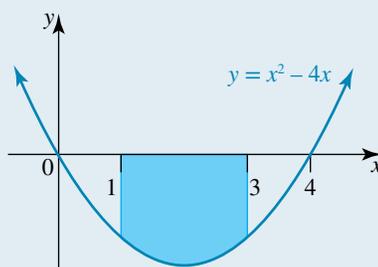
**THINK**

**WRITE**

1. Express the area in definite integral notation, showing a negative sign in front of the integral as the region is below the $x$-axis.

$$Area = -\int_{1}^{3} (x^2 - 4x)\, dx$$

| | |
|---|---|
| **2.** Antidifferentiate the integrand. | $= -\left[\frac{1}{3}x^3 - 2x^2\right]_1^3$ |
| **3.** Evaluate. | $= -\left[\left(\frac{1}{3}(3)^3 - 2(3)^2\right) - \left(\frac{1}{3}(1)^3 - 2(1)^2\right)\right]$ |
| | $= -\left[(9 - 18) - \left(\frac{1}{3} - 2\right)\right]$ |
| | $= -\left[-9 - \left(-1\frac{2}{3}\right)\right]$ |
| | $= -\left[-9 + 1\frac{2}{3}\right]$ |
| | $= -\left(-7\frac{1}{3}\right)$ |
| | $= 7\frac{1}{3}$ |
| **4.** State the solution. | The area is $7\frac{1}{3}$ square units. |

## WORKED EXAMPLE 9 Determining the area under a curve

**Calculate the area bound by the curve $y = (x^2 - 1)(x^2 - 4)$ and the $x$-axis from $x = -2$ to $x = 2$.**

**THINK**

**1.** Make a careful sketch of the given function. Shade the required region.

**WRITE**

The graph cuts the $y$-axis where $x = 0$.
∴ the $y$-intercept is $(0, 4)$.
∴ the graph cuts the $x$-axis where $y = 0$:
$$(x^2 - 1)(x^2 - 4) = 0$$
$$(x - 1)(x + 1)(x - 2)(x + 2) = 0$$
$$x = \pm 1, x = \pm 2$$



$y = (x^2 - 1)(x^2 - 4)$
$(0, 4)$
$(-2, 0)$ $(-1, 0)$ $(1, 0)$ $(2, 0)$
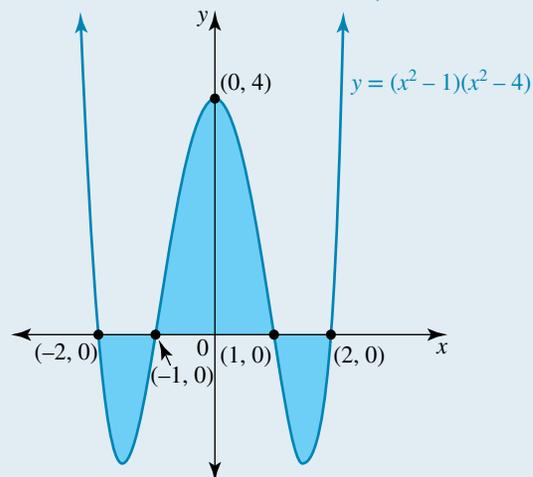
**2.** Express the area using definite integrals. Account for the negative regions by subtracting these from the positive areas. Note that the region from $x = -2$ to $x = -1$ is the same as the region from $x = 1$ to $x = 2$ due to the symmetry of the graph.

$$A = \int_{-1}^{1} (x^2 - 1)(x^2 - 4) \, dx - 2 \int_{1}^{2} (x^2 - 1)(x^2 - 4) \, dx$$

$$= \int_{-1}^{1} (x^4 - 5x^2 + 5) \, dx - 2 \int_{1}^{2} (x^4 - 5x^2 + 5) \, dx$$

**3.** Antidifferentiate and evaluate.

$$= \left[ \frac{1}{5}x^5 - \frac{5}{3}x^3 + 5x \right]_{-1}^{1} - 2 \left[ \frac{1}{5}x^3 - \frac{5}{3}x^3 + 5x \right]_{1}^{2}$$

$$= \left( \frac{1}{5}(1)^5 - \frac{5}{3}(1)^3 + 5(1) \right)$$

$$- \left( \frac{1}{5}(-1)^5 - \frac{5}{3}(-1)^3 + 5(-1) \right)$$

$$-2 \left[ \left( \frac{1}{5}(2)^5 - \frac{5}{3}(2)^3 + 5(2) \right) \right.$$

$$\left. - \left( \frac{1}{5}(1)^2 - \frac{5}{3}(1)^3 + 5(1) \right) \right]$$

$$= \left( \frac{1}{5} - \frac{5}{3} + 5 \right) - \left( -\frac{1}{5} + \frac{5}{3} - 5 \right)$$

$$-2 \left[ \left( \frac{32}{5} - \frac{40}{3} + 10 \right) - \left( \frac{1}{5} - \frac{5}{3} + 5 \right) \right]$$

$$= \frac{1}{5} - \frac{5}{3} + 5 + \frac{1}{5} - \frac{5}{3} + 5$$

$$-2 \left( \frac{32}{5} - \frac{40}{3} + 10 - \frac{1}{5} + \frac{5}{3} - 5 \right)$$

$$= \frac{2}{5} - \frac{10}{3} + 10 - \frac{64}{5} + \frac{80}{3} - 20 + \frac{2}{5} - \frac{10}{3} + 10$$

$$= -12 + 20$$

$$= 8$$

**4.** Write the answer.

The area is 8 square units.

| **8.4 Exercise** | **8.4 Exam questions** on | These questions are even better in jacPLUS! |
|---|---|---|

| **Simple familiar** | **Complex familiar** | **Complex unfamiliar** |
|---|---|---|
| 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 | 13, 14, 15, 16 | 17, 18, 19, 20 |

These questions are even better in jacPLUS!
- Receive immediate feedback
- Access sample responses
- Track results and progress

**Find all this and MORE in jacPLUS** ▶

**Simple familiar**

1. **WE7** For each of the following graphs:
   **i.** express the shaded area as a definite integral
   **ii.** hence, determine the shaded area, giving your answer in exact form.
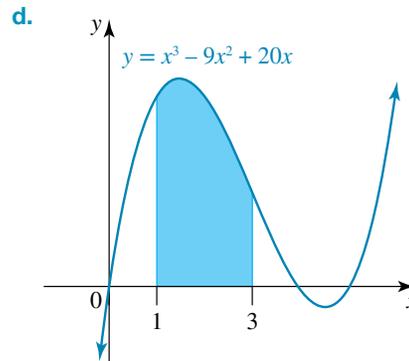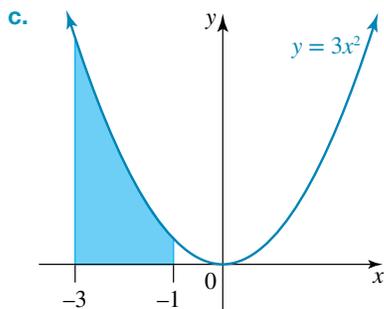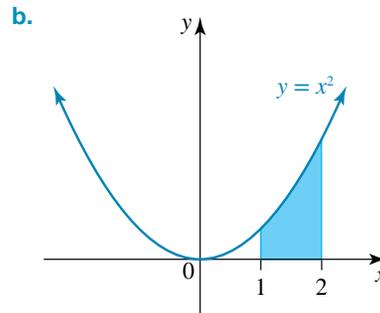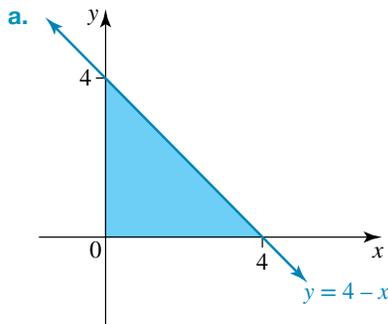
   a.
   

   b.
   
   $y = x^2$

   c.
   
   $y = 3x^2$

   d.
   
   $y = x^3 - 9x^2 + 20x$

2. For each of the following graphs:
   **i.** express the shaded area as a definite integral
   **ii.** hence, calculate the shaded area, giving your answer in exact form.

   a.
   
   $y = e^x$

   b.
   
   $y = e^{-2x}$

**c.**



$y = 2\sin 2x$

**d.**



$y = \cos\dfrac{x}{3}$

3. **WE8** Express the following shaded areas as definite integrals. Hence, calculate the shaded areas, giving your answers in exact form.

**a.**



$y = -4 - 2x$

**b.**



$y = x^2 - 4$

**c.**



$y = 1 - x^2$

**d.**



$y = x^3$
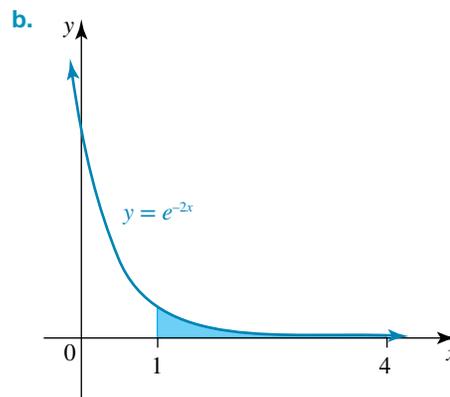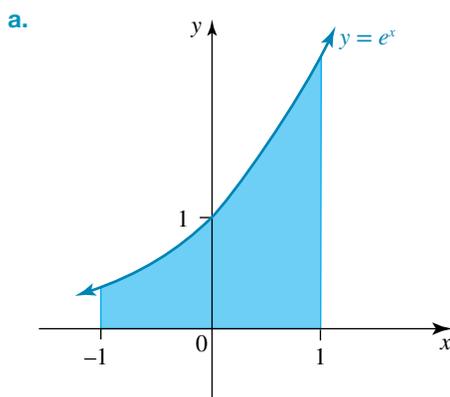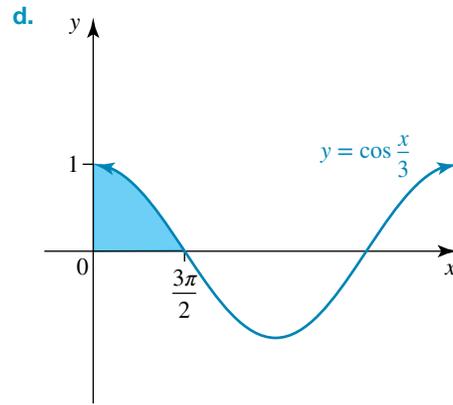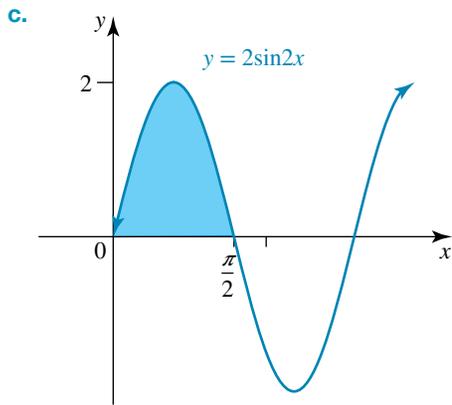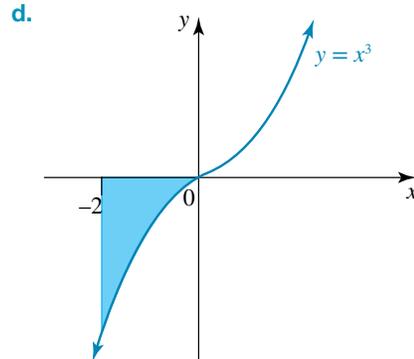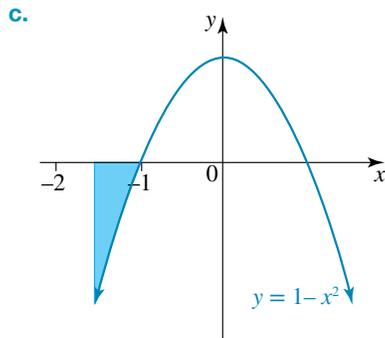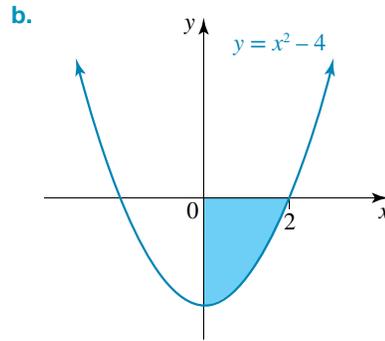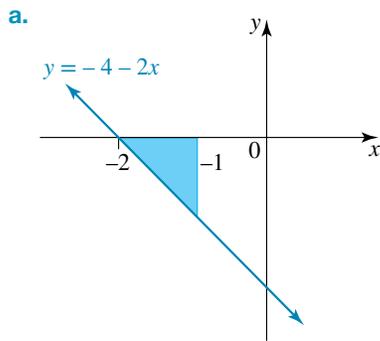
4. Express the following shaded areas as definite integrals. Hence, calculate the shaded areas, giving your answer in exact form.

**a.**



$y = -e^x$

**b.**



$y = -e^{-2x}$

**c.**


$y = \sin x$

**d.**


$y = 2 \cos \dfrac{x}{2}$
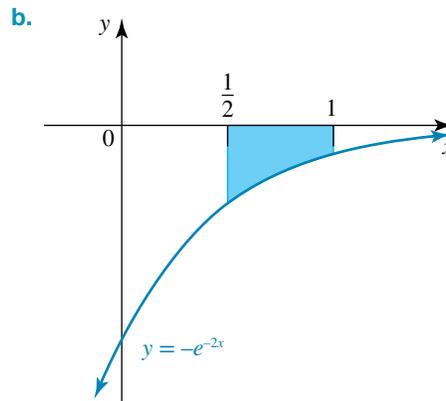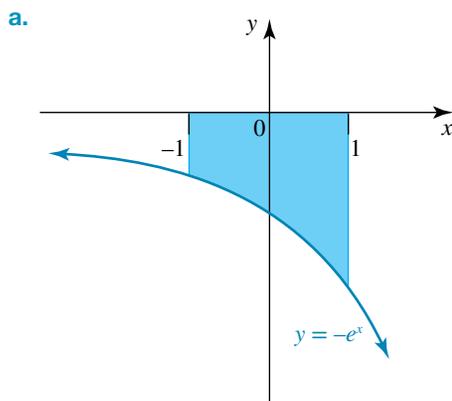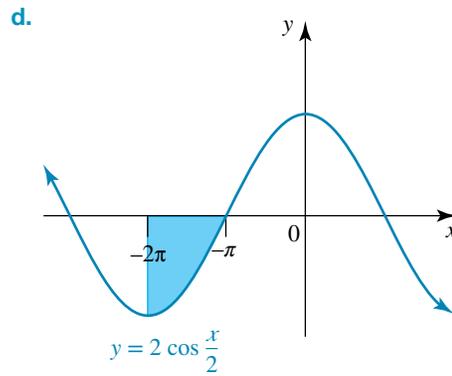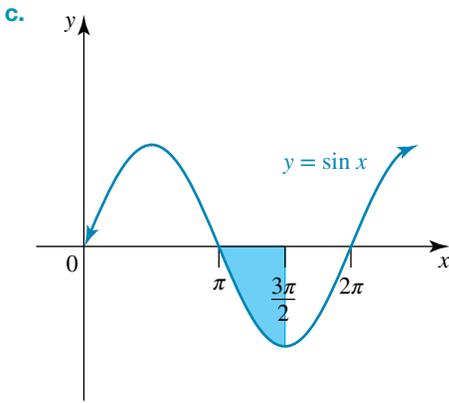
5. **WE9** Consider the function $y = (x^2 - 1)(x^2 - 9)$.

   a. Sketch the graph of the function, stating all axis intercepts.

   b. Calculate the area enclosed by the function, the lines $x = -3$ and $x = 3$, and the $x$-axis.

6. For the curve shown, calculate the area between the curve and the $x$-axis from:

   a. $x = -2$ to $x = 1$

   b. $x = 1$ to $x = 3$

   c. $x = -2$ to $x = 3$.


$y = x^3 - 2x^2 - 5x + 6$

7. For each of the following functions:

   i. sketch the function, showing clearly all axis intercepts

   ii. calculate the area bounded by the graph of the function and the $x$-axis.

   a. $g(x) = 8 - x^2$

   b. $g(x) = x^3 - 4x^2$

   c. $f(x) = x(x - 2)(x - 3)$

   d. $h(x) = (x - 1)(x + 2)(x + 5)$
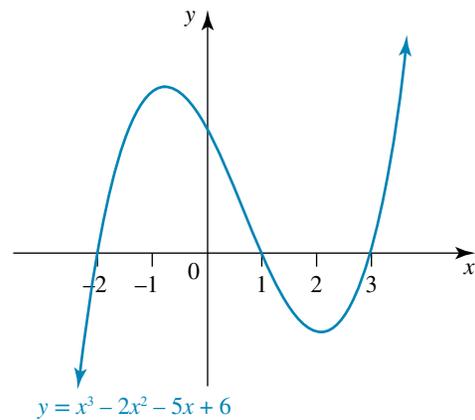
8. Determine the exact area between the curve $y = \dfrac{1}{x}$, the $x$-axis and the lines $x = \dfrac{1}{2}$ and $x = 2$. Express your answer in simplest form.

9. Calculate the exact area of the region enclosed by the $x$-axis, $y = e^{3x}$ and the lines $x = 1$ and $x = 2$.

10. Calculate the exact area of the region enclosed by the $x$-axis, $y = -\cos(x)$ and the lines $x = \dfrac{\pi}{3}$ and $x = \dfrac{5\pi}{6}$. (Use a sketch graph to assist your calculation.)

11. Determine the area bound by the curve defined by the rule $y = 2\sqrt{x}$, $x \ge 0$ and the $x$-axis from $x = 0$ to $x = 25$.

12. Calculate the area bounded by the curve $y = 2\sin(2x) + 3$, the $x$-axis and the lines $x = 0$ and $x = \pi$.

**Complex familiar**

13. Sketch the graph of $y = 1 - e^{-x}$ and hence calculate the exact area between the curve and the $x$-axis from $x = -1$ to $x = 1$.

**14.** The graph of $y = \dfrac{1}{x^2}$, $x < 0$ is shown. Calculate the area of the shaded region (that is for $-2.5 \le x \le -0.5$). Verify your answer using technology.

$y = \dfrac{1}{x^2}$
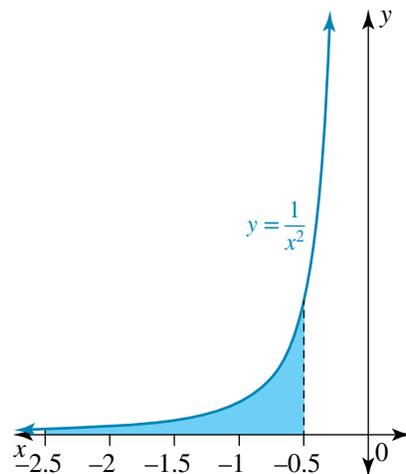
**15.** The graph of the function $y = 2\sin(x)$ is shown. Using calculus, calculate the area between the curve and the $x$-axis from $x = 0$ to $x = 4\pi$.

$y = 2\sin(x)$
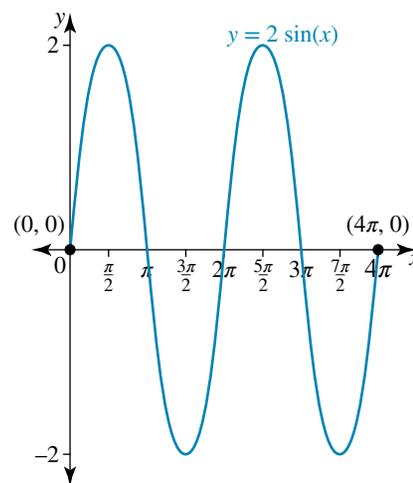
**16. a.** Differentiate $x\log_e(x)$ for $x > 0$.
  **b.** Hence, determine an antiderivative of $\log_e(x)$.
  **c.** Determine the area bounded by the graph of $y = \log_e(x)$, the $x$-axis, $x = 1$ and $x = 4$, giving your answer in exact form.
  **d.** Verify your answer using technology.

$(0, 0)$   $(4\pi, 0)$

**Complex unfamiliar**

**17.** The graph of $y = 2xe^{x^2}$ is shown.
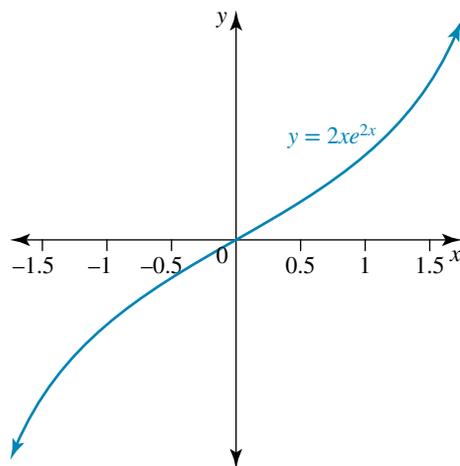
$y = 2xe^{2x}$

Determine the area, correct to 2 decimal places, between the curve and the $x$-axis from $x = -1$ to $x = 1$.

**18.** The graph of $y = \dfrac{x}{x^2 + 2}$ is shown.
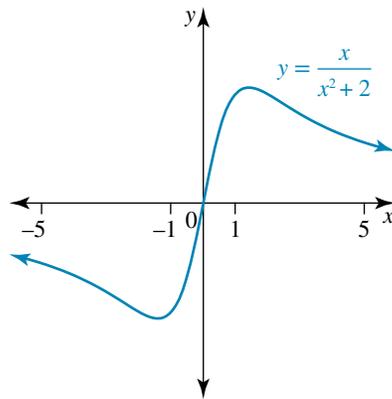


Determine the area, correct to 2 decimal places, between the curve and the $x$-axis from $x = -1$ to $x = 1$.

**19.** The graph of $f: \mathbb{R} \to \mathbb{R}, f(x) = 3x^3$ is shown.
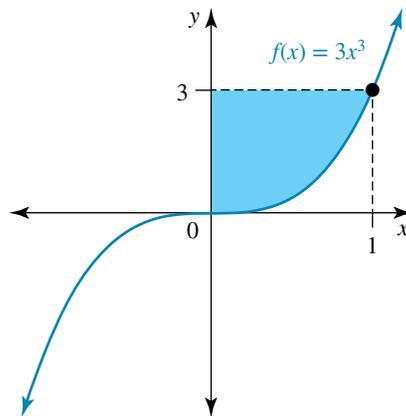


Determine the area of the shaded region.

**20.** The graph of $y = 2 \sin(x), -\dfrac{3\pi}{2} \le x \le \dfrac{3\pi}{2}$ is shown.
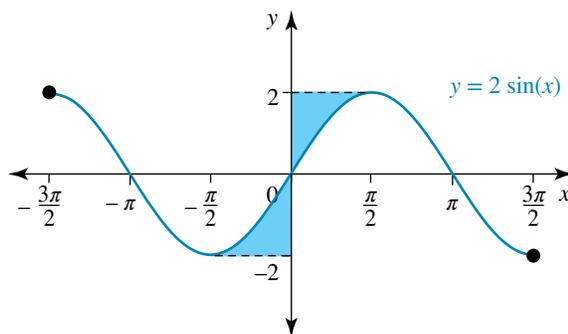


Determine the area of the shaded region.

**Fully worked solutions for this chapter are available online.**

# LESSON
## 8.5 Areas between curves

When we determine areas between two continuous functions, $f(x)$ and $g(x)$, for an interval $a \leq x \leq b$, our approach depends on whether the curves intersect or not in this interval.

### 8.5.1 When $f(x)$ and $g(x)$ do not intersect in the interval $a \leq x \leq b$

Example 1: If the region is above the $x$-axis, the lower function is subtracted from the higher function to ensure a positive answer.

Blue shaded area $= \displaystyle\int_a^b \left[ f(x) - g(x) \right] dx$
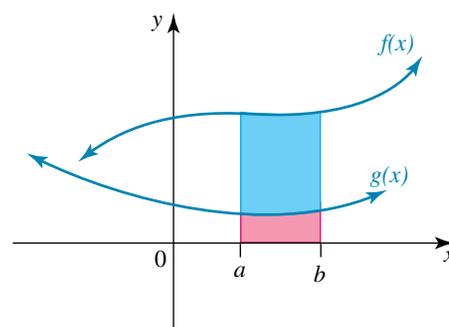
Example 2: If the region is below the $x$-axis, the lower function is subtracted from the higher function to again ensure a positive answer.

Blue shaded area $= \displaystyle\int_a^b \left[ f(x) - g(x) \right] dx$
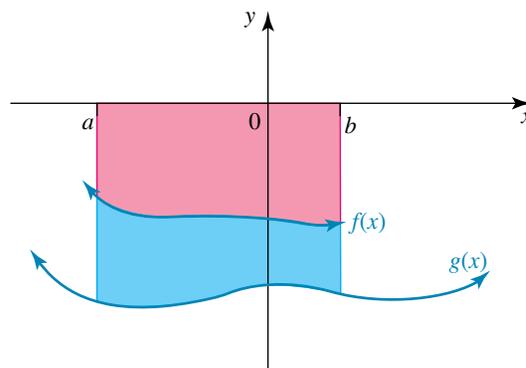
Example 3: If the region crosses the $x$-axis, the lower function is again subtracted from the higher function to ensure a positive answer.

Shaded area $= \displaystyle\int_a^b \left[ f(x) - g(x) \right] dx$
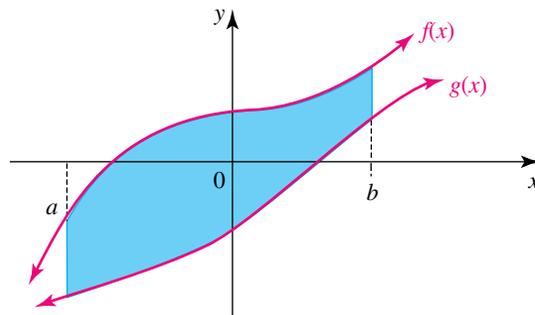
In the case where the region crosses the $x$-axis, there is no need to consider the $x$-intercepts because, if the two functions were both translated vertically by the same factor, $k$ units, to ensure they are above the $x$-axis, the constants would cancel each other. That is, $\displaystyle\int_a^b \left[ (f(x) + k) - (g(x) + k) \right] dx = \int_a^b \left[ f(x) - g(x) \right] dx$.

## WORKED EXAMPLE 10 Determining the area between two curves

The functions $f: \mathbb{R} \to \mathbb{R}, f(x) = (x+2)(2-x)$ and
$g: \mathbb{R} \to \mathbb{R}, g(x) = x + 2$ are shown.

a. The graphs intersect at $(m, 0)$ and $(p, q)$. Determine the values of the constants $m, p$ and $q$.

b. Calculate the area bound by the curve and the line.
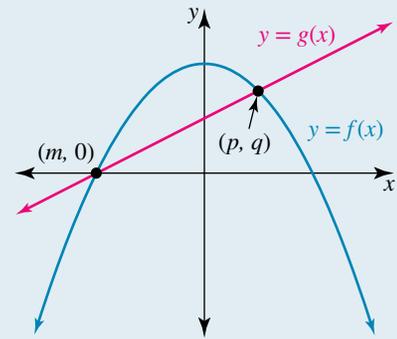


**THINK**

**a. 1.** Points of intersection are found by solving the equations simultaneously, so equate the equations and solve for $x$.

**2.** Determine the corresponding $y$-values.

**3.** State the solution.

**b. 1.** Determine whether $f > g$ or $f < g$.

**2.** Express the area in definite integral notation and simplify the expression within the integral.

**3.** Antidifferentiate and evaluate.

**4.** Write the answer.

**WRITE**

**a.**
$$x + 2 = (x+2)(2-x)$$
$$x + 2 = 4 - x^2$$
$$x^2 + x - 2 = 0$$
$$(x+2)(x-1) = 0$$
$$x = -2, x = 1$$

When $x = -2$, $y = -2 + 2 = 0$.
When $x = 1$, $y = 1 + 2 = 3$.

$m = -2, p = 1, q = 3$

**b.** As $f(x) = 4 - x^2$ lies above $g(x) = x + 2$, $f > g$.

$$A = \int_{-2}^{1} (f(x) - g(x)) \, dx$$

$$= \int_{-2}^{1} \left(4 - x^2 - (x+2)\right) dx$$

$$= \int_{-2}^{1} \left(-x^2 - x + 2\right) dx$$

$$= \left[-\frac{1}{3}x^3 - \frac{1}{2}x^2 + 2x\right]_{-2}^{1}$$

$$= \left(-\frac{1}{3}(1)^3 - \frac{1}{2}(1)^2 + 2(1)\right)$$

$$\quad - \left(-\frac{1}{3}(-2)^3 - \frac{1}{2}(-2)^2 + 2(-2)\right)$$

$$= -\frac{1}{3} - \frac{1}{2} + 2 - \frac{8}{3} + 2 + 4$$

$$= -3 - \frac{1}{2} + 8$$

$$= 4\frac{1}{2}$$

The area is 4.5 square units.

| TI | THINK | DISPLAY/WRITE | CASIO | THINK | DISPLAY/WRITE |
|---|---|---|---|---|

**a. 1.** On a Graphs page, complete the entry line for function 1 as: $f1(x) = (x+2)(2-x)$ then press ENTER.

**a. 1.** On a Graph screen, complete the entry line $y1 = (x+2)(2-x)$ then press EXE. Complete the entry line $y2 = x+2$ and press EXE. Press F6 to DRAW.
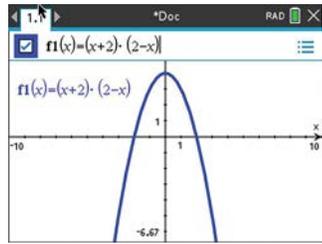
**2.** Complete the entry line for function 2 as: $f2(x) = x+2$ then press ENTER.
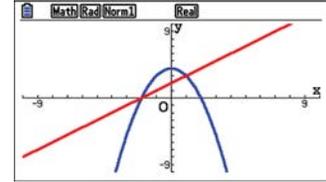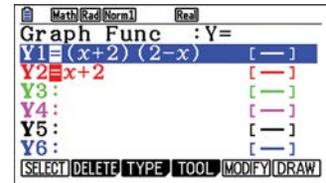
**2.** To determine the points of intersection, select G-Solv by pressing F5, then F5 again for Intersect. Select an intersection and press EXE. Repeat the process for the second intersection.

**3.** To determine the points of intersection, press MENU, then select:
6: Analyze Graph
4: Intersection
When prompted, select a point before and after an intersection point to identify coordinates. Repeat this process to determine the other point of intersection.
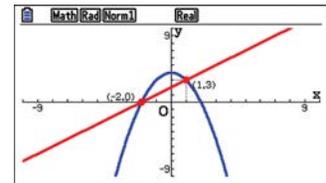
**b. 1.** To determine the area bounded by the two curves, select G-Solv by pressing F5, then F6 for more options.
Select $\int dx$ by pressing F3 then select Intsect by pressing F3 again. Select an intersection and press EXE. Repeat the process for the second intersection.

**b. 1.** To find the area bounded by the two curves, press MENU, then select:
6: Analyze Graph
7: Bounded Area
Click on the left-most point of intersection, then click on the right-most point of intersection.

**2.** The answer appears on the screen. The area is 4.5 square units.

**2.** The answer appears on the screen. The area is 4.5 square units.

CHAPTER 8 Further integration 387
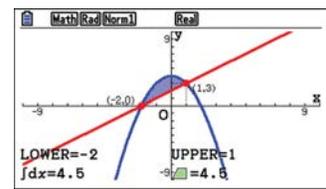
# 8.5.2 When $f(x)$ and $g(x)$ intersect in the interval $a \le x \le b$

Let the points of intersection of $f(x)$ and $g(x)$ in the interval $a \le x \le b$
be $x = c$ and $x = d$, where $c < d$.

The area is found by considering the separate intervals:

$a \le x \le c$, $c \le x \le d$ and $d \le x \le b$

For each section, the lower function is subtracted from the higher
function to ensure a positive answer.

Shaded area =

$$\int_a^c \left[ g(x) - f(x) \right] dx + \int_c^d \left[ f(x) - g(x) \right] dx + \int_d^b \left[ g(x) - f(x) \right] dx$$
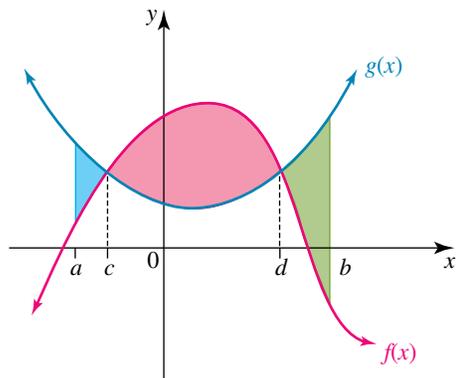


## WORKED EXAMPLE 11 Determining area with graphs intersect in the interval

Consider the functions $f(x) = \dfrac{4}{x}$ and $g(x) = x$.

**a.** Determine the values of $x$ where the functions intersect.
**b.** Sketch the graphs of the functions on the same axes. Shade the region between the two functions
and $x = 1$ and $x = 3$.
**c.** Hence, calculate the area between $f(x)$ and $g(x)$ from $x = 1$ to $x = 3$.

**THINK**

**a. 1.** State the two functions.

**2.** Let $f(x) = g(x)$ to calculate the
values of $x$ where the graphs
intersect.

**3.** Solve for $x$.

**b.** Sketch $f(x)$ and $g(x)$ on the
same axis and shade the region
between the two curves from
$x = 1$ to $x = 3$.

**WRITE**

**a.** $f(x) = \dfrac{4}{x}$, $g(x) = x$

For points of intersection, $x = \dfrac{4}{x}$.

$$x^2 = 4$$
$$x^2 - 4 = 0$$
$$(x + 2)(x - 2) = 0$$
$$x = -2 \text{ and } x = 2$$

**b.**
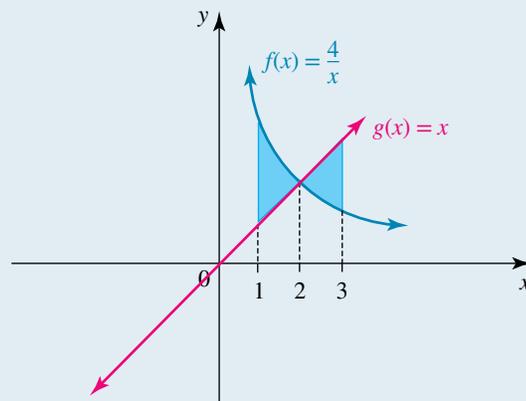
**c. 1.** State the area as the sum of two integrals for the two subintervals.

**c.** $\text{Area} = \int_1^2 \left( \dfrac{4}{x} - x \right) dx + \int_2^3 \left( x - \dfrac{4}{x} \right) dx$

**2.** Antidifferentiate.

$$= \left[ 4 \log_e (x) - \dfrac{1}{2}x^2 \right]_1^2 + \left[ \dfrac{1}{2}x^2 - 4 \log_e (x) \right]_2^3$$

**3.** Evaluate the two integrals.

$$= \left[ 4 \log_e (2) - \dfrac{1}{2}(2)^2 \right] - \left[ 4 \log_e (1) - \dfrac{1}{2}(1)^2 \right]$$
$$+ \left\{ \left[ \dfrac{1}{2}(3)^2 - 4 \log_e (3) \right] - \left[ \dfrac{1}{2}(2)^2 - 4 \log_e (2) \right] \right\}$$
$$= \left[ 4 \log_e (2) - 2 \right] - \left[ 4 \log_e (1) - \dfrac{1}{2} \right]$$
$$+ \left\{ \left[ \dfrac{9}{2} - 4 \log_e (3) \right] - \left[ 2 - 4 \log_e (2) \right] \right\}$$
$$= 4 \log_e (2) - 2 - 0 + \dfrac{1}{2} + \dfrac{9}{2} - 4 \log_e (3) - 2 + 4 \log_e (2)$$

**4.** Simplify.

$$= 4 \log_e \left( \dfrac{4}{3} \right) + 1$$

**5.** State the area.

The area is $4 \log_e \left( \dfrac{4}{3} \right) + 1$ or approximately 2.151 square units.

---

## WORKED EXAMPLE 12 Determining area with graphs intersect in the interval

The graphs of $f(x) = 3 \sin(2x)$ and $g(x) = 3 \cos(2x)$ are shown for $x \in [0, \pi]$.

**a.** Determine the coordinates of the point(s) of intersection of $f$ and $g$ for the interval $\left[ 0, \dfrac{\pi}{2} \right]$.

**b.** Using calculus, determine the area enclosed between the curves on the interval $\left[ 0, \dfrac{\pi}{2} \right]$.
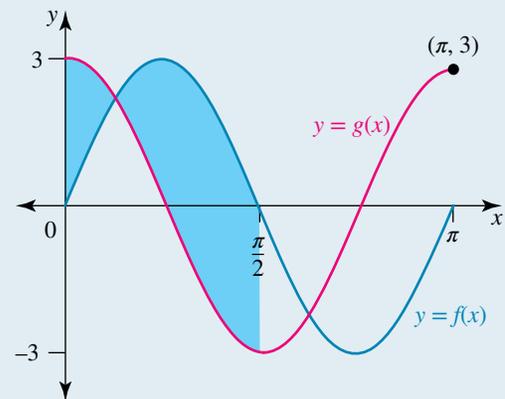


**THINK**

**a. 1.** Use simultaneous equations to determine where the graphs intersect, and equate the two equations.

**WRITE**

**a.** $3 \sin(2x) = 3 \cos(2x)$

$\dfrac{3 \sin(2x)}{3 \cos(2x)} = 1, 0 \leq x \leq \dfrac{\pi}{2}.$

$\tan(2x) = 1, 0 \leq x \leq \dfrac{\pi}{2}$

**2.** Solve for $2x$ for $x \in \left[0, \dfrac{\pi}{2}\right]$.

$2x = \dfrac{\pi}{4}$

$\therefore x = \dfrac{\pi}{8}$

**3.** Calculate the corresponding $y$-value.

$f\left(\dfrac{\pi}{8}\right) = 3\sin\left(\dfrac{\pi}{4}\right)$

$= 3 \times \dfrac{1}{\sqrt{2}}$

$= \dfrac{3\sqrt{2}}{2}$

**4.** Write the solution.

The coordinates are $\left(\dfrac{\pi}{8}, \dfrac{3\sqrt{2}}{2}\right)$.

**b. 1.** Determine when $f > g$ and $f < g$.

**b.** When $0 < x < \dfrac{\pi}{8} >, g > f$.

When $\dfrac{\pi}{8} < x < \dfrac{\pi}{2}$, $f > g$.

**2.** Express each area individually in definite integral notation.

The area is equal to:

$$A = \int_{0}^{\frac{\pi}{8}} (3\cos(2x) - 3\sin(2x))\, dx + \int_{\frac{\pi}{8}}^{\frac{\pi}{2}} (3\sin(2x) - 3\cos(2x))\, dx$$

**3.** Use calculus to antidifferentiate and evaluate.

$$= \left[\dfrac{3}{2}\sin(2x) + \dfrac{3}{2}\cos(2x)\right]_{0}^{\frac{\pi}{8}} + \left[-\dfrac{3}{2}\cos(2x) - \dfrac{3}{2}\sin(2x)\right]_{\frac{\pi}{8}}^{\frac{\pi}{2}}$$

$$= \dfrac{3}{2}\sin\left(\dfrac{\pi}{4}\right) + \dfrac{3}{2}\cos\left(\dfrac{\pi}{4}\right) - \left(\dfrac{3}{2}\sin(0) + \dfrac{3}{2}\cos(0)\right)$$

$$-\dfrac{3}{2}\cos(\pi) - \dfrac{3}{2}\sin(\pi) - \left(-\dfrac{3}{2}\cos\left(\dfrac{\pi}{4}\right) - \dfrac{3}{2}\sin\left(\dfrac{\pi}{4}\right)\right)$$

$$= \dfrac{3}{2} \times \dfrac{\sqrt{2}}{2} + \dfrac{3}{2} \times \dfrac{\sqrt{2}}{2} - 0 - \dfrac{3}{2} + \dfrac{3}{2} - 0 + \dfrac{3}{2} \times \dfrac{\sqrt{2}}{2} + \dfrac{3}{2} \times \dfrac{\sqrt{2}}{2}$$

$$= \dfrac{3\sqrt{2}}{4} + \dfrac{3\sqrt{2}}{4} + \dfrac{3\sqrt{2}}{4} + \dfrac{3\sqrt{2}}{4}$$

$$= 3\sqrt{2}$$

**4.** Write the answer.

The area is $3\sqrt{2}$ square units.

**learn on**

| **8.5 Exercise** | **8.5 Exam questions** on |
|---|---|

| **Simple familiar** | **Complex familiar** | **Complex unfamiliar** |
|---|---|---|
| 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 | 13, 14, 15, 16 | 17, 18, 19, 20 |

These questions are even better in jacPLUS!
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**Simple familiar**

1. **WE10** Calculate the shaded area in each of the following diagrams.

a.



$y = 3x$
$y = 4 - x^2$

b.



$y = x^2$
$y = 8 - x^2$

2. Determine the shaded area in each of the following diagrams.

a.



$y = x^3$
$y = 3x$
$\sqrt{3}$

b.



$y = e^x$
$y = 9 - x^2$
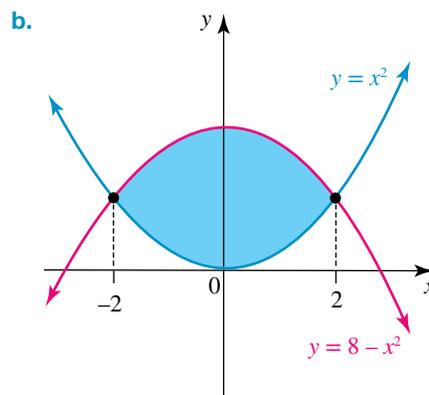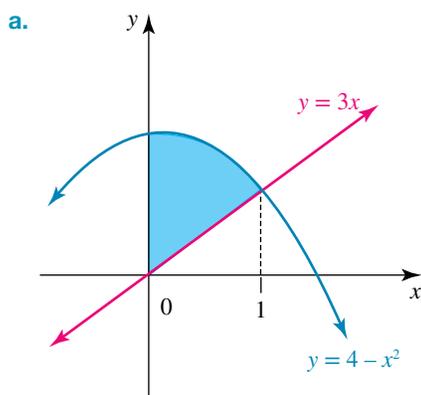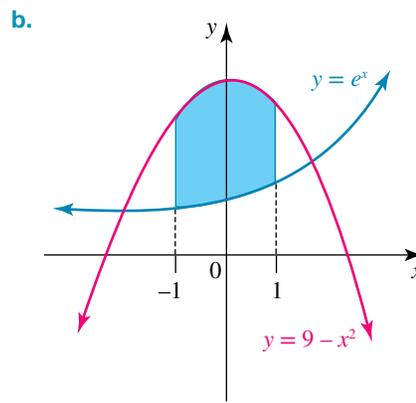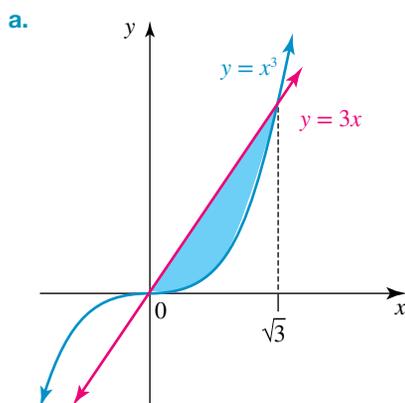
3. Determine the shaded area in each of the following diagrams.

a.



$y = x$
$y = -e^x$

b.



$y = x^2 - 5$
$y = -4$
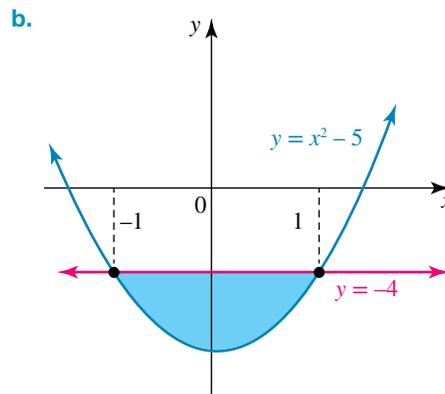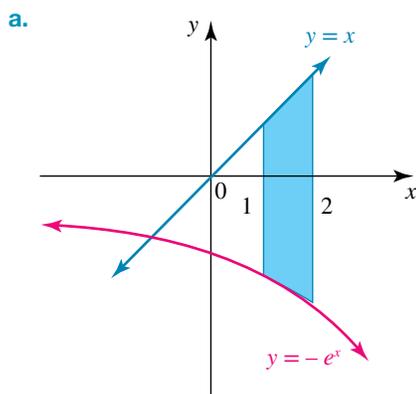
**4.** **MC** Identify which one of the following does not equal the shaded area shown.

A. $\displaystyle\int_1^5 g(x)\,dx - \int_1^5 f(x)\,dx$

B. $\displaystyle\int_1^5 g(x)\,dx + \int_5^1 f(x)\,dx$

C. $\displaystyle\int_1^5 f(x)\,dx - \int_1^5 g(x)\,dx$

D. $\displaystyle\int_1^5 \left[g(x) - f(x)\right]dx$
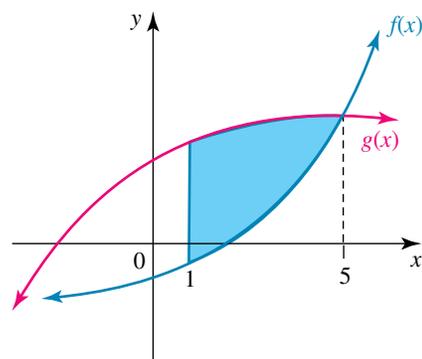


**5.** **MC** The area bounded by the curves $f(x)$, $g(x)$ and the lines $x = -3$ and $x = -1$ is equal to:

A. $\displaystyle\int_{-1}^{-3} \left[f(x) - g(x)\right]dx$

B. $\displaystyle\int_{-3}^{-1} \left[f(x) + g(x)\right]dx$

C. $\displaystyle\int_{-3}^{-1} \left[g(x) - f(x)\right]dx$

D. $\displaystyle\int_{-3}^{-1} \left[f(x) - g(x)\right]dx$
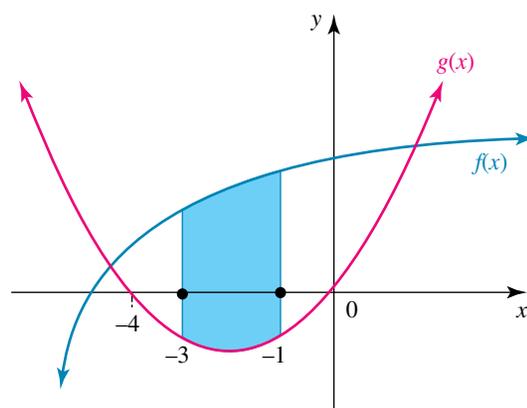


**6.** **MC** The shaded area shown is equal to:

A. $\displaystyle\int_0^4 \left[f(x) - g(x)\right]dx$

B. $\displaystyle\int_0^3 \left[g(x) - f(x)\right]dx + \int_3^4 \left[f(x) - g(x)\right]dx$

C. $\displaystyle\int_0^4 \left[g(x) - f(x)\right]dx$

D. $\displaystyle\int_0^3 \left[f(x) - g(x)\right]dx + \int_3^4 \left[g(x) - f(x)\right]$
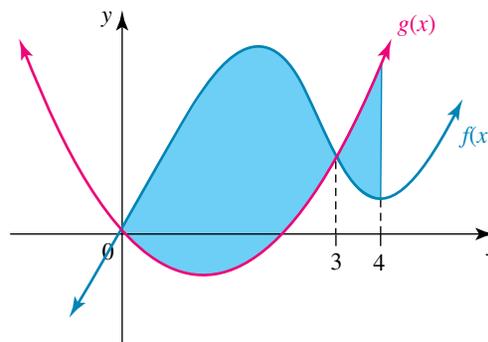


**7.** **WE11** Consider the functions $f(x) = x^3$ and $g(x) = x$.

a. Determine the values of $x$ where the functions intersect.
b. Sketch the graphs of the functions on the same axes.
c. Hence, calculate the area between $f(x)$ and $g(x)$, giving an exact answer.

8. Consider the functions $f(x) = x^3 + 2x$ and $g(x) = 3x^2$.
   a. Determine the values of $x$ where the functions intersect.
   b. Sketch the graphs of the functions on the same axes.
   c. Hence, calculate the area between $f(x)$ and $g(x)$, giving an exact answer.

9. The graphs of $g(x) = \sqrt{x}$ and the line $f(x) = 4$ are shown. Determine the coordinates of the point of intersection between $f$ and $g$, and hence calculate the area of the shaded region.
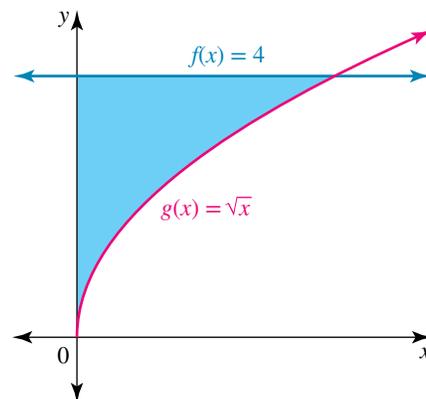
10. Determine the area enclosed between the curve $f(x) = (x - 3)^2$ and the line $g(x) = 9 - x$.

11. **WE12** Consider the functions $f(x) = \sin(x)$ and $g(x) = -\cos(x)$ for $x \in [0, \pi]$.
   a. Determine the coordinates of the points of intersection of $f$ and $g$ for the given domain.
   b. Using calculus, determine the area enclosed between the curves for the given interval.

12. **MC** The graphs of $y = 0.25x^2 (x - 3)(x + 3)$ and $y = -0.5x^2 (x - 3)(x + 3)$ are shown.
   The area of the region enclosed between the curves and the lines $x = -3$ and $x = 3$ is:
   A. 60.75 units$^2$
   B. 48.6 units$^2$
   C. 121.5 units$^2$
   D. 24.3 units$^2$
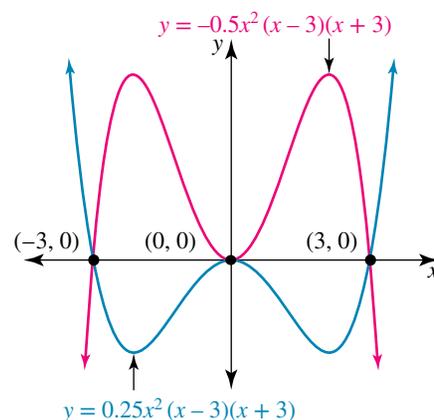
**Complex familiar**

13. Determine the area between the curves $y = \sqrt{3} - \sin(2x)$ and $y = \sin(2x)$ from $x = 0$ to $x = \dfrac{\pi}{4}$.

14. Determine the exact area bounded by the curves $y = e^x$ and $y = 3 - 2e^{-x}$.

15. The graphs of $f(x) = 3x^3 - x^4$ and $g(x) = -x + 3$ are shown.
   a. The graphs intersect at the points $(a, b)$ and $(c, 0)$. Determine the constants $a$, $b$ and $c$.
   b. Determine the area enclosed between the curves from $x = a$ to $x = c$.
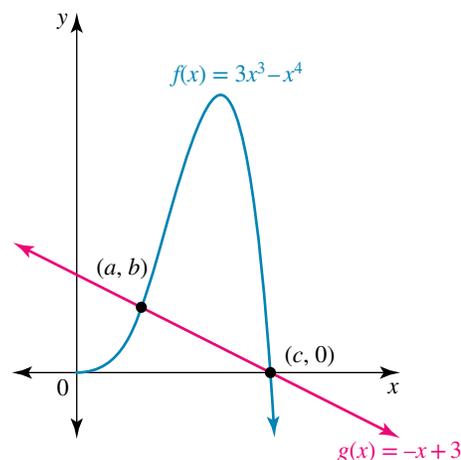
**16.** The graph shows the cross-section of a bricked archway. (All measurements are in metres.)

   **a.** Determine the $x$-intercepts of $f(x)$.

   **b.** Determine the $x$-intercepts of $g(x)$.

   **c.** Determine the cross-sectional area of the brickwork.



$$f(x) = 4 - \frac{1}{4}x^2$$

$$g(x) = 3 - \frac{1}{3}x^2$$

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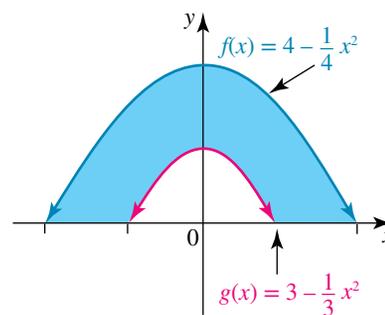
**Complex unfamiliar**

**17.** The edge of a garden bed can be modelled by the rule

$$y = 0.5 \sin\left(\frac{x}{2}\right) + 2$$

The bed has edges defined by $y = 0$, $x = 0$ and $x = 4\pi$.
All measurements are in metres.
Topsoil is going to be used on the garden bed in preparation for new planting for spring. The topsoil is to be spread so that it is uniformly 50 cm thick. Determine the amount of soil, to the nearest cubic metre, that will be needed for the garden bed.



**18.** A stone footbridge over a creek is shown along with the mathematical profile of the bridge. The arch of the footbridge can be modelled by a quadratic function for $x \in [-5, 5]$, with all measurements in metres.
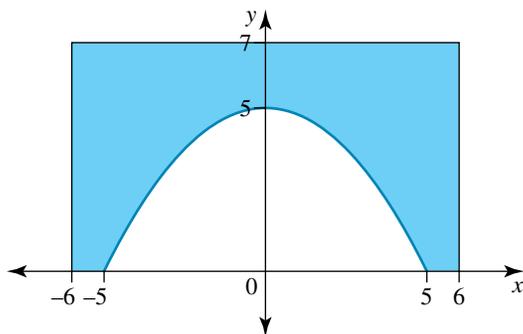




The shaded area represents the side of the stone footbridge. The width of the footbridge is 3 metres.
Determine the volume of stones used in the construction of the footbridge.

**19.** The graph of the function $y = \dfrac{1}{x} + x^3 - 4$ is shown. The tangent to the curve at $x = 1$ is also shown.

Determine the area of the shaded region.



$$y = \frac{1}{x} + x^3 - 4$$

**20.** The graph of $y = \dfrac{5x}{x^2 + 1}$ is shown. The tangent to the curve at $x = -0.5$ is also shown.

Determine the area of the shaded region. Give your answer correct to 2 decimal places.



**Fully worked solutions for this chapter are available online.**

# LESSON
## 8.6 Modelling and solving with definite integrals

## 8.6.1 Definite integrals and the trapezoidal rule

Earlier in this chapter, we used areas of rectangles, using both left end-point and right end-point rectangles, and trapeziums to approximate the area under a curve. We also found that using trapeziums gives a better approximation to the area under the curve, if the same width as the rectangles is used.

Consider the function $y = f(x)$, where $w$ is the width of each trapezium.

Area using two trapeziums

$$= \frac{w}{2}\left((fx_0) + f(x_1)\right) + \frac{w}{2}\left((fx_1) + f(x_2)\right)$$

$$= \frac{w}{2}\left[f(x_0) + 2f(x_1) + f(x_2)\right]$$
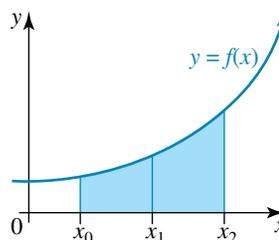
Area using three trapeziums

$$= \frac{w}{2}((fx_0) + f(x_1)) + \frac{w}{2}((fx_1) + f(x_2)) + \frac{w}{2}(f(x_2) + f(x_3))$$

$$= \frac{w}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + f(x_3)\right]$$

$$= \frac{w}{2}\left[f(x_0) + 2(f(x_1) + f(x_2)) + f(x_3)\right]$$
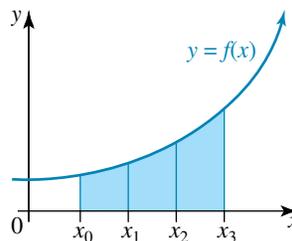


This pattern can be continued to obtain the general trapezoidal rule to estimate the area under a curve. In this method, the region of the function $y = f(x)$ between $x = a$ and $x = b$ is divided into $n$ intervals, where $n$ is an even number. The width of an interval, $w$, is found using $w = \dfrac{b-a}{n}$ and $x_0 = a$ and $x_n = b$.

Using the trapezoidal rule, the area is approximated by fitting trapeziums across each interval.
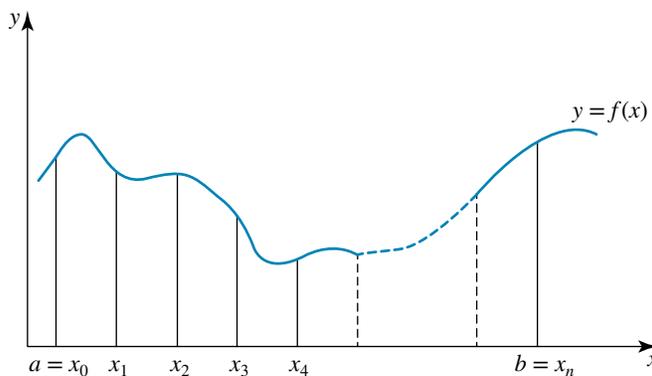


This gives an approximate value of the definite integral $\displaystyle\int_a^b f(x)\,dx$.

---

**The trapezoidal rule and definite integrals**

$$\int_a^b f(x)\,dx \approx \frac{w}{2}\left[f(x_0) + 2(f(x_1) + f(x_2) + \dots f(x_{n-1})) + f(x_n)\right]$$

where

$$w = \frac{b-a}{n}.$$

---

### WORKED EXAMPLE 13 Using the trapezoidal rule and definite integrals

Use the trapezoidal rule with 4 strips to determine the approximate value of the definite integral $\displaystyle\int_1^3 \ln x\,dx$. Give your answer correct to 4 decimal places.

**THINK**

1. There are 4 strips, so $n = 4$ (even).
   The integration is for $1 \le x \le 3$, so $a = 1$ and $b = 3$ to calculate $w$.

