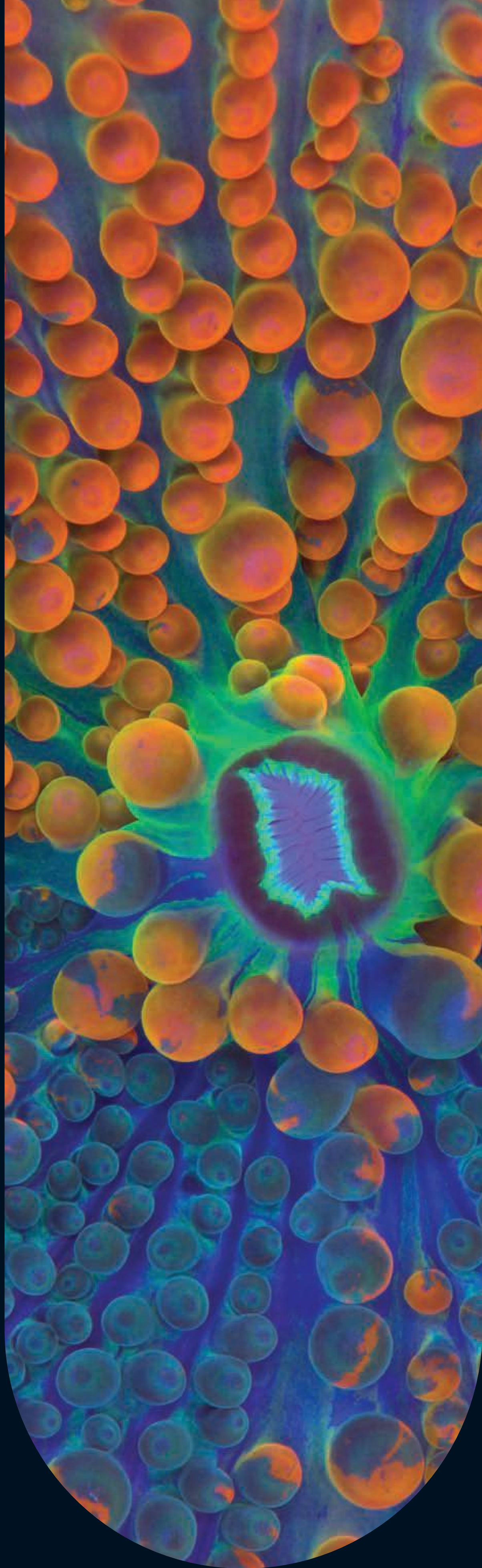
nelson maths.

12

General Mathematics

Units **3** & **4**

Lani Killalea
Tracey MacBeth-Dunn
Anna Wethereld
Series editor
Stephen Swift



QLD

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General Mathematics

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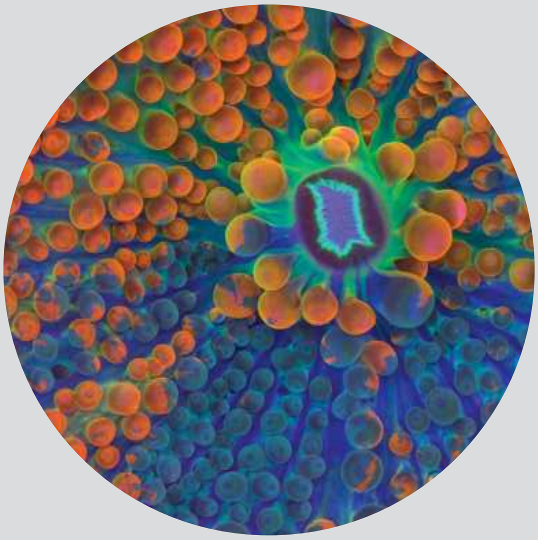
Ross Brodie
Regina Edwards
Stephen Swift

LEARNING DISCOVERY

SEA CORALS



Corals are animals, not plants, that grow on the ocean floor. Hard corals extract calcium from the water to build limestone skeletons, which form the colourful structures of coral reefs. The shape and growth patterns of these corals can be modelled using mathematical equations.

QLD

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UNIT 4

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Preface

Nelson Maths 11–12 QLD is a new QCE mathematics series that is backed by research into the science of learning. The design and structure of the series has been informed by teacher advice and evidence-based pedagogy, with the focus on preparing QCE students for their exams and maximising their learning achievement.

- Using **backwards learning design**, this series has been built by analysing past QCE exam questions and ensuring that all theory and examples are precisely mapped to the QCAA syllabus and exams.
- To reduce the **cognitive load** for 21st century learners, explanations are clear and concise, using the technique of **chunking** text with accompanying diagrams and infographics.
- The exercise structure of **Recap**, **Mastery** and **Exam practice** leads students from procedural fluency to **higher-order thinking** using the learning techniques of **interleaving**.
- **Exam practice** includes exam-style questions and past QCE exam questions with **examiner advice and feedback**.
- The cumulative structure of Exercise **Recaps** and **Cumulative examinations** in every chapter is built on the learning and memory techniques of **spacing** and **retrieval**, as recommended by the new syllabus.

About the authors

Lani Killalea has taught Mathematics and Technology for over 20 years and was Head of Department and Deputy Principal in Brisbane and regional schools. She teaches at Aspley State High School, was a district panellist for Brisbane North and has taught a range of subjects from learning support to senior courses.

Tracey MacBeth-Dunn has taught Mathematics for over 30 years, including experience as Head of Mathematics and external exam marker for General Mathematics in Queensland. She teaches at Canterbury College in Waterford and was a lead author on the Australian Curriculum series *Nelson Senior Maths 11–12 General Mathematics*.

Anna Wethereld has been a leader in schools, curriculum, assessment and teacher training for over 25 years, including at Brisbane State High School, QCAA and Queensland College of Teachers. She was a panellist for Mathematics C (Specialist Maths) for Brisbane North, a Head of Mathematics and Deputy Principal. As the state's Mathematics Curriculum Manager at NESA, Anna led Mathematics K–12 in NSW and coordinated the development of its Years 11–12 syllabuses.

Series editor **Stephen Swift** taught Mathematics, Science and Computing in urban and regional schools, and was Head of Mathematics at Wellington Point State High School in Brisbane. He was the lead author of *Nelson QMaths 7–12* and *Nelson Senior Maths 11–12* (Australian Curriculum).

Contributing authors

Tracey MacBeth-Dunn wrote the worked solutions.
Sharon Kirkby wrote many of the worksheets.
Scott Smith, **Katie Jackson**, **John Drake**, and **Joanne Magner** created the video tutorials.
Joel Speranza created the *Exam question analysis* videos.
Kahlia Dreyer and **Trisha Goss** wrote the topic tests.

Syllabus reference grid

Topics and subtopics		*Nelson Maths 12 General Mathematics QLD* chapter
UNIT 3: BIVARIATE DATA AND TIME SERIES ANALYSIS, SEQUENCES AND EARTH GEOMETRY		
1 Bivariate data analysis 1		
Identifying and describing associations between two categorical variables	1	Associations between variables
Identifying and describing associations between two numerical variables	1	Associations between variables
2 Bivariate data analysis 2		
Fitting a linear model to numerical data	3	Associations and linear modelling
Association and causation	3	Associations and linear modelling
3 Time series analysis		
Describing and interpreting patterns in time series data	4	Time series analysis
Analysing time series data	4	Time series analysis
4 Growth and decay in sequences		
The arithmetic sequence	2	Arithmetic and geometric sequences
The geometric sequence	2	Arithmetic and geometric sequences
5 Earth geometry and time zones		
Locations on the Earth	5	Earth geometry and time zones
Time zones	5	Earth geometry and time zones
UNIT 4: INVESTING AND NETWORKING		
1 Loans, investments and annuities 1		
Compound interest loans and investments	6	Loans, investments and annuities
Present value of ordinary annuities	6	Loans, investments and annuities
2 Loans, investments and annuities 2		
Perpetuities and future value of ordinary annuities	8	Annuities and perpetuities
3 Graphs and networks		
Graphs, associated terminology and the adjacency matrix	7	Graphs and networks
Planar graphs, paths and cycles	7	Graphs and networks
4 Networks and decision mathematics 1		
Trees and minimum connector problems	9	Spanning trees and critical paths
Project planning and scheduling using critical path analysis	9	Spanning trees and critical paths
5 Networks and decision mathematics 2		
Flow networks	10	Flow and assignment problems
Assigning order and the Hungarian algorithm	10	Flow and assignment problems

About this book

At the beginning of each chapter

Syllabus coverage and extracts are shown at the front of the chapter, along with a listing of **Nelson MindTap** chapter resources.

Syllabus coverage

UNIT 3, TOPIC 3: TIME SERIES ANALYSIS

Describing and interpreting patterns in time series data

- Construct and use time series plots.
- Describe time series plots by identifying features, including trend (long-term direction, e.g. increasing/decreasing), seasonality (systematic, calendar-related movements) and irregular fluctuations (unsystematic, short-term fluctuations).

Analysing time series data

- Smooth time series data by calculating a simple moving average using the mean or median for an odd number of data, including the use of spreadsheets.
- Deseasonalise a time series by calculating the seasonal indices using the average percentage method, including the use of spreadsheets.
- Fit a least-squares line to model long-term trends in time series data.
- Solve practical problems that involve the analysis of time series data.

Videos (2):
4.3 Seasonal indices
Exam question analysis Time series analysis

Prior learning (1):
4.1 Time series analysis

Worksheets (7):
4.1 Time series plots
4.2 Smoothing time series data • Moving means • Moving medians
4.3 Deseasonalisation • Seasonality
Cumulative exams General Maths formula sheet

Nelson MindTap

To access resources above, visit **cengage.com.au/nelsonmindtap**

Terminology previews the keywords and phrases from within the chapter.

Terminology

cyclical	deseasonalisation	irregular	mean
median	moving average	quarter	reseasonalisation
seasonal	seasonal index	smoothing	time series
time series plot	trend		

In each chapter

Important facts and formulas are highlighted in a shaded box.

Important words and phrases are printed in blue and listed in the **Glossary and index** at the back of the book.

Time series

- A time series is a set of data collected at regular periods of time.
- A **time series plot** is a graph of a time series with time on the horizontal axis.
- Time series graphs are used to detect patterns and trends in the data.

Cognitive verbs are highlighted in light green. See QCAA's glossary of cognitive verbs for more verbs and details.

Worked examples are explained clearly step-by-step, with the mathematical working shown on the right-hand side.

WORKED EXAMPLE 1 Plotting time series graphs

This table shows the number of patrons at a restaurant every day over 4 weeks.

	Mon	Tue	Wed	Thu	Fri	Sat	Sun
Week 1	0	21	40	42	80	79	68
Week 2	0	14	38	38	66	80	46
Week 3	0	20	23	67	80	80	33
Week 4	0	29	27	50	80	80	34

a **Construct** a time series plot from the data.

b **Describe** features observed in the data.

Steps | **Working**

a 1 Place the days of the week on the x-axis and create a scale for the number of patrons on the y-axis.

2 Plot the number of customers each day for the four weeks in order.

b Write observations from the time series.

The data shows that the restaurant did not have more than 80 patrons on any given day and there were no patrons on Mondays. It could have a maximum capacity of 80 and be shut on Mondays. The highest trade days are Fridays and Saturdays.

Exam hacks highlight valuable exam hints and common student errors.

Exam hack

When drawing a time series plot, make the time axis appropriate for the data.

9780170484824

In each chapter exercise, **Recap** questions revise skills from the previous exercise (useful for lesson starters), **Mastery** questions provide skill practice linked to worked examples, while **Exam practice** applies learned skills to exam-style problems and past QCE questions, including multiple choice, short response, problem-solving, and simple and complex questions.

EXERCISE 6.2 Effective interest rates ANSWERS p. 454

Recap

1 Ethan placed $15 000 in a term deposit returning 3.5% p.a. compounded yearly for 5 years. Calculate

a the total value of the investment at the end of the 1st and 5th year

b the interest earned in the 1st year

c the total interest earned on the investment.

2 How much should Renee invest in an account paying 6% p.a. interest calculated half-yearly to have $12 500 in 7 years' time?

Mastery

3 WORKED EXAMPLE 6 Mia invests $10 000 at 8.2% p.a. compounded weekly. What is the effective rate of interest?

4 Find the effective interest rate for each of the following.

a 7.5% p.a. compounding monthly

b 13.4% p.a. compounding daily

c 8.4% p.a. compounding quarterly

d 8% p.a. compounding fortnightly

5 Liam has a choice of 3 investments at his bank. He can invest at

- 7.6% p.a. compounded quarterly
- 7.5% p.a. compounded monthly or
- 7.4% p.a. compounded daily.

Calculate the effective interest rate for each option and decide which choice is the best.

6 An investor is looking for the best return from 3 different offers. Which of the following options would be best for the investor?

Option 1: 7.51% p.a. compounded weekly

Option 2: 7.45% p.a. compounded daily

Option 3: 7.59% p.a. compounded monthly

Exam practice

7 Which of the following has the highest effective interest rate?

A 7.1% p.a. compounding daily

B 7.4% p.a. compounding monthly

C 7.3% p.a. compounding quarterly

D 7.5% p.a. compounding annually

8 The effective rate of interest for an investment earning 8% p.a. compounded monthly is closest to

A 8.1% B 8.2% C 8.3% D 8.4%

9 (2 marks) Andy has selected a term deposit account with an interest rate of 5.4% p.a. compounded quarterly. What is the difference between the effective interest rate and the nominal interest rate?

10 ©QCAA 2020 1Q25 (5 marks) A financial institution offers two investment options:

Option 1: 7% p.a. compounding quarterly

Option 2: 6.8% p.a. compounding monthly

Use the effective interest rate formula to determine the option that will provide the better return.

Past QCE exam questions are clearly tagged. For example, © QCAA 2023 1Q18 means

2023 Exam paper 1 Question 18.

For multiple choice questions, the percentage of students who chose the correct answer is shown (except for the 2020 exam),

for example, 68%.

KEY

2021	2021 exam year
2020S	QCAA 2020 sample exam
1	Exam paper 1
2	Exam paper 2

Complex familiar CF and complex unfamiliar CU questions are clearly tagged and colour-coded.

11 (4 marks) CF The Queensland fisheries catch (not including aquaculture) and the amount gambled in Queensland casinos and clubs from 2013 to 2023 have a fairly strong association, represented by the equation $G = -1.169C + 5.726$, with C measured in hundreds of thousands, G in millions of dollars and $R^2 = 0.6$. Discuss the reasonableness of the association and validity of predictions made from it.

12 (4 marks) CU The table below shows the number of marriages in Australia for 2011–2019 and the average Brisbane wedding venue hire cost per person for those years.

Marriages	121 754	123 243	118 959	121 197	113 595	118 401	112 954	119 186	113 815
Cost $ pp	96	94	112	105	140	142	153	180	185

ABS (Australian Bureau of Statistics) (2024) Marriages and Divorces Australia, ABS website, accessed 26 February, 2025

Technology promotes ICT in the classroom, using spreadsheets and the internet.

TECHNOLOGY Investment annuity recurrence relations calculations

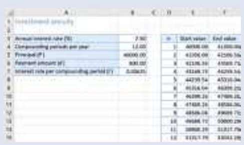
You can make a spreadsheet for recurrence relation calculations. The procedure below sets out the steps for the initial annuity situation in Worked example 1. An investment annuity of $40 000, compounding monthly with periodic payments of $800 at 7.5% p.a.

1 Set up the spreadsheet as shown below.

- Format cells B3:B7 as numbers with 2 decimal places.
- Format B7 as a number with 5 decimal places.
- Format cells E4:F27 as numbers with 2 decimal places.
- n = each period of the investment (month for this example)
- Start value = the value of the investment at the start of the period.
- End value = the value of the investment at the end of the period.

2 Enter the formulas shown.

3 Copy down formulas from D5, E5 and F4 to row 27. This will calculate the values for the first 24 periods.



© Microsoft Corporation

You can extend the table further down if required.

Use the spreadsheet to carry out investment annuity calculations.

Investigations explore the syllabus in more detail, providing ideas for modelling activities and assessment tasks.

Investigation Using a spreadsheet to explore an investment annuity

Use the spreadsheet you have just created or access the spreadsheet 'Investment annuity' on Nelson MindTap to explore the parameters of a loan. Consider an investment annuity of $40 000, compounding monthly with periodic payments of $800 at 7.5% p.a. (Worked example 1).

What is the value of the annuity at the end of 2 years?

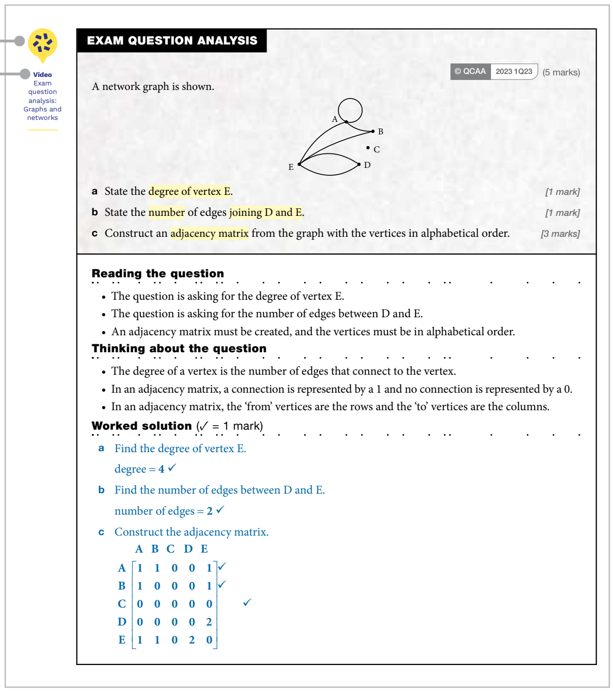
How long will it be before the investment annuity will be $80 000?

What effect does changing the compound periods and payments per year have on the investment?

At the end of each chapter

Exam question analysis leads students through a past QCE or exam-style question that exemplifies the chapter, discusses how to approach the question, provides advice on interpreting the question, highlights keywords, describes common student errors and presents a fully worked solution and marking key.

Each exam question analysis is accompanied by a **video** walkthrough explained by Joel Speranza.



Video Exam question analysis: Graphs and networks

EXAM QUESTION ANALYSIS

© QCAA 2023 1Q23 (5 marks)

A network graph is shown.

a State the degree of vertex E. [1 mark]

b State the number of edges joining D and E. [1 mark]

c Construct an adjacency matrix from the graph with the vertices in alphabetical order. [3 marks]

Reading the question

- The question is asking for the degree of vertex E.
- The question is asking for the number of edges between D and E.
- An adjacency matrix must be created, and the vertices must be in alphabetical order.

Thinking about the question

- The degree of a vertex is the number of edges that connect to the vertex.
- In an adjacency matrix, a connection is represented by a 1 and no connection is represented by a 0.
- In an adjacency matrix, the 'from' vertices are the rows and the 'to' vertices are the columns.

Worked solution (✓ = 1 mark)

a Find the degree of vertex E.

degree = 4 ✓

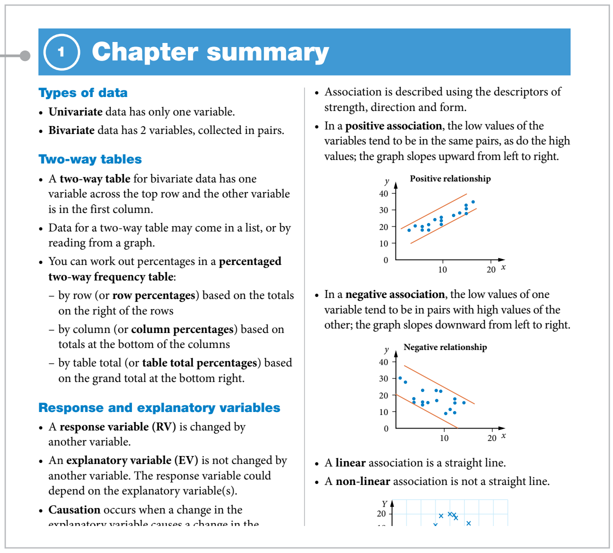
b Find the number of edges between D and E.

number of edges = 2 ✓

c Construct the adjacency matrix.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 1 & 1 & 0 & 2 & 0 \end{bmatrix} \end{array} \checkmark\checkmark\checkmark$$

Chapter summary for easy reference.



1 Chapter summary

Types of data

- **Univariate** data has only one variable.
- **Bivariate** data has 2 variables, collected in pairs.

Two-way tables

- A **two-way table** for bivariate data has one variable across the top row and the other variable is in the first column.
- Data for a two-way table may come in a list, or by reading from a graph.
- You can work out percentages in a **percentaged two-way frequency table**:
  - by row (or **row percentages**) based on the totals on the right of the rows
  - by column (or **column percentages**) based on totals at the bottom of the columns
  - by table total (or **table total percentages**) based on the grand total at the bottom right.

Response and explanatory variables

- A **response variable (RV)** is changed by another variable.
- An **explanatory variable (EV)** is not changed by another variable. The response variable could depend on the explanatory variable(s).
- **Causation** occurs when a change in the explanatory variable causes a change in the

- Association is described using the descriptors of strength, direction and form.
- In a **positive association**, the low values of the variables tend to be in the same pairs, as do the high values; the graph slopes upward from left to right.

Positive relationship

- In a **negative association**, the low values of one variable tend to be in pairs with high values of the other; the graph slopes downward from left to right.

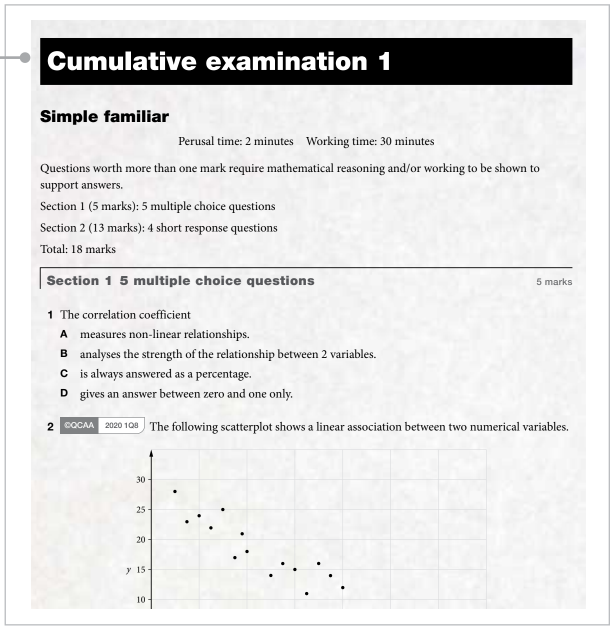
Negative relationship

- A **linear** association is a straight line.
- A **non-linear** association is not a straight line.

Cumulative examinations 1 and **2** are mini exams based on the format of the QCE examination papers 1 and 2, with around 50% of questions focusing on the chapter in which they appear and 50% on previous chapters.

Cumulative examination 1 contains simple familiar questions in multiple choice and short response format: 30 minutes, 18 marks.

Cumulative examination 2 contains complex familiar and complex unfamiliar questions in short response format: 30 minutes, 12 marks.



Cumulative examination 1

Simple familiar

Perusal time: 2 minutes Working time: 30 minutes

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

Section 1 (5 marks): 5 multiple choice questions

Section 2 (13 marks): 4 short response questions

Total: 18 marks

Section 1 5 multiple choice questions 5 marks

1 The correlation coefficient

A measures non-linear relationships.

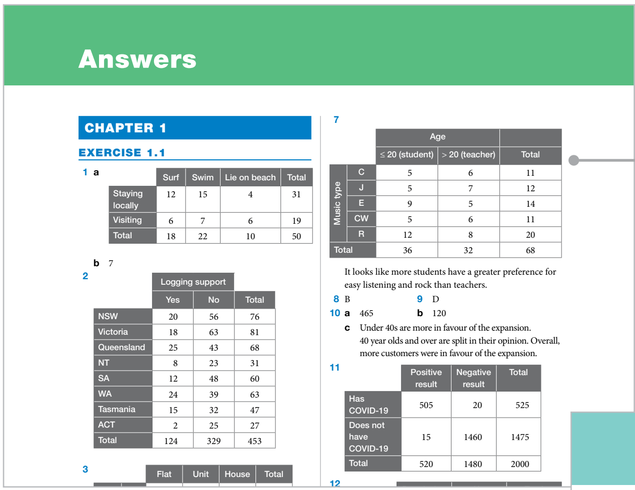
B analyses the strength of the relationship between 2 variables.

C is always answered as a percentage.

D gives an answer between zero and one only.

2 ©QCAA 2020 1Q8 The following scatterplot shows a linear association between two numerical variables.

At the end of the book



Answers

CHAPTER 1

EXERCISE 1.1

1 a

	Surf	Swim	Lie on beach	Total
Staying locally	12	15	4	31
Visiting	6	7	6	19
Total	18	22	10	50

b 7

2

	Logging support		
	Yes	No	Total
NSW	20	56	76
Victoria	18	63	81
Queensland	25	43	68
NT	8	23	31
SA	12	48	60
WA	24	39	63
Tasmania	15	32	47
ACT	2	25	27
Total	124	329	453

3

	Flat	Unit	House	Total

7

		Age		
		≤ 20 (student)	> 20 (teacher)	Total
Music type	C	5	6	11
	J	5	7	12
	E	9	5	14
	CW	5	6	11
	R	12	8	20
Total		36	32	68

It looks like more students have a greater preference for easy listening and rock than teachers.

8 B 9 D

10 a 465 b 120

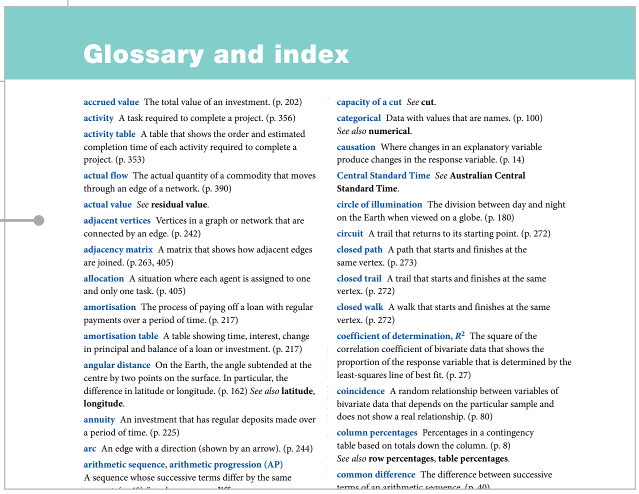
c Under 40s are more in favour of the expansion. 40 year olds and over are split in their opinion. Overall, more customers were in favour of the expansion.

11

	Positive result	Negative result	Total
Has COVID-19	505	20	525
Does not have COVID-19	15	1460	1475
Total	520	1480	2000

12

Answers (with worked solutions provided on Nelson MindTap for teachers to allocate to students).



Glossary and index

accrued value The total value of an investment. (p. 202)
activity A task required to complete a project. (p. 356)
activity table A table that shows the order and estimated completion time of each activity required to complete a project. (p. 353)
actual flow The actual quantity of a commodity that moves through an edge of a network. (p. 390)
actual value *See* **residual value**.
adjacent vertices Vertices in a graph or network that are connected by an edge. (p. 242)
adjacency matrix A matrix that shows how adjacent edges are joined. (p. 263, 405)
allocation A situation where each agent is assigned to one and only one task. (p. 405)
amortisation The process of paying off a loan with regular payments over a period of time. (p. 217)
amortisation table A table showing time, interest, change in principal and balance of a loan or investment. (p. 217)
angular distance On the Earth, the angle subtended at the centre by two points on the surface. In particular, the difference in latitude or longitude. (p. 162) *See also* **latitude, longitude**.
annuity An investment that has regular deposits made over a period of time. (p. 225)
arc An edge with a direction (shown by an arrow). (p. 244)
arithmetic sequence, arithmetic progression (AP) A sequence whose successive terms differ by the same

capacity of a cut *See* **cut**.
categorical Data with values that are names. (p. 100) *See also* **numerical**.
causation Where changes in an explanatory variable produce changes in the response variable. (p. 14)
Central Standard Time *See* **Australian Central Standard Time**.
circle of illumination The division between day and night on the Earth when viewed on a globe. (p. 180)
circuit A trail that returns to its starting point. (p. 272)
closed path A path that starts and finishes at the same vertex. (p. 273)
closed trail A trail that starts and finishes at the same vertex. (p. 272)
closed walk A walk that starts and finishes at the same vertex. (p. 272)
coefficient of determination, R^2 The square of the correlation coefficient of bivariate data that shows the proportion of the response variable that is determined by the least-squares line of best fit. (p. 27)
coincidence A random relationship between variables of bivariate data that depends on the particular sample and does not show a real relationship. (p. 80)
column percentages Percentages in a contingency table based on totals down the column. (p. 8) *See also* **row percentages, table percentages**.
common difference The difference between successive terms of an arithmetic sequence. (p. 40)

A combined **Glossary and index**.

Nelson MindTap

Nelson MindTap is an online learning space that provides students with interactive learning and assessment experiences. Margin links in the student book signpost multimedia student resources found on Nelson MindTap*.

Videos

Skillsheets

Worksheets

For students:

- **Engage** with the **online eBook** by adding notes, highlights, bookmarks, and using the **Search** and **Read Aloud** (in Australian voice) functions.
- **Watch videos** featuring expert teacher advice to unpack worked examples and deepen your understanding.
- **Revise** using pre-chapter **prior learning sheets, skillsheets** and **worksheets** to practise your skills and build your confidence.
- **Navigate** your own learning path, accessing the content and support as you need it.

For teachers:

- **Access topic tests, teaching plans** and **worked solutions** to each exercise set.
- **Use course customisation** to tailor content and assign tasks to groups of students or the whole class.
- **Integrate** content directly within your school's LMS for ease of access.
- Help build your students' exam readiness with **Cognero Assess** – a test bank containing hundreds of questions and answers to create, assign or export formative and summative tests.

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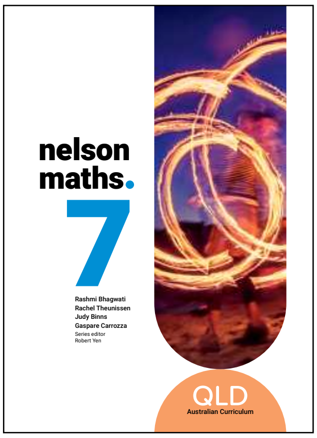
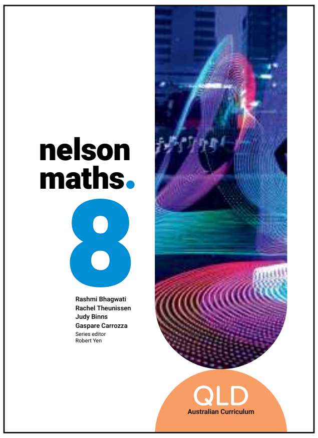
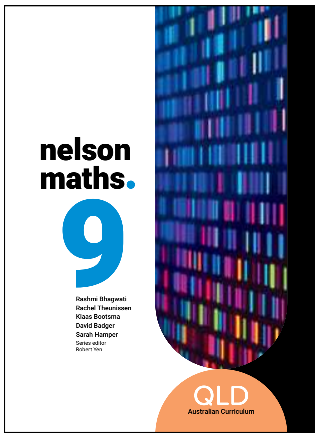
Security & privacy:

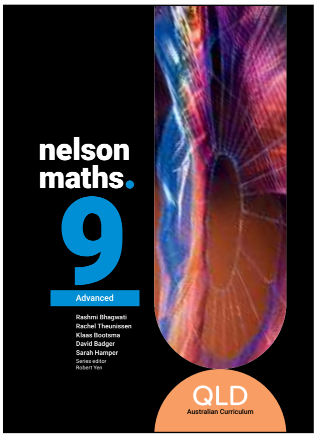
Nelson MindTap joined the Safer Technologies 4 Schools (ST4S) Product Badge Program in 2024. The annual ST4S assessment is part of our commitment to supporting the online security and safety of students and schools. Learn more at st4s.edu.au

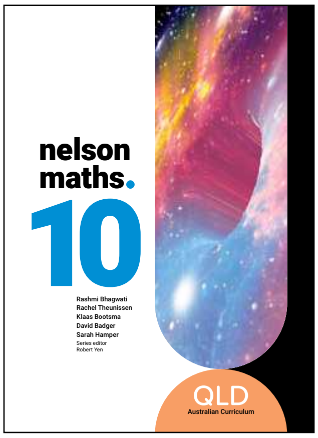
Nelson Maths 7–12 QLD series

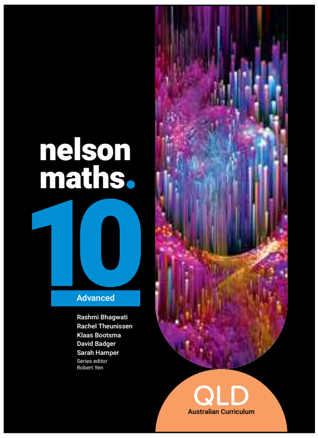


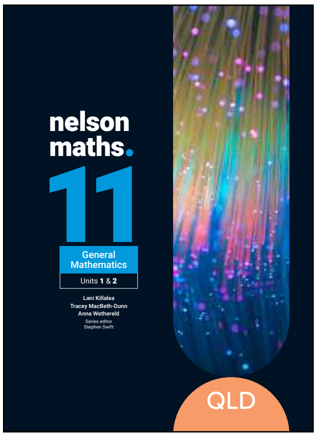


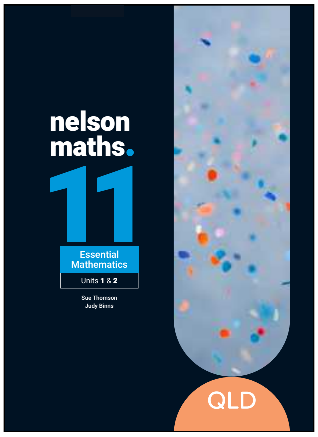


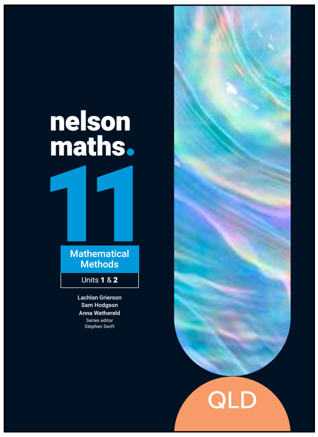


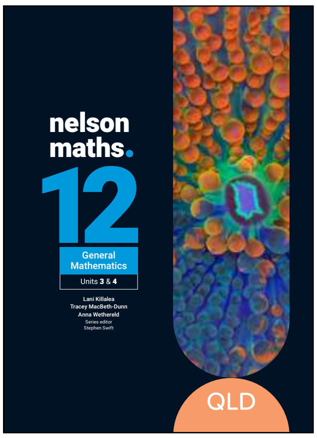


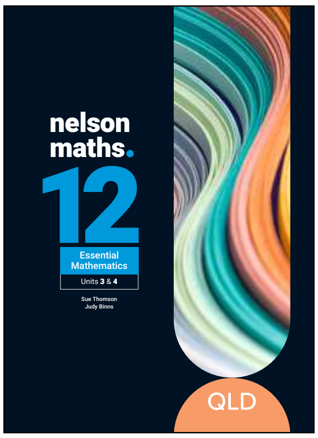


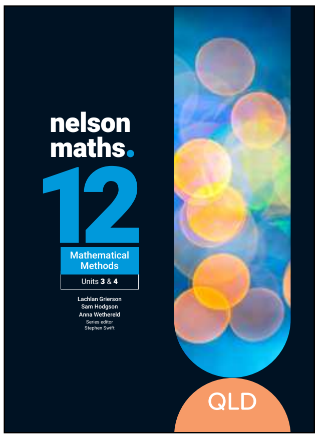












List of symbols and abbreviations

$=$	is equal to
$\neq$	is not equal to
$\approx$	is approximately equal to
$<$	is less than
$>$	is greater than
$\leq$	is less than or equal to
$\geq$	is greater than or equal to
t_n	nth term of a sequence
FV	future value
PV	present value
i	interest rate
$\pm$	plus or minus
π	pi ≈ 3.14
$^\circ$	degree
$'$	minute
$\angle$	angle
$\triangle$	triangle
$\therefore$	therefore
$\sqrt{\ }$	square root, radical sign
S	surface area
S 37° W	a compass bearing
217° T	a true bearing
AUD	Australian dollar
GST	goods and services tax
LHS	left-hand side
RHS	right-hand side
m	gradient of a straight line
c	y-intercept of a straight line
(x, y)	Cartesian coordinates, ordered pair
%	percentage
A	matrix A
a_{34}	element in 3rd row and 4th column of matrix A
I	identity matrix
cos	cosine ratio
sin	sine ratio
tan	tangent ratio
$\bar{x}$	mean
Σ	the sum of
Q_1	1st quartile or lower quartile
Q_2	median (2nd quartile)
Q_3	3rd quartile or upper quartile
IQR	interquartile range
SD	standard deviation
s_x	standard deviation of a sample
σ	standard deviation of a population
r	Pearson's correlation coefficient
R^2, r^2	coefficient of determination
EV	explanatory variable
RV	response variable
EST	earliest starting time
LST	latest starting time

List of mathematical verbs

A glossary of 'doing words' commonly found in mathematics problems

Cognitive verbs are highlighted in light green. See QCAA's glossary of cognitive verbs for more details.

analyse: study and state in detail the relationship of parts of a situation

apply: use knowledge or a procedure in a given situation

calculate: find a numerical value

comment: express an opinion or judgement about a statement or calculation

communicate: transmit information to others

compare: state similarities and differences and their significance

consider: take into account; think carefully

construct: draw an accurate diagram or logically arrange items or ideas

convert: change from one form to another

classify: identify and sort into categories

deduce: use logic and reason to prove something

define: give the meaning of or identify in exact terms

demonstrate: show to be correct

describe: state the features of a situation, object, pattern, event, etc.

determine: find the answer or make a decision using evidence

discuss: give reasons based on evidence for and against a position or proposal

establish: introduce and develop a result

evaluate: find the value of or state the application, strengths and limitations of a solution

examine: investigate the details and assumptions of a situation

expand: remove brackets or change a product to an extended sum of terms

explain: state the meaning in logical detail

explore: examine or state the details and assumptions of a situation

factorise: convert to factors or change a sum of terms to a product of factors

formulate: give mathematical expression to an idea or situation

hence find/prove: find an answer or prove a result using previous answers or information supplied

identify: state the type, name or distinguishing feature of an item or situation

interpret: state a conclusion or trend from given information

investigate: establish facts, trends or conclusions from collected information

isolate a variable: express a formula or equation with one variable only, usually on the left-hand side (LHS)

justify: give reasons or evidence for an answer or conclusion

modify: change to accommodate different information

obtain: find an answer or conclusion

predict: use given information or results to guess what will happen under different conditions

prove: use logical steps to establish the truth of a result, statement or assumption

recall: remember (and state)

recognise: use knowledge to identify features of a situation

show that: in questions where the answer is given, use mathematical reasoning to prove that the answer is true

simplify: reduce the size of numbers in a fraction, or reduce the size of an algebraic expression

sketch: draw a diagram that shows the general shape and includes relevant features

solve: find the answer or explanation for a problem, particularly the values of variables in equations or inequalities

substitute: replace a variable by a number to calculate an answer

test: check that a statement or result is correct

translate (to mathematical form): express a situation as mathematical relationships

verify: check a solution or result, usually referring to the given situation

write/state: give the answer, formula or result without showing any working or explanation (This usually means that the answer can be found mentally, or in one step)

CHAPTER

1 ASSOCIATIONS BETWEEN VARIABLES

Prior learning
Associations between variables

Syllabus coverage

UNIT 3, TOPIC 1: BIVARIATE DATA ANALYSIS 1

Identifying and describing associations between two categorical variables

- Understand the meaning of bivariate data.
- Construct two-way frequency tables and determine the associated row and column sums and percentages.
- Use an appropriately percentaged two-way frequency table to identify patterns that suggest the presence of an association.
- Understand an association in terms of differences observed in percentages across categories in a systematic and concise manner, and interpret this in the context of the data.

Identifying and describing associations between two numerical variables

- Identify the explanatory variable and the response variable.
- Construct and use a scatterplot to identify the association between two numerical variables.
- Describe an association between two numerical variables in terms of direction (positive/negative), form (linear/non-linear) and strength (strong/moderate/weak).
- Calculate Pearson's correlation coefficient, r, from raw data using technology, and interpret it to quantify the strength of a linear association.
- Calculate the coefficient of determination, R^2, from raw data using technology, and intepret it to assess the strength of a linear association in terms of the explained variation.
- Use the correlation coefficient, r, to determine the coefficient of determination, R^2, and vice versa.

Videos (3):
1.2 Percentaged two-way tables
1.5 Correlation
Exam question analysis Associations between variables

Prior learning (1):
1.1 Associations between variables

Worksheets (10):
1.1 Two-way tables 1 • Two-way tables 2
1.2 Percentage tables
1.4 Scatterplot and associations • A page of scatterplots • Correlations matching game • Height vs shoe size • Body measurements • Relationships between variables
Cumulative exams General Maths formula sheet

Nelson MindTap

To access resources above, visit **cengage.com.au/nelsonmindtap**

Terminology

association	bivariate	causation	coefficient of determination
column percentage	correlation	(Pearson's) correlation coefficient	
explanatory variable	linear	moderate association	negative association
non-linear	percentaged two-way frequency table		positive association
response variable	row percentage	scatterplot	strong association
two-way table	univariate	weak association	

1.1 Bivariate data

You have analysed data with a single variable, such as the hair colour of everyone in your class. There is only one variable with values like blonde, red, black or brown. This chapter will analyse data that has 2 variables, such as hair colour *and* height of everyone in your class. The data values could be (black, 175 cm) or (red, 157 cm).

Types of data

- **Univariate data** has only one variable, such as the number of sales in a month.
- **Bivariate data** has 2 variables, collected in pairs. A study of the relationship between a person's age and their height is an example of bivariate data.

You can use a frequency table to analyse univariate data. You can use a **two-way table** to analyse bivariate data, where one variable goes across the top row and the other variable is in the first column.

Worksheets
Two-way tables 1
Two-way tables 2

WORKED EXAMPLE 1 Constructing two-way tables

A market researcher surveyed 50 people about their favourite ice-cream flavour.
Of the 30 females, 20 said they preferred chocolate, while 14 of the males preferred other flavours.

a **Construct** a two-way table of the data.

b **Determine** how many males preferred chocolate.

Steps | **Working**

a **1** Show gender in the first column and favourite flavour across the top row. There are only 2 values for each variable.

2 Fill in the known cells from the question.

	Chocolate	Other flavours	Total
Male		14	
Female	20		30
Total			50

b **1** Complete the table by filling in the missing cells.

In the Females row:

Other flavours = total – chocolate

= 30 – 20

= 10

Continue adding and subtracting to complete the table.

Having totals for rows and columns helps to calculate the missing values.

	Chocolate	Other flavours	Total
Male	6	14	20
Female	20	10	30
Total	26	24	50

The total in the bottom right cell represents the total number of responses (data values). This should match the number in the question.

2 Write the answer.

There were 6 males who preferred chocolate.

Two-way tables can be used to identify patterns by using a small number of classes to group **discrete** or **continuous** values.

WORKED EXAMPLE 2 Two-way table from a raw data list

The heights of a sample of Year 12 students and the numbers of electronic devices in their homes are shown below.

Adam 185 cm, 3	Asha 168 cm, 4	Bonnie 157 cm, 2	Charmaine 172 cm, 5
Chris 194 cm, 1	Colin 165 cm, 3	David 178 cm, 2	Elena 176 cm, 1
Harry 169 cm, 0	Helga 158 cm, 3	Janita 162 cm, 4	Lee 183 cm, 2
Mary 164 cm, 1	Mustafa 167 cm, 2	Mirza 179 cm, 2	Pam 160 cm, 2
Peter 177 cm, 1	Phoong 166 cm, 3	Roberto 185 cm, 4	Robyn 156 cm, 3
Sam 164 cm, 4	Stefan 172 cm, 3	Thomas 167 cm, 2	Xing 155 cm, 1

a **Construct** a two-way table with height in 3 classes.

b **Identify** any patterns.

Steps | **Working**

a **1** Choose the height classes.

Use short (< 166 cm), average (166–175 cm) and tall (> 175 cm) for heights.

2 Use tally marks to fill in the table to avoid mistakes.

Working across the data rows, place the tally marks in order, i.e. Adam, Asha, Bonnie, Charmaine, and so on.

<table>
<tr><th></th><th></th><th colspan="3">Height (cm)</th></tr>
<tr><th></th><th></th><th>Short</th><th>Average</th><th>Tall</th></tr>
<tr><th rowspan="6">Number of devices</th><th>0</th><td></td><td></td><td></td></tr>
<tr><th>1</th><td></td><td></td><td></td></tr>
<tr><th>2</th><td>|</td><td></td><td>|</td></tr>
<tr><th>3</th><td></td><td></td><td></td></tr>
<tr><th>4</th><td></td><td>|</td><td></td></tr>
<tr><th>5</th><td></td><td>|</td><td></td></tr>
</table>

3 Complete the table and then replace the tally marks by the frequencies.

<table>
<tr><th></th><th></th><th colspan="4">Height (cm)</th></tr>
<tr><th></th><th></th><th>Short</th><th>Average</th><th>Tall</th><th>Total</th></tr>
<tr><th rowspan="6">Number of devices</th><th>0</th><td>0</td><td>1</td><td>0</td><td>1</td></tr>
<tr><th>1</th><td>2</td><td>0</td><td>3</td><td>5</td></tr>
<tr><th>2</th><td>2</td><td>2</td><td>3</td><td>7</td></tr>
<tr><th>3</th><td>3</td><td>2</td><td>1</td><td>6</td></tr>
<tr><th>4</th><td>2</td><td>1</td><td>1</td><td>4</td></tr>
<tr><th>5</th><td>0</td><td>1</td><td>0</td><td>1</td></tr>
<tr><th colspan="2">Total</th><td>9</td><td>7</td><td>8</td><td>24</td></tr>
</table>

b Note any patterns.

While shorter people had the highest device count, the totals for each height group were similar. It can be seen from the table that most households, regardless of height had between 1 and 3 devices.

What the table does not show is the relationship between height and the number of devices. Logically, height would not dictate the number of devices a household had. Therefore, neither variable could cause a change in the other.

9780170484824

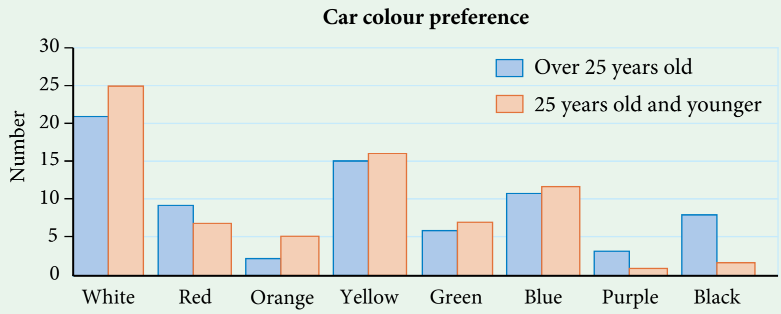
Investigation	Subject choices and careers

Try to categorise the subject choices of students in your class as 'mainly science', 'mainly arts', 'mainly humanities' or 'other'.
Choose some career pathways for next year, such as 'TAFE', 'apprentice', 'university', 'employment'.
Gather information from your class and draw up a two-way table. You may want to extend your data gathering to another class if it can be arranged by your teacher.
Can you draw any conclusions from your table?
Discuss the difficulties of classifying the information in a meaningful way.

You can use a side-by-side column graph to represent the data in a two-way table, or the other way round.

WORKED EXAMPLE 3 Using graphs to create a two-way table

The column graph below shows car colour preferences. **Construct** a two-way table for the data.



Steps

1 Put age group across the top row and the car colours in the first column.

2 Read the frequencies for each cell from the column graph.

Working

		Over 25	25 and younger	Total
Colour	White	21	25	46
	Red	9	7	16
	Orange	2	5	7
	Yellow	15	16	31
	Green	6	7	13
	Blue	11	12	23
	Purple	3	1	4
	Black	8	2	10
Total		75	75	150

EXERCISE 1.1 Bivariate data

ANSWERS p. 436

Mastery

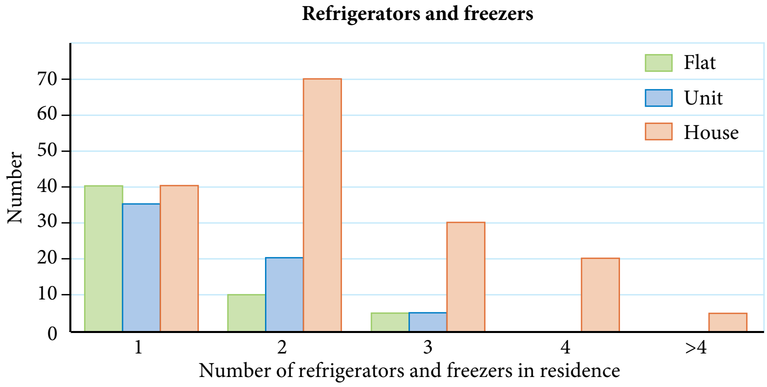
1 WORKED EXAMPLE 1 At Kirra beach, 50 people were asked whether they were staying locally or visiting and whether they intended to surf, swim or sunbake on the beach. Of the 31 people staying locally, 12 were surfers and 4 were sunbakers. Altogether, there were 18 surfers and 10 sunbakers.

- **a** Construct a two-way table.
- **b** Determine the number of swimmers visiting.

2 WORKED EXAMPLE 2 The data below shows the results of a survey on the amount of support for logging of old native hardwood timber in state forests. Construct a two-way table for this data.

NSW:	20 for, 56 against	SA:	12 for, 48 against
Vic:	18 for, 63 against	WA:	24 for, 39 against
Qld:	25 for, 43 against	Tas:	15 for, 32 against
NT:	8 for, 23 against	ACT:	2 for, 25 against

3 WORKED EXAMPLE 3 Construct a two-way table from this side-by-side column graph showing the results of a survey about types of residences and the number of refrigerators and freezers in them.



4 A two-way table was constructed as shown below, but only partially completed.

	Male	Female
Book club member	20	32
Non-member	8	15

a How many non-members were there?

b How many females were there?

5 Construct a side-by-side column graph from the following two-way table showing the numbers of people driving different types of vehicles and their opinions about whether there should be more police cars patrolling the roads or not.

	More patrols needed	No more needed
Car	30	15
Motorcycle	8	6
Truck	20	15
Delivery van	6	18

6 The results of a survey of 40 city people and 40 country people on how they use sunscreen are given in the two-way table below. Construct a side-by-side column graph from the results.

	At work	At home	On the beach
City people	7	12	21
Country people	4	8	28

7 The data below shows the results of a survey of the preferred music of teachers and students in a school. The first letter shows whether the preferred music is Classical (C), Jazz (J), Easy listening (E), Country and Western (CW) or Rock (R). The age of the person is shown next. Choose suitable age groupings and show the information as a two-way table. Are there any patterns?

C, 17	E, 13	E, 48	R, 51	C, 26	E, 18	CW, 30
R, 14	R, 50	J, 54	R, 15	R, 16	E, 18	CW, 15
CW, 53	R, 13	E, 43	E, 37	J, 18	CW, 16	R, 18
J, 33	R, 15	E, 16	E, 30	CW, 27	R, 15	R, 33
C, 16	R, 14	E, 17	E, 18	J, 38	CW, 30	J, 33
C, 55	CW, 51	J, 18	R, 53	C, 51	E, 18	CW, 15
J, 25	R, 34	R, 17	J, 14	C, 16	C, 29	E, 38
R, 17	R, 46	R, 39	CW, 32	J, 17	CW, 16	J, 15
J, 51	C, 16	E, 15	R, 13	CW, 16	E, 17	C, 15
J, 30	R, 18	C, 35	R, 52	C, 38		

Exam practice

Use the following two-way table to answer Questions 8 and 9.

The table shows the number of people who wear a smartwatch during different exercises.

	Wears smartwatch	Does not wear a smartwatch	Total
Swimming	25	47	72
Running	70	18	88
Total	95	65	160

8 How many people do not wear a smartwatch while running?

A 70 **B** 18 **C** 88 **D** 160

9 How many people wear smartwatches?

A 25 **B** 70 **C** 65 **D** 95

10 (4 marks) During a survey at a local shopping centre, customers were asked whether they would support an expansion of the centre to allow for a restaurant precinct. Data was collected and summarised in the two-way table below.

Would you support an expansion to allow for a new restaurant precinct?

	Age of customer		
	Under 40	40 years old and over	Total
Yes	200	120	320
No	43	102	145
Total	243	222	465

a How many customers were surveyed? *[1 mark]*

b Of the respondents, how many 40 years old and over want the new expansion to occur? *[1 mark]*

c Explain what the data shows in relation to the question. *[2 marks]*

11 (2 marks) During winter, a university wanted to study the accuracy of home Rapid Antigen Test (RAT) kits in diagnosing COVID-19. They surveyed 2000 people who undertook a home RAT and a PCR lab test to confirm the diagnosis. Of the 525 participants found to have COVID-19, 505 RAT results were deemed accurate. The remaining 1475 participants, who were not infected, showed 15 false positives from their RAT kits. Display this information in a two-way table.

12 (3 marks) CF 220 students apply to travel to different venues for school sport and need to have a signed permission slip to go. 85 students selected ten-pin bowling, 90 students selected ice-skating and 45 students selected basketball. On the first day, only 80% of participants in each sport had a permission slip. Construct a two-way table displaying this data and determine how many students will be allowed to travel to the sports venues.

Worksheet
Percentage tables

1.2 Percentaged two-way tables

This table represents the results of a survey of 250 pet owners and whether they have pet insurance.

	Dog owner	Cat owner	Total
Insured	43	32	83
Not insured	82	93	167
Total	125	125	250

Sometimes it is easier to see a relationship using **percentaged two-way frequency tables**. You need to understand what the percentages represent. This is useful when making comparisons or finding probabilities for decision making.

	Dog owner %	Cat owner %
Insured	34.4	25.6
Not insured	65.6	74.4
Total	100	100

Percentaged two-way frequency tables

You can work out percentages in a percentaged two-way frequency table:

- **by row** (or **row percentages**) based on the totals on the right of the rows
- **by column** (or **column percentages**) based on the totals at the bottom of the columns
- **by table total** (or **table percentages**) based on the grand total at the bottom right.

Exam hack

Carefully consider the question to ensure you are finding the correct percentage.

WORKED EXAMPLE 4 Finding percentages from two-way tables

The table shows the preferences of people for ice-cream flavours.

	Chocolate	Other flavour	Total
Adults	6	14	20
Children	20	10	30
Total	26	24	50

Calculate

a the percentage of children preferring chocolate ice cream.

b the percentage of people who are children preferring chocolate ice cream.

c the percentage of chocolate ice-cream lovers who are children.

Steps	Working
a **1** Calculate the percentage of 20 children who prefer chocolate ice cream 'out of' 30 children in total.	% by row $= \frac{20}{30} \times 100\% \approx 67\%$
2 Write the answer.	About 67% of children prefer chocolate ice cream.
b **1** This is 'out of' the total surveyed.	% by table total $= \frac{20}{50} \times 100\% = 40\%$
2 Write the answer.	40% are children chocolate lovers.
c **1** This is 'out of' all chocolate ice-cream lovers.	% by column $= \frac{20}{26} \times 100\% \approx 77\%$
2 Write the answer.	About 77% of those who prefer chocolate ice cream are children.

1.2

Video
Percentaged two-way tables

Exam hack

You must identify the total correctly to get the right answers. Cross check the totals of columns and rows to confirm this.

When interpreting percentaged two-way tables, it is important to consider the reasonableness of any patterns in the data. This includes considering **association** between the variables. Is a pattern visible between groups within the data? Does one variable lead to a change in the other?

WORKED EXAMPLE 5 Constructing percentaged two-way tables

Queenslanders were asked about whether they approved of daylight saving in summer.

50 people from each of 5 main regions were interviewed. The results were collated in a two-way table.

		Response		
		Yes	No	Total
Region	Far north	16	34	50
	West	6	44	50
	Mid-north coast	11	39	50
	Greater Brisbane	27	23	50
	Gold Coast	41	9	50
	Total	101	149	250

a **Represent** the results as a percentaged two-way frequency table and find the percentage of each response by region.

b Draw conclusions of this survey by **examining** the results.

Steps | **Working**

a Construct a percentaged two-way table by finding the percentage of each regional response.

Work out percentages by rows.

	Yes %	No %	Total %
Far north	$\frac{16}{50} \times 100 = 32$	$\frac{34}{50} \times 100 = 68$	100
West	12	88	100
Mid-north coast	22	78	100
Greater Brisbane	54	46	100
Gold Coast	82	18	100

Due to the need for rounding of some questions, the total of percentages may not necessarily equal 100% exactly.

b **1** Comment on any patterns.

The survey results show that people outside the south-east corner of Queensland were more likely to be opposed to daylight saving. The western region had the highest level of opposition with 88% of participants voting no.

In the south-east corner, it could be seen that the greater Brisbane respondents were almost evenly split, with 54% in favour and 46% opposed. This was in contrast to 82% of the Gold Coast respondents voting in favour of daylight saving.

2 Comment on the reasonableness of the results.

While the data showed clear results in some regions, the survey only covered 250 people. As there are millions of residents in Queensland, the survey may not be fully reflective of the opinions of the greater population.

EXERCISE 1.2 Percentaged two-way tables

ANSWERS p. 436

Recap

1 Which situation involves bivariate data?

A Measuring the heart rate of marathon runners after an event.

B Reviewing ticket sales of a theme park for each month in a year.

C Recording the average speed of cars at a specific point on a highway.

D Measuring the effect of blood alcohol readings of car drivers and their reaction time.

2 Copy and complete this table about entertainment preferences.

		Gender		
		Female	Male	Total
Preference	Music concert	155		
	Comedy show		80	165
	Total	240		420

Mastery

3 WORKED EXAMPLE 4 This table shows the number of recorded traffic offences in Queensland over 2 financial years.

Rate of recorded traffic offences		
Type	Years	
	2021–2022	2022–2023
Dangerous operation of a vehicle	3 054	2 920
Drink/drug driving	26 818	28 199
Disqualified driving	10 551	12 998
Interfere with mechanism of motor vehicle	37	45
Total	40 460	44 162

Source: Queensland Government Statistician's Office, Queensland Treasury, Crime report, Queensland, 2022–2023

Use the table to find the percentage, correct to 2 decimal places, of

a drink/drug driving offences in 2021–2022

b dangerous operation offences in 2022–2023

c interference offences in 2021–2022

d disqualified driving offences over 2 years

e the highest recorded offence in the 2 years.

4 The table shows the results of a survey of employees' preferences for the length of their lunch break (with no changes to number of hours worked).

	Lunch break		
	$\frac{1}{2}$ hour	$\frac{3}{4}$ hour	1 hour
Workshop	14	20	12
On-site	20	14	6
Office	6	8	13

Use the table to find the percentage, to the nearest whole number, of

a workshop workers who prefer 1 hour

b office workers who prefer $\frac{1}{2}$ hour

c those who prefer $\frac{1}{2}$ hour who are office workers

d all workers who prefer $\frac{1}{2}$ hour and are office workers

e on-site workers who prefer 1 hour.

5 The table shows the results of quality testing samples of toys produced by different production lines in a factory.

	Quality control	
	Pass	Fail
Line A	35	6
Line B	42	8
Line C	59	8

Use the table to find the percentage of

a failures that are from Line A

b Line A that fail quality control

c total production from Line A

d good production that is from Line A

e Line C that fail quality control.

6 WORKED EXAMPLE 5 Voters were asked for their feelings about Australia becoming a republic, as well as their voting intentions for the next election. Of the Labor voters, 28 were in favour of a republic, 17 were against and 4 didn't care. Of the Liberal/National voters, 25 were in favour, 27 were against and 3 didn't care. Of those who favoured minor parties, 8 were in favour, 6 were against and 7 didn't care.

a Construct a two-way table of this information.

b Calculate percentages and state any noticeable patterns in the results.

7 The data below shows the results of a survey of people in different states regarding their support for oil exploration on the Great Barrier Reef.

NSW:	Y Y Y N N N N N N	SA:	Y Y N N N N N N
Vic:	Y Y N N N N N N N	WA:	Y Y Y Y N N N N N
Qld:	Y Y Y Y N N N N N N N	Tas:	Y N N N N
NT:	Y Y Y N N N N	ACT:	Y N N N N

Show the information as a percentaged two-way frequency table and interpret the data.

Exam practice

8 The two-way table displays the results of a student survey about their participation in playing computer games or sport.

	Plays sport	Does not play sport
Plays games	29%	61%
Does not play games	56%	44%

In the table, 44% represents

A students who do not play sport or computer games.

B students who play sport but not computer games.

C students who play both sport and computer games.

D students who only play computer games.

9 The results of a survey of 130 people about their highest level of education is shown in this table.

		Location		
		Suburban	Inner-city	Total
Education	Certificate III	20	29	49
	Diploma	17	19	36
	Degree	24	21	45
	Total	61	69	130

The percentage of inner-city people with a degree is closest to

A 16.2% **B** 30.4% **C** 46.7% **D** 53.1%

10 (4 marks) To better inform their sports programming, a TV station wanted to examine the sports viewing preference of males and females. The percentaged two-way table displays the results.

		Gender	
		Female	Male
Sport	AFL	42%	31%
	Soccer	22%	17%
	Rugby League	36%	52%

a State which sport is the most popular for these viewers. *[1 mark]*

b Which is the most popular sport for female viewers? *[1 mark]*

c If the station can only show 2 of the sports, examine the data and justify which 2 sports should be shown. *[2 marks]*

11 (6 marks) Two hundred patients participated in a vaccine trial. Of the 85 patients who were given the vaccine, 78 tested positive for antibodies after the dose. Five people who did not receive the vaccine also tested positive for antibodies.

a Use the data to draw up and complete a two-way table. *[2 marks]*

b Convert the table into a percentage frequency table. *[2 marks]*

c Examine the results to determine the error rate of the vaccine in the trial. *[2 marks]*

12 (5 marks) CU Some people in Cairns gave information about their ages and the number of brothers and sisters they had. The following information was collected, where the age is given first and then the number of siblings.

Alanna 28, 2	Allan 55, 1	Andrew 18, 1	Anita 29, 0	Ann-Marie 41, 2
Astrid 43, 1	Bjorn 16, 0	Bruce 24, 2	Carlo 35, 3	Chris 53, 2
Denise 28, 3	Duke 37, 3	Elaine 35, 0	Eleni 34, 1	Jenny 19, 0
Jodie 28, 6	Joy 33, 2	Kandai 34, 2	Kayo 16, 1	Ken 19, 3
Louise 22, 2	Lynne 17, 4	Michael 17, 1	Miguel 22, 2	Paul 34, 2
Peter 35, 0	Rebecca 46, 5	Sandra 58, 4	Shaia 23, 6	Shayne 16, 0
Tom 16, 4	Trudy 44, 0	Wendy 32, 5		

Analyse the data to determine the reasonableness of this statement:

'Due to population trends, the older someone is, the more likely they are to have 3 or more siblings.'

13 ©QCAA 2024 2Q1 (5 marks) CF Each of the 60 performers in a music and dance concert is either a Year 11 or Year 12 student and either a musician or a dancer.

There are four more Year 11 students than Year 12 students. One quarter of the Year 11 students are dancers and half of the Year 12 students are dancers.

Copy and complete the two-way frequency table to calculate the percentage of students who are musicians.

	Year 11	Year 12	Total
Musician			
Dancer			
Total			60

1.3 Response and explanatory variables

Does studying before a test improve your performance? Or does performing well in tests make you more likely to study beforehand? When you examine changes in variables, you need to decide which comes first. Which variable could cause the change? Which variable is changed as a result?

Response and explanatory variables

- A **response variable** (**RV**) is changed by another variable.
- An **explanatory variable** (**EV**) is not changed by another variable. The response variable could depend on the explanatory variable(s).
- **Causation** occurs when a change in the explanatory variable causes a change in the response variable.

Exam hack

You can often work out which variable is which by putting the variables in the sentence: 'Changes of the [*explanatory variable*] cause changes in the [*response variable*]'.

WORKED EXAMPLE 6 Determining the variables

For each pair of variables, **determine** which is the response variable and which is the explanatory variable.

a Temperature during the day and number of people at the beach.

b The overall mass of tomatoes picked and the time pickers spend working.

Steps	Working
a **1** Try putting the variables in each position of the test sentence.	Changes of the temperature cause changes in the number of people at the beach. OR Changes in the number of people at the beach cause changes of the temperature.
2 Pick the answer that makes the most sense.	The first option is the most appropriate. The response variable is the number of people at the beach. The explanatory variable is the temperature.

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b	**1** Try putting the variables in each position of the test sentence.	The overall mass of tomatoes picked causes changes in the time pickers spend working. or The time employees spend working cause changes in the overall mass of tomatoes picked.
	2 Pick the answer that makes more sense.	The second option is the most appropriate. The response variable is the overall mass of tomatoes picked. The explanatory variable is the time pickers spend working.

Scatterplots

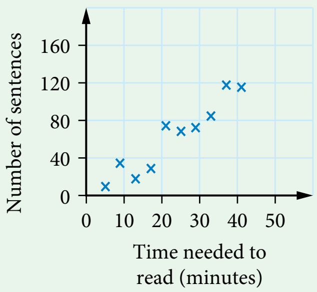
With a **scatterplot** (also called a **scatter graph**):

- use ordered pairs of numerical bivariate data.
- place the explanatory variable on the horizontal axis and the response variable on the vertical axis.
- choose scales so the axes are about the same length.

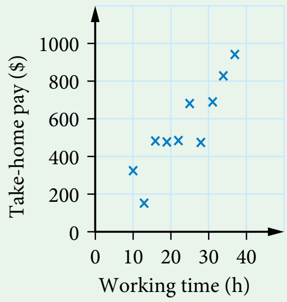
WORKED EXAMPLE 7 Identifying variables in scatterplots

State whether or not each scatterplot has been correctly drawn.

a



b



	Steps	**Working**
a	**1** Which is the response variable?	The time taken to read something depends on the number of sentences, so the time is the response variable.
	2 Write the answer.	The variables on the scatterplot are the wrong way around.
b	**1** Which is the response variable?	Take-home pay depends on the time spent working, so take-home pay is the response variable.
	2 Write the answer.	The scatterplot is correctly drawn.

EXERCISE 1.3 Response and explanatory variables

ANSWERS p. 437

Recap

1 A group of university students were asked if they drank coffee. If the percentage of coffee drinkers was 40% out of 30 coffee drinkers, how many university students drank coffee?

A 9 **B** 10 **C** 12 **D** 18

2 The percentaged two-way table displays information about Arts subject choices in Year 12.

	Male %	Female %
Drama	45	37
Art	28	52
Music	27	11

In which way are the percentages calculated?

Mastery

3 WORKED EXAMPLE 6 In each pair of variables, decide which is more likely to be the response variable and which is the explanatory variable. Give the explanatory variable first.

a evaporation rate, temperature

b time, mass of ice melted

c cost, mass of flour

d number of hens, number of eggs laid

e erosion of beach, number of king tides

4 In each pair of variables, identify the response variable (RV) and explanatory variable (EV).

a amount of rainfall, flow rate of a river

b number of seedlings eaten, number of birds

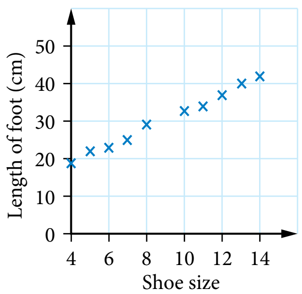
c time spent training, time taken to run 100 m

d amount of sawdust, number of logs milled

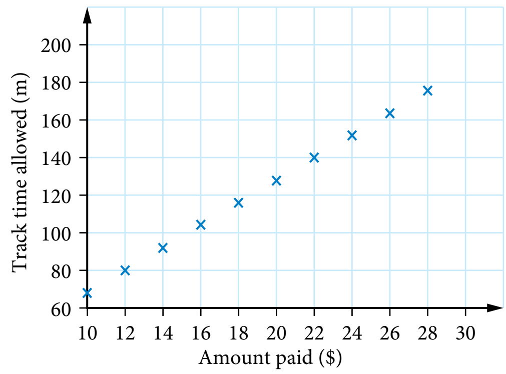
e amount of sawdust, amount of time taken to sweep

5 WORKED EXAMPLE 7 State whether or not the scatterplots below have been correctly drawn.

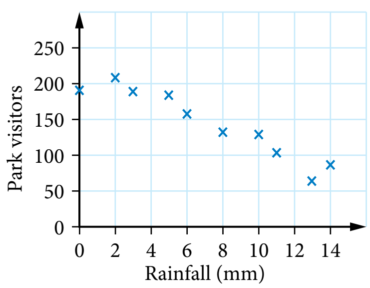
a



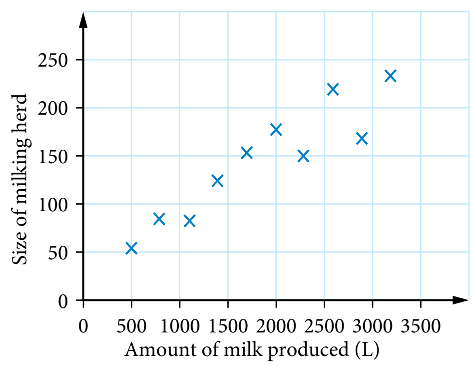
b



c



d



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6 Choose at least 5 pairs of variables from the following list. Each pair must be an explanatory and response variable, in that order.

age of student	cost of vegetable	speed of yacht
mass of vegetable	height of student	type of vegetable
floor area tiled	hair colour of student	number of tiles available
length of yacht	size of army	size of tiles
cost of army	sail area of yacht	crew size of army tank

7 State which variables in question **6** could not be explanatory or response variables.

Exam practice

8 ©QCAA 2023 1Q7 83% Which statement is always true for a causal relationship between an explanatory variable and a response variable?

A One of the variables is a confounding variable.

B The relationship is explained by a third variable.

C There is a positive association between the variables.

D The response variable is dependent on the explanatory variable.

9 Two studies were undertaken to examine the relationship between age and average resting heart rate, as well as resting heart rate and gender. Which of the following pairs represent the 2 explanatory variables from the studies?

A Heart rate and resting heart rate

B Age and resting heart rate

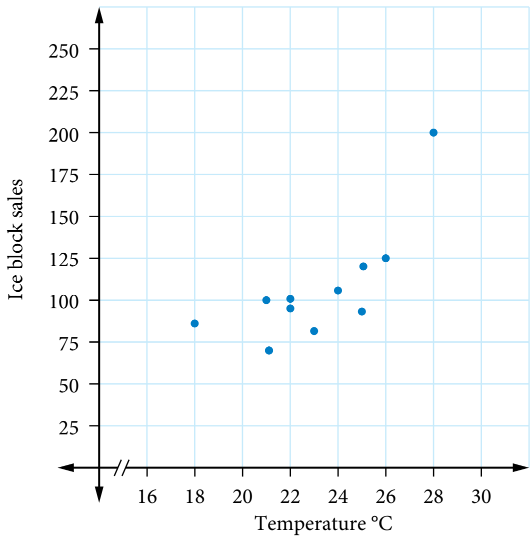
C Age and gender

D Gender and resting heart rate

10 (1 mark) Identify the response variable for this study:

'The study of the effect of caffeine consumption on heart rate.'

11 (2 marks) A student made the following scatterplot in an exam. Determine if it is correctly constructed.



12 (2 marks) CF If the variable of plant height is being assessed, select an appropriate variable pairing that has a logical relationship and explain why you made your selection.

Worksheets
Scatterplot and associations
A page of scatterplots
Correlations matching game
Height vs shoe size
Body measurements
Relationships between variables

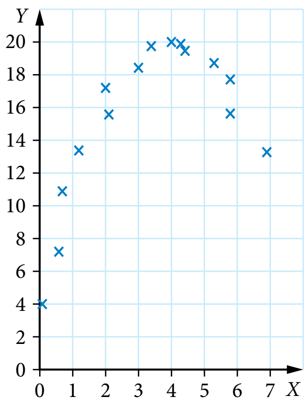
1.4 Associations

For bivariate data where both variables are **numerical**, a scatterplot can help you find patterns.

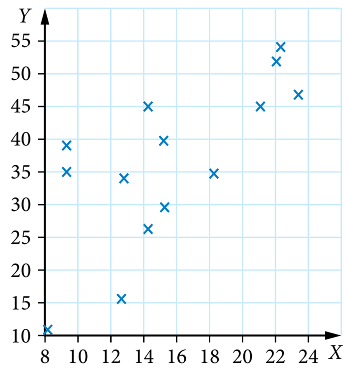
A pattern in a scatterplot suggests an **association or correlation** (**relationship**) between variables. Association does not imply causation between the variables: change in one variable does not necessarily *cause* change in the other.

Consider these scatterplots.

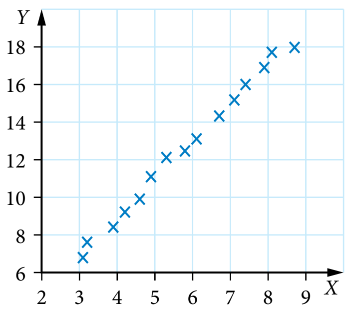
A



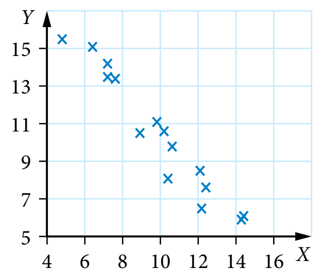
B



C



D



Graph **A** has a clear association, but it is not a straight line. Graphs **C** and **D** have clear associations that approximate straight lines, but the line is much clearer in graph **C**. Graph **B** could be a straight line going up from left to right, but it is not very clear.

Describing associations

When describing associations, you should comment on strength, direction and form.

Strength

- A **strong association** has points that approximate an almost perfect line.
- A **moderate association** has an upward or downward trend that can be shown by a straight line through the points. Almost all the points are close to such a line.
- A **weak association** has a pattern, but no clear line.

Direction

- A **positive association** goes upward from left to right.
- A **negative association** goes downward from left to right.

Form

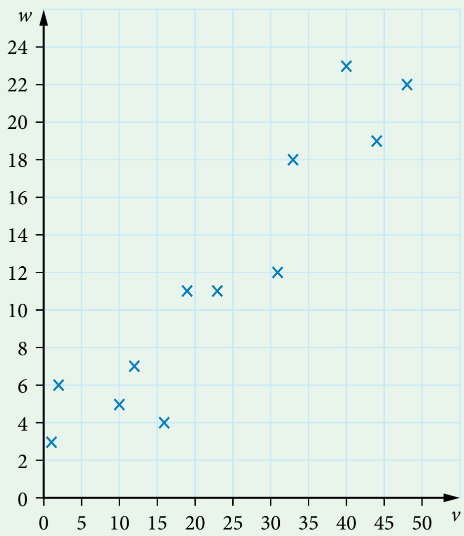
- A **linear** association is a straight line.
- A **non-linear** association is not a straight line. It will usually follow a curve.

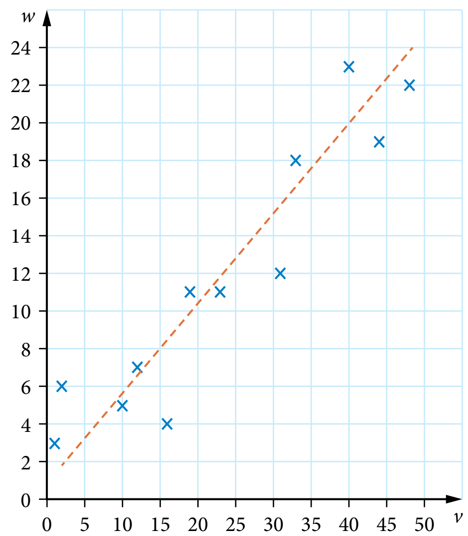
Where there is no defined strength, direction or form to a scatterplot, it is said to have **no association**.

WORKED EXAMPLE 8 Describing association of scatterplots

The scatterplot shows the values of some bivariate data.

Describe any apparent association between v and w, where v is the explanatory variable and w is the response variable.



Steps	Working
1 Describe the points.	The points have a clear trend upwards from left to right.
2 Try drawing a line of best fit (by eye) to consider how close points are to decide strength.	
3 Describe the result.	All the points are fairly close to a straight line sloping upwards.
4 Write the answers, ensuring strength, direction and form are stated.	There is a moderate positive linear association between v and w.

Exam hack

In exams, sometimes only 2 of the 3 descriptors (strength, direction, form) are required in the answer. Read the question carefully to ensure you answer correctly.

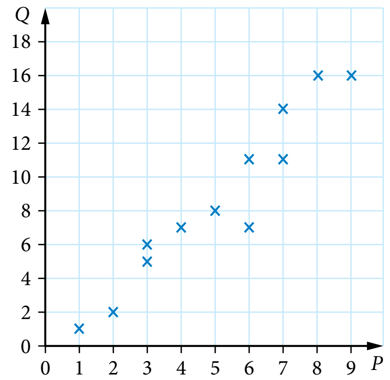
WORKED EXAMPLE 9 Positive scatterplots

The table below shows the values of some bivariate data with variables P and Q, where P is the explanatory variable and Q is the response variable.

P	1	3	6	7	2	3	4	5	6	7	8	9
Q	1	6	7	11	2	5	7	8	11	14	16	16

a **Construct** a scatterplot of the data with P on the horizontal axis.

b **Describe** the association of the data in the plot.

Steps	Working
a **1** Choose appropriate scales. Go up in 1s for P and up in 2s for Q so the scales are about the same size. **2** Plot the points.	
b Describe the association.	As the dots are fairly close to a line, the graph shows a **strong positive linear** association.

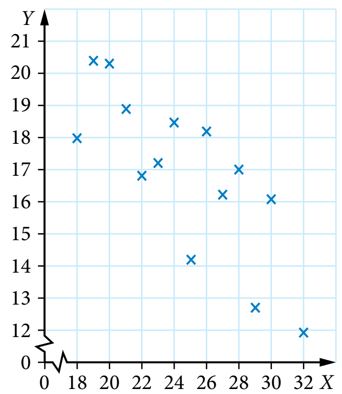
WORKED EXAMPLE 10 Negative scatterplots

The table below shows the values of some bivariate data with variables X and Y, where X is the explanatory variable and Y is the response variable.

X	18	19	20	21	22	23	24	25	26	27	28	29	30	32
Y	18	20.4	20.3	18.9	16.8	17.2	18.5	14.2	18.2	16.2	17	12.7	16.1	11.9

a **Construct** a scatterplot of the data with X on the horizontal axis.

b **Describe** the association of the data in the plot.

Steps	Working
a **1** Choose appropriate scales. Go up in 2s for X and up in 1s for Y so the scales are about the same size. **2** Plot the points.	
b Comment on the graph and describe the strength, direction and form.	The points slope roughly downward from left to right. As the points are not close to a line, the data has a **moderate negative linear** association.

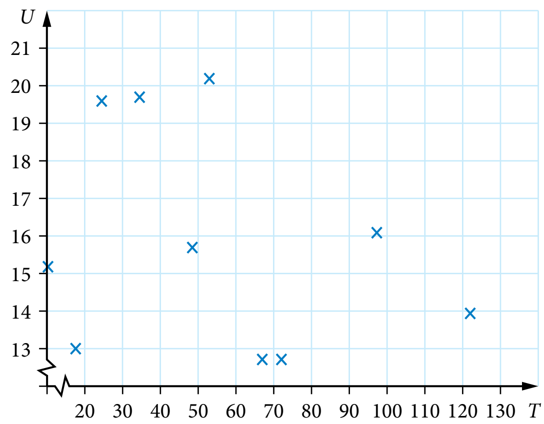
WORKED EXAMPLE 11 No relationship scatterplots

The table below shows the values of some bivariate data with variables T and U, where T is the explanatory variable and U is the response variable.

T	10.4	17.6	24.5	34.5	48.7	53.1	67.1	72	97.3	122
U	15.2	13	19.6	19.7	15.7	20.2	12.7	12.7	16.1	13.9

a **Construct** a scatterplot of the data with T on the horizontal axis.

b **Describe** the association of the data in the plot.

Steps	Working
a 1 Choose appropriate scales. Go up in 10s for T and up in 1s for U so the scales are about the same size. **2** Plot the points with T on the horizontal axis.	
b Comment on the graph and describe the strength, direction and form.	There is no clear strength, direction or form to the data points. The data has **no association** between T and U.

Investigation Class associations

Collect the following data from your class:

Shoe size	height	forearm length (elbow to wrist)
age in months	number of siblings	hair length
number of people living in home		

There are many possible scatterplots that you can draw from the data.

Work in groups of 2 or 3 to draw the scatterplots; your teacher will assign the scatterplots for each group.

Comment on any patterns in the scatterplots.

EXERCISE 1.4 Scatterplots and associations

ANSWERS p. 438

Recap

1 State the response variable in each situation.

a Studying the effect of study time on academic results.

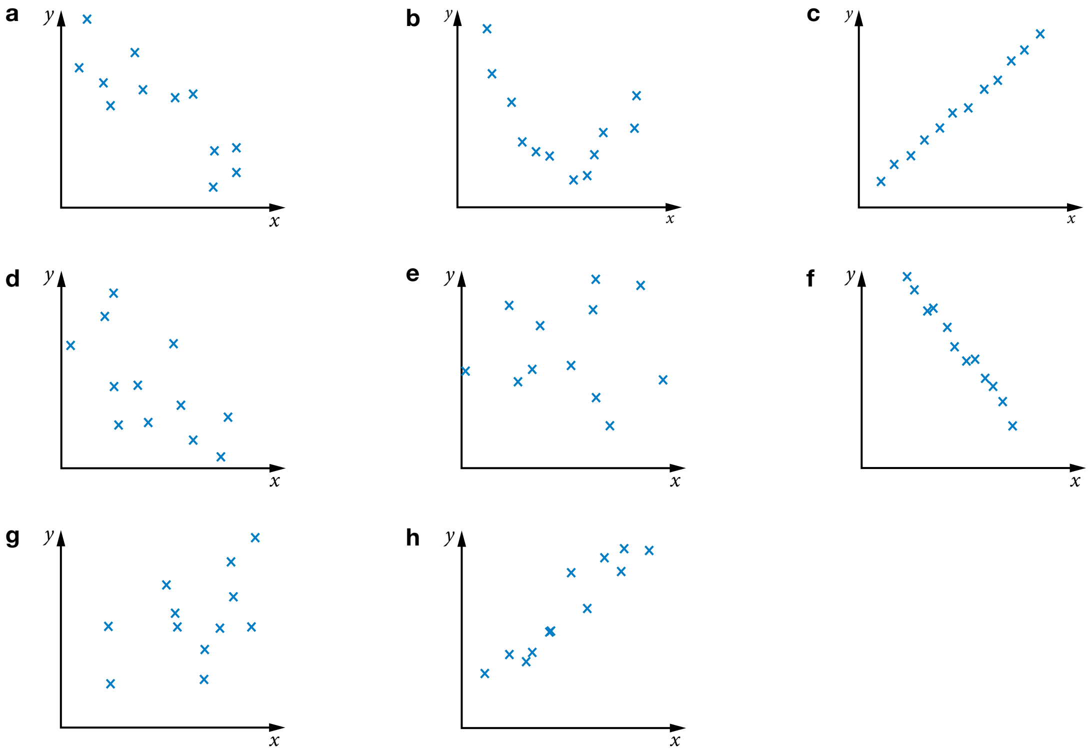
b Measuring whether hearing loss is linked to the sound level of headphones.

2 Determine how many people were surveyed in the following table.

	Children	Adults
Vaccinated	103	232
Not vaccinated	27	68

Mastery

3 WORKED EXAMPLE 8 Describe any apparent associations in each scatterplot.



4 WORKED EXAMPLE 9 This table shows the values of some bivariate data.

P	5.6	6.9	6.8	7.4	7.7	9.6	10.6	9.5	9.3	11.6	12	12.2
Q	11.5	11.7	12.7	12.5	12.7	14.2	13	13.9	13.8	14.8	15.4	14.7

a Construct a scatterplot of the data with P on the horizontal axis.

b Comment on the association.

5 WORKED EXAMPLE 10 This table shows the values of some bivariate data.

X	10.3	14.5	16.4	13.7	18.1	14.3	17.6	19	24.7	26.3	27.6	23.6
Y	51	53.1	41.5	42.4	41.3	26.9	30.6	22.1	24.5	13.4	10.2	16.9

a Sketch a scatterplot of the data with X on the horizontal axis.

b Describe the association between the variables.

6 WORKED EXAMPLE 11 This table shows the values of some bivariate data.

A	27	35	34	32	47	44	56	45	49	58
B	56	33	13	34	64	45	27	15	24	44

a Make a scatterplot of the data with A on the horizontal axis.

b Describe the association of the data in the plot. Comment on any patterns.

7 For each table, construct a scatterplot and describe the association between the variables.

a

V	0.7	1.2	1.4	2	2.1	2.4	2.8	2.9	2.9	3.9	4
W	12.7	13.3	17.6	14.1	15.1	13.1	23.4	18.5	17.6	26.7	25.9

b

P	0.6	1.6	3.1	4.1	5.3	6.2	6.7	7.7	8.7	10.3	10.8	11.8	13.1	14.2
V	39.1	36.3	33.5	33.2	31.1	28.2	24.9	23.5	23	19.9	18.5	17.1	13.7	11.3

c

x	3.5	0.8	4.2	10.5	7.7	13.3	14.9	15.9	15.7	21.4
y	1.2	11	10.8	5.1	15.3	5	12.5	16.8	19.9	8.5

d

T	2	3	8	5	4	6	8	7	5	10
N	20	24	20	31	31	30	25	24	36	8

Exam practice

8 For the variables, temperature and public pool entries, the most appropriate association description should be a

A moderate positive linear association
B moderate negative linear association
C weak negative linear association
D no association.

9 For the variables, amount of food sales at a football stadium and their prices, the most appropriate association description should be a

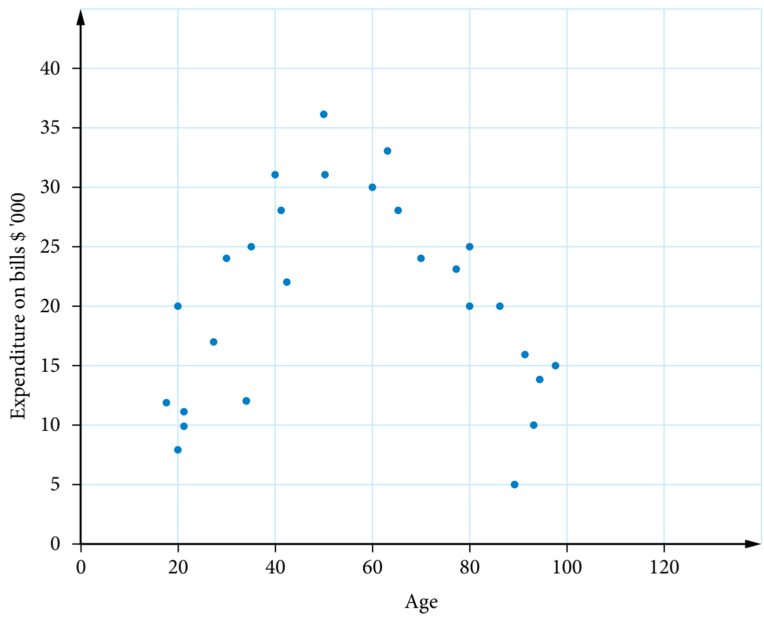
A strong positive association
B moderate negative association
C weak non-linear association
D no association.

10 A recent study looked at height of a person and the hours spent listening to audiobooks. The expected strength of the association should be a

A strong association
B moderate association
C weak association
D no association.

11 (2 marks) A recent study looked at whether drinking "zero sugar" soft drinks caused weight gain. State which is the response variable and which is the explanatory variable.

12 (2 marks) Describe the strength and form of this scatterplot.



13 (4 marks) CF 20 people were asked how many takeaway meals they had the previous week. The data below shows their ages and the number of takeaway meals they reported.

35, 2	55, 1	58, 2	51, 0	39, 1
22, 3	24, 4	48, 0	41, 4	19, 5
39, 2	21, 5	65, 2	59, 0	46, 2
24, 2	26, 4	38, 3	59, 2	18, 4

Analyse the data to determine whether a person's age has an association with the number of takeaways they will have in a week.

14 ©QCAA 2024 1Q11 93% The scatterplot shows an association between two numerical variables.

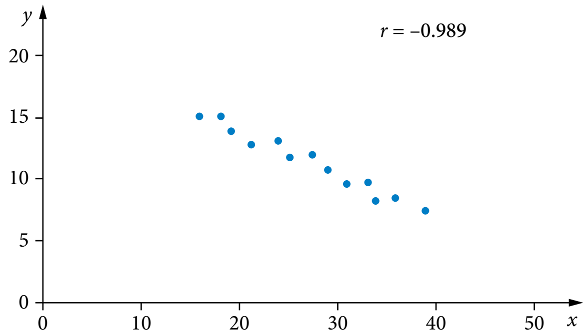
The association is best described as

A negative and weak.

B negative and linear.

C positive and strong.

D non-linear and weak.



1.5 Correlation

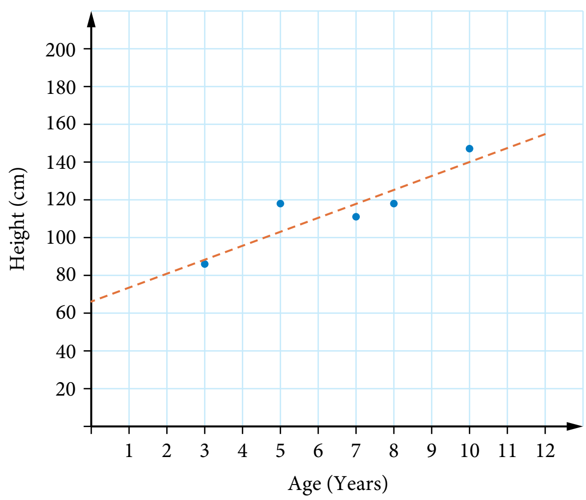
Video
Correlation

While drawing a scatterplot and interpreting association by sight may be useful, it may not be accurate, and users may interpret the association differently. Using a statistical measurement is a more accurate way of describing the strength of an association.

This table shows the ages and heights of a group of girls.

	Skye	Zoe	Mila	Hope	Arwen
Age (years)	3	5	7	8	10
Height (cm)	86	116	111	118	147

A scatterplot shows how close bivariate data is to a straight line.



From the graph the association can be determined as positive and linear, but as the number of values is small, it is difficult to determine strength. Using a statistical measure will make this more accurate.

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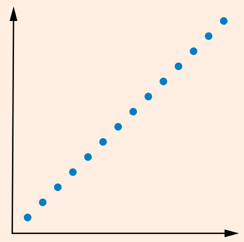
Pearson's correlation coefficient

The **Pearson's correlation coefficient** (r) is a measure of the strength and direction of the linear relationship between a pair of variables.

- The **correlation coefficient** is always a number from -1 to 1.
- A positive correlation coefficient means that there is a positive association.
- A negative correlation coefficient means that there is a negative association.
- A correlation coefficient of -1 or 1 means that there is a perfect straight line.
- While it measures strength and direction, it does not prove causation between the 2 variables.

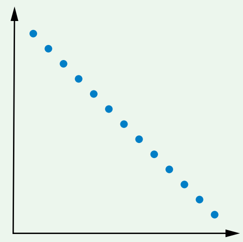
Perfect positive association

$r = 1$



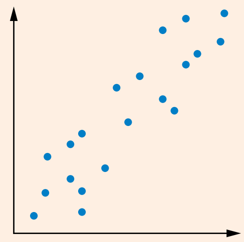
Perfect negative association

$r = -1$



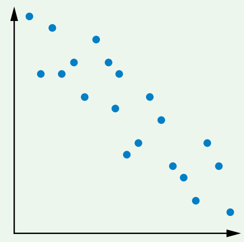
Strong positive linear association

$0.75 \le r < 1$



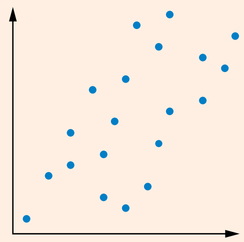
Strong negative linear association

$-1 < r \le -0.75$



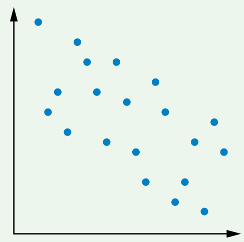
Moderate positive association

$0.5 \le r < 0.75$



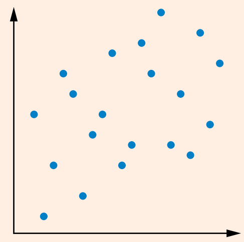
Moderate negative linear association

$-0.75 < r \le -0.5$



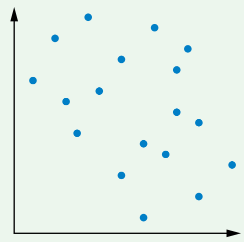
Weak positive linear association

$0.25 \le r < 0.5$



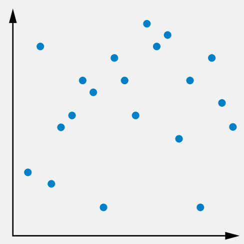
Weak negative linear association

$-0.5 < r \le -0.25$



No linear association

$-0.25 < r < 0.25$



When applying these scales, it is assumed that:

- the association is linear
- there are no outliers

TECHNOLOGY	Correlation coefficient

In this course you will use a scientific calculator, spreadsheet or other technology to find the correlation coefficient.

Name	Skye	Zoe	Mila	Hope	Arwen
Age (years)	3	5	7	8	10
Height (cm)	86	116	111	118	147

Enter the data in the table to calculate the correlation coefficient (r).

Casio (fx-82AU PLUS II)

1 Press MODE, **2**: STAT and **2**: A+BX.

2 Enter the values of age and height in the X and Y columns respectively. You can enter the data into each column by hitting = after each value. Use the arrows to move to the next column.

3 Press **AC** then press SHIFT, **1** STAT, **5**: Reg, **3**: r and press = to read the result of 0.911 98...

4 While in stats mode, use SHIFT, **1** to bring up the menu to find other values, such as $\bar{x}$, $\bar{y}$, sx and sy.

5 Press MODE, **1**: COMP to exit the statistics mode. This will also clear the existing statistics list from your calculator.

Sharp (EL-531TH)

1 Press MODE, **1** and choose **1** (LINE): Linear regression calculation.

2 Enter the values of age and height in pairs. To enter the first pair in the table, press **3**, STO →, **86**, **M+**, the second pair follows with **5**, STO →, **116**, **M+**, etc. until you have entered all the values.

3 Then press ALPHA,) to read the result for r of 0.911 98...

4 Use ALPHA, and other buttons on the calculator to find statistical values, such as ($\bar{x}$), ALPHA, **4** or sy, ALPHA, **8**.

5 Reset your calculator to clear the datasets saved in the memory.

TI-30XB

1 Press (**DATA**) and **enter** the values of age and height in the L1 and L2 columns respectively. This can be done vertically, hitting **Enter** to move down the column. Use arrows to move sideways.

2 Then press the 2nd button, then data to get the STAT features. Select **2**: Var Stats.

3 Make sure it has XDATA: L1 and YDATA: L2 and then use the cursor to go down to **CALC** and press **enter**.

4 Use the arrows to move down to **r =** to read the result 0.91198...

5 You can use the same section to find other variables for your questions, such as $\bar{x}$, $\bar{y}$, sx and sy.

6 Press 2nd MODE **5** to exit the statistics calculations.

7 Remember to clear your calculator between questions to remove existing datasets. Do this by pressing 2nd, **0** and selecting option 2 for YES.

1.5

WORKED EXAMPLE 12 Calculating correlation coefficient

A study is undertaken to test whether there is an association between a person's age and the average number of minutes they listen to a music app in a day.

Calculate the correlation coefficient for the following data and interpret the result.

Age	14	18	25	36	41	52	63	75
Time (min)	75	151	135	57	92	30	98	54

Steps	Working
1 Use your scientific calculator to find the correlation coefficient. See the calculator instructions.	You should get a value of –0.5222… if you have entered the data correctly.
2 Discuss the association.	There is a moderate negative correlation between a person's age and the number of minutes they spend listening to a music app in a day. While this result is moderate, it does not prove causation. Looking at the individual data pairs, as someone gets older, they may actually listen to a music app more, such as the 63-year-old.

The correlation coefficient, r, measures the strength of an association. You can use the square of the correlation coefficient, r^2 or $\boldsymbol{R^2}$, to measure how much the response variable is explained by the explanatory variable.

The coefficient of determination

- The value of R^2 (r^2) is called the **coefficient of determination**.
- R^2 will always be a positive number as it is a square of the correlation coefficient.
- R^2 measures how much the response variable is determined by the explanatory variable. It is the proportion of the response variable that changes in the same way as the explanatory variable.
- $R^2 = 0$ means that the points are randomly scattered, and the response variable has no relationship with the explanatory variable.
- $R^2 = 1$ means that the points are exactly on the line, so the explanatory variable completely determines the value of the response variable.
- R^2 can be interpreted by converting it to a percentage. $R^2 \times 100$ will give you the percentage of the change in the response variable that is explained by the explanatory variable. For example, if $R^2 = 0.78$ for the relationship between height and shoe size, 78% of the surveyed peoples' shoe sizes can be explained by their height.

WORKED EXAMPLE 13 Calculating coefficient of determination

Calculate the coefficient of determination if the correlation coefficient is

a 0.32

b –0.79

Steps	Working
a 1 Calculate the coefficient of determination (R^2) by squaring the correlation coefficient.	$R^2 = 0.32^2$
2 Write the answer.	$= 0.1024$
b 1 Calculate (R^2).	$R^2 = (-0.79)^2$
2 Write the answer.	$= 0.6241$

WORKED EXAMPLE 14 Interpreting coefficient of determination

The coefficient of determination between the English and Mathematics grades of Year 12 students is about 0.45. **Explain** what this means in comparison to the correlation coefficient.

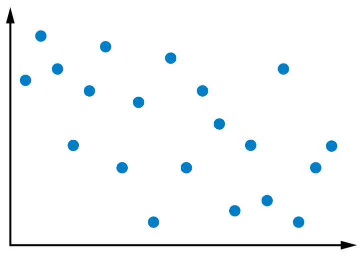
Steps	Working
1 Give the numerical meaning.	The English and Mathematics grades have 45% in common. Looking at it the other way around, 55% of the English grade is *not* determined by the Mathematics grade.
2 Calculate the correlation coefficient.	$r = \pm\sqrt{0.45}$ $\approx \pm 0.671$ Without an indicator of slope or raw data, the answer for r may be positive or negative.
3 Interpret the mathematical meaning.	The correlation coefficient shows that the strength is moderate, however most of the English and Mathematics grades are independent and an increase or decrease in one may not relate to the other.

EXERCISE 1.5 Correlation

ANSWERS p. 439

Recap

1 What strength of association is shown in this scatterplot?



A strong **B** moderate **C** weak **D** no association

2 State the explanatory variable when analysing the association between temperature and cinema ticket sales.

Mastery

3 Describe the strength and direction of each correlation coefficient.

a –0.347 **b** 1 **c** 0.58 **d** –0.89

e 0.443 **f** –1 **g** 0.94 **h** 0.685

4 WORKED EXAMPLE 12 Use your calculator to find the correlation coefficient of this table of data.

m	20	35	42	54	73	90
E	64	28	52	28	19	6

5 Use your calculator to find the correlation coefficient of p and q in this table.

p	9	10	11	10	12	12	16	14
q	20	25	18	19	30	18	31	15

6 Use your calculator to find the correlation coefficient from the table below and describe the association.

x	17.7	44.2	94.7	91.2	85.1	120.4	163.7	124	149	173.7
y	17.5	15	27.4	18.9	26.9	27.2	21.6	24.3	21	37.5

1.5

7 Use your calculator to find the correlation coefficient from the table below and describe the association.

a	9.9	10.8	11.3	11.7	12.3	12.8	13.4	13.2	14.4	14.4
b	127	129	130	128	130.2	130.1	131	130.7	131.9	130.3

8 WORKED EXAMPLE 13 Calculate the coefficient of determination for each value of r.

a $r = 0.4$ **b** $r = 0.156$ **c** $r = -0.85$ **d** $r = -1$

e $r = 0.76$ **f** $r = -0.39$ **g** $r = 0.55$ **h** $r = -0.616$

9 WORKED EXAMPLE 14 Calculate the correlation coefficient with reference to the given direction.

a $R^2 = 0.0625$ (positive) **b** $R^2 = 0.8281$ (negative) **c** $R^2 = 0.4356$ (negative)

d $R^2 = 0.7225$ (positive) **e** $R^2 = 0.3136$ (positive) **f** $R^2 = 0.021\,025$ (negative)

10 The mouse population in a field of wheat was estimated by trapping them at night and counting them. The population was monitored every 3 days for a month and the following results were obtained.

Determine if there is any association and comment on your results.

Time (days)	0	3	6	9	12	15	18	21	24	27	30
Population ('00)	0.6	0.9	1.1	1	1.2	1.4	1.6	2	2.2	2.2	2.5

Exam practice

11 A coefficient of determination was calculated to be 0.4096. The value of the correlation coefficient is

A –0.64 **B** 0.1024 **C** 0.2048 **D** –0.372

12 If the correlation coefficient is –0.67, the association could be best described as

A strong negative **B** moderate negative

C weak negative **D** no association

13 A comparison of daily temperature and slushy sales has a coefficient of determination of 0.83. This suggest that

A 83% of the variation of the slushy sales is in response to a variation of temperature.

B 83% of the variation of the temperature is explained by the explanatory variable.

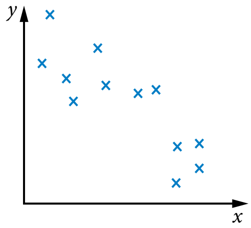
C 83% of the variation of explanatory variable is explained by the response variable.

D 83% of the variation of temperature is explained by the response variable.

14 (1 mark) Using your calculator, determine the correlation coefficient from the following data.

X	7	10	15	4	8	13	13	12	8
Y	32	31	56	21	40	35	35	30	46

15 (3 marks) The scatterplot has a coefficient of determination of 0.3025.



a Calculate the correlation coefficient. *[1 mark]*

b Interpret the association and state the effect of a variation of the explanatory variable on the response variable. *[2 marks]*

16 (4 marks) CF The loss of yield from mouse plagues in regional Australian wheat crops were estimated as follows. The mouse plagues occurred at different stages of the crop cycle.

Loss ('000s tons)	3	29	34	52	78	114	112	134	162	168
Population (mice/ha)	6	4	18	28	14	42	36	55	67	73

The Department of Primary Industries put out a statement that 'The size of a mouse plague only has a moderate impact on wheat production.' Examine the data to justify if the statement is reasonable.

17 ©QCAA 2024 1Q3 25% The coefficient of determination, R^2, is equal to 0.36 for the linear association between x (explanatory variable) and y (response variable).

Which statement is correct?

A 36% of the variation in x can be explained by the variation in y.

B 36% of the total variation can be explained by the linear association.

C 36% of the predicted outcomes can be explained by the variation in x.

D 36% of the variation in x can be predicted by the linear association.

EXAM QUESTION ANALYSIS

©QCAA 2023 1Q22 (4 marks)

1.5

A person used a fitness tracker to monitor their hours of sleep and the distance they travelled each day.

Sleep (hours)	7	7.5	8	6.5	9.5	6	8.5
Distance (km)	5	7	5	7	4	8	6

a Using distance as the response variable, display the data in a scatterplot with labelled axes. *[3 marks]*

b Identify the direction of the association between hours of sleep and distance travelled. *[1 mark]*

Reading the question

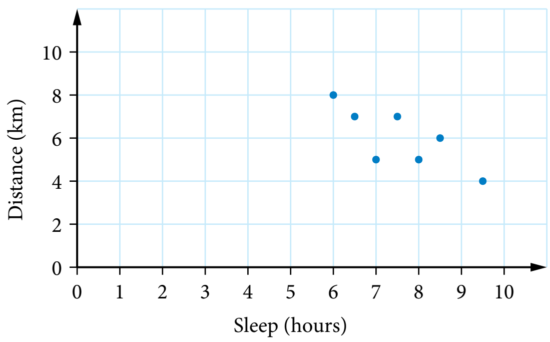
- Construct a scatterplot.
- Identify direction of the association.

Thinking about the question

- Identify the explanatory variable (sleep) and the response variable (distance).
- The numbers are small enough that each axis may use intervals of 1.
- Both axes only need to go up to 10.
- Ensure the data pairs are appropriately placed with a dot on the graph.
- The question only asks for direction, not strength or form.

Worked solution (✓ = 1 mark)

a



✓ correctly identifies the axis for the response variable

✓ formats scatterplot with appropriate scaling and labelling for both axes

✓ accurately plots all points

b The association is **negative.** ✓

Video
Exam question analysis: Associations between variables

1 Chapter summary

Types of data

- **Univariate** data has only one variable.
- **Bivariate** data has 2 variables, collected in pairs.

Two-way tables

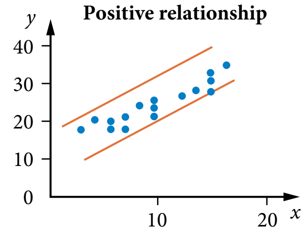
- A **two-way table** for bivariate data has one variable across the top row and the other variable is in the first column.
- Data for a two-way table may come in a list, or by reading from a graph.
- You can work out percentages in a **percentaged two-way frequency table**:
  - by row (or **row percentages**) based on the totals on the right of the rows
  - by column (or **column percentages**) based on totals at the bottom of the columns
  - by table total (or **table total percentages**) based on the grand total at the bottom right.

Response and explanatory variables

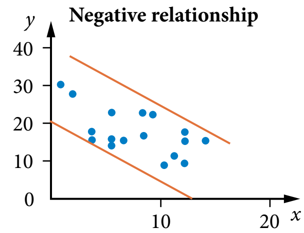
- A **response variable (RV)** is changed by another variable.
- An **explanatory variable (EV)** is not changed by another variable. The response variable could depend on the explanatory variable(s).
- **Causation** occurs when a change in the explanatory variable causes a change in the response variable.

Scatterplots

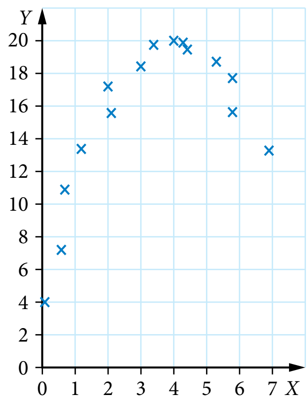
- Use ordered pairs of numerical bivariate data for a **scatterplot**.
- The explanatory variable is placed on the horizontal axis and the response variable on the vertical axis.
- Choose scales so the axes are about the same length.
- A pattern in a scatterplot suggests an **association** between bivariate numerical variables.
- Association is described using the descriptors of strength, direction and form.
- In a **positive association**, the low values of the variables tend to be in the same pairs, as do the high values; the graph slopes upward from left to right.



- In a **negative association**, the low values of one variable tend to be in pairs with high values of the other; the graph slopes downward from left to right.



- A **linear** association is a straight line.
- A **non-linear** association is not a straight line.



- A **strong association** has points that make an almost perfect line.

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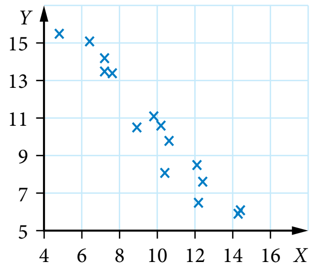
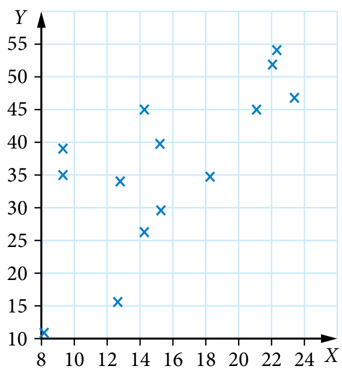
- A **moderate association** has an upward or downward trend that can be shown by a straight line through the points. Almost all the points are close to such a line.



- A **weak association** has a pattern, but no clear line.



- **Pearson's correlation coefficient**, r, calculates the strength and direction of the relationship between 2 variables.
- **Correlation coefficient** is assumed to be linear and not include outliers.
- The correlation coefficient is always a number from -1 to 1.

  $r = 1$: perfect positive association, a perfect straight line

  $0.75 \leq r < 0.5$: strong positive association

  $0.5 \leq r < 0.75$: moderate positive association

  $0.25 \leq r < 0.5$: weak positive association

  $-0.25 < r < 0.25$: little or no association

  $-0.5 < r \leq -0.25$: weak negative association

  $-0.75 < r \leq -0.5$: moderate negative association

  $-1 < r \leq -0.75$: strong negative association

  $r = -1$: perfect negative association, a perfect straight line
- While the correlation coefficient measures strength and direction, it does not prove causation between the 2 variables.

The coefficient of determination

- The value of $\boldsymbol{R^2}$ (r^2) is called the **coefficient of determination**.
- R^2 is a positive number.
- R^2 measures how much the response variable is determined by the explanatory variable.
- $R^2 = 0$ means that the response variable has no relationship with the explanatory variable.
- $R^2 = 1$ means that the explanatory variable completely determines the value of the response variable.
- R^2 can be interpreted by converting it to a percentage, $R^2 \times 100$.

Worksheet
General Maths formula sheet

Cumulative examination 1

Simple familiar

Perusal time: 2 minutes   Working time: 30 minutes

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

Section 1 (5 marks): 5 multiple choice questions

Section 2 (13 marks): 4 short response questions

Total: 18 marks

Section 1 5 multiple choice questions
5 marks

1 The correlation coefficient

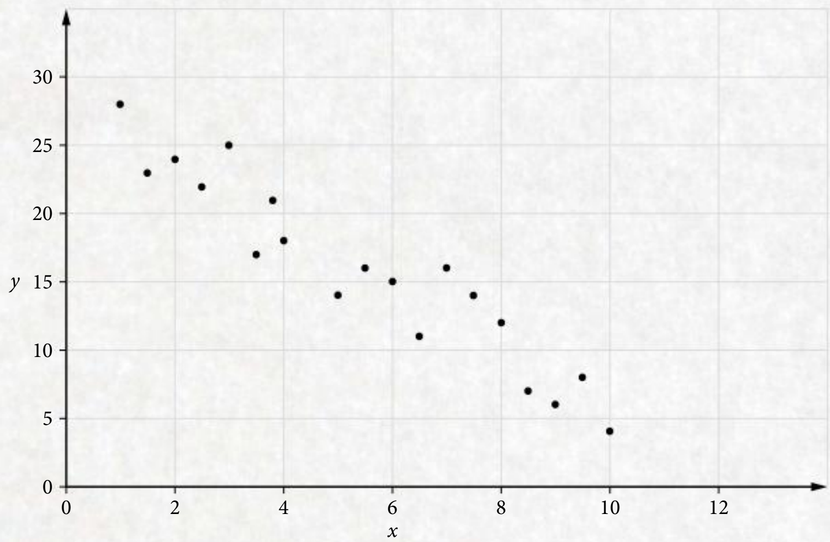
A measures non-linear relationships.

B analyses the strength of the relationship between 2 variables.

C is always answered as a percentage.

D gives an answer between zero and one only.

2 ©QCAA 2020 1Q8 The following scatterplot shows a linear association between two numerical variables.



Choose the best description for the direction and strength of the association.

A strong positive

B strong negative

C weak positive

D weak negative

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3 ©QCAA 2020 1Q14 A sample of university staff and students was asked whether they preferred catching public transport or driving their own car to university. The data collected is shown in the table.

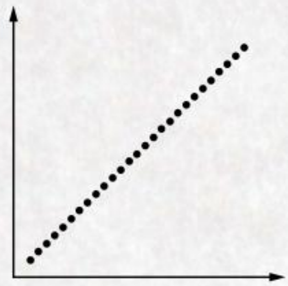
	Public transport	Drive own car
Staff	2	18
Students	48	12

What percentage of university students prefer to drive their own car?

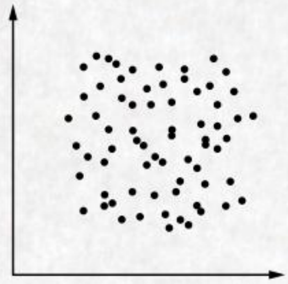
A 12% **B** 15% **C** 20% **D** 40%

4 ©QCAA 2022 1Q12 Identify the scatterplot that best demonstrates a strong negative association.

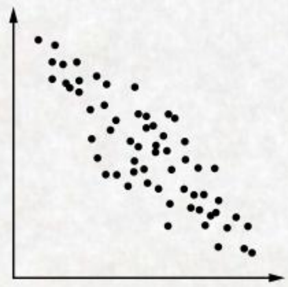
A



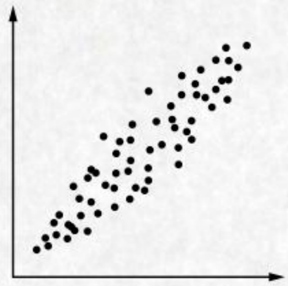
B



C



D



5 ©QCAA 2021 1Q11 Which option is an example of bivariate data?

A The rating given to a brand of meat pies as poor, fair or good.

B The number of people in a household and amount of water used.

C The number of cars passing through a particular set of traffic lights.

D The time a person spends using a mobile phone on a Friday evening.

CUMULATIVE EXAMINATION 1

Section 2 4 short response questions 13 marks

6 (3 marks) A nursery monitored the growth of potted flowers after 5 weeks. The table shows data comparing growth time (in days) and plant height (in cm).

Time (days)	36	37	37	38	38	42	43	44	44	45
Height (cm)	42	43	44	43	44	41	42	43	41	42

a Determine the correlation coefficient for the data to 3 decimal places. *[1 mark]*

b Describe what this correlation suggests about the association between growth time and plant height. *[2 marks]*

7 (2 marks) State an example that would be an appropriate explanatory variable for the response variable 'Income ($)' and explain why it would be appropriate.

8 ©QCAA 2022 1Q24 (5 marks) The maximum temperature and the number of pies sold each day at a bakery are provided in the table.

Maximum temperature (°C)	29	20	31	27	23	25	22	33
Number of pies sold	32	39	25	33	37	35	37	30

a Construct a scatterplot to display the data. *[3 marks]*

b Describe the association between the maximum temperature and the number of pies sold in terms of direction and strength. *[2 marks]*

9 (3 marks) A study was undertaken on the use of popular social media platforms by age. The percentaged two-way table displays the results.