**WRITE**

$$w = \frac{b-a}{n}$$
$$= \frac{3-1}{4}$$
$$= \frac{1}{2}$$

2. Construct a table of values for $x_0$ to $x_n$.

| $n$ | $x_n$ | $f(x) = \ln x$ |
|---|---|---|
| 0 | 1 | 0 |
| 1 | 1.5 | $\ln(1.5)$ |
| 2 | 2 | $\ln(2)$ |
| 3 | 2.5 | $\ln(2.5)$ |
| 4 | 3 | $\ln(3)$ |

3. State the trapezoidal rule.

$$\int_a^b f(x)\,dx \approx \frac{w}{2}\left[f(x_0) + 2\left(f(x_1) + f(x_2) + \dots f(x_{n-1})\right) + f(x_n)\right]$$

4. Substitute from the table of values.

$$\text{Area} \approx \frac{\left(\frac{1}{2}\right)}{2}\left[0 + 2\left(\ln(1.5) + \ln(2) + \ln(2.5)\right) + \ln(3)\right]$$

5. Simplify using log laws and approximate to 4 decimal places.

$$\approx \frac{1}{4}\left[0 + 2\ln(1.5 \times 2 \times 2.5) + \ln(3)\right]$$

$$\approx \frac{1}{4}\left[0 + 2\ln(7.5) + \ln(3)\right]$$

$$\approx 1.2821 \quad \text{(to 4 decimal places)}$$

## 8.6.2 Total change as the integral of instantaneous change

If we are given the equation for the rate of change and we want to determine the amount that has changed over a particular time period, we would integrate the rate of change equation using the starting and finishing times as the terminals.

For example, if we know the rate of water flowing, $\frac{dV}{dt}$ in L/min, and we want to determine the amount of liquid that has flowed in the first 30 minutes, we would evaluate $\displaystyle\int_0^{30} \frac{dV}{dt}\,dt$.

### WORKED EXAMPLE 14 Determining total change

It is common practice to include heating in concrete slabs when new residential homes or units are being constructed, because it is more economical than installing heating later. A typical reinforced concrete slab, 10–15 centimetres thick, has tubing installed on top of the reinforcement, then concrete is poured on top. When the system is complete, hot water runs through the tubing. The concrete slab absorbs the heat from the water and releases it into the area above.

The amount of water in litres/minute flowing through the tubing over $t$ minutes can be modelled by the rule

$$\frac{dV}{dt} = 2\left(\cos\left(\frac{\pi t}{3}\right) + \sin\left(\frac{\pi t}{9}\right) + 3\right)$$

**The graph of this function is shown.**

**a.** Calculate the rate of flow of water, correct to
 2 decimal places, at:
 **i.** 4 minutes
 **ii.** 8 minutes.
**b.** Determine the period of the given function.
**c.** Determine the volume of water that flows through the
 tubing during the time period for one
 whole cycle.

$$\frac{dV}{dt} = 2\left[\cos\left(\frac{\pi t}{3}\right) + \sin\left(\frac{\pi t}{9}\right) + 3\right]$$

(0, 8)

| THINK | WRITE |
|---|---|
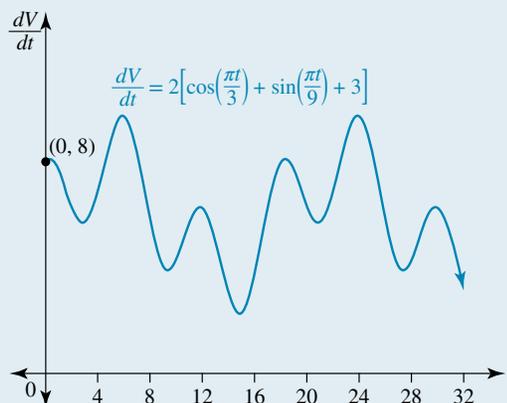
**a.** **i.** Substitute $t = 4$ into the given
 equation and evaluate.

**a.** **i.** $\dfrac{dV}{dt} = 2\left(\cos\left(\dfrac{\pi t}{3}\right) + \sin\left(\dfrac{\pi t}{9}\right) + 3\right)$

When $t = 4$, $\dfrac{dV}{dt} = 2\left(\cos\left(\dfrac{4\pi}{3}\right) + \sin\left(\dfrac{4\pi}{9}\right) + 3\right)$

$= 6.97$

The rate at 4 minutes is 6.97 litres/minute.

**ii.** Substitute $t = 8$ into the given
 equation and evaluate.

**ii.** When $t = 8$,

$\dfrac{dV}{dt} = 2\left(\cos\left(\dfrac{8\pi}{3}\right) + \sin\left(\dfrac{8\pi}{9}\right) + 3\right)$

$= 5.68$

The rate at 8 minutes is 5.68 litres/minute.

**b.** Determine the cycle for the
 function by analysing the
 shape of the graph.

**b.** A complete cycle for the function occurs between $t = 6$ and
 $t = 24$, so the period is $24 - 6 = 18$ minutes.

**c.** **1.** The area under the curve of
 the equation of the rate of flow
 gives the total volume that has
 flowed through the tubing.

**c.** $A = \displaystyle\int_6^{24} 2\left(\cos\left(\dfrac{\pi t}{3}\right) + \sin\left(\dfrac{\pi t}{9}\right) + 3\right) dt$

$= 2\displaystyle\int_6^{24} \left(\cos\left(\dfrac{\pi t}{3}\right) + \sin\left(\dfrac{\pi t}{9}\right) + 3\right) dt$

**2.** Antidifferentiate and evaluate.

$= 2\left[\dfrac{3}{\pi} \sin\left(\dfrac{\pi t}{3}\right) - \dfrac{9}{\pi} \cos\left(\dfrac{\pi t}{9}\right) + 3t\right]_6^{24}$

$= 2\left(\left(\dfrac{3}{\pi} \sin(8\pi) - \dfrac{9}{\pi} \cos\left(\dfrac{8\pi}{3}\right) + 72\right)\right.$

$\left. - \left(\dfrac{3}{\pi} \sin(2\pi) - \dfrac{9}{\pi} \cos\left(\dfrac{2\pi}{3}\right) + 18\right)\right)$

$= 2\left(-\dfrac{9}{\pi} \cos\left(\dfrac{2\pi}{3}\right) + 72 + \dfrac{9}{\pi} \cos\left(\dfrac{2\pi}{3}\right) - 18\right)$

$= 2 \times 54$

$= 108$

**3.** Write the answer.

The volume of water that passes through the tubing during
 one cycle is 108 litres.

In economics, the instantaneous rate of change with respect to the number of items is also referred to as the marginal rate of change.

For example, if the revenue function for selling $x$ units is given as $R(x)$ dollars, then the marginal revenue is given by $\dfrac{dR}{dx}$ dollars per unit. The marginal revenue is the extra revenue received for selling one more unit after a particular number of units have been sold.

---

### WORKED EXAMPLE 15 Determining marginal rates of change

A manufacturer of a game knows that the revenue, $R$, for selling $x$ games is $R = 500\sqrt{x}$. The costs, $C$, to produce $x$ games is $C = 2000 + x\sqrt{x}$.
a. Calculate the profit made when 25 games were sold.
b. Determine the average profit per game when 25 games were sold.
c. For 100 games, calculate and interpret:
   i. the marginal revenue
   ii. the marginal cost
   iii. the marginal profit.

| THINK | WRITE |
|---|---|
| **a. 1.** State the revenue equation. | **a.** $R = 500\sqrt{x}$ |
| **2.** Substitute $x = 25$ and evaluate. | $R = 500\sqrt{25} = 2500$ |
| **3.** State the cost equation. | $C = 2000 + x\sqrt{x}$ |
| **4.** Substitute $x = 25$ and evaluate. | $C = 2000 + 25\sqrt{25} = 2125$ |
| **5.** Profit = revenue − cost | Profit = $2500 - 2125$ <br> Profit = $\$375$ |
| **b.** Average profit per game $= \dfrac{\text{profit}}{\text{number}}$ | **b.** Profit per game $= \dfrac{375}{25}$ <br> $= \$15$ |
| **c. i. 1.** State the revenue equation. | **c. i.** $R = 500\sqrt{x}$ |
| **2.** Differentiate with respect to $x$. | $R = 500x^{\frac{1}{2}}$ |
| **3.** Substitute $x = 100$ and evaluate. | $\dfrac{dR}{dx} = 500 \times \dfrac{1}{2}x^{\frac{-1}{2}}$ <br> $\dfrac{dR}{dx} = \dfrac{250}{\sqrt{x}}$ <br> $\dfrac{dR}{dx} = \dfrac{250}{\sqrt{100}}$ <br> $= 25$ <br> The marginal revenue at $x = 100$ is $\$25$, so the approximate revenue from selling the 101st game is $\$25$. |

**ii. 1.** State the cost equation.

**ii.** $C = 2000 + x\sqrt{x}$

**2.** Differentiate with respect to $x$.

$C = 2000 + x^{\frac{3}{2}}$

**3.** Substitute $x = 100$ and evaluate.

$\dfrac{dC}{dx} = \dfrac{3}{2}x^{\frac{1}{2}}$

$\dfrac{dC}{dx} = \dfrac{3\sqrt{x}}{2}$

$\dfrac{dC}{dx} = \dfrac{3\sqrt{100}}{2}$

$\dfrac{dC}{dx} = 15$

The marginal cost at $x = 100$ is \$15, so the approximate cost for making the 101st game is \$15.

**iii. 1.** Write an equation for profit.

**iii.** Profit = revenue − costs

**2.** Differentiate with respect to $x$.

$= 500\sqrt{x} - (2000 + x\sqrt{x})$

**3.** Substitute $x = 100$ and evaluate.

$= 500x^{\frac{1}{2}} - 2000 - x^{\frac{3}{2}}$

$\dfrac{dP}{dx} = 500 \times \dfrac{1}{2}x^{\frac{-1}{2}} - \dfrac{3}{2}x^{\frac{1}{2}}$  $\dfrac{dP}{dx} = \dfrac{250}{\sqrt{x}} - \dfrac{3\sqrt{x}}{2}$

$\dfrac{dP}{dx} = \dfrac{250}{\sqrt{100}} - \dfrac{3\sqrt{100}}{2}$

$\dfrac{dP}{dx} = 25 - 15$

$= 10$

*Note:* Observe that marginal profit = marginal revenue − marginal cost.

The marginal profit at $x = 100$ is \$10, so the approximate profit for selling the 101st game is \$10.

---

## WORKED EXAMPLE 16 Applying rates of change to solving problems

On any day, the cost per item for a machine producing $x$ items is given by $\dfrac{dC}{dx} = 50 - 4e^{0.02x}$, where $x \in [0,\ 200]$ and $C$ is the cost in dollars.

**a.** Use the rate to calculate the marginal cost of producing the 100th item.
**b.** Determine the total cost of producing the first 100 items.
**c.** Determine the average cost of production for the first 100 items.

**THINK**

**WRITE**

**a. 1.** State the marginal rate of change.

**a.** $\dfrac{dC}{dx} = 50 - 4e^{0.02x}$

**2.** Substitute $x = 100$ and evaluate.

$\dfrac{dC}{dx} = 50 - 4e^{0.02 \times 100}$

$= 50 - 4e^2$

$= 20.4438$

**3.** Answer the question.

The marginal cost of producing the 100th item is approximately \$20.44.

**b. 1.** Area under the rate of change equation of cost gives the total cost for a given interval.

**b.** Total cost $= \displaystyle\int_0^{100} \left(50 - 4e^{0.02x}\right) dx$

**2.** Antidifferentiate and evaluate.

$= \left[50x - 4 \times \dfrac{1}{.02} e^{0.02x}\right]_0^{100}$

$= \left[50x - 200e^{0.02x}\right]_0^{100}$

$= \left(50 \times 100 - 200e^2\right) - \left(0 - 200e^0\right)$

$= 5000 - 200e^2 + 200$

$= 3722.19$

**3.** Answer the question.

The total cost of producing the first 100 items is \$3722.19.

**c. 1.** Average cost of production
= total cost / number produced

**c.** Average cost of production $= \dfrac{\$3722.19}{100}$

$= \$37.22$

**2.** Answer the question.

The average cost of production for the first 100 items is approximately \$37.22 each.

## 8.6.3 Displacement, velocity and acceleration

The relationships between displacement, velocity and acceleration have been discussed in chapters 4 and 5.

**Applications using kinematics**
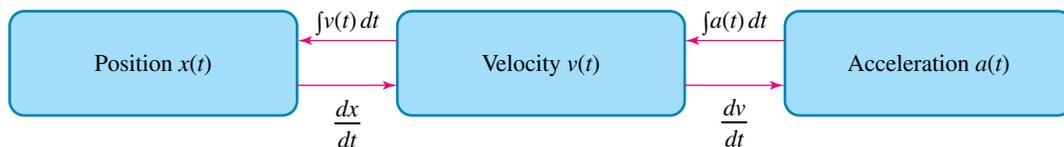


Your knowledge about the definite integral and the area under curves gives you additional skills for the calculation of facts related to kinematics.

From equations or graphs of velocity as a function of time, $v = f(t)$, the following may be obtained:
- displacement is the definite integral or the signed area
- distance travelled is the area under the graph.

A particle starting from rest accelerates according to the rule $a = 3t(2 - t)$.
a. Determine a relationship between the velocity of the particle, $v$ metres/second, and the time, $t$ s.
b. Determine the displacement of the particle after 4 s.
c. Sketch the graph of velocity versus time for the first 4 s of the motion.
d. Calculate the distance travelled by the particle in the first 4 s.

**THINK**

**WRITE**

a. 1. Antidifferentiate the acceleration equation to determine the velocity equation.

a. $v = \int a(t)dt$

$\qquad = \int (3t(2 - t))dt$

$\qquad = \int (6t - 3t^2)dt$

$\qquad = 3t^2 - t^3 + c$

2. Apply the initial conditions to determine $v$ in terms of $t$.

When $t = 0$, $v = 0$, so $c = 0$.
$\therefore v = 3t^2 - t^3$

b. 1. Integrate $v$ between $t = 0$ and $t = 4$. As we are determining displacement, there is no need to sketch the graph.

b. $x = \int_0^4 (3t^2 - t^3)dt$

$\qquad = \left[ t^3 - \frac{1}{4}t^4 \right]_0^4$

$\qquad = \left( 4^3 - \frac{1}{4}(4)^4 \right) - \left( 0^3 - \frac{1}{4}(0)^4 \right)$

$\qquad = 0$

2. Write the answer.

After 4 s the displacement is 0.

c. Sketch a graph of $v$ versus $t$.

c. $y$-intercept: $(0, 0)$   $t$-intercepts: $0 = 3t^2 - t^3$
$\qquad\qquad\qquad\qquad\qquad\qquad = t^2(3 - t)$
$\qquad\qquad\qquad\qquad\qquad\qquad t = 0, 3$
When $t = 4$, $v = 3 \times 4^2 - 4^3 = -16$.



d. 1. The area under the curve of a velocity–time graph gives the distance covered. Set up the integrals and subtract the negative region.
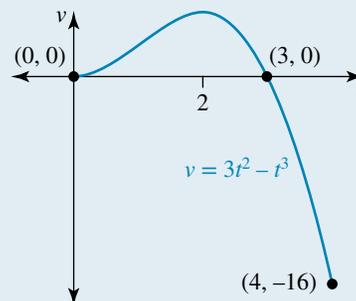
d. $D = \int_0^3 (3t^2 - t^3)dt - \int_3^4 (3t^2 - t^3)dt$

**2.** Antidifferentiate and evaluate.

$$= \left[t^3 - \frac{1}{4}t^4\right]_0^3 - \left[t^3 - \frac{1}{4}t^4\right]_3^4$$

$$= \left(\left(3^3 - \frac{3^4}{4}\right) - \left(0^3 - \frac{0^4}{4}\right)\right)$$

$$- \left(\left(4^3 - \frac{4^4}{4}\right) - \left(3^3 - \frac{3^4}{4}\right)\right)$$

$$= 27 - \frac{81}{4} - 0 - 64 + 64 + 27 - \frac{81}{4}$$

$$= 54 - \frac{162}{4}$$

$$= 13.5$$

**3.** Write the answer.

The distance travelled by the particle in 4 seconds is 13.5 metres.

**Alternative working for the distance travelled in the first 4 seconds:**

$$x = \int_0^4 (3t^2 - t^3)dt$$

$$= \left[t^3 - \frac{1}{4}t^4\right]_0^4$$

$$= \left(4^3 - \frac{1}{4}(4)^4\right) - \left(0^3 - \frac{1}{4}(0)^4\right)$$

$$= 0$$

**1.** Determine where the particle is at rest.

At rest: $v = 0$
$3t^2 - t^3 = 0$

$t^2(3 - t) = 0$
$t = 0$ or $3$
The particle is at rest initially and after 3 seconds, so it changes direction after 3 seconds.

**2.** Determine the displacement of the particle initially and at $t = 3$ and 4.
*Note:* We were not told where the particle was initially, so we cannot find the constant, $c$.

$x = t^3 - \frac{1}{4}t^4 + c$
At $t = 0$: $x = c$
At $t = 3$: $x = 27 - \frac{81}{4} + c$
$x = \frac{27}{4} + c$
At $t = 4$: $x = 64 - 64 + c$  $x = c$

**3.** Draw a motion diagram to represent the displacement of the particle during the first 4 seconds.



Distance travelled $= \frac{27}{4} + \frac{27}{4}$

$$= \frac{27}{2} = 13.5 \text{ metres}$$

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**Simple familiar**

1. **WE13** Use the trapezoidal rule with 4 strips to determine the approximate value of the definite integral
$\int_{1}^{3} (x \ln x)\, dx$. Give your answer correct to 2 decimal places.

2. Use the trapezoidal rule with 4 strips to determine the approximate value of the definite integral
$\int_{0}^{4} \ln (x + 1)\, dx$. Give your answer correct to 4 decimal places.

3. **WE14** Heat escapes from a storage tank at a rate of kilojoules per day. This rate can be modelled by

$$\frac{dH}{dt} = 1 + \frac{\pi^2}{9} \sin \left( \frac{\pi t}{45} \right), 0 \le t \le 100$$

where $H(t)$ is the total accumulated heat loss in kilojoules, $t$ days after 1 June.

a. Calculate the rate of escape of the heat, correct to 2 decimal places, at:

i. 15 days                  ii. 60 days.

b. Determine the period of the function.

c. Calculate the total accumulated heat loss after 45 days. Give your answer in exact form.

4. The average rate of increase, in cm/month, in the length of a baby boy from birth
until age 36 months is given by the rule

$$\frac{dL}{dt} = \frac{4}{\sqrt{t}}$$

where $t$ is the time in months since birth and $L$ is the length in centimetres.
Determine the average total increase in length of a baby boy from 6 months of
age until 36 months of age. Give your answer correct to 1 decimal place.

5. A number of apprentice bricklayers are competing in a competition in which
they are required to build a fence. The competitors must produce a fence that is
straight, neatly constructed and level. The winner will also be judged on how
many bricks they have laid during a 30-minute period.
The winner laid bricks at a rate defined by the rule

$$\frac{dN}{dt} = 0.8t + 2$$

where $N$ is the number of bricks laid after $t$ minutes.

a. Sketch the graph of the given function for $0 \le t \le 30$.
b. Shade the region defined by $10 \le t \le 20$.
c. Determine how many bricks in total the winner laid in the
10-minute period defined by $10 \le t \le 20$.

6. The rate of growth of mobile phone subscribers with a particular company in the United Kingdom can be modelled by the rule

$$\frac{dN}{dt} = 0.853e^{0.1333t}$$

where $N$ million is the number of subscribers with the company since 1998 and $t$ is the number of years since 1998, the year the company was established. Determine how many millions of mobile phone subscribers have joined the company between 1998 and 2015, correct to 1 decimal place.

7. **WE15** A manufacturer of a game knows that the revenue, $R, for selling $x$ games is $R = 100(\sqrt{x+4} - 2)$ The costs, $C, to produce $x$ games is $C = 50 + x\sqrt{x}$.
   a. Calculate the profit made when 10 games were sold.
   b. Determine the average profit per game when 10 games were sold.
   c. For 20 games, calculate and interpret:
      i. the marginal revenue
      ii. the marginal cost
      iii. the marginal profit.

8. The weekly profit of a factory, $P$ (in hundreds of dollars), is given by $P = 8n - n\sqrt{n}$, where $n$ is the number of employees.
   a. Calculate the weekly profit of a factory with 16 employees.
   b. Determine the average weekly profit per employee when there are 16 employees.
   c. Calculate the marginal weekly profit, in dollars per employee, when the number of employees is:
      i. 10 employees
      ii. 25 employees.

9. **WE16** A manufacturer has found that the cost per item to produce $x$ items is given by $\frac{dC}{dx} = 20 + x + e^{-0.05x}$, where $x \in [0, 50]$ and $C$ is the cost in dollars.
   a. Use the rate to calculate the marginal cost of producing the 10th item.
   b. Determine the total cost of producing the first 10 items.
   c. Determine the average cost of production for the first 10 items.

10. On any day the cost per item for a machine producing $n$ items is given by $\frac{dC}{dn} = 40 - 2e^{0.01n}$, where $n \in [0.200]$ and $C$ is the cost in dollars.
    a. Use the rate to determine the cost of producing the 100th item.
    b. Express $C$ as a function of $n$.
    c. Determine the total cost of producing the first 100 items.
    d. Determine the average cost of production for the first 100 items.

11. **WE17** A particle moves in a line so that its velocity, $v$ metres/second, from a fixed point, O, is defined by $v = 1 + 3\sqrt{t+1}$, where $t$ is the time in seconds.
    a. Determine the initial velocity of the particle.
    b. Determine the acceleration of the particle when:
       i. $t = 0$          ii. $t = 8$.
    c. Sketch the graph of $v$ versus $t$ for the first 10 s.
    d. Calculate the distance covered by the particle in the first 8 s.

**12.** An object travels in a line so that its velocity, $v$ metres/second, at time $t$ seconds is given by

$$v = 3 \cos \left( \frac{t}{2} - \frac{\pi}{4} \right), t \geq 0$$

Initially the object is $-3\sqrt{2}$ metres from the origin.

**a.** Determine the relationship between the displacement of the object, $x$ metres, and time, $t$ seconds.

**b.** Calculate the displacement of the object when time is equal to $3\pi$ seconds.

**c.** Sketch the graph of $v$ versus $t$ for $0 \leq t \leq 4\pi$.

**d.** Determine the distance travelled by the object after $3\pi$ seconds. Give your answer in metres, correct to 2 decimal places.

**e.** Determine a relationship between the acceleration of the object, $a$ metres/second$^2$, and time, $t$ seconds.

**f.** Calculate the acceleration of the object when $t = 3\pi$ seconds.

**Complex familiar**

**13.** Consider the function $f(x) = \sqrt{9 - x^2}$.

**a.** Use the trapezoidal rule with 6 strips to approximate $\int_{-3}^{3} \sqrt{9 - x^2} \, dx$.

**b.** Sketch the function $y = f(x)$ and state its domain.

**c.** Determine the exact value of $\int_{-3}^{3} \sqrt{9 - x^2} \, dx$.

**d.** Determine the percentage error in the trapezoidal rule, to the nearest integer.

**14.** Part of the graph of the function $y = 2 \ln (x + 5)$ is shown. Use the trapezoidal rule with 4 strips to approximate the area enclosed by the curve, $x = 0$ and $y = 0$. Give your answers in the form:

$$p \ln (2) + q \ln (3) + r \ln (5)$$

where $p$, $q$, $r$ are constants.

**15.** The rate of change of position (velocity) of a racing car travelling down a straight stretch of road is given by

$$\frac{dx}{dt} = t(16 - t)$$

where $x$ is measured in metres and $t$ in seconds.

**a.** Determine the velocity when:

   **i.** $t = 0$

   **ii.** $t = 4$.

**b.** Determine:

   **i.** when the maximum velocity occurs

   **ii.** the maximum velocity.

**c.** Sketch the graph of $\dfrac{dx}{dt}$ against $x$ for $0 \leq t \leq 16$.

**d.** Determine the area under the graph between $t = 0$ and $t = 10$.

**e.** Explain what this area represents.
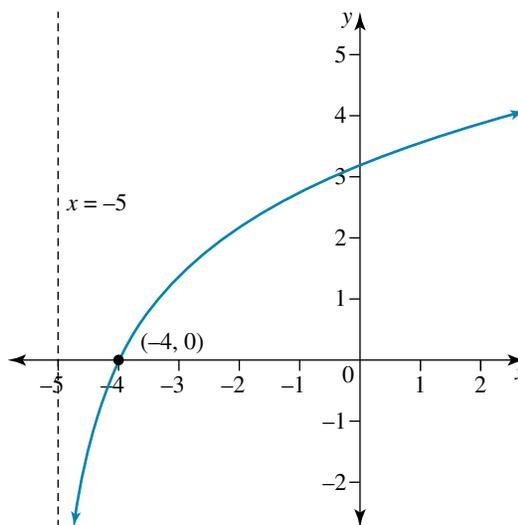
**16.** The rate of flow of water into a hot water system during a 12-hour period on a certain day is thought to be

$$\frac{dV}{dt} = 10 + \cos\left(\frac{\pi t}{2}\right)$$

where $V$ is in litres and $t$ is the number of hours after 8 am.

a. Sketch the graph of $\frac{dv}{dt}$ against $t$.

b. Determine the length of time for which the rate is above 10.5 L/h.

c. Determine the volume of water that has flowed into the system between:

   **i.** 8 am and 2 pm

   **ii.** 3 pm and 8 pm.

### Complex unfamiliar

**17.** The maintenance costs for a car increase as the car gets older. It has been suggested that the increase in maintenance costs of dollars per year could be modelled by

$$\frac{dC}{dt} = 15t^2 + 250$$

where $t$ is the age of the car in years and $C$ is the total accumulated cost of maintenance for $t$ years. Determine the total accumulated cost of maintenance for $t = 5$ to $t = 10$ years.

**18.** A particle moves in a straight line. At time $t$ seconds its velocity, $v$ metres per second, is defined by the rule

$$v = e^{-0.5t} - 0.5, t \geq 0$$

The graph of the motion is shown.
Determine the exact distance covered by the particle in the first 4 seconds.



(0, 0.5)

(2 log$_e$ (2), 0)

$v = e^{-0.5t} - 0.5$
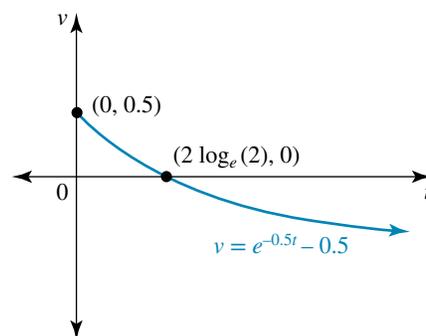
**19.** Heat escapes from a storage tank so that the rate of heat loss, in kilojoules per hour, is given by

$$\frac{dH}{dt} = 1 + \frac{1}{2}\sin\left(\frac{\pi t}{12}\right)$$

for $0 \leq t \leq 24$. Determine the amount of heat lost in one day.

**20.** A particle moves in a straight line with acceleration, $a$ m/s$^2$, given by $a = \dfrac{1}{(t+2)^2}$.

If the particle starts from rest at the origin, determine the distance travelled in the first 10 seconds. Express your answer in the form $p + q \ln(r)$, where $p$, $q$ and $r$ are constants.

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# LESSON
## 8.7 Review

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|---|---|---|
| 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 | 13, 14, 15, 16 | 17, 18, 19, 20 |

**Simple familiar**

1. Using the figure shown, calculate the approximate area under the curve from $x = 2$ to $x = 4$, using the left-hand, or lower, rectangles.
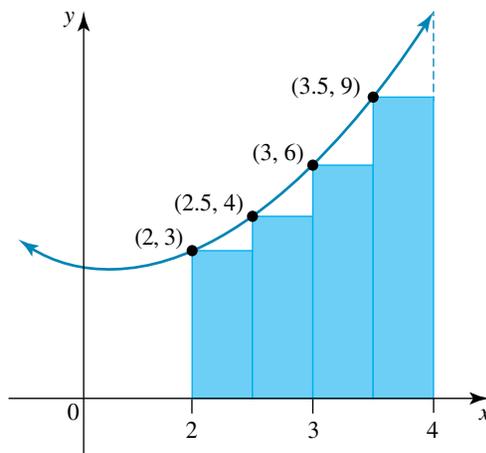
**2.** Using the figure shown, calculate the approximate area under the curve from $x=-5$ to $x=-1$, using the upper rectangles.
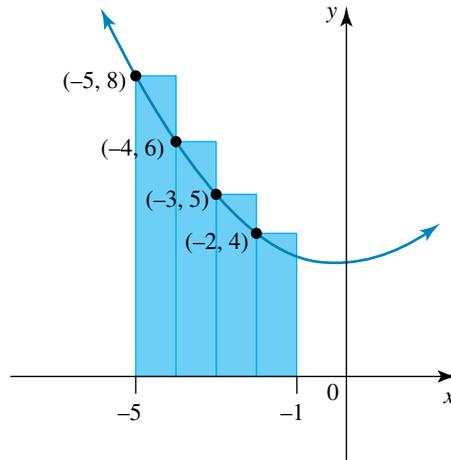


**3.** A student is using the trapezoidal rule to approximate the area under the curve shown from $x=0$ to $x=4$. State what their answer would be.
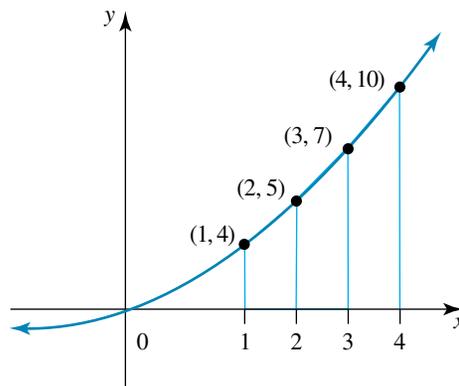


**4.** Apply the trapezoidal method to determine the area between $y=e^{2x-1}$ and the x-axis from $x=0$ to $x=4$, using intervals that are 1 unit wide. (Give your answer correct to 2 decimal places.)

**5.** Consider the area under the curve $y=\log_e(x)$ from $x=2$ to $x=4$. With interval widths of 0.5 units, determine an approximation to the area using:

**a.** left end-point rectangles

**b.** right end-point rectangles

**c.** the average of the rectangles.

Give your answers in exact form.

**6.** Determine:

**a.** $\displaystyle\int_0^2 \left(3x+6\sqrt{x}+1\right) dx$

**b.** $\displaystyle\int_0^{\frac{1}{2}} (e^x+1)(e^x-1)\, dx$

**c.** $\displaystyle\int_{-1}^0 \frac{9}{(2x+3)^4}\, dx$

**d.** $\displaystyle\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \cos 2x\, dx$

**7.** Given that $\displaystyle\int_{0}^{k}(4x-5)\,dx=-2$, determine two possible values for $k$.

**8.** Given that $\displaystyle\int_{1}^{5}f(x)\,dx=4$ and $\displaystyle\int_{1}^{5}g(x)\,dx=3$, determine:

**a.** $\displaystyle\int_{1}^{5}(4f(x)+1)\,dx$    **b.** $\displaystyle\int_{1}^{5}(2f(x)-g(x))\,dx$    **c.** $\displaystyle\int_{1}^{5}(3f(x)+2g(x)-5)\,dx$

**9. a.** Sketch the graph of the function $f(x)=\dfrac{1}{x-2}$.

 **b.** Calculate the exact area between the graph of $f(x)$, the $x$-axis and the lines $x=3$ and $x=6$.

**10.** Determine the area bounded by the curve $y=x(x+2)(x-3)$ and the $x$-axis.
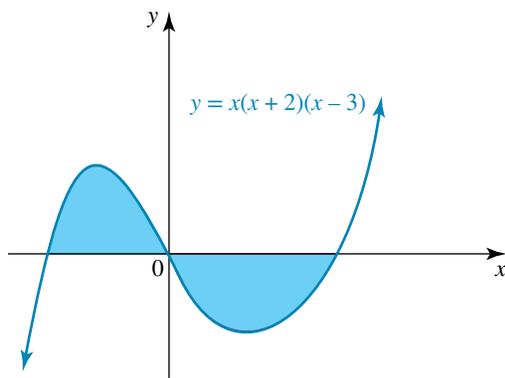


**11.** Determine $m$ if $\displaystyle\int_{\frac{1}{2}}^{m}6\,(2x-1)^2\,dx=1$.

**12.** A particle starts at the origin and travels in a straight line with a velocity, $v$ m/s, modelled by
$v=t^2-t-2$, where $t$ is the time in seconds.

 **a.** State the equation for the acceleration of the particle.
 **b.** Determine when the particle is at rest.
 **c.** Determine the displacement of the particle after 3 s.
 **d.** Determine the distance covered by the particle in the first 3 s.
 **e.** Hence, determine the average speed of the particle for the first 3 s.

**Complex familiar**

**13. a.** Determine any point(s) of intersection between the two curves $f(x)=x^3-3x+2$ and $g(x)=x+2$.
 **b.** Sketch $f(x)$ and $g(x)$ on the same set of axes. Label the point(s) of intersection and any $x$- and $y$-intercepts.
 **c.** Evaluate the area between the two curves.

**14.** A manufacturer of a new gadget knows that the revenue, $R$, for selling $x$ gadgets is $R = e^{\frac{x}{20}} - 1$.
The costs, $C$, to produce $x$ gadgets is $C = 40 + x - \sqrt{x}$.

   **a.** Calculate the profit or loss made when:

      **i.** 50 gadgets are sold

      **ii.** 100 gadgets are sold.

   **b.** Determine the average profit per gadget when 100 gadgets are sold.

   **c.** For 120 gadgets, calculate and interpret:

      **i.** the marginal revenue

      **ii.** the marginal cost

      **iii.** the marginal profit.

**15.** The velocity of a particle is given by $v = 2\sin(2t) + 3$, where $x$ is the displacement in metres and $t$ is the time in seconds. Initially the particle is at the origin.

   **a.** Show that the displacement is given by $x = -\cos(2t) + 3t + 1$.

   **b.** Determine the displacement when $t = \dfrac{\pi}{2}$ s.

   **c.** Determine the distance that the particle has travelled in the first $\dfrac{\pi}{2}$ s.

**16.** The graphs of $y = 2\sin(x) + k$ and $y = k\cos(x)$ are shown. The shaded region is equal to $(3\pi + 4)$ square units.
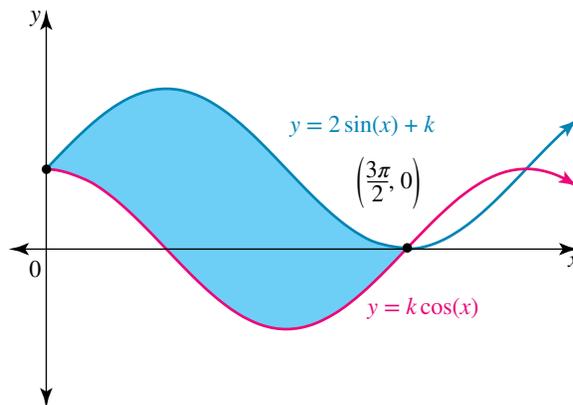


Determine the value of the constant $k$.

**17.** The diagram shows a feeding trough for horses, with dimensions given in centimetres. The parabola is a cross-sectional profile of each end of the trough.
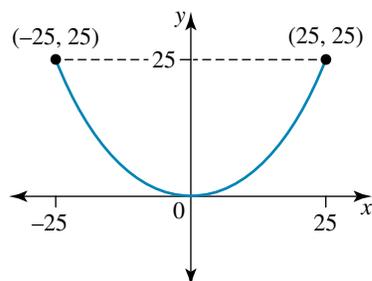


Determine the volume of the trough if it is known to be 1.2 m long.

**17.** The graphs with equations $y = \sqrt{x}$ and $y = 2 - x$ are shown.
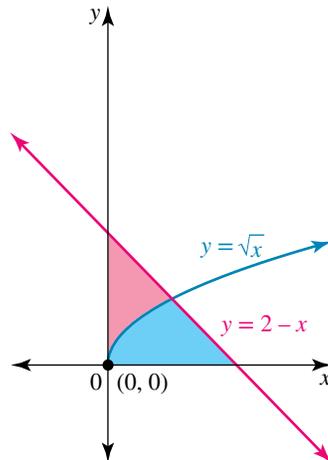


Determine the ratio of the pink shaded area to the blue shaded area.

**19.** A ground-cover plant can cover the ground at a rate modelled by $\dfrac{dA}{dt} = 2t + 6t^2 - \dfrac{1}{4}t^3$, where $A$ is the area in square centimetres and $t$ is the time in weeks.

Initially, the plant covered $10 \text{ cm}^2$. Determine how long, to the nearest week, it will take to cover $0.6 \text{ m}^2$, and the maximum area the plant will cover, correct to 2 decimal places.

**20.** The cross-section of a platform is shown. (All dimensions are in metres.)
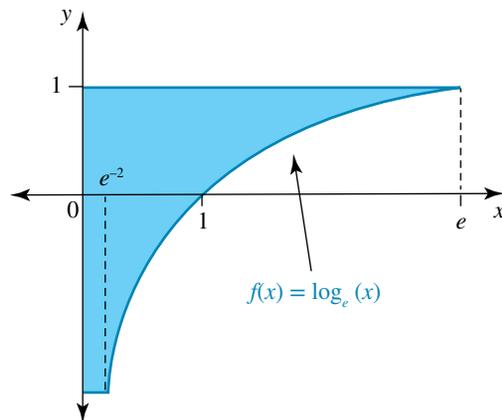


Determine the differential of $x \ln(x)$ and the volume of concrete required to build this platform if it is 20 m long. Give your answer to the nearest cubic metre.

# 8.7 Past QCAA exam questions

**Question 1 (1 mark)**

*Source:* QCAA 2023 Mathematical Methods, Paper 1, Section 1, Q5; © State of Queensland (QCAA) 2024.

Determine $\int_1^3 \frac{1}{2x} dx$.

**A.** $\frac{1}{2} \ln 6$                       **B.** $\frac{1}{2} \ln 5$

**C.** $\frac{1}{2} \ln 4$                       **D.** $\frac{1}{2} \ln 3$

**Question 2 (1 mark)**

*Source:* QCAA 2023 Mathematical Methods, Paper 2, Section 1, Q6; © State of Queensland (QCAA) 2024.

$\int_a^{5a} \frac{1}{x+a} dx, a \neq 0$ is

**A.** 1.7918                       **B.** 1.6094

**C.** 1.3863                       **D.** 1.0986

**Question 3 (1 mark)**

*Source:* QCAA 2023 Mathematical Methods, Paper 1, Section 1, Q9; © State of Queensland (QCAA) 2024.

Determine $\int_0^3 \pi \sin(\frac{\pi}{3}x)\, dx$.

**A.** 3                       **B.** 6

**C.** −3                       **D.** −6

**Question 4 (4 marks)**

*Source:* QCAA 2023 Mathematical Methods, Paper 1, Section 2, Q12; © State of Queensland (QCAA) 2024.

The region bounded by the $x$-axis and the curve of $f(x) = x(4-x)$ represents the plan of a garden bed.

All measurements are in metres.

**a.** Estimate the area of the garden bed using sums of the form $\sum_i f(x_i)\delta x_i$ where $x_1 = 1, x_2 = 2, x_3 = 3$ and $x_4 = 4$.    **[1 mark]**

**b.** Use a definite integral to determine the area of the garden bed.    **[3 marks]**

A company makes windows using glass that has a mass of $5.6\,\text{kg}$ per square metre. A customer orders an unusual window in a partial parabolic shape, as shown.

Not to scale
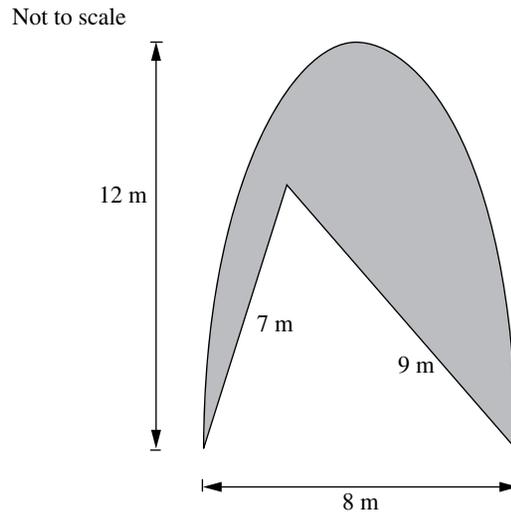


Determine the mass of the window.

**Fully worked solutions for this chapter are available online.**

# Answers

## Chapter 8 Further integration

### 8.2 Estimating the area under a curve

#### 8.2 Exercise

**1.** 10 sq. units

**2. a.** 8 sq. units  **b.** 42 sq. units

**3. a.** $\dfrac{25}{12}$ sq. units or approx. 2.08 sq. units

    **b.** $\dfrac{77}{60}$ sq. units or approx. 1.28 sq. units

**4. a.** 30 sq. units  **b.** 22 sq. units

**5.** 9 sq. units

**6. a.** 8 sq. units  **b.** 13 sq. units

**7. a.** 26 sq. units  **b.** 41 sq. units  **c.** $33\dfrac{1}{2}$ sq. units

**8. a.** $(1 + e + e^{-1})$ sq. units

    **b.** $\log_e (24)$ sq. units

**9. a.** 7.25 sq. units  **b.** 100 sq. units

**10. a.** $19\dfrac{1}{4}$ sq. units  **b.** $\dfrac{1}{2}\left(1 + 2e + 2e^2 + e^3\right)$ sq. units

**11. a.** 22.5 sq. units  **b.** 20.8 sq. units

**12. a.** 1.87 sq. units  **b.** 1.68 sq. units

**13.** 12 sq. units

**14.** 21 sq. units

**15. a.** 4  **b.** 7.56 sq. units

### 8.3 The fundamental theorem of calculus

#### 8.3 Exercise

**1. a.** $\dfrac{1}{3}$  **b.** $20\dfrac{1}{4}$  **c.** $5\dfrac{1}{3}$

    **d.** $\dfrac{1}{3}$  **e.** 8

**2. a.** $37\dfrac{1}{3}$  **b.** $\dfrac{3}{5}\ln(5) \approx 0.966$

    **c.** $\dfrac{85}{256}$  **d.** 2

    **e.** 3.65 (to 2 decimal places)

**3. a.** 1  **b.** 0  **c.** $20 - 10\sqrt{2} \approx 5.86$

    **d.** $-6$  **e.** 0

**4. a.** $\dfrac{1}{4}\left(e^8 - 1\right)$  **b.** $3\left(1 - e^{\frac{-2}{3}}\right)$

    **c.** $2\left(e^{-2} - e^2\right)$  **d.** $\dfrac{1}{2}\left(e^{12} + 15 - e^6\right)$

    **e.** $5 \ln(4) + 2e^2 - 2e^{\frac{1}{2}}$

**5. a.** 27  **b.** $9\dfrac{1}{6}$  **c.** 0

    **d.** 0  **e.** $\dfrac{4}{3}\left(\sqrt{10} - 2\right)$

**6. a.** $\dfrac{65}{4} = 16\dfrac{1}{4}$  **b.** $2 + \dfrac{1}{2}e^2 - \dfrac{1}{2}e^{-2}$

**7. a.** 0  **b.** $\sqrt{2} - \dfrac{3\sqrt{3}}{4}$

**8. a.** 21  **b.** 11  **c.** $-16$  **d.** 8

**9. a.** $-15$  **b.** $-12.5$  **c.** 32.5

    **d.** 20  **e.** 25  **f.** 12.5

**10.** 2

**11.** 3

**12.** D

**13.** $k = \pm 1$

**14.** $2 \log_e (3)$

**15.** 4

**16.** $k = \sqrt{5}$

**17.** $(\pi + 2)$

**18.** $\dfrac{1}{3}\left(e^{-2} - 1\right)$

**19.** $\dfrac{dy}{dx} = \dfrac{2e^x}{(e^x + 1)}$ and $\displaystyle\int_1^5 \dfrac{e^x}{e^x + 1}\, dx = 3.6935$

**20.** $a = 1;\ 21\dfrac{1}{3}$ units$^2$

### 8.4 Areas under a curve

#### 8.4 Exercise

**1. a. i.** $\displaystyle\int_0^4 (4 - x)\, dx$

    **ii.** 8 sq. units

    **b. i.** $\displaystyle\int_1^2 (x^2)\, dx$

    **ii.** $2\dfrac{1}{3}$ sq. units

    **c. i.** $\displaystyle\int_{-3}^{-1} (3x^2)\, dx$

    **ii.** 26 sq. units

    **d. i.** $\displaystyle\int_1^3 (x^3 - 9x^2 + 20x)\, dx$

    **ii.** 22 sq. units

**2. a. i.** $\displaystyle\int_{-1}^1 (e^x)\, dx$

    **ii.** $(e - e^{-1})$ sq. units

    **b. i.** $\displaystyle\int_1^4 (e^{-2x})\, dx$

    **ii.** $\dfrac{1}{2}\left(e^{-2} - e^{-8}\right)$ sq. units

**c.** **i.** $\displaystyle\int_{\frac{\pi}{2}}^{0} (2\sin(2x))\,dx$

**ii.** 2 sq. units

**d.** **i.** $\displaystyle\int_{0}^{\frac{3\pi}{2}} \cos\frac{x}{3}\,dx$

**ii.** 3 sq. units

**3. a.** 1 sq. unit

**b.** $5\dfrac{1}{3}$ sq. units

**c.** $\dfrac{7}{24}$ sq. units

**d.** 4 sq. units

**4. a.** $\left(e - e^{-1}\right)$ sq. units

**b.** $\dfrac{1}{2}\left(e^{-1} - e^{-2}\right)$ sq. units

**c.** 1 sq. unit

**d.** 4 sq. units

**5. a.** $x$-intercepts: $x = 1, -1, 3, -3$; $y$-intercept: $y = 9$
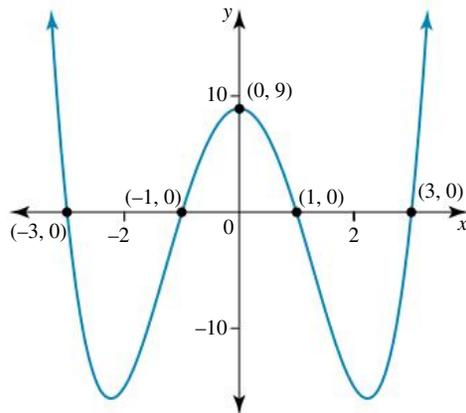


**b.** $52\dfrac{4}{15}$ sq. units

**6. a.** $15\dfrac{3}{4}$ sq. units   **b.** $5\dfrac{1}{3}$ sq. units   **c.** $21\dfrac{1}{12}$ sq. units

**7. a.** **i.** $y = 8 - x^2$
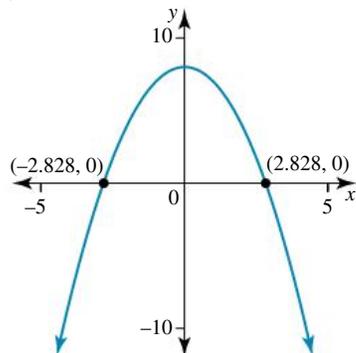


$x$-intercepts: $x = \pm\sqrt{8}$ or $\pm 2\sqrt{2}$

**ii.** $\dfrac{64\sqrt{2}}{3}$ sq. units

**b.** **i.** $y = x^3 - 4x^2$



$x$-intercepts: $x = 0, 4$
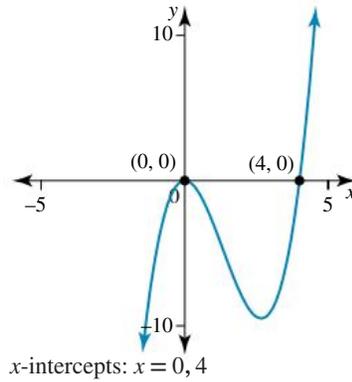
**ii.** $21\dfrac{1}{3}$ sq. units

**c.** **i.** $y = x(x - 2)(x - 3)$



$x$-intercepts: $x = 0, 2, 3$
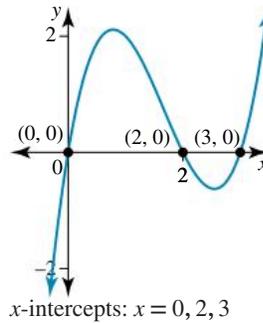
**ii.** $3\dfrac{1}{12}$ sq. units

**d.** **i.** $y = (x - 1)(x + 2)(x + 5)$



$x$-intercepts: $x = -5, -2, 1$
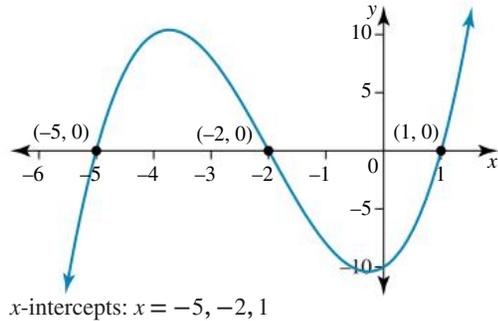
**ii.** $40\dfrac{1}{2}$ sq. units

**8.** $2\ln(2)$ sq. units

**9.** $\dfrac{1}{3}\left(e^6 - e^3\right)$ sq. units

**10.** $\dfrac{3 - \sqrt{3}}{2}$ sq. units

**11.** $166\dfrac{2}{3}$ sq. units

**12.** $3\pi$ sq. units

**13.** $\left(e + e^{-1} - 2\right)$ sq. units

**14.** 1.6 units$^2$

**15.** 16 sq. units

**16. a.** $\dfrac{dy}{dx} = 1 + \ln(x)$

**b.** $\displaystyle\int \ln(x)\,dx = x\ln(x) - x + c$

**c.** $(4\ln(4) - 3)$ sq. units

**d.** Verified using technology, the area is approximately 2.55 sq. units.

**17.** 3.44 sq. units

**18.** 0.41 sq. units

**19.** $\dfrac{9}{4}$ sq. units

**20.** $2(\pi - 2)$ sq. units

## 8.5 Areas between curves

### 8.5 Exercise

**1. a.** $2\dfrac{1}{6}$ sq. units  **b.** $21\dfrac{1}{3}$ sq. units

**2. a.** $2\dfrac{1}{4}$ sq. units

**b.** $\dfrac{52}{3} + \dfrac{1}{e} - e \approx 14.98$ sq. units

**3. a.** $e^2 - e + \dfrac{3}{2} \approx 6.17$ sq. units

**b.** $1\dfrac{1}{3}$ sq. units

**4.** C

**5.** D

**6.** D

**7. a.** $x = 0, 1$ or $-1$

**b.**
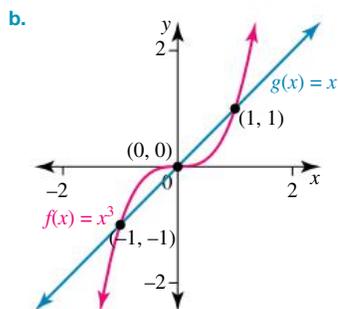


**c.** $\dfrac{1}{2}$ sq. units

**8. a.** $x = 0, 1$ or $2$

**b.**
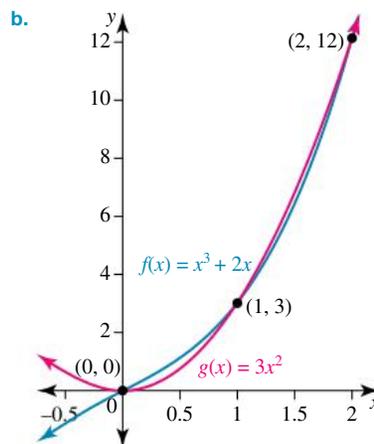


**c.** $\dfrac{1}{2}$ sq. units

**9.** $(16, 4)$; $21\dfrac{1}{3}$ sq. units

**10.** $20\dfrac{5}{6}$ sq. units

**11. a.** $\left(\dfrac{3\pi}{4}, \dfrac{1}{\sqrt{2}}\right)$  **b.** $2\sqrt{2}$ sq. units

**12.** B

**13.** $\dfrac{\pi\sqrt{3}}{12}$ sq. units ($\approx 0.45$ sq. units)

**14.** $(3\ln(2) - 2)$ sq. units

**15. a.** $a = 1, b = 2, c = 3$

**b.** 9.6 sq. units

**16. a.** $(-4, 0), (4, 0)$

**b.** $(-3, 0), (3, 0)$

**c.** $9\dfrac{1}{3}$ m$^2$

**17.** 13 cubic metres of soil

**18.** 152 cubic metres

**19.** $\dfrac{3}{4} + \ln(2)$ units$^2$

**20.** 1.63 units$^2$

## 8.6 Modelling and solving with definite integrals

### 8.6 Exercise

**1.** 2.97

**2.** 3.9828

**3. a. i.** 1.95 kJ/day

**ii.** 0.05 kJ/day

**b.** 90 days

**c.** Accumulated heat loss after 45 days is $(45 + 10\pi)$ kJ, or approx. 76.42 kJ.

**4.** 28.4 cm

**5.**



$\frac{dN}{dt} = 0.8t + 2$

(0, 2)

(30, 26)
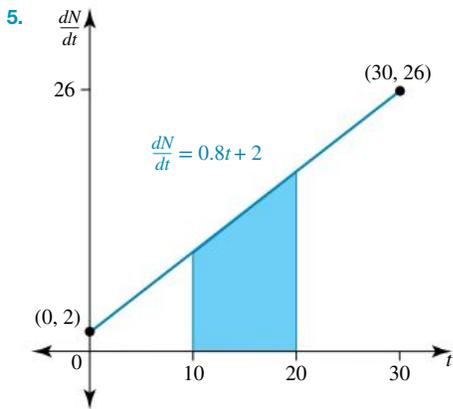
**c.** 140 bricks

**6.** 55.3 million

**7. a.** $92.54

**b.** $9.25

**c.  i.** Marginal revenue at $x = 20$ is $10.21, so the approximate revenue from selling the next game is $10.21.

**ii.** Marginal cost at $x = 20$ is $6.71, so the approximate cost of manufacturing the next game is $6.71.

**iii.** Marginal profit at $x = 20$ is $3.50, so the approximate profit from selling the next game is $3.50.

**8. a.** $6 400

**b.** $400

**c.  i.** $32.57/employee

**ii.** $2/employee

**9. a.** $30.61        **b.** $257.87        **c.** $25.79/item

**10. a.** $34.56

**b.** $C = 40n - 200\,e^{0.01n} + 200$

**c.** $3 656.34

**d.** $36.56/item

**11. a.** 4 m/s

**b.  i.** 1.5 m/s$^2$

**ii.** 0.5 m/s$^2$

**c.**



$y = 1 + 3\sqrt{x+1}$

(10, 10.95)

(0, 4)
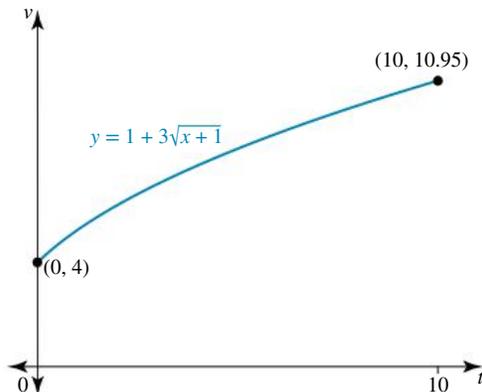
**d.** 60 m

**12. a.** $x = 6 \sin\left(\dfrac{t}{2} - \dfrac{\pi}{4}\right)$

**b.** $-3\sqrt{2}$ m



$v = 3 \cos\left(\dfrac{t}{2} - \dfrac{\pi}{4}\right)$

$\left(0, \dfrac{3\sqrt{2}}{2}\right)$

$\left(4\pi, \dfrac{3\sqrt{2}}{2}\right)$

$\left(\dfrac{3\pi}{2}, 0\right)$

$\left(\dfrac{7\pi}{2}, 0\right)$
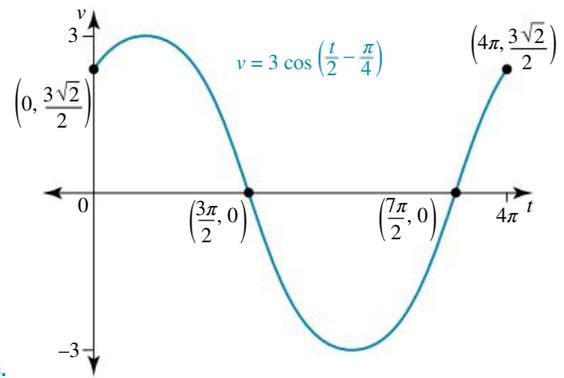
**c.**

**d.** 20.49 m

**e.** $a = \dfrac{dv}{dt} = -\dfrac{3}{2} \sin\left(\dfrac{t}{2} - \dfrac{\pi}{4}\right)$

**f.** $\dfrac{3\sqrt{2}}{4}$ m/s$^2$

**13. a.** 15.365

**b.**



(0, 3)

(−3, 0)

(3, 0)
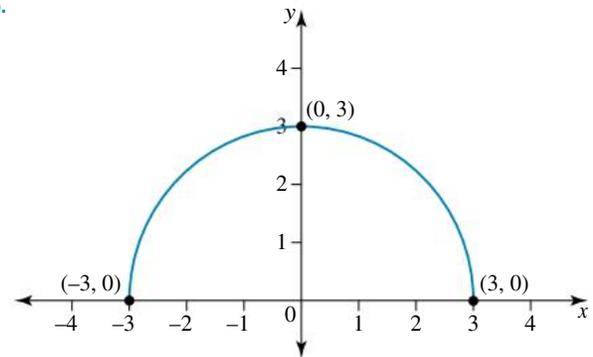
Domain: $-3 \le x \le 3$ or $x \in \left[-3,\ 3\right]$

**c.** $\dfrac{9\pi}{2}$ sq. units

**d.** Larger by 9%

**14.** $6 \ln (2) + 2 \ln (3) + \ln (5)$

**15. a.  i.** 0 m/s

**ii.** 48 m/s

**b.  i.** 8 s

**ii.** 64 m/s

**c.**



$\dfrac{dx}{dt}$ (m/s)

64

$\dfrac{dx}{dt} = t(16 - t)$
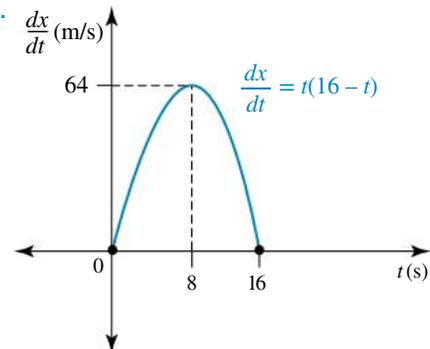
**d.** $466\dfrac{2}{3}$ m

**e.** The area represents the distance travelled in the first 10 s.

**16. a.**



$$\frac{dV}{dt} = 10 + \cos\left(\frac{\pi t}{2}\right)$$
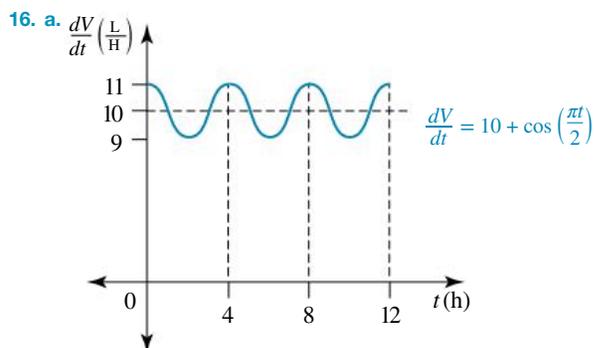
**b.** 4 h

**c. i.** 60 L            **ii.** 50.6 L

**17.** $5625

**18.** $2\left(1 - \ln(2) + e^{-2}\right)$ m or 0.8844 m to 4 decimal places

**19.** 24 kJ

**20.** $5 - \ln(6)$ m

## 8.7 Review

## 8.7 Exercise

**1.** 11 sq. units

**2.** 23 sq. units

**3.** 21 sq. units

**4.** 719.72 sq. units

**5. a.** $\frac{1}{2}\ln\left(\frac{105}{2}\right)$ sq. units

**b.** $\frac{1}{2}\ln(105)$ sq. units

**c.** $\frac{1}{4}(2\ln(105) - \ln(2))$ sq. units

**6. a.** $8 + 8\sqrt{2}$    **b.** $\frac{1}{2}e - 1$    **c.** $1\frac{4}{9}$

**d.** $-\frac{\sqrt{3}}{2}$

**7.** $\frac{1}{2}, 2$

**8. a.** 20       **b.** 5       **c.** $-2$

**9. a.**
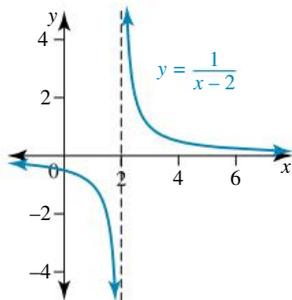


**b.** $\log_e(4)$ sq. units

**10.** $21\frac{1}{12}$ sq. units

**11.** 1

**12. a.** $a = \dfrac{dv}{dt} = 2t - 1$

**b.** The particle is at rest at $t = 2$ s.

**c.** After 3 s, the displacement is $-\dfrac{3}{2}$ m, or $\dfrac{3}{2}$ m to the left of the origin.

**d.** The distance travelled in the first 3 s is $5\dfrac{1}{6}$ m.

**e.** The average speed for the first 3 s is $1\dfrac{13}{18}$ m/s.

**13. a.** $(-2, 0), (0, 2), (2, 4)$

**b.**
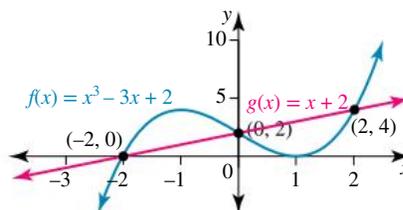


**c.** The area between the curves is 8 sq. units.

**14. a. i.** A loss of $71.75

    **ii.** A profit of $17.41

**b.** $0.17/gadget

**c. i.** The marginal revenue at $x = 120$ is $20.17, so the approximate revenue from selling the next gadget is $20.17.

    **ii.** The marginal cost at $x = 120$ is $0.95, so the approximate cost of manufacturing the next gadget is $0.95.

    **iii.** The marginal profit at $x = 120$ is $19.22, so the approximate profit from selling the next gadget is $19.22.

**15. a.** Sample responses can be found in the worked solutions in the online resources.

**b.** $\left(2 + \dfrac{3\pi}{2}\right)$ m

**c.** $\left(2 + \dfrac{3\pi}{2}\right)$ m

**16.** 2

**17.** 50 000 cm$^3$

**18.** 5:7

**19.** 19 weeks, 0.75 m$^2$

**20.** 52 m$^3$

## 8.7 Past QCAA exam questions

**1.** D

**2.** D

**3.** B

**4. a.** 10 m$^2$       **b.** $10\frac{2}{3}$ m$^2$

**5.** 208.136 kg

# 9 Cosine and sine rules

## LESSON SEQUENCE

**Fully worked solutions of this chapter are available online.**

## EXAM PREPARATION

Access exam-style questions in every lesson, available online and access past QCAA exam questions in every review.

**On** Resources

| | | |
|---|---|---|
| **Solutions** | Solutions — Chapter 9 |
| **Exam questions** | Exam question booklet — Chapter 9 |
| **Digital documents** | Learning matrix — Chapter 9 |
| | Chapter summary — Chapter 9 |

# LESSON
## 9.1 Overview

### 9.1.1 Introduction

The significance of understanding Trigonometry extends beyond classroom theory. Historically, the sine and cosine rules have been used in various fields, from ancient architectural marvels to modern engineering. The sine rule, which relates the ratios of the lengths of sides of a triangle to the sines of their opposite angles, is crucial in solving problems involving non-right-angled triangles, particularly in navigation and astronomy. The cosine rule, on the other hand, generalises the Pythagorean theorem to all types of triangles, making it essential for calculating unknown sides or angles in complex structures.

Professionals such as civil engineers, architects and computer graphics designers rely on these principles to perform precise calculations and design robust structures. Civil engineers use the sine and cosine rules to ensure the structural integrity and safety of bridges and buildings. Architects apply these rules to create accurate blueprints and determine the dimensions of various architectural elements. In meteorology, these trigonometric principles aid in understanding weather patterns and interpreting geographical data.

### 9.1.2 Syllabus Links

| Lesson | Lesson title | Syllabus links |
|---|---|---|
| 9.2 | **The sine rule** | ○ Use the sine rule (ambiguous case is required), $\dfrac{a}{\sin(A)} = \dfrac{b}{\sin(B)} = \dfrac{c}{\sin(C)}$, where $a$, $b$ and $c$ are the side lengths of the triangle and $A$, $B$ and $C$ are the corresponding opposite angles. |
| 9.3 | **The cosine rule** | ○ Use the cosine rule, $c^2 = a^2 + b^2 - 2ab\,\cos(C)$. |
| 9.4 | **Area of a triangle** | ○ Use the formula $area = \dfrac{1}{2}bc\sin(A)$ to calculate the area of a triangle. |
| 9.5 | **Modelling and solving with the sine and cosine rules** | ○ Model and solve problems that involve the sine rule, cosine rule and the area formula in two- and three-dimensional contexts (including bearings, directions and angles of elevation and depression), with and without technology. |

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# LESSON
## 9.2 The sine rule

## 9.2.1 Review of the Pythagorean identity

- The mathematical equation $\sin^2(A) + \cos^2(A) = 1$ is a trigonometric identity because it holds true for all values of $A$.
- It is called the Pythagorean identity given its relationship to Pythagoras theorem, $a^2 + b^2 = c^2$.
- In Unit 2, this identity was used to solve trigonometric equations such as $1 - \sin(x) = 2\cos^2(x)$.
- In this unit, it will be useful for simplifying simple expressions such as $\sin^2(7x) + \cos^2(7x) = 1$, and may also play a role in solving more complex or unfamiliar trigonometric problems.

### The Pythagorean identity

$$\sin^2(A) + \cos^2(A) = 1$$

The trigonometry we have studied so far has been applicable to only right-angled triangles. The **sine rule** allows us to calculate the lengths of sides and the size of angles in non-right-angled triangles. Consider the triangle shown.

## 9.2.2 Derivation of the sine rule

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$A$, $B$ and $C$ represent the three angles in the triangle ABC, and $a$, $b$ and $c$ represent the three sides. Remember that each side is named with the lower-case letter of the opposite vertex.

Construct a line from C to a point, D, perpendicular to AB. CD is the perpendicular height of the triangle, $h$.

Now consider $\triangle$ACD and $\triangle$BCD separately.

Using the formula for the sine ratio:

$\sin(\theta) = \dfrac{\text{opp}}{\text{hyp}} \qquad \sin(\theta) = \dfrac{\text{opp}}{\text{hyp}}$

$\sin(A) = \dfrac{h}{b} \qquad \sin(B) = \dfrac{h}{a}$

$\quad h = b\sin(A) \qquad h = a\sin(B)$

We are now able to equate these two expressions for $h$.

$$a\,\sin(B) = b\,\sin(A)$$
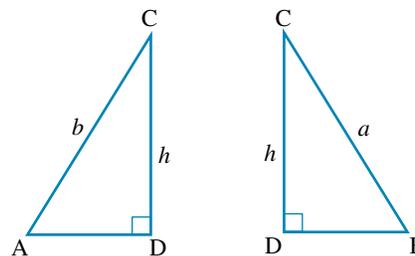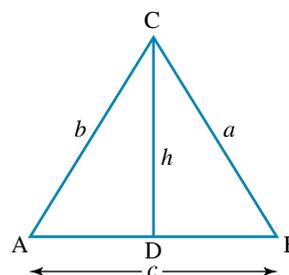
Dividing both sides by $\sin(A)\sin(B)$, we get:

$$\frac{a\sin(B)}{\sin(A)\sin(B)} = \frac{b\sin(A)}{\sin(A)\sin(B)}$$

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)}$$

Similarly, we are able to show that each of these is also equal to $\dfrac{c}{\sin(C)}$.

The sine rule states that in any triangle, ABC, the ratio of each side to the sine of its opposite angle will be equal.
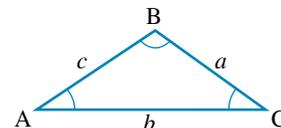


### The sine rule

To determine the side length of a non-right angle triangle:

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

where $a$, $b$ and $c$ are the side lengths of the triangle and $A$, $B$ and $C$ are the corresponding opposite angles.

## 9.2.3 Using the sine rule to calculate side lengths

The sine rule formula allows us to calculate the length of a side in any triangle if we are given the length of one other side and two angles. When using the formula, we need to use only two parts of it.

### WORKED EXAMPLE 1 Calculating side length using the sine rule

Calculate the length of the side marked $x$ in the triangle shown, correct to 1 decimal place.
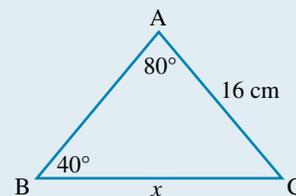


THINK

1. Label the sides of the triangle.

WRITE
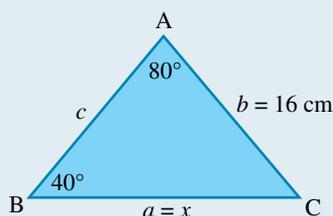
| | |
|---|---|
| **2.** Write the formula. | $\dfrac{a}{\sin(A)} = \dfrac{b}{\sin(B)}$ |
| **3.** Substitute $a = x$, $b = 16$, $A = 80°$ and $B = 40°$. | $\dfrac{x}{\sin(80°)} = \dfrac{16}{\sin(40°)}$ |
| **4.** Make $x$ the subject of the equation by multiplying by $\sin(80°)$. | $x = \dfrac{16\sin(80°)}{\sin(40°)}$ |
| **5.** Calculate and round to 1 decimal place. | $x \approx 24.5$ cm |

To use the sine rule we need to know the angle opposite the side we are calculating and the angle opposite the side we are given. In some cases these are not the angles we are given. In such cases we need to use the fact that the angles in a triangle add to 180° to calculate the required angle.

## WORKED EXAMPLE 2 Calculating side length using the sine rule

**Calculate the length of the side labelled $y$ in the figure shown, correct to 2 decimal places.**
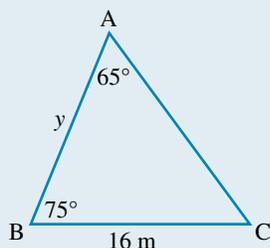


**THINK**

**WRITE**

| **THINK** | **WRITE** |
|---|---|
| **1.** Calculate the size of angle $C$. | $C = 180° - 65° - 75°$ <br> $\quad = 40°$ |
| **2.** Write the formula. | $\dfrac{a}{\sin(A)} = \dfrac{c}{\sin(C)}$ |
| **3.** Substitute $a = 16$, $c = y$, $A = 65°$ and $C = 40°$. | $\dfrac{16}{\sin(65°)} = \dfrac{y}{\sin(40°)}$ |
| **4.** Make $y$ the subject of the equation. | $y = \dfrac{16\sin(40°)}{\sin(65°)}$ |
| **5.** Calculate and round to 2 decimal places. | $\quad = 11.35$ m |

Using the sine rule allows us to solve a number of more complex problems. As with our earlier trigonometry problems, we begin with a diagram and give a written answer to each question.

## WORKED EXAMPLE 3 Solving problems involving the sine rule

**To calculate the height of a building, Kevin measures the angle of elevation to the top as 52°. He then walks 20 m closer to the building and measures the angle of elevation as 60°. Calculate the height of the building.**

| THINK | WRITE |
|---|---|
| 1. Draw a labelled diagram of the situation and fill in the given information. |  |
| 2. Check that one of the criteria for the sine rule has been satisfied for triangle ABC. | The sine rule can be used for triangle ABC since two angles and one side length have been given. |
| 3. Determine the value of angle ACB, using the fact that the angle sum of any triangle is 180°. | $\angle ACB = 180° - (52° + 120°)$ <br> $\quad\quad\quad = 8°$ |
| 4. Write down the sine rule to determine $b$. | To determine side length $b$ of triangle ABC: $\dfrac{b}{\sin(B)} = \dfrac{c}{\sin(C)}$ |
| 5. Substitute the known values into the rule. | $\dfrac{b}{\sin(120°)} = \dfrac{20}{\sin(8°)}$ |
| 6. Make $b$ the subject of the equation. | $b = \dfrac{20 \times \sin(120°)}{\sin(8°)}$ |
| 7. Calculate and round the answer to 2 decimal places and include the appropriate unit. | $\approx 124.45 \text{ m}$ |
| 8. Draw a diagram of the situation, that is triangle ADC, labelling the required information. Also label the sides of the triangle. |  |
| 9. Choose the sine ratio as we are determining the opposite side and have been given the hypotenuse. | $\sin(\theta) = \dfrac{\text{opp}}{\text{hyp}}$ |
| 10. Substitute for $\theta$ and the hypotenuse. | $\sin(52°) = \dfrac{h}{124.45}$ |
| 11. Make $h$ the subject of the equation. | $124.45 \sin(52°) = h$ <br> $h = 124.45 \sin(52°)$ |
| 12. Calculate and round appropriately. | $\approx 98.07$ |
| 13. Give a written answer. | The height of the building is 98.07 m. |
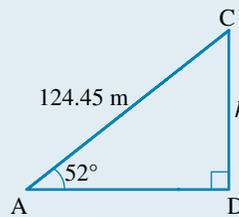
## 9.2.4 Using the sine rule to determine angle sizes

Using the sine rule we are able to calculate angle sizes as well. To do this, we need to be given the length of two sides and the angle opposite one of them. For simplicity, in solving the triangle we invert the sine rule formula when we are using it to determine an angle.

The formula is written:

> **The sine rule**
>
> **To determine the angle size in a non-right angle triangle:**
>
> $$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}$$
>
> **where $a$, $b$ and $c$ are the side lengths of the triangle and $A$, $B$ and $C$ are the corresponding opposite angles.**

As with determining side lengths, we use only two parts of the formula.

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### WORKED EXAMPLE 4 Calculating the angle size using the sine rule

**Calculate the size of the angle, $\theta$, in the figure shown, correct to the nearest degree.**
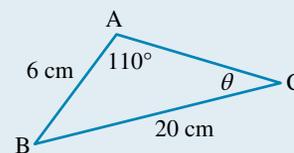


| THINK | WRITE |
|---|---|
| **1.** Write the formula. | $\dfrac{\sin(A)}{a} = \dfrac{\sin(C)}{c}$ |
| **2.** Substitute $A = 110°$, $C = \theta$, $a = 20$ and $c = 6$. | $\dfrac{\sin(110°)}{20} = \dfrac{\sin(\theta)}{6}$ |
| **3.** Make $\sin\theta$ the subject of the equation. | $\sin(\theta) = \dfrac{6\sin(110°)}{20}$ |
| **4.** Calculate a value for $\sin(\theta)$. | $\sin(\theta) = 0.2819$ <br> $\theta = \sin^{-1}(0.2819)$ |
| **5.** Calculate $\sin^{-1}(0.2819)$ to determine $\theta$. | $\theta = 16°$ |

---

## 9.2.5 The ambiguous case of the sine rule

When we are given two side lengths of a triangle and an acute angle opposite one of these sides, there are two different triangles we can draw. So far we have only dealt with triangles in which all angles are acute; however, it is also possible to draw triangles with obtuse angles. This is known as the ambiguous case of the sine rule.

For example, take the triangle ABC, where $a = 12$, $c = 8$ and $C = 35°$.

When we solve this for angle $A$, we determine an acute angle as shown:

$$\frac{8}{\sin(35)} = \frac{12}{\sin(A)}$$

$$8\sin(A) = 12\sin(35)$$

$$\sin(A) = \frac{12\sin(35)}{8}$$

$$A = \sin^{-1}\left(\frac{12\sin(35)}{8}\right)$$

$$= 59.36°$$
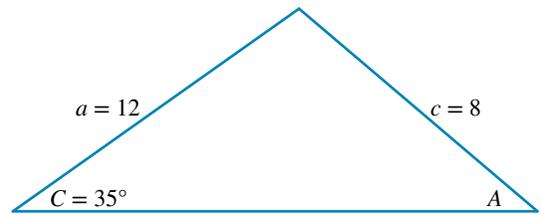
However, there is also an obtuse-angled triangle that can be drawn from this given information.

In this case, the size of the obtuse angle is the supplement of the acute angle calculated previously.

$$A' = 180° - A$$
$$= 180° - 59.36°$$
$$= 120.64°$$
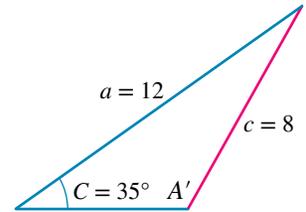
## 9.2.6 Determining when to use the ambiguous case

The ambiguous case of the sine rule does not work for every example. This is due to the way the ratios are set in the development of the sine rule, since side length $a$ must be longer than $h$, where $h$ is the length of the altitude from angle $B$ to the base line $b$.

For the ambiguous case to be applicable, the following conditions must be met:
- The given angle must be acute.
- The adjacent side must be greater than the opposite side.
- The opposite side must be greater than the adjacent side multiplied by the sine of the given angle.

In general, when given a triangle with two known sides and an angle opposite one of them (SSA), you should check for the ambiguous case unless instructed otherwise.

For example, in triangle ABC, where $a = 12$, $c = 8$ and $C = 35°$, if $\sin(A) < 1$, then both $A$ and its supplementary angle should be considered as possible solutions.

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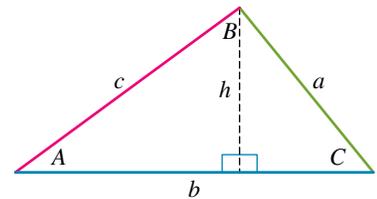
### WORKED EXAMPLE 5 The ambiguous case of the sine rule

**Calculate the two possible values of angle $A$ for triangle ABC, given $a = 15$, $c = 8$ and $C = 25°$.**

**THINK**

1. Draw a non-right-angled triangle, labelling with the given information. Angle $A$ is opposite to side $a$. Angle $C$ is opposite to side $c$. Note that two triangles can be drawn, with angle $A$ being either acute or obtuse.

**WRITE**
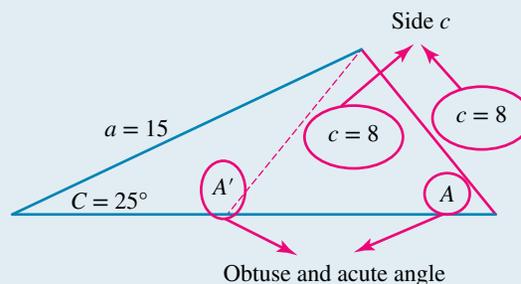
| 2. Substitute the known values into the sine rule. | $\dfrac{a}{\sin(A)} = \dfrac{c}{\sin(C)}$ |
|---|---|

3. Rearrange the equation to make $\sin(A)$ the subject.

$$\dfrac{15}{\sin(A)} = \dfrac{8}{\sin(25°)}$$
$$15\sin(25°) = 8\sin(A)$$
$$\sin(A) = \dfrac{15\sin(25°)}{8}$$
$$\sin(A) = 0.7924...$$

4. The sin of $A$ is less than 1. Therefore, two angles should be considered as possible solutions.

$\sin(A) < 1$
$\therefore$ consider two possible solutions.

5. Make sure your calculator is in degree mode. The calculator will only give the acute angle value.

$$A = \sin^{-1}\left(\dfrac{15\sin(25°)}{8}\right)$$
$$= 52.41°$$

6. Solve for the obtuse angle $A'$.

$$A' = 180° - A$$
$$= 180° - 52.41°$$
$$= 127.59°$$

7. Write the answer.

The two possible value for $A$ are $52.41°$ and $127.59°$.

## Exercise 9.2 The sine rule

| 9.2 Exercise | 9.2 Exam questions on |
|---|---|

| Simple familiar | Complex familiar | Complex unfamiliar |
|---|---|---|
| 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14 | 15, 16, 17, 18, 19, 20 | – |

### Simple familiar

1. Write down the sine rule formula as it applies to each of the triangles below.

a.
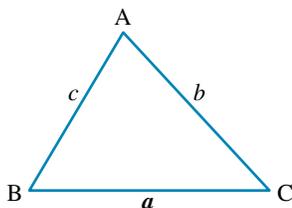
b.
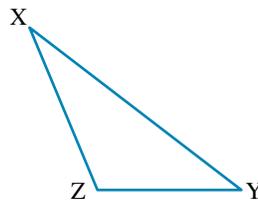
c.
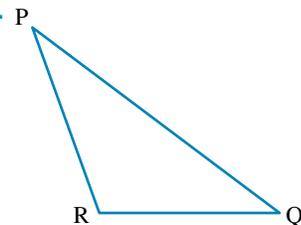
2. In a triangle, the length of side $a$ is 8 units, and angles $A$ and $B$ are $60°$ and $45°$ respectively. Calculate the exact value of side $b$.

3. In $\triangle ABC$, $b = 12$ cm, $\angle ABC = 30°$ and $\angle ACB = 45°$. Calculate the exact value of side $c$.

**4.** In $\triangle ABC$, $a = \dfrac{10}{\sqrt{2}}$ cm, $c = 10$ cm and $\angle ACB = 45°$. Calculate the magnitude of angle $\angle CAB$.

**5.** WE1 Use the sine rule to calculate the length of the side marked with the pronumeral in each of the following, correct to 3 significant figures.

a.
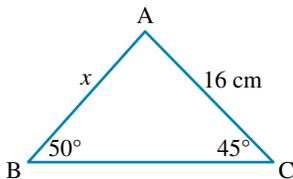


b.
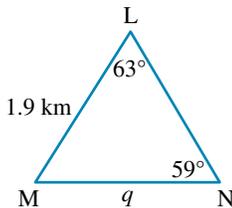


c.
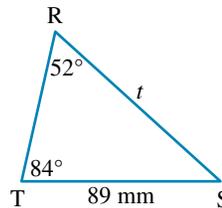


**6.** WE2 In each of the following, use the sine rule to calculate the length of the side marked with the pronumeral, correct to 1 decimal place, by first calculating the size of the third angle.

a.
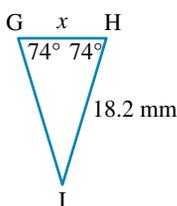


b.
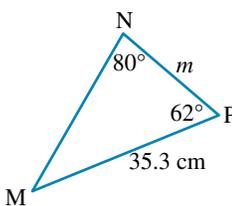


c.
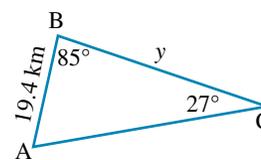


**7.** ABC is a triangle in which BC $= 9$ cm, $\angle BAC = 54°$ and $\angle ACB = 62°$. Calculate the length of side AB, correct to 1 decimal place.

**8.** XYZ is a triangle in which $y = 19.2$ m, $\angle XYZ = 42°$ and $\angle XZY = 28°$. Calculate $x$, correct to 3 decimal places.

**9.** X and Y are two trees 30 m apart on one side of a river, as shown in the diagram. Z is a tree on the opposite side of the river. It is found that $\angle XYZ = 72°$ and $\angle YXZ = 59°$. Calculate the distance XZ, correct to 1 decimal place.
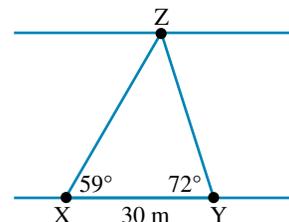


**10.** WE4 Calculate the size of the angle marked with a pronumeral in each of the following, correct to the nearest degree.

a.
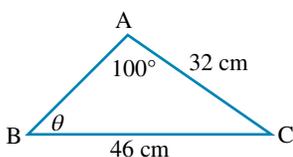


b.
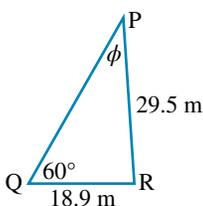


c.
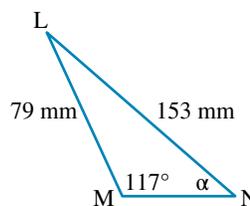


**11.** In $\triangle PQR$, $q = 12$ cm, $r = 16$ cm and $\angle PRQ = 56°$. Calculate the size of $\angle PQR$, correct to the nearest degree.

**12.** In $\triangle KLM$, LM $= 4.2$ m, KL $= 5.6$ m and $\angle KML = 27°$. Calculate the size of $\angle LKM$, correct to the nearest degree.

**13.** WE5 In a triangle ABC, $a = 9$ cm, $c = 8$ cm and $C = 42°$. Calculate the two possible values of angle A.

**14.** Calculate the possible values for angle A in a triangle ABC for which $c = 15$ cm, $b = 12$ cm and $C = 35°$.

**15.** In a triangle, the lengths of side *b* is 24 units, and angles *A* and *B* are 30° and 120° respectively. Calculate the exact value for the perimeter of the triangle.

**16.** From a point, M, the angle of elevation to the top of a building, B, is 34°. From a point, N, 20 m closer to the building, the angle of elevation is 49°.

    **a.** Draw a diagram of this situation.
    **b.** Calculate the distance NB, correct to 1 decimal place.
    **c.** Calculate the height of the building, correct to the nearest metre.

**17.** **WE3** To calculate the height of a building, Kevin measures the angle of elevation to the top as 48°. He then walks 18 m closer to the building and measures the angle of elevation as 64°. Calculate the height of the building.

**18.** A surveyor marks three points, X, Y and Z, in the ground. The surveyor measures XY to be 13.7 m and XZ to be 14.2 m. ∠XYZ is 60°.

    **a.** Calculate ∠XZY to the nearest degree.
    **b.** Calculate ∠YXZ to the nearest degree.

**19.** A cliff is 37 m high. The rock slopes outward at an angle of 50° to the horizontal, then cuts back at an angle of 25° to the vertical, meeting the ground directly below the top of the cliff.

Carol wishes to abseil from the top of the cliff to the ground as shown in the diagram. Her climbing rope is 45 m long, and she needs 2 m to secure it to a tree at the top of the cliff. Determine if the rope will be long enough to allow her to reach the ground.
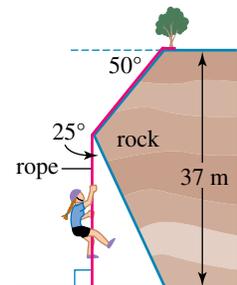
**20.** Two wires support a flagpole. The first wire is 8 m long and makes a 65° angle with the ground. The second wire is 9 m long. Calculate, to the nearest degree, the angle that the second wire makes with the ground.

**Fully worked solutions for this chapter are available online.**

# LESSON
## 9.3 The cosine rule

The sine rule will not allow us to solve all triangles. Depending on the information provided about the triangle, we may need to use the **cosine rule**.

### 9.3.1 Derivation of the cosine rule

In any non-right-angled triangle, ABC, a perpendicular line can be drawn from angle $B$ to side $b$. Let D be the point where the perpendicular line meets side $b$, and let the length of the perpendicular line be $h$. Let the length AD $= x$ units. The perpendicular line creates two right-angled triangles, ADB and CDB.
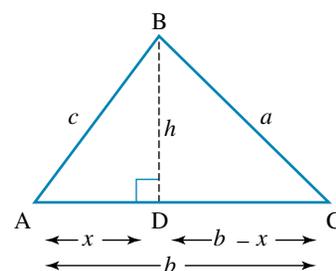
Using triangle ADB and Pythagoras' theorem, we obtain:

$$c^2 = h^2 + x^2 \qquad [1]$$

Using triangle CDB and Pythagoras' theorem, we obtain:

$$a^2 = h^2 + (b - x)^2 \qquad [2]$$

Expanding the brackets in equation [2]:

$$a^2 = h^2 + b^2 - 2bx + x^2$$

Rearranging equation [2] and using $c^2 = h^2 + x^2$ from equation [1]:

$$\begin{aligned} a^2 &= h^2 + x^2 + b^2 - 2bx \\ &= c^2 + b^2 - 2bx \\ &= b^2 + c^2 - 2bx \end{aligned}$$

From triangle ABD, $x = c\cos(A)$. Therefore, $a^2 = b^2 + c^2 - 2bx$ becomes

$$a^2 = b^2 + c^2 - 2bc\cos(A)$$

This is called the cosine rule and is a generalisation of Pythagoras' theorem.

In a similar way, if a perpendicular line is drawn from angle $A$ to side $a$ or from angle $C$ to side $c$, the two right-angled triangles give $c^2 = a^2 + b^2 - 2ab\cos(C)$ and $b^2 = a^2 + c^2 - 2ac\cos(B)$ respectively.

In any triangle ABC,
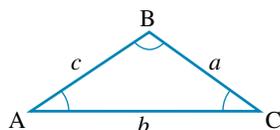


the following rule can be stated.

## The cosine rule (sides)

**The cosine rule in this form can calculate the length of each of the following sides:**

$$a^2 = b^2 + c^2 - 2bc\cos(A)$$
$$b^2 = a^2 + c^2 - 2ac\cos(B)$$
$$c^2 = a^2 + b^2 - 2ab\cos(C)$$

## 9.3.2 Using the cosine rule to calculate side lengths

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The cosine rule can be used to solve non-right-angled triangles if we are given either of the following:

- three sides of the triangle
- two sides of the triangle and the included angle (the angle between the given sides).

### WORKED EXAMPLE 6 Using the cosine rule to determine a side length

**Determine the third side of triangle ABC given $a = 6$, $c = 10$ and $B = 76°$.**

| THINK | WRITE |
|---|---|
| 1. Draw a labelled diagram of the triangle ABC and fill in the given information. |  |
| 2. Write the appropriate cosine rule to determine $b$. | $b^2 = a^2 + c^2 - 2ac\cos(B)$ |
| 3. Substitute the given values into the rule. | $= 6^2 + 10^2 - 2 \times 6 \times 10 \times \cos(76°)$ |
| 4. Evaluate. | $\approx 106.969\,372\,5$ $b = \sqrt{106.969\,372\,5}$ |
| 5. Round the answer to 2 decimal places. | $\approx 10.34$ |
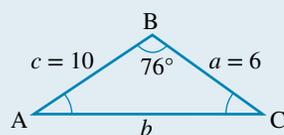
The cosine rule also allows us to solve a wider range of practical problems. The important part of solving such problems is marking the correct information on your diagram. If you can identify two side lengths and the included angle, you can use the cosine rule.

A surveyor standing at point X sights point M, 50 m away and point N, 80 m away. If the angle between the lines XM and XN is 45°, calculate the distance between the points M and N, correct to 1 decimal place.

| THINK | WRITE |
|---|---|
| 1. Draw a diagram and mark all given information on it. |  |
| 2. Write the formula with $x^2$ as the subject. | $x^2 = m^2 + n^2 - 2mn\cos(X)$ |
| 3. Substitute $m = 80$, $n = 50$ and $X = 45°$. | $= 80^2 + 50^2 - 2 \times 80 \times 50 \times \cos(45°)$ |
| 4. Calculate the value of $x^2$. | $= 3243.15$ |
| 5. Calculate $x$ by taking the square root of $x^2$. | $x = \sqrt{3243.15}$ |
| 6. Round your answer to 1 decimal place. | $\approx 56.9\,\text{m}$ |
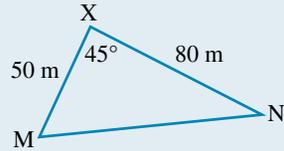
## 9.3.3 Using the cosine rule to calculate angle sizes

We can use the cosine rule to calculate the size of the angles within a triangle. Consider the cosine rule formula.

$$a^2 = b^2 + c^2 - 2bc\cos(A)$$

We now make $\cos(A)$ the subject of this formula.

$$a^2 = b^2 + c^2 - 2bc\cos(A)$$
$$a^2 + 2bc\cos(A) = b^2 + c^2$$
$$2bc\cos(A) = b^2 + c^2 - a^2$$
$$\cos(A) = \frac{b^2 + c^2 - a^2}{2bc}$$

In this form, we can use the cosine rule to calculate the size of an angle if we are given all three side lengths. We can write the cosine rule in three different forms, depending on which angle we wish to calculate.

### The cosine rule (angles)

The cosine rule in this form can calculate the size of each of the following angles:

$$\cos(A) = \frac{b^2 + c^2 - a^2}{2bc}$$
$$\cos(B) = \frac{a^2 + c^2 - b^2}{2ac}$$
$$\cos(C) = \frac{a^2 + b^2 - c^2}{2ab}$$

## WORKED EXAMPLE 8 Using the cosine rule to determine an angle size

**Calculate the size of angle $B$ in the triangle shown, correct to the nearest degree.**
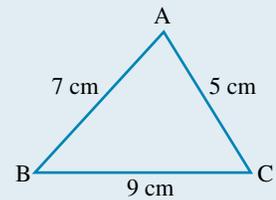


| THINK | WRITE |
|---|---|
| 1. Write the formula with $\cos(B)$ as the subject. | $\cos(B) = \dfrac{a^2 + c^2 - b^2}{2ac}$ |
| 2. Substitute $a = 9$, $b = 5$ and $c = 7$. | $\cos(B) = \dfrac{9^2 + 7^2 - 5^2}{2 \times 9 \times 7}$ |
| 3. Calculate the value of $\cos(B)$. | $\cos(B) = \dfrac{105}{126}$<br>$= 0.8333$ |
| 4. Make $B$ the subject of the equation. | $B = \cos^{-1}(0.8333)$ |
| 5. Calculate $B$. | $B = 34°$ |

As we found earlier, the cosine ratio for an obtuse angle will be negative. So, when we get a negative result to the calculation for the cosine ratio, this means that the angle we are calculating is obtuse. Your calculator will give the obtuse angle when you take the inverse.

## WORKED EXAMPLE 9 Using the cosine rule to determine an angle size 2

**Calculate the size of angle $Q$ in the triangle shown, correct to the nearest degree.**
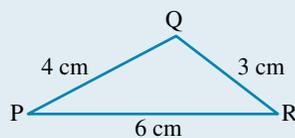


| THINK | WRITE |
|---|---|
| 1. Write the formula with $\cos(Q)$ as the subject. | $\cos(Q) = \dfrac{p^2 + r^2 - q^2}{2pr}$ |
| 2. Substitute $p = 3$, $q = 6$ and $r = 4$. | $\cos(Q) = \dfrac{3^2 \times 4^2 - 6^2}{2 \times 4 \times 3}$ |
| 3. Calculate the value of $\cos(Q)$. | $\cos(Q) = \dfrac{-11}{24}$<br>$= -0.4583$ |
| 4. Make $Q$ the subject of the equation. | $Q = \cos^{-1}(-0.4583)$ |
| 5. Calculate $Q$. | $Q = 117°$ |

In some cosine rule questions, you have to work out which angle you need to determine. For example, you could be asked to calculate the size of the largest angle in a triangle. To do this you do not need to calculate all three angles. The largest angle in any triangle will be the one opposite the longest side. Similarly, the smallest angle will lie opposite the shortest side.

## WORKED EXAMPLE 10 Using the cosine rule to determine the largest angle

**Calculate the size of the largest angle in the triangle shown, correct to the nearest degree.**
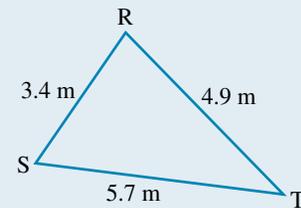


| THINK | WRITE |
|---|---|
| **1.** ST is the longest side. Therefore, angle $R$ is the largest angle. | |
| **2.** Write the formula with $\cos(R)$ as the subject. | $\cos(R) = \dfrac{s^2 + t^2 - r^2}{2st}$ |
| **3.** Substitute $r = 5.7$, $s = 4.9$ and $t = 3.4$. | $\cos(R) = \dfrac{4.9^2 + 3.4^2 - 5.7^2}{2 \times 4.9 \times 3.4}$ |
| **4.** Calculate the value of $\cos(R)$. | $\cos(R) = \dfrac{3.08}{33.32}$ <br> $= 0.0924$ |
| **5.** Make $R$ the subject of the equation. | $R = \cos^{-1}(0.0924)$ |
| **6.** Calculate $R$. | $R = 85°$ |
| **7.** Give a written answer. | The largest angle in the triangle is 85°. |

Many problems that require you to determine an angle are solved using the cosine rule. As always, these begin with a diagram and are finished off by giving a written answer.

## WORKED EXAMPLE 11 Problem solving using the cosine rule 2

**Two paths diverge from point A. The first path goes for 1.25 km to point B. The second path goes for 1.4 km to point C. Points B and C are exactly 2 km apart. Calculate the angle at which the two paths diverge, correct to the nearest degree.**

| THINK | WRITE |
|---|---|
| **1.** Draw a diagram. |  |
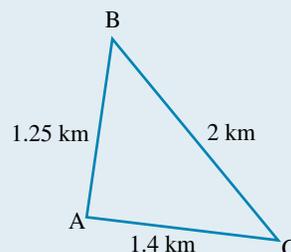
**2.** Write the formula with $\cos(A)$ as the subject.

$$\cos(A) = \frac{b^2 + c^2 - a^2}{2bc}$$

**3.** Substitute $a = 2$, $b = 1.4$ and $c = 1.25$.

$$\cos(A) = \frac{1.4^2 + 1.25^2 - 2^2}{2 \times 1.4 \times 1.25}$$

**4.** Calculate the value of $\cos(A)$.

$$\cos(A) = \frac{-0.4775}{3.5}$$
$$= -0.1364$$

**5.** Make $A$ the subject of the equation.

$$A = \cos^{-1}(-0.1364)$$

**6.** Calculate the value of $A$.

$$= 98°$$

**7.** Give a written answer.

The roads diverge at an angle of 98°.

## Exercise 9.3 The cosine rule

**learn**on

| 9.3 Exercise | 9.3 Exam questions **on** |
|---|---|

| Simple familiar | Complex familiar | Complex unfamiliar |
|---|---|---|
| 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 12, 13, 14, 15, 16 | 17, 18, 19, 20, 21, 22 | 23, 24 |

**Simple familiar**

**1.** Write down the cosine rule formula as it applies to each of the triangles below. In each case, make the boldfaced pronumeral the subject.

**a.**
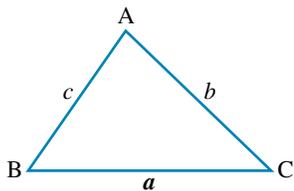


**b.**
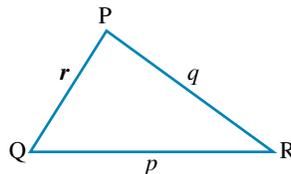


**c.**
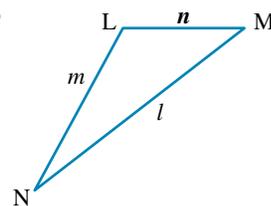


**2.** Consider a triangle with side lengths measuring, $a = 11$ mm, $b = 17$mm and $c = 13$mm. Without calculating any angles, arrange angles $A, B$ and $C$ in ascending order of magnitude.

**3.** In triangle ABC, $A = 120°$ and the respective side lengths of $b$ and $c$ are 7 cm and 6 cm. Calculate the length of side $a$ correct to two decimal places.

**4.** In triangle ABC, $a = 5$ m, $c = 4$ m and $B = 60°$. Calculate side length $b$.

**5.** Calculate $\angle CAB$ in $\triangle ABC$ given $a = 7, b = 8$ and $c = 5$.

**6.** Calculate the third side of triangle ABC given $a = 3.4, b = 7.8$ and $C = 80°$.

**7.** **WE6** Calculate the length of the side marked with a pronumeral in each of the following, correct to 3 significant figures.

a.



b.



c.
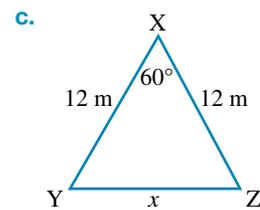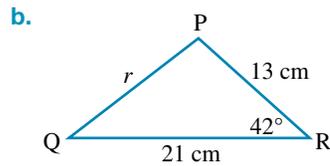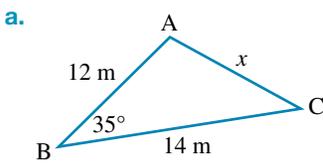


**8.** In each of the following obtuse-angled triangles, calculate the length of the side marked with the pronumeral, correct to 1 decimal place.

a.



b.



c.
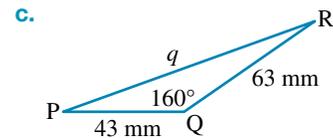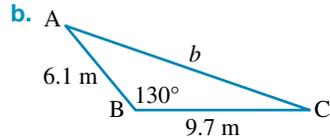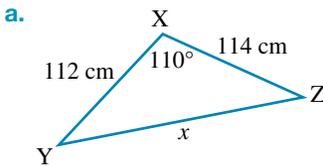


**9.** In triangle ABC, $b = 64.5$, $c = 38.1$ and $A = 58°34'$. Calculate $a$.

**10.** In triangle ABC, $a = 17$, $c = 10$ and $B = 115°$. Calculate $b$, and hence calculate $A$ and $C$ in degrees and minutes.

**11.** **WE7** Len and Morag walk separate paths that diverge from one another at an angle of 48°. After 3 hours Len has walked 7.9 km and Morag has walked 8.6 km. Calculate the distance between the two walkers at this time, correct to the nearest metre.

**12.** **WE8** Calculate the size of the angle marked with the pronumeral in each of the following triangles, correct to the nearest degree.

a.



b.



c.
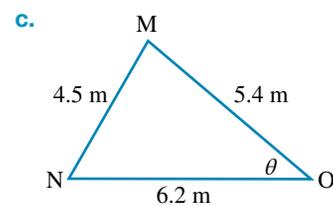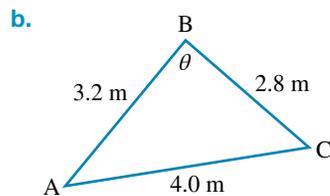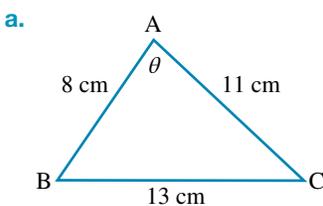


**13.** **WE9** In each of the obtuse-angled triangles below, calculate the size of the angle marked with the pronumeral, to the nearest degree.

a.



b.



c.
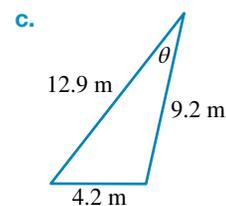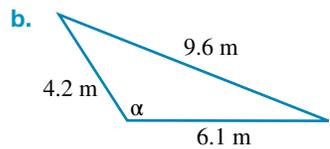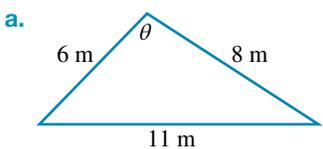


**14.** **WE10** In triangle ABC, $a = 356$, $b = 207$ and $c = 296$. Calculate the largest angle in degrees and minutes.

**15.** Calculate, in degrees and minutes, the smallest angle in the triangle with sides 6 cm, 4 cm and 8 cm.

**16.** In triangle ABC, $a = 23.6$, $b = 17.3$ and $c = 26.4$. Calculate the size of all the angles in degrees and minutes.

**Complex familiar**

**17.** A cricketer is fielding 20 m from the batsman and at an angle of 35° to the pitch. The batsman hits a ball 55 m and straight behind the bowler. Calculate how far the fielder must run to field the ball. (Give your answer to the nearest metre.)

**18.** Calculate the size of all three angles (correct to the nearest degree) in a triangle with side lengths 12 cm, 14 cm and 17 cm.

**19.** **WE11** Two roads diverge from point P. The first road is 5 km long and leads to point Q. The second road is 8 km long and leads to point R. The distance between Q and R is 4.6 km. Calculate the angle at which the two roads diverge.

**20.** From the top of a vertical cliff 68 m high, an observer notices a yacht at sea. The angle of depression to the yacht is 47°. The yacht sails directly away from the cliff, and after 10 minutes the angle of depression is 15°. Calculate the distance the yacht has sailed.

**21.** The cord supporting a picture frame is 58 cm long. It is hung over a single hook in the centre of the cord and the cord then makes an angle of 145° as shown in the diagram. Calculate the length of the backing of the picture frame, to the nearest centimetre.
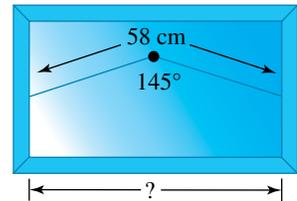
**22.** An advertising balloon is attached to two ropes 120 m and 100 m long. The ropes are anchored to level ground 35 m apart. Calculate how high the balloon can fly.

**Complex unfamiliar**

**23.** Sophie is playing hockey and is about to score a goal. She is positioned 7 m from one goalpost and 5.2 m from the other, with the hockey goal being 3 m wide. Determine the angle, to the nearest degree, within which Sophie must shoot to score a goal.

**24.** Three circles of radii 5 cm, 6 cm and 8 cm are positioned so that they just touch one another. Their centres form the vertices of a triangle. Calculate the largest angle in the triangle.
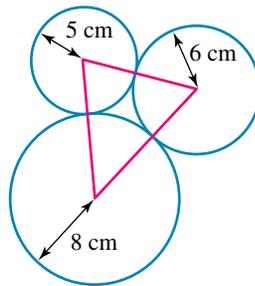
**Fully worked solutions for this chapter are available online.**

# LESSON
## 9.4 Area of a triangle

### 9.4.1 Area of triangles

int-6483

The area of a right-angled triangle is given by the formula below.

---

**Area of a right-angled triangle**

$$\textbf{Area} = \frac{1}{2}\,\textbf{(base)} \times \textbf{(height)}$$

---

For a triangle that is not right-angled, if two sides and the angle included between these two sides are known, it is also possible to calculate the area of the triangle from that given information.

Consider the triangle ABC shown, where the convention of labelling the sides opposite the angles $A$, $B$ and $C$ with lower-case letters $a$, $b$ and $c$ respectively has been adopted in the diagram.

In triangle ABC, construct the perpendicular height, $h$, from B to a point D on AC. As this is not necessarily an isosceles triangle, D is not the midpoint of AC.

In the right-angled triangle BCD, $\sin(C) = \dfrac{h}{a} \Rightarrow h = a\sin(C)$.

This means the height of triangle ABC is $a\sin(C)$ and its base is $b$.
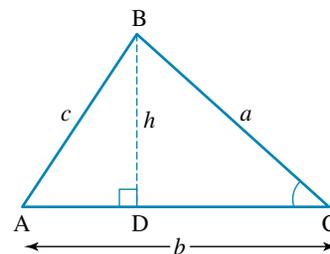
The area of the triangle ABC can now be calculated.

$$
\begin{aligned}
\text{Area} &= \frac{1}{2}(\text{base}) \times (\text{height})\\
&= \frac{1}{2}b \times a\sin(C)\\
&= \frac{1}{2}ab\sin(C)
\end{aligned}
$$

The formula for the area of the triangle ABC, $A_\Delta = \dfrac{1}{2}ab\sin(C)$, is expressed in terms of two of its sides and the angle included between them.

Alternatively, using the height as $c\sin(A)$ from the right-angled triangle ABD on the left of the diagram, the area formula becomes $A_\Delta = \dfrac{1}{2}bc\sin(A)$.

It can also be shown that the area is $A_\Delta = \dfrac{1}{2}ac\sin(B)$.

Hence, the area of a triangle is $\dfrac{1}{2} \times$ (product of two sides) $\times$ (sine of the angle included between the two given sides).
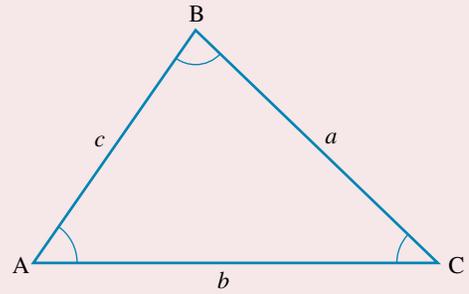
## Area of any triangle

Each of the formulas below can be used to calculate the area of any triangle:

$$\text{Area} = \frac{1}{2}bc \sin(A)$$

$$\text{Area} = \frac{1}{2}ac \sin(B)$$

$$\text{Area} = \frac{1}{2}ab \sin(C)$$



## WORKED EXAMPLE 12 Calculating the area of a triangle

Calculate the exact area of the triangle ABC for which $a = \sqrt{62}, b = 5\sqrt{2}, c = 6\sqrt{2}$ cm and $A = 60°$.
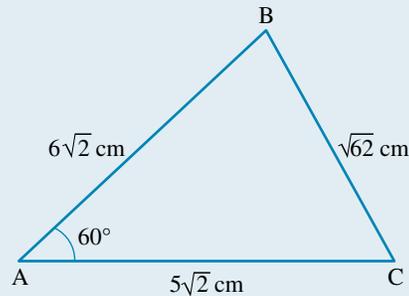
| THINK | WRITE |
|---|---|
| 1. Draw a diagram showing the given information.<br>*Note:* The naming convention for labelling the angles and the sides opposite them with upper- and lower-case letters is commonly used. |  |
| 2. State the two sides and the angle included between them. | The given angle $A$ is included between the sides $b$ and $c$. |
| 3. State the appropriate area formula and substitute the known values. | The area formula is:<br>$A_\Delta = \frac{1}{2}bc \sin(A), b = 5\sqrt{2}, c = 6\sqrt{2}, A = 60°$<br>$\therefore A = \frac{1}{2} \times 5\sqrt{2} \times 6\sqrt{2} \times \sin(60°)$ |
| 4. Evaluate, using the exact value for the trigonometric ratio. | $\therefore A = \frac{1}{2} \times 5\sqrt{2} \times 6\sqrt{2} \times \frac{\sqrt{3}}{2}$<br>$= \frac{1}{2} \times 30 \times 2 \times \frac{\sqrt{3}}{2}$<br>$= 15\sqrt{3}$ |
| 5. State the answer. | The area of the triangle is $15\sqrt{3}$ cm². |

## WORKED EXAMPLE 13 Calculating the area of a triangle 2

**Calculate the areas of the following triangles. Give both answers correct to 2 decimal places.**

**a.**



7 cm
63°
5 cm
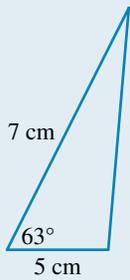
**b. A triangle with sides of length 8 cm and 7 cm, and an included angle of 55°.**

| THINK | WRITE |
|---|---|
| **a. 1.** Label the vertices of the triangle. | **a.**  B, 7 cm, 63°, A, 5 cm, C |
| **2.** Write down the known information. | $b = 5$ cm<br>$c = 7$ cm<br>$A = 63°$ |
| **3.** Substitute the known values into the formula to calculate the area of the triangle. | $\text{Area} = \dfrac{1}{2}bc\sin(A)$<br>$= \dfrac{1}{2} \times 5 \times 7 \times \sin(63°)$<br>$= 15.592\dots$<br>$= 15.59$ (to 2 d.p.) |
| **4.** Write the answer, remembering to include the units. | The area of the triangle is 15.59 cm$^2$, correct to 2 decimal places. |
| **b. 1.** Draw a diagram to represent the triangle. | **b.**  C, 8 cm, 55°, A, 7 cm, B |
| **2.** Write down the known information. | $b = 8$ cm<br>$c = 7$ cm<br>$A = 55°$ |
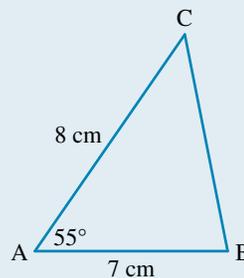
3. Substitute the known values into the formula to calculate the area of the triangle.

$$\text{Area} = \frac{1}{2}bc\sin(A)$$
$$= \frac{1}{2} \times 8 \times 7 \times \sin(55°)$$
$$= 22.936\ldots$$
$$= 22.94 \text{ (to 2 d.p.)}$$

4. Write the answer, remembering to include the units.

The area of the triangle is 22.94 cm², correct to 2 decimal places.

## Exercise 9.4 Area of a triangle

| 9.4 Exercise | 9.4 Exam questions on |
| --- | --- |

| Simple familiar | Complex familiar | Complex unfamiliar |
| --- | --- | --- |
| 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 | 11, 12, 13, 14 | 15 |

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• Access sample responses
• Track results and progress

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### Simple familiar

1. Consider the triangle ABC for which $b = 16$ mm, $c = 12$ mm and $A = 30°$. Calculate the exact area of $\triangle ABC$.

2. **WE12** Calculate the exact area of the triangle ABC for which $a = 10$ cm, $b = 6\sqrt{2}$ cm, $c = 2\sqrt{13}$ cm and $C = 45°$.

3. Calculate the exact area of the triangle ABC for which $a = 8$, $c = 9$ and $\angle ABC = 120°$.

4. The area of triangle ABC is $3\sqrt{3}$ cm². Calculate the length of line segment $c$ given $A = 60°$ and $b = 3$ cm.

5. The area of triangle ABC is 30 cm². Calculate the length of line segment $a$ given $B = 150°$ and $c = 10$ cm.

6. **WE13a** Calculate the area of the following triangle, correct to 2 decimal places.
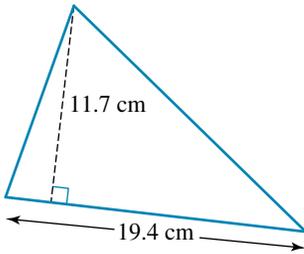
11.9 mm
38°
14.4 mm
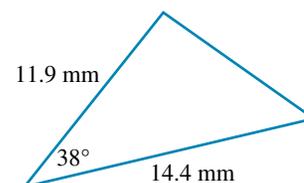
7. **WE13b** Calculate the area of a triangle with sides of length 14.3 mm and 6.5 mm, and an inclusive angle of 32°.

8. Horses graze over a triangular area XYZ where Y is 4 km east of X and Z is 3 km from Y on a bearing of N20°W. Calculate the area, correct to 2 decimal places, the horses can graze in.
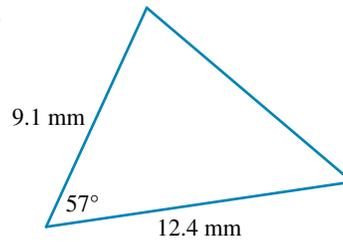
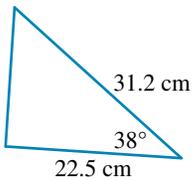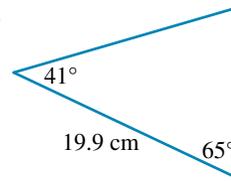**9.** Calculate the areas of the following triangles.

**a.**

11.7 cm

19.4 cm

**b.**

9.1 mm

57°

12.4 mm

**10.** Calculate the areas of the following triangles.

**a.**

31.2 cm

38°

22.5 cm

**b.**

41°

19.9 cm

65°

**Complex familiar**

**11. a.** An isosceles triangle ABC has sides BC and AC of equal length, 5 cm. If the angle enclosed between the equal sides is 20°, calculate:

  **i.** the area of the triangle to 3 decimal places

  **ii.** the length of the third side AB to 3 decimal places.

**b.** An equilateral triangle has a vertical height of 10 cm. Calculate the exact perimeter and area of the triangle.

**c.** Calculate the exact area of the triangle ABC if, using the naming convention, $a = 4\sqrt{2}$ cm, $b = 6$ cm and $C = 30°$.

**12.** A triangle has two sides of length 9.5 cm and 13.5 cm, and one angle of 40.2°. Calculate all three possible areas of the triangle.

**13.** Calculate the areas of the regions shown, to the nearest square metre. The diagrams are not to scale.

**a.**

109 m

62 m

78 m

**b.**

40 m

50 m

30°

40°

35 m

**c.**

121 m

214 m

32°

34°

190 m

41°

156 m
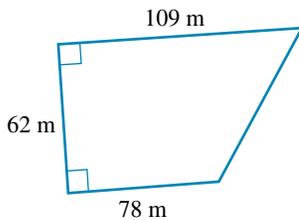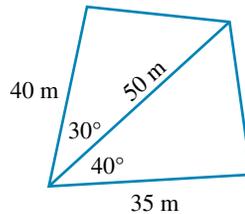
**14.** A field in the shape of a quadrilateral is shown. Calculate the amount of grass seed required to cover the dry field with a 1 mm layer.

104°

87 m

100°

111°

125 m

45°

15. A section is removed from a block of wood 20 cm in length, 10 cm in width and having a height of 6 cm in such a way that the triangle AFH is formed.
Determine the total surface area of the block of wood after the removal of the section.

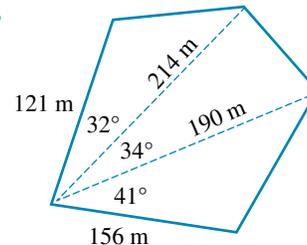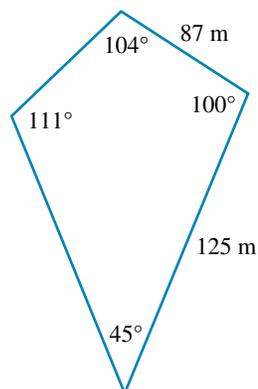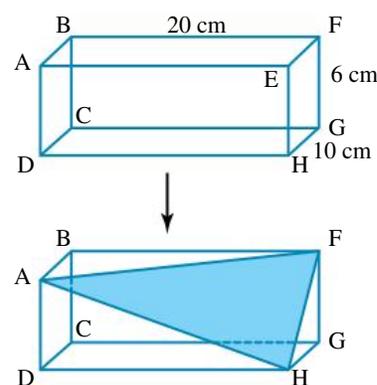Fully worked solutions for this chapter are available online.