		Age	
		Under 16	16 +
Social Media Platform	InstaTube	20%	45%
	Tikgram	36%	38%
	YouTok	44%	17%

a State which platform is the most used by under-16s. *[1 mark]*

b Which is the most popular platform for all viewers? *[1 mark]*

c If the government wanted to work with platforms about enforcing age restrictions on their content, explain which platforms should be targeted. *[1 mark]*

Cumulative examination 2

Complex familiar and unfamiliar

Perusal time: 2 minutes   Working time: 30 minutes

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

Worksheet
General Maths formula sheet

2 short response questions

11 marks

1 (6 marks) A survey was undertaken at a school about their school's sports uniform. Seventy students were interviewed from the junior school and eighty from the senior school. The students were asked whether there should be a new uniform design, changes made to the existing uniform or no change to the uniform. The results were as follows:

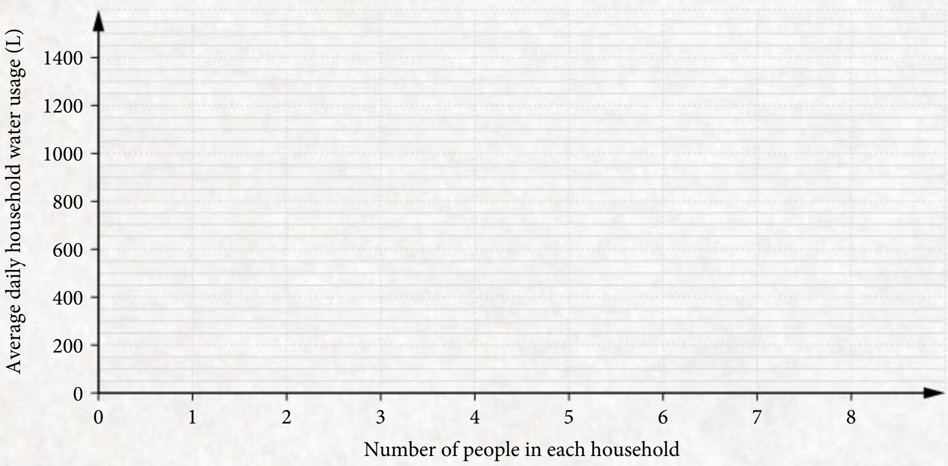
- 20 juniors wanted a new uniform and 11 wanted to change the existing uniform
- 27 seniors wanted a new uniform and 33 wanted to change the existing uniform

Analyse the results to determine if there is an association between the age of the student and their view about the school uniform and whether the school should consider a discussion about potential uniform changes.

2 ©QCAA 2020 2Q2 (5 marks) The number of people living in each household and the average daily household water usage, measured in litres (L), were recorded for 10 households.

Number of people in each household	6	2	4	5	5	4	3	1	6	7
Average daily household water usage (L)	990	160	320	480	410	280	240	130	940	1340

Calculate Pearson's correlation coefficient and then evaluate the appropriateness of using this coefficient for the association between daily water usage and the number of people living in a household. Copy and use the graph below.



CHAPTER

2 ARITHMETIC AND GEOMETRIC SEQUENCES

9780170484824

Prior learning
Arithmetic and geometric sequences

2.1

Syllabus coverage

UNIT 3, TOPIC 4: GROWTH AND DECAY IN SEQUENCES

The arithmetic sequence

- Use recursion to generate an arithmetic sequence.
- Display the terms of an arithmetic sequence in both tabular and graphical form and demonstrate that arithmetic sequences can be used to model linear growth and decay in discrete situations.
- Use the rule for the n^{th} term of an arithmetic sequence.
    - $t_n = t_1 + (n - 1)d$ where t_n is n^{th} term, t_1 is first term, n is term number and d is common difference
- Use arithmetic sequences to model and analyse practical situations involving linear growth or decay, e.g. analysing a simple interest loan or investment, calculating a taxi fare based on the flag fall and the charge per kilometre, calculating the value of an item using the straight–line method of depreciation.

The geometric sequence

- Use recursion to generate a geometric sequence.
- Display the terms of a geometric sequence in both tabular and graphical form and demonstrate that geometric sequences can be used to model exponential growth and decay in discrete situations.
- Use the rule for the n^{th} term of a geometric sequence.
    - $t_n = t_1 r^{(n-1)}$ where t_n is n^{th} term, t_1 is first term, n is term number and r is common ratio
- Use geometric sequences to model and analyse practical situations involving geometric growth and decay (use of logarithms not required), e.g. modelling the growth of a bacterial population that doubles in size each hour, calculating the value of an item using the diminishing–value method of depreciation.

General Mathematics 2025 v1.2 General senior syllabus pp. 24–25,

Videos (3):
2.2 Arithmetic sequences
2.5 Geometric sequences
Exam question analysis Arithmetic and geometric sequences

Prior learning (1):
2.1 Arithmetic and geometric sequences

Worksheets (8):
2.1 Arithmetic progressions
2.2 Arithmetic sequences
2.4 Classifying sequences
2.5 Geometric sequences • Geometric progressions
2.6 Modelling with sequences • Modelling arithmetic and geometric sequences
Cumulative exams General Maths formula sheet

Nelson MindTap

To access resources above, visit
cengage.com.au/nelsonmindtap

Terminology

arithmetic sequence	common difference	common ratio
depreciation	diminishing–value depreciation	exponential function
exponential growth and decay	general term (n^{th} term)	geometric growth and decay
geometric sequence	linear decay	linear growth
recurrence relation	recursion	salvage (scrap) value
sequence	simple interest	straight–line depreciation
term		

Worksheet
Arithmetic progressions

2.1 Arithmetic sequences

A **mathematical sequence** is a list of numbers in order, for example, 5, 8, 11, 14, …
This sequence starts at 5 and has 3 added to each term.

Investigation Sequence rules

Work out a rule for each sequence below. There could be more than one answer for some of them.

5, 6, 8, 11, 15, 20, 26, …	37, 72, 26, 61, 15, 50, 4, 39, 92, …
5, 10, 20, 40, 80, …	29, 27, 25, 23, 21, 19, …
1, 3, 6, 10, 15, 21, 28, …	8, 4, 2, 1, $\frac{1}{2}$, $\frac{1}{4}$, …
1, 1, 2, 3, 5, 8, 13, 21, 34, …	7, 11, 15, 19, 23, 27, 31, …
48, 85, 59, 96, 70, 8, 81, 19, 92, …	1, −5, −11, −17, −23, −29, …

Make up some sequences of your own for other students to try. Try theirs!

Arithmetic sequences

- A **sequence** is a list of numbers in a particular order.
- Each member of the sequence is called a **term**, usually written as $t_1, t_2, t_3, t_4, \ldots$
  The subscript tells you which term it is.
- The **first term** is written as t_1.
- The **n^{th} term** is written as t_n.
- If the difference between successive terms is a constant, it is an **arithmetic sequence**.
- The difference is called the **common difference**, given by

$$d = t_2 - t_1 = t_3 - t_2 = t_4 - t_3 = \ldots \quad \text{or} \quad d = t_n - t_{n-1}$$

WORKED EXAMPLE 1 Arithmetic sequences

For each sequence

i **state** if it is an arithmetic sequence and give the common difference if it is.

ii **calculate** the 8th term.

a 5, 9, 13, 17, 21, … **b** 5, 10, 20, 40, 80, …

c 1, 4, 9, 16, 25, … **d** 17, 11, 5, −1, −7, …

Steps	Working
a i 1 Look for a pattern.	Each term is 4 more than the previous term.
2 Write the answer.	It is an arithmetic sequence with a common difference of 4.
ii 1 Continue the pattern to find the 8th term.	t_1 t_2 t_3 t_4 t_5 t_6 t_7 t_8 5 9 13 17 21 25 29 33 + 4 + 4 + 4
2 Write the answer.	The 8th term is 33.
b i 1 Look for a pattern.	Each term is twice the previous term.
2 Write the answer.	There is no constant difference, so it is not an arithmetic sequence.
ii 1 Continue the pattern to find the 8th term.	t_1 t_2 t_3 t_4 t_5 t_6 t_7 t_8 5 10 20 40 80 160 320 640 × 2 × 2 × 2
2 Write the answer.	The 8th term is 640.

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c	**i**	**1**	Look for a pattern.	Each term is the square of the term number.
		2	Write the answer.	It is not an arithmetic sequence.
	ii	**1**	Continue the pattern to find the 8th term.	t_1 t_2 t_3 t_4 t_5 t_6 t_7 t_8 1 4 9 16 25 36 49 64
		2	State the answer.	The 8th term is 64.
d	**i**	**1**	Look for a pattern.	Each term is 6 less than the previous term.
		2	Write the answer.	It is an arithmetic sequence with a common difference of −6.
	ii	**1**	Continue the pattern to find the 8th term.	t_1 t_2 t_3 t_4 t_5 t_6 t_7 t_8 17 11 5 −1 −7 −13 −19 −25 −6 −6 −6
		2	Write the answer.	The 8th term is −25.

Arithmetic sequences and recursion

Recursion shows how each term is calculated from the one before it.
The first term, t_1, is usually given as well.
The rule is written as a **recurrence relation.**

The recurrence relation of an arithmetic sequence is written as

$$t_{n+1} = t_n + d, \quad t_1 = a$$

where d is the common difference and a is the first term.

WORKED EXAMPLE 2 Recurrence relation

Determine the recurrence relation for the sequence: 25, 30, 35, 40, …

Steps	Working
1 Identify the common difference and the first term.	$d = 5$ and $t_1 = 25$
2 Write the recurrence relation.	$t_{n+1} = t_n + 5,\ t_1 = 25$

WORKED EXAMPLE 3 Generate an arithmetic sequence

Generate the first 5 terms of the arithmetic sequence defined by the recurrence relation $t_{n+1} = t_n - 3$, with $t_1 = 7$.

Steps	Working
1 Identify the first term and the common difference.	$t_1 = 7$ and $d = -3$
2 Add the common difference, −3, to the first term.	$t_2 = 7 - 3 = 4$
3 Continue to get the next terms.	$t_3 = 4 - 3 = 1$ $t_4 = 1 - 3 = -2$ $t_5 = -2 - 3 = -5$
4 Answer the question.	The first 5 terms are 7, 4, 1, −2, −5.

You can show an arithmetic sequence in a table and as a graph.
An arithmetic sequence will always give a **linear** graph. Its graph is always a straight line.

WORKED EXAMPLE 4 Arithmetic sequence in table and graph form

An arithmetic sequence has a first term of 3 and a common difference of 6. Show the first 8 terms

a in a table

b as a graph.

Steps | **Working**

a 1 Make a table and put in the term numbers, n, and the first term.

n	1	2	3	4	5	6	7	8
t_n	3							

2 Add 6 to each term to get the next one.

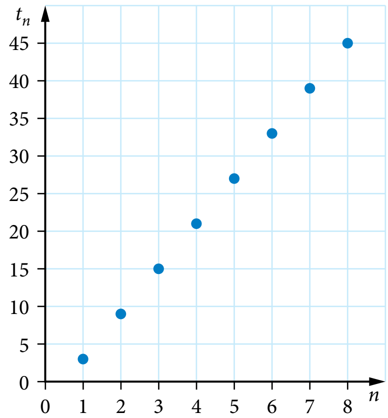
n	1	2	3	4	5	6	7	8
t_n	3	9	15	21	27	33	39	45

b 1 Draw up the axes with n horizontal and t_n vertical.

2 Plot the points from the table:

(1, 3), (2, 9), etc.

Notice that the points of the sequence make a straight line!



Linear growth and decay

An arithmetic sequence with a common difference that is:

- positive shows **linear growth**.
- negative shows **linear decay**.

WORKED EXAMPLE 5 Linear growth

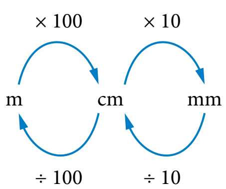
A grass tree on the east coast of Queensland grows by 9 mm each year.
When first measured, it has a height of 0.6 m. **Determine** how high it will be in another 5 years.

Steps | **Working**

1 Write the common difference in metres.
This is the change from one year to the next.

$d = 9\text{ mm} \div 10 \div 100$

$= 0.009\text{ m}$



2 Write the first term. This is the height at the beginning.

$t_1 = 0.6\text{ m}$

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3	Another 5 years means looking for the 6th term. Continue the sequence to find the 6th term.	$t_1 = 0.6\text{ m}$ $t_2 = 0.6\text{ m} + 0.009\text{ m} = 0.609\text{ m}$ $t_3 = 0.609\text{ m} + 0.009\text{ m} = 0.618\text{ m}$ $t_4 = 0.618\text{ m} + 0.009\text{ m} = 0.627\text{ m}$ $t_5 = 0.627\text{ m} + 0.009\text{ m} = 0.636\text{ m}$ $t_6 = 0.636\text{ m} + 0.009\text{ m} = 0.645\text{ m}$
4	Write the answer.	The grass tree is estimated to be 0.645 metres high in another 5 years.

TECHNOLOGY Recurrence calculations on a calculator

A simple shortcut to perform repetitive calculations is to use the ANS function. The following illustrates how this would be done for the tree height in Worked example 5.

Steps	Casio fx-82AU PLUS II	Sharp EL-531TH	TI-30XB
1 Enter t_1 and push equals.	Enter 0.6 and press =.	Enter 0.6 and press =.	Enter 0.6 and press =.
2 Set up the recursive rule and calculate t_2.	Enter 0.009 + and press ANS, then =.	Enter 0.009 + and press ALPHA, = then = again.	Enter 0.009 + and press 2nd, (–) then **enter** again.
	The result should be 0.609, which is t_2.		
3 Now, get the calculator to repeat the calculation to find t_3.	Press =.	Press =.	Press ENTER.
	The result should be 0.618, which is t_3.		
4 Repeat the calculation to find t_4.	Press =.	Press =.	Press ENTER.
	The result should be 0.627, which is t_4.		

Repeat this process as many times as you need to. Just make sure to write down each result so you know how many times you have performed the calculation.

EXERCISE 2.1 Arithmetic sequences

ANSWERS p. 440

Mastery

1 WORKED EXAMPLE 1 For each sequence

i state if it is an arithmetic sequence and give the common difference if it is.

ii calculate the 8th term if arithmetic.

a 1.2, 2.4, 3.6, 4.8, 6.0, …

b 3, 6, 12, 24, 48, …

c 2, 5, 4, 7, 6, …

d –20, –18, –16, –14, –12, …

e 40, 60, 90, 135, 202.5, …

f 9, 8.3, 7.6, 6.9, 6.2, …

2 WORKED EXAMPLE 2 Write the recurrence relation for each sequence.

a 11, 14, 17, 20, 23, …

b 98, 95, 92, 89, 86, …

c $5, 5\frac{1}{4}, 5\frac{1}{2}, 5\frac{3}{4}, 6, \ldots$

d 34, 46, 58, 70, 82, …

e 9, 8.6, 8.2, 7.8, 7.4, …

f 1.7, 1.4, 1.1, 0.8, 0.5, …

3 WORKED EXAMPLE 3 Generate the first 5 terms of each arithmetic sequence described.

a $t_{n+1} = t_n - 8$, with $t_1 = 75$.

b $t_{n+1} = t_n + 3$, with $t_1 = 11$.

c $t_{n+1} = t_n + 1.5$, with $t_1 = -2$.

d $t_{n+1} = t_n - 35$, with $t_1 = 110$.

e $t_{n+1} = t_n + 2.1$, with $t_1 = 0.3$.

f $t_{n+1} = t_n - \frac{2}{3}$, with $t_1 = 4$.

4 WORKED EXAMPLE 4 For each arithmetic sequence described, show the first 8 terms

i in a table

ii as a graph.

a $t_1 = 35$ and common difference of -3.

b $t_{n+1} = t_n + 0.2$, with $t_1 = 1.6$.

c $t_{n+1} = t_n - 0.5$, with $t_1 = 2$.

d $t_1 = 7$ and a common difference of 1.2.

e $t_{n+1} = t_n + 5$, with $t_1 = 11$.

f $t_{n+1} = t_n - 8$, with $t_1 = 70$.

5 WORKED EXAMPLE 5 A dam west of Winton reduces in level by 1.6 cm a week due to evaporation. Use by cattle reduces it by 3.2 cm a week as well. The depth of the dam after a recent rain was 2.4 m. How long will it take to reduce to half this level if cattle continue to use it and there is no rain?

6 A granite boulder in Girraween National Park has a diameter of 4.3 m. It loses 4 cm a year due to exfoliation ('onion–skin' weathering).

a Show in a table its diameter, in metres, for the first 8 years.

b How long would it take to reduce to a diameter of 3.9 m?

Exam practice

7 © QCAA 2020 1Q3 For the sequence 4, 2, 0, −2, −4, … the common difference is

A 4 **B** 2 **C** −2 **D** −4

8 © QCAA 2023 1Q9 MODIFIED Determine the 6th term of the arithmetic sequence that begins 2, 9, …

A 35 **B** 37 **C** 42 **D** 44

9 A sequence is defined by $T_{n+1} = T_n + 13$. Given that $T_5 = 65$, the value of T_2 is

A 13 **B** 26 **C** 39 **D** 78

10 (2 marks) State the first 3 terms of the sequence $T_{n+1} = T_n + 7$, $T_1 = 11$.

11 (3 marks) Show the first 6 terms of the sequence $T_{n+1} = T_n - 3.5$, $T_3 = 9$ as a graph.

12 © QCAA 2020 1Q18 MODIFIED (4 marks) Exhibition organisers believe that the number of attendees increases each day as an arithmetic sequence. The organisers know that 483 people attended the first day and 831 people attended the fourth day.

a Determine the common difference. *[2 marks]*

b Use the result from part **a** to predict the number of people who will attend the sixth day. *[2 marks]*

2.2 The general term of an arithmetic sequence

The sequence 7, 13, 19, 25, … is an arithmetic sequence with a common difference of 6.

Notice that

$$\begin{aligned} t_1 &= 7 && = 7 + 0 \times 6 \\ t_2 &= 7 + 6 && = 7 + 1 \times 6 \\ t_3 &= 7 + 6 + 6 && = 7 + 2 \times 6 \\ t_4 &= 7 + 6 + 6 + 6 && = 7 + 3 \times 6 \end{aligned}$$

and so on. In this sequence $t_1 = 7$ and $d = 6$. Notice that $t_3 = t_1 + 2d$ and $t_4 = t_1 + 3d$.

General term of a arithmetic sequence

The **general term** (**n^{th} term**) of an arithmetic sequence with first term t_1 and common difference d is given by

$$t_n = t_1 + (n - 1)d$$

Worksheet Arithmetic sequences

Video Arithmetic sequences

WORKED EXAMPLE 6 Finding a general term of an arithmetic sequence

Determine

a the 15th term of the sequence with first term 6 and common difference 4.

b the 12th term of the sequence 28, 23, 18, 13, ...

Steps	Working
a 1 Write the known variables.	$t_1 = 6$, $d = 4$, $n = 15$
2 Write the formula.	$t_n = t_1 + (n - 1)d$
3 Substitute values and calculate the answer.	$t_{15} = 6 + (15 - 1) \times 4$ $= 62$
4 Write the answer.	The 15th term is 62.
b 1 Write the known variables.	$t_1 = 28$, $d = -5$, $n = 12$
2 Write the formula.	$t_n = t_1 + (n - 1)d$
3 Substitute values and calculate the answer.	$t_{12} = 28 + (12 - 1) \times (-5)$ $= -27$
4 Write the answer.	The 12th term is −27.

You can use the formula for the general term to find the number of terms to reach or pass a particular number.

WORKED EXAMPLE 7	Using the general term formula to solve a problem
How many terms of the sequence 14, 17, 20, 23, … does it take to exceed 300?	
Steps	**Working**
1 Write the known variables.	$t_1 = 14, d = 3, t_n = 300$
2 Write the formula.	$t_n = t_1 + (n-1)d$
3 Substitute values and solve the equation.	$300 = 14 + (n-1) \times 3$ $300 - 14 = 3(n-1)$ $286 = 3(n-1)$ $\frac{286}{3} = n - 1$ $95.3\ldots = n - 1$ $95.3\ldots + 1 = n$ $n = 96.3\ldots$ Notice that the 96th term will not be over 300, so must round up to the 97th. If unsure, check by working out the 96th and 97th terms to see which is over 300.
4 Write the answer.	97 terms are required to exceed 300.

You might be given 2 different terms of an arithmetic sequence. Solve simultaneous equations to find the first term and common difference.

WORKED EXAMPLE 8	Using simultaneous equations to solve problems
The 3rd term of an arithmetic sequence is 8 and the 13th term is 5. **Determine** the first term and the common difference.	
Steps	**Working**
1 Write the known variables.	$t_3 = 8, t_{13} = 5$
2 Write the formula.	$t_n = t_1 + (n-1)d$
3 Substitute values. This will give 2 different equations.	$8 = t_1 + (3-1)d$ $8 = t_1 + 2d \quad \ldots[1]$ $5 = t_1 + (13-1)d$ $5 = t_1 + 12d \quad \ldots[2]$
4 Rearrange equation [1].	$8 - 2d = t_1 \quad \ldots[3]$
5 Substitute equation [3] into equation [2] and solve for d.	Sub [3] into [2]: $5 = 8 - 2d + 12d$ $5 = 8 + 10d$ $5 - 8 = 10d$ $-3 = 10d$ $\frac{-3}{10} = d$ $d = -0.3$
6 Substitute $d = -0.3$ into equation [3] to find t_1.	Sub $d = -0.3$ into [3]: $8 - 2 \times (-0.3) = t_1$ $t_1 = 8.6$
7 Answer the question.	The first term is 8.6 and the common difference is –0.3.

TECHNOLOGY Checking your answers

Use your scientific calculator to check your answers.
In the previous example, you can check that your answers give the right terms using the formula $t_n = t_1 + (n-1)d$.

For t_3, enter 8.6 + (3 − 1) × (−) 0.3 → 8
For t_{13}, enter 8.6 + (13 − 1) × (−) 0.3 → 5

EXERCISE 2.2 The general term of an arithmetic sequence

ANSWERS p. 441

Recap

1 Which of the following sequences is an arithmetic sequence?

A 1.7, 1.3, 1.0, 0.8, 0.7, …
B 7, 70, 700, 7000, …
C 4.1, 4.7, 5.3, 5.9, 6.5, …
D 3, −6, 9, −12, 15, …

2 Generate the first 6 terms of the arithmetic sequence $t_{n+1} = t_n - 17$, with $t_1 = 25$.

Mastery

3 WORKED EXAMPLE 6 Determine

a the 10th term of the sequence with first term 11 and common difference −4.
b the 28th term of the sequence with first term −5 and common difference 1.8.
c the 21st term of the sequence with first term $2\frac{1}{2}$ and common difference 7.
d t_{40} of the sequence with first term 1.9 and common difference 0.06.

4 Calculate

a t_{11} of the sequence 4, 10, 16, 22, …
b 30th term of the sequence −5, −12, −19, −26, …
c t_{16} of the sequence 3.4, 5.3, 7.2, 9.1, …
d 22nd term of the sequence $\frac{2}{15}, \frac{1}{3}, \frac{8}{15}, \frac{11}{15}, \ldots$

5 An arithmetic sequence is defined by the recurrence relation $t_{n+1} = t_n + 0.6$, $t_1 = 9$.

a Write the rule for the n^{th} term.
b Find the 12th term of the sequence.

6 An arithmetic sequence is defined by the recurrence relation $t_{n+1} = t_n - 1.4$, $t_1 = 24$.
Calculate the 25th term of the sequence.

7 WORKED EXAMPLE 7 Determine the first term of the arithmetic sequence 3, 10, 17, 24, … that is more than 400.

8 Calculate the first term of the arithmetic sequence

a 103, 99, 95, 91, … that is less than 30.
b 7.8, 8.2, 8.6, 9.0, … that is greater than 42.
c with a first term of −11 and a common difference of 0.75 that is positive.
d with a first term of 128 and a common difference of −7 that is less than 10.

9 Calculate the number of terms of the arithmetic sequence $t_{n+1} = t_n + 2.5$, $t_1 = 14$ that it takes for the term to be greater than 50.

10 What is the first term of the arithmetic sequence $t_{n+1} = t_n + 0.8$, $t_1 = -16$ that will exceed 14?

11 Determine the number of terms of the arithmetic sequence $t_{n+1} = t_n - 1.2$, $t_1 = 4.5$ that are needed for the term to be less than 2.

12 WORKED EXAMPLE 8 The 9th term of an arithmetic sequence is 12 and the 34th term is 57. Determine the first term and the common difference.

13 The 12th term of an arithmetic sequence is 24 and the 4th term is 40. Determine the first term and the common difference.

14 Find the first term and the common difference of an arithmetic sequence with the 10th term 34.5 and the 17th term 26.8.

Exam practice

15 A sequence is defined by $t_{n+1} = t_n - 5$, $t_1 = 11$.

The rule for the n^{th} term of the sequence is

A $t_n = 6 - 5n$ **B** $t_n = 6 + 11n$ **C** $t_n = -16 + 11n$ **D** $t_n = 16 - 5n$

16 © QCAA 2021 1Q9 17% A staircase is to be extended by installing n additional stairs. Each stair is 25 cm high. If the existing staircase reaches 1.2 m off the ground, which rule models the total height the stairs will reach in centimetres?

A $t_n = 25 + (n - 1) \times 1.45$ **B** $t_n = 1.2 + (n - 1) \times 25$

C $t_n = 145 + (n - 1) \times 25$ **D** $t_n = 25 + (n - 1) \times 120$

17 (3 marks) Consider the sequence 12, 7, 2, −3, …

By deducing a rule for the n^{th} term, or otherwise, determine which term of the sequence is −168.

18 © QCAA 2020 2Q1 (5 marks) A water tank contains 12 500 L of water. The tap is accidently left on and the tank loses 135 L per minute. The tap is turned off when the tank has 5000 L of water left.

Use a mathematical model to determine how long the tap was left on to the nearest minute.

19 © QCAA 2024 1Q16 (3 marks) The number of seats in each row of a theatre forms the terms of the arithmetic sequence $t_{n+1} = t_n + 8$, where $t_1 = 25$.

a How many seats are in the second row of the theatre?

b Copy and complete the table and then calculate the total number of seats in the first four rows of the theatre.

Row	1	2	3	4
Number of seats				

2.3 Modelling with arithmetic sequences

2.3

Simple interest is an example of a situation involving linear growth. The same amount of **interest** is added regularly to the investment.

Straight-line depreciation is an example of a situation involving linear decay. The value of the asset decreases by the same amount each period.

Simple interest

An investment at **simple interest** is an example of linear growth. You can use a recurrence relation to model simple interest

$$A_{n+1} = A_n + d$$

where A_0 = principal

d = interest for each year.

WORKED EXAMPLE 9 Simple interest

Emily has an investment account of $500 that pays 4.5% p.a. simple interest.

a Find the interest for 1 year.

b Find an expression for the value of the investment after n years.

c **Calculate** how much she will have in the account after 10 years.

d **Determine** how long it will take for Emily to have more than $1000 in the account.

Steps	Working
a 1 Calculate the interest.	Interest for 1 year = 4.5% of \$500 $= 0.045 \times \$500$ $= \$22.50$ per year
2 Write the answer.	The interest for one year is $22.50.
b 1 Interpret how the value of the investment changes.	The value of the investment starts at \$500 and then increases by \$22.50 for each year.
2 Define the variables.	Let the value of the investment be A and the number of years be n.
3 Write the expression.	$A = \$500 + \$22.50 \times n$
c 1 Substitute $n = 10$ and find the value.	$A = \$500 + \$22.50 \times 10$ $= \$725$
2 Write the answer.	Emily will have \$725 in the account after 10 years (assuming she does not deposit or withdraw anything, and there are no fees).
d 1 Substitute in $A = \$1000$ and solve for n.	$\$1000 = \$500 + \$22.50 \times n$ $\$1000 - \$500 = \$22.50 \times n$ $\$500 = \$22.50 \times n$ $\frac{\$500}{\$22.50} = n$ $n = 22.2\ldots$
2 Answer the question.	It will take Emily 23 years to have \$1000 in her account.

Straight–line depreciation

Depreciation is the loss in value of an asset over its working life. When modelling straight–line depreciation, the loss is a set amount per year. This is a linear decay.

You can use a recurrence relation to model **straight–line depreciation**

$$V_{n+1} = V_n - d$$

where V_0 = initial value of the asset

d = depreciation each year

The value **(book value)** of the asset is given by

$$V_n = \text{initial value} - \text{depreciation per year} \times n$$

The **salvage** (or **scrap**) **value** is the value of the asset at the end of its working life.

WORKED EXAMPLE 10 Straight–line depreciation

Matthew bought a bulldozer for \$350 000. It depreciates by 4.5% of the initial value every year.

a Use recursion to calculate the value of the bulldozer in 5 years.

b The salvage value is \$200 000. How long will it take for the bulldozer to reach the salvage value?

Steps	Working
a 1 Calculate the annual depreciation.	depreciation = 4.5% of \$350 000 = 0.045 × \$350 000 = \$15 750
2 Write down the given values.	initial value = \$350 000 depreciation = \$15 750 per year $n = 5$
3 Write the recursion formula.	V_n = initial value – depreciation per year × n
4 Use the recursion formula to work out the value for each year until the 5th year.	$V_5 = 350\,000 - 15\,750 \times 5$ = \$271 250
5 Answer the question.	The bulldozer will be worth \$271 250 in 5 years time.
b 1 Write down the given values.	V_n = \$200 000 initial value = \$350 000 depreciation = \$15 750 per year
2 Write the formula.	V_n = initial value – depreciation per year × n
3 Substitute in the given values and solve.	$\$200\,000 = \$350\,000 - \$15\,750 \times n$ $\$200\,000 - \$350\,000 = -\$15\,750 \times n$ $-\$150\,000 = -\$15\,750 \times n$ $\frac{-\$150\,000}{-\$15\,750} = n$ $n = 9.52\ldots$ years
4 Answer the question.	It will take the bulldozer 10 years to reach the salvage value of \$200 000.

Remember that a constant increase or decrease shows an arithmetic sequence.

WORKED EXAMPLE 11 Application of linear growth

A taxi company charges a flag fall of \$5.50 plus \$2.48 for each kilometre travelled.

a Write an expression for the cost of a fare x km long.

b Becky took a taxi from work to home. If work gave her \$50, what is the furthest distance the taxi could take her?

Steps	Working
a **1** Write the fixed amount and the cost per km.	fixed = \$5.50, \$2.48/km
2 Using recursion.	fare $= 5.50 + 2.48 \times x$ $= 5.50 + 2.48x$
3 Answer the question.	The cost of the fare x km long is $\$(5.50 + 2.48x)$.
b **1** Substitute fare = 50 and evaluate for x.	$50 = 5.50 + 2.48 \times x$ $50 - 5.50 = 2.48 \times x$ $44.50 = 2.48 \times x$ $\frac{44.50}{2.48} = x$ $x = 17.94\ldots$
2 Answer the question. In this instance you need to round down to 17 km as Becky wouldn't have enough money to cover the extra distance to travel 18 km.	Becky could travel a maximum of 17 km.

EXERCISE 2.3 Modelling with arithmetic sequences

ANSWERS p. 441

Recap

1 Calculate the 35th term of an arithmetic sequence that has a first term of 7.4 and a common difference of –0.6.

2 For the sequence 33, 36.5, 40, 43.5, …

a Which term will be the first one over 300?

b Determine the value of the term from part **a**.

Mastery

3 WORKED EXAMPLE 9 Joshua has \$2300 in a special savings account that pays 5.3% interest.

a Calculate the interest for 1 year.

b Determine an expression for the value of the investment after n years.

c How much will Joshua have in his account after 7 years?

4 Cruz borrows \$75 000 and needs to pay 4.5% p.a. simple interest for each year that he borrows the money.

a Determine the expression for the total value he will have to pay back if the loan was repaid after n years.

b Calculate the amount owed after 6 years.

c If Cruz paid off over \$100 000, what is the minimum number of years he took to pay off the loan?

5 Alice is buying a small used car for \$8590. She gets finance from the dealer at 13.5% flat–rate interest over 3 years with 5% of the car's value to be paid as loan insurance. She pays equal monthly instalments.

a How much interest does she pay?

b How much is the loan insurance?

c How much does she pay altogether?

d Determine her monthly instalment.

e Write an expression for the amount she owes after n months.

f How much does she still owe after 2 years?

6 WORKED EXAMPLE 10 Akira is in a band and bought a new guitar for \$3350. It depreciates by 12% of the original value every year.

a Write an expression for its value after n years.

b Calculate its value after 3 years.

7 A coin collection has been valued at \$42 500. It increases in value by 15% of its original value each year.

a Write an expression for its value after n years.

b Find its value after 7 years.

c Determine how long until the coin collection is valued at more than \$160 000?

8 WORKED EXAMPLE 11 The flag fall (starting fee) at night for a taxi is \$6.30 and the cost rate is \$2.17/km.

a Write an expression for the fare for a trip of n km at night.

b Calculate the cost of a 26 km trip starting at 1 am.

9 An electrician charges a callout fee of \$120 and then \$75/hour.

a Write an expression for the cost of the electrician for a job that took n hours.

b Determine the cost for a repair that took 7 hours.

10 Mila saves \$20 a week. She currently has \$240 saved. How long will it take her to reach her target of \$800?

11 A water tank can hold 8000 litres, but after a long dry spell there are only 1300 litres in it. It started raining steadily at 10 am and the tank was taking in 13 litres every minute. At what time will the tank be full? State any assumptions you make.

12 Chloe has borrowed \$6590 to buy a car. The flat–rate interest is 14.6% p.a. and loan insurance is 4.5% of the amount borrowed. She has signed a contract to pay it off over 4 years with monthly instalments. Write an expression for the amount she owes after n months and find what she owes after 3 years.

13 An office printer costing \$3600 has a useful life of 4 years or 120 000 pages. It has no scrap value.

a i Find the annual depreciation.

ii Write an expression for its value after n years of use.

b i Find the depreciation per 10 000 pages.

ii Write an expression for its value after printing n pages.

14 Ashutosh bought a van to do deliveries. The van cost \$48 000 and must be replaced every 5 years under his delivery contract. Its salvage value is estimated to be \$18 000 after 5 years.

Calculate its value after 2 years.

Exam practice

15 Which of the following rules could be used to find the balance after n years of a \$20 000 investment at a simple interest rate of 3% p.a.?

A $T_{n+1} = T_n + 600,\ T_1 = 20\,000$

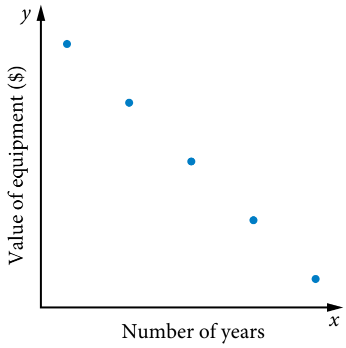
B $T_n = 19\,400 + 600 \times n$

C $T_n = 20\,600 + 600 \times n$

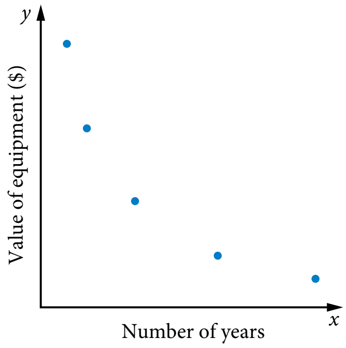
D $T_n = 20\,000 + 600 \times n$

16 The value of the equipment in a factory is depreciated using a set amount per year. Which one of the following graphs could show the value of the equipment as it depreciates?

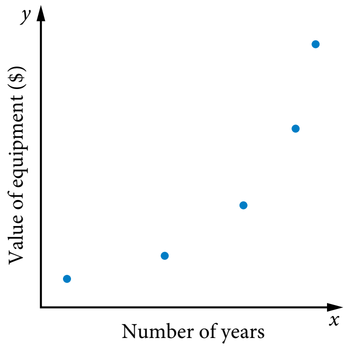
A



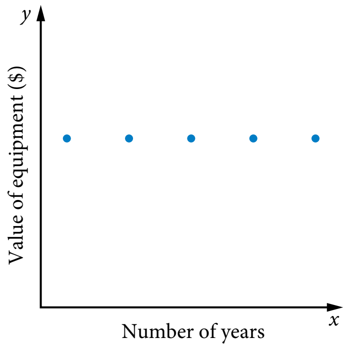
B



C



D



17 (5 marks) A water tank is full. When a tap at the bottom of the tank is opened, 84 litres run out in the first minute, 78 litres in the second minute and 72 litres in the third minute. This pattern continues until the tank is empty.

a Write a rule for the n^{th} term of a sequence in the form $T_n = A + Bn$, which will model this situation where T_n is the amount of water that runs out in the n^{th} minute. *[2 marks]*

b How many litres run out in the seventh minute? *[1 mark]*

c How long until the tank is empty? *[2 marks]*

18 © QCAA 2021 1Q20 (4 marks) A personal trainer bought a business vehicle for \$53 000 at the start of 2017. The insurance company depreciates it by \$4200 each year. Identify and use a mathematical model to determine the value of the vehicle at the start of 2025.

Worksheet
Classifying sequences

2.4 Geometric sequences

Some sequences are generated by multiplying the previous term by the same amount.
These are called **geometric sequences**.

For the sequence: 2, 6, 18, 54, 162, … each term of the sequence is 3 times larger than the previous term with a first term of 2. This is a geometric sequence as each term is found by multiplying the previous term by the same amount.

$$2,\ \underset{\times 3}{\frown}\ 6,\ \underset{\times 3}{\frown}\ 18,\ \underset{\times 3}{\frown}\ 54,\ \underset{\times 3}{\frown}\ 162,\ \ldots$$

Geometric sequences

- A sequence where every term is the same multiple of the previous one is called a geometric sequence.
- The multiple factor is called the **common ratio** and abbreviated as r.
- The common ratio is found by dividing any term by the previous term: $r = \frac{t_{n+1}}{t_n}$.
  You can test if a sequence is geometric by checking for a common ratio.

WORKED EXAMPLE 12 Geometric sequences

Determine whether each sequence is a geometric sequence and, if it is, state the common ratio.

a $75, 15, 3, \frac{3}{5}, \ldots$

b $2, -12, 72, -432, \ldots$

c $4, -8, -16, 32, \ldots$

d $200, 210, 220.5, 231.525, \ldots$

Steps	Working
a 1 Check the ratios of successive terms.	$\frac{t_2}{t_1} = \frac{15}{75} = \frac{1}{5}$ $\frac{t_3}{t_2} = \frac{3}{15} = \frac{1}{5}$ ✓ $\frac{t_4}{t_3} = \frac{\frac{3}{5}}{3} = \frac{1}{5}$ ✓
2 Write the answer.	There is a common ratio of $r = \frac{1}{5}$. So, it is a geometric sequence.
b 1 Check the ratios of successive terms.	$\frac{t_2}{t_1} = \frac{-12}{2} = -6$ $\frac{t_3}{t_2} = \frac{72}{-12} = -6$ ✓ $\frac{t_4}{t_3} = \frac{-432}{72} = -6$ ✓
2 Write the answer.	There is a common ratio of $r = -6$. So, it is a geometric sequence.

c	**1** Check the ratios of successive terms.	$\frac{t_2}{t_1} = \frac{-8}{4} = -2$ $\frac{t_3}{t_2} = \frac{-16}{-8} = 2$ ✗
	2 Write the answer.	There is no common ratio. So, it is not a geometric sequence.
d	**1** Check the ratios of successive terms.	$\frac{t_2}{t_1} = \frac{210}{200} = 1.05$ $\frac{t_3}{t_2} = \frac{220.5}{210} = 1.05$ ✓ $\frac{t_4}{t_3} = \frac{231.525}{220.5} = 1.05$ ✓
	2 Write the answer.	There is a common ratio of $r = 1.05$. So, it is a geometric sequence.

- If $r > 1$, the sequence is increasing.
- If $0 < r < 1$, it is decreasing.
- If $r < 0$, the sign of each term is the opposite of the previous term.

Geometric recursion

The recurrence relation for a geometric sequence is written as:

$$t_{n+61} = rt_n, \quad t_1 = a$$

where r is the common ratio and a is the first term.

WORKED EXAMPLE 13 Recurrence relation

Determine the recurrence relation for the sequence: 60, 12, 2.4, 0.48, ….

Steps	Working
1 Check if it is an arithmetic or geometric sequence by looking for a common difference or ratio.	$60 - 12 = 48$ $12 - 2.4 = 9.6$ No common difference. So, not an arithmetic sequence. $\frac{12}{60} = \frac{1}{5}$ $\frac{2.4}{12} = \frac{1}{5}$ $\frac{0.48}{2.4} = \frac{1}{5}$ There is a common ratio. So, it is a geometric sequence.
2 Identify the common ratio and the first term.	$r = \frac{1}{5}$ and $t_1 = 60$
3 Write the recurrence relation $t_{n+1} = r \times t_n$, $t_1 = a$. The next term is found by multiplying $\frac{1}{5}$ to the current term.	The recurrence relation is $t_{n+1} = \frac{1}{5} \times t_n$, $t_1 = 60$.

You can show a geometric sequence in a table and plotted as a graph. A geometric sequence is *not* a linear function, it is an **exponential function**. Its graph is always a curve.

WORKED EXAMPLE 14 Geometric sequence graph

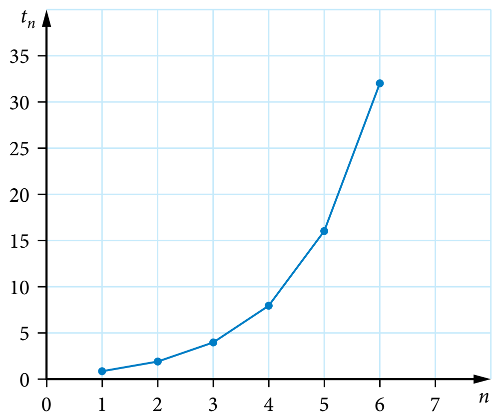
a **Generate** the first 6 terms of the geometric sequence defined by the recurrence relation $t_{n+1} = 2t_n$, with $t_1 = 1$.

b Show the first 6 terms in a table.

c Plot a graph to show the first 6 terms of the sequence.

Steps	Working
a **1** Identify the first term and the common ratio.	$t_1 = 1$ and $r = 2$
2 Start with first term and multiply by 2 to get the next term.	$t_2 = 1 \times 2 = 2$
3 Continue to generate the next terms.	$t_3 = 2 \times 2 = 4$ $t_4 = 4 \times 2 = 8$ $t_5 = 8 \times 2 = 16$ $t_6 = 16 \times 2 = 32$
4 Answer the question.	The first 6 terms are 1, 2, 4, 8, 16 and 32.
b Make a table with the term numbers, n, in the first row and the values of each term, t_n in the second row.	see table below
c **1** Draw up the axes with n on the horizontal axis and t_n on the vertical axis. **2** Plot the points from the table: $(1, 1), (2, 2), (3, 4), (4, 8), (5, 16), (6, 32)$	see graph below

n	1	2	3	4	5	6
t_n	1	2	4	8	16	32



WORKED EXAMPLE 15 Geometric (exponential) growth

Arohi decides to save $20 in the first week of her new job and then save 20% extra each week.

a Write a recursive rule to show the amount Arohi will save each week.

b **Calculate** what she will save in the 5th week.

Steps	Working
a **1** Write the common ratio. Amount is increasing by 20%.	$r = 100\% + 20\%$ $= 120\%$ $= 1.2$
2 Write the first term. This is the amount she saves in the first week.	$t_1 = \$20$
3 Geometric sequence is $t_{n+1} = r \times t_n,\ t_1 = a$.	$t_{n+1} = 1.2 \times t_n,\ t_1 = \$20$
b **1** 'In the 5th week' means we need to find the 5th term.	$t_1 = \$20$ $t_2 = 1.2 \times 20 = \$24$ $t_3 = 1.2 \times 24 = \$28.80$ $t_4 = 1.2 \times 28.80 = \$34.56$ $t_5 = 1.2 \times 34.56 = \$41.47$
2 Write the answer.	Arohi will save $41.47 in the 5th week.

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TECHNOLOGY Recurrence calculations on a calculator

A simple shortcut to perform repetitive calculations is to use the ANS function. The following illustrates how this would be done for the investment in Worked example 15.

Steps	Casio fx-82AU PLUS II	Sharp EL-531TH	TI-30XB
1 Enter t_1 and push equals.	Enter 20 and press = .	Enter 20 and press = .	Enter 20 and press ENTER .
2 Set up the recursive rule and calculate t_2.	Enter 1.2 × and press ANS, then =.	Enter 1.2 × and press ALPHA, = then = again.	Enter 1.2 × and press 2nd, (−) then ENTER again.
	The result should be 24, which is t_2.		
3 Now, get the calculator to repeat the calculation to find t_3.	Press =.	Press =.	Press ENTER.
	The result should be 28.8, which is t_3.		
4 Repeat the calculation to find t_4.	Press =.	Press =.	Press ENTER.
	The result should be 34.56, which is t_4.		

Repeat this process as many times as you need to. Just make sure to write down each result so you know how many times you have performed the calculation.

EXERCISE 2.4 Geometric sequences

ANSWERS p. 442

Recap

1 The flag fall for a taxi is \$5.80 and the cost rate is \$2.36/km.

a Write an expression for the fare for a trip distance of n km.

b Calculate the cost of a taxi trip that is 18 km long.

2 Maia bought office equipment for \$65 000. It depreciates at \$4700 per year.

a Write an expression for its value after n years.

b Calculate its value after 6 years.

Mastery

3 WORKED EXAMPLE 12 Determine whether each sequence is a geometric sequence and, if it is, state the common ratio.

a 2, 4, 8, 16, …

b 1, 4, 9, 25, …

c 8, −4, 2, −1, …

d $6, 2, \frac{1}{2}, \frac{1}{6}, \ldots$

e 120, 252, 529.2, 1111.32, …

f $\frac{1}{2}, -\frac{1}{6}, \frac{1}{18}, -\frac{1}{54} \ldots$

4 State the common ratio and calculate the next 3 terms of the geometric sequence

$$27, 18, 12, \ldots$$

5 WORKED EXAMPLE 13 Determine the recurrence relation for the sequence

$$5, 15, 45, 135, \ldots$$

6 Determine whether each sequence is a geometric sequence and, if it is, state the recurrence relation.

a $-5, 25, -125, 625, \ldots$

b $2\frac{1}{2}, 4\frac{1}{4}, 8\frac{1}{8}, 16\frac{1}{16}, \ldots$

c $80, 40, 20, -10, \ldots$

d $6, -9, 13.5, -20.25, \ldots$

e

n	1	2	3	4
t_n	16	4	1	$\frac{1}{4}$

f

n	1	2	3	4
t_n	1.8	5.4	16.2	48.6

7 A geometric sequence is defined by $t_{n+1} = -\frac{1}{2}t_n$, $t_1 = 8$. Find the first 5 terms.

8 WORKED EXAMPLE 14 A geometric sequence is defined by the recurrence relation $t_{n+1} = 2t_n$, with $t_1 = 7$.

a Complete a table showing the first 5 terms of the sequence.

b Plot a graph to show the first 5 terms of the sequence.

9 A geometric sequence has a common ratio of 0.8 and a first term of 200.

a Define the recurrence relation.

b Complete a table showing the first 5 terms of the sequence.

c Plot a graph to show the first 5 terms of the sequence.

10 WORKED EXAMPLE 15 A painting was bought for \$75 000. It is increasing in value by 15% per year.

a What is the common ratio?

b Write a recursive rule to show the amount the painting will be worth at the end of each year.

c What will the painting be worth in 3 years?

11 There is concern that the population of little blue penguins on an island is declining. There were initially 13 500 little blue penguins on the island. The numbers have been recorded at the beginning of each year for 3 years.

Years	1	2	3
Population	13 500	10 800	8640

a Show that the numbers form a geometric sequence.

b What is the common ratio?

c Define the recurrence relation to show the predicted population at the end of each year.

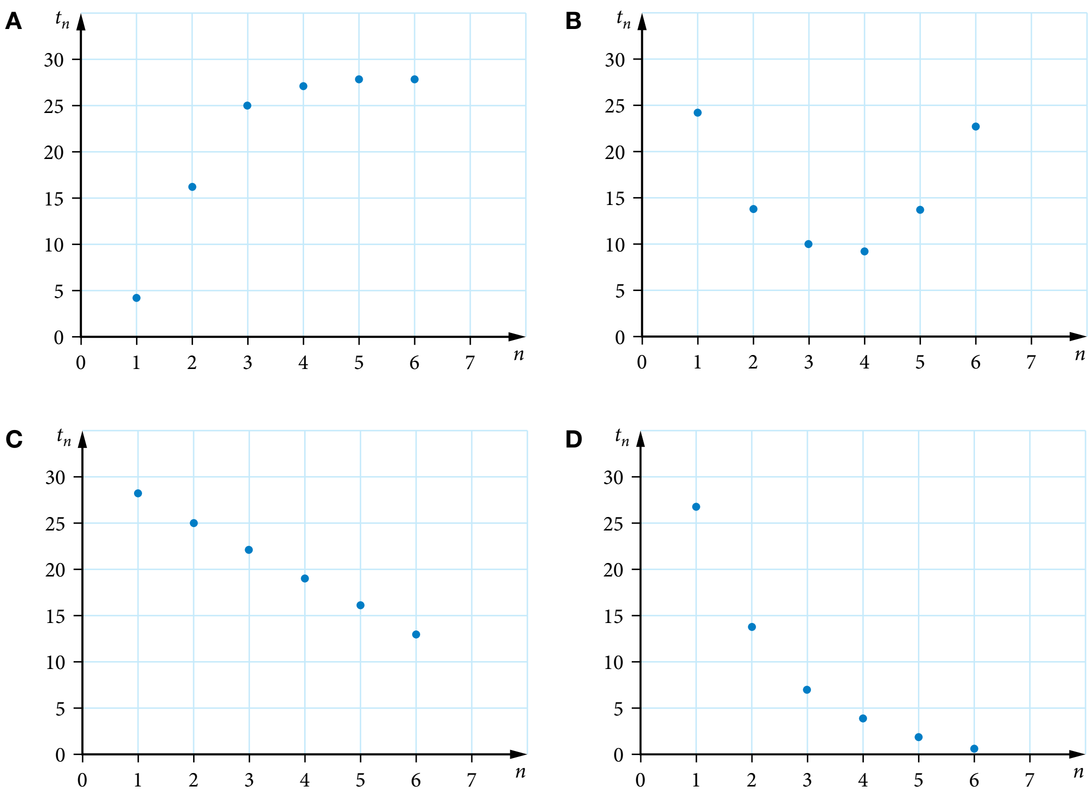
d What will the prediction be for the population of little blue penguins at the end of 6 years?

Exam practice

12 A sequence of numbers is generated by the recurrence relation $t_{n+1} = 4t_n$, with $t_1 = 2$.
What is the value of t_3?

A 12 **B** 24 **C** 32 **D** 64

13 © QCAA 2020 1Q4 Which of the following graphs could be modelled using a geometric sequence?



14 © QCAA 2023 1Q15 41% The resale value of a boat shows geometric decay.

Years after purchase	0	1	2
Resale value ($)	50 000	40 000	32 000

Determine the resale value four years after purchase.

A $16 000 **B** $20 480 **C** $22 000 **D** $25 600

15 (2 marks) State the first 3 terms of the sequence $t_{n+1} = 1.5t_n$, $t_1 = 7.5$.

16 © QCAA 2024 1Q9 54% Determine the 4th term for the geometric sequence that begins 1000, –900, …

A 729 **B** 700 **C** –700 **D** –729

2.4

Worksheets Geometric sequences

Geometric progressions

2.5 The general term of a geometric sequence

The sequence 4, 10, 25, 62.5, … is a geometric sequence with a common ratio of 2.5.

Notice that

$$\begin{aligned} t_1 &= 4 &&= 4 \times 2.5^0 \\ t_2 &= 4 \times 2.5 &&= 4 \times 2.5^1 \\ t_3 &= 4 \times 2.5 \times 2.5 &&= 4 \times 2.5^2 \\ t_4 &= 4 \times 2.5 \times 2.5 \times 2.5 &&= 4 \times 2.5^3 \end{aligned}$$

Remember that $a^0 = 1$.

and so on.

Each term is 4 × 2.5 to the power of (one less than the term number).

General term of a geometric sequence

The general term (n^{th} term) of a geometric sequence with first term t_1 and common ratio r is given by

$$t_n = t_1\, r^{(n-1)}$$

Video Geometric sequences

WORKED EXAMPLE 16 Finding a general term of a geometric sequence

Calculate the indicated term in each geometric sequence.

a 5th term of the sequence with first term 10 and common ratio 1.2.

b 6th term of the sequence with $t_1 = 3.25$ and $t_4 = 0.208$.

Steps	Working
a **1** Write the known variables.	$t_1 = 10, r = 1.2, n = 5$
2 Write the formula.	$t_n = t_1\, r^{(n-1)}$
3 Substitute values and calculate the answer.	$t_5 = 10 \times 1.2^{(5-1)}$ $t_5 = 10 \times 1.2^4$ $t_5 = 20.736$
4 Write the answer.	The 5th term is 20.736.
b **1** Write the known variables.	$t_1 = 3.25, t_4 = 0.208, n = 4$
2 Write the formula.	$t_n = t_1\, r^{(n-1)}$
3 Substitute in values for 4th term.	$0.208 = 3.25 \times r^{(4-1)}$
4 Solve for r.	$0.208 = 3.25 \times r^3$ $\frac{0.208}{3.25} = r^3$ $r^3 = 0.064$ $r^3 = \sqrt[3]{0.064}$ $r = 0.4$
5 Write the known values.	$t_1 = 3.25, r = 0.4, n = 6$
6 Solve to find t_6.	$t_6 = 3.25 \times 0.4^{(6-1)}$ $t_6 = 3.25 \times 0.4^5$ $t_6 = 0.033\,28$
7 Write the answer.	The 6th term is 0.033 28.

It may be easier to write answers as fractions instead of recurring decimals.

You can find any term of a geometric sequence from knowing any 2 terms.

WORKED EXAMPLE 17 **Finding a geometric sequence given 2 terms**

The 2nd and 5th terms of a geometric sequence are 48 and 0.75 respectively.

Determine the first term and common ratio.

Steps	Working
1 Write the known variables.	$t_2 = 48,\ n = 2$ and $t_5 = 0.75,\ n = 5$
2 Write the formula.	$t_n = t_1 r^{(n-1)}$
3 Substitute values.	$48 = t_1 \times r^{(2-1)}$ and $0.75 = t_1 \times r^{(5-1)}$ $48 = t_1 \times r$ ...[1] $0.75 = t_1 \times r^4$ ...[2]
4 Solve the simultaneous equations using substitution.	Rearrange [1]: $48 = t_1 \times r$ $\frac{48}{r} = t_1$ ...[3] Substitute [3] into [2]: $0.75 = \frac{48}{r} \times r^4$ $0.75 = 48 \times r^3$ $\frac{0.75}{48} = r^3$ $0.015\,625 = r^3$ $r = \sqrt[3]{0.015\,625}$ $r = 0.25$ Substitute $r = 0.25$ into [3]: $t_1 = \frac{48}{r}$ $= \frac{48}{0.25}$ $= 192$
5 Check values.	Check: Sub $t_1 = 192$ and $r = 0.25$ [1] LHS = 48 RHS = 192×0.25 = 48 So, LHS = RHS. [2] LHS = 0.75 RHS = 192×0.25^4 = 0.75 So, LHS = RHS.
6 Write the answer.	The first term is 192 and the common ratio is 0.25.

Exam hack

If you make an error in an exam, check quickly (no more than a minute). If you can't see the error straight away, come back later. You will often see it straight away when you come back.

WORKED EXAMPLE 18 Finding the first term to exceed a given value

Find the first term to exceed 800 in the geometric sequence with a first term of 500 and a common ratio of 1.053.

Steps	Working
1 Write the first term and common ratio.	$t_1 = 500,\ r = 1.053$
2 Write the formula.	$t_n = t_1\, r^{(n-1)}$
3 Substitute values.	$t_n = 500 \times 1.053^{n-1}$
4 Use your calculator to try some values of n.	Try:
Try values of n until the value first exceeds 800.	$n = 5,\quad t_5 = 500 \times 1.053^4 = 614.728\ldots$ too small
	$n = 10,\quad t_{10} = 500 \times 1.053^9 = 795.839\ldots$ just below
	$n = 11,\quad t_{11} = 500 \times 1.053^{10} = 838.018\ldots$ over 800
5 Write the answer.	The first term of the sequence to exceed 800 is the 11th term, which is $\approx 838.018\ldots$

TECHNOLOGY Geometric sequence

A spreadsheet is very useful for sequences. The calculations for questions like the above example are particularly tedious.

Open a new spreadsheet and complete these steps.

1. Type 'First term =' into cell A1.
2. Type 'Common ratio =' into cell A2.
3. Type 500 into cell B1.
4. Type 1.053 into cell B2.
5. Widen column A so both '=' signs are visible.
6. Type 'Term number' into cell A3.
7. Type 'Term value' into cell B3.
8. Type 1 into cell A4.
9. Type '= B1' into cell B4.
10. Type '= A4 + 1' into cell A5.
11. Type '= B4*$B$2' into cell B5. The dollar signs make the cell reference absolute. It won't change when you copy and paste it.
12. Now, highlight cells A5 and B5 and copy them.
13. Highlight cells A6 and B6 down to, say, A20 and B20 and paste.
14. Save your spreadsheet as, say, 'GP Terms' to use it in the future.

Try changing the values in cells B1 and B2.

	A	B
1	First term =	500
2	Common ratio =	1.053
3	Term number	Term value
4	1	500
5	2	526.5
6	3	554.4045
7	4	583.787939
8	5	614.728699
9	6	647.30932
10	7	681.616714
11	8	717.7424
12	9	755.782747
13	10	795.839233
14	11	838.018712
15	12	882.433704
16	13	929.20269

EXERCISE 2.5 The general term of a geometric sequence

ANSWERS p. 442

Recap

1 A geometric sequence is defined by $t_{n+1} = 2.5t_n$, $t_1 = 4$. Find the first 5 terms.

2 The population of an island is growing. There were initially 7500 people on the island. The numbers have been recorded at the end of each year for the last 2 years.

Years	0	1	2
Population	7500	8100	8748

a Show that the numbers form a geometric sequence.

b What is the common ratio?

c Define the recurrence relation to show the predicted population at the end of each year.

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Mastery

3 WORKED EXAMPLE 16 Calculate the indicated term in each geometric sequence.

a 6th term of the sequence with first term 6 and common ratio 3.

b t_8 for the sequence 5, –10, 20, –40, ….

c 6th term of the sequence with $t_1 = -20$ and common ratio 0.3.

d t_5 if $t_1 = -6$ and $r = \frac{1}{3}$.

e 6th term of the sequence described by the recurrence relation $t_{n+1} = 1.05t_n$, $t_1 = 500$.

f 10th term of the sequence with first term –7 and common ratio $\frac{1}{2}$.

4 Determine the indicated term in each geometric sequence.

a 10th term of the sequence with first term 12 and 5th term 192.

b t_6 if $t_1 = 1.08$ and $t_4 = -0.04$.

c 4th term of the sequence with $t_1 = -36$ and 6th term 3515.625.

d 5th term if $t_1 = 162$ and 3rd term is 8.

e t_5 if $t_1 = 10$ and 4th term is 5.12.

f 4th term of the sequence with first term $\frac{8}{625}$ and 6th term $1\frac{1}{4}$.

5 Find the indicated term in each geometric sequence.

a 7th term of the sequence with common ratio of 2 and 3rd term = 48.

b t_5 if $r = 1.5$ and $t_4 = 81$.

c 5th term of the sequence with $r = -\frac{1}{4}$ and 3rd term of $1\frac{1}{2}$.

d 7th term if $r = 5$ and 4th term is 10.

e t_2 if t_5 is $\frac{4}{9}$ and common ratio is $\frac{1}{3}$.

f 6th term of the sequence with 3rd term 540 and common ratio 1.2.

6 WORKED EXAMPLE 17 A geometric sequence has $t_3 = 20$ and $t_6 = 160$.

a Write the equation representing the n^{th} term using the values from $t_3 = 20$.

b Write the equation representing the n^{th} term using the values from $t_6 = 160$.

c Solve the two equations simultaneously to find the first term and common ratio.

7 The 2nd and 7th terms of a geometric sequence are 0.4 and 1250 respectively.
Find the first term and common ratio.

8 The 3rd and 5th terms of a geometric sequence are $2\frac{1}{4}$ and $\frac{9}{16}$ respectively.
Find the first 3 terms of the sequence.

9 WORKED EXAMPLE 18 Find the first term to exceed 300 in the geometric sequence with a first term of 200 and a common ratio of 1.04.

10 A geometric sequence has a first term of 6 and a common ratio of 3. Find the first term to exceed 1200.

11 Find the first term to be below 1 in the geometric sequence 80, 40, 20, ….

12 Moss on some wet paving increases in area by 20% each week during winter. At the beginning of winter, it covers an area of 15 cm^2.

a For the geometric sequence formed by the area of moss each week, what is the

i first term?

ii common ratio?

iii expression for the n^{th} term?

b How many weeks does it take to reach a size of 35 cm^2?

13 The population of mice in a wheat field increases geometrically as the crop ripens. When first checked, there are estimated to be 30 mice per hectare, but 2 weeks later there are thought to be 39 mice/ha. A significant drop in yield occurs if the population exceeds 100 mice/ha. How much longer will it take for this to happen?

Exam practice

14 A geometric sequence is defined by the recurrence relation $t_{n+1} = \frac{1}{4}t_n$, $t_1 = 120$.
The formula for the n^{th} term is

A $t_n = 120 \times \left(\frac{1}{4}\right)^n$ **B** $t_n = 120 \times \left(\frac{1}{4}\right)^{n-1}$ **C** $t_n = \frac{1}{4} \times 120^n$ **D** $t_n = \frac{1}{4} \times 120^{n-1}$

15 The current population of sloths on an island is 7210. It is predicted to decline steadily at an annual rate of 7%. The population after 10 years is approximately

A 3752 **B** 6705 **C** 7715 **D** 13 255

16 (4 marks) Weilong starts on a salary of $72 000 in January 2025 and his salary increases by 3.1% each year.

a Write a rule for his salary, t_n, after n years. *[2 marks]*

b Use the rule to find when his salary will be above $100 000. *[2 marks]*

17 © QCAA 2024 1Q8 66% After n bounces, the rebound height (cm) of a ball, t_n, is modelled by the rule

$$t_n = 240 \times (0.5)^{(n-1)}.$$

Calculate the difference in rebound height (cm) between the first bounce and the third bounce.

A 90 **B** 120 **C** 180 **D** 210

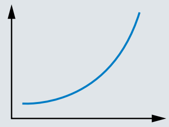
2.6 Modelling with geometric sequences

The growth of a bacteria population that doubles in size each hour is **geometric growth**.
Calculating the value of an item that reduces by a percentage of its value each year is **geometric decay**.

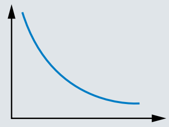
Exponential growth and decay

A geometric sequence with a common ratio that is:

- greater than 1 shows **exponential growth**.



- between 0 and 1 shows **exponential decay**.



You can use a recurrence relation to model geometric growth and decay

$$t_{n+1} = rt_n$$

where t_0 = initial value of the asset

r = common ratio

The value of a particular term in the sequence can be found using the formula for the n^{th} term of a geometric sequence, $t_n = t_1 r^{(n-1)}$.

Geometric growth and decay are also called **exponential growth and decay**.

Population growth and decay

A population growing or reducing at a set rate forms a geometric sequence.

When the rate is given as a percentage, the common ratio is found by adding or subtracting the percentage from 100%.

For example, if it increases by 15%,

$r = 100\% + 15\%$

$= 115\%$

$= 1.15$

If it decreases by 15%,

$r = 100\% - 15\%$

$= 85\%$

$= 0.85$

2.6

Worksheets
Modelling with sequences

Modelling arithmetic and geometric sequences

WORKED EXAMPLE 19 Geometric growth

A research scientist is looking at the growth rate of a population of bacteria.
The initial population is 35 000 and he discovers that the population is doubling in size each hour.

a For the geometric sequence formed by the population of bacteria after n hours, what is the

i first term?

ii common ratio?

b Copy and complete the table showing the population of bacteria, t_n, at the beginning of each hour, n, for the first 4 hours.

Number of hours, n	0	1	2	3	4
Population, t_n	35 000				

c Write a rule for the number of bacteria after n hours.

d Use the rule to find the number of bacteria after 6 hours.

Steps	Working
a **i** First term is the initial population.	first term, $t_1 = 35\,000$
ii The population doubles.	common ratio, $r = 2$
b **1** Write the recursive rule. Each population is double the last.	$t_{n+1} = 2 \times t_n$
2 Work out each term using the rule.	$t_1 = 35\,000$ (for 0 hours) $t_2 = 2 \times 35\,000 = 70\,000$ $t_3 = 2 \times 70\,000 = 140\,000$ $t_4 = 2 \times 140\,000 = 280\,000$ $t_5 = 2 \times 280\,000 = 560\,000$
3 Complete table.	(see table below)
c Write the formula for the n^{th} term.	$t_n = t_1 r^{(n-1)}$
d **1** The population at the end of 6 hours is the 7th term, so find t_7.	$t_7 = 35\,000 \times 2^{(7-1)}$ $= 35\,000 \times 2^6$ $= 2\,240\,000$
2 Answer the question.	The number of bacteria after 6 hours is 2 240 000.

Number of hours	0	1	2	3	4
Population	35 000	70 000	140 000	280 000	560 000

Diminishing-value depreciation

Diminishing-value depreciation reduces the value of an asset by a set percentage of the previous year's value. This is a geometric decay.

You can use a recurrence relation to model diminishing–value depreciation

$$t_{n+1} = rt_n$$

where t_1 = initial value of the asset

and $r = 100\% -$ depreciation percentage.

The **salvage (or scrap) value** is the value of the asset at the end of its working life.

WORKED EXAMPLE 20 Diminishing–value method of depreciation

Office furniture cost \$12 000 and it depreciates in value by 15% each year.

a Write a recursive rule that represents the value of the furniture at the beginning of each year.

b **Determine** how much the furniture will be worth at the end of the 5th year.

c The furniture is written off when its value falls below \$2500. When will the furniture be written off?

Steps	Working
a 1 Recognise type of sequence.	Decreasing by 15% each year, so the values form a geometric sequence.
2 Write the first term, t_1 and common ratio, r.	$t_1 = 12\,000$ $\quad r = 100\% - 15\%$ $= 85\%$ $= 0.85$
3 Write the recursive formula.	$t_{n+1} = t_n \times 0.85,\ t_1 = 12\,000$
b 1 Write the formula for the n^{th} term, where n is the value of the office furniture at the beginning of each year.	$t_n = t_1 r^{(n-1)}$
2 The value at the end of the 5th year is the same as the value at the beginning of the 6th year so find t_6.	$t_6 = 12\,000 \times 0.85^{(6-1)}$ $= 12\,000 \times 0.85^5$ $= \$5324.463\ldots$
3 Answer the question.	The furniture will be worth \$5324.46 at the end of the 5th year.
c 1 Write the formula for the n^{th} term, where n is the value of the office furniture at the beginning of each year.	$t_n = t_1 r^{(n-1)}$
2 Looking for n when $t_n = 2500$.	$2500 = 12\,000 \times 0.85^{(n-1)}$
3 Can't solve, so guess and check some values of n. From part **b**, we know n must be more than 6. REMEMBER: The goal is to find the first t_n that is below 2500.	Try $n = 8$, $t_8 = 12\,000 \times 0.85^{(8-1)} = \$3836.925\ldots$ above \$2500 $n = 10$, $t_{10} = 12\,000 \times 0.85^{(10-1)} = \$2779.403\ldots$ just above $n = 11$, $t_{11} = 12\,000 \times 0.85^{(11-1)} = \$2362.492\ldots$ just below
4 Answer the question.	The furniture will be written off at the beginning of the 11th year (or the end of the 10th year).

WORKED EXAMPLE 21 Modelling with geometric sequences

A tennis ball dropped onto concrete rebounds to 60% of its previous height. It starts from a height of 2 m.

a What height does it reach after 3 bounces?

b What height does it reach after 10 bounces?

c How many bounces will it take to reduce to a height of 20 cm?

Steps	**Working**
a **1** State that the heights form a geometric sequence and how you know this.	rebound height = 60% of previous height = 0.6 × previous height So, the heights form a geometric sequence.
2 Write the first term, t_1 and common ratio, r.	$t_1 = 2$ m $r = 0.6$ $= 200$ cm
3 Write the formula for the n^{th} term.	$t_n = t_1 r^{(n-1)}$
4 Looking for rebound height *after* 3 bounces, so looking for t_4. Initial height = t_1 Height after 1st bounce = t_2 Height after 2nd bounce = t_3 Height after 3rd bounce = t_4	$t_4 = 200 \times 0.6^{(4-1)}$ $= 200 \times 0.6^3$ $= 43.2$ cm It is important to read the question carefully and know what each term represents. A common mistake here would have been to see 3 bounces and assume it was t_3.
5 Answer the question.	After 3 bounces, the height reaches 43.2 cm.
b **1** Write the formula for the n^{th} term.	$t_n = t_1 r^{(n-1)}$
2 After 10th bounce → find t_{11}.	$t_{11} = 200 \times 0.6^{(11-1)}$ $= 200 \times 0.6^{10}$ $= 1.209\ldots$ cm
3 Answer the question.	After 10 bounces, the height reaches 1.2 cm.
c **1** Write the formula for the n^{th} term.	$t_n = t_1 r^{(n-1)}$
2 Looking for n when $t_n = 20$ cm.	$20 = 200 \times 0.6^{(n-1)}$
3 Can't solve, so guess and check some values of n. From parts **a** and **b**, we know n must be between 3 and 10. REMEMBER: The goal is to find the first t_n that is below 20.	Try $n = 5$, $t_5 = 200 \times 0.6^{(5-1)} = 25.92$ cm just above 20 $n = 6$, $t_6 = 200 \times 0.6^{(6-1)} = 15.52$ cm just below 20 t_6 is the height of the ball after the 5th bounce.
4 Answer the question.	It takes 5 bounces before the rebound height reduces to (less than) 20 cm.

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EXERCISE 2.6 Modelling with geometric sequences

ANSWERS p. 443

Recap

1 Determine the 4th term of the sequence described by the recurrence relation $t_{n+1} = 0.75t_n$, $t_1 = 6$.

2 A geometric sequence has a first term of 10 and a common ratio of 4. Find the first term to exceed 1500.

Mastery

3 WORKED EXAMPLE 19 It is estimated that there are 14 000 quolls left in the wild. A study showed that the population is decreasing by 8% each year.

- a For the geometric sequence formed by the population of quolls after n years, what is the
  - i first term?
  - ii common ratio?
- b Copy and complete the table showing the population of quolls at the beginning of each year for the first 4 years.

Number of years	1	2	3	4
Population	14 000			

- c Write a rule for the number of quolls after n years.
- d Use the rule to find the number of quolls after 8 years.
- e If this rate continues, how long until there are less than 1000 quolls left in the wild?

4 WORKED EXAMPLE 20 Silvia paid $8500 for an office computer system and used depreciation of 30% each year to calculate its book value.

- a Determine the first term and the common ratio of the geometric sequence formed by the book value of the office computer system each year.
- b Write a rule for the n^{th} term of the geometric sequence.
- c What is the book value after 5 years?

5 WORKED EXAMPLE 21 Mahima is beginning a fitness program. She starts at 2 km and is planning to increase the distance she walks by 15% each week.

- a For the geometric sequence formed by the distance Mahima walks in n weeks, what is the
  - i first term?
  - ii common ratio?
- b Write a rule for the distance she walks after n weeks.
- c Use the rule to find the distance she walks after 6 weeks.
- d Is it reasonable to use this model to predict how far she is walking in the 20th week? Justify your decision.

6 The value of an art piece appreciates at an average 8% p.a. The piece was purchased for $45 000.

- a Determine the first term and the common ratio of the geometric sequence formed by the appreciation value of the art piece each year.
- b Write a rule for the n^{th} term of the geometric sequence.
- c If it continues to appreciate at the same rate, predict the value of the art piece 10 years after it was purchased?

7 In efforts to save the quoll from extinction, a population was relocated to a national park. The number of quolls living in the park is modelled by the recurrence relation: $t_{n+1} = 1.16t_n$, where $t_1 = 580$.

a Graph the number of quolls at the beginning of each year for 5 years, by copying and completing the table of values.

Number of years	1	2	3	4	5
Population					

b Comment on the size of the population over time.

c If the same rate continues, predict the population of quolls in the park after 15 years.

8 Shubham bought a new truck for \$124 000. The accountant gave him 2 options to calculate the depreciation.

Option 1 – A fixed rate deduction of \$12 300 per year.

Option 2 – A reducing balance depreciation of 14% of the value each year.

a For option 1, write a rule for the n^{th} term of the sequence formed by the value of the truck each year.

b For option 2, write a rule for the n^{th} term of the sequence formed by the value of the truck each year.

c Which option will give him the highest depreciation value if he knows he is keeping the truck for 5 years? Justify your answer.

9 At the start of one year, a loaf of bread is priced at \$4.50. The inflation rate is constant at 2.3% p.a.

a Does the price of the bread each year form an arithmetic or geometric sequence? Justify your answer.

b Write a rule for the n^{th} term of the sequence.

c At this inflation rate, predict the price of the loaf of bread in 6 years time.

10 A rubber ball is dropped from a tall building. On its first bounce it reaches 11 metres. The height of each bounce is half the height of the previous bounce.

a Write a recurrence relation that defines the geometric sequence formed by the bounce heights.

b On which bounce does the height of the bounce first fall below 1 metre?

11 Fiafia starts on a salary of \$46 000 and her salary increases by a set percentage each year.

a Determine the percentage increase each year if at the end of 5 years she is earning \$63 750.

b In which year will she first earn over \$100 000?

12 A microphone is placed too close to the speaker, so it begins to emit a squeal. The feedback loop increases the volume by 3% each millisecond.

a For the geometric sequence formed by the volume after n milliseconds, what is the

i first term?

ii common ratio?

iii expression for the n^{th} term?

b Hence, determine the percentage increase in volume after 20 milliseconds.

13 Lani put a weight on the end of a piece of elastic. When she hung it from a door frame, the weight was at a height of 0.85 m. She pulled it down to a height of 0.5 m. She let it go and it went up and then fell down to a height of 0.6 m. If the heights form a geometric sequence, how many cycles would it take to settle to within 1 cm of the resting position? Justify your answer.

14 © QCAA 2023 1Q6 MODIFIED 36% In January 2024, 80 fish were released into a new dam that has the capacity to support 15 000 fish. If the fish population triples each year, which sequence models the number of fish in the dam in January 2030?

A $t_n = t_1 r^{(n-1)}$, where $t_1 = 80$, $r = 3$, $n = 6$

B $t_n = t_1 r^{(n-1)}$, where $t_1 = 80$, $r = 3$, $n = 7$

C $t_n = t_1 + (n-1)d$, where $t_1 = 80$, $d = 3$, $n = 6$

D $t_n = t_1 + (n-1)d$, where $t_1 = 80$, $d = 3$, $n = 7$

15 A farmer bought a tractor for \$45 000 and used reducing balance depreciation of 18% p.a. The tractor will reach scrap value in 10 years. The scrap value will be

A less than \$5000

B between \$5000 and \$6000

C between \$6000 and \$7000

D more than \$7000

16 (6 marks) A new nature reserve has been built and is being stocked with Tasmanian devils. At the beginning of 2025 the reserve holds 170 Tasmanian devils. Monitoring has shown that the number is increasing at a consistent rate of 4% per month.

a Write a recursive rule to give the number of Tasmanian devils on the island at the beginning of each month from the beginning of 2025. *[2 marks]*

b Determine a rule for the n^{th} term of this sequence. *[2 marks]*

c Transfers to other reserves will be allowed at the beginning of the next month after the Tasmanian devil population first reaches 550. Determine how many months after the beginning of 2025 transferring can begin. *[2 marks]*

17 (6 marks) CF A pump removes one–quarter of the contents of a 20 000–litre water tank every 15 minutes. Determine how long the pump needs to run to reduce the contents of a full tank to 500 litres.

18 © QCAA 2021 2Q7 (6 marks) CU The table shows the total number of times a new song is played on a music service in the days following its first release.

Number of days since first release	5	10	15	20
Total number of times played ('000s)	8	12	18	27

The songwriter is paid 0.175 cents every time their song is played and will be paid after 60 days. They predict that by that time, they will be owed at least \$1000.

Given that the number of times the song is played is increasing exponentially, evaluate the reasonableness of this prediction.

19 © QCAA 2024 2Q6 (6 marks) CU The daily cost (\$) for a person for meals and accommodation is predicted to change according to the cost models shown.

Category	2021 daily cost (\$)	Cost model, where n = number of years after 2020
Meals	c	$m_n = m_1 + 3(n-1)$
Accommodation	$2c$	$a_n = a_1 \times 1.1^{(n-1)}$

In 2021, the daily cost for a person's meals was \$60.

In 2025, the total cost for a person for seven days for meals and accommodation is estimated to be between \$1500 and \$2000. Evaluate the reasonableness of the estimate.

Video
Exam question analysis: Arithmetic and geometric sequences

EXAM QUESTION ANALYSIS

©QCAA 2022 2Q6 (7 marks)

The first three lines in a pattern have the equations given. Their slopes form the terms of one sequence and their y-intercepts form the terms of another sequence. Each sequence is either arithmetic or geometric.

Line 1: $y = -0.8x + 1.2$

Line 2: $y = 0.4x + 2.7$

Line 3: $y = -0.2x + 4.2$

Determine the coordinates of the point where Line 5 in the pattern intersects Line 1.

Reading the question

- Key concepts you need to be clear on is arithmetic and geometric sequences.
- You also need to know slope and y–intercept from $y = mx + c$.
- It asks for coordinates of point, so the answer needs x and y values.

Thinking about the question

- Need to find the 2 sequences formed by the slopes and y–intercepts.
- Find the formula for the n^{th} term of each sequence.
- Use this to find the 5th term in both sequences. This will give the equation of Line 5.
- Find the point of intersection of Line 5 and Line 1 using simultaneous equations.

Worked solution (✓ = 1 mark)

Slopes

The 3 slopes give the sequence

$-0.8, 0.4, -0.2, \ldots$

No common difference, so not arithmetic.

$\frac{0.4}{-0.8} = -0.5, \ \frac{-0.2}{0.4} = -0.5$

There is a common ratio of -0.5.

$\therefore$ this forms a geometric sequence with

$\mathbf{t_1 = -0.8}$ **and** $\mathbf{r = -0.5}$. ✓

$t_n = t_1 r^{(n-1)}$

$t_5 = -0.8 \times (-0.5)^{(5-1)}$

$\mathbf{t_5 = -0.05}$ ✓

y–intercepts

The 3 y–intercepts give the sequence

$1.2, 2.7, 4.2, \ldots$

1.5 1.5

There is a common difference of 1.5.

$\therefore$ this forms an arithmetic sequence with

$\mathbf{t_1 = 1.2}$ **and** $\mathbf{d = 1.5}$. ✓

$t_n = t_1 + (n-1)d$

$t_5 = 1.2 + (5-1) \times 1.5$

$\mathbf{t_5 = 7.2}$ ✓

$\therefore$ Line 5 is $y = -0.05x + 7.2$

Solve Line 1 and Line 5 simultaneously:

$y = -0.8x + 1.2 \quad \ldots[1]$

$y = -0.05x + 7.2 \quad \ldots[2]$

Sub [1] into [2] as both equal to y:

$-0.8x + 1.2 = -0.05x + 7.2$

$-0.75x + 1.2 = 7.2$

$-0.75x = 6$

$x = \frac{6}{-0.75}$

$\mathbf{x = -8}$ ✓

Sub $x = -8$ into [1]:

$y = -0.8 \times (-8) + 1.2$

$\mathbf{y = 7.6}$ ✓

Solution is $x = -8$ and $y = 7.6$.

Check: Sub $x = -8$ and $y = 7.6$ into [1] and [2]:

[1] LHS $= 7.6$

RHS $= -8.0 \times (-8) + 1.2 = 7.6$

So, LHS = RHS.

[2] LHS $= 7.6$

RHS $= -0.05 \times (-8) + 7.2 = 7.6$

So, LHS = RHS.

✓ for logical organisation communicating key steps

$\therefore$ the coordinates of the point where Line 5 in the pattern intersects Line 1 is $(-8, 7.6)$.

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2 Chapter summary

Arithmetic sequences

- If a term is generated by adding or subtracting the previous term by the same amount, then it is an **arithmetic sequence**.
- The difference between one term and its previous term is called the **common difference**.
- The **recurrence relation** is written as

$$t_{n+1} = t_n + d,\ t_1 = a$$

where d is the common difference and a is the first term.
- Once an arithmetic sequence has been generated, it can be shown in a table and then plotted as a graph. An arithmetic sequence will always give a **linear** graph.
- The **general term (n^{th} term)** of an arithmetic sequence with **first term t_1** and **common difference d** is given by

$$t_n = t_1 + (n-1)d$$

- Arithmetic sequences are used to **model linear growth and decay** such as **simple interest** and **straight-line depreciation**.
- In straight-line depreciation the value of the asset is given by

$$V_n = \text{initial value} - \text{depreciation per year} \times n$$

- The **salvage** (or **scrap**) **value** is the value of the asset at the end of its working life.

Geometric sequences

- If a term is generated by multiplying or dividing the previous term by the same amount, then it is a **geometric sequence**.
- The ratio (quotient) of one term to (divided by) its previous term is called the **common ratio**.
- The **recurrence relation** is written as

$$t_{n+1} = rt_n,\ t_1 = a$$

where r is the common ratio and a is the first term.
- Once a geometric sequence has been generated, it can be shown in a table and then plotted as a graph. A geometric sequence will always give a **curved** graph.
- The general term (n^{th} term) of a geometric sequence with first term t_1 and common ratio r is

$$t_n = t_1 r^{(n-1)}$$

- Geometric sequences are used to model **exponential growth and decay**, such as population growth or decay by a percentage and **diminishing-value depreciation**.

Worksheet
General Maths formula sheet

Cumulative examination 1

Simple familiar

Perusal time: 2 minutes   Working time: 30 minutes

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

Section 1 (5 marks): 5 multiple choice questions

Section 2 (13 marks): 3 short response questions

Total: 18 marks

Section 1 5 multiple choice questions

5 marks

1 © QCAA 2021 1Q11 Which option is an example of bivariate data?

A The rating given to a brand of meat pies as poor, fair or good.

B The number of people in a household and amount of water used.

C The number of cars passing through a particular set of traffic lights.

D The time a person spends using a mobile phone on a Friday evening.

2 © QCAA 2022 1Q13 The two–way table summarises the semester 1 results for students enrolled in two courses, Machinery and Electrical. Students achieved either satisfactory (S) or unsatisfactory (U).

		Machinery	
		S	U
Electrical	S	80%	10%
	U	20%	90%

The 10% cell in the table indicates that

A 10% of all students achieved satisfactory in Electrical.

B 10% of all students achieved unsatisfactory in Machinery.

C 10% of all students who achieved satisfactory in Electrical achieved unsatisfactory in Machinery.

D 10% of all students who achieved unsatisfactory in Machinery achieved satisfactory in Electrical.

3 © QCAA 2023 1Q6 In January 2022, 40 fish were released into a new dam that has the capacity to support 10 000 fish. It is predicted that the dam will reach its capacity in January 2030 if the fish population doubles every year.

Which sequence rule models the prediction?

A $t_n = t_1 r^{(n-1)}$, where $t_1 = 40$, $r = 2$, $n = 8$

B $t_n = t_1 r^{(n-1)}$, where $t_1 = 40$, $r = 2$, $n = 9$

C $t_n = t_1 + (n-1)d$, where $t_1 = 40$, $d = 2$, $n = 8$

D $t_n = t_1 + (n-1)d$, where $t_1 = 40$, $d = 2$, $n = 9$

4 © QCAA 2023 1Q5 A scatterplot is created to identify the nature of the relationship between two variables: vehicle age and distance travelled.

Which statement is correct?

A The vertical axis should show vehicle age as the response variable.

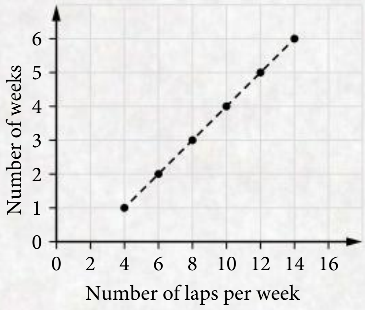
B The horizontal axis should show vehicle age as the explanatory variable.

C The horizontal axis should show distance travelled as the response variable.

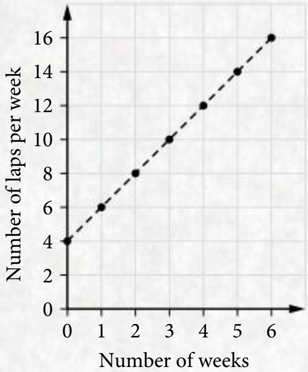
D The vertical axis should show distance travelled as the explanatory variable.

5 © QCAA 2022 1Q4 A swimmer has a weekly training routine to improve their fitness as modelled by the recursive function $T_{n+1} = T_n + 2$, where T_n is the number of laps they swim in week n and $T_1 = 4$. Which graph best represents the swimmer's routine?

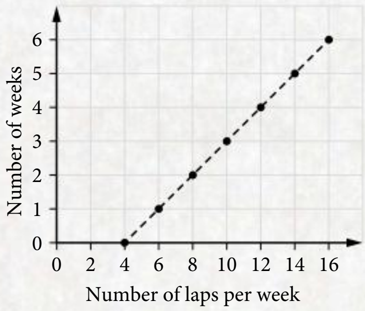
A



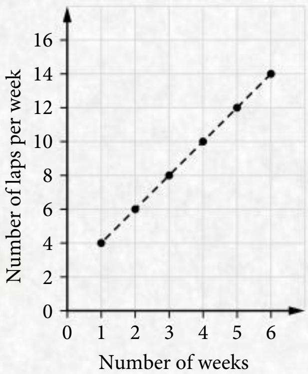
B



C

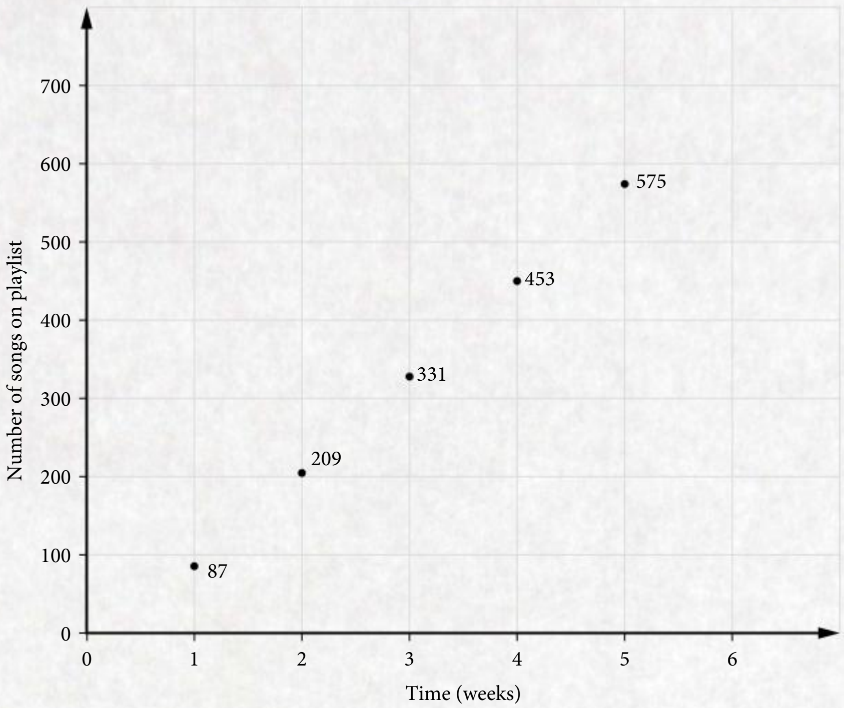


D



Section 2 3 short response questions 13 marks

6 © QCAA 2022 1Q18 (4 marks) The number of songs on a person's playlist, n, in each week since joining a music streaming service, t, forms an arithmetic sequence, as shown by the graph.



Use the arithmetic sequence to predict the number of songs on this person's playlist 25 weeks after joining the streaming service.

7 © QCAA 2023 1Q22 MODIFIED (4 marks) A person tracked the hours they slept and the calories they ate each day.

Sleep (hours)	7	6.5	9	9.5	8	7.5	6
Calories	2000	2300	1350	1200	1800	1800	2200

a Using calories as the response variable, display the data in a scatterplot with labelled axes. *[3 marks]*

b Identify the direction of the association between hours of sleep and calories eaten. *[1 mark]*

8 © QCAA 2020 1Q26 (5 marks) A scientist observed that the population of a specific bird species is decreasing by 17% each year and that at the beginning of 2016, there were 483 birds.

a Use a geometric sequence to model the bird population. *[2 marks]*

b Using the model from part **a**, predict the number of birds remaining at the beginning of 2021. *[3 marks]*

9780170484824

Cumulative examination 2

Complex familiar and unfamiliar

Perusal time: 2 minutes   Working time: 30 minutes

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

2 short response questions — 11 marks

Worksheet
General Maths formula sheet

1 © QCAA 2020S 2Q3 (4 marks) Music students in a school were asked how many hours they practised their musical instrument each week.

Seventy–two Year 7 music students were surveyed. The results are shown below.

- 15 students practised for 2 hours or less.
- 53 students practised for more than 2 hours and up to 4 hours.
- 4 students practised for more than 4 hours per week.

Seventeen Year 12 music students were surveyed. The results are shown below.

- 2 students practised for 2 hours or less.
- 11 students practised for more than 2 hours and up to 4 hours.
- 4 students practised for more than 4 hours per week.

Determine any patterns that suggest the presence of an association, providing reasons for your conclusion.

2 (7 marks) When an empty garbage truck with a maximum carrying capacity of 1400 kilograms, starts its daily round, it stops at a school where it collects 210 kilograms of garbage. It then travels along a suburban street, where the amount of rubbish collected from each house is the same. After the 10th house, the truck is carrying 490 kilograms of garbage.

Will the truck be able to collect garbage at the 44th house?

CHAPTER

3 ASSOCIATIONS AND LINEAR MODELLING

9780170484824

Syllabus coverage

UNIT 1, TOPIC 2: BIVARIATE DATA ANALYSIS 2

Fitting a linear model to numerical data

- Model a linear relationship by using technology to fit a least-squares line to the data, in the form of $y = mx + c$ where m is slope (gradient) and c is y-intercept.
- Understand and use $m = r\frac{s_y}{s_x}$ and $c = \bar{y} - m\bar{x}$ to determine the equation of a least-squares line, where m is slope (gradient), r is correlation coefficient, s_y is (sample) standard deviation of y values, s_x is (sample) standard deviation of x values, c is y-intercept, $\bar{y}$ is mean of y values and $\bar{x}$ is mean of x values.
- Construct a residual plot and use it to assess the appropriateness of fitting a linear model to the data.
- Interpret the y-intercept and slope (gradient) of the fitted line.
- Distinguish between interpolation and extrapolation.
- Use the equation of the least-squares line to make predictions.
- Recognise and explain the potential dangers of extrapolation.

Association and causation

- Recognise and explain that an observed association between two variables (categorical and/or numerical) does not necessarily mean that there is a causal relationship between them.
- Identify and communicate possible non-causal explanations for an association, including coincidence or the influence of another variable.
- Solve practical problems by identifying, analysing and describing associations between two variables (categorical and/or numerical).

3.1

Prior learning Associations and linear modelling

Video (1):
Exam question analysis Associations and linear modelling

Prior learning (1):
3.1 Associations and linear modelling

Worksheets (6):
3.2 Lines of fit • Least-squares regression line • Interpolation and extrapolation
3.3 Residual plots • Predictions and residuals
Cumulative exams General Maths formula sheet

Nelson MindTap

To access resources above, visit **cengage.com.au/nelsonmindtap**

Terminology

actual value	association	causation	coincidence
confounding variable	extrapolation	interpolation	least-squares line
outlier	prediction	residual	residual plot
residual value	slope	standard deviation	y-intercept

3.1 Association and causation

Heart disease kills many Australians each year. Doctors and medical researchers spend a lot of time and money on research in the hope of finding the causes. They use statistics to find associations between different variables.

However, just because there is a correlation between two variables, does not mean one variable causes a change in the other. In the graph below, $r = 0.827$ and $R^2 = 0.797$, which indicates a strong correlation between the variables. Adult shoe sizes do not change much, but their weights may. You know that shoe size does not determine weight. This could be an example of limited research, participants, or a coincidence.

Association and causation

- The strength of an **association** between variables is measured through correlation. This value does not measure **causation**.
- For causation to occur, a change in the **explanatory variable** must *cause* a change in the **response variable**.
- If a causation is unclear, a third variable called a **confounding variable** may be the reason. This is a variable that is not measured and may not have been known when gathering the initial data. It can also cause a change in the response variable and might be the reason for a high correlation between the measured variables.
- Associations can also be a **coincidence**, particularly over a short period of time.

WORKED EXAMPLE 1 Determining causation

Over the period from 1950 to 1980, there was a very strong relationship between the consumption of carrots in Australia and the rate of lead poisoning. However, since 2000, the association has significantly declined in strength.

a **State** if you think this is likely to be a causal relationship.

b **Consider** and list reasons for your answers, including reasons for the decline.

[Hint: Before 1980, lead was used in petrol for motor vehicles.]

Steps	Working
a State your opinion about causation.	It seems unlikely that there is a causal relationship in either direction.
b List justified reasons for your causal statement.	During this time the population increased. This would cause a natural increase in the consumption of carrots. Lead poisoning would also increase due to the growing use of lead in petrol-driven vehicles and through exposure to lead products, such as paint, used in the building and renovating of houses.

WORKED EXAMPLE 2 Identifying non-causal explanations for association

Since about 1990, the populations of frogs and honeybees in Europe have both declined. There is an association between the populations. **Identify** some non-causal reasons that could account for this association.

Steps	Working
1 The size of cities has increased all over the world.	Both populations could have been affected by habitat destruction. This would be a confounding variable.
2 The number of people has increased.	They could both have declined as a result of the increase in human activity. The human population would be a confounding variable.
3 They may have both been affected by infectious diseases.	They could both be affected by diseases in the population. This would be a coincidence.

EXERCISE 3.1 Association and causation

ANSWERS p. 443

Mastery

1 Which one of the following statements is true about causation?

A An explanatory variable will always have a strong association with a response variable.

B Both response and explanatory variables will cause changes in each other.

C The response variable causes changes in the explanatory variable.

D A change in the explanatory variable will cause a change in the response variable.

Exam hack

Think of what is logical when considering causation.

2 WORKED EXAMPLE 1 For each association between pairs of variables

i state if you think there is a causal association.

ii suggest some possible reasons for the association.

a A strong negative association between the decreasing population of the Southern Cassowary since the 1980s and an increase in the number of flights into Cairns.

b A positive association between the number of 4WDs visiting K'gari and the number of dingo attacks at campsites.

c A moderate positive association in the increased flights by the Royal Flying Doctors Service (RFDS) and the increase of the bilby population near Charleville since the early 2000s.

d A negative association between the increased number of cruise ships visiting the Whitsundays and the decreased availability of long-term rental accommodation.

e A negative association between the increase in cattle numbers in Australia and the loss of rainforest.

f A strong association between the number of industrial accidents and the amount of coal exported from Australia.

3 WORKED EXAMPLE 2 For each correlation between pairs of variables, suggest a confounding variable that could be the underlying cause of the correlation between the two.

a A strong positive association between the number of vehicles on roads and the number of doctor consultations in Australia.

b A positive association between the increased number of students in a school and the increased wins of the school volleyball team.

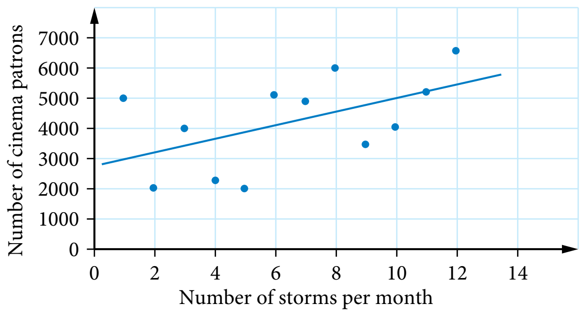
c A negative association between the number of successful treatments of stomach ulcer cases and the speed of treatments of sports injuries in Australia in the 2000s.

d A moderate positive association between the number of weddings held in Queensland and the sales of Bundaberg Ginger Beer overseas.

e A negative association between the number of workers in the Brisbane CBD and the number of people climbing the Story Bridge.

f A positive association between the number of rescues at Gold Coast beaches and the traffic volume of the M1 motorway between Brisbane and the Gold Coast.

4 The graph shows the association between the number of storms in South-East Queensland and the number of cinema patrons over a 12-month period.



a Describe the strength of the association.

b State if you think there is a causal relationship between the variables.

c For this data, $R^2 = 0.28$. Explain whether this indicator proves causation between the variables or whether another variable may influence the results in the graph.

5 A study found a relationship between the number of cockroaches detected in a house each night for a week and the amount of rain each day.

a Explain which variable you believe should be the explanatory variable.

b Estimate the strength of association between the two variables.

c Determine whether there is a causal relationship between the two variables.

6 It was found that there is a strong association between school enrolments in Central Queensland and new boat sales in the same location.

a State if you think there is a causal relationship.

b Suggest some possible reasons for the association.

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Exam practice

7 The most appropriate explanatory variable that would have a causal effect on the response variable 'weight' is

A gender

B daily calorie intake

C income

D birth weight

8 A confounding variable is a variable that

A proves causality.

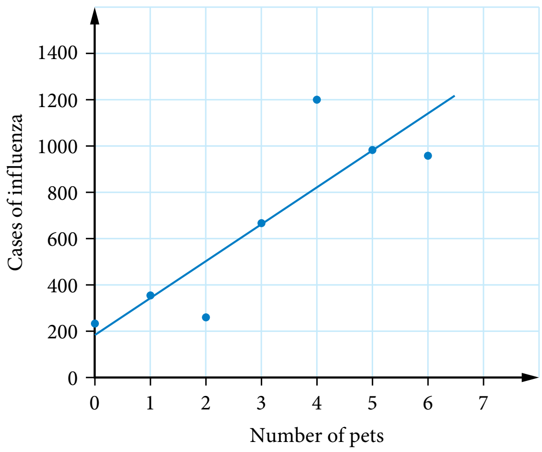
B relates to the strength of the association between variables.

C is an external factor that may influence the response or explanatory variables.

D is considered in all research and experiments where data is gathered.

9 (2 marks) Giving an example, explain the difference between association and causation.

10 (3 marks) CF The graph displays the results of a study looking at the relationship between the average number of pets per household and the number of influenza cases in the Darling Downs region in the middle of the year. The study found there was a correlation coefficient of 0.86.



Justify whether it is reasonable for Queensland Health to make planning decisions for future influenza seasons based on this data.

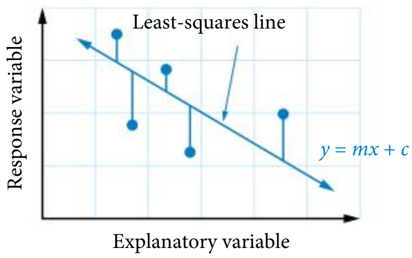
Worksheets
Lines of fit

Least-squares regression line

3.2 Least-squares line

You may be able to draw a line of best fit to a graph of bivariate data. In this chapter, the **least-squares line** is used.

The least-squares line makes the squares of the **vertical distances** of the points from the line the smallest they can possibly be. You need to identify the **explanatory variable** and **response variable** to find the equation of the least-squares line.



Least-squares line formulas

The equation of the least-squares line is the linear equation $y = mx + c$.

The **slope** (**gradient**) is calculated using $m = r\frac{s_y}{s_x}$ and the **y-intercept** is found using $c = \bar{y} - m\bar{x}$

where

- $\bar{x}$ and $\bar{y}$ are the means of the x and y data values
- r is Pearson's correlation coefficient
- s_x and s_y are the sample **standard deviations** of the x and y data values.

Calculate m first, as it is needed for the calculation of the y-intercept (c).

The least-squares line may also be written as $y = a + bx$, where b represents the gradient and a represents the y-intercept. Calculators use this form.

WORKED EXAMPLE 3 Calculating the equation of the least-squares line

Calculate the equation of least-squares line if

$\bar{x} = 26.25$ $\quad s_x = 4.559$ $\quad r = -0.905$

$\bar{y} = 33.5$ $\quad s_y = 4.536$

Steps	Working
1 Calculate the gradient.	$m = r\frac{s_y}{s_x}$ $= -0.905 \times \frac{4.536}{4.559}$ $= -0.9004\ldots$ ≈ -0.9
2 Calculate the y-intercept.	$c = \bar{y} - m\bar{x}$ $= 33.5 - (-0.9) \times 26.25$ $= 57.125$ ≈ 57.13
3 Substitute the gradient and y-intercept into the equation to get the answer.	$y = mx + c$ $y = -0.9x + 57.13$

Keep accurate values on your calculator to make sure you get the correct answers.

WORKED EXAMPLE 4 Using a calculator to find the equation of a least-squares line

a Determine the equation of the least-squares line from the data.

x	10	14	24	27	35
y	16	26	32	42	48

b Sketch a scatterplot and fit a least-squares line.

Steps	Working
a **1** Identify the variables.	x = explanatory variable y = response variable
2 Enter the data into the calculator. Find the required variables for the formulas.	**Casio fx-82AU PLUS II** Press MODE, **2**: STAT and **2**: A+BX Enter the values in the x and y columns respectively. Press AC, then press SHIFT **1** (STAT): s_x 4: Var, 4:s_x = 10.075 s_y 4: Var, 7:s_y = 12.696 $\bar{x}$ 4: Var, 2: $\bar{x}$ = 22 $\bar{y}$ 4: Var, 4: $\bar{y}$ = 32.8 r 5: Reg, 3:r = 0.973 y-intercept (c) 5: Reg, 1:A = 5.81 gradient (m) 5: Reg, 2:B = 1.23 Press MODE, **1**: COMP to exit the statistics mode. **Sharp EL-531TH** Press MODE, **1** and choose **1** (LINE): Linear regression calculation. Enter the values in pairs by pressing 10, STO, 16, M+. You will see Data Set = 1. Continue entering the remaining data pairs in the same way until you have entered all the values. To get the information, press s_x ALPHA 5 = 10.075 s_y ALPHA 8 = 12.696 $\bar{x}$ ALPHA 4 = 22 $\bar{y}$ ALPHA 7 = 32.8 r ALPHA ÷ = 0.973 y-intercept (c) ALPHA (= 5.81 gradient (m) ALPHA) = 1.23 Press MODE **0** to exit statistics calculations or press the reset button. **TI-30XB** Press data and enter the values in the L1 and L2 columns respectively. Then press 2nd DATA (STAT) and choose **2**: 2-Var Stats. Make XDATA: L1 and YDATA: L2, if they are not already, and use the cursor to go down to **CALC** and press ENTER. Move the cursor down to get the values. s_x = 10.075 s_y = 12.696 $\bar{x}$ = 22 $\bar{y}$ = 32.8 r = 0.973 y-intercept (c) = 5.81 gradient (m) = 1.23 Select 2nd **0** to reset your calculator.
3 Substitute the gradient and y-intercept into the equation to write the answer.	$y = mx + c$ $y = 1.23x + 5.74$

Exam hack

While the calculator can give you the gradient and y-intercept, check the marks allocated to the question to see if the working for m and c are required.

3.2

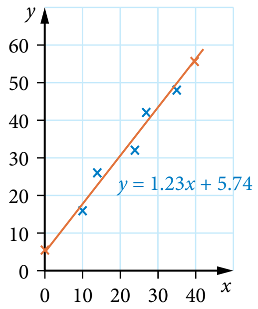
b **1** Work out a couple of points on the line.

When $x = 0$, $y = 5.81$. $(0, 5.81)$
When $x = 40$, $y = 55$. $(40, 55)$

2 Construct the scatterplot.

Plot the 2 points to draw in the least-squares line.

Write the equation next to the line on the graph.



Worksheet Interpolation and extrapolation

Interpolation and extrapolation

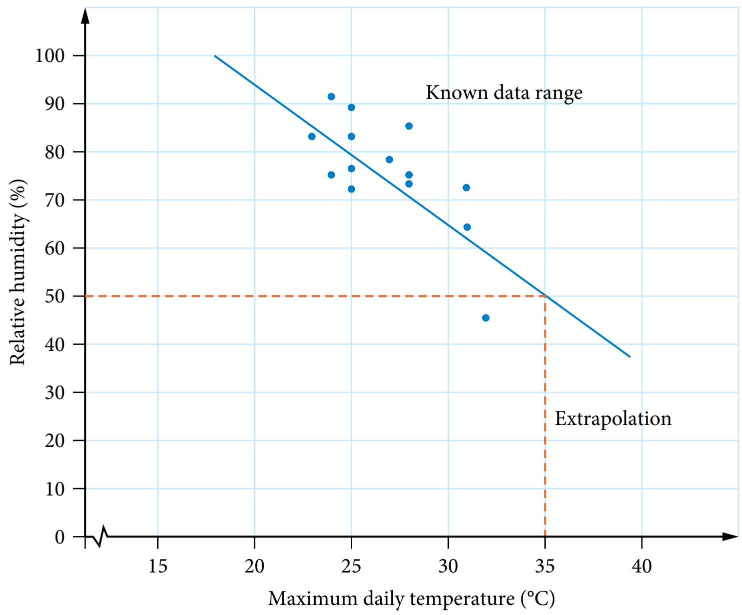
You can make **predictions** using the least-squares equation or graph.

Interpolation and extrapolation

- **Interpolation** means using a fitted line to predict a value within the range of the data.
- **Extrapolation** means using a fitted line to predict a value outside of the range of the data.

Extrapolation assumes that existing trends will continue. This is not necessarily accurate. Using extrapolated predictions may lead to incorrect results and decisions.

For example, the graph below shows the relationship between temperature (EV) and relative humidity (RV) for a coastal Queensland town during spring. The equation for the least-squares line is $y = -2.88x + 153.24$, the correlation coefficient $r = -0.73$ and coefficient of determination $R^2 = 0.53$.



Using the least-squares line on the graph, the relative humidity on a 35°C day would be 50%. The formula is more accurate, giving 52.44%.

The line suggests that the warmer the weather, the lower the humidity.

However, this is not always the case, particularly in summer. It is also possible for a 15°C rainy day in winter to have a high relative humidity.

For values outside of the range of data, you need to consider the strength of the relationship and what is logically possible.

WORKED EXAMPLE 5 Use the least-squares line to make a prediction

The table lists the number of visitors to a public pool during the school holidays in April and the temperature on those days.

Temperature (°C)	25	27	29	26	23	24	26
Number of visitors	215	200	305	250	190	185	170

Use the least-squares line to

a interpolate the number of visitors when the temperature is 28°C

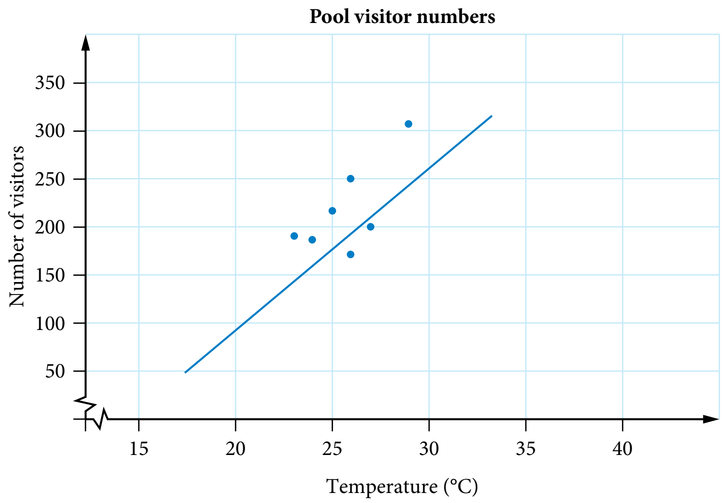
b extrapolate the number of visitors when the temperature is 32°C

Steps | **Working**

a 1 Sketch the scatterplot for the data.

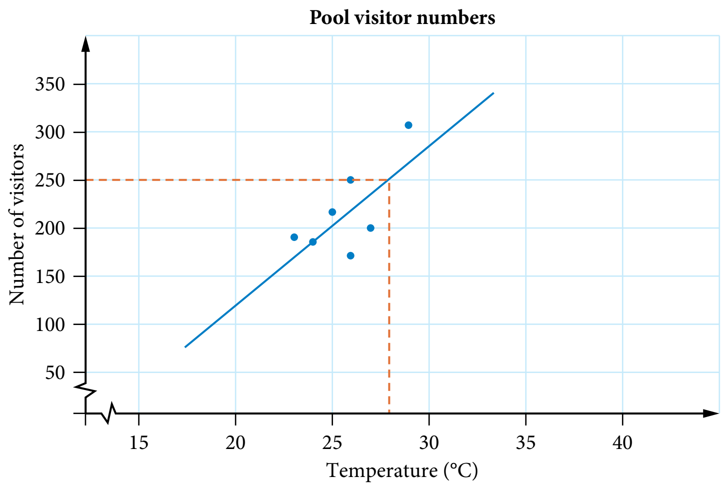
Temperature = explanatory variable (x)

Number of visitors = response variable (y)



2 Use the line to predict the number of visitors when $x = 28$°C.

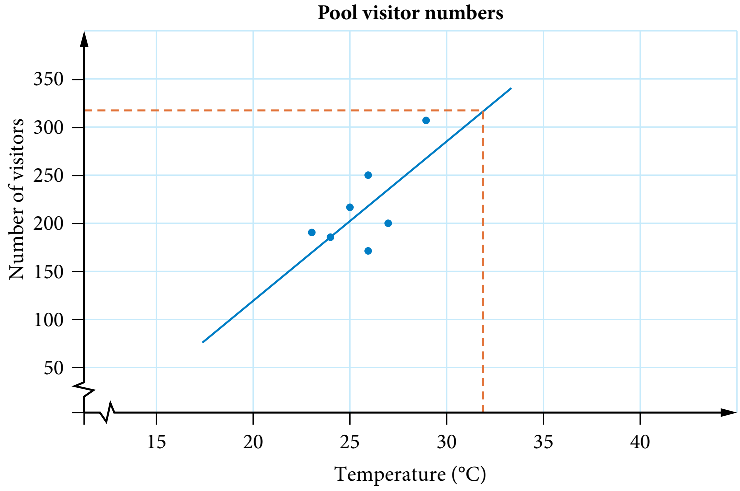
This is interpolation as the predicted value is within the known actual data values.



When the temperature is 28°C, approximately 250 people will visit the pool.

b Use the line to predict the number of visitors when $x = 32$°C.

This is extrapolation as the data predicted value is beyond the known data.



When the temperature is 32°C, approximately 320 people will visit the pool, *assuming that this number of people will fit*.

WORKED EXAMPLE 6 Using the equation of a least-squares line to make a prediction

The table lists the number of visitors to a public pool during the school holidays in April and the temperature on those days.

Temperature (°C)	25	27	29	26	23	24	26
Number of visitors	215	200	305	250	190	185	170

Use a calculator to **determine** the equation of the least-squares line and **predict** how many visitors the pool would have for the temperatures

a 28°C

b 32°C

Steps	Working
a 1 Identify the variables.	temperature = explanatory variable (x) number of visitors = response variable (y)
2 Enter the data into a scientific calculator to find the gradient and y-intercept.	$m = 16.77$ $c = -214.77$
3 Substitute to find the least-squares equation.	$y = mx + c$ $y = 16.77x - 214.77$
4 Substitute $x = 28$°C to find y.	$y = 16.77 \times 28 - 214.77$ $= 254.79$ ≈ 255 visitors
b Substitute $x = 32$°C to find y.	$y = 16.77 \times 32 - 214.77$ $= 321.87$ ≈ 322 visitors

EXERCISE 3.2 Least-squares line

ANSWERS p. 444

Recap

1 For the response variable 'income', which of the following explanatory variables would have the greatest causal link to the response variable?

A hair colour
B number of pets owned
C age of car
D years of work experience

2 A third variable that can influence the association between two comparative variables is

A an explanatory variable.
B a confounding variable.
C an exterior variable.
D a causal variable.

Mastery

3 WORKED EXAMPLE 3 Calculate the equation of the least-squares line using the given information. Round m and c to 2 decimal places.

a $s_x = 3.05$
$s_y = 1.6$
$\bar{x} = 11.4$
$\bar{y} = 11.7$
$r = 0.94$

b $s_x = 2.39$
$s_y = 5.32$
$\bar{x} = 5.2$
$\bar{y} = 27.4$
$r = -0.93$

c $s_x = 2.36$
$s_y = 7.11$
$\bar{x} = 6$
$\bar{y} = 43.8$
$r = 0.63$

d $s_x = 13.17$
$s_y = 11.4$
$\bar{x} = 42.14$
$\bar{y} = 152$
$r = 0.33$

Exam hack

Ensure you check the question for the number of decimal points required. Remember, the more you round, the less accurate your results become.

WORKED EXAMPLE 4 For each table of values in Questions **4** to **8**:

a find the equation of the least-squares line using the statistics mode on your calculator.

b plot a scatterplot and fit a least-squares line.

4

x	0	2	5	7	10	12	15
y	32	32	35	58	81	65	83

5

x	0	2	4	6	8	9	11	13	15
y	74	68	60	66	64	60	47	47	47

6

x	5	7	8	10	11	13	14	16	17	19
y	9.7	6	10.5	12.3	12.7	12.6	15	20.7	19.2	25

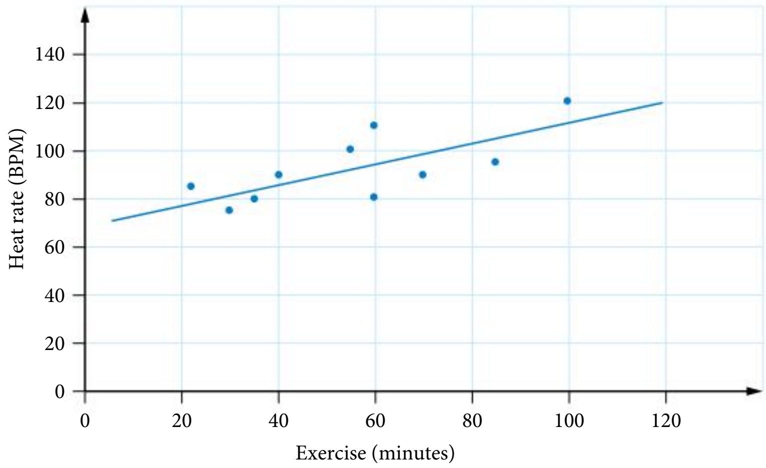
7

x	2	3	5	6	8	9	10	12	13	15
y	4.4	3.2	3.9	4.4	4.3	6.3	5.2	7.2	7	6.7

8

x	12	15	18	22	25	28	31	34	37	41	44	47
y	31	27.1	28.2	33.7	32.2	23.4	20.4	18.2	16.2	17.9	17	18.7

9 WORKED EXAMPLE 5 This graph shows the relationship between the number of minutes of exercise and heart rate in beats per minute.



Use the equation of the least-squares line to predict the heart rate after

a 10 mins of exercise.
b 50 mins of exercise.
c 80 mins of exercise.
d 110 mins of exercise.

Exam hack

Don't forget to include **units** on your answer if they are given in the question.

10 WORKED EXAMPLE 6 A stadium reported the number of noise complaints after 7 concerts throughout the year.

Audience size (People)	32 900	36 300	46 150	45 800	39 200	39 750	40 100
Number of complaints	3	5	17	71	15	10	8

a Use a calculator to determine the equation of the least-squares line and predict the number of complaints for each audience size. Round your answers to the nearest whole number.

i 20 500 **ii** 38 200 **iii** 45 500 **iv** 51 000

b Discuss the reasonableness of your answers to part **a**.

Exam practice

11 Which combination of variables does the least-squares line use to determine the equation of the line?

A x, y, s_x, s_y, R^2 **B** $r, \bar{x}, \bar{y}, s_x, s_y, R^2$ **C** r, x, y, s_x, s_y **D** $r, \bar{x}, \bar{y}, s_x, s_y$

12 The least-squares line $y = 3.5x + 210$ predicts that when $x = 20$, y will be

A 190 **B** 210 **C** 280 **D** 910

13 Given that $r = 0.63$, $s_x = 1.85$, $s_y = 5.9$, the slope is approximately

A 0.19 **B** 2.07 **C** 3.24 **D** 6.87

14 (4 marks) Use formulas to find the values of the slope (m) and the y-intercept (c), given $r = 0.927$, $\bar{x} = 26.6$, $\bar{y} = 219$, $s_x = 2.875$, $s_y = 112.784$.