# LESSON
## 9.5 Modelling and solving with the sine and cosine rules

### SYLLABUS LINKS

- Model and solve problems that involve the sine rule, cosine rule and the area formula in two- and three-dimensional contexts (including bearings, directions and angles of elevation and depression), with and without technology.

*Source:* Mathematical Methods Senior Syllabus 2024 ©State of Queensland (QCAA) 2024; licensed under CCBY 4.0.

The sine and cosine rules can be used to calculate distances, heights and bearings in both two-dimensional and three-dimensional contexts.

As with determining side lengths, some questions will be problems that require you to draw a diagram to extract the required information and then give the answer in written form.

### 9.5.1 Bearings

A **bearing** gives the direction of travel from one point to another, measured as an acute angle from a fixed north–south line. As shown in the diagrams, a bearing of S20°E indicates a direction 20 degrees east of due south. The bearing N40°W is a direction 40 degrees west of north.
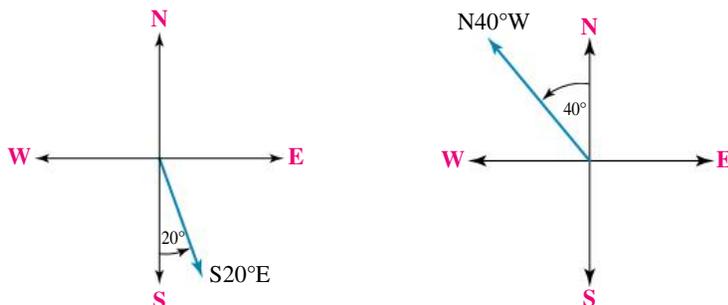
True bearings are measured clockwise from north and are represented using three digits. The diagrams below illustrate the true bearings of 025° and 250°, typically written as 025°T and 250°T.
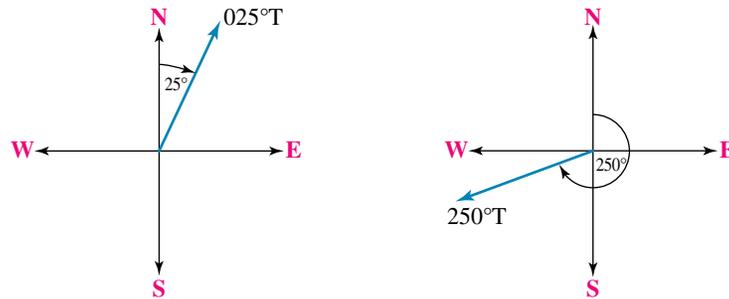


A direction of travel can be expressed as a compass or true bearing as shown below.

$$N42°E = 042°T$$
$$N35°W = 325°T$$
$$S15°W = 195°T$$

## WORKED EXAMPLE 14 Determining bearing using the sine rule

**From point P, a ship (S) is sighted 12.4 km away on a bearing of 137°. Point Q is due south of P and is a distance of 31.2 km from the ship. Calculate the bearing of the ship from Q, correct to the nearest degree.**
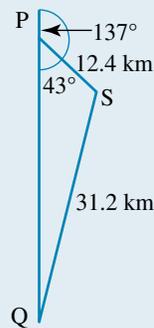
| THINK | WRITE |
|---|---|
| 1. Draw a diagram. |  |
| 2. Write the formula. | $\dfrac{\sin(Q)}{q} = \dfrac{\sin(P)}{p}$ |
| 3. Substitute for $p$, $q$ and $P$. | $\dfrac{\sin(Q)}{12.4} = \dfrac{\sin(43°)}{31.2}$ |
| 4. Make $\sin(Q)$ the subject. | $\sin(Q) = \dfrac{12.4\sin(43°)}{31.2}$ |
| 5. Calculate a value for $\sin(Q)$. | $\sin(Q) = 0.271$ |
| 6. Calculate $\sin^{-1}(0.271)$ to calculate $Q$. | $Q = 16°$ |
| 7. Give a written answer. | The bearing of the ship from $Q$ is 016°T or N16°E. |

Two rowers set out from the same point. One rows N70°E for 2000 m and the other rows S15°W for 1800 m. Calculate the distance between the two rowers.

| THINK | WRITE |
|---|---|
| 1. Draw a labelled diagram of the triangle, call it ABC and fill in the given information. |  |
| 2. Write down the appropriate cosine rule to calculate side $c$. | $c^2 = a^2 + b^2 - 2ab\cos(C)$ |
| 3. Substitute the given values into the rule. | $= 2000^2 + 1800^2 - 2 \times 2000 \times 1800\cos(125°)$ |
| 4. Evaluate. | $\approx 11\,369\,750.342$ $c = \sqrt{11\,369\,750.342}$ |
| 5. Round the answer to 2 decimal places. | $\approx 3371.91$ |
| 6. Give a written answer. | The rowers are 3371.91 m apart. |
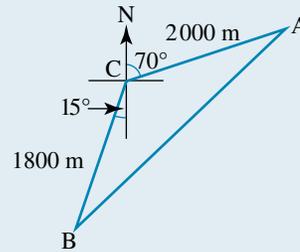
## 9.5.2 Angles of elevation and depression

The line from an observer's eye to an object is called the line of sight. If the object is above the horizontal, the angle between the line of sight and the horizontal is the **angle of elevation**; if the object is below, it is the **angle of depression**.

A helicopter pilot flying over a straight, flat road notices two signposts, A and B, that are 7 km apart. Signpost A is in front of the helicopter, while signpost B is behind. The angles of depression from the helicopter to the signposts are 42 degrees to signpost A and 67 degrees to signpost B. Calculate the distance from the helicopter to signpost A.

| THINK | WRITE |
|---|---|
| 1. Define points A, B and C. | Let, A = signpost A B = signpost B C = helicopter |

2. Construct a labelled diagram of
   the problem.
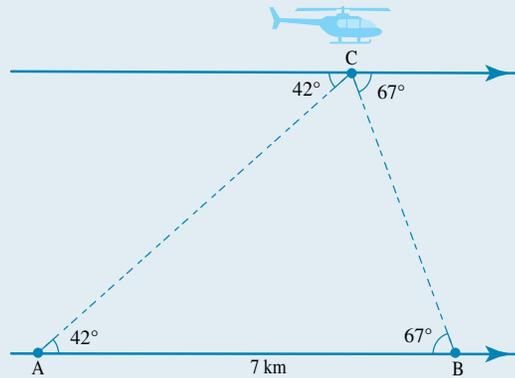


3. Calculate ∠ACB using the
   supplementary angle law.

$$\angle ACB = 180° - 42° - 67°$$
$$= 71°$$

4. Apply the sine rule to calculate
   side length $b$.

$$\frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$
$$\frac{b}{\sin(67°)} = \frac{7}{\sin(71°)}$$
$$b = \frac{7 \times \sin(67°)}{\sin(71°)}$$
$$b = 6.81 \text{ km}$$

5. Conclude with a written
   statement.

The straight-line distance between the
helicopter and signpost A is 6.81 km.

## WORKED EXAMPLE 17 Solving problems involving angles of elevation and depression

**A surveyor records that the top of a nearby church steeple has an angle of elevation of 5°, and the top of a clock tower has an 8° angle of elevation. From his position in the middle of a paddock, the church steeple is located at N10°W, while the clock tower is at N15°E. If the tops of the church steeple and the clock tower are both 30 m above the level of the surveyor, determine the distance between them.**

**THINK**

1. There are two different planes that must be
   considered in this problem: the vertical plane
   in which the angles of elevation are taken, and
   the horizontal plane where the bearings of the
   steeple and the tower have been observed.
   First, draw diagrams showing the information
   in the vertical plane for the two structures,
   assigning appropriate letters for positions and
   unknown distances.

**WRITE**
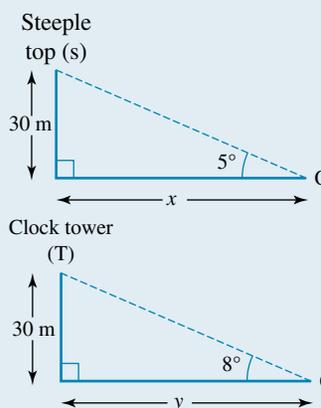
**2.** Use the tan ratio to determine the horizontal distance, $x$, between the surveyor and the church.

$$\tan(\theta) = \frac{\text{opp}}{\text{adj}}$$

$$\tan(5°) = \frac{30}{x}$$

$$x = \frac{30}{\tan(5°)}$$

$$= 342.9 \text{ m}$$

The church is located 342.9 m from the surveyor

**3.** Similarly, determine the horizontal distance, $y$, between the surveyor and the clock tower.

$$\tan(\theta) = \frac{\text{opp}}{\text{adj}}$$

$$\tan(8°) = \frac{30}{y}$$

$$y = \frac{30}{\tan(8°)}$$

$$= 213.5 \text{ m}$$

The clock tower is located 213.5 m from the surveyor.

**4.** Draw a third diagram showing the bearings of the two structures and their horizontal distances from the surveyor.
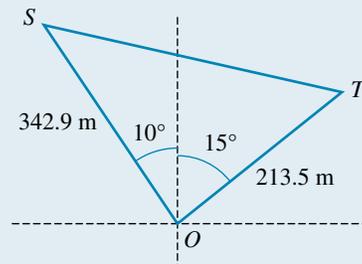


**5.** Use the cosine rule to set up an equation to solve for the distance between the church and the clock tower.

$$ST^2 = OS^2 + OT^2 - 2(OS)(OT)\cos(\angle SOT)$$

**6.** Calculate the distance and round appropriately.

$$x^2 = (342.9)^2 + (213.5)^2 - 2(342.9)(213.5)\cos(25°)$$

$$x^2 = 30\,470.7$$

$$x^2 = \sqrt{30\,470.7}$$

$$x = 174.6$$

**7.** Write the answer.

The horizontal distance between the church and the clock tower is 174.6 m.

# Exercise 9.5 Modelling and solving with the sine and cosine rules

**learn on**

| Simple familiar | Complex familiar | Complex unfamiliar |
|---|---|---|
| 1, 2 | 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17 | 18, 19, 20, 21 |

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## Simple familiar

1. Charini is standing at point A on a river's edge. She knows that point B is directly across the river, $4\sqrt{3}$ metres away, with both sides of the river being parallel. Point C is located 4 metres further along the riverbank from B. Charini currently aligns her line of sight from A to B. Calculate the angle Charini needs to rotate her line of sight if she wishes to swim directly from A to C.

2. **WE14** A, B and C are three towns marked on a map. Judy calculates that the distance between A and B is 45 km and the distance between B and C is 32 km. ∠CAB is 45°. Calculate ∠ACB, correct to the nearest degree.

## Complex familiar

3. **WE15** Ship A is 16.2 km from port on a bearing of N53°E and ship B is 31.6 km from the same port on a bearing of S63°E. Calculate the distance between the two ships, correct to 1 decimal place.

4. David and Harry leave from base camp C on separate straight-line paths. David travelled 4 kilometres on a bearing of N14°W whereas Harry covered 5 kilometres on the bearing N16°E. Calculate the exact straight-line distance between their final resting locations.

5. Tara begins her bushwalking journey from orienteering marker A and travels 2 km N40°E north to marker B. She then travels another 3 km to marker C on the bearing S20°E. Calculate the exact straight-line distance Tara must walk to return to marker A.

6. In a three-dimensional space, DC represents a 5-metre vertical flagpole. A and B are two points in the same horizontal plane as the foot C of the pole. The angle above the horizontal from A to D is 30° and the angle above the horizontal from B to D is 45°. Given ∠ACB = 90°, determine the distance between A and B to the nearest metre.

7. In travelling across flat land, you notice a mountain directly in front of you. Its angle of elevation (to the peak) is 30 degrees. After you drive 8 kilometres closer to the mountain, the angle of elevation is 45 degrees. Calculate the exact height of the mountain.

8. **WE16** Matthew is flying a light aircraft over a straight, flat road. While in the air, he observes two signposts, A and B, which are 12 km apart. Signpost A is in front of the aircraft, while signpost B is behind. The angles of depression from the aircraft to the signposts are 48 degrees to point A and 71 degrees to point B. Calculate the distance from the aircraft to signpost A.

9. A river has parallel banks that run directly east–west. Kylie takes a bearing to a tree on the opposite side. The bearing is 047°T. She then walks 10 m due east, and takes a second bearing to the tree. This is 305°T. Calculate:
   a. her distance from the second measuring point to the tree
   b. the width of the river, to the nearest metre.

**10.** A cross-country runner runs at 8 km/h on a bearing of 150°T for 45 minutes, then changes direction to a bearing of 053°T and runs for 80 minutes until she is due east of the starting point.
Calculate:

**a.** the distance of the second part of the run
**b.** her speed for this section
**c.** the distance to run back to the starting point.

**11.** From a fire tower, A, a fire is spotted on a bearing of N42°E. From a second tower, B, the fire is on a bearing of N12°W. The two fire towers are 23 km apart, and A is N63°W of B. Calculate how far the fire is from each tower.

**12.** An observer sights the top of a building at an angle of elevation of 20°. From a point 30 m closer to the building, the angle of elevation is 35° as shown in the diagram.
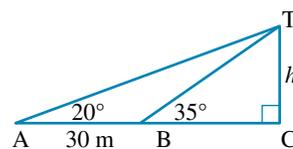
**a.** Calculate the size of ∠ATB.

**b.** Show that the distance BT can be given by the expression $BT = \dfrac{30\sin(20°)}{\sin(15°)}$.

**c.** Show that the height of the building can be given by the expression $h = \dfrac{30\sin(20°)\sin(35°)}{\sin(15°)}$.

**d.** Calculate the height of the building, correct to 1 decimal place.

**13.** Every car should carry a jack. One type of jack used to raise a car is a scissor-jack. A simple diagram of a scissor-jack is shown. The threaded rod is rotated to increase or decrease the length of the line segment BD.
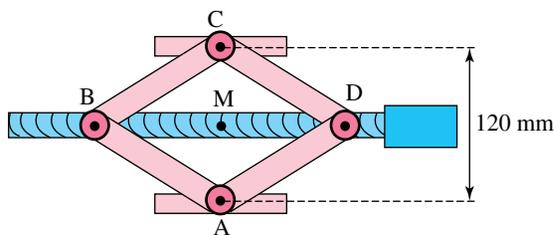
AB = BC = CD = AD = 200 mm

**a.  i.** In △BCD, M is the midpoint of BD. Calculate the length of CM.
**ii.** If ∠BCD = 160°, calculate the length of BD correct to the nearest millimetre.
**iii.** Calculate the size of ∠MBC.
**b.** The jack is raised by reducing the length of the line segment BD.

**i.** If the height of the jack, AC, is raised to 250 mm, calculate the length of BD, correct to the nearest millimetre.
**ii.** If ∠MBC is 70°, calculate the length of BD and what is the height of the jack.

**14.** A ship sails on a bearing of S20°W for 14 km, then changes direction and sails for 20 km and drops anchor. Its bearing from the starting point is now N65°W.

**a.** Calculate the distance the ship is from the starting point.
**b.** Determine the bearing the ship sailed on the 20 km leg.

**15.** Two rowers set out from the same point. One rows N30°E for 1500 m and the other rows S40°E for 1200 m. Calculate the distance between the two rowers, correct to the nearest metre.

**16.** Mario cycles 12 km in a direction N68°W, then 7 km in a direction of N34°E.

   **a.** Calculate how far he is from the starting point.
   **b.** Calculate the bearing of the starting point from his finishing point.

**17.** A plane flies in a direction of N70°E for 80 km, then on a bearing of S10°W for 150 km. Calculate:

   **a.** how far the plane is from its starting point
   **b.** which direction the plane is from its starting point.

**Complex unfamiliar**

**18.** A plane takes off at 10:00 am from an airfield and flies at 120 km/h on a bearing of N35°W. A second plane takes off at 10:05 am from the same airfield and flies on a bearing of S80°E at a speed of 90 km/h. Calculate the distance between the planes at 10:25 am.

**19.** **WE17** The pilot of a helicopter hovering 100 metres above the ocean observes a dinghy at a 30° angle of depression on a bearing of N40°E and a yacht at an angle of depression of 5° that is located at S20°E. Calculate the distance between the yacht and the dinghy. Give your answer to the nearest metre.

**20.** A mine shaft travels north, slanting down into the earth for 2 km at an angle of 10° to the earth's surface. At the end of the slope the shaft travels horizontally east for 1 km and then veers 45° towards the south for a further 800 m before ending. A new shaft is to be dug that travels in a straight line directly from this point back to the entrance on the surface.
Calculate the length of the new mine shaft and determine the angle the new mine shaft will make with the earth's surface at the mine entrance.

**21.** Seth is beginning to build a large geometric sculpture using steel poles and wire. So far, pole A and pole B have been cemented into position and tensioned wires run from the tops and bases of the poles to an anchor point, O, on the ground. The anchor point is located 4 m from the base of pole A, and the wire connecting the anchor point to the top of pole A makes an angle of 30° with the ground. The wire connecting the anchor point to the top of pole B makes an angle of 23° with the ground. Pole B is 8 m high. The wires connecting the bases of the poles with the anchor point make an angle of 42°.
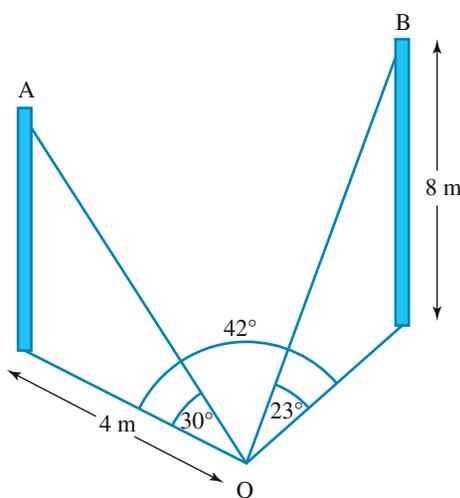


A triangular sail will fill the area bounded by the tops of poles A and B and the anchor point. Calculate the area of sail cloth required.

**Fully worked solutions for this chapter are available online.**

# LESSON
## 9.6 Review

### 9.6.1 Summary

### 9.6 Exercise

**learn on**

| 9.6 Exercise | 9.6 QCAA questions on |
|---|---|

| Simple familiar | Complex familiar | Complex unfamiliar |
|---|---|---|
| 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 | 11, 12, 13, 14, 15, 16, 17 | 18, 19, 20 |

#### Simple familiar

1. Apply the Pythagorean identity, $\sin^2(x) + \cos^2(x) = 1$, to evaluate the following.
   a. $\sin^2(20°) + \cos^2(20°)$
   b. $\sin^2(\pi) + \cos^2(\pi)$
   c. $\sin^2(2.5) + \cos^2(2.5)$

2. Apply the Pythagorean identity to evaluate the following.
   a. $\dfrac{1}{3}\sin^2\left(\dfrac{\pi}{2}\right) + \dfrac{1}{3}\cos^2\left(\dfrac{\pi}{2}\right)$
   b. $2\sin^2(\alpha) + 2\cos^2(\alpha)$
   c. $5\sin^2(\beta) + 5\cos^2(\beta)$

3. In $\triangle ABC$, $b = 6$ cm, $\angle ABC = 60°$ and $\angle ACB = 45°$. Calculate the exact value of side $c$.

4. In $\triangle ABC$, $a = 3$ m, $b = \sqrt{19}$ m and $c = 5$ m. Calculate $\angle ABC$.

5. Calculate the exact area of an equilateral triangle that has a side length of $\sqrt{12}$ m.

6. **MC** In a triangle ABC, $a = 5$, $b = 6$ and $C = 105°$. The length of $c$ is:
   A. 3.04
   B. 5.15
   C. 8.75
   D. 7.83

**7.** Calculate the following, correct to 2 decimal places.

**a.** Side length $b$
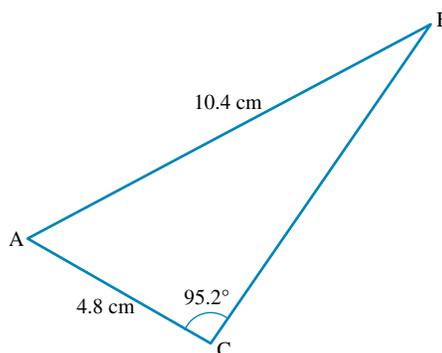
A

44.5°

8.51 cm

81°

C        5.97 cm        B

**b.** ∠ABC

B

10.4 cm

A

4.8 cm        95.2°

C

**8.** Calculate the following, correct to 2 decimal places.

**a.** Side length $x = BC$

B

5.35 cm

$x$

A   40°

9 cm        C

**b.** ∠ACB

B

4.57 cm

6.93 cm

A

8.41 cm

C
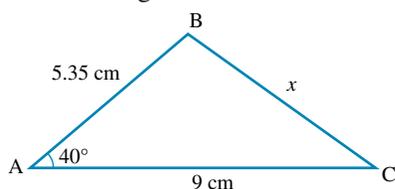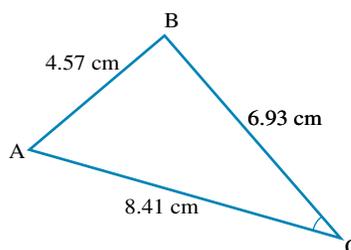
**9.** In △XYZ, $x = 9.2$ cm, ∠XYZ $= 56°$ and ∠YXZ $= 38°$. Determine $y$, correct to 1 decimal place.

**10.** The area of triangle ABC is 126.8 cm². Given $a = 14$ cm, $b = 24$ cm, calculate angle C to the nearest whole degree.

**Complex familiar**

**11.** From a port P, a ship Q is 20 km away on a bearing of S60°E, and a ship R is 12 km away on a bearing of N60°E. Calculate the exact distance between the two ships.

**12.** Consider the following quadrilateral consisting of triangles ACB and ABD.

A

42°        6.5 cm

D

5.2 cm

3.4 cm

102.5°        B

C
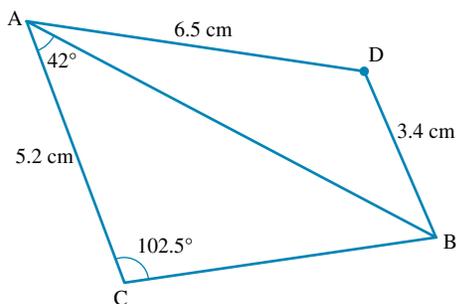
**a.** Calculate the length of the line segment AB, correct to 1 decimal place.
**b.** Use the answer from **a** to calculate the largest angle in triangle ABD, correct to 1 decimal place.

**13.** Determine all possible values of ∠ABC in the triangle ABC, if $A = 43.2°$, $a = 18.5$ and $b = 24.9$.

**14.** A person on a jet ski is travelling due east. The bearing of a floatable buoy is N68°E. After the jet ski has travelled 480 metres the bearing of the buoy is N49°E. Calculate the distance from the jet ski to the buoy.

**15.** A ship sails 52 kilometres from port A to port B on a bearing of N63°E, then sails 124 km from port B to port C on a bearing of N70°W.

    **a.** Calculate to the nearest kilometre, the distance of port A from port C.

    **b.** Determine the bearing of port A from port C, correct to the nearest degree.

**16.** From point A on level ground, the angle of elevation to a plane is 72°. From point B on the ground, due west of A, the angle of elevation is 47°. If A and B are 3500 m apart, determine the height of the plane off the ground.

**17.** A soccer goal is 8 m wide.

    **a.** A player is directly in front of the goal such that he is 12 m from each post. Determine the angle within which he must kick the ball to score a goal.

    **b.** A second player takes an angled shot. This player is 12 m from the nearest post and 17 m from the far post. Determine the angle within which he must kick the ball to score a goal.



**Complex unfamiliar**

**18.** A block of land is known to be in the shape of an isosceles triangle. The unequal side is 4 km in length and the equal angles are $\beta$ where $\cos(\beta) = \dfrac{\sqrt{5}}{3}$. Calculate the exact area of this site.

**19.** Two circular pulleys with radii 3 cm and 8 cm have their centres 13 cm apart. Calculate the length of the belt required to pass tightly around the pulleys, giving the answer to 1 decimal place.
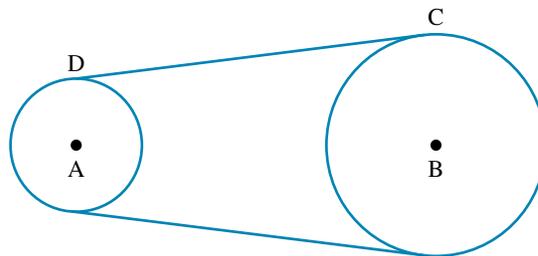


**20.** At a popular island getaway, the main tourist attractions are two resort towers of heights measuring 256.1 metres and 144.3 metres respectively. The towers are located along a north easterly esplanade, with the taller tower A, situated to the southwest of the shorter tower B. Tad and Kellie walk $d$ metres directly north from the base of B to a café located at P. The corresponding angles of elevation from P to the top of towers A and B are 54.0° and 74.5°. Calculate the straight-line distance between the top of towers A and B to the nearest metre.

## 9.6 Past QCAA exam questions

### Question 1 (1 mark)

*Source:* QCAA 2024 Mathematical Methods, Paper 2, Section 1, Q7; © State of Queensland (QCAA) 2024.

Identify the possible values for $a$ in the triangle.

A. $13.5°$ or $126.5°$
B. $53.5°$ or $126.5°$
C. $53.5°$ or $86.5°$
D. $13.5°$ or $86.5°$



### Question 2 (5 marks)

*Source:* QCAA 2024 Mathematical Methods, Paper 1, Section 2, Q18; © State of Queensland (QCAA) 2024.

The diagram shows some dimensions of a large storage container that is a rectangular prism. The angle ABC is $60°$.

A person requires a container that is at least $4$ metres in height.
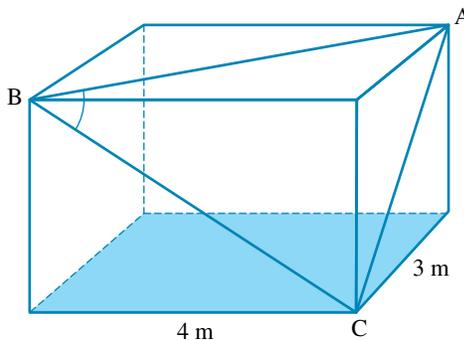


Make a justified decision about whether this storage container meets the person's requirements.

### Question 3 (7 marks)

*Source:* QCAA 2023 Mathematical Methods, Paper 2, Section 2, Q14; © State of Queensland (QCAA) 2024.

A fence divides a paddock into two triangular sections as shown.
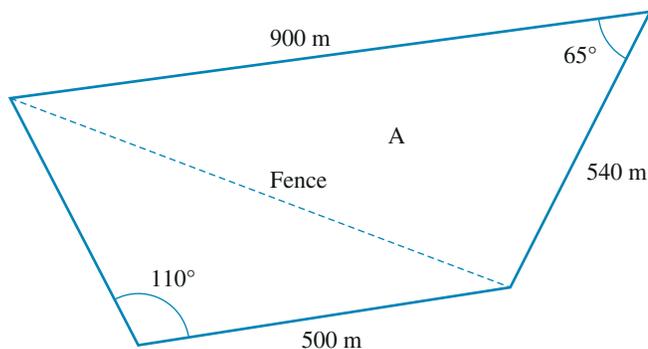


a. Determine the length of the fence. [1 mark]
b. Calculate the area of triangular section A. [1 mark]
c. Determine the total area of the paddock. [5 marks]

**Question 4 (7 marks)**

*Source:* QCAA 2022 Mathematical Methods, Paper 2, Section 2, Q15; © State of Queensland (QCAA) 2024.

A hiker begins her journey at a youth hostel ($H$) and walks for $8$ km on a bearing of $052°$T to her lunch stop ($L$). She then walks on a bearing of $210°$T for $5.2$ km until she reaches a campsite ($C$).

Determine the direction she would need to walk in a straight line to return directly to the youth hostel.
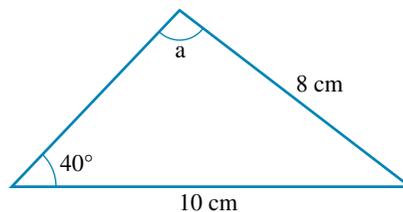
**Question 5 (1 mark)**

*Source:* QCAA 2022 Mathematical Methods, Paper 2, Section 1, Q9; © State of Queensland (QCAA) 2024.

Determine the length of side AB in triangle ABC.
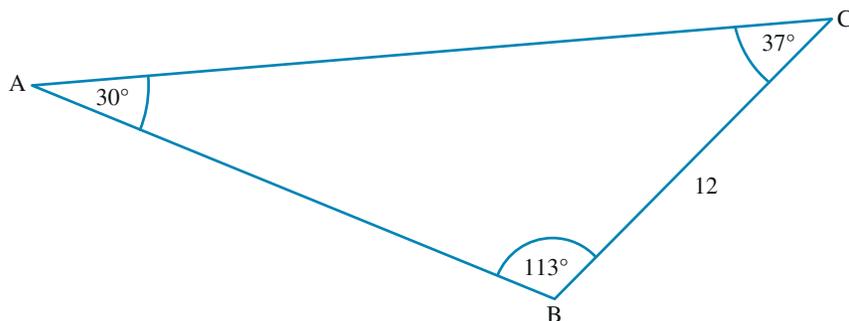


**A.** 22.13

**B.** 14.44

**C.** 9.97

**D.** 7.82

**Fully worked solutions for this chapter are available online.**

# Answers

## Chapter 9 Cosine and sine rules

### 9.2 The sine rule

#### 9.2 Exercise

1. a. $\dfrac{a}{\sin(A)} = \dfrac{b}{\sin(B)} = \dfrac{c}{\sin(C)}$

   b. $\dfrac{x}{\sin(X)} = \dfrac{y}{\sin(Y)} = \dfrac{z}{\sin(Z)}$

   c. $\dfrac{p}{\sin(P)} = \dfrac{q}{\sin(Q)} = \dfrac{r}{\sin(R)}$

2. $\dfrac{8\sqrt{2}}{\sqrt{3}}$ units

3. $12\sqrt{2}$ cm

4. $30°$

5. a. 14.8 cm    b. 1.98 km    c. 112 mm

6. a. 10.0 mm    b. 22.1 cm    c. 39.6 km

7. 9.8 cm

8. 27.0 m

9. 37.8 m

10. a. $43°$    b. $34°$    c. $27°$

11. $38°$

12. $20°$

13. $48.8°$ or $131.2°$

14. $27.3°$

15. $\left(16\sqrt{3} + 24\right)$ units

16. a.



   b. 43.2 m

   c. 33 m

17. 43.62 m
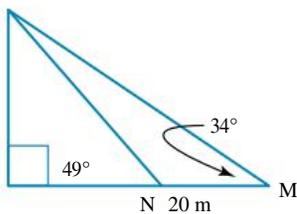
18. a. $57°$    b. $63°$

19. 45 m is enough as only 43 m is required.

20. $54°$

### 9.3 The cosine rule

#### 9.3 Exercise

1. a. $a^2 = b^2 + c^2 - 2bc\cos(A)$
   b. $r^2 = p^2 + q^2 - 2pq\cos(R)$
   c. $n^2 = l^2 + m^2 - 2lm\cos(N)$

2. $A, C, B$

3. 11.27 cm

4. $\sqrt{21}$ metres

5. $60°$

6. 7.95

7. a. 8.05 m    b. 14.3 cm    c. 12.0 m

8. a. 185.1 cm    b. 14.4 m    c. 104.4 mm

9. 55.22

10. $b = 23.08$, $A = 41°53'$, $C = 23°7'$

11. 6.742 km

12. a. $85°$    b. $83°$    c. $45°$

13. a. $103°$    b. $137°$    c. $10°$

14. $88°15'$

15. $28°57'$

16. $A = 61°15'$, $B = 40°$, $C = 78°45'$

17. 40 m

18. $82°$, $54°$, $44°$

19. $32°$

20. 1.14 km/h

21. 55 cm

22. 89.12 m

23. $23°$

24. $70°49'$ (or $70.82°$)

### 9.4 Area of a triangle

#### 9.4 Exercise

1. $48\text{ mm}^2$

2. $30\text{ cm}^2$

3. $18\sqrt{3}\text{ units}^2$

4. 4 cm

5. 12 cm

6. $52.75\text{ mm}^2$

7. $24.63\text{ mm}^2$

8. $5.64\text{ km}^2$

9. a. $113.49\text{ cm}^2$    b. $47.32\text{ mm}^2$

10. a. $216.10\text{ cm}^2$    b. $122.48\text{ cm}^2$

11. a.   i. $4.275\text{ cm}^2$
        ii. $1.736$ cm

    b. Perimeter $= 20\sqrt{3}$ cm; area $= \dfrac{100\sqrt{3}}{3}\text{ cm}^2$

    c. $6\sqrt{2}\text{ cm}^2$

12. $41.39\text{ cm}^2$, $61.41\text{ cm}^2$, $59.12\text{ cm}^2$

13. a. $5797\text{ m}^2$    b. $1062\text{ m}^2$    c. $27\,952\text{ m}^2$

14. $8.14\text{ m}^3$

15. $690.4\text{ cm}^2$

### 9.5 Modelling and solving with the sine and cosine rules

#### 9.5 Exercise

1. $30°$

2. $84°$

3. 28.5 km

4. $\sqrt{41 - 20\sqrt{3}}$ kilometres

5. $\sqrt{7}$ km

6. 10 metres

7. $\dfrac{8\sqrt{3}}{3-\sqrt{3}}$ kilometres

8. 12.97 kilometres.

9. **a.** 6.97 m    **b.** 4 m

10. **a.** 8.63 km    **b.** 6.48 km/h    **c.** 9.90 km

11. 22.09 km from A, 27.46 km from B

12. **a.** 15°

 **b.** Sample responses can be found in the worked solutions in the online resources.

 **c.** Sample responses can be found in the worked solutions in the online resources.

 **d.** 22.7 m

13. **a.** **i.** 60 mm

 **ii.** 394 mm

 **iii.** 10°

 **b.** **i.** 312 mm

 **ii.** BD = 137 mm; AC = 376 mm

14. **a.** 13.11 km    **b.** N20°47′W

15. 2218 m

16. **a.** 12.57 km    **b.** S35°E

17. **a.** 130 km    **b.** S22°12′E

18. 74.3 km

19. 1239 m

20. 2131 m, 9.4

21. 28.99 m²

## 9.6 Review

### 9.6 Exercise

1. **a.** 1    **b.** 1    **c.** 1

2. **a.** $\dfrac{1}{3}$    **b.** 2    **c.** 5

3. $\dfrac{6\sqrt{2}}{\sqrt{3}}$ cm

4. 60°

5. $3\sqrt{3}$ m²

6. C

7. **a.** 6.93 cm    **b.** 27.36°

8. **a.** 5.99 cm    **b.** 32.9°

9. 12.4 cm

10. 49°

11. $\sqrt{304}$ km

12. **a.** 8.7 cm    **b.** $D = 119.7°$

13. 67.13°, 112.87°

14. 552.3 metres

15. **a.** 96 km    **b.** S7.24°E

16. 2783 m

17. **a.** 39°    **b.** 25°

18. $\dfrac{8\sqrt{5}}{5}$ km²

19. 62.5 cm

20. 192 metres.

## 9.6 Past QCAA exam questions

1. B

2. The height of the box meets the person's requirements as the height is $\dfrac{\sqrt{624}}{5} > 4$

3. **a.** 831.153 m

 **b.** 220 232.8 m²

 **c.** 341 123.96 m²

4. A bearing of 263°30′ T

5. B

# 10 General continuous random variables

## LESSON SEQUENCE

**Fully worked solutions for this chapter are available online.**

## EXAM PREPARATION

Access exam-style questions in every lesson, available online and access past QCAA exam questions in every review.

**on Resources**

| | |
|---|---|
| **Solutions** | Solutions — Chapter 10 |
| **Exam questions** | Exam question booklet — Chapter 10 |
| **Digital documents** | Learning matrix — Chapter 10 |
| | Chapter summary — Chapter 10 |

# LESSON
## 10.1 Overview

### 10.1.1 Introduction

Continuous random variables such as height have values that depend upon the precision of the devices used to measure them. Modern technology has allowed standardised measurements of length to a reasonable level of precision; however, the first 'standardised' units of length used over five thousand years ago were a bit unreliable. Length was described in terms of cubits, where one cubit was equal to the distance between the elbow and the tips of the finger. Obviously, with so much variation between human beings, an 'official cubit' had to be decided upon, but even then there were problems. Cubit rods found by archaeologists have varied from 47.2 to 52.5 cm in length, with each cubit broken up into 5 to 7 'palms' and a palm broken up into anywhere between 4 and 7 'fingers'. So, depending on the cubit rod used, the same person could be described as being 3 cubits, 2 palms and 1 finger tall; 3 cubits, 5 palms and 5 fingers tall; or somewhere in between.

Today, when people measure their height they can be confident that they will get the same value each time when rounded to the nearest whole centimetre. The size of this centimetre is standardised as one-hundredth of a metre.

The metre itself is defined as being the distance travelled by light in a vacuum in $\dfrac{1}{299\,792\,458}$ of a second.

### 10.1.2 Syllabus links

| Lesson | Lesson title | Syllabus links |
|--------|-------------|----------------|
| 10.2 | **Continuous random variables and the probability density function** | ○ Use relative frequencies and histograms obtained from data to estimate probabilities associated with a continuous random variable. <br> ○ Understand the concepts of a probability density function, cumulative distribution function, and probabilities associated with a continuous random variable given by integrals; examine simple types of continuous random variables and use them in appropriate contexts. |
| 10.3 | **Cumulative distribution functions** | ○ Understand the concepts of a probability density function, cumulative distribution function, and probabilities associated with a continuous random variable given by integrals; examine simple types of continuous random variables and use them in appropriate contexts. |
| 10.4 | **Measures of centre and spread** | ○ Calculate the expected value, $E(X) = \mu = \displaystyle\int_{-\infty}^{\infty} xp(x)\,dx$, of a continuous random variable where $p(x)$ is the probability density function. <br> ○ Calculate the variance, $\mathrm{Var}(X) = \sigma^2 = \displaystyle\int_{-\infty}^{\infty} (x - \mu)^2 p(x)\,dx$, and standard deviation $\sigma$, of a continuous random variable. |

*Source:* Mathematical Methods Senior Syllabus 2024 © State of Queensland (QCAA) 2024; licensed under CC BY 4.0.

*Note:* The following dot point will be covered in Chapter 11.

- Understand standardised normal variables (*z*-values, *z*-scores) and use these to compare samples.

# LESSON
## 10.2 Continuous random variables and the probability density function

In Chapter 6, we dealt with discrete random variables — data that is finite or countable. The number of white cars in a car park, the number of students in a class and the number of lollies in a jar are all examples of discrete random variables. A **continuous random variable**, however, can assume any value within a given range.

### 10.2.1 Using relative frequencies and histograms to estimate probabilities

Continuous random variables can represent any value within a given **interval**. Examples of continuous random variables include time, height and weight — all quantities that must be measured rather than counted.

A **histogram** can be used to graph continuous data within specified intervals, with each interval being of uniform width. The *x*-axis is the variable being analysed, the *y*-axis represents the frequency and a bar is drawn for each interval with its height indicating the frequency of values obtained for that specific class interval.

Consider an Australian health study that was conducted. The study targeted young people aged 5 to 17 years old. They were asked to estimate the average number of hours of physical activity they participated in each week. The results of this study are shown in the following histogram.

Let *X* be the continuous random variable for the average number of hours of physical activity a young person participated in each week.

| Hours of physical activity (h) | Frequency |
|:---:|:---:|
| $0 \leq X < 1$ | 364 |
| $1 \leq X < 2$ | 347 |
| $2 \leq X < 3$ | 156 |
| $3 \leq X < 4$ | 54 |
| $4 \leq X < 5$ | 32 |
| $5 \leq X < 6$ | 10 |
| $6 \leq X < 7$ | 7 |



Physical activity
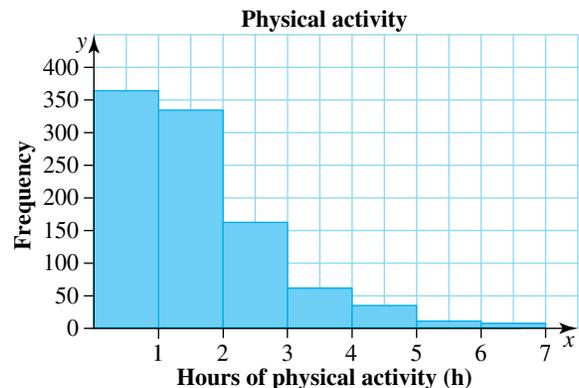
The frequencies of individual activity times cannot be determined due to the fact that the times have been grouped into class intervals. This limits the information we are able to extract from the histogram. For example, for the class interval $0 \le X < 1$ we know that 364 of the young people surveyed participated in less than one hour of physical activity on average each week, however we cannot tell how many of them completed no physical activity (0 hours) each week. As there are an infinite number of values that the variable can have within that interval, the probability of a continuous random variable assuming an exact value is zero.

Histograms can be used to determine the **relative frequency** of specified intervals. The relative frequency of an interval is the frequency of that interval divided by the total frequency for the entire study. The relative frequency for a particular interval can also be considered as the probability of a variable lying within that particular interval.

**Calculating probability**

$$P(a < x < b) = \frac{f(a < x < b)}{\sum f}$$

So, the probability that a randomly selected person in the study did between 1 hour and less than 2 hours of activity can be determined:

$$
\begin{aligned}
P(1 \le x < 2) &= \frac{f(1 \le x < 2)}{\sum f} \\
&= \frac{347}{364 + 347 + 156 + 54 + 32 + 10 + 7} \\
&= \frac{347}{970} \\
&= 0.358
\end{aligned}
$$

Similarly, we could determine the probability that a person did 4 hours or more of activity per week by adding the interval frequencies for all intervals higher than and including 4:

$$
\begin{aligned}
P(x \ge 4) &= \frac{f(4 \le x < 5) + f(5 \le x < 6) + f(6 \le x < 7)}{\sum f} \\
&= \frac{32 + 10 + 7}{970} \\
&= \frac{49}{970} \\
&= 0.051
\end{aligned}
$$

A limitation is that we can only determine relative frequencies for intervals that start and/or end on the end points of the specified class intervals. For example, we would not be able to determine the probability that a person did less than 1.5 hours of exercise as this value lies within a class interval rather than being an end point.

In a study, the nose lengths, $X$ millimetres, of 75 adults were measured. The results of the study are shown in the table and histogram.

| Nose length (mm) | Frequency |
|---|---|
| $27.5 < X \leq 32.5$ | 2 |
| $32.5 < X \leq 37.5$ | 5 |
| $37.5 < X \leq 42.5$ | 17 |
| $42.5 < X \leq 47.5$ | 21 |
| $47.5 < X \leq 52.5$ | 11 |
| $52.5 < X \leq 57.5$ | 7 |
| $57.5 < X \leq 62.5$ | 6 |
| $62.5 < X \leq 67.5$ | 5 |
| $67.5 < X \leq 72.5$ | 1 |
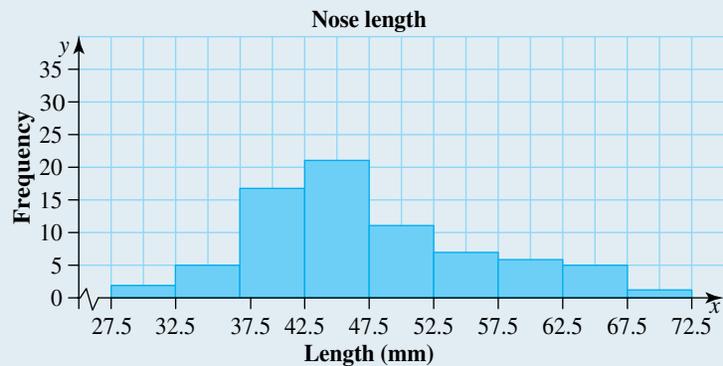
Nose length



Determine the probability of someone in the study having a nose length that is:
a. greater than 42.5 mm and less than or equal to 47.5 mm
b. less than or equal to 47.5 mm.

THINK

WRITE

a. 1. Use the table to determine the frequency of the interval and the sum of the frequencies for the study.

a. $f(42.5 < x \leq 47.5) = 21$
$$\sum f = 2 + 5 + 17 + 21$$
$$+ 11 + 7 + 6 + 5 + 1$$
$$= 75$$

2. Substitute values into the equation and evaluate.

$$P(42.5 < x \leq 47.5) = \frac{f(42.5 < x \leq 47.5)}{\sum f}$$
$$= \frac{21}{75}$$
$$= 0.28$$

3. Answer the question.

The probability of a person having a nose length greater than 42.5 mm and less than or equal to 47.5 mm is 0.28.

b. 1. Determine the total of the frequencies for all of the intervals such that $x \leq 47.5$.

b. $f(x \leq 47.5) = 2 + 5 + 17 + 21$
$$= 45$$

2. Substitute values into the equation and evaluate.

$$P(x \leq 47.5) = \frac{f(x \leq 47.5)}{\sum f}$$
$$= \frac{45}{75}$$
$$= 0.6$$

3. Answer the question.

The probability of a person having a nose length less than or equal to 47.5 mm is 0.6.

# 10.2.2 Modelling continuous random variables

A histogram is used to plot the frequencies of specified intervals for a continuous random variable.

The histogram shown displays the masses (in kg) of members joining the Toowong branch of a popular weight loss program.

**Masses of people joining a weight loss program**
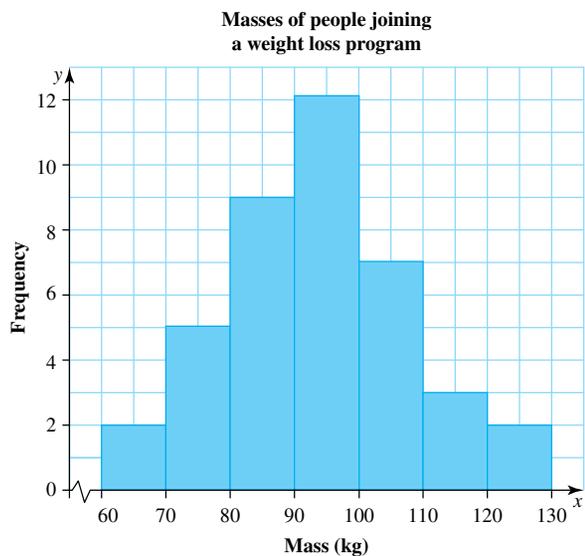


A histogram can be converted to a density distribution by dividing the frequency by the sample size × interval width for each interval. In a density distribution, the area of the rectangle for each interval is equal to the probability for that interval. Therefore, the total area of the rectangles equals 1, as the sum of all probabilities must equal 1.

**Masses of people joining a weight loss program**
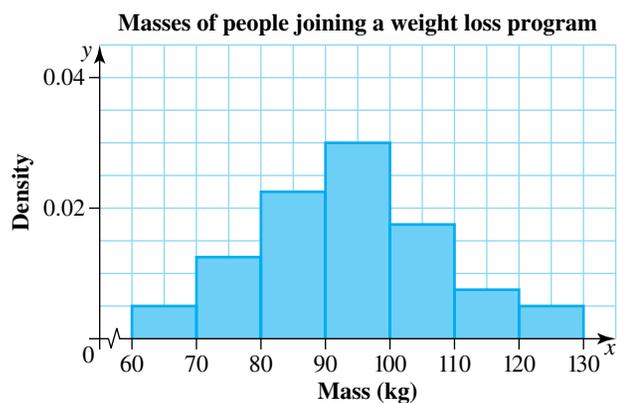


If data points are placed at the midpoint of the top of each rectangle and the data points are joined, the general shape of the distribution can be observed.

**Masses of people joining a weight loss program**
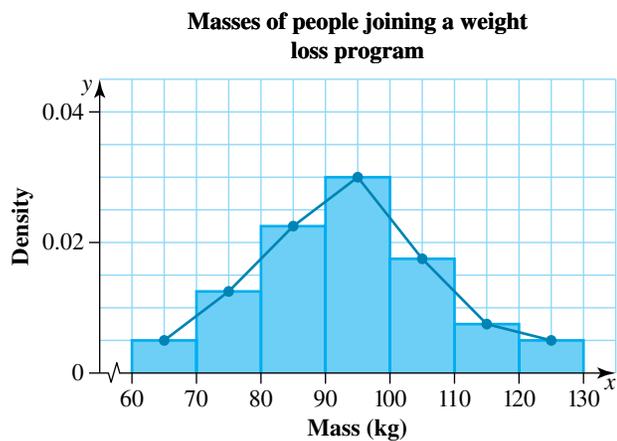


This is a relatively small set of data, as the study only involved 40 people. However, if we increased the size of the study so that the members of more branches of the weight loss program were included and the class intervals were made smaller, the line joining the data points would become a smooth curve with an area under the curve equal to 1.

This smooth curve can be represented by a mathematical function $f(x)$. This function is called the **probability density function** (pdf).

**Masses of people joining a weight loss program**
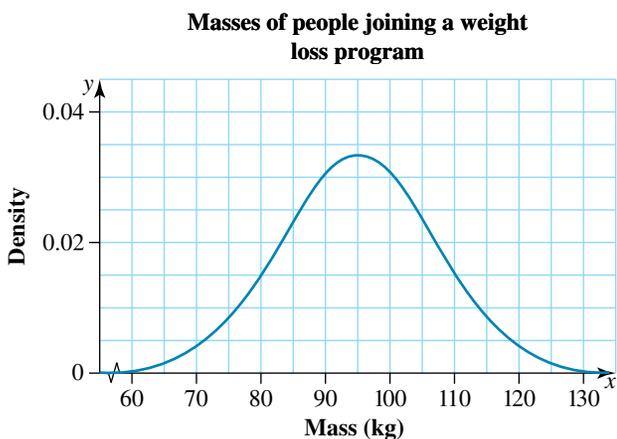
# 10.2.3 Probability density functions

## Probability density functions

**A probability density function has the following characteristics:**
1. **As probability is always a positive value, $f(x)$ must always be positive.**
   **$f(x) \geq 0$ for all $x$.**
2. **The area under the curve must be equal to 1.**

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

3. **When the function $f(x)$ has an interval $[m, n]$ where $\int_{m}^{n} f(x)\, dx = 1$ then $f(x) = 0$ for all $x$ values outside the interval $[m, n]$.**

All three characteristics listed above must be fulfilled in order for a function to be classified as a probability density function.

---

### WORKED EXAMPLE 2 Determining if a function is a probability density function

**Sketch each of the following functions and determine whether each one is a probability density function.**

**a.** $f(x) = \begin{cases} 2(x-1), & 1 \leq x \leq 2 \\ 0, & \text{elsewhere} \end{cases}$

**b.** $f(x) = \begin{cases} 2e^{-x} & 0 \leq x \leq 2 \\ 0, & \text{elsewhere} \end{cases}$

| THINK | WRITE |
|---|---|
| **a. 1.** Describe the shape of the function. | **a.** Shape: increasing, linear |
| **2.** Determine the endpoints of the function. | $f(x) = 2(x-1)$ <br> $f(1) = 2(1-1)$ <br> $f(1) = 0$ <br> $f(2) = 2(2-1)$ <br> $f(2) = 2$ |
| **3.** Sketch the function. Remember that a closed circle indicates that the data point is included in the function, while an open circle means the data point is not included. Make sure to include the horizontal lines for $y = 0$ either side of the interval $1 \leq x \leq 2$. |  |
| **4.** Identify if all the $f(x)$ values are positive. | Yes, $f(x) \geq 0$ for all $x$-values. |
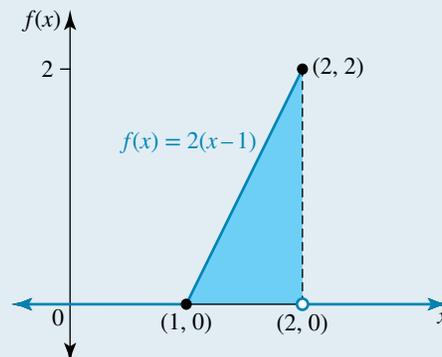
**5.** Determine if the area under the curve is equal to 1.

Method 1: Using the area of triangles

Area of shaded region $= \dfrac{1}{2} \times \text{base} \times \text{height}$

$$= \dfrac{1}{2} \times 1 \times 2$$

$$= 1$$

Method 2: Using calculus

Area of shaded region $= \displaystyle\int_1^2 2(x-1)\,dx$

$$= \int_1^2 (2x-2)\,dx$$

$$= \left[x^2 - 2x\right]_1^2$$

$$= (2^2 - 2(2)) - (1^2 - 2(1))$$

$$= 0 - 1 + 2$$

$$= 1$$

**6.** Identify if all the $f(x)$ values outside the interval $1 \le x \le 2$ are equal to 0.

Yes, $f(x) = 0$ outside the interval $1 \le x \le 2$.

**7.** If all three requirements above are satisfied, then it is a probability density function. State the result.

All three requirements are satisfied; therefore, this is a probability density function.

**b. 1.** Describe the shape of the function.

**b.** Shape: decreasing, exponential

**2.** Determine the endpoints of the function.

$f(x) = 2e^{-x}$

$f(0) = 2e^{-0}$

$f(0) = 2$

$f(2) = 2e^{-2}$

$f(2) = \dfrac{2}{e^2}$

**3.** Sketch the function. Make sure to include the horizontal lines for $y = 0$ either side of the interval $0 \le x \le 2$.
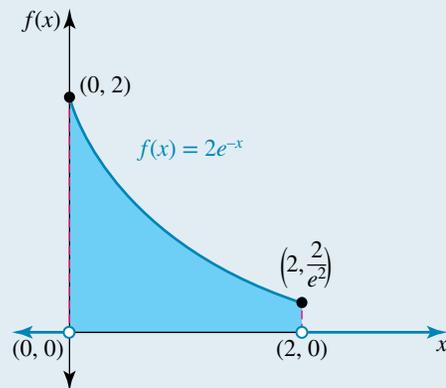


**4.** Identify if all the $f(x)$ values are positive.

Yes, $f(x) \ge 0$ for all $x$-values.

**5.** Determine if the area under the curve is equal to 1.

$\displaystyle\int_0^2 2e^{-x}\,dx = 2\int_0^2 e^{-x}\,dx$

$$= 2[-e^{-x}]_0^2$$

$$= 2\left(-e^{-2} + e^0\right)$$

$$= 2\left(-e^{-2} + 1\right)$$

$$= 1.7293$$

**6.** Identify if all the $f(x)$ values outside the interval $0 \leq x \leq 2$ are equal to 0.

Yes, $f(x) = 0$ outside the interval $0 \leq x \leq 2$.

**7.** If any of the three requirements above are not satisfied, then it is not a probability density function. State the result.

The area under the curve is not equal to 1. Therefore, this is not a probability density function.

The area under the curve of a probability density function for a specific interval is equal to the probability for that interval.

For the interval $[a, b]$:

$$P(a \leq X \leq b) = \int_a^b f(x)\, dx$$

---

### WORKED EXAMPLE 3 Determining probabilities in probability density functions

**For the following probability density function:**

$$f(x) = \begin{cases} -\dfrac{2}{9}(x - 5), & 2 \leq x \leq 5 \\ 0, & \text{elsewhere} \end{cases}$$

**a.** calculate $P(X < 3)$ 

**b.** calculate $P(3 < X < 4)$.

| THINK | WRITE |
|---|---|
| **a. 1.** The probability for $X < 3$ will be the area under the curve between the lower bound of 2 and the upper bound of $\dfrac{3}{8}$. | **a.** $P(X < 3) = \displaystyle\int_2^3 -\dfrac{2}{9}(x - 5)\, dx$ <br><br> $= -\dfrac{2}{9}\displaystyle\int_2^3 (x - 5)\, dx$ |
| **2.** Integrate the function between these bounds and solve. | $= -\dfrac{2}{9}\left[\dfrac{1}{2}x^2 - 5x\right]_2^3$ <br><br> $= -\dfrac{2}{9}\left[\left(\dfrac{1}{2}(3)^2 - 5\,(3)\right) - \left(\dfrac{1}{2}(0)^2 - 5\,(0)\right)\right]$ <br><br> $= -\dfrac{2}{9}\left[\dfrac{9}{2} - 15 - 2 + 10\right]$ <br><br> $= -\dfrac{2}{9} \times -\dfrac{5}{2}$ <br><br> $= \dfrac{5}{9}$ |

**b. 1.** The probability for $3 < X < 4$ will be the area under the curve between the lower bound of 3 and the upper bound of 4.

**b.** $P(3 < X < 4) = \int_3^4 -\frac{2}{9}(x-5)\,dx$

$$= -\frac{2}{9}\int_3^4 (x-5)\,dx$$

**2.** Integrate the function between these bounds and solve.

$$= -\frac{2}{9}\left[\frac{1}{2}x^2 - 5x\right]_3^4$$

$$= -\frac{2}{9}\left[\left(\frac{1}{2}(4)^2 - 5(4)\right) - \left(\frac{1}{2}(3)^2 - 5(3)\right)\right]$$

$$= -\frac{2}{9}\left[8 - 20 - \frac{9}{2} + 15\right]$$

$$= -\frac{2}{9} \times -\frac{3}{2}$$

$$= \frac{1}{3}$$

---

## WORKED EXAMPLE 4 Determining unknown values in probability density functions

**Given that the following functions are probability density functions, determine the value of $a$ in each function.**

**a.** $f(x) = \begin{cases} a(x-1)^2, & 0 \le x \le 4 \\ 0, & \text{elsewhere} \end{cases}$

**b.** $f(x) = \begin{cases} ae^{-4x}, & x > 0 \\ 0, & \text{elsewhere} \end{cases}$

**THINK**

**WRITE**

**a. 1.** As the function has already been defined as a probability density function, this means that the area under the graph is 1.

**a.** $\int_0^4 f(x)\,dx = 1$

$$\int_0^4 a(x-1)^2\,dx = 1$$

**2.** Remove $a$ from the integral, as it is a constant.

$$a\int_0^4 (x-1)^2\,dx = 1$$

**3.** Antidifferentiate and substitute in the terminals.

$$a\int_0^4 (x-1)^2\,dx = 1$$

$$a\left[\frac{(x-1)^3}{3}\right]_0^4 = 1$$

$$a\left[\frac{3^3}{3} - \frac{(-1)^3}{3}\right] = 1$$

**4.** Solve for $a$.

$$a\left(9 + \frac{1}{3}\right) = 1$$

$$a \times \frac{28}{3} = 1$$

$$a = \frac{3}{28}$$

**b. 1.** As the function has already been defined as a probability density function, this means that the area under the graph is 1.