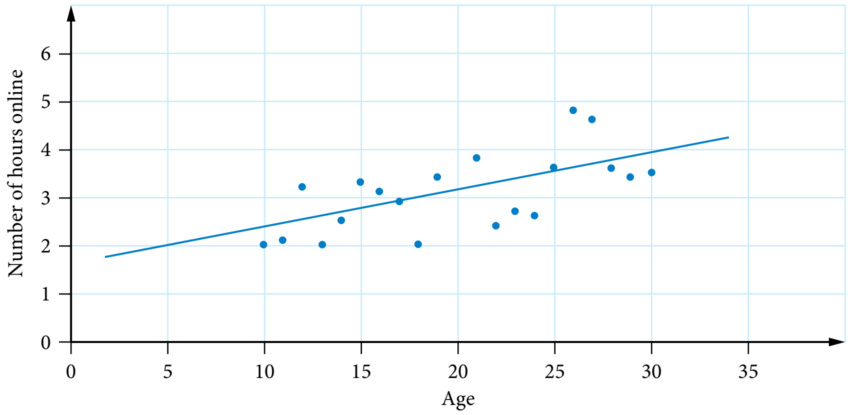
15 (4 marks) The number of operating hours and day's takings of a shop are shown in the table.

Hours opened, h	3	4	5	6	7	8	9	10	11	12
Takings \$'000, t	1.5	1.8	1.9	2.4	2.2	2.4	3	3	3.4	3.3

a Determine the equation of the least-squares line. *[1 mark]*

b Predict the number of takings on day 25. *[3 marks]*

16 © QCAA 2020 1Q24 MODIFIED (4 marks) The following data for the age of a person and the average number of hours they spend online in a day was recorded and a least-squares line was developed and graphed.



a Use the least-squares line to estimate how many hours a 21-year-old would likely spend online during a day. *[1 mark]*

b Classify the prediction for part **a** as either interpolation or extrapolation. *[1 mark]*

c Based on the graph, the following statement was made:

'The older a person is, the more time they will spend online.'

Comment on the reasonableness and possible dangers of this statement. *[2 marks]*

17 (3 marks) CF Calculate the average number of hours participants, between 10–18 years old, play sport during the week given the equation of the least-squares line is $y = -0.177x + 7.265$ and $\bar{x} = 18.5$.

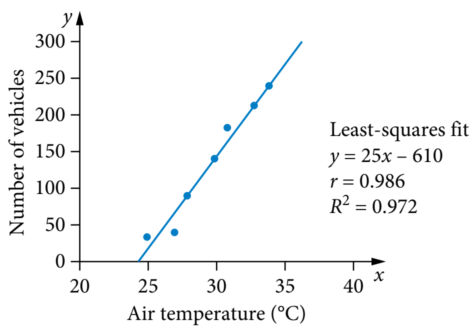
18 © QCAA 2024 1Q20 (4 marks) The graph shows the association between the air temperature, x, and the number of vehicles parked at a train station, y.

a Identify Pearson's correlation coefficient and use it to describe the strength of the association between x and y.

It is suggested that the time of day, t, could be a confounding variable in this situation.

b Define *confounding variable*.

c Explain why t could be a confounding variable in this situation.

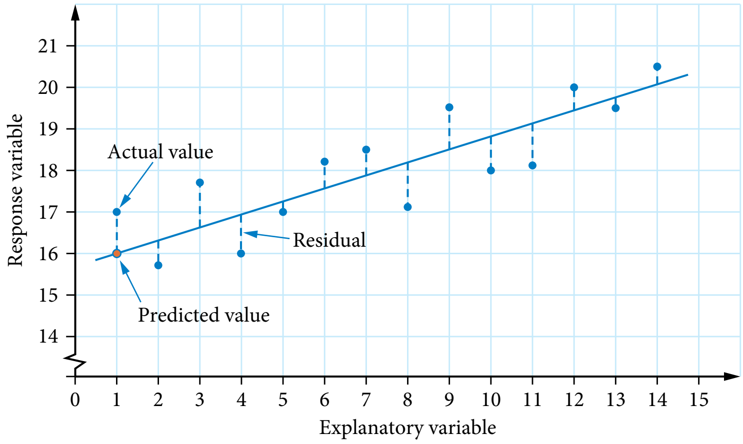


Worksheets
Residual plots

Predictions
and residuals

3.3 Residual plots

The points in a scatterplot will not usually align perfectly with the least-squares line.
A **residual** is the vertical distance between a point on a scatterplot and the line of best fit.



Residual values

To find **residual values** (vertical distances):

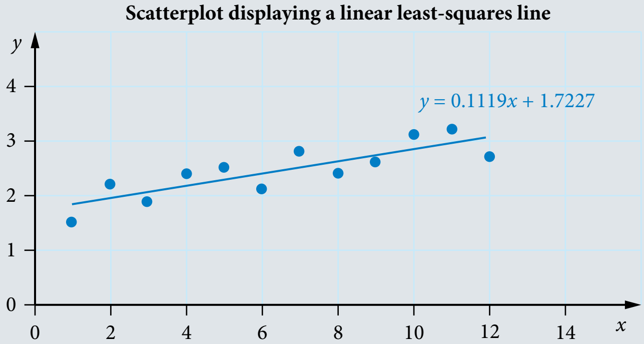
residual value = **actual value** – predicted value

If the actual value is

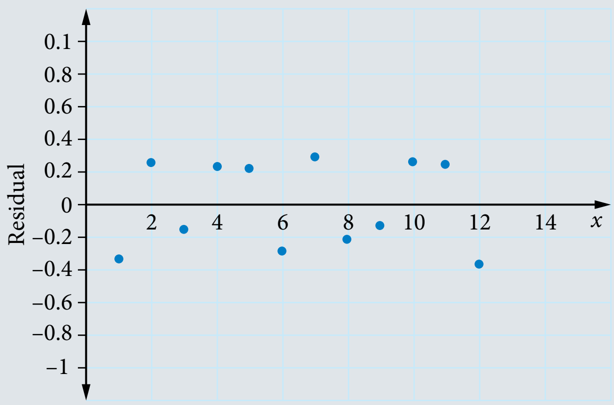
- above the line, the residual value will be positive.
- below the line, the residual value will be negative.

A **residual plot** shows all the residual values against the explanatory variable.
If the residual plot has dots spread around zero, the data follows a linear relationship.

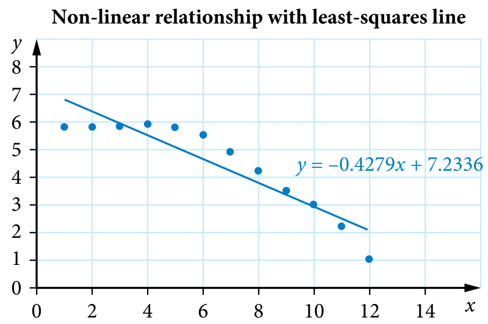
Scatterplot displaying a linear least-squares line

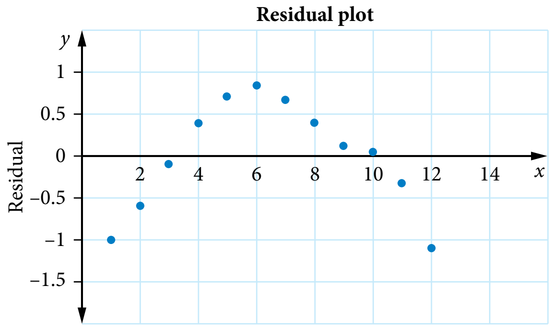


Residual plot



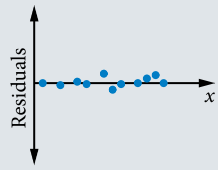
Where the residual plot dots show a pattern, the data will most likely follow a non-linear relationship. In this case, a linear model is not appropriate for making predictions.



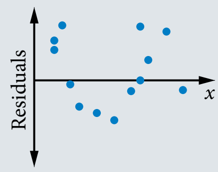


Determining if a residual plot suits a linear model

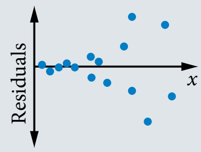
- A residual plot with points scattered close to both sides of the axis shows that the association between the variables is most likely linear.



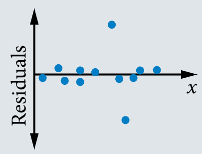
- A residual plot with points in a curved line or points in a line on one side of the axis shows that the association between the variables is non-linear.



- A residual plot with points diverging from the axis in one direction shows that association between the variables is non-linear.

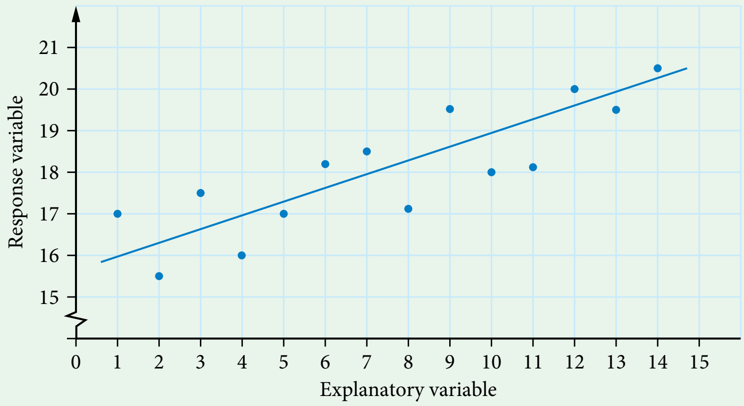


- A residual plot with one or several **outliers** indicates that there are outliers in the original data.
  If they are valid points, the association between the variables may not be linear.



WORKED EXAMPLE 7 Creating residual plots from graphs

Using the information in the scatterplot, **calculate** the residual values for $1 \le x \le 4$ and **sketch** a residual plot.



Steps	Working

1 Draw up a table of values and record information from the graph.

Let $y_a = (y \text{ actual})$

$y_p = (y \text{ prediction})$

x	1	2	3	4
y_a	17	15.5	17.5	16
y_p	16	16.3	16.7	17

2 Calculate the residual value for each x value.

When

$x = 1$, RV $= 17 - 16$

$= 1$

$x = 2$, RV $= 15.5 - 16.3$

$= -0.8$

$x = 1$, RV $= 17.5 - 16.7$

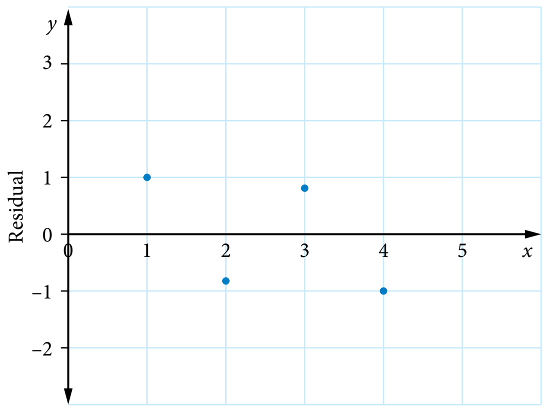
$= 0.8$

$x = 1$, RV $= 16 - 17$

$= -1$

x	1	2	3	4
y_a	17	15.5	17.5	16
y_p	16	16.3	16.8	17
Residual value	1	−0.8	0.8	−1

3 Sketch the residual plot showing the 4 residual values.



WORKED EXAMPLE 8 Creating residual plots from least-square line equations

The data below has a least-squares model of $y = 1.84x + 3.97$.

x	2	3	4	5	6	7	8	9
y	8.2	8.4	9.2	12.6	14.5	16.3	19.9	19.8

a **Calculate** the residuals, correct to one decimal place.

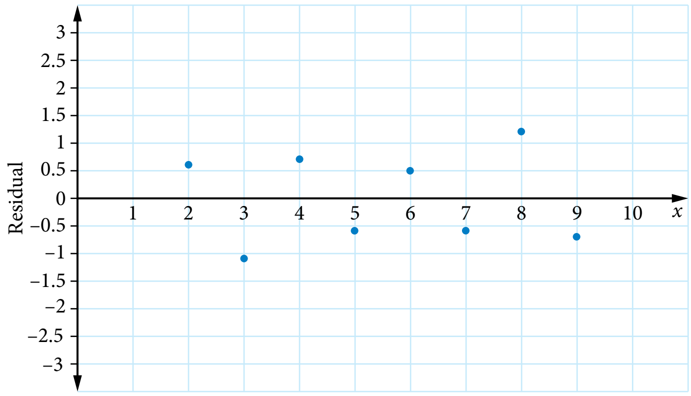
b **Construct** the residual plot.

c **Comment** on the results.

Steps	Working
a **1** Substitute $x = 2$ into the equation to find the predicted value.	When $x = 2$ $y = 1.84 \times 2 + 3.97$ $= 7.65$
2 Calculate the residual value.	RV $= 8.2 - 7.65$ $= 0.55$ ≈ 0.6
3 Continue the process to complete the table of values.	see table below

x	2	3	4	5	6	7	8	9
y_a	8.2	8.4	12	12.6	15.5	16.3	19.9	19.8
y_p	7.65	9.49	11.33	13.17	15.01	16.85	18.69	20.53
RV	0.6	−1.1	0.7	−0.6	0.5	−0.6	1.2	−0.7

b Use the x and RV values to sketch the residual plot.



c Comment on the residual plot.

As the dots are scattered close to both sides of the axis, the plot suggests a linear model is appropriate.

WORKED EXAMPLE 9 Calculating actual values from residual values

Use the equation of the least-squares line $y = -2.8x + 185$ to find the missing values in the table.

x	10	20	30	d
y (actual)	154	b	103.5	72
y (predicted)	157	129	c	73
Residual	a	0.5	2.5	−1

Steps	Working
a Find the residual value.	residual value = actual value − predicted value $= 154 - 157$ $= -3$
b Find the actual value.	actual value = residual value + predicted value $= 0.5 + 129$ $= 129.5$

c Find the predicted value.	predicted value = actual value − residual value = 103.5 − 2.5 = 101
d 1 Find the x value. Reverse the equation of the least-squares line by using the predicted value for y. y (predicted) = 73	$y = -2.8x + 185$ $73 = -2.8x + 185$ $\frac{73-185}{-2.8} = x$ $40 = x$
2 Answer the question.	The value of x is 40.

EXERCISE 3.3 Residual plots

ANSWERS p. 445

Recap

1 If $r = 0.68$, $s_y = 24.5$ and $s_x = 3.8$, the slope of the line is closest to

A 1.03 **B** 4.38 **C** 4.39 **D** 6.45

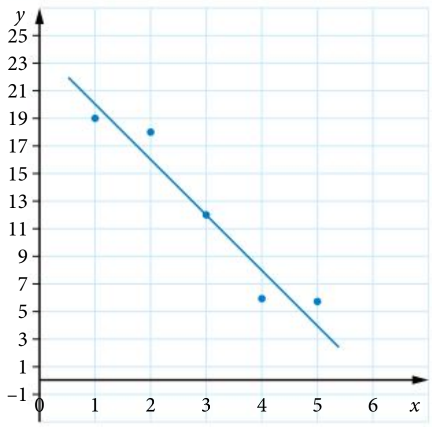
2 The formula used for calculating the y-intercept of a least-squares line is

A $\bar{y} = \bar{x} + mr$ **B** $c = m + \bar{x}$ **C** $\bar{y} = c + m\bar{x}$ **D** $c = \bar{y} - m\bar{x}$

Mastery

3 WORKED EXAMPLE 7 Construct a residual plot from each scatterplot.

a



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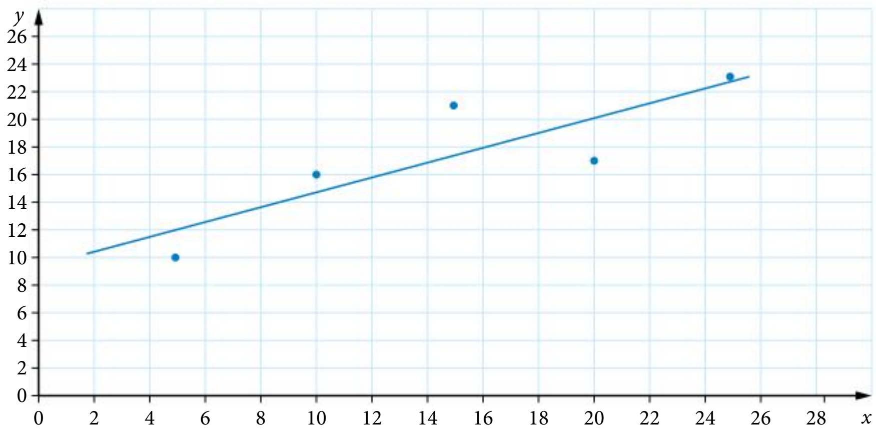
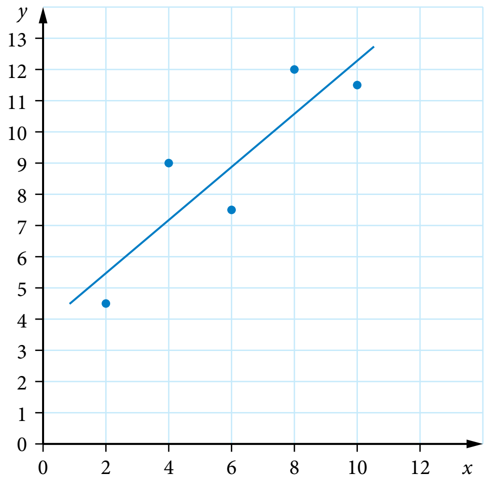
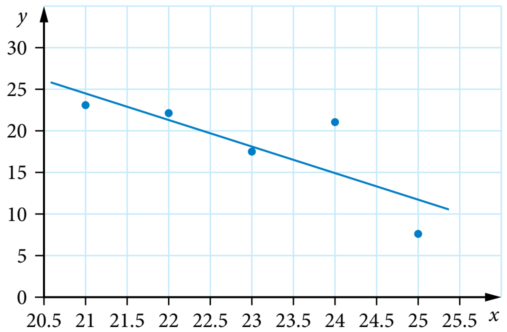


9780170484824

c



d



Exam hack

The use of a table showing x, y_a, y_p and residuals will help you to organise the information and your thoughts when calculating with residuals.

4 WORKED EXAMPLE 8 The data below has a least-squares line of $y \approx 30.473 - 0.5489x$.

x	4	5	6	8	10	11	13	15	16	19
y	28	28	28	26	24	25	23	22	21	21

a Calculate the residuals, correct to 2 decimal places.

b Draw the residual plot.

c Comment on the results.

5 The data below has a least-squares line of $y \approx 0.5361x - 8.1186$.

x	24	26	27	29	30	31	34	35	36	38
y	5	8	7	9	1	9	10	10	13	13

a Calculate the residuals, correct to 2 decimal places.

b Draw the residual plot.

c Comment on the results.

6 The data below has a least-squares model of $y \approx 0.3445x + 14.049$.

x	7	8	10	11	14	14	15	17	19	20
y	18	15	14	30	5	20	36	–1	3	47

a Calculate the residuals, correct to 2 decimal places.

b Draw the residual plot.

c Comment on the results.

7 The data below has a least-squares model of $y \approx 25.619x - 143.02$.

x	4	5	8	9	13	14	15	18	20	22
y	12	33	93	43	147	117	180	303	394	527

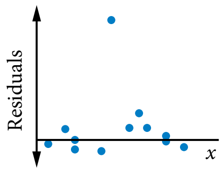
a Calculate the residuals, correct to 2 decimal places.

b Draw the residual plot.

c Comment on the results.

8 For each residual plot, state whether a straight-line model is appropriate. Assume that all points are valid.

a



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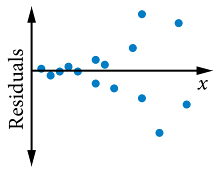

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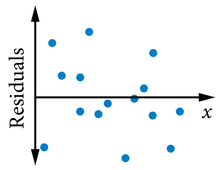

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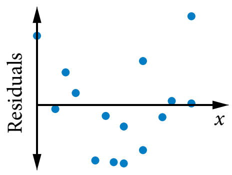
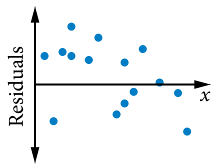
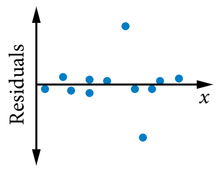


e



f



9 WORKED EXAMPLE 9 Complete the table by determining the missing values, given $y = 0.75x + 2.5$.

x	2	4	c	8	10	12
y (actual)	0.5	b	9	7.5	7	f
y (predicted)	a	5.5	7	8.5	e	11.5
Residual	−3.5	3	2	d	−3	3

Exam practice

10 The purpose of a residual plot is to

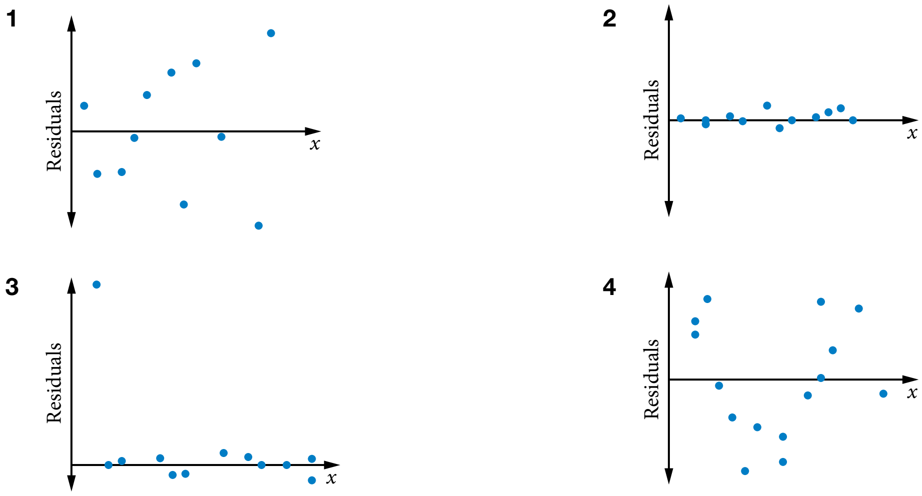
A display a scatterplot in a different orientation.

B display the predicted values against the explanatory variable values in a scatterplot.

C display the difference between the observed data and the predicted data from a linear line of best fit on a scatterplot.

D determine whether a linear line of best fit was appropriately drawn onto a scatterplot.

11 Refer to the residual plots to answer the question.



The best display of a residual plot that shows a non-linear relationship is

A 1 and 2 **B** 1 and 3 **C** 1 and 4 **D** 2 and 3

12 (5 marks) Using the tabulated data

x	1	2	5	7	10	11	12	14
y	25.1	20.5	17.7	17.4	15.8	10.9	21.3	8.8

a calculate the equation of the least-squares line for the data. *[1 mark]*

b draw the residual plot and use it to state if the model is appropriate. *[4 marks]*

13 (6 marks) CU A food truck operator decided to analyse the relationship between the maximum daily temperature and the amount of revenue they made from the sale of hot chips during spring. The following data was collected.

Temperature °C	Sales $'00
25	296
26	290
27	282
28	285
29	280
30	280
31	270

The operator has a break-even sales amount of $26 000. A sales amount of less than this will result in a loss on the hot chips. If there is a predicted residual amount of –5.48, calculate the highest daily temperature he can sell chips, without making a loss.

3.4 Practical problems with associations

You use two-way tables to identify associations between **categorical variables**.

You use scatterplots to identify associations between **numerical variables**.

In both cases, you want to find how changes in one variable affect another.

Remember that a statistical finding does not *prove* that a change in one variable causes a change in another.

WORKED EXAMPLE 10 Determining associations in categorical data

The table records the occurrence of skin cancer among a group of older people with different skin types.

Analyse the table to **identify** any significant differences between skin types.

	Light skin	Dark skin
Cancer	31	34
No skin cancer	65	280

Steps / **Working**

1 As cancer by skin type is being asked, total the columns in the table.

	Light	Dark
Cancer	31	34
No cancer	65	280
Totals	96	314

2 Convert each result into a percentage.

	Light	Dark
Cancer	32.3%	10.8%
No cancer	67.7%	89.2%
Totals	100%	100%

3 Compare the percentages for the different skin types.

32.3% of people with light skin had skin cancer, but only 10.8% of those with dark skin had one.

4 Calculate a probability comparison.

$32.3\% \div 10.8\% \approx 3$

5 Write the answer.

People with light skin were 3 times as likely to get skin cancer than those with dark skin.

WORKED EXAMPLE 11 Determining associations in numerical data

The difference, D, in °C between the average surface temperature and the long-term average surface temperature in a northern coastal region of Queensland every 10 years is shown below.

Y	1920	1930	1940	1950	1960	1970	1980	1990	2000	2010
D	−0.5	−0.4	−0.4	0.6	−0.2	0.2	0.4	0.3	0	0.8

a **Use** the table to find a relationship for the data and predict the average surface temperature change by 2100.

b **Discuss** the reasonableness of the answer to part **a**.

Steps / **Working**

a 1 Find the rule of the least-squares line.

$D = 0.0108Y - 21.118$, with $R^2 = 0.52$

2 Use the equation to predict the temperature variation in 2100.

$= 0.0108 \times 2100 - 21.118$

≈ 1.6

3 Write the answer.

On current trends, the average surface temperature will be 1.6°C higher in 2100.

b Explain the reasonableness of making a prediction in 2100.	The correlation coefficient shows a moderate strength of 0.721 and the coefficient of determination suggests that the year has a 52% impact on the temperature variance. Years, however, do not directly cause weather. Using the least-squares line to extrapolate data this far in the future is unreasonable.

EXERCISE 3.4 Practical problems with associations

ANSWERS p. 446

Recap

1 Given $\bar{y} = 79.5$, $\bar{x} = 9.5$, $r = 0.89$, $s_x = 3.03$ and $s_y = 15.27$, the equation to find the y-intercept for the least-squares line is

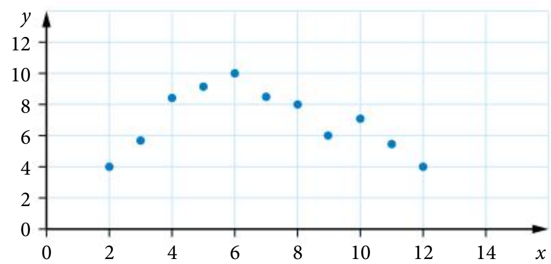
A $c = 9.5 - 0.18 \times 79.5$

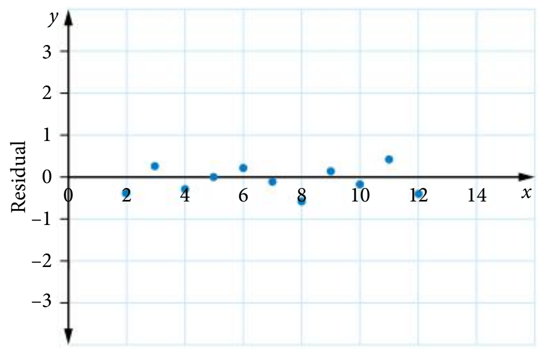
B $c = 9.5 - 0.18 \times 79.5$

C $c = 79.5 - 4.49 \times 9$

D $c = 79.5 - 4.49 \times 9.5$

2 Explain why the residual plot below does not match the scatterplot.





Mastery

3 WORKED EXAMPLE 10 A survey of whether people in Springvale have asthma is shown.

Asthma	Male	Female
Yes	23	25
No	214	181

a Use a percentaged two-way table to identify the differences in asthma prevalence between males and females.

b The government wants to ensure funding for health programs is targeting the correct groups. Using your answer to part **a**, discuss the direction the government should be focusing their funding for asthma education.

4 The table below shows the results of an Australian survey of whether people were overweight or obese.

	Not overweight	Overweight	Obese
Males	94	89	53
Females	108	75	42

Analyse the table to identify and discuss the reasons for any differences between male and female obesity results.

5 WORKED EXAMPLE 11 The Australian Institute of Health and Welfare recorded the numbers of people in Australia with dementia over 2017–2023. The result are shown.

Year	2017	2018	2019	2020	2021	2022	2023
People (thousands)	345	356	367	375	386	397	409

a Use the data to predict the numbers of people with dementia in 2050.

b The institute predicts that the number of people living with dementia in Australia in 2050 will be approximately 772 000. Discuss reasons for any differences between your answer from part **a** and the institute's prediction.

6 The number of Year 12 students in a primary school with blue eyes, dark eyes, fair hair and dark hair are shown in the table.

	Blue eyes	Dark eyes
Fair hair	63	55
Dark hair	32	86

a Use a percentaged two-way table to analyse eye colour and the likelihood of a particular hair colour.

b Using your answer to part **a**, discuss the association of eye colour and hair colour and the reasonableness of the findings from the study.

Exam practice

7 (2 marks) Which of the following variables could not be response or explanatory variables? Explain your reasoning.

eye colour   hours worked   cost of a meal   size of a fridge   type of metal

8 (2 marks) Give two reasons why the use of least-square lines to extrapolate data is hazardous when making predictions.

9 (3 marks) The relationship between overnight temperature (n°C) and the maximum temperature the next day (M°C) in Stanthorpe is $M = 1.33n + 5.83$. Round answers to the nearest degree.

a Use the relationship to predict the maximum temperature when the overnight temperature was −2°C. *[1 mark]*

b Calculate the minimum temperature when the maximum temperature the next day was 23°C. *[1 mark]*

c Determine if the answers from parts **a** and **b** are reasonable. *[1 mark]*

10 (4 marks) CF The percentage of mobile phone users in Australia by age is shown in the table.

Age	20	25	30	35	40	45	50
Phone users	12.4	13.7	14.3	14.0	13.0	12.0	12.7

Phone companies are predicting the number of retirees using a mobile phone will be the lowest of all the surveyed age groups. Use the data to predict the number of users in the 75-year-old age group and discuss the reasonableness of your answer.

3.4

11 (4 marks) CF The Queensland fisheries catch (not including aquaculture) and the amount gambled in Queensland casinos and clubs from 2013 to 2023 have a fairly strong association, represented by the equation $G = -1.169C + 5.726$, with C measured in hundreds of thousands, G in millions of dollars and $R^2 = 0.6$. Discuss the reasonableness of the association and validity of predictions made from it.

12 (4 marks) CU The table below shows the number of marriages in Australia for 2011–2019 and the average Brisbane wedding venue hire cost per person for those years.

Marriages	121 754	123 243	118 959	121 197	113 595	118 401	112 954	119 186	113 815
Cost $ pp	96	94	112	105	140	142	153	180	185

ABS (Australian Bureau of Statistics) (2024) Marriages and Divorces Australia, ABS website, accessed 26 February, 2025

In 2021, the number of registered marriages in Australia was only 89 167. Use linear models to analyse the data and explain reasons for the results.

13 © QCAA 2024 2Q3 (5 marks) CF Table 1 shows the latitude, x, and ultraviolet index, y, for Australian locations at noon on the first day of autumn. Table 2 categorises the ultraviolet index.

Table 1		
Location	**Latitude (° S)**	**Ultraviolet index**
Brisbane	27	12
Darwin	12	13
Melbourne	38	6
Perth	32	11
Sydney	34	9

Table 2	
Ultraviolet index	**Category**
11+	extreme
8, 9, 10	very high
6, 7	high
3, 4, 5	moderate
1, 2	low

A person in Hobart (43° S 147° E) at noon on the first day of autumn receives a phone app notification that the ultraviolet index is high.

Use the equation for the least-squares line for the data in table 1 and the information in table 2 to evaluate the reasonableness of the phone app notification.

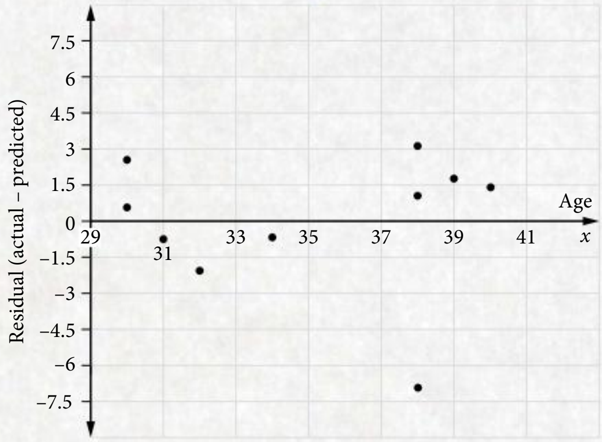
Video Exam question analysis: Associations and linear modelling

EXAM QUESTION ANALYSIS

©QCAA 2021 2Q5 (6 marks)

Researchers gathered a set of data to determine if a model could reliably predict systolic blood pressure given a person's age. One candidate was 31 years old and had a systolic blood pressure of 119.

For this data, the correlation coefficient (r) is 0.875, the standard deviation for the person's age (s_x) is 4, and the standard deviation for the systolic blood pressure (s_y) is 6. A residual plot was produced for the model.



Determine the actual systolic blood pressure, to the nearest whole number, for the oldest person in the sample (40 years old).

Reading the question

- The values for correlation coefficient (r), standard deviations (s_x) (s_y) and the actual value for a 31-year-old is given.
- The question is asking for an actual figure for a 40-year-old.
- x and y (actual) values are not directly given.
- The x values can be inferred from the residual graph.

Thinking about the question

- If working a question forward to get to a residual graph, the x and y (actual) data would be used to find the least-squares line, before calculating predictions and residuals.
- This question begins with residuals and needs to work backwards to find (y) actual values.
- There is enough information to find the gradient of the least-squares line.
- Some information is given for the 31-year-old, that can be substituted into the residual formula and the least-squares equation, to find the value of c (y-intercept).

Worked solution (✓ = 1 mark)

Find the predicted data for $x = 31$:

$y_a - y_p = -0.75$

$119 - y_p = -0.75$

$\therefore \mathbf{y_p = 119.75}$ ✓

Find m:

$m = r\dfrac{s_y}{s_x}$

$= 0.875 \times \dfrac{6}{4}$

$= \mathbf{1.3125}$ ✓

Find c:

$y = mx + c$

$119.75 = 1.3125 \times 31 + a$

$\mathbf{79.0625 = a}$ ✓

$\therefore$ the least-squares equation is $y = 1.3125x + 79.0625$.

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Calculate the actual (y_a) value for the oldest patient for $x = 40$:

$y_p = 1.3125 \times 40 + 79.0625$

$= \mathbf{131.5625}$ ✓

Use the residual = 1.4 to find y_a:

$y_a = 131.5625 + 1.4$

$= 132.9625$

State the answer with the systolic blood pressure rounded to a whole number.

The oldest person in the sample has a systolic blood pressure of 133. ✓

✓ ensure logical organisation and the communication of key steps is shown

3.4

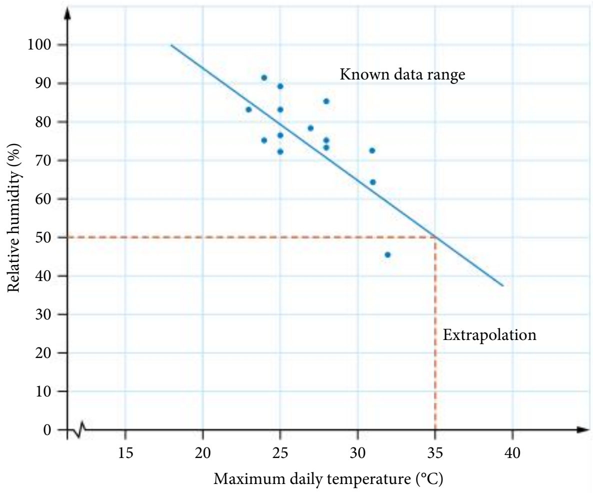
3 Chapter summary

Association and causation

- An **association** occurs between two variables, the explanatory variable (x value) and the response variable (y value). An association or measured correlation does not indicate a causal relationship between the two variables.
- **Causation** occurs when the explanatory variable causes a change in the response variable.
- **Confounding variables** are a third variable that can cause a change in variables and affect the association.
- A **coincidence** may also be the reason for 2 unrelated variables having an association.

Least-squares line

- To find a least-squares model, the equation of the linear **least-squares line** is $y = mx + c$.
- The **slope** (gradient) is calculated by $m = r\frac{s_y}{s_x}$ and the **y-intercept** by $c = \bar{y} - m\bar{x}$.
- Calculate m first, as it is needed for the calculation of the y-intercept (c).
- $\bar{x}$ and $\bar{y}$ are the means of the x and y data values.
- r is Pearson's correlation coefficient.
- s_x and s_y are the sample **standard deviations** of the x and y data values.
- The least-squares line may be expressed as $y = a + bx$, where b represents the gradient and a represents the y-intercept. Calculators use this form.

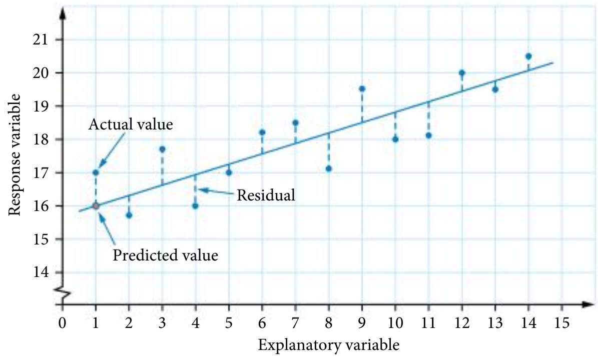


Interpolation and extrapolation

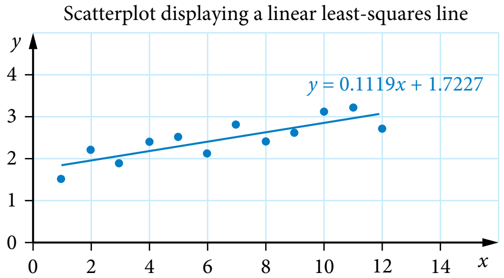
- **Predictions** made within the known data range is called **interpolation**.
- Predictions made outside of the known data range is called **extrapolation**.
- Making predictions using extrapolation is not as reliable as interpolated predictions. Variables such as time, location, societal changes, and/or weather patterns could be reasons why extrapolated predictions may not be accurate in future time periods.

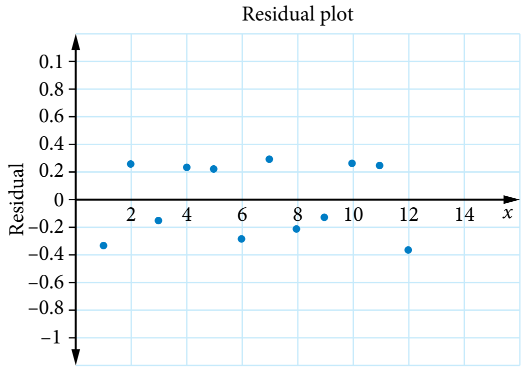
Residuals

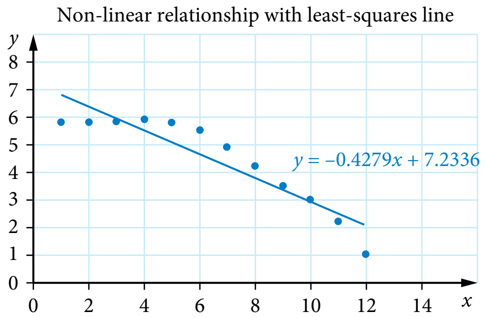
- A **residual** is the difference between a data point and the value predicted by the line of best fit (least-squares line).
- A plot of the residuals against the explanatory variable is called a **residual plot**. Residual value = actual value (y_a) – predicted value (y_p)

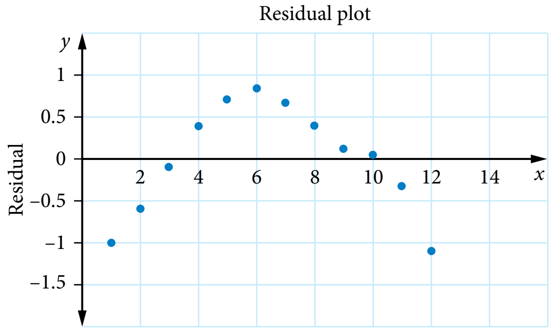


Determining linear modelling from residual plots









- A residual plot with points scattered close to both sides of the axis shows that the association between the variables is most likely linear.
- A residual plot with points in a curved line or points in a line on one side of the axis shows that the association between the variables is non-linear.
- A residual plot with points diverging from the axis in one direction shows the association between the variables is non-linear.
- A residual plot with one or several **outliers** indicates that there are outliers in the original data. If they are valid points, the association between the variables may not be linear.

Worksheet
General Maths formula sheet

Cumulative examination 1

Simple familiar

Perusal time: 2 minutes Working time: 30 minutes

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

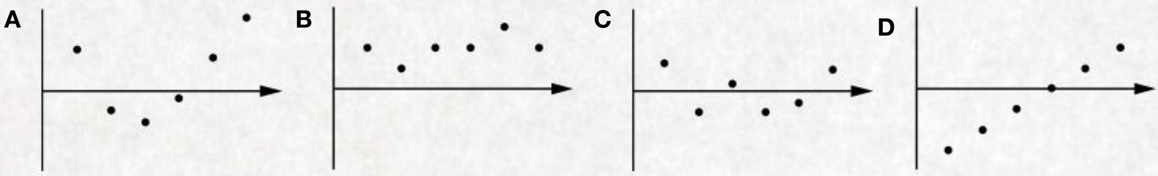
Section 1 (5 marks): 5 multiple choice questions

Section 2 (13 marks): 4 short response questions

Total: 18 marks

Section 1 5 multiple choice questions 5 marks

1 © QCAA 2020 1Q1 Four linear models have been developed for a data set. Identify the residual plot that indicates that the developed linear model is justified.

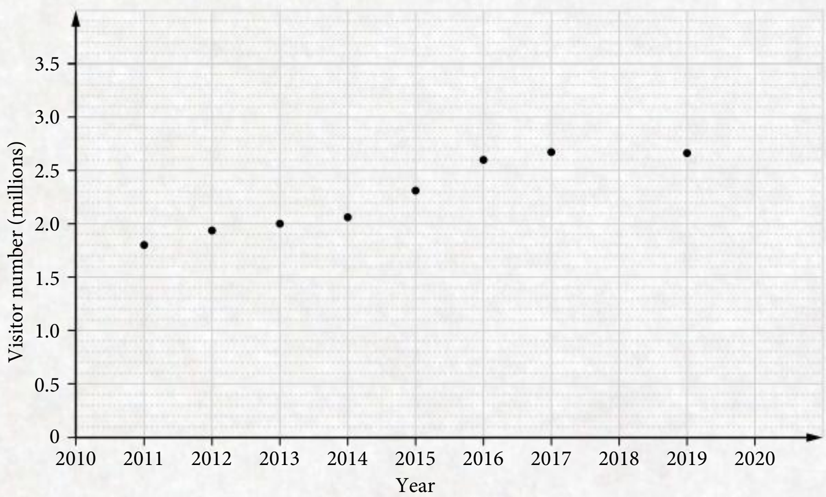


2 © QCAA 2022 1Q11 The equation of a fitted line for the number of free throws in basketball, t, and the number of hours in a training session, h, is $t = 26.781 + 12.974h$.

The predicted number of free throws for a 5-hour training session, when rounded to the nearest whole number, is

A 64 **B** 65 **C** 91 **D** 92

3 © QCAA 2022 1Q8 The scatterplot shows the annual number of visitors to the Great Barrier Reef Marine Park.



For 2018, the annual number of visitors could best be

A interpolated as 2.7 million.
B extrapolated as 2.7 million.
C interpolated as 3.2 million.
D extrapolated as 3.2 million.

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4 © QCAA 2023 1Q9 Determine the 6th term of the arithmetic sequence that begins 3, 9, …

A 21 **B** 33 **C** 45 **D** 729

5 © QCAA 2020S 1Q5 Bivariate data involves

A two values.

B two variables.

C two data displays.

D two data varieties.

Section 2 4 short response questions 13 marks

6 © QCAA 2020 1Q18 (4 marks) Exhibition organisers believe that the number of attendees increases each day as an arithmetic sequence. The organisers know that 353 people attended the first day and 439 people attended the third day.

a Determine the common difference. *[2 marks]*

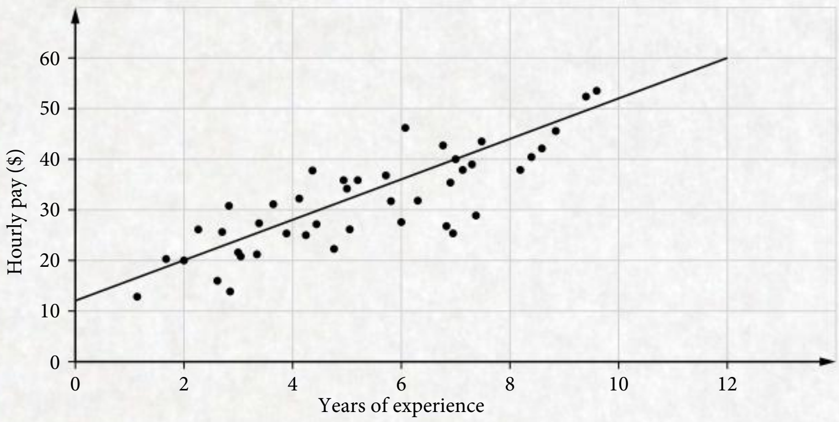
b Use the result from part **a** to predict the number of people who will attend on the sixth day. *[2 marks]*

7 © QCAA 2020 1Q19 (4 marks) Data was collected relating the number of hours spent fishing and the total number of fish caught. The linear model for this data was found to be $y = 2.3x + 31.4$, where x is the number of hours spent fishing, and y is the total number of fish caught.

a Use the model to predict the number of fish caught if 12 hours were spent fishing. *[1 mark]*

b The correlation coefficient for this data is 0.688 and the coefficient of determination is 0.473. Use each of these to describe the strength of the linear association between the two variables and decide if your prediction is valid. *[3 marks]*

8 © QCAA 2022 1Q23 MODIFIED (2 marks) The least-squares line has been provided for a scatterplot that shows the association between an employee's years of experience, n, and their hourly pay, p.



Given that the least-squares line passes directly through the points $(2, 20)$ and $(7, 40)$, determine its equation.

9 © QCAA 2020S 1Q27 (3 marks) Use the recursive rule $t_{n+1} = 0.65t_n$, $t_1 = 120$ to complete the following.

a Calculate the percentage by which each term decreases. *[1 mark]*

b Calculate t_4. *[2 marks]*

CUMULATIVE EXAMINATION 1

Worksheet
General Maths formula sheet

Cumulative examination 2

Complex familiar and unfamiliar

Perusal time: 2 minutes Working time: 30 minutes

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

2 short response questions — 11 marks

1 © QCAA 2020 2Q3 (6 marks) The least-squares line for a sample of five data points was found to be $y = 2.1875x + 0.0625$, with a correlation coefficient of $r = 0.875$.

Determine a set of values for p and q, given that these values differ by 3.

x	4	3	8	4	6
y	p	4	16	8	q

2 © QCAA 2022 2Q3 (5 marks) In a company's first 10 years of operation, the average annual profit ($\overline{y}$) was \$9660 with a standard deviation (s_y) of \$3010. Fitting a least-squares line to the data comparing annual profit (y) to the year of operation (x) produced a correlation coefficient of 0.9987.

Show that the predicted profit, to the nearest dollar, for this company in the 11th year of operation will be \$15 121.

CHAPTER

4 TIME SERIES ANALYSIS

Prior learning
Time series analysis

Syllabus coverage

UNIT 3, TOPIC 3: TIME SERIES ANALYSIS

Describing and interpreting patterns in time series data

- Construct and use time series plots.
- Describe time series plots by identifying features, including trend (long-term direction, e.g. increasing/decreasing), seasonality (systematic, calendar-related movements) and irregular fluctuations (unsystematic, short-term fluctuations).

Analysing time series data

- Smooth time series data by calculating a simple moving average using the mean or median for an odd number of data, including the use of spreadsheets.
- Deseasonalise a time series by calculating the seasonal indices using the average percentage method, including the use of spreadsheets.
- Fit a least-squares line to model long-term trends in time series data.
- Solve practical problems that involve the analysis of time series data.

General Mathematics 2025 v1.2 General senior syllabus p. 24,

Videos (2):
4.3 Seasonal indices
Exam question analysis Time series analysis

Prior learning (1):
4.1 Time series analysis

Worksheets (7):
4.1 Time series plots
4.2 Smoothing time series data • Moving means • Moving medians
4.3 Deseasonalisation • Seasonality
Cumulative exams General Maths formula sheet

Nelson MindTap

To access resources above, visit
cengage.com.au/nelsonmindtap

Terminology

cyclical	deseasonalisation	irregular	mean
median	moving average	quarter	reseasonalisation
seasonal	seasonal index	smoothing	time series
time series plot	trend		

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4.1 Time series plots

Many businesses record daily, weekly and monthly sales and expenses so they can plan for the future. Information that is recorded every day, every month or at any regular period is called a **time series**. Most of the information collected by the Australian Bureau of Statistics (ABS) is time series information. This is also true of the weather and climate data collected by the Bureau of Meteorology (BOM).

Time series

- A time series is a set of data collected at regular periods of time.
- A **time series plot** is a graph of a time series with time on the horizontal axis.
- Time series graphs are used to detect patterns and trends in the data.

Worksheet Time series plots

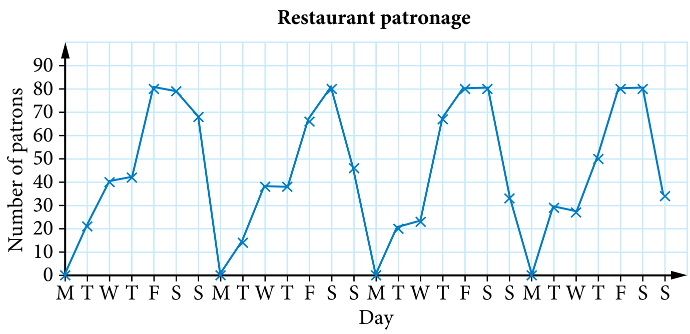
WORKED EXAMPLE 1 Plotting time series graphs

This table shows the number of patrons at a restaurant every day over 4 weeks.

	Mon	Tue	Wed	Thu	Fri	Sat	Sun
Week 1	0	21	40	42	80	79	68
Week 2	0	14	38	38	66	80	46
Week 3	0	20	23	67	80	80	33
Week 4	0	29	27	50	80	80	34

a **Construct** a time series plot from the data.

b **Describe** features observed in the data.

Steps	Working
a **1** Place the days of the week on the x-axis and create a scale for the number of patrons on the y-axis. **2** Plot the number of customers each day for the four weeks in order.	
b Write observations from the time series.	The data shows that the restaurant did not have more than 80 patrons on any given day and there were no patrons on Mondays. It could have a maximum capacity of 80 and be shut on Mondays. The highest trade days are Fridays and Saturdays.

Exam hack

When drawing a time series plot, make the time axis appropriate for the data.

WORKED EXAMPLE 2 Plotting time series graphs using technology

The blood pressure of a middle-aged man over a number of years is shown below. The first number is the systolic pressure and the second is the diastolic pressure (both measured in mm Hg). Systolic pressure is the maximum pressure generated by the heart and diastolic pressure indicates the pressure in the arteries when the heart rests between beats.

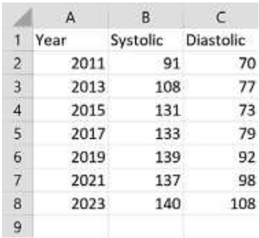
Year	2011	2013	2015	2017	2019	2021	2023
BP	91/70	108/77	131/73	133/79	139/92	137/98	140/108

a Use technology to plot the data to create a time series plot.

b Draw conclusions from the data.

Steps | **Working**

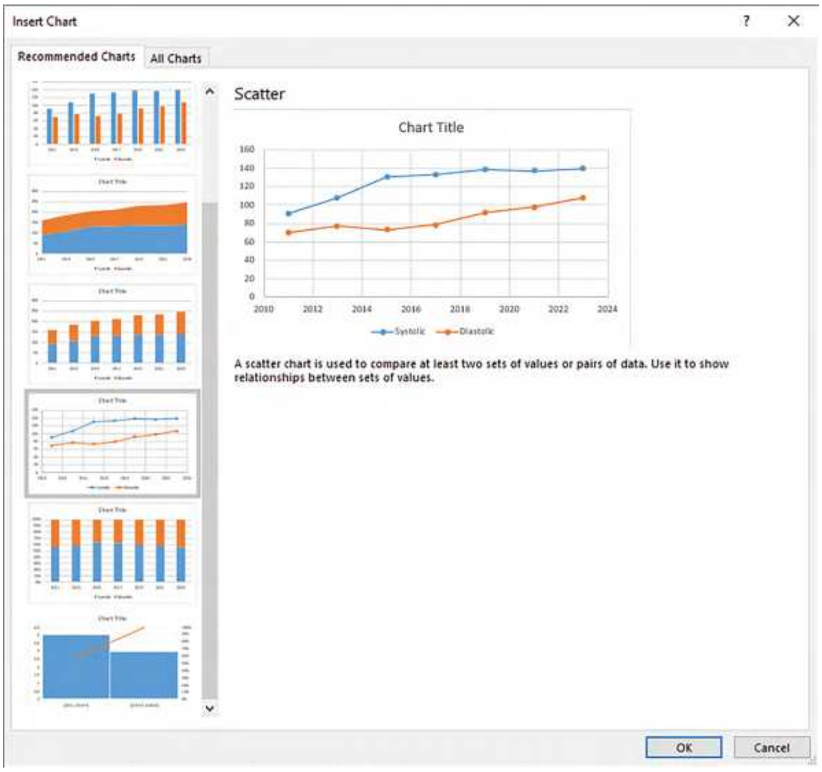
a 1 Open a new spreadsheet and type up a table showing the data split into the two readings.



	A	B	C
1	Year	Systolic	Diastolic
2	2011	91	70
3	2013	108	77
4	2015	131	73
5	2017	133	79
6	2019	139	92
7	2021	137	98
8	2023	140	108
9			

2 Highlight the 3 columns of data and select Insert → Recommended Charts → Scatter.

Ensure it is the version showing connected dots.



3 Label the axes.

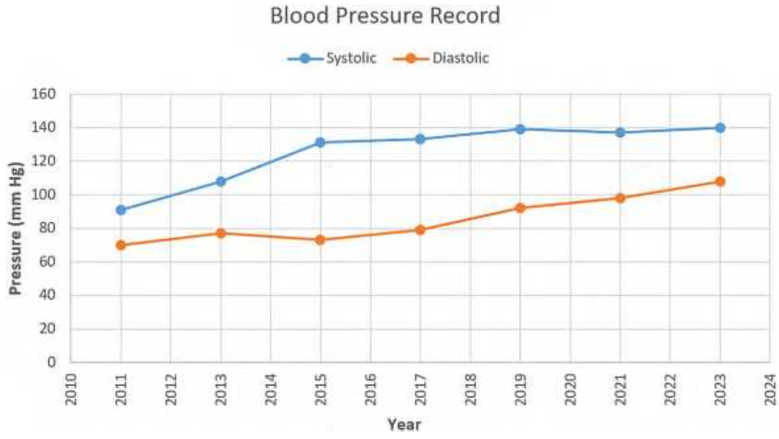
Label the x-axis 'Year'.

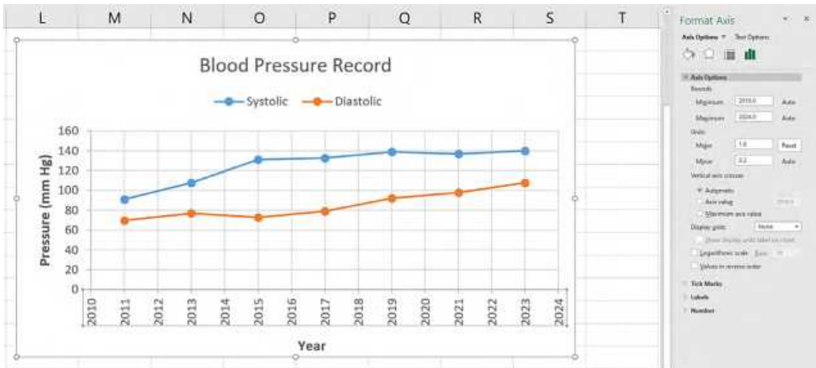
Changing the direction of the years to vertical will create space. This can be done by right-clicking on the x-axis numbers and select 'Format Axis' to change parameters.

Label the y-axis 'Pressure (mm Hg)'.

Interval limits can be changed using the 'Format Axis' function.

4 A legend may be used to identify each line.





Exam hack

Consider the intervals used for the y-axis to best display the data. Do not limit the highest values to numbers in the given data.

b Write your conclusions from the data.

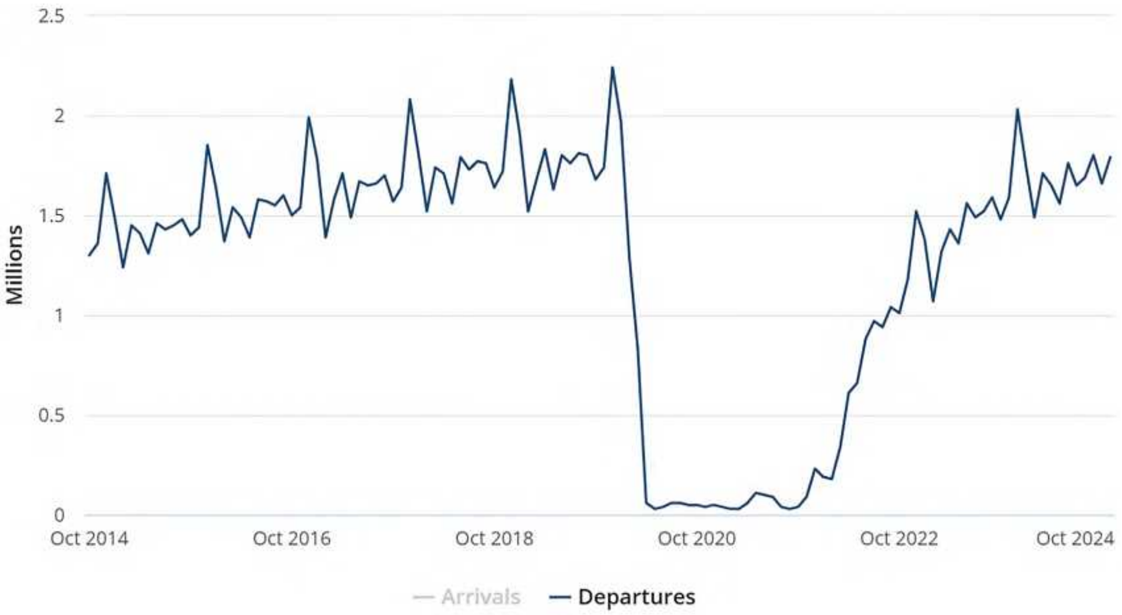
There has been a gradual increase in blood pressure over the 13 years of recording.

Time series patterns

- The 3 basic patterns are trend, seasonal and irregular variation.
- A **trend** shows a gradual increase or decrease in the data over time. This is usually seen in a time series graph over several months or years. It could be linear (i.e. a straight line) or non-linear.
- A **seasonal** variation rises and falls in the same way each year. The changes through the year are often related to the seasons. Seasonality includes systematic cycles, usually related to seasons, months or financial **quarters**.
- An **irregular** variation that does not follow a pattern.
- **Cyclical** variation has rises and falls over a varying period (cycle). The cycle is usually more than a year.
- An **outlier** is an observation that is dramatically different from the rest of the time series.

Real time series usually combine several patterns, such as seasonal and trend variations. Even with a strong trend, seasonal or cyclic variation will have irregularities. Consider the following time series graph about people departing Australia.

Total overseas arrivals and departures – October 2014 to October 2024



Source: Australian Bureau of Statistics (2024), Overseas Arrivals and Departures, Australia, ABS website, accessed 31 December 2024

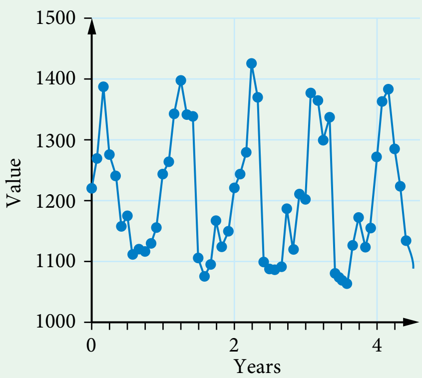
The following can be observed from this graph:

- From 2014 to 2019, a seasonal pattern showing an increasing trend was evident.
- In 2020, a sudden drop in the data highlights the onset of COVID-19 and lockdowns.
- From 2021, there is an upwards trend in the data as borders re-opened and overseas travel became more accessible.
- From 2023, the original seasonal cycle seems to be returning.

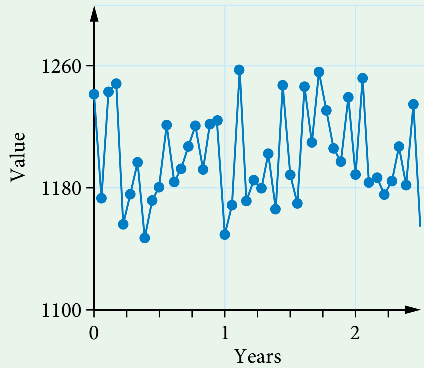
WORKED EXAMPLE 3 Identify time series patterns

Describe each time series.

a

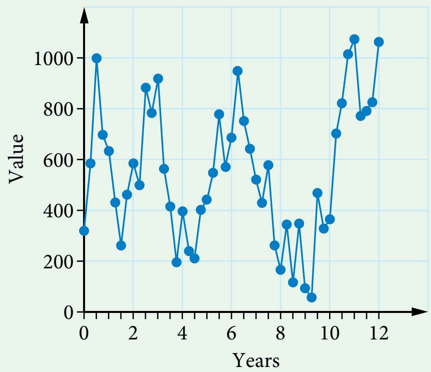


b



c

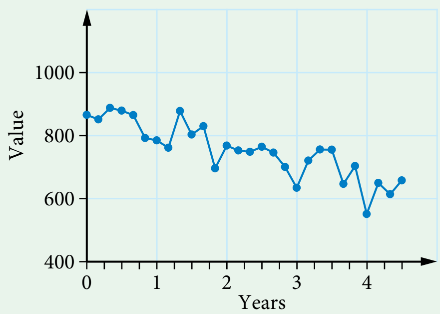
d



Steps	Working
a Each year has the same pattern of variation.	There is seasonal variation.
b There is no pattern in the variation.	There is irregular variation.
c The values gradually increase.	There is an upward trend.
d There are peaks and troughs over the years.	There is a cyclical pattern.

WORKED EXAMPLE 4 Interpreting a time series plot

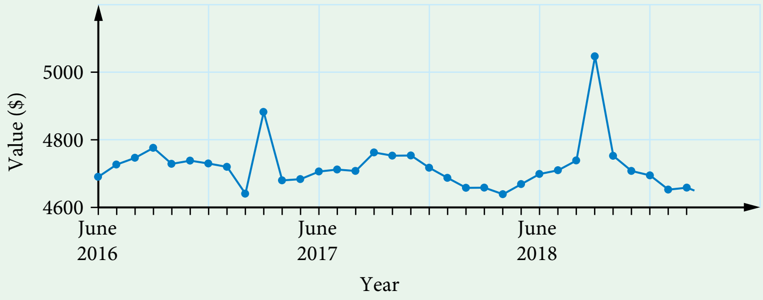
Describe the features of this time series.



Steps	Working
1 Look at the general direction of the graph.	There is a downward trend.
2 Look at the pattern in each year.	There is an irregular cyclical variation.
3 Write the conclusion.	The time series has an irregular cyclical variation with an overall downward trend.

WORKED EXAMPLE 5 Interpreting plots with outliers

Identify any outliers in this time series.



Steps	Working
Identify the patterns and features in the data.	There is a seasonal variation across each year, but there are 2 extraordinary data points. There are outliers in March 2017 and September 2018.

Exam hack

Unless asked in the question, do not assume or guess what caused a fluctuation, just identify it.

EXERCISE 4.1 Time series plots

ANSWERS p. 447

Mastery

1 WORKED EXAMPLE 1 The average maximum and minimum temperatures in °C for each month in 2024 in Mackay are shown.

Month	J	F	M	A	M	J	J	A	S	O	N	D
Maximum	32.6	30.9	29.7	27.5	24.5	22.7	21.8	24.1	26.5	28.8	30.9	31.1
Minimum	25.7	24.3	24.3	21.0	18.6	13.7	13.9	16.1	18.7	20.9	23.1	23.9

Data sourced from Bureau of Meteorology (2024), BOM website, accessed 1 January 2025

a Construct a time series plot from the data.

b Describe what the data in the graph shows.

2 The average incomes of wage and salary earners of 2 places in Queensland over 6 years is shown.

Year	2005	2006	2007	2008	2009	2010
Longreach	33 559	35 191	37 373	40 255	40 831	43 245
Mackay	43 659	46 522	49 005	53 109	55 320	60 163

a Construct a time series plot from the data.

b Describe what the data in the graph shows.

3 WORKED EXAMPLE 2 The annual number of births in Queensland over the period 2000–2023 is shown.

Year	2000	2001	2002	2003	2004	2005	2006	2007
Births	47 278	47 678	47 771	48 342	49 940	51 707	52 695	61 306
Year	2008	2009	2010	2011	2012	2013	2014	2015
Births	63 168	66 149	64 523	63 253	63 837	63 354	63 066	61 745
Year	2016	2017	2018	2019	2020	2021	2022	2023
Births	61 841	61 185	61 913	61 735	59 490	64 111	62 094	58 458

Data: Queensland Government Statistician's Office, Queensland Treasury, QGSO website, accessed 21 January 2025

a Use technology to construct a time series plot.

b Draw conclusions about the data in this graph.

Exam hack

When manually plotting a graph, not all numbers can be plotted at exactly the right points. Be careful with your rounding to best position the *y* value.

4 A courier driver has kept records of weekly income over 6 months (26 weeks).

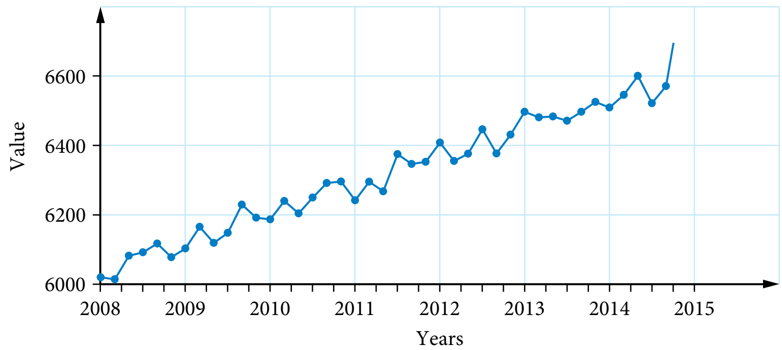
Week	1	2	3	4	5	6	7	8	9
Income $	690	644	552	529	782	1012	1104	989	736
Week	10	11	12	13	14	15	16	17	18
Income $	667	46	345	460	782	805	805	782	644
Week	19	20	21	22	23	24	25	26	
Income $	621	874	920	966	828	713	621	690	

a Use the data to construct (using technology or manual method) a time series plot.

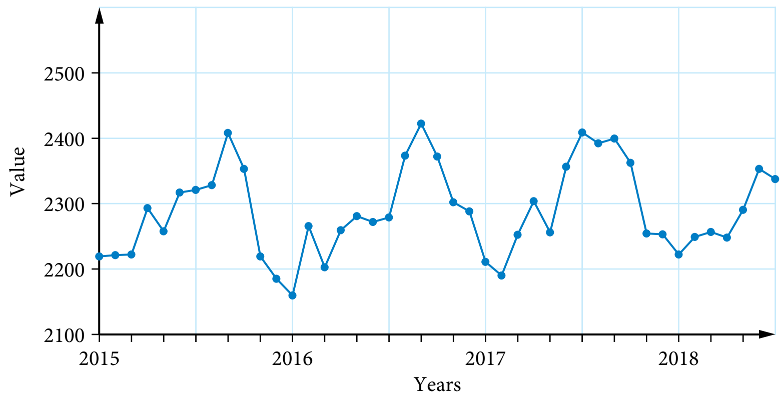
b Draw conclusions about the data in this graph.

5 WORKED EXAMPLE 3 Describe the patterns and features of each time series graph.

a



b



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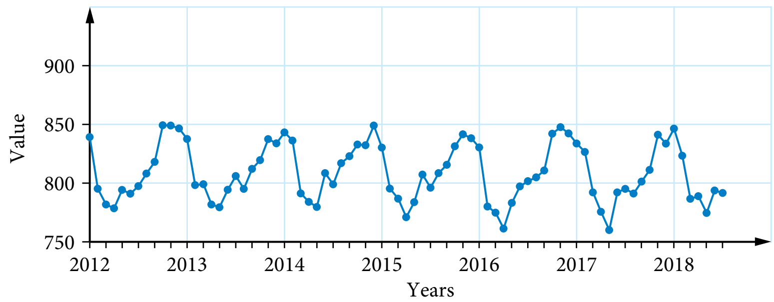
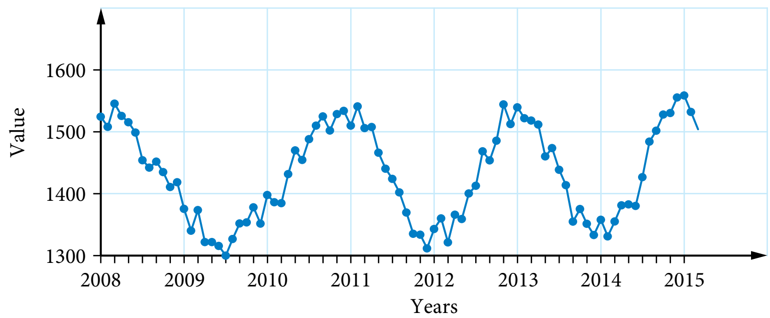
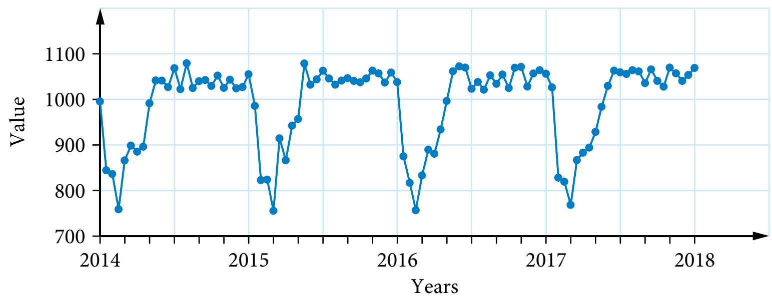


e



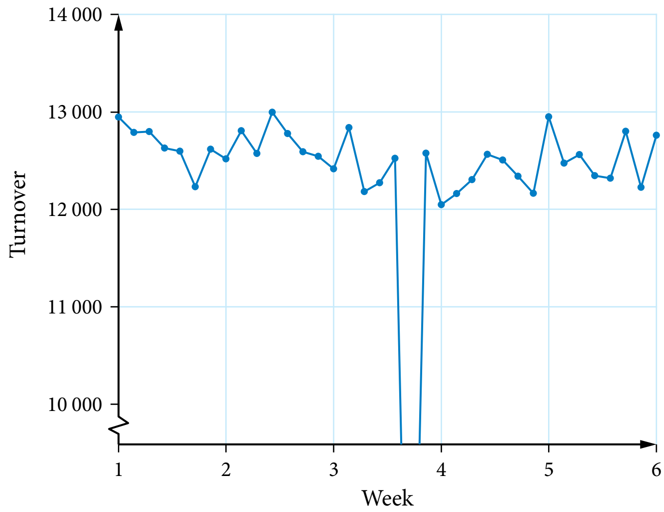
f



6 WORKED EXAMPLE 4 Identify any patterns in this time series that shows the numbers of victims of motor vehicle theft in Queensland over a 30-year period.

Source: Australian Bureau of Statistics (2023), Recorded Crime – Victims 1993–2023, ABS website, accessed 31 December 2024

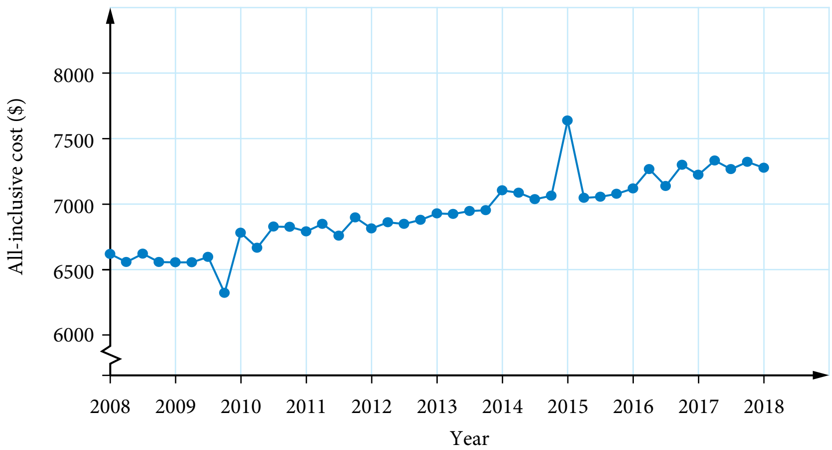
7 WORKED EXAMPLE 5 The time series graph shows daily takings of a petrol station over a 5-week period. Each week starts on Monday.



a Describe the patterns displayed in the time series plot.

b State a possible reason for the change in Week 3–4.

8 Identify any patterns or features in this time series of the average price of all-inclusive 7-day packaged holidays in Asia. The data was collected quarterly, starting in March 2008.



Exam practice

9 The height and weight of a baby is measured each month for the first 3 years of her life. The most appropriate graph to show her growth is a

A histogram

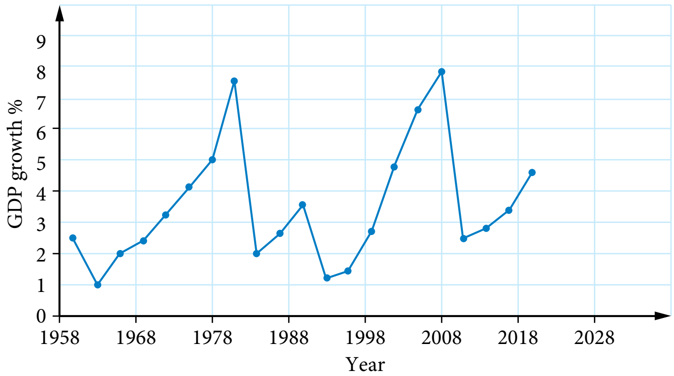
B dot plot

C time series plot

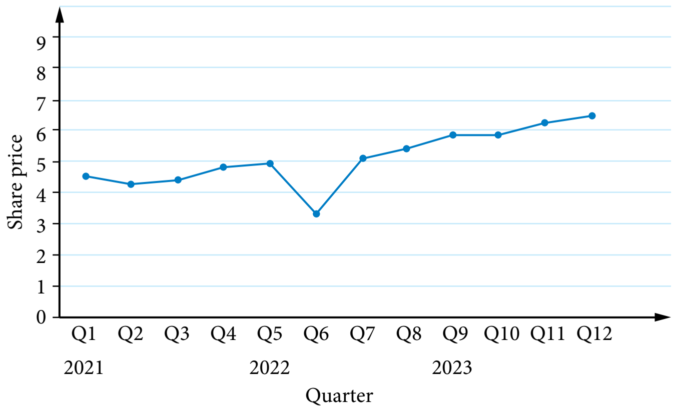
D stem-and-leaf plot

10 Identify the time series plot that best displays a seasonal pattern.

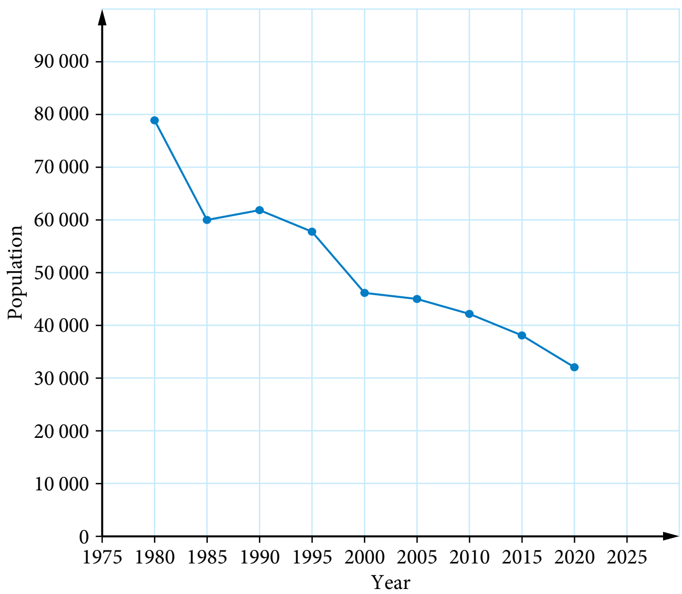
A



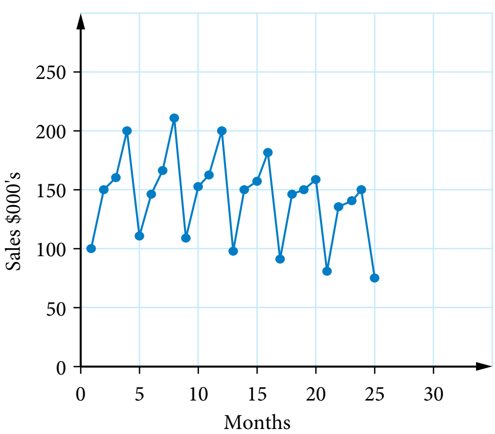
B



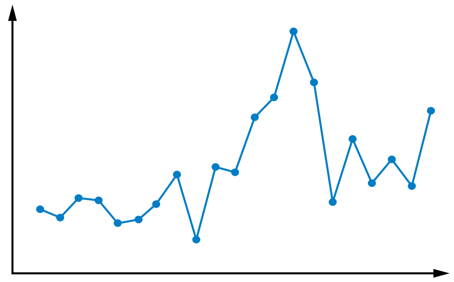
C



D



11



The pattern in this time series plot is best described as

A irregular fluctuations with an increasing trend

B seasonal with an increasing trend

C seasonal with a decreasing trend

D cyclical with an increasing trend

12 (4 marks) Delia checked her pulse every minute during and after exercising for 8 minutes. The results are shown below in beats per minute.

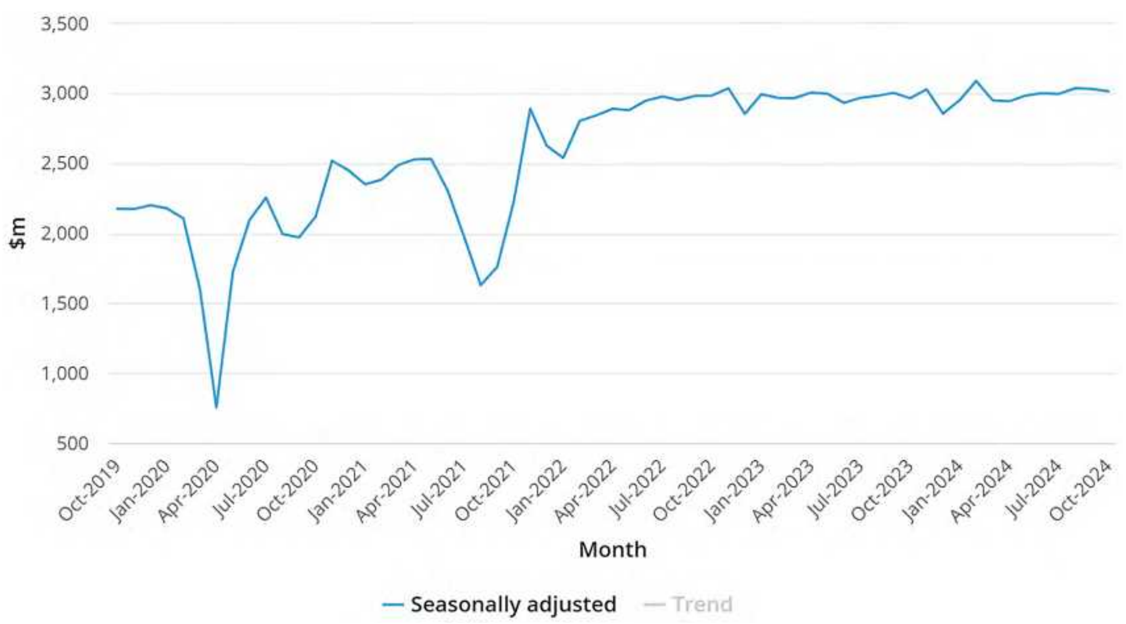
Time	0	1	2	3	4	5	6	7	8	9	10	11	12	13
Pulse	60	70	78	90	110	120	134	135	136	118	80	70	65	63

a Construct a time series plot. *[2 marks]*

b Describe the time series plot. *[2 marks]*

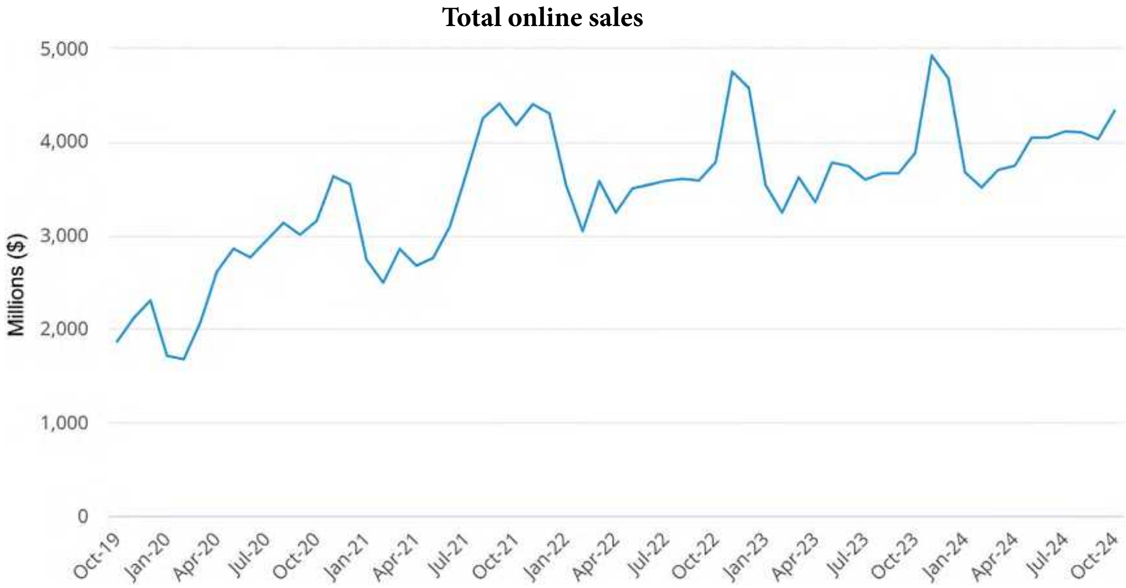
13 (2 marks) This time series plot shows the total sales of clothing and accessories in Australia by month from 2019 to 2024. Describe two features of the time series plot.

Clothing, footwear and personal accessory retailing



Source: Australian Bureau of Statistics (2024), Retail Trade, Australia, ABS website, accessed 31 December 2024

14 (3 marks) Describe this time series plot by identifying key features of the total online sales in Australia from October 2019 to October 2024.



Source: Australian Bureau of Statistics (2024), Retail Trade, Australia, ABS website, accessed on 31 December 2024

Worksheets
Smoothing time series data
Moving means
Moving medians

4.2 Smoothing time series data

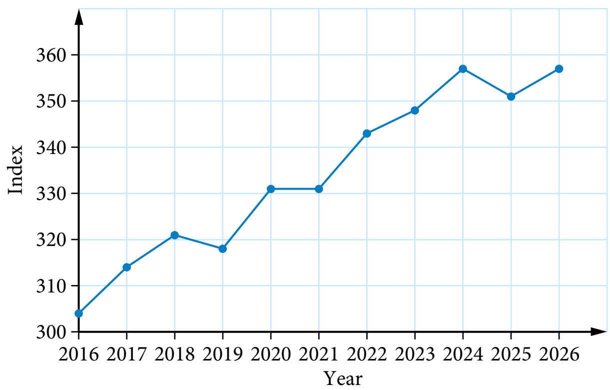
Smoothing means removing variations from time series data to reveal trends.

The **moving average** of odd groups of data is the simplest method.

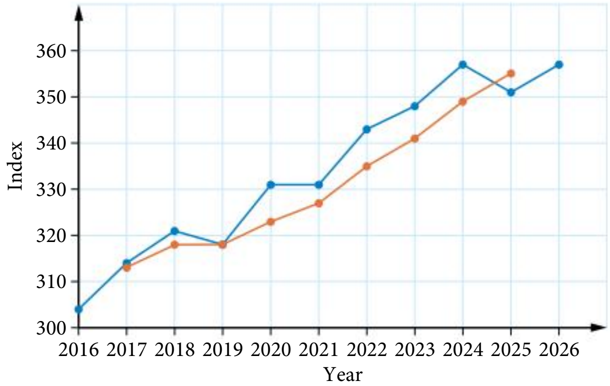
Use a series of **means** or **medians** of an *odd* number of data points.

Consider the data shown in the table and graph.

Year	2016	2017	2018	2019	2020	2021	2022	2023	2024	2025	2026
Index	304	314	321	318	331	331	343	348	357	351	357



It is obvious that the index value is gradually increasing. If the mean (average) of 3 data values at a time are calculated and graphed, the result is the red line below.



The red line is much smoother than the original data.

Moving averages using means

A **3-point moving average** calculates averages using 3 values. You can also smooth data using **5-point** and **7-point moving averages**. When smoothing data, the number of points is reduced.

WORKED EXAMPLE 6 Using means to find moving averages

The table displays the monthly bookings for a restaurant over 9 months.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep
Bookings	240	185	165	270	170	160	190	205	265

Use means to **calculate**

a the 3-point moving average for April

b the 5-point moving average for June

c the 7-point moving average for May.

Steps	Working
a Average the 3 values where April is the middle value. ie. Mar, Apr, May	$\frac{165+270+170}{3} = 201.67$
b Average the 5 values where June is the middle value. ie. Apr, May, Jun, Jul, Aug	$\frac{270+170+160+190+205}{5} = 199$
c Average the 7 values where May is the middle value. ie. Feb, Mar, Apr, May, Jun, Jul, Aug	$\frac{185+165+270+170+160+190+205}{7} = 192.14$

WORKED EXAMPLE 7 Smoothing using 3-point moving mean

a **Use** means to **calculate** a 3-point moving average for the data below. (Round to one decimal place.)

Year	2016	2017	2018	2019	2020	2021	2022
Index	331	331	343	348	357	351	357

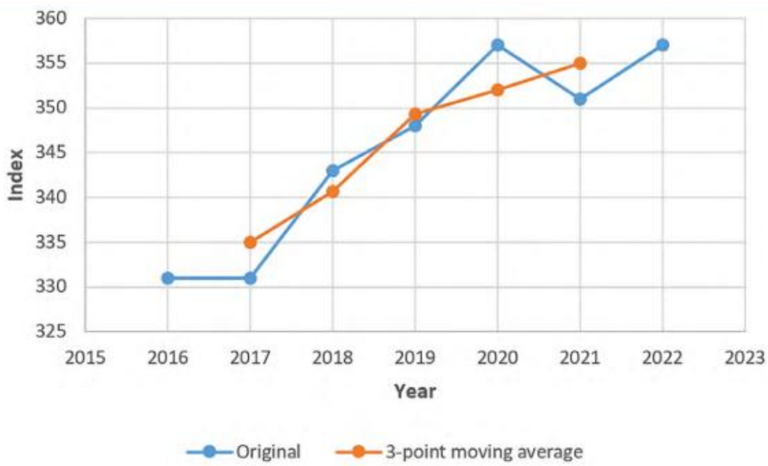
b **Construct** a graph showing the original and smoothed data.

c **Compare** the lines.

Steps	Working
a **1** Set up a table to show the variable headings as well as the calculations for the moving average. **2** Work out the average for each group of 3, moving down one value to begin the second set of 3. Complete the table.	(see table below)

Year	Index	3-point moving average
2016	331	
2017	331	$\frac{331+331+343}{3} = 335$
2018	343	$\frac{331+343+348}{3} = 340.7$
2019	348	$\frac{343+348+357}{3} = 349.3$
2020	357	$\frac{348+357+351}{3} = 352$
2021	351	$\frac{357+351+357}{3} = 355$
2022	357	

b Construct a time series graph, showing both the original and smoothed data.



c Compare the lines.

The smoothed data shows the trend much more clearly than the original information. The fluctuation in 2020 has been smoothed.

WORKED EXAMPLE 8 Smoothing using 5-point moving average with a spreadsheet

a **Use** means to **calculate** a 5-point moving average. (Round to one decimal place.)

Year	2016	2017	2018	2019	2020	2021	2022
Index	331	331	343	348	357	351	357

b **Use** the spreadsheet graphing tool to **construct** a time series plot showing the original and smoothed data.

c **Compare** the lines.

Steps	Working

a Set up a table to show the variable headings as well as the calculations for the moving average.

Work out the average for each group of 5, moving down one value to begin the second set of 5.

Complete the table.

Year	Index	5-point moving average
2016	331	
2017	331	
2018	343	$\frac{331+331+343+348+357}{5}=342$
2019	348	$\frac{331+343+348+357+351}{5}=346$
2020	357	$\frac{343+348+357+351+357}{5}=351.2$
2021	351	
2022	357	

b **1** Use the same process as the manual version but find the mean using the formula.

Position each mean to line up with the centre value in the set of 5.

Spreadsheet formulas

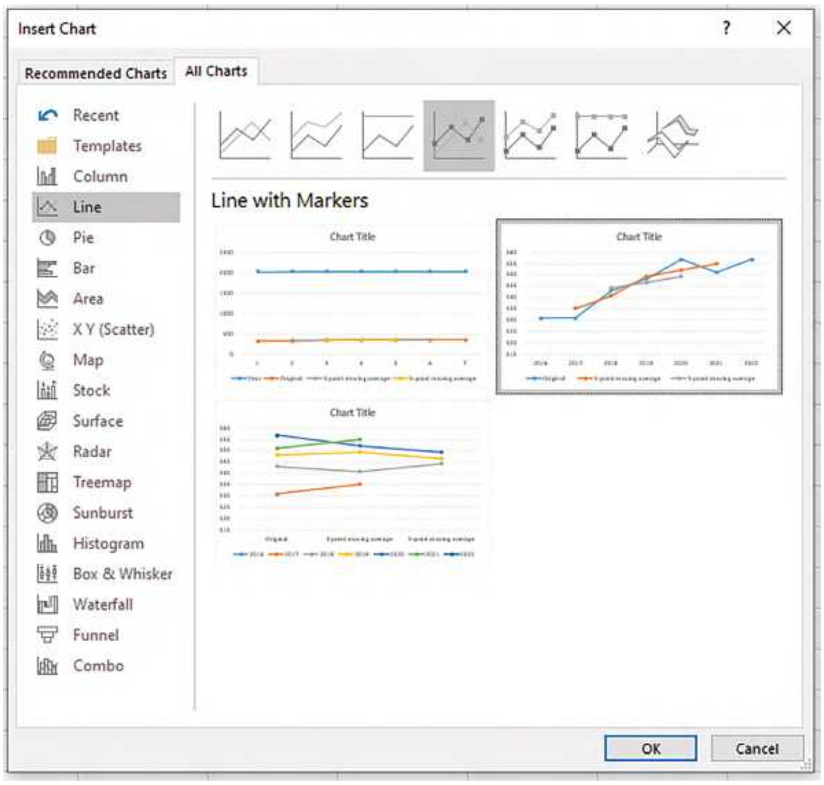
	A	B	D
1	Year	Original Index	5-point moving average
2	2016	331	
3	2017	331	
4	2018	343	=AVERAGE(B2:B6)
5	2019	348	=AVERAGE(B3:B7)
6	2020	357	=AVERAGE(B4:B8)
7	2021	351	
8	2022	357	
9			

Spreadsheet values

	A	B	D
1	Year	Original index	5-point moving average
2	2016	331	
3	2017	331	
4	2018	343	342
5	2019	348	346
6	2020	357	351.2
7	2021	351	
8	2022	357	

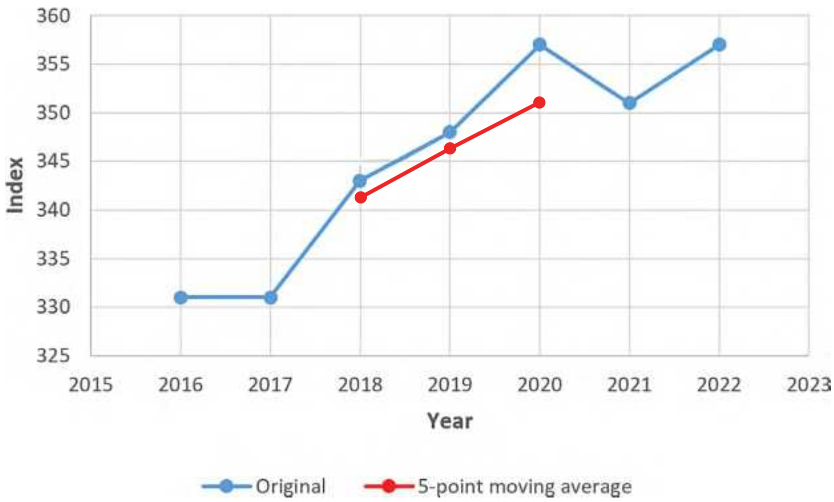
2 Start by highlighting the full table.

Select the Insert tab → Recommended charts → All Charts → Line.



© Microsoft Corporation

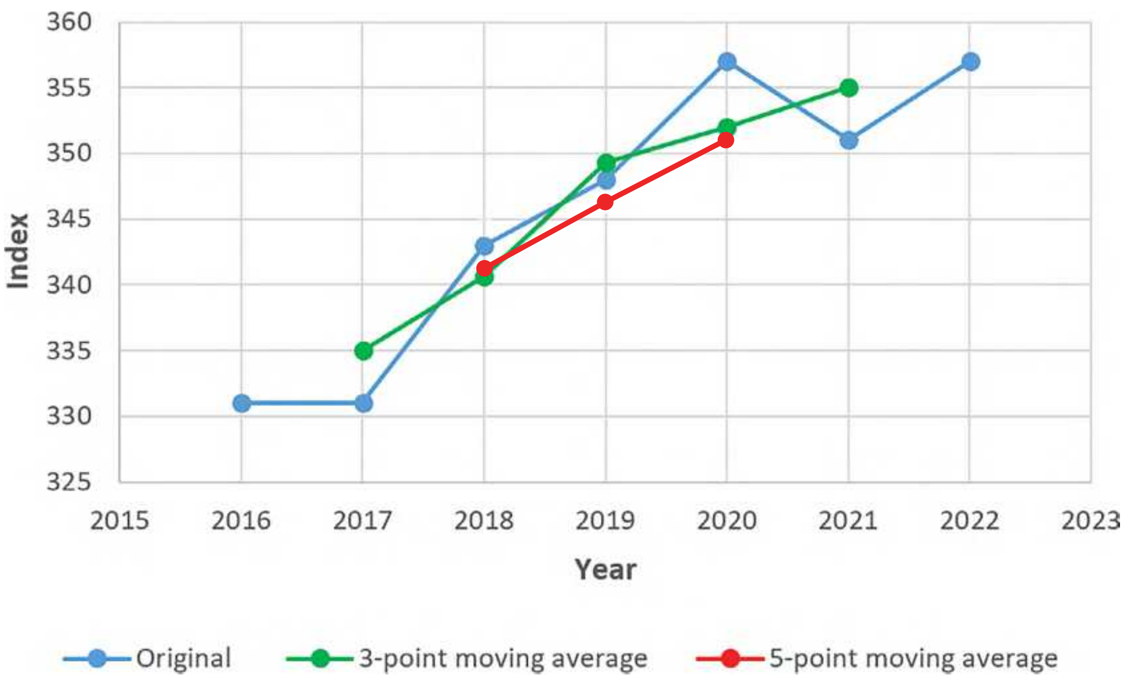
3 Once selected, add on axis titles, a legend and a title if the question calls for it.



c Compare the lines. Both lines continue to show an upwards trend. Whilst the 5-point moving average line is straighter, it has too few data points to be useful and to accurately draw conclusions from.

4.2

In the previous example, if both moving averages are plotted together on a graph, it is easier to see that the 5-point moving average smooths the data too much. While the trend looks more linear, there are not enough data points to reasonably interpolate the data.



Moving averages using medians

Another averaging smoothing technique is to use the **median** value rather than the mean. This method is more appropriate for smaller sets of data with fluctuations. Odd-point data groups are used for this method.

WORKED EXAMPLE 9 Smoothing using 3-point moving average using medians

The data below displays the number of cycling accidents recorded in a regional city over a 10-year period.

a **Calculate** a 3-point moving average using medians for the data below.

Year	2013	2014	2015	2016	2017	2018	2019	2020	2021	2022
Height (cm)	150	45	71	110	160	130	180	280	200	300

b **Construct** a graph showing the original and smoothed data.

Steps

a **1** Draw up a table to show the data with headings for numerical order and 3-point moving median.

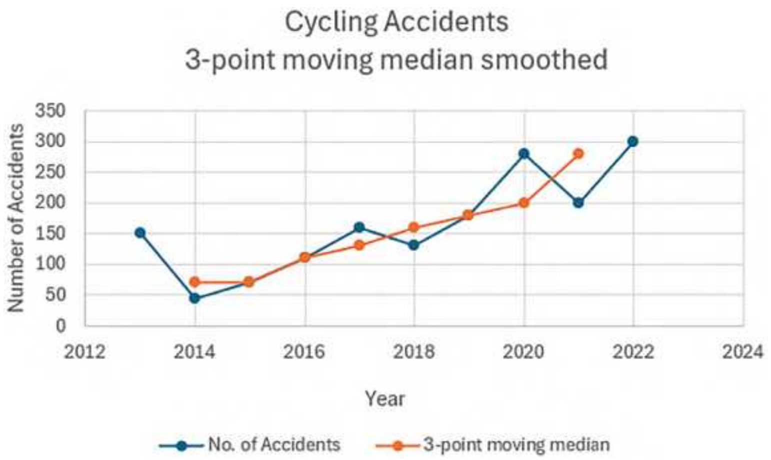
2 For each group of 3 data points, list the values in numerical order.

3 Identify the median (middle number) and place it in the final column.

Working

Year	Number of accidents	Numerical order	3-point moving median
2013	150		
2014	45	45, 71, 150	71
2015	71	45, 71, 110	71
2016	110	71, 110, 160	110
2017	160	110, 130, 160	130
2018	130	130, 160, 180	160
2019	180	130, 180, 280	180
2020	280	180, 200, 280	200
2021	200	200, 280, 300	280
2022	300		

b Construct a time series graph, displaying both the original and smoothed data.



WORKED EXAMPLE 10 Smoothing using 5-point moving median with a spreadsheet

The data below displays the number of cycling accidents recorded in a regional city over a 10-year period.

a **Calculate** a 5-point moving median for the same set of cycling data below.

Year	2013	2014	2015	2016	2017	2018	2019	2020	2021	2022
Heigh (cm)	150	45	71	110	160	130	180	280	200	300

b **Use** a spreadsheet to **construct** a graph of the original and smoothed data.

c **Compare** the lines.

Steps

a **1** Set up a table to show the variable headings as well as the calculations for the moving median.

2 For each group of 5 data points, list the values in numerical order.

3 Identify the median (middle number) and place it in the final column.

Working

Year	No. of Accidents	Numerical Order	5-point moving median
2013	150		
2014	45		
2015	71	45, 71, 110, 150, 160	110
2016	110	45, 71, 110, 130, 160	110
2017	160	71, 110, 130, 160, 180	130
2018	130	110, 130, 160, 180, 280	160
2019	180	130, 160, 180, 200, 280	180
2020	280	130, 180, 200, 280, 300	200
2021	200		
2022	300		

b **1** Enter the data into a spreadsheet using the same headings.

Position each median to line up with the centre value in the set of 5.

Spreadsheet formulas

	A	B	C
1	Year	No. of Accidents	5-point moving median
2	2013	150	
3	2014	45	
4	2015	71	=MEDIAN(B2:B6)
5	2016	110	=MEDIAN(B3:B7)
6	2017	160	=MEDIAN(B4:B8)
7	2018	130	=MEDIAN(B5:B9)
8	2019	180	=MEDIAN(B6:B10)
9	2020	280	=MEDIAN(B7:B11)
10	2021	200	
11	2022	300	
12			

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Spreadsheet values

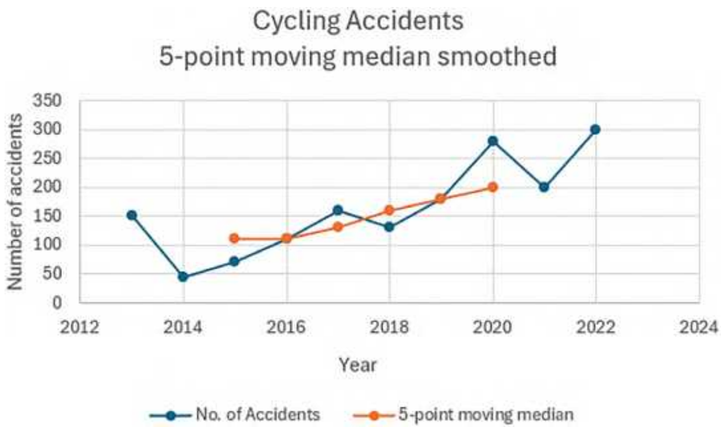
	A	B	C
1	Year	No. of Accidents	5-point moving median
2	2013	150	
3	2014	45	
4	2015	71	110
5	2016	110	110
6	2017	160	130
7	2018	130	160
8	2019	180	180
9	2020	280	200
10	2021	200	
11	2022	300	
12			

2 Highlight the full table (values version).

Select the Insert tab → Recommended charts → Line.

Right-click on the graph to bring up the tools. Select ⊞ to bring up the Tools menu.

Label the graph with title, axis labels and a legend.



c Compare the lines.

Both lines show an increasing trend. The 5-point moving median has fewer points and is much straighter. It has smoothed out the fluctuations in 2017 and 2020.

Unless specifically stated otherwise, moving averages are based on means.
Using medians instead is usually because of the nature or size of the data set.

EXERCISE 4.2 Smoothing time series data

ANSWERS p. 448

Recap

1 The most likely time series plot pattern for the study of 'Money earned by nurses' would be

A fluctuating
B long-term increase
C seasonal
D cyclical

2 A time series plot displays

A a series of data over many years
B the response variable data over time
C data collected at regular time intervals
D data that is only seasonal in nature

Mastery

3 WORKED EXAMPLE 6 The table below displays the number of participants in an online course throughout one week.

Day	Saturday	Sunday	Monday	Tuesday	Wednesday	Thursday	Friday
Participants	208	160	85	310	122	405	249

For the data in the table, use means to calculate

a the 3-point average for Sunday

b the 3-point average for Monday

c the 3-point average for Thursday

d the 5-point average for Monday

e the 5-point average for Tuesday

f the 7-point average for Tuesday.

4 WORKED EXAMPLE 7 The table below shows the annual turnover of a local retailer over 12 years.

Year	2009	2010	2011	2012	2013	2014
Turnover $'000	1206	1229	1234	1259	1243	1275
Year	2015	2016	2017	2018	2019	2020
Turnover $'000	1282	1283	1292	1293	1319	1255

a Calculate a 3-year moving average using means, rounded to the nearest whole number.

b Draw a graph showing the original and smoothed sets of data.

5 WORKED EXAMPLE 8 A theatre at the Gold Coast recorded the following ticket sales over 12 months.

Month	Jan	Feb	Mar	Apr	May	Jun
Sales $'000	4375	3200	2850	4105	2540	4720
Month	Jul	Aug	Sep	Oct	Nov	Dec
Sales $'000	3970	2060	3405	3790	2040	4930

a Calculate a 5-year moving average using means, rounded to the nearest whole number.

b Draw a graph showing the original and smoothed sets of data.

6 The table below shows the average annual profit of an Australian confectionery maker over a 15-year period. Use a spreadsheet to answer parts **a**–**c** below.

Year	2011	2012	2013	2014	2015	2016	2017	2018
Profit $'000	3406	3514	3459	3495	3571	3534	3537	3551
Year	2019	2020	2021	2022	2023	2024	2025	
Profit $'000	3635	3673	3700	3702	3697	3681	3704	

a Calculate a 3-year moving average using means.

b Calculate a 7-year moving average using means.

c Draw a graph showing the original and both smoothed sets of data.

d Comment on the graphs.

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7 WORKED EXAMPLE 9 This table shows the total monthly distance, in thousands of kilometres, driven by drivers for a Brisbane-based courier over a 1-year period.

Month	J	F	M	A	M	J	J	A	S	O	N	D
Distance	456	457	455	477	469	492	490	511	509	528	520	530

a Smooth the data using a 3-point moving average using medians.

b Draw a graph showing the original and smoothed data.

8 WORKED EXAMPLE 10 A study was conducted on the number of visitors to a Sunshine Coast surf club each quarter over 3 years.

Quarter	1 (2021)	2	3	4	1 (2022)	2
Visitors ('000s)	12.5	8.6	6.7	11.6	13.9	9.1
Quarter	3	4	1 (2023)	2	3	4
Visitors ('000s)	5.4	10.7	14.6	9.6	7.7	11.2

a Use a spreadsheet to smooth the data using a 5-point moving average using medians.

b Draw a graph showing the original and smoothed data.

9 This table shows the number of rail passengers stopping at Bundaberg from 1993 to 2019.

Year	1993	1994	1995	1996	1997	1998	1999	2000	2001
Passengers	543 731	548 531	556 112	567 134	575 106	584 625	594 584	602 912	610 810
Year	2002	2003	2004	2005	2006	2007	2008	2009	2010
Passengers	619 587	629 771	638 955	647 254	657 797	697 903	706 462	717 988	724 956
Year	2011	2012	2013	2014	2015	2016	2017	2018	2019
Passengers	733 652	747 682	761 411	773 309	784 224	794 815	806 555	807 682	804 705

Use a spreadsheet to construct a graph showing the 3- and 5-point moving median smoothing.

Exam practice

10 © QCAA 2021 1Q1 MODIFIED The fourth smoothed value for the 3-point moving average using means is closest to

Day	1	2	3	4	5	6	7
Value	5	10	18	32	52	70	90

A 34 **B** 51 **C** 71 **D** 107

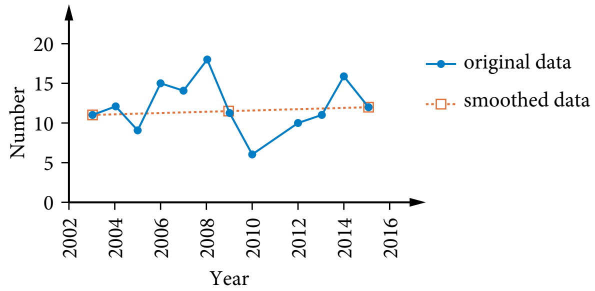
11 The 5-point moving average using means for Quarter 3 of 2022 is

Q1 (2022)	Q2	Q3	Q4	Q1 (2023)	Q2	Q3	Q4
250	389	401	300	230	367	470	294

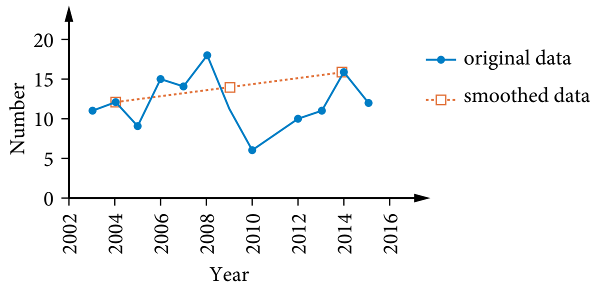
A 314 **B** 344 **C** 363 **D** 1386

12 © QCAA 2020 1Q10 Which of the following graphs represents a time series plot with a 3-point moving average?

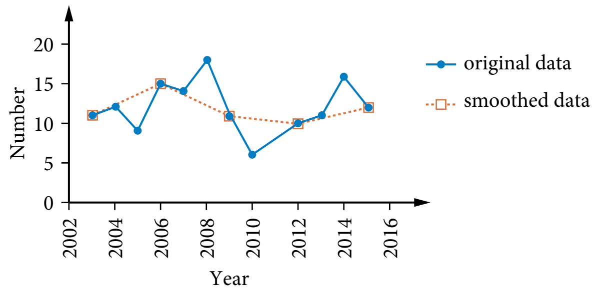
A



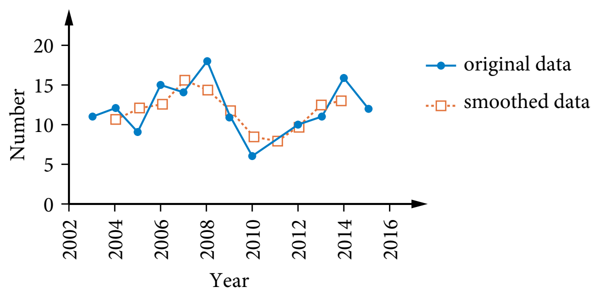
B



C



D



13 (2 marks) Explain the differences and difficulties using a 7- or 9-point moving average instead of a 3- or 5-point moving average.

14 (5 marks) The average maximum and minimum daily temperatures each month for Thargomindah are below.

Month	J	F	M	A	M	J	J	A	S	O	N	D
Maximum temp (°C)	36.9	35.6	33.2	28.4	23.3	19.7	19.2	21.6	26.0	30.1	33.4	36.2

a Plot the temperatures for Thargomindah. *[2 marks]*

b Smooth the temperatures using a 5-point moving median and plot the results against the raw data. *[2 marks]*

c Comment on the effect of the smoothing on the temperatures. *[1 mark]*

15 (5 marks) CF This table shows the annual production, in thousands of tonnes, of a coal mine. It has been in operation for a long time and needs to produce at least 200 000 tonnes per year to remain viable. Use 5-point moving averages to predict when it will close and reason if this prediction is accurate.

Year	2006	2007	2008	2009	2010	2011	2012	2013	2014	2015	2016
Production	1589	1421	1598	1110	1346	1007	847	852	667	631	670

4.3 Seasonal indices and deseasonalising

Seasonal patterns can cycle through days, years, or even weather seasons. Smoothing techniques are not useful with seasonal patterns.

Worksheets
Deseasonalisation
Seasonality

Seasonal indices

Seasonal indices use annual **averages** to make adjustments.

$$\text{seasonal index} = \frac{\text{actual value}}{\text{average for the season}}$$

Seasonal index for one year

Follow the steps below to make a seasonal index from monthly data for one year.

1 Find the average for the year.

2 Divide the amount for each period by the average for the year.

Properties of seasonal indices

- A seasonal index more than 1 is above average.
- A seasonal index less than 1 is below average.
- The sum of all seasonal indices should be equal to the number of seasonal indices in the year.

Monthly data: seasonal indices add to 12; quarterly data: seasonal indices add to 4.

Video
Seasonal indices

WORKED EXAMPLE 11 Calculating seasonal indices for one year

The monthly sales ($'000s) for a pool supplies shop for one year are shown in the table.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Value	635	618	563	477	396	351	359	332	404	479	552	601

a **Calculate** the seasonal index for each month.

b **Identify** which months are above average, rounding to 2 decimal places.

Steps | **Working**

a **1** Calculate the average sales for the year.

$$\frac{635+618+563+477+396+351+359+332+404+479+552+601}{12}$$

$$= \frac{5767}{12}$$

$$\approx 480.6$$

2 Divide each month by the annual average.

Check your calculations by adding the seasonal indices. This should equal 12, as there are 12 months.

$$\text{Jan} = \frac{635}{480.6} = 1.32$$

Month		Seasonal Index
Jan	$\frac{635}{480.6}$	1.321
Feb	$\frac{618}{480.6}$	1.286
Mar	$\frac{563}{480.6}$	1.171
Apr	$\frac{477}{480.6}$	0.993
May	$\frac{396}{480.6}$	0.824
Jun	$\frac{351}{480.6}$	0.730
Jul	$\frac{359}{480.6}$	0.747
Aug	$\frac{332}{480.6}$	0.691
Sep	$\frac{404}{480.6}$	0.841
Oct	$\frac{479}{480.6}$	0.997
Nov	$\frac{552}{480.6}$	1.149
Dec	$\frac{601}{480.6}$	1.251
Total		12

b List the seasons that have a seasonal index above 1.

The months above average are January (1.32), February (1.29), March (1.17), November (1.15) and December (1.25).

Seasonal index for multiple years

1. Find the seasonal indices for each year.
2. Find the average of each season's indices.

4.3

WORKED EXAMPLE 12 Calculating seasonal indices for multiple years

This table shows the quarterly values of a variable over 3 years.

Year	Q1	Q2	Q3	Q4
2020	763	754	724	670
2021	783	759	721	684
2022	781	761	738	693

Calculate the seasonal index for each quarter, rounded to 2 decimal places.

Steps | **Working**

1 Calculate the annual averages for each year.

$$2020 \text{ average} = \frac{763+754+724+670}{4} = 727.75$$

$$2021 \text{ average} = \frac{783+759+721+684}{4} = 736.75$$

$$2022 \text{ average} = \frac{781+761+738+693}{4} = 743.25$$

2 Divide each quarter by the annual average for that year.

Year	Q1	Q2	Q3	Q4
2020	$\frac{763}{727.75}$	$\frac{754}{727.75}$	$\frac{724}{727.75}$	$\frac{670}{727.75}$
2021	$\frac{783}{736.75}$	$\frac{759}{736.75}$	$\frac{721}{736.75}$	$\frac{684}{736.75}$
2022	$\frac{781}{743.25}$	$\frac{761}{743.25}$	$\frac{738}{743.25}$	$\frac{693}{743.25}$

3 List the answers.

Year	Q1	Q2	Q3	Q4
2020	1.048	1.036	0.995	0.921
2021	1.063	1.030	0.979	0.928
2022	1.051	1.024	0.993	0.932

4 Calculate each quarter's seasonal index.

$$Q1 = \frac{1.048+1.063+1.051}{3} = 1.054$$

$$Q2 = \frac{1.036+1.03+1.024}{3} = 1.03$$

$$Q3 = \frac{0.995+0.979+0.993}{3} = 0.989$$

$$Q4 = \frac{0.921+0.928+0.932}{3} = 0.927$$

5 List the seasonal index for each quarter.

The quarterly indices are Q1 = 1.05, Q2 = 1.03, Q3 = 0.99 and Q4 = 0.93.

WORKED EXAMPLE 13 Using spreadsheets to calculate seasonal indices

This table shows the seasonal values for the number of air conditioners installed by an installer.

Year	Summer	Autumn	Winter	Spring
2021	240	105	92	184
2022	235	114	100	156
2023	262	110	85	177

Use a spreadsheet to **calculate** the seasonal index for each season, rounded to 2 decimal places.