**b.**
$$\int_0^\infty f(x)\,dx = 1$$

$$\int_0^\infty ae^{-4x}\,dx = 1$$

**2.** Remove $a$ from the integral, as it is a constant.

$$a\int_0^\infty e^{-4x}\,dx = 1$$

**3.** To evaluate an integral containing infinity as one of the terminals, we determine the appropriate limit.

$$a \times \lim_{k \to \infty} \int_0^k e^{-4x}\,dx = 1$$

**4.** Antidifferentiate and substitute in the terminals.

$$a \times \lim_{k \to \infty} \left[ -\frac{1}{4}e^{-4x} \right]_0^k = 1$$

$$a \times \lim_{k \to \infty} \left( -\frac{e^{-4x}}{4} + \frac{1}{4} \right) = 1$$

$$a \times \lim_{k \to \infty} \left( -\frac{1}{4e^{4k}} + \frac{1}{4} \right) = 1$$

**5.** Solve for $a$. Remember that a number divided by an extremely large number is effectively 0, so $\lim_{k \to \infty} \left( \dfrac{1}{e^{4k}} \right) = 0$.

$$a\left( 0 + \frac{1}{4} \right) = 1$$

$$\frac{a}{4} = 1$$

$$a = 4$$

## 10.2.4 Piecewise probability density functions

Consider a continuous variable $X$ such that $X = x$ where $x \in R$.

$$\begin{aligned} P(X = x) &= \int_x^x f(x)\,dx \\ &= [F(x)]_x^x \\ &= F(x) - F(x) \\ P(X = x) &= 0 \end{aligned}$$

Given that the probability value of a continuous random variable $X$ lying in the interval between $a$ and $b$ is described by

$$P(a < x < b) = \int_a^b f(x)\,dx$$

and that the probability of $X$ having an exact value is 0, that is,

$$P(X = a) = P(X = b) = 0,$$

it follows that

$$P(a \le X \le b) = P(a < X \le b) = P(a \le X < b) = P(a < X < b)$$

and

$$P(a \le X \le b) = P(a \le X \le c) + P(c < X \le b), \text{ where } a \le c \le b.$$

These properties are particularly useful when the probability density function is a piecewise function and the required probability encompasses two functions.

**A continuous random variable, $Y$, has a probability density function, $f$, defined by**

$$f(y) = \begin{cases} -ay, & -3 \le y \le 0 \\ ay, & 0 < y \le 3 \\ 0, & \text{elsewhere} \end{cases}$$

**where $a$ is a positive constant.**
**a. Sketch the graph of $f$.**
**b. Determine the value of the constant, $a$.**
**c. Determine $P(1 \le Y \le 3)$.**
**d. Determine $P(Y < 2|Y > -1)$.**

**THINK**

**a.** The piecewise function contains three sections. The first graph, $f(y) = -ay$, is a straight line with end points of $(0, 0)$ and $(-3, 3a)$.

The second graph is also a straight line and has end points of $(0, 0)$ and $(3, 3a)$. Don't forget to include the $f(y) = 0$ lines for $x > 3$ and $x < -3$.

**WRITE**

**a.** $f(-3) = 3a$ and $f(3) = 3a$



**b.** Use the fact that $\int_{-3}^{3} f(y)\,dy = 1$ to solve for $a$.

**b.** $\int_{-3}^{3} f(y)\,dy = 1$

Using the area of a triangle, we find:

$$\frac{1}{2} \times 3 \times 3a + \frac{1}{2} \times 3 \times 3a = 1$$

$$\frac{9a}{2} + \frac{9a}{2} = 1$$

$$9a = 1$$

$$a = \frac{1}{9}$$

**c.** Identify the part of the function that the required $y$-values sit within: the values $1 \le Y \le 3$ are within the region where $f(y) = \frac{1}{9}y$.

**c.** $P(1 \le Y \le 3) = \int_{1}^{3} f(y)\,dy$

$$= \int_{1}^{3} \left(\frac{1}{9}y\right) dy$$

$$= \left[\frac{1}{18}y^2\right]_{1}^{3}$$

$$= \frac{1}{18}(3)^2 - \frac{1}{18}(1)^2$$

$$= \frac{8}{18}$$

$$= \frac{4}{9}$$

*Note:* The method of finding the area of a trapezium could also be used.

**d. 1.** State the rule for the conditional probability.

**d.** $P(Y < 2 | Y > -1) = \dfrac{P(Y < 2 \cap Y > -1)}{P(Y > -1)}$

$\qquad\qquad\qquad = \dfrac{P(-1 < Y < 2)}{P(Y > -1)}$

**2.** Determine $P(-1 < Y < 2)$. As the interval is across two functions, the interval needs to be split.

$P(-1 < Y < 2) = P(-1 < Y < 0) + P(0 \le Y < 2)$

**3.** To calculate the probabilities we need to determine the areas under the curve.

$P(-1 < Y < 2) = \displaystyle\int_{-1}^{0} -\frac{1}{9}\, y\, dy + \int_{0}^{2} \frac{1}{9}\, y\, dy$

$\qquad\qquad\qquad = -\displaystyle\int_{-1}^{0} \frac{1}{9}\, y\, dy + \int_{0}^{2} \frac{1}{9}\, y\, dy$

**4.** Antidifferentiate and evaluate after substituting the terminals.

$\qquad\qquad\qquad = -\left[\dfrac{1}{18} y^2\right]_{-1}^{0} + \left[\dfrac{1}{18} y^2\right]_{0}^{2}$

$\qquad\qquad\qquad = -\left(\dfrac{1}{18}(0)^2 - \dfrac{1}{18}(-1)^2\right) + \dfrac{1}{18}(2)^2 - \dfrac{1}{18}(0)^2$

$\qquad\qquad\qquad = \dfrac{1}{18} + \dfrac{4}{18}$

$\qquad\qquad\qquad = \dfrac{5}{18}$

**5.** Determine $P(Y > -1)$. As the interval is across two functions, the interval needs to be split.

$P(Y > -1) = P(-1 < Y < 0) + P(0 \le Y \le 3)$

**6.** To determine the probabilities we need to determine the areas under the curve. As $P(0 \le Y \le 3)$ covers exactly half the area under the curve, $P(0 \le Y \le 3) = \dfrac{1}{2}$. (The entire area under the curve is always 1 for a probability density function.)

$P(Y > -1) = \displaystyle\int_{-1}^{0} -\frac{1}{9}\, y\, dy + \int_{0}^{3} \frac{1}{9}\, y\, dy$

$\qquad\qquad\quad = -\displaystyle\int_{-1}^{0} \frac{1}{9}\, y\, dy + \frac{1}{2}$

**7.** Antidifferentiate and evaluate after substituting the terminals.

$\qquad\qquad\quad = -\left[\dfrac{1}{18} y^2\right]_{-1}^{0} + \dfrac{1}{2}$

$\qquad\qquad\quad = -\left(\dfrac{1}{18}(0)^2 - \dfrac{1}{18}(-1)^2\right) + \dfrac{1}{2}$

$\qquad\qquad\quad = \dfrac{1}{18} + \dfrac{9}{18}$

$\qquad\qquad\quad = \dfrac{10}{18}$

$\qquad\qquad\quad = \dfrac{5}{9}$

**8.** Now substitute into the formula to determine

$$P(Y < 2|Y > -1) = \frac{P(-1 < Y < 2)}{P(Y > -1)}.$$

$$P(Y < 2|Y > -1) = \frac{P(-1 < Y < 2)}{P(Y > -1)}$$

$$= \frac{5}{18} \div \frac{5}{9}$$

$$= \frac{5}{18} \times \frac{9}{5}$$

$$= \frac{1}{2}$$

---

# Exercise 10.2 Continuous random variables and the probability density function

**learn on**

| 10.2 Exercise | 10.2 Exam questions **on** |
|---|---|

| Simple familiar | Complex familiar | Complex unfamiliar |
|---|---|---|
| 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 | 13, 14, 15, 16 | 17, 18, 19, 20 |

### Simple familiar

**1.** Identify which of the following are continuous random variables.

   **a.** The population of your town or city
   **b.** The types of motorbike in a parking lot
   **c.** The heights of people in an identification line-up
   **d.** The masses of babies in a group
   **e.** The languages spoken at home by students in your class
   **f.** The time spent watching TV
   **g.** The number of children in the families in your suburb
   **h.** The air pressure in your car's tyres
   **i.** The number of puppies in a litter
   **j.** The types of radio program listened to by teenagers
   **k.** The times for swimming 50 metres
   **l.** The quantity of fish caught in a net
   **m.** The number of CDs you own
   **n.** The types of shops in a shopping centre
   **o.** The football competition ladder at the end of each round
   **p.** The lifetimes of torch batteries
   **q.** The number of people attending a rock concert
   **r.** Exam grades
   **s.** The types of magazine sold at a news agency
   **t.** Hotel accommodation ratings

**2.** **WE1** The frequency histogram shows the distribution of masses (in kilograms) of 60 students in Year 7 at Northwood State High School. Determine the probability that a random student has a mass:
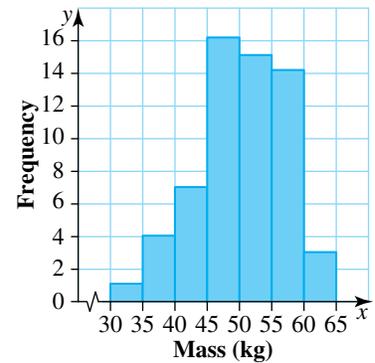
a. between 40 and 60 kilograms
b. less than 45 kilograms
c. greater than 55 kilograms.



**3.** A small car-hire firm keeps note of the age and kilometres covered by each of the cars in their fleet. Generally, cars are no longer used once they have either covered 350 000 kilometres or are more than 5 years old. The following information describes the ages of the cars in their current fleet.

| Age | Frequency |
|-----|-----------|
| $0 < x \le 1$ | 10 |
| $1 < x \le 2$ | 26 |
| $2 < x \le 3$ | 28 |
| $3 < x \le 4$ | 20 |
| $4 < x \le 5$ | 11 |
| $5 < x \le 6$ | 4 |
| $6 < x \le 7$ | 1 |



Age of rental car

a. Determine:

   i. $P(X \le 2)$             ii. $P(X > 4)$
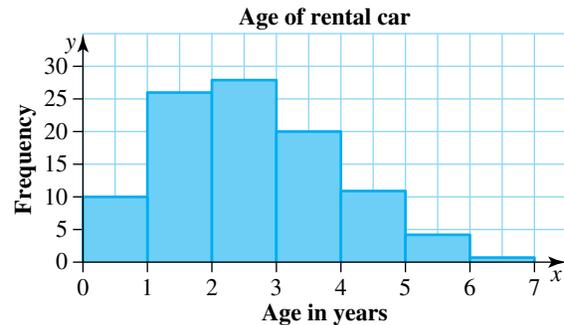
b. Determine:

   i. $P(1 < X \le 4)$        ii. $P(X > 1 | X \le 4)$

**4.** The battery life for batteries in television remote controls was investigated in a study. The results are shown in the table and histogram.

| Hours of life | Frequency |
|-----|-----------|
| $0 < x \le 15$ | 15 |
| $15 < x \le 30$ | 33 |
| $30 < x \le 45$ | 23 |
| $45 < x \le 60$ | 26 |
| $60 < x \le 75$ | 3 |



Remote battery life
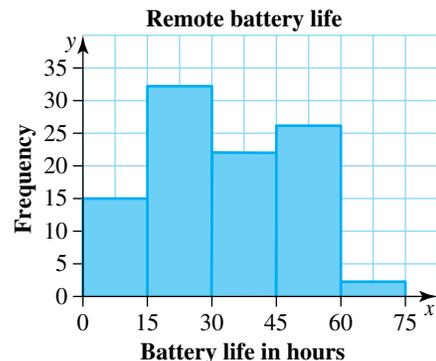
a. Determine how many remote control batteries were included in the study.
b. Determine the probability that a battery will last more than 45 hours.
c. Determine the probability that a battery will last between 15 and 60 hours.
d. A new battery producer is advocating that their batteries have a long life of 60+ hours. If it is known that this is just advertising hype because these batteries are no different from the batteries in the study, determine the probability that these new batteries will have a life of 60+ hours.

**5.** **WE2** Sketch each of the following functions and determine whether each one is a probability density function.

a. $f(x) = \begin{cases} \dfrac{1}{4}e^{2x}, & 0 \le x \le \log_e(3) \\ 0, & \text{elsewhere} \end{cases}$

b. $f(x) = \begin{cases} 0.25, & -2 \le x \le 2 \\ 0, & \text{elsewhere} \end{cases}$

6. Sketch each of the following functions and determine whether each one is a probability density function.

a. $f(x) = \begin{cases} \dfrac{1}{2} \cos(x), & -\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2} \\ 0, & \text{elsewhere} \end{cases}$

b. $f(x) = \begin{cases} \dfrac{1}{2\sqrt{x}}, & \dfrac{1}{2} \le x \le 4 \\ 0, & \text{elsewhere} \end{cases}$

7. **WE3** For the following probability density function:

$$f(x) = \begin{cases} \dfrac{1}{9}x^2, & 0 \le x \le 3 \\ 0, & \text{elsewhere} \end{cases}$$

a. calculate $P(X > 2)$

b. calculate $P(1 < X < 3)$.

8. The continuous random variable $Z$ has a probability density function given by

$$f(z) = \begin{cases} -z + 1, & 0 \le z < 1 \\ z - 1, & 1 \le z \le 2 \\ 0, & \text{elsewhere.} \end{cases}$$

a. Sketch the graph of $f$.
b. Determine $P(Z < 0.75)$.
c. Determine $P(Z > 0.5)$.

9. **WE4** Given that the following function is a probability density function, determine the value of $n$.

$$f(x) = \begin{cases} n(x^3 - 1), & 1 \le x \le 3 \\ 0, & \text{elsewhere} \end{cases}$$

10. The continuous random variable $X$ has a probability density function given by

$$f(x) = \begin{cases} 4x^3, & 0 \le x \le a \\ 0, & \text{elsewhere} \end{cases}$$

where $a$ is a constant.

a. Determine the value of the constant $a$.
b. Sketch the graph of $f$.
c. Determine $P(0.5 \le X \le 1)$.

11. For the following probability density function:

$$f(x) = \begin{cases} \dfrac{1}{2}x - 2, & 4 \le x \le k \\ 0, & \text{elsewhere} \end{cases}$$

a. determine the value of k
b. calculate $P(X < 4.5)$
c. calculate $P(X > 5 \mid X < 5.7)$.

12. Determine the value of the constant $m$ in each of the following if each function is a probability density function.

a. $f(x) = \begin{cases} m(6 - 2x), & 0 \le x \le 2 \\ 0, & \text{elsewhere} \end{cases}$

b. $f(x) = \begin{cases} me^{-2x}, & x \ge 0 \\ 0, & \text{elsewhere} \end{cases}$

c. $f(x) = \begin{cases} me^{2x}, & 0 \le x \le \log_e(3) \\ 0, & \text{elsewhere} \end{cases}$

**13.** A number of experienced shot-putters were asked to aim for a line 10 metres away.

After each of them put their shot, its distance from the 10-metre line was measured. All of the shots were on or between the 8- and 10-metre lines. The results of the measurements are shown, where $X$ is the distance in metres from the 8-metre line.

| Metres | Frequency |
|---|---|
| $0 < x \leq 0.5$ | 75 |
| $0.5 < x \leq 1$ | 63 |
| $1 < x \leq 1.5$ | 45 |
| $1.5 < x \leq 2$ | 17 |
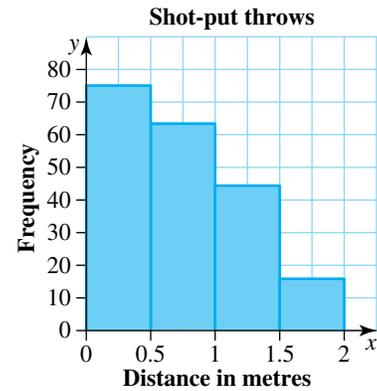
Shot-put throws



**a.** Determine how many shot-put throws were measured.

**b.** Calculate:

  **i.** $P(X > 0.5)$

  **ii.** $P(1 < X \leq 2)$

**c.** A guest shot-putter is visiting the athletics club where the measurements are being conducted. His shot-putting ability is equivalent to the abilities of the club members. Determine the probability that he puts the shot between 50 cm and 1 m of the 10-metre line if it is known that he put the shot within 1 metre of the 10-metre line.

**14.** **WE5** Given that the following function is a probability density function, determine the value of $a$.

$$f(x) = \begin{cases} -ax, & -2 \leq x \leq 0 \\ 2ax, & 0 < x \leq 3 \\ 0, & \text{elsewhere} \end{cases}$$

**15.** The rectangular function, $f$, is defined by the rule

$$f(x) = \begin{cases} c, & 0.25 < x < 1.65 \\ 0, & \text{elsewhere.} \end{cases}$$

Determine the value of the constant $c$, given that $f$ is a probability density function.

**16.** Let $X$ be a continuous random variable with the probability density function

$$f(x) = \begin{cases} x^2 + 2kx + 1, & 0 \leq x \leq 3 \\ 0, & \text{elsewhere.} \end{cases}$$

Show that the value of $k$ is $-\dfrac{11}{9}$.

**17.** The graph of a function, $f$, is shown. If $f$ is known to be a probability density function, show that the value of $z$ is $\dfrac{1}{3}$.
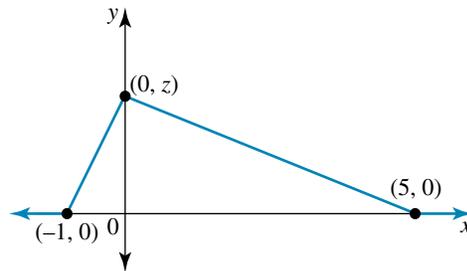


**18.** $X$ is a continuous random variable such that

$$f(x) = \begin{cases} \dfrac{1}{2} \log_e \left( \dfrac{x}{2} \right), & 2 \le x \le a \\ 0, & \text{elsewhere} \end{cases} \quad \text{and} \quad \int_2^a f(x)\, dx = 1.$$

The graph of this function is shown.

Differentiate $x \log_e \left( \dfrac{x}{2} \right)$ and use this to determine the value of the constant $a$.
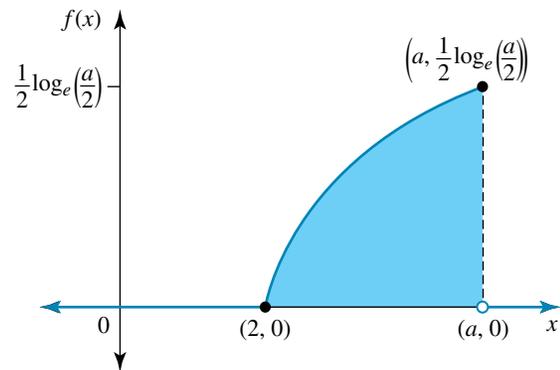


**19.** The graph of a probability density function is displayed below. Determine the $y$-intercept for the function.
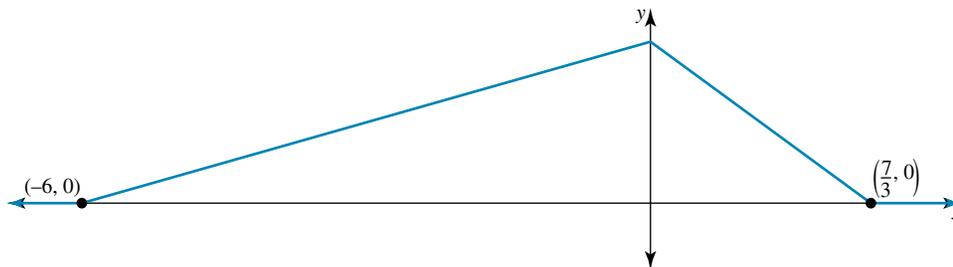


**20.** The following graph depicts a probability density function.
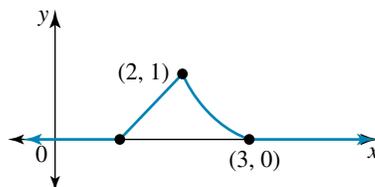


The function is of the form:

$$f(x) = \begin{cases} ax + b, & c \le x < 2 \\ (x-d)^2, & 2 \le x \le 3. \\ 0, & \text{elsewhere} \end{cases}$$

Determine $P\left( X < \dfrac{5}{3} \right)$.

**Fully worked solutions for this chapter are available online.**

# LESSON
## 10.3 Cumulative distribution functions

### 10.3.1 The cumulative distribution function

When considering the probability density function $f(x)$, the corresponding **cumulative distribution function** (cdf) is denoted with a capital letter as $F(x)$. The cumulative distribution function cumulates (adds together) the probabilities of all $x$-values less than the value being considered. As the $x$-value being considered increases, more probabilities are added, and because probabilities are always positive values, it means that the function $F(x)$ will never decrease.

Consider the probability distribution function $f(x)$ such that

$$f(x) = \begin{cases} 0.5x, & 0 \leq x \leq 2 \\ 0, & \text{elsewhere} \end{cases}.$$

The probabilities for consecutive intervals across the distribution can be tabulated as shown.

| $a \leq x \leq b$ | $P(a \leq x \leq b)$ |
|---|---|
| $0 \leq x < 0.25$ | 0.015 625 |
| $0.25 < x \leq 0.5$ | 0.046 875 |
| $0.5 < x \leq 0.75$ | 0.078 125 |
| $0.75 < x \leq 1$ | 0.109 375 |
| $1 < x \leq 1.25$ | 0.140 625 |
| $1.25 < x \leq 1.5$ | 0.171 875 |
| $1.5 < x \leq 1.75$ | 0.203 125 |
| $1.75 < x \leq 2$ | 0.234 375 |

As $P(a \leq X \leq b) = P(a \leq X \leq c) + P(c < X \leq b)$, $a < c < b$,

$$P(X \leq 0.5) = P(0 \leq X \leq 0.25) + P(0.25 < X \leq 0.5)$$
$$P(X \leq 0.75) = P(X \leq 0.5) + P(0.5 < X \leq 0.75)$$
$$P(X \leq 1) = P(X \leq 0.75) + P(0.75 < X \leq 1)$$

and so on.

We can tabulate the cumulative probability $P(x \le c)$ across the interval $0 \le c \le 2$.

| $x \le c$ | $P(x \le c)$ |
|-----------|--------------|
| $x \le 0.25$ | 0.015 625 |
| $x \le 0.5$ | 0.062 5 |
| $x \le 0.75$ | 0.140 625 |
| $x \le 1$ | 0.25 |
| $x \le 1.25$ | 0.390 625 |
| $x \le 1.5$ | 0.562 5 |
| $x \le 1.75$ | 0.765 625 |
| $x \le 2$ | 1 |

When $F(x)$ is graphed against $x$, the curve formed is continuous across the range of the probability function.
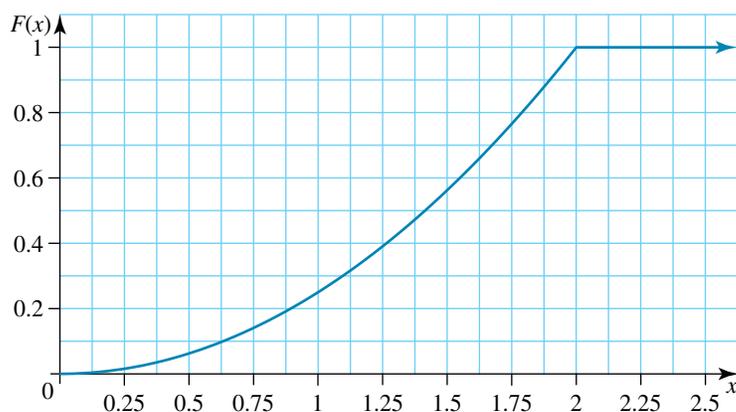


## Cumulative distribution functions

**As $F(a)$ represents the cumulative probabilities up to $x = a$, this represents the area under the curve of the probability between $-\infty$ and $a$.**

$$F(a) = P(X \le a) = \int_{-\infty}^{a} f(x)\, dx$$

**Therefore, the cumulative distribution function can be found by integrating the probability density function.**

**If a probability distribution function $f(x)$ has an interval $[m, n]$ where $\int_{m}^{n} f(x)\, dx = 1$ and $f(x) = 0$ for all $x$-values outside the interval $[m, n]$, this means that the cumulative distribution function $F(x)$ must equal 0 when $x < m$ and $F(x)$ must equal 1 when $x > n$ (as the sum of all probabilities equals 1).**

For example, consider the probability density function:

$$f(x) = \begin{cases} 2x, & 0 \le x \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

$$F(x) = \int_0^x 2x \, dx \ \text{ for } 0 \le x \le 1$$

$$F(x) = \left[ x^2 \right]_0^x$$

$F(x) = x^2$ for $0 \le x \le 1$

The corresponding cumulative distribution function is:

$$F(x) = \begin{cases} 0, & x < 0 \\ x^2, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$$

The graphs of these functions are:

**Probability density function**

**Cumulative distribution function**
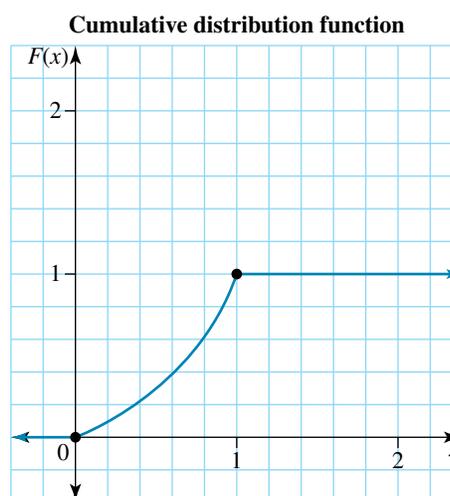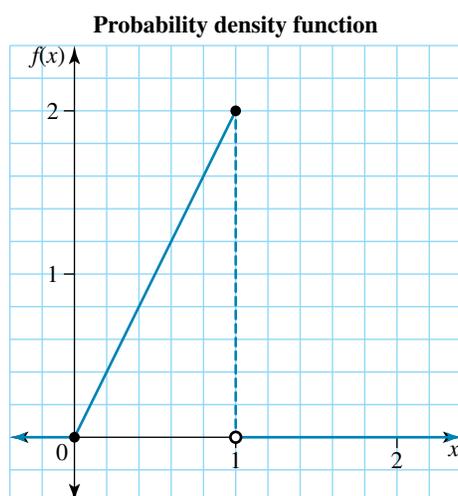
In the graph of this probability density function, the area under the curve from the lower bound up to a specified $x$-value represents the cumulative probability up to this $x$-value. This cumulative probability is the $F(x)$ value, or $y$-value, for the specified $x$-value in the corresponding cumulative distribution function.

For example, when considering $x = 0.5$ for the function above, the area under the curve of the probability density function between $0 \le x \le 0.5$ is:

$$\int_0^{0.5} 2x \, dx$$

$$= 0.25$$

As the area under the curve of the probability density function represents the probability of a value being within the interval $0 \le x \le 0.5$, this means:

$$P(0 \le x \le 0.5) = \int_0^{0.5} 2x \, dx = 0.25$$
$$= 0.25$$

This corresponds to the cumulative probability from the lower bound of 0 up to the upper bound of 0.5.
Therefore, in the cumulative distribution function:

$$F(x) = x^2$$
$$F(0.5) = 0.5^2$$
$$= 0.25$$

**Probability density function**



**Cumulative distribution function**
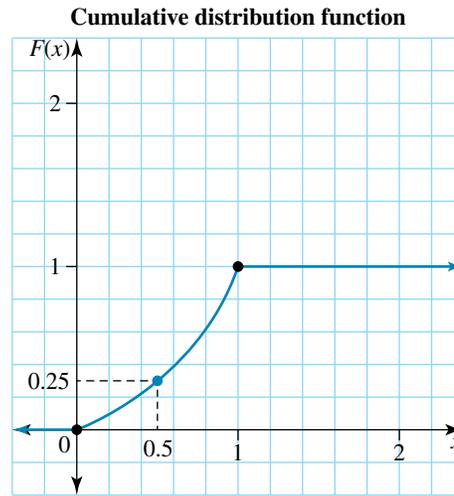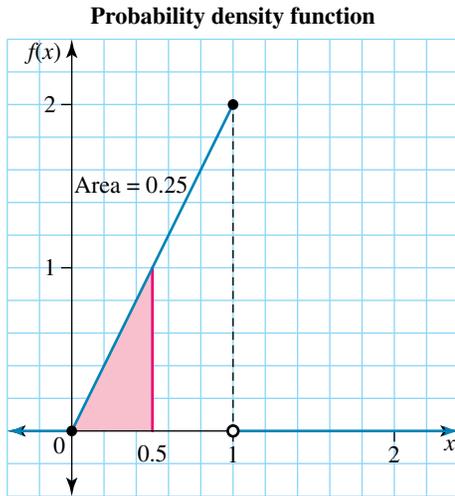
Similarly, when considering the interval $0 \leq x \leq 0.8$ for the function above, the probability density function can be used to determine the cumulative probability for the interval:

$$(0 \leq x \leq 0.8) = \int_0^{0.8} 2x \, dx$$
$$= 0.64$$

Alternatively, the cumulative distribution function can be used to determine the cumulative probability for the interval:

$$F(x) = x^2$$
$$F(0.8) = 0.8^2$$
$$= 0.64$$

**Probability density function**



**Cumulative distribution function**
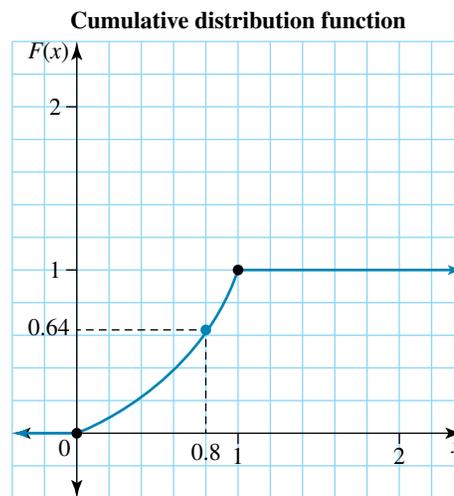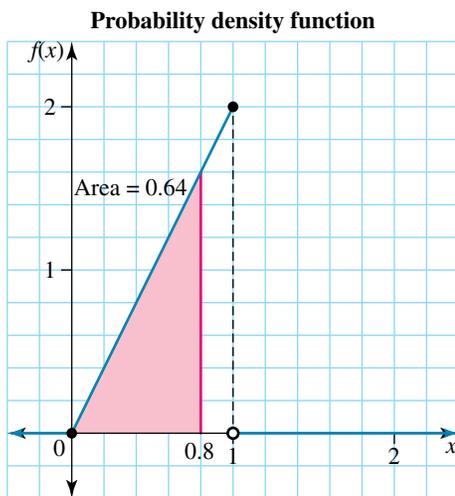
It is important to remember that the area under the probability density curve for an interval will be equivalent to a point on the cumulative distribution curve.

The probability of a variable falling within a particular interval is easily determined by using the cumulative distribution function.

Given that $P(X \le b) = P(X \le a) + P(a < X \le b)$,

$$P(a < X \le b) = P(X \le b) - P(X \le a)$$

Thus, $P(a < X \le b) = F(b) - F(a)$.

---

### WORKED EXAMPLE 6 Determining cumulative distribution functions and probabilities

**The continuous random variable $X$ has a probability density function defined by**

$$f(x) = \begin{cases} \dfrac{1}{8}x - \dfrac{1}{8}, & 1 \le x \le 5 \\ 0, & \text{elsewhere.} \end{cases}$$

a. **Determine the cumulative distribution function, $F(x)$, for $X$.**
b. **Determine:**
   i. $P(X < 2)$                      ii. $P(3 < X < 4)$

| THINK | WRITE |
|---|---|
| a. 1. Integrate the probability density function to determine the cumulative distribution function between $1 \le x \le 5$. As it is a function that needs to be determined and not a specific value, the upper value of the integral is not defined. Therefore, the variable $x$ is included as the upper bound. This means that your output will be a function where any $x$-value can be used in order to determine the cumulative probability from $x = 1$ up to that $x$-value. | a. For $1 \le x \le 5$: $F(x) = \displaystyle\int_1^x \dfrac{1}{8}x - \dfrac{1}{8}\,dx$ $F(x) = \left[\dfrac{1}{16}x^2 - \dfrac{1}{8}x\right]_1^x$ $F(x) = \left(\dfrac{1}{16}x^2 - \dfrac{1}{8}x\right) - \left(\dfrac{1}{16}(1)^2 - \dfrac{1}{8}(1)\right)$ $F(x) = \dfrac{1}{16}x^2 - \dfrac{1}{8}x + \dfrac{1}{16}$ |
| 2. State the cumulative distribution function. | $F(x) = \begin{cases} 0, & x < 1 \\ \dfrac{1}{16}x^2 - \dfrac{1}{8}x + \dfrac{1}{16}, & 1 \le x \le 5 \\ 1, & x > 5 \end{cases}$ |
| b. i. 1. Identify the appropriate equation and substitute the appropriate values. | b. i. $P(X < 2) = F(X < 2)$ |
| 2. Evaluate. | $= \dfrac{1}{16}(2)^2 - \dfrac{1}{8}(2) + \dfrac{1}{16}$ $= \dfrac{1}{16}$ |
| 3. Answer the question. | The probability of $X$ being less than 2 is $\dfrac{1}{16}$. |

| | | |
|---|---|---|
| **ii.** | **1.** Identify the appropriate equation and substitute the appropriate values. | **ii.** $P(3 < X < 4) = F(3 < X < 4) = F(4) - F(3)$ |
| | **2.** Evaluate. | $= \left(\dfrac{1}{16}(4)^2 - \dfrac{1}{8}(4) + \dfrac{1}{16}\right) - \left(\dfrac{1}{16}(3)^2 - \dfrac{1}{8}(3) + \dfrac{1}{16}\right)$ |
| | | $= \dfrac{5}{16}$ |
| | **3.** Answer the question. | The probability of $X$ being 3 and 4 is $\dfrac{5}{16}$. |

## WORKED EXAMPLE 7 Determining complex cumulative distribution functions and probabilities

**A continuous random variable $Y$ has a probability density function such that**

$$f(y) = \begin{cases} y, & 0 \le y \le 1 \\ 2 - y, & 1 < y \le 2 \\ 0, & \text{elsewhere.} \end{cases}$$

**a. Determine the cumulative distribution function, $F(y)$, of $Y$.**
**b. Determine:**
   **i. $P(Y \le 0.8)$**                     **ii. $P(0.6 \le Y < 1.2)$**

**THINK**

**a. 1.** Integrate $f(y)$ to determine $F(y)$ over the domains $0 \le y \le 1$ and $1 < y \le 2$.

**2.** Write $F(y)$ as a piecewise function for all $y \in R$.

**WRITE**

**a.** For $0 \le y \le 1$, $F(y) = \displaystyle\int_0^y y \, dy = \dfrac{y^2}{2}$.

For $1 < y \le 2$,

$F(y) = P(Y \le y) = P(Y \le 1) + P(1 < Y \le y)$

$F(y) = \displaystyle\int_0^1 y \, dy + \int_1^y (2 - y) \, dy$

$= \dfrac{1}{2} + \left[2y - \dfrac{y^2}{2}\right]_1^y$

$= \dfrac{1}{2} + \left[\left(2y - \dfrac{y^2}{2}\right) - \left(2 \times 1 - \dfrac{1^2}{2}\right)\right]_1^y$

$= 2y - \dfrac{y^2}{2} - 1$

$$F(y) = \begin{cases} 0, & y \le 0 \\ \dfrac{y^2}{2}, & 0 < y \le 1 \\ 2y - \dfrac{y^2}{2} - 1, & 1 < y \le 2 \\ 1, & y > 2 \end{cases}$$

| | |
|---|---|
| **b. i. 1.** Identify the appropriate equation and substitute the appropriate values. | **b. i.** $P(Y \leq 0.8) = F(Y \leq 0.8)$ $= \dfrac{(0.8)^2}{2}$ |
| **2.** Evaluate. | $= 0.32$ |
| **3.** Answer the question. | The probability of $Y$ being less than 0.8 is 0.32. |
| **ii. 1.** Identify the appropriate equation. | **ii.** $P(a \leq Y < b) = F(b) - F(a)$ $P(0.6 \leq Y < 1.2) = F(1.2) - F(0.6)$ |
| **2.** Evaluate. | $= \left[ 2(1.2) - \dfrac{(1.2)^2}{2} - 1 \right] - \left[ \dfrac{(0.6)^2}{2} \right]$ $= 0.68 - 0.18$ $= 0.5$ |
| **3.** Answer the question. | The probability of $Y$ being between 0.6 and 1.2 is 0.5. |

# Exercise 10.3 Cumulative distribution functions

**learn on**

| **10.3 Exercise** | **10.3 Exam questions** on |
|---|---|

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| Simple familiar | Complex familiar | Complex unfamiliar |
|---|---|---|
| 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 | 13, 14, 15, 16 | 17, 18, 19, 20 |

**Simple familiar**

**1.** **WE6** The continuous random variable $X$ has a uniform rectangular probability density function defined by

$$f(x) = \begin{cases} \dfrac{1}{5}, & 1 \leq x \leq 6 \\ 0, & \text{elsewhere.} \end{cases}$$

**a.** Determine the cumulative distribution function, $f(x)$, for $X$.
**b.** Determine:
   **i.** $P(x \leq 4)$
  **ii.** $P(2.2 < x \leq 4.5)$

**2.** Determine the cumulative distribution function for the following probability density function.

$$f(x) = \begin{cases} 4x^3, & 0 \leq x \leq 1 \\ 0, & \text{elsewhere.} \end{cases}$$

**3.** Let $X$ be a continuous random variable with a probability density function defined by

$$f(x) = \begin{cases} \dfrac{1}{2}\sin(x), & 0 \le x \le \pi \\ 0, & \text{elsewhere.} \end{cases}$$

**a.** Determine the cumulative distribution function $F(x)$ for $X$.

**b.** Determine $P\left(X \le \dfrac{\pi}{2}\right)$.

**c.** Determine $P\left(\dfrac{\pi}{4} < X < \dfrac{3\pi}{4}\right)$.

**d.** Determine $P\left(X > \dfrac{\pi}{4} \mid X < \dfrac{3\pi}{4}\right)$.

**4. a.** Determine the probability density function for the following cumulative distribution function.

$$F(x) = \begin{cases} 0, & x < -1 \\ \dfrac{1}{4}(x+1)^2, & -1 \le x \le 1 \\ 1, & x > 1 \end{cases}$$

**b.** Calculate $P(X < 0.4)$.

**5. a.** Determine the probability density function for the following cumulative distribution function.

$$F(x) = \begin{cases} 0, & x < 0 \\ \dfrac{1}{8}x^3, & 0 \le x \le 2 \\ 1, & x > 2 \end{cases}$$

**b.** Calculate $P(X < 0.8)$.
**c.** Calculate $P(X > 1.3)$.
**d.** Calculate $P(X < 1.5 \mid X > 0.4)$

**6.** The amount of petrol sold daily by a busy service station is a uniformly distributed probability density function. A minimum of 18 000 litres and a maximum of 30 000 litres are sold on any given day. The graph of the function is shown.

**a.** Determine the value of the constant $k$.
**b.** Determine the formula for the probability density function $f(x)$.
**c.** Determine the formula for the cumulative distribution function $F(x)$.
**d.** Determine the probability that between 20 000 and 25 000 litres of petrol are sold on a given day.
**e.** Determine the probability that as much as 26 000 litres of petrol were sold on a particular day, given that it was known that at least 22 000 litres were sold.
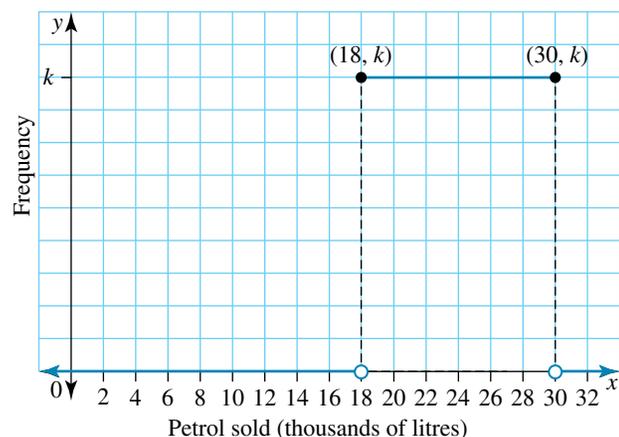
**7.** The continuous random variable $X$ has a probability density function defined by

$$f(x) = \begin{cases} \dfrac{3}{8}x^2, & 0 \le x \le 2 \\ 0, & \text{elsewhere} \end{cases}.$$

**a.** Determine the cumulative distribution function for $f(x)$.
**b.** Determine $P(X > 1.2)$.
**c.** Determine the value of $n$ such that $P(X \le n) = 0.75$.

**8.** The continuous random variable $X$ has a probability density function defined by

$$f(x) = \begin{cases} \dfrac{3}{4}(1 - x^2), & -1 \le x \le 1 \\ 0, & \text{elsewhere.} \end{cases}$$

**a.** Determine the cumulative distribution function for $f(x)$.
**b.** Determine $P(-0.6 < X < 0.3)$.
**c.** Determine the value $k$ such that $P(X > k) = 0.475$.

**9.** The continuous random variable $X$ has a cumulative distribution function defined by

$$F(x) = \begin{cases} 0, & x < -1 \\ m(x + 1), & -1 \le x \le 1 \\ 1, & x > 1. \end{cases}$$

**a.** Determine the value of the constant $m$.
**b.** Calculate $P(X < 0.8 \,|\, X > -0.6)$.

**10.** The continuous random variable $Y$ has a probability density function defined by

$$f(y) = \begin{cases} ny^2, & 0 \le y \le 3 \\ 0, & \text{elsewhere.} \end{cases}$$

**a.** Determine the value of the constant $n$.
**b.** Determine the cumulative distribution function for $f(y)$.
**c.** Calculate $P(Y < 2.2 \,|\, Y > 1.4)$.

**11.** The continuous random variable $Y$ has a probability density function defined by

$$f(y) = \begin{cases} ky(1 - y), & 0 \le y \le 1 \\ 0, & \text{elsewhere.} \end{cases}$$

**a.** Determine the value of the constant $k$.
**b.** Determine the cumulative distribution function for $f(y)$.
**c.** Determine $P(Y > 0.7)$.

**12.** The continuous random variable $X$ has a cumulative distribution function defined by

$$F(x) = \begin{cases} 0, & x < 0 \\ 0.25x^2 - 2x + 4, & 4 \le x \le c \\ 1, & x > c. \end{cases}$$

**a.** Determine the value of the constant $c$.
**b.** Determine the probability density function for $F(x)$.
**c.** Determine the value of $m$ such that $P(X > m) = \dfrac{7}{8}$.

**13.** [WE7] A continuous random variable $X$ has a probability density function such that

$$f(x) = \begin{cases} 6x + 2, & -\dfrac{1}{3} \leq x \leq 0 \\ -8x^2 + 2, & 0 < x \leq \dfrac{1}{2} \\ 0, & \text{elsewhere.} \end{cases}$$

**a.** Determine the cumulative distribution function, $F(x)$, of $X$.
**b.** Determine:

  **i.** $P(X < 0.3)$
  **ii.** $P(-0.2 < X < 0.1)$

**14.** A continuous random variable $Y$ has a probability density function such that

$$f(y) = \begin{cases} \dfrac{2}{3}y, & 0 \leq y < 1 \\ \dfrac{1}{3}(3 - y), & 1 \leq y \leq 3 \\ 0, & \text{elsewhere.} \end{cases}$$

**a.** Determine the cumulative distribution function, $F(y)$, of $Y$.
**b.** Determine:

  **i.** $P(Y > 2.5)$
  **ii.** $P(0.4 < Y < 1.7)$

**15.** A probability density function is defined by the rule

$$f(x) = \begin{cases} k(2 + x), & -2 \leq x < 0 \\ k(2 - x), & 0 \leq x \leq 2 \\ 0, & \text{elsewhere} \end{cases}$$

where $X$ is a continuous random variable and $k$ is a constant.

**a.** Sketch the graph of $f$.
**b.** Determine the value of $k$.
**c.** Determine the cumulative distribution function $F(x)$.
**d.** Determine $P(-1 \leq X \leq 1)$.
**e.** Determine $P(X \geq -1 | X \leq 1)$.

**16.** The continuous random variable $X$ has a probability density function defined by

$$f(x) = \begin{cases} 0.1x - 0.5, & 5 \leq x \leq 7 \\ 0.2, & 7 < x \leq 10 \\ -0.1x + 1.2, & 10 < x \leq 12 \\ 0, & \text{elsewhere.} \end{cases}$$

Determine the cumulative distribution function $F(x)$.

17. Determine the cumulative distribution function for the following probability density function.

$$f(x) = \begin{cases} \dfrac{-22}{15}x, & -1 \leq x \leq 0 \\ \dfrac{1}{10}x^2, & 0 < x \leq c \\ 0, & \text{elsewhere} \end{cases}$$

18. The probability density function $f(x)$ can be defined by

$$f(x) = \begin{cases} mx - 2, & 8 \leq x \leq 10 \\ -mx + n, & 10 < x \leq 12 \\ 0, & \text{elsewhere} \end{cases}.$$

The graph of $f(x)$ is shown below.
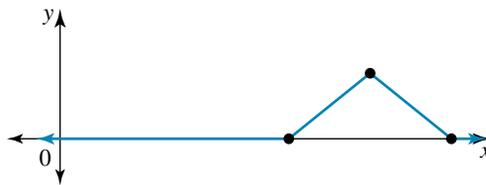


Determine the cumulative distribution function.

19. The probability density function $f(x)$ can be defined by

$$f(x) = \begin{cases} c(x^2 - 4), & 2 \leq x \leq a \\ 12c, & a < x \leq \dfrac{187}{36} \\ 0, & \text{elsewhere}. \end{cases}$$

$P(X > 4.5) = \dfrac{1}{3}$ and $a < 4.5$.

Determine the cumulative distribution function.

20. The continuous random variable $T$ represents the time in hours that office workers in a particular company spend writing emails during a day. The probability density function of $T$ is

$$f(t) = \begin{cases} \dfrac{8}{27}(t^2 - k), & 1 \leq t \leq m \\ 0, & \text{elsewhere}. \end{cases}$$

The probability that an office worker spends more than 2 hours writing emails during a day is $\dfrac{49}{81}$.

Determine the cumulative distribution function for $T$.

**Fully worked solutions for this chapter are available online.**

# LESSON
## 10.4 Measures of centre and spread

### 10.4.1 Measure of centre: the expected value

The expected value, or mean, is a measure of centre that can be determined for continuous random variables that are modelled by probability density functions.

Remember that for a discrete random variable,

$$E(X) = \mu = \sum p_i x_i$$

This definition can also be applied to a continuous random variable.

**Calculating expected value (mean)**

**When the probability density function is given by $p(x)$, the expected value (mean) can be calculated using**

$$E(X) = \mu = \int_{-\infty}^{\infty} x p(x)\, dx$$

Consider the continuous random variable, $X$, that has a probability density function defined by

$$f(x) = \begin{cases} x^2, & 0 \le x \le 1 \\ 0, & \text{elsewhere} \end{cases}.$$

For this function,

$$
\begin{aligned}
E(X) = \mu &= \int_0^1 x\, p(x)\, dx \\
&= \int_0^1 x(x^2)\, dx \\
&= \int_0^1 x^3\, dx \\
&= \left[ \frac{x^4}{4} \right]_0^1 \\
&= \frac{1^4}{4} - 0 \\
&= \frac{1}{4}
\end{aligned}
$$

**A continuous random variable, $Y$, has a probability density function, $f$, defined by**

$$f(y) = \begin{cases} ky, & 0 \le y \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

**where $k$ is a constant.**
**a. Sketch the graph of $f$.**
**b. Determine the value of the constant $k$.**
**c. Determine the mean of $Y$.**

**THINK**

**a.** The graph $f(y) = ky$ is a straight line with end points at $(0, 0)$ and $(1, k)$. Remember to include the lines $f(y) = 0$ for $y > 1$ and $y < 0$.

**b.** Solve $\displaystyle\int_0^1 ky \, dy = 1$ to determine the value of $k$.

**c. 1.** State the rule for the mean.

**WRITE**

**a.**
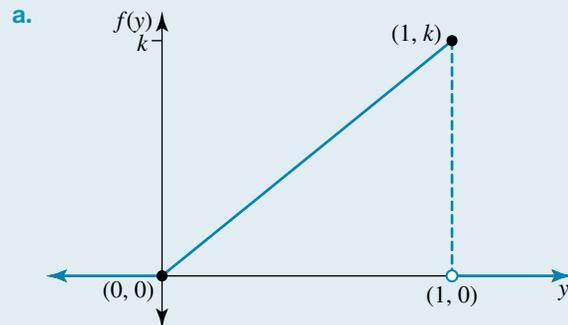


**b.** $\displaystyle\int_0^1 ky \, dy = 1$

$k \displaystyle\int_0^1 y \, dy = 1$

$k \left[ \dfrac{y^2}{2} \right]_0^1 = 1$

$\dfrac{k(1)^2}{2} - 0 = 1$

$\dfrac{k}{2} = 1$

$k = 2$

Using the area of a triangle also enables you to find the value of $k$.

$\dfrac{1}{2} \times 1 \times k = 1$

$\dfrac{k}{2} = 1$

$k = 2$

**c.** $\mu = \displaystyle\int_0^1 y(2y) \, dy$

$= \displaystyle\int_0^1 2y^2 \, dy$

**2.** Antidifferentiate and simplify.

$$= \left[\frac{2}{3}y^3\right]_0^1$$

$$= \frac{2(1)^3}{3} - 0$$

$$= \frac{2}{3}$$

## 10.4.1 Measures of spread: variance and standard deviation

The variance and standard deviation are important measures of spread in statistics.

**Calculating variance and standard deviation**

When the probability density function is given by $p(x)$ and the expected value $(\mu)$ is known, the variance can be calculated using

$$\text{Var}(X) = \sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 p(x)\, dx$$

The standard deviation $(\sigma)$ can be calculated using

$$\text{SD}(X) = \sigma = \sqrt{\text{Var}(X)}$$

**WORKED EXAMPLE 9 Determining the variance and standard deviation**

For a continuous random variable, $X$, with a probability density function, $f$, defined by

$$f(x) = \begin{cases} \dfrac{1}{2}x + 2, & -4 \le x \le -2 \\ 0, & \text{elsewhere,} \end{cases}$$

determine:
a. the mean
b. the variance
c. the standard deviation, correct to 4 decimal places.

| THINK | WRITE |
|---|---|
| a. 1. State the rule for the mean and simplify. | a. $\mu = \displaystyle\int_{-4}^{-2} xf(x)\, dx$ |
| | $= \displaystyle\int_{-4}^{-2} x\left(\frac{1}{2}x + 2\right) dx$ |
| | $= \displaystyle\int_{-4}^{-2} \left(\frac{1}{2}x^2 + 2x\right) dx$ |

**2.** Antidifferentiate and evaluate.

$$= \left[\frac{1}{6}x^3 + x^2\right]_{-4}^{-2}$$

$$= \left(\frac{1}{6}(-2)^3 + (-2)^2\right)$$

$$- \left(\frac{1}{6}(-4)^3 + (-4)^2\right)$$

$$= \frac{4}{3} + 4 + \frac{32}{3} - 16$$

$$= -\frac{8}{3}$$

$$= -2\frac{2}{3}$$

**b. 1.** Write the rule for variance.

**b.** $\mathrm{Var}(X) = \displaystyle\int_{-4}^{-2} (x - \mu)^2 f(x)\, dx$

**2.** Substitute in the appropriate values.

$$= \int_{-4}^{-2} \left(x - \left(-\frac{8}{3}\right)\right)^2 \left(\frac{1}{2}x + 2\right) dx$$

**3.** Evaluate.

$$= \int_{-4}^{-2} \left(x + \frac{8}{3}\right)^2 \left(\frac{1}{2}x + 2\right) dx$$

$$= \int_{-4}^{-2} \left(x^2 + \frac{16}{3}x + \frac{64}{9}\right) \left(\frac{1}{2}x + 2\right) dx$$

$$= \int_{-4}^{-2} \frac{1}{2}x^3 + \frac{8}{3}x^2 + \frac{32}{9}x + 2x^2 + \frac{32}{3}x + \frac{128}{9}\, dx$$

$$= \int_{-4}^{-2} \frac{1}{2}x^3 + \frac{14}{3}x^2 + \frac{128}{9}x + \frac{128}{9}\, dx$$

$$= \left[\frac{1}{8}x^4 + \frac{14}{9}x^3 + \frac{64}{9}x^2 + \frac{128}{9}x\right]_{-4}^{-2}$$

$$= \left(\frac{1}{8}(-2)^4 + \frac{14}{9}(-2)^3 + \frac{64}{9}(-2)^2 + \frac{128}{9}(-2)\right) -$$

$$\left(\frac{1}{8}(-4)^4 + \frac{14}{9}(-4)^3 + \frac{64}{9}(-4)^2 + \frac{128}{9}(-4)\right)$$

$$= \frac{2}{9}$$

**c. 1.** Write the rule for standard deviation.

**c.** $\sigma = \sqrt{\mathrm{Var}(X)}$

**2.** Substitute in the appropriate value.

$$\sigma = \sqrt{\frac{2}{9}}$$

**3.** Evaluate.

$$\sigma = 0.4714$$

| **TI | THINK** | **WRITE** | **CASIO | THINK** | **WRITE** |
|---|---|---|---|---|

**a. 1.** On a Calculator page. Press MENU, then select:
4: Calculus
5: Numerical Integral.
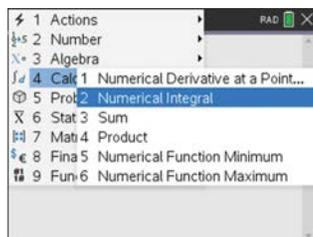
**a. 1.** On a Run-Matrix page, select:
F4 (MATH) then F6 (arrow)

then $\int dx$.

**2.** Complete the entry line as:

$$\int_{-4}^{-2} x\left(\frac{1}{2}x+2\right) dx$$

then press the ENTER button.

**2.** Complete the entry line as:

$$\int_{-4}^{-2} x\left(\frac{1}{2}x+2\right) dx$$

then press the EXE button.

**3.** The answer appears on the screen.
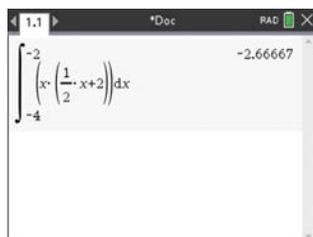
$-2.6666\ldots$ or $-\dfrac{8}{3}$

**3.** The answer appears on the screen.

$-2.6666\ldots$ or $-\dfrac{8}{3}$

**b. 1.** On a Calculator page. Press MENU, then select:
4: Calculus
2: Numerical Integral.
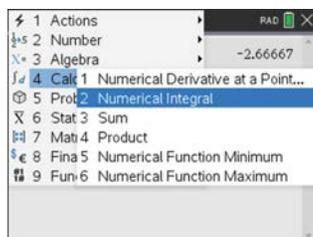
**b. 1.** On a Run-Matrix page, select:
F4 (MATH) then F6 (arrow) then

$\int dx$.

**2.** Complete the entry line as:

$$\int_{-4}^{-2} (x-(-\frac{8}{3}))^2$$

$$\left(\frac{1}{2}x+2\right) dx$$

then press the ENTER button.

**2.** Complete the entry line as:

$$\int_{-4}^{-2} (x-(-\frac{8}{3}))^2$$

$$\left(\frac{1}{2}x+2\right) dx$$

then press the EXE button.

**3.** The answer appears on the screen.
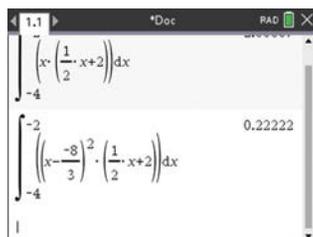
$0.222\ldots$ or $\dfrac{2}{9}$

**3.** The answer appears on the screen.

$0.222\ldots$ or $\dfrac{2}{9}$

**c. 1.** Complete the entry line as:

$$\sqrt{\frac{2}{9}}$$

then press the ENTER button.

**c. 1.** Complete the entry line as:

$$\sqrt{\frac{2}{9}}$$

then press the EXE button.

# Exercise 10.4 Measures of centre and spread

learnon

**10.4 Exercise** | **10.4 Exam questions** on | These questions are even better in jacPLUS!
- Receive immediate feedback
- Access sample responses
- Track results and progress

| Simple familiar | Complex familiar | Complex unfamiliar |
|---|---|---|
| 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 | 13, 14, 15, 16 | 17, 18, 19, 20 |

**Find all this and MORE in jacPLUS** ▶

## Simple familiar

**1.** WE8 The continuous random variable $Z$ has a probability density function of

$$f(z) = \begin{cases} 4, & 1 \le z \le a \\ 0, & \text{elsewhere} \end{cases}$$

where $a$ is a constant.

**a.** Determine the value of the constant $a$.
**b.** Determine the mean of $Z$.

**2.** The continuous random variable, $Y$, has a probability density function of

$$f(y) = \begin{cases} 2y, & 0 \le y \le a \\ 0, & \text{elsewhere} \end{cases}$$

where $a$ is a constant.
Determine:

**a.** the value of the constant $a$  **b.** E($Y$).

**3.** WE9 For the continuous random variable $Z$, the probability density function is

$$f(z) = \begin{cases} 2z - 4, & 2 \le z \le 3 \\ 0, & \text{elsewhere} \end{cases}.$$

Determine the mean, variance and standard deviation.

**4.** Let $X$ be a continuous random variable with a probability density function of

$$f(x) = \begin{cases} \dfrac{1}{2\sqrt{x}}, & 0 \le x \le 1 \\ 0, & \text{elsewhere} \end{cases}.$$

**a.** Show that $f$ is a probability density function.
**b.** Determine E($X$).

**5.** The continuous random variable $Y$ has a probability density function defined by

$$f(y) = \begin{cases} \dfrac{y^2}{3}, & 0 \le y \le \sqrt[3]{9} \\ 0, & \text{elsewhere} \end{cases}.$$

Determine, correct to 4 decimal places, the expected value of $Y$.

**6.** Consider the continuous random variable $X$ with a probability density function of

$$f(x) = \begin{cases} h(2-x), & 0 \leq x \leq 2 \\ h(x-2), & 2 < x \leq 4 \\ 0, & \text{elsewhere} \end{cases}$$

where $h$ is a positive constant.

**a.** Determine the value of the constant $h$.
**b.** Determine E($X$).
**c.** Determine Var($X$).

**7.** The continuous random variable $Z$ has a probability density function defined by

$$f(y) = \begin{cases} \dfrac{a}{z}, & 1 \leq z \leq 8 \\ 0, & \text{elsewhere} \end{cases}$$

where $a$ is a constant.