Steps / **Working**

1 Calculate the annual averages for each year.

	A	B	C	D	E	F
1	Year	Summer	Autumn	Winter	Spring	Annual Average
2	2021	240	105	92	184	=AVERAGE(B2:E2)
3	2022	235	114	100	156	=AVERAGE(B3:E3)
4	2023	262	110	85	177	=AVERAGE(B4:E4)
5						

	A	B	C	D	E	F
1	Year	Summer	Autumn	Winter	Spring	Annual Average
2	2021	240	105	92	184	155.25
3	2022	235	114	100	156	151.25
4	2023	262	110	85	177	158.5
5						

2 Create a second table and divide each season's actual value by the annual average for that year.

By placing a $ with the cell reference for each annual average, the 'Fill Tool' (bottom right of each cell) may be used to drag the formula to other cells. This needs to be done separately for each line.

	A	B	C	D	E	F
1	Year	Summer	Autumn	Winter	Spring	Annual Average
2	2021	240	105	92	184	155.25
3	2022	235	114	100	156	151.25
4	2023	262	110	85	177	158.5
5						
6	Year	Summer	Autumn	Winter	Spring	
7	2021	1.55	0.68	0.59	1.19	
8	2022	1.55	0.75	0.66	1.03	
9	2023	1.65	0.69	0.54	1.12	
10	Seasonal Index	1.58	0.71	0.60	1.11	
11						

Formulas

5					
6	Year	Summer	Autumn	Winter	Spring
7	2021	=B2/$F$2	=C2/$F$2	=D2/$F$2	=E2/$F$2
8	2022	=B3/$F$3	=C3/$F$3	=D3/$F$3	=E3/$F$3
9	2023	=B4/$F$4	=C4/$F$4	=D4/$F$4	=E4/$F$4
10	Seasonal Index	=AVERAGE(B7:B9)	=AVERAGE(C7:C9)	=AVERAGE(D7:D9)	=AVERAGE(E7:E9)
11					

3 List the seasonal index for each quarter, to 2 decimal places.

Summer = 1.58
Autumn = 0.71
Winter = 0.60
Spring = 1.11

4 Check they add to 4.

1.58 + 0.71 + 0.60 + 1.11 = 4

5 Give the answer.

The seasonal indices are summer 1.58, autumn 0.71, winter 0.60 and spring 1.11.

Deseasonalisation

Deseasonalisation uses seasonal indices to remove the seasonal component from the data. You can use a graph of deseasonalised data to make long-term predictions.

$$\text{deseasonalised value} = \frac{\text{actual value}}{\text{seasonal index}}$$

Once a prediction value is made, **reseasonalisation** is used to calculate the actual value for that specific time period.

$$\text{reseasonalised value} = \text{deseasonalised value} \times \text{seasonal index}$$

4.3

WORKED EXAMPLE 14 Deseasonalising data

a **Use** the seasonal index for each quarter to deseasonalise the quantity of coffee sales ('000) for a café from the time series data below. Round deseasonalised values to the nearest whole number.

Quarter	Q1	Q2	Q3	Q4
Index	0.665	1.164	1.292	0.879

2020				2021			
Q1	Q2	Q3	Q4	Q1	Q2	Q3	Q4
134	237	258	171	133	235	251	182

b **Construct** a scatterplot of the original data in part **a** and the deseasonalised data on the same axes.

c The business predicts that the deseasonalised sales of coffee will be 210 000 in quarter 3 of 2022. **Calculate** the predicted actual sales for that quarter.

Steps | **Working**

a Use the formula to find the deseasonalised figure for each quarter.

$$\text{deseasonalised figure} = \frac{\text{actual value}}{\text{seasonal index}}$$

Quarter	Actual value		Deseasonalised value
2020 Q1	134	$\frac{134}{0.665}$	202
Q2	237	$\frac{237}{1.164}$	204
Q3	258	$\frac{258}{1.292}$	200
Q4	171	$\frac{171}{0.879}$	195
2021 Q1	133	$\frac{133}{0.665}$	200
Q2	235	$\frac{235}{1.164}$	202
Q3	251	$\frac{251}{1.292}$	194
Q4	182	$\frac{182}{0.879}$	207

b Label the x-axis 'Quarter' and the y-axis 'Quantity sold'.

Plot both the actual and deseasonalised values to create two time series plots.

c **1** Identify key information.

deseasonalised value = 210 000 (210)
seasonal index Q3 = 1.292

2 Substitute the values into the reseasonalisation formula to find the actual figure.

reseasonalised value = deseasonalised value × seasonal index
= 210 × 1.292
= 271.32

3 State the answer.

The predicted actual sales would be 271 320 coffees.

WORKED EXAMPLE 15 Using a spreadsheet to deseasonalise data

a **Use** the seasonal index for each quarter to deseasonalise the total rainfall data. Use a spreadsheet and round values to the nearest whole number.

Season	Summer	Autumn	Winter	Spring
Index	0.868	0.996	1.427	0.709

2022				2023			
Summer	Autumn	Winter	Spring	Summer	Autumn	Winter	Spring
140	165	237	102	138	154	220	125

b **Construct** a scatterplot of the original data and the deseasonalised data on the same axes.

c **Comment** on the results.

Steps | **Working**

a **1** List the data in the spreadsheet.

	A	B	C
1	Season	Actual value (mm)	Seasonal Index
2	2022 Summer	140	0.868
3	Autumn	165	0.996
4	Winter	237	1.427
5	Spring	102	0.709
6	2023 Summer	138	0.868
7	Autumn	154	0.996
8	Winter	220	1.427
9	Spring	125	0.709
10			

9780170484824

2 Add in a deseasonalised value column and use the formula to calculate each value.

If calculating seasonal indices in an earlier part of a question, the seasonal index values may be copied with more decimal places.

Please note, these values will need to be copied without their original formula. Doing so would create an error.

The seasonal indices are repeated for each year.

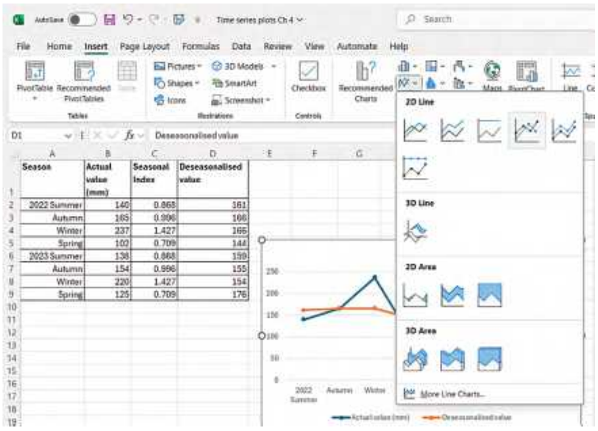
	A	B	C	D
1	Season	Actual value (mm)	Seasonal Index	Deseasonalised value
2	2022 Summer	140	0.868	=B2/C2
3	Autumn	165	0.996	=B3/C3
4	Winter	237	1.427	=B4/C4
5	Spring	102	0.709	=B5/C5
6	2023 Summer	138	0.868	=B6/C6
7	Autumn	154	0.996	=B7/C7
8	Winter	220	1.427	=B8/C8
9	Spring	125	0.709	=B9/C9
10				

	A	B	C	D
1	Season	Actual value (mm)	Seasonal Index	Deseasonalised value
2	2022 Summer	140	0.868	161
3	Autumn	165	0.996	166
4	Winter	237	1.427	166
5	Spring	102	0.709	144
6	2023 Summer	138	0.868	159
7	Autumn	154	0.996	155
8	Winter	220	1.427	154
9	Spring	125	0.709	176
10				

b 1 Highlight columns A, B and D. Hold down the Ctrl button while highlighting to skip the Seasonal Index column.

	A	B	C	D
1	Season	Actual value	Seasonal Index	Deseasonalised value
2	2022 Summer	140	0.868	161
3	Autumn	165	0.996	166
4	Winter	237	1.427	166
5	Spring	102	0.709	144
6	2023 Summer	138	0.868	159
7	Autumn	154	0.996	155
8	Winter	220	1.427	154
9	Spring	125	0.709	176
10				

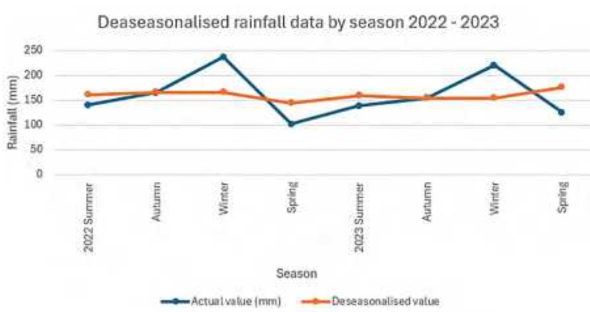
2 Select the Insert tab → then the down arrow for the plot 'Line with markers'. Plot both the actual and deseasonalised values to create two time series plots.



Label the x-axis 'Season' and the y-axis 'Rainfall (mm)'.

3 To rotate the x-axis, right-click on the x-axis and select Format Axis from the pop-up menu.

On the right-margin, select the size and properties icon from the Axis Options menu. This will give you the rotational directions to select from.



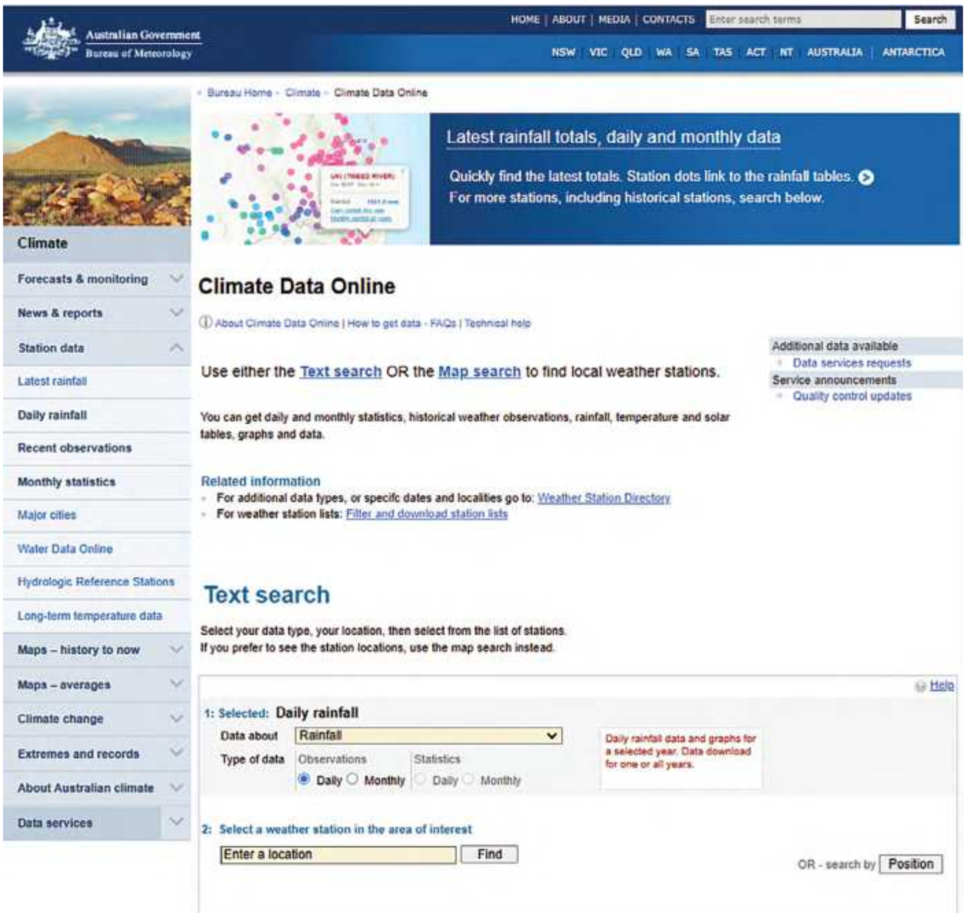
c Comment on the graph.

The deseasonalised series shows summer, winter and spring have been smoothed to show a straighter smoothed line. Autumn's deseasonalised value was closer to the actual.

Investigation Local rainfall

The Bureau of Meteorology collects large quantities of weather data for analysis and record keeping. The Bureau's Climate Data Online section contains recorded data on temperature, rainfall, humidity and so on. Data can be exported in .csv form, ready to go into a spreadsheet.

Investigate rainfall patterns in your city or town using this data. Use a spreadsheet program to organise, calculate seasonal indices, deseasonalise or form trend lines to analyse rainfall patterns over time. By using a spreadsheet, large quantities of data may be transferred. Consider analysing 100 years of data.



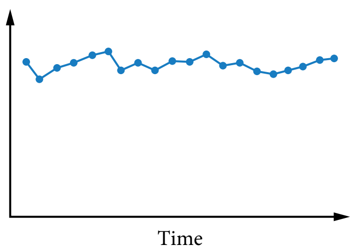
Source: Bureau of Meteorology (2025), BOM website https://reg.bom.gov.au/climate/data/, accessed 1 January 2025

EXERCISE 4.3 Seasonal indices and deseasonalising

ANSWERS p. 450

Recap

1 What pattern is shown in this time series plot?



A increasing trend **B** decreasing trend **C** no trend **D** seasonal trend

2 Based on the series data, the 5-point moving average for Wednesday is

Day	Sun	Mon	Tue	Wed	Thu	Fri	Sat
Value	5.7	2.1	1.8	4.9	5.9	6.4	5.8

A 4.22 **B** 4.65 **C** 6.38 **D** 15.98

Mastery

3 WORKED EXAMPLE 11

a The quarterly values of a variable for 2019 are shown. Calculate the quarterly index value.

Quarter	1st	2nd	3rd	4th
Value	480	586	608	537

b The quarterly values of a variable for 2021 are shown. Calculate the quarterly index.

Quarter	Q1	Q2	Q3	Q4
Value	246	235	223	242

c The monthly values of a variable for one year are shown. Determine the seasonal index for each month.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Value	234	214	220	207	222	227	216	239	248	253	263	255

d The monthly values of a variable for one year are shown. Calculate the seasonal index for each month.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Value	125	128	145	148	165	175	163	170	154	151	130	138

4 Determine the missing value in each seasonal index table.

a

Summer	Autumn	Winter	Spring
1.025	0.98		1.08

b

Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
0.702	0.712	0.886		1.115	1.331	1.324	1.255	1.181	0.971	0.814	0.811

c

Quarter	1	2	3	4
Seasonal index	0.67	1.52	1.04	

5 WORKED EXAMPLE 12

a The quarterly values of a variable over 2 years are shown. Calculate the seasonal index for each quarter.

Quarter	Q1	Q2	Q3	Q4
2016	451	419	405	454
2017	462	424	405	449

b The quarterly values of a variable over 3 years are shown. Calculate the seasonal index for each quarter.

Season	Summer	Autumn	Winter	Spring
2021	176	183	174	178
2022	193	169	167	179
2023	191	167	165	177

6 WORKED EXAMPLE 13 Use a spreadsheet to calculate the monthly values of the variable shown below for 3 years to find the seasonal index for each month.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
2019	352	381	344	353	326	332	308	324	329	359	359	365
2020	378	384	382	340	335	338	343	303	330	322	339	342
2021	324	378	380	335	305	301	323	287	300	302	320	362

7 WORKED EXAMPLE 14 The seasonal index for rainfall (mm) is shown for a town in Far North Queensland below.

Season	Summer	Autumn	Winter	Spring
Index	0.952	1.028	1.043	0.977

a Use the index to deseasonalise the time series of the same variable below.

2021	Summer	Autumn	Winter	Spring
Value	242	265	261	253

2022	Summer	Autumn	Winter	Spring
Value	236	262	267	249

b Make a plot of the original data and the smoothed data on the same axes.

c The Bureau of Meteorology predicts that the deseasonalised rainfall in Autumn of 2023 should be approximately 283 mm. Reseasonalise the data to determine the actual amount of rain for that season.

8 The quarterly index for a particular cost variable is shown below.

Quarter	Q1	Q2	Q3	Q4
Index	1.094	0.957	0.888	1.062

a Use the index to deseasonalise the time series of the same variable below.

Q1 2020	Q2	Q3	Q4	Q1 2021	Q2	Q3	Q4
163	142	133	164	169	142	136	157

b Make a time series plot of the original data and the smoothed data on the same axes.

c Comment on the graph.

9 WORKED EXAMPLE 15 The monthly index for a particular variable is shown below.

Month	Jan	Feb	Mar	Apr	May	Jun
Index	1.12	1.093	1.061	0.99	0.943	0.905
Month	Jul	Aug	Sep	Oct	Nov	Dec
Index	0.879	0.903	0.934	1.005	1.064	1.101

a Use a spreadsheet to deseasonalise, to the nearest whole number, the time series of the same variable below.

Month	J	F	M	A	M	J	J	A	S	O	N	D
2023	232	233	225	205	199	191	188	192	198	211	220	236
2024	236	235	227	208	201	188	188	188	197	215	227	231

b Make a time series plot of the original data and the deseasonalised data on the same axes.

c Comment on the results.

Use this information to answer Questions 10–12.

The table shows the weekly sales ($ '000) for a retail store.

Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
5.8	4.9	8.7	15.4	12.3	16.2	13.5

10 The weekly sales average, rounded to the nearest whole number, is

A 9700 **B** 10 210 **C** 10 790 **D** 10 971

11 The seasonal index for Friday is

A 0.528 **B** 0.79 **C** 1.021 **D** 1.121

12 The seasonalised value for Monday is closest to

A 0.28 **B** 0.53 **C** 5.20 **D** 5.29

13 (2 marks) Determine the seasonal index for a month if the actual value for the month is 402 and the total values for 12 months is 4972.

14 (3 marks) The data displays the numbers of passengers passing through Brisbane Airport over a 3-month period.

	Apr	May	Jun
Number of passengers	1 321 934	1 337 849	1 341 813
Deseasonalised passenger numbers	1 362 819	1 365 152	1 355 367

Calculate the seasonal indices for each month.

15 (4 marks) CF The actual number of visitors ('000) to a theme park over 2 years are shown.

Season	Summer	Autumn	Winter	Spring
2020	505	342	424	483
2021	610	401	526	570

Calculate the seasonal index for winter.

16 © QCAA 2022 2Q1 (4 marks) CF A company's deseasonalised profits ($ '000) for 2 years were recorded below.

Quarter	Q1	Q2	Q3	Q4
2021	96.91	90.73	88.72	98.34
2022	84.13	89.73	91.61	81.79

The seasonal indices for Q2, Q3 and Q4 were 1.003, 1.037 and 1.027 respectively. Determine the actual profits for Q1 in 2022.

4.4 Modelling long-term trends

A linear model (an example of a **trend line**) can be used in making predictions.
In Chapter 3, *Associations and linear modelling*, the **least-squares line** for a linear association was used to make predictions.

You can use the least-squares line to find a trend line. Many other techniques are also used in practice.

In time series, the x-axis is *not* an **explanatory variable**.

WORKED EXAMPLE 16 Using a trend line to make a prediction

A retailer analysed the monthly sales of mobile phones. After gathering 24 months of data, they calculated a trend line of $s = 23.54 \times \textit{month} + 483.21$. Their annual average for the current year is 567.25 phones per month.

a **Predict** how many phones will be sold in 12 months' time.

b **Comment** on the reasonableness of the result from part **a**.

Steps	Working
a **1** Calculate the month number.	$24 + 12 = 36$
2 Substitute 36 into the equation.	$s = 23.54 \times 36 + 483.21$ $= 1330.65$ ≈ 1331 phones
b Comment on the prediction.	The equation has a positive gradient, which would lead to an increased value in the 36th month. The likelihood of there being 1331 sales of phones however is unlikely, as the current monthly average is approximately 567 phones. The retailer would need to at least double sales within a year to achieve this. As with most long-term predictions, the further away from the actual data, the less accurate the prediction.

WORKED EXAMPLE 17 Determining a least-squares line to make predictions

The data below shows the number of performances held in a theatre during the year. The theatre can only accommodate one performance in a day.

Year	2013	2014	2015	2016	2017	2018	2019
Number of performances	108	200	205	271	283	307	298

a **Determine** the equation of the least-squares line.

b **Predict** the number of performances held in 2026.

c **Comment** on the accuracy of the prediction.

Steps | **Working**

a **1** To assist in counting years for the prediction, convert the years to 1–7.

Year	1	2	3	4	5	6	7
Value	108	200	205	271	283	307	298

Exam hack

Changing the numbers like this is a method used in the marking guide for the external assessments.

2 Use your calculator to find the least-squares line.

Casio fx-82AU PLUS II

Press MODE, **2**: STAT and **2**: A + BX

Enter the values in the x and y columns respectively.

Press AC then press SHIFT **1** (STAT):

s_x	4:Var, 4:s_x =
s_y	4:Var, 7:s_y =
$\bar{x}$	4:Var, 2: $\bar{x}$ =
$\bar{y}$	4:Var, 4: $\bar{y}$ =
r	5:Reg, 3:r =
y-intercept (c)	5:Reg, 1:A = 115.71
gradient (m)	5: Reg, 2:B = 30.79

Press MODE, **1**: COMP to exit the statistics mode.

Sharp EL-531TH

Press MODE, **1** and choose **1** (LINE): Linear regression calculation.

Enter the values in pairs by pressing 10, STO, 16, M+. You will see Data Set = 1. Continue entering the remaining data pairs in the same way until you have entered all the values.

To get the information, press

s_x	ALPHA 5 =
s_y	ALPHA 8 =
$\bar{x}$	ALPHA 4 =
$\bar{y}$	ALPHA 7 =
r	ALPHA ÷ =
y-intercept (c)	ALPHA (= 115.71
gradient (m)	ALPHA) = 30.79

Press MODE **0** to exit statistics calculations or press the reset button.

TI-30XB

Press data and enter the values in the L1 and L2 columns respectively.

Then press 2nd DATA (STAT) and choose **2**: 2-Var Stats.

Make XDATA: L1 and YDATA: L2, if they are not already, and use the cursor to go down to CALC and press ENTER.

Move the cursor down to get the values.

$s_x = 10.075$

$s_y = 12.696$

$\bar{x} = 22$

$\bar{y} = 32.8$

$r = 0.973$

y-intercept (c) = 115.71

gradient (m) = 30.79

Select 2nd **0** to reset your calculator.

3 List the gradient and y-intercept to form the equation.

The least-squares line is $y = 30.79x + 115.71$.

b **1** Determine the period number 2026 will be.

2026 – 2019 = 7 years difference

∴ 2026 will be the 14th period.

2 Substitute $x = 14$ into the equation.

$y = 30.79 \times 14 + 115.71$

$= 546.77$

≈ 547 performances

c Comment on the accuracy of the prediction.

While performance number of 547 follows the progression of the positive gradient of 30.79, the value is not reasonable. The theatre can only accommodate one performance per day. Because of this, the highest number of performances the theatre can have is 365 in a regular year.

WORKED EXAMPLE 18 Using deseasonalised data to predict future values

The table displays the deseasonalised annual rainfall data for Moreton Island from 1870 to 1875.

Year	Actual	Deseasonalised value
1870	185	2087.67
1871	128	1505.50
1872	159	1763.16
1873	155	1986.32
1874	120	1438.60
1875	150	1957.83

a **Use** a spreadsheet program to plot a time series graph showing the deseasonalised values.

b **Use** the spreadsheet functions to add a linear trend line and display the equation of the line.

c **Use** the equation and the index of 0.859 to **predict** the actual rainfall for 1876.

d **Comment** on the graph and the accuracy of long-term rainfall forecasting.

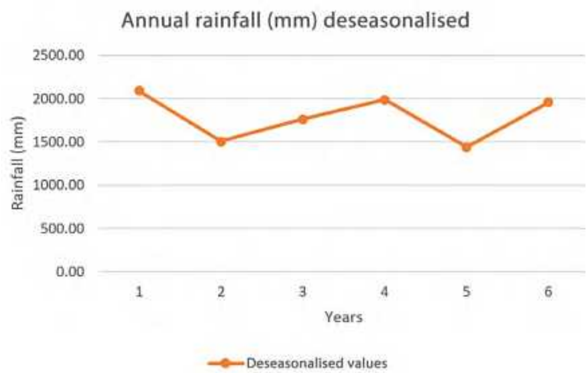
Steps | **Working**

a 1 Enter the table of data into a new spreadsheet. It is already in the correct format.

	A	B	C	D
1	Year	Actual	Year Number	Deseasonalised Value
2	1870	185	1	2087.67
3	1871	128	2	1505.50
4	1872	159	3	1763.16
5	1873	155	4	1986.32
6	1874	120	5	1438.60
7	1875	150	6	1957.83
8				

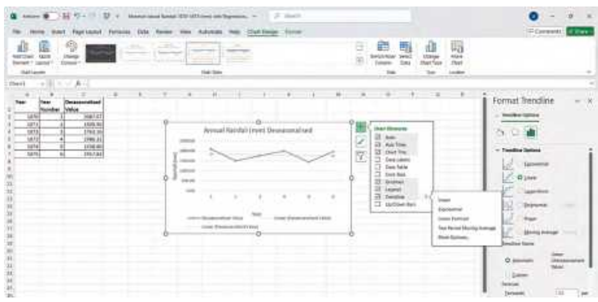
2 Highlight the year number and deseasonalised value columns before selecting Insert → Recommended Charts → select → Line with markers.

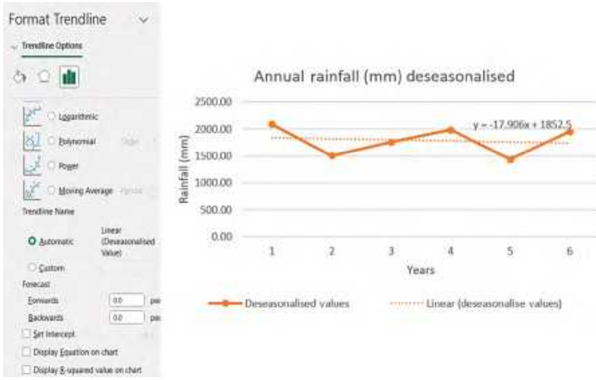
3 Add a title and axis labels.



b 1 Right-click on the plot and select the plus sign.

Tick the trendline box and use the arrow next to it to open the menu.



	2 In the Format Trendline menu, tick Display Equation on chart. This will add the trendline and display the equation on the graph. Move the equation so it can be easily seen.	 © Microsoft Corporation
c	**1** As 1876 is the next year, substitute $x = 7$ into the equation to find the predicted rainfall figure.	$y = -17.906 \times 7 + 1852.5$ $\approx 1727\text{ mm}$
	2 Reseasonalise the result to find the actual rainfall figure.	actual value $= 1727 \times 0.859$ $\approx 1483\text{ mm}$
d	Comment on the graph.	The rainfall graph has a negative gradient. This means the trend line would eventually reach zero and then negative values. This is not realistic. The use of the trendline for predicting the rainfall for 1876 could be classed as accurate as the calculation is close to the actual recorded values. Beyond this, predictions are less accurate the further away from the original data the prediction is calculated.

EXERCISE 4.4 Modelling long-term trends

ANSWERS p. 451

Recap

1 A seasonal index

A adds up to the number of years

B has pairs of values that each add to one

C adds up to 4 if measured quarterly

D exaggerates seasonal trends.

2 The data for 4 quarters is 19.8, 26.1, 42.4 and 20.3 respectively.
Calculate the 3-point moving average for the second quarter.

Mastery

3 WORKED EXAMPLE 16 A business analysed the number of complaints each month for one year.
The least-squares line based on this data was $y = -0.6503x + 11.727$, where y represents the number of complaints and x represents the month number.

a Describe the gradient of the trendline.

b Predict how many complaints would be received in August.

c **i** Predict how many complaints there would be after 2 years (24th month).

ii Comment on the reasonableness of your answer in part **i**.

d Calculate which month number would have the prediction of zero complaints.

4 Althea started an exercise regime and logged the number of minutes she exercised for over 7 days. From the data, a least-squares line was determined as $time = 10.55 \times day\ number + 2.85$.

a Using this equation, predict the number of whole minutes Althea will exercise for on

i day 1 **ii** day 7 **iii** day 10 **iv** day 30

b Comment on the reasonableness of using the equation to make predictions about the number of minutes of exercise.

5 WORKED EXAMPLE 17 The data below shows the values of a variable over 5 years.

Year	2016	2017	2018	2019	2020
Value	102	117	116	118	122

a Find the least-squares line using the statistics mode of your calculator.

b Predict the value in 2023.

c Is the prediction likely to be accurate?

6 The data below shows the operational costs ($ ‘000) for a retailer for 8 quarters over 2 years.

Quarter	1	2	3	4	5	6	7	8
Cost	184	194	203	212	217	223	225	234

a Find the least-squares model using the statistics mode of your calculator.

b Predict the operating cost in the 20th quarter.

c How accurate is this prediction?

7 The data below shows the number of days in a month that a person played tennis.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Days	10	8	10	12	10	9	7	8	5	5	6	4

a Use your calculator to find the least-squares model.

b Predict the number of days played in the 15th month.

c Is the prediction likely to be accurate?

8 WORKED EXAMPLE 18 The data below shows deseasonalised values of a variable over 20 years.

Year	1997	1998	1999	2000	2001	2002	2003	2004	2005	2006
Value	108	113	116	118	123	129	133	136	140	141
Year	2007	2008	2009	2010	2011	2012	2013	2014	2015	2016
Value	146	151	153	157	162	167	171	176	181	183

a Use a spreadsheet to construct a time series plot and to place a trend line for this data. Leave your x-axis values as years, do not convert to year number.

b State the equation for the trend line.

c Predict the value in 2020.

9 A supermarket recorded the number of customers ('000) passing through its doors each quarter for 4 years. The table shows the deseasonalised values for each quarter.

Quarter	1	2	3	4
2020	202	140	196	185
2021	219	247	216	233
2022	261	274	258	249
2023	234	275	250	254

a Construct a time series plot showing a trend line against the deseasonalised values.

b Use the graph to determine the equation of the trend line.

c Using the seasonal indices below, calculate the actual number of customers in the 4th quarter of 2024.

	Q1	Q2	Q3	Q4
Seasonal index	0.959	0.857	1.038	1.146

Exam practice

10 Given cost = 2.75 × months + 305.6, the cost in the 30th month is closest to

A 287 **B** 388 **C** 398 **D** 402

11 The table displays the monthly sales ($) for 6 months.

Month (M)	1	2	3	4	5	6
Monthly sales (S)	25 000	26 500	23 400	31 000	32 200	28 500

The equation of the least-squares line for this data is

A $M = 1205.71 \times S + 23\,546.67$

B $M = 23\,546.67 \times S + 1205.81$

C $S = 23\,546.67 \times M + 1205.81$

D $S = 1205.71 \times M + 23\,546.67$

12 In Q2 of 2021, the deseasonalised value for a variable was 405. If the seasonal index for that quarter is 0.89, what is the actual value for that quarter?

A 204.62 **B** 280.41 **C** 360.45 **D** 402.22

13 (4 marks) A variable had a value of 236 at the beginning of 2015 and a long-term trend equation of $V = 1.4Y - 2485$, where V is the value and Y is the year. The quarterly seasonal indices are shown below.

Quarter	Q1	Q2	Q3	Q4
Index	0.91	1.06	1.09	0.94

a Use the trend to find the value at the end of 2020, correct to the nearest whole number. *[1 mark]*

b Use the seasonal index to find the actual value at the end of March in 2021. *[1 mark]*

c Use the trend to find the value at the beginning of 2025. *[1 mark]*

d Use the seasonal index to find the actual value at the beginning of October in 2025. *[1 mark]*

14 (3 marks) CF The slope of the long-term trend of a variable is 1.24 and the value at the beginning of 2017 was 246. Predict the value at the beginning of 2025, correct to the nearest whole number.

Video
Exam question analysis: Time series analysis

EXAM QUESTION ANALYSIS

©QCAA 2020S 2Q9 (4 marks)

The table below shows the seasonally adjusted rainfall data (in mm) for a town in Australia.

	2015	2016
Summer	96.77	101.61
Autumn	98.04	101.96
Winter	100.00	101.39
Spring	100.97	102.91

The seasonal indices are shown in the table below.

Summer	1.24
Autumn	1.02
Winter	0.72
Spring	1.03

In 2017, the actual rainfall in summer was 127 mm, autumn was 106 mm, winter was 75 mm and spring was 107 mm.

Analyse the data to predict the total rainfall for each season in 2025.

Reading the question

- The deseasonalised data has already been given for 2015 and 2016.
- Seasonal indices have been given for each season.
- The actual values for 2017 are given.
- The question is asking for predictions for each season in 2025.

Thinking about the question

- 2017's data will need to be deseasonalised. Each value will need to be divided by the index for that season.
- The question doesn't state to graph the data, therefore a least-squares line approach needs to be taken. To do this, each deseasonalised data value will need to be given a number value instead of using the words summer, autumn, winter and spring. Start with 1. Use a calculator to find the equation.
- 2025 is 8 years in the future. The number of periods between spring 2017 and summer 2025 is 28. Summer 2025 will be period 41. This will need to be substituted into the equation to find the predicted value for that season in 2025.
- Reseasonalising the data is necessary to find the actual rainfall totals.

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Worked solution (✓ = 1 mark)

Seasonally adjust 2017 data

	2015	2016	2017
Summer	96.77	101.61	102.42
Autumn	98.04	101.96	103.92
Winter	100.00	101.39	104.17
Spring	100.97	102.91	103.88

✓

Use the above as y data and increment each season from 1 to 12 (Summer 2015 to Spring 2017) as the x data to find the following rule:

$y = 0.5992x + 97.609$ ✓

Mean rainfall by season in 2025	Multiply by SI (round to whole no.)
$y(41) = 122.1762$	151
$y(42) = 122.7754$	125
$y(43) = 123.3746$	89
$y(44) = 123.9738$	+ 128
	493 ✓

For 2025, the forecast is 151 mm of rain in summer, 125 mm of rain in autumn, 89 mm of rain in winter and 128 mm of rain in spring. ✓

4.4

4 Chapter summary

Time series

- A **time series** is a set of data collected at regular periods of time.
- A **time series plot** is a graph of a time series with time on the horizontal axis. You normally join the points to make a line graph.

Time series patterns

- A **trend** has a gradual increase or decrease. The trend could be linear (a straight line) or non-linear.
- A **seasonal** variation rises and falls in the same way each year. Seasonality includes systematic cycles, usually related to seasons, months or financial quarters.
- **Irregular** variation does not follow a pattern.
- **Cyclical** variation has rises and falls over a varying period (cycle). The cycle is usually more than a year.
- An **outlier** is an observation that is dramatically different from the rest of the time series.

Smoothing time series data

- **Smoothing** removes variations from data to make important trends more obvious.
- **Moving averages** are the simplest method to smooth data. **Means** or **medians** are grouped using odd sets of data points. The more values in a group, the shorter and smoother the line becomes.
- Moving averages are usually calculated in groupings of 3 points, 5 points or 7 points. Other odd groupings may be used.
- Moving averages using means may be called **moving means**. The term **moving medians** may also be used.

Seasonal indices

Seasonal indices use annual averages to make the adjustments.

$$\text{seasonal index} = \frac{\text{actual value}}{\text{average for the season}}$$

Seasonal index for one year

1. Find the average for the year.
2. Divide the amount for each period by the average for the year.

Properties of seasonal indices:

- A seasonal index more than 1 is above average.
- A seasonal index less than 1 is below average.
- All seasonal indices should total to the number of seasonal indices in the year. For example, monthly data: seasonal indices add to 12; quarterly data: seasonal indices add to 4.

Seasonal index for multiple years

1. Find the seasonal indices for each year.
2. Find their averages.

Deseasonalisation is a form of smoothing, which uses the seasonal indices to remove the seasonal component from the data. This method allows a linear model to be placed on the graph to make long-term predictions from the data.

$$\text{deseasonalised value} = \frac{\text{actual value}}{\text{seasonal index}}$$

Once a prediction value is made, **reseasonalisation** is used to calculate the actual value for that specific period of time.

$$\text{reseasonalised value} = \text{deseasonalised value} \times \text{seasonal index}$$

Trend lines and predictions

- A line plotted on a time series graph may show a trend used for making future **predictions**.
- A **least-squares line** equation may be used to calculate the equation of the line of a linear trend with or without a graph. This is the easiest way to substitute values for predictions.
- The closer to the actual data time the prediction is, the more accurate it will be. Long-term predictions may not be valid.

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Cumulative examination 1

Simple familiar

Perusal time: 2 minutes Working time: 30 minutes

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

Section 1 (5 marks): 5 multiple choice questions

Section 2 (14 marks): 4 short response questions

Total: 19 marks

Section 1 5 multiple choice questions

5 marks

Worksheet General Maths formula sheet

1 © QCAA 2023 1Q7 Which statement is always true for a causal relationship between an explanatory variable and a response variable?

A One of the variables is a confounding variable.

B The relationship is explained by a third variable.

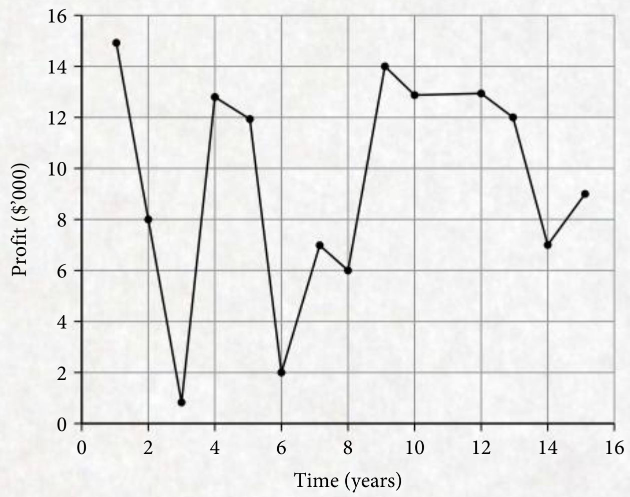
C There is a positive association between the variables.

D The response variable is dependent on the explanatory variable.

2 © QCAA 2021 1Q14 A town's current population of 15 480 is predicted to grow steadily at an annual rate of 12%. The predicted population after 10 years is approximately

A 48 079 **B** 34 056 **C** 18 576 **D** 17 338

3 © QCAA 2023 1Q2 A time series plot is shown.



It could best be described as

A cyclical **B** seasonal **C** irregular **D** increasing

4 © QCAA 2023 1Q10 Annual sales data and related quarterly indices are shown. The quarterly indices were calculated by applying the average percentage method using the mean.

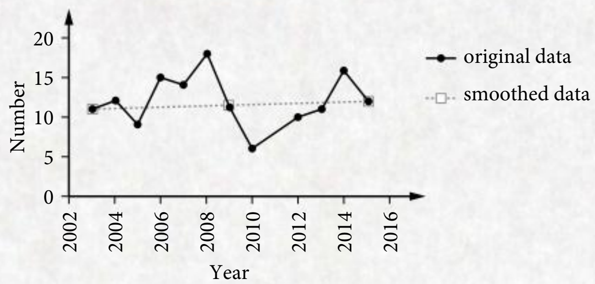
Quarter	Q1	Q2	Q3	Q4
Sales	160	x	128	200
Index	1.0	0.95	y	1.25

Determine the values for x and y.

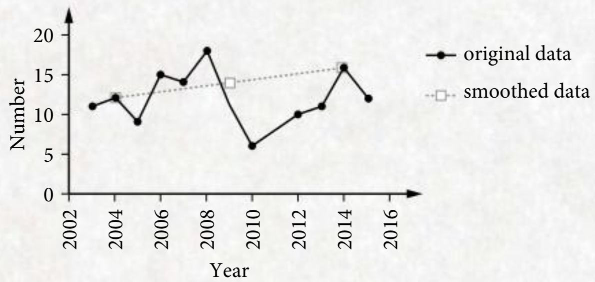
	x	y
A	122	0.8
B	122	3.2
C	152	0.8
D	152	3.2

5 © QCAA 2020 1Q10 Which of the following graphs represents a time series plot with a 3-point moving average?

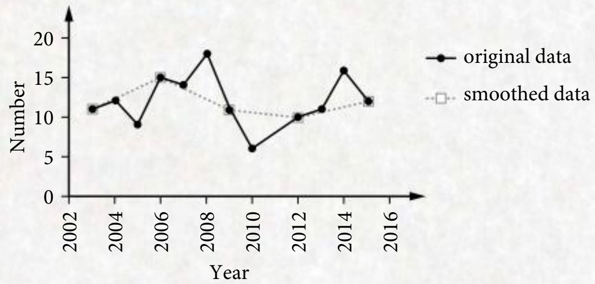
A



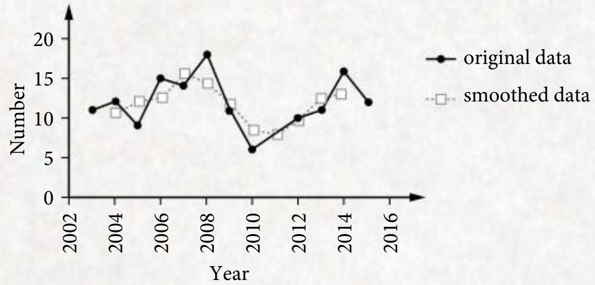
B



C



D



Section 2 4 short response questions 14 marks

6 © QCAA 2022 1Q16 (3 marks) The table shows the number of sales for a small business in their first six months of trading.

Time in months, t	Number of sales, n
1	86
2	180
3	160
4	226
5	240
6	335

a Use your calculator to determine the equation of the least-squares line. *[1 mark]*

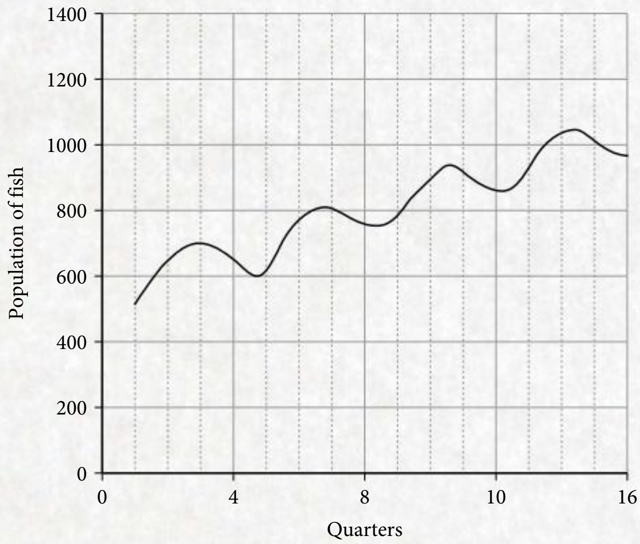
b Use the equation from part **a** to predict the number of sales in the 21st month. *[2 marks]*

7 © QCAA 2020 1Q18 MODIFIED (4 marks) Exhibition organisers believe that the number of attendees increases each day as an arithmetic sequence. The organisers know that 310 people attended the first day and 439 people attended the fourth day.

a Determine the common difference. *[2 marks]*

b Use the result from part **a** to predict the number of people who will attend the seventh day. *[2 marks]*

8 © QCAA 2021 1Q24 (3 marks) The population of fish in a fish farm pond was recorded every three months. The graph shows the data.



Describe the time series plot by identifying three key features.

9 (4 marks) A local boutique counts the number of customers visiting the store each day. The results are given below.

Day	Mon	Tue	Wed	Thu	Fri	Sat	Sun
Number of customers	65	80	70	140	100	180	96

a Construct a time series graph using this data. *[2 marks]*

b Use a 3-point moving average to smooth the data and plot this on the same set of axes as the original data in the graph. *[2 marks]*

Worksheet
General Maths formula sheet

Cumulative examination 2

Complex familiar and unfamiliar

Perusal time: 2 minutes Working time: 30 minutes

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

3 short response questions — 11 marks

1 © QCAA 2022 2Q1 (4 marks) The table shows a swimwear company's seasonally adjusted swimsuit sales (in thousands).

	Season			
	Spring	Summer	Autumn	Winter
Seasonally adjusted swimsuit sales (in thousands)	33.3	34.8	36.4	35.8

The long-term seasonal indices for spring, summer and winter are 1.11, 1.42 and 0.62 respectively.

Determine the actual swimsuit sales for autumn.

2 © QCAA 2023 2Q2 (4 marks) Buffalo fly bites cause skin wounds on cattle. The table shows the average number of skin wounds per animal in a herd for two years.

	Autumn	Winter	Spring	Summer
2021	285	28	195	460
2022	276	22	170	392

Deseasonalise the data.

3 © QCAA 2020S 2Q8 (3 marks) A teacher wants to know the best way for their students to improve their marks. They surveyed a sample of students who graduated last year and asked them three questions:

- What was the overall mark you achieved on the final assessment?
- On a typical night, how many hours sleep would you get?
- During a typical school term, how many classes did you miss?

They obtained the following data:

Overall percentage mark (%)	Hours of sleep	Classes missed
96	10	0
85	9	2
76	8	8
65	7	5
42	5	6

Construct a mathematical argument to determine which explanatory variable is the better predictor for the overall mark.

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CHAPTER

5 EARTH GEOMETRY AND TIME ZONES

Prior learning
Earth geometry and time zones

Syllabus coverage

UNIT 3, TOPIC 5: EARTH GEOMETRY AND TIME ZONES

Locations on the Earth

- Understand the meaning of great circles.
- Understand the meaning of angles of latitude and longitude (in decimal degrees, and degrees and minutes) in relation to the equator and the prime meridian respectively.
- Locate positions on Earth's surface given latitude and longitude, e.g. using a globe, map, GPS and other digital technologies.
- State latitude and longitude for positions on Earth's surface, e.g. investigating a map of Australia and locating boundary positions for Aboriginal peoples' and Torres Strait Islander peoples' language groups, Australian landmarks or local land boundaries.
- Calculate angular distance and distance between two places on Earth on the same meridian.
  $D = 111.2 \times$ angular distance where D is distance in kilometres
- Calculate angular distance and distance between two places on Earth on the same parallel of latitude.
  $D = 111.2 \cos\theta \times$ angular distance where D is distance in kilometres and θ is latitude
- Solve practical problems involving latitude, longitude, angular distance and distance.

Time zones

- Understand the meaning of Greenwich Mean Time (GMT), International Date Line and Coordinated Universal Time (UTC).
- Understand the link between longitude and time.
- Determine the number of degrees of longitude for a given time difference.
- Calculate time differences between two places on Earth.
- Solve practical problems involving time zones, making allowances for daylight saving where necessary, e.g. seasonal time systems used by Aboriginal peoples and Torres Strait Islander peoples, making phone calls, broadcasting events, travelling, preparing an itinerary.

Videos (5):
5.2 Distances between places in different hemispheres
5.5 International time zones 1 • International time zones 2 • Time zones with daylight saving
Exam question analysis Earth geometry and time zones

Prior learning (1):
5.1 Earth geometry and time zones

Worksheets (10):
5.1 Latitude and longitude • Australian coordinates • Positions on the globe • Map of the world • Position coordinates
5.2 Distances on the Earth
5.5 Applications of time zones • Table of time zones • Map of the world
Cumulative exams General Maths formula sheet

Puzzle (1):
Chapter summary World crossword

Nelson MindTap

To access resources above, visit **cengage.com.au/nelsonmindtap**

Terminology

angular distance	Australian Central Standard Time (ACST)	
Australian Eastern Standard Time (AEST)	Australian Western Standard Time (AWST)	
circle of illumination	Coordinated Universal Time (UTC)	daylight saving
equator	great circles	Greenwich Mean Time (GMT)
International Date Line	latitude	longitude
meridian	meridian of longitude	parallels of latitude
prime meridian	small circles	standard time zone
time zone		

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5.1 Latitude and longitude

Earth rotates in a complete circle once a day, from west to east. It is the effect of gravity on Earth that has created its shape – approximately the shape of a sphere.

Great circles and small circles

A **great circle** is the largest circle that can be drawn around the surface of a sphere. A great circle has:

- its centre located at the centre of the sphere
- the same diameter (and radius) as the sphere.

Great circles on the Earth's surface have their centre located in the centre of the Earth.

The diameter of the Earth joining the North and South poles is referred to as the Earth's axis.

A **small circle** is a circle drawn on a sphere that is smaller than a great circle. A small circle:

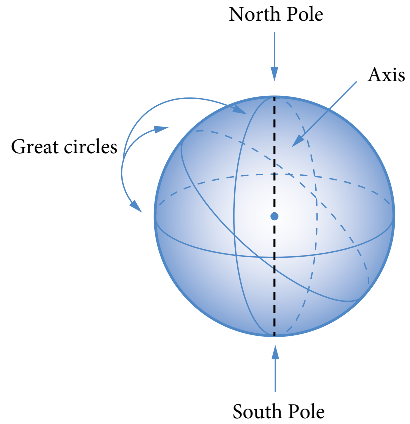
- does *not* have its centre located at the centre of the sphere
- has a different diameter (and radius) to the sphere.

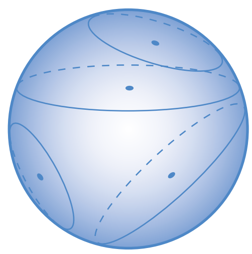
You can find the circumference of a great circle or small circle by using the formula:

$C = 2\pi r$, where r is the radius of the sphere

OR

$C = \pi d$, where d is the diameter of the sphere.




Small circles

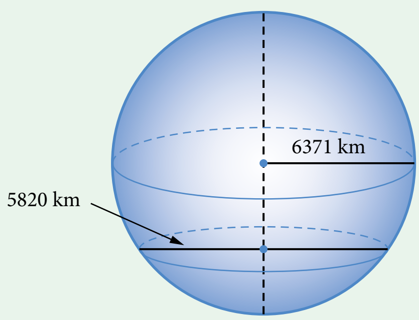
Worksheets
Latitude and longitude
Australian coordinates
Positions on the globe
Map of the world
Position coordinates

WORKED EXAMPLE 1 Great circle and small circle

Calculate the length of

a the great circle around the Earth, with a radius of 6371 km

b the small circle around the Earth with a diameter of 5820 km.



Steps	Working
a **1** Identify the radius, r, and the circumference formula to use.	$r = 6371$ $C = 2\pi r$
2 Substitute in r.	$= 2\pi \times 6371$
3 Evaluate and round appropriately.	$\approx 40\,030.2$ km
4 State the result.	The length of the great circle around the Earth is 40 030.2 km.

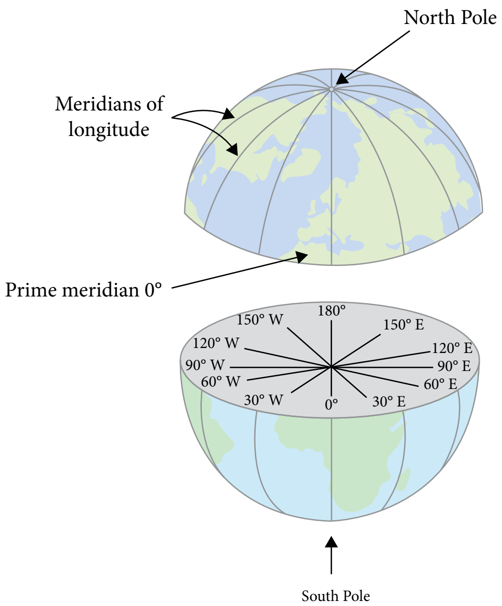
b **1** Identify the diameter, d, and the circumference formula to use.	$d = 5820$ $C = \pi d$ Hint: Alternatively, you can use $C = 2\pi r$.
2 Substitute in d.	$= \pi \times 5820$
3 Evaluate and round where appropriate.	$\approx 18\,284.1$ km
4 State the result.	The length of the small circle around the Earth is 18 284.1 km.

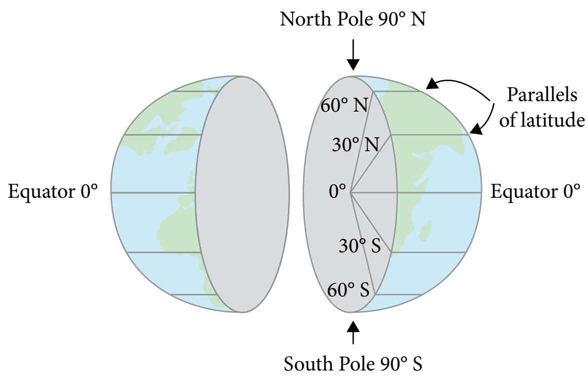
Positions on the Earth's surface

The location of points on the Earth's surface can be described using great circles and small circles.

The **meridians of longitude** are semi-great circles that pass between the North and South poles. The **prime meridian** of longitude (0°) is the meridian passing through Greenwich, England.

The **equator** is the great circle, which is located halfway between the North Pole and the South Pole and is perpendicular to the Earth's axis.





Parallels of latitude are small circles parallel to the equator with their centres at the axis.

Any point on the Earth's surface can be located by its **latitude** and **longitude** coordinates, both measured in degrees.

Latitude is the angle between the equator and the parallel of latitude through a given position. The **angular distance** is measured north or south of the equator. The maximum latitude for any point on the Earth is 90° N or 90° S.

Longitude is the angle between the prime meridian and the meridian passing through a given position. The angular distance is measured east or west of the prime meridian. The meridian located directly opposite the prime meridian is called the **International Date Line** and has a longitude of either 180° E or 180° W depending on which way the angular distance is measured.

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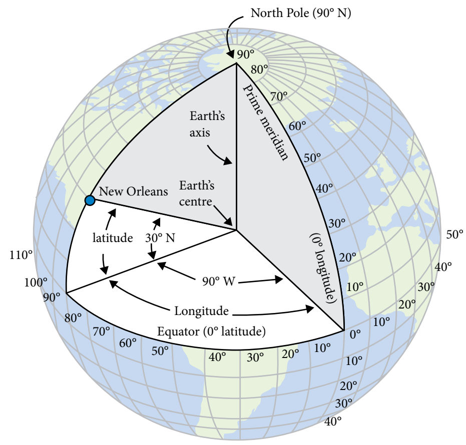
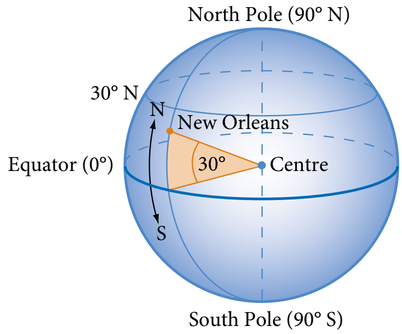
The parallels of latitude and the meridians of longitude create a grid on the Earth that gives the coordinates of latitude and longitude.

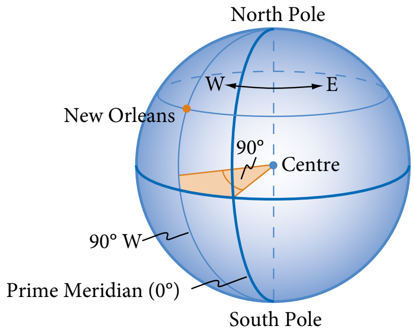
For example, New Orleans, USA, is located at 30° N 90° W. This means it is located on:

- the parallel of latitude that forms a 30° angle north of the equator.
- the meridian of longitude that forms a 90° angle west of the prime meridian.



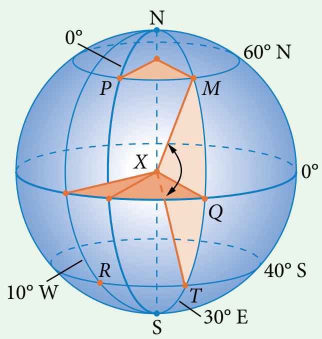
An alternate way to view this is using the following type of projection.





WORKED EXAMPLE 2	**Latitude and longitude**

State the positions of *M*, *P*, *Q*, *R* and *T*.



Steps	**Working**
1 Locate the point on the diagram. Write down the latitude, including N or S as required. Write down the longitude, including E or W as required.	*M* is 60° N 30° E
2 Repeat for all points. Note: *P* doesn't need E or W for 0°—it's on the prime meridian. *Q* doesn't need N or S for 0°—it's on the equator.	*P* is 60° N 0° *Q* is 0° 30° E *R* is 40° S 10° W *T* is 40° S 30° E

Angle measurements

The coordinates of a position (latitude and longitude) are usually expressed as either decimal degrees (e.g. 15.50°) or degrees and minutes (e.g. 15° 30′).

One degree is divided into 60 minutes. Note that these minutes are not related to time. They are just a fraction of an angle.

Degrees and minutes

- 1 degree (1°) = 60 minutes (60′)

For example, the position of the Port of Brisbane, expressed in degrees and minutes is 27° 23′ S 153° 10′ E. You would say this as 27 degrees, 23 minutes south, 153 degrees, 10 minutes east.

Exam hack

You can express your angles of latitude and longitude in decimal degrees or degrees and minutes, e.g. 15.50° or 15°30′ unless directed to express them in a particular format. If you are not sure use the format of the angles in the question.

WORKED EXAMPLE 3 Converting between decimal degrees and degrees and minutes

a Convert 25.8° to degrees and minutes.

b Convert 43° 55′ to decimal form, correct to 2 decimal places.

Steps	Working
a **1** To convert the decimal part to minutes multiply it by 60.	$0.8 \times 60 = 48'$
2 Write the full angle.	$25.8° = 25° 48'$
b **1** To convert the minutes part to a decimal, divide it by 60. Round to 2 decimal places.	$55 \div 60 = 0.92$
2 Write the full angle.	$43° 55' = 43.92°$

Hint: Check your answers with your calculator using the ° ′ ″ function.

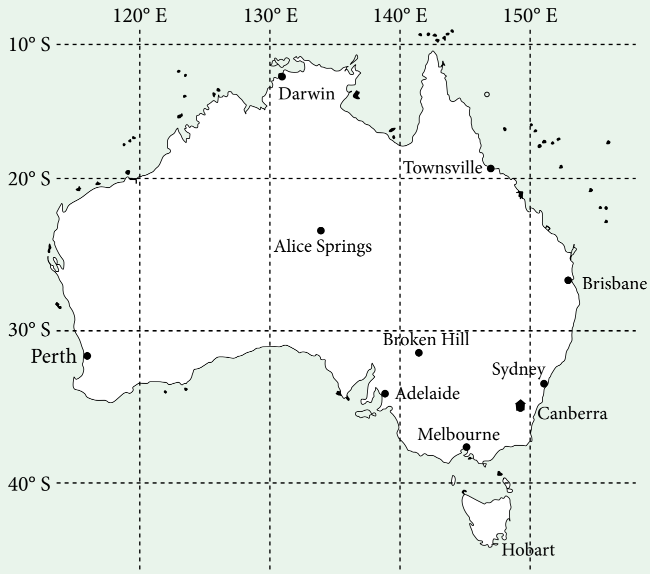
Locations and coordinates

WORKED EXAMPLE 4 Finding positions and locations using a map

Use the map to determine

a the latitude and longitude of Sydney

b the city located at 24° S 134° E.



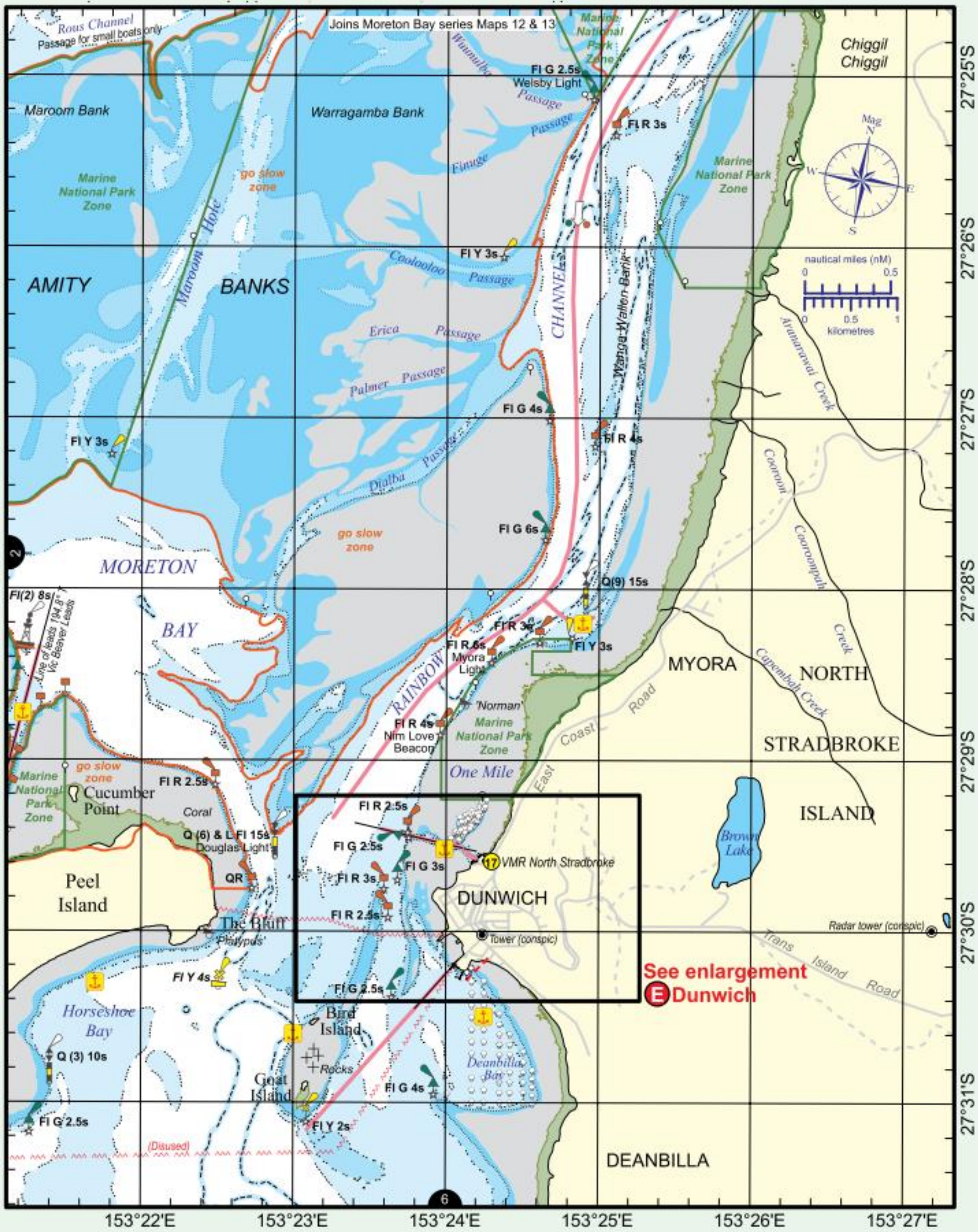
Steps	Working
a Locate Sydney on the map. The values for latitude are displayed on the left of the map and longitude along the top.	Hint: If the map has further intervals shown, use those for greater accuracy rather than approximating.
Sydney is located about a third of the way between 30° S and 40° S latitude a little to the right of the 150° E longitude.	
Identify the approximate latitude and longitude. State the answer.	Sydney is located at approximately 33° S 151° E.
b 24° S is located about halfway between the lines 20° S and 30° S.	Remember to write the coordinates in the correct order – latitude (N/S) first, then longitude (E/W).
134° E is located about halfway between the lines 130° E and 140° E.	
Identify the city located at the intersection.	Alice Springs is located at 24° S 134° E.

WORKED EXAMPLE 5 Maritime chart

Use the maritime chart below to **determine**

a the position of Dunwich

b the landmark located near 27° 25′ S 153° 25′ E.



Steps	Working
a **1** Locate Dunwich on the map. Read the latitude value on the right of the map and the longitude value on the bottom.	Dunwich is located at approximately 27° 30′ S 153° 24′ E.
2 Identify the approximate latitude and longitude.	
b **1** 27° 25′ S is the top line of latitude on the chart. 153° 25′ E is the 4th line of longitude from the left on the chart.	
2 Identify the landmark located near the intersection.	Welsby Light is located near 27° 25′ S 153° 25′ E.

Investigation Using Google Earth™

Google Earth™ can be used in many ways including finding the coordinates of a place or finding a place given its coordinates.

Finding the coordinates of a place

Digital Image Library/Alamy Stock Photo

- Open Google Earth™ on your device and type 'London' into the 'Search' field at the top left. If you are given options, select the most appropriate such as *London, UK*.
- Right click on 'London' on the map and select 'Get info'. This will give you the position coordinates of London (51° 31′ N 0° 08′ W).
  The latitude and longitude are also shown in the bottom right-hand corner. This is the location of the pointer. Move the pointer around and notice how the position shown changes. The position shown is accurate to within a metre.
- Try zooming in and out to change the detail shown.
  Try finding the coordinates of some places such as Brisbane, New York and Singapore.
  See whether you can find your own house and note its precise location.

Finding a place with given coordinates

- Enter the coordinates of the place, such as 25° 12′ S 153° 08′ E in the 'Search' field. Type the coordinates in without the degree (°) symbol. Google Earth™ will show the corresponding position.
- Try identifying the locations or landmarks located on or near the following coordinates: 33° 53.5′ S 151° 16.5′ E and 37° 49′ N 122° 29′ W.

EXERCISE 5.1 Latitude and longitude

ANSWERS p. 453

Mastery

1 **a** Are all small circles the same length?

b Which parallel of latitude has the greatest length?

c What is at latitude 90° S?

d Are all meridians of longitude the same length?

2 WORKED EXAMPLE 1 Calculate the length of the great circle on a sphere with

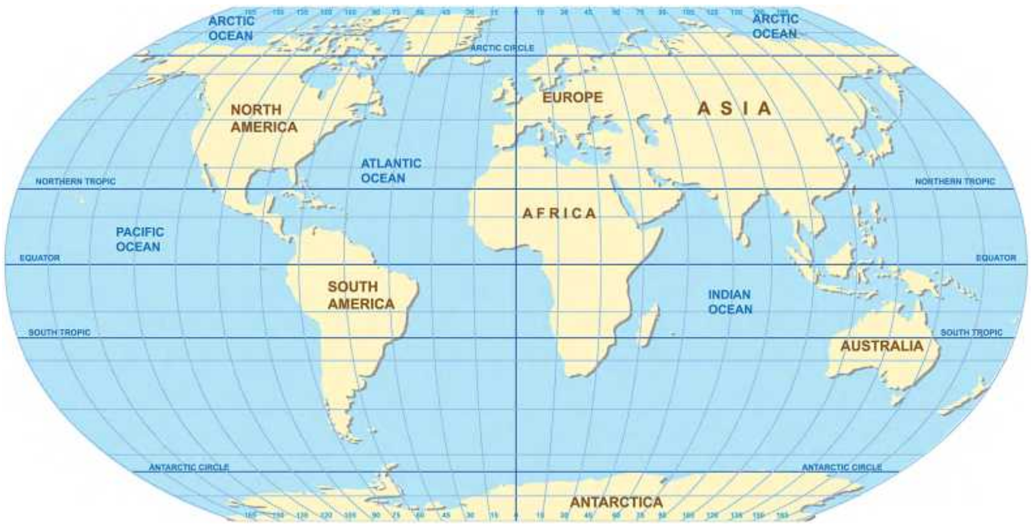
a a radius of 400 km **b** a diameter of 10 000 km

3 Calculate the length of the small circle on a sphere with

a a radius of 3150 m **b** a diameter of 7500 m

4 The 80° meridian of longitude passes through the Indian Ocean. Would it be east or west?

Refer to the following map for Questions 5 and 6.



5 Match each country to its position latitude from the following list.

40° N 30° S 35° N 40° S 65° N 10° S 45° N 0°

a Australia **b** Canada **c** Greenland **d** Japan

e New Zealand **f** Papua New Guinea **g** Indonesia **h** Spain

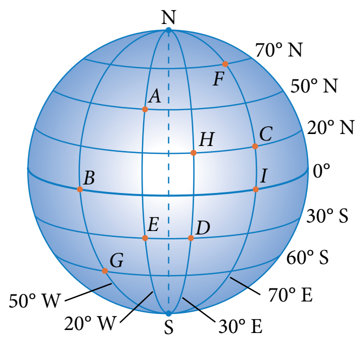
6 Match each country to its position longitude from the following list.

80° W 10° E 20° W 0° E 60° W 115° E 100° W 30° E

a America **b** Argentina **c** China **d** England

e Egypt **f** Iceland **g** Italy **h** Mexico

7 WORKED EXAMPLE 2 State the coordinates (latitude and longitude) of each point shown on the diagram.



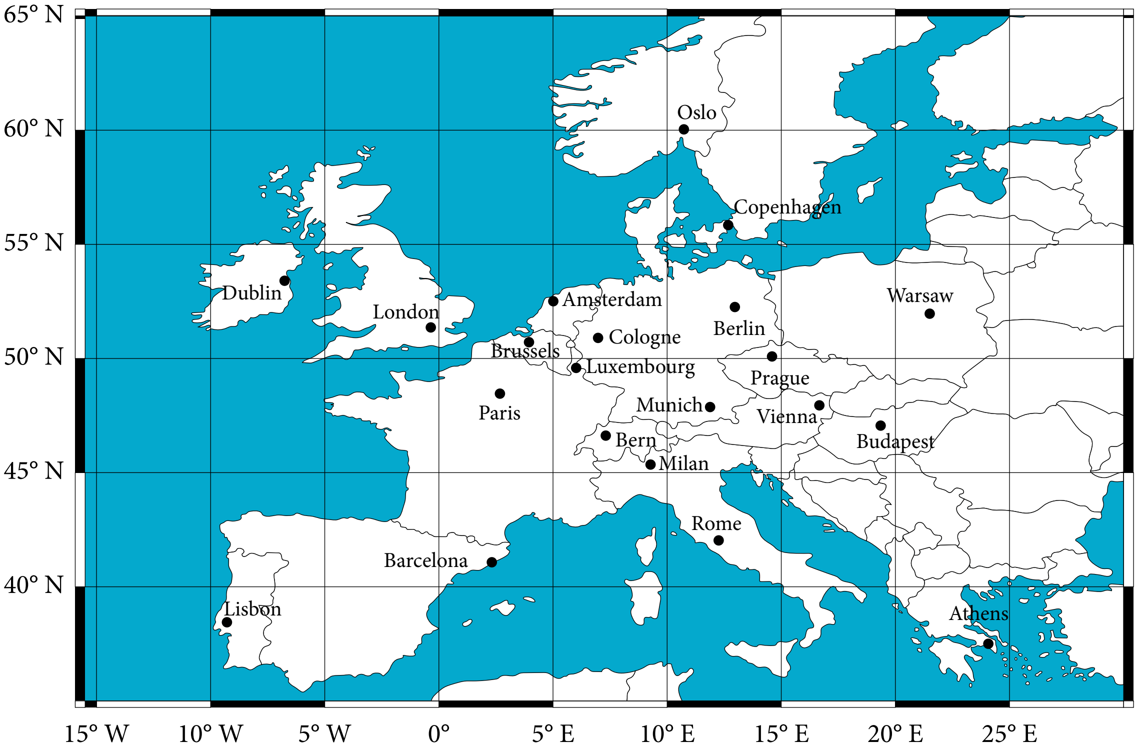
8 WORKED EXAMPLE 3 Convert each angle size from decimal degrees to degrees and minutes.

a 23.40° **b** 86.15°

9 Convert each angle size from degrees and minutes to decimal degrees.

a 102° 52′ **b** 35° 25′

Refer to the following map for Questions 10 and 11.



10 WORKED EXAMPLE 4 Write the position coordinates of

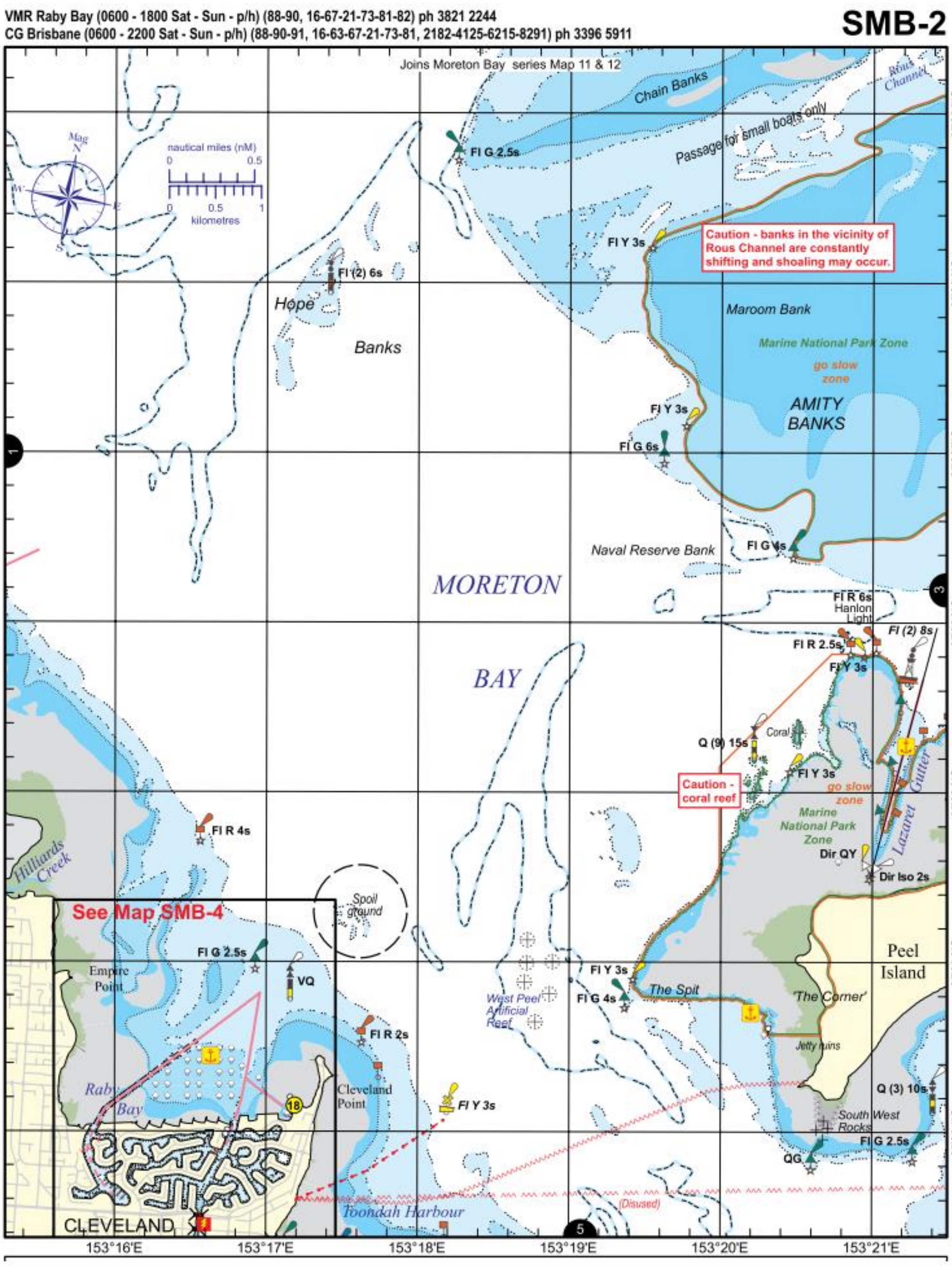
a Prague **b** Lisbon **c** London **d** Rome

11 Name the city located at

a 48° 50′ N 02° 20′ E **b** 59° 55′ N 10° 45′ E

c 37° 58′ N 23° 46′ E **d** 53° 21′ N 06° 15′ W

Refer to the following maritime chart for Questions 12 and 13.



12 WORKED EXAMPLE 5 Write the position of each place in degrees and minutes.

a Peel Island **b** Hanlon Light **c** Empire Point

13 Determine what is located at each position.

a 27° 30′ S 153° 19′ E **b** 27° 31′ S 153° 16′ E

14 Use Google Earth™ or another application to find the position in degrees and minutes of

a Uluru
b Helsinki
c Brisbane
d Gibraltar
e Sydney Opera House
f Trevi Fountain
g Niagara Falls (Canada)
h Lake Mungo (Australia)

15 Use Google Earth™ or another application to name the place situated at

a 25° 11′ S 153° 08′ E
b 33° 58′ S 18° 26′ E
c 32° S 115° 32′ E
d 40° 44′ N 74° 01′ W
e 29° 59′ N 31° 08′ E
f 50° 51′ N 4° 21′ E
g 22° 20′ N 114° 10′ E
h 51° 30′ N 0° 08.5′ W

Exam practice

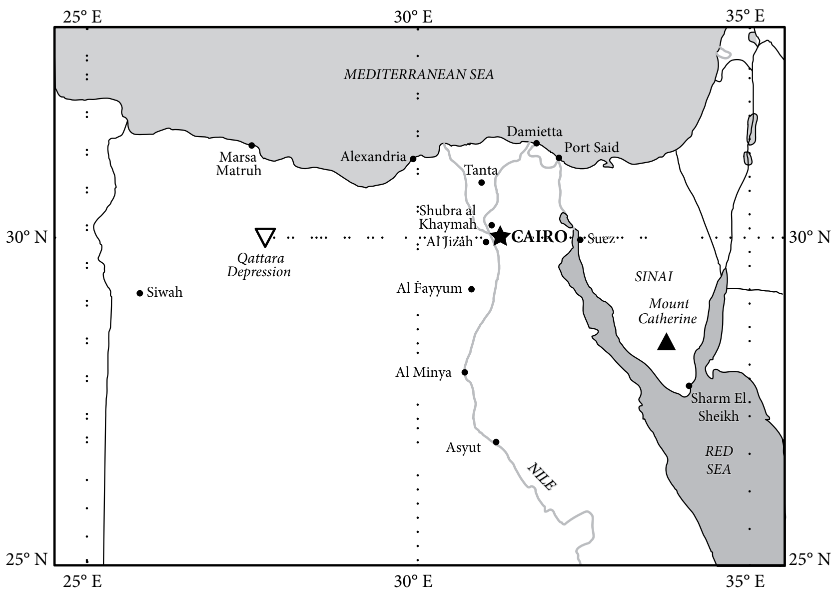
16 Convert 78.15° to degrees and minutes.

A 78° 9′
B 78° 10′
C 78° 15′
D 78° 25′

17 Calculate the length of a great circle on a sphere with a radius of 40 cm.

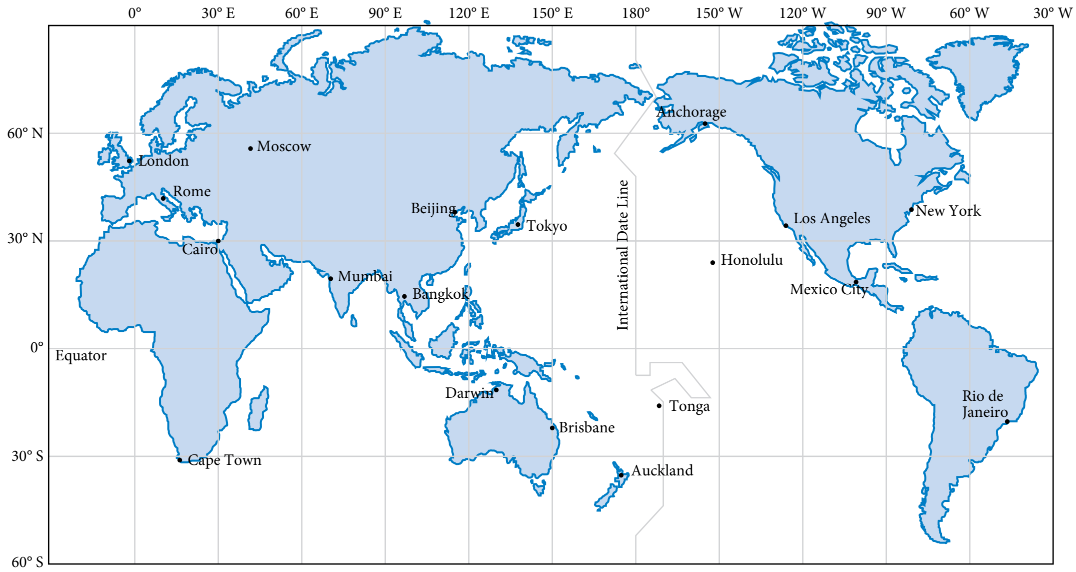
A 251.3 cm
B 200 cm
C 125.7 cm
D 320 cm

18 © QCAA 2020S 1Q6 Use the map to determine which city is located closest to (31° N 31° E).



A Tanta
B Damietta
C Alexandria
D Al Fayyum

19 (4 marks) On this map



a which 2 cities are approximately on the same parallel of latitude and what is the latitude? *[2 marks]*

b which city has the coordinates 35° N 140° E? *[1 mark]*

c what are the coordinates of Rio de Janeiro to the nearest 5°? *[1 mark]*

20 © QCAA 2024 1Q1 70% A location with coordinates (28° N 16° W) is positioned.

A 28° north of the prime meridian and 16° west of the equator.

B 28° north of the equator and 16° west of the prime meridian.

C 28° north of the International Date Line and 16° west of the equator.

D 28° north of the equator and 16° west of the International Date Line.

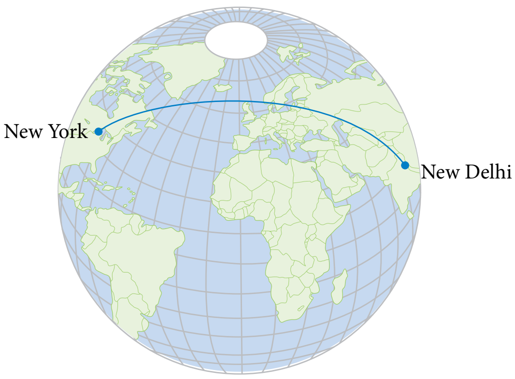
Worksheet Distances on the Earth

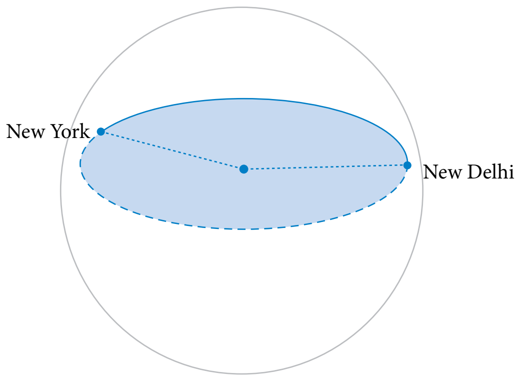
5.2 Distances on the Earth

Distances on the same meridian

The shortest distance between 2 points on a flat surface is a straight line, but the Earth is not flat. It is a sphere, so the shortest distance between 2 locations on the Earth's surface is along the arc of a great circle that connects the locations.

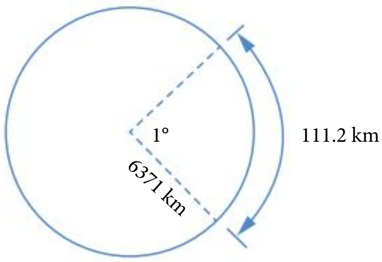
For example, to find the distance from New York, USA, to New Delhi, India, you would find the arc of the great circle that connects the 2 cities.





Great circles on the Earth have an average radius of 6371 km, so their circumference is $2 \times \pi \times 6371 = 40\,030.173...$ km.

If you travelled through an angle of 1° along a great circle on Earth, the arc length would be: $\frac{1}{360} \times 40\,030.173 = 111.194...$ km. This is $\frac{1}{360}$ of the circumference of the Earth.



5.2

Points that lie on the same meridian are on a great circle. To calculate the distance between 2 points on the same meridian, use the difference in their latitudes (angular distance).

Distances between places on the same meridian

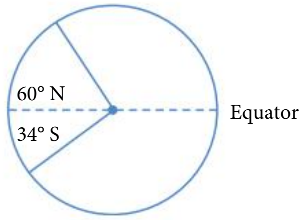
- The shortest distance between any 2 points on the Earth is along a great circle.
- On a great circle, $1° \approx 111.2$ km.
- The distance, D km, between 2 points on the same meridian is $D = 111.2 \times$ angular distance.

WORKED EXAMPLE 6 Distance between places in different hemispheres

Stockholm in Sweden is located at 60° N 18° E (northern hemisphere) and Cape Town in South Africa is located at 34° S 18° E (southern hemisphere).

a **Determine** the angular distance between Stockholm and Cape Town.

b **Calculate** the distance between Stockholm and Cape Town.

Steps	Working
a 1 Check the places lie on the same meridian.	Both cities lie on the meridian at 18° E.
2 Draw a diagram to visualise their position relative to the equator.	
3 As the cities are on different sides of the equator, add the latitudes.	angular distance = 60° + 34° = 94°
4 State the result.	The angular distance is 94°.
b 1 Calculate the distance to the nearest kilometre.	$D \approx 111.2 \times 94°$ $\approx 10\,453$ km
2 State the result.	The distance between Stockholm and Cape Town is about 10 453 km.

Video
Distances between places in different hemispheres

WORKED EXAMPLE 7 Distance between places in the same hemisphere

Calculate the distance between Auckland Island (51° S 166° E) and Noumea (22° S 166° E).

Steps	Working
1 Check the places lie on the same meridian.	Both cities lie on the meridian at 166° E.
2 As both places are on the same side of the equator (southern hemisphere), subtract the latitudes.	angular distance = 51° – 22° = 29°
3 Calculate the distance to the nearest kilometre.	$D \approx 111.2 \times 29°$ ≈ 3225 km
4 State the result.	The distance between Auckland Island and Noumea is about 3225 km.

Exam hack

To find the angular distance of places on the same meridian:

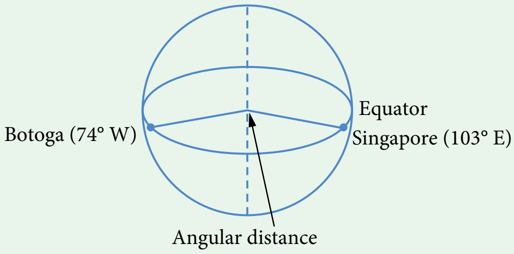
- if they are on the same side of the equator, find the difference between the latitudes
- if they are on different sides of the equator, add the latitudes.

Distances on the same parallel of latitude

The equator is the only parallel of latitude that is a great circle. To find the distance between two points on the equator, use the same formula used in Examples 6 and 7, except that the angular distance is not between parallels of latitude but instead between meridians of longitude.

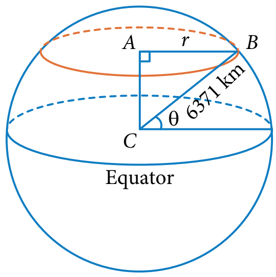
WORKED EXAMPLE 8 Distance between places on the equator

Determine the shortest distance between Singapore (103° E) and Botoga, Columbia (74° W).



Steps	Working
1 Find the angular distance between 74° W and 103° E. Because they are on opposite sides of the prime meridian, add the longitudes.	angular distance = 74° + 103° = 177°
2 Calculate the distance to the nearest kilometre.	D ≈ 111.2 × 177° ≈ 19 682 km
3 State the result.	The distance between Singapore and Botoga is about 19 682 km.

If two places lie on the same parallel of latitude (other than the equator), they lie on the circumference of a small circle.



The radius of a small circle that is a parallel of latitude is obviously less than 6371 km and its value decreases the further away the circle is from the equator.

In the diagram, r is the radius of the small circle of latitude of θ from the equator.

By alternate angles between parallel lines, $\angle B$ is also θ.

By trigonometry, $\cos\theta = \dfrac{r}{6371}$

$$r = 6371\cos\theta$$

You can use this radius of a small circle of latitude to calculate the distance between 2 points on the same parallel of latitude, using the difference in their longitudes (angular distance).

The circumference of a small circle is $2\pi r = 2\pi \times 6371\cos\theta = 12\,742\pi\cos\theta$.

The arc length of 1° of this small circle is $\dfrac{1}{360} \times 12\,742\pi\cos\theta = 111.194\ldots\cos\theta$

$$\approx 111.2\cos\theta$$

Distances between places on the same parallel of latitude

- On a parallel of latitude, $1° \approx 111.2\cos\theta$ km, where θ is the angle of latitude.
- The distance, D km, between 2 points on the same parallel of latitude is $D = 111.2\cos\theta \times$ angular distance, where θ is the angle of latitude.

WORKED EXAMPLE 9	**Distances between places on the same parallel of latitude**
Calculate the distance along the parallel of latitude between Bowen, QLD (20° S 148° E) and Port Hedland, WA (20° S 119° E).	
Steps	**Working**
1 Calculate the angular distance between 148° E and 119° E. Because they are on the same (eastern) side of the prime meridian, subtract the longitudes.	angular distance = 148° − 119° = 29°
2 Write the formula.	$D = 111.2\cos\theta \times$ angular distance
3 Substitute in the latitude and angular distance.	$D \approx 111.2 \times \cos(20°) \times 29$
4 Evaluate and round.	≈ 3030 km
5 State the result.	It is about 3030 km along the parallel of latitude from Bowen to Port Hedland.

5.2

Exam hack

If you are asked to find the shortest distance between 2 places, the angular distance must be less than 180°.

To find the angular distance of places on the same latitude:

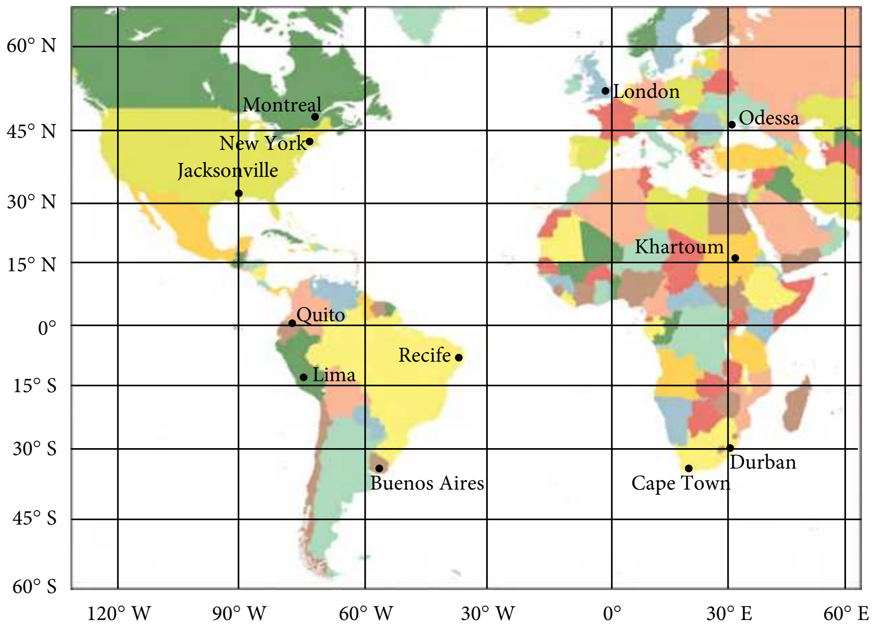
- if they are on the same side of the prime meridian, find the difference between the longitudes.
- if they are on different sides of the prime meridian, add the longitudes.

EXERCISE 5.2 Distances on the Earth

ANSWERS p. 453

Recap

Refer to the map below for Questions 1 and 2.



1 The approximate coordinates for Durban are

A 30° E 32° S **B** 30° E 32° N **C** 30° S 32° E **D** 30° N 32° E

2 Which city is located at 46° N 74° W?

A London **B** Montreal **C** New York **D** Odessa

Mastery

Express answers to the nearest whole number.

3 WORKED EXAMPLE 6 Perth is located at 31° S 116° E and Beijing in China is located at 40° N 116° E.

a Determine the angular distance between Perth and Beijing.

b Determine the distance between Perth and Beijing.

4 WORKED EXAMPLE 7 Cooktown in Queensland and Kyabram in Victoria lie on the same meridian. What is the distance between the two towns if Cooktown is located at 15° 47′ S 145° E and Kyabram at 36° 32′ S 145° E?

5 What is the distance from Boston (42° N 71° W) to Santiago, Chile (33° S 71° W)?

6 WORKED EXAMPLE 8 Kampala (32° E) in Uganda and Quito (78° W) in Ecuador both lie approximately on the equator. Find the shortest distance between Kampala and Quito to the nearest kilometre.

7 WORKED EXAMPLE 9 What is the distance from Brisbane (27° S 153° W) to Corrientes, Argentina (27° S 59° W) along the parallel of latitude?

8 What is the distance between Montevideo, Uruguay (35° S 56° W) and Canberra (35° S 149° E) along the parallel of latitude?

Exam practice

Questions 9 and 10 refer to Puerto Toro, Chile (55° 05′ S 67° 05 W), the farthest south that people permanently live on Earth (excluding Antarctic Stations).

9 Approximately how far is Puerto Toro from the equator?

A 4876 km **B** 5923 km **C** 6125 km **D** 7459 km

10 Approximately how far is Puerto Toro from the South Pole?

A 3781 km **B** 3883 km **C** 3892 km **D** 16 133 km

11 © QCAA 2020S 1Q28 (3 marks) The approximate coordinates of Geelong, Australia, are 38° 09′ S, 144° 21′ E and Kushiro, Japan, are 43° 01′ N, 144° 21′ E.

Calculate the distance between Geelong and Kushiro to the nearest kilometre.

12 © QCAA 2020S 1Q29 (4 marks) Seoul, South Korea, is located at approximately 37.6° N, 127.0° E. San Francisco, USA, is located at approximately 37.6° N, 122.4° W. Determine the shortest distance between these two cities to the nearest kilometre.

13 (6 marks) CF At which latitude(s) is the radius of a small circle on Earth half the radius of the equator, and what is the circumference of the parallels of latitude?

5.3 Practical problems with distances

Latitude, longitude and distance calculations are present in many practical applications. They are used for navigating a ship, flying an aircraft, or simply finding your way around using a GPS device.

WORKED EXAMPLE 10 Finding the time taken to travel between places on land

Longreach and Alice Springs both lie on the Tropic of Capricorn (23° S). If you could drive directly between Longreach (144° 16′ E) and Alice Springs (133° 53′ E) at 110 km/h, how long would it take?

Steps	Working
1 Determine if the towns lie on the same meridian or parallel of latitude.	Both towns lie on the parallel of latitude 23° S.
2 Find the angular distance.	angular distance = 144°16′ − 133°53′ = 10°23′
3 Calculate the distance for a difference of 10°23′ of longitude at a latitude of 23°.	$D \approx 111.2 \times \cos(23°) \times 10° 23'$ ≈ 1063 km
4 Write the rule for speed.	$\text{speed} = \dfrac{\text{distance}}{\text{time}}$
5 Substitute known values.	$110\text{ km/h} = \dfrac{1063\text{ km}}{\text{time}}$
6 Rearrange to solve for time and evaluate.	$\text{time} = \dfrac{1063}{110}$ ≈ 9.67 hours
7 State the result.	It is about 9.67 hours to travel from Longreach to Alice Springs.

WORKED EXAMPLE 11 Distances along a parallel of latitude

San Francisco, USA, and Athens, Greece, are located on the same latitude (38° N).
Their longitudes are 122° W and 23° E respectively. **Determine** the distance between the 2 cities if

a flying west from Athens to San Francisco

b flying east from Athens to San Francisco.

Steps	Working
a 1 Find the angular distance between 122° W and 23° E. Add them as they are on opposite sides of the prime meridian. San Francisco $\longleftarrow$ Athens 122° W 23° E	angular distance = 122° + 23° = 145°
2 Calculate the distance for a difference of 145° of longitude at a latitude of 38°.	$D \approx 111.2 \times \cos(38°) \times 145°$ $\approx 12\,705.89$ km
3 State the result.	The distance flying west from Athens to San Francisco is 12 705.89 km.
b 1 Flying east from Athens means taking the longer way around the Earth in the opposite direction, so the angular distance will be 360° − 145°. Athens $\longrightarrow$ San Francisco 122° W 23° E	alternate angular distance = 360° − 145° = 215°
2 Calculate the distance for a difference of 215° of longitude at a latitude of 38°.	$D \approx 111.2 \times \cos(38°) \times 215°$ $\approx 18\,839.76$ km
3 State the result.	The distance flying east from Athens to San Francisco is 18 839.76 km.

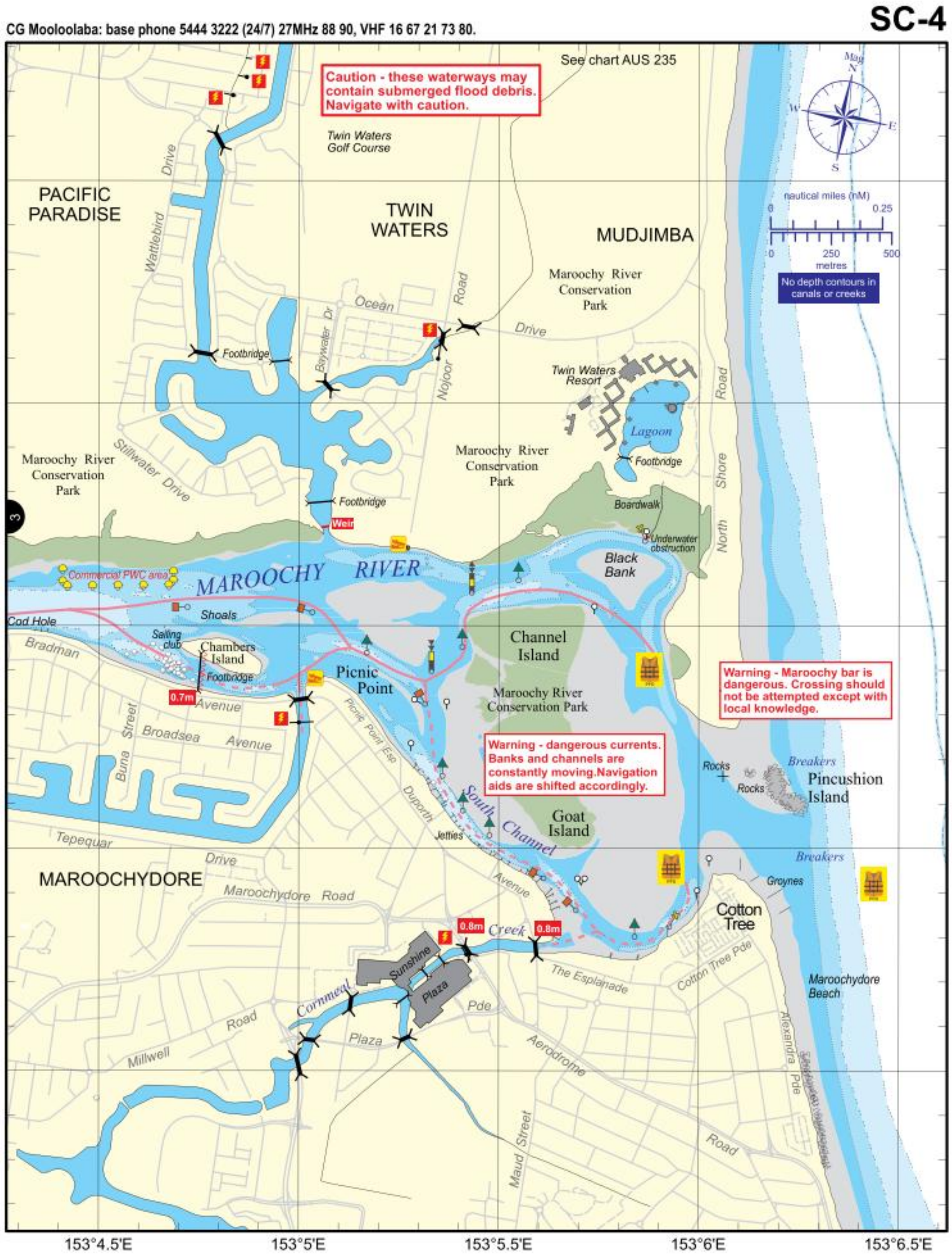
Recap

1 Find the distance between Melbourne (38° S 145° E) and Wewak, Papua New Guinea (4° S 145° E).

2 What is the distance between the Indian cities Aurangabad (20° N 75° E) and Puri (20° N 86° E), along the parallel of latitude?

Mastery

Refer to the following maritime chart for Questions 3 and 4.



3 Find the difference in latitude and longitude between Chambers Island and Black Bank.

4 Identify the latitudes and longitudes which contain the Lagoon at Twin Waters Resort.

5 WORKED EXAMPLE 10 How long would it take a light aircraft to fly at 210 km/h from Barcelona (41° N 2° E) to Tirana, Albania (41° N 20° E) along the parallel of latitude?

6 Find the time taken for a migrating bird to fly from Townsville (19° S 147° E) to Chimoio, Mozambique (19° S 33° E) if it flies along the parallel of latitude at 19 km/h.

7 WORKED EXAMPLE 11 Beijing and Madrid are located on the same latitude (40° N). Their longitudes are 116.4° E and 3.7° W respectively. Determine correct to 2 decimal places

a the distance flying west from Madrid to Beijing.

b the distance flying east from Madrid to Beijing.

c how long each flight would take if the speed of the plane is 600 km/h.

Exam practice

8 What is the circumference of a parallel of latitude at 45° N on Earth?

A 40 075 km B 28 307 km C 31 830 km D 20 038 km

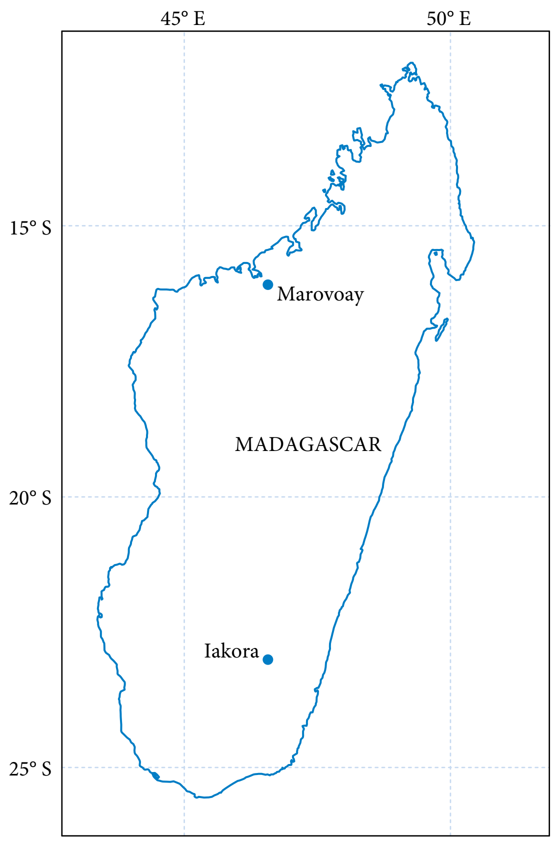
9 Perth (Australia) is at approximately 31° S 115° E while Ulaanbaatar (Mongolia) is at approximately 47° N 115° E. What is the approximate distance between Perth and Ulaanbaatar along the meridian?

A 7000 km B 8700 km C 9500 km D 10 400 km

10 © QCAA 2022 1Q22 (4 marks) Marovoay and Iakora are located on the same meridian at 46.6° E, as shown on the map of Madagascar.

a Determine the latitudes of Marovoay and Iakora. *[1 mark]*

b Use the result from part **a** to determine the shortest distance between Marovoay and Iakora. *[3 marks]*



11 (4 marks) CF A plane flying at 400 km/h travelled directly north for 6 hours and 40 minutes before making an emergency landing. If the plane took off from Hobart (43° S 147° E), what is the coordinates of where it landed?

12 (5 marks) CF An aircraft flies from City A (38° S 146° E) due north to City B (17° S 146° E). It then flies due west to City C (17° S 124° E). How far has the plane flown?

13 (6 marks) CU For this question, the radius of Earth is 6371 km and one nautical mile is approximately 1.852 km.

1 knot is a nautical mile per hour.

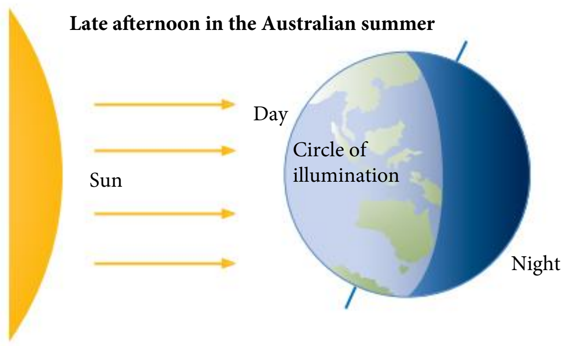
Darwin (Australia) is located at 12° S 131° E and Sorong (Indonesia) at 1° S 131° E. A fishing vessel can travel at an average speed of 15 knots (nautical miles per hour), from Darwin to Sorong. It needs to complete this trip in 48 hours or less. Will this be possible?

5.4 Longitude and time zones

As the Earth rotates, half of the Earth faces the sun and experiences day, while the other half experiences night at the same time.

During every 24-hour period, the **circle of illumination** sweeps around the Earth, bringing dawn at one edge and sunset at the other.

Time and **time zones** are largely dependent on the meridians of longitude. There are some exceptions, and these relate to local requirements.



Longitude and time

The Earth rotates 360° every 24 hours, so:

- dividing both sides by 24, each hour corresponds to a difference in longitude of 15°
- dividing both sides by 360, each degree corresponds to a 4-minute time difference.

Longitude and time

- 15° of longitude = 1-hour time difference
- 1° of longitude = 4-minute time difference

WORKED EXAMPLE 12 Time differences between meridians of longitude

Determine the time difference between longitudes separated by

a 60° **b** 78°

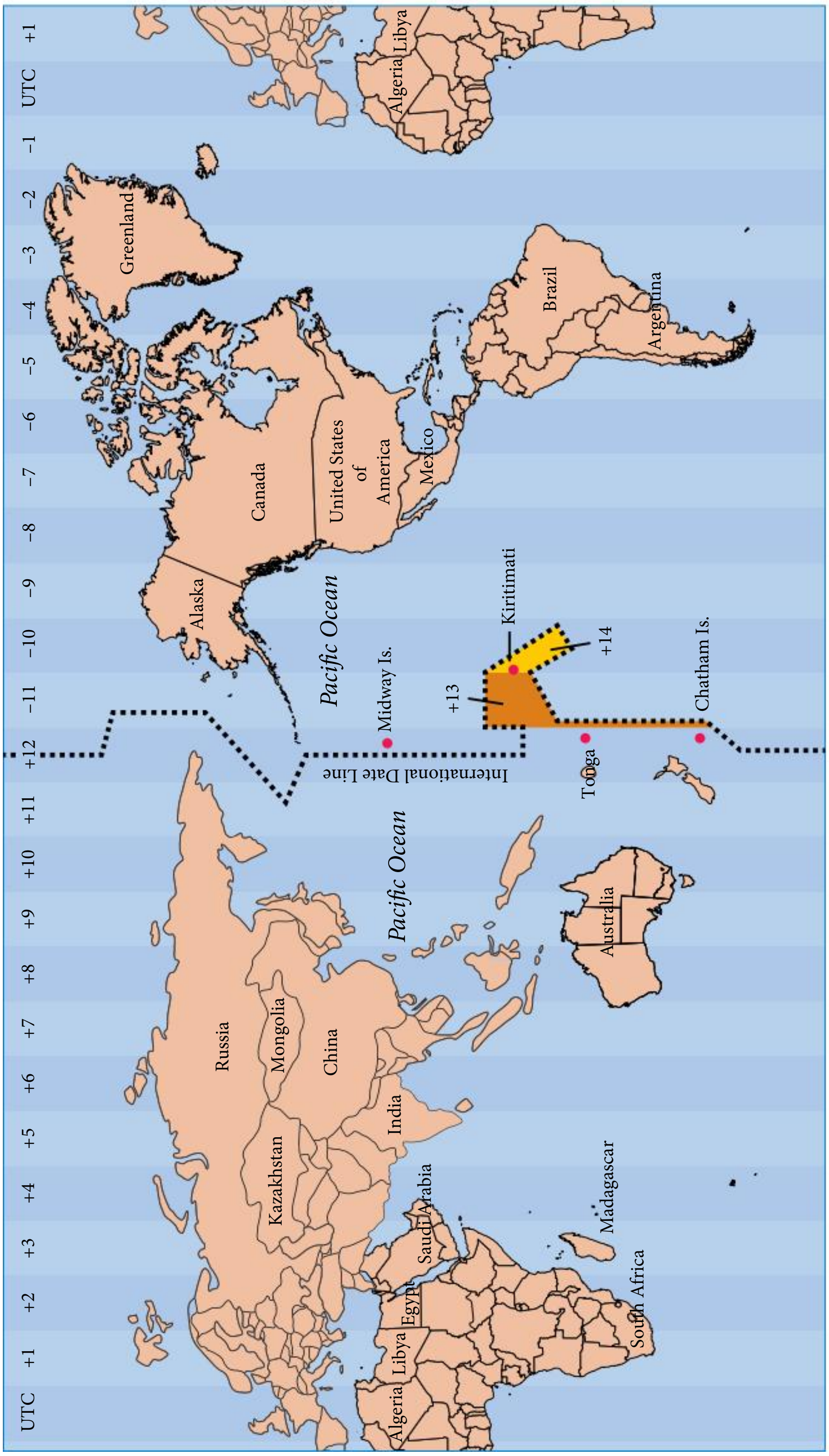
Steps	Working
a 1 Find the number of hours by dividing 60° by 15°.	$\frac{60^\circ}{15^\circ} = 4$
2 State the result.	There are 4 hours between longitudes separated by 60°.
b 1 Divide 78° by 15° to determine the hours.	$\frac{78^\circ}{15^\circ} = 5$ with remainder 3°
2 Multiply the remainder 3° by 4 to determine the minutes.	$3^\circ \times 4 = 12$ Hint: You can check your answer with your calculator using the ° ' " function.
3 Combine and state the result.	There are 5 hours and 12 minutes between longitudes separated by 78°.

GMT, UTC and standard time zones

Times around the Earth are related to the time along the prime meridian (also called the Greenwich Meridian).

Greenwich Mean Time (GMT) and **Coordinated Universal Time (UTC)** are both time standards used to regulate timekeeping around the world. GMT is based on the mean solar time at the prime meridian in England, whereas UTC is adjusted to take account of changes in the Earth's rotation. While GMT and UTC are almost identical, UTC is the more precise and widely used time standard. The abbreviation is not 'CUT' because 'UTC' is used internationally to cover all languages.

The Earth is divided into **standard time zones**, which are all calculated in relation to the prime meridian using longitude. Time zones are placed approximately in 15° increments of longitude, with the clocks set 1 hour apart in neighbouring time zones.



For example, between Perth (116° E) and Brisbane (153° E), there are 153° – 114° = 37° of longitude, which equates to approximately 2 time zones (2 × 15° = 30°). These cities are therefore in time zones that are 2 hours apart.

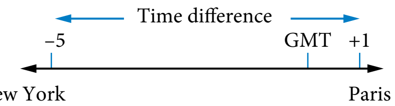
Time zones and places that are east of the prime meridian are ahead of GMT (UTC), while time zones and places west of the prime meridian are behind GMT (UTC).

WORKED EXAMPLE 13 Finding time difference between places using GMT or UTC

Eastern Standard Time in New York is GMT –5 and Central European Time in Paris is GMT +1.

a When GMT is 12 noon, what time is it in New York and Paris?

b **Determine** how many hours ahead or behind Paris is to New York.

Steps	Working
a **1** Identify how many hours ahead or behind each city is to GMT.	New York is GMT –5, meaning that New York is 5 hours behind GMT. Paris is GMT +1, meaning that Paris is 1 hour ahead of GMT.
2 Work out what time it is in each city.	When GMT is 12 noon: Eastern Standard Time in New York is 7:00 am. (12 – 5 = 7 am) Central European Time in Paris is 1:00 pm.
b **1** Subtract the relative times.	 Time difference = 1 – (–5) = 6 hours
2 State the result.	Paris is 6 hours ahead of New York.

Australian time zones

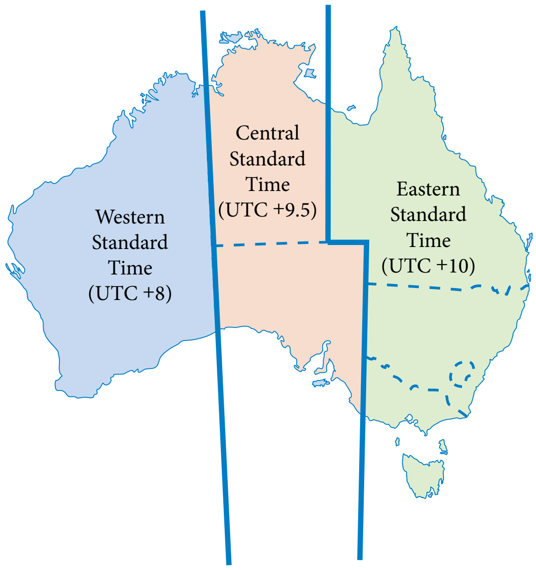
Australia is divided into 3 time zones, Eastern Standard Time, Central Standard Time and Western Standard Time.

The **Australian Eastern Standard Time (AEST)** zone covers the whole of Queensland, New South Wales, the ACT, Victoria and Tasmania. It is 10 hours ahead of UTC (or GMT).

The **Australian Central Standard Time (ACST)** zone covers South Australia and the Northern Territory, but it is only half an hour behind Eastern Standard Time. It is 9.5 hours ahead of UTC.

The **Australian Western Standard Time (AWST)** zone is 2 hours behind Eastern Standard Time and covers Western Australia. It is 8 hours ahead of UTC.

NSW, Victoria, Tasmania, South Australia and the ACT add one hour to their time zones between October and April for **daylight saving** to take advantage of the longer periods of sunlight during summer.

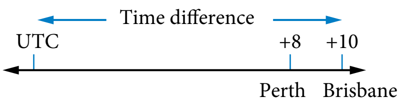


WORKED EXAMPLE 14 Australian time zones

If it is 9 am in Brisbane (UTC +10), **determine** what time is it in

a Perth (UTC +8)?

b Sydney (UTC +10) when daylight saving is applied?

Steps	Working
a 1 Work out the relative time difference.	 Brisbane (UTC +10) is 2 hours ahead of Perth (UTC +8).
2 Work out what time it is in Perth.	If it is 9 am in Brisbane, then: 9 am − 2 hours = 7 am Perth is 2 hours behind Brisbane.
3 State the answer.	If it is 9 am in Brisbane, then it is 7 am in Perth.
b 1 Work out the relative time difference.	Brisbane remains at UTC +10 all year round, while Sydney moves to UTC +11 during daylight saving time (DST). So, Sydney will be 1 hour ahead.
Work out what time it is in Sydney.	If it is 9 am in Brisbane (UTC +10): 9 am + 1 hour = 10 am
2 State the result.	If it is 9 am in Brisbane, then it is 10 am in Sydney.

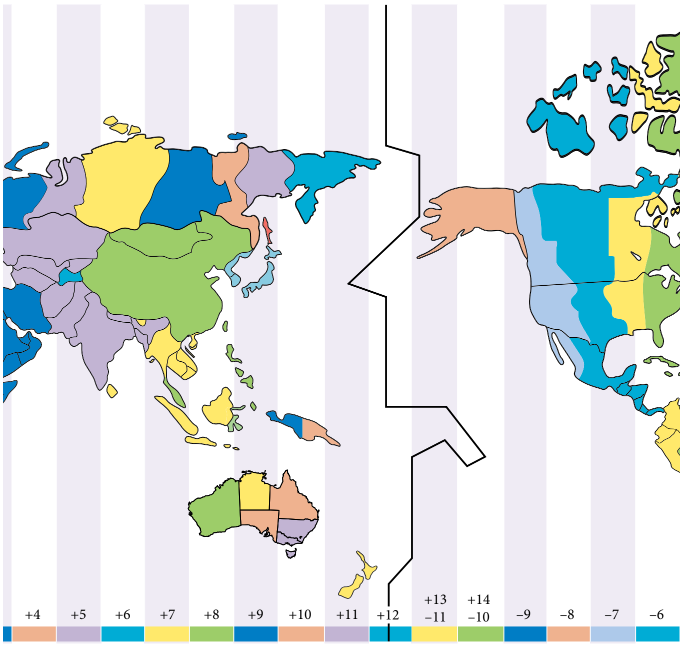
Exam hack

When a question mentions daylight saving, make sure you add 1 hour to the UTC value of the location.

International Date Line

The International Date Line (IDL) is located opposite the prime meridian at the 180° meridian.