**a.** Determine the value, correct to 4 decimal places, of the constant $a$.
**b.** Determine E($Z$) correct to 4 decimal places.
**c.** Determine Var($Z$) and SD($Z$).

**8.** Consider the continuous random variable $X$ with a probability density function of

$$f(x) = \begin{cases} k, & a \leq x \leq b \\ 0, & \text{elsewhere} \end{cases}$$

where $a$, $b$ and $k$ are positive constants.

**a.** Sketch the graph of the function $f$.
**b.** Show that $k = \dfrac{1}{b-a}$.
**c.** Determine E($X$) in terms of $a$ and $b$.

**9.** $X$ is a continuous random variable. The graph of the probability density function $f(x) = \dfrac{1}{\pi}(\sin(2x) + 1)$ for $0 \leq x \leq \pi$ is shown with $f(x) = 0$ when $x < 0$ and $x > \pi$.

**a.** Show that $f(x)$ is a probability density function.
**b.** Calculate E($X$) correct to 4 decimal places.
**c.** Calculate, correct to 4 decimal places:

  **i.** Var($X$)
  **ii.** SD($X$)
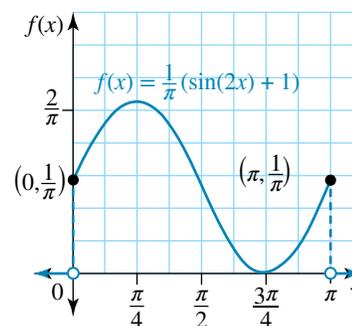


**10.** The continuous random variable $Z$ has a probability density function of

$$f(z) = \begin{cases} \dfrac{3}{z^2}, & 1 \leq z \leq a \\ 0, & \text{elsewhere} \end{cases}$$

where $a$ is a constant.

**a.** Show that the value of $a$ is $\dfrac{3}{2}$.
**b.** Determine the mean value and variance of $f$ correct to 4 decimal places.

**11.** The continuous random variable $Y$ has a probability density function

$$f(y) = \begin{cases} 0.2 \log_e \left( \dfrac{y}{2} \right), & 2 \leq y \leq 7.9344 \\ 0, & \text{elsewhere} \end{cases}.$$

   **a.** Using technology, verify that $f$ is a probability density function.
   **b.** Using technology, determine $E(Y)$ correct to 4 decimal places.
   **c.** Using technology, determine $\text{Var}(Y)$ and $\text{SD}(Y)$ correct to 4 decimal places.

**12.** The continuous random variable $Z$ has a probability density function

$$f(z) = \begin{cases} \sqrt{z-1}, & 1 \leq z \leq a \\ 0, & \text{elsewhere} \end{cases}$$

where $a$ is a constant.

   **a.** Determine the value of the constant $a$ correct to 4 decimal places.
   **b.** Determine, correct to 4 decimal places:
      **i.** $E(Z)$                    **ii.** $\text{Var}(Z)$              **iii.** $\text{SD}(Z)$

**Complex familiar**

**13.** The function defines the probability density function for the continuous random variable $X$.

$$f(x) = \begin{cases} 3e^{-3x}, & x \geq 0 \\ 0, & \text{elsewhere} \end{cases}$$

   **a.** Differentiate $xe^{-3x}$ and hence determine an antiderivative of $3xe^{-3x}$.
   **b.** Determine the mean using the answer from part **a**.

**14.** The time in minutes that an individual must wait in line to be served at the local bank branch is defined by

$$f(t) = 2e^{-2t}, \; t \geq 0$$

where $T$ is a continuous random variable.

   **a.** Differentiate $xe^{-2x}$ and hence determine an antiderivative of $2xe^{-2x}$.
   **b.** Calculate the mean waiting time for a customer in the queue.

**15. a.** Determine the derivative of $\sqrt{4 - x^2}$.
   **b.** Hence, determine the mean value of the probability density function defined by

$$f(x) = \begin{cases} \dfrac{3}{\pi \sqrt{4 - x^2}}, & 0 \leq x \leq \sqrt{3} \\ 0, & \text{elsewhere.} \end{cases}$$

**16.** The probability density function $f(x)$ can be defined by

$$f(x) = \begin{cases} kx - 0.8, & 7 \leq x \leq 11 \\ 0, & \text{elsewhere.} \end{cases}$$

Determine the standard deviation of $X$.

17. The continuous random variable $X$ has a probability density function defined by

$$f(x) = \begin{cases} ax - bx^2, & 0 \le x \le 2 \\ 0, & \text{elsewhere.} \end{cases}$$

Determine the values of the constants $a$ and $b$ if $E(X) = 1$.

18. The time shoppers spend in line at a retail store, in minutes, is represented by the continuous random variable $T$. The probability distribution function of $T$ is

$$f(x) = \begin{cases} -mt + \dfrac{33}{200}, & 0 \le t \le c \\ 0, & \text{elsewhere.} \end{cases}$$

The mean time in line for shoppers is $\dfrac{1072}{3}m$.

Determine the variance of $T$.

19. The continuous random variable $X$ is uniformly distributed and is a probability density function for $m \le x \le n$ where $m$ and $n$ are positive real numbers. The mean of the distribution is $\dfrac{92}{7}$ and $P(X < 8.8) = 0.196$.

Determine $P(X > 11.7)$.

20. The continuous random variable $X$ has a cumulative distribution function defined by

$$F(x) = \begin{cases} 0, & x < a \\ 0.015x^2 - kx + \dfrac{196}{75}, & a \le x \le 22 \\ 1, & x > 22 \end{cases}$$

where $k$ is a positive value and $E(X) = 19.54$.

Determine $\sigma(X)$.

**Fully worked solutions for this chapter are available online.**

# LESSON
## 10.5 Review

### 10.5.1 Summary

### 10.5 Exercise

learn**on**

### Simple familiar

1. Identify which of the following represent continuous random variables.

   a. The number of goals scored at a football match
   b. The heights of students in a Maths B class
   c. Shoe sizes
   d. The number of girls in a five-child family
   e. The time taken to run a distance of 10 kilometres in minutes

2. $X$ is a continuous random variable with a probability function defined by

$$f(x) = \begin{cases} 2\sin(2x), & a \leq x \leq b \\ 0, & \text{elsewhere.} \end{cases}$$

Given that $a = 0$ and $0 \leq b \leq \pi$, determine the value of $b$.

3. A survey was taken to determine the amount of time, $X$ hours, that teenagers spend interacting with digital devices during a 24-hour period. The table of findings and histogram are shown.

**Teenagers' interaction with a digital device**

| Time in hours | Frequency |
| --- | --- |
| $0 \leq x \leq 1$ | 20 |
| $1 < x \leq 2$ | 50 |
| $2 < x \leq 3$ | 60 |
| $3 < x \leq 4$ | 20 |



a. Identify how many teenagers were surveyed.
b. Determine $P(X \leq 3)$.

**4.** The graph of a rectangular or uniform probability density function, $f(x)$, is shown.
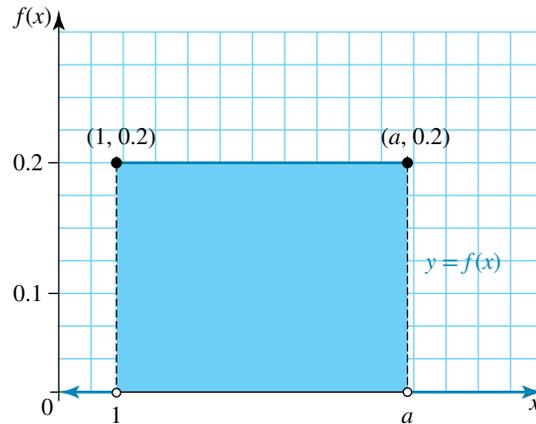


Determine the value of the constant $a$.

**5.** $Y$ is a continuous random variable with a probability density function of

$$f(y) = \begin{cases} 3y^2, & 0 \le y \le 1 \\ 0, & \text{elsewhere.} \end{cases}$$

**a.** Determine the cumulative distribution function for $f(y)$.
**b.** Determine $P(0.2 \le Y < 1)$.

**6.** A histogram was compiled as shown based upon the weights of luggage taken onto a flight from Rockhampton.

**Histogram of weights of airline luggage**
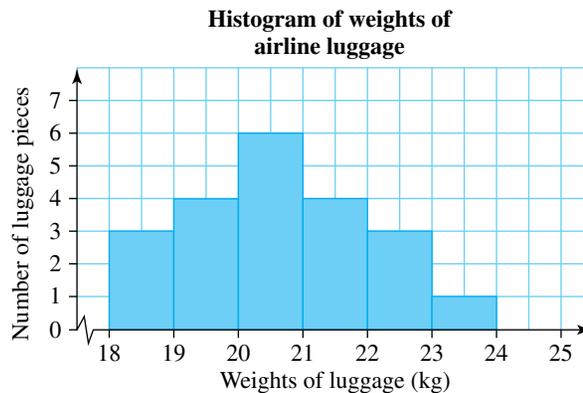


**a.** Explain why it is not possible to determine how many items of luggage had a weight of 19.5 kg.
**b.** Determine the probability that a piece of luggage chosen at random from the flight weighed less than 19 kg.

**7.** A continuous probability density function is defined by

$$f(x) = \begin{cases} 2x, & 0 \le x \le 1 \\ 0, & \text{elsewhere.} \end{cases}$$

Calculate:

**a.** the mean of the distribution

**b.** the variance of the distribution.

**8. a.** Sketch the graph of

$$f(x) = \begin{cases} \dfrac{1}{4}(1-x), & -1 \le x < 1 \\ \dfrac{1}{4}(x-1), & 1 \le x \le 3 \\ 0, & \text{elsewhere.} \end{cases}$$

**b.** Show that $f(x)$ is a probability density function.

**9.** Explain the difference between the probability density function and the cumulative distribution function.

**10.** $X$ is a continuous random variable with a probability density function defined by

$$f(x) = \begin{cases} \dfrac{1}{2}x & 0 \le x \le 2 \\ 0 & \text{elsewhere.} \end{cases}$$

**a.** Determine the cumulative distribution function for $f(x)$.
**b.** Sketch the cumulative distribution function, indicating on the graph where the value of $P(X < 1.5)$ can be found.

**11.** Determine a formula for the probability density function that is drawn here, given that the function is equal to 0 for $x < 0$ and $x > 8$.
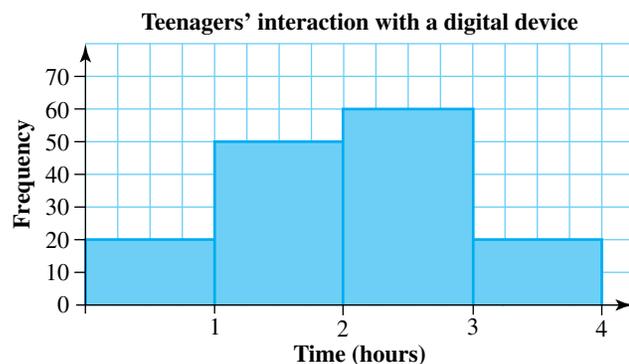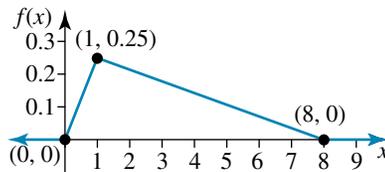


**12.** Let $X$ be a continuous random variable with a probability density function defined by

$$f(x) = \begin{cases} ax^2, & 0 \le x \le 3 \\ 0, & \text{elsewhere} \end{cases}$$

where $a$ is a constant.

**a.** Determine the value of $a$.
**b.** Determine $P(1 \le X \le 2)$.

**Complex familiar**

**13.** The graph of the probability function $f(x) = \dfrac{x}{x^2 + 1}$, $x \ge 0$ for the continuous random variable $X$ is shown.



**a.** Differentiate $\log_e(x^2 + 1)$ and hence determine an antiderivative of $\dfrac{x}{x^2 + 1}$.
**b.** Determine $P(X \le 2)$.
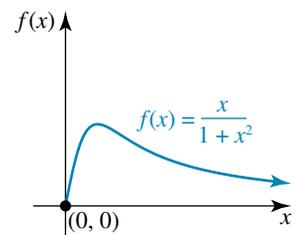
**14.** Given that the following function is a probability density function, determine the value of $a$.

$$f(x) = \begin{cases} ax + 0.5, & -2 \le x < 0 \\ -ax + 1, & 0 \le x \le 1 \\ 0, & \text{elsewhere.} \end{cases}$$

**15.** The continuous random variable $T$ has a probability density function defined by

$$f(t) = \begin{cases} 5e^{-5t}, & t \ge 0 \\ 0, & \text{elsewhere} \end{cases}$$

where $T$ is a continuous random variable.

**a.** Differentiate $xe^{-5x}$ and hence determine an antiderivative of $5xe^{-5x}$.
**b.** Determine $E(T)$.

**16.** The continuous random variable $X$ has a probability density function defined by

$$f(x) = \begin{cases} k \cos(2x), & 0 \le x \le \dfrac{\pi}{4} \\ 0, & \text{elsewhere} \end{cases}$$

where $k$ is a constant.
Determine the standard deviation of $f$.

**Complex unfamiliar**

**17.** For the following probability density function, determine the values of $a$ and $b$ when the sum of $a$ and $b$ is 0.05.

$$f(x) = \begin{cases} \dfrac{1}{5}x^2, & a \le x \le 0 \\ bx, & 0 < x \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

**18.** $X$ is a continuous random variable such that

$$f(x) = \begin{cases} n \sin(3x)\cos(3x), & 0 \le x \le \dfrac{\pi}{12} \\ 0, & \text{elsewhere.} \end{cases}$$

If $f$ is known to be a probability density function, determine the value of the constant, $n$.

19. Patrick has just spread lawn seed on his nature strip. With constant watering and plenty of sunshine, the time it takes for the lawn seed to germinate, $T$ days after the seeding, can be determined by the probability density function

$$f(t) = \begin{cases} ke^{-0.15t}, & t \geq 0 \\ 0, & \text{elsewhere} \end{cases}$$

where $k$ is a constant.

Differentiate $te^{-0.15t}$ and use this to determine the expected period of time for the germination of the lawn seed. Give your answer correct to the nearest day.



20. The graph of a probability density function, $f(x)$, is displayed below. The probability that $X < -4.7$ is 0.051. Determine the probability that $X > -3.6$.
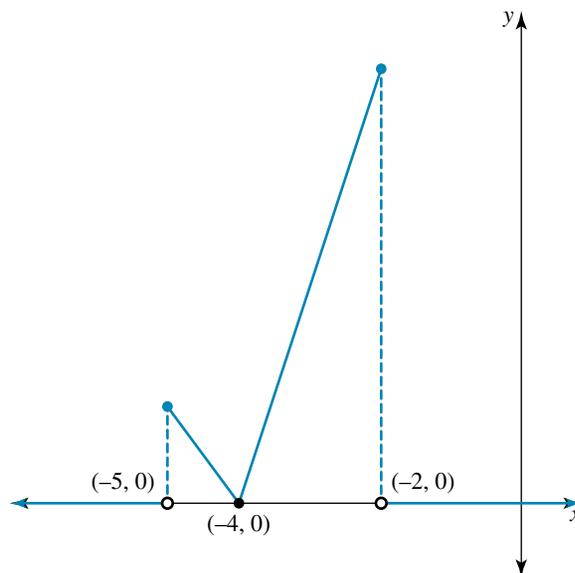
## 10.5 Past QCAA exam questions

**Question 1 (1 mark)**

*Source:* QCAA 2023 Mathematical Methods, Paper 1, Section 1, Q10; © State of Queensland (QCAA) 2024.

The continuous random variable $Y$ has the probability density function

$$f(y) = \begin{cases} 1 + y, & 0 \le y \le \sqrt{3} - 1 \\ 0, & \text{otherwise} \end{cases}$$

Determine $P\left(0 \le y \le \dfrac{1}{2}\right)$.

**A.** $\dfrac{1}{5}$      **B.** $\dfrac{3}{8}$      **C.** $\dfrac{5}{8}$      **D.** $\dfrac{3}{4}$

**Question 2 (1 mark)**

*Source:* QCAA 2023 Mathematical Methods, Paper 1, Section 1, Q7; © State of Queensland (QCAA) 2024.

Determine the mean of the continuous random variable $X$ with the probability density function

$$f(x) = \begin{cases} \dfrac{1}{8}x, & 0 \le x \le 4 \\ 0, & \text{otherwise} \end{cases}$$

**A.** $\dfrac{1}{8}$      **B.** $\dfrac{3}{8}$      **C.** $\dfrac{1}{2}$      **D.** $\dfrac{8}{3}$

**Question 3 (4 marks)**

*Source:* QCAA 2022 Mathematical Methods, Paper 2, Section 2, Q16; © State of Queensland (QCAA) 2024.

The time spent waiting in a queue at a certain supermarket is given by $(X + 11)$ minutes, where $X$ is a random variable with the probability density function

$$f(x) = \begin{cases} \dfrac{a(4 - x^2)}{32}, & -2 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$$

Determine the probability of waiting between 10 and 12 minutes in a queue at this supermarket.

**Question 4 (1 mark)**

*Source:* QCAA 2022 Mathematical Methods, Paper 2, Section 1, Q5; © State of Queensland (QCAA) 2024.

The continuous random variable $X$ has the probability density function

$$f(x) = \begin{cases} \dfrac{\cos(x)}{2}, & \dfrac{-\pi}{2} \le x \le \dfrac{\pi}{2} \\ 0, & \text{otherwise} \end{cases}$$

The standard deviation of $X$ is

**A.** 0.467      **B.** 0.684      **C.** 1.211      **D.** 1.467

**Question 5 (4 marks)**

*Source:* QCAA 2022 Mathematical Methods, Paper 1, Section 2, Q18; © State of Queensland (QCAA) 2024.

A percentile is a measure in statistics showing the value below which a given percentage of observations occur.

The continuous random variable $X$ has the probability density function

$$f(x) = \begin{cases} 2x - 2, & 1 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

Determine the $36$th percentile of $X$.

**Fully worked solutions for this chapter are available online.**

# Answers

## Chapter 10 General continuous random variables

### 10.2 Continuous random variables and the probability density function

#### 10.2 Exercise

**1.** c, d, f, h, k and p are continuous random variables.

**2. a.** $\dfrac{13}{15}$     **b.** $\dfrac{1}{5}$     **c.** $\dfrac{17}{60}$

**3. a. i.** $\dfrac{9}{25}$     **ii.** $\dfrac{4}{25}$

    **b. i.** $\dfrac{37}{50}$     **ii.** $\dfrac{37}{42}$

**4. a.** 100 batteries     **b.** $\dfrac{29}{100}$

    **c.** $\dfrac{41}{50}$     **d.** $\dfrac{3}{100}$
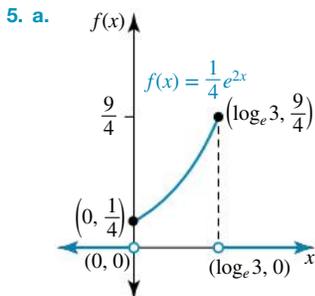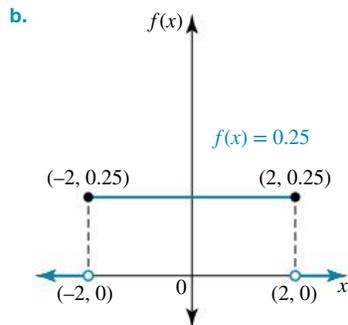
**5. a.**



This is a probability density function.

    **b.**



This is a probability density function.

**6. a.**



This is a probability density function.

**b.**
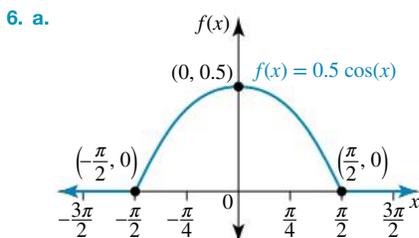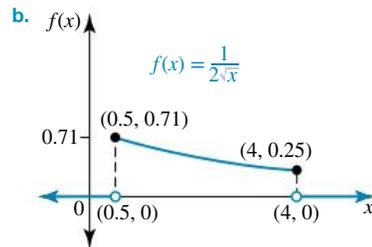


This is not a probability density function.

**7. a.** $\dfrac{19}{27}$     **b.** $\dfrac{26}{27}$

**8. a.**
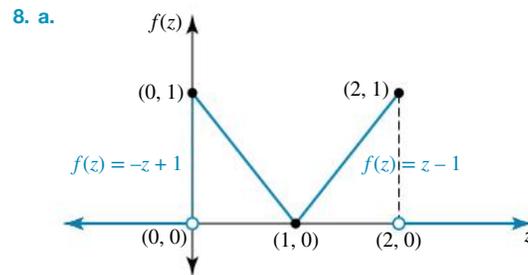


    **b.** $\dfrac{15}{32}$

    **c.** $\dfrac{5}{8}$

**9.** $n = \dfrac{1}{18}$

**10. a.** 1

    **b.**
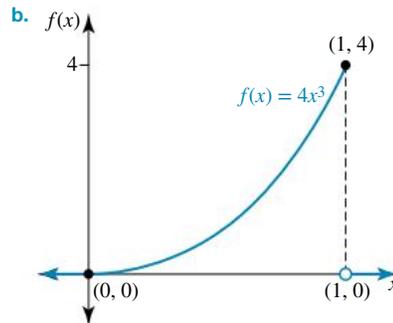


    **c.** $\dfrac{15}{16}$

**11. a.** $k = 6$

    **b.** $P(X < 4.5) = \dfrac{1}{16}$

    **c.** $P(X > 5 \mid X < 5.7) = 0.6540$

**12. a.** $\dfrac{1}{8}$     **b.** 2     **c.** $\dfrac{1}{4}$

**13. a.** 200

    **b. i.** $\dfrac{5}{8}$     **ii.** $\dfrac{31}{100}$

    **c.** $\dfrac{21}{46}$

**14.** $a = \dfrac{1}{11}$

**15.** $\dfrac{5}{7}$

**16.** Sample responses can be found in the worked solutions in the online resources.

**17.** Sample responses can be found in the worked solutions in the online resources.

**18.** $\displaystyle\int \log_e\left(\dfrac{x}{2}\right) dx = x\log_e\left(\dfrac{x}{2}\right) - x$

$a = 2e$

**19.** $y$-intercept $= (0, 0.24)$

**20.** $\dfrac{3}{8}$

## 10.3 Cumulative distribution functions

### 10.3 Exercise

**1. a.** $F(x) = \begin{cases} 0, & x < 1 \\ \dfrac{x}{5} - \dfrac{1}{5}, & 1 \le x \le 6 \\ 1, & x > 6 \end{cases}$

**b. i.** $0.6$

**ii.** $0.46$

**2.** $F(x) = \begin{cases} 0, & x < 0 \\ x^4, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$

**3. a.** $F(x) = \begin{cases} 0, & x < 0 \\ \dfrac{1}{2}(1 - \cos(x)), & 0 \le x \le \pi \\ 1, & x > \pi \end{cases}$

**b.** $\dfrac{1}{2}$  **c.** $\dfrac{\sqrt{2}}{2}$  **d.** $2\sqrt{2} - 2$

**4. a.** $f(x) = \begin{cases} \dfrac{1}{2}(x + 1), & -1 \le x \le 1 \\ 0, & \text{elsewhere} \end{cases}$

**b.** $P(X < 0.4) = 0.49$

**5. a.** $f(x) = \begin{cases} \dfrac{3}{8}x^2, & 0 \le x \le 2 \\ 0, & \text{elsewhere} \end{cases}$

**b.** $P(X < 0.8) = 0.064$

**c.** $P(X > 1.3) = 0.7254$

**d.** $P(X < 1.5 \mid X > 0.4) = 0.4172$

**6. a.** $\dfrac{1}{12}$

**b.** $f(x) = \begin{cases} \dfrac{1}{12}, & 18 \le x \le 30 \\ 0, & \text{elsewhere} \end{cases}$

**c.** $F(x) = \begin{cases} 0, & x < 18 \\ \dfrac{x}{12} - \dfrac{3}{2}, & 18 \le x \le 30 \\ 1, & x > 30 \end{cases}$

**d.** $\dfrac{5}{12}$

**e.** $\dfrac{1}{2}$

**7. a.** $F(x) = \begin{cases} 0, & x < 0 \\ \dfrac{x^3}{8}, & 0 \le x \le 2 \\ 1, & x > 2 \end{cases}$

**b.** $0.784$

**c.** $1.817$

**8. a.** $F(x) = \begin{cases} 0, & x < -1 \\ \dfrac{3}{4}\left(x - \dfrac{1}{3}x^3 + \dfrac{2}{3}\right), & -1 \le x \le 1 \\ 1, & x > 1 \end{cases}$

**b.** $P(-0.6 < X < 0.3) = 0.614\,25$

**c.** $k = 0.497\,22$

**9. a.** $m = \dfrac{1}{2}$

**b.** $P(X < 0.8 \mid X > -0.6) = 0.875$

**10. a.** $n = \dfrac{1}{9}$

**b.** $F(y) = \begin{cases} 0, & y < 0 \\ \dfrac{1}{27}y^3, & 0 \le y \le 3 \\ 1, & y > 3 \end{cases}$

**c.** $P(Y < 2.2 \mid Y > 1.4) = 0.3259$

**11. a.** $k = 6$

**b.** $F(y) = \begin{cases} 0, & y < 0 \\ 3y^2 - 2y^3, & 0 \le y \le 1 \\ 1, & y > 1 \end{cases}$

**c.** $P(Y > 0.7) = 0.216$

**12. a.** $c = 6$

**b.** $f(x) = \begin{cases} 0.5x - 2, & 4 \le x \le 6 \\ 0, & \text{elsewhere} \end{cases}$

**c.** $m = 4.71$

**13. a.** $F(x) = \begin{cases} 0, & x < -\dfrac{1}{3} \\ 3x^2 + 2x + \dfrac{1}{3}, & -\dfrac{1}{3} \le x \le 0 \\ -\dfrac{8}{3}x^3 + 2x + \dfrac{1}{3}, & 0 < x \le \dfrac{1}{2} \\ 1, & x > \dfrac{1}{2} \end{cases}$

**b. i.** $P(X < 0.3) = 0.8613$

**ii.** $P(-0.2 < X < 0.1) = 0.576$

**14. a.** $F(y) = \begin{cases} 0, & y < 0 \\ \dfrac{1}{3}y^2, & 0 \le y < 1 \\ y - \dfrac{1}{6}y^2 - \dfrac{1}{2}, & 1 \le y \le 3 \\ 1, & y > 3 \end{cases}$

**b. i.** $P(Y > 2.5) = 0.041\,67$

**ii.** $P(0.4 < Y < 1.7) = 0.665$

**15. a.**



Graph of $f(x)$ showing a triangle with peak at $(0, 2k)$, $f(x) = k(2+x)$ on the left, $f(x) = k(2-x)$ on the right, with base from $(-2, 0)$ to $(2, 0)$.
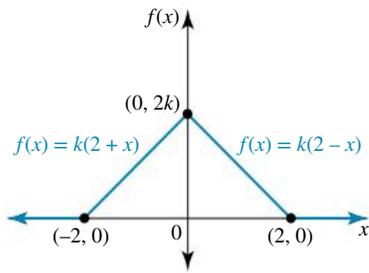
**b.** $\dfrac{1}{4}$

**c.** $F(x) = \begin{cases} 0, & x < 2 \\ \dfrac{x}{2} + \dfrac{x^2}{8} + \dfrac{1}{2}, & -2 \le x < 0 \\ \dfrac{1}{2} + \dfrac{x}{2} - \dfrac{x^2}{8}, & 0 \le x \le 2 \\ 1 & x > 2 \end{cases}$

**d.** $\dfrac{3}{4}$

**e.** $\dfrac{6}{7}$

**16.** $F(x) = \begin{cases} 0, & x < 5 \\ 0.05x^2 - 0.5x + 1.25, & 5 \le x \le 7 \\ 0.2x - 1.2, & 7 < x \le 10 \\ -0.05x^2 + 1.2x - 6.2, & 10 < x \le 12 \\ 1, & x > 12 \end{cases}$

**17.** $F(x) = \begin{cases} 0, & x < -1 \\ -\dfrac{11}{15}x^2 + \dfrac{11}{15}, & -1 \le x \le 0 \\ \dfrac{1}{30}x^3 + \dfrac{11}{15}, & 0 < x \le 2 \\ 1, & x > 2 \end{cases}$

**18.** $F(x) = \begin{cases} 0, & x < 8 \\ \dfrac{1}{8}x^2 - 2x + 8, & 8 \le x \le 10 \\ -\dfrac{1}{8}x^2 + 3x - 17, & 10 < x \le 12 \\ 1, & x > 12 \end{cases}$

**19.** $F(x) = \begin{cases} 0, & x < 2 \\ \dfrac{1}{75}x^3 - \dfrac{4}{25}x + \dfrac{16}{75}, & 2 \le x \le 4 \\ \dfrac{12}{25}x - \dfrac{112}{75}, & 4 < x \le \dfrac{187}{36} \\ 1, & x > \dfrac{187}{36} \end{cases}$

**20.** $F(t) = \begin{cases} 0, & t < 1 \\ \dfrac{8}{81}t^3 - \dfrac{8}{27}t + \dfrac{16}{81}, & 1 \le t \le 2.5 \\ 1, & t > 2.5 \end{cases}$

## 10.4 Measures of centre and spread

### 10.4 Exercise

**1. a.** $a = \dfrac{5}{4}$      **b.** $E(Z) = \dfrac{9}{8}$

**2. a.** $a = 1$      **b.** $E(Y) = \dfrac{2}{3}$

**3.** $E(Z) = \dfrac{8}{3} = 2\dfrac{2}{3}$; $Var(Z) = \dfrac{1}{18}$; $SD(Z) = \dfrac{\sqrt{2}}{6}$

**4. a.** Sample responses can be found in the worked solutions in the online resources.

     **b.** $E(X) = \dfrac{1}{3}$

**5.** $E(Y) = 1.5601$

**6. a.** $h = \dfrac{1}{4}$      **b.** $E(X) = 2$      **c.** $Var(X) = 2$

**7. a.** $a = 0.4809$

     **b.** $E(Z) = 3.3663$

     **c.** $Var(Z) = 3.8164$; $SD(Z) = 1.9536$

**8. a.**



Graph of $f(x)$ showing a horizontal line $f(x) = k$ between points $(a, k)$ and $(b, k)$, with dashed lines down to $(a, 0)$ and $(b, 0)$.
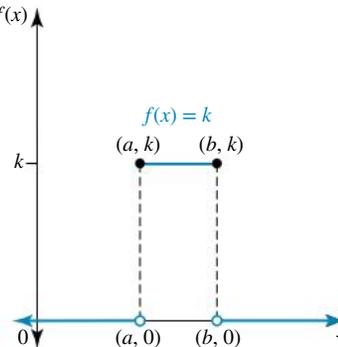
     **b.** Sample responses can be found in the worked solutions in the online resources.

     **c.** $E(X) = \dfrac{b + a}{2}$

**9. a.** Sample responses can be found in the worked solutions in the online resources.

     **b.** $E(X) = 1.0708$

     **c. i.** $Var(X) = 0.5725$

       **ii.** $SD(X) = 0.7566$

**10. a.** Sample responses can be found in the worked solutions in the online resources.

     **b.** $E(Z) = 1.2164$; $Var(Z) = 0.0204$

**11. a.** $\displaystyle\int_2^{7.9344} f(y)\, dy = 1$

     **b.** $5.7278$

     **c.** $Var(Y) = 2.1579$; $SD(Y) = 1.4690$

**12. a.** $a = 2.3104$

     **b. i.** $E(Z) = 1.7863$

       **ii.** $Var(Z) = 0.1178$

       **iii.** $SD(Z) = 0.3432$

**13. a.** $\displaystyle\int 3xe^{-3x}\, dx = -\dfrac{1}{3}e^{-3x} - xe^{-3x} + c$

     **b.** $\dfrac{1}{3}$

**14. a.** $\int 2xe^{-2x}dx = -\dfrac{1}{2}e^{-2x} - xe^{-2x} + c$

**b.** 0.5 minutes

**15. a.** $\dfrac{dy}{dx} = -\dfrac{x}{\sqrt{4-x^2}}$

**b.** $E(X) = \dfrac{3}{\pi}$

**16.** $\sigma(X) = 0.9727$

**17.** $a = \dfrac{3}{2}; b = \dfrac{3}{4}$

**18.** $\mathrm{Var}(T) = 5.1513$

**19.** $P(X > 11.7) = 0.601$

**20.** $\sigma(X) = 1.6457$

## 10.5 Review

## 10.5 Exercise

**1.** **b** and **e** are continuous random variables.

**2.** $\dfrac{\pi}{4}$

**3. a.** 150      **b.** $\dfrac{13}{15} = 0.8\dot{6}$

**4.** 6

**5. a.** $F(y) = \begin{cases} 0, & x < 0 \\ y^3, & 0 \le y \le 1 \\ 1, & y > 1 \end{cases}$

**b.** 0.992

**6. a.** The value of 19.5 kg does not lie at the end of an interval. As the weight is a continuous variable, the bags in the interval $19 \le W \le 20$ can have an infinite number of values.

**b.** $\dfrac{1}{7} \approx 0.1429$

**7. a.** $E(X) = \dfrac{2}{3}$

**b.** $\mathrm{Var}(X) = \dfrac{1}{18} = 0.0\dot{5}$

**8. a.**
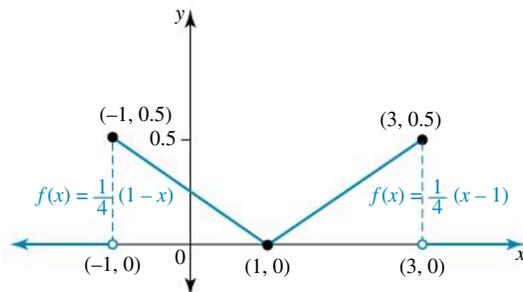
**b.** Sample responses can be found in the worked solutions in the online resources.

**9.** Sample responses can be found in the worked solutions in the online resources.

**10. a.** $F(x) = \begin{cases} 0, & x < 0 \\ \dfrac{x^2}{4}, & 0 \le x \le 2 \\ 1, & x > 2 \end{cases}$

**b.**
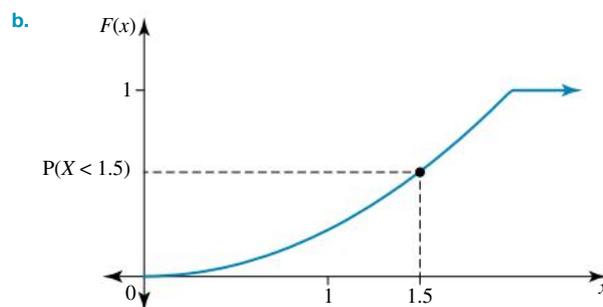
**11.** $f(x) = \begin{cases} \dfrac{1}{4}x, & 0 \le x \le 1 \\ \dfrac{2}{7} - \dfrac{x}{28}, & 1 < x \le 8 \\ 0, & \text{elsewhere} \end{cases}$

**12. a.** $\dfrac{1}{9}$      **b.** $\dfrac{7}{27}$

**13. a.** $\int \dfrac{x}{x^2 + 1}dx = \dfrac{1}{2}\log_e(x^2 + 1) + c$

**b.** $\dfrac{1}{2}\log_e(5)$

**14.** $\dfrac{2}{5}$

**15. a.** $\int 5xe^{-5x}dx = -\tfrac{1}{5}e^{-5x} - xe^{-5x} + c$

**b.** $\dfrac{1}{5}$

**16.** $\sigma = 0.1881$

**17.** $a = -1.5, b = 1.55$

$f(x) = \begin{cases} \dfrac{1}{5}x^2, & -1.5 \le x \le 0 \\ 1.55x, & 0 < x \le 1 \\ 0, & \text{elsewhere} \end{cases}$

**18.** 12

**19.** 7 days

**20.** $P(X > -3.6) = 0.864$

## 10.5 Past QCAA exam questions

**1.** C

**2.** D

**3.** 0.6875

**4.** B

**5.** $m = 1.6$

# 11 Normal distributions

## LESSON SEQUENCE

**Fully worked solutions of this chapter are available online.**

## EXAM PREPARATION

Access exam-style questions in every lesson, available online and access past QCAA exam questions in every review.

**on** Resources

| | |
|---|---|
| **Solutions** | Solutions — Chapter 11 |
| **Exam questions** | Exam question booklet — Chapter 11 |
| **Digital documents** | Learning matrix — Chapter 11 |
| | Chapter summary — Chapter 11 |

# LESSON
## 11.1 Overview

### 11.1.1 Introduction

The pattern now referred to as the normal distribution was first referred to by Galileo Galilei in 1632. Writing on the nature of scientific errors in observing astronomical motion, he noted that small errors in astronomical measurements occurred more frequently than larger errors and that the measurements were distributed symmetrically about the true value. In the intervening centuries, it has been found that the bell-shaped curve of the normal distribution turns up everywhere that the measurement of natural phenomena is involved. The heights of human beings in a population, for example, follow a normal distribution. For heights of human beings, it should be noted that the exact geometry of the bell curve varies between population groups, as different genetic and socioeconomic factors result in different values for the average and standard deviation.

Although the normal distribution for human height has no theoretical limits as such, the probability of an adult human being 4 metres tall is so incredibly small that it could be regarded as zero. So far, the tallest height recorded for an adult human is 272 cm and the shortest a mere 55 cm.

### 11.1.2 Syllabus links

| Lesson | Lesson title | Syllabus links |
|--------|-------------|----------------|
| 11.2 | **The normal distribution** | ○ Identify contexts, e.g. naturally occurring variations, that are suitable for modelling by normal variables.<br>○ Recognise features of the graph of the probability density function of the normal distribution with mean $\mu$ and standard deviation $\sigma$ and the use of the standard normal distribution. |
| 11.3 | **Standardised normal variables** | ○ Recognise and use the link between the normal distribution and the notation $X \sim N\left(\mu, \sigma^2\right)$.<br>○ Understand standardised normal variables ($z$-values, $z$-scores) and use these to compare samples. |
| 11.4 | **The inverse normal distribution** | ○ Calculate probabilities and quantiles associated with a given normal distribution, using technology. |
| 11.5 | **Modelling and solving with normal distributions** | ○ Model and solve problems that involve normal distributions, with and without technology (distribution tables are not required). |

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# LESSON
## 11.2 The normal distribution

### 11.2.1 Introduction

The probability distribution of many naturally occurring continuous random variables such as heights and weights in large populations has a distinctive bell shape, with the most frequently occurring values clustered closely around the mean. This type of distribution is most commonly referred to as the normal distribution, although you may encounter the terms 'bell curve' or 'Gaussian curve' being used to describe it.

Apart from continuous random variables such as height and weight, the **normal distribution curve** can be reliably used to model a wide variety of frequency distributions. Examples include examination results, the intelligence quotients of children in a particular age group, the usable lifetimes of light bulbs and even the ages of stars.

The degree to which a normal curve spreads out depends upon the values of the mean and the standard deviation of the data that it models. The diagram shows three different normal distributions.

Graph 1 has a mean of −1 and a standard deviation of 0.5.

Graph 2 has a mean of 0 and a standard deviation of 1.

Graph 3 has a mean of 3 and a standard deviation of 0.25.

As you can see, the larger the value of the standard deviation is, the more spread out the bell curve appears. The central peak is always positioned at $x = \mu$.

In general, if $X$ is a continuous random variable that follows a normal distribution with a mean of $\mu$ and a variance of $\sigma^2$, it is written as $X \sim N(\mu, \sigma^2)$ and its probability density function is given by the equation

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}.$$
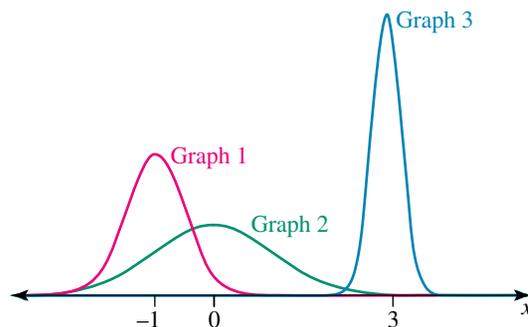
### 11.2.2 Properties of the normal distribution

int-6438

The normal distribution has five important characteristics.

1. Normal distributions are defined by two parameters — the mean, $\mu$, and the standard deviation, $\sigma$.
2. A normal distribution is symmetrical about the mean.
3. The mean, median and mode are equal.
4. The area under the curve is equal to 1. That is, $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1$.
5. The majority of the values cluster around the centre of the curve, with fewer values at the tails of the curve.

As the mean and standard deviation can vary, and the area under the graph must be constant and equal to 1, changing the mean and the standard deviation transforms the normal curve. Changing the standard deviation dilates the curve by a factor of $\dfrac{1}{\sigma}$ parallel to the $y$-axis and by a factor of $\sigma$ parallel to the $x$-axis. Changing the mean translates the curve horizontally along the $x$-axis.
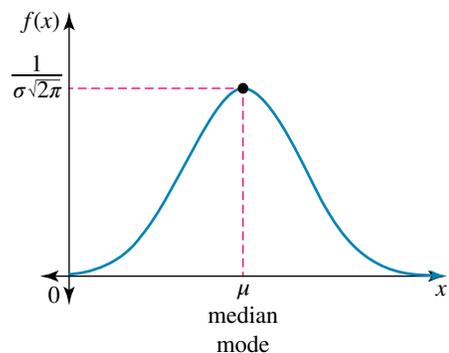


## 11.2.3 Important intervals and their properties

int-6439

Often we are required to determine the proportion of a population for a given interval. Using the property that the symmetry of the normal distribution is about the mean, we are able to establish the following facts.

- Approximately 68% of the population will fall within 1 standard deviation of the mean:

$$P(\mu - \sigma < X < \mu + \sigma) \approx 0.68.$$

- Approximately 95% of the population will fall within 2 standard deviations of the mean:

$$P(\mu - 2\sigma < X < \mu + 2\sigma) \approx 0.95.$$

We say that a randomly chosen member of the population will most probably be or is highly likely to be within 2 standard deviations of the mean.

- Approximately 99.7% of the population will fall within 3 standard deviations of the mean:

$$P(\mu - 3\sigma < X < \mu + 3\sigma) \approx 0.997.$$

We say that a randomly chosen member of the population will almost certainly be within 3 standard deviations of the mean.

This is shown on the following graphs.
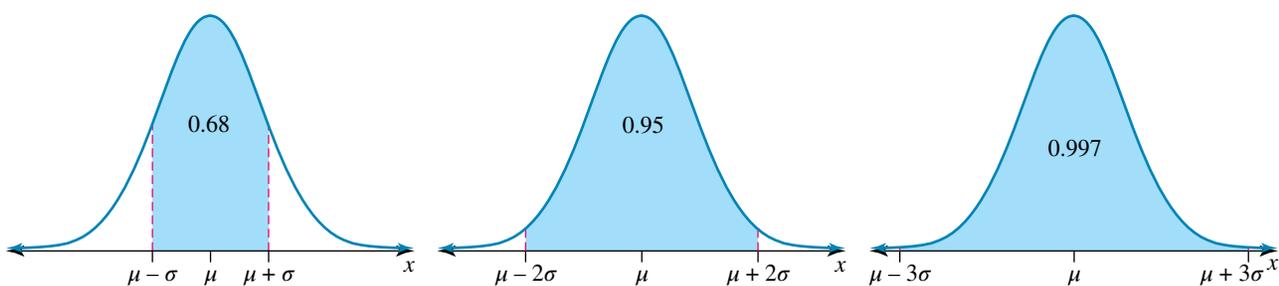


These facts are collectively known as the **empirical rule** (or the 68–95–99.7% rule).

A more comprehensive breakdown of the proportion of the population for each standard deviation is shown on the graph.

## The empirical rule

- **Approximately 68% of the population will fall within 1 standard deviation of the mean:**

$$P(\mu - \sigma < X < \mu + \sigma) \approx 0.68.$$

- **Approximately 95% of the population will fall within 2 standard deviations of the mean:**

$$P(\mu - 2\sigma < X < \mu + 2\sigma) \approx 0.95.$$

- **Approximately 99.7% of the population will fall within 3 standard deviations of the mean:**

$$P(\mu - 3\sigma < X < \mu + 3\sigma) \approx 0.997.$$



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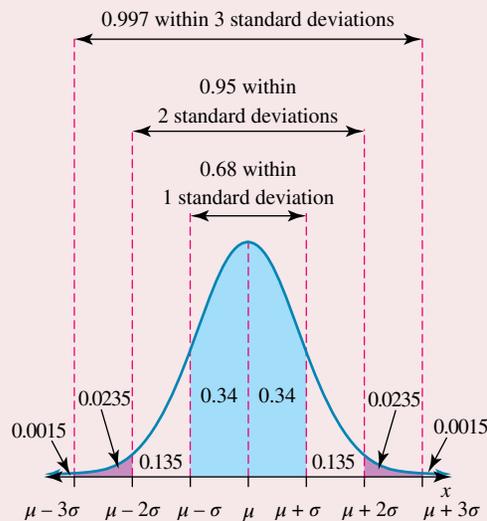
### WORKED EXAMPLE 1 Using the properties of a normal distribution

The heights of the women in a particular town are normally distributed with a mean of 165 cm and a standard deviation of 9 cm.
a. Determine the approximate probability that a woman chosen at random has a height that is between 156 cm and 174 cm.
b. Determine the approximate probability that a woman chosen at random is taller than 174 cm.
c. Determine the approximate percentage of the women in this particular town who are shorter than 147 cm.

| THINK | WRITE |
|---|---|
| **a.** Determine how many standard deviations from the mean the 156–174 cm range is. | **a.** Let $X$ be the height of women in this particular town.<br>$\mu + \sigma = 165 + 9$<br>$\qquad = 174$<br>$\mu - \sigma = 165 - 9$<br>$\qquad = 156$<br>Since the range is 1 standard deviation from the mean, $P(156 \leq X \leq 174) \approx 0.68$. |

**b.** Use the fact that $P(156 \leq X \leq 174) \approx 0.68$ to calculate the required probability. Sketch a graph to help.

**b.**
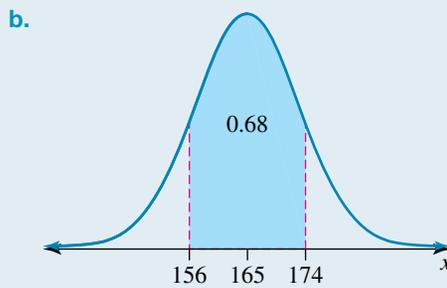


Since $P(156 \leq X \leq 174) \approx 0.68$,
$$P(X < 156) \cup P(X > 174) \approx 1 - 0.68$$
$$= 0.32$$
Because of symmetry,
$$P(X < 156) = P(X > 174)$$
$$= \frac{0.32}{2}$$
$$= 0.16$$
Thus, $P(X > 174) \approx 0.16$.

**c. 1.** Determine how many standard deviations 147 cm is from the mean.

**c.** $\mu - \sigma = 165 - 9$
$$= 156$$
$$\mu - 2\sigma = 165 - 2 \times 9$$
$$= 147$$
147 cm is 2 standard deviations from the mean. The corresponding upper value is $183(165 + 2 \times 9)$.
$$P(147 \leq X \leq 183) \approx 0.95$$

**2.** Using symmetry, calculate $P(X < 147)$.
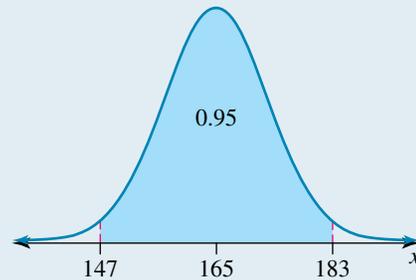


Thus, $P(X < 147) \cup P(X < 183) \approx 1 - 0.95$
$$= 0.05$$
and by symmetry,
$$P(X < 147) = P(X > 183) \approx \frac{0.05}{2}$$
$$= 0.025$$
Thus, approximately 2.5%, of the population of women in this particular town are shorter than 147 cm.

The probability density function for a normal distribution is given by

$$f(x) = \frac{2}{\sqrt{2\pi}}e^{-\frac{1}{2}(2(x-1))^2}, \ x \in \mathbb{R}.$$

**a.** State the mean and standard deviation of the distribution, given that the probability density function of a normal distribution is given by $f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$.

**b.** Sketch the graph of the function.

**THINK**

**a.** Use $f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ to determine $\mu$ and $\sigma$.

**b.** Sketch the graph with a mean of 1 and a standard deviation of 0.5. The $x$-axis needs to be scaled with markings at $\mu$, $\mu \pm \sigma$, $\mu \pm 2\sigma$ and $\mu \pm 3\sigma$. The peak of the graph must also be labelled with its coordinates.

**WRITE**

**a.** $f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

$= \frac{2}{\sqrt{2\pi}}e^{-\frac{1}{2}(2(x-1))^2}$

$\frac{1}{\sigma} = 2$, so $\sigma = \frac{1}{2}$ and $\mu = 1$.

**b.**



# Exercise 11.2 The normal distribution

learn **on**

| 11.2 Exercise | 11.2 Exam questions **on** |
|---|---|

| Simple familiar | Complex familiar | Complex unfamiliar |
|---|---|---|
| 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 | 12, 13, 14, 15, 16 | 17 |

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### Simple familiar

**1.** Identify which of the following contexts are suitable for modelling normal random variables:
  **a.** Heights of women in the Australian population
  **b.** IQ scores of children aged 10
  **c.** Results in a Year 9 Maths class
  **d.** Usable lifetime of light bulbs

2. A continuous random variable, $X$, is known to be normally distributed with a mean of 15 and a standard deviation of 5. Determine the range between which approximately:

   a. 68% of the values lie
   b. 95% of the values lie
   c. 99.7% of the values lie.

3. **WE1** The results of a Mathematical Methods test are normally distributed with a mean of 72 and a standard deviation of 8.

   a. Determine the approximate probability that a student who sat the test has a score which is greater than 88.
   b. Determine the approximate proportion of the students who sat the test and had a score that was less than 48.
   c. Determine the approximate percentage of the students who sat the test and scored less than 80.

4. The length of pregnancy for a human is normally distributed with a mean of 275 days and a standard deviation of 14 days. A mother gave birth in less than 233 days. Calculate the approximate probability of this happening for the general population.

5. Scores on a commonly used IQ test are known to be normally distributed with a mean of 120 and a standard deviation of 20.

   a. Determine:

      i. $\mu \pm \sigma$        ii. $\mu \pm 2\sigma$        iii. $\mu \pm 3\sigma$

   b. Determine:

      i. $P(X < 80)$        ii. $P(X > 180)$

6. The results of a Year 12 Biology examination are known to be normally distributed with a mean of 70 and a standard deviation of 6. Determine the approximate percentage of students sitting for this examination who can be expected to achieve a score that is greater than 88.

7. **WE2** The probability density function of a normal distribution is given by

$$f(x) = \frac{1}{3\sqrt{2\pi}} \, e^{-\frac{1}{2}\left(\frac{x-2}{3}\right)^2}.$$

   a. State the mean and the standard deviation of the distribution, given that the probability density function of a normal distribution is given by $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$.
   b. Sketch the graph of the probability function.

8. A normal distribution has a probability density function of

$$f(x) = \frac{1}{\sqrt{2\pi}} \, e^{-\frac{1}{2}(x-3)^2}.$$

   a. State $\mu$ and $\sigma$, given that the probability density function of a normal distribution is given by $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$.
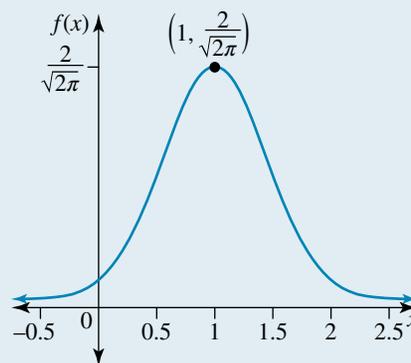   b. Sketch the graph of the probability function.

9. Consider the normal probability density function

$$f(x) = \frac{1}{4\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x+2}{4}\right)^2},\ x \in \mathbb{R}.$$

Identify $\mu$, given that the probability density function of a normal distribution is given by $f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$.

10. A normal probability density function is defined by

$$f(x) = \frac{10}{3\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{10(x-1)}{3}\right)^2},\ x \in \mathbb{R}.$$

   a. Determine the values of $\mu$ and $\sigma$, given that the probability density function of a normal distribution is given by $f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$.

   b. State what effect the mean and standard deviation have on the graph of the normal distribution.

   c. Sketch the graph of the function, $f$.

11. A normal probability density function is given by

$$f(x) = \frac{1}{10\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x+4}{10}\right)^2},\ x \in \mathbb{R}.$$

   a. Determine the values of $\mu$ and $\sigma$, given that the probability density function of a normal distribution is given by $f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$.

   b. State what effect the mean and standard deviation have on the graph of the normal distribution.

   c. Determine:

   i.  Var($X$)
   ii. E$\left(X^2\right)$

   d. Using technology, verify that this is a probability density function.

### Complex familiar

12. $f(x) = \frac{5}{2\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{5(x-2)}{2}\right)^2},\ x \in \mathbb{R}$ defines a normal probability density function.

   a. Determine the values of $\mu$ and $\sigma$, given that the probability density function of a normal distribution is given by $f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$.

   b. Calculate E$\left(X^2\right)$.

   c. Determine:

   i.  E($5X$)
   ii. E$\left(5X^2\right)$

**13.** The number of pears harvested from each tree in a large orchard is normally distributed with a mean of 230 and a standard deviation of 25. Determine the approximate probability that the number of pears harvested from a randomly selected tree is:

**a.** less than 280
**b.** between 180 and 280
**c.** is greater than 180, given that less than 280 pears were harvested.

**14.** The annual rainfall in a particular area of Australia, $X$ mm, is known to be normally distributed with a mean of 305 mm and a standard deviation of 50 mm.

**a.** Calculate the approximate value of $P(205 < X < 355)$.
**b.** Determine $k$ such that $P(X < k) \approx 0.025$.
**c.** Determine $h$ such that $P(X < h) \approx 0.0015$.

**15.** A normally distributed probability density function is given by

$$f(x) = \frac{5}{\sqrt{2\pi}} e^{-\frac{1}{2}(5(x-1))^2}, x \in \mathbb{R}.$$

**a.** Calculate $\text{Var}(X)$, giving your answer correct to 2 decimal places.
**b.** Calculate $E(X^2)$, giving your answer correct to 2 decimal places.
**c.** Determine:

   **i.** $E(2X + 3)$
   **ii.** $E((X + 1)(2X - 3))$

**16.** A continuous random variable, $X$, is normally distributed with a mean of 72.5 and a standard deviation of 8.4. Determine the approximate values for:

**a.** $P(64.1 < X < 89.3)$
**b.** $P(X < 55.7)$
**c.** $P(X > 47.3 \mid X < 55.7)$
**d.** $m$ such that $P(X > m) \approx 0.16$.

**17.** The intelligence quotient (IQ) of a large population is normally distributed with a mean of 104 and a standard deviation of 15. Eight individuals are chosen at random from the population. Determine the probability that no more than three of the individuals have an IQ of at least 124.

**Fully worked solutions for this chapter are available online.**

# LESSON
## 11.3 Standardised normal variables

### 11.3.1 The standard normal distribution

int-6441

Suppose we are comparing the results of two students who took similar Maths tests. Michelle obtained 92 on one test, for which the results were known to be normally distributed with a mean of 80 and a standard deviation of 6. Samara obtained 88 on her test, for which the results were known to be normally distributed with a mean of 78 and a standard deviation of 10. Which student was more successful?

This question is very difficult to answer unless we have some common ground for a comparison. This can be achieved by using a transformed or standardised form of the normal distribution called the **standard normal distribution**. The variable in a standard normal distribution is always denoted by $Z$, so that it is immediately understood that we are dealing with the standard normal distribution. The standard normal distribution always has a mean of 0 and a standard deviation of 1, so $z$ in the following formula indicates how many standard deviations the corresponding $X$-value is from the mean. To calculate the value of $z$, we determine the difference between the $x$-value and the mean, $x - \mu$. To determine how many standard deviations this equals, we divide by the standard deviation, $\sigma$. The result is known as the $z$-value or $z$-score.

> **The $z$-value formula**
> $$z = \frac{x - \mu}{\sigma}$$

Therefore, if $z = \dfrac{x - \mu}{\sigma}$, $\mu = 0$ and $\sigma = 1$, the probability density function is given by

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}, z \in \mathbb{R}$$

Remember that $\mu \pm 3\sigma$ encompasses approximately 99.7% of the data, so for the standard normal curve, these figures are $0 \pm 3 \times 1 = 0 \pm 3$. Therefore, approximately 99.7% of the data lies between $-3$ and 3.

For the standard normal distribution, we say $Z \sim N(0, 1)$.

For Michelle: $X \sim N\left(80, 6^2\right)$

$$\begin{aligned}
z &= \frac{x - \mu}{\sigma} \\
&= \frac{92 - 80}{6} \\
&= \frac{12}{6} \\
&= 2
\end{aligned}$$
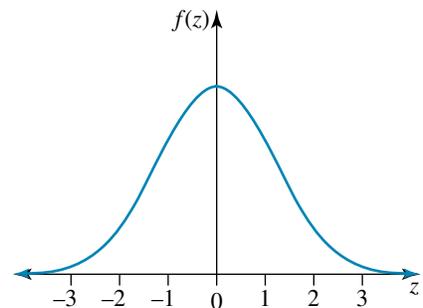
For Samara: $X \sim N\left(78, 10^2\right)$

$$z = \frac{x - \mu}{\sigma}$$
$$= \frac{88 - 78}{10}$$
$$= \frac{10}{10}$$
$$= 1$$

Michelle's mark lies within 2 standard deviations of the mean, so it lies in the top 2.5%, whereas Samara's mark is 1 standard deviation from the mean, so it is in the top 16%. Hence, Michelle performed better than Samara.

Obviously, not all data values will lie exactly 1, 2 or 3 standard deviations from the mean. In these cases, once the $z$-value is obtained, the corresponding probability may be found using the cumulative normal distribution (CND) function on your graphics calculator.

### WORKED EXAMPLE 3 Calculating probabilities using the standard normal distribution

**a.** **Calculate the values of the following probabilities, correct to 4 decimal places.**
   **i.** $P(Z < 2.5)$
   **ii.** $P(-1.25 \le Z \le 1.25)$
**b.** $X$ **is a normally distributed random variable such that** $X \sim N\left(25, 3^2\right)$**.**
   **i.** **Calculate** $P(X > 27)$ **correct to 4 decimal places.**
   **ii.** **Determine the corresponding standard normal variable,** $Z$**.**

**THINK**

**a. i. 1.** Sketch a graph to help understand the problem.

**2.** Use your graphics calculator to calculate the probability.
The upper limit is 2.5 and the lower limit is $-\infty$. The mean is 0 and the standard deviation is 1.
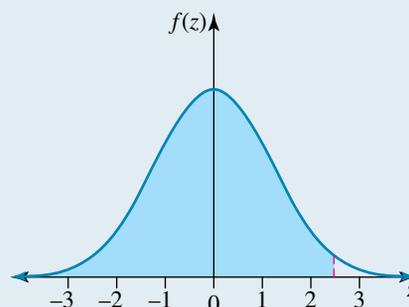
**ii. 1.** Sketch a graph to help understand the problem.
It is important to remember that $P(-1.25 \le Z \le 1.25) = P(-1.25 < Z < 1.25)$ since $P(Z = z) = 0$ for all values of $z$. This is because we cannot measure a continuous random variable exactly.
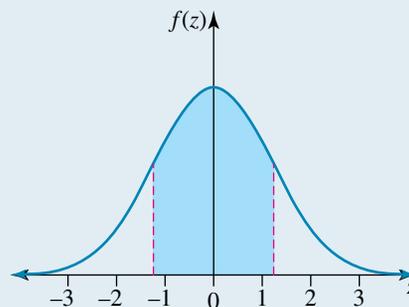
**2.** Use your graphics calculator to calculate the probability.
The upper limit is 1.25 and the lower limit is $-1.25$.

**WRITE**

**a. i.**



$P(z < 2.5) = 0.9938$

**ii.**



$P(-1.25 < Z < 1.25) = 0.7887$

**b. i. 1.** Sketch a graph to help understand the problem.

**2.** Use your graphics calculator to calculate the probability.
The upper limit is ∞ and the lower limit is 27. The mean is 25 and the standard deviation is 3.

**b. i.**
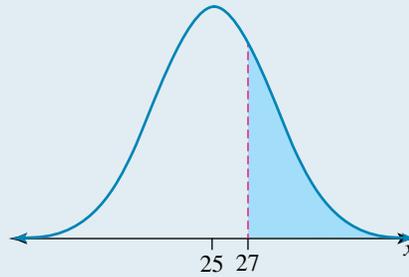


$$P(X > 27) = 0.2525$$

**ii. 1.** Write the rule to standardise $X$.

**2.** Substitute the mean and standard deviation.

**ii.** $z = \dfrac{x - \mu}{\sigma}$

$$z = \dfrac{27 - 25}{3}$$

$$= \dfrac{2}{3}$$

---

**TI | THINK**

**a. 1.** On a Calculator page, select:
MENU
6: Statistics
5: Distributions
2: Normal Cdf…

**WRITE**



**2.** Complete the entry lines as:
Lower bound: −9e999 Upper bound: 2.5
$\mu$: 0
$\sigma$: 1
Press the OK button.



**3.** The answer appears on the screen.



normCdf(-9.ε999,2.5,0,1)          0.99379
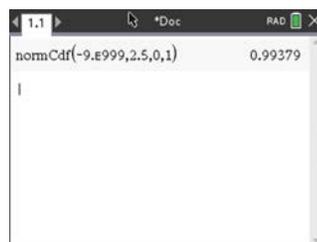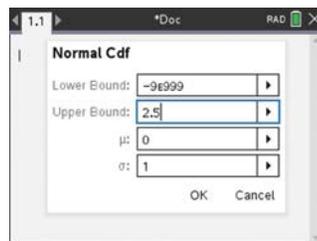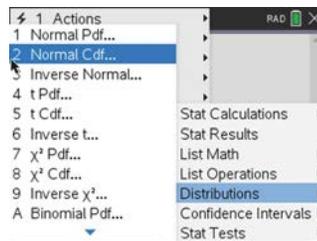
**CASIO | THINK**

**a. 1.** On a Statistic screen, select:
DIST
NORM
Ncd

**WRITE**



**2.** Complete the entry lines as:
Variable Lower: $-1 \times 10^{99}$
Upper: 2.5
$\sigma$: 1
$\mu$: 0
Press the EXE button.



**3.** The answer appears on the screen.

# Exercise 11.3 Standardised normal variables
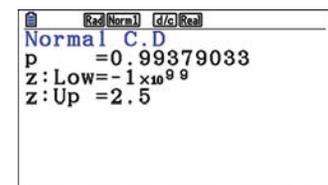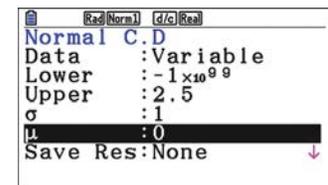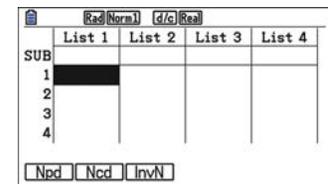
| **11.3 Exercise** | **11.3 Exam questions** on |
|---|---|

| Simple familiar | Complex familiar | Complex unfamiliar |
|---|---|---|
| 1, 2, 3, 4, 5, 6 | 7, 8, 9, 10, 11, 12, 13 | 14, 15, 16 |

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## Simple familiar

1. **WE3** Calculate the values of the following probabilities correct to 4 decimal places.

   **a. i.** $P(Z < 1.2)$
   **ii.** $P(-2.1 < Z < 0.8)$
   **b.** $X$ is a normally distributed random variable such that $X \sim N\left(45, 6^2\right)$.

   **i.** Calculate $P(X > 37)$ correct to 4 decimal places.
   **ii.** Determine the corresponding standard normal variable, $Z$.

2. If $Z \sim N(0, 1)$, determine:

   **a.** $P(Z \leq 2)$
   **b.** $P(Z \leq -2)$
   **c.** $P(-2 < Z \leq 2)$
   **d.** $P(Z > 1.95) \cup P(Z < -1.95)$

3. Calculate the values of the following probabilities correct to 4 decimal places.

   **a.** $P(Z \leq 1)$
   **b.** $P(Z \leq 1.234)$
   **c.** $P(Z \geq -1.139)$
   **d.** $P(Z > -1.3)$
   **e.** $P(-1.6 \leq Z \leq 1.4)$
   **f.** $P(-0.645 \leq Z \leq 0.645)$

4. If $X$ is normally distributed with $\mu = 50$ and $\sigma = 8$, determine the equivalent $z$-value(s) and hence determine:

   **a.** $P(X > 55)$
   **b.** $P(28 \geq X \leq 65)$
   **c.** $P(X < 40 \,|\, X < 70)$

5. If $X \sim N(20, 25)$, determine:

   **a.** $P(X > 27)$
   **b.** $P(X \geq 18)$
   **c.** $P(X \leq 8)$
   **d.** $P(7 \leq X \leq 12)$
   **e.** $P(X < 17 \,|\, X \leq 25)$
   **f.** $P(X < 17 \,|\, X < \mu)$

6. Light bulbs have a mean life of 125 hours and a standard deviation of 11 hours.
   Determine the probability that a randomly selected light bulb lasts:

   **a.** longer than 140 hours
   **b.** less than 100 hours
   **c.** between 100 and 140 hours.

## Complex familiar

7. The heights jumped by Year 9 high-jump contestants follow a normal distribution with a mean jump height of 152 cm and a variance of 49 cm. Determine the probability that a competitor jumps:

   **a.** at least 159 cm
   **b.** less than 150 cm
   **c.** between 145 cm and 159 cm
   **d.** between 140 cm and 160 cm
   **e.** between 145 cm and 150 cm, given that they jumped over 140 cm.

**8.** $X$ is a continuous random variable and is known to be normally distributed.

  **a.** If $P(X < a) = 0.35$ and $P(X < b) = 0.62$, determine:

    **i.** $P(X > a)$                             **ii.** $P(a < X < b)$

  **b.** If $P(X < c) = 0.27$ and $P(X < d) = 0.56$, determine:

    **i.** $P(c < X < d)$                      **ii.** $P(X > c \mid X < d)$

  **c.** A random variable, $X$, is normally distributed with a mean of 20 and a standard deviation of 5.

    **i.** Determine $k$ if $P(X > 32) = P(Z > k)$.

    **ii.** Determine $n$ if $P(X < 12) = P(Z > n)$.

**9.** For a particular type of laptop computer, the length of time, $X$ hours, between charges of the battery is normally distributed such that $X \sim N\left(50, 15^2\right)$. Calculate $P(50 < X < 70)$.

**10.** Convert the variable in each of the following expressions to a standard normal variable, $Z$, and use it to write an equivalent expression. Use your calculator to evaluate each probability.

  **a.** $P(X < 61), X \sim N(65, 9)$

  **b.** $P(X \geq 110), \ X \sim N(98, 225)$

  **c.** $P(-2 < X \leq 5), \ X \sim N(2, 9)$

**11.** The volume of milk in a 1-litre carton is normally distributed with a mean of 1.000 litres and a standard deviation of 0.006 litres. A randomly selected carton is known to have more than 1.004 litres. Determine the probability that it has less than 1.011 litres.

**12.** A radar gun is used to measure the speeds of cars on a freeway. The speeds are normally distributed with a mean of 98 km/h and a standard deviation of 6 km/h. Calculate the probability that a car picked at random is travelling at:

  **a.** more than 110 km/h

  **b.** less than 90 km/h

  **c.** a speed between 90 km/h and 110 km/h.



**13.** Julie completed a range of internal exams in Year 12. In English she received a mark of 20 out of 25, and in Mathematical Methods she received 12 out of 15. The standard deviations of the subjects were 2.1 and 1.2, and the means were 18 and 10.8 respectively. Determine in which subject Julie performed better, using mathematical reasoning to justify your response.

### Complex unfamiliar

**14.** The battery life of mobile phones is considered to be normally distributed. The mean battery lives of a Bobo phone and a Gigi phone are 32 hours and 34 hours respectively, and the standard deviations are 2.6 and 2.8. If Judy's Gigi phone lasted 30 hours and is widely considered to be better than Fred's Bobo phone, determine the expected length of time that Fred's phone battery will last.

**15.** John is gathering data on two species of beetle: the longhorn beetle and the carrion beetle. Both species are equally likely to be caught in a particular region. One technique for telling them apart is by measuring the lengths of their antennae. For the longhorn beetle, the antennae are distributed normally with a mean of 22 mm and a standard deviation of 1.5 mm. John knows that 8% of carrion beetles have antennae that are shorter than 15.5 mm, and 8% of longhorn beetles have antennae that are longer than 22.5 mm. Assuming that the antenna lengths are normally distributed, calculate the mean and standard deviation of the antenna lengths of carrion beetles, giving your answers correct to the nearest 0.1 mm.

**16.** Determine the probability that the quadratic equation $x^2 + 2x + F = 0$ has real roots, given the random variable $F$ is normally distributed with a mean of 0 and a standard deviation of 1.

**Fully worked solutions for this chapter are available online.**

# LESSON
# 11.4 The inverse normal distribution

## 11.4.1 Introduction to the inverse normal distribution

The inverse normal distribution is a very useful concept, particularly when you need to determine a value in a normally distributed dataset based on a specific probability. Imagine you have a set of data, like test scores, that follow a bell-shaped curve (the normal distribution). If you know the probability, say the top 10% of scores, the inverse normal distribution helps you figure out the score that separates the top 10% from the rest. In simpler terms, it answers questions like, 'What score do I need to be in the top 10% of my class?' This is done using a statistical function that essentially 'flips' the normal distribution to provide the corresponding value for any given percentile, making it a handy tool for decision-making and assessments.

Technology provides an easy way to determine a $Z$ or $X$ value, given a probability for a normal distribution. Suppose $X$ is normally distributed with a mean of 32 and a standard deviation of 5. We wish to determine $P(X \leq a) = 0.72$.

The key information to enter into your calculator is the known probability, that is, the area under the curve. It is essential to input the correct area so that your calculator knows if you are inputting the 'less than' area or the 'greater than' area.
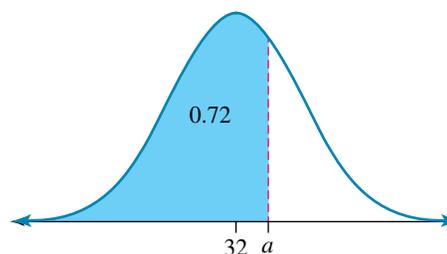
**If $X$ is a normally distributed random variable, determine:**
**a. $m$ given that $P(X \le m) = 0.85, X \sim N\left(15.2, 1.5^2\right)$**
**b. $n$ given that $P(X > n) = 0.37, X \sim N\left(22, 2.75^2\right)$**
**c. $p$ given that $P(37.6 - p \le X \le 37.6 + p) = 0.65, X \sim N\left(37.6, 12^2\right)$.**

| THINK | WRITE |
|---|---|
| **a.** Use the probability menus on your graphics calculator to determine the required $X$ value. | **a.** $P(X \le m) = 0.85, \mu = 15.2, \sigma = 1.5$ <br> $m = 16.7547$ |
| **b.** Use the probability menus on the graphics calculator to determine the required $X$ value. <br> *Note:* It may be a requirement to input the 'less than' area, so <br> $P(X < n) = 1 - 0.37$ <br> $\qquad\qquad = 0.63$ | **b.** $P(X > n) = 0.37, \mu = 22, \sigma = 2.75$ <br> $n = 22.9126$ |
| **c. 1.** Sketch a graph to visualise the problem. Due to symmetry, the probabilities either side of the upper and lower limits can be calculated. | **c.**  <br> $1 - 0.65 = 0.35$ <br> $P(X < 37.6 - p) = P(X > 37.6 + p)$ <br> $\qquad\qquad = \dfrac{0.35}{2}$ <br> $\qquad\qquad = 0.175$ |
| **2.** Determine $p$ by determining $X$ given that $P(X < 7.6 - p) = 0.175$. <br> *Note:* $p$ could also be found by using the upper limit. | $P(X < 37.6 - p) = 0.175$ <br> $37.6 - p = 26.38$ <br> $\qquad\quad = 37.6 - 26.38$ <br> $\qquad\quad = 11.22$ |

| TI | THINK | WRITE | CASIO | THINK | WRITE |
|---|---|---|---|
| **a. 1.** On a Calculator page, select: MENU 6: Statistics 5: Distributions 3: Inverse Normal... | | **a. 1.** On a Statistic screen, select: DIST NORM InvN. | |
| **2.** Complete the entry lines as: Area: 0.85 $\mu$: 15.2 $\sigma$: 1.5 Press the OK button. | | **2.** Complete the entry lines as: Data: Variable Tail: Left Area: 0.85 $\sigma$: 1.5 | |

| 1.1 ▶ | •Doc | RAD ⬛ ✕ |
|---|---|---|
| invNorm(0.85,15.2,1.5) | | 16.7547 |
| | | |
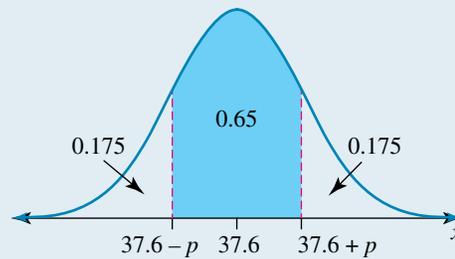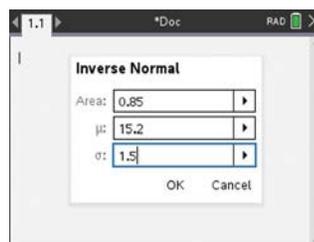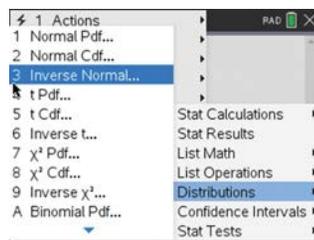
```
            [Rad Norm1] [d/c Real]
Inverse Normal
    xInv=16.7546501
```
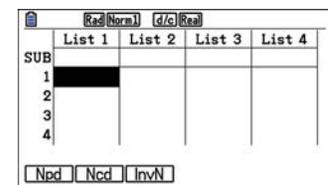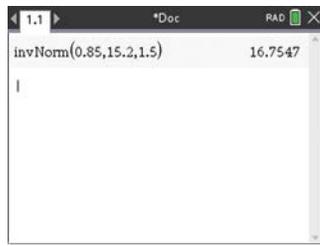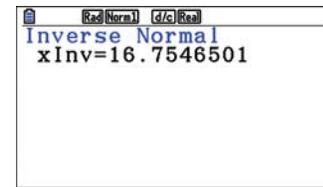
## 11.4.2 Quantiles and percentiles

Quantiles and percentiles are terms that enable us to convey information about a distribution. Quantiles refer to the value below which there is a specified probability that a randomly selected value will fall. For example, to determine the 0.7 quantile of a standard normal distribution, we determine $a$ such that $P(Z < a) = 0.7$.

Percentiles are very similar to quantiles. For the example of $P(Z < a) = 0.7$, we could also be asked to determine the 70th percentile for the standard normal distribution.

---

### WORKED EXAMPLE 5 Using quantiles and percentiles

**a.** **For the normally distributed variable $X$, the 0.15 quantile is 1.9227 and the mean is 2.7. Determine the standard deviation of the distribution.**

**b.** **$X$ is normally distributed so that the 63rd percentile is 15.896 and the standard deviation is 2.7. Determine the mean of $X$.**

**THINK**

**a. 1.** Write the probability statement.

   **2.** Determine the corresponding standardised value, $Z$, by using your calculator.

   **3.** Write the standardised formula connecting $z$ and $x$.

   **4.** Substitute the appropriate values and solve for $\sigma$.

**b. 1.** Write the probability statement.

   **2.** Calculate the corresponding standardised value, $Z$. by using your calculator.

   **3.** Write the standardised formula connecting $z$ and $x$.

   **4.** Substitute in the appropriate values and solve for $\mu$.

**WRITE**

**a.** The 0.15 quantile is 1.9227.
$$P(X < 1.9227) = 0.15$$

$$P(Z < z) = 0.15$$
$$z = -1.0364$$

$$z = \frac{x - \mu}{\sigma}$$

$$-1.0364 = \frac{1.9227 - 2.7}{\sigma}$$
$$-1.0364 = -0.7773$$
$$\sigma = 0.75$$

**b.** The 63rd percentile is 15.896.
$$P(X < 15.896) = 0.63$$

$$P(Z < z) = 0.63$$
$$z = 0.3319$$

$$z = \frac{x - \mu}{\sigma}$$

$$0.3319 = \frac{15.896 - \mu}{2.7}$$
$$0.8960 = 15.896 - \mu$$
$$\mu = 15$$

---

# Exercise 11.4 The inverse normal distribution

| 11.4 Exercise | 11.4 Exam questions on |
|---|---|

| Simple familiar | Complex familiar | Complex unfamiliar |
|---|---|---|
| 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 | 12, 13, 14, 15 | 16, 17, 18 |

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## Simple familiar

1. **MC** If $P(z \leq c) = 0.8$, then $c$ is equal to:
   A. $-0.842$
   B. $-0.253$
   C. $0.253$
   D. $0.842$

2. **MC** If $P(z \leq c) = 0.7$, the best description of $c$ is:
   A. 70th percentile
   B. 30th percentile
   C. 70th percentile or 0.70 quantile
   D. 30th percentile or 0.30 quantile

3. **WE4** Calculate the value of $a$, correct to 2 decimal places, if $X$ is normally distributed and:
   a. $P(X \leq a) = 0.16, X \sim N\left(41, 6.7^2\right)$
   b. $P(X > a) = 0.21, X \sim N\left(12.5, 2.7^2\right)$
   c. $P(15 - a \leq X \leq 15 + a) = 0.32, X \sim N\left(15, 4^2\right)$

4. Calculate the values of $m$ and $n$ if $X$ is normally distributed and $P(m \leq X \leq n) = 0.92$ when $\mu = 27.3$ and $\sigma = 8.2$. The specified interval is symmetrical about the mean.

5. **WE5** $X$ is distributed normally with a mean of 112, and the 42nd percentile is 108.87. Calculate the standard deviation of the distribution, correct to 1 decimal place.

6. $X$ is a normally distributed random variable such that $X \sim N\left(\mu, 4.45^2\right)$. If the 0.11 quantile is 32.142, calculate the value of $\mu$, correct to 1 decimal place.

7. If $X$ is distributed normally with $\mu = 43.5$ and $\sigma = 9.7$, calculate:
   a. the 0.73 quantile
   b. the 24th percentile.

8. $X$ is distributed normally with a standard deviation of 5.67, and $P(X > 20.952) = 0.09$. Calculate the mean of $X$, giving your answer correct to 2 decimal places.

9. $X$ is distributed normally with a standard deviation of 3.5, and $P(X < 23.96) = 0.28$. Calculate the mean for $X$, rounded to the nearest whole number.

10. $X \sim N\left(115, \sigma^2\right)$ and the 76th percentile is 122.42. Calculate the value of $\sigma$, giving your answer correct to 1 decimal place.

11. $X$ is distributed normally with $\mu = 41$ and $P(X > 55.9636) = 0.11$. Calculate $\sigma$, giving your answer correct to 1 decimal place.

**12.** *X* is distributed normally such that $P(X < 33.711) = 0.36$ and $P(X < 34.10) = 0.42$. Calculate the mean and the standard deviation of *X*, giving your answers correct to 1 decimal place.

**13.** *X* is distributed normally such that $P(X > 18.376) = 0.31$ and the 45th percentile is 15.15. Calculate $\mu$ and $\sigma$ for *X*, giving your answers correct to 1 decimal place.

**14.** *X* is normally distributed with a mean of $\mu$ and a standard deviation of 3. If 35% of *X*-values are at least 27, calculate the mean.

**15.** The time taken for Grade 4 students to complete a small jigsaw puzzle follows a normal distribution with a standard deviation of 30 seconds. If 70% of Grade 4 students complete the puzzle in 4 minutes or less, calculate the mean completion time for Grade 4 students.

**16.** In a university course, exam scores for a particular subject are reported as follows: the mean score of a sample of 50 students is 70. The scores are known to be normally distributed, and the 90th percentile is reported as 82. Jasmine scored 84. Determine, using mathematical reasoning whether she has reached the minimum score required to gain honours recognition — a score associated with the top 5% of students.

**17.** Pasta bridges were constructed in a school Engineering course, and those that could support the heaviest load were recognised with certificates in Assembly. The load results were normally distributed with a mean of 1.54 kg and a standard deviation of 0.15 kg. Two award categories were used: distinction for the top 10% of students and credit for the next 10%. Bart's bridge just qualified for a credit and June's bridge just missed out on a distinction. Determine the difference between the two students' loads to the nearest gram.

**18.** Although some variation is expected, the volume of a cordial bottle should have a mean of 582 mL. Bottles at or below the 15th percentile are rejected. Thirty-eight percent of the bottles contain 585 mL or more of cordial. Assuming the volumes are normally distributed, determine the smallest volume (in mL) that will be accepted.

**Fully worked solutions for this chapter are available online.**

# LESSON
## 11.5 Modelling and solving with normal distributions

## 11.5.1 Modelling and solving with normal distributions

Application problems involving the normal distribution cover a wide range of topics. Such questions will not only incorporate theory associated with the normal distribution but may also include other areas of probability you have previously studied.

### WORKED EXAMPLE 6 Solving problems using the normal distribution

**The amount of instant porridge oats in packets packed by a particular machine is normally distributed with a mean of $\mu$ grams and a standard deviation of 6 grams. The advertised weight of a packet is 500 grams.**

**a.** **Calculate the proportion of packets that will be underweight (less than 500 grams) when $\mu = 505$ grams.**

**b.** **Calculate the value of $\mu$ required to ensure that only 1% of packets are underweight.**

**c.** **As a check on the setting of the machine, a random sample of 5 boxes is chosen and the setting is changed if more than one of them is underweight. Calculate the probability that the setting on the machine is changed when $\mu = 505$ grams.**

| THINK | WRITE |
|---|---|
| **a. 1.** Rewrite the information in the question using appropriate notation. | **a.** $X$ is the amount of instant porridge oats in a packet and $X \sim N\left(505, 6^2\right)$. |
| **2.** Use your graphics calculator to calculate $P(X < 500)$. | $P(X < 500) = 0.2023$ |
| **b. 1.** State the known probability. | **b.** $P(X < 500) = 0.01$ |
| **2.** Calculate the corresponding standardised value, $Z$, by using a graphics calculator. | $P(Z < z) = 0.01$ <br> $z = -2.3263$ |
| **3.** Write the standardised formula connecting $z$ and $x$. | $Z = \dfrac{x - \mu}{\sigma}$ |
| **4.** Substitute the appropriate values and solve for $\mu$. | $-2.3263 = \dfrac{500 - \mu}{6}$ <br> $-13.9581 = 500 - \mu$ <br> $\mu = 513.96$ g |

## Exercise 11.5 Modelling and solving with normal distributions  learnon

| **11.5 Exercise** | **11.5 Exam questions** on |
|---|---|

| Simple familiar | Complex familiar | Complex unfamiliar |
|---|---|---|
| 1 | 2, 3, 4, 5, 6, 7, 8, 9, 10 | 11, 12, 13, 14, 15 |

These questions are even better in jacPLUS!
- Receive immediate feedback
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### Simple familiar

**1.** **WE6** Packages of butter with a stated weight of 500 g have an actual weight of $W$ g, which is normally distributed with a mean of 508 g.

**a.** If the standard deviation of $W$ is 3.0 g, calculate:

**i.** the proportion of packages that weigh less than 500 g

**ii.** the weight that is exceeded by 99% of the packages.

**b.** If the probability that a package weighs less than 500 g is not to exceed 0.01, calculate the maximum allowable standard deviation of $W$.

### Complex familiar

**2.** Victoria is selling hampers of handmade soaps online. The masses of her hampers are normally distributed, with a mean $\mu = 5.0$ kg and a standard deviation $\sigma$.
She considers her hampers are not up to par with her quality goals if their mass is smaller than 4.5 kg.
She picks a hamper at random. The probability that it is heavier than 4.5 kg is 0.96. Determine the value of the standard deviation to the nearest gram.

**3.** Chocolate Surprise is a toy that is packed inside an egg-shaped chocolate. A certain manufacturer provides four different types of Chocolate Surprise toy — a car, an aeroplane, a ring and a doll — in the proportions given in the table.

| Toy | Proportion |
|---|---|
| Car | $3k^2 + 2k$ |
| Aeroplane | $6k^2 + 2k$ |
| Ring | $k^2 + 2k$ |
| Doll | $3k$ |

a. Show that $k$ must be a solution to the equation $10k^2 + 9k - 1 = 0$.
b. Calculate the value of $k$.

In response to customer demand, the settings on the machine that produce Chocolate Surprise have been changed so that 25% of all Chocolate Surprises produced contain rings. A sample of 8 Chocolate Surprises is randomly selected from a very large number produced by the machine.

c. Calculate the expected number of Chocolate Surprises in the sample that contain rings. Give your answer correct to the nearest whole number.
d. Calculate the probability, correct to 4 decimal places, that this sample has exactly 2 Chocolate Surprises that contain rings.

**4.** A particular brand of car speedometer was tested for accuracy. The error measured is known to be normally distributed with a mean of 0 km/h and a standard deviation of 0.76 km/h. Speedometers are considered unacceptable if the error is more than 1.5 km/h. Calculate the proportion of speedometers that are unacceptable.

**5.** The heights of adult males in Perth can be taken as normally distributed with a mean of 174 cm and a standard deviation of 8 cm. Suppose the Western Australia Police Force accepts recruits only if they are at least 180 cm tall.

a. Determine the percentage of Perth adult males who satisfy the height requirement for the Western Australia Police Force.
b. Determine the minimum height, to the nearest centimetre, that the Western Australia Police Force would have to accept if it wanted a quarter of the Perth adult male population to satisfy the height requirement.

**6. a.** Farmer David grows avocados on a farm on Mount Tamborine, Queensland. The average weight of his avocados is known to be normally distributed with a mean weight of 410 g and a standard deviation of 20 g.



    **i.** Calculate the probability that an avocado chosen at random weighs less than 360 g.

    **ii.** Calculate the probability that an avocado that weighs less than 360 g weighs more than 340 g.

**b.** Farmer Jane grows avocados on a farm next to farmer David's. If $Y$ represents the average weight of Jane's avocados, the weights of which are also normally distributed where $P(Y < 400) = 0.4207$ and $P(Y > 415) = 0.3446$, calculate the mean and standard deviation of the weights of Jane's avocados. Give answers correct to the nearest integer.

**7.** A manufacturer produces metal rods whose lengths are normally distributed with a mean of 145.0 cm and a standard deviation of 1.4 cm.

**a.** Calculate the probability, correct to 4 decimal places, that a randomly selected metal rod is longer than 146.5 cm.

**b.** A metal rod has a size fault if its length is not within $d$ cm either side of the mean. The probability of a metal rod having a size fault is 0.15. Calculate the value of $d$, giving your answer correct to 1 decimal place.

**c.** A random sample of 12 metal rods is taken from a crate containing a very large number of metal rods. Calculate the probability that there are exactly 2 metal rods with a size fault, giving your answer correct to 4 decimal places.

**d.** The sales manager is considering what price, $\$x$, to sell each of the metal rods for, whether they are good or have some kind of fault. The materials cost is $\$5$ per rod. The metal rods are sorted into three bins. The staff know that 15% of the manufactured rods have a size fault and another 17% have some other fault. The profit, $\$Y$, is a random variable whose probability distribution is shown in the following table.

| Bin | Description | Profit ($y$) | $P(Y = y)$ |
|-----|-------------|--------------|------------|
| A | Good metal rods that are sold for $x$ dollars each | $x - 5$ | $a$ |
| B | Metal rods with a size fault — these are not sold but recycled. | $0$ | $0.15$ |
| C | Metal rods with other faults — these are sold at a discount of $\$3$ each. | $x - 8$ | $0.17$ |

    **i.** Calculate the value of $a$, correct to 2 decimal places.

    **ii.** Calculate the mean of $Y$ in terms of $x$.

    **iii.** Hence or otherwise, calculate, correct to the nearest cent, the selling price of good rods so that the mean profit is 0.

    **iv.** The metal rods are stored in the bins until a large number is ready to be sold. Determine what proportion of the rods ready to be sold are good rods.

8. A company sells two different products, $X$ and $Y$, for $5.00 and $6.50 respectively. Regular markets exist for both products, with sales being normally distributed and averaging 2500 units (standard deviation 700) and 3000 units (standard deviation 550) respectively each week. It is company policy that if in any one week the sales for a particular product fall below half the average, that product is advertised as a 'special' for the following week.

   a. Calculate the probability, correct to 4 decimal places, that product $X$ will be advertised as a 'special' next week.

   b. Calculate the probability, correct to 4 decimal places, that product $Y$ will be advertised as a 'special' next week.

   c. Calculate the probability, correct to 4 decimal places, that both products will be advertised as a 'special' next week.

   d. If 40% of the company's product is product $X$ and 60% is product $Y$, calculate the probability that:

      i. one product is a 'special'

     ii. if one product is advertised as 'special', then it is product $X$.

9. The height of plants sold at a garden nursery supplier are normally distributed with a mean of 18 cm and a standard deviation of 5 cm.



   a. Complete the following table by calculating the proportions for each of the three plant sizes, correct to 4 decimal places.

| Description of plant | Plant size (cm) | Cost in $ | Proportion of plants |
|---|---|---|---|
| Small | Less than 10 cm | 2.00 | |
| Medium | 10–25 cm | 3.50 | |
| Large | Greater than 25 cm | 5.00 | |

   b. Calculate the expected cost, to the nearest dollar, for 150 plants chosen at random from the garden nursery.

10. A fruit grower produces peaches whose weights are normally distributed with a mean of 185 g and a standard deviation of 20 g.
Peaches whose weights exceed 205 g are sold to the cannery, yielding a profit of 60c per peach. Peaches whose weights are between 165 g and 205 g are sold to wholesale markets at a profit of 45 cents per peach. Peaches whose weights are less than 165 g are sold for jam at a profit of 30c per peach.



   a. Calculate the percentage of peaches sold to the canneries.

   b. Calculate the percentage of peaches sold to the wholesale markets.

   c. Calculate the mean profit per peach.

**11.** The Lewin Tennis Ball Company makes tennis balls whose diameters are distributed normally with a mean of 70 mm and a standard deviation of 1.5 mm. The tennis balls are packed and sold in cylindrical tins that each hold five tennis balls. A tennis ball fits in the tin if its diameter is less than 71.5 mm. Lewin management wants engineers to change the manufacturing process so that 99.5% of all balls produced have a diameter between 68.6 mm and 71.4 mm. The mean is to stay at 70 mm but the standard deviation is to be changed. Determine what the new standard deviation should be, correct to 4 decimal places.

**12.** The Apache Orchard grows a very juicy apple called the Fuji apple. Fuji apples are picked and then sorted by diameter in three categories:
- small — diameter less than 60 mm
- jumbo — the largest 15% of the apples
- standard — all other apples.

Diameters of Fuji apples are found to be normally distributed with a mean of 71 mm and a standard deviation of 12 mm.

**a.** The Apache Orchard receives the following prices for Fuji apples:
- small — 12 cents each
- standard — 15 cents each
- jumbo — 25 cents each.

Determine the expected income, correct to the nearest dollar, for a container of 2500 unsorted apples.

**b.** Some apples are selected before sorting and are packed into bags of six to be sold at the front gate of the orchard. Calculate the probability, correct to 4 decimal places, that one of these bags contains at least two jumbo apples.

**13.** A brand of disinfectant is sold in two sizes: standard and large. For each size, the contents, in litres, of a randomly chosen bottle is normally distributed with a mean and standard deviation as shown in the following table.

| Bottle size | Mean | Standard deviation |
|---|---|---|
| Standard | 0.765 L | 0.007 L |
| Large | 1.015 L | 0.009 L |

Calculate the probability that a box of 12 randomly chosen large bottles contains at least 4 bottles whose contents are each less than 1 L.

**14.** Amalie is gathering data on two particular species of yellow butterflies: the lemon emigrant and the yellow emigrant. These two species can be very difficult to tell apart. Both species are equally likely to be caught in a particular area of Australia. One technique for telling them apart is by measuring the lengths of their antennae. For the lemon emigrant, the antennae are distributed normally with a mean of 22 mm and a standard deviation of 1.5 mm.
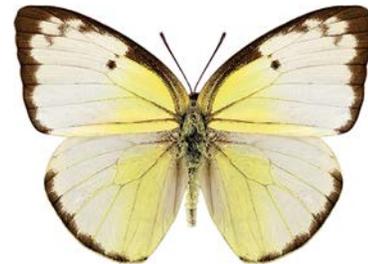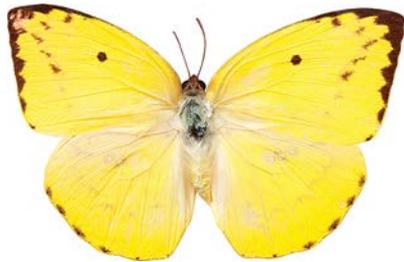


In the region where Amalie is hunting for yellow butterflies, 45% of the yellow butterflies are lemon emigrants and 55% are lemon emigrants. Calculate the probability, correct to 4 decimal places, that a random sample of 12 butterflies from the region will contain 5 yellow emigrant butterflies.

**15.** The daily error (in seconds) of a particular brand of clock is known to be normally distributed. Only those clocks with an error of less than 3 seconds are acceptable. If 2.5% of the clocks are rejected for losing time and 2.5% of the clocks are rejected for gaining time.
Determine the probability that fewer than 2 clocks are rejected in a batch of 12 such clocks.

**Fully worked solutions for this chapter are available online.**

# LESSON
## 11.6 Review

### 11.6.1 Summary

### 11.6 Exercise

| 11.6 Exercise | 11.6 QCAA questions on |
| --- | --- |

| Simple familiar | Complex familiar | Complex unfamiliar |
| --- | --- | --- |
| 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 | 13, 14, 15, 16 | 17, 18, 19, 20 |

**Simple familiar**

1. The diagram shows two normal distributions with means $\mu_1$ and $\mu_2$ and variances $\sigma_1^2$ and $\sigma_2^2$.
   Blue: $X_1 \sim N\left(\mu_1, \sigma_1^2\right)$
   Pink: $X_2 \sim N\left(\mu_2, \sigma_2^2\right)$
   Determine which of the two distributions has:

   a. the higher mean
   b. the smaller standard deviation.
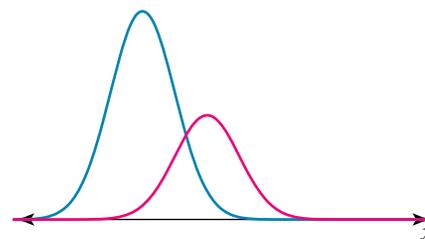
2. If $X \sim N(20, 25)$, calculate:

   a. $P(X > 27)$
   b. $P(X \geq 18)$
   c. $P(7 \leq X \leq 12)$
   d. $P(X < 17 \mid X \leq 25)$

3. The variable $X$ is normally distributed with mean $\mu = 9$ and standard deviation $\sigma = 3$. Determine the equivalent standard $Z$-score for the following $X$-values.

   a. $X = 10$
   b. $X = 7.5$
   c. $X = 12.4$

4. $X$ is normally distributed with a mean of 10 and a standard deviation of 2. Calculate $X_1$ if:

   a. $P(X \leq x_1) = 0.72$
   b. $P(X < x_1) = 0.4$

5. Calculate the value of $c$ given that $P(-c \leq Z \leq c) = 0.38$.

6. Tennis balls are dropped from a height of 2 m. The rebound height of the balls is normally distributed with a mean of 1.4 m and a standard deviation of 0.1 m. Determine the probability that a ball rebounds more than 1.25 m.

**7.** The results by Justine in Chemistry, Mathematical Methods and Physics are shown in the table. The marks, $X$, the mean, $\mu$, and standard deviation, $\sigma$, for each examination are given.

| Subject | Mark, $X$ | Mean, $\mu$ | Standard deviation, $\sigma$ | Standardised mark, $Z$ |
|---|---|---|---|---|
| Chemistry | 72 | 68 | 5 | |
| Mathematical Methods | 75 | 69 | 7 | |
| Physics | 68 | 61 | 8 | |

Complete the table by calculating Justine's standardised mark for each subject and use this to determine in which subject she did best when compared to her peers.

**8.** If $Z \sim N(0, 1)$, identify the 0.35 quantile.

**9.** The results of a state-wide Science exam are normally distributed. If 68% of entrants scored between 8 and 12 points on the exam, determine the approximate values of the mean and the standard deviation.

**10.** The heights of Year 9 students are known to be normally distributed with a mean of 160 cm and a standard deviation of 8 cm.
   **a.** Determine how tall Theo is if he is taller than 95% of Year 9 students.
   **b.** Determine how tall Luisa is if she is shorter than 80% of Year 9 students.

**11.** If $X \sim N\left(20, \sigma^2\right)$ and $P(X \geq 19) = 0.7$, calculate the standard deviation, $\sigma$.

**12.** Describe the relationship between the mean, mode and median of a standard normal distribution.

**13.** Peter has a strawberry farm in Stanthorpe. The average length of a strawberry is normally distributed with a mean of 3.5 cm and a standard deviation of 0.8 cm.
   Strawberries that are longer than 4.5 cm are sold to a restaurant supplier for $6.50 per kilogram. Strawberries that are between 2.5 cm and 4.5 cm long are sold to a supermarket supplier for $4.50 per kilogram, and strawberries that are less than 2.5 cm long are sold to a jam manufacturer for $1.75 per kilogram.
   **a.** Calculate the percentage of strawberries that are sold to the supermarket supplier.
   **b.** Calculate the percentage of strawberries that are sold to the jam manufacturer.
   **c.** Calculate the mean profit for a kilogram of strawberries.

**14.** Jing Jing scored 85 on the mathematics section of a scholarship examination, the results of which were known to be normally distributed with a mean of 72 and a standard deviation of 9. Rani scored 18 on the mathematics section of a similar examination, the results of which were normally distributed with a mean of 15 and a standard deviation of 4. Assuming that both tests measure the same kind of ability, determine which student has the higher score.

**15.** $X$ is a normally distributed variable for which $P(X < 47) = 0.3694$ and $P(X > 56) = 0.3385$. Calculate the mean and standard deviation of $X$.

**16.** The lengths of fish caught in a certain river follow a normal distribution, with mean 32 cm and standard deviation 4 cm. Fish that are less than 27 cm long are considered to be undersized and must be returned to the river. Calculate the expected number of fish that a fisherman could take home if he catches 20 fish in one afternoon and follows the rules for undersized fish.

**Complex unfamiliar**

**17.** The lengths of certain sunflower stems follow a normal distribution with a mean of 75 cm and a standard deviation of 8 cm. Stems are measured and awarded grades depending on their lengths. The top 10% receive an A grade, the next 10% a B grade and the third 10% a C grade. Give the ranges of sunflower stem lengths to 2 decimal places for which an A, B and C grade are awarded, respectively.



**18.** Under certain circumstances, a random variable $X$ that follows a binomial distribution with $n$ trials and a probability of success $p$ can be approximated to a normal distribution. The mean of this normal distribution approximation is defined by $\mu = np$, the variance $\sigma^2 = npq$, and standard deviation $\sigma = \sqrt{npq}$. Use a normal approximation to determine the probability that an archery contestant will score at least 240 bullseyes out of 500 shots given that she hits the bullseye 49% of the time during practice.

**19.** $P(a < X < b) = 0.52$ and the specified interval is symmetrical about the mean. If $X$ is normally distributed with a mean of 42.5 and a standard deviation of 10.3, calculate $P(X > a \mid X < b)$.

**20.** $X$ is distributed normally such that $P(X < 39.9161) = 0.579$ and $P(X > 38.2491) = 0.4798$. Calculate the mean and the standard deviation of $X$, giving your answers correct to 2 decimal places.

# 11.6 Past QCAA exam questions

**Question 1 (1 mark)**

*Source:* QCAA 2023 Mathematical Methods, Paper 2, Section 1, Q3; © State of Queensland (QCAA) 2024.

In a certain normal distribution curve, 95% of the area lies between the values 50.32 and 113.68.

The mean of this distribution is 82.

Determine the standard deviation.

- **A.** 16.16
- **B.** 21.12
- **C.** 31.68
- **D.** 63.36

## Question 2 (1 mark)

*Source:* QCAA 2023 Mathematical Methods, Paper 2, Section 1, Q10; © State of Queensland (QCAA) 2024.

A student is trying to determine which subject they performed best in compared to other students. Results from recent tests in four subjects (A to D) are shown. Assume student results in each subject are normally distributed.

In which subject did the student perform best compared to other students?

| | Class mean | Class standard deviation | Student's result |
|---|---|---|---|
| **A.** | 62 | 22 | 77 |
| **B.** | 55 | 25 | 74 |
| **C.** | 61 | 15 | 70 |
| **D.** | 73 | 20 | 82 |

## Question 3 (5 marks)

*Source:* QCAA 2023, Mathematical Methods, Paper 2, Section 2, Q17; © State of Queensland (QCAA) 2024.

Model bridges were constructed for a competition. The models that could support the heaviest loads before collapsing were given awards.

The load results of the competition were normally distributed, with a mean of $1.36 \text{ kg}$ and a standard deviation of $0.12 \text{ kg}$.

Three award categories were used: honours for the top $15\%$ of load results; distinction for the next $15\%$; and commendation for the next $15\%$.

The model bridge constructed by Finley only just missed out on a commendation. Kirby's model bridge only just qualified for honours. Determine the difference, to the nearest gram, between the loads supported by Finley and Kirby's models.

## Question 4 (1 mark)

*Source:* QCAA 2022 Mathematical Methods, Paper 1, Section 1, Q4; © State of Queensland (QCAA) 2024.

The weekly amount of money a company spends on repairs is normally distributed, with a mean of $\$1200$ and a standard deviation of $\$100$.

Given that $P(Z \leq -2.5) = 0.0062$ and $P(Z > 1) = 0.1587$, where $Z$ is a standard normal random variable, determine the probability that the weekly repair costs will be between $\$950$ and $\$1300$.
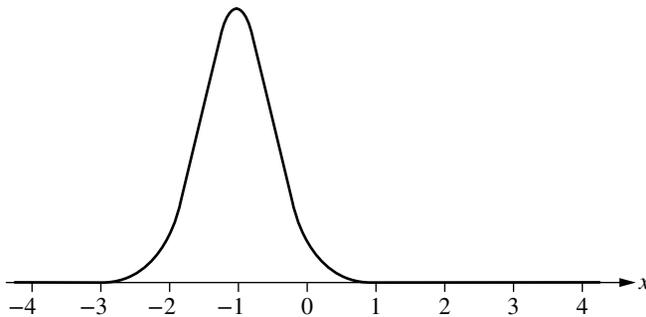
A. $0.6525$
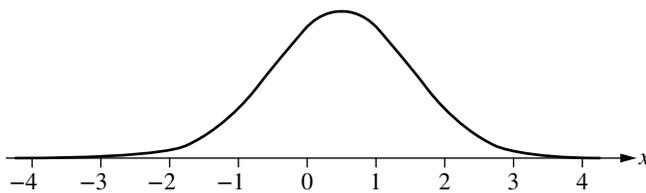B. $0.6587$
C. $0.8351$
D. $0.841$

*Source:* QCAA 2022 Mathematical Methods, Paper 1, Section 1, Q5; © State of Queensland (QCAA) 2024.

Which normal distribution curve best represents a normal distribution with a mean of $1$ and a standard deviation of $0.5$?
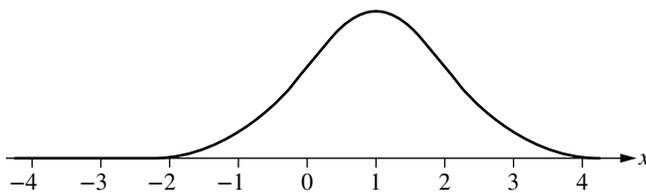
**A.**



**B.**



**C.**



**D.**
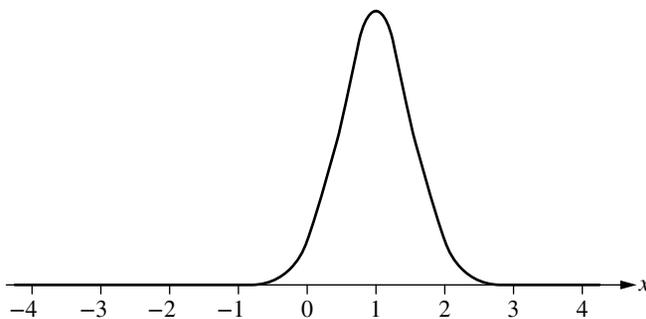


**Fully worked solutions for this chapter are available online.**

# Answers

## 11 Normal distributions

### 11.2 The normal distribution

#### 11.2 Exercise

**1. a.** Heights of women in the Australian population
**b.** IQ scores of children aged 10
**c.** No, a single class is unlikely to be normally distributed.
**d.** Usable lifetime of light bulbs

**2. a.** 10 and 20
**b.** 5 and 25
**c.** 0 and 30

**3. a.** 0.025
**b.** 0.0015
**c.** 0.84

**4.** 0.0015

**5. a. i.** 100 and 140
**ii.** 80 and 160
**iii.** 60 and 180
**b. i.** 0.025
**ii.** 0.0015

**6.** 0.15%

**7. a.** $\mu = 2; \sigma = 3$
**b.**
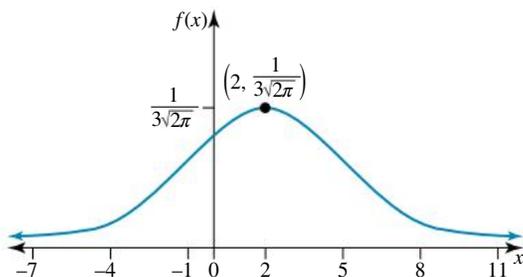


**8. a.** $\mu = 3, \sigma = 1$
**b.**
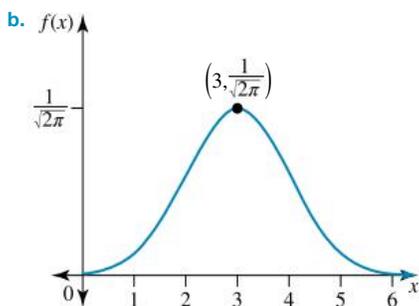


**9.** $\mu = -2$

**10. a.** $\mu = 1; \sigma = 0.3$ or $\dfrac{3}{10}$
**b.** Dilation of factor $\dfrac{10}{3}$ parallel to the $y$-axis, dilation of factor $\dfrac{3}{10}$ parallel to the $x$-axis, translation of 1 unit in the positive $x$-direction
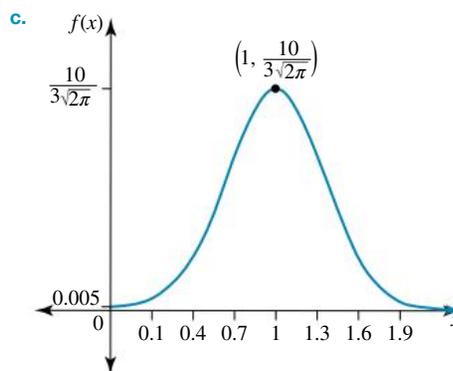
**c.**



**11. a.** $\mu = -4; \sigma = 10$
**b.** Dilation of factor $\dfrac{1}{10}$ from the $x$-axis, dilation of factor 10 from the $y$-axis, translation of 4 units in the negative $x$-direction
**c. i.** 100     **ii.** 116
**d.** $\displaystyle\int_{\infty}^{-\infty} \dfrac{1}{10\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x+4}{10}\right)^2} dx = 0.9999 \approx 1$
$f(x) \geq 0$ for all values of $x$, and the area under the curve is 1. Therefore, this function is a probability density function.

**12. a.** $\mu = 2; \sigma = \dfrac{2}{5} = 0.4$
**b.** $\dfrac{104}{25} = 4.16$
**c. i.** 10     **ii.** 20.8

**13. a.** 0.975     **b.** 0.95     **c.** 0.9744

**14. a.** 0.815     **b.** 205     **c.** 155

**15. a.** 0.04
**b.** 1.04
**c. i.** 5     **ii.** $-1.92$

**16. a.** 0.815     **b.** 0.025     **c.** 0.94     **d.** 80.9

**17.** 0.9964 or 99.64%

### 11.3 Standardised normal variables

#### 11.3 Exercise

**1. a.i.** 0.8849     **ii.** 0.7703
**b.i.** 0.9088     **ii.** $-\dfrac{4}{3}$

**2. a.** 0.9772     **b.** 0.0228     **c.** 0.9545     **d.** 0.0512

**3. a.** $P(Z \leq 1) = 0.8413$
**b.** $P(Z \leq 1.234) = 0.8914$
**c.** $P(Z \geq -1.139) = 0.8726$
**d.** $P(Z < -1.3) = 0.0968$
**e.** $P(-1.6 \leq Z \leq 1.4) = 0.8644$
**f.** $P(-0.645 \leq Z \leq 0.645) = 0.4811$

**4. a.** $P(z > 0.625) = 0.2660$
**b.** $P(-2.75 \leq z \leq 1.875) = 0.9667$
**c.** $\dfrac{P(z < -1.25)}{P(z < 2.5)} = 0.1063$

**5. a.** 0.0808     **b.** 0.6554     **c.** 0.0082
**d.** 0.0501     **e.** 0.3260     **f.** 0.5486

**6. a.** 0.0863     **b.** 0.0115     **c.** 0.9022

**7. a.** 0.1587     **b.** 0.3874     **c.** 0.6826
   **d.** 0.8302     **e.** 0.2391

**8. a.i.** 0.65           **ii.** 0.27
   **b.i.** 0.29          **ii.** 0.5179
   **c.i.** 2.4           **ii.** 1.6

**9.** 0.4088

**10. a.** 0.0912     **b.** 0.2119     **c.** 0.7501

**11.** 0.8676

**12. a.** 0.0228     **b.** 0.0912     **c.** 0.8860

**13.** Mathematical Methods

**14.** Less than 28 hours 17 minutes

**15.** $\sigma = 2.5\,\text{mm}$, $\mu = 19.0\,\text{mm}$

**16.** $P(F \le 1) = 0.3413$

## 11.4 The inverse normal distribution

### 11.4 Exercise

**1.** D

**2.** C

**3. a.** 34.34     **b.** 14.68     **c.** 1.65

**4.** $m = 12.9444$; $n = 41.6556$

**5.** 15.5

**6.** 37.6

**7. a.** 49.4443     **b.** 36.6489

**8.** 13.35

**9.** 26

**10.** 10.5

**11.** 12.2

**12.** $\mu = 34.6$; $\sigma = 2.5$

**13.** $\mu = 15.8$; $\sigma = 5.2$

**14.** 25.844

**15.** 3 min 44 s

**16.** Has not reached Honours; ~1.42% short.

**17.** 66 grams

**18.** $x = 579\,\text{mL}$

## 11.5 Modelling and solving with normal distributions

### 11.5 Exercise

**1. a.i.** 0.0038          **ii.** 501.0210 g
   **b.** 3.4389 g

**2.** 286 g

**3. a.** $3k^2 + 2k + 6k^2 + 2k + k^2 + 2k + 3k = 1$
     $10k^2 + 9k - 1 = 0$
   **b.** $\dfrac{1}{10}$
   **c.** 2
   **d.** 0.3115

**4.** 0.0484 or 4.84%

**5. a.** 22.66%                  **b.** 179 cm

**6. a.i.** 0.0062                **ii.** 0.9625
   **b.** $\mu = 405$; $\sigma = 25$

**7. a.** 0.1420     **b.** 2.0          **c.** 0.2924
   **d. i.** 0.68            **ii.** $0.85x - 4.76$
      **iii.** \$5.60           **iv.** 80%

**8. a.** 0.0371     **b.** 0.0032     **c.** 0.0001
   **d. i.** 0.0167              **ii.** 0.8856

**9. a.** See the table at the bottom of the page.*
   **b.** \$531

**10. a.** 15.87%     **b.** 68.27%     **c.** 45c

**11.** 0.4987

**12. a.** \$399     **b.** 0.2236

**13.** 0.0019

**14.** 0.2225

**15.** 0.8816

## 11.6 Review

### 11.6 Exercise

**1. a.** Pink          **b.** Blue

**2. a.** 0.0808    **b.** 0.6554    **c.** 0.0501    **d.** 0.3260

**3. a.** $\dfrac{1}{3} = 0.\dot{3}$    **b.** $-0.5$     **c.** $1.1\dot{3}$

**4. a.** 11.166     **b.** 9.494

**5.** 0.496

**6.** 0.9332

**7.** Chemistry: 0.8; Mathematical Methods: 0.857;
   Physics: 0.875. Justine did best in Physics.

**8.** $-0.385$

**9.** $\mu = 10$; $\sigma = 2$

**10. a.** Theo is 173.16 cm tall.
   **b.** Luisa is 153.26 cm tall.

**11.** 1.908

**12.** Mean = median = mode

**13. a.** 10.56%     **b.** 10.56%     **c.** \$4.42/kg

**14.** Jing Jing did better.

---

**\* 9. a.**

| Description of plant | Plant size (cm) | Cost in $ | Proportion of plants |
| --- | --- | --- | --- |
| Small | Less than 10 cm | 2.00 | 0.0548 |
| Medium | 10−25 cm | 3.50 | 0.8644 |
| Large | Greater than 25 cm | 5.00 | 0.0808 |

**15.** $\mu = 51; \sigma = 12$

**16.** 17

**17.** A grade for stems longer than 85.25 cm,
B grade for stem lengths between 81.73 cm and 85.25 cm,
C grade for stem lengths between 79.19 cm and 81.73 cm.

**18.** 0.6725

**19.** 0.6842

**20.** $\mu = 37.68; \sigma = 11.21$

## 11.6 Past QCAA exam questions

**1.** A

**2.** B

**3.** 109 g

**4.** C

**5.** D

**UNIT 4** Further calculus, trigonometry and statistics
**TOPIC 4** Sampling and proportions
**TOPIC 5** Interval estimates for proportions

# 12 Sampling, proportions and interval estimates

## LESSON SEQUENCE

**Fully worked solutions of this chapter are available online.**

## EXAM PREPARATION

Access exam-style questions in every lesson, available online and access past QCAA exam questions in every review.

**on** Resources

| | | |
|---|---|---|
| **Solutions** | Solutions — Chapter 12 | |
| **Exam questions** | Exam questions booklet — Chapter 12 | |
| **Digital documents** | Learning matrix — Chapter 12<br>Chapter summary — Chapter 12 | |

## 12.1.1 Introduction

Many of the statistics about life in Australia are drawn from the Census of Population and Housing conducted by the Australian Bureau of Statistics every five years. A census is a survey of a total population. Although it is meaningful to gather data using a census, it is quite impractical, and sometimes impossible, so a sample is usually taken. Unfortunately, as soon as we use a sample to make decisions about a population, we run the risk of introducing bias.



Moreover, sometimes companies or governing bodies can intentionally introduce bias in their sampling to make a more compelling argument. Consequently, understanding sampling and its intended or accidental consequences is an important part of making informed decisions. Sampling methods, such as random sampling, stratified sampling and cluster sampling, aim to reduce bias and improve the accuracy of the data collected. However, even with these methods, it is crucial to be aware of potential biases and limitations. By critically evaluating the sampling process and the data obtained, we can make better-informed decisions and avoid being misled by biased information. This understanding is essential for interpreting statistics and making sound judgements in various fields, including public policy, business and research.

# 12.1.2 Syllabus links

| Lesson | Lesson title | Syllabus links |
|--------|--------------|----------------|
| 12.2 | Random sampling | ○ Understand the concept of a random sample.<br>○ Understand sources of bias in samples, and procedures to ensure randomness.<br>○ Identify and use procedures to ensure randomness.<br>○ Recognise and use graphical displays of real and simulated data of random samples from various types of distributions, including uniform, Bernoulli, binomial and normal. |
| 12.3 | Sample proportions | ○ Understand the concept of the sample proportion $\hat{p}$ as a random variable whose value varies between samples, and the formulas for the mean $p$ and standard deviation $\sqrt{p(1-p)/n}$ of the sample proportion $\hat{p}$, where $n$ is the sample size.<br>○ Recognise and use the approximate normality of the distribution of $\hat{p}$ for large samples.<br>○ Use repeated random sampling data, for a variety of values of $p$ and a range of sample sizes, to examine the distribution of $\hat{p}$ and the approximate standard normality of $\dfrac{\hat{p}-p}{\sqrt{\hat{p}(1-\hat{p})/n}}$, where the closeness of the approximation depends on both $n$ and $p$. |
| 12.4 | Confidence intervals for proportions | ○ Understand the concept of an interval estimate for a parameter associated with a random variable.<br>○ Understand and use the approximate confidence interval, $\left(\hat{p}-z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}},\hat{p}+z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}\right)$, as an interval estimate for $p$, the population proportion, where $z$ is the appropriate quantile for the standard normal distribution.<br>○ Understand and use the approximate margin of error, $z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$<br>○ Understand and use the relationship between margin of error, level of confidence and sample size.<br>○ Understand that there are variations in confidence intervals between samples and that most, but not all, confidence intervals contain $p$.<br>○ Model and solve problems that involve interval estimates for proportions, with and without technology. |

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# LESSON
## 12.2 Random sampling

### 12.2.1 Introduction

Suppose you were interested in the percentage of Year 12 graduates who plan to study Mathematics once they complete school. It is probably not practical to question every student. There must be a way that we can ask a smaller group and then use this information to make generalisations about the whole group.