It runs from the North Pole to the South Pole, but it is not a straight line as it is adjusted around country and political borders. The date changes at the International Date Line.



Aman Bhagat/Quora

The International Date Line

When you cross the International Date Line:

- from west to east, you subtract a day (for example, when flying from Australia to Hawaii, Saturday becomes Friday)
- from east to west, you add a day (for example, when flying from Hawaii to Australia, Monday becomes Tuesday).

WORKED EXAMPLE 15 Finding times and days in different time zones

If it is 11:36 pm on Sunday in Alice Springs (GMT +9.5), **determine** the time and day in Melbourne (GMT +10).

Steps	Working
1 Work out the relative time difference.	Melbourne (GMT +10) is 0.5 hours ahead of Alice Springs (GMT +9.5).
2 Work out what time it is in Melbourne.	11:36 pm Sunday + 0.5 hours = 12:06 am Monday
3 State the result.	If it is 11:36 pm on Sunday in Alice Springs, then it is 12:06 am on Monday in Melbourne.

EXERCISE 5.4 Longitude and time zones

ANSWERS p. 453

5.4

Recap

1 Calculate the circumference of the parallel of latitude at 35° N.

2 Find the distance between Perth (32° S 116° E) and Taree (32° S 152° E) along the parallel of latitude.

Mastery

3 WORKED EXAMPLE 12 What is the time difference between longitudes separated by

a 90° b 74° c 32° d 153°

4 WORKED EXAMPLE 13 If it is 6:00 am Tuesday in Dubai (UTC + 4), what time is it in Singapore (GMT + 8)?

5 WORKED EXAMPLE 14 If it is 4:00 pm in Sydney (UTC +10), what time is it in Darwin (UTC +9.5)?

6 WORKED EXAMPLE 15 Calculate the time in

a Tokyo (UTC +9) when it is 10:00 pm in Perth (UTC +8).

b Berlin (UTC +1) when it is 4:00 am Saturday in New York (UTC –5).

c Los Angeles (UTC –8) when it is 7:00 am Wednesday in Cape Town (UTC +2).

d Hobart (UTC +10) when it is 11:00 am in London (UTC +0).

Exam practice

7 What is the time difference between 2 cities separated by 45 degrees of longitude?

A 2 hours B 3 hours C 4 hours D 5 hours

8 Australia spans several time zones due to its vast geographical extent. Which city in Australia is located in the time zone known as Australian Eastern Standard Time (AEST)?

A Perth B Sydney C Darwin D Adelaide

9 If it is 12:00 pm in Melbourne (UTC +10), what time is it in Darwin (UTC +9.5)?

A 10:00 am B 10:30 am C 11:00 am D 11:30 am

10 (1 mark) What is the time difference in hours between Madrid (UTC +1) and Tokyo (UTC +9)?

11 (3 marks) If it is 5 am in Ottawa, Canada (UTC –5), what time would it be in Istanbul, Türkiye (UTC +2)?

12 (3 marks) City X is located at 30° S 100° W and City Y is located at 20° S 50° W. In relation to the sun rise in City Y, when will the sun rise in City X?

13 (3 marks) CF Sally is in Auckland, New Zealand (UTC +12) on holidays. She decides to call her parents who live in Honolulu, Hawaii (UTC –10). If Sally calls on Sunday at 8:15 pm, what time will it be at her parents' house?

14 © QCAA 2024 1Q10 46% The local time in Osaka (35° N 135° E) is two hours ahead of the local time in Phnom Penh. What is the most likely longitude for Phnom Penh?

A 5° N B 65° N C 105° E D 165° E

Videos
International time zones 1
International time zones 2

Worksheets
Applications of time zones
Table of time zones
Map of the world

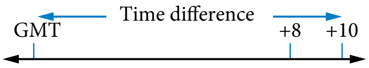
5.5 Practical problems with time zones

WORKED EXAMPLE 16 Time zones without daylight saving

It takes about 5 hours to fly from Brisbane (GMT +10) to Perth (GMT +8).

a If you take off at 8 am in Brisbane, at what time will you arrive in Perth?

b If you take off at 8 am in Perth, at what time will you arrive in Brisbane?

Steps	Working
a 1 Calculate the arrival time in Perth in terms of Brisbane time.	arrival time = 8 am + 5 hours = 1 pm
2 Work out the relative time difference.	 Perth (GMT +8) is 2 hours behind Brisbane (GMT +10).
3 Work out the actual time it is in Perth when the plane arrives.	1 pm – 2 hours = 11 am in Perth
4 State the result.	The flight will arrive in Perth at 11 am local time.
b 1 Calculate the arrival time in Brisbane in terms of Perth time.	arrival time = 8 am + 5 hours = 1 pm
2 Work out the actual time it is in Brisbane when the plane arrives.	1 pm + 2 hours = 3 pm in Brisbane
3 State the result.	The flight will arrive in Brisbane at 3 pm local time.

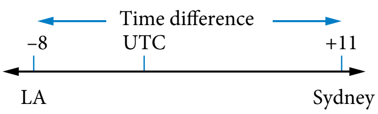
Video
Time zones with daylight saving

WORKED EXAMPLE 17 Time zones with daylight saving

A flight from Sydney (UTC + 10) to Los Angeles (UTC – 8) takes about 13 hours. The flight leaves Sydney at 1:30 pm on Sunday while daylight saving is in effect. When will the flight arrive in Los Angeles

a in Sydney time?

b in Los Angeles time?

Steps	Working
a 1 Calculate the arrival time in Los Angeles in terms of Sydney time.	arrival time = 1:30 pm Sunday + 13 hours = 2:30 am Monday
2 State the result.	The flight will arrive in Los Angeles at 2:30 am the next day (Monday), Sydney time.
b 1 Work out the relative time difference. Remember to adjust for DST.	Sydney moves to UTC + 11 during DST.  time difference = 11 + 8 = 19 hours Los Angeles is 19 hours *behind* Sydney.
2 Work out the actual time it is in Los Angeles when the plane arrives.	arrival time = 2:30 am (Monday) – 19 hours = 7:30 am (Sunday)
3 State the result.	The flight will arrive in Los Angeles at 7:30 am (Sunday) local time.

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The previous example demonstrates a paradox where you arrive at your destination 6 hours *before* you left (left at 1:30 pm, arrived at 7:30 am). This is because the 13-hour flight is negated by the 19-hour time difference. The opposite would be true if you were flying from Los Angeles to Sydney: the 13-hour flight would have the 19-hour time difference added to it, resulting in the Sydney arrival time being 32 hours after the Los Angeles time you left.

EXERCISE 5.5 Practical problems with time zones

ANSWERS p. 453

Recap

1 What is the time difference between two cities separated by 70° of longitude?

A 4 hours **B** 4.10 hours **C** 4 hours and 40 minutes **D** 5 hours

2 If it is 1:25 pm in London (UTC +0), what time is it in Cairns (UTC +10)?

A 9:25 am **B** 11:25 am **C** 9:25 pm **D** 11:25 pm

Mastery

3 WORKED EXAMPLE 16 Calculate the time in Brisbane (GMT +10) when it is

a 9:00 pm in Mogadishu, Somalia (GMT +3)
b 3:20 pm in Suva, Fiji (GMT +12)
c 2:45 pm in Belfast, Northern Ireland (GMT +0)
d 10:15 am in Rome, Italy (GMT +1).

4 An email is sent at 11:00 am on Wednesday from Sydney (GMT +10). Assuming that there are no transmission delays, work out the time and day when it will arrive in

a Bangkok, Thailand (GMT +7)
b Santiago, Chile (GMT –3)
c Lagos, Nigeria (GMT +1)
d Lima, Peru (GMT –5).

5 Sam works in Melbourne (UTC +10) and needs to organise an online meeting with Ruchi in Chicago (UTC –6). Given that Melbourne is currently in daylight saving time, if Sam schedules the meeting for 10 am Tuesday her time, what time would it be in Chicago for Ruchi?

6 WORKED EXAMPLE 17 A plane leaves Brisbane (UTC +10) at 7:00 am on Saturday for Narita airport, Tokyo (UTC +9). The direct flight takes 9 hours and 30 minutes. When it arrives, what will be the time in

a Brisbane?
b Tokyo?

7 A plane leaves Sydney (UTC +10) at 1:20 pm on Tuesday for Heathrow airport, London (UTC +0). The flight takes 25 hours and 45 minutes, including a stopover at Changi airport, Singapore (UTC +8). When it arrives, what will be the day and time in

a Sydney?
b London?

Exam practice

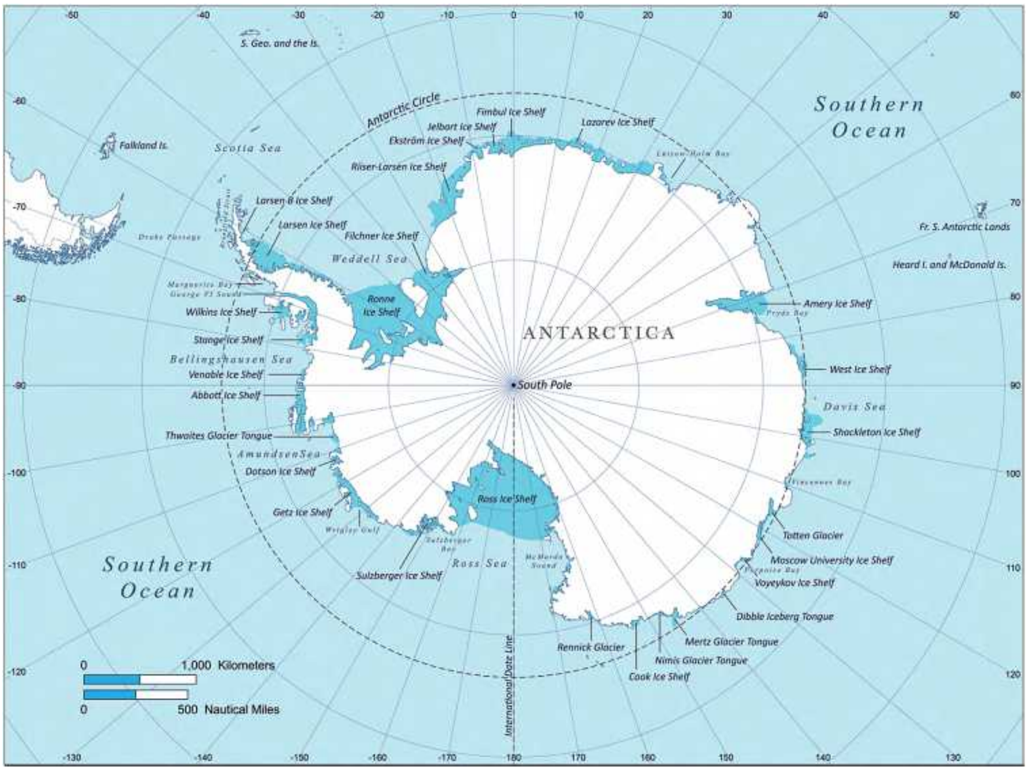
8 The location of Town X is 25° N 45° E. The location of Town Y is 10° N 105° E.

Which of the following is true? (Ignore time zones.)

A Town X is 4 hours behind Town Y.
B Town X is 4 hours ahead of Town Y.
C Town X is 1 hour behind Town Y.
D Town X is 1 hour ahead of Town Y.

9 (4 marks) A flight leaves Brisbane (UTC +10) at 4:30 pm on Tuesday and flies to Rome (UTC +1). Determine the local time and day the plane arrives in Rome if the flight takes 23 hours and 45 minutes (including stopovers).

10 (9 marks) There are around 70 permanent base stations on the Antarctic. The map below shows a few of these.



Gary Riegel/Shutterstock.com

a The 75° E meridian of longitude forms one half of a great circle. What meridian of longitude forms the other half of the great circle? *[1 mark]*

b What is the shortest distance between Fossil Bluff (71° S 70° W) and Casey (66° S 110° E) stations? *[3 marks]*

c What is the time difference between the 2 stations? *[1 mark]*

For **d** and **e**, consider a small plane that flies from Fossil Bluff to Casey:

d what landmark will it fly over and what are the coordinates of this landmark? *[2 marks]*

e how long will the flight take if the average speed of the plane is 200 km/h? *[2 marks]*

11 (4 marks) It takes $24\frac{3}{4}$ hours, including stopovers, to fly from Frankfurt (UTC +2) in Germany to Brisbane (UTC +10). When would a flight leaving Frankfurt at 7 am on Wednesday arrive in Brisbane?

12 (3 marks) CF Find the line of longitude that Marseille (France) sits on if the time difference between Marseille and Brisbane (27° S 153′ E) is 9 hours and 52 minutes.

13 © QCAA 2020S 2Q1 (4 marks) CF Sara departs from Sydney, Australia, on Saturday at 6:15 am on a flight to Johannesburg, South Africa. The flight takes 14 hours and 18 minutes.

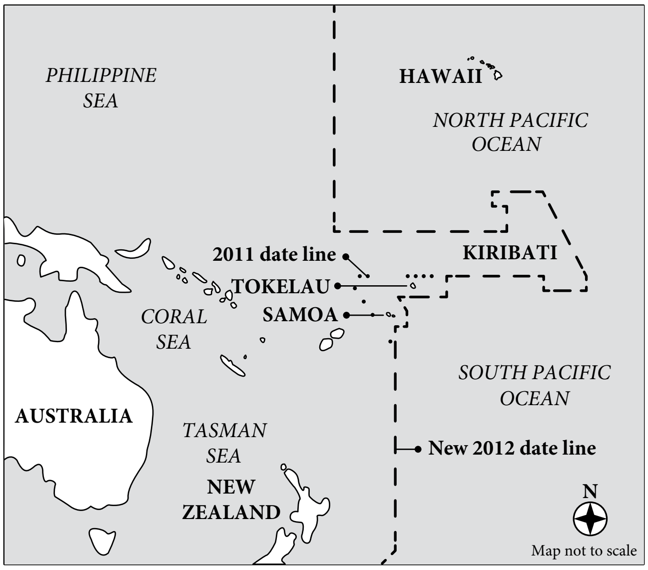
Her friend Marcus arrives in Johannesburg from Lima, Peru, at the same time, on a flight that took 18 hours and 15 minutes.

The time zones are shown in the following table.

City	Country	UTC
Sydney	Australia	+10
Lima	Peru	–5
Johannesburg	South Africa	+2

Determine the local time and day that Marcus left Lima.

14 © QCAA 2020S 2Q6 (3 marks) CU At midnight on 29 December 2011, the International Date Line was moved. This resulted in Samoa and Tokelau changing from the eastern side of the date line to the western side (see map below).



Determine what time, date and year it was in Samoa one minute after this change.

15 © QCAA 2024 2Q7 (7 marks) CU A non-stop flight departs Sydney (UTC +10) at 9:50 pm Tuesday local time and arrives in Los Angeles (UTC –8) at 6:50 pm Tuesday local time. Flight speed is assumed to be constant. Determine the local time and day in Sydney when the flight distance travelled is 4828 km, with 7242 km remaining.

Video
Exam question analysis: Earth geometry and time zones

EXAM QUESTION ANALYSIS

Exam-style question (6 marks)

Find the shortest distance (the diagonal distance) in km between Manila, The Philippines, (15° N 121° E) and Nadi, Fiji, (18° S 177° E) correct to the nearest kilometre.

Reading the question

- The question is asking for the diagonal distance.
- Locations of Manila and Nadi have been given.
- The answer must be rounded to the nearest kilometre.

Thinking about the question

- The distance north-south between Nadi and Manila is needed.
- The distance east-west between Nadi and Manila is needed.
- To find the diagonal distance, use Pythagoras' theorem.

Worked solution (✓ = 1 mark)

Find north-south distance between Nadi and Manila:

angular distance $= 15° + 18°$

$= \mathbf{33°}$ ✓

distance $= 33 \times 111.2$

$= \mathbf{3670\,km}$ ✓

Find east-west distance between Nadi and Manila:

angular distance $= 177° - 121°$

$= \mathbf{56°}$ ✓

distance $= 56 \times 111.2\cos 18°$

$= \mathbf{5922\,km}$ ✓

Find diagonal distance:

$c^2 = a^2 + b^2$

$\mathbf{c^2 = 3670^2 + 5922^2}$ ✓

$c = \sqrt{48\ 538\ 984}$

$c = 6966.9924\ldots$

The diagonal distance between Manila and Nadi is approximately 6967 km. ✓

5 Chapter summary

Latitude and longitude

- A **great circle** is the largest circle that can be drawn around the surface of a sphere.
- Great circles on the Earth's surface have their centre located in the centre of the Earth.
- The **equator** is a great circle.
- The **meridians of longitude** are semi-great circles that pass between the North and South poles.
- The **prime meridian** of longitude (0°) is the **meridian** passing through Greenwich, England.
- A **small circle** is a circle drawn on a sphere that is smaller than a great circle and its centre is not located in the centre of the sphere.
- **Parallels of latitude** are small circles parallel to the equator with their centres at the axis.
- The **latitude** of a point is the angle between the equator and the parallel of latitude passing through the point. It is measured north or south from the equator.
- The **longitude** of a point on the Earth is the angle between the prime meridian and a meridian passing through the point. It is measured east or west of the prime meridian.
- A position's coordinates are stated with latitude first and then longitude in the form 50° N 20° W.

Distances on the Earth

- The shortest distance between any 2 points on the Earth is part of a great circle.
- The distance, D km, between two points on the same meridian is $D = 111.2 \times$ **angular distance**.
- The distance, D km, between 2 points on the same parallel of latitude is $D = 111.2 \cos\theta \times$ angular distance, where θ is the angle of latitude.

Longitude and time zones

- 15° of longitude = 1 hour time difference.
- 1° of longitude = 4 minute time difference.
- **GMT (Greenwich Mean Time)** and **UTC (Coordinated Universal Time)** are used to regulate timekeeping around the world. UTC is more commonly used.
- **Time zones** are placed approximately in 15° increments of longitude, with the clocks set 1 hour apart in neighbouring time zones.
- Time zones are determined with reference to the prime meridian: UTC + zones to the east and UTC – zones to the west.
- The **International Date Line (IDL)** is the line at which the date is changed.

Puzzle
World crossword

Worksheet
General Maths formula sheet

Cumulative examination 1

Simple familiar

Perusal time: 2 minutes   Working time: 30 minutes

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

Section 1 (5 marks): 5 multiple choice questions

Section 2 (13 marks): 3 short response questions

Total: 18 marks

Section 1 5 multiple choice questions

5 marks

1 © QCAA 2020S 1Q15 The prime meridian is located at

A 0° latitude.

B 0° longitude.

C the intersection of 0° latitude and 0° longitude.

D the intersection of 0° latitude and 180° longitude.

2 © QCAA 2020 1Q2 The standard Australian time zones are shown on the map.

All states and territories, except Western Australia (WA), Queensland (QLD) and the Northern Territory (NT), have daylight saving in summer. Daylight saving time is 1 hour ahead of standard time.

When it is 10:00 am daylight saving time in New South Wales (NSW), it is

A 9:30 am in South Australia (SA) and 9:00 am in QLD.

B 9:30 am in SA and 11:00 am in QLD.

C 10:30 am in SA and 9:00 am in QLD.

D 10:30 am in SA and 11:00 am in QLD.

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3 © QCAA 2022 1Q15 The actual distance between two cities has been correctly calculated as 556 km. The latitude and longitude respectively of these two cities could be

A 2° N 104° W and 3° S 104° W.

B 2° N 104° W and 3° N 104° W.

C 25° N 150° E and 30° S 150° E.

D 25° N 145° E and 25° N 150° E.

4 © QCAA 2021 1Q3 The table shows the results of a student survey about their preferred movie genre.

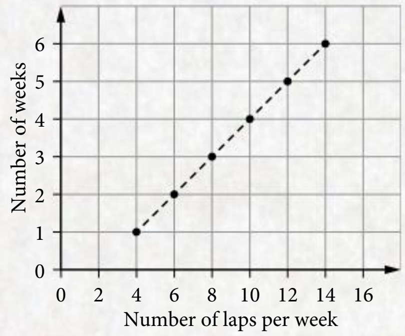
	Genre		
Year level	Comedy	Action	Science fiction
7–8	20	25	21
9–10	24	53	21
11–12	36	28	12

Of the students who preferred comedy, what percentage were in Year 9 or higher?

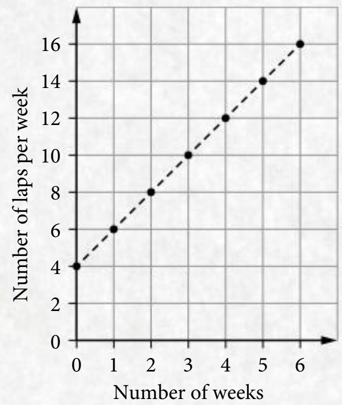
A 25% **B** 30% **C** 60% **D** 75%

5 © QCAA 2022 1Q4 A swimmer has a weekly training routine to improve their fitness as modelled by the recursive function $T_{n+1} = T_n + 2$, where T_n is the number of laps they swim in week n and $T_1 = 4$. Which graph best represents the swimmer's routine?

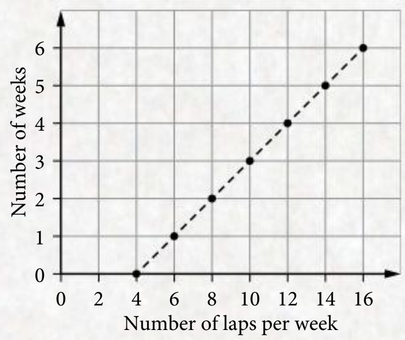
A



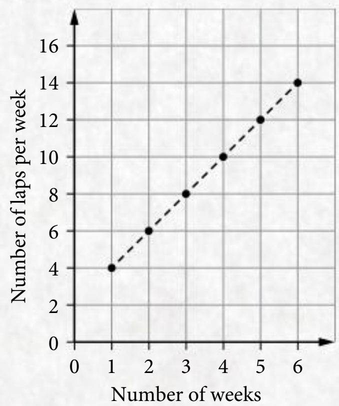
B



C

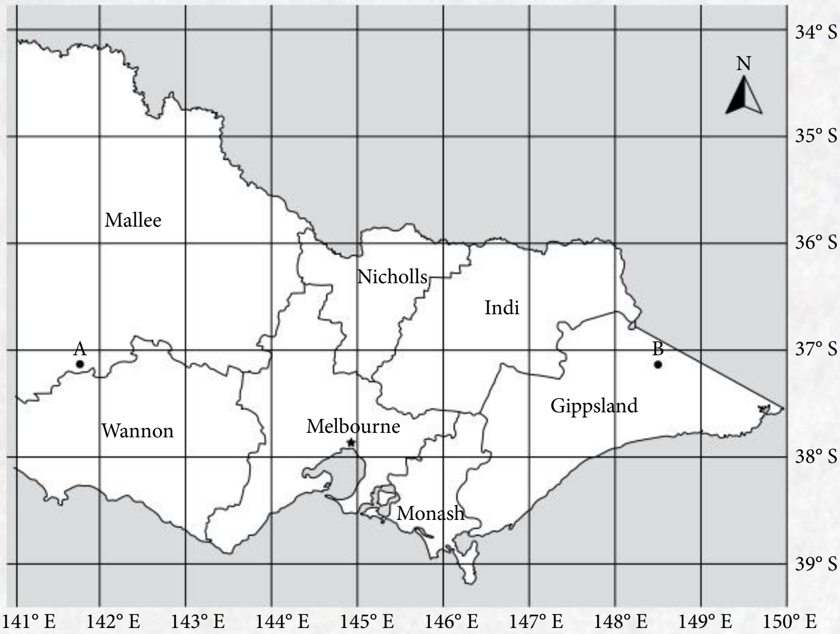


D



Section 2 3 short response questions 13 marks

6 © QCAA 2021 1Q21 MODIFIED (5 marks) The map shows regional federal electorates in Victoria.



Determine the distance between point A in the electorate of Mallee and point B in the electorate of Gippsland along the same parallel of latitude, to the nearest 100 km.

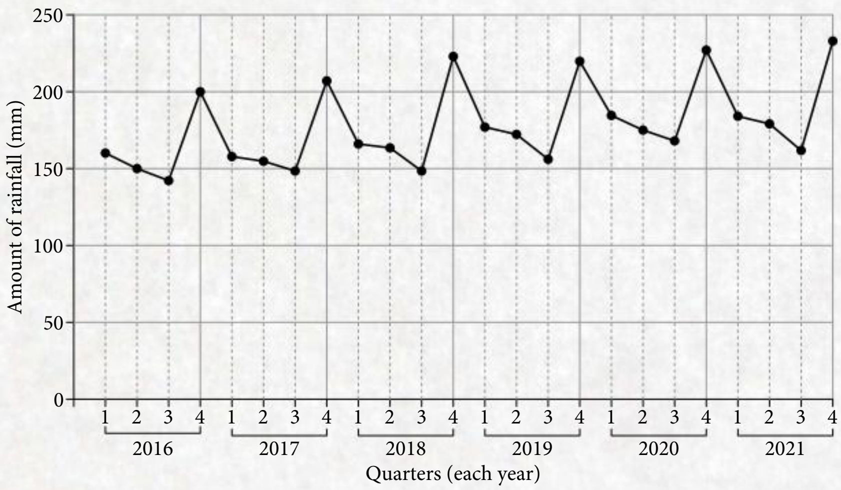
7 © QCAA 2021 1Q16 (4 marks) The table shows the values of bivariate data with variables g and h.

g	9	2.5	4.5	8	6	3.5	1
h	3.5	6.5	10	6	9.5	9	3

a Construct a scatterplot of the data with g as the explanatory variable. *[2 marks]*

b Describe the association in terms of form and strength. *[2 marks]*

8 © QCAA 2022 1Q19 (4 marks) The graph shows the amount of rainfall (in mm) for each quarter from 2016 to 2021.



a Describe the long-term trend and seasonality of the time series data. *[2 marks]*

b A least-squares line was fitted to the data, with y representing the amount of rainfall and x representing the number of quarters since the beginning of 2016 (e.g. $x = 5$ for the first quarter of 2017).

$$y = 1.763x + 156.5$$

Interpret the y-intercept and slope of the fitted line. *[2 marks]*

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Cumulative examination 2

Complex familiar and unfamiliar

Perusal time: 2 minutes   Working time: 30 minutes

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

2 short response questions

12 marks

Worksheet
General Maths formula sheet

1 © QCAA 2023 2Q4 (5 marks) Hiroki believes that more fish are caught on warmer days. Jiro believes that the number of fish caught in a day is more dependent on the number of people fishing.

Bivariate datasets for six days are shown.

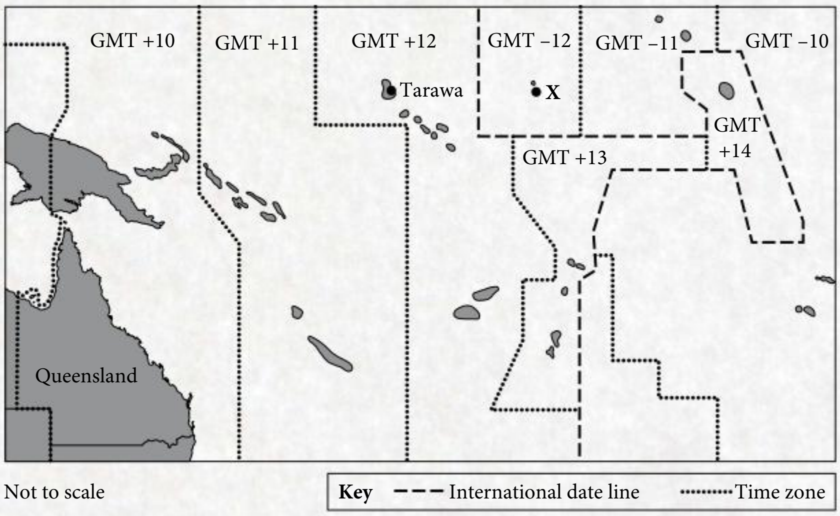
Temperature, t (°C)	32	26	20	27	23	29
Number of fish caught, f	530	400	320	220	180	120

Number of people fishing, p	46	58	38	34	30	28
Number of fish caught, f	530	400	320	220	180	120

Calculate the correlation coefficient for each dataset and use the results to identify the explanatory variable for the stronger linear association. Use the least-squares line equation for the stronger linear association to predict the number of fish caught on a 25°C day when 50 people are fishing.

2 © QCAA 2022 2Q7 (7 marks) You live in Queensland and your friend is on a cruise ship holiday.

As their ship departs from X to travel 1350 km due west to Tarawa, your friend sends you a message saying, 'Local time 6:12 am Wednesday and enjoying the sunrise as our ship begins its trip to Tarawa.'



You plan to phone your friend as soon as they arrive in Tarawa.

Assuming their ship is travelling at 50 km/h, determine the time in Queensland when you will phone your friend.

CHAPTER

6 LOANS, INVESTMENTS AND ANNUITIES

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Syllabus coverage

UNIT 4, TOPIC 1: LOANS, INVESTMENTS AND ANNUITIES 1

Compound interest loans and investments

- Use a recurrence relation to model a compound interest loan or investment.

  $A_{n+1} = rA_n$ where A_{n+1} is total amount at the beginning of the $(n + 1)^{\text{th}}$ period, A_n is total amount at the beginning of the n^{th} period, and $r = 1 + i$ where i is interest rate per compounding period
- Use the compound interest formula to model a compound interest loan or investment.

  $A = P(1 + i)^n$ where A is total amount, P is principal, i is interest rate per compounding period and n is number of compounding periods
- Calculate the effective annual rate of interest, $i_{\text{effective}}$, and use the results to compare interest on loans or investments when interest is paid or charged for different compounding periods, including daily, monthly, quarterly and six-monthly.

  $i_{\text{effective}} = (1 + i)^k - 1$, where i is interest rate per compounding period and k is number of compounding periods per year
- Solve practical problems involving compound interest loans or investments, including determining the total amount of the loan or investment, total interest, principal, interest rate per year and per compounding period, and the effect of the interest rate and number of compounding periods on the total amount.

Present value of ordinary annuities

- Use a recurrence relation to model the present value of an ordinary annuity, e.g. reducing balance loan or retirement pension with periodic payments where interest is calculated before the periodic payment is made.

  $A_{n+1} = rA_n - d$ where A_{n+1} is total amount at the beginning of the $(n + 1)^{\text{th}}$ period, A_n is total amount at the beginning of the n^{th} period, d is periodic payment, and $r = 1 + i$ where i is interest rate per compounding period
- Use the present value annuity formula to model the present value of an ordinary annuity, e.g. reducing balance loan or retirement pension with periodic payments where interest is calculated before the periodic payment is made.

  $A_{PV} = d\left(\dfrac{1-(1+i)^{-n}}{i}\right)$ where A_{PV} is total amount, d is periodic payment, i is interest rate per compounding period and n is number of compounding periods
- Solve practical problems involving the present value of an ordinary annuity, including determining the total amount of the annuity, periodic payment, total payments and total interest.

6.1

Prior learning
Loans, investments and annuities

Videos (2):

6.1 Principal and interest rate from compound interest formula

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Nelson MindTap

To access resources above, visit **cengage.com.au/nelsonmindtap**

Terminology

accrued value	amortisation	amortisation table	annuity (annuities)
compound interest	compounding period	effective interest rate	future value
interest	interest rate	loan schedule	nominal interest rate
per annum (p.a.)	periodic interest rate	periodic payments	present value
principal	recurrence relation	recursive rule	reducing balance loan
repayment	simple interest		

6.1 Compound interest

Recurrence relations

Chapter 2, *Arithmetic and geometric sequences*, established the use of **recursion** to generate arithmetic and geometric sequences.

A **recurrence relation** (or a **recursive rule**) is a mathematical rule used to generate a sequence. It has 2 parts:

- a starting point or term. This could also be an equation identifying the initial term.
- an equation which explains how to generate the following term from any term.

Interest is the 'fee' for using someone's money. When you borrow money from the bank, you are 'using' the bank's money, so you pay the bank interest. If you invest money in the bank, the bank is 'using' your money and pays you interest.

Simple interest is a fixed percentage of the amount borrowed or invested, calculated using the formula

$$I = Pin$$

where I is interest, P is **principal** (the amount borrowed or invested), i is annual **interest rate** as a decimal, and n is time period in years.

Most investments and loans use **compound interest**. This is where the interest is added to the principal and the interest for the next period is calculated using this new balance. The interest is calculated regularly, at the end of a set time, which is called the **compounding period**.

Compound interest is a recursive relation, as the amount invested or borrowed forms a geometric sequence for each consecutive compounding period.

Compound interest recurrence relation

The value of a compound interest investment at the end of the n^{th} compounding period is given by the recurrence relation

$$A_{n+1} = rA_n$$

where A_0 is the principal

A_n is total amount at the beginning of the n^{th} period

A_{n+1} is total amount at the beginning of the $(n+1)^{\text{th}}$ period

and $r = 1 + i$ where i is interest rate per compounding period.

- The total interest earned or charged after n compounding periods $= A_n - A_0$.
- The interest earned or charged after the n^{th} compounding period $= A_{n+1} - A_n$.

WORKED EXAMPLE 1 Finding the recursive relation

\$30 000 is invested at 5% p.a. compounded yearly. Write the recurrence relation that shows the value of the investment at the end of each year.

Steps	Working
1 State the value of A_0.	$A_0 = 30\,000$
2 Calculate r using $r = 1 + i$.	$r = 1 + 0.05$
	$r = 1.05$
3 Write the recursive relation using $A_{n+1} = rA_n$.	$A_{n+1} = 1.05A_n$, where $A_0 = 30\,000$

The recursive relation can be used to model the balance of a compound interest loan or investment.

WORKED EXAMPLE 2 Using the recursive relation to model compound interest

$45 000 is invested at 6.3% p.a. compounded yearly. **Determine**:

a the recursive relation that models the value of this investment at the end of each year

b how much the investment is worth at the end of 1, 2 and 3 years

c how much interest was earned in the 3rd year of the investment.

Steps	Working
a **1** State the value of A_0.	$A_0 = 45\,000$
2 Calculate r using $r = 1 + i$.	$r = 1 + 0.063$ $r = 1.063$
3 Write the recursive relation using $A_{n+1} = rA_n$.	$A_{n+1} = 1.063A_n$, where $A_0 = 45\,000$
b **1** Determine A_1.	$A_1 = 1.063 \times 45\,000$ $= 47\,835$
2 Repeat the process to find A_2 and A_3. In calculations, use the unrounded values. For money, round to 2 decimal places unless told otherwise.	$A_2 = 1.063 \times 47\,835$ $= 50\,848.605\ldots$ $A_3 = 1.063 \times 50\,848.60\ldots$ $= 54\,052.067\ldots$
3 State the results.	The investment is worth $47 835 after 1 year, $50 848.61 after 2 years and $54 052.07 after 3 years.
c **1** Calculate the interest in the 3rd year by finding the difference between A_2 and A_3.	Interest $= A_3 - A_2$ $= 54\,052.067\ldots - 50\,848.605\ldots$ $= 3203.462\ldots$
2 State the answer.	The interest earned in the 3rd year is $3203.46.

TECHNOLOGY Recurrence calculations on a calculator

A simple shortcut to perform repetitive calculations is to use the ANS function.
These steps show how this would be done for the investment in the previous example.

Steps	Casio fx-82AU PLUS II	Sharp EL-531TH	TI-30XB
1 Enter A_0.	Enter 45 000 and press =.	Enter 45 000 and press =.	Enter 45 000 and press ENTER.
2 Set up the recursive rule and calculate A_1.	Enter 1.063 × and press ANS, =.	Enter 1.063 × and press ALPHA, = =.	Enter 1.063 × and press 2nd, (−) ENTER.
	The result should be 47 835, which is A_1.		
3 Now, get the calculator to repeat the calculation to find A_2.	Press =.	Press =.	Press ENTER.
	The result should be 50 848.60…, which is A_2.		
4 Repeat the calculation to find A_3.	Press =.	Press =.	Press ENTER.
	The result should be 54 052.06…, which is A_3.		

You can repeat this process as many times as you need to. Just make sure to write down each result so you know how many times you have performed the calculation.

Compounding periods and interest rates

6.1

Compound interest is always stated as a rate per year, however the compounding periods often vary.

- Daily – means the interest is calculated every day and added to the account.
- Weekly – means the interest is calculated every week and added to the account.
- Other common compounding periods are shown in the table.

Compounding period	Number of periods per year
Yearly	1
Six-monthly	2
Quarterly	4
Monthly	12
Fortnightly	26
Weekly	52
Daily	365

Changing the number of compounding periods per annum affects the total interest earned on the principal amount invested. Shorter compounding periods mean that interest is added back to the principal more often and so the interest earned is higher.

Interest rate (i) per compound period

periodic interest rate, i, $= \dfrac{\text{annual interest rate as a decimal}}{\text{number of compounding periods per year}}$

WORKED EXAMPLE 3 Working with compound interest periods

\$5000 is invested at 7.1% p.a. interest **per annum (p.a.)** compounding monthly.

Determine

a the number of compounding periods in a year

b the number of compounding periods in 4 years

c the periodic interest rate for the investment.

Steps	Working
a State the number of compounding periods in a year.	There are 12 monthly compounding periods per year.
b Multiply the number of periods per year by 4 and state the answer.	$12 \times 4 = 48$ There are 48 monthly compounding periods over 4 years.
c Divide the annual rate by compounding periods per year. State the answer.	$i = \dfrac{0.071}{12} \approx 0.0059$ The periodic interest rate is 0.0059.

Consider the value of an investment A, with a principal of P and interest rate i over several compounding periods. The table shows the recursive relation calculations in general terms for each period.

Compounding period	A_n
A_0: Start	P
A_1: Period 1	$P(1 + i)$
A_2: Period 2	$P(1 + i) \times (1 + i) = P(1 + i)^2$
A_3: Period 3	$P(1 + i)^2 \times (1 + i) = P(1 + i)^3$
A_4: Period 4	$P(1 + i)^3 \times (1 + i) = P(1 + i)^4$
$\vdots$	$\vdots$
A_n: Period n	$P(1 + i)^{n-1} \times (1 + i) = P(1 + i)^n$

This illustrates how the general rule for compound interest was developed.

Compound interest formula

$$A = P(1 + i)^n$$

where A = total value of the investment (or loan) after n compounding periods (A_n). This is also called the **accrued value**.

P = principal or initial value (A_0)

i = periodic interest rate as a decimal

n = number of compounding periods

The total amount of interest = $A - P$.

WORKED EXAMPLE 4 Using the compound interest formula

On the day their grandchild is born, Tom and Geraldine deposit \$10 000 in a term deposit account that pays 8% p.a. interest, compounded six-monthly.

Calculate the amount in the account on the child's 18th birthday.

Steps	Working
1 Find the values of P, i and n.	$P = 10\,000$ There are 2 six-monthly periods in a year. $i = \frac{0.08}{2} = 0.04$ $n = 18 \times 2 = 36$
2 Substitute the values of P, i and n into the formula $A = P(1 + i)^n$ and evaluate.	$A = P(1 + i)^n$ $A = 10\,000(1 + 0.04)^{36}$ $= 41\,039.325\ldots$
3 State the result.	On the child's 18th birthday, the account will be worth approximately \$41 039.33.

Exam hack

If a question asks for how much an account is worth at a given time, you need to use the compound interest formula. If the question asks for the amount of interest over a given period, you need to use the compound interest recursive relation.

The compound interest formula can also be used in reverse to find an unknown principal or interest rate.

WORKED EXAMPLE 5 **Using the compound interest formula to find *P* and *i***

a Sam can invest some money at 9% p.a. interest compounded monthly.
How much should Sam invest if she wants to have $10 000 available in 5 years?

b Two years ago, Tanh placed $8000 in an investment, paying interest compounded quarterly.
If his money has grown to $9500, what interest rate p.a. did he earn?

Steps	**Working**
a 1 Need to find the value of P.	$A = 10\,000$
Find the values of A, i and n.	There are 12 monthly periods in a year.
	$i = \frac{0.09}{12} = 0.0075$
	$n = 5 \times 12 = 60$
2 Substitute the values of A, i and n into the formula $A = P(1+i)^n$ and simplify.	$A = P(1+i)^n$
	$10\,000 = P(1 + 0.0075)^{60}$
	$10\,000 = P \times 1.565\ldots$
3 Rearrange to isolate P and evaluate.	$P = \frac{10\,000}{1.565\ldots}$
	$= 6386.996\ldots$
4 State the result.	Sam should invest $6387.
b 1 Need to find the value of i.	$P = 8000$
Find the values of A, P and n.	$A = 9500$
	There are 4 quarterly periods in a year. $n = 2 \times 4 = 8$
2 Substitute the values of A, P and n into the formula $A = P(1+i)^n$.	$A = P(1+i)^n$
	$9500 = 8000(1+i)^8$
3 Rearrange and simplify.	$(1+i)^8 = \frac{9500}{8000}$
	$(1+i)^8 = 1.1875$
4 Use the $\sqrt[n]{\ }$ on the calculator to find the 8th root.	$1 + i = \sqrt[8]{1.1875}$
	$= 1.0217\ldots$
5 Rearrange to isolate i and evaluate.	$i = 1.0217\ldots - 1$
	$= 0.0217\ldots$ per quarter
6 Calculate the annual rate.	annual rate $= 0.0217\ldots \times 4 \times 100$
	$\approx 8.69\%$
7 State the result.	Tanh earned approximately 8.69% p.a.

6.1

Video
Principal and interest rate from compound interest formula

TECHNOLOGY Compound interest calculations

You can make a spreadsheet for compound interest calculations.

1 Put the following headings in the cells indicated.

A3: Annual interest rate (R)   A4: Principal (P)   A5: Periods per year (k)

A7: End of year   B7: Total amount   C7: Yearly interest   D7: Total interest

You can use cell formatting, colours and column widths to make it look professional.

2 Put in, say, 0.08 for the interest rate, 1000 for the principal, 365 for the periods per year in cells B3 to B5.

3 Enter the years for which interest will be calculated by placing a **1** into cell A8. In cell A9, enter the formula **=A8+1**. Copy this formula down from A9 to cell A19.

4 The total amount at the end of each year will be different because it will have interest earned for the year added in. Begin by entering the formula **=C$3*(1+C$2/C$4)^(A7*C$4)** into cell B8 and copying this formula down from B8 to B19.

NOTE: To format the cells to show the answers as dollar amounts, round to 2 decimal places.

a Highlight the cell range you want to format – in this case select cells B8 to D19.

b Using the *number formatting* drop down box on the top ribbon, select currency.

5 Enter the yearly interest calculation by placing **=B8–B4** into cell C8, **=B9-B8** into C9 and copying cell C9 down to C19.

6 Finally, enter the calculation for total interest by placing **=B8–B$4** into cell D8 and copying cell D8 down to D19.

	A	B	C	D
1	Compound interest			
2				
3	Annual interest rate (R)	0.08		
4	Principal (P)	1000		
5	Compounding periods per year (k)	365		
6				
7	End of year	Total amount	Yearly interest	Total interest
8	1	$1,083.28	$83.28	$83.28
9	2	$1,173.49	$90.21	$173.49
10	3	$1,271.22	$97.73	$271.22
11	4	$1,377.08	$105.86	$377.08
12	5	$1,491.76	$114.68	$491.76
13	6	$1,615.99	$124.23	$615.99
14	7	$1,750.57	$134.58	$750.57
15	8	$1,896.35	$145.78	$896.35
16	9	$2,054.27	$157.92	$1,054.27
17	10	$2,225.35	$171.07	$1,225.34
18	11	$2,410.67	$185.32	$1,410.66
19	12	$2,611.42	$200.75	$1,611.41

Spreadsheet
Compound interest 2

Use the spreadsheet to carry out compound interest calculations by entering an annual rate as a decimal into cell B3, a principal into B4 and the number of compounding periods per year into B5. Try different values for interest and principal and note the effect.

You can change the years over which the calculations are made by entering the year of your choice into cell A8. Try beginning at a year other than year 1.

A more sophisticated version of this spreadsheet can be downloaded from Nelson MindTap. Use it to check your answers to the following exercise.

EXERCISE 6.1 Compound interest

ANSWERS p. 454

Mastery

1 WORKED EXAMPLE 1 $10 000 is invested at 6.8% p.a. compounded yearly. Write the recurrence relation that shows the value of the investment at the end of each year.

2 The recurrence relation $A_{n+1} = 1.08A_n$, where $A_0 = 12\,000$, represents the yearly value of an investment of

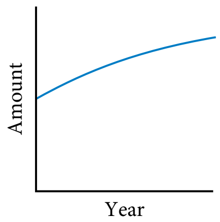
- **A** $12 000 invested at 0.08% p.a. compounded yearly
- **B** $12 000 invested at 1.08% p.a. compounded yearly
- **C** $12 000 invested at 8% p.a. compounded yearly
- **D** $12 000 invested at 10.8% p.a. compounded yearly

3 WORKED EXAMPLE 2 Liesel invested $19 000 at 6% p.a. compounded annually for 4 years. Calculate

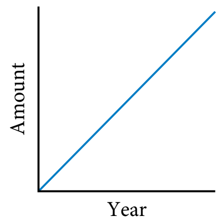
- **a** the total value of the investment at the end of the 1st, 2nd and 4th year
- **b** the interest earned in the second year
- **c** the total interest earned on the investment.

4 An investment of $20 000 is made at a fixed rate of interest compounding monthly over a number of years. Which graph best represents the amount of the investment at the end of each year?

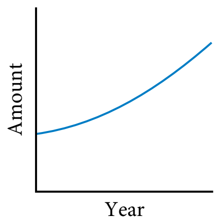
A



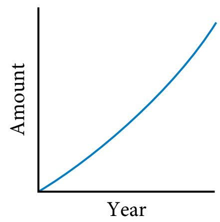
B



C



D



5 WORKED EXAMPLE 3 Sophie invests $12 000 at 8% p.a. compounding six-monthly.

- **a** What is the investment worth at the end of 2 years?
- **b** How much interest was earned at the end of 2 years?

6 WORKED EXAMPLE 4 $1500 is invested in a term deposit account paying 7% p.a. compounding monthly. How much will the investment be worth after 2 years and how much interest will be earnt?

7 If $9000 is invested for 8 years at 11% p.a. compounding quarterly, how much interest would the investment earn over that time?

8 WORKED EXAMPLE 5 If it takes 4 years for $2000 to accumulate to $4000 compounded annually, find the compound interest rate per annum.

9 The amount of money that should be invested at 8% p.a. interest, compounding annually, if you require $10 000 in 3 years' time, is closest to

A $5000 **B** $8000 **C** $12 000 **D** $15 000

10 The final amount if \$15 000 is invested at 7% p.a. compounded quarterly for 6 years is

A \$21 442 **B** \$22 510 **C** \$22 747 **D** \$22 801

11 Grace has \$10 000 to invest for 5 years. Which of the following will give her the greatest return?

A 8.4% p.a. compounded daily
B 8.4% p.a. compounded weekly
C 8.4% p.a. compounded quarterly
D 8.4% p.a. compounded annually

12 Which of the following would be the best investment? Justify your answer.

Investment A: \$7500 at 8.8% p.a. for 2.5 years compounded monthly

Investment B: \$7500 at 8.8% p.a. for 2.5 years compounded quarterly

Investment C: \$7500 at 15% p.a. for 1.5 years compounded weekly

13 Sue can currently invest money at 6.5% p.a. compounded monthly. How much should she invest now in order to have \$10 500 available in 4 years?

14 How long will it take \$9000 to accumulate to at least \$15 000 at 16% p.a. compounded annually?

Exam practice

15 ©QCAA 2020S 1Q19 An investment of \$10 000 is made at an interest rate of 7.5% per annum compounding half yearly.

The number of years it will take for the value of the investment to exceed \$18 000 is

A 7 **B** 8 **C** 15 **D** 16

16 Which investment has its yearly value described by the recurrence relation $A_{n+1} = 1.12A_n$, where $A_0 = 20\,000$?

A \$20 000 invested at 0.012% p.a. compounded yearly
B \$20 000 invested at 0.12% p.a. compounded yearly
C \$20 000 invested at 1.12% p.a. compounded yearly
D \$20 000 invested at 12% p.a. compounded yearly

17 Which expression gives the amount an investment of \$30 000 at 9% p.a. compounded quarterly will grow to in 5 years?

A $30\,000\left(1+\frac{0.9}{100}\right)^{20}$
B $30\,000\left(1+\frac{9}{100}\right)^{5}$
C $30\,000\left(1+\frac{0.9}{400}\right)^{20}$
D $30\,000\left(1+\frac{9}{400}\right)^{20}$

18 (4 marks) \$24 000 is invested at 5.8% p.a. compounding yearly.

a What is the investment worth at the end of 5 years? *[2 marks]*
b How much interest was earned in the 3rd year of the investment? *[2 marks]*

19 (4 marks) \$800 is invested at 12.75% p.a. compounding weekly.

a What is the investment worth at the end of 4 years? *[2 marks]*
b How much interest was earned in the 4 years? *[2 marks]*

20 (4 marks) CF Tyler invests \$26 000 at 6.4% p.a. compounding quarterly. How much interest was earned in the second year of the investment?

21 (4 marks) CF At what compound interest rate p.a. should $7500 be invested to achieve a total value of $10 500 after 5 years compounded quarterly?

22 (4 marks) CF Ashok has $20 000 to invest for a period of 5 years and is offered 2 options by a financial institution.

Option A: 7.8% p.a. compounding annually.

Option B: 7.6% p.a. compounding weekly.

Which of the investments would provide a better return for Ashok? Give reasons for your answer.

23 (4 marks) CF Nikita had $5000 in an investment and, after 4 years compounding monthly, it had grown to $6300. What interest rate was Nikita getting?

24 (4 marks) CF Zoe wants to invest some money for a period of 7 years. She can invest her funds at 5.9% p.a. compounded annually, or at 5.2% p.a. compounded daily. Which of these investments would provide a better return? Give reasons for your answer.

25 ©QCAA 2024 1Q18 (4 marks) When a child is born, their parent deposits $3000 to open an investment account earning interest at 4.2% p.a. compounding monthly. If there are no further transactions and the interest rate does not change, calculate the amount of interest earned by the child's 18th birthday.

6.2 Effective interest rates

The interest rate that is stated for a loan or investment is called the **nominal interest rate**. These rates are provided as an annual rate, usually with the compounding period.

Investigation — Comparing interest payments

Imagine if you had $100 000 that you were going to place in a term deposit account paying 12% compounding interest over 2 years. You have the option of selecting your compounding period: annual, monthly, weekly or daily.

1. Work out how much interest you would earn for each option over the 2 years.
2. Draw a graph with all options, showing the number of compounding periods in 1 year on the horizontal axis and the amount of interest on the vertical axis.
3. Consider the effect of a more frequent compound interest payment at the same nominal rate.

When comparing nominal interest rates, you must consider the compounding periods. The **effective interest rate** takes the number of compounding periods into account. It is the rate, that if compounded annually, would produce the same interest payment as a rate that is compounded more frequently.

You can use the effective rate to compare rates of interest for which the compounding periods are different.

- For an investment, the best option will have the highest effective interest rate.
- For a loan, the best option will have the lowest effective interest rate.

Effective interest rate formula

$$i_{\text{effective}} = (1 + i)^k - 1$$

where i = the interest rate per compounding period as a decimal

k = the number of compounding periods per year

WORKED EXAMPLE 6	Effective interest rate

Ella is saving for a new car and wants to invest the money she has saved so far in an interest-bearing account. She is offered the following rates from different banks.

Offer 1: 5.4% p.a. compounded quarterly.

Offer 2: 5.3% p.a. compounded daily.

Offer 3: 5.6% p.a. compounded monthly.

Offer 4: 5.6% p.a. compounded six-monthly.

Which offer will give Ella the best return?

Steps	Working
1 For each option, substitute the known variables into the effective interest rate formula ($i_{\text{effective}} = (1 + i)^k - 1$) and solve, rounding to 2 decimal places.	**Offer 1** $k = 4, i = \frac{0.054}{4} = 0.0135$ $i_{\text{effective}} = (1 + 0.0135)^4 - 1$ $= 0.0551\ldots$ The effective interest rate for Offer 1 is approximately 5.51%. **Offer 2** $k = 365, i = \frac{0.053}{365} = 0.000\,145\ldots$ $i_{\text{effective}} = (1 + 0.000\,145\ldots)^{365} - 1$ $= 0.0544\ldots$ The effective interest rate for Offer 2 is approximately 5.44%. **Offer 3** $k = 12, i = \frac{0.056}{12} = 0.004\,66\ldots$ $i_{\text{effective}} = (1 + 0.004\,66\ldots)^{12} - 1$ $= 0.05745\ldots$ The effective interest rate for Offer 3 is approximately 5.75%. **Offer 4** $k = 2, i = \frac{0.056}{2} = 0.028$ $i_{\text{effective}} = (1 + 0.028)^2 - 1$ $= 0.05678\ldots$ The effective interest rate for Offer 4 is approximately 5.68%.
2 Compare the effective interest rates and select the largest.	Ella should choose Offer 3 as it pays the higher effective interest rate. It will earn the most interest.

Exam hack

If an interest rate compounds annually, then the effective interest rate is always the same as the nominal interest rate.

9780170484824

EXERCISE 6.2 Effective interest rates

ANSWERS p. 454

Recap

1 Ethan placed $15 000 in a term deposit returning 3.5% p.a. compounded yearly for 5 years. Calculate

a the total value of the investment at the end of the 1st and 5th year

b the interest earned in the 1st year

c the total interest earned on the investment.

2 How much should Renee invest in an account paying 6% p.a. interest calculated half-yearly to have $12 500 in 7 years' time?

Mastery

3 WORKED EXAMPLE 6 Mia invests $10 000 at 8.2% p.a. compounded weekly. What is the effective rate of interest?

4 Find the effective interest rate for each of the following.

a 7.5% p.a. compounding monthly

b 13.4% p.a. compounding daily

c 8.4% p.a. compounding quarterly

d 8% p.a. compounding fortnightly

5 Liam has a choice of 3 investments at his bank. He can invest at

- 7.6% p.a. compounded quarterly
- 7.5% p.a. compounded monthly or
- 7.4% p.a. compounded daily.

Calculate the effective interest rate for each option and decide which choice is the best.

6 An investor is looking for the best return from 3 different offers. Which of the following options would be best for the investor?

Option 1: 7.51% p.a. compounded weekly

Option 2: 7.45% p.a. compounded daily

Option 3: 7.59% p.a. compounded monthly

Exam practice

7 Which of the following has the highest effective interest rate?

A 7.1% p.a. compounding daily

B 7.4% p.a. compounding monthly

C 7.3% p.a. compounding quarterly

D 7.5% p.a. compounding annually

8 The effective rate of interest for an investment earning 8% p.a. compounded monthly is closest to

A 8.1% **B** 8.2% **C** 8.3% **D** 8.4%

9 (2 marks) Andy has selected a term deposit account with an interest rate of 5.4% p.a. compounded quarterly. What is the difference between the effective interest rate and the nominal interest rate?

10 ©QCAA 2020 1Q25 (5 marks) A financial institution offers two investment options:

Option 1: 7% p.a. compounding quarterly

Option 2: 6.8% p.a. compounding monthly

Use the effective interest rate formula to determine the option that will provide the better return.

11 (5 marks) Evie inherits \$30 000 from her aunt. She is offered three investment options:

- a term deposit at 7.5% p.a. compounded daily
- insurance bonds at 7.7% p.a. compounded monthly
- government bonds at 7.9% p.a. compounded annually.

Which investment should Evie choose if she wants to earn the most interest?

12 (4 marks) CF Determine which investment would provide the better return: 6.08% p.a. compounding quarterly or 5.94% p.a. compounding daily.

13 ©QCAA 2024 1Q14 44% Which option will **not** change the effective annual rate of interest for a loan?

A changing the nominal annual rate of interest

B changing the period when interest is charged

C changing the repayment amount for each period

D changing the number of compounding periods per year

6.3 Practical problems with compound interest

Consider an investment account which just received a deposit of \$5000. The amount of money in the account right now is \$5000 and this is called the **present value**. The amount that the investment has grown to in 4 years is called the **future value**.

You can use the compound interest formula to find present and future values.

Given $A = P(1 + i)^n$, A is the value of the investment or loan in the *future*. P is the principal which is the value of the investment or loan in the *present*.

So, the compound interest formula could be written as:

$$FV = PV(1 + i)^n$$

where $FV(= A)$ = the future value of the investment or loan

$PV\ (= P)$ = the present value of the investment or loan.

> You do not need to know these versions of the compound interest formula; however, they are commonly used and are an accepted form of the compound interest formula.

WORKED EXAMPLE 7 Future value

Ollie has just received an inheritance of \$8000. He decides to invest the money in an account paying 7% p.a. compound interest, compounding monthly. **Determine** the value of Ollie's inheritance in 11 years.

Steps	Working
1 Need to find the future value A. Find the values of P, i and n. **Exam hack** If compounding period is not stated in a question, assume it is 1 year.	$P = 8000$ There are 12 monthly periods in a year. $i = \dfrac{0.07}{12} \approx 0.005\,83\ldots$ $n = 11 \times 12 = 132$
2 Substitute the values of P, i and n into the $A = P(1 + i)^n$ and evaluate.	$A = 8\,000(1 + 0.00583\ldots)^{132}$ $= 17\,239.519\ldots$
3 State the result.	Ollie's inheritance will have a value of approximately \$17 239.52 in 11 years.

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WORKED EXAMPLE 8	**Present value**
Toni is planning a bathroom renovation, which she anticipates will cost \$7800 in 4 years' time. How much should Toni invest at 5% p.a. compound interest to ensure she has the required amount in the future?	
Steps	**Working**
1 Need to find the present value P. Find the values of A, i and n.	$A = 7800$ As no compound period is stated, assume it is 1. $i = 0.05$ $n = 4$
2 Substitute the values of A, i and n into the formula $A = P(1 + i)^n$.	$7800 = P(1 + 0.05)^4$
3 Rearrange to isolate P and evaluate.	$P = \dfrac{7800}{(1+0.05)^4}$ $= 6417.079\ldots$
4 State the result.	Toni will need to invest at least \$6417.08 to have \$7800 in 4 years' time.

Compound interest is applicable to many financial situations, such as wage-related employment agreements and insurance policies.

WORKED EXAMPLE 9	**Compound interest problem**
Luka is 25 and has just started paying \$5000 a year into a life insurance fund. She is promised a lump sum of \$450 000 when she turns 65 and the policy matures. If the average interest rate over this period is 5.5% p.a., **determine** the present value of the payout.	
Steps	**Working**
1 Need to find the present value P. Find the values of A, i and n.	$A = 450\,000$ As no compound period is stated, assume it is 1. $i = 0.055$ $n = 40$
2 Substitute the values of A, i and n into the formula $A = P(1 + i)^n$.	$450\,000 = P(1 + 0.055)^{40}$
3 Rearrange to isolate P and evaluate.	$P = \dfrac{450\,000}{(1+0.055)^{40}}$ $= 52\,858.414\ldots$
4 State the result and identify what it represents. The present value of the future payout (or investment) is the single amount you could invest *now* at the same interest rate and term to achieve that future amount.	The present value of the payout is about \$52 858.

EXERCISE 6.3 Practical problems with compound interest

ANSWERS p. 454

Recap

1 What is the effective rate of interest for an investment earning 7% p.a. compounded monthly?

2 A financial institution offers 2 investment options:

Option 1: 5.6% p.a. compounding quarterly

Option 2: 6.1% p.a. compounding monthly

Use the effective interest rate formula to determine the option that will provide the better return.

Mastery

3 WORKED EXAMPLE 7 Connor has $3000 invested at 10% p.a. interest compounded quarterly. What will be the value of his investment in 12 years?

4 WORKED EXAMPLE 8 What is the present value of $7600 in 8 years if the interest rate is 8.4% p.a.?

5 Calculate the future value of each investment.

a $9000 invested at 5% p.a. compounding monthly for 7 years

b $12 700 invested at 6.8% p.a. compounding weekly for 12 years

c $23 000 invested at 8.2% p.a. compounding quarterly for 6.5 years

d $5000 invested at 7% p.a. compounding annually for 10 years

6 Calculate the present value of each future value.

a $8000 paid after 10 years invested at 3.5% p.a. compounding fortnightly

b $25 000 paid after 11 years invested at 6.7% p.a. compounding weekly

c $19 500 paid after 8 years 6 months invested at 5.9% p.a. compounding monthly

d $10 000 paid after 6 years invested at 5% p.a. compounding annually

7 WORKED EXAMPLE 9 Scarlett has an insurance policy that promises a payout of $300 000 in 35 years' time. If the average interest rate over this period is 6.8% p.a., what is the present value of the payout?

8 Hannah signs an employment contract that guarantees an annual increase of 6% p.a. subject to satisfactory performance of duties. If Hannah now earns $570 a week, how much will she earn in 5 years' time if she performs at a satisfactory level?

Exam practice

9 $5000 is invested at 6% p.a. compounding quarterly. The value of the investment in 5 years' time is given by

A $5000(1+0.06)^5$ **B** $5000(1+0.06)^{20}$ **C** $5000(1+0.015)^{20}$ **D** $5000(1+0.015)^4$

10 An investment that was made 4 years ago is now worth $8000. The money was invested at 4.8% p.a. compounding monthly. The original value of the investment is given by

A $\frac{8000}{(1+0.048)^{48}}$ **B** $\frac{8000}{(1+0.004)^{12}}$ **C** $\frac{8000}{(1+0.004)^{4}}$ **D** $\frac{8000}{(1+0.004)^{48}}$

11 (2 marks) Sienna wants to obtain $12 500 in 6 years' time. If the compound interest rate available is 8.3% p.a., how much must she invest now to reach her goal?

12 (4 marks) Emily expects to get a \$15 000 payout from some government bonds in 5 years' time. If the interest rate is 11.5% p.a., determine

a the present value of this payout *[2 marks]*

b how much interest was earned. *[2 marks]*

13 (4 marks) Cooper is planning a trip overseas and knows that he will need \$15 000 to pay for the trip. He wants to be able to pay for his holiday in 1.5 years' time.

a If Cooper can earn 8.05% p.a. calculated monthly, how much does he need to invest now? *[2 marks]*

b How much does he need to invest if he can earn only 4.25% p.a. interest calculated monthly? *[2 marks]*

14 (4 marks) **CF** An inheritance of \$30 000 is held in an interest-bearing trust account for Harrison, who is currently $7\frac{1}{2}$ years old. The trust is guaranteed to earn 4.55% p.a. compound interest calculated quarterly. How much will Harrison collect from the trust when he is entitled to claim it on his 21st birthday?

15 (4 marks) **CF** Kylie had \$6000 in an investment and after 5 years compounding quarterly, it had grown to \$7998. What interest rate was Kylie getting?

16 (4 marks) **CF** Hunter pays \$75 a month into an insurance policy. He will retire after 40 years, at which time the policy will pay a total of \$250 000. If the average interest rate for that period is 7.2% p.a., what is the present value of the payout?

17 (4 marks) **CU** How long would it take Jarrod to triple his investment at 7.5% p.a. interest, calculated monthly?

6.4 Reducing balance loans

Loan recurrence relations

A **reducing balance loan** is a compound interest loan with regular **periodic payments**. Personal loans and home loans are examples of reducing balance loans. As each payment is made, the amount owing (the balance) reduces.

The **balance** of the loan changes in 2 ways each compounding period. The balance:

- increases, as the compound interest being charged by the bank is added to the loan.
- decreases, as a fixed **repayment** amount is paid into the account.

The interest is calculated using the amount still owing at the beginning of each compounding period, then the repayment is made.

The change in the balance of the loan is a combination of geometric growth and linear decay.

Reducing balance loan recurrence relation

The value of a reducing balance loan is given by the recurrence relation

$$A_{n+1} = rA_n - d$$

where A_0 is the principal

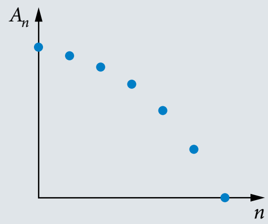
A_n is the total amount at the beginning of the n^{th} period

A_{n+1} is the total amount at the beginning of the $(n + 1)^{\text{th}}$ period

d is the periodic payment

and $r = 1 + i$ where i is interest rate per compounding period.

- The reduction in the principal after n compounding periods $= A_0 - A_n$.
- The total repayments after n compounding periods $= n \times d$.
- The total interest (I) after n compounding periods is I = total repayments made – reduction in the principal.



The graph of a reducing balance loan recurrence relation looks like this.

Exam hack

Exam questions will not always state that the loan is a reducing balance loan. You can identify a reducing balance loan as it will involve compound interest and a regular payment.

WORKED EXAMPLE 10 Loan recurrence relation 1

Charlotte has taken out a personal loan of \$15 000 with an interest rate of 11.4% per annum compounding monthly. Charlotte makes regular monthly payments of \$530.

a **State** the recurrence relation for the balance of the loan.

b **Determine** how much interest Charlotte will pay in the first 2 months of her loan.

Steps	Working
a **1** Identify the compounding periods per year.	$n = 12$
2 Determine A_0, i and d.	$A_0 = 15\,000$ $i = \dfrac{0.114}{12} = 0.0095$ $d = 530$
3 Calculate r using $r = 1 + i$.	$r = 1 + 0.0095$ $= 1.0095$
4 Write the recurrence relation using $A_{n+1} = rA_n - d$.	$A_{n+1} = 1.0095A_n - 530$, where $A_0 = 15\,000$
b **1** Determine the balance of the loan for the first 2 months using the recurrence relation.	$A_0 = 15\,000$ $A_1 = 1.0095A_0 - 530$ $= 1.0095 \times 15\,000 - 530$ $= 14\,612.5$ $A_2 = 1.0095A_1 - 530$ $= 1.0095 \times 14\,612.5 - 530$ $= 14\,221.318\ldots$
2 Identify A_0, A_2, n and d.	$A_0 = 15\,000$, $A_2 = 14\,221.318\ldots$ $n = 2$, $d = 530$

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3 Calculate the interest after 2 months (2 repayments).	reduction in principal = \$15 000 – \$14 221.318... = \$778.681... repayments = \$530 × 2 = \$1060 interest = \$1060 – \$778.681... = \$281.318…
4 State the result.	After two months, Charlotte will have paid approximately \$281.32 in interest.

WORKED EXAMPLE 11 Loan recurrence relation 2

Harry has taken out a reducing balance loan compounding quarterly with regular quarterly payments according to the recurrence relation

$$A_{n+1} = 1.0215A_n - 715, \text{ where } A_0 = 18\,000.$$

a How much did Harry borrow?

b What are Harry's regular loan repayments?

c Using recursion, show that Harry will owe \$16 995 (to the nearest dollar) after 3 quarters.

d **Determine** the annual compounding interest rate for this loan.

e After how many quarters will the balance of Harry's loan be below \$15 000?

Steps	**Working**
a Identify A_0.	Harry borrowed \$18 000.
b Identify d.	The loan repayments are \$715 per quarter.
c Use the recurrence relation to find A_3 to the nearest dollar.	$A_0 = 18\,000$ $A_1 = 1.0215A_0 - 715$ $= 1.0215 \times 18\,000 - 715$ $= 17\,672$ $A_2 = 1.0215A_1 - 715$ $= 1.0215 \times 17\,672 - 715$ $= 17\,336.95$ $A_3 = 1.0215A_2 - 715$ $= 1.0215 \times 17.336.95 - 715$ $= 16\,994.69$ $\approx 16\,995$
d **1** Use the recurrence relation to identify r, and then use $r = 1 + i$ to find i.	$r = 1.0215 = 1 + i$ $i = 0.0215$ The quarterly interest rate is 2.15%.
2 Use i to find the annual interest rate by multiplying by 4.	annual rate = 4 × 2.15 = 8.6%
3 State the result.	The annual compounding interest rate is 8.6%.

e **1** Using trial and error, determine how many quarters it takes for the balance to go below \$15 000.	$A_0 = 18\,000$ $A_1 = 17\,672$ $A_2 = 17\,336.948$ $A_3 = 16\,994.692\ldots$ $A_4 = 16\,645.078\ldots$ $A_5 = 16\,287.947\ldots$ $A_6 = 15\,923.138\ldots$ $A_7 = 15\,550.485\ldots$ $A_8 = 15\,169.821\ldots$ $A_9 = 14\,780.972\ldots$
2 State the result.	It will take 9 quarters for the balance of Harry's loan to fall below \$15 000.

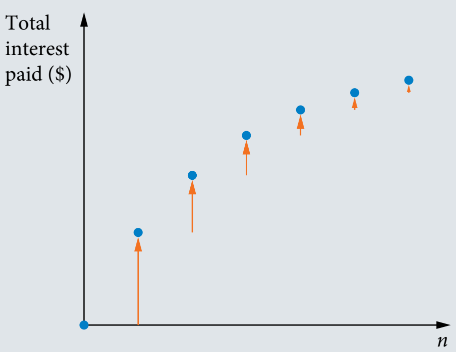
Loan graphs

For a reducing balance loan, the regular payment is made up of part interest and part principal.

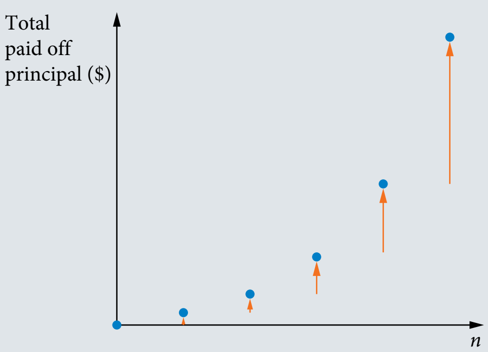
- Each compounding period, the amount of interest (\$) paid will *decrease*.
  - In the early part of the loan, most of the payment goes towards paying off the interest.
  - Towards the end of the loan, only a small part of the payment is interest.
- Each compounding period, the amount paid off the principal (\$) will *increase*.
  - In the early part of the loan a small amount of the payments goes towards the principal, whereas towards the end of the loan most of the payment goes to the principal.

Reducing balance loan payment graphs

Graph shape of total interest paid (\$):



Graph shape of total amount paid off principal (\$):



If a payment is missed, this will cause extra interest to be incurred. Even if the payment is made up later in the loan, the value of the loan will not reach a balance of \$0 when it originally should have.

Loan amortisation tables

6.4

Amortisation is the process of paying off a loan with regular payments over a period of time. An **amortisation table** (also called a **loan schedule**) provides a way to track the time, interest, the principal reduction and the balance of the loan over each compounding period.

Reducing balance loan amortisation table

Payment number (n)	Opening balance ($)	Payment ($)	Interest ($)	Principal reduction ($)	Closing balance* ($)
1	A_0	d	iA_0	$d - iA_0$	A_1
⋮	⋮	⋮	⋮	⋮	⋮
$n+1$	A_n	d	iA_n	d – interest	$A_{n+1} = rA_n - d$

where A_0 is the principal

A_n is the total amount at the beginning of the n^{th} period

A_{n-1} is the previous total balance

d is the periodic payment

and i is interest rate per compounding period.

* Formula for closing balance:

closing balance = opening balance – principal reduction

OR

closing balance = opening balance – payment + interest.

When completing an amortisation table (or schedule), always round to the nearest cent. The usual process of not rounding until the very end of a calculation does not apply.

WORKED EXAMPLE 12 Using reducing balance loan amortisation tables

Kate and Will have just borrowed \$200 000 from a bank to complete some renovations on their home. Interest is calculated monthly and they must make monthly instalments of \$1600 to repay the loan. The amortisation table shows the first month's calculation for the loan.

Payment number	Opening balance	Payment	Interest	Principal reduction	Closing balance
1	200 000	1600.00	1090	510	199 490
2					
3					
4					

a Use the table to **determine** the monthly interest rate i.

b **Determine** the nominal interest rate for the loan.

c Copy and complete the amortisation table.

d How much was owing on the loan after 4 months?

e How much interest had been paid during this period?

Steps	Working
a The interest rate is calculated by dividing the interest by the principal.	$i = \dfrac{1090}{200\,000}$ $= 0.005\,45$
b **1** Use the compounding period to determine the nominal interest rate.	There are 12 compounding periods. nominal interest rate $= 12 \times 0.00545 \times 100$ $= 6.54\%$ p.a.
2 State the result.	The nominal interest rate is 6.54% p.a.

c Complete the table using:

interest = $i \times$ opening balance

principal reduction = d – interest

closing balance = opening balance – principal reduction

Or alternatively:

closing balance = opening balance – repayment + interest

Payment number	Opening balance	Payment	Interest	Principal reduction	Closing balance
1	200 000	1600.00	1090	510	199 490
2	199 490	1600.00	$0.00545 \times 199\,490$ = 1087.22	1600 – 1087.22 = 512.78	199 490 – 512.78 = 198 977.22
3	198 977.22	1600.00	1084.43	515.57	198 461.65
4	198 461.65	1600.00	1081.62	518.38	197 943.27

d Identify the closing balance for 4 months.

After 4 months, there is $197 943.27 owing on the loan.

e **1** The total interest paid is the difference between the total repayments and the principal reduction.

First, find the total payments made.

total payments = 4×1600
= 6400

2 Determine the principal reduction by calculating $A_0 - A_4$.

principal reduction = 200 000 – 197 943.27
= 2056.73

3 Calculate the interest paid.

interest paid = 6400 – 2056.73
= 4343.27

This answer can also be calculated by adding the 4 values in the interest column.

4 State the result.

A total of $4343.27 in interest has been paid in the first 4 months.

TECHNOLOGY Amortisation tables (loan schedules)

You can make a spreadsheet for amortisation calculations. The procedure below sets out the steps for the loan situation in the previous example. A loan of $200 000, compounding monthly with periodic payments of $1600.

1 Set up the spreadsheet as shown below.

- Format cells B3:B7 as numbers with 2 decimal places.
- Format B7 as a number with 5 decimal places.
- Format cells B10:E21 as numbers with 2 decimal places. This spreadsheet will explore the first year of payments.

	A	B	C	D	E
1	Loan schedule				
2					
3	Annual interest rate (R)	6.54			
4	Compounding periods per year (n)	12			
5	Principal (P)	200000			
6	Payment amount ($)	1600			
7	Interest rate per period (i)	0.00545			
8					
9	Payment number	Opening balance	Payment	Interest	Closing balance

2 Enter the formulas shown below.

	A	B	C	D	E
1	Loan schedule				
2					
3	Annual interest rate (*R*)	6.54			
4	Compounding periods per year (n)	12			
5	Principal (*P*)	200000			
6	Payment amount ($)	1600			
7	Interest rate per period (*i*)	=B3/B4/100			
8					
9	**Payment number**	**Opening balance**	**Payment**	**Interest**	**Closing balance**
10	1	=B5	=$B$6	=B10*$B$7	=B10+D10-C10
11	=A10+1	=E10	=$B$6	=B11*$B$7	=B11+D11-C11

3 Copy row 11 (A11:E11) down from row 11 to row 21. This will calculate the values for the first 12 months.

	A	B	C	D	E
1	Loan schedule				
2					
3	Annual interest rate (*R*)	6.54			
4	Compounding periods per year (n)	12			
5	Principal (*P*)	200000			
6	Payment amount ($)	1600			
7	Interest rate per period (*i*)	0.00545			
8					
9	**Payment number**	**Opening balance**	**Payment**	**Interest**	**Closing balance**
10	1	200000	1600.00	1090	199490.00
11	2	199490.00	1600.00	1087.22	198977.22
12	3	198977.22	1600.00	1084.43	198461.65
13	4	198461.65	1600.00	1081.62	197943.26
14	5	197943.26	1600.00	1078.79	197422.05
15	6	197422.05	1600.00	1075.95	196898.00
16	7	196898.00	1600.00	1073.09	196371.10
17	8	196371.10	1600.00	1070.22	195841.32
18	9	195841.32	1600.00	1067.34	195308.66
19	10	195308.66	1600.00	1064.43	194773.09
20	11	194773.09	1600.00	1061.51	194234.60
21	12	194234.60	1600.00	1058.58	193693.18

Use the spreadsheet to carry out reducing balance calculations.

Investigation — Using amortisation tables to explore changes

Using the spreadsheet you have just created or access the spreadsheet on Nelson MindTap, explore the effects of changing the parameters of a loan. Start with a loan of $200 000, compounding monthly with periodic payments of $1600 (Worked example 12).

- Explore what happens if the annual interest rate is changed. What happens when the interest rate is greater than 12?
- Restore the original annual interest rate (6.54% p.a.). Explore the effect when the number of compounding periods and hence payments each year is changed. Comment on the results.
- Restore the original number of compounding periods (12). Explore the effect of increasing and decreasing the size of the periodic payments made each compound period. What is the smallest regular payment that must be made so that the balance of the loan reduces?
- Restore the original payment value (1600). Adjust the spreadsheet to make payments every fortnight at half the amount (26 payments a year). Check the total after the same amount of time.
- **Extension:** Explore the effect of an extra payment (of a large amount) on the length of the loan and on the total amount of interest paid.

WORKED EXAMPLE 13 Analysing loan amortisation tables

Tess has taken out a 1-year short-term personal loan of $10 000 from a bank to re-insulate her apartment. Her loan has an interest rate of 14% p.a. compounding quarterly with quarterly payments of $2720. The incomplete amortisation table started by Tess is shown below.

Payment number	Opening balance	Payment	Interest	Principal reduction	Closing balance
1	10 000	2720	350	2370	7630
2	7630	2720	267.05	2452.95	*A*
3	*B*	2720	*C*	2538.80	*D*

a **Determine** the missing values *A*, *B*, *C* and *D*.

b By the end of the third quarter, how much of the principal has been repaid?

c What should the amount of the final payment be if Tess wants to ensure the loan balance is zero at the end of the loan term?

d **Determine** the total amount paid for the loan.

e How much interest did Tess pay?

Steps	Working
a 1 Find the closing balance (A) = opening balance − principal reduction.	$A = 7630 - 2452.95$ $= 5177.05$
2 Identify the value of the opening balance (B).	$B = A = 5177.05$
3 Find the interest (C) = i × opening balance.	$C = \frac{0.14}{4} \times 5177.05$ $= 181.20$
4 Find the closing balance (D).	$D = 5177.05 - 2538.80 = 2638.25$
b 1 Principal repaid = principal − closing balance.	Amount repaid at the end of quarter 3 = 10 000 − 2638.25 = $7361.75
2 State the result.	At the end of the third quarter, $7361.75 has been paid off the principal.

c 1 The table is one row short, so calculate the final row values.

Payment number	Opening balance	Payment	Interest	Principal reduction	Closing balance
4	2638.25	2720	92.34	2627.66	10.59

Steps	Working
2 Work out the final payment value. The final adjusted payment = payment + residual amount.	final payment = 2720 + 10.59 = 2730.59
3 State the result.	The final payment should be $2730.59 if Tess wants a zero balance at the end of 1 year.
d 1 Calculate the total amount paid = total full repayments + adjusted final payment.	The total amount paid = 4 × 2720 + 10.59 = 10 890.59
2 State the result.	The loan cost Tess $10 890.59.
e 1 Find the total interest. The total interest paid = total amount paid − principal.	total interest = 10 890.59 − 10 000 = 890.59
2 State the result.	Tess paid a total of $890.59 in interest.

EXERCISE 6.4 Reducing balance loans

ANSWERS p. 455

Recap

1 An amount of \$10 000 is invested at 7.25% p.a. calculated weekly. Assuming that there are 52 weeks in a year, what will be the total amount of the investment after 4.5 years?

2 Annika needs to have \$3700 available in 5 years' time. She can get 7.8% p.a. interest.
How much should she invest now?

Mastery

3 WORKED EXAMPLE 10 Newton has taken out a personal loan of \$18 000 with an interest rate of 12.5% per annum, compounding six-monthly. He makes regular six-monthly payments of \$5200.

- **a** Write a recurrence relation for the balance of the loan.
- **b** Determine how much interest Newton will pay in the first year of his loan.
- **c** Use your calculator to determine how long it will take for Newton to pay off the loan completely, and hence determine the size of the final payment.

4 Emma has taken out a loan of \$25 000 to buy her first car. The loan has an interest rate of 10.8% per annum, compounding quarterly. She makes regular quarterly payments of \$2500.

- **a** Write a recurrence relation for the balance of the loan.
- **b** Use your calculator to determine the balance of her loan after 12 months.

5 WORKED EXAMPLE 11 Imran has taken out a reducing balance loan, compounding monthly with regular monthly payments according to the recurrence relation

$$A_{n+1} = 1.008A_n - 600 \text{ where } A_0 = 20\,000.$$

- **a** How much did Imran borrow?
- **b** What are the regular monthly payments?
- **c** Using recursion, show that Imran will owe \$18 219 (to the nearest dollar) after 4 months.
- **d** Determine the nominal interest rate for this loan.
- **e** After how many months will the balance of the loan be below \$17 000?

6 WORKED EXAMPLE 12 Wally has just borrowed \$850 000 from a bank to buy a house. Interest is calculated monthly and they must make monthly instalments of \$6476 to repay the loan.
The amortisation table shows the first month's calculation for the loan.

Payment number	Opening balance	Payment	Interest	Principal reduction	Closing balance
1	850 000	6476	5950	526	849 474
2		6476		529.68	
3					
4					

- **a** Use the table to determine the monthly interest rate *i*.
- **b** Determine the nominal interest rate for the loan.
- **c** Copy and complete the amortisation table.
- **d** How much was owing on the loan after 4 months?
- **e** How much interest had been paid during this period?

7 Draw up an amortisation table showing the progress of a loan of

a $25 000 at 8.5% compounded monthly with monthly payments of $1130 for 4 months

b $63 000 at 12% compounded fortnightly with fortnightly payments of $800 for 6 months.

8 WORKED EXAMPLE 13 Luke takes out a short-term 1-year personal loan of $9000 to pay for his upcoming holiday overseas. The loan has an interest rate of 13.7% p.a. compounding monthly with monthly payments of $800. The incomplete amortisation table is shown below.

Payment number	Opening balance	Payment	Interest	Principal reduction	Closing balance
1	9000.00	800.00	102.75	697.25	8302.75
2	8302.75	800.00	94.79	705.21	A
3	B	800.00	C	713.26	D

a Determine the missing values A, B, C and D.

b By the end of the third month, how much of the principal has been repaid?

c What should the amount of the final payment be if Luke wants to ensure the loan balance is zero at the end of the loan term?

d What was the total amount paid for the loan?

e How much interest did Luke pay?

9 Nada borrowed $24 000 to buy new computers for her business. The terms of the loan are that interest is charged at 9.8% p.a. compounded monthly and regular monthly repayments of $600 will be made by Nada. She decides to review her loan after 5 months.

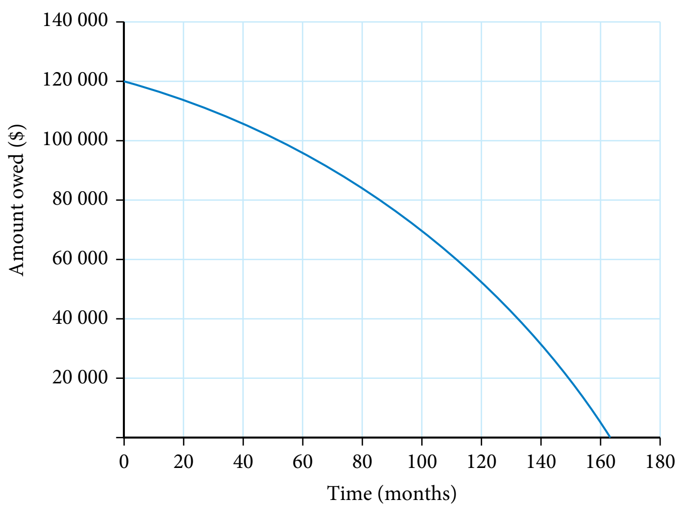
a How much does she still owe?

b By how much has Nada reduced the loan?

c How much interest has Nada paid?

Exam practice

10 ©QCAA 2020S 1Q2 The graph below shows the relationship between the amount owed and time taken to repay a reducing balance loan.



The owed amount will be reduced to half the original amount borrowed in

A 60 months B 81 months C 112 months D 162 months

11 ©QCAA 2020S 1Q13 For a personal loan of \$350, a payment of \$60 is made after interest is added each month. The following recurrence relation shows the remaining balance at the end of consecutive months. The recurrence relation is

$$A_{n+1} = 1.0243A_n - 60.$$

The remaining balance after two months is

A \$191.73 **B** \$242.81 **C** \$245.76 **D** \$298.51

12 A loan is repaid in 60 monthly payments. Which of the following payments will reduce the principal by the greatest amount?

A 1st **B** 10th **C** 30th **D** 60th

13 An amount of \$18 000 is borrowed at a rate of 8.4% p.a. interest calculated monthly. The loan is repaid using monthly payments of \$350. If A_n is the amount owing after n payments, which of the following is the recurrence relation for this loan?

A $A_{n+1} = 0.007A_n - 350$, where $A_0 = 18\,000$

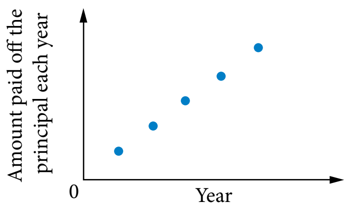
B $A_{n+1} = 0.06A_n - 18\,000$, where $A_0 = 350$

C $A_{n+1} = 0.084A_n - 350$, where $A_0 = 18\,000$

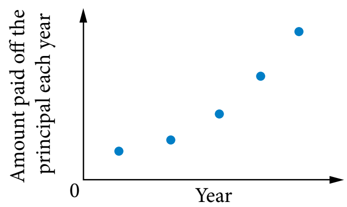
D $A_{n+1} = 0.7A_n - 350$, where $A_0 = 18\,000$

14 Five years ago Megan took out a reducing balance loan to buy her home. She has been paying it off in equal payments each year. She has correctly graphed the amount paid off the principal each year for the first 5 years. Which of the graphs below best represents the amount she has paid off the principal over the last 5 years?

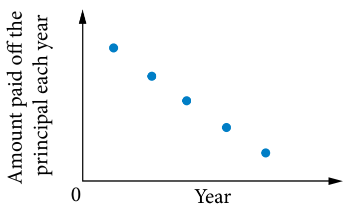
A



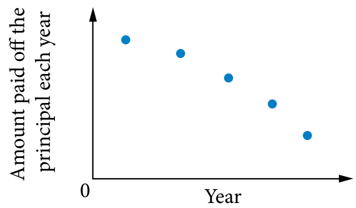
B



C



D



15 An amount is borrowed with interest charged at 7.5% p.a. compounded monthly. Which of the following loan terms would result in the greatest amount of interest being paid?

A 5 years **B** 10 years **C** 15 years **D** 20 years

16 (4 marks) Blake has a reducing balance loan. Interest is charged on this loan at the rate of 5.6% per annum. Interest is calculated fortnightly and Blake makes fortnightly payments of $750.

The first 5 lines of the amortisation table for Blake's loan are shown below.

Payment number	Payment	Interest	Principal reduction	Closing balance
0	0	0	0.00	120 000
1	750.00	258.46	491.54	119 508.46
2	750.00	257.40	492.60	119 015.86
3	750.00	256.34		118 522.21
4	750.00	255.28	494.72	

a Using the values in the amortisation table, calculate

i the principal reduction associated with payment 3 *[1 mark]*

ii the balance of the loan after payment 4. *[1 mark]*

b Determine the recurrence relation that models the balance of the loan. *[2 marks]*

17 (10 marks) Amanda has just taken out a loan of $5400 for a new electric motorcycle. She makes payments of $715 per quarter. The schedule below shows the progress of her loan.

Payment number	Opening balance	Payment	Interest	Principal reduction	Closing balance
1	5400.00	715.00	124.20	590.80	4809.20
2	4809.20	715.00	110.61	604.39	4204.81
3	4204.81	715.00	96.71	618.29	3586.52
4					

a Show that the nominal interest rate is approximately 9.2% p.a. *[2 marks]*

b Write a recursive rule to determine the closing balance of the loan at the end of each quarter. *[2 marks]*

c Copy and complete the last row of the loan schedule spreadsheet above. *[2 marks]*

d Determine how many quarters it will take Amanda to pay off the loan, if her final payment includes an additional payment of $275.11. *[1 mark]*

e Calculate how much interest is paid over the duration of the loan. *[3 marks]*

18 (4 marks) CF Liam borrowed $30 000 on 3 June to buy a new car. The loan has an interest rate of 12.7% p.a. with interest calculated monthly and monthly repayments of $620. Determine the amount of interest paid and the remaining loan amount on 3 November.

19 (4 marks) CF Tammy borrows $1000 from a short-term lender at a rate of 17% p.a. with interest calculated weekly. She repays $55.12 per week.

Calculate the amount owing and the interest paid after 6 weeks of payments.

20 ©QCAA 2024 1Q22 (6 marks) A reducing balance loan for $15 000 has an interest rate of 8.4% p.a. calculated monthly with a $250 repayment at the end of every month.

a Use the monthly interest rate to write a recurrence relation for the loan balance after n months. *[2 marks]*

b Calculate the loan balance after two months. *[1 mark]*

c Use the reduction in the loan balance and the total repayments to determine the amount of interest paid in the first two months. *[3 marks]*

6.5 Present value of ordinary annuities

6.5

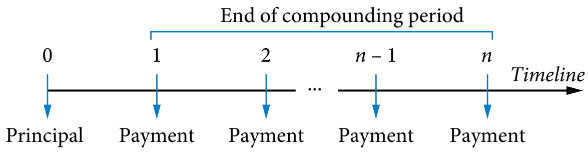
Annuity recurrence relations

An investment or loan that involves compound interest and a series of regular equal periodic payments over time is called an **annuity**. The payments are made to the investor.

A reducing balance loan is an example of an annuity, where the investor is the bank.

A retirement pension is also an annuity. In a retirement pension, the investor (retiree) provides the bank with a lump sum (principal) and the bank then makes regular payments back to investor for a fixed period of time until the balance is $0. It is just like a reducing balance loan, except the bank is 'borrowing' the money from the investor and is paying the investor interest.

Most annuities are structured so that the regular payment is made at the end of each period after the interest has been added. This type of annuity is called an **ordinary annuity**.



Worksheet
Annuities

The recurrence relation for an annuity is the same as a reducing balance loan.

Annuity recurrence relation

The present value of an ordinary annuity is given by the recurrence relation

$$A_{n+1} = rA_n - d$$

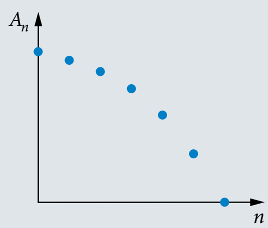
where A_0 is the principal (the cost of the annuity)

A_n is the total amount at the beginning of the n^{th} period

A_{n+1} is the total amount at the beginning of the $(n+1)^{\text{th}}$ period

d is the periodic payment

and $r = 1 + i$ where i is interest rate per compounding period.



The graph of an annuity recurrence relation looks like this.

WORKED EXAMPLE 14 Annuity recurrence relation

Matt plans to invest \$50 000 in an annuity at a rate of 8.7% per annum compounding quarterly, where he makes regular quarterly withdrawals of \$3500.

a Write a recurrence relation for the balance of the investment.

Matt changes his mind and decides to invest the \$50 000 into an annuity which compounds six-monthly, with six-monthly withdrawals. The annuity has the recurrence relation

$$A_{n+1} = 1.06A_n - 3500, \text{ where } A_0 = 50\,000.$$

b **Determine** the annual interest rate for this investment.

c **Determine** how much the investment has changed between the second and third six-monthly periods (to the nearest cent).

d **Determine** how long it will be before Matt's investment has a value below \$45 000.

Steps			Working
a	**1**	Identify A_0, n and d.	$n = 4$ $A_0 = 50\,000$ $d = 3500$
	2	Calculate i.	$i = \frac{0.087}{4} = 0.021\,75$
	3	Calculate r using $r = 1 + i$.	$r = 1 + 0.021\,75$ $r = 1.02175$
	4	Write the recurrence relation using $A_{n+1} = rA_n - d$.	$A_{n+1} = 1.02175A_n - 3500$, where $A_0 = 50\,000$
b	**1**	Use the recurrence relation to identify r, and then use $r = 1 + i$ to find i.	$r = 1.06 = 1 + i$ $i = 0.06$ The six-monthly interest rate is 6%.
	2	Use i to find the annual interest rate by multiplying by 4.	annual rate $= 2 \times 6$ $= 12\%$
	3	State the result.	The annual interest rate is 12%.
c	**1**	Determine the balance of the loan for the first 3 periods using the recurrence relation.	$A_0 = 50\,000$ $A_1 = 1.06A_0 - 3500$ $= 1.06 \times 50\,000 - 3500$ $= 49\,500$ $A_2 = 1.06A_1 - 3500$ $= 1.06 \times 49\,500 - 3500$ $= 48\,970$ $A_3 = 1.06A_2 - 3500$ $= 1.06 \times 48\,970 - 3500$ $= 48\,408.20$
	2	Calculate $A_3 - A_2$.	$A_2 - A_3 = 48\,970 - 48\,408.20$ $= 561.80$
	3	State the result.	The investment has reduced by \$561.80 between the second and third six-monthly period.

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d **1** To find the number of payments, use the recurrence relation $A_{n+1} = rA_n - d$ and calculator recursion to find the number of payments.

2 Use calculator recursion (or a spreadsheet).

Steps	**Casio fx-82AU PLUS II**	**Sharp EL-531TH**	**TI-30XB**
Enter A_0 and push =.	Enter 50 000 and press =.	Enter 50 000 and press =.	Enter 50 000 and press ENTER.
Set up the recursive rule and calculate A_1.	Enter 1.06× ANS, then − 3500 =.	Enter 1.06× ALPHA = then − 3500 =.	Enter 1.06× 2nd, (−) then − 3500 ENTER again.
	The result should be \$49 500, which is A_1.		
Now get the calculator to repeat the calculation to find A_2.	Press =.	Press =.	Press ENTER.
	The result should be \$48 970, which is A_2.		

Repeat this process, making sure to count and record how many payments are needed for a balance below \$45 000.	$A_3 = \$48\,408.20$, $A_4 = \$47\,812.69$ $A_5 = \$47\,181.45$, $A_6 = \$46\,512.34$ $A_7 = \$45\,803.08$, $A_8 = \$45\,051.27$, $A_9 = \$44\,254.34$.
3 State the result.	After 4 years, Matt will have less than \$45 000 in his account.

When considering investments such as the one in the above example, note:

- higher interest rates result in a longer duration for an annuity
- higher regular withdrawals result in a shorter duration for an annuity.

Annuity amortisation tables

Amortisation tables for annuities work the same way as for a reducing balance loan. The only difference is that *interest* may be relabelled as *investment gain*. Both titles are equally valid.

Annuity amortisation table

Payment number (n)	Opening balance (\$)	Payment (\$)	Investment gain (\$)	Principal reduction (\$)	Closing balance* (\$)
1	A_0	d	iA_0	$d - iA_0$	A_1
⋮	⋮	⋮	⋮	⋮	⋮
$n + 1$	A_n	d	iA_n	d − interest	$A_{n+1} = rA_n - d$

where A_0 is the principal

A_n is the closing balance after n compounding periods

d is the periodic payment

and i is interest rate per compounding period.

WORKED EXAMPLE 15 Annuity amortisation table

The amortisation table shows the first 4 payments of an ordinary annuity which compounds monthly and has equal monthly withdrawals. Some of the entries are missing.

Payment number	Opening balance	Payment	Investment gain	Principal reduction	Closing balance
1	150 000.00	1450.00	1125.00	325.00	149 675.00
2	149 675.00	1450.00	1122.56	327.44	
3	149 347.56	1450.00	1120.11	329.89	149 017.67
4	149 017.67	1450.00			148 685.30

Use the amortisation table to **determine**

a the amount invested and the monthly withdrawals

b the monthly interest rate i

c the nominal interest rate for the investment

d the balance after two payments

e the interest paid at the fourth month

f the amount the principal is reduced in the fourth month.

Steps	Working
a Identify the principal and the payments from the table.	The amount invested was $150 000 and the monthly withdrawals are $1450.
b Find i by dividing the interest (investment gain) in the first month by the opening balance.	$i = \frac{1125}{150\,000}$ $= 0.0075$ The monthly interest rate is 0.0075%.
c **1** Use the compounding periods to determine the nominal interest rate.	There are 12 compounding periods. nominal interest rate $= 12 \times 0.0075 \times 100$ $= 9\%$ p.a.
2 State the result.	The nominal interest rate is 9% p.a.
d Calculate the closing balance using closing balance = opening balance + interest – payment.	closing balance $= 149\,675 + 1122.56 - 1450$ $= \$149\,347.56$
e Calculate the interest in the fourth month using the formula: interest $= iA_3$.	interest $= 0.0075 \times 149\,017.67$ $= \$1117.63$
f Determine the principal reduction using the formula: reduction = *payment* – *interest*.	principal reduction $= 1450 - 1117.63$ $= \$332.37$

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Present value annuity formula

When a present value annuity situation involves a small number of payments or compound periods, recurrence relations and amortisation tables can easily be used to model the loan or investment. In more complex situations, the annuity formula is useful in calculating unknown values including the principal, periodic payment and balance.

6.5

Present value annuity formula

$$A_{PV} = d\left(\frac{1-(1+i)^{-n}}{i}\right)$$

where A_{PV} = the total amount of the investment (or loan)
d = the periodic payment
i = interest rate per compounding period
n = number of compounding periods.

WORKED EXAMPLE 16 Finding the payment using the annuity formula

Ellie invests \$220 000 in an annuity that pays 6.9% p.a. interest compounded monthly. After interest is added each month, she receives a payment. How much will each payment be if the annuity lasts for 20 years?

Steps	Working
1 Identify A_0 and n.	$n = 12 \times 20 = 240$ $A_0 = A_{PV} = 220\,000$
2 Calculate i.	$i = \frac{0.069}{12} = 0.005\,75$
3 Substitute A_{PV}, i and n into the formula and simplify. $A_{PV} = d\left(\frac{1-(1+i)^{-n}}{i}\right)$	$220\,000 = d\left(\frac{1-(1+0.005\,75)^{-240}}{0.00575}\right)$ $220\,000 = d \times 129.986\ldots$
4 Rearrange and evaluate to find d.	$d = \frac{220000}{129.986\ldots}$ $= 1692.477\ldots$ ≈ 1692.48
5 State the answer.	Ellie will receive monthly payments of \$1692.48 for 20 years.

WORKED EXAMPLE 17 Using the annuity formula for a reducing balance loan

Aiden decides to borrow \$80 000 from his credit union at 11.7% p.a. compounded monthly to buy his dream car. Calculate

a the monthly repayment if the loan is paid off after 12 years

b the total amount repaid

c the total interest paid.

Steps	Working
a **1** Identify A_0 and n.	$n = 12 \times 12 = 144$ $A_0 = A_{PV} = 80\,000$
2 Calculate i.	$i = \frac{0.117}{12} = 0.009\,75$
3 Substitute A_{PV}, i and n into the formula and simplify. $A_{PV} = d\left(\frac{1-(1+i)^{-n}}{i}\right)$	$80\,000 = d\left(\frac{1-(1+0.009\,75)^{-144}}{0.009\,75}\right)$ $80\,000 = d \times 77.201\ldots$
4 Rearrange and evaluate to find d.	$d = \frac{80\,000}{77.201\ldots}$ $= 1036.253\ldots \approx 1036.25$
5 State the answer.	Aiden's monthly repayment will be \$1036.25.
b **1** Aiden makes 144 payments of \$1036.25.	total paid $= 144 \times 1036.25$ $= 149\,220$
2 State the result.	Aiden pays a total of \$149 220.
c **1** Calculate the interest using: I = total paid – principal.	$I = 149\,220 - 80\,000$ $= 69\,220$
2 State the result.	Aiden pays \$69 220 in interest.

At this stage if you are asked to determine the balance after a given period or the number of payments before a loan will be paid out, you can only do so using the recurrence relation. The use of a spreadsheet or technology is also recommended.

EXERCISE 6.5 Present value of ordinary annuities

ANSWERS p. 455

Recap

1 On 7 April, Lucas and Lily borrow \$220 000 from a bank to buy a home unit. They borrow the money at 8.3% p.a. Interest is calculated monthly and they must pay \$1881.45 each month for 20 years to repay the loan. Use a recurrence relation to calculate

a the amount still owing on the loan on 7 September in the same year

b the total interest that has been paid to the bank during this period.

2 Flynn borrowed \$19 000 at 9% p.a. interest compounding monthly with monthly repayments of \$251.

a Create an amortisation table showing the progress of the loan over the first 6 months.

b State the total interest paid, the total repayments made, the reduction of the principal and the balance of the loan after 6 months.

Mastery

3 WORKED EXAMPLE 14 Alexis invests \$27 000 in an annuity at a rate of 6.5% p.a. compounding monthly. Regular monthly withdrawals of \$340 are made. The recurrence relation for the balance of the investment where $A_0 = 27\,000$ is

A $A_{n+1} = 1.09A_n - 340$

B $A_{n+1} = 1.0075A_n - 340$

C $A_{n-1} = 1.0075A_n + 340$

D $A_n = 1.0075A_{n+1} - 340$

4 Margot invests \$90 000 in an annuity that pays at a rate of 5.7% per annum compounding annually where she receives regular annual payments of \$9000.

a Write a recursive relation for the balance of the investment.

b What is the balance of the annuity after 5 payments?

c Determine how much the investment has changed between the fifth and sixth periods (to the nearest cent).

d When will Margot's balance fall below \$40 000?

5 WORKED EXAMPLE 15 Sarah has recently invested an inheritance she received in an annuity which compounds six-monthly and pays her regular equal six-monthly payments. The amortisation table shows the first 3 payments. Some of the entries are missing.

Payment number	Opening balance	Payment	Interest	Principal reduction	Closing balance
1	300 000.00	20 000.00	9450.00	10 550.00	289 450.00
2	289 450.00	20 000.00	9117.68	10 882.33	
3	278 567.68	20 000.00			267 342.56

Use the table to determine

a the amount invested and the six-monthly repayments

b the six-monthly interest rate i

c the nominal interest rate for the investment

d the balance after 2 payments

e the interest paid at the third period

f the amount the principal is reduced in the third period.

6 WORKED EXAMPLE 16 Hugo invests \$109 000 in an annuity that pays 5.5% p.a. interest compounded annually. After interest is added each year, he receives a payment. Use the formula for the payment of an ordinary annuity to calculate how much each payment will be if the annuity lasts for 10 years.

7 Joshua invests \$80 000 at 7.3% p.a. compounded monthly.

a What is the greatest amount that he can withdraw at the end of each month for the investment to last 12 years?

b If he wants the investment to last 15 years, what is the greatest monthly amount he can withdraw?

c If he needs the investment to last only 7 years, what is the greatest monthly amount he can withdraw?

8 WORKED EXAMPLE 17 Ryan takes out a loan for $20 000 so he can buy a car. The finance company offers Ryan a loan at 12.9% p.a. compounded monthly. Calculate

- **a** the monthly repayment if the loan is paid off after 5 years
- **b** the total amount repaid
- **c** the total interest paid.

9 Addison has a reducing balance loan of $14 000 with interest charged at 9.8% p.a., calculated monthly.

- **a** Find the monthly repayment and the total interest charged if the loan is fully repaid in
  - **i** 2 years
  - **ii** 3 years
  - **iii** 4 years
  - **iv** 4.5 years.
- **b** What happens to the repayment value and the total interest repaid as the time taken to repay the loan increases?

Exam practice

10 ©QCAA 2020S 1Q17 A reducing balance loan of $12 000 was taken out at 10.5% per annum with interest calculated monthly for 5 years. Determine the monthly repayment required to pay off the loan.

A $152.93 **B** $162.18 **C** $257.93 **D** $267.18

11 Lilly invests $12 000 in an annuity at a rate of 6.5% p.a. compounding annually. Regular withdrawals of $1400 are made. The recurrence relation for the balance of the investment where $A_0 = 12\,000$ is

A $A_n = 1.065A_{n+1} - 1400$

B $A_{n+1} = 1.065(A_n - 1400)$

C $A_{n+1} = 1.065A_n + 1400$

D $A_{n+1} = 1.065A_n - 1400$

12 (3 marks) Sam invested $210 000 in an annuity. This investment earns interest at the rate of 5.9% per annum, compounding monthly. Immediately after the interest has been added to the account each month, Sam withdraws a payment of $5000.

- **a** Find a recurrence relation that can be used to determine the value of Sam's investment after n months. *[2 marks]*
- **b** When will Sam's annuity have a balance below $190 000? *[1 mark]*

13 (2 marks) Geoff decides to invest $25 000 into an annuity which compounds quarterly, with quarterly withdrawals. The annuity has the recurrence relation

$$A_n = 1.0325A_{n+1} - 1200, \text{ where } A_0 = 25\,000.$$

- **a** Determine the annual interest rate for this investment. *[1 mark]*
- **b** Determine how much the investment has changed between the third and fourth quarterly periods (to the nearest cent). *[1 mark]*

14 (6 marks) Tish invests $45 000 into an annuity paying 4.8% per annum. The annuity is designed to pay her annual payments of $5771 for 10 years. The amortisation table for the annuity is shown below. Some of the entries are missing.

Payment number	Opening balance	Payment	Interest	Closing balance
1	45 000.00	5771.00	2160.00	
2		5771.00	1986.67	37 604.67
3	37 604.67	5771.00	1805.02	33 638.70
4	33 638.70	5771.00	1614.66	29 482.35
5	29 482.35	5771.00	1415.15	25 126.51
6	25 126.51	5771.00	1206.07	20 561.58
7	20 561.58	5771.00	986.96	15 777.53
8	15 777.53			10 763.86
9	10 763.86	5771.00	516.67	5509.52
10	5509.52	5771.00	264.46	2.98

a Find the balance of the annuity after one payment has been made. *[2 marks]*

b Find the reduction in the principal of the annuity in the fifth year. *[2 marks]*

c Find the amount of payment number 8 that is the interest earned. *[2 marks]*

15 Robbie has just inherited $100 000 and invested it at 6.2% p.a. compounded annually. How much should Robbie withdraw at the end of each year for the next 10 years to end up with a zero balance?

16 ©QCAA 2022 1Q25 (5 marks) A couple borrow money to complete home renovations. Their bank has loaned the amount at 2.4% p.a. compounding monthly with repayments of $993.14 each month for 15 years.

a Determine the amount of money borrowed. *[3 marks]*

b Write a recurrence relation for the amount owing after n months. *[2 marks]*

17 (3 marks) CF Yuki won the lottery. He will receive $65 000 per month for the next 20 years. What single amount would need to be deposited now in an annuity earning 7.4% p.a. compounded monthly by the lottery owner to fund Yuki payments?

18 (4 marks) CU Michael has just won a 'set for life' annuity which will pay him $10 000 at the end of each year for 10 years. An acquaintance offers to give Michael $60 000 right now for the annuity. Michael thinks that a 'fair' rate of return for this investment would be 6% p.a. Should he accept his acquaintance's offer? Justify your answer.

19 (4 marks) CU Yasmin is involved in a car accident that was someone else's fault. She wins a settlement of $25 000 each year for the next 25 years, for a total of $625 000. The insurance company offers Yasmin a one-payment settlement of $300 000 now instead of the annual payments. If the money can be invested at 7% p.a. compounded annually, should she accept the lump sum offer?

Video
Exam question analysis: Loans, investments and annuities

EXAM QUESTION ANALYSIS

©QCAA 2023 1Q17 (4 marks)

Terome paid a $50 000 deposit on a house valued at $570 000 and borrowed the remainder as a reducing balance loan at 6.6% p.a. compounding monthly. Determine the monthly repayment required to pay off the loan over 25 years.

Reading the question

- The question is asking for the monthly repayment.
- The question involves a reducing balance loan compounding monthly for 25 years.
- There is a deposit of $50 000 and a house valued at $570 000.

Thinking about the question

- The annuity formula (A_{PV}) can be used to find d.
- The compounding period is monthly.
- Must find n and i.
- The amount borrowed is not $570 000 as there is a deposit of $50 000.

Worked solution (✓ = 1 mark)

Compounding periods per year = 12.

$n = 25 \times 12$

$= \mathbf{300}$

$i = \frac{0.066}{12}$

$= \mathbf{0.0055}$ ✓ for n and i

Deposit = $50 000

Amount borrowed = 570 000 – 50 000

= **520 000** ✓

$$A_{PV} = d\left(\frac{1-(1+i)^{-n}}{i}\right)$$

$$\mathbf{520000 = d\left(\frac{1-(1+0.0055)^{-300}}{0.0055}\right)} \quad ✓$$

$520\,000 = d \times 146.741\ldots$

$d = \frac{520\,000}{146.741\ldots}$

$= 3543.64$

The monthly repayment is **$3543.64**. ✓

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6 Chapter summary

Compound interest

- The value of a **compound interest** investment at the end of the n^{th} **compounding period** is given by the recurrence relation

$$A_{n+1} = rA_n$$

where A_0 is the principal

A_n is total amount at the beginning of the n^{th} period

A_{n+1} is total amount at the beginning of the $(n+1)^{\text{th}}$ period

and $r = 1 + i$ where i is interest rate per compounding period.

- The **periodic interest rate**, i,

$$= \frac{\text{annual interest rate as a decimal}}{\text{number of compounding periods per year}}$$

- Compound interest formula:

$$A = P(1+i)^n$$

where A = total value of the investment (or loan) after n compounding periods (A_n)

P = principal or initial value (A_0)

i = periodic interest rate as a decimal

n = number of compounding periods.

Effective interest rate

- The interest rate that is stated for a loan or investment is called the **nominal interest rate**.
- The **effective interest rate** takes into account the number of compounding periods:

$$i_{\text{effective}} = (1+i)^k - 1$$

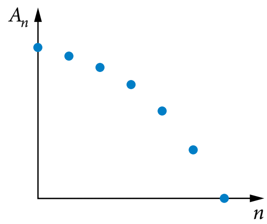
where i = the interest rate per compounding period as a decimal

k = the number of compounding periods per year.

- The amount of money in an account right now is called the **present value**.
- The amount that the investment will grow to in the future is called the **future value**.

Reducing balance loans

- A **reducing balance loan** is a compound interest loan with regular periodic payments.



- The value of a reducing balance loan is given by the recurrence relation

$$A_{n+1} = rA_n - d$$

where A_0 is the principal

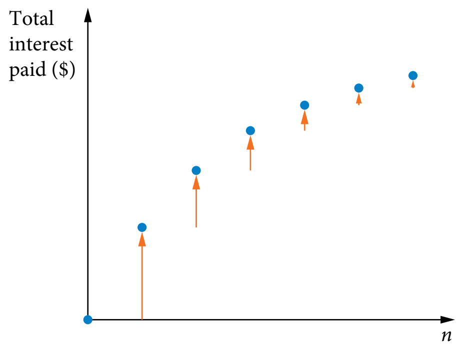
A_n is the total amount at the beginning of the n^{th} period

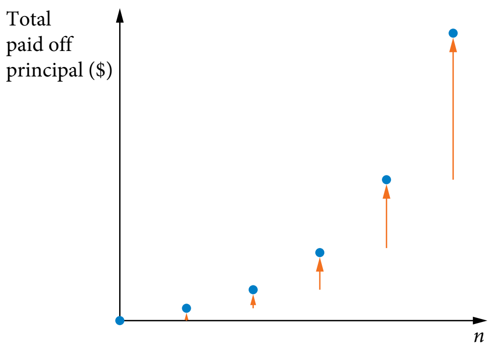
A_{n+1} is the total amount at the beginning of the $(n+1)^{\text{th}}$ period

d is the periodic payment

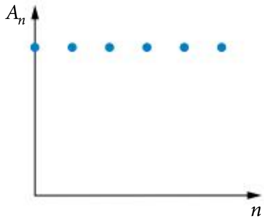
and $r = 1 + i$ where i is interest rate per compounding period.

- Reducing balance loan payment graphs:



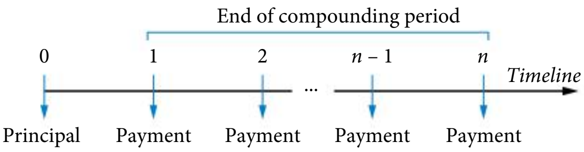


- **Amortisation** is the process of paying off a loan with regular payments over a period of time. An **amortisation table** (also called a **loan schedule**) provides a way to track the time, interest, the principal reduction and the balance of the loan over each compounding period.

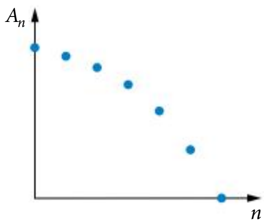


Ordinary annuities

- An **annuity** involves compound interest and a series of regular equal periodic payments over a given time.



- An **ordinary annuity** has a regular payment made at the end of each period after the interest has been added.



- The present value of an ordinary annuity is given by the recurrence relation

$$A_{n+1} = rA_n - d$$

where A_0 is the principal (the cost of the annuity)
A_n is the total amount at the beginning of the n^{th} period
A_{n+1} is the total amount at the beginning of the $(n+1)^{\text{th}}$ period
d is the periodic payment
and $r = 1 + i$ where i is interest rate per compounding period.

- The present value of an ordinary annuity is given by the formula

$$A_{PV} = d\left(\frac{1-(1+i)^{-n}}{i}\right)$$

where A_{PV} = the total amount of the investment (or loan)
d = the period payment
i = periodic interest rate per compounding period
n = number of compounding periods.

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Cumulative examination 1

Simple familiar

Perusal time: 2 minutes   Working time: 30 minutes

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

Section 1 (5 marks): 5 multiple choice questions

Section 2 (13 marks): 4 short response questions

Total: 18 marks

Section 1 5 multiple choice questions

5 marks

Worksheet General Maths formula sheet

1 ©QCAA 2020 1Q5 Determine the equation of the least-squares line where $r = 0.926$, $\bar{x} = 5.2$, $s_x = 2.3$, $\bar{y} = 68.6$ and $s_y = 41.7$.

A $y = 16.79x - 1146.51$

B $y = 16.79x - 18.70$

C $y = 0.05x + 68.33$

D $y = 0.05x + 1.70$

2 ©QCAA 2021 1Q10 City A is located at 55° N 120° E and City B is located at 40° N 165° E. The sun will rise in City A approximately

A 1 hour before it rises in City B.

B 1 hour after it rises in City B.

C 3 hours before it rises in City B.

D 3 hours after it rises in City B.

3 ©QCAA 2021 1Q15 Which of the following investment options gives the best return?

A 5.93% p.a. compounding daily

B 5.95% p.a. compounding monthly

C 5.97% p.a. compounding quarterly

D 5.99% p.a. compounding six-monthly

4 ©QCAA 2023 1Q12 A reducing balance loan with an initial balance of \$6000 is modelled by the recurrence relation $A_{n+1} = \left(1 + \frac{0.03}{12}\right)A_n - 400$, where n is the number of months.

The loan balance at the end of two months is closest to

A \$5100   **B** \$5200   **C** \$5215   **D** \$5230

5 ©QCAA 2020 1Q6 A loan of \$10 000 has interest charged on a reducing balance at 6% p.a. compounding quarterly with quarterly repayments of \$700. The balance after 6 months is

A \$8696.75   **B** \$8891.75   **C** \$8900.00   **D** \$9794.00

Section 2 4 short response questions

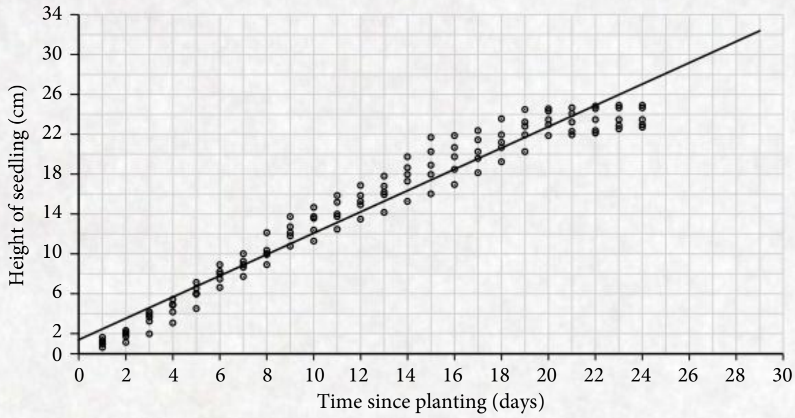
13 marks

6 ©QCAA 2023 1Q19 (4 marks) Ngarra compares two investment options and decides option A will provide the better return.