### 12.2.2 Sample statistics and population parameters

A **population** is a group that you want to know something about, and a **sample** is the group within the population that you collect the information from. Normally, a sample is smaller than the population; the exception is a census, where the whole population is the sample.

The number of members in a sample is called the **sample size** (symbol $n$), and the number of members of a population is called the **population size** (symbol $N$). Sometimes the population size is unknown.

A **parameter** is a characteristic of a population, whereas a **statistic** is a characteristic of a sample. This means that a statistic is always known exactly (because it is measured from the sample that has been selected). A parameter is usually estimated from a sample statistic. (The exception is if the sample is a census, in which case the parameter is known exactly.)

The sample and population size are typically given as numbers, whereas the sample statistic and population parameter are expressed as a fraction or decimal.

#### The relationship between populations and samples

Population: size $N$
(this is the group we want to know about)

Parameter: $p$
(population proportion)

Random selection

Sample: size $n$
(this is the group that we can measure)

Statistic: $\hat{p}$
(sample proportion)
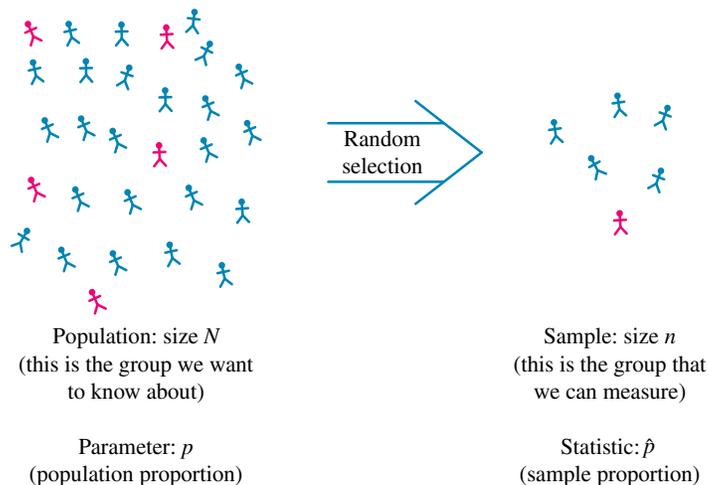
## Probability distributions and sampling variability

The proportion of successes in a population is called $p$ and is a constant value.

$$p = \frac{\text{number in the population with the favourable attribute}}{\text{population size}}$$

The sample size is called $n$.

The number of successes in the sample is called $X$.

The proportion of successes in the sample, $\hat{p}$, will vary from one sample to another.

$$\hat{p} = \frac{\text{number in the sample with the favourable attribute}}{\text{sample size}}$$

$$= \frac{X}{n}$$

It is known that there are 2.5 million people living in Brisbane and 1.27 million identify as female. In a sample of 100 people, 53 identified as female. This dataset could be described using the following notation:

$$N = 2\,500\,000$$

$$p = \frac{1\,270\,000}{2\,500\,000} = 0.507$$

$$n = 100$$

$$\hat{p} = \frac{53}{100} = 0.53$$

### WORKED EXAMPLE 1 Population parameters and sample statistics

**In a city with 500 000 households, it is known that 63% have a pet. In a sample of 180 households, 109 were found to have a pet. Identify the sample size, population size, population parameter and sample statistic for this dataset.**

| THINK | WRITE |
|---|---|
| a. The population is the total number of households. | a. $N = 500\,000$ |
| b. The population parameter is the number of households with a pet, expressed as a decimal or a fraction. | b. $p = \frac{63}{100}$ <br> $p = 0.63$ |
| c. The sample size is the number of households sampled. | c. $n = 180$ |
| d. The sample statistic is the number of these households with a pet, expressed as a decimal or a fraction. | d. $\hat{p} = \frac{109}{180} = 0.61$ |

## 12.2.3 Random samples

A good sample should be representative of the population. If we consider our initial interest in the proportion of Year 12 graduates who intend to study Mathematics once they finished school, it would be unsuitable to use a Mathematical Methods class as a sample, as this would skew the data towards a group that were mathematically minded. Similarly, we would not want to select classes that all had the same teacher, as this is another very specific group.

A **random sample** means that every member of the population must have the same probability of being selected. We can use technology to generate a random sample.

A **systematic sample** is often more convenient and almost as good as a random sample. In a systematic sample, every $k$th member of the population is sampled. For example, on a school database, you could survey every 20th name on the school roll. There is an assumption that choosing who you will survey systematically should generate a sample that is close enough to 'random', although there is still a chance that there will be bias within the chosen group.

Unfortunately, in the real world, we often must use a **self-selected sample**, where participants opt in to participating. This is almost never truly representative of the total population.

## 12.2.4 Using technology to select a sample

If you know the population size, it should also be possible to produce a list of population members. Assign each population member a number (from 1 to $N$). Use the random number generator on your calculator to generate a random number between 1 and $N$. The population member who was allocated that number becomes the first member of the sample. Continue generating random numbers until the required number of members has been picked for the sample. If the same random number is generated more than once, ignore it and continue selecting members until the required number has been chosen.

---

### WORKED EXAMPLE 2 Identifying sources of random or biased samples

**For each of the samples, decide whether they would represent a random sample. If not, explain the source of bias.**
**a. Commuters at a train station were interviewed about their preferred method of transportation.**
**b. Every 40th name on a school roll is selected and interviewed about extracurricular offerings at the School.**
**c. A random number generator is used to select a sample of 10 employees at a company who are interviewed about staff wellbeing.**
**d. Year 9 students were interviewed about the items at the tuckshop.**

| THINK | WRITE |
|---|---|
| a. Are commuters at a train station reflective of all people commuting to work? | a. No, because this sample would overrepresent people who catch the train to work. |
| b. Is every 40th person reflective of all students at the school? | b. Yes, a random sample. |
| c. Is a random number generator an effective way to select a sample? | c. Yes, a random sample. |
| d. Are Year 9 students reflective of all students at the school? | d. No, because this sample does not represent students in other year levels. |

Use a random number generator to select a committee of 4 from those listed below.

| John | Peter | Damian | Adele |
|------|-------|--------|-------|
| Laura | Jessica | Sally | Gemma |
| Emilia | Ashton | Steven | Andrew |
| Tim | Emma | Ben | Terry |
| Michael | Owen | Elizabeth | Simon |

**THINK**

1. Everyone needs to be assigned an integer value from 1 to 20. Replace the names in the table with the numbers 1 to 20.

2. A calculator can be used to generate four random numbers. If a number appears twice, do not record the repeated value.

3. List the names of the people in the committee.

**WRITE**

| 1 | 2 | 3 | 4 |
|---|---|---|---|
| 5 | 6 | 7 | 8 |
| 9 | 10 | 11 | 12 |
| 13 | 14 | 15 | 16 |
| 17 | 18 | 19 | 20 |

8, 16, 19, 11

Gemma, Terry, Elizabeth, Steven

**TI | THINK**

**WRITE**

1. Open a calculator page.

2. Press MENU
5 Probability
4 Random
2 Integer

3. Input 'randInt(1, 20)' to generate a random number between 1 and 20 and press ENTER.

4. Continue to press ENTER to generate more numbers.

**CASIO | THINK**

**WRITE**
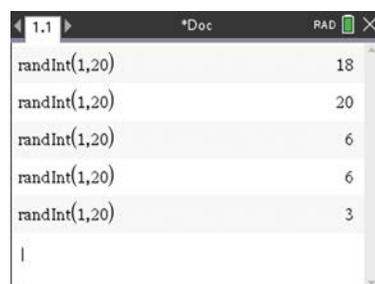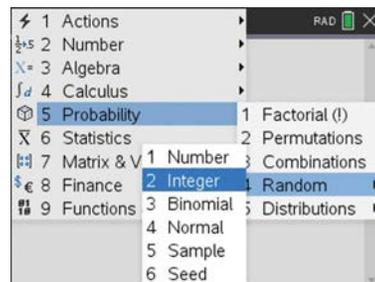
1. Open a Run-Matrix screen.

2. Press OPTN then navigate across to the Probability menu by pressing F6 then F3 for Prob.

3. Select the Random integer function by pressing F4 to select RanInt#.

4. Input the range RanInt#(1, 20) and press EXE.

5. To continue generating numbers, press EXE.
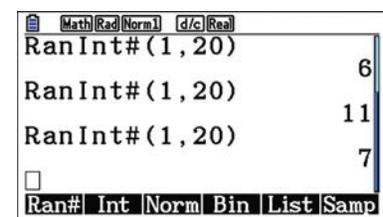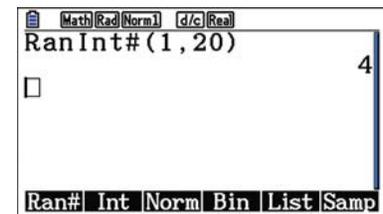
## 12.2.5 Errors in sampling

Even if every effort is made to select a random sample, there is no guarantee that it is suitably reflective of the total population. This is easiest to consider with discrete distributions, but the same is true for continuous.

Randomly selecting samples from uniform and normally distributed datasets will typically give more accurate data due to their symmetry, as shown in the distributions below.

Discrete uniform
distribution

Discrete normal
distribution

However, randomly selecting samples from datasets that are skewed, like the discrete distributions shown here, have a high risk of being biased.

Positively skewed distribution

Negatively skewed distribution
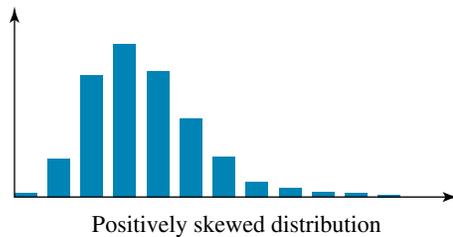
In a positively skewed distribution, a sample might include more lower values, suggesting the data clusters around the lower end. Conversely, a random sample might pick up larger outliers, implying higher values overall. This applies similarly to negatively skewed distributions.
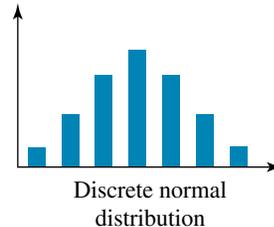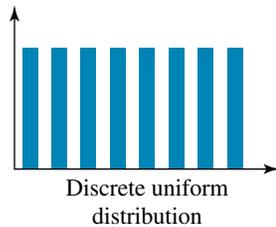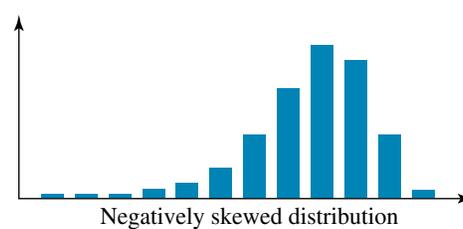
Though sampling from uniform and normal distributions usually reflects the total dataset better, the randomness can yield anomalies, skews the results. Without the entire dataset, determining its exact distribution is challenging, complicating adjustments for sample biases.

Despite these issues, sampling is often necessary. Thus, making informed choices about sampling methods and sizes, and critically evaluating results, is essential.

---

### WORKED EXAMPLE 4 Effect of skewed distributions on sampling

**A positively skewed distribution has a mean of 56 and a median of 48. A student takes a random sample of 10 values from this distribution.**
**a. Is it likely that the sample mean will be greater than or less than the population mean? Explain your reasoning.**
**b. If the sample mean turns out to be 62, what can you infer about the values included in the sample?**

**THINK**

**a.** If the data is mostly clustered towards the lower end, with a long tail, it is likely that 10 random values will overrepresent the lower end of the data set.

**b.** If the sample mean turns out to be larger than the population mean, then some of the large outliers must have been selected in the sample.

**WRITE**

**a.** The sample mean will likely be smaller than the population mean, as most of the data is clustered around the lower end.

**b.** The values randomly included in the sample must include some of the large outliers in the dataset.

| **12.2 Exercise** | **12.2 Exam questions** on |
|---|---|

| Simple familiar | Complex familiar | Complex unfamiliar |
|---|---|---|
| 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 | 11, 12, 13, 14, 15 | 16 |

These questions are even better in jacPLUS!
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### Simple familiar

1. **WE1** In a 99-g bag of lollies, there were 6 green lollies out of the 15 that were counted. Determine the sample proportion, $\hat{p}$.

2. Hang is interested in seedlings that can grow to more than 5 cm tall in the month of her study period. She begins with 20 seedlings and finds that 6 of them are more than 5 cm tall after the month. Determine the sample proportion, $\hat{p}$.

3. In a school with 1500 students, 1051 play Football. In a sample of 50 students chosen at random, 32 indicated that they played Football. Identify the sample size, population size, population parameter and sample statistic for this dataset.

4. In an electorate with 120 000 registered voters, it is known that 3% did not cast their vote in the most recent election. When 150 of these voters were interviewed, 4 admitted they had not voted, and had instead been fined. Identify the sample size, population size, population parameter and sample statistic for this dataset.

5. According to the 2011 census, there is an average of 2.6 people per household. Determine whether this a sample statistic or a population parameter.

6. A morning television show conducts a viewer phone-in poll and announces that 95% of listeners believe that Australia should become a republic. Comment on the validity of this type of sample.

7. Tony took a survey by walking around the playground at lunch and asking fellow students questions. Explain why this is not the best sampling method.

8. **WE2** For each of the samples, decide whether they would represent a random sample. If not, explain the source of bias.
   a. Students in the basketball program are interviewed about their favourite sports.
   b. Everyone in your methods class is interviewed about their after-school study intentions.
   c. A random number generator is used to select 10 lucky door prizes at a function.

**9.** **WE3** Use technology to select a committee of six from those listed below.

| John | Peter | Damian | Adele |
|------|-------|--------|-------|
| Laura | Jessica | Sally | Gemma |
| Emilia | Ashton | Steven | Andrew |
| Tim | Emma | Ben | Terry |
| Michael | Owen | Elizabeth | Simon |

**10.** Abby, Adam, Adelaide, Adeline, Alex, Amelia, Andrew, Anthony, Astor and Ava volunteer to help at their school's open day. Use technology to randomly select four students for the role.

**Complex familiar**

**11.** A standard warranty lasts for 1 year. It is possible to buy an extended warranty for an additional 2 years. The insurer decides to use the sales figures from Tuesday to estimate the proportion of extended warranties sold. If 537 units were sold and 147 of them included extended warranties, estimate the proportion of sales that will include extended warranties.

**12.** It is known that in a sample of 1350 Queenslanders, 762 said that they regularly use greenspaces. If there are 5.4 million people currently living in Queensland, determine how many you would expect would regularly utilise greenspaces.

**13.** **WE4** A negatively skewed distribution has a mean of 112 and a median of 135. A student takes a random sample of 10 values from this distribution.

**a.** Is it likely that the sample mean will be greater than or less than the population mean? Explain your reasoning.
**b.** If the sample mean turns out to be 109, what can you infer about the values included in the sample?

**14.** A distribution of unknown shape is known to have a mean of 85 and a median of 86. A researcher takes a sample of 10 scores and finds a mean of 84 and a median of 85. She is concerned that this still does not adequately describe the dataset, so she takes another sample of 25 scores. This time she calculates a mean of 89 and a median of 85. Describe the likely shape of the data and explain your answer mathematically.

**15.** Consider a positively skewed distribution and a normal distribution, both with a mean of 30.

**a.** Sketch and label each distribution.
**b.** Predict which sample mean is likely to be higher and explain why you would expect them to be different.
**c.** Explain for which distribution a larger sample size is more important.

**Complex unfamiliar**

**16.** A battery supply company believes that the lifespan of their batteries follows a normal distribution with a mean of 600 hours and a standard deviation of 40 hours. A quality control team takes two samples both of size ten. The first sample has a mean lifespan of 605 hours, but the second sample has a mean lifespan of 578 hours. Confused, the team decides to take a sample of 50 batteries, and finds this mean is 592 hours. Comment on the reasonableness of their belief that the distribution is normal providing full mathematical justification.

**Fully worked solutions for this chapter are available online.**

# LESSON
## 12.3 Sample proportions

## 12.3.1 Sampling distribution of $\hat{p}$

In most of the questions in 13.2, we calculated sample statistics even though we already had the population parameter. This is unrealistic. Typically, we are taking a sample because the population is too large to gather data from. Although $p$ is a constant value, the value of $\hat{p}$ is variable, and depends on the values within the sample. In this lesson, we will explore what would happen if you took lots of samples of the same size from a larger dataset (assume you return each sample to the population before selecting again).

Consider a company with 1000 employees whose salaries range from $70 000 to $250 000. As the Chief Financial Officer, you know that the average salary is $149 500, with a standard deviation of $30 465. To validate your budget and gain further insights, you engage an external auditing firm to conduct a salary review.

The audit team consists of 10 independent reviewers, who each interview 15 employees. This results in 10 unique samples, providing different perspectives on employee salaries.

The table below summarises the findings from each reviewer:

| Reviewer | Sample mean ($\hat{p}$) | Sample standard deviation |
|:---:|:---:|:---:|
| 1 | $149 218 | $39 532 |
| 2 | $155 113 | $22 506 |
| 3 | $144 390 | $29 086 |
| 4 | $151 420 | $40 131 |
| 5 | $159 461 | $17 024 |
| 6 | $139 797 | $25 851 |
| 7 | $152 917 | $38 778 |
| 8 | $150 315 | $20 735 |
| 9 | $147 881 | $28 879 |
| 10 | $144 421 | $31 487 |

Although the range of $\hat{p}$ values is over $20 000, the average of these values is $149 493.30.

We can also map these means using a number line:



| 135 | 140 | 145 | 150 | 155 | 160 | 165 |

Salaries ($'000)
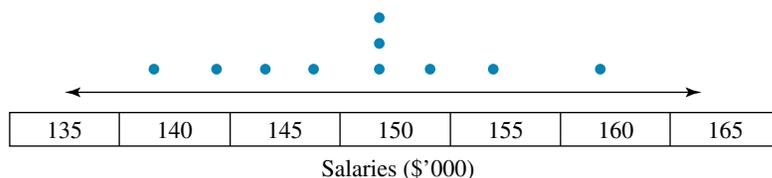
Even this small-scale example illustrates that if we continue to take samples of the same size, and determining sample means for each one, the distribution of $\hat{p}$ values will follow a normal distribution with a mean of $p$. Consequently, when we take enough samples from a large dataset, we find:



The further $\hat{p}$ is below $\mu_{\hat{p}}$, the less likely

$\mu_{\hat{p}}$

The further $\hat{p}$ is above $\mu_{\hat{p}}$, the less likely.

The most likely value of $\hat{p}$

---
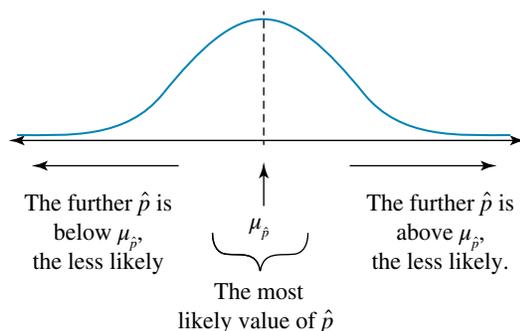
### The sample proportion $\hat{p}$

**The formulas for the mean and standard deviation of the sample proportion $\hat{p}$:**

| Mean | $p$ |
|---|---|
| Standard deviation | $\sqrt{\dfrac{p(1-p)}{n}}$ |

**where $n$ is the sample size.**

---

Given that we often do not know the value of $p$, we must instead use our sample statistic value, $\hat{p}$, as this is our best approximate for $p$.

For the normal approximation to hold, it is important to have a large enough sample size, $n$, as this reduces the standard deviation. Moreover, if the true population mean is very close to 0 or 1, the distribution of $\hat{p}$ can be skewed, making the normal approximation invalid.

Given that the value of $\hat{p}$ is normally distributed, we can adapt our standard $z$-score formula to suit this context.

---

### Standard $z$-score formula

$$z = \frac{x - \mu}{\sigma}$$

### Adapted $z$-score formula for sample proportions

$$z = \frac{\hat{p} - p}{\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}}$$

---

This adapted $z$-score formula can be used to assess the accuracy of the sample statistic.

## WORKED EXAMPLE 5 Calculating sample statistics

If $N = 600$, $n = 60$ and $p = 0.3$:

a. calculate the mean of the distribution
b. calculate the standard deviation of the distribution, correct to 2 decimal places.

| THINK | WRITE |
|---|---|
| a. The mean is $p$. | a. $\mu_{\hat{p}} = p$ <br> $\quad = 0.3$ |
| b. 1. Write the rule for the standard deviation. | b. $\sigma_{\hat{p}} = \sqrt{\dfrac{p(1-p)}{n}}$ |
| 2. Substitute the appropriate values and simplify. | $= \sqrt{\dfrac{0.3 \times (1-0.3)}{60}}$ <br> $= 0.06$ |

## WORKED EXAMPLE 6 Using the normal approximation

Assume that 72% of Year 12 students will go on to tertiary education. Using the normal approximation, determine the probability that, in a random sample of 150 Year 12 students, less than 70% will go on to tertiary education.

| THINK | WRITE |
|---|---|
| 1. The mean is $p$. | $\mu_{\hat{p}} = p$ <br> $p = 0.72$ |
| 2. Write the rule for the standard deviation. | $\sigma_{\hat{p}} = \sqrt{\dfrac{p(1-p)}{n}}$ |
| 3. Substitute the appropriate values and simplify. | $= \sqrt{\dfrac{0.72(1-0.28)}{150}}$ <br> $= 0.0367$ |
| 4. Use a calculator to determine the probability. | $P(\hat{p} < 0.7) = 0.2927$ |

## WORKED EXAMPLE 7 Using sample statistic formulae

A distribution for $\hat{p}$ has a standard deviation of 0.0204. If the sample size was 480 and $\hat{p} < 0.4$, determine the population proportion, correct to 2 decimal places.

| THINK | WRITE |
|---|---|
| 1. The standard deviation is 0.0204 and the sample size is 480. | $\sigma_{\hat{p}} = 0.0204$ <br> $n = 480$ |
| 2. Write the rule for the standard deviation. | $\sigma_{\hat{p}} = \sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$ |

**3.** Substitute the appropriate values and simplify.

$$0.0204 = \sqrt{\frac{\hat{p}(1-\hat{p})}{480}}$$

$$0.000\,416\,16 = \frac{\hat{p}(1-\hat{p})}{480}$$

$$0.000\,416\,16 = \frac{\hat{p}(1-\hat{p})}{480}$$

$$0.199\,756\,8 = \hat{p}(1-\hat{p})$$

**4.** Set up a quadratic equation and solve for both values of $\hat{p}$.

$$\hat{p}^2 - \hat{p} + 0.199\,756\,8 = 0$$

**5.** Use the condition $\hat{p} < 0.4$ to determine the correct value of $\hat{p}$.

$$\hat{p} = 0.7242 \text{ or } \hat{p} = 0.2758$$
$$\therefore \hat{p} = 0.2758$$

## Exercise 12.3 Sample proportions

**learn**on

| **12.3 Exercise** | **12.3 Exam questions** on |
|---|---|

| **Simple familiar** | **Complex familiar** | **Complex unfamiliar** |
|---|---|---|
| 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 | 11, 12, 13, 14, 15 | 16 |

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### Simple familiar

**1.** **WE5** If $N = 500$, $n = 50$ and $p = 0.5$:

    **a.** State the mean of the distribution.
    **b.** Calculate the standard deviation of the distribution, correct to 2 decimal places.

**2.** If $N = 1000$, $n = 100$ and $p = 0.8$:

    **a.** State the mean of the distribution.
    **b.** Calculate the standard deviation of the distribution, correct to 2 decimal places.

**3.** If $N = 1500$, $n = 150$ and $p = 0.15$, calculate the mean and standard deviation for the distribution of $\hat{p}$. Give your answers correct to 3 decimal places where appropriate.

**4.** If $N = 1200$, $n = 100$ and $p = 0.75$, calculate the mean and standard deviation for the distribution of $\hat{p}$. Give your answers correct to 3 decimal places where appropriate.

**5.** **WE6** In a large inner-city office building, 36% of employees bring their own lunch on any given day. Using the normal approximation, determine the probability that, in a random sample of 75 employees more than 40% of them brought their own lunch that day.

**6.** Using the normal approximation, determine the probability that, when a die is rolled 100 times, the proportion of 2s rolled will be less than or equal to 0.2.

**7.** A study shows that 63% of Queenslanders travel to the beach for their holidays. Using the normal approximation, determine the probability that, in a random sample of 180 people, fewer than 60% will go to the beach.

8. A distribution for $\hat{p}$ has a mean of 0.12 and a standard deviation of 0.0285. State the population proportion and calculate the sample size.

9. A distribution for $\hat{p}$ has a mean of 0.81 and a standard deviation of 0.0253. State the population proportion and calculate the sample size.

10. **WE7** If $N = 1500$, $n = 150$ and $p = 0.15$, use technology to graph the distribution for $\hat{p}$.

11. A distribution for $\hat{p}$ has a standard deviation of 0.015. If the sample size was 510 and $\hat{p} > 0.5$, determine the population proportion, correct to 2 decimal places.

12. A distribution for $\hat{p}$ has a standard deviation of 0.0255. If the sample size was 350, determine the population proportion, correct to 2 decimal places.

13. It is known that 72% of people who visit a shopping centre make a purchase. In a random sample of 150 shoppers, 110 said that they made a purchase that day. Determine the approximate probability that, in another random sample of 150 shoppers, the proportion who made a purchase that day is greater than in the original sample.

14. Nidya is a top goal shooter. The probability of her getting a goal is 0.8. To keep her skills up, each night she has 200 shots on goal. Applying the normal approximation, determine the probability that on Monday the proportion of times she scores a goal is between 0.8 and 0.9, given that it is more than 0.65.

15. In the latest referendum, approximately 87.5% of registered voters cast their ballots in the election. A government analysis focuses on an electorate with 115 000 registered voters. If it is known that fewer than 101 000 people voted in this electorate, calculate the probability that the sample proportion of voters exceeds the national average of 87.5%.

16. A school principal wants to understand the overall proportion of students at her school who feel engaged in their classes. To investigate this, she surveys two different samples each of size $n$ across the student body. Interestingly, she finds that the sample proportion in one is 60%, with a $z$-score of 0.2155, and in the other is 55% with a $z$-score of $-0.8621$. Determine the true population parameter and the size of both samples.

**Fully worked solutions for this chapter are available online.**

# LESSON
## 12.4 Confidence intervals for proportions

### 12.4.1 Introduction to confidence intervals

You have just learned that different samples will likely have different values for $\hat{p}$. So how helpful are any of these values?

Given that we know that our $\hat{p}$ values are normally distributed with a mean of $p$, we can use our knowledge of $z$-scores and sampling to determine a range of values we can reasonably expect the true population parameter to lie within.

Working with our sample statistic formulae:

$$\mu_{\hat{p}} = p$$

$$\sigma_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}}$$

As we do not know the exact value for $p$, the best estimate is $\hat{p}$. This means that the best estimate of the standard deviation is $\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$.

We also learnt that for normal distributions, $z = \dfrac{x-\mu}{\sigma}$, which we can adapt to $z = \dfrac{\hat{p} \pm p}{\sigma_{\hat{p}}}$ to determine the lower and uppermost acceptable values of $z$.

Rearranging this gives rise to our formula for calculating confidence intervals: $\hat{p} \pm z\sigma_{\hat{p}}$.

> **Approximate confidence interval**
>
> **An approximate confidence interval for a population proportion is given by**
>
> $$\left(\hat{p} - z\sqrt{\frac{\hat{p}\,(1-\hat{p})}{n}},\ \hat{p} + z\sqrt{\frac{\hat{p}\,(1-\hat{p})}{n}}\right)$$

The distance between the endpoints of the confidence interval and the sample estimate is called the **margin of error**, $E$.

> ### Margin of error
>
> $$E = z\sqrt{\frac{\hat{p}\,(1-\hat{p})}{n}}$$
>
> **and the confidence interval for a population proportion is therefore given by $\hat{p} \pm E$.**

The empirical rule for normal distributions tells us that 95% of data lies within two standard deviations of the mean, however, this is an approximation. If you use technology, you can see that a more accurate value is that 95% of the data lies within 1.96 standard deviations of the mean. Hence, for a 95% confidence interval, we use the $z$-score 1.96.

> ### 95% confidence interval
>
> **A 95% confidence interval for a population proportion is given by $\hat{p} \pm E$ where $E = z\sigma_{\hat{p}}$.**
>
> **Therefore, the interval is $(\hat{p} - 1.96\sigma_{\hat{p}},\ \hat{p} + 1.96\sigma_{\hat{p}})$.**

We can calculate other confidence intervals by determining the $z$-score that corresponds to the desired percentage of data.

## 12.4.2 Interpreting a confidence interval

A common misconception is that a 95% confidence interval implies that we are 95% confident that the true population parameter lies within 1.96 standard deviations of the sample statistic. However, this is not accurate. The image below shows how we **should** interpret a confidence interval. Any single sample could be anomalous, and it's confidence interval may not contain the true population parameter. However, if you take repeated samples, we expect 19 of the 20 to contain the true value. Again, this stresses the importance of repeated sampling.
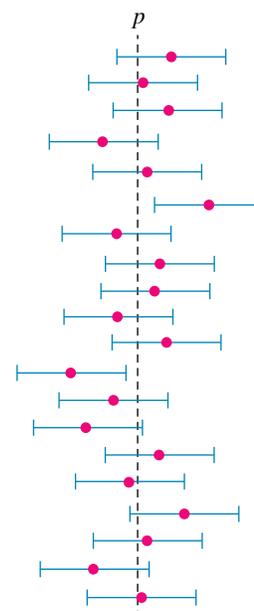
## 12.4.3 The relationship between margin of error, $z$-scores and confidence intervals

Note that to reduce the margin of error for a confidence interval, you can increase the sample size.

There is a trade-off between confidence level and the size of the margin of the error. If you use the $z$-score of 2.58 you can raise the confidence level to 99%, which indicates that 99 out of 100 sample intervals will include the true population parameter. However, this higher confidence level also leads to an increased margin of error, potentially decreasing the practical utility of the approximation.

## WORKED EXAMPLE 8 Calculating confidence intervals

There are 20 people in your class and 25% are planning on buying their lunch. Estimate the proportion of the school population that will purchase their lunch today. Determine a 95% confidence interval for your estimate, given $z = 1.96$.

| THINK | WRITE |
|---|---|
| 1. There are 20 people in the class. This is the sample size. 25% are buying their lunch. This is the sample proportion. | $n = 20$<br>$\hat{p} = 0.25$ |
| 2. For a 95% confidence interval, $z = 1.96$. | $z = 1.96$ |
| 3. The confidence interval is $\left(\hat{p} - z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}\right).$ Determine $z\sigma_{\hat{p}}$. | $z\sigma_{\hat{p}} = z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$<br>$= 1.96\sqrt{\dfrac{0.25 \times 0.75}{20}}$<br>$= 0.1898$ |
| 4. Identify the 95% confidence interval by determining the upper and lower values. | $\hat{p} - z\sigma_{\hat{p}} = 0.25 - 0.1898$<br>$= 0.0602$<br>$\hat{p} + z\sigma_{\hat{p}} = 0.25 + 0.1898$<br>$= 0.4398$ |
| 5. Write the answer. | The confidence interval, $\text{CI} = (0.06, \ 0.44)$. |

## WORKED EXAMPLE 9 Using confidence intervals to determine a sample size

Grow Well are 95% sure that 30% to 40% of shoppers prefer their mulch. Determine the sample size needed for this level of confidence.

| THINK | WRITE |
|---|---|
| 1. The confidence interval is symmetric about $\hat{p}$ : $(\hat{p} - z\sigma_{\hat{p}}, \hat{p} + z\sigma_{\hat{p}})$, so the value of $\hat{p}$ must be halfway between the upper and lower values of the confidence interval. | $\hat{p} = \dfrac{30 + 40}{2}$<br>$= 35\%$<br>$= 0.35$ |
| 2. State the $z$-value related to the 95% confidence interval. | $z = 1.96$ |
| 3. The lower value of the confidence interval, 30%, is equivalent to $\hat{p} - z\sigma_{\hat{p}}$. Substitute the appropriate values. *Note:* The equation $0.4 = \hat{p} + z\sigma_{\hat{p}}$ could also have been used. | $0.3 = \hat{p} - z\sigma_{\hat{p}}$<br>$= \hat{p} - z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$<br>$= 0.35 - 1.96\sqrt{\dfrac{0.35(1-0.35)}{n}}$ |
| 4. Solve for $n$. | $n = 349.586$ |
| 5. Write the answer. | The sample size needed was 350 people. |

Calculate the sample size needed to achieve a 3% margin of error for an approximate 95% confidence interval for the proportion $p$ of households in Australia that recycle regularly, given that a sample proportion $\hat{p}$ was estimated to be 0.6.

| THINK | WRITE |
|---|---|
| 1. The margin of error needs to be 3%, this mean $E = 0.03$. A 95% confidence interval means $z = 1.96$. The value of $\hat{p} = 0.6$. | $E = 0.03$ <br> $z = 1.96$ <br> $\hat{p} = 0.6$ <br><br> $E = z\sigma_{\hat{p}} = z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$ |
| 2. Write the rule for the margin of error. | $0.03 = 1.96\sqrt{\dfrac{0.6(1-0.6)}{n}}$ |
| 3. Substitute the appropriate values and calculate. | $\left(\dfrac{0.03}{1.96}\right)^2 = \dfrac{0.6 \times 0.4}{n}$ |
| 4. Round the value of $n$ to determine a suitable sample size. | $n = \dfrac{0.921984}{0.0009}$ <br> $n = 1024.43$ <br> $\therefore$ A sample size of 1025 is required to attain a 3% margin of error |

# Exercise 12.4 Confidence intervals for proportions

learn**on**

| 12.4 Exercise | 12.4 Exam questions **on** |
|---|---|

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| Simple familiar | Complex familiar | Complex unfamiliar |
|---|---|---|
| 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 | 12, 13, 14, 15, 16, 17 | 18 |

### Simple familiar

1. A local council wants to estimate the proportion of its citizens who contribute to political parties. They survey a random sample of 500 people and find that 27 of them make contributions.
   a. State the value of $\hat{p}$.
   b. Calculate a 95% confidence interval for the estimate.

2. A survey of 1000 high school students found that 876 of them used social media.
   a. State the value of $\hat{p}$.
   b. Calculate a 95% confidence interval for the estimate.

3. **WE8** Of 30 people surveyed, 78% said that they like breakfast in bed. Estimate the proportion of the population that like breakfast in bed. Determine a 95% confidence interval for the estimate.

**4.** Of the 53 people at swimming training today, 82% said that their favourite stroke is freestyle. Estimate the proportion of the population whose favourite stroke is freestyle. Determine a 95% confidence interval for the estimate.

**5.** Of the 25 people in an English classroom, 21 also study mathematics.
   **a.** Estimate the proportion of students who study maths.
   **b.** Determine a 95% confidence interval for the estimate.

**6.** **WE9** Jenny samples 116 people and finds that 86% of them plan to go swimming over the summer holidays. Estimate the proportion of the population that plan to go swimming over the summer holidays. Determine a 99% confidence interval for your estimate.

**7.** Yuki samples 95 people and finds that 30% of them eat chocolate daily. Estimate the proportion of the population that eats chocolate daily. Determine a 90% confidence interval for your estimate.

**8.** Two hundred Queenslanders are surveyed about sports and 53 state that their favourite sport is Rugby League.
   **a.** Calculate a 95% confidence interval for the estimate.
   **b.** Calculate and compare this value to a 90% and 99% confidence interval.

**9.** Teleco is being criticised for its slow response time when handling complaints. The company claims that it will respond within 1 day. Of the 3760 complains in a given week, a random sample of 250 was selected. Of these, it was found that 20 of them had not been responded to within 1 day.
   **a.** Determine the 95% confidence interval for the proportion of claims that take more than 1 day to resolve.
   **b.** Determine the 99% confidence interval for the proportion of claims that take more than 1 day to resolve.

**10.** In a country town, the owners of Edie's Eatery are 95% sure that 35% to 45% of their customers love their homemade apple pie. Determine the sample size needed for this level of confidence.

**11.** If Parkers want to be 90% confident that between 75% and 85% of their customers will shop in their store for more than 2 hours, determine the sample size that will be needed.

**Complex familiar**

**12.** **WE10** Tatiana is conducting a survey to estimate the proportion of Year 12 students who will take a gap year after they complete their studies. Previous surveys have shown the proportion to be approximately 15%. Determine the required size of the sample so that the margin of error for the survey is 3% in a confidence interval of approximately 95% for this proportion.

**13.** Calculate the sample size needed to achieve a 3% margin of error for an approximate 95% confidence interval when estimating the probability $p$ of getting heads in a coin toss, given that a sample proportion was estimated to be 0.53.

**14.** Smooth Writing are 95% sure that 25% to 35% of shoppers prefer their pen. Determine the sample size needed for this level of confidence.

**15.** An online tutoring company is 99% sure that 20% to 30% of students prefer to use their company. Determine the sample size needed for this level of confidence.

**16.** Barton's Dentistry want to be able to claim that 90% to 98% of people floss daily. They would like 99% confidence about their claim. Determine how many people they need to survey.

**17.** Bentons' claim that between 85% and 95% of their customers stay for more than 2 hours when they shop. If they surveyed 100 people, determine how confident they are about this claim.

### Complex unfamiliar

**18.** A research team is studying the effectiveness of a new drug and observes 100 participants, finding that 17 have no improvement in their symptoms. The team wants to determine the true efficacy of the drugs, which is the percentage of the population whose symptoms will improve. To ensure that their results are considered reliable, they need to achieve a margin of error of 5% with a 95% confidence interval. Determine whether the pilot sample size of 100 participants is sufficient to meet these requirements, and if not, calculate the appropriate sample size needed.



**Fully worked solutions for this chapter are available online.**

# LESSON
## 12.5 Review

## 12.5.1 Summary

Hey students! Now that it's time to revise this topic, go online to:

- Access the topic summary
- Review your results
- Watch teacher-led videos
- Practise exam questions

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## 12.5 Exercise

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| 12.5 Exercise | 12.5 QCAA questions **on** |
|---|---|

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| Simple familiar | Complex familiar | Complex unfamiliar |
|---|---|---|
| 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 | 13, 14, 15, 16 | 17, 18, 19, 20 |

### Simple familiar

1. Glen is interested in the proportion of internet users in his local town. He surveys 150 households and determines that 132 of them have internet access. What proportion of households in Glen's town have internet access?

2. A football club has 16 000 members, 7500 of whom are women. A random sample of 200 members was selected, and 96 of them were women. Identify the sample size, population size, population parameter and sample statistic for this dataset.

3. A yoghurt company wants to understand people's flavour preferences for their product. They know that of the 5000 cartons they sold last month 2100 were vanilla. In a random sample of 100 people, 43 purchased vanilla last month. Identify the sample size, population size, population parameter and sample statistic for this dataset.

4. If a population proportion is believed to be 0.25 and samples of size 30 are chosen, determine the standard deviation of $\hat{p}$.

5. It is known that 7 out of 12 people prefer Coke to Pepsi. If samples of 50 people are chosen, determine the standard deviation of $\hat{p}$.

6. In a recent voter survey, an approximate 90% confidence interval for the proportion of people who will vote for a particular party is (0.58, 0.66).
   a. Determine the value of $\hat{p}$ for this confidence interval.
   b. Calculate the value of the margin of error.

**7.** The formal committee of Brickwall State High School is trying to choose between strawberry cheesecake and passionfruit cheesecake for dessert at the formal dinner. There will be 423 students at the dinner. Of the 52 senior students sampled, 23 prefer the strawberry cheesecake. Determine the proportion of students who chose passionfruit cheesecake.

**8.** If $N = 2000$, $n = 120$ and $p = 0.37$, calculate:

a. the mean of the population
b. the standard deviation of the population, correct to 3 decimal places.

**9.** A textbook publishing company is 95% sure that 65%–75% of students prefer to use their resources. Determine the sample size needed for this level of confidence.

**10.** On a particular Friday night, 52 000 people went to Suncorp Stadium to watch the football. Every 25th person entering the stadium was asked who they thought would win. Out of the people asked, 1600 people believed that the Queensland Reds would win.

a. Determine the population size.
b. Determine the sample size.
c. Determine $\hat{p}$.

**11.** **MC** Which of the following could be a distribution for $\hat{p}$ for large samples? Justify your answer.

**A.**

**B.**

**C.**

**D.**
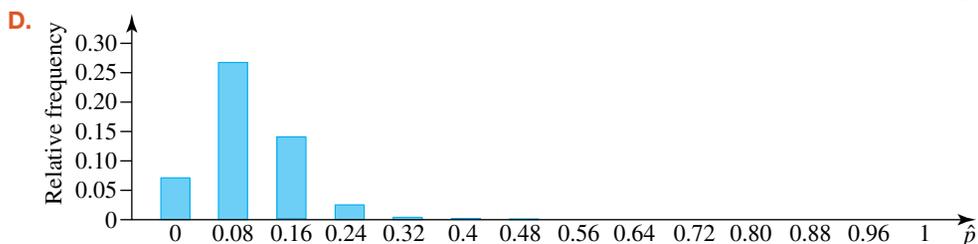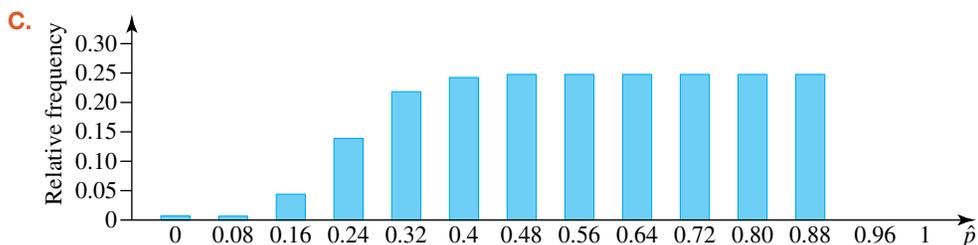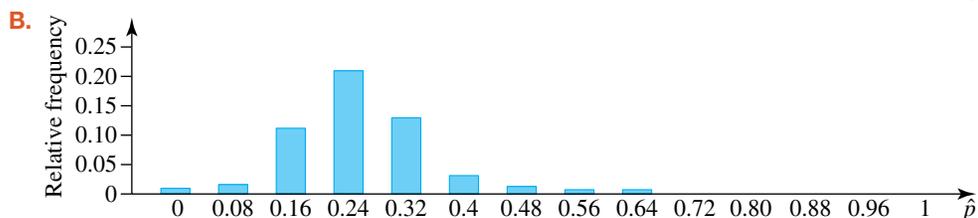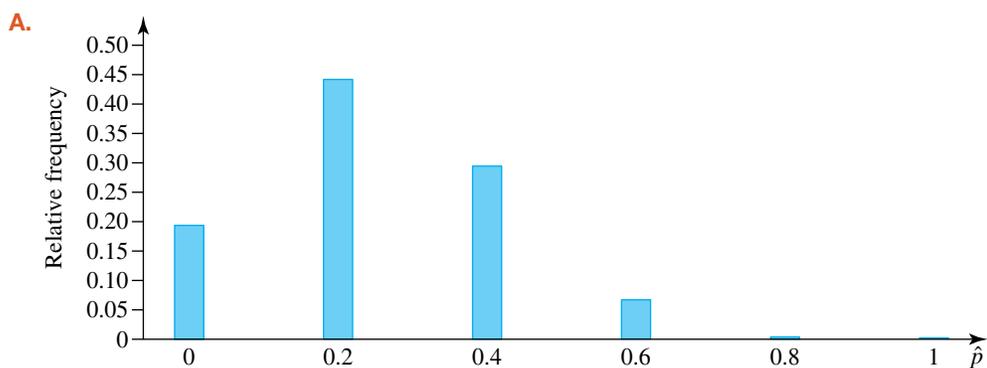
12. A local radio station surveys 130 people and determines that 17 of them listen to the radio in the car.

    **a.** Estimate the proportion of the population that listen to the radio in the car.

    **b.** Determine a 95% confidence interval for this estimate.

### Complex familiar

13. A positively skewed distribution has a mean of 87 and a median of 65. A student takes a random sample of 10 values from this distribution.

    **a.** Is it likely that the sample mean will be greater than or less than the population mean? Explain your reasoning.

    **b.** If the sample mean turns out to be 92, what can you infer about the values included in the sample?

14. Each year the Year 7 class from Gympie State High visits a theme park. One hundred of the students decided to go on the monster rollercoaster, and 10 of them complained of feeling dizzy afterwards.

    **a.** Determine the value of the sample proportion.

    **b.** Write an expression for the 85% confidence interval for the likelihood of feeling dizzy.

    **c.** Determine the margin of error, $E$, for the 85% confidence interval.

    **d.** If only 50 people had decided to go on the rollercoaster, determine the effect on the margin of error.

15. A distribution for $\hat{p}$ has a standard deviation of 0.05. If the sample size is 50, determine the population proportion, correct to 2 decimal places.

16. Maxwell Industries surveyed 100 customers and believe that between 85% and 90% of their customers are satisfied with their level of service. How confident are they about that claim?

### Complex unfamiliar

17. Krypton Industries are 99% certain that between 67% and 83% of people prefer their product. Determine how many people were sampled for this level of confidence.

18. The state government wants to estimate the proportion of Queenslanders who regularly use toll roads. The data collection team wants to ensure that the margin of error is within 2% for an approximate 95% confidence interval (assume $z = 2$). Determine for what value of $\hat{p}$ the standard deviation is minimised, and based on this, make a recommendation about the size of the sample the government should use.

19. It is known that 26% of Year 12 students have their license by the time they graduate. If a random sample of 50 Year 12 students is chosen from the population, determine the probability that the sample proportion lies within one standard deviation of the population proportion. Use mathematical reasoning to justify your response.

20. A research group wants to estimate the proportion of adults in Queensland who use public transport to get to work. A survey of 500 people resulted in a 95% confidence interval for this proportion of [0.42, 0.50]. The group's leader believes this data to be too variable and asks for the variance to be no greater than 0.035%. Determine the smallest sample that can be selected to ensure this criteria is met.

# 12.5 Past QCAA exam questions

## Question 1 (1 mark)

*Source:* QCAA 2023 Mathematical Methods, Paper 1, Section 1, Q8; © State of Queensland (QCAA) 2024.

A sample of size $n$ was used to estimate a population proportion. An approximate margin of error of $3\%$ was calculated using $z = 1.96$. Given the sample proportion was $0.6$, determine $n$.

**A.** $n = \dfrac{\left(\frac{0.03}{1.96}\right)^2}{0.24}$

**B.** $n = \dfrac{0.24}{\left(\frac{0.03}{1.96}\right)^2}$

**C.** $n = \dfrac{\left(\frac{0.03}{1.96}\right)^2}{2.4}$

**D.** $n = \dfrac{2.4}{\left(\frac{0.03}{1.96}\right)^2}$

## Question 2 (4 marks)

*Source:* QCAA 2023 Mathematical Methods, Paper 2, Section 2, Q11; © State of Queensland (QCAA) 2024.

A researcher found that 17 out of 50 randomly selected people had used public transport in the past week.

**a.** Determine the sample proportion of people who had used public transport in the past week. **[1 mark]**

**b.** Determine an approximate $95\%$ confidence interval for the proportion of people who had used public transport in the past week. **[2 marks]**

**c.** Someone claims that: *50% of people use public transport each week.* Use your answer from Question **b.** to explain whether the data can or cannot support this claim. **[1 mark]**

## Question 3 (1 mark)

*Source:* QCAA 2022 Mathematical Methods, Paper 1, Section 1, Q8; © State of Queensland (QCAA) 2024.

In a survey, 80 respondents exercised daily, while 120 did not. When calculating the approximate $95\%$ confidence interval for the proportion of people who exercise daily, the margin of error is

**A.** $1.96\sqrt{\dfrac{0.4(1-0.4)}{200}}$

**B.** $0.95\sqrt{\dfrac{0.4(1-0.4)}{200}}$

**C.** $1.96\sqrt{\dfrac{0.67(1-0.67)}{120}}$

**D.** $0.95\sqrt{\dfrac{0.67(1-0.67)}{120}}$

*Source:* QCAA 2022 Mathematical Methods, Paper 1, Section 1, Q10; © State of Queensland (QCAA) 2024.

A survey plans to draw conclusions based on a random sample of $1\%$ of Queensland's adult population. To be regarded as a random sample, every

**A.** adult in the population will be placed in an alphabetical list and every 100th person will be selected for the sample.
**B.** adult in the population can choose to participate until the sample size has been reached.
**C.** subgroup within the population will be represented in a similar proportion in the sample.
**D.** adult in the population will have an equal chance of being selected for the sample.

**Question 5 (8 marks)**

*Source:* QCAA 2022 Mathematical Methods, Paper 2, Section 2, Q14; © State of Queensland (QCAA) 2024.

Ravi randomly sampled $200$ different pet owners in Brisbane and found that $50$ celebrate their pet's birthday. **[2 marks]**

**a.** Determine an approximate $95\%$ confidence interval for the proportion of Brisbane pet owners who celebrate their pet's birthday.

Two of Ravi's friends also randomly sampled Brisbane pet owners. The results are shown in the table.

| Friend's name | Number sampled | Number who celebrate their pet's birthday |
|---|---|---|
| Khadija | 100 | 26 |
| Tim | 150 | 34 |

Khadija suggested a more precise estimate for the proportion of Brisbane pet owners who celebrate their pet's birthday could be obtained by combining their results.

**b.** Using all available data, determine an approximate $95\%$ confidence interval for the proportion of Brisbane pet owners who celebrate their pet's birthday. **[2 marks]**
**c.** Use the results from part **a** and **b** to evaluate the reasonableness of Khadija's suggestion.
The proportion of all Brisbane pet owners who celebrate their pet's birthday is $0.24$. **[2 marks]**
**d.** Using the normal approximation, determine the probability that in a randomly selected sample of size $200$, more than $30\%$ of pet owners celebrate their pet's birthday. **[2 marks]**

**Fully worked solutions for this chapter are available online.**

# Answers

## Chapter 12 Sampling, proportions and interval estimates

### 12.2 Random sampling

#### 12.2 Exercise

1. $\dfrac{2}{5} = 0.4$

2. $\dfrac{3}{10} = 0.3$

3. Population size, $N = 1500$,
   sample size, $n = 50$,
   population parameter, $p = 0.7$,
   sample statistic, $\hat{p} = 0.64$

4. Population size, $N = 120\,000$,
   sample size, $n = 150$,
   population parameter, $p = 0.03$,
   sample statistic, $\hat{p} = 0.0267$

5. Population parameter

6. The sample is not random; therefore, the results are not likely to be random.

7. It is probably not random. Tony is likely to ask people who he knows or people who approach him.

8. **a.** No, these students are more likely to select basketball than students not in the basketball program.

   **b.** No, these students are more likely to pursue mathematics courses after school.

   **c.** Yes

9. Answers will vary.

10. Answers will vary.

11. $\dfrac{147}{537} \approx 0.274$

12. We would expect 3 048 000 Queenslanders to regularly utilise greenspaces.

13. **a.** The sample mean will likely be greater than the population mean because most of the data is at the upper end of the distribution. It is likely that the lower tail-end values may not be represented in a small sample.

   **b.** This could imply that the small sample did pick up on one of the anomalous tail values, painting an unrealistic picture of the data set.

14. Given that the small sample had a mean and median below the population, but the larger sample had a mean decidedly above, this implies that the dataset might feature mostly lower values, but then have a longer tail at the high end, which is positively skewed. However, it is impossible to know the shape of the whole dataset, as all samples are random.
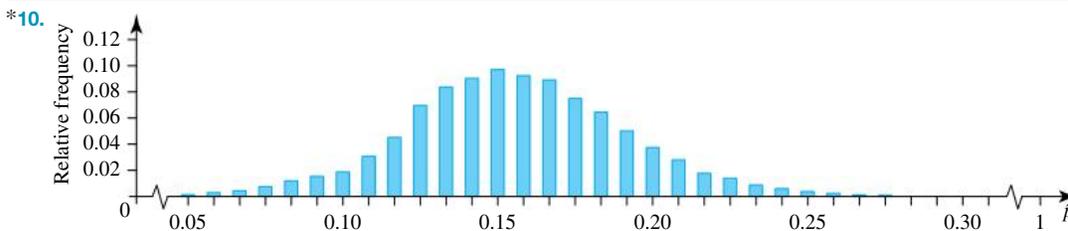
15. **a.**



Normal (no skew)     Positively skewed
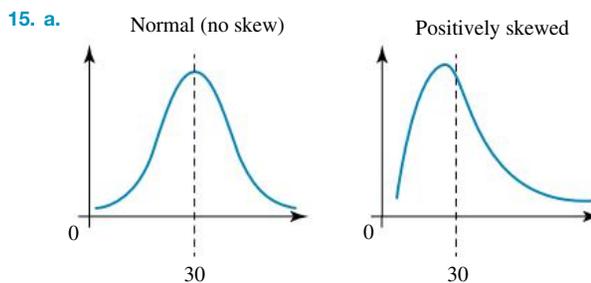
   **b.** The sample mean for the normal distribution is likely to be higher as a small random sample of the positively skewed distribution will likely overrepresent smaller values.

   **c.** A larger sample size would be more important in the positively skewed distribution, in hopes that this captures the full picture of the data, including some of the higher anomalous values.

16. The two small samples had very different means, which already implies that the data might not be normally distributed. The much larger sample had a mean between these two values but below the population mean. This would imply that smaller values are more common than higher values, indicating that the dataset might be positively skewed.

### 12.3 Sample proportions

#### 12.3 Exercise

1. **a.** 0.5      **b.** 0.07

2. **a.** 0.8      **b.** 0.04

3. $\mu_{\hat{p}} = 0.15$, $\sigma_{\hat{p}} = 0.029$

4. $\mu_{\hat{p}} = 0.75$, $\sigma_{\hat{p}} = 0.043$

5. $P(\hat{p} > 0.4) = 0.2351$

6. $P(\hat{p} < 0.2) = 0.9858$

7. $P(\hat{p} < 0.6) = 0.2023$

8. $p = 0.12$, $n = 130$

9. $p = 0.81$, $n = 240$

10. See the image at the bottom of the page.*

11. $p = 0.87$

12. $p = 0.35$ or $p = 0.65$

13. $P(\hat{p} > 0.7333) = 0.3585$

14. 0.4998

*10.

**15.** $P(\hat{p} > 0.875 \,|\, \hat{p} < 0.8783) = 0.4999$

**16.** $n = 115$ and $p = 0.59$

## 12.4 Confidence intervals for proportions

### 12.4 Exercise

**1. a.** $\hat{p} = 0.054$

   **b.** $CI = (0.0342, \ 0.0738)$

**2. a.** $\hat{p} = 0.876$

   **b.** $CI = (0.8556, \ 0.8964)$

**3.** $(0.63, \ 0.93)$

**4.** $(0.72, \ 0.92)$

**5. a.** $\hat{p} = 0.84$

   **b.** $CI = (0.6963, \ 0.9837)$

**6.** $(0.78, \ 0.94)$

**7.** $(0.22, 0.38)$

**8. a.** 95% $CI = (0.2039, \ 0.3261)$

   **b.** 90% $CI = (0.2137, \ 0.3163)$
   99% $CI = (0.1846, \ 0.3454)$
   As the value of the confidence interval approaches 100, the margin of error gets larger, resulting in a wider confidence interval. However, a larger confidence interval also enhances the likelihood that the true population mean falls within its range.

**9. a.** 95% $CI = (0.0463, \ 0.1137)$

   **b.** 99% $CI = (0.0357, \ 0.1243)$

**10.** 369

**11.** 173

**12.** 544

**13.** $n = 1064$

**14.** 323

**15.** 499

**16.** 235

**17.** 90%

**18.** The sample of size 100 is insufficient, and a sample of size 217 is required to achieve a 5% margin of error.

### 12.5 Review

### 12.5 Exercise

**1.** 0.88

**2.** Population size, $N = 16\,000$,
sample size, $n = 200$,
population parameter, $p = \dfrac{15}{32}$,
sample statistic, $\hat{p} = 0.48$

**3.** Population size, $N = 5000$,
sample size, $n = 100$,
population parameter, $p = 0.42$,
sample statistic, $\hat{p} = 0.43$

**4.** 0.079

**5.** $\sigma_{\hat{p}} = 0.0697$

**6. a.** 0.62      **b.** 0.04

**7.** 0.56

**8. a.** 0.37      **b.** 0.044

**9.** 323

**10. a.** 52 000     **b.** 2080     **c.** 0.77

**11.** B

**12. a.** $\hat{p} = 0.1308$

    **b.** 95% $CI = (0.0728, \ 0.1888)$

**13. a.** The sample mean will likely be less than the population mean because most of the data is at the lower end of the distribution. It is likely that the higher tail-end values may not be represented in a small sample.

    **b.** This could imply that the small sample did pick up on one of the anomalous larger values, causing the mean to be higher than expected, painting an unrealistic picture of the data set.

**14. a.** 0.1

    **b.** $0.1 \pm 0.0432$

    **c.** 0.0432

    **d.** $E$ increases by a factor of $\sqrt{2}$.

**15.** $p = 0.15$ or $0.85$

**16.** There is a 55% likelihood that the true proportion lies within this interval.

**17.** 195

**18.** $\hat{p} = \dfrac{1}{2}$ and $n \geq 2500$

**19.** There is an 87% chance that the sample proportion lies within one standard deviation of the population proportion.

**20.** The smallest sample size that will ensure a variance no greater than 0.035% is 710.

## 12.5 Past QCAA exam questions

**1.** B

**2. a.** 0.34

    **b.** $CI = (0.20869, 0.47130)$

    **c.** The claim is not supported.

**3.** A

**4.** D

**5. a.** $(0.19, \ 0.31)$

    **b.** $(0.2047, \ 0.2842)$

    **c.** By combining the results, the sample size is increased and the confidence interval width is reduced. The new sample statistic provides a better estimate for the population parameter.

    **d.** 0.0235