- Option A: 5.60% p.a. compounding monthly
- Option B: 5.62% p.a. compounding quarterly

Use the effective annual rate of interest formula to evaluate the reasonableness of Ngarra's decision.

7 ©QCAA 2020 1Q24 MODIFIED (3 marks) The following data for the height of five seedlings was collected and the least-squares line was developed and graphed.



a Use the least-squares line to estimate the height of a nine-day-old seedling. *[1 mark]*

b Based on the graph, the following statement was made: 'A seedling will reach a height of about 32 cm by day 29.' Comment on the reasonableness and the possible dangers of this statement. *[2 marks]*

8 ©QCAA 2021 1Q17 (4 marks) Determine the monthly repayment on a reducing balance loan of \$720 000 at 4.8% p.a. over 25 years. Give your answer to the nearest dollar.

9 ©QCAA 2021 1Q25 MODIFIED (2 marks) A conference is being held in Singapore (UTC + 8). A conference attendee got a flight from Brisbane (UTC + 10) at 10:30 am Brisbane time on Monday 7 December.

If the flight from Brisbane to Singapore took 7 hours and 40 minutes, determine the time, day and date in Singapore when the flight lands.

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Cumulative examination 2

Complex familiar and unfamiliar

Perusal time: 2 minutes   Working time: 30 minutes

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

3 short response questions 12 marks

1 (2 marks) Without performing any calculations, determine which investment would have a higher effective rate of interest: 5.7% p.a. compounded fortnightly or 5.7% p.a. compounded every 6 months. Give reasons for your answer.

Worksheet General Maths formula sheet

2 (3 marks) Lily is very ambitious and thinks that she should double her income in the next 5 years. What percentage increase must she get each year to achieve her objective?

3 ©QCAA 2023 2Q6 (7 marks) The table shows the average superannuation account balance for workers of various ages in two different industries. The coefficient of determination, R^2, for age versus account balance is 0.95 for industry A and 0.96 for industry B. 40-year-old Leigh works in the industry for which age explains a higher percentage of the account balance variation. Tony is 10 years older than Leigh and works in the other industry.

Age (years)	Account balance ($)	
	Industry A	Industry B
22	7500	8100
32	42 000	60 000
42	98 000	120 000
52	160 000	210 000
62	290 000	360 000
72	400 000	480 000

Use linear models to predict the difference in current superannuation account balances for Leigh and Tony.

CHAPTER

7 GRAPHS AND NETWORKS

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Syllabus coverage

UNIT 4, TOPIC 3: GRAPHS AND NETWORKS

Graphs, associated terminology and the adjacency matrix

- Understand the meaning of graph, vertex (node), edge (arc), loop, degree of a vertex, subgraph, simple graph, complete graph, bipartite graph, directed graph (digraph), weighted graph and network.
- Construct a network diagram to represent practical situations, e.g. tracks connecting camp sites in a national park, a social network, a transport network with one-way streets, the results of a round-robin sporting competition.
- Construct an adjacency matrix from a given graph or digraph.
- Construct a graph or digraph from a given adjacency matrix.

Planar graphs, paths and cycles

- Understand the meaning of planar graph and face.
- Apply Euler's formula to solve problems relating to planar graphs.
  - $v + f - e = 2$ where v is number of vertices, f is number of faces and e is number of edges
- Understand the meaning of walk, trail, path, open walk, open trail, open path, closed walk, closed trail (circuit), closed path (cycle), connected graph and bridge.
- Solve practical problems to determine the shortest path between two vertices in a weighted graph (by trial-and-error methods only).
- Understand the meaning of Eulerian trail, semi-Eulerian graph, Eulerian circuit and Eulerian graph, and the conditions for their existence.
- Solve practical problems involving semi-Eulerian graphs and Eulerian graphs.
- Understand the meaning of Hamiltonian path, semi-Hamiltonian graph, Hamiltonian cycle and Hamiltonian graph.
- Solve practical problems involving semi-Hamiltonian graphs and Hamiltonian graphs (by trial-and-error methods only).

7.1

Prior learning
Graphs and networks

Videos (2):
7.3 Adjacency matrices
Exam question analysis Graphs and networks

Prior learning (1):
7.1 Graphs and networks

Worksheets (9):
7.1 Applications of networks
7.2 Types of graphs • Planar graphs
7.3 Adjacency matrices 1 • Adjacency matrices 2
7.5 Eulerian graphs • Eulerian trails and circuits
7.6 Hamiltonian paths and cycles
Cumulative exams General Maths formula sheet

Nelson MindTap

To access resources above, visit
cengage.com.au/nelsonmindtap

Terminology

adjacency matrix	arc	bipartite graph	circuit
connected graph	cycle	degree (of a vertex)	directed graph
edge	Eulerian graph	Euler's formula	face
Hamiltonian graph	loop	path (open and closed)	planar graph
semi-Eulerian graph	semi-Hamiltonian graph	shortest path	subgraph
trail (open and closed)	vertex (vertices)	walk	weighted graph

Worksheet
Applications of networks

7.1 Graphs and networks

A **network** is a group of interconnected items such as people, places or things. They can be represented as a **graph** (also called a **network diagram**). The study of networks is also called **graph theory**.

Features of graphs and networks

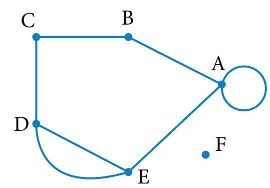
A graph consists of a set of points called **vertices** (or **nodes**) connected with lines called **edges** (or **arcs**).

A **vertex** is usually labelled by a capital letter but can be labelled as a lowercase letter or not labelled at all.
In the graph shown, the vertices are A, B, C, D, E and F.

An edge generally connects 2 vertices and is named by the 2 vertices it connects. The edges in the graph are AB, BC, CD, DE, EA and AA.

Adjacent vertices are 2 vertices connected by an edge.

If 2 vertices are connected by 2 or more edges, the edges are called **multiple edges**. Vertices D and E are connected with multiple edges.



Exam hack

Edges are often drawn as straight lines, but they can also be drawn as curves. Drawing them as curves may make it easier to represent or visualise a network or graph.

If an edge connects a vertex to itself, it is called a **loop**, as shown at vertex A.

If a vertex is not connected to any other vertex, it is called an *isolated vertex*, such as vertex F.

The number of edges that are connected to a vertex is called the **degree (of the vertex)** and it is written as deg (V), where V is the label of the vertex. For the graph above:

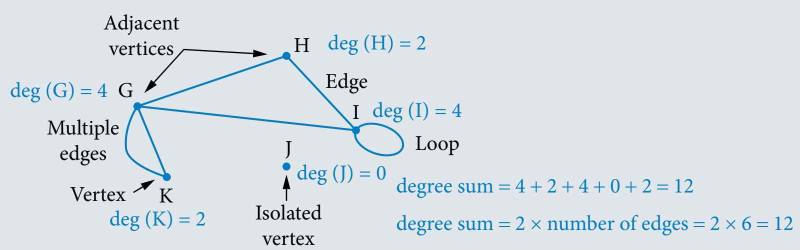
- F has degree 0. This is written as $\deg(F) = 0$
- C has degree 2, that is, $\deg(C) = 2$
- E has degree 3, that is, $\deg(E) = 3$
- A has degree 4, that is, $\deg(A) = 4$. A loop adds 2 edges as it connects to the vertex twice.

If you add all the degrees of the vertices in a graph (called the *degree sum*, *S*), you will find that:

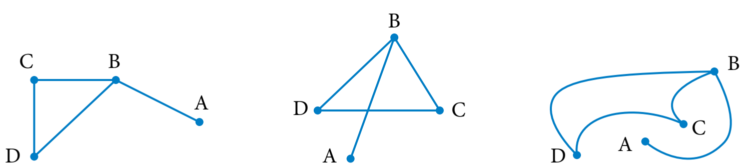
$$S = 2 \times \text{number of edges}.$$

This means the degree sum is always an even number (see below).

Features of a graph



Graphs can often be drawn in multiple ways. For example, the graphs below are equivalent even though they look different.



The edges of a graph can intersect without there being a vertex. When drawing a graph, it is helpful to not have edges intersecting as it makes it easier to identify the vertices.

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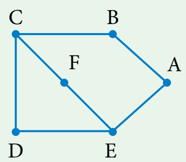
WORKED EXAMPLE 1 Identifying features of graphs

For each graph

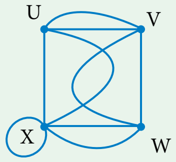
i count and list the vertices and edges

ii show that the degree sum is twice the number of edges.

a



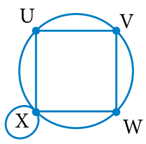
b



Steps	Working
a i 1 Count and list the number of vertices.	6 vertices: A, B, C, D, E, F
2 Count and list the number of edges.	7 edges: AB, BC, CD, CF, DE, EF, EA
ii 1 Find the degree of each vertex and then add them.	see table below
2 Show that multiplying the number of edges by 2 gives the same result.	2 × number of edges = 2 × 7 = 14 = degree sum
3 State findings.	The sum of the degrees is equal to twice the number of edges.

Vertex	A	B	C	D	E	F	**Sum**
Degree	2	2	3	2	3	2	**14**

b i 1 Redraw the graph to uncross the intercepting edges – this verifies there is no vertex at the point of intersection.

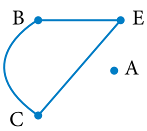


Steps	Working
2 Count and list the number of vertices.	4 vertices: U, V, W, X
3 Count and list the number of edges.	9 edges: 2 × UV, 2 × VW, 2 × WX, 2 × XU, XX
ii 1 Find the degree of each vertex and then add them.	see table below
2 Show that multiplying the number of edges by 2 gives the same result.	2 × number of edges = 2 × 9 = 18 = degree sum
3 State findings.	The sum of the degrees is equal to twice the number of edges.

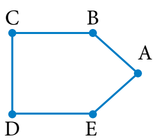
Vertex	U	V	W	X	**Sum**
Degree	4	4	4	6	**18**

A **subgraph** is a graph that is formed using a subset of a larger graph's edges and vertices.

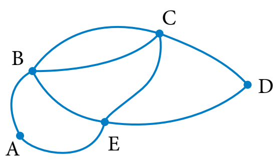
A subgraph cannot include any new vertices or edges that were not in the original, larger graph.



and



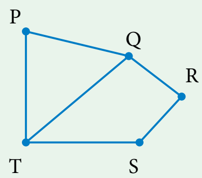
are both subgraphs of



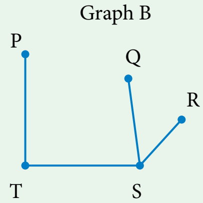
WORKED EXAMPLE 2 Identifying and drawing subgraphs

Consider Graph A and Graph B.

Graph A

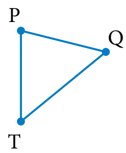


Graph B



a **State** with reasons if Graph B is a subgraph of Graph A.

b Draw a subgraph of Graph A which has 3 vertices all with degree 2.

Steps	Working
a Check that Graph B only contains vertices and edges from Graph A.	Graph B is not a subgraph of Graph A as Graph B contains an additional edge QS.
b **1** Identify 3 vertices in Graph A that are connected and have degree 2 or more.	Vertices P, Q and T are connected and have degree 2 or more. In a subgraph joining these 3 vertices, the degree of each vertex would be 2.
2 Draw the subgraph.	

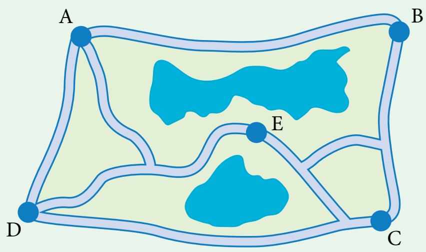
Creating graphs and networks

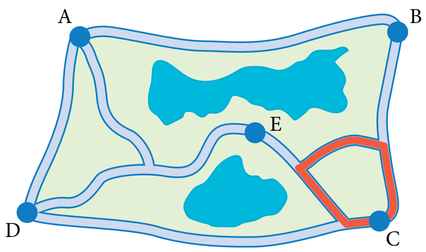
Graphs are used to represent different types of connections in real-world situations such as road systems, maps, rail networks, friendship and social networks, and sporting competitions. When using a graph to represent a practical situation, you must identify what the vertices and edges represent. For example, in a graph representing a small social network, the vertices could represent each person and the edges a social connection or friendship.

WORKED EXAMPLE 3 Representing road systems as graphs

The map of a bush reserve shows the gates and fire trails (roads) through the area.

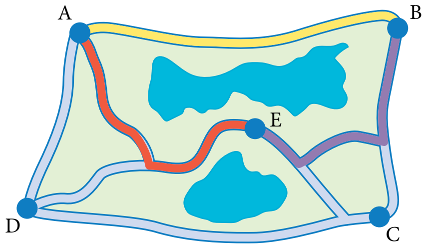
Draw a graph to represent the possible ways to travel through the bush reserve.



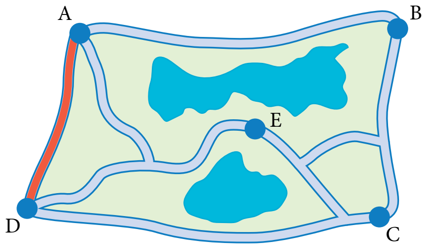
Steps	Working
a **1** Find loops by identifying roads from a gate back to itself. There is one path from C to C. So, the graph has a loop at C.	

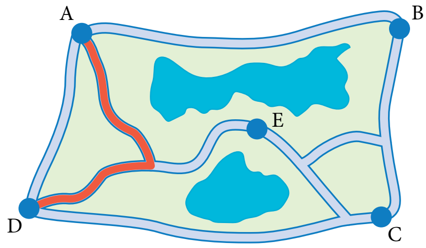
2 Find edges by identifying paths between gates.

There is only **one** path from A–B, A–E and B–E.

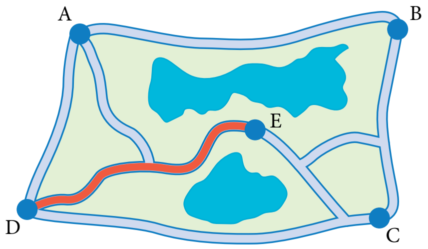


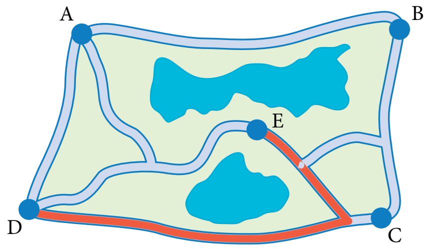
There are 2 paths from A–D that do not go through any other gates. So, the graph has 2 **multiple edges** joining A–D.



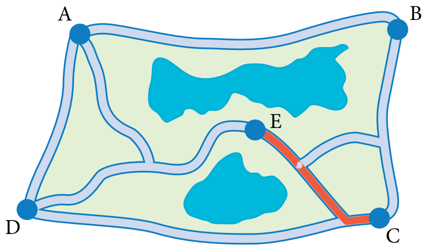


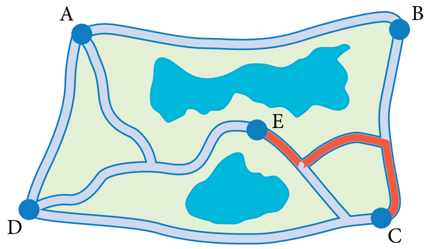
There are 2 paths from D–E. So, there are 2 **multiple edges** joining D–E.



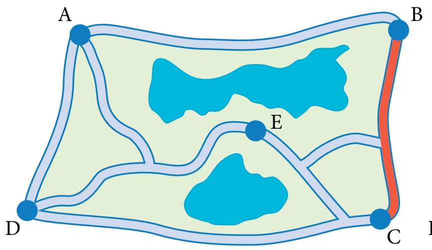


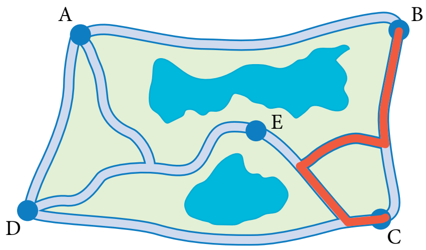
There are 2 paths from C–E. So, there are 2 **multiple edges** joining C–E.



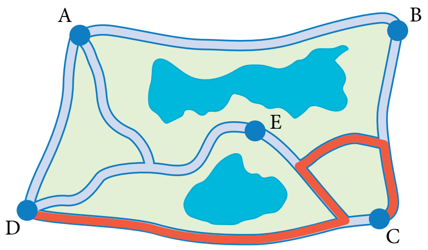


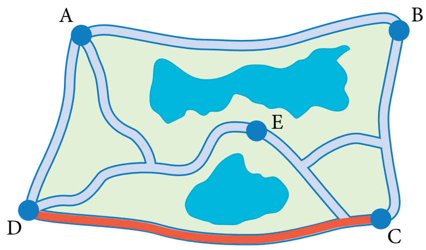
There are 2 paths from C-B. So there will be 2 **multiple edges** joining C-B.



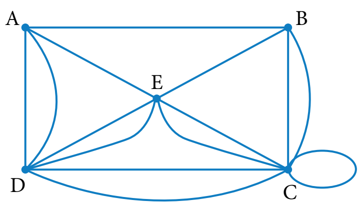


There are 2 paths from D-C. So there will be 2 **multiple edges** joining D-C.



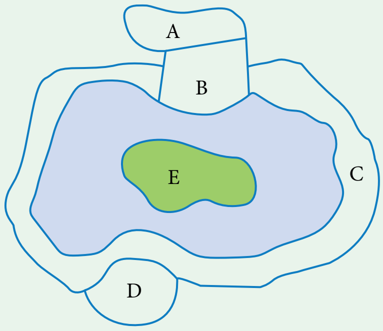


3 Draw the graph.



WORKED EXAMPLE 4 Representing maps as graphs

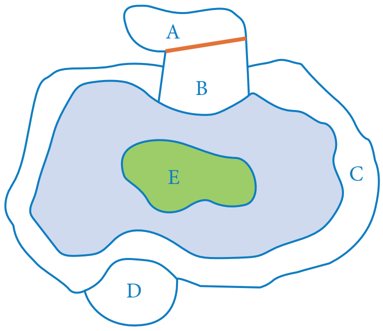
A local community space has 5 zones, labelled as A to E on the map. Area E is surrounded by a small lake and is only accessible by water. Draw a network showing the land connections between the 5 zones.



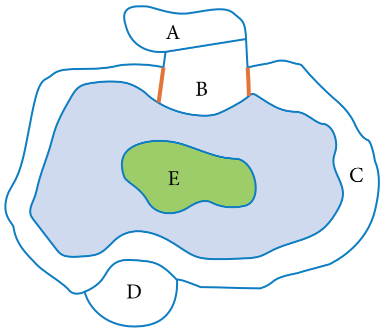
Steps | **Working**

1 Identify and list all zone land connections.

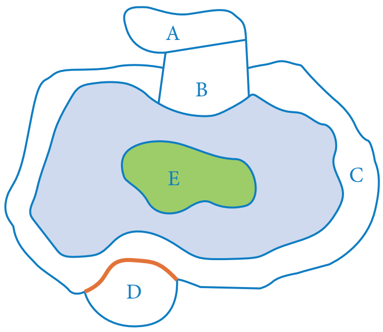
Zones A and B have one land connection.



Zones B and C have 2 land connections.

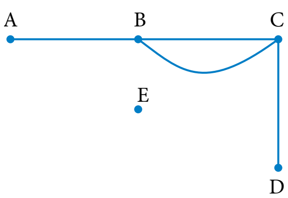


Zones C and D have one land connection.



Zone E has no land connections.

2 Let A, B, C, D and E be the vertices of the graph and the land connections the edges. Draw the graph.



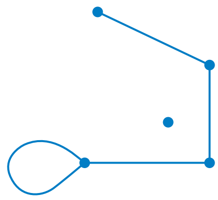
EXERCISE 7.1 Graphs and networks

ANSWERS p. 456

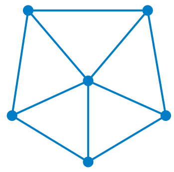
Mastery

1 WORKED EXAMPLE 1 How many vertices and edges does each graph have?

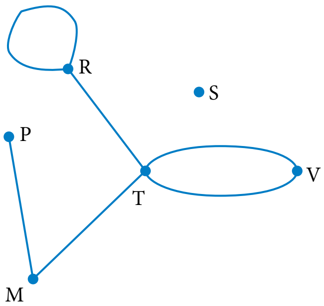
a



b



2 Consider the graph shown.



a Name the vertex adjacent to P.

b The loop connects which vertex to itself?

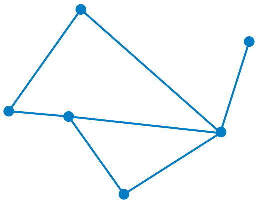
c Name the isolated vertex.

d Name the vertices connected by multiple edges.

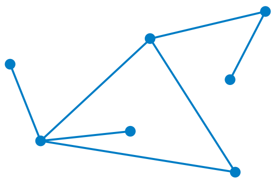
e Name the vertex adjacent to M.

3 For each graph, verify that the sum of the degrees of the vertices is twice the number of edges.

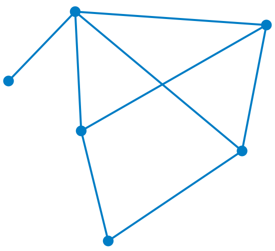
a



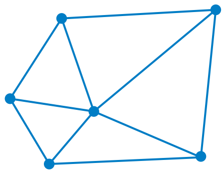
b



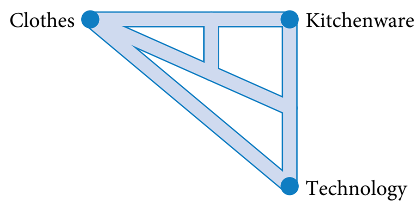
c



d

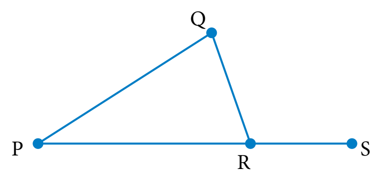


4 WORKED EXAMPLES 2, 3 The pathway system shows how 3 areas in a department store, Clothes (C), Kitchenware (K) and Technology (T), are wheelchair accessible.



- **a** Determine if the graph that represents this pathway system has loops or multiple edges. Justify your answers.
- **b** Create a graph representing this pathway system.
- **c** Construct, if possible, a subgraph containing all 3 sections (C, K and T), with an isolated vertex and 3 edges. If not, justify why it is not possible.

5 The graph shown here can be described using the vertices (V) and edges (E) as:

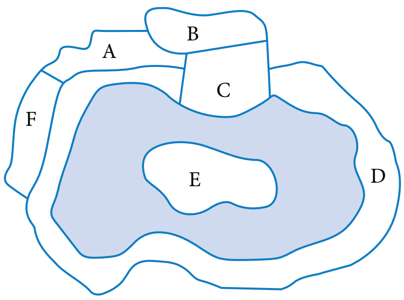


$$V = \{P, Q, R, S\} \text{ and } E = \{PQ, QR, PR, RS\}.$$

Draw a graph for each listing of vertices and edges.

- **a** $V = \{A, B, C\}$ and $E = \{AB, BC\}$
- **b** $V = \{W, X, Y, Z\}$ and $E = \{WX, XY, YW\}$
- **c** $V = \{H, K, M, L, N\}$ and $E = \{HK, KM, HL, LM, LN, NM\}$
- **d** $V = \{M, N, O, P, Q, R\}$ and $E = \{MN, NO, OP, PQ, QM, MR, NR, OR, PR, QR\}$
- **e** $V = \{F, K, J, L, M, P\}$ and $E = \{FJ, FL, JK, LK, PM\}$
- **f** $V = \{A, H, Q, M, V\}$ and $E = \{AH, QH, MQ, MM\}$

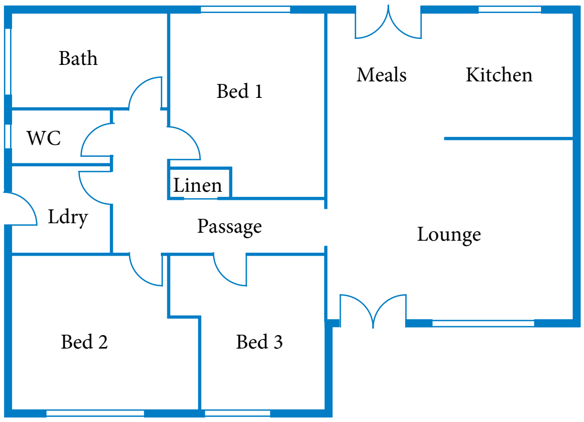
6 WORKED EXAMPLE 4 A theme park is divided into 6 zones, labelled A to F in the map. Zone E can only be accessed via a boat ride. Draw a network to represent the land connections between the zones.



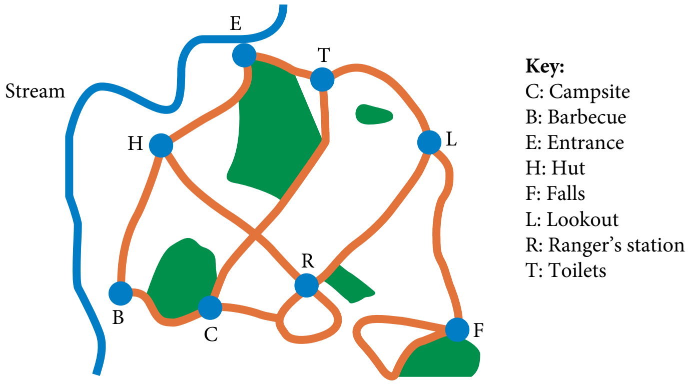
7 A netball carnival has 6 teams competing over 2 days. The roster for the tournament is shown below. Draw a graph to represent this information.

	Tigers	Bullets	Raiders	Chargers	Lightning	Victory
Tigers			✓		✓	✓
Bullets			✓	✓		✓
Raiders	✓	✓		✓		
Chargers		✓	✓		✓	
Lightning	✓			✓		✓
Victory	✓	✓			✓	

8 Draw the house plan shown here as a network, showing the doors (doorways) as edges and the rooms and outside as vertices.

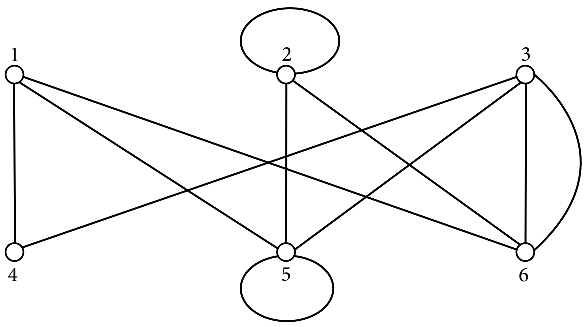


9 Draw a graph to represent this tourist map.



Exam practice

10 © QCAA 2020S 1Q3 The graph below has 6 vertices.



The vertex with the smallest degree is

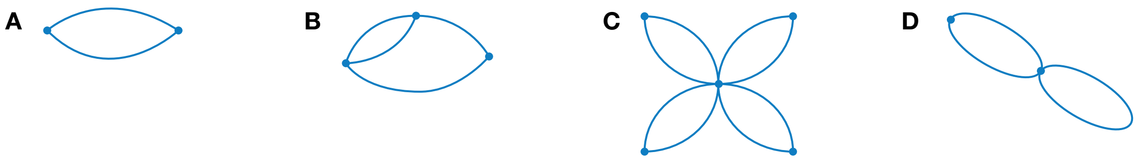
A 1

B 2

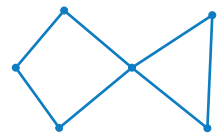
C 3

D 4

11 Which graph contains a loop?



12 Which statement is **not** true for this graph?



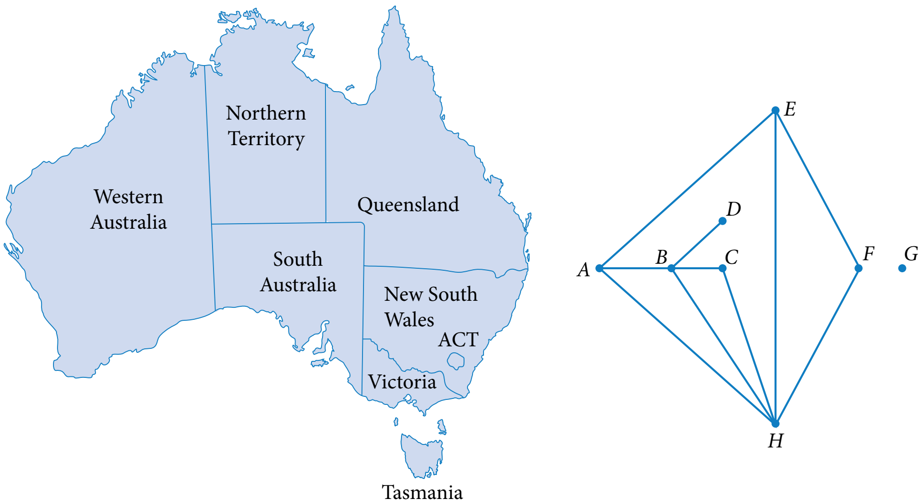
A There are no isolated vertices.

B All vertices have an even degree.

C The sum of the degrees is 12.

D The graph has 6 vertices.

13 (7 marks) The map of Australia shows the Northern Territory, the Australian Capital Territory (ACT) and the 6 states. In the network diagram, each of the vertices A to H represents one of the states or territories shown on the map. The edges represent a border shared by either 2 states or by a state and territory.



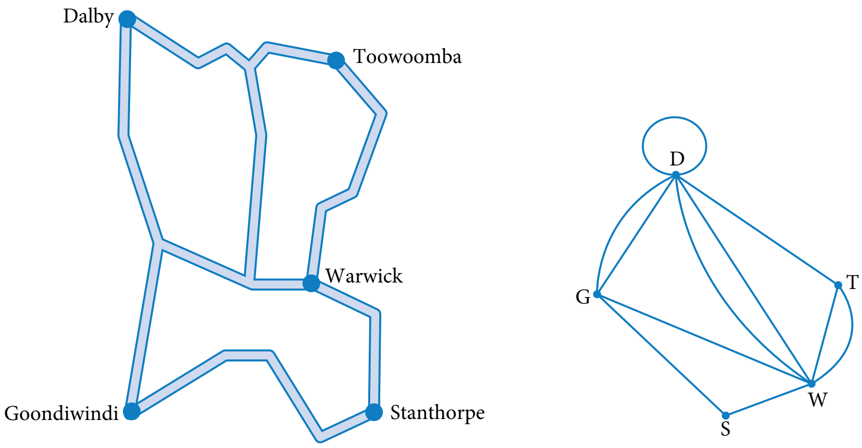
a Which vertex represents Queensland? *[1 mark]*

b Which vertex represents the Australian Capital Territory (ACT)? Justify your answer. *[2 marks]*

c Which vertex represents Tasmania and state its degree? *[2 marks]*

d State the degree sum of the network and explain its significance in the context of the map of Australia. *[2 marks]*

14 (3 marks) CF The map roughly illustrates key road connections between 5 towns, Stanthorpe (S), Warwick (W), Goondiwindi (G), Toowoomba (T) and Dalby (D). A network was created to represent the map.



a Explain what the loop at D represents in terms of a driver departing Dalby. *[1 mark]*

b A mistake was made in the construction of the network. Identify the mistake and explain how it should be fixed. *[2 marks]*

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15 (2 marks) CF A camp site used by scouts has various tracks for walking. The tracks connect various places at the camp site as follows:

Main hut to beach	Beach to flying fox
Flying fox to main hut	Main hut to barbecue site
Barbecue site to flying fox	Barbecue site to main gate
Main gate to main hut	Main hut to Scoutmaster's hut
Scoutmaster's hut to beach	

Draw a network showing the paths.

16 (3 marks) CU The map of mainland USA shows the 5 common regions it can be divided into. Use a network to find the smallest number of colours required to colour the map given no 2 regions that share a common border can be the same colour.

Worksheets
Types of graphs

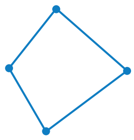
Planar graphs

7.2 Types of graphs

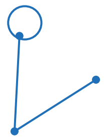
Graphs can have different features and be represented and described in several ways.

Simple and complete graphs

A **simple graph** is a graph that does not contain any loops or multiple edges between adjacent vertices.

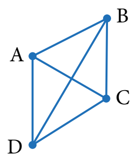


simple graph

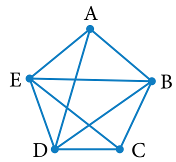


not a simple graph as it has a loop

A **complete graph** is a simple graph where every vertex is connected to every other vertex.



complete graph



not a complete graph as A and C are not connected

Given that each vertex in a complete graph with n vertices is connected to every other vertex, then the degree of each vertex is $n - 1$.

Simple and complete graphs

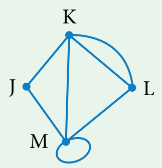
- A simple graph is a graph without loops or multiple edges.
- A complete graph is a simple graph where every vertex is connected to every other vertex.
- In a complete graph with n vertices, the degree of each vertex is $n - 1$.

WORKED EXAMPLE 5 Simple and complete graphs

Justify whether the graph is

a a simple graph

b a complete graph.



Steps	Working
a Identify the existence of any loops or multiple edges.	There is a loop at M and multiple edges between vertices K and L. This graph is not a simple graph.
b Recall that the first condition of a complete graph is that it must be simple.	As the graph is not simple, it cannot be complete.

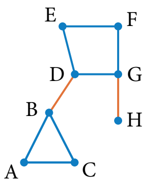
Connected graphs

A **connected graph** is a graph where there is a path from a vertex to any other vertex in the graph along edges. If there is not a path along edges between all vertices in a graph, it is a **disconnected graph**.

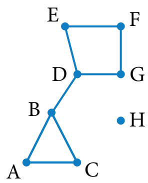
An edge that keeps a graph connected is called a **bridge**. If you delete a bridge, the graph becomes disconnected.

> **Connected graphs**
>
> - A connected graph is a graph where there is a path from any vertex to any other vertex.
> - A bridge is an edge that keeps a graph connected.

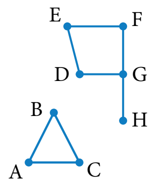
The graph on the left is a connected graph that has 2 bridges, BD and GH.



connected graph



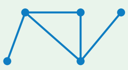
disconnected graph – the bridge GH has been removed

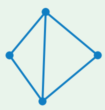


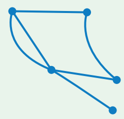
disconnected graph – the bridge BD has been removed

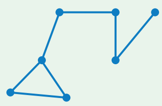
WORKED EXAMPLE 6 Identifying bridges

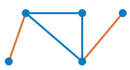
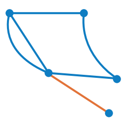
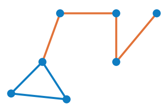
Determine the number of bridges in each of these connected graphs.

a 

b 

c 

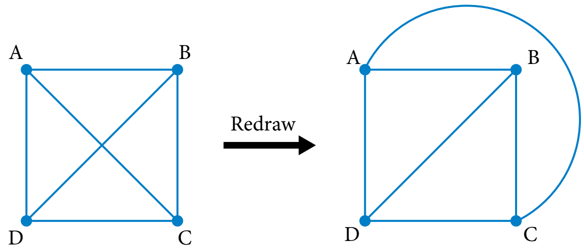
d 

Steps	Working
a–d Decide which edge(s) will make the graph disconnected if deleted. Hint: Highlight or colour edges that may be bridges to help confirm if the graph becomes disconnected.	**a**  There are 2 bridges. **b** There are no bridges. **c**  There is one bridge. **d**  There are 4 bridges.

Planar graphs and Euler's formula

Planar graphs are connected graphs that can be drawn without any edges crossing.

Consider the left graph below. It is a planar graph as it can be redrawn so that the 2 edges AC and BD are not crossing.

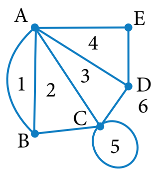


If you imagine that the edges are like elastic bands, you can visualise how edge AC is able to be stretched so that it still joins A and C but does not cross the edge BD. Both graphs shown here represent the same information regarding vertices and connections.

There is no easy method for working out if a graph with crossing edges can be drawn as a planar graph. You can, however, use trial and error. Whatever you do, make sure that the connections remain the same.

Faces are regions of a planar graph that are enclosed or bound by edges. Before you count the faces on a graph, check there are no edges that are crossing; if there are, redraw the graph with no crossings if possible.

This graph has 6 faces.



Note that:

- the loop at C, where an edge starts and ends at the same vertex, creates a face (face 5)
- the 2 multiple edges between A and B creates a face (face 1)
- the infinitely large region outside the graph counts as a face (face 6).

Planar graphs

- A planar graph is a connected graph that can be drawn with no edges crossing.
- Faces are regions of a planar graph that are enclosed or bound by edges, including the region outside the graph.

Euler's formula applies to graphs that are connected and planar.

Euler's formula

For planar graphs

$$v + f - e = 2$$

where v = number of vertices

f = number of faces

e = number of edges.

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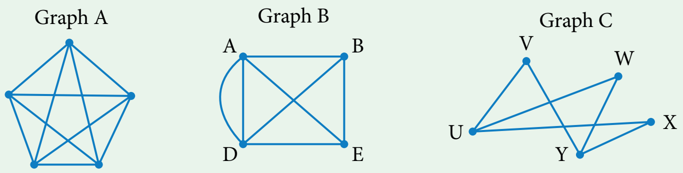
WORKED EXAMPLE 7 Identifying planar graphs and verifying Euler's formula

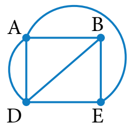
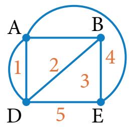
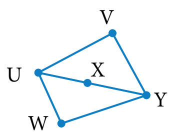
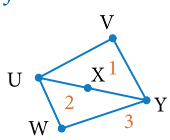
For each graph

i identify if the graph is connected

ii redraw the graph, if possible, to show it is a planar graph

iii verify if Euler's formula holds true.



Steps	Working
a **i** Identify if the graph is connected.	Graph A is connected.
ii Try to uncross all the edges.	It is not possible to uncross the edges. Graph A is not planar.
iii Justify why Euler's formula is not true.	Euler's formula will not hold true as the graph is not planar.
b **i** Identify if the graph is connected.	Graph B is connected.
ii Try to uncross all the edges. Move edge AE outside of the graph.	 Graph B is planar as it can be redrawn with no edges crossing.
iii **1** Count the number of vertices and edges.	$v = 4$, $e = 7$
2 Identify the number of faces.	$f = 5$
Hint: It may be helpful to identify each face on the graph so you do not miss any.	
3 Write Euler's formula.	$v + f - e = 4 + 5 - 7$
4 Substitute and evaluate.	$= 2$
5 State whether Euler's formula holds true.	Euler's formula holds true for Graph B.
c **i** Identify if the graph is connected.	Graph C is connected.
ii Try to uncross all the edges.	 Graph C is planar as it can be redrawn with no edges crossing.
iii **1** Count the number of vertices, edges and faces.	$v = 5$ $e = 6$ $f = 3$ 
2 State Euler's formula, substitute and evaluate.	$v + f - e = 5 + 3 + 6$ $= 2$
3 State whether Euler's formula holds true.	Euler's formula holds true for Graph C.

Euler's formula can be used to find the number of vertices, edges or faces for any planar graph.

WORKED EXAMPLE 8 **Using Euler's formula**

a A planar graph has 10 edges and 4 faces. Determine the number of vertices.

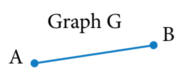
b A planar graph has 12 vertices and 6 faces. Determine the number of edges.

Steps	Working
a **1** State Euler's formula.	$v + f - e = 2$
2 Substitute for e and f. Solve for v.	$v + 4 - 10 = 2$ $v - 6 = 2$ $v = 8$
3 State the result.	The graph has 8 vertices.
b **1** Substitute into Euler's formula.	$12 + 6 - e = 2$
2 Solve for e.	$18 - e = 2$
3 State the result.	$e = 16$

Bipartite graphs

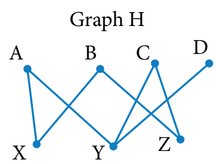
A **bipartite graph** is a graph with vertices that can be divided into 2 distinct sets so that each edge of the graph connects a vertex from one subset to the other.

A very simple bipartite graph is a connected graph with 2 vertices and one edge, such as Graph G.

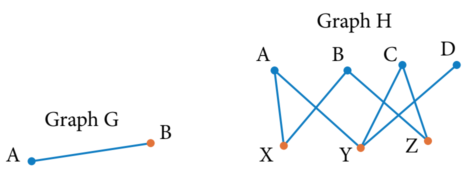


Graph H is also an example of a bipartite graph as vertices A, B, C and D could form one set and X, Y and Z could form another.

Notice how there are no edges that connect vertices from the same set.



An easy way to check if a graph is bipartite is to use the vertex two-colouring technique. Simply colour the vertices in 2 distinct colours and check that no adjacent vertices are of the same colour. For example, Graph G and Graph H can be shown to be bipartite using a blue-red colouring technique.



Bipartite graphs

A bipartite graph has vertices that can be divided into 2 distinct sets and no edges connect vertices from the same set.

A **complete bipartite graph** is one where every vertex in one set is connected to every vertex in the second set.

WORKED EXAMPLE 9 Identifying bipartite graphs

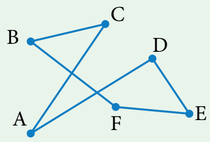
Consider Graph K.

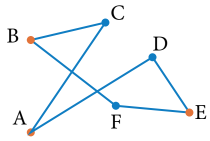
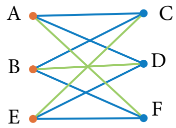
Determine if Graph K is bipartite.

If so, identify what edges would need to be added to make Graph K a complete bipartite graph.

If not, identify the features which make the graph non-bipartite.

Graph K

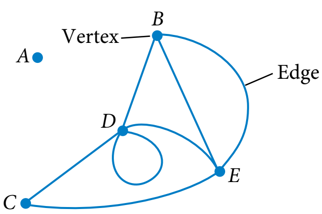


Steps	Working
1 Starting at any vertex, use the two-colour technique to colour adjacent vertices.	
2 Determine and state if the graph is bipartite.	No adjacent vertices are the same colour, so Graph K is bipartite.
3 Redraw Graph K so that each set is in a different 'column'. To make the bipartite graph complete, add edges so that each vertex in one set is connected to all vertices in the other set.	
4 State the new required edges.	The required edges are AF, BD and EC.

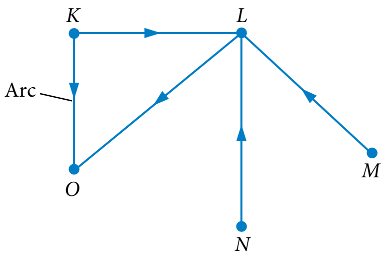
Directed and weighted graphs

A **directed graph**, also called a **digraph**, is a graph where each edge has a direction, shown as an arrowhead, between the 2 adjacent vertices it connects. Directed edges are commonly referred to as **arcs**. You can only travel along an arc in the direction of the arrowhead.

A graph that has edges with no direction is called an **undirected graph**.

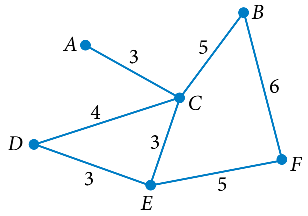


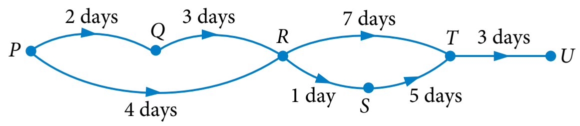
undirected graph



directed graph

A **weighted graph** (often just called a **network**) is a graph that has additional information labelled on each edge like a length. This information can represent things such as distances, time, cost or other quantities.





Directed and weighted graphs

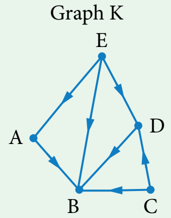
- A directed graph has edges with direction.
- An undirected graph has edges with no direction.
- A weighted graph has a **weight** labelled on each edge.

WORKED EXAMPLE 10 Exploring directed graphs

Consider Graph K.

a **Identify** which vertex cannot be reached from E.

b Which vertex cannot be used a starting point to reach any other vertex? **Justify** your answer.



Steps	Working
a Start at E and see which vertices can be reached.	A can be reached. B can be reached. D can be reached. C cannot be reached.
b Look for a vertex that only has arrows coming into it not travelling out of it.	B cannot be used as a starting point as all edges travel into B and none go out of B.

Investigation The handshake problem

Consider a group of 5 people in a room who are meeting for the first time. Each person shakes hands with every other person once. How many handshakes are there in total?

1 Draw a graph to represent this situation. What type of graph is this?

2 How many handshakes are there (count the number of edges)?

3 Explain why the number of handshakes is not $5 \times 4 = 20$.

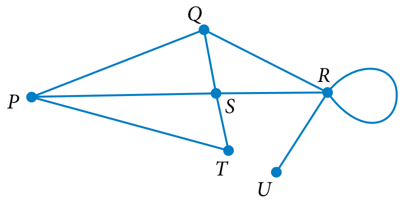
4 Create a simple formula that will determine the actual number of handshakes for a group of n people.

EXERCISE 7.2 Types of graphs

ANSWERS p. 456

Recap

Refer to this graph for Questions 1 and 2.



1 The degree of vertex S in the figure is

A 1 **B** 2 **C** 3 **D** 4

2 The sum of the degrees of the vertices is

A 14 **B** 15 **C** 16 **D** 18

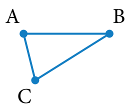
Mastery

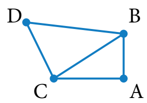
3 WORKED EXAMPLE 5 Justify whether each graph is

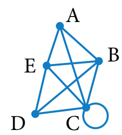
i a simple graph

ii a complete graph.

a


b


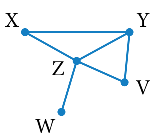
c


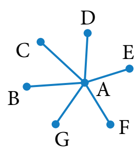
d


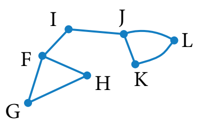
4 A complete graph has 5 vertices. How many edges does it have?

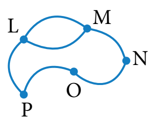
A 7 **B** 8 **C** 9 **D** 10

5 WORKED EXAMPLE 6 For each connected graph, identify any bridges.

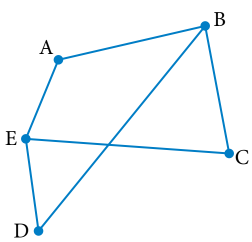
a


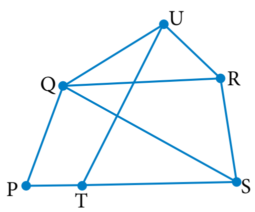
b


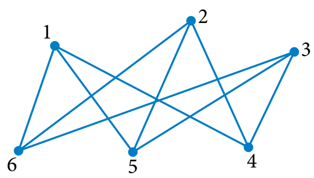
c


d


6 WORKED EXAMPLE 7 Redraw each graph, if possible, to show it is a planar graph.

a


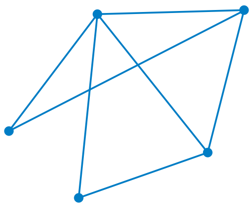
b


c


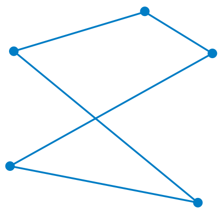
9780170484824

7 Which of these graphs is a planar graph with 5 vertices and 2 faces?

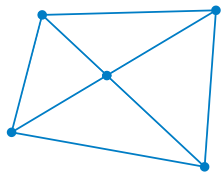
A



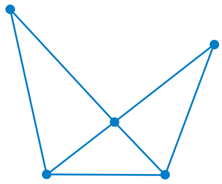
B



C



D



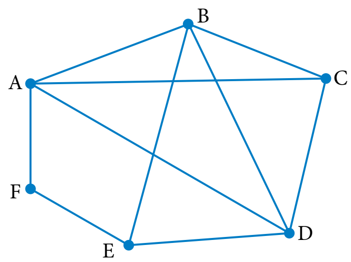
8 WORKED EXAMPLE 8

a A connected planar graph has 3 edges and 3 faces. How many vertices does it have?

b A connected planar graph has 3 vertices and 4 faces. How many edges does it have?

c A connected planar graph has 7 edges and 5 vertices. How many faces does it have?

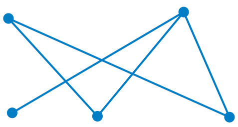
9 **a** Redraw this graph to show it is planar.



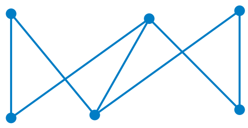
b Verify if Euler's formula holds true.

10 WORKED EXAMPLE 9 Which of these is not a bipartite graph?

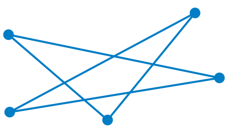
A



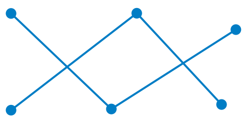
B



C



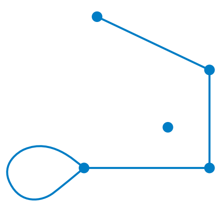
D



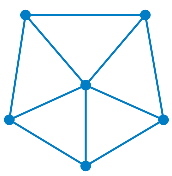
11 WORKED EXAMPLE 10 List which of the following graphs are

a connected
b complete
c directed
d simple
e bipartite
f disconnected.

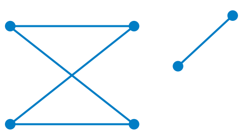
A



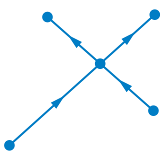
B



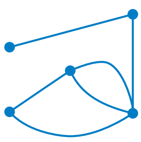
C



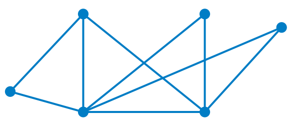
D



E



F



9780170484824

12 A medical practice consists of 4 doctors: Aria, Benjamin, Hannah and Oscar. The practice does not operate on a Sunday. The availability of the doctors to work on particular days is:

- Aria is available on Tuesday, Thursday and Saturday.
- Benjamin is available every day.
- Hannah is available on Wednesday, Thursday and Friday.
- Oscar is available on Monday and Saturday.

Draw a graph to model this situation and identify what type of graph it is.

13 A squash club is holding a round robin tournament with their top 6 players. In a round robin tournament, each player competes against every other player once only.

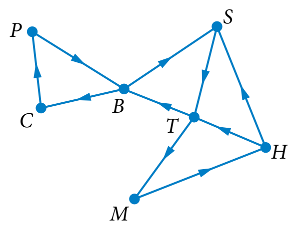
a Number the players 1 through 6 and draw a graph to represent this information.

b What type of graph is this?

c How many games will be played in total?

d Consider a round robin tournament with 5 teams. Without drawing a graph, how many games will be played?

14 One-way streets connect various points of interest in a city as shown below.



Key:
S: School
B: Bus stop
C: Cinema
T: Train station
P: Petrol station
H: Hospital
M: Museum

a Is it possible to travel from the cinema to every other location without passing a location more than once?

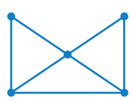
b Describe 2 routes from the museum to the bus stop.

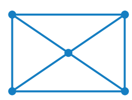
c Is it possible to travel from the school to every other location without passing a location more than once?

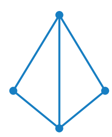
d How many different routes are there from the bus stop to the hospital?

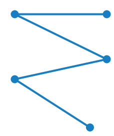
Exam practice

15 © QCAA 2020 1Q13 Which of the following is a planar graph with 5 vertices and 4 faces?

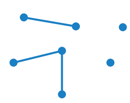
A


B


C


D


16 What is the smallest number of edges that need to be added to make this a connected graph?

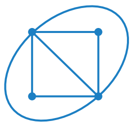
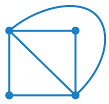
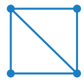
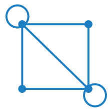


A 2 **B** 3 **C** 4 **D** 5

17 To make a complete graph with 4 vertices, the number of edges needed is

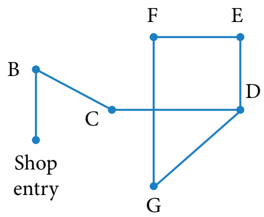
A 3 **B** 4 **C** 5 **D** 6

18 © QCAA 2020S 1Q14 Identify the planar graph that has 4 faces.

A

B

C

D


19 (2 marks) Is it possible for a planar graph to have 6 vertices, 10 edges and 5 faces? Justify your answer.

20 (4 marks) The graph below shows the covered walkways through a carpark to a shop's entry.



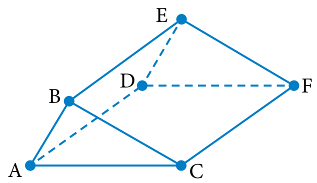
a Using Euler's formula, state the number of vertices, edges and faces, and show that the graph is planar. *[2 marks]*

The covered walkways need to be replaced and the owner would like to decrease the replacement cost by removing one edge (one walkway).

b Identify any edge that cannot be removed. While keeping the graph connected. *[1 mark]*

c What is the name given to the type of edge identified in part **b**? *[1 mark]*

21 (3 marks) CF Verify whether this prism can be represented as a planar graph.



22 (4 marks) CF A graph has 6 vertices with degrees 2, 2, 3, 4, 4, 5.

If this graph is planar, how many faces would it have?

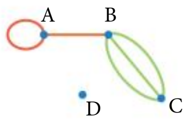
7.3 Graphs and matrices

Adjacency matrices

Matrices are often used when dealing with graphs as they provide a simple way to represent and work with the graph data.

An **adjacency matrix** for a non-directed graph describes the number of edges between **adjacent vertices** on the graph. An adjacency matrix for a graph with n vertices will be an $n \times n$ matrix (**square matrix**).

Consider the non-directed graph and its adjacency matrix shown below.

Graph	Adjacency matrix	Description
 Number of vertices = 4 Number of edges = 5	$\begin{array}{cc} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$ Number of rows = 4	• One loop at A to A. • One edge from A to B. • 3 edges from B to C. • No edges to D or between A and C.

Worksheets
Adjacency matrices 1
Adjacency matrices 2

Comparing the graph to its adjacency matrix, note:

- the row and column labels match the vertices of the graph
- the number of vertices is equal to the number of rows in the matrix (4 vertices = 4 rows). This number is also the dimension of the matrix (4×4)
- where there are no edges between adjacent vertices, a 0 is listed as the element
- where an edge(s) exists between adjacent vertices, the corresponding element in the matrix is equal to the number of edges between the vertices
- the number of edges is equal to the sum of the elements on or below the **leading diagonal** ($5 = 1 + 1 + 3$)

$$\begin{array}{cc} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

HINT: The leading diagonal of a square matrix is the diagonal line of elements that run from the top left corner to the bottom right corner of the matrix.

- the degree of a vertex is equal to the sum of the elements of the row (+1 for each loop).

$$\begin{array}{cc} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$\deg(A) = 1 + 1 + 1 = 3$

$\deg(B) = 1 + 3 = 4$

$\deg(C) = 3$

$\deg(D) = 0$

Adjacency matrices of undirected graphs

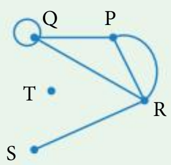
An adjacency matrix is a square matrix representing a graph where:

- each row and column are labelled as a vertex
- the number of vertices is equal to number of rows
- the elements show the number of edges between vertices
- the elements are symmetric about its leading diagonal
- the leading diagonal shows the number of loops
- the number of edges is equal to sum of the elements on and below the leading diagonal
- the degree of a vertex is equal to sum of the elements of the row (+1 for each loop).

Video
Adjacency matrices

WORKED EXAMPLE 11 Constructing an adjacency matrix

Represent the graph as an adjacency matrix.



Steps | **Working**

1 List the connections in terms of the number of edges between vertices.

One edge from P to Q.
2 edges from P to R.
A loop from Q to Q.
One edge from Q to R.
One edge from R to S.
No edges to T.

2 Label the rows and columns of a blank matrix with the vertex names.

$$\begin{array}{cc} & \begin{matrix}P & Q & R & S & T\end{matrix} \\ \begin{matrix}P\\Q\\R\\S\\T\end{matrix} & \begin{bmatrix}\square & \square & \square & \square & \square\\ \square & \square & \square & \square & \square\\ \square & \square & \square & \square & \square\\ \square & \square & \square & \square & \square\\ \square & \square & \square & \square & \square\end{bmatrix}\end{array}$$

3 Fill in the matrix based on the edges listed.

HINT: For an undirected graph, the elements of the matrix are symmetrical about the leading diagonal. So, an edge from P to R is also an edge from R to P.

$$\begin{array}{cc} & \begin{matrix}P & Q & R & S & T\end{matrix} \\ \begin{matrix}P\\Q\\R\\S\\T\end{matrix} & \begin{bmatrix}\square & 1 & 2 & \square & 0\\ 1 & 1 & 1 & \square & 0\\ 2 & 1 & \square & 1 & 0\\ \square & \square & 1 & \square & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}\end{array}$$

4 Complete the matrix by writing 0 for all the remaining elements.

$$\begin{array}{cc} & \begin{matrix}P & Q & R & S & T\end{matrix} \\ \begin{matrix}P\\Q\\R\\S\\T\end{matrix} & \begin{bmatrix}0 & 1 & 2 & 0 & 0\\ 1 & 1 & 1 & 0 & 0\\ 2 & 1 & 0 & 1 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}\end{array}$$

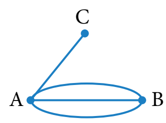
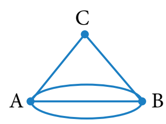
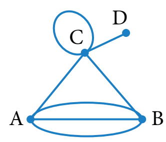
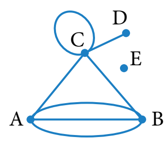
7.3

WORKED EXAMPLE 12 **Constructing a graph from an adjacency matrix**

a Determine if this adjacency matrix represents a simple graph. Justify your answer.

b Represent the adjacency matrix as a graph.

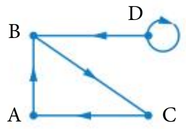
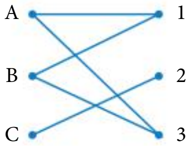
$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array}\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 3 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Steps	Working
a **1** Look at the leading diagonal to determine if there are any loops, or if there are any elements larger than 1 which indicates multiple edges.	There is a 1 for C–C, which indicates a loop at C. There is a 3 for A–B and B–A, which indicates there are multiple edges.
2 State the answer.	This matrix does not represent a simple graph.
b **1** Pick a vertex to start at. Identify vertices it connects to and number of edges. Represent this in a simple diagram. Starting with vertex A. A has 3 edges connected to B and one edge connected to C. Draw A, B and C with connections.	
2 Move to the next vertex and repeat the process. HINT: You do not need to consider connections to vertices you have already drawn in other than as a checking tool. B connects to C (1 edge). Draw in the connection to C.	
3 C connects to D (1 edge) and has a loop. Draw in the connection to D and the loop.	
4 D only connects to C, so move on to E. E does not connect to any vertex, it is an isolated vertex.	

Adjacency matrices for directed graphs and bipartite graphs are slightly different. Rather than the elements of the matrix representing the sum of edges between vertices, the elements indicate the type of connection between vertices.

The vertices labelled along the rows are the 'from' vertices and the vertices labelled in the columns are the 'to' vertices. The numbers 0 and 1 are used to indicate the type or existence of a connection. For example, in a digraph an arc is heading from vertex X into vertex Y. A 1 would be used to indicate that an arc is heading *from* the X *to* the Y. In a bipartite graph, a 1 indicates the existence of a connection and a 0 indicates that there is no connection.

Consider the directed and bipartite graphs and their adjacency matrices shown below.

Graph	Adjacency matrix	Description
	$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$	• An arc goes *from* A *to* B, this is indicated by a 1. • Arcs also head *from* and *to*: B–C, C–A, D–B, indicated by 1s. • A loop goes *from* D to D, this is indicated by a 1.
	$\begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$	• A connects to 1 and 3, indicated by 1s. • B connects to 1 and 3, indicated by 1s. • C only connects to 2 indicated by a 1.

Adjacency matrices of directed graphs

An adjacency matrix is a square matrix representing a graph where:

- each row and column are labelled as a vertex, where the rows are the *from* vertices and the columns are the *to* vertices
- the elements show the number of arcs (or directed edges) between vertices
- the elements are not necessarily symmetrical about the leading diagonal
- the leading diagonal shows the number of loops.

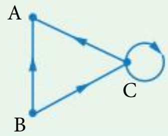
Adjacency matrices of bipartite graphs

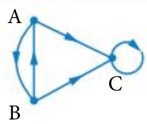
- Each row and column are labelled as a vertex.
- A 1 represents a connection and a 0 indicate that there is no connection.

WORKED EXAMPLE 13 Constructing an adjacency matrix from a digraph

a Represent this digraph as an adjacency matrix.

b What would happen to the adjacency matrix if an arc was added from A to B? Justify your answer.

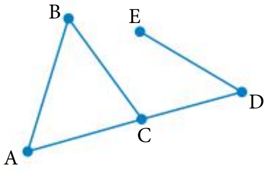


Steps	Working
a 1 Label the rows and columns of a blank matrix with the vertex names.	$\begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{bmatrix} \end{array}$
2 Start from vertex A and systematically complete each row of the matrix.	$\begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$
b 1 Sketch the graph with the new arc.	
2 Identify the change that would need to happen to the adjacency matrix and describe the change.	The cell in the first row, second column, which represents 'from A to B', would change from a 0 to a 1.

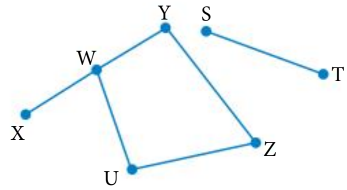
7.3

Recap

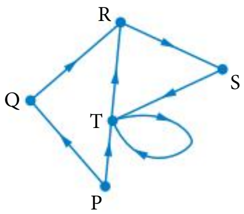
1 **I**



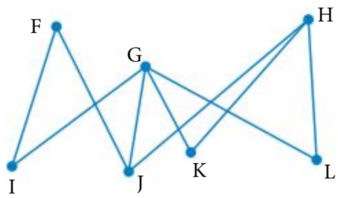
II



III



IV



a Identify the number of vertices and edges in each graph.

b Identify any connected graph(s).

c Identify any simple graph(s).

d Identify any directed graph(s).

e Identify any bipartite graph(s).

2 A complete graph with 8 vertices has

A 25 edges **B** 28 edges **C** 40 edges **D** 56 edges

Mastery

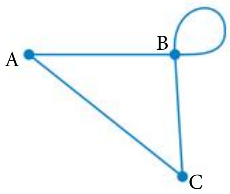
3 For each graph

i confirm that the associated adjacency matrix is correct

ii calculate the sum of the degrees of the vertices

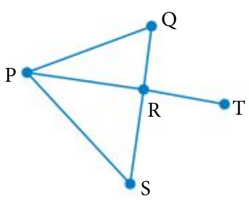
iii compare each answer to part **ii** with the sum of the cells of the associated adjacency matrix.

a



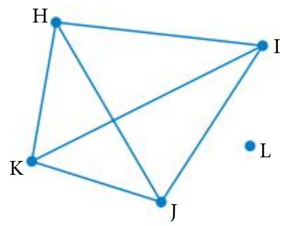
$$\begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array}$$

b



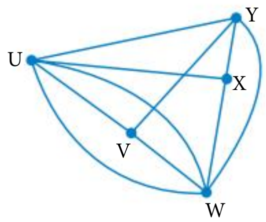
$$\begin{array}{c} \\ P \\ Q \\ R \\ S \\ T \end{array}\begin{array}{c} \begin{array}{ccccc} P & Q & R & S & T \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

c



$$\begin{array}{c|c} & \begin{array}{ccccc} H & I & J & K & L \end{array} \\ \begin{array}{c} H \\ I \\ J \\ K \\ L \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

d



$$\begin{array}{c|c} & \begin{array}{ccccc} U & V & W & X & Y \end{array} \\ \begin{array}{c} U \\ V \\ W \\ X \\ Y \end{array} & \begin{bmatrix} 0 & 1 & 2 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 2 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

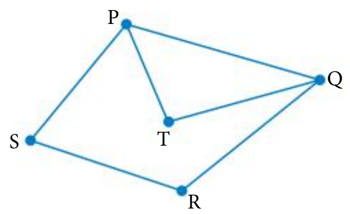
4 Without constructing a graph, the number of edges in the graph represented by this adjacency matrix is

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

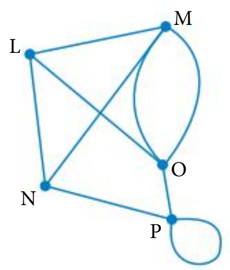
A 2 **B** 3 **C** 4 **D** 5

5 WORKED EXAMPLE 11 Represent each graph using an adjacency matrix.

a



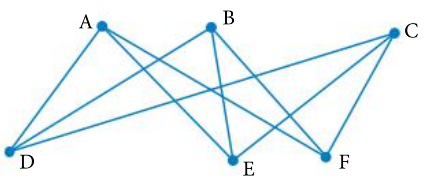
b



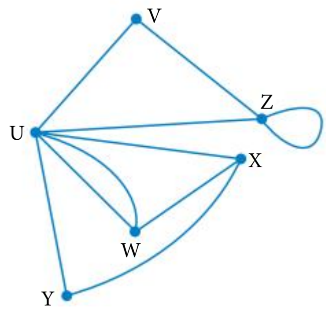
Exam hack

Consider marking edges that you have recorded to keep track of where you have been.

c



d



6 WORKED EXAMPLE 12 For each adjacency matrix

i determine if the matrix represents a simple graph

ii construct a possible graph.

a $\begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array}$

b $\begin{array}{c} \\ P \\ Q \\ R \\ S \end{array} \begin{array}{c} \begin{array}{cccc} P & Q & R & S \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$

c $\begin{array}{c} \\ R \\ S \\ T \\ U \\ V \end{array} \begin{array}{c} \begin{array}{ccccc} R & S & T & U & V \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$

d $\begin{array}{c} \\ H \\ I \\ J \\ K \\ L \\ M \end{array} \begin{array}{c} \begin{array}{cccccc} H & I & J & K & L & M \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$

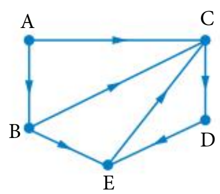
7 This is the adjacency matrix for a planar graph with 4 vertices.

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Without drawing the graph, the number of faces on the graph is

A 2 **B** 3 **C** 4 **D** 5

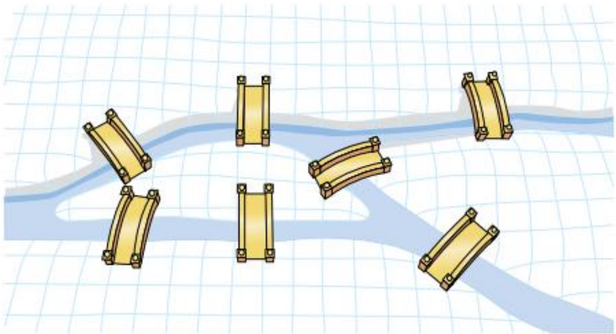
8 WORKED EXAMPLE 13



a Represent this digraph using an adjacency matrix.

b Describe the effect on the matrix if a loop was added at B.

9 A famous graph problem originated in the medieval town of Königsberg. The Pregel river flowed through the town around a central island before dividing into 2. The 4 parts of the town were linked by 7 bridges, as shown on the diagram.



The problem was to determine if it was possible to find a route through Königsberg, beginning and ending at the same point, that crossed each bridge exactly once.

a Draw a graph of this situation using the vertices of the graph to represent the parts of the city and the edges of the graph to represent the bridges.

b Construct an adjacency matrix for the graph.

c Work out a method of using the graph to find a solution to the problem.

d What do you conclude about the solution of this problem?

Exam practice

Use the adjacency matrices below to answer Questions 10–12.

Matrix 1
$$\begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 \end{bmatrix}$$

Matrix 2
$$\begin{bmatrix} 0 & 2 & 1 & 1 \\ 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Matrix 3
$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Matrix 4
$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

10 The matrix that corresponds to a graph that has a loop is

A Matrix 1 **B** Matrix 2 **C** Matrix 3 **D** Matrix 4

11 The matrix that corresponds to a complete graph is

A Matrix 1 **B** Matrix 2 **C** Matrix 3 **D** Matrix 4

12 The matrix that corresponds to a disconnected graph is

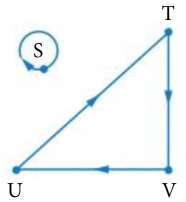
A Matrix 1 **B** Matrix 2 **C** Matrix 3 **D** Matrix 4

13 This is the adjacency matrix for a planar graph with 5 vertices. Without drawing the graph, the number of faces on the graph is

$$\begin{bmatrix} 0 & 1 & 1 & 2 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

A 2
B 4
C 6
D 7

14 © QCAA 2020S 1Q18 Which adjacency matrix below represents the same information that is given in the network diagram?



A

$$\text{From}\quad \begin{array}{c} \\ S \\ T \\ U \\ V \end{array}\begin{array}{c} \text{To} \\ \begin{array}{cccc} S & T & U & V \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

B

$$\text{From}\quad \begin{array}{c} \\ S \\ T \\ U \\ V \end{array}\begin{array}{c} \text{To} \\ \begin{array}{cccc} S & T & U & V \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

C

$$\text{From}\quad \begin{array}{c} \\ S \\ T \\ U \\ V \end{array}\begin{array}{c} \text{To} \\ \begin{array}{cccc} S & T & U & V \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

D

$$\text{From}\quad \begin{array}{c} \\ S \\ T \\ U \\ V \end{array}\begin{array}{c} \text{To} \\ \begin{array}{cccc} S & T & U & V \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

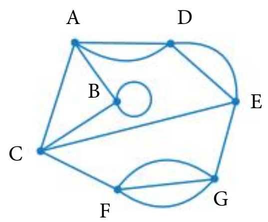
15 (4 marks) A cruise ship has 5 specialist operations areas and 5 employees who work in those areas. The adjacency matrix below shows each of the 5 employees (A, B, C, D, E) and the 5 specialist areas they are trained to work in. Employees can only work in areas they have been trained for.

$$\begin{array}{c} \text{Specialist} \\ \text{operations area} \end{array}\quad \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \text{Employee} \\ \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

a Draw a bipartite graph to represent the adjacency matrix. *[2 marks]*

b Is the bipartite graph a simple graph? Justify your answer. *[2 marks]*

16 (3 marks) CF The network map shows the road connections between 7 tourist destinations in a city.



Complete the adjacency matrix below for the network.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \\ G \end{array}\begin{array}{c} \begin{array}{ccccccc} A & B & C & D & E & F & G \end{array} \\ \begin{bmatrix} ? & ? & ? & ? & 0 & 0 & 0 \\ ? & ? & ? & ? & 0 & 0 & 0 \\ ? & ? & ? & ? & 1 & 1 & 0 \\ ? & ? & ? & ? & 2 & 0 & 0 \\ ? & ? & ? & ? & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 3 & 0 \end{bmatrix} \end{array}$$

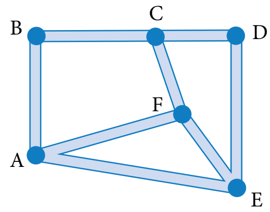
7.4 Exploring and travelling using graphs

Graphs are often used to model, analyse and solve situations involving exploring or travelling. This includes problems such as finding the shortest distance between places or minimising the time taken to travel between locations using different **routes**.

A **walk** is a **route** through a graph, from one vertex to another, along a sequence of edges.

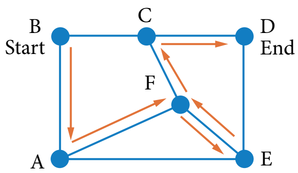
Walks can include repeated edges and vertices.

A map of the covered walkways between buildings in a school is shown.



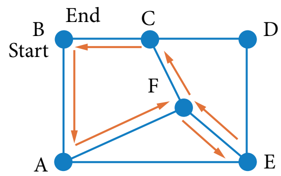
Two possible walks along the covered walkways are shown below. In both Walk 1 and Walk 2 the edge FE is travelled along twice, which, in this case, means vertex F is visited twice.

Walk 1: B–A–F–E–F–C–D



open walk

Walk 2: B–A–F–E–F–C–B



closed walk

In undirected graphs, the direction of travel is not important and that also means edge FE and EF are considered the same edge.

A walk that starts and ends at different vertices is called an **open walk**. A walk that starts and ends at the same vertex is called a **closed walk**.

Types of walks

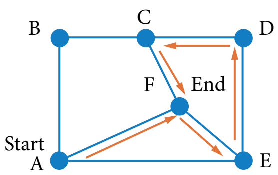
There are 2 special types of walks: a **trail** and a **path**.

A trail is a walk with no repeated edges but can have repeated vertices. Like a walk, a trail can be either open or closed.

Walk 3 illustrates an **open trail** where the trail starts and ends at different vertices. It also shows how a vertex (F) can be visited more than once.

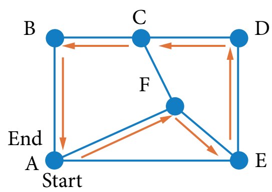
Walk 4 illustrates a **closed trail** where the trail starts and ends at the same vertex.
A closed trail is also called a **circuit**.

Walk 3: A–F–E–D–C–F



open trail

Walk 4: A–F–E–D–C–B–A



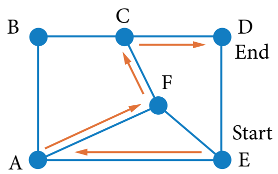
closed trail (circuit)

A path is a walk with no repeated edges and no repeated vertices. Like a walk and a trail, a path can be either open or closed.

Walk 5 illustrates an **open path** where the path starts and finishes at different vertices and has no repeated edges or vertices.

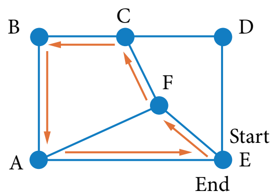
Walk 6 illustrates a **closed path** where the path starts and finishes at the same vertex and has no repeated edges or vertices except for the start/end vertex. A closed path is also called a **cycle**.

Walk 5: E–A–F–C–D



open path

Walk 6: E–F–C–B–A–E

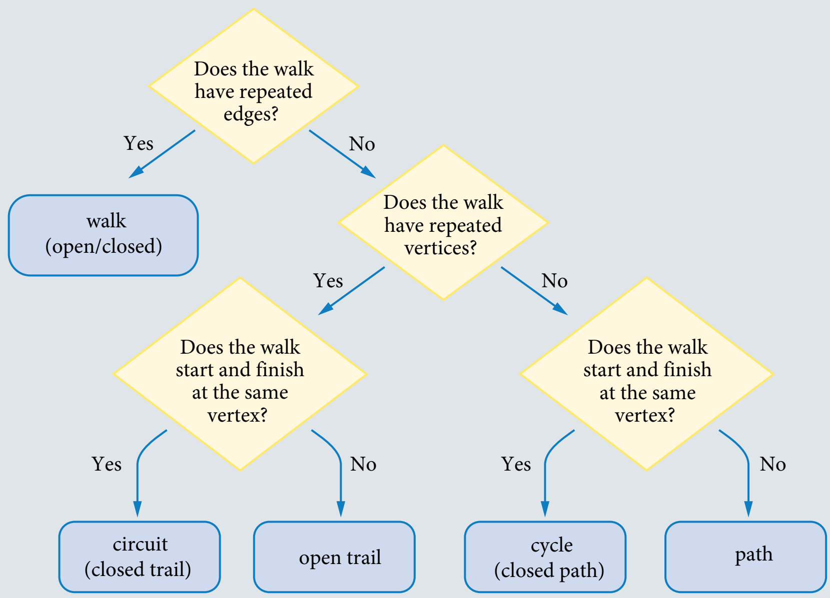


closed path (cycle)

Types of walks

- A walk is a route through a graph, from one vertex to another, along a sequence of edges.
- A trail is a walk with no repeated edges but can have repeated vertices.
- A path is a walk with no repeated edges and no repeated vertices.
- Walks, trails and paths can be open (start and end at different vertices) or closed (starts and ends at the same vertex).

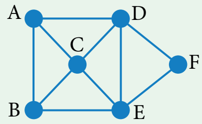
Walk classification chart



WORKED EXAMPLE 14 Types of walks

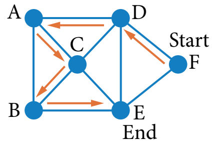
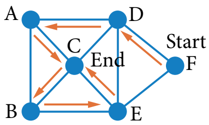
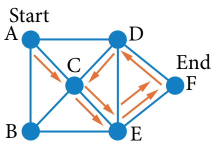
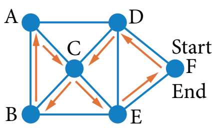
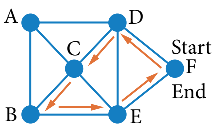
The graph shows different hiking tracks through a local bush reserve.

Classify each hiking track as a walk, trail, path, open/closed. Justify your answer.



a F–D–A–C–B–E **b** F–D–A–C–B–E–C **c** A–C–E–F–D–C–E–F

d F–D–C–B–A–C–E–F **e** F–D–C–B–E–F

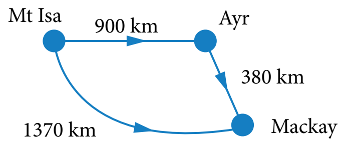
Steps	Working
a **1** Identify the movement through the graph.	
2 Use the 3 classification questions to classify the walk, in this order: Does the walk have repeated edges? Does the walk have repeated vertices? Does the walk start and finish at the same vertex?	This walk has no repeated edges. It has no repeated vertices. It does not start and finish at the same vertex, so it's an open path.
3 State the answer.	F–D–A–C–B–E is a open path.
b The steps from part **a** are repeated for B to E. 	This walk has no repeated edges. It has a repeated vertex. It does not start and finish at the same vertex. F–D–A–C–B–E–C is an open trail.
c 	This walk has 2 repeated edges. A–C–E–F–D–C–E–F is an open walk.
d 	This walk has no repeated edges. It has a repeated vertex, C. It starts and finishes at the same vertex, F. F–D–C–B–A–C–E–F is a circuit (closed trail).
e 	This walk has no repeated edges. It has no repeated vertices (except the first and last, F). It starts and finishes at the same vertex, F. F–D–C–B–E–F is a cycle (closed path).

Shortest paths

Understanding the types and strength of connections represented by edges in a graph can be useful in solving everyday situations. Weighted graphs have extra information labelled on each edge, called the **weight**. On the graph shown, the weight represents the distances by road in kilometres between each town.

Weighted graphs can be used to solve shortest path problems. Such problems can involve finding the shortest distance, quickest time or minimum cost.

To find the **shortest path** from a starting vertex to an end vertex, identify the shortest sequence of edges between those 2 points.

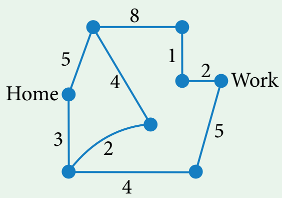


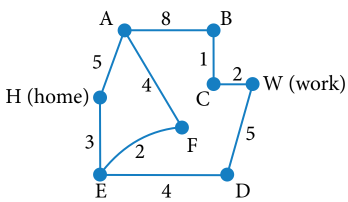
The length of an edge does not need to match the weight. An edge representing 20 km does not have to be twice the size of an edge of weight 10 km.

For example, you could use the graph to find the shortest distance from Mt Isa to Mackay. While Mt Isa–Mackay appears to be the most direct route (1370 km), Mt Isa–Ayr–Mackay is the shortest route (1280 km).

WORKED EXAMPLE 15 **Finding the shortest path by trial-and-error**

The network shows the time in minutes to travel along a series of roads. Find the fastest time to travel from Home to Work by listing all the reasonable options.



Steps	Working
1 Add labels to the vertices.	
2 List all reasonable paths and calculate the time taken along each path.	H–A–B–C–W = 5 + 8 + 1 + 2 = 16 min H–A–F–E–D–W = 5 + 4 + 2 + 4 + 5 = 20 min H–E–D–W = 3 + 4 + 5 = 12 min
3 Identify the fastest path and state the answer.	The shortest time to travel from Home to Work will be 12 minutes.

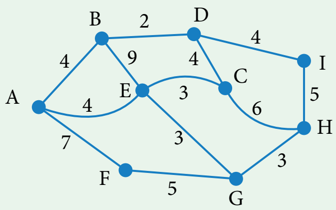
There are formal algorithms that can be used to find shortest paths in more complex networks such as Dijkstra's algorithm; however, in General Mathematics trial-and-error or inspection are all that's required.

Shortest paths with/without a particular vertex

- To find a shortest path **including** a particular vertex, find the shortest paths from the starting point to that vertex, and from that vertex to the end point.
- To find the shortest path **not including** a particular vertex, remove that vertex and its connecting edges from the network, then find the shortest path using the remaining network.

WORKED EXAMPLE 16 Finding the shortest path by inspection with conditions

The network represents the distance (in kilometres) between different suburbs in a city.



Find the shortest distance

a from D to H, including E

b from B to C, not including D.

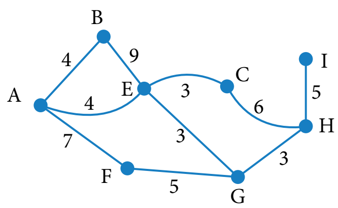
Steps	**Working**
a **1** List possible paths from D to E and calculate the distance of each path.	D–C–E = 4 + 3 = 7 D–B–E = 2 + 9 = 11

Exam hack

You do not need to find all paths once you can see that all other paths WILL be longer than the paths already identified.

2 List possible paths from E to H.	E–G–H = 3 + 3 = 6 E–C–H = 3 + 6 = 9

Hint: You should not backtrack when identifying paths.

3 Combine the shortest paths D–E and E–H to calculate the shortest path from D–H including E.	shortest path = D–C–E + E–G–H = 7 + 6 = 13
4 State the answer.	The shortest path from D–H including E is D–C–E–G–H and is 13 km long.
b **1** Remove (or cross out) the D and all its edges as it cannot be included.	
2 List possible paths from B to C and calculate the distance of each path.	B–E–C = 9 + 3 = 12 B–A–E–C = 4 + 4 + 3 = 11
3 State the answer.	The shortest path from B–C excluding D is B–A–E–C and has a distance of 11 km.

EXERCISE 7.4 Exploring and travelling using graphs

ANSWERS p. 458

Recap

1 Without constructing a graph, the number of edges in the graph represented by this adjacency matrix is

$$\begin{bmatrix} 0 & 1 & 0 & 2 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 2 & 0 & 3 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

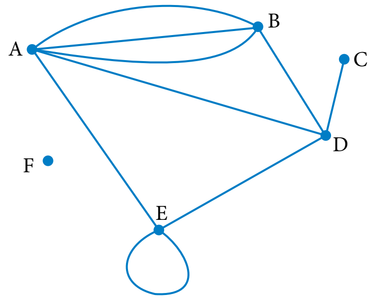
A 9

B 10

C 12

D 17

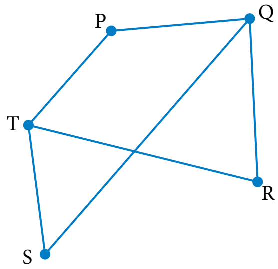
2 Represent this graph as an adjacency matrix.



Mastery

Refer to this graph for Questions 3–5.

3 WORKED EXAMPLE 14 The route P–T–S–T–R is best described as



A a closed walk **B** an open walk **C** a cycle **D** an open trail

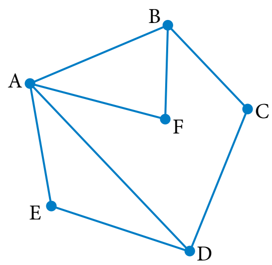
4 The route S–Q–R–T–P is best described as

A a closed walk **B** a circuit **C** a cycle **D** an open path

5 The route S–T–P–Q–S is best described as

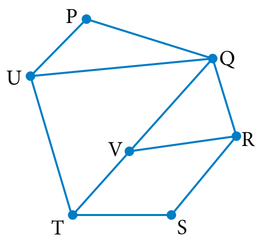
A a closed walk **B** an open walk **C** a cycle **D** a circuit

6 Use this graph to identify the type of each route.



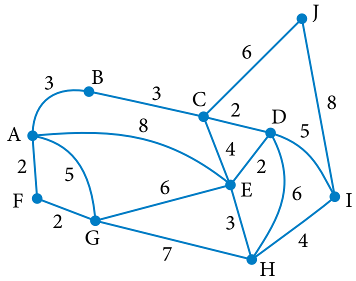
a C–D–A–E–D–C **b** A–E–D–C–B **c** A–B–C–D–A
d C–D–A–F–B–C **e** E–A–B–C–D–E **f** B–A–F–B–C

7 For this network, name one example of

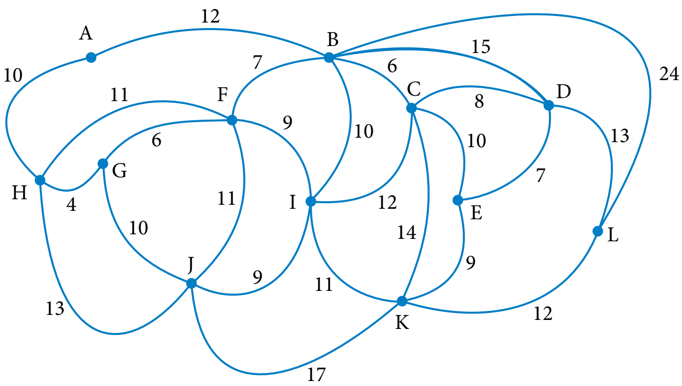


a a path **b** an open walk **c** a cycle
d an open trail **e** a circuit.

8 WORKED EXAMPLE 15 Use trial-and-error to find the shortest path between A and H in the network shown, where all distances are in kilometres.



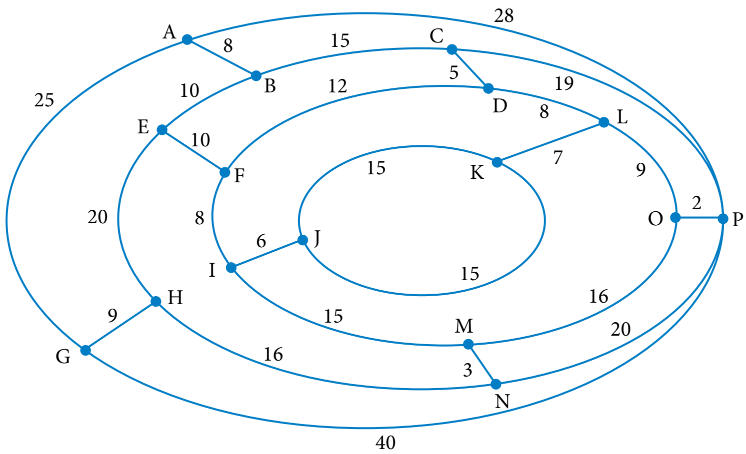
9 WORKED EXAMPLE 16 In the network shown, find the shortest path from



a H to C **b** D to J **c** A to L

d G to B, including K **e** H to B, not including F.

10 In the network shown, find the shortest path from



a P to G **b** J to A **c** C to N

d E to O, including A **e** I to L, not including D.

11 A telephone network in a remote area has microwave, landline and radio links between sites. The cost to the telephone company of a link varies according to the cost of link maintenance and the volume of calls. The costs, in cents, of direct links are shown in the table.

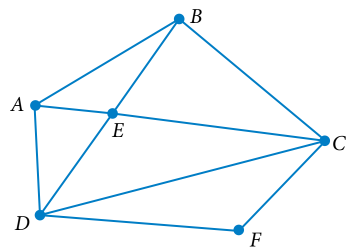
Site	A	B	C	D	E	F	G	H	I
A	0	12			5	15	18		
B	12	0					10		7
C			0	12		8		18	
D			12	0	8	11	15		9
E	5			8	0	14			12
F	15		8	11	14	0	8	5	
G	18	10		15		8	0		
H			18			5		0	
I		7		9	12				0

Find the cheapest link between sites

a A and H **b** D and B **c** F and I **d** H and E.

Exam practice

Refer to the graph for Questions 12–14.



12 The route A–B–E–B–C is best described as

A a closed walk **B** an open walk **C** a cycle **D** an open trail

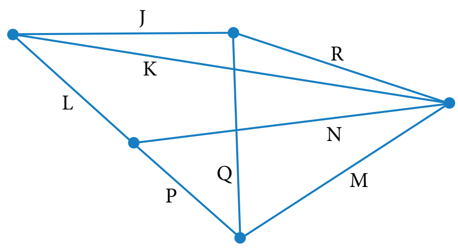
13 The route A–D–E–C–D–F is best described as

A a closed walk **B** an open walk **C** a cycle **D** an open trail

14 The route A–E–C–F–D–A is best described as

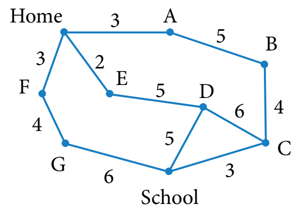
A a closed walk **B** an open trail **C** a cycle **D** a circuit

15 © QCAA 2020S 1Q9 The edges J, R, M, P, P, Q on the network diagram below make



A a path **B** a cycle **C** a closed trail **D** an open walk

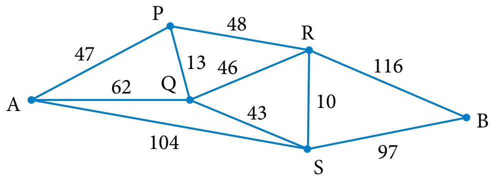
16 Charlie rides her bike to school each day. The network represents the roads Charlie can use between her house to school. The distances are in kilometres.



The shortest distance Charlie can ride between school and home is

A 12 **B** 13 **C** 15 **D** 16

17 (2 marks) In this graph of a road map, the distances are in kilometres. Find the shortest path from A to B.

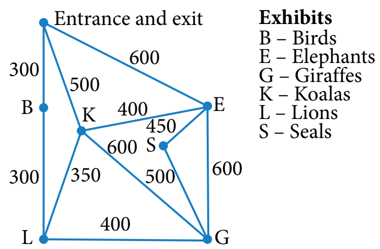


18 © QCAA 2022 2Q4 (5 marks) CF The table shows the current road length (in kilometres) between 6 towns.

	Manon	Veria	Bolint	Farra	Recen	Alin
Manon		16	34	–	–	33
Veria			12	–	–	15
Bolint				–	10	–
Farra					15	23
Recen						15
Alin						

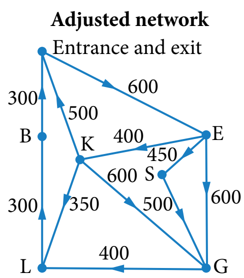
The government plans to build a direct road between Manon and Farra. Use a network diagram to determine the length of the direct road if it is to be 4 km shorter than the length of the current shortest road route between Manon and Farra.

19 (4 marks) CF The network shows the key animal exhibits in a zoo and the paths connecting them. The lengths of each road are in metres.



The zoo manager is looking at ways to minimise congestion on each path and is considering making them all one-way only. The original network can now be represented by the weighted directed graph below.

Analyse the graph and justify whether the manager has achieved his goal.



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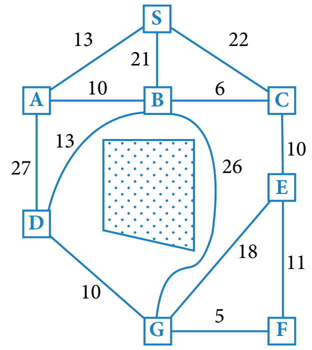
20 (6 marks) CU The road system on a farm connecting the storage shed (S) to the seven key crops areas (A, B, C, D, E, F and G) is shown on the right. The distances of each road are in kilometres.

Fertiliser must be delivered from the storage shed to crop area G using the shortest path.

Road BG currently goes around an old unused field. It is proposed that a direct road (through the unused field) be constructed that will reduce the distance between crop areas B and G.

By how much can the direct road between B and G be reduced, so that the shortest path from the storage shed to crop area G includes the direct road BG?



7.5 Eulerian graphs

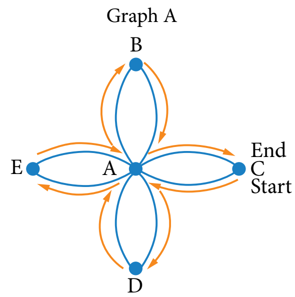
Sometimes in life the most efficient route is not the shortest path, but rather the route that passes along every edge.

An **Eulerian trail** (or **Eulerian path**) is a trail that travels along every edge of a graph only once. It may include repeated vertices.

An **Eulerian circuit** (or **Eulerian cycle** or **closed Eulerian trail**) is a trail that travels along every edge of a graph exactly once and starts and ends at the same vertex.

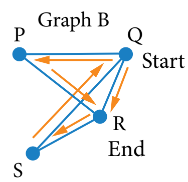
A connected graph that contains an Eulerian circuit is called an **Eulerian graph**.

For example, the trail C–A–D–A–E–A–B–A-C in Graph A is an Eulerian circuit. Graph A is also an Eulerian graph as it is connected and contains an Eulerian circuit.



An Eulerian trail that starts and ends at different vertices is called an **open Eulerian trail** (also called a **semi-Eulerian trail**). A connected graph that contains an open Eulerian trail is called a **semi-Eulerian graph**.

For example, the trail Q–P–R–S–Q–R in Graph B is an open Eulerian trail, so Graph B is a semi-Eulerian graph.



An easy way to identify if a graph is Eulerian is to check if all the vertices have an even degree. This also means that an Eulerian circuit will also exist if there are no vertices with an odd degree.

A graph is semi-Eulerian if exactly 2 vertices have an odd degree. This means that an open Eulerian trail will also exist if there are exactly 2 odd vertices.

Worksheets
Eulerian graphs

Eulerian trails and circuits

Exam hack

If a question asks if an open Eulerian trail or an Eulerian circuit exists, find the degree of all the vertices in the graph to quickly identify whether the correct conditions apply.

Exam hack

An open Eulerian trail will always start at one of the 2 odd degree vertices and end at the other odd degree vertex.

Eulerian trails and circuits

- An Eulerian trail is a walk with no repeated edges but includes every edge in the graph. It can include repeated vertices.
- An open Eulerian trail is an Eulerian trail that starts and finishes at different vertices.
- An Eulerian circuit is an Eulerian trail that starts and ends at the same vertex.

Properties of Eulerian and semi-Eulerian graphs

For an Eulerian graph:

- all vertices must be of even degree
- an Eulerian circuit exists.

For a semi-Eulerian graph:

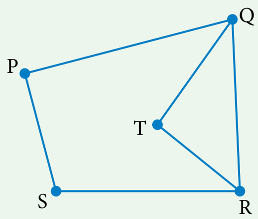
- exactly 2 vertices are of odd degree
- an open Eulerian trail exists
- the open Eulerian trail will start and finish at a vertex of odd degree.

WORKED EXAMPLE 17 Eulerian and semi-Eulerian graphs

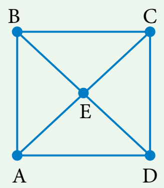
i By first considering the degree of each vertex, **justify** whether each graph is Eulerian, semi-Eulerian or neither.

ii If the graph is Eulerian or semi-Eulerian, **state** a possible corresponding closed or open Eulerian trail.

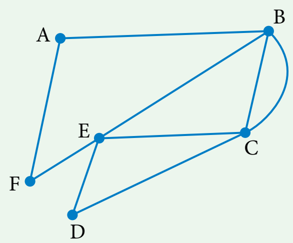
a



b



c



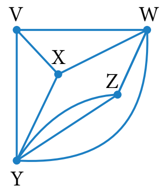
Steps				Working
a	**i**	**1**	Write the degree of each vertex.	deg (P) = 2 deg (Q) = 3 deg (R) = 3 deg (S) = 2 deg (T) = 2
		2	Identify which conditions are met and conclude the result.	The graph is semi-Eulerian because it has exactly 2 vertices of odd degree.
	ii	**1**	The graph is semi-Eulerian.	An Eulerian path exists.
		2	The Eulerian path must start and end at a vertex of odd degree.	The Eulerian path must start at Q and finish at R or start at R and finish at Q.
		3	The trail must use each edge exactly once. Vertices can be visited more than once.	Q–P–S–R–T–Q–R is an Eulerian path.
		4	Other Eulerian trails may exist.	Q–T–R–S–P–Q–R and R–S–P–Q–T–R–Q are also Eulerian paths.
b	**i**	**1**	Write the degree of each vertex.	deg (A) = 3 deg (B) = 3 deg (C) = 3 deg (D) = 3 deg (E) = 4
		2	Identify which conditions are met and conclude the result.	The graph is neither Eulerian nor semi-Eulerian as there is neither 0 nor 2 vertices of odd degree.
c	**i**	**1**	Write the degree of each vertex.	deg (A) = 2 deg (B) = 4 deg (C) = 4 deg (D) = 2 deg (E) = 4 deg (F) = 2
		2	Identify which conditions are met and conclude the result.	The graph is Eulerian because all vertices are of even degree.
	ii	**1**	The graph is Eulerian.	An Eulerian circuit exists.
		2	Select any vertex to start and trace out a trail that uses each edge once and ends at the vertex where you started.	A–F–E–D–C–E–B–C–B–A is an Eulerian cycle.
		3	Other Eulerian circuits may exist.	B–E–F–A–B–C–E–D–C–B and C–D–E–F–A–B–E–C–B–C are also Eulerian circuits.

EXERCISE 7.5 Eulerian graphs

ANSWERS p. 459

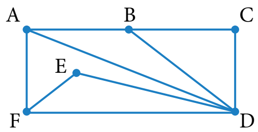
Recap

1 Which walk is not a path?



A V–X–W–Y–Z **B** V–W–X–Y–Z **C** V–X–Y–Z–W **D** V–Y–W–Z–X

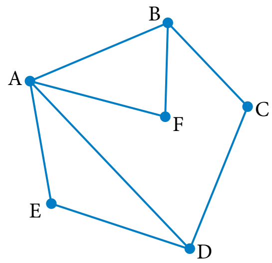
2 Which term best describes the walk A–B–C–D–E–F–D–A?



A closed trail **B** path **C** trail **D** walk

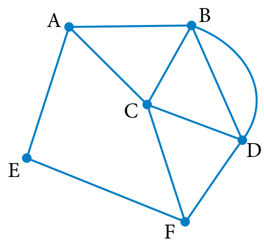
Mastery

3 WORKED EXAMPLE 17 Consider the graph shown.



a State the degree of each vertex.

b Justify whether the graph is Eulerian, semi-Eulerian or neither.



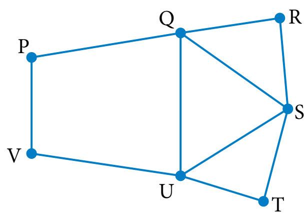
4 a Use the degrees of the vertices to show that this graph is semi-Eulerian.

b Name the vertices that could be the start of the Eulerian trail for this graph.

c Describe an Eulerian trail for this graph.

5 a Use the degrees of the vertices to show that this graph is Eulerian.

b Describe an Eulerian circuit for this graph.



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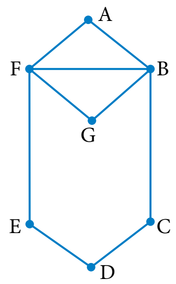
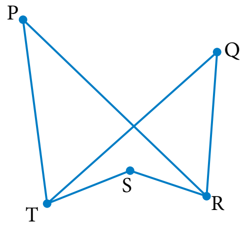
6 **i** Classify each graph as Eulerian or semi-Eulerian.

ii Identify the Eulerian circuit or Eulerian trail that exists in each case.

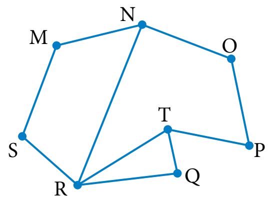
a



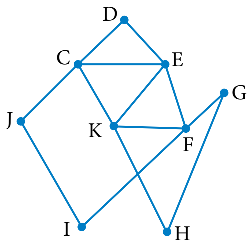
b



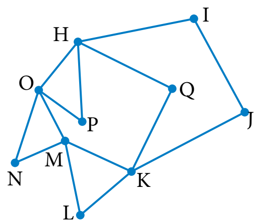
c



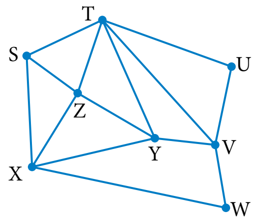
d



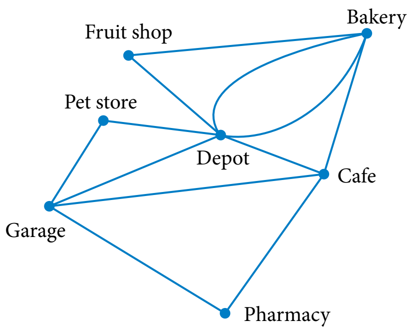
e



f



7 Emily delivers pamphlets along the streets that connect the various locations around a central depot. She must deliver pamphlets along each street and start and end each delivery run at the depot.

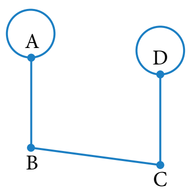


a What type of graph is formed by the street map?

b What is the most efficient route that Emily could take? Explain your answer.

Exam practice

8 © QCAA 2022 1Q6 84% This semi-Eulerian graph can be changed to an Eulerian graph by



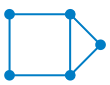
A adding a loop to vertex B

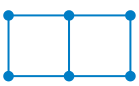
B removing the loop at vertex A

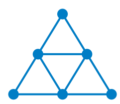
C adding an edge between vertices A and D

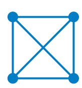
D removing the edge between vertices B and C

9 Which one of these graphs is Eulerian?

A 

B 

C 

D 

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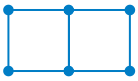
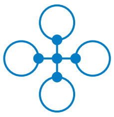
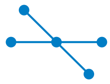
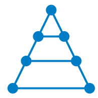
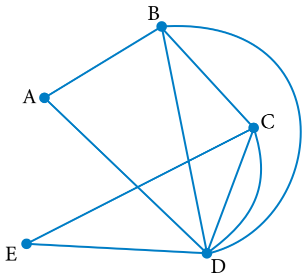
10 Which one of these graphs is semi-Eulerian?

A  **B**  **C**  **D** 

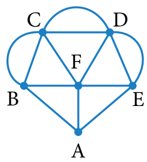
11 (3 marks)

a Verify that this graph is Eulerian. *[2 marks]*

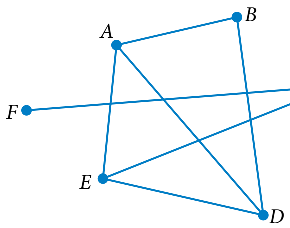


b Describe an Eulerian circuit for this graph. *[1 mark]*

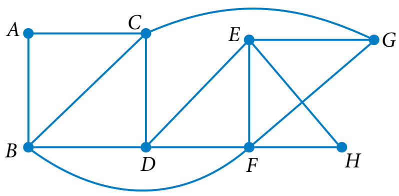
12 (2 marks) For this graph an open Eulerian trail is possible if only one edge is removed.
Determine the number of ways this can be achieved, stating the edge removed in each case.



13 (2 marks) CF Add 2 edges to transform this graph so that it has an Eulerian circuit.

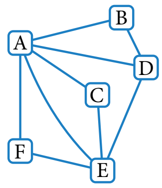


14 (3 marks) CF The streets of a town are represented by this map. The various intersections of streets are indicated by the letters A, B, C, etc. A security firm is employed to drive along a route that includes each street. The owner of the security company wants the route to cover each street once only.



Determine whether a route that starts and finishes at the same intersection and suits the owner's request is possible. Explain your answer.

15 (3 marks) CF A community college campus consists of 6 buildings (A, B, C, D, E and F). Each building is connected by pathways as shown in the diagram.



A college campus tour is being planned and you have been asked to design the tour route. The tour must meet the following conditions:

- the tour must start and finish in the Administration block (A)
- the tour must travel along every pathway exactly once
- the tour can visit a building more than once.

Justify whether such a tour route is possible. If so, state the order of the tour.

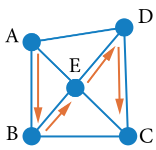
Worksheet
Hamiltonian paths and cycles

7.6 Hamiltonian graphs

In some situations, it is more practical for all vertices to be visited but not all edges. The best example of this is the travelling salesperson problem, where the salesperson wishes to visit all towns on a route but only wants to visit each town once.

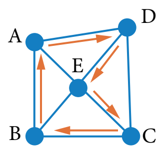
A **Hamiltonian path** is a path that includes each vertex in a graph exactly once and has no repeated vertices. If a Hamiltonian path starts and finishes at the same vertex, it is called a **Hamiltonian cycle**.

If a connected graph contains a Hamiltonian cycle, it is called a **Hamiltonian graph**. If a connected graph contains a Hamiltonian path, it is called a **semi-Hamiltonian graph**.



Hamiltonian path (A–B–E–D–C)

semi-Hamiltonian graph



Hamiltonian cycle (A–D–E–C–B–A)

Hamiltonian graph

Unlike Eulerian graphs and semi-Eulerian graphs, Hamiltonian graphs and semi-Hamiltonian graphs do not have any existence conditions. This means to determine whether a graph is Hamiltonian, semi-Hamiltonian or neither, trial-and-error or inspection methods need to be used.

Exam hack

A way to remember the difference between Eulerian and Hamiltonian is to remember Eulerian trails and circuits must travel on every **E**dge, and Hamiltonian paths and cycles are like visiting every **H**ouse (vertex) on a map.

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Hamiltonian paths and cycles

A Hamiltonian path:

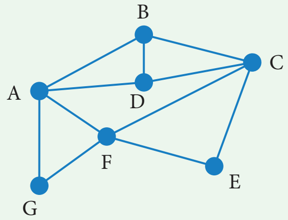
- has no repeated edges or vertices
- includes every vertex exactly once
- starts and finishes at different vertices.

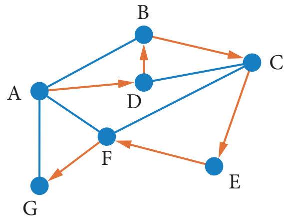
A Hamiltonian cycle:

- has no repeated edges or vertices
- includes every vertex exactly once
- starts and finishes at the same vertex.

WORKED EXAMPLE 18 Hamiltonian path

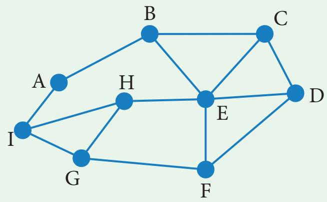
Find a Hamiltonian path for this graph.

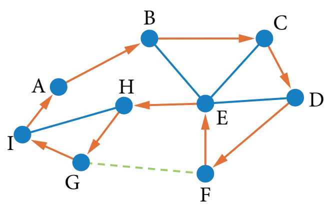


Steps	Working
1 Start at any vertex, such as A. Move from A to an adjacent vertex, say D. Now, move to another adjacent vertex (but not A). Continue this process until every vertex has been visited. No vertex can be revisited. Hint: If you can't find a Hamiltonian path starting at a particular vertex, select a different one and try again.	
2 Name the path.	A Hamiltonian path is A–D–B–C–E–F–G.

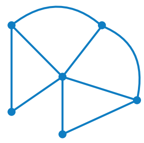
WORKED EXAMPLE 19 Hamiltonian cycle

Name a Hamiltonian cycle for this graph that begins at C and does not include edge FG.



Steps	Working
1 Edge FG can't be used, so either delete it or cross it out. Start at C. Move to an adjacent vertex, say D. Select another adjacent vertex, say F. As edge FG cannot be used, move to E. Continue in this way until all vertices have been visited.	
2 Name the path.	A Hamiltonian cycle is C–D–F–E–H–G–I–A–B–C.

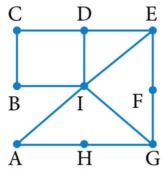
Recap



1 The minimum number of edges that must be added to this graph so an Eulerian circuit is possible is

A 0 **B** 1

C 2 **D** 3



2 For this network diagram, an Eulerian trail can be found by

A adding an edge that joins C to I

B adding an edge that joins E to H

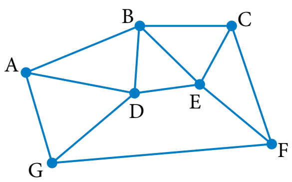
C removing the edge that joins F to G

D removing the edge that joins D to E

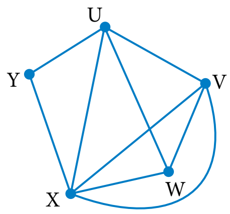
Mastery

3 WORKED EXAMPLE 18 For each graph shown, list a Hamiltonian path that

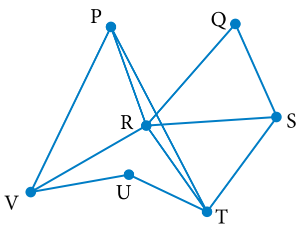
a commences at B and finishes at C



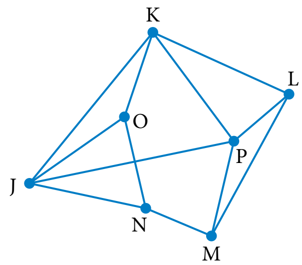
b commences at V and finishes at X



c commences at R and finishes at Q

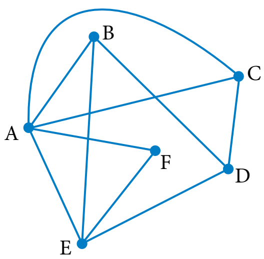


d commences at P and finishes at K.

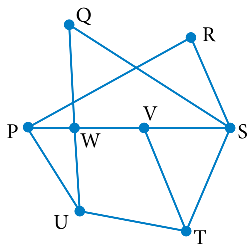


4 For each graph, list a Hamiltonian cycle that commences and finishes at

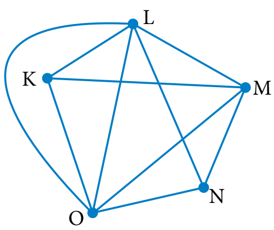
a F



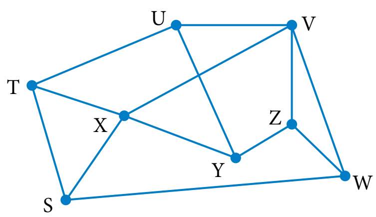
b V



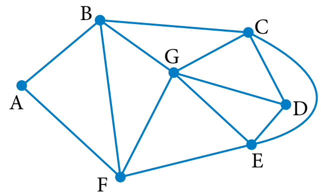
c O



d W



5 WORKED EXAMPLE 19 For this graph, name



a a Hamiltonian cycle that begins at G and does not include edge CD

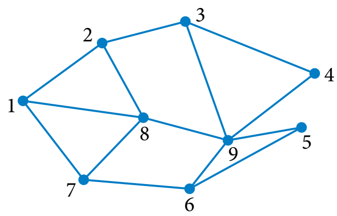
b a Hamiltonian path that ends at A and does not include edge FE

c a Hamiltonian cycle that begins at E and does not include edge ED or BC

d a Hamiltonian path that begins at G and does not include edge CE or AF

e a Hamiltonian cycle that begins at A and does not include edge BF or FE.

6 Sarena is a salesperson who travels to different suburbs to sell supplies to health food stores. A map of the suburbs and roads with direct linkages between the suburbs is shown.



Key:
1: Bellbowrie
2: Mount Ommaney
3: Indooroopilly
4: Rocklea
5: Archerfield
6: Richlands
7: Wacol
8: Jindalee
9: Corinda

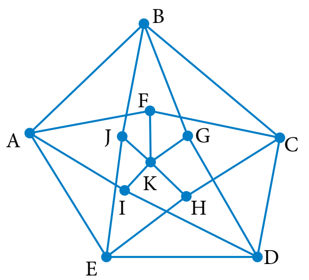
Sarena wants to travel from Indooroopilly and visit each of the other suburbs once before returning to Indooroopilly.

a What type of route is this called?

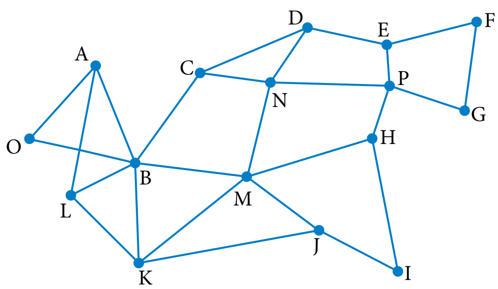
b Describe a route that Sarena could take.

c List the connections that Sarena does not use.

7 Name a Hamiltonian cycle for this graph.

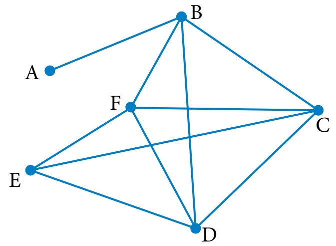


8 An orienteering course is set out according to the map shown.



Competitors start and finish at point A and need to visit every one of the other checkpoints. Find a route that does not involve visiting any checkpoint more than once.

Refer to this diagram for Questions 9 and 10.



9 Which walk is a Hamiltonian path?

A A–B–F–E–D–C **B** B–D–C–F–E–A **C** A–B–C–F–E–C–D **D** D–E–F–C–F–B–A

10 Which statement is true?

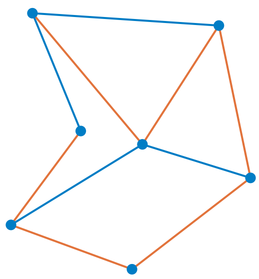
A C–D–E–F–B–A is a Hamiltonian cycle.

B Only one Hamiltonian path exists.

C A Hamiltonian path cannot commence at D.

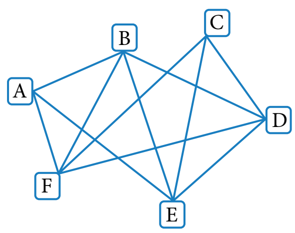
D No Hamiltonian cycle exists.

11 The highlighted route in orange on the graph is best described as



A an Eulerian path **B** a Hamiltonian path **C** a Hamiltonian cycle **D** none of the above

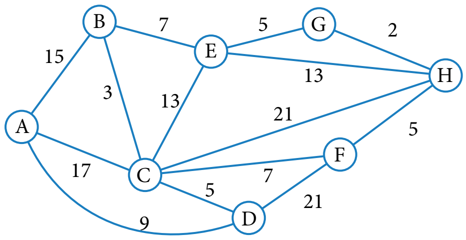
12 (3 marks) The organiser of the upcoming GM Music Festival is looking at ways to improve the event. Last year the various stages were set up as shown in the graph. The vertices are the stages, and the edges are the walkways connecting the stages.



a A major complaint of the last event was the congestion that occurred as festival goers moved between stages. The organiser has decided to reposition the stages and walkways so that no walkways intersect. Draw a possible graph which meets these conditions. *[2 marks]*

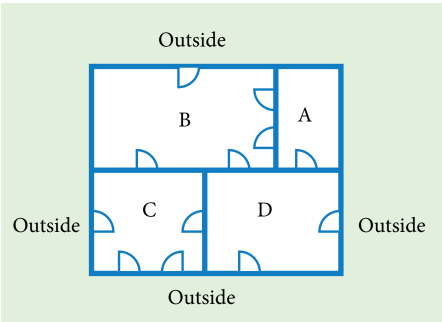
b If a festival goer wanted to visit all the stages exactly once, determine a pathway they could follow if they started at stage E. *[1 mark]*

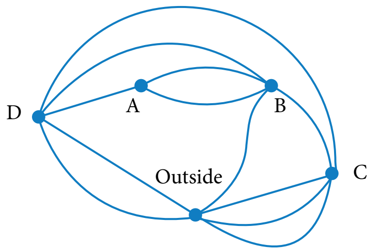
13 (3 marks) CF The graph below represents the part of the ferry network in Brisbane spanning from St Lucia (A) to Hamilton (H) and 6 other stops along the river (B, C, D, E, F and G). The numbers on each edge represent the time, in minutes, taken to depart a ferry stop and dock at the next stop.



Identify the shortest Hamiltonian path from St Lucia (A) and state its duration.

14 (3 marks) CU James' holiday rental home has 4 rooms (A, B, C and D). Below is the floor plan of the house which shows the rooms, the 12 doors and the outside yard area. A graph of the floor plan is also provided. On the graph the edges represent the doors, and the vertices represent the rooms and the outside area.





James hires a cleaner to clean the house. He wants the cleaner to enter the house from the outside area, and walk through each room exactly once, cleaning each room as they go and finishing in the outside area.

If 3 internal doors are locked and only unlocked doors are considered in James's route for the cleaner, identify which 3 doors can be locked and still allow the cleaner to walk their route. Explain your answer.

15 © QCAA 2024 2Q2 (4 marks) CF The table shows the travel time (minutes) between five islands in the Torres Strait for a ferry service.

	Waiben	Palilug	Ngurapai	Keriri	Gealug
Waiben		—	18	—	14
Palilug			—	25	16
Ngurapai				20	28
Keriri					—
Gealug					

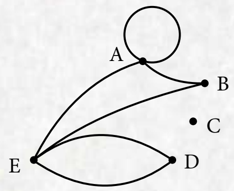
Construct a weighted graph and use it to calculate the total travel time for a ferry that completes a Hamiltonian cycle beginning at Waiben.

Video
Exam question analysis: Graphs and networks

EXAM QUESTION ANALYSIS

© QCAA 2023 1Q23 (5 marks)

A network graph is shown.



a State the degree of vertex E. *[1 mark]*

b State the number of edges joining D and E. *[1 mark]*

c Construct an adjacency matrix from the graph with the vertices in alphabetical order. *[3 marks]*

Reading the question

- The question is asking for the degree of vertex E.
- The question is asking for the number of edges between D and E.
- An adjacency matrix must be created, and the vertices must be in alphabetical order.

Thinking about the question

- The degree of a vertex is the number of edges that connect to the vertex.
- In an adjacency matrix, a connection is represented by a 1 and no connection is represented by a 0.
- In an adjacency matrix, the ‘from’ vertices are the rows and the ‘to’ vertices are the columns.

Worked solution (✓ = 1 mark)

a Find the degree of vertex E.

degree = **4** ✓

b Find the number of edges between D and E.

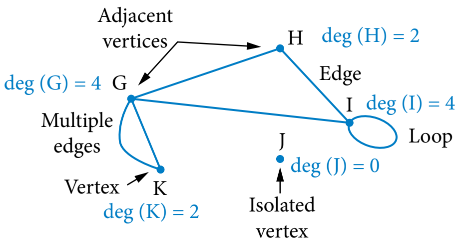
number of edges = **2** ✓

c Construct the adjacency matrix.

$$\begin{array}{c c l}
 & \begin{matrix} \text{A} & \text{B} & \text{C} & \text{D} & \text{E} \end{matrix} & \\
\begin{matrix} \text{A} \\ \text{B} \\ \text{C} \\ \text{D} \\ \text{E} \end{matrix} & \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 1 & 1 & 0 & 2 & 0 \end{bmatrix} & \begin{matrix} \checkmark \\ \checkmark \\ \qquad\checkmark \\ \\ \\ \end{matrix}
\end{array}$$

7 Chapter summary

Features of a graph



degree sum = 4 + 2 + 4 + 0 + 2 = 12

degree sum = 2 × number of edges = 2 × 6 = 12

- A **subgraph** is a graph formed using a subset of a larger graph's edges and vertices.

Types of graphs

- A **simple graph** is a graph without loops or multiple edges.
- A **complete graph** is a simple graph where every vertex is connected to every other vertex.
- Every vertex in a complete graph has *degree* = *number of vertices* – 1.
- In a **connected graph**, there is a path from any vertex to any other vertex. If there is not a path between all vertices, it is a **disconnected graph**.
- A **bridge** is an edge that keeps a graph connected.
- A **planar graph** is a connected graph that can be drawn with no edges crossing.
- **Faces** are regions of a planar graph that are enclosed or bound by edges.
- **Euler's formula** for planar graphs $v + f - e = 2$

  where v = the number of vertices

  f = the number of faces

  e = the number of edges.
- A **bipartite graph** has vertices that can be divided into 2 distinct sets. There are no edges connecting vertices from the same set.
- A **directed graph** (also called a **digraph**) has edges (also called **arcs**) with direction.
- An **undirected graph** has edges with no direction.
- A **weighted graph** has a **weight** labelled on each edge. The weight is extra information, such as distances, times or costs, labelled on the edges. These quantities (weights) are often used to solve **shortest path problems** by inspection.

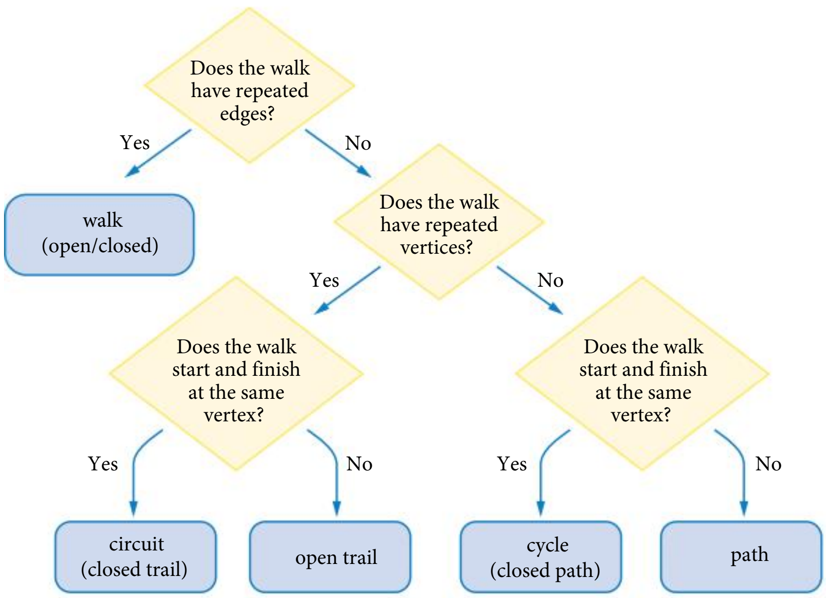
Graphs and matrices

- An **adjacency matrix** is a square matrix representing a graph where:
  - each row and column are labelled as a vertex
  - the number of vertices = number of rows
  - the elements show the number of edges between vertices
  - for undirected graphs, the elements are symmetric about its leading diagonal
  - the leading diagonal shows the number of loops
  - the number of edges = sum of the elements on and below the leading diagonal
  - the degree of a vertex = sum of the elements of the row (+1 for each loop).

Types of walks

- A **walk** is a route through a graph, from one vertex to another, along a sequence of edges.
- A **trail** is a walk with no repeated edges but can have repeated vertices.
- A **path** is a walk with no repeated edges and no repeated vertices.
- Walks, trails and paths can be **open** (starts and ends at different vertices) or **closed** (starts and ends at the same vertex).
  - A closed trail is also called a **circuit**.
  - A closed path is also called a **cycle**.

Walk classification chart



Eulerian trails and circuits

- An **Eulerian trail** is a walk with no repeated edges that includes every edge in the graph.
- An **open Eulerian trail** (**semi-Eulerian trail**) starts and finishes at different vertices. It will only exist if the graph has exactly 2 vertices of odd degree.
  - An open Eulerian trail will always start at one of the 2 odd degree vertices and end at the other odd degree vertex.
  - A connected graph that contains an open Eulerian trail is called a **semi-Eulerian graph**.
- An **Eulerian circuit** is an Eulerian trail that starts and ends at the same vertex.
  - An Eulerian circuit will only exist if all vertices have an even degree.
  - A connected graph that contains an Eulerian circuit is called an **Eulerian graph**.

Hamiltonian paths and cycles

- A **Hamiltonian path** is a walk that includes each vertex exactly once, and there are no repeated vertices.
  - A connected graph containing a Hamiltonian path is called a **semi-Hamiltonian graph**.
- A **Hamiltonian cycle** is a walk that includes each vertex exactly once, there are no repeated vertices, and it starts and finishes at the same vertex.
  - A connected graph containing a Hamiltonian cycle is called a **Hamiltonian graph**.

Cumulative examination 1

Simple familiar

Perusal time: 2 minutes   Working time: 30 minutes

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

Section 1 (5 marks): 5 multiple choice questions

Section 2 (13 marks): 4 short response questions

Total: 18 marks

Worksheet
General Maths formula sheet

Section 1 5 multiple choice questions

5 marks

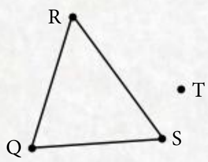
1 © QCAA 2021 1Q8 A basketball competition has six teams that have completed three rounds of competition as shown.

	Bears	Eagles	Lions	Meerkats	Tigers	Wombats
Bears	–	–	✓	–	✓	✓
Eagles	–	–	✓	✓	–	✓
Lions	✓	✓	–	✓	–	–
Meerkats	–	✓	✓	–	✓	–
Tigers	✓	–	–	✓	–	✓
Wombats	✓	✓	–	–	✓	–

The graph to represent this information has

A 6 edges. **B** 9 edges. **C** 15 edges. **D** 18 edges.

2 © QCAA 2020 1Q11 Determine the adjacency matrix that represents this graph.



A

$$\begin{array}{cc} & \begin{array}{ccc} Q & R & S \end{array} \\ \begin{array}{c} Q \\ R \\ S \end{array} & \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array}$$

B

$$\begin{array}{cc} & \begin{array}{cccc} Q & R & S & T \end{array} \\ \begin{array}{c} Q \\ R \\ S \\ T \end{array} & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

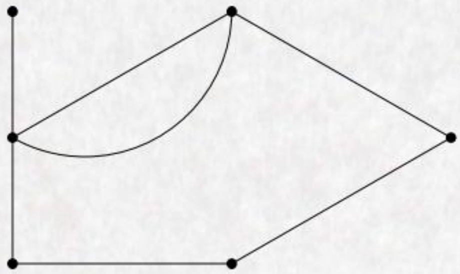
C

$$\begin{array}{cc} & \begin{array}{cccc} Q & R & S & T \end{array} \\ \begin{array}{c} Q \\ R \\ S \\ T \end{array} & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

D

$$\begin{array}{cc} & \begin{array}{cccc} Q & R & S & T \end{array} \\ \begin{array}{c} Q \\ R \\ S \\ T \end{array} & \begin{bmatrix} 0 & 2 & 2 & 0 \\ 2 & 0 & 2 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

3 © QCAA 2022 1Q2 The total number of vertices in this graph is



A 3 **B** 5 **C** 6 **D** 7

4 © QCAA 2023 1Q6 In January 2022, 40 fish were released into a new dam that has the capacity to support 10 000 fish. It is predicted that the dam will reach its capacity in January 2030 if the fish population doubles every year.

Which sequence rule models the prediction?

A $t_n = t_1 r^{(n-1)}$, where $t_1 = 40$, $r = 2$, $n = 8$

B $t_n = t_1 r^{(n-1)}$, where $t_1 = 40$, $r = 2$, $n = 9$

C $t_n = t_1 + (n-1)d$, where $t_1 = 40$, $d = 2$, $n = 8$

D $t_n = t_1 + (n-1)d$, where $t_1 = 40$, $d = 2$, $n = 9$

5 © QCAA 2021 1Q15 MODIFIED Which of the following investment options gives the best return?

A 5.95% p.a. compounding daily

B 5.96% p.a. compounding monthly

C 5.97% p.a. compounding quarterly

D 5.99% p.a. compounding six-monthly

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Section 2 4 short response questions

13 marks

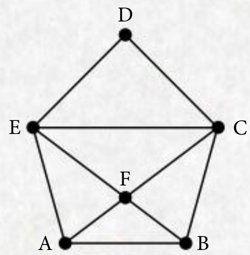
6 © QCAA 2020 1Q23 (4 marks) A plane leaves Brisbane (UTC +10) at 10:45 pm on Monday and takes 14 hours and 35 minutes to fly to Dubai (UTC +4). Determine the local time and day in Dubai when the plane arrives.

7 © QCAA 2022 1Q17 (4 marks) An investment of $50 000 that compounds interest monthly is modelled by the recurrence relation $A_{n+1} = 1.00375A_n$ where $A_0 = 50\,000$.

a What would be the advertised interest rate per annum, compounding monthly? *[2 marks]*

b How many months would it take for the value of the investment to exceed $51 000? *[2 marks]*

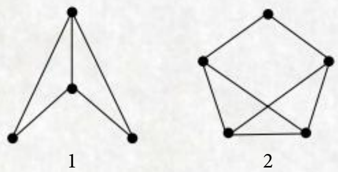
8 © QCAA 2020 1Q16 MODIFIED (2 marks) Use the graph to identify whether each of the following is a cycle, an open walk, an open trail or a closed trail.



a AFCFB *[1 mark]*

b ABCDEFA *[1 mark]*

9 © QCAA 2023 1Q20 MODIFIED (3 marks) Graphs 1 and 2 are shown.



a Show that Euler's formula applies to graph 1. *[2 marks]*

b Redraw graph 2 to represent it as a planar graph. *[1 mark]*

Worksheet
General Maths formula sheet

Cumulative examination 2

Complex familiar and unfamiliar

Perusal time: 2 minutes   Working time: 30 minutes

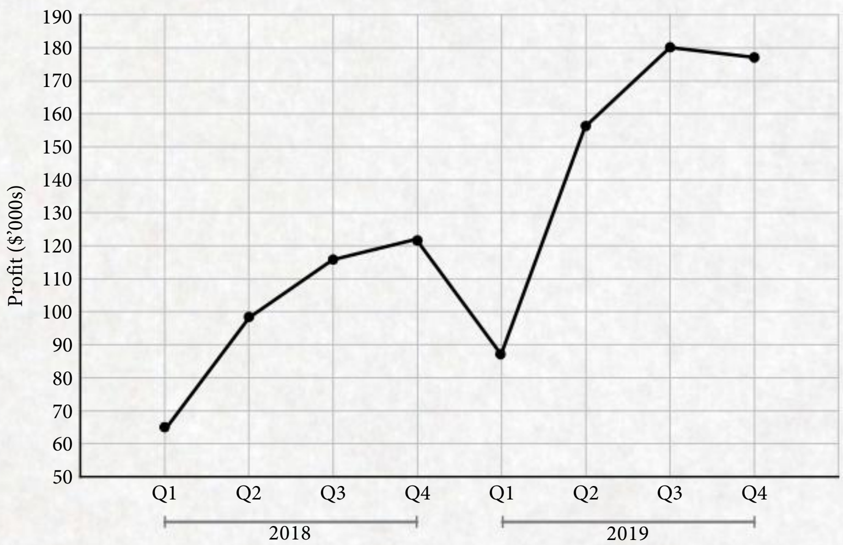
Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

2 short response questions 12 marks

1 © QCAA 2020 2Q4 (5 marks) The following data shows the profits per quarter for a company for the last two years.

	Quarter	Profit ($'000s)
2018	1	64
	2	98
	3	116
	4	122
2019	1	87
	2	156
	3	180
	4	177

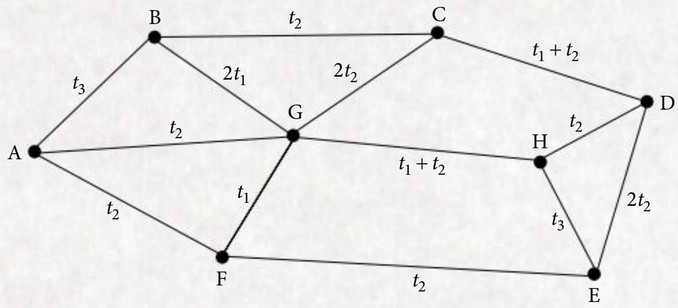
Copy the graph below showing the original data. Deseasonalise the data and plot this on the same set of axes as the original data.



2 © QCAA 2023 2Q5 (7 marks) At 9:00 am, a security guard begins their patrol of the eight work sites represented in the network diagram, starting and ending at site A. They drive at 40 km/h on the roads between sites and check every site once for 15 minutes.

The length (km) of each road corresponds to the terms of the arithmetic sequence $t_n = t_1 + 2(n - 1)$, where $t_1 = 1$.

Determine the earliest possible time the security guard can finish their patrol and identify the route they must follow.



CHAPTER

8 ANNUITIES AND PERPETUITIES

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Syllabus coverage

UNIT 4, TOPIC 2: LOANS, INVESTMENTS AND ANNUITIES 2

Perpetuities and future value of ordinary annuities

- Use a recurrence relation to model the future value of an ordinary annuity, e.g. compound interest investment with periodic payments where interest is calculated before the periodic payment is made.
  - $A_{n+1} = rA_n + d$ where A_{n+1} is total amount at the beginning of the $(n + 1)^{\text{th}}$ period, A_n is total amount at the beginning of the n^{th} period, d is periodic payment and $r = 1 + i$ where i is interest rate per compounding period
- Use the future value annuity formula to model the future value of an ordinary annuity, e.g. compound interest investment with periodic payments where interest is calculated before the periodic payment is made.
  - $A_{FV} = d\left(\dfrac{(1+i)^n - 1}{i}\right)$ where A_{FV} is total amount, d is periodic payment, i is interest rate per compounding period and n is number of compounding periods
- Solve practical problems involving the future value of an ordinary annuity, including determining the total amount of the annuity, periodic payment, total payments and total interest.
- Use the perpetuity formula, $A = \dfrac{d}{i}$ where A is total amount, d is periodic payment and i is interest rate per compounding period.
- Solve practical problems involving perpetuities, including determining the total amount of the perpetuity, periodic payment and interest rate per compounding period.

8.1

Prior learning
Annuities and perpetuities

Video (1):
Exam question analysis Annuities and perpetuities

Prior learning (1):
8.1 Annuities and perpetuities

Spreadsheet (1):
8.1 Investment annuity

Worksheets (3):
8.2 Superannuation
8.4 Annuity problems
Cumulative exams General Maths formula sheet

Puzzle (1):
8.3 Perpetuities

Nelson MindTap

To access resources above, visit
cengage.com.au/nelsonmindtap

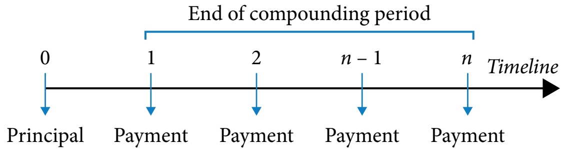
Terminology

fixed-term loan	flexible-term loan	investment annuity
perpetuity	sinking fund	superannuation

8.1 Future value of ordinary annuities

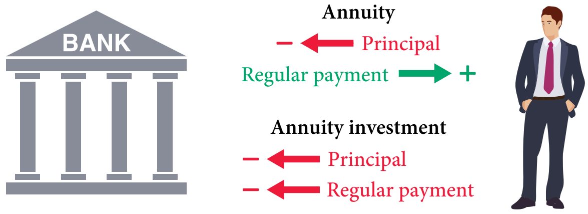
Investment annuity recurrence relations

Chapter 6 established that an **annuity** is a financial situation that involves compound interest and a series of regular equal periodic payments. The payments are taken *out of* the account.



An **ordinary annuity** has the regular payment made at the end of each period after the interest has been added.

An annuity that involves an initial investment amount with regular deposits and no withdrawal of funds is called an **investment annuity**. Payments are made *into* the account. Superannuation is an example of an investment annuity.



The recurrence relation for an investment annuity is the same as the annuity recurrence relation except the payment is added rather than subtracted.

Investment annuity recurrence relation

The future value of an ordinary annuity is given by the recurrence relation

$$A_{n+1} = rA_n + d$$

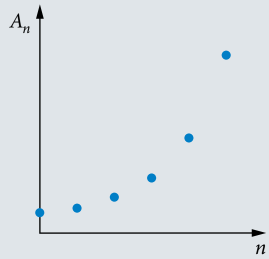
where A_n is the total amount at the beginning of the n^{th} period

A_{n+1} is the total amount at the beginning of the $(n + 1)^{\text{th}}$ period

d is the periodic payment

and $r = 1 + i$ where i is the interest rate per compounding period.

The graph of an annuity investment looks like a compound interest graph except it increases faster.



Exam hack

Be careful not to confuse annuities and annuity investments. Both are investments, but with annuities the regular payments are withdrawals and with annuity investments the regular payments are added.

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8.1

WORKED EXAMPLE 1 Annuity recurrence relations

Abbey receives an inheritance of $40 000. She wants to save a deposit for a home, so she plans to invest the money at 7.5% p.a. compounded monthly. In addition, Abbey plans to make a monthly deposit of $800.

a **Construct** a recurrence relation for this investment annuity.

Abbey's bank offers her a second different investment annuity opportunity. This offer compounds six-monthly with six-monthly deposits that has the recurrence relation

$$A_{n+1} = 1.04A_n + 800, \text{ where } A_0 = 40\,000$$

b **Determine** the nominal interest rate for this investment.

c **Calculate** the value of the annuity at the end of 3 years.

d **Determine** how much the investment has changed in the first 3 years (to the nearest cent).

Steps	Working
a **1** Identify A_0, n and d.	$A_0 = 40\,000$ $n = 12$ $d = 800$
2 Calculate i.	$i = \dfrac{0.075}{12} = 0.006\,25$
3 Calculate r using $r = 1 + i$.	$r = 1 + 0.006\,25$ $= 1.006\,25$
4 Write the recursive relation using $A_{n+1} = rA_n + d$.	$A_{n+1} = 1.006\,25A_n + 800$, where $A_0 = 40\,000$.
b **1** Use the recurrence relation to identify r, and then use $r = 1 + i$ to find i.	$r = 1.04 = 1 + i$ $i = 0.04$ The six-monthly interest rate is 4%.
2 Use i to find the annual interest rate by multiplying by 4.	annual rate $= 2 \times 4$ $= 8\%$
3 State the result.	The nominal interest rate is 8%.
c **1** Determine the balance of the annuity for the first 6 periods using the recurrence relation.	$A_0 = 40\,000$ $A_1 = 1.04A_0 + 800$ $= 1.04 \times 40\,000 + 800$ $= 42\,400$ $A_2 = 1.04A_1 + 800$ $= 1.04 \times 42\,400 + 800$ $= 44\,896$ $A_3 = 47\,491.84$ $A_4 = 50\,191.51$ to the nearest cent. $A_5 = 52\,999.17$ $A_6 = 55\,919.14$
2 State the result.	The value of the investment after 3 years is $55 919.14.
d **1** Calculate $A_6 - A_0$.	$A_6 - A_0 = 55\,919.14 - 40\,000$ $= 15\,919.14$
2 State the result.	The investment has grown by $15 919.14 in the first 3 years.

TECHNOLOGY Investment annuity recurrence relations calculations

You can make a spreadsheet for recurrence relation calculations. The procedure below sets out the steps for the initial annuity situation in Worked example 1. An investment annuity of $40 000, compounding monthly with periodic payments of $800 at 7.5% p.a.

1 Set up the spreadsheet as shown below.

- Format cells B3:B7 as numbers with 2 decimal places.
- Format B7 as a number with 5 decimal places.
- Format cells E4:F27 as numbers with 2 decimal places.
- n = each period of the investment (month for this example)
- Start value = the value of the investment at the start of the period.
- End value = the value of the investment at the end of the period.

	A	B	C	D	E	F
1	Investment annuity					
2						
3	Annual interest rate (%)	7.5		*n*	Start value	End value
4	Compounding periods per year	12		1		
5	Principal (*P*)	40000				
6	Payment amount (*d*)	800				
7	Interest rate per compounding period (*i*)					
8						
9						

2 Enter the formulas shown.

	A	B	C	D	E	F
1	Investment annuity					
2						
3	Annual interest rate (%)	7.5		*n*	Start value	End value
4	Compounding periods per year	12		1	=B5	=E4*(1+$B$7)+$B$6
5	Principal (*P*)	40000		=D4+1	=F4	
6	Payment amount (*d*)	800				
7	Interest rate per compounding period (*i*)	=B3/B4/100				
8						
9						

3 Copy down formulas from D5, E5 and F4 to row 27. This will calculate the values for the first 24 periods.

	A	B	C	D	E	F
1	Investment annuity					
2						
3	Annual interest rate (%)	7.50		*n*	Start value	End value
4	Compounding periods per year	12.00		1	40000.00	41050.00
5	Principal (*P*)	40000.00		2	41050.00	42106.56
6	Payment amount (*d*)	800.00		3	42106.56	43169.73
7	Interest rate per compounding period (*i*)	0.00625		4	43169.73	44239.54
8				5	44239.54	45316.04
9				6	45316.04	46399.26
10				7	46399.26	47489.26
11				8	47489.26	48586.06
12				9	48586.06	49689.73
13				10	49689.73	50800.29
14				11	50800.29	51917.79
15				12	51917.79	53042.28

You can extend the table further down if required.
Use the spreadsheet to carry out investment annuity calculations.

Spreadsheet
Investment annuity

Investigation Using a spreadsheet to explore an investment annuity

Use the spreadsheet you have just created or access the spreadsheet 'Investment annuity' on Nelson MindTap to explore the parameters of a loan. Consider an investment annuity of $40 000, compounding monthly with periodic payments of $800 at 7.5% p.a. (Worked example 1).

What is the value of the annuity at the end of 2 years?

How long will it be before the investment annuity will be $80 000?

What effect does changing the compound periods and payments per year have on the investment?

Investment annuity amortisation tables

Amortisation tables for investment annuities work in a similar way to annuity or reducing balance loan tables. The two key differences are that the payment is added rather than subtracted after the interest is calculated and the principal grows rather than reduces.

8.1

Annuity amortisation table

Payment number (n)	Opening balance ($)	Payment ($)	Investment gain ($)	Principal addition ($)	Closing balance ($)
1	A_0	d	iA_0	$d + iA_0$	A_1
⋮	⋮	⋮	⋮	⋮	⋮
$n + 1$	A_n	d	iA_n	d + interest	$A_{n+1} = rA_n + d$

where A_0 is the principal

A_n is the closing balance after n compounding periods

d is the periodic payment

and i is interest rate per compounding period.

WORKED EXAMPLE 2 Investment annuity amortisation tables

Tai has invested $5500 into an investment annuity which compounds monthly. He makes regular monthly payments of $100 into the account each month after the initial deposit.

The amortisation table shows the first few payments and associated calculations. Some entries are missing.

Payment number	Opening balance	Payment	Investment gain	Principal addition	Closing balance
1	5500.00	100.00	38.50	138.50	5638.50
2	5638.50	100.00	39.47	139.47	5777.97
3					
4			41.43	341.43	6259.84

a Use the amortisation table to **determine**

i the monthly interest rate i

ii the nominal interest rate for the investment.

b Complete the row in the table for payment number 3, stating all values to the nearest cent.

c Tai deposited a bigger payment in the fourth month. How much was the payment?

Steps	Working
a i Find i using the interest in the first month and the formula: interest $= iA_0$.	$38.50 = i \times 5500$ $i = 0.007$
ii 1 Use the compounding periods to determine the nominal interest rate.	There are 12 compounding periods. nominal interest rate $= 12 \times 0.007 \times 100$ $= 8.4\%$ p.a.
2 State the result.	The nominal interest rate is 8.4% p.a.

b Complete the table using:

interest = i × opening balance

principal addition = d + interest

closing balance = opening balance + principal addition

Payment number	Opening balance	Payment	Investment gain	Principal addition	Closing balance
1	5500.00	100.00	38.50	138.50	5638.50
2	5638.50	100.00	39.47	139.47	5777.97
3	5777.97	100.00	0.007 × 5777.97 = 40.45	100 + 40.45 = 140.45	5777.97 + 140.45 = 5918.42

c **1** Calculate the additional payment using principal addition = d + interest	$341.43 = d + 41.43$ $d = \$300$
2 State the result.	Tai made a payment of \$300.

Future value annuity formula

Chapter 6, *Loans, investments and annuities*, established the present value annuity formula. You can find the value of an investment annuity using a similar formula called the future value annuity formula.

The difference between the present value annuity formula and the future value annuity formula is that the value of the investment (A_n) *increases* with the *addition* of a regular deposit (d).

Future value annuity formula with no initial deposit

$$A_{FV} = d\left(\frac{(1+i)^n - 1}{i}\right)$$

where A_{FV} = the total amount of the investment

d = the period payment at the end of the compounding period with no initial investment

i = periodic interest rate per compounding period

n = number of compounding periods.

WORKED EXAMPLE 3 **Finding the payment using the future value annuity formula**

Cathie plans to save \$10 000 over the next 2 years so she can go on an overseas holiday. How much will she need to pay each month into her annuity account if the interest is 9.6% p.a. compounding monthly?

Steps	Working
1 Identify A_{FV} and n.	$A_{FV} = 10\,000$ $n = 12 \times 2 = 24$
2 Calculate i.	$i = \frac{0.096}{12} = 0.008$

3 Substitute A_{FV}, i and n into the formula and simplify. $A_{FV}=d\left(\frac{(1+i)^n-1}{i}\right)$	$10\,000=d\left(\frac{(1+0.008)^{24}-1}{0.008}\right)$ $10\,000=d\times 26.343\ldots$
4 Rearrange and evaluate to find d.	$d=\frac{10\,000}{26.343\ldots}$ $=379.605\ldots$ ≈ 379.61
5 State the answer.	Cathie will need to make monthly payments of \$379.61 to have \$10 000 in 2 years.

If an initial deposit is made, the future value annuity formula must be adjusted to account for the deposit.

Future value annuity formula with an initial deposit

$$A_{FV}=A_0(1+i)^n+d\left(\frac{(1+i)^n-1}{i}\right)$$

where A_{FV} = the total amount of the investment
A_0 = the initial deposit
d = the period payment
i = periodic interest rate per compounding period
n = number of compounding periods.

WORKED EXAMPLE 4 Future value annuity formula

Amelia currently has \$6000. She plans to buy a car in 5 years' time, so she invests her money in an investment annuity that is averaging an interest rate of 7.2% p.a. compounded annually.

a If she deposits \$3500 each year, find the amount available for her new car in 5 years.

b The car Amelia wants cost close to \$40 000. How long will it take for her to have at least \$40 000 in her account?

Steps	Working
a 1 Identify A_0, d and n.	$A_0=6000$ $d=3500$ $n=5$
2 Calculate i.	$i=0.072$
3 Substitute A_0, d, i and n into the formula and simplify. $A_{FV}=A_0(1+i)^n+d\left(\frac{(1+i)^n-1}{i}\right)$	$A_{FV}=6000(1+0.072)^5+3500\left(\frac{(1+0.072)^5-1}{0.072}\right)$ $A_{FV}=8494.252\ldots+20\,208.065\ldots$ $=28\,702.318\ldots$
4 State the answer.	Amelia will have \$28 702.32 for her car in 5 years.

b To find the number of payments, use the recurrence relation $A_{n+1} = rA_n + d$ and calculator recursion to find the number of payments.

1 Create the recurrence relation.	$A_{n+1} = 1.072 \times A_n + 3500$, where $A_n = 6000$
2 Use calculator recursion (or a spreadsheet).	

Steps	Casio fx-82AU PLUS II	Sharp EL-531TH	TI-30XB
Enter A_0 and push equals.	Enter 6000 and press =.	Enter 6000 and press =.	Enter 6000 and press ENTER.
Set up the recursive rule and calculate A_1.	Enter 1.072 × and press ANS, then + 3500 =.	Enter 1.0072 × and press ALPHA, = then + 3500 =.	Enter 1.072 × and press 2nd, (–) then + 3500 ENTER.
	The result should be \$9932.00, which is A_1.		
Now, get the calculator to repeat the calculation to find A_2.	Press =.	Press =.	Press ENTER.
	The result should be \$14 147.10, which is A_2.		
Repeat this process making sure to count and record how many payments are needed for a balance over \$40 000.		$A_3 = \$18\,665.70$, $A_4 = \$23\,509.63$ $A_5 = \$28\,702.32$, $A_6 = \$34\,268.89$ $A_7 = \$40\,236.25$	
3 State the result.		After 7 years, Amelia will have over \$40 000 in her account.	

EXERCISE 8.1 Future value of ordinary annuities

ANSWERS p. 460

Mastery

1 WORKED EXAMPLE 1 Chiara receives a tax refund of \$7500. She wants to save to build an extension to her barn, so she decides to invest the money at 6.9% p.a. compounded monthly. In addition, Chiara also makes a monthly deposit of \$650.

a Write a recurrence relation for this investment annuity.

b How much will Chiara's account be worth after 4 months?

2 Yuri has accepted an investment annuity opportunity from his bank. The annuity compounds quarterly with quarterly deposits and is modelled by the recurrence relation

$$A_{n+1} = 1.012A_n + 1800, \text{ where } A_0 = 200\,000$$

a Determine the nominal interest rate for this investment.

b Calculate the value of the annuity at the end of 4th period.

c Determine how much the investment has changed in the first year (to the nearest cent).

3 WORKED EXAMPLE 2 Kay made an initial deposit of \$8000 on an investment taken out over 6 years at a rate of 5.5% p.a. with interest calculated annually. An additional deposit of \$1200 is made each year.

a Create an amortisation table for the first 5 years of the investment.

b Calculate how much interest had been earned after 5 years.

c What is the value of the annuity at the end of this period?

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4 Huon has inherited \$15 000 and deposits it into an account that pays interest at the rate of 8% p.a. with interest calculated monthly. He decides to make regular monthly deposits of \$350 at the end of each month after the initial deposit, apart from a larger deposit in the fourth month.

The amortisation table shows the first few payments and associated calculations. Some entries are missing.

Payment number	Opening balance	Payment	Investment gain	Principal addition	Closing balance
1	15 000.00	350.00	100.00	450.00	15 450.00
2	15 450.00	350.00	103.00	453.00	
3		350.00			16 359.02
4	16 359.02			809.06	17 168.08

Use the amortisation table to determine

a the balance of the investment at the end of the second month

b the principal addition in the third month

c the amount of Huon's larger payment in the fourth month

d the amount of interest Huon earnt in the first 4 months.

5 WORKED EXAMPLE 3 An account pays interest at 5.3% p.a. compounded weekly. \$120 is deposited into this account each week. How much will be in the account at the end of 3 years?

6 WORKED EXAMPLE 4 Oscar currently has \$5000. He plans to use this money to buy some replacement equipment for his landscaping business. He invests this money in an investment account that is averaging an interest rate of 8.6% p.a. compounded monthly. He also deposits \$450 each month into the account. How much will Oscar have available to spend on equipment when it needs replacement in 4 years' time?

7 \$25 000 is placed in an investment annuity paying 5.8% p.a. compounded quarterly and quarterly deposits of \$4500 are made. Use a table or spreadsheet to find the value of the annuity at the end of 2 years.

Exam practice

8 Matilda deposits \$5000 in an investment annuity paying 6.8% p.a. compounded annually and also deposits \$1000 at the end of each year. If A_n is the value of the investment after n years, a recurrence relation for A_n is

A $A_{n+1} = 1.068(A_n + 1000)$, where $A_0 = 5000$

B $A_{n+1} = 1.068A_n + 1000$, where $A_0 = 5000$

C $A_{n+1} = 1.068A_n - 5000$, where $A_0 = 1000$

D $A_{n+1} = 0.068A_n + 1000$, where $A_0 = 5000$

Use the following for Questions 9 and 10.

The value of an annuity investment, in dollars, after n years, A_n can be modelled by the recurrence relation

$$A_{n+1} = 1.052A_n + 1460, \text{ where } A_0 = 75\,000$$

9 What is the value of the regular payment added to the principal of this investment annuity?

A \$520 **B** \$1052 **C** \$1460 **D** \$2000

10 The increase in value of the investment between the third and fourth year is closest to

A \$5638.72 **B** \$5931.93 **C** \$6240.39 **D** \$23 171.05

11 (4 marks) At the start of each year, Liam deposits \$2400 into an account earning 7% p.a. interest compounded annually. Interest is paid at the end of the year before the next deposit is made.

a Write a recurrence relation for this investment. *[2 marks]*

b Determine the balance of the account at the end of the fifth year. *[2 marks]*

12 (4 marks) Anastacia deposits \$4000 into an account that pays interest at 4.9% p.a. compounded monthly. She also makes a deposit of \$250 at the end of each month. How much will Anastacia have in this account at the end of 5 years?

13 (3 marks) David aims to save \$60 000 over the next 10 years. What regular quarterly contributions will David need to make if the account pays 6.1% per annum, with interest compounding quarterly?

14 (6 marks) CU Georgia makes a deposit of \$1200 into an account paying 7.8% p.a. compounded monthly. She also deposits \$400 at the end of each month. What single sum could she have invested at the start to achieve the same financial result at the end of 3 years?

Worksheet
Superannuation

8.2 Practical problems with future value of ordinary annuities

Superannuation

Superannuation is an example of an investment annuity. In many situations, an employer will make contributions into an employee's superannuation account on a regular basis. The minimum contribution that must be made is 12% of an employee's ordinary earnings. The annuity is set aside for the employee for their working life and used to help fund their retirement.

In Australia, retirement income is often funded through a mix of personal savings, a government pension (Age Pension) and superannuation (super). Currently, the retirement age is 67 years old. This is the age when you can qualify for the Age Pension.

In practice, the amount of money accumulated depends on the size of any initial deposit, the value and frequency of the regular payments into the super fund, the interest earned, inflation and taxation that is paid. To simplify calculations in this section of the book, it is assumed that the interest rate remains the same and the effects of inflation and taxation will not be considered.

Superannuation

As superannuation is an investment annuity, it can be modelled by

- the recurrence relation $A_{n+1} = rA_n + d$
- the formula $A_{FV} = d\left(\frac{(1+i)^n - 1}{i}\right)$ (no initial deposit)
- the formula $A_{FV} = A_0(1+i)^n + d\left(\frac{(1+i)^n - 1}{i}\right)$.

After n compounding periods:

- the increase in the principal $= A_n - A_0$.
- the total deposited $= n \times d$.

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8.2

WORKED EXAMPLE 5 Using recurrence relations for superannuation

Ashton starts work in a new job. His employer sets up a superannuation account for him and deposits $450 at the start of each month. The account earns 8.1% p.a. compounded monthly.

a **Construct** a recurrence relation to model the balance of Ashton's superannuation account.

b What will be the balance of Ashton's account in 6 months?

c How much interest will Ashton's account earn in the first 6 months?

Steps	Working
a **1** Identify A_1, n and d.	$A_1 = 450$ $n = 12$ $d = 450$
2 Calculate i.	$i = \frac{0.081}{12} = 0.006\,75$
3 Calculate r using $r = 1 + i$.	$r = 1 + 0.006\,75$ $r = 1.006\,75$
4 Write the recursive relation using $A_{n+1} = rA_n + d$.	$A_{n+1} = 1.006\,75A_n + 450$, where $A_1 = 450$
b **1** Determine the balance of the fund, to the nearest cent, for the first 6 periods using the recurrence relation.	$A_0 = 0$ $A_1 = 1.006\,75 \times 0 + 450$ $= 0 + 450$ $= 450$ $A_2 = 1.006\,75A_1 + 450$ $= 1.006\,75 \times 450 + 450$ $= 903.04$ $A_3 = 1359.13$ $A_4 = 1818.31$ $A_5 = 2280.58$ $A_6 = 2745.97$
2 State the answer.	Ashton will have $2745.97 in his super fund in 6 months.
c **1** Calculate the total interest earned using $I = (A_n - A_0) - n \times d$.	$(2745.97 - 0) - 6 \times 450 = 45.97$
2 State the result.	In the first 6 months, Ashton will earn $45.97 in interest.

WORKED EXAMPLE 6 Using the future value annuity formula for superannuation

Imogen is aged 41 and is planning to retire at 67 years of age. She currently has a superannuation fund with a balance of $40 000 and is earning 6.9% p.a. compounded monthly.

a If her employer makes regular monthly contributions of $320, how much will be in her account after 10 years?

b Imogen estimates that she will need a lump sum of $550 000 to provide for her retirement. Find the regular monthly contributions she will need to make to achieve her lump sum target.

Steps	Working
a **1** Identify A_0, d and n.	$A_0 = 40\,000$ $d = 320$ $n = 10 \times 12 = 120$
2 Calculate i.	$i = \frac{0.069}{12} = 0.005\,75$

	3 Substitute A_0, d, i and n into the formula and simplify. $A_{FV} = A_0(1+i)^n + d\left(\frac{(1+i)^n - 1}{i}\right)$	$A_{FV} = 40\,000(1+0.005\,75)^{120} + 320\left(\frac{(1+0.005\,75)^{120} - 1}{0.00575}\right)$ $A_{FV} = 79\,591.179\ldots + 55\,083.38\ldots$ $= 134\,674.56$
	4 State the answer.	Imogen will have \$134 674.56 in her super account in 10 years.
b	**1** Identify known values.	$A_0 = 40\,000$ $i = \frac{0.069}{12} = 0.005\,75$ $A_{FV} = 550\,000$
	2 Find n by working out how many years until Imogen retires.	Years before retirement $= 67 - 41$ $= 26$ $\therefore\ n = 26 \times 12 = 312$
	3 Substitute A_0, A_{FV}, i and n into the formula and simplify. $A_{FV} = A_0(1+i)^n + d\left(\frac{(1+i)^n - 1}{i}\right)$	$550\,000 = 40\,000(1+0.005\,75)^{312} + d\left(\frac{(1+0.005\,75)^{312} - 1}{0.005\,75}\right)$ $550\,000 = 239\,305.598\ldots + d \times 866.546\ldots$ $310\,694.4013 = d \times 866.546\ldots$ $d = 358.543\ldots$
	4 Calculate the difference between the required payment and her employer contribution.	$358.543\ldots - 320 = 38.543\ldots$
	5 State the answer.	Imogen will need to contribute an extra \$38.54 to each deposit to ensure she has \$550 000 at retirement.

Sinking funds

Another common type of investment annuity is a fund set up to receive periodic payments to save for an item to be purchased in the future. This type of investment is called a **sinking fund**. The periodic payments, together with the interest earned by the payments, are designed to produce a given sum in the future, called the **future value**.

As a sinking fund is an investment annuity, it can be modelled by

- the recurrence relation $A_{n+1} = rA_n + d$
- the formula $A_{FV} = d\left(\frac{(1+i)^n - 1}{i}\right)$ (no initial deposit)
- the formula $A_{FV} = A_0(1+i)^n + d\left(\frac{(1+i)^n - 1}{i}\right)$.

After n compounding periods

- the increase in the principal $= A_n - A_0$
- the total deposited $= n \times d$.

WORKED EXAMPLE 7	**Using the future value annuity formula for a sinking fund**
a A sinking fund is established to buy a piece of equipment that needs to be replaced in a year. A deposit of \$900 per month is placed into an account that pays 9.6% interest, compounded monthly. How much money will be in the sinking fund after 1 year?	
b Jaiden's parents set up a sinking fund to pay for his education. They would like to have \$80 000 after 15 years. The account they choose pays 7.8% p.a. compounded quarterly. How much must each quarterly deposit be to reach their goal?	
Steps	**Working**
a 1 Identify d, A_0 and n.	$d = 900$ $A_0 = 0$ $n = 12$
2 Calculate i.	$i = \frac{0.096}{12} = 0.008$
3 Substitute d, A_0, i and n into the formula and simplify. $A_{FV} = d\left(\frac{(1+i)^n - 1}{i}\right)$	$A_{FV} = 900\left(\frac{(1+0.008)^{12} - 1}{0.008}\right)$ $A_{FV} = 900 \times 12.542\ldots$ $= 11\,288.103\ldots$
4 State the answer.	The sinking fund will be worth \$11 288.10 after 1 year.
b 1 Identify known values.	$A_{FV} = 80\,000$ $A_0 = 0$ $n = 15 \times 4 = 60$
2 Calculate i.	$i = \frac{0.078}{4} = 0.0195$
3 Substitute A_{FV}, A_0, i and n into the formula and simplify. $A_{FV} = d\left(\frac{(1+i)^n - 1}{i}\right)$	$80\,000 = d\left(\frac{(1+0.0195)^{60} - 1}{0.0195}\right)$ $80\,000 = d \times 112.098\ldots$ $d = \frac{80\,000}{112.098\ldots}$ $= 713.66$
4 State the answer.	Jaiden's parents need to make quarterly payments of \$713.66 to reach their goal.

All the worked examples in this chapter, including the previous example, could be solved using a spreadsheet.

EXERCISE 8.2 Practical problems with future value of ordinary annuities

ANSWERS p. 461

Recap

1 Lyle makes an initial deposit of $10 000 into an investment annuity taken out over 5 years with an interest rate of 6% p.a. The account compounds monthly and Lyle makes additional deposits of $500 each month, except in the final month when they add a larger amount.

The amortisation table shows the first few payments and associated calculations. Some entries are missing.

Payment number	Opening balance	Payment	Investment gain	Principal addition	Closing balance
1	10 000.00	500.00	50.00	550.00	10 550.00
2	10 550.00	500.00	52.75	552.75	11 102.75
3	11 102.75	500.00	55.51	555.51	
4		500.00		558.29	12 216.56
5	12 216.56			961.08	13 177.64

Use the amortisation table to determine

a the balance of the investment at the end of the third month

b the investment gain in the fourth month

c the amount of the larger payment Lyle deposited in the fifth month

d the interest Lyle earnt in the first 5 months.

2 Yasmin makes regular quarterly deposits of $1960 in an account with an interest rate of 6.5% p.a. compounded quarterly. She wants to use her investment to provide a deposit for a townhouse.

a Write a recurrence relation for this investment.

b Calculate the value of her account at the end of 4 years.

Mastery

3 WORKED EXAMPLE 5 Asam's employer establishes a superannuation account for him by depositing $380 at the start of each month. The account earns 8.2% p.a. compounded monthly.

a Construct a recurrence relation to model the balance of Asam's superannuation account.

b What will the balance of Asam's account be in 4 months?

c How much interest will Asam's account earn in the first 4 months?

4 WORKED EXAMPLE 6 Luca is aged 37 and is planning to retire at 60 years of age. The current balance of his superannuation account is $38 000 and the fund earns 8.3% p.a. compounded monthly.

a If Luca's employer makes regular monthly contributions of $320, how much will be in his account after 10 years?

b He estimates that he will need a lump sum of $550 000 to provide for his retirement. Find the regular monthly contributions he will need to make to reach his target.

5 Jake is 52 and is planning to retire at 65 years of age. His current superannuation fund has a balance of $85 000 and is earning 6.5% p.a. compounded monthly. He recently reviewed his superannuation arrangements and estimates that he needs $450 000 to retire at age 65. Find the monthly contributions needed to meet his retirement target.

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6 WORKED EXAMPLE 7 Harriet has decided to deposit \$600 each month for 20 years in an account that pays interest of 7.2% p.a. compounded monthly.

a How much will be in the account at the end of 20 years?

b Harriet believes she needs to accumulate \$450 000 in the 20-year period to have enough for retirement. If she can maintain the initial interest rate for the entire period, what monthly contribution should she make to achieve her goal?

7 A sinking fund is set up to pay for the replacement of office equipment. The equipment will need to be replaced in 5 years' time and will cost \$65 000. The sinking fund is invested in an account that pays 5.2% p.a. compounded monthly. How much must each monthly deposit be to reach their goal?

8.2

Exam practice

8 \$400 is invested at the start of each month in an account that earns 6.6% p.a. compounded monthly. At the end of each month, interest is added. This process is repeated each month. If the value of the investment at the start of the nth month is A_n, which of the following represents a recurrence relation for the value of the investment?

A $A_n = 1.0055(A_{n+1} - 400)$, where $A_1 = 400$

B $A_n = 1.066A_{n+1} + 400$, where $A_1 = 400$

C $A_{n+1} = 1.0055(A_n + 400)$, where $A_1 = 400$

D $A_{n+1} = 1.0055A_n + 400$, where $A_1 = 400$

9 Jordan deposits \$25 000 in an account paying 5.4% p.a. compounded monthly. At the end of each month, interest is added and then a deposit of \$440 is made. If the value of the investment at the start of month n is A_n, which of the following represents a recurrence relation for the value of the investment?

A $A_{n+1} = 1.0045A_n + 440$, where $A_1 = 25\,000$

B $A_{n+1} = 1.054A_n + 440$, where $A_1 = 25\,000$

C $A_n = 1.0045A_{n+1} - 440$, where $A_1 = 25\,000$

D $A_n = 1.0045A_{n+1} + 440$, where $A_1 = 25\,000$

10 Indiana has \$50 000 in a superannuation account earning an interest rate of 9% p.a. with interest compounded monthly. Her employer makes monthly contributions of \$250. Indiana is aged 42 and wishes to retire at 60 years of age. Which of the following equations should she use to find the final value of her superannuation account?

A $A_{18} = 50\,000 \times 1.09^{18} + 250\left(\frac{1.09^{18} - 1}{0.09}\right)$

B $A_{216} = 50\,000 \times 1.0075^{216} + 250\left(\frac{1.0075^{216} - 1}{0.0075}\right)$

C $A_{216} = 50\,000 \times 1.0075^{216} - 250\left(\frac{1.0075^{216} + 1}{2.0075}\right)$

D $A_{216} = 50\,000 \times 1.09^{216} + 250\left(\frac{1.09^{216} - 1}{0.09}\right)$

11 (3 marks) Paige is 53 and is planning to retire at 65 years of age. She estimates that she needs \$640 000 to fund her retirement. Her current superannuation fund has a balance of \$143 000 and is earning 8.5% p.a. compounded monthly. Find the monthly contributions needed to meet Paige's retirement lump sum target.

12 (3 marks) A sinking fund is set up to pay for the replacement of a bulldozer. The bulldozer will need to be replaced in 6 years' time and will cost \$236 000. The sinking fund is invested in an account that pays 6.1% p.a. compounding monthly. How much must each monthly deposit be to replace the bulldozer?

13 (3 marks) Ethan wants to travel overseas in 5 years' time. He currently has \$3000 that he invests in an account that pays 7.5% p.a. compounded monthly. In addition to his initial investment, he also deposits \$1200 each month. How much will Ethan have to spend on his trip?

14 (5 marks) CU Alice is 46 and is planning to retire at 65 years of age. At her current contribution rate, she estimates she will have \$480 000 when she retires. Her current superannuation fund has a balance of \$65 000 and is earning 7% p.a. compounded monthly.

In her final 10 years before retirement, Alice doubles her monthly contribution. Determine the amount (to the nearest \$100) she will have for her retirement.

15 (4 marks) CU Peter is starting a new job, where he will earn \$3540 per fortnight. His employer has guaranteed to pay 12% of Peter's fortnightly salary into his super fund. Peter's super fund earns an average of 5.4% p.a. compounding fortnightly. If the interest rate of the super fund remains consistent, determine the balance of his account after 12 months of work.

Puzzle
Perpetuities

8.3 Perpetuities

A **perpetuity** (or **investment perpetuity** or **perpetual annuity**) is a type of annuity where the regular periodic payments are exactly equal to the interest for each compounding period. This means that the balance of the investment will never change, and the periodic payments can continue forever. Scholarships or grants are often set up as perpetuities.

Perpetuity recurrence relations and formula

Perpetuities are an investment which produces periodic payments from the interest. The payments are given by the formula $d = Ai$, where i is the interest rate for the payment period.

Perpetuities

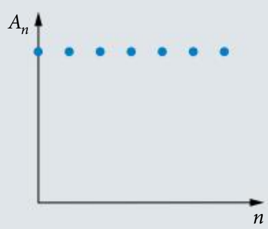
For perpetuity with a value of $A_n = A$ after n compounding periods

$$A = \frac{d}{i}$$

where A is the total amount

d is the periodic payment

and i is interest rate per compounding period.



The graph of perpetuity looks like this.

The perpetuity formula can be rearranged to calculate the interest rate required to provide a particular withdrawal or the payment per period.

$$i = \frac{d}{A} \text{ and } d = Ai$$

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8.3

WORKED EXAMPLE 8 Perpetuities 1

a Kristy has taken out an investment whose recurrence relation for A_n, where the value of the investment after n compounding periods, is

$$A_{n+1} = 1.008A_n - 350, \text{ where } A_0 = 43\,750.$$

Show that Kristy's investment is a perpetuity.

b Anna wishes to set up a perpetuity to fund a neonatal research grant of \$12 000 each year. If interest on the investment is at a guaranteed rate of 7.5% p.a. compounding annually, how much will Anna need to invest?

c Praveen has taken out an \$800 000 perpetuity to fund his grandchildren's education. He invests the money in bonds that pay a return of 6.5% per annum, compounding quarterly. How much will Praveen receive each quarter to fund his grandchildren's education from his investment?

Steps	Working
a **1** Find A_1 and A_2.	$A_0 = 43\,750$ $A_1 = 1.008A_0 - 350$ $= 1.008 \times 43\,750 - 350$ $= 43\,750$ $A_2 = 1.008A_1 - 350$ $= 1.008 \times 43\,750 - 350$ $= 43\,750$
2 For perpetuities $A_n = A$.	$A_1 = A_2 = A$ The value of the investment stays the same as the initial investment after each compounding period. This shows the investment is a perpetuity.
b **1** Identify known values.	$d = 12\,000$ $i = 0.075$
2 Calculate A using $A = \frac{d}{i}$.	$A = \frac{12\,000}{0.075}$ $= 160\,000$
3 State the result.	Anna must invest \$160 000 in the perpetuity.
c **1** Identify known values.	$A = 800\,000$ $i = \frac{0.065}{4} = 0.016\,25$
2 Calculate d, using $d = Ai$.	$d = 800\,000 \times 0.016\,25$ $= 13\,000$
3 State the result.	Praveen will receive quarterly payments of \$13 000.

WORKED EXAMPLE 9 Perpetuities 2

Martina is a successful soccer player who uses \$150 000 to establish a perpetuity for the junior club where she played as a child. The money is invested in a managed fund that returns 7.2% p.a. compounded every 6 months.

a **Calculate** the amount of the bi-annual grant.

'bi-annual' means twice a year.

b What interest rate (compounded every 6 months) would be required for the perpetuity to provide a six-monthly grant of \$8000?

Steps	Working
a **1** Identify known values.	$A = 150\,000$ $i = \dfrac{0.072}{2} = 0.036$
2 Calculate d, using $d = Ai$.	$d = 150\,000 \times 0.036$ $= 5400$
3 State the result.	A grant of \$5400 can be provided six-monthly.
b **1** Identify known values.	$A = 150\,000$ $d = 8000$
2 Calculate i, using $i = \dfrac{d}{A}$.	$i = \dfrac{8000}{150\,000}$ $= 0.0533\ldots$
3 Find the annual rate.	annual rate $= 2 \times 0.0533\ldots \times 100$ $= 10.67\%$
4 State the result.	A grant of \$8000 biannually would require an interest rate of 10.67% p.a. compounded six-monthly.

EXERCISE 8.3 Perpetuities

ANSWERS p. 461

Recap

1 Levi starts work in a new job. His employer starts a superannuation account for him by depositing \$350 at the start of each month. The account earns 7.5% p.a. compounded monthly.
What will be the balance of Levi's account after 25 years?

2 A company will need to replace its copiers in 4 years' time. The company deposits \$2400 at the start of each month into a sinking fund account that pays 5.1% p.a. interest, compounding monthly.
How much will the company have in the sinking fund when the copier needs to be replaced?

Mastery

3 WORKED EXAMPLE 8 A famous actor invests \$200 000 to establish a drama scholarship to provide a monthly allowance for a student to study acting. If the money is invested in a bond that returns 6.9% p.a. compounded monthly, how much is the monthly allowance?

4 A company puts \$300 000 in an investment that pays a return of 7.2% p.a. compounded annually to set up a perpetuity. The perpetuity provides an annual grant for an employee to do study in an area of personal interest. What is the value of the grant?

5 WORKED EXAMPLE 9 Hamish is a blogger who uses \$220 000 to establish a perpetuity. The perpetuity is used to fund a monthly grant to study ways to reduce cyber bullying. The money is invested in utility bonds that return 7.5% p.a. compounded monthly.

a Calculate the amount of the monthly grant.

b What interest rate (compounded monthly) would be required for the perpetuity to provide a grant of \$2250 a month?

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6 Dr Allan wants to create an annual Mathematics prize for the top General Maths student in Year 12 at his school. He has taken out an investment whose recurrence relation for A_n, where the value of the investment after n compounding periods is

$$A_{n+1} = 1.01A_n - 300, \text{ where } A_0 = 30\,000.$$

Show that Dr Allan's investment is a perpetuity.

7 A perpetuity is established to provide a scholarship that consists of a monthly payment of \$1240. The money is invested in a managed fund that pays a guaranteed return of 6.5% p.a. compounded monthly. How much (to the nearest dollar) needs to be invested to provide the scholarship?

8 Nathan is a successful rugby league player who uses \$300 000 to establishes a perpetuity for his first club he played for as a child. What interest rate (compounded quarterly) would be required for the perpetuity to provide a quarterly grant of \$5000?

Exam practice

9 A sum of \$180 000 is used to establish a perpetuity that funds an annual study grant. If the money is invested with a long-term return of 5.5% p.a. compounded annually, the value of the grant is

A \$5500 **B** \$9900 **C** \$19 800 **D** \$99 000

10 A perpetuity provides an annual scholarship of \$27 000. If the money is invested in a bond that returns 6% p.a. compounded annually, then the amount invested is

A \$162 000 **B** \$324 000 **C** \$450 000 **D** \$600 000

11 (2 marks) Aaliyah puts her retirement savings of \$540 000 into a perpetuity that pays 6.2% p.a. compounded monthly. How much does she receive each month?

12 (2 marks) The House Builders Association (HBA) establishes a perpetuity to provide a building industry apprenticeship. The HBA can secure an investment that earns an interest rate of 5.5% p.a. compounded annually. If the annual scholarship is \$27 000, how much needs to be invested?

13 (8 marks) A benefactor has \$150 000 to set up a perpetuity as a grant to help people with diabetes. The benefactor invests in government bonds that return 6.9% p.a. compounded annually.

a Find the amount of the annual grant. *[2 marks]*

b What interest rate (compounded annually) would be required if the perpetuity is to provide \$12 500 each year? *[2 marks]*

The benefactor decides to look at alternative investment options for the perpetuity.

c How much extra would the annual grant grow by if the original interest rate was compounded monthly? *[2 marks]*

d What interest rate (compounded quarterly) would be required to provide quarterly payments of \$3200? *[2 marks]*

14 (4 marks) CF Holly and Luke's grandparents want to establish a perpetuity to provide university scholarships for their two grandchildren. They would like the perpetuity to provide an annual amount of \$6000 for each grandchild. They are able to invest their money in bonds that have a long-term interest rate of 8% p.a. compounded annually. How much will the grandparents need to invest?

15 (4 marks) CU A philanthropist wants to establish a fund that pays out a scholarship of \$10 000 every year in perpetuity. The first scholarship will be paid out in 5 years' time. Assuming an interest rate of 8% p.a. compounded annually, how much will the philanthropist need to pay into the fund?

16 ©QCAA 2024 1Q21 (3 marks) A perpetuity earns interest quarterly at 5.2% p.a. and pays \$975 each quarter.

a Determine the quarterly interest rate. *[1 mark]*

b Calculate the value of the perpetuity. *[2 marks]*

Worksheet Annuity problems

8.4 Practical problems with annuities and perpetuities

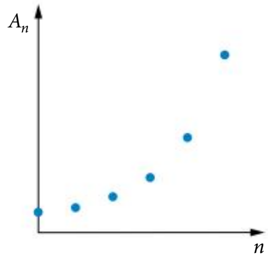
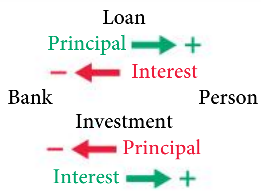
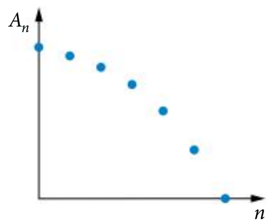
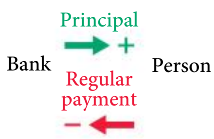
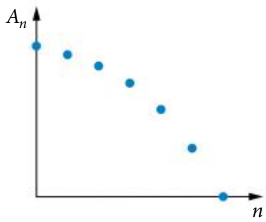
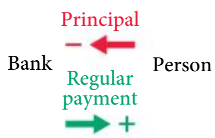
Types of investments and loans

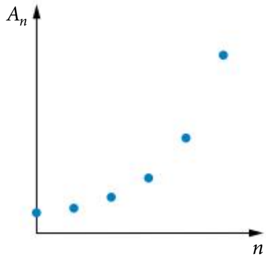
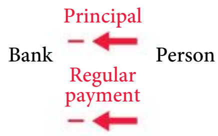
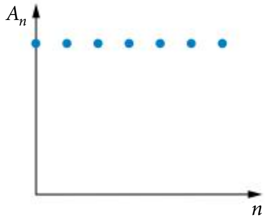
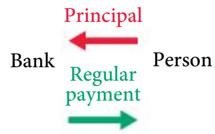
When solving problems involving loans and investments, it is a good idea to determine the type of annuity or account based on the circumstances of the situation. For instance, if the situation involves

- only compound interest and no regular periodic payments (withdrawals or deposits), then the situation is not an annuity but rather a compound interest account only
- compound interest and regular periodic deposits, then it is an investment annuity
- compound interest and regular periodic withdrawals, then it is an ordinary annuity.

Identifying the type of loan or investment will help you decide how to solve the problem and what formulas or recurrence relation you will use.

Loans and investment summary

Type	Recurrence relation / formula	Graph	Principal / payment
Compound interest	$A_{n+1} = rA_n$ $A = P(1+i)^n$ $FV = PV(1+i)^n$ $PV = FV(1+i)^{-n}$		
Reducing balance loan	$A_{n+1} = rA_n - d$ $A_{PV} = d\left(\frac{1-(1+i)^{-n}}{i}\right)$		
Annuity (Present value)	$A_{n+1} = rA_n - d$ $A_{PV} = d\left(\frac{1-(1+i)^{-n}}{i}\right)$		

Type	Recurrence relation / formula	Graph	Principal / payment
Investment annuity (Future value)	$A_{n+1} = rA_n + d$ $A_{FV} = d\left(\frac{(1+i)^n - 1}{i}\right)$ (no initial deposit) $A_{FV} = A_0(1+i)^n + d\left(\frac{(1+i)^n - 1}{i}\right)$		
Perpetuity	$A_n = A$ $A = \frac{d}{i}$		

WORKED EXAMPLE 10 Type of annuity

Zac currently has \$5000. He wants to buy a car in 3 years' time, so he invests this money in an account that is averaging an interest rate of 8.1% p.a. compounded monthly. Zac also deposits \$900 each month into the account.

a What type of annuity does this represent?

b How much will Zac have to spend on a car?

Steps	Working
a This situation involves an initial deposit and regular deposits. There are no withdrawals.	This is an investment annuity.
b **1** Identify known values A_0 and d.	$A_0 = 5000$ $d = 900$ The investment is compounding monthly.
2 Calculate i.	$i = \frac{0.081}{12} = 0.006\,75$
3 Determine the number of compounding periods, n.	$n = 12 \times 3 = 36$
4 Substitute A_0, d, i and n into the formula and simplify. $A_{FV} = A_0(1+i)^n + d\left(\frac{(1+i)^n - 1}{i}\right)$	$A_{FV} = 5000(1+0.006\,75)^{36} + 900\left(\frac{(1+0.006\,75)^{36} - 1}{0.006\,75}\right)$ $= 6370.140\ldots + 36\,537.068\ldots$ $= 42\,907.21$
5 State the result.	After 3 years, Zac will have \$42 907.21 to buy a car.

Reducing balance loans

Most of the reducing balance loans that you have met so far have been **fixed-term loans**. **Flexible-term loans** are reducing balance loans where the interest rate, payment value or the frequency of payment may vary. The most common change to the terms of a loan is the interest rate.

Reducing balance loans can be modelled by recurrence relations and amortisation tables. As they are a type of ordinary annuity, they can also be modelled using annuity formulas.

Reducing balance loan formulas

Initial value of loan (A_{PV})

$$A_{PV} = d\left(\frac{1-(1+i)^{-n}}{i}\right)$$

Loan repayment (d)

$$d = \frac{A_{PV}\,i}{1-(1+i)^{-n}}$$

Balance of the loan (B_n)

$$B_n = A_0(1+i)^n - d\left(\frac{(1+i)^n - 1}{i}\right)$$

where A_{PV} = the total amount of the loan (= principal (A_0))
B_n = the balance of the loan after n payments
d = the periodic payment
i = interest rate per compounding period
n = number of compounding periods.

You can also use the formula for B_n for an ordinary annuity.

To find the initial value of a reducing balance loan, use the present value annuity formula.
To find the value of the loan later, use the future value of an annuity formula.

WORKED EXAMPLE 11 Using an annuity formula for a reducing balance loan

To buy her first car, Felicia borrowed \$25 000 as a reducing balance loan at 4.8% p.a. compounding monthly, with monthly repayments of \$500. How much will Felicia owe after 4 years?

Steps	Working
1 Identify known values A_0 and d.	$A_0 = 25\,000$ $d = 500$ The investment is compounding monthly.
2 Calculate i.	$i = \frac{0.048}{12} = 0.004$
3 Determine the number of compounding periods, n.	$n = 12 \times 4 = 48$
4 Substitute A_0, d, i and n into the formula and evaluate. $A_{FV} = A_0(1+i)^n - d\left(\frac{(1+i)^n - 1}{i}\right)$	$A_{FV} = 25\,000(1+0.004)^{48} - 500\left(\frac{(1+0.004)^{48} - 1}{0.004}\right)$ $= 30\,280.164\ldots - 26\,400.820\ldots$ $= 3879.34$
5 State the result.	After 4 years, Felicia will owe \$3879.34.

WORKED EXAMPLE 12 **Modified repayments**

Zoe borrows \$80 000 at 7.5% p.a. compounding monthly over 9 years with monthly repayments of \$1020. After 4 years, the interest rate is reduced to 6% p.a.

a What is the new repayment amount required for Zoe to pay off the loan in the same period of time?

b How much does Zoe save due to the reduction in the interest rate?

Steps	Working
a 1 Identify known values A_0, d and n.	$A_0 = 80\,000$ $d = 1020$ The investment is compounding monthly over 4 years. $n = 12 \times 4 = 48$
2 Calculate i.	$i = \frac{0.075}{12} = 0.006\,25$
3 Substitute A_0, d, i and n into the formula and evaluate. $A_{FV} = A_0\left(1+i\right)^n - d\left(\frac{\left(1+i\right)^n - 1}{i}\right)$	$A_{FV} = 80\,000\left(1 + 0.006\,25\right)^{48} - 1020\left(\frac{\left(1 + 0.006\,25\right)^{48} - 1}{0.006\,25}\right)$ $= 107\,887.9321 - 56\,891.3815$ $= 50\,996.55$
4 Identify known values for the second 5 years of the loan.	$A_0 = 50\,996.55$ $A_{FV} = 0$ The investment is compounding monthly over 5 years. $n = 12 \times 5 = 60$
5 Calculate i.	$i = \frac{0.06}{12} = 0.005$
6 Substitute A, i and n into the formula, simplify and evaluate. $A_{FV} = A_0\left(1+i\right)^n - d\left(\frac{\left(1+i\right)^n - 1}{i}\right)$	$0 = 50\,996.55\left(1+0.005\right)^{60} - d\left(\frac{\left(1+0.005\right)^{60} - 1}{0.005}\right)$ $-68\,786.705\ldots = -d \times 69.770\ldots$ $d = 985.906\ldots$
7 State the result.	The new repayment is \$985.91.
b 1 Calculate the difference between the old and new repayments.	Saving per payment $= 1020 - 985.91$ $= 34.09$
2 Calculate the total savings over 5 years.	total savings $= 34.09 \times 60$ $= 2045.40$
3 State the result.	Zoe will save a total of \$2045.40.

Superannuation and retirement pensions

It is common for people to build their superannuation throughout their careers and upon retirement *roll* it into a retirement pension account.

The time before retirement is called the *deposit phase* as you (and/or your employer) deposit money into your superannuation fund.

When your super is rolled into a retirement pension fund, you start to receive payments from this fund. This period is called the *withdrawal phase*.

Deposit phase	Withdrawal phase
The deposit phase can be modelled by: • the recurrence relation $A_{n+1} = rA_n + d$ • the 2 formulas $A_{FV} = d\left(\frac{(1+i)^n - 1}{i}\right)$ (no initial deposit, $A_0 = 0$) $A_{FV} = A_0(1+i)^n + d\left(\frac{(1+i)^n - 1}{i}\right)$ (where $A_0 > 0$). After n compounding periods: • the increase in the principal $= A_n - A_0$ • the total deposited $= n \times d$ • the interest earned $= (A_n - A_0) - n \times d$.	The withdrawal phase can be modelled by: • the recurrence relation, $A_{n+1} = rA_n - d$ • the formula, $A_{FV} = A_0(1+i)^n - d\left(\frac{(1+i)^n - 1}{i}\right)$ After n compounding periods: • the reduction in the principal $= A_0 - A_n$ • the total withdrawn $= n \times d$ • the interest earned $= (A_n - A_0) - n \times d$.

WORKED EXAMPLE 13 Connecting superannuation to retirement pensions

Melissa is 47 and has just started a new job. She transfers her old superannuation account worth \$328 000 to the new one her employer has set up. The new account earns 6.6% p.a. compounding monthly and her employer deposits \$900 per month into the account.

At 67, Melissa retires and rolls her super account into a retirement pension. The retirement pension is set up to earn 5.2% p.a. compounding quarterly and have a zero balance at 30 years.

a What is the balance of Melissa's superannuation account at retirement?

b How much will Melissa's retirement pension pay her?

Steps	Working
a **1** For the deposit phase identify known values A_0, d and n.	$A_0 = 328\,000$ $d = 900$ Melissa will retire in 20 years. The investment is compounding monthly. $\therefore\ n = 12 \times 20 = 240$
2 Calculate i.	$i = \frac{0.066}{12} = 0.0055$
3 Substitute A_0, d, i and n into the formula and simplify. $A_{FV} = A_0(1+i)^n + d\left(\frac{(1+i)^n - 1}{i}\right)$	$A_{FV} = 328\,000(1 + 0.0055)^{240} + 900\left(\frac{(1 + 0.0055)^{240} - 1}{0.0055}\right)$ $= 1\,223\,409.440... + 446\,712.027...$ $\approx 1\,670\,121.47$
4 State the result.	Melissa's superannuation account has a value of \$1 670 121.47 at retirement.

b	**1** For the withdrawal phase, identify known values.	$A_0 = 1\,670\,121.47 = A_{PV}$ The pension will have a zero balance in 30 years. The investment is compounding quarterly. $\therefore\ n = 4 \times 30 = 120$
	2 Calculate i.	$i = \dfrac{0.052}{4} = 0.013$
	3 Substitute A_0, d and n into the formula and simplify to find d. $A_{PV} = d\left(\dfrac{1-(1+i)^{-n}}{i}\right)$	$1\,670\,121.47 = d\left(\dfrac{1-(1+0.013)^{-120}}{0.013}\right)$ $1\,670\,121.47 = d \times 60.595\ldots$ $d = 27\,561.831\ldots$
	4 State the result.	Melissa will receive quarterly payments of \$27 561.83.

This type of approach can also be applied to situations involving an investment annuity which is then *rolled* into an annuity or perpetuity.

EXERCISE 8.4 Practical problems with annuities and perpetuities

ANSWERS p. 461

Recap

1 A successful artist uses \$360 000 to establish a perpetuity. The perpetuity provides the funds for a monthly grant to stage exhibitions of landscape paintings. If the money is invested in bonds that return 5.5% p.a. compounded monthly, then the monthly allowance is

A \$1650 **B** \$1870 **C** \$3300 **D** \$16 500

2 \$240 000 is used to establish a perpetuity to provide a monthly payment to a community health centre. The fund in which the money is invested earns 5.5% p.a. compounded monthly.

a Calculate the amount of the grant.

b What interest rate (compounded quarterly) would be required for the perpetuity to provide a monthly grant of \$3600?

Mastery

3 WORKED EXAMPLE 10 Aaliyah wants to take an extended holiday in 5 years' time. She estimates she will need \$45 000 for the trip. She currently has savings of \$4000 and she puts this in an account that pays 7.6% p.a. interest, compounded monthly. She intends to make regular monthly deposits to the account for the next 5 years.

a What type of annuity is represented by this situation?

b If Aaliyah deposits \$400 a month, will she reach her savings goal?

c Calculate the value of the regular deposit required for Aaliyah to reach her target.

4 Milan wants to set up an account to supplement his current income. He wants to deposit a sum of money in an account that pays interest at the rate of 8.5% p.a. compounded monthly so that he can draw payments of \$1250 a month for the next 18 years.

a What type of annuity is this arrangement?

b How much should he deposit now?

5 WORKED EXAMPLE 11 Asha borrowed \$55 000 at 7.8% p.a. interest compounding fortnightly to start his own business. If Asha made fortnightly repayments of \$430, what is the balance of the loan after 2 years?

6 WORKED EXAMPLE 12 Oliver borrows \$25 000 at 6.4% p.a. compounding monthly over 7 years with monthly repayments.

- **a** How much is Oliver's monthly repayment?
- **b** How much does he repay in total?
- **c** How much interest does Oliver pay?
- **d** After 3 years, the interest rate is increased to 7.9% p.a. and remains the same for the rest of the loan. What is the new repayment amount required for Oliver to pay off the loan in the same period of time?
- **e** How much more does it cost Oliver to pay out the loan due to the increase in the interest rate?

7 WORKED EXAMPLE 13 Ros has a current balance of \$421 355.82 in her superannuation account. She has just started a new job and her employer deposits \$395 into this account every fortnight. The super fund earns 4.3% per annum, compounding fortnightly. Ros plans to retire in 22 years when she is 67 and will then no longer receive deposits from her employer. At that time, Ros will place the balance of her super account into a retirement pension account, which will pay interest at 3.8% per annum compounding monthly. Her account will have a zero balance after 246 payments. What is the value of Ros' pension payments?

Exam practice

8 An amount of \$10 000 is borrowed at an interest rate of 6.7% p.a. compounding monthly. The loan is repaid with monthly payments of \$196.60 over 5 years. Which of the following statements is incorrect?

- **A** Increasing the monthly repayment to \$220 will decrease the length of the loan.
- **B** If the interest rate increases to 7% p.a. and the repayment stays the same, the length of the loan will decrease.
- **C** If the length of the loan is increased to 7 years, the monthly payment required will decrease.
- **D** If the length of the loan is increased to 7 years, the total amount of interest paid will increase.

9 (2 marks) Jessica places \$200 000 in an account earning 6.4% p.a. compounded quarterly. She wants to make quarterly withdrawals so that the account has a zero balance at the end of 6 years. What is the value of the withdrawal she should make, rounded to the nearest \$10?

10 (3 marks) Harrison owns a delicatessen. He knows that he will have to replace a meat slicer, coffee roaster and other pieces of equipment in 5 years' time. The total cost of the machinery will be \$38 000. Harrison decides to deposit a sum of money into an account at the end of each month to build up enough money to pay for the equipment. If the account where he deposits his money pays 7.3% p.a. compounded monthly, what is the minimum amount that each deposit must be?

11 (4 marks) Olivia needed to buy furniture for her flat, so she borrowed \$5000 at 8.6% p.a. interest compounding fortnightly. If she paid fortnightly repayments of \$75, what is the balance of the loan after 2 years?

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12 (4 marks) A services club wants to set up an account to provide an annual grant for local improvement programs. The club has raised \$180 000 and the members of the club want to ensure that the amount invested remains the same. The club can invest in energy bonds that return 6.4% p.a. compounded annually.

- **a** What is the value of the annual grant? *[2 marks]*
- **b** What interest rate (compounded annually) would be required if the investment is to provide \$13 500 each year? *[2 marks]*

13 (4 marks) An initial deposit of \$12 000 was made in an investment account over 5 years at a rate of 9.25% p.a. with interest calculated monthly. An additional deposit of \$1500 is made at the end of each month after interest has been added.

- **a** What is the investment worth at the end of 3 years? *[2 marks]*
- **b** How much interest did the investment earn in the last month of the investment? *[2 marks]*

14 (2 marks) Anh is currently aged 43 and he wants to be able to retire at 62 years of age. Right now, he has an account with a balance of \$55 000 and it is earning 5.8% p.a. compounded monthly.

- **a** If Anh makes regular monthly contributions of \$450, how much will be in his account after 10 years? *[1 mark]*
- **b** If Anh needs a lump sum of \$490 000 to fund his retirement, what is the value of the regular monthly contribution he will need to make? *[1 mark]*

15 ©QCAA 2021 1Q22 (4 marks) Rosa borrowed \$32 000 as a reducing balance loan at 4.8% p.a. compounding monthly, with monthly repayments of \$278.

How much will Rosa owe after 2 months, to the nearest cent?

16 (5 marks) **CF** Poppy borrows \$108 000 at 8.8% p.a. compounding quarterly over 12 years with quarterly repayments. After 5 years, the interest rate decreases to 6.5% p.a. and remains the same for the rest of the loan. What is the new repayment amount required for Poppy to pay off the loan in the same period of time?

17 (4 marks) **CF** Thea contributed \$2500 per quarter into an account earning 7.25% p.a. compounded quarterly for a period of 10 years. After 10 years, she increased her quarterly payments to \$3200 for another 10 years. What is the final value of her account?

18 (2 marks) **CU** As a settlement of an insurance claim, Beau is offered regular payments of \$15 000 each year for the next 15 years. Beau would prefer a one-time payment, now. Assuming the lump sum could be invested at 7.7% p.a. compounded annually, what would be a fair amount that should be offered to Beau?

19 (5 marks) **CF** Heidi is a professional athlete who believes that her playing career will last 9 years. She is considering 2 possible options to prepare for her future.

Option 1: She deposits \$48 000 at the end of each year for 9 years in an account paying 8% p.a. compounded annually.

Option 2: She deposits \$4000 at the end of each month for 9 years in an account paying 8% compounded monthly.

Which option should Heidi choose? Justify your answer.

20 (3 marks) **CU** A loan of \$25 000 is borrowed at 11.5% p.a. interest compounded annually. The loan is repaid with 3 annual instalments leaving a balance of \$6714.46 at the end of the 4 years. Determine the value of the annual payments.

21 (2 marks) CU Some lenders recommend the use of fortnightly payments for mortgages. They claim that making half the monthly payment each fortnight considerably reduces the interest and time taken to pay off the mortgage. Evaluate the reasonableness of this claim.

22 (4 marks) CU Ella borrowed \$38 000 at 11.3% p.a. compounded monthly. She elects to repay the loan over 8 years with monthly instalments of \$603.09. After 3 years, Ella decides that she would like to repay the loan in full in the next 3.5 years. Find the monthly repayment needed to achieve her goal.

23 (5 marks) CU Molly is a 48-year-old woman. At the end of each quarter, she puts \$4000 in a retirement account that pays 5.4% p.a. interest compounded quarterly. Molly withdraws the entire amount when she turns 60, and deposits it in a different account that pays 7.2% interest compounded monthly. From then on she deposits \$450 into the account at the end of each month. How much is in the account when Molly reaches age 65?

24 (3 marks) CU An investment account currently has a balance of \$191 653. Annual contributions of \$4800 are made and the account earns 5.5% p.a. compounded annually. What was the initial amount in the account 9 years ago?

25 (7 marks) CU Reuben started his savings by investing an inheritance of \$250 000 for 15 years in an account earning 4.8% p.a. interest compounding monthly.

After the 15 years he put the balance into a superannuation account for his retirement in 35 years. The superannuation account earns 6.1% p.a. interest compounding monthly and he made regular monthly payments of \$1000 for 35 years until he retired.

When Reuben retired, he invested the money in an annuity which earns 5.2% p.a. compounding monthly. He has been taking out regular monthly payments of \$30 000 from this account. How much money, to the nearest cent, is in Reuben's account after he has been retired for 10 years?

26 (7 marks) CU James plans to retire from work 10 years from now. His retirement goal is to have a balance of \$1 500 000 in an annuity investment at that time. The present value of this annuity investment is \$591 247.51, on which he earns interest at the rate of 5.34% per annum, compounding monthly. To make this investment grow faster, James will add a \$3000 payment at the end of every month. Two years from now, he expects the interest rate of this investment to fall to 5.20% per annum, compounding monthly. It is expected to remain at this rate until James retires. When the interest rate drops, he must increase his monthly payment if he is to reach his retirement goal. Find the value of this new monthly payment (round to the nearest 10 cents).

EXAM QUESTION ANALYSIS

©QCAA 2023 2Q7 (5 marks)

Five years ago, a retiree invested \$100 000 in a compound interest account earning 3.8% p.a. compounding monthly. They now intend to use the balance of the account to begin a perpetuity that will return 4% p.a. compounding annually and pay them \$6000 each year.

Provide advice to the retiree about whether their compound interest investment is large enough to finance the perpetuity.

Reading the question

- The question is asking for the retiree's deposit value for a future perpetuity.
- The retiree's deposits earn interest and are transferred to a perpetuity.
- The investment is compounding monthly and the perpetuity is compounding annually.

Thinking about the question

- Will have to work backwards from the perpetuity to the account to find the value.
- Need to use the perpetuity formula, to find the amount invested from the retiree's account.
- The final amount of the compound interest account will need to equal or exceed the amount needed for the perpetuity.

Worked solution (✓ = 1 mark)

Consider the perpetuity.

$d = 6000$

$i = 0.04$

$A = ?$

$A = \frac{d}{i}$

$\mathbf{A = \frac{6000}{0.04}}$ ✓

$\mathbf{= 150\,000}$ ✓

Consider the investment annuity.

$A_{FV} = 150\,000$

$i = \frac{0.038}{12} = 0.00316\ldots$

$n = 5 \times 12 = 60$

$A_0 = P = ?$

$A_{FV} = P(1+i)^n$

$\mathbf{150\,000 = P(1 + 00316\ldots)^{60}}$ ✓

$150\,000 = P \times 1.208\ldots$

$P = \frac{150\,000}{1.208\ldots}$

$\mathbf{P = 124\,081.1136 \approx 124\,081.11}$ ✓

The principal needs to be \$124 081.11. The compound interest investment will not provide enough money to finance the perpetuity (\$100 000 < \$124 081.11). ✓

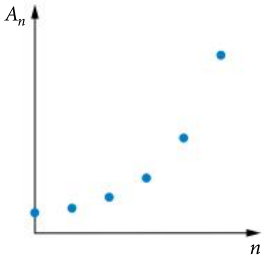
8.4

Video
Exam question analysis: Annuities and perpetuities

8 Chapter summary

Investment annuities

- An annuity that involves an initial investment amount with regular deposits and no withdrawal of funds is called an **investment annuity**. Payments are made *into* the account.
- The future value of an investment annuity is given by the recurrence relation



$$A_{n+1} = rA_n + d$$

where A_n is the total amount at the beginning of the n^{th} period
A_{n+1} is the total amount at the beginning of the $(n+1)^{th}$ period
d is the periodic payment
and $r = 1 + i$ where i is interest rate per compounding period.

- The future value of an annuity (no initial deposit) is given by

$$A_{FV} = d\left(\frac{(1+i)^n - 1}{i}\right).$$

- The future value of an annuity (with an initial deposit) is given by

$$A_{FV} = A_0(1+i)^n + d\left(\frac{(1+i)^n - 1}{i}\right)$$

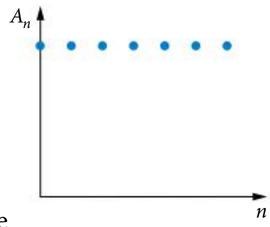
where A_{FV} is the total amount of the investment
d is the period payment at the end of the compounding period with no initial investment
i is the periodic interest rate per compounding period
n is the number of compounding periods.

- Personal **superannuation** is money that is saved during an individual's working life to provide a regular income in retirement.
- Superannuation and sinking funds are both investment annuities and can be modelled by:
  - the recurrence relation $A_{n+1} = rA_n + d$
  - the formula $A_{FV} = d\left(\frac{(1+i)^n - 1}{i}\right)$

    (no initial deposit)
  - the formula $A_{FV} = A_0(1+i)^n + d\left(\frac{(1+i)^n - 1}{i}\right)$.

Perpetuities

- A **perpetuity** (or **investment perpetuity** or **perpetual annuity**) is a type of annuity where the regular periodic payments are exactly equal to the interest for each compounding period.



- For perpetuity with a value of $A_n = A$ after n compounding periods

$$A = \frac{d}{i}$$

where A is the total amount
d is the periodic payment
and i is interest rate per compounding period.

- Contributing payments to an investment annuity is sometimes referred to as the deposit phase.
- Withdrawing payments from an annuity or perpetuity is sometimes referred to as the withdrawal phase.

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Deposit phase	Withdrawal phase
The deposit phase can be modelled by: • the recurrence relation $A_{n+1} = rA_n + d$ • the formula $A_{FV} = d\left(\frac{(1+i)^n - 1}{i}\right)$, (no initial deposit, $A_0 = 0$) $A_{FV} = A_0(1+i)^n + d\left(\frac{(1+i)^n - 1}{i}\right)$, (where $A_0 > 0$). After n compounding periods: • the increase in the principal $= A_n - A_0$ • the total deposited $= n \times d$ • the interest earned $= (A_n - A_0) - n \times d$.	The withdrawal phase can be modelled by: • the recurrence relation $A_{n+1} = rA_n - d$ • the formula $A_{FV} = A_0(1+i)^n - d\left(\frac{(1+i)^n - 1}{i}\right)$. After n compounding periods: • the reduction in the principal $= A_0 - A_n$ • the total withdrawn $= n \times d$ • the interest earned $= (A_n - A_0) - n \times d$.

Cumulative examination 1

Worksheet
General Maths formula sheet

Simple familiar

Perusal time: 2 minutes Working time: 30 minutes

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

Section 1 (5 marks): 5 multiple choice questions

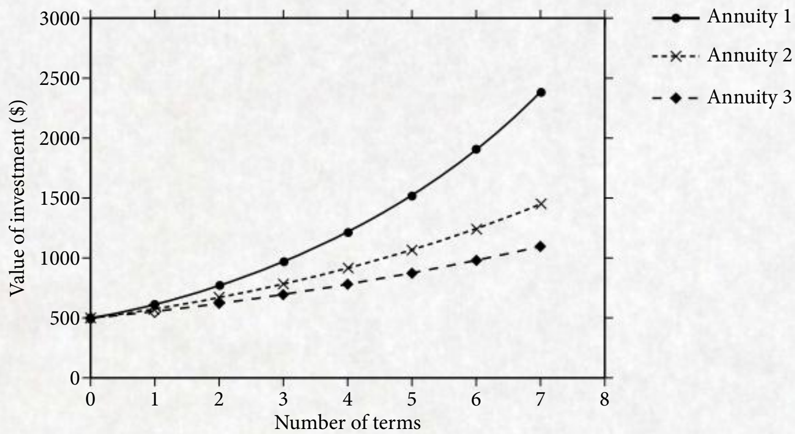
Section 2 (12 marks): 3 short response questions

Total: 17 marks

Section 1 5 multiple choice questions

5 marks

1 © QCAA 2020 1Q15 The graph shows the value of three different annuities over time.



Which of the following statements gives a plausible explanation for the different values after seven terms?

A Annuity 1 and Annuity 2 have higher regular deposits than Annuity 3.

B Annuity 1 and Annuity 2 have shorter interest terms than Annuity 3.

C Annuity 2 and Annuity 3 have a lower initial value than Annuity 1.

D Annuity 2 and Annuity 3 have higher interest rates than Annuity 1.

2 © QCAA 2023 1Q11 An annuity with an initial zero balance has $500 deposited at the end of every month. The annuity earns 4.8% p.a. interest, compounding monthly.
At the end of the fourth month, the balance is closest to

A $2002 **B** $2008 **C** $2012 **D** $2014

3 A perpetuity provides an annual prize of $1500. If the money is invested in an account that returns 3.75% p.a. compounded annually, then the amount invested is

A $25 000 **B** $32 000 **C** $38 000 **D** $40 000

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4 © QCAA 2020 1Q9 It is observed that as the number of ice blocks sold each month increases, the number of fans sold also increases. Which of these statements is therefore true?

A There is a negative causation between the number of ice blocks sold and the number of fans sold each month.

B There is a positive causation between the number of ice blocks sold and the number of fans sold each month.

C There is a negative association between the number of ice blocks sold and the number of fans sold each month.

D There is a positive association between the number of ice blocks sold and the number of fans sold each month.

5 © QCAA 2020S 1Q7 To calculate the number of degrees of longitude across the Earth's surface for a time zone difference of one hour, use

A $\frac{90}{12}$ **B** $\frac{90}{24}$ **C** $\frac{360}{12}$ **D** $\frac{360}{24}$

Section 2 3 short response questions — 12 marks

6 © QCAA 2020 1Q20 (4 marks) Determine the monthly repayment on a \$350 000 home loan over 25 years with 6.5% p.a. fixed interest compounded monthly.

7 © QCAA 2020 1Q17 (3 marks) Calculate the distance along the parallel of latitude between Mount Gambier, South Australia (37° 50' S, 140° 47' E) and Bairnsdale, Victoria (37° 50' S, 147° 37' E) to the nearest kilometre.

8 (5 marks) Tam has a choice of investments over four years. She can:

Option A: deposit \$900 a month into an account paying 8.6% p.a. compounding monthly, or

Option B: deposit \$450 each fortnight into an account paying 8.6% p.a. compounding fortnightly.

a How much will she have after four years with Option A? *[2 marks]*

b How much will she have after four years with Option B? *[2 marks]*

c Which option should Tam choose? *[1 mark]*

Worksheet
General Maths formula sheet

Cumulative examination 2

Complex familiar and unfamiliar

Perusal time: 2 minutes   Working time: 30 minutes

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

3 short response questions — 11 marks

1 © QCAA 2021 2Q3 (5 marks) Jo contributes \$2500 per quarter to an annuity earning 3.6% p.a. compounding quarterly.

At the end of 4 years, Jo makes a one-off extra contribution of \$10 000 and continues with the regular quarterly contributions.

Determine the value of the annuity at the end of 6 years, to the nearest dollar.

2 © QCAA 2020S 2Q4 (3 marks) You have a savings account with an interest rate of 2.4% per annum paid quarterly. Your bank makes you a new offer. You are entitled to either an increase in the interest rate of 0.5% per annum, or to your current interest paid monthly.

Determine which option is the most profitable.

3 © QCAA 2020S 2Q8 (3 marks) A teacher wants to know the best way for their students to improve their marks. They surveyed a sample of students who graduated last year and asked them three questions:

- What was the overall mark you achieved on the final assessment?
- On a typical night, how many hours sleep would you get?
- During a typical school term, how many classes did you miss?

They obtained the following data:

Overall percentage mark (%)	Hours of sleep	Classes missed
96	10	0
85	9	2
76	8	8
65	7	5
42	5	6

Construct a mathematical argument to determine which explanatory variable is the better predictor for the overall mark.

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CHAPTER

9 SPANNING TREES AND CRITICAL PATHS

Prior learning
Spanning trees and critical paths

Syllabus coverage

UNIT 4, TOPIC 4: NETWORKS AND DECISION MATHEMATICS 1

Trees and minimum connector problems

- Understand the meaning of tree, spanning tree and minimum spanning tree.
- Determine a minimum spanning tree in a weighted connected graph.
- Solve practical problems involving minimum spanning trees, e.g. minimising the length of cable needed to provide power from a single power station to substations in several towns.

Project planning and scheduling using critical path analysis (CPA)

- Construct a project network diagram (activity on arc) to represent the durations and interdependencies of activities that must be completed during the project (excluding dummy activities).
- Use forward and backward scanning to determine the earliest starting time (EST) and latest starting time (LST) for each activity in the project.
- Use ESTs and LSTs to locate the critical path/s for a project.
- Use the critical path to determine the minimum time for a project to be completed.
- Calculate float times for non-critical activities.
- Solve small-scale practical problems involving critical path analysis.

General Mathematics 2025 v1.2 General senior syllabus p. 30,

Videos (7):
9.1 Trees and adjacency matrices • Prim's algorithm
9.3 Project networks
9.4 Forward scanning • Latest starting times • Critical path analysis
Exam question analysis Spanning trees and critical paths

Prior learning (1):
9.1 Spanning trees and critical paths

Worksheets (9):
9.1 Shortest paths and trees • Minimum spanning tree
9.3 Project networks • Drawing directed graphs
9.4 Calculating EST and LST • Critical paths • Critical paths and activity float times • Critical path analysis
Cumulative exams General Maths formula sheet

Puzzle (1):
Chapter summary Networks find-a-word

Nelson MindTap

To access resources above, visit **cengage.com.au/nelsonmindtap**

Terminology

activities	activity table	backwards scanning	concurrent activities
connector problems	critical activities	critical path	critical path analysis
earliest start time (EST)	float time	forward scanning	immediate predecessor
latest start time (LST)	minimum spanning tree	non-critical activities	prerequisite
Prim's algorithm	project network diagram	slack time	
spanning tree	tree		

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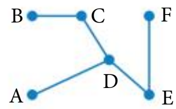
9.1 Minimum spanning trees

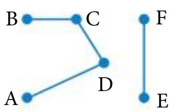
Trees and spanning trees

A **tree** is a connected graph with no **loops**, multiple **edges** between adjacent **vertices**, or **cycles**.

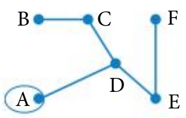
This graph is a tree.



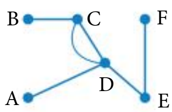
These graphs are not trees.



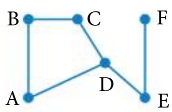
not a connected graph



has a loop



has multiple edges



contains a cycle

The number of edges in a tree will always be one less than the number of vertices.

Every connected graph will contain at least one **subgraph** that is a tree.

Video
Trees and adjacency matrices

Worksheets
Shortest paths and trees

Minimum spanning tree

WORKED EXAMPLE 1 Identifying trees

For each adjacency matrix, draw the graph that corresponds to the matrix and **determine** if it is a tree.

a

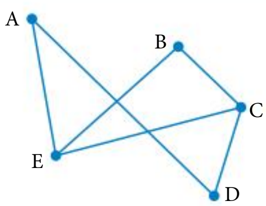
$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array}\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

b

$$\begin{array}{c} \\ P \\ Q \\ R \\ S \\ T \end{array}\begin{array}{c} \begin{array}{ccccc} P & Q & R & S & T \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Steps | **Working**

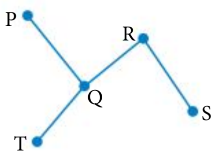
a **1** Draw the 5 vertices and corresponding edges.



2 The graph contains the cycle A–D–C–B–E–A.

The graph is not a tree.

b **1** Draw the 5 vertices and corresponding edges.



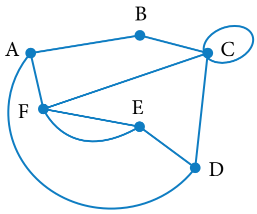
2 This graph is connected and contains no loops, cycles or multiple edges.

This graph is a tree.

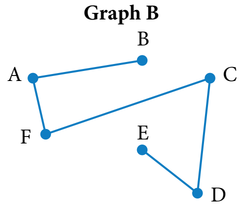
A **spanning tree** is a tree subgraph that connects all the vertices of the original graph. Every connected graph has at least one spanning tree. The spanning tree does not contain any edge that is not in the original graph.

Consider Graph A below. Graphs B and C are both subgraphs and spanning trees of Graph A.

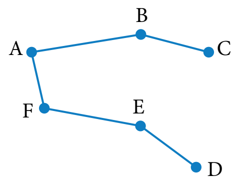
Graph A



Graph B



Graph C



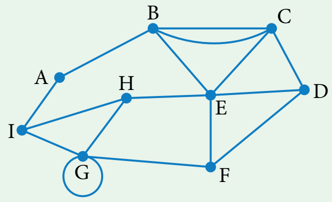
spanning trees

WORKED EXAMPLE 2 Identifying spanning trees

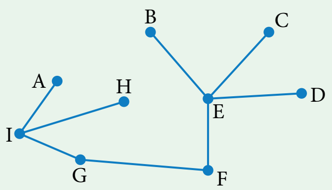
Which of the following graphs are a spanning tree of the graph shown on the right?

For those that are spanning trees, verify that the number of edges is one less than the number of vertices.

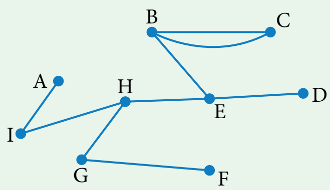
For those that aren't spanning trees, provide a reason.



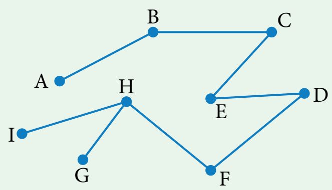
a



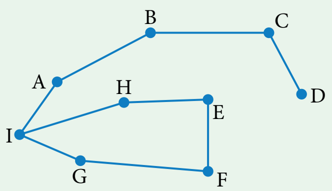
b



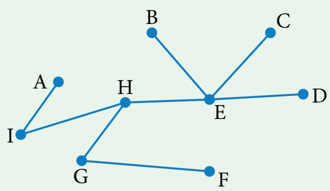
c



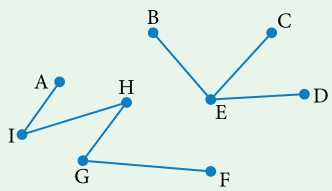
d



e



f



Steps	Working
For each graph, consider the following:	**a** This graph is a spanning tree.
Is it connected?	It has 9 vertices and 8 edges.
Does it have all the vertices of the original graph?	**b** This graph is not a spanning tree.
Does it have no loops?	It has 2 edges between B and C.
Does it have no multiple edges between any	**c** This graph is not a spanning tree.
2 vertices?	It has an edge, HF, that isn't in the original graph.
Does it have no cycles?	**d** This graph is not a spanning tree.
	It has a cycle: I–H–E–F–G–I.
	e This graph is a spanning tree.
	It has 9 vertices and 8 edges.
	f This graph is not a spanning tree.
	It is not connected.

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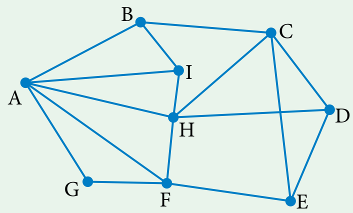
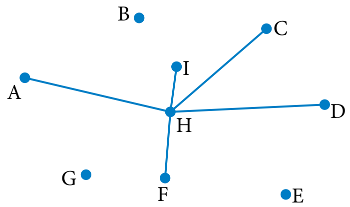
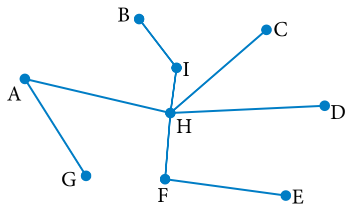
WORKED EXAMPLE 3 Creating a spanning tree

Draw a spanning tree for this graph.



Steps	Working
1 Start at any vertex, say H. Now, connect H to every vertex it is adjacent to.	
2 Continue connecting adjacent vertices until all vertices are connected. Make sure that there are no cycles. Hint: Another way of creating a spanning tree is to remove any edges that form a cycle, one at a time.	

Trees

- A tree is a connected graph with no loops, multiple edges between any adjacent vertices, or cycles.
- The number of edges in a tree is always one less than the number of vertices.

Spanning trees

- A spanning tree is a tree subgraph that connects all the vertices of the original graph.
- Every connected graph has at least one spanning tree.

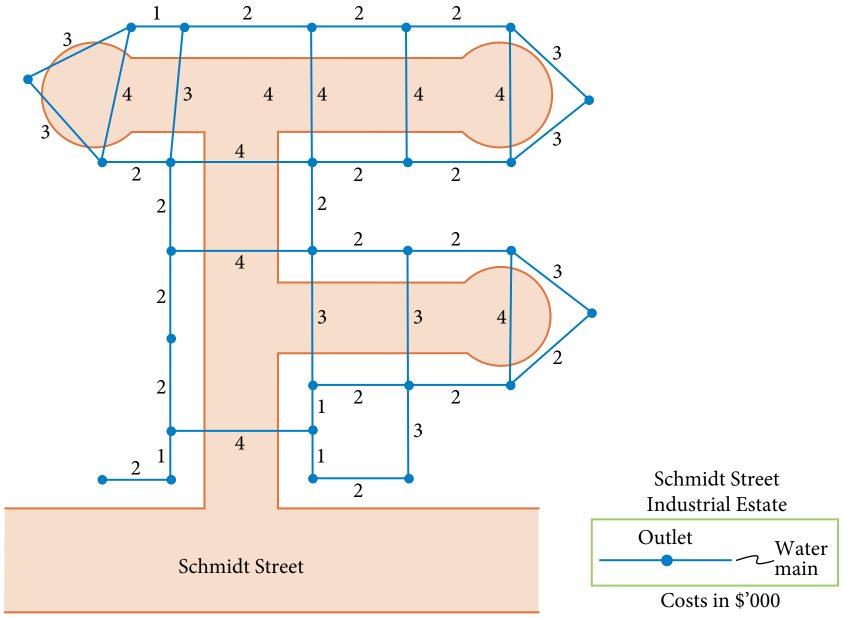
Connector problems

For any graph, there may be more than one possible spanning tree. In a weighted graph, if the weight of every spanning tree is found, the tree or trees with the smallest weight is called the **minimum spanning tree**.

Connector problems involve situations where you (generally) want to minimise the connections between all vertices for an overall minimum total weight. To solve a minimum connector problem, you need to find the minimum spanning tree. Minimum spanning trees are often used when connecting services such as water, electricity and NBN to homes, or in software such as real-time facial recognition.

Investigation Water services

The network shows the possible water mains layout for an industrial estate. The cost of laying each part is shown next to the section. Find the cheapest way to connect the pipes so that every location (outlet) has water.



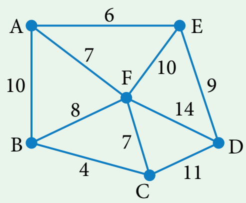
Can you work out a strategy for finding the cheapest method?

Minimum spanning trees

A minimum spanning tree for a network is a spanning tree with the total weight of its edges at a minimum. All vertices in the original network are connected, but the total weight of the reduced network is a minimum.

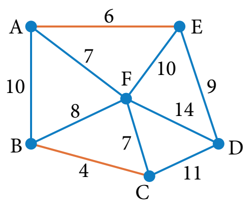
WORKED EXAMPLE 4 Finding the minimum spanning tree by trial-and-error

Consider the network.



a **Use** trial-and-error to identify the minimum spanning tree for this network.

b **Calculate** the total weight of the minimum spanning tree.

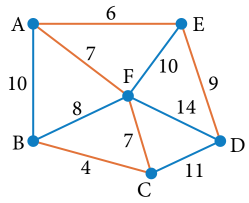
Steps	Working
a **1** List the edges in ascending order of weight (length).	BC = 4, AE = 6, AF = 7, FC = 7, FB = 8, ED = 9, AB = 10, EF = 10, CD = 11, FD = 14
2 Select the shortest edge, BC = 4. Next, select the next shortest edge that does not form a cycle, AE = 6.	

3 Now, select the next shortest edge that does not form a cycle, AF = 7.

Repeat the procedure to select FC and finally ED.

Check that all vertices have been included.

The tree is the minimum spanning tree.



4 If possible, state the answer. — The minimum spanning tree is BC, CF, FA, AE, ED.

b 1 Calculate the total weight of the minimum spanning tree.

Distance of minimum spanning tree

= BC + CF + FA + AE + ED

= 4 + 7 + 7 + 6 + 9

= 33

2 State the result. — The total weight of the minimum spanning tree is 33.

For small networks, the minimum spanning tree can be found by trial-and-error, however for larger networks an algorithm should be used.

Prim's algorithm

Prim's algorithm provides a series of steps to follow to find a minimum spanning tree.

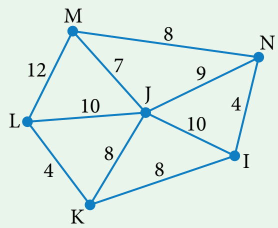
Video
Prim's algorithm

Prim's algorithm

1. Choose any vertex to be the first vertex of the tree. Identify the edge with the lowest weight connected to the first vertex. Add this edge and its other vertex to the first vertex to form a tree.
2. Consider *all* the edges connected to the tree. Select the edge with the lowest weight that is not already in the tree. Add this edge and its vertex to the tree. If there are several edges with the same lowest weight, choose one of them.
3. Repeat step 2 until all vertices in the original graph are included in the minimum spanning tree.

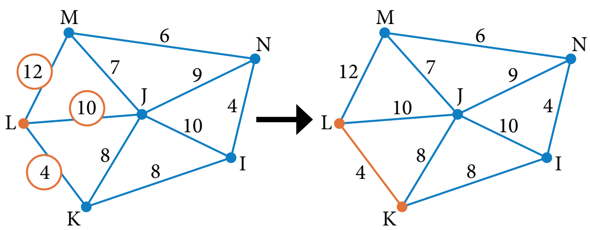
WORKED EXAMPLE 5 Prim's algorithm

Use Prim's algorithm to **determine** the minimum spanning tree for this graph and **calculate** its total weight.



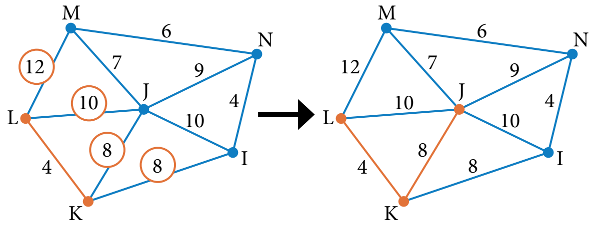
Steps | **Working**

1 Choose any vertex – say L. Select the edge with the lowest weight (LK) that connects to it and highlight it.

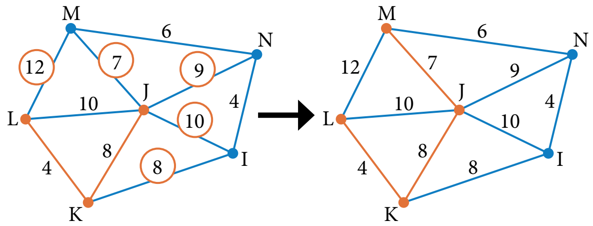


Hint: It doesn't matter which vertex is the starting vertex as the spanning tree will end up with all vertices in it.

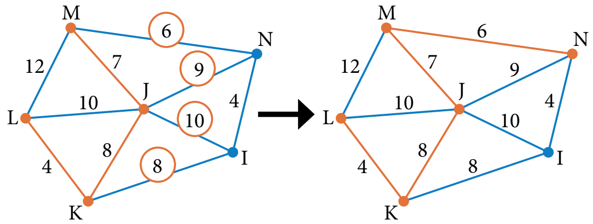
2 Look at all the edges connected to L and K. Select the edge with the lowest weight (KJ or KI as they have the equal lowest weight) and highlight it.



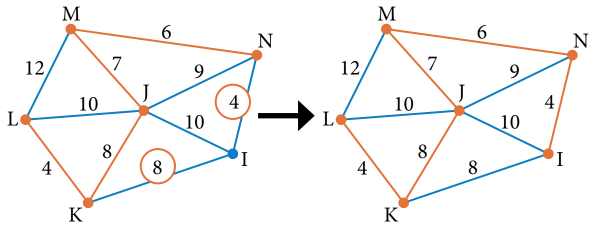
3 Look at all the edges connected to L, K and J. Select the edge with the lowest weight (JM) and highlight it.



4 Look at all the edges connected to L, K, J and M. Select the edge with the lowest weight (MN) and highlight it.



5 With the next step all vertices will be selected. Look at all the edges connected to L, K, J, M and N. Select the edge with the lowest weight (NI) and highlight it.



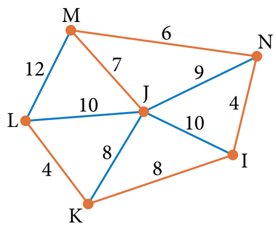
6 State the minimum spanning tree.

The minimum spanning tree is L–K–J–M–N–I.

7 Find the total weight of the minimum spanning tree.

The total weight of the minimum spanning tree is $4 + 8 + 7 + 6 + 4 = 29$.

Recall that in step 2, edge KJ was chosen over KI. If KI had been selected, the minimum spanning tree would have been slightly different BUT have had the same weight.



EXERCISE 9.1 Minimum spanning trees

ANSWERS p. 462

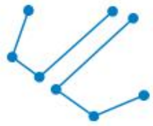
Mastery

1 Determine whether each graph is a tree, giving reasons.

a

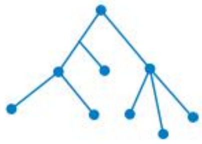
b

c

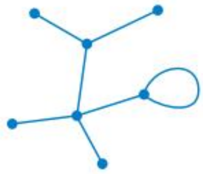


d

e



f



2 WORKED EXAMPLE 1 For each adjacency matrix shown, draw its corresponding graph and determine if the graph is a tree.

a
$$\begin{array}{cc} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

b
$$\begin{array}{cc} & \begin{array}{cccc} P & Q & R & S \end{array} \\ \begin{array}{c} P \\ Q \\ R \\ S \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

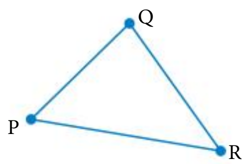
c
$$\begin{array}{cc} & \begin{array}{cccc} E & F & G & H \end{array} \\ \begin{array}{c} E \\ F \\ G \\ H \end{array} & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

d
$$\begin{array}{cc} & \begin{array}{ccccc} E & F & G & H & I \end{array} \\ \begin{array}{c} E \\ F \\ G \\ H \\ I \end{array} & \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

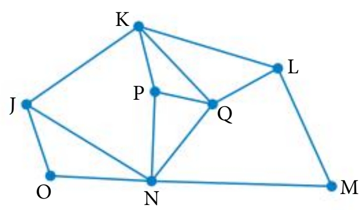
e
$$\begin{array}{cc} & \begin{array}{ccccc} U & V & W & X & Y \end{array} \\ \begin{array}{c} U \\ V \\ W \\ X \\ Y \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

f
$$\begin{array}{cc} & \begin{array}{cccccc} K & L & M & N & O & P \end{array} \\ \begin{array}{c} K \\ L \\ M \\ N \\ O \\ P \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

3 WORKED EXAMPLES 2, 3 Many graphs have several different possible spanning trees. Sketch all the distinct spanning trees for this graph.



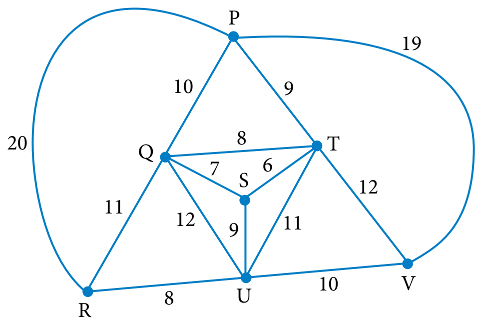
4 **a** Draw a spanning tree for this graph.



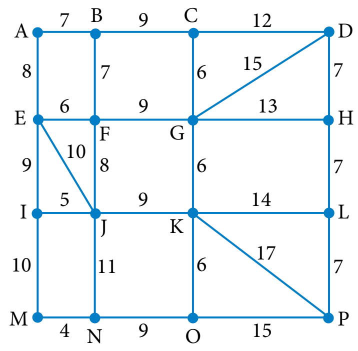
b Use the spanning tree to name the edges in the original graph that are not required to connect all vertices.

5 WORKED EXAMPLE 4 Use trial-and-error to find the minimum spanning tree for each graph.

a

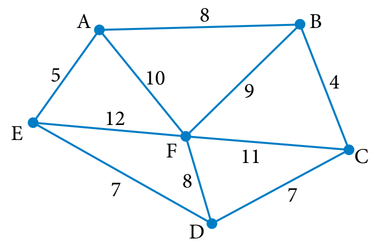


b



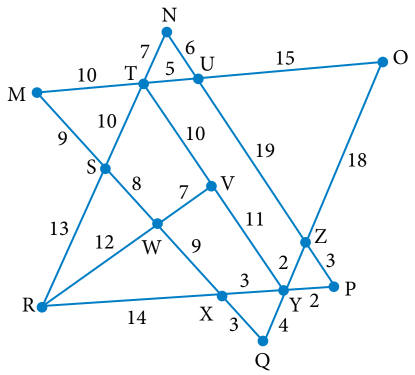
6 **a** Use trial-and-error to identify the minimum spanning tree for this graph.

b What is the total weight of the minimum spanning tree?

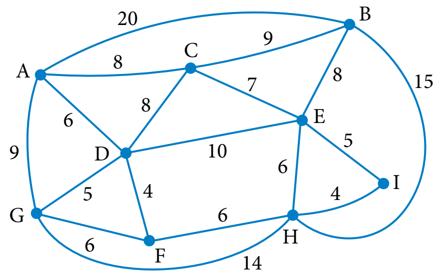


7 WORKED EXAMPLE 5 Use Prim's algorithm to find a minimum spanning tree for each graph.

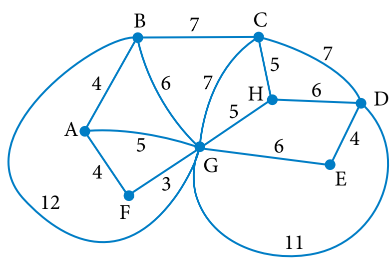
a



b

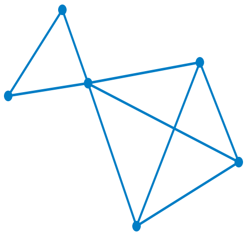


8 This graph shows the distance in kilometres between bus stops in a town. Use Prim's algorithm to find a minimum spanning tree for this network and state its length.

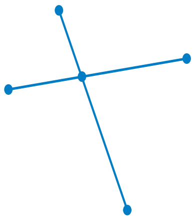


9780170484824

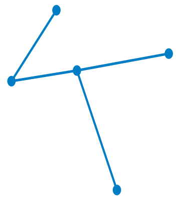
9 Which of the graphs **A–D** is not a spanning tree for the graph below?



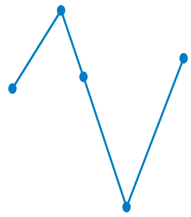
A



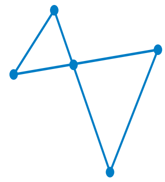
B



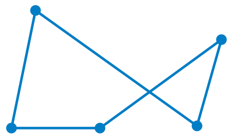
C



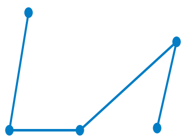
D



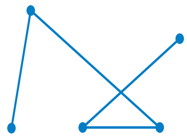
10 Which of the graphs **A–D** is not a spanning tree for the graph below?



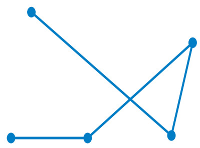
A



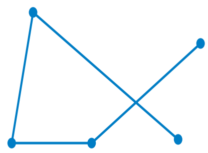
B



C

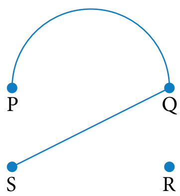


D

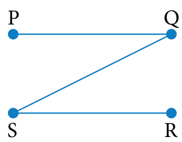


11 © QCAA 2022 1Q9 73% Identify the graph that is a spanning tree.

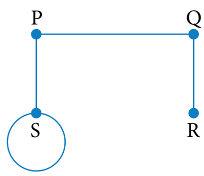
A



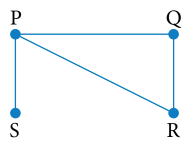
B



C

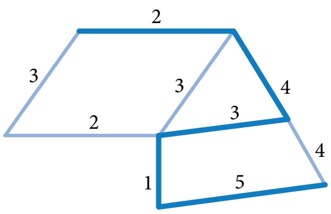


D

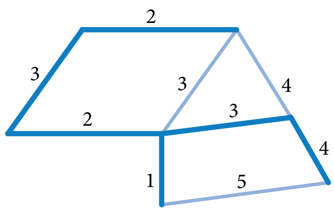


12 © QCAA 2020S 1Q1 Identify a minimum spanning tree from the diagrams below.

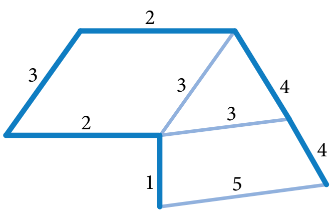
A



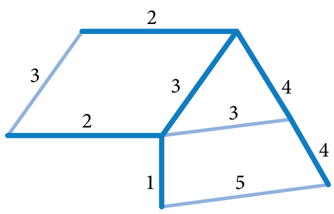
B



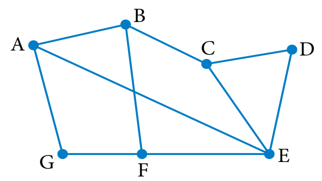
C



D



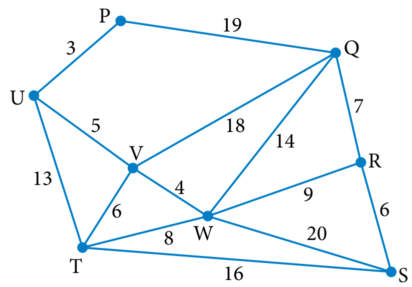
13 (2 marks) Name 4 edges that could be removed from this graph to form a spanning tree.



14 (2 marks) Draw the graph that corresponds to the adjacency matrix and determine if the graph is a tree.

$$\begin{array}{c} \\ P \\ Q \\ R \\ S \\ T \end{array} \begin{array}{c} \begin{array}{ccccc} P & Q & R & S & T \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

15 (2 marks) Find the minimum spanning tree for this graph and state its total weight.



9.2 Practical problems with minimum spanning trees

Practical applications of spanning trees are all around you. Applications include bioinformatics, Geographic Information Systems (GIS), transport networks, water distribution networks and electrical grids.

> Practical applications can involve both minimum and maximum spanning trees. You can easily adjust Prim's algorithm to find the maximum spanning tree by identifying the longest edges rather than the shortest.

WORKED EXAMPLE 6 Solving practical problems using Prim's algorithm

The sewerage pipes between 7 houses in an estate need to be replaced. The table summarises the length, in metres, of the sewer pipes that connect the houses (A, B, C, D, E, F and G). The replacement sewerage pipes need to link each house.

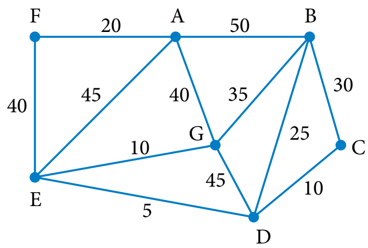
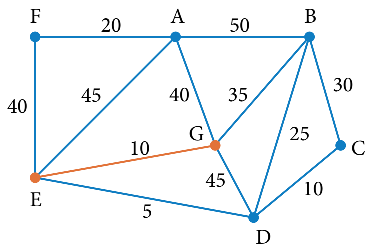
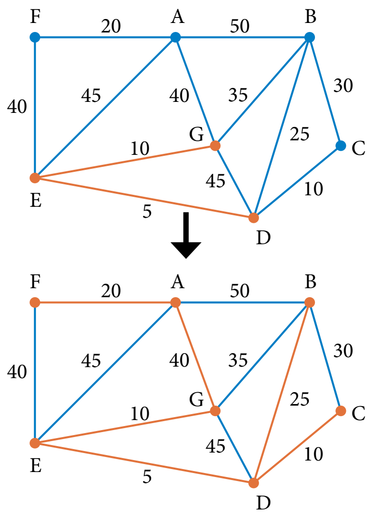
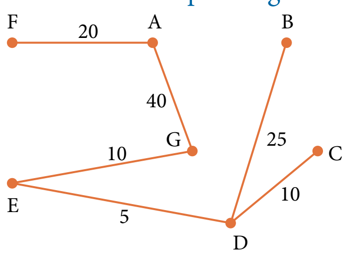
House	A	B	C	D	E	F	G
A		50			45	20	40
B	50		30	25			35
C		30		10			
D		25	10		5		45
E	45			5		40	10
F	20				40		
G	40	35		45	10		

a **Determine** the minimum spanning tree for this situation.

b **Calculate** the minimum length of sewerage pipe required to connect all the houses.

Exam hack

Problems will not always tell you to use Prim's algorithm. You can use it in any situation involving minimum spanning trees unless told to use trial-and-error.

Steps	Working
a 1 Draw a graph of the situation.	
2 Choose a starting vertex, say G. Select the edge with the lowest weight connected to it (GE) and highlight it to form a tree.	
3 Consider **all** the edges connected to the tree. Select the edge with the lowest weight that is not already in the tree (ED). Add this edge and its vertex to the tree. Repeat this step until all vertices in the original graph are included in the minimum spanning tree.	
4 Identify the minimum spanning tree.	The minimum spanning tree is 

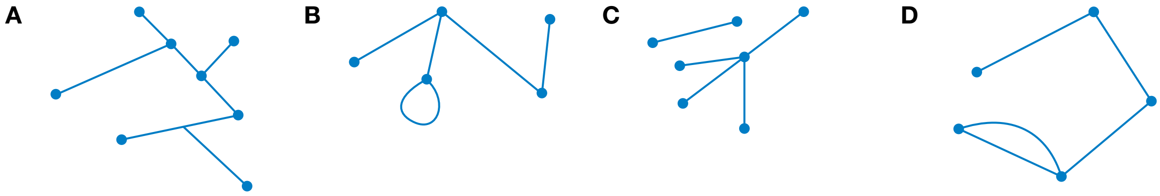
b **1** Find the minimum length of sewerage pipe by adding all the edges in the minimum spanning tree.	length = 20 + 40 + 10 + 5 + 25 + 10 = 110 m
2 State the minimum length.	The minimum length of sewerage pipe required to connect all the houses is 110 m.

EXERCISE 9.2 Practical problems with minimum spanning trees

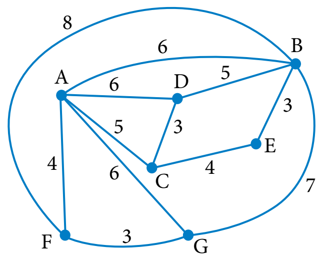
ANSWERS p. 463

Recap

1 Identify the graph that is a tree.



2 **a** Identify the minimum spanning tree for this network.

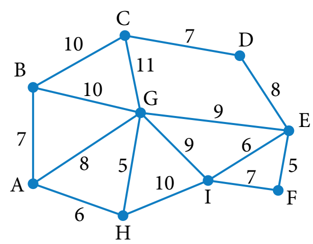


b Determine the total weight of the minimum spanning tree from part **a**.

c Name the edges in the original graph that are not required to connect all the vertices.

Mastery

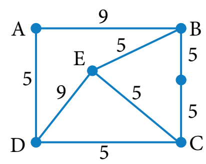
3 WORKED EXAMPLE 6 This network shows the distances, in metres, between powered caravan sites at a caravan park. The vertices represent the caravan sites.



The minimum length of electricity cable required to connect all the powered caravan sites is

A 48 m **B** 53 m **C** 55 m **D** 60 m

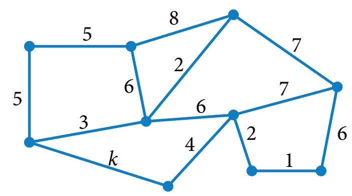
4 Consider the weighted graph which shows the distance, in metres, between houses in a community.



How many different minimum spanning trees are possible?

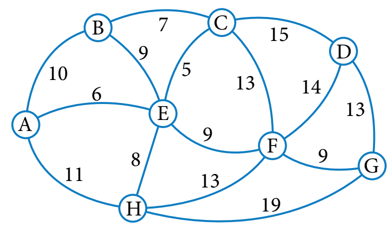
A 1 **B** 2 **C** 3 **D** 4

5 The network shows the distance in metres between viewing points on a sensory trail in a local nature reserve. The minimum spanning tree for the network includes the edge with weight labelled k.
If the total weight of all edges for the minimum spanning tree is 33 m, calculate the value of k.



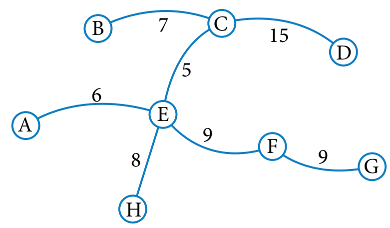
Exam practice

6 (5 marks) A network of hiking tracks connects eight points across Mt Cootha, as shown. The number on each edge represents the time, in minutes, that an average hiker should take to hike along each track.



a What is the shortest path and hence time an average hiker should take to go from point B to point G? *[3 marks]*

b A spanning tree of the hiking network has been created. Determine if the tree is a minimum spanning tree. Justify your answer. *[2 marks]*



7 © QCAA 2020S 2Q2 (4 marks) CF A cable company is calculating the cost of laying cable between five houses (A, B, C, D and E). The unit cost of laying the cable is $2.50 per metre.

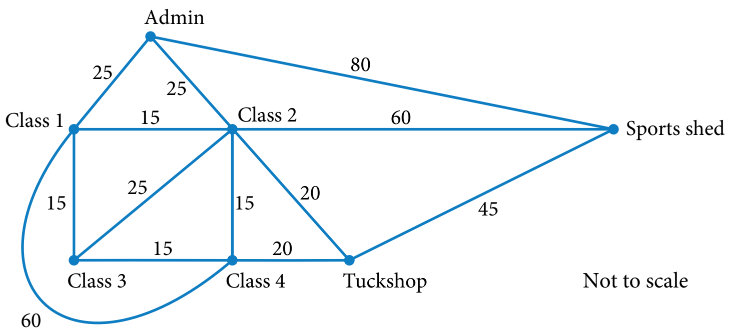
The shortest distances (in metres) between the houses are shown below.

	A	B	C	D	E
A	0	20	400	35	8000
B	20	0	800	70	3000
C	400	800	0	500	2000
D	35	70	500	0	8000

a Construct a network diagram to represent the distances between the houses *[1 mark]*

b Determine the minimum cost to connect all the houses. *[3 marks]*

8 © QCAA 2021 2Q2 (4 marks) CF All buildings in a school are connected by underground electricity cables, indicated by the network. All measurements are in metres.

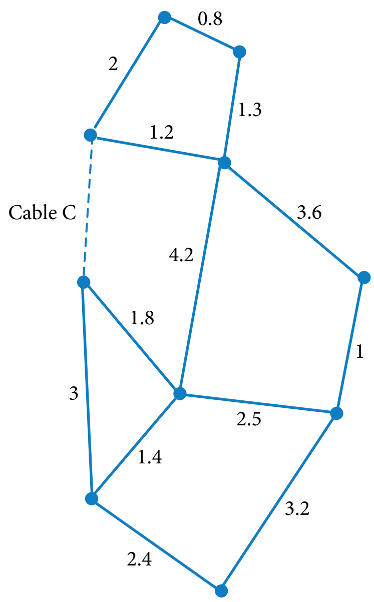


The electricity cables need replacing and will cost $1200 per metre. The school wants to minimise costs by replacing the shortest length of cable necessary to connect all buildings.

If the school has a budget of $155 000, evaluate whether they can afford this project.

9 © QCAA 2023 2Q3 (5 marks) CF The diagram represents a network of 10 ski stations connected by chairlift cables. The length (km) of each cable is shown, except for cable C, which is closed for maintenance. When cable C reopens, the minimum total cable length required to connect all stations will decrease by 1 km.

Determine the length of cable C and the minimum total cable length required to connect all stations when cable C reopens.



10 (5 marks) CU A plumbing company is calculating the minimum length of pipe required to supply eight camp sites (S, T, U, V, W, X, Y and Z) in a national park with drinking water. The main water source is at S. The waterpipes can only be laid alongside walking tracks.

The shortest distances (in kilometres) alongside walking tracks connecting campsites are shown below.

	S	T	U	V	W	X	Y	Z
S		4						
T	4		6			7		
U		6		8	4	5		
V			8		5			6
W			4	5		2		5
X		7	5		2		1	
Y						1		9
Z				6	5		9	

A pipe cannot be laid alongside the track connecting U to W.
What is the shortest path the pipes can be laid from U to Z avoiding the path connecting U to W?

11 (4 marks) CU The government is looking to replace the existing powerlines between 6 areas in Southeast Queensland. All of the areas must be connected. They asked a contractor to provide a quote for the cost of each build. The contractor provided the following table which identifies the 6 areas and the cost, in millions of dollars, to build new powerlines connecting each area.

Area	Brisbane	Ipswich	Logan	Moreton Bay	Redlands	Somerset
Brisbane		7	6	9	6	11
Ipswich	7		9			12
Logan	6	9			8	
Moreton Bay	9				8	10
Redlands	6		8	8		
Somerset	11	12		10		

Cost = \$37 million (min.) to \$50 million (max.).

Determine where the contractor would suggest building the powerlines if they want to charge the maximum cost for the build and what the maximum cost would be.

Video
Project networks

Worksheets
Project networks

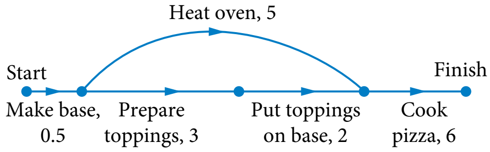
Drawing directed graphs

9.3 Project networks

Project network diagrams

Networks have many uses including situations that involve scheduling of tasks to complete a project. Tasks that are required to complete a project are called **activities**.

A **project network (diagram)** is a directed network that shows the sequence of activities and the time required for the activity's completion. For example, the project network on the right shows the activities required to prepare and cook a pizza.



The edges (arcs) of a project network represent the activity and its duration. They are arranged to show the order in which activities must be completed.

The **nodes** (vertices) represent the start and finish of a particular event.

An activity that must be completed immediately before another activity can start is called an **immediate predecessor** or **prerequisite**.

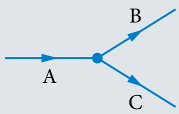
Types of connections in a project network

One preceding event

Event B is preceded by A.

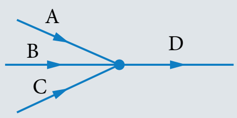
Multiple activities with the same preceding event

Events B and C are preceded by A.



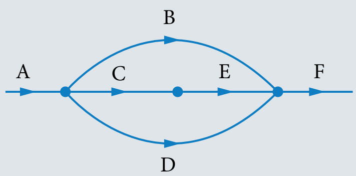
Multiple preceding events

Event D is preceded by A, B and C.



Concurrent events

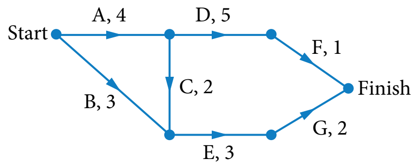
Event F is preceded by B, E and D. Activities B and D can occur at the same time and are called **concurrent activities**.



Activity tables

An **activity table** shows the order and estimated completion time of each activity. Often activities are coded by a letter. The activity table below shows 7 activities (A to G) required for a project. The project network diagram corresponding to the activity table is shown next to it.

Activity	Duration (days)	Immediate predecessors
A	4	—
B	3	—
C	2	A
D	5	A
E	3	B, C
F	1	D
G	2	E



Note: '-' indicates no predecessor

Guideline to follow when drawing a project network diagram

- The start of the network must be at a vertex labelled 'Start'.
- Identify activities that do not have predecessors. These are the starting activities and connect to the 'Start' vertex.
- Multiple predecessors to an activity will end at the same vertex.
- An activity must only be represented by one arc in the network.
- 2 vertices can only be connected by a single arc.
- The network must end at a vertex labelled 'Finish', which shows the completion of the project.

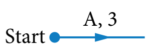
WORKED EXAMPLE 7 Drawing a project network from an activity table

Draw a project network for the activity table.

Activity	Time (min)	Immediate predecessor
A	3	None
B	2	A
C	7	A
D	5	B
E	4	B
F	1	D
G	3	E
H	8	A
I	1	C
J	2	F, G, H, I

Steps | **Working**

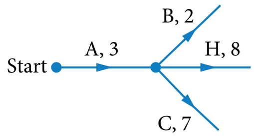
1 Activity A has no prerequisites.
Label the first vertex 'Start' and from it draw an arc labelled 'A, 3'.



When labelling activities, you do not have to use 'A, 3' you could use 'A 3'. In the exam, use the form given in the question.

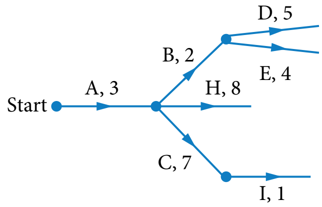
2 Activities B, C and H are all preceded by A.

Add a vertex at the end of arc A and from it draw 3 arcs labelled 'B, 2', 'H, 8' and 'C, 7'.



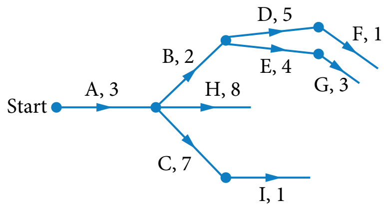
3 Activity D and E are preceded by B. Add a vertex at the end of arc B. From it draw 2 arcs for D and E and label them.

Activity I is preceded by C. Add a vertex at the end of arc C, draw in an arc for I and label.



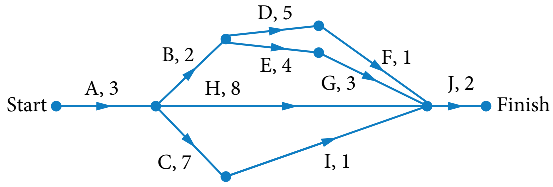
4 Activity F is preceded by D. Add a vertex at the end of arc D, draw in an arc for F and label.

Activity G is preceded by E. Add a vertex at the end of arc E, draw in an arc for G and label.



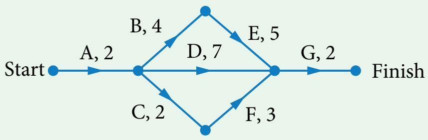
5 Activity J is preceded by F, G, H and I. This means activities F, G, H and I all end at the same vertex. Add the end vertex, draw in an arc for J and label.

Add a vertex at the end of arc J and label 'Finish'.



WORKED EXAMPLE 8 Constructing an activity table from a project network

Construct an activity table for the project network shown.



Steps

1 Create a table with a row for every activity and add in the activity durations.

2 For every activity, look at what its immediate predecessor(s) are and note these in the table.

Working

Activity	Time (min)	Immediate predecessor
A	2	–
B	4	A
C	2	A
D	7	A
E	5	B
F	3	C
G	2	D, E, F

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EXERCISE 9.3 Project networks

ANSWERS p. 463

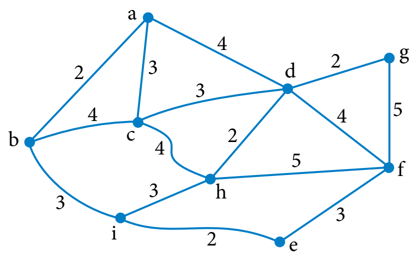
Recap

1 A large open plains wildlife sanctuary is setting up driving trails between observation points where visitors can see different animals. The table below shows the distances, in kilometres, between these observation points.

	A	B	C	D	E	F
A		4	6			
B	4		5	7		
C	6	5		8	9	
D		7	8		6	5
E			9	6		7
F				5	7	

Determine the minimum total distance of driving trails required to connect all the observation points.

2 The water pipes between 8 regional towns (a, b, c, d, e, f, g, h and i) need to be replaced. The graph shows the distance in kilometres between the towns. Replacement water pipes need to link each town.

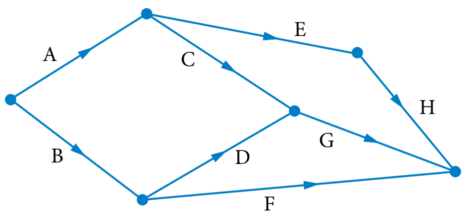


a Identify the minimum spanning tree for the graph.

b What is the minimum length of water pipe required to connect all the towns?

Mastery

Refer to this diagram for Questions 3 and 4.



3 Which of the following statements is true?

A Activity A is an immediate predecessor of H.

B Activity C is an immediate predecessor of G.

C Activity G must be done before activity D.

D Activity H must be done before activity A.

4 Which of the following statements is true?

A Activities A and C are both prerequisites of G.

B Activities A and B must be done before activity C.

C Activities C and E must be done before activity H.

D Activities C and D are both prerequisites of G.

5 WORKED EXAMPLE 7 Draw a project network for each activity table.

a

Activity	Time	Prerequisites
A	3	None
B	2	A
C	5	None
D	6	B, C

b

Activity	Time	Prerequisites
A	4	None
B	3	None
C	6	B
D	5	A
E	2	D, C

c

Activity	Time	Prerequisites
A	5	None
B	4	A
C	6	B
D	7	A
E	5	D
F	2	D
G	3	F
H	2	C
I	4	H, E, G

d

Activity	Time	Prerequisites
A	2	None
B	1	A
C	5	A
D	3	A
E	8	B
F	3	D
G	6	F
H	7	C
I	4	E
J	3	G, H, I

6 Draw a project network for preparing and serving breakfast in bed for carers day.

Code	Activity	Time (min)	Prerequisites
A	Cut bread for toast	1	None
B	Make toast	3	A
C	Spread toast	1	B
D	Fetch newspaper	2	None
E	Boil water in jug	5	None
F	Make tea	1	E
G	Cook sausages and eggs	15	None
H	Put crockery, cutlery and newspaper on tray	2	D
I	Put food on tray	2	C, F, G, H
J	Take tray to bedroom	1	I

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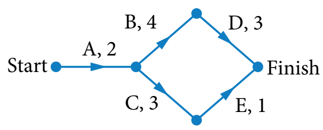
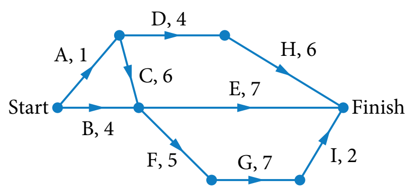
7 WORKED EXAMPLE 8 Construct an activity table for the project networks, where the time is in days.

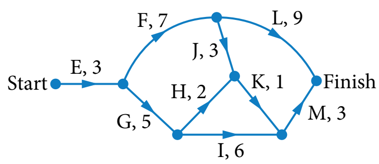
a



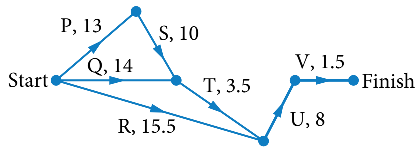
b



c

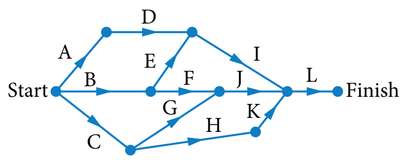


d



Exam practice

Use the project network diagram below to answer Questions 8, 9 and 10.



8 How many activities in the project network have exactly 2 immediate predecessors?

A 0 **B** 1 **C** 2 **D** 3

9 What are the predecessors of L?

A I, J **B** B, F, J **C** I, J, K **D** I, J, M

10 The total number of activities that need to be completed before J is

A 2 **B** 4 **C** 6 **D** 8

11 (3 marks) Caterers have been booked for an event. The table shows the required activities, together with the times taken (in hours) and the immediate predecessors for each activity.

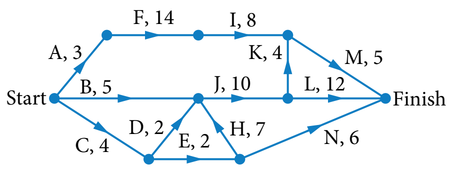
Create a project network diagram for the event showing all activities and durations.

Activity	Time (hours)	Prerequisites
A	2	None
B	3.5	None
C	3	None
D	1.5	A
E	1	B, D, F
F	2	C
G	2	E, I
H	3	A
I	4	C

12 (4 marks) CF The activities for an IT project are summarised in the table below.

Activity	A	B	C	D	E	F	G	H	I	J	K	L	M
Duration (days)	3	5	4	2	2	14	5	7	8	10	4	12	5
Predecessor(s)	–	–	–	C	C	A	B, D, H	E	F, G	B, D, H	J	J	I, K

The project network diagram for the project was produced and is shown below. It has been identified that there are several errors or omissions in the project network.



Identify the errors or omissions and justify any changes that need to be made to the project network diagram to correct these issues.

Videos
Forward scanning

Latest starting times

Worksheets
Calculating EST and LST

Critical paths

Critical paths and activity float times

Critical path analysis

9.4 Critical path analysis

Projects usually involve multiple activities that need to be completed in a certain amount of time. **Critical path analysis** is the process of analysing project activities and how each activity affects the completion time for the project. This enables the overall minimum time to complete the project to be determined.

Forward scanning to determine the EST

Forward scanning through a network enables you to determine the **earliest starting time (EST)** for all activities in the network. The EST is the earliest possible time to start an activity based on prerequisite activities.

Forward scanning process to find the EST

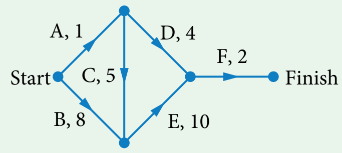
To determine the ESTs:

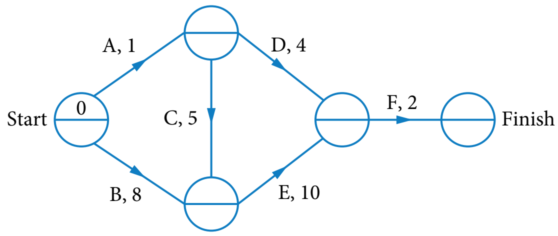
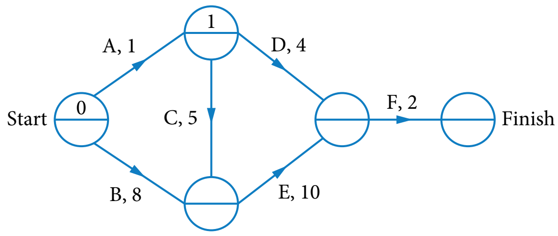
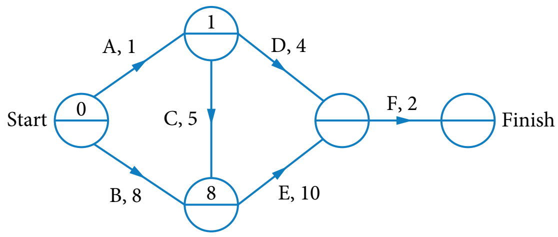
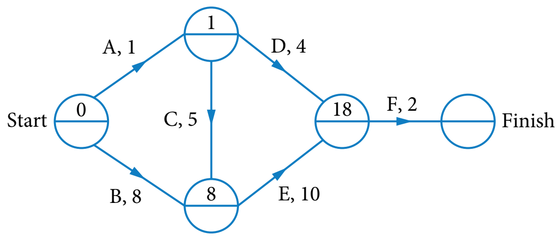
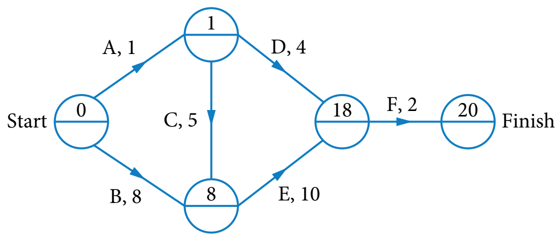
1. Change each vertex in the network to an open circle with a horizontal diameter, ⊖. (Alternatively, a circle with a vertical diameter ⦶ or a double box can be used ⊟.)
2. The ESTs will be calculated from left to right in the network.
3. In the circle at the start of the network, put 0 in the top semicircle. This represents the start of the project. It also represents the EST. The EST for all activities will be written in the top semicircle.
4. Add the activity time to the EST of the previous vertex to find the EST of the next activity. If multiple activities lead into a vertex, the highest total time becomes the EST.
5. Continue until the finish circle of the network is reached.

WORKED EXAMPLE 9 Finding the earliest starting times

Determine the earliest starting times (EST) for each activity in the network shown.

Activity times shown are in minutes.



Steps	Working
1 Draw an open circle with a horizontal diameter at each vertex. Put a 0 as the EST at the start vertex.	
2 To find the EST for activity C and D, add the activity time for A to the EST for the start vertex. This gives an EST for C and D of 1 minute. This is written in the top of the semicircle for the start of those activities.	
3 Activity E has 2 prerequisites, activity B and C. Adding the activity time for C gives an EST for E as 1 + 5 = 6. Adding the activity time for B gives an EST for E of 8. Use the higher value as the EST.	
4 Activity F has 2 prerequisites, activity D and E. Adding the activity time for D gives an EST for F as 1 + 4 = 5. Adding the activity time for E gives an EST for F of 18. Use the higher value.	
5 To calculate the EST for the Finish, find the total for the path including activity F. Activity F = 18 + 2 = 20. 20 minutes is the final EST.	

The 'Finish' EST in a project network represents the overall minimum completion time for the project. In the above example, the minimum completion time is 20 minutes.

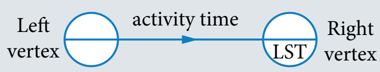
Backward scanning to determine the LST

Backward scanning is the process that enables you to determine the **latest starting time (LST)** for all activities in a network. The LST is the latest possible time an activity can start without delaying the completion of the project.

Backward scanning process to find the LST

The LST value for any activity can be determined by:

LST for an activity (Left vertex) = LST at Right vertex – activity time



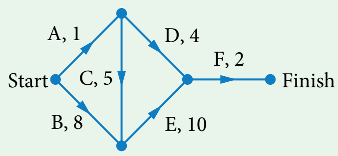
To determine the LSTs:

1. Begin with a network in which the EST for each activity has been calculated.
2. The LSTs will be calculated from right to left in the network.
3. Start the LST calculations at the 'Finish' vertex. At the "Finish' vertex, LST = EST. This value is written in the bottom semicircle for the last vertex. The LST for all activities will be written in the bottom semicircle.
4. Working backwards, to find the LST at the left vertex, subtract the activity time from the LST at the right vertex. If multiple activities join at a vertex (left vertex), the lowest total time becomes the LST.
5. Continue until the start of the network is reached. The LST for the 'Start' vertex must be 0.

WORKED EXAMPLE 10 Finding the latest starting times

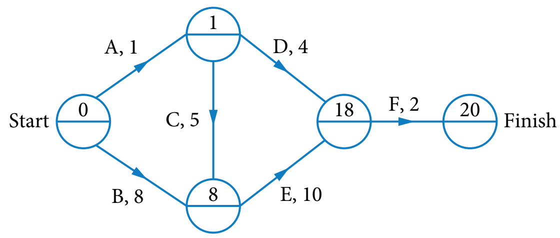
Determine the latest starting times (LST) for each activity in the network shown.

Activity times shown are in minutes.



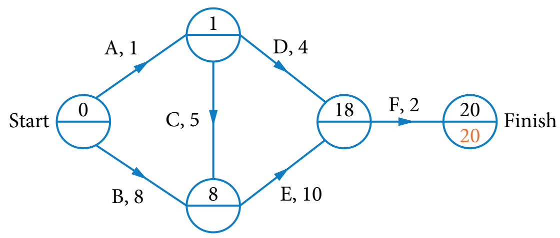
Steps | **Working**

1 Find the EST for each activity. These were found in Worked example 9.



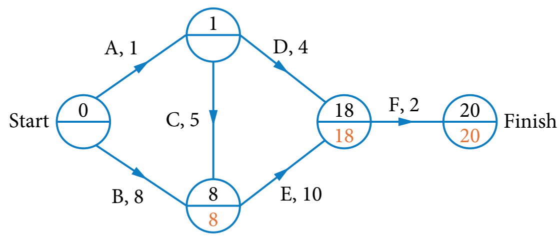
2 Work backwards from right to left in the network.

The LST for 'Finish' is equal to the EST.



3 The LST for activity F is 20 – 2 = 18. This is written in the bottom semicircle at the start of activity F.

The LST for activity E is 18 – 10 = 8.

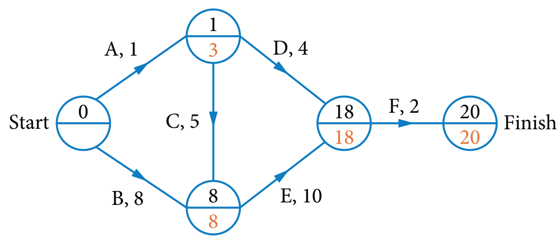


4 There are 2 paths backwards to the start of Activity C and D. The LST for each is:

LST for C = 8 – 5 = 3

LST for D = 18 – 4 = 14

The lowest total time of 3 is used.

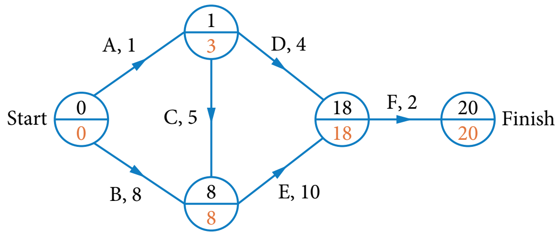


5 There are 2 paths backwards to the start of activity A and B. The LST for each is:

LST for A = 3 − 1 = 2

LST for B = 8 − 8 = 0

The lowest total time of 0 is used.

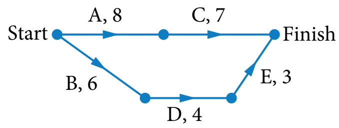


Identifying the critical path

There are often several different possible paths from the start to the finish vertices in a project network. The **critical path** is the longest path from start to finish and determines the project completion time.

The project network on the right shows 5 activities (A, B, C, D and E) and the time in hours to complete each activity.

There are 2 paths from start to finish: A–C and B–D–E.



The completion time for each path is: A–C = 8 + 7 = 15 hours and B–D–E = 6 + 4 + 3 = 13 hours.

This means that the critical path is A–C as it takes longer to complete than B–D–E. So, the project completion time is 15 hours.

The activities that are on the critical path are called the **critical activities**. All other activities are called **non-critical activities.**

The critical path

For any network, the critical path is the path with the longest total time.
It is used to determine the project completion time.

On a critical path, the activities have the same EST and LST values.

In simple networks the critical path can be found by trial-and-error. In more complex situations, the critical path can be found by using the forward and backwards scanning processes.

Process for finding the critical path

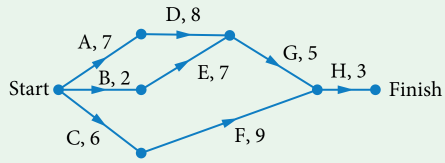
To determine the critical path:

1. Use forward scanning to determine the EST for each activity.
2. Use backward scanning to determine the LST for each activity.
3. Identify activities for which EST = LST. These activities form the critical path.

Video
Critical path analysis

WORKED EXAMPLE 11 Finding the critical path

This project network shows the required project activities and the time taken to complete them in hours.



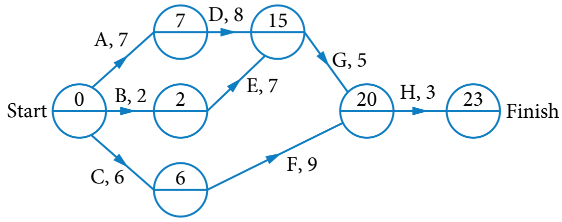
a **Determine** the EST and LST for each activity.

b **Identify** the critical path.

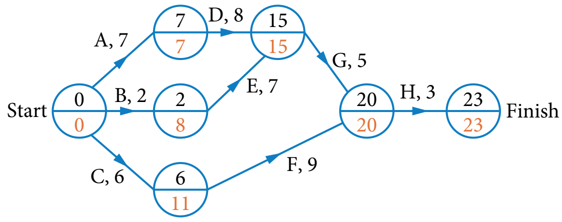
c **Determine** the shortest time taken to complete the project.

Steps | **Working**

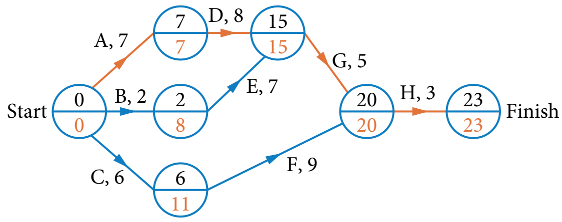
a 1 Determine the EST for each activity using the forward scanning process.



2 Determine the LST for each activity using the backwards scanning process.



b 1 Identify critical activities where EST = LST.



2 Identify the critical path.

The critical path is A–D–G–H.

c Identify the time taken for the critical path.

The critical path is the shortest completion time.

Critical path = 23 hours

The shortest time to complete this project is 23 hours.

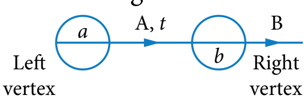
Activity float times

The critical path contains all the critical activities which cannot be delayed without causing the project completion time to be increased.

Non-critical activities may have some flexibility in the time they can be delayed without affecting the whole project. The amount of leeway before the activity impacts the project is called the **float time** (or **slack time**). Critical activities have float times of zero.

In a project where there are not multiple activities leading out from a vertex, the float time is simply the LST – EST. However, when project networks contain multiple activities coming from a vertex you must use a more general approach which will work in any situation.

For this diagram:



a = EST at left vertex, b = LST at right vertex

t = activity time for activity A

LST for activity A = $b - t$

float for activity A = $b - a - t$

$= \text{LST}_{\text{right}} - \text{EST}_{\text{left}} - \text{activity time}$

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9.4

Float time for project activities

The float time is the maximum time that an activity can be delayed or extended without affecting the project completion time.

- For non-critical activities: float = $LST_{right} - EST_{left}$ – activity time
- For critical activities: float = 0.

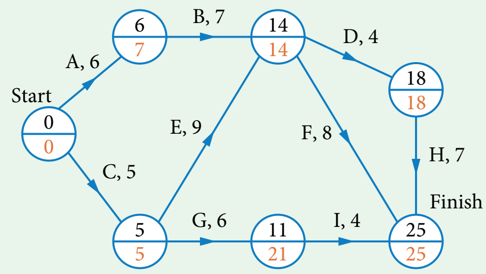
WORKED EXAMPLE 12 Finding the critical path and float times

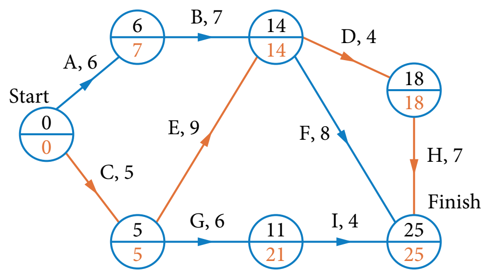
The forward scan and backward scan for this project network have been completed. The activity times are shown in days.

Determine

a the critical path

b the float times for all non-critical activities.



Steps	Working
a 1 Identify critical activities where EST = LST.	
2 Identify the critical path.	Critical path = C–E–D–H
b 1 Identify all non-critical activities.	Non-critical activities are A, B, F, G and I.
2 State the float time formula.	float = $LST_{right} - EST_{left}$ – activity time
3 Find the float time for each non-critical activity by substituting into the formula and evaluating.	Float for activity A = 7 – 0 – 6 = 1 day Float for activity B = 14 – 6 – 7 = 1 day Float for activity F = 25 – 14 – 8 = 3 days Float for activity G = 21 – 5 – 6 = 10 days Float for activity I = 25 – 11 – 4 = 10 days

Hint: Notice when looking at the graph that it appears float time only exists for A and G. However, multiple activities lead out of 3 vertices, meaning more non-critical activities may have float time.

Exam hack

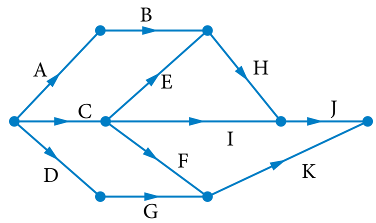
Always check the float time for all non-critical tasks using $LST_{right} - EST_{left}$ – activity time as it will ensure you calculate the correct float time.

EXERCISE 9.4 Critical path analysis

ANSWERS p. 464

Recap

1 Which statement is true about this network?



A Activities E and F are both prerequisites of K.

B Activities A and D must be done before activity G.

C Activities B and E are both prerequisites of H.

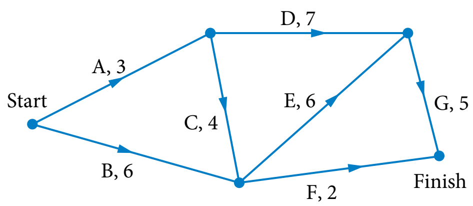
D Activities C and E must be done before activity H.

2 Draw a project network diagram for this activity table.

Activity	Time (days)	Prerequisites
A	3	None
B	6	None
C	5	None
D	4	A, B
E	7	C
F	3	C
G	2	D, E

Mastery

Use the project network below for Questions 3 to 6.



The project network shows activities A, B, C, D, E, F and G and the times for each activity in minutes.

3 What is the earliest starting time for activity G?

A 10 mins **B** 12 mins **C** 13 mins **D** 15 mins

4 What is the latest starting time for activity E?

A 6 mins **B** 7 mins **C** 8 mins **D** 10 mins

5 What is the earliest completion time for the project?

A 8 mins **B** 12 mins **C** 15 mins **D** 18 mins

6 If the time taken to complete activity A is increased to 5 minutes, how much will the completion time for the project change?

A 0 mins **B** 1 mins **C** 2 mins **D** 3 mins

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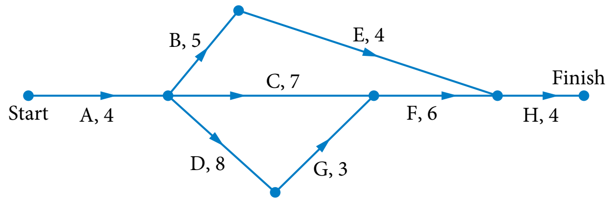
7 Use this project network diagram to find the earliest starting time for the vertex at



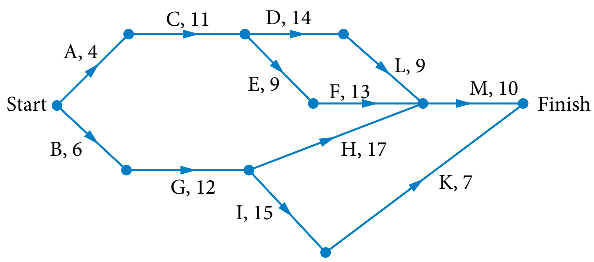
- **a** the start
- **b** the end of activity D
- **c** the start of activity F
- **d** the end of activities E and F.

8 WORKED EXAMPLES 9, 10 The activity table for a project is shown.

Activity	Time (hrs)	Prerequisites
A	4	None
B	10	None
C	7	A
D	12	None
E	8	B
F	4	C
G	5	F, D, E
H	7	F, D, E
I	3	G
J	6	H, I

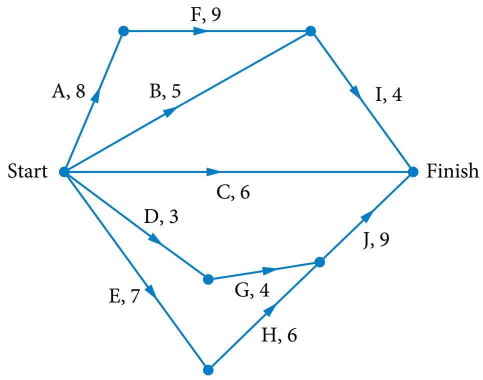
- **a** Draw the project network diagram that corresponds to this table.
- **b** Use forward scanning to calculate the earliest starting time for each activity.
- **c** Use backward scanning to calculate the latest starting time for each activity.

9 This project network shows the time taken, in days, for several activities.



- **a** Use forward scanning to calculate the earliest starting time for each activity.
- **b** Use backward scanning to calculate the latest starting time for each activity.
- **c** If the time taken for activity G increased to 15 days, how would this affect the earliest starting time for the last vertex?
- **d** If the time taken for activity H increased to 20 days, how would this affect the earliest starting time for the last vertex?

10 This project network shows the time taken in hours for a number of activities.



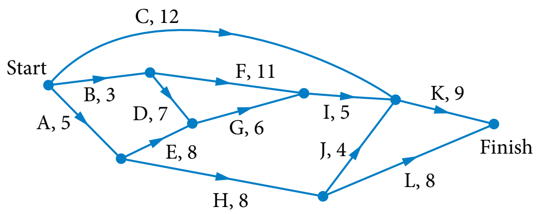
a Use forward scanning to calculate the earliest starting time for each activity.

b Use backward scanning to calculate the latest starting time for each activity.

c If the time taken for activity D increased to 8 hours, how would this affect the earliest starting time for the last vertex?

d If the time taken for activity H increased to 8 hours, how would this affect the time taken to complete the project?

11 WORKED EXAMPLE 11 This project network shows several activities and the time taken to complete them in weeks.



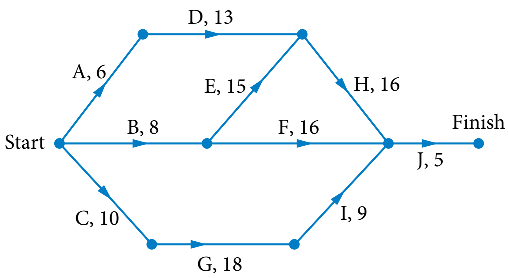
a Find the EST and LST for each activity.

b Identify the critical path.

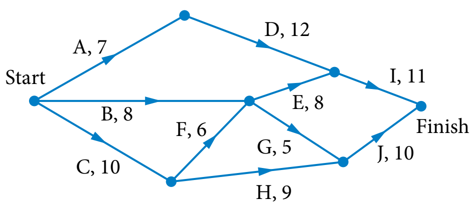
c Determine the shortest time taken to complete the project.

12 For each project network, find the critical path and shortest time taken to complete the project.

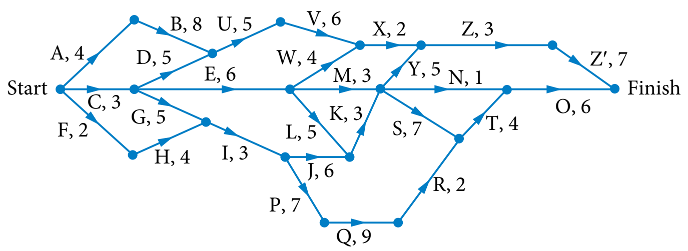
a



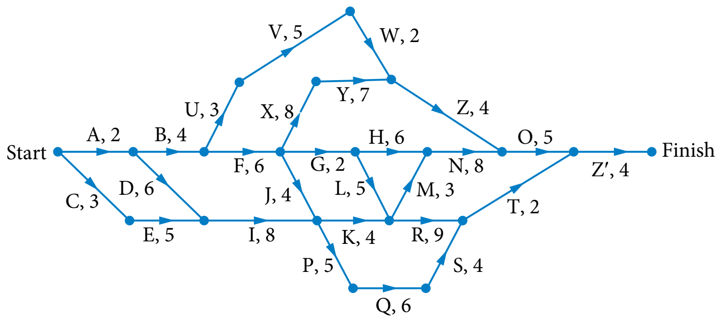
b



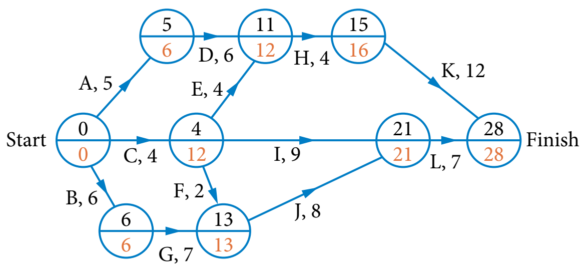
c



d



13 WORKED EXAMPLE 12 This project network shows project activities and associated times in days.



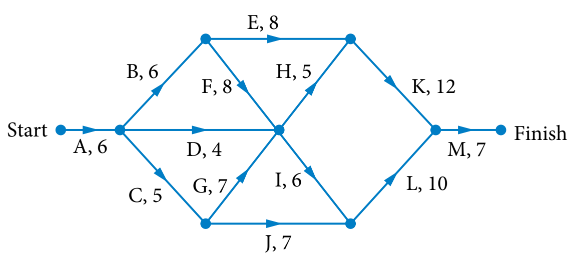
a Identify the critical path.

b Calculate the float time for

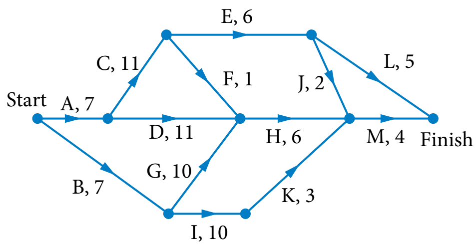
i K **ii** D **iii** C **iv** I

14 Use float times to find the critical path of each project network.

a

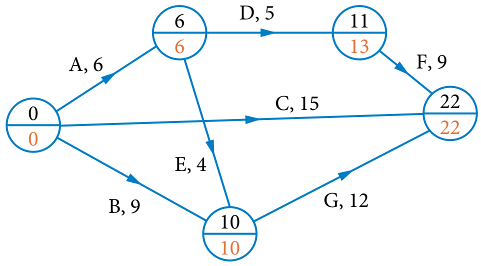


b



Exam practice

Use the project network shown for Questions 15 to 18. The times given are in days.



15 The critical path is

A A–D–F **B** A–E–G **C** B–E–D–F **D** A–E–D–F

16 The non-critical activities are

A only D **B** A and B **C** B, D, C and F **D** A, E and G

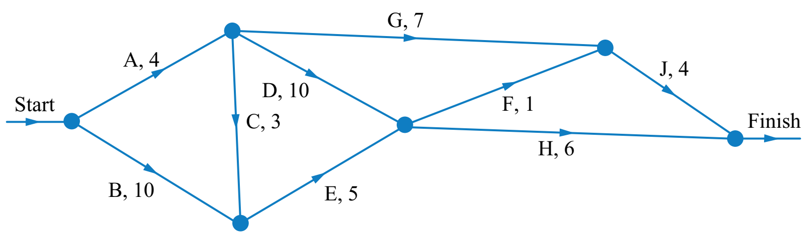
17 If activity E is delayed by 3 weeks, what is the shortest completion time for the project?

A 18 weeks **B** 22 weeks **C** 24 weeks **D** 25 weeks

18 If activity D is delayed by 2 weeks, what is the shortest completion time for the project?

A 15 weeks **B** 18 weeks **C** 22 weeks **D** 25 weeks

19 © QCAA 2020S 1Q8 Activities A, B, C, D, E, F, G, H and J must be completed for the project shown in the network. Activity times are in minutes.



The earliest starting time in minutes for Activity J on the network diagram above is

A 16 **B** 15 **C** 13 **D** 11

20 © QCAA 2023 1Q3 80% The duration, in minutes, of all activities in a project are shown.

Activity	P	Q	R	S	T	U	V
Duration	38	42	32	34	16	14	26

The critical path for the project is PRSV.

What is the earliest completion time for the project if it starts at 11:00 am?

A 12:30 pm **B** 1:10 pm **C** 1:30 pm **D** 2:10 pm

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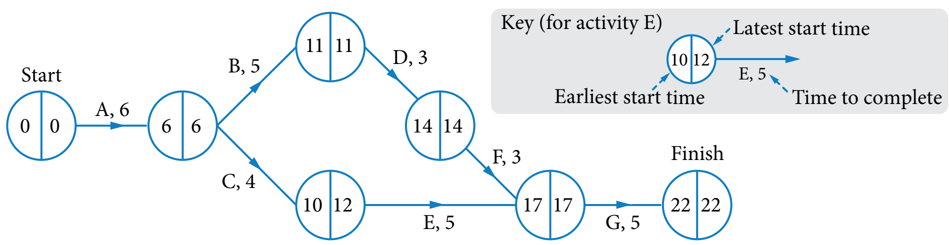
21 © QCAA 2023 1Q8 49% Activities P and Q are the critical activities for a project.

Activity	Duration	Prerequisite activity
P	3	–
Q	6	P

What are the earliest starting time (EST) and latest starting time (LST) for Activity Q?

	EST	LST
A	3	3
B	3	6
C	6	6
D	6	9

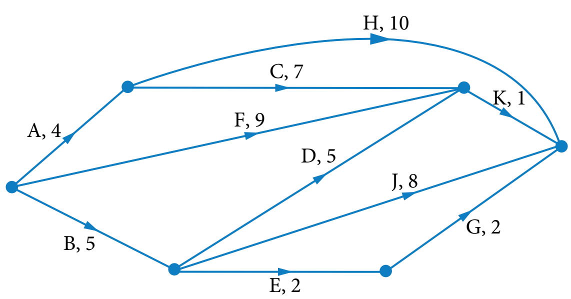
22 © QCAA 2020 1Q12 The activity times in the project network shown are in days.



The greatest float time for a non-critical activity in this network is

A 2 days **B** 4 days **C** 5 days **D** 12 days

23 © QCAA 2020S 1Q30 (4 marks) The network diagram below illustrates the time taken in days for 10 different activities, labelled A–K, involved in the completion of a task.



a Identify the critical path for the diagram above. *[1 mark]*

b Calculate the latest starting time for Activity G. *[1 mark]*

c Calculate the float time for Activity E. *[2 marks]*

24 © QCAA 2021 1Q19 (5 marks) The activity table for a project is shown.

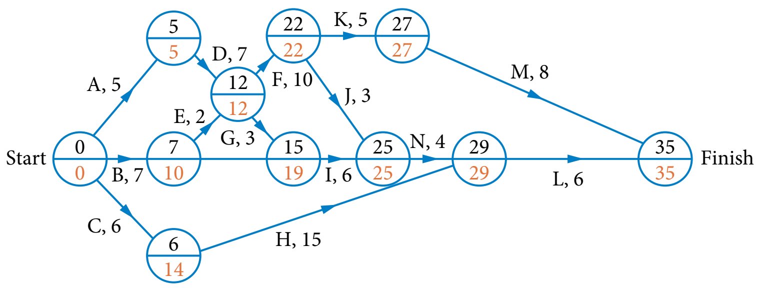
Activity	Prerequisite activity	Time (days)
A	—	2
B	—	4
C	A	3
D	B	6
E	D	3
F	C, E	4
G	D	8
H	F, G	4

a Use the activity table to construct a network diagram, including earliest and latest starting times. *[3 marks]*

b Determine the critical path. *[1 mark]*

c Determine the shortest completion time for the project. *[1 mark]*

25 (7 marks) CF The activity times in this project network are shown in hours.



What are the effects of delaying activity F by 5 hours or I by 3 hours? If an extra hour can be allocated to one of activities H, F or M, which should be chosen? Give reasons for your answer.

26 (6 marks) CU The activity table for a project is shown below.

Activity	Time (days)	Immediate predecessor(s)
A	7	None
B	2	None
C	15	None
D	10	A, B
E	10	A, B
F	5	D
G	8	E
H	5	D
I	2	H, G
J	8	F
K	2	C, I
L	3	K
M	4	J, L

Due to scheduling issues, activity G is delayed by 2 days and activity J is able to be completed in 2 less days. Compare the differences between the original and new project times.

9.5 Practical problems with critical path analysis

Critical path analysis enables you to plan and implement projects, so they are completed in the minimum amount of time and at the minimum cost.

Analysing project activities and their relationships to each other helps you to understand how changes may affect the completion time and cost for the project. For example, it may save money to delay non-critical activities. Using float times, it is also possible for you to determine where any additional resources should be directed so that they produce the greatest effect.

WORKED EXAMPLE 13 Considering changes to a project network

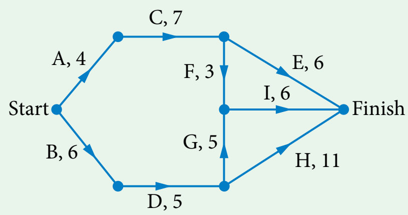
The project network for a certain project is shown on the right. The activity times are in hours.

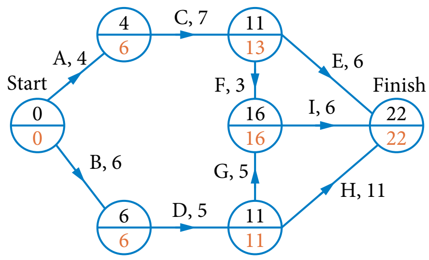
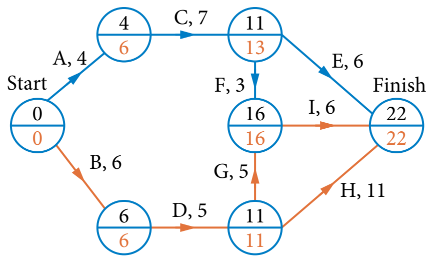
a **Determine** the critical path and minimum time the project to be completed in.

b If activity G is delayed by 3 hours, what effect does this have on the duration of the project and the critical activities?

c **Calculate** the float time for the non-critical activities.

d Extra resources can be used to shorten the duration of one of the activities B, F or I (on the original network) by 1 hour. Which of these activities should be shortened, and why?



Steps	**Working**
a **1** Determine the EST for each activity using the forward scanning process. **2** Determine the LST for each activity using the backwards scanning process.	
3 Identify critical activities where EST = LST.	
4 Identify the critical paths.	There are 2 critical paths. Critical path = B–D–G–I and Critical path = B–D–H.
5 State the minimum project completion time.	Completion time is 22 hours.
b Activity G is part of a critical path. Any delay in a critical activity will delay the project.	A delay of 3 hours in activity G will cause a delay of 3 hours in the project completion time.
c **1** Identify all non-critical activities.	Non-critical activities are A, C, E and F.
2 State the float time formula.	float = LST_{right} – EST_{left} – activity time
3 Find the float time for each non-critical activity by substituting into the formula and evaluating.	Float for activity A = 6 – 0 – 4 = 2 hours Float for activity C = 13 – 4 – 7 = 2 hours Float for activity E = 22 – 11 – 6 = 5 hours Float for activity F = 16 – 11 – 3 = 2 hours

d	**1** Consider the impact of reducing each activity time by 1 hour. B is a critical activity.	Reducing the duration of activity B by 1 hour will shorten the project by 1 hour.
	2 F is a non-critical activity.	Reducing the duration of activity F will not shorten the project as F has a float time of 2 hours.
	3 I is a critical activity.	Even though I is a critical activity, there are 2 critical paths (BDGI and BDH), so reducing I would not have any effect unless H was reduced as well.
	4 State the conclusion.	Shorten the duration of B as this will reduce the duration of the project.

Project change considerations

When considering changes to a project it is useful to keep the following in mind:

- If a critical activity is delayed by a given amount of time this will increase the project completion time by the same amount.
- If a critical activity time is reduced, this will reduce the project completion time by the same amount provided there is ONLY one critical path.
- If a non-critical activity is delayed by less than its float time, this will have no effect on the project completion time.
- If a non-critical activity is delayed by its float time, the activity will become critical and any further delay will increase the project completion time.

WORKED EXAMPLE 14 Adjusting a project because of delays

A builder is renovating a bathroom. All activities involved in the renovation are shown below.

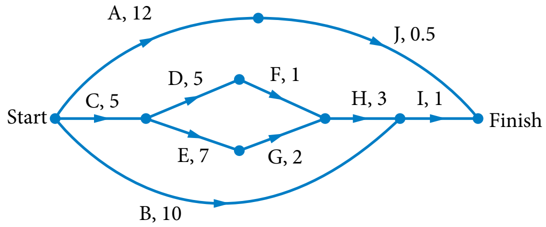
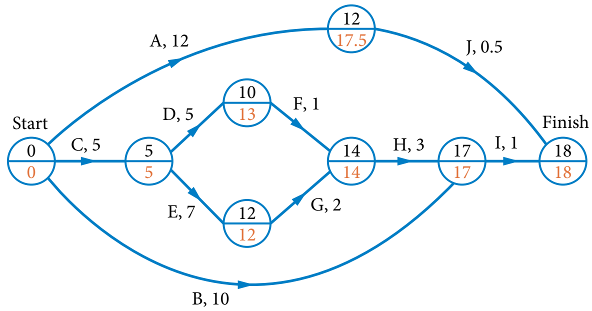
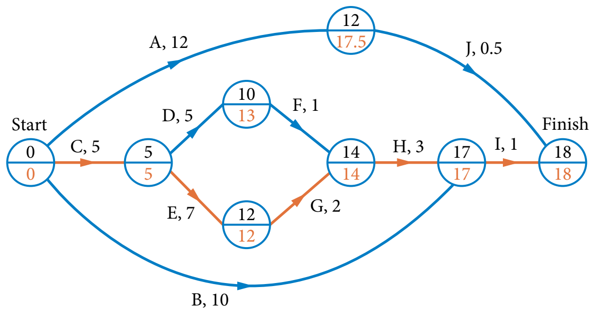
Code	Activity	Time (days)	Prerequisites
A	Order wall mirror	12	None
B	Order drawer handles	10	None
C	Make vanity unit frame	5	None
D	Make vanity top	5	C
E	Make drawers	7	C
F	Fit top	1	D
G	Fit drawers	2	E
H	Finish and paint the vanity	3	F, G
I	Fit drawer handles	1	B, H
J	Fit mirror to the wall	0.5	A

a **Construct** a project network for the bathroom renovation.

b **Determine** the critical path and shortest time the renovation can be completed in.

c The delivery of the drawer handles is delayed. For how long can the delivery be delayed before it affects the completion time for the project?

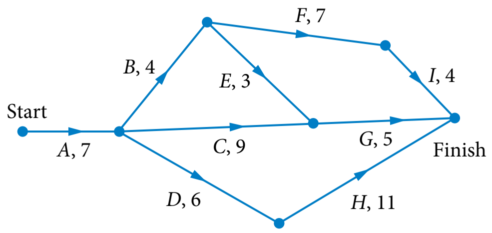
d The weather is unsuitable for the finishing and painting of the vanity to occur. If finishing and painting is delayed by 2 days, how does this affect the completion time?

Steps	Working
a Construct a project network.	
b **1** Determine the EST for each activity using the forward scanning process. Determine the LST for each activity using the backwards scanning process.	
2 Identify critical activities where EST = LST.	
3 Identify the critical path.	Critical path = C–E–G–H–I
4 State the minimum project completion time.	The shortest time for completion of the renovation is 18 days.
c **1** The ordering (and delivery) of the drawer handles (activity B) is a non-critical activity. Calculate the float time.	float = $LST_{right} - EST_{left}$ – activity time Float for activity B = 17 – 0 – 10 = 7 days
2 State the result.	The delivery of the drawer handles can be delayed by up to 7 days before the completion time for the project is affected.
d Finishing and painting the vanity (activity H) is a critical activity.	A delay of 2 days for finishing and painting the vanity will increase the project completion time to 20 days.

9.5

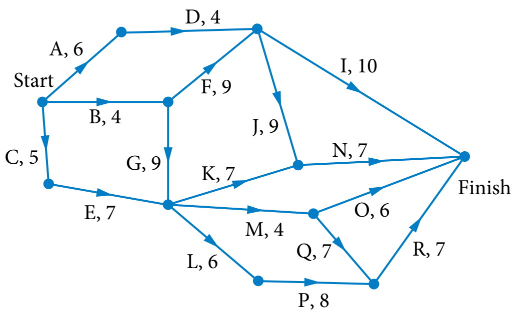
Recap

1 The path with the longest duration is



A A–B–F–I **B** A–C–G **C** A–B–E–F–I **D** A–D–H

2 The activities that make up a project and the time taken to complete each in days are shown.



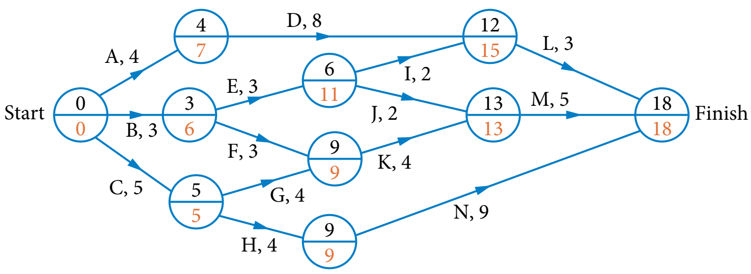
a Find the EST and LST for each activity.

b Identify the critical path.

c Determine the shortest time taken to complete the project.

Mastery

3 WORKED EXAMPLE 13 The activity times in this project network are shown in weeks.



a Calculate the float time for each of the non-critical activities.

b D is delayed by 5 weeks. Calculate the critical path and the shortest completion time for the project.

c E is delayed by 6 weeks. Calculate the critical path and the shortest completion time for the project.

d Additional resources are provided for the project. Due to scheduling, only one of the activities C, J or K (on the original network) can be shortened by 1 week. Which of these activities should be shortened, and why?

4 This activity table lists the steps for making a raised garden bed using sleepers.

Code	Activity	Time (hours)	Prerequisites
A	Take the trailer to the dump	2	F
B	Put the new plants into the bed	3	D, H
C	Mark out the site of the garden bed	1	None
D	Put the new soil into the bed	3	G, M
E	Water the plants in	0.5	B
F	Remove the existing vegetation and put in trailer	4	C
G	Put the side sleepers in and bolt into place	3	J
H	Buy the plants to put in the bed	2	L
I	Concrete corner posts into place	1.5	K
J	Allow concrete to dry	10	I
K	Dig holes for the corner posts	3.5	C
L	Plan the layout of the plants to be placed in the bed	2	None
M	Use the trailer to get the new soil	2.5	None

a Draw a project network for this project. Note that the use of the trailer makes several networks possible. Justify your choice.

b Find the earliest starting time and latest starting time for each activity.

c If the plant nursery experiences an 8-hour delay in the supply of the plants for the project, how will this affect the earliest starting time for the last vertex?

5 WORKED EXAMPLE 14 The table below describes the steps involved in producing a dance clip for social media by a dance company.

Activity	Description	Duration (days)	Predecessor
A	Decide on the style and content	2	None
B	Prepare visual storyboard and determine choreography	10	A
C	Obtain equipment and props	5	A
D	Cast dancers	7	B
E	Organise production workers	3	C
F	Prepare props and outfits	5	E
G	Record scenes	12	D, F
H	Edit recorded scenes	8	G
I	Combine scenes and finalise video	2	H
J	Prepare website and upload video	1	I

a Construct a project network for this task.

b Determine the critical path and the shortest time to complete the project.

c How will it affect the completion time for the project if it takes 3 extra days to hire the production workers? Explain your answer.

d It is decided that the editing of scenes and finalisation of the video will be done as a single activity that will be completed in 6 days. How does this affect the project completion time?

6 A publisher decides to print calendars that advertise local businesses. The publisher will arrange the printing and distribution of the calendars and will charge businesses to advertise on the calendar. The activity table for the calendar's production is given below.

Code	Activity	Time (days)	Prerequisites
A	Publicise calendar	30	None
B	Sell advertising space	40	None
C	Design calendar layout	30	None
D	Typeset calendar with advertisements	10	C
E	Organise contract for delivery of calendars	10	None
F	Print calendars	30	B, D
G	Send calendars to distributors	5	E, F

a Use a project network to find the critical path and the total time taken.

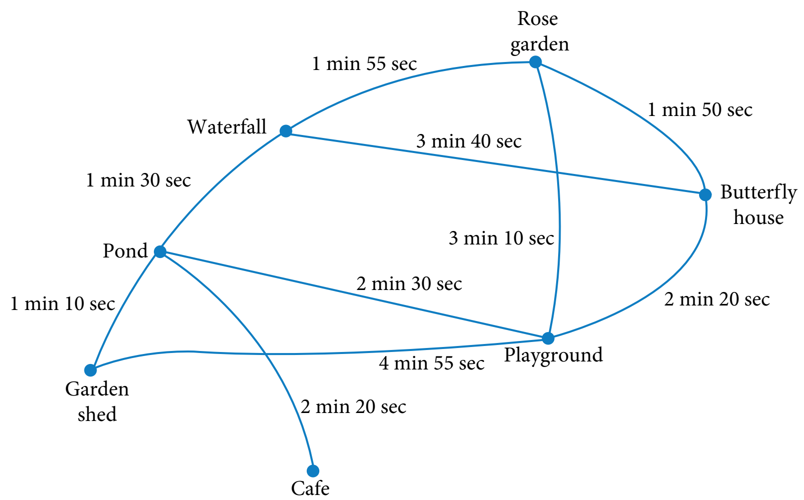
b What is the effect of a delay of 20 days in publicising the calendar? Justify your answer.

c What is the effect of a delay of 20 days in selling advertising space? Justify your answer.

d What is the effect of a delay of 20 days in designing the calendar layout? Justify your answer.

Exam practice

7 © QCAA 2020S 1Q25 (3 marks) The connected weighted graph below shows the time taken in minutes and seconds to walk between places in a park.



a Identify which 2 places are joined by a network bridge. *[1 mark]*

b Calculate the earliest time that a gardener could arrive at the butterfly house if they leave the garden shed at 8:55 am. *[2 marks]*

8 (6 marks) The activity table below shows the various tasks required to assemble and use a 'flat-pack' barbecue.

a Draw a project network for assembling and using a 'flat-pack' barbecue. *[2 marks]*

Code	Activity	Time (min)	Prerequisites
A	Read the instructions	5	None
B	Unpack the pieces from the box	6	None
C	Screw the metal parts together	10	A, B
D	Put the wheels on	5	C
E	Screw the wooden parts to the frame	10	D
F	Place the heat beads in the tray	3	E
G	Fill the gas bottle	20	E
H	Connect the gas bottle	2	G
I	Light the barbecue and heat the beads	15	F, H
J	Prepare the food	20	None
K	Cook the food	15	I, J

b Find the greatest delay possible in each activity below without affecting the completion time of the project.

i reading the instructions

ii putting the wheels on

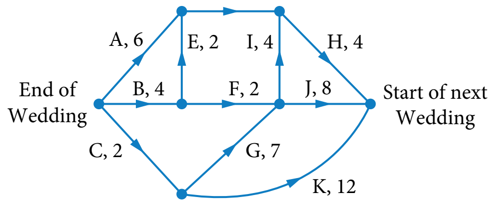
iii placing the heat beads in the tray

iv preparing the food *[4 marks]*

9 (4 marks) CF The various activities and associated times required for the task of putting on a local art exhibition are shown in this table. The organisers have set aside 8 weeks to run the project. Will this be enough time? Justify your answer.

Code	Activity	Time (days)	Prerequisites
A	Invite artists to submit photographs of work	10	None
B	Select paintings for exhibition	5	A
C	Book hall	1	None
D	Have catalogue printed	30	B, C
E	Invite local celebrity to open exhibition	3	C
F	Arrange catering for opening of exhibition	2	C
G	Arrange rosters for volunteers to manage exhibition	1	C
H	Set up hall with display screens for paintings	2	C
I	Collect paintings	3	B
J	Mount paintings	2	H, I
K	Hold exhibition	10	D, E, F, G, J
L	Return unsold paintings to artists	5	K
M	Pay artists for work sold	6	K

10 (5 marks) CU A wedding venue hires staff to ensure all necessary tasks can be done between the end of one wedding and the start of the next. The network diagram shows tasks and time needed for each to be completed.



To increase profits the business manager has determined that the minimum time between weddings must be reduced by 4 hours. They are hiring one additional staff member to help with **one** of the tasks, so it takes less time to finish.

Which tasks should the new staff member help with and how long should it take if it is to reduce the minimum time required between weddings by 4 hours?

EXAM QUESTION ANALYSIS

©QCAA 2020 2Q6 (7 marks)

A company needs to complete the following project as quickly as possible. Each task can only be completed by a single employee and must be completed before that employee can start the next task.

Task	Time (days)	Prerequisite
A	3	—
B	4	—
C	2	A
D	8	C
E	5	C
F	4	B
G	3	B
H	1	E, F
I	2	G
J	3	H, I

The owner believes that this project can be completed in minimal time with only three employees. Evaluate the reasonableness of this belief.

Reading the question

- The question is asking for the minimum time for the project completion.
- 3 employees are available to cover all tasks.
- The reasonableness of the owner's belief must be checked.

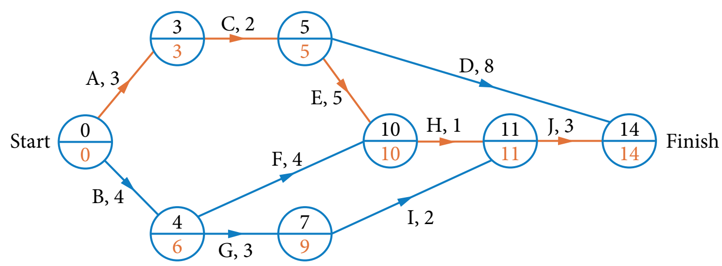
Thinking about the question

- A project network is needed.
- The critical paths are needed.
- The minimum completion time is needed.
- Need to check if 3 employees are enough to allocate all tasks.
- Need to evaluate the reasonableness of the belief.

Worked solution (✓ = 1 mark)

Draw project network. ✓

Conduct forward ✓ and backward ✓ scans to identify critical path.



Determine minimum completion time.

The shortest path is **14**. ✓

Determine minimum number of staff required to complete job on 14th day.

If the company employed 3 people as suggested, the following task allocation could be used.

Worker 1 completes all jobs on the critical path and completes the job on day 14.

Worker 2 completes non-critical tasks that are available on days 1 to 4.

9.5

Video Exam question analysis: Spanning trees and critical paths

Worker 3 is needed from **Day 5**. ✓

Evaluate the reasonableness of the brief.

Worker	Day 1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	A	A	A	C	C	E	E	E	E	E	H	J	J	J
2	B	B	B	B	F	F	F	F						
3					G	G	G	I	I					
4							D	D	D	D	D	D	D	D

With 3 workers activity D is not completed. ✓

The owner's belief is incorrect, at least 4 employees are required for the job to be completed on day 14. ✓

9 Chapter summary

Minimum spanning trees

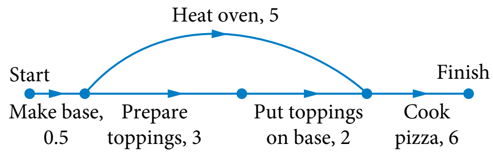
- A **tree** is a connected graph with no loops, multiple edges between adjacent vertices or cycles. The number of edges in a tree will always be one less than the number of vertices.
- A **spanning tree** is a tree subgraph that connects all the vertices of the original graph. Every connected graph has at least one spanning tree.
- A **minimum spanning tree** is a spanning tree with the total weight of its edges at a minimum. All vertices in the original network are connected but the total weight of the reduced network is a minimum.

Prim's algorithm

1. Choose any vertex to be the first vertex of the tree. Identify the edge with the lowest weight connected to the first vertex. Add this edge and its other vertex to the first vertex to form a tree.
2. Consider *all* the edges connected to the tree. Select the edge with the lowest weight that is not already in the tree. Add this edge and its vertex to the tree. If there are several edges with the same lowest weight, choose one of them.
3. Repeat step 2 until all vertices in the original graph are included in the minimum spanning tree.

Project networks

- **Activities** are the tasks required to complete a project.
- A **project network diagram** is a directed network that shows the sequence of activities and the time required for the activity's completion.
- The edges of a project network represent the activity and its duration. They show the order in which activities must be completed.

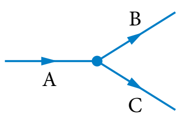
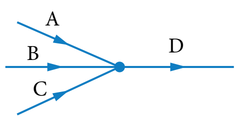
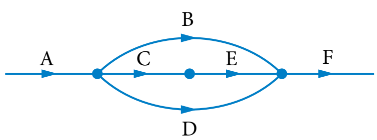


The nodes (vertices) represent the start and finish of a particular event.

- An **immediate predecessor** or **prerequisite** is an activity that must be completed immediately before another activity can commence.
- An **activity table** shows the order and estimated completion time for each **activity**.

Puzzle
Networks find-a-word

There are several types of connections.

One preceding event		Event B is preceded by A.
Multiple activities with the same preceding event		Events B and C are preceded by A.
Multiple preceding events		Event D is preceded by A, B and C.
Concurrent activities		Event F is preceded by B, E and D. Activities B and D can occur at the same time and are called concurrent activities.

- When drawing a project network diagram:
  - the 'Start' of the network must be at a vertex.
  - identify activities that do not have predecessors. These are the starting activities and connect to the 'Start' vertex.
  - multiple predecessors to an activity will end at the same vertex.
  - an activity must only be represented by one arc.
  - 2 vertices can only be connected by a single arc.
  - the network must end at the 'Finish' vertex which shows the completion of the project.

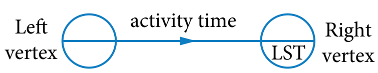
Forward scanning to determine the EST

- **Forward scanning** through a network enables you to determine the **earliest starting time (EST)** for all activities in the network.
- The EST is the earliest possible time to start an activity based on prerequisite activities.
- To determine the ESTs:
  1. Change each vertex in the network to an open circle with a horizontal diameter, ⊖. (Alternatively, a circle with a vertical diameter ⦶ or a double box can be used ⊟.)
  2. The ESTs will be calculated from left to right.
  3. In the circle at the start of the network put 0 in the top semicircle. This represents the start of the project. It also represents the EST. The EST for all activities will be written in the top semicircle.
  4. Add the activity time to the EST of the previous vertex to find the EST next activity. If multiple activities lead into a vertex, the highest total time becomes the EST.
  5. Continue until the finish circle of the network is reached.

Backward scanning to determine the LST

- **Backward scanning** through a network enables you to determine the **latest starting time (LST)** for all activities in a network.
- The LST is the latest possible time an activity can start without delaying the completion of the project.
- The LST value for any activity can be determined by:

  LST for an activity (Left vertex) = LST at Right vertex – activity time



- To determine the LSTs:
  1. Begin with a network in which the EST for each activity has been calculated.
  2. The LSTs will be calculated from right to left in the network.
  3. Start the LST calculations at the 'Finish' vertex. At the "Finish' vertex, LST = EST. This value is written in the bottom semicircle for the last vertex. The LST for all activities will be written in the bottom semicircle.
  4. Working backwards, to find the LST at the left vertex subtract the activity time from the LST at the right vertex. If multiple activities join at a vertex (Left vertex), the lowest total time becomes the LST.
  5. Continue until the start of the network is reached. The LST for the 'Start' vertex must be 0.

Critical path analysis

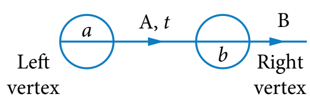
- The **critical path** is the longest path from start to finish and determines the project completion time.
- The activities that are on the critical path are called the **critical activities**. All other activities are called **non-critical activities**.
- On a critical path, the activities have the same EST and LST values.
- To determine the critical path:
  1. Use forward scanning to determine the EST for each activity.
  2. Use backward scanning to determine the LST for each activity.
  3. Identify activities for which EST = LST. These activities form the critical path.

Activity float times

- The **float time** (or **slack time**) is the maximum time that an activity can be delayed or extended without affecting the project completion time.
- In the diagram

  a = EST at left vertex, b = LST at right vertex

  t = activity time for activity A



  LST for activity A = $b - t$

  float for activity A = $b - a - t$

- For non-critical activities:
  float = $\text{LST}_{\text{right}} - \text{EST}_{\text{left}}$ – activity time
- For critical activities: float = 0

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Cumulative examination 1

Simple familiar

Perusal time: 2 minutes Working time: 30 minutes

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

Section 1 (5 marks): 5 multiple choice questions

Section 2 (13 marks): 3 short response questions

Total: 18 marks

Worksheet General Maths formula sheet

Section 1 5 multiple choice questions

5 marks

1 © QCAA 2021 1Q12 Which statement is correct?

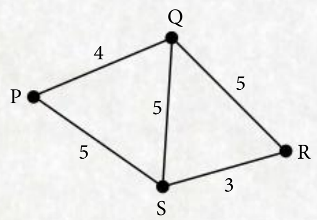
A A minimum spanning tree must contain a loop.

B A minimum spanning tree must contain a cycle.

C Every network has only one minimum spanning tree.

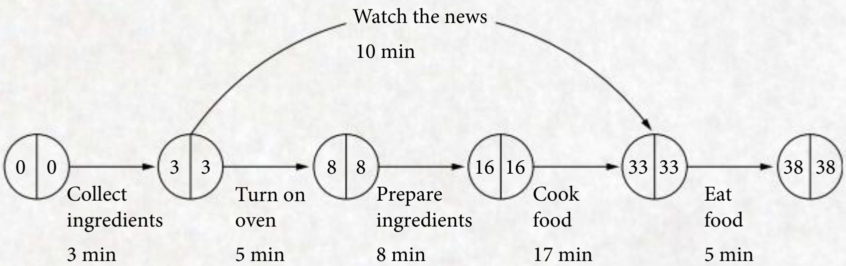
D A minimum spanning tree has one more vertex than the number of edges.

2 © QCAA 2020 1Q7 What is the length of the minimum spanning tree for this network?
All distances are in kilometres (km).



A 22 km **B** 14 km **C** 12 km **D** 3 km

3 © QCAA 2022 1Q1 The float time, in minutes, for the non-critical activity of this project network is



A 10 **B** 20 **C** 23 **D** 30

4 © QCAA 2022 1Q14 A rugby fan in Perth (Australia) plans to watch a live match played in New Zealand next winter. The time zone for Perth is UTC +8. The time zone for New Zealand is UTC +13 in winter and UTC +12 in summer. If the match is played at 6:30 pm New Zealand time, what time will the match be viewed in Perth?

A 1:30 pm **B** 2:30 pm **C** 10:30 pm **D** 11:30 pm

5 © QCAA 2023 1Q12 A reducing balance loan with an initial balance of $6000 is modelled by the recurrence relation $A_{n+1} = \left(1 + \frac{0.03}{12}\right)A_n - 400$, where n is the number of months.

The loan balance at the end of two months is closest to

A $5100 **B** $5200 **C** $5215 **D** $5230

Section 2 3 short response questions

13 marks

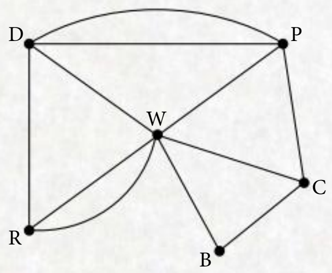
6 © QCAA 2022 1Q16 (3 marks) The table shows the number of sales for a small business in their first six months of trading.

Time in months, t	Number of sales, n
1	86
2	180
3	160
4	226
5	240
6	335

a Use your calculator to determine the equation of the least-squares line. *[1 mark]*

b Use the equation from part **a** to predict the number of sales in the 21st month. *[2 marks]*

7 © QCAA 2022 1Q21 (5 marks) The paths connecting various landmarks in a park are shown.



Key	
B	Bus stop
C	Coffee shop
D	Duck pond
P	Playground
R	Rose garden
W	Water feature

a Identify one cycle that passes through the rose garden and the playground. *[1 mark]*

b Identify whether the graph is Eulerian or semi-Eulerian. Justify your response. *[2 marks]*

c Construct an adjacency matrix from the graph, using the vertex order listed in the key. *[2 marks]*

8 © QCAA 2023 1Q24 (5 marks) Jed is preparing and serving a meal of curry and rice with naan bread. The table shows the duration and prerequisites for all the tasks they must complete.

Task	Task description	Duration (minutes)	Prerequisite
A	Assemble equipment	2	—
B	Boil rice	20	A
C	Prepare curry ingredients	6	A
D	Make naan bread	8	A
E	Simmer curry	40	C
F	Fry naan bread	4	D
G	Serve meal	2	B, E, F

a Construct a network diagram to show the sequence of tasks from start to finish, labelling all tasks and durations. Use forward and backward scanning to show the earliest and latest starting times for all tasks. *[3 marks]*

b Determine the critical activities and minimum completion time for preparing and serving the meal. *[2 marks]*

CUMULATIVE EXAMINATION 1

Cumulative examination 2

Worksheet
General Maths formula sheet

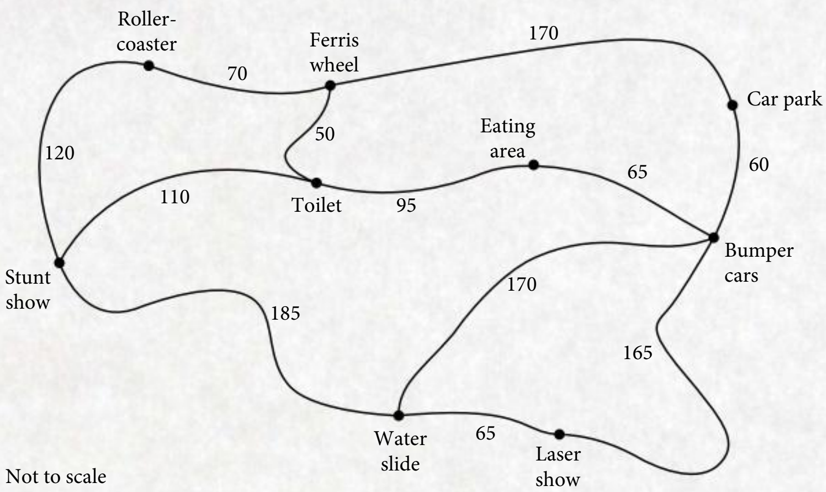
Complex familiar and unfamiliar

Perusal time: 2 minutes   Working time: 30 minutes

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

2 short response questions   12 marks

1 © QCAA 2022 2Q5 (5 marks) The map details the length (in metres) of paths between nine key locations in a theme park. The annual cost to maintain the paths is $214 per metre. The theme park manager believes at least $138 000 can be saved each year if some paths are removed, while still allowing visitors to access every key location using paths.



Evaluate the reasonableness of the manager's belief.

2 © QCAA 2020 2Q7 (7 marks) A couple saved for their retirement by making the same monthly payments for 20 years into an account that earned 4.2% p.a. compounded monthly.

At the age of 65, the couple retired and used all their savings to purchase a perpetuity with an interest rate of 5.76% p.a. compounded monthly, paying $3600 each month.

How much did they save each month to prepare for their retirement?

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CHAPTER

10 FLOW AND ASSIGNMENT PROBLEMS

Prior learning
Flow and assignment problems

Syllabus coverage

UNIT 4, TOPIC 5: NETWORKS AND DECISION MATHEMATICS 2

Flow networks

- Understand the meaning of source node, sink node, cut, minimum cut and maximum flow.
- Use a flow network diagram to identify a cut.
- Determine the capacity of a cut.
- Solve small-scale practical problems involving flow networks (up to 8 possible cuts), including determining the minimum cut and the maximum flow.

Assigning order and the Hungarian algorithm

- Use a bipartite graph and its tabular or matrix form to represent possible assignments for an allocation problem.
- Determine the optimum (minimum and maximum) assignment/s for small-scale practical problems by inspection.
- Use the Hungarian algorithm (3×3 up to 5×5 square matrices) to determine the optimum (minimum and maximum) assignment/s for larger practical problems.

Videos (2):
10.1 Capacity of a cut
Exam question analysis Flow and assignment problems

Prior learning (1):
10.1 Flow and assignment problems

Worksheets (3):
10.1 Network flow capacity
10.4 The Hungarian algorithm
Cumulative exams General Maths formula sheet

Puzzle (1):
Chapter summary Networks crossword

Nelson MindTap

To access resources above, visit **cengage.com.au/nelsonmindtap**

Terminology

actual flow	adjacency matrix	allocation	assignment
bipartite graph	capacity of a cut	cost matrix	cut
dummy	flow capacity (capacity)	flow network	Hungarian algorithm
inflow capacity	maximum flow	minimum cut	optimum allocation
outflow capacity	row and column reduction	sink	source
valid cut			

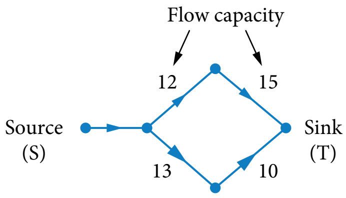
10.1 Flow networks

Situations which involve the flow of things, such as water through a pipe system, traffic on roads or internet traffic on a website, can be represented using flow networks.

Flow capacity and maximum flow

A **flow network** is a directed graph (digraph). The first **node** (**vertex**) in the network is called the **source (S)** node. This is where all flows start. The flow travels through the network to the final node called the **sink (T)** node.

Each **edge** (arc) in the network has a **flow capacity** (**capacity**) which is the maximum flow the edge can hold.



Worksheet Network flow capacity

Flow capacity and maximum flow

- The **inflow capacity** is the total of the flow capacities of all edges leading into the node.
- The **outflow capacity** is the total of the flow capacities of the edges leaving the node.
- The **maximum flow** through a node is the smaller of the total inflow capacity and the total outflow capacity of the node.

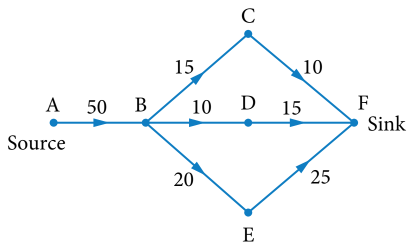
Consider this flow network.

All flow starts at A, the source node. All flow ends at F, the sink node.

The inflow capacity of B is 50.

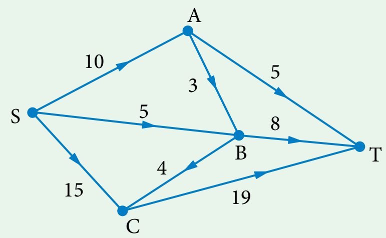
The outflow capacity of B is 45 (15 + 10 + 20).

The maximum flow through B is 45 (50 > 45).



WORKED EXAMPLE 1 Inflow, outflow and maximum capacity of a node

This flow network represents pipes carrying water from a source (S) to a sink (T) via a number of nodes. The flow capacity for each pipe is shown in litres per minute.



Determine

a the inflow capacity of B

b the outflow capacity of B

c the maximum flow through B.

Steps		Working
a	**1** The inflow is the sum of the arcs leading into B.	Inflow into B = 3 + 5 = 8 L/min
	2 State the result.	The inflow into B is 8 L/min.
b	**1** The outflow is the sum of the arcs leading out of B.	Outflow from B = 8 + 4 = 12 L/min
	2 State the result.	The outflow from B is 12 L/min.

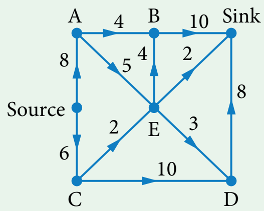
c	**1** The maximum flow out of B is the smaller of the inflow and outflow capacity.	$8 < 12$ Inflow is smaller.
	2 State the result.	The maximum flow through B is 8 L/min.

In flow networks, it is important to remember:

- all flows start from the source
- all outflows end at the sink
- the flow along an edge must be less than or equal to the capacity of the edge
- the outflow of a node cannot be larger than the inflow capacity of the node
- the maximum flow is the same as the flow capacity.

WORKED EXAMPLE 2 Flow through a network by inspection

The network shows the flow capacity of water pipes in litres per minute (L/min).



Identify a possible flow through the network from source node to sink node.

> HINT: There may be situations where the **actual flow** is less than the capacity. In these situations, it is useful to show the actual flow and the flow capacity on an arc.
>
> For example, 4/8 or 4,8 would indicate that the actual flow through an edge is 4, while the capacity of the edge is 8.

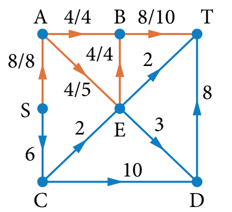
Steps | **Working**

1 Starting at the source pick an existing arc, say S–A, and follow the flow of water along S–A and through the network.

8 L/min is pushed from S to A.

From A there is capacity for 4 L/min to move along A–B, leaving 4 L/min to move along A–E and E–B to B.

This means 8 L/min flows into B and is pushed from B to the T.

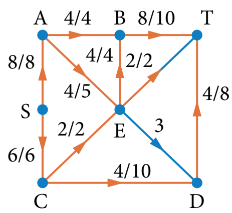


2 Repeat this process along other arcs, exiting the source.

6 L/min is pushed from S to C.

From C, there is capacity for 2 L/min to move along C–E and E–T. This leaves 4 L/min to move along C–D and D–T.

This means 6 L/min flows into T from D and E.



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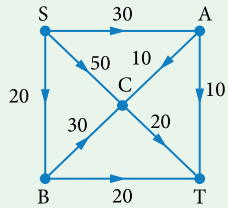
In most networks, the objective is to maximise the flow rate. By calculating the flow through a network, you can find the maximum flow.

WORKED EXAMPLE 3 The maximum flow through a network by inspection

The digraph shows the traffic capacity, in vehicles per minute, of roads in and out of three intersections (A, B and C). **Determine** the maximum flow of traffic through this network.

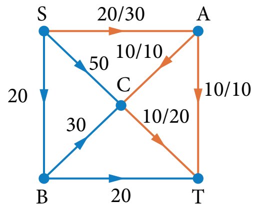


Steps | **Working**

1 The outflow from A is 20, so start by pushing 20 from S to A.

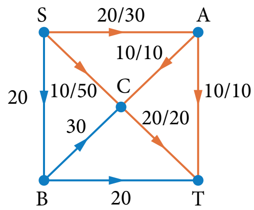
From A, 10 flows along A–T to T and 10 flows along A–C to C and then on to T.

HINT: It is useful to do a quick scan of the network and identify the outflow capacity as this helps determine the inflow capacity.



2 Repeat this process for arc S–C.

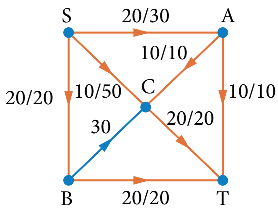
The available capacity of C–T is now 10, so this is the maximum outflow from C. So, push 10 from S to C and on to T.



3 Repeat this process for arc S–B.

The arc B–C can't be used as there is no spare capacity in C–T.

The capacity of B–T is 20, so push 20 from S to B and on to T.



4 There is no unused capacity in the arcs that flow into the sink, so the maximum flow has been achieved. Determine the total of the flows into T to find the maximum flow.

$$\text{maximum flow} = 10 + 20 + 20$$
$$= 50$$

5 State the result.

The maximum flow for this network is 50 vehicles per minute.

Video
Capacity of a cut

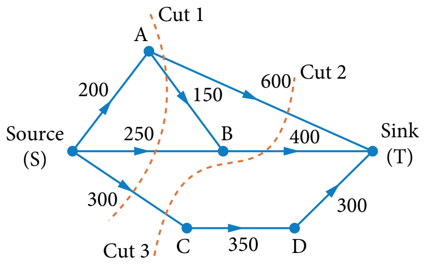
Capacity of a cut

An alternate method to inspection for finding the maximum flow through a network involves **cuts**.

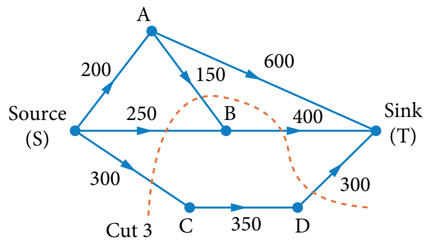
A cut is a line that divides a network into 2 parts, separating the source from the sink.

The maximum flow through a network can be found using the minimum capacity of the cuts.

Consider the network below. Cuts 1 and 2 are both **valid cuts** as they stop flow from the source to the sink. Cut 3 is not valid as it does not stop all flow between the source and sink. Flow is still possible along S–A–T.



Valid cuts



Invalid cut

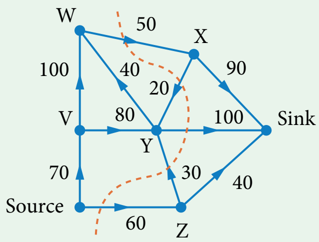
The **capacity of a cut** is the sum of all the flows that pass across the cut in the direction *from* the source to the sink. In the above network, the capacity of Cut 1 is 1300 (600 + 150 + 250 + 300).

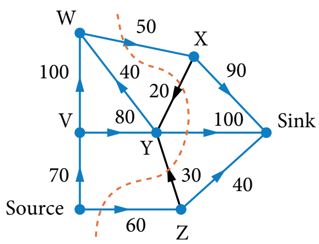
Capacity of a cut

- A cut must stop all possible flow from the source node to the sink node.
- The capacity of a cut is the sum of all flow capacities that pass across the cut in the direction going *from* the source node towards the sink node. Any flow in the direction from the sink to the source side is ignored.

WORKED EXAMPLE 4 Finding the capacity of a cut

Determine the capacity of the cut in this network.



Steps	Working
1 Check the direction of all flows across the cut. Identify any that need to be ignored as they flow from the sink back towards the source.	Arcs X–Y and Z–Y are not counted as they flow from the sink towards the source over the cut.
HINT: It can be useful to highlight arcs that flow from the sink towards the source to ensure you don't add them into the capacity of the cut.	
2 Add the flows that cross the cut from the source towards the sink.	capacity of the cut = 50 + 100 + 60 = 210
3 State the result.	The capacity of the cut is 210.

In any network, there may be many possible cuts. In this course, the maximum number of cuts could be up to 8 in a network.

Maximum flow and minimum cut

The maximum flow in a network from the source to the sink is related to the **minimum cut**, which is the cut with the minimum capacity.

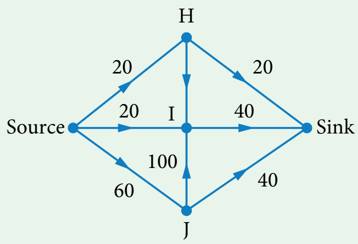
Process to find the maximum flow using the minimum cut

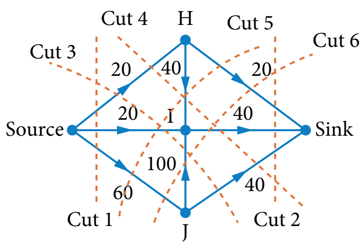
To determine the maximum flow for a network:

1. Identify the valid cuts through the network.
2. Find the capacity of each cut.
3. Maximum flow of the network = capacity of the minimum cut.

WORKED EXAMPLE 5 The maximum flow of a network using the cut capacities

Calculate the maximum flow for this network.



Steps	Working
1 Identify valid cuts which stop all flow from the source to the sink. **Exam hack** Cut lines can be straight or curved. Use whatever helps clearly identify valid cuts.	
2 Calculate the capacity of each cut. For Cut 4, do not count H–I given the flow is in the wrong direction. For Cut 6, do not count J–I.	capacity of Cut 1 = 20 + 20 + 60 = 100 capacity of Cut 2 = 20 + 40 + 40 = 100 capacity of Cut 3 = 20 + 20 + 100 + 40 = 180 capacity of Cut 4 = 20 + 40 + 40 = 100 capacity of Cut 5 = 20 + 40 + 20 + 60 = 140 capacity of Cut 6 = 20 + 40 + 60 = 120
3 Identify minimum cut (= maximum flow).	minimum cut = 100
4 The minimum cut has the maximum flow.	The maximum flow for the network is 100.

Mastery

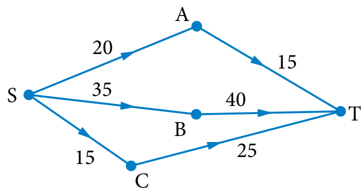
1 WORKED EXAMPLE 1 For each flow network, find

i the inflow capacity of each node including the sink node, T (excluding S)

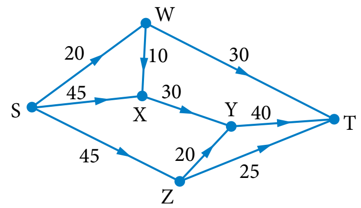
ii the outflow capacity of each node including the source node, S (excluding T)

iii the maximum flow through each node (excluding S and T).

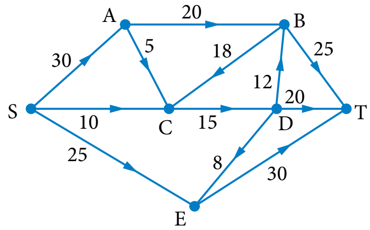
a



b



Refer to this flow network for Questions 2–4.



2 The maximum flow into C is

A 5 **B** 15 **C** 20 **D** 33

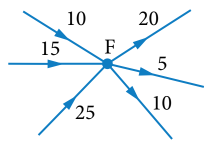
3 The maximum flow out of D is

A 12 **B** 20 **C** 32 **D** 40

4 The maximum flow through B is

A 20 **B** 25 **C** 32 **D** 43

5 This digraph shows the traffic capacity, in vehicles per minute, of roads in and out of an intersection, labelled F.



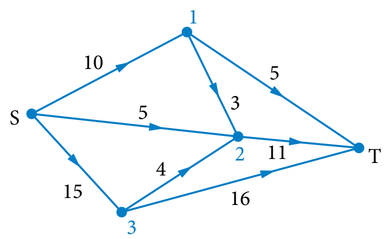
Determine

a the inflow capacity of F

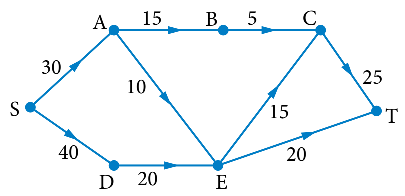
b the outflow capacity of F

c the maximum flow through F

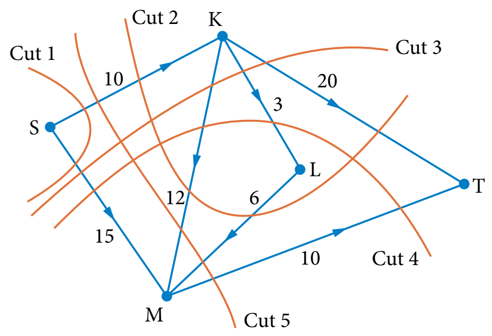
6 WORKED EXAMPLE 2 Copy this network and show the flow from the source (S) to the sink (T).



7 WORKED EXAMPLE 3 Find the maximum flow through this network using inspection.



Questions 8 to 11 refer to the flow network that shows a flow from the source (S) to the sink (T).



8 Which of the cuts are not valid?

A Cuts 1 and 2 **B** Cuts 2 and 4

C Cuts 1, 2 and 4 **D** Cuts 1, 3 and 5

9 The capacity of Cut 3 is

A 25 **B** 35 **C** 48 **D** 50

10 The capacity of Cut 5 is

A 20 **B** 25 **C** 30 **D** 32

11 The maximum flow from source to sink is

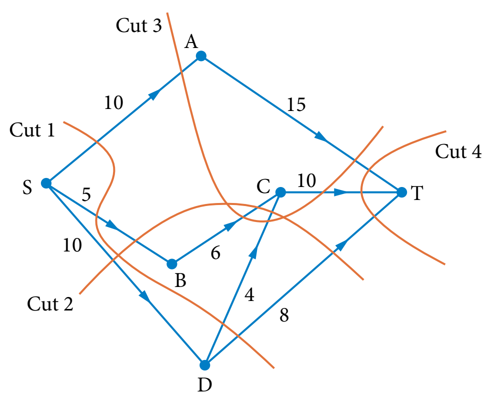
A 20 **B** 30 **C** 40 **D** 50

12 WORKED EXAMPLE 4 For each flow network

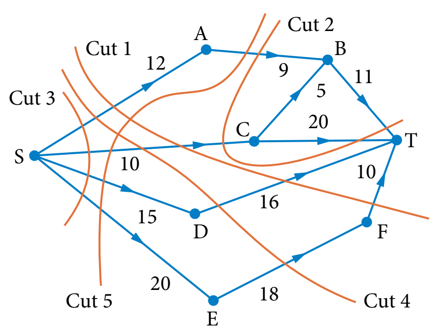
i identify any invalid cuts

ii calculate the capacity of valid cuts.

a

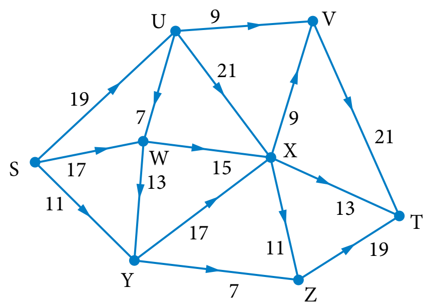


b

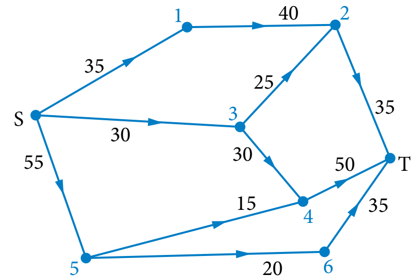


13 WORKED EXAMPLE 5 Determine the maximum flow for each network.

a

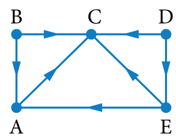


b



Exam practice

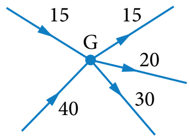
14 The flow network shows a water pipe system that connects 5 points (A, B, C, D and E) underground. The direction of the flow of water between the points is shown.



In the network, water can flow from

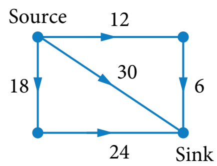
A A to B **B** A to D **C** D to A **D** D to B

15 The directed graph shows the flow of data through a router labelled G in bits per second. What is the maximum flow through G?



A 50 **B** 55 **C** 60 **D** 65

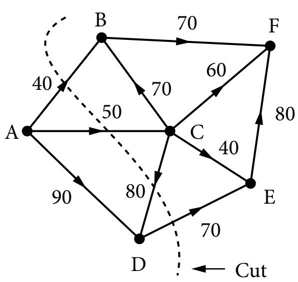
16 The digraph shows the flow of waste, in litres per hour, through a system of pipes connecting the source to the sink.



The maximum flow in litres per hour from the source to the sink is

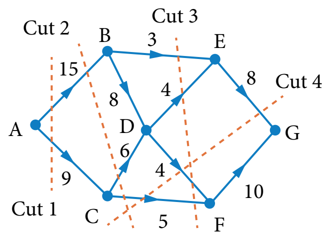
A 30 **B** 50 **C** 54 **D** 60

17 ©QCAA 2020S 1Q16 Determine the value of the cut made on the network diagram below.



A 160 **B** 170 **C** 180 **D** 240

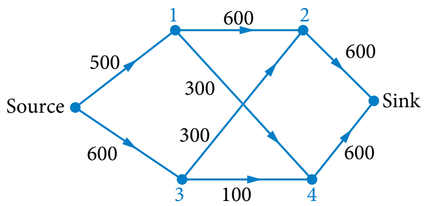
18 On the digraph below the weights on the arcs give the maximum flow between nodes. Six cuts have been made on the diagram. Determine which cut allows you to find the maximum flow from point A to G.



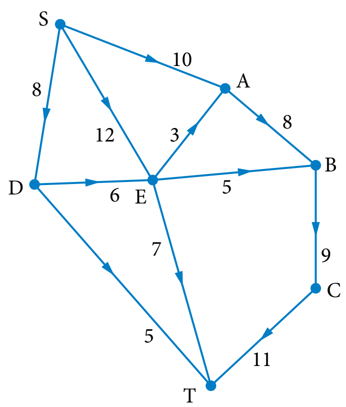
A Cut 1 **B** Cut 2 **C** Cut 3 **D** Cut 4

19 (3 marks) This is the flow network for pipes carrying oil from a refinery (the source) to a storage facility (the sink). The source has a maximum outflow of 900 litres/second (L/s).

Show the flow through this network from the source (S) to the sink (T).



20 (6 marks) The network shown here represents pipes carrying water from a source (S) to a sink (T) via a number of nodes. The flow capacity for each pipe is shown in litres per second.



a Determine

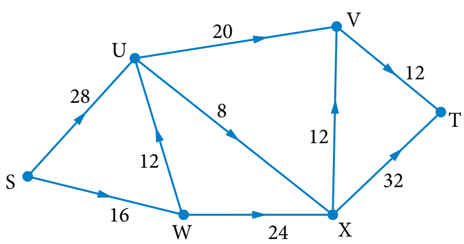
i the maximum flow into E *[1 mark]*

ii the maximum flow out of E *[1 mark]*

b Show the flow through the network from source to sink. *[3 marks]*

c Using inspection, what is the maximum flow through the network? *[1 mark]*

21 (3 marks) CF Examine the network shown.



Show the flow through the network when the flow capacity is 36.

10.2 Practical problems with flow networks

Practical applications of flow networks are everywhere. They include water supply and drainage systems to your home, metro train and bus services, and internet traffic and data routing.

WORKED EXAMPLE 6 **Finding the maximum flow from an activity table**

Water flows from a dam (source, S) to a reservoir (sink, T) via several nodes, as shown in the following table. The flow capacity of the pipes between each node is shown.

a **Represent** the information in the table as a flow network.

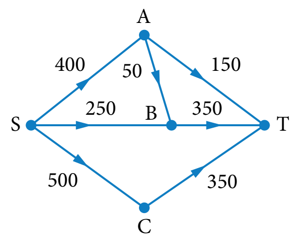
b **Determine** the maximum flow for the network.

From	To	Capacity (L/min)
S	A	400
A	T	150
A	B	50
S	B	250
B	T	350
S	C	500
C	T	350

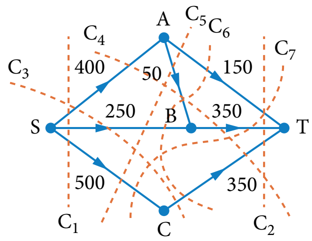
Steps | **Working**

a Construct a flow network.

Label the nodes (A, B, C, S and T) and arcs with capacity and direction.



b **1** Identify valid cuts.



Exam hack

When labelling cuts, use the notation given in the question. If no notation is given, write Cut 1 or C_1.

Cut 1 shows that the maximum flow out of the source is 1150. Cut 2 shows that the inflow into the sink is 850. This tells you the maximum capacity can be no more than 850. Further cuts will identify if there are other restrictions to the flow.

2 Calculate the capacity of each cut.

For Cut 4, do not count A–B given the flow is in the wrong direction.

For Cut 6, do not count J–I.

capacity of Cut 1 = 400 + 250 + 500 = 1150

capacity of Cut 2 = 150 + 350 + 350 = 850

capacity of Cut 3 = 400 + 250 + 350 = 1000

capacity of Cut 4 = 400 + 350 + 350 = 1100

capacity of Cut 5 = 150 + 50 + 250 + 500 = 950

capacity of Cut 6 = 150 + 50 + 250 + 350 = 800

capacity of Cut 7 = 150 + 350 + 500 = 1000

3 Identify minimum cut (= maximum flow).

minimum cut = 800

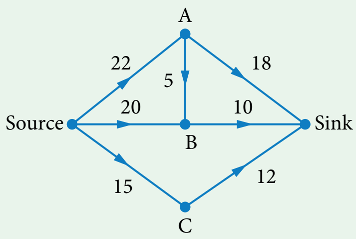
4 The minimum cut has the maximum flow.

The maximum flow for the network is 800 L/min.

9780170484824

WORKED EXAMPLE 7 Finding the maximum flow using cut capacities

The network shows the flow of patients through an emergency medical centre. The arcs represent the various areas of the centre, and the weight represents the maximum number of patients per hour who can be dealt with in that area. The nodes represent patient data processing points.

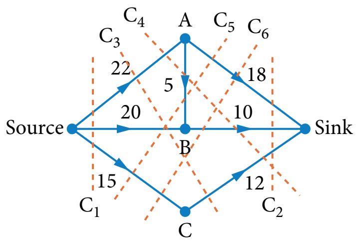


a **Determine** the maximum flow of patients.

b **Identify** how the flow should be directed along each arc to achieve the maximum flow.

Steps | **Working**

a 1 Identify valid cuts.



2 Calculate the capacity of each cut.

The vertical edge with a capacity of 5 is not counted as the flow is in the wrong direction.

capacity of $C_1 = 22 + 20 + 15 = 57$

capacity of $C_2 = 18 + 10 + 12 = 40$

capacity of $C_3 = 22 + 20 + 12 = 52$

capacity of $C_4 = 22 + 10 + 12 = 42$

capacity of $C_5 = 18 + 5 + 20 + 15 = 58$

capacity of $C_6 = 18 + 10 + 15 = 43$

3 Identify minimum cut = maximum flow.

minimum cut = 40.

4 State the result.

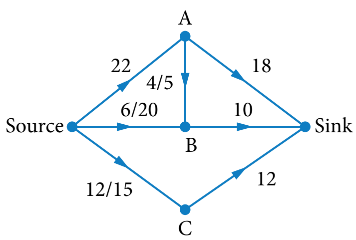
The maximum flow is 40 patients per hour.

b 1 Calculate the maximum flow through each node.

Node	A	B	C
Inflow	22	$20 + 5 = 25$	15
Outflow	$18 + 5 = 23$	10	12
Maximum flow	22	10	12

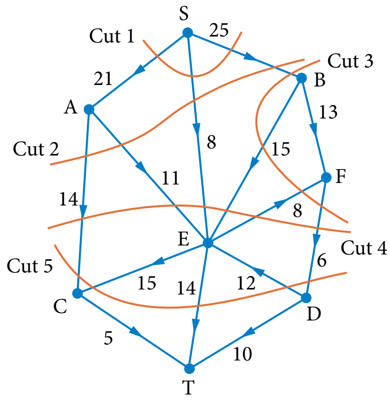
2 As the maximum flow is 40, the flow leaving the source and entering the sink must be 40.

Adjust the inflow or outflow at each vertex to achieve the maximum flow.



Recap

Refer to the flow network shown for Questions 1 and 2.



1 Which cut is not valid?

A Cut 1 **B** Cut 2 **C** Cut 3 **D** Cut 4

2 What is the capacity of Cut 5 for the network?

A 46 **B** 49 **C** 54 **D** 58

Mastery

3 WORKED EXAMPLE 6 Electricity moves from a generator (source) to a substation (sink) via a number of nodes, as shown in the table. The flow capacity of the wires, measured in megawatts (MW), between each node is shown.

a Represent the information in the table on a flow network diagram.

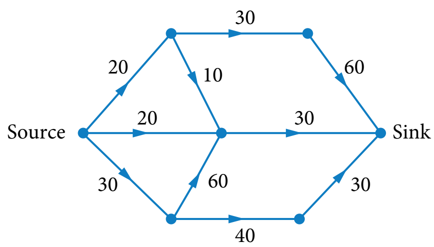
b Determine the maximum flow for the network.

From	To	Capacity (MW)
S	A	1100
A	B	1500
B	T	1200
S	C	800
C	E	200
C	T	300
S	D	500
D	E	900
E	T	1000

4 WORKED EXAMPLE 7 The flow of oil through a series of pipelines, in litres per minute is shown in the network.

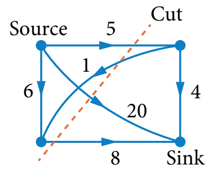
a Determine the maximum flow for the network.

b Identify how the flow should be directed along each arc to achieve the maximum flow.



Exam practice

Questions 5 and 6 relate to this graph which shows the flow of gas along a pipe system, in litres per minute.



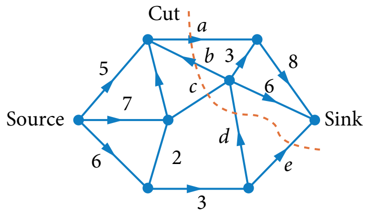
5 The capacity of the cut, in litres per minute, is

A 31 **B** 32 **C** 33 **D** 34

6 The maximum flow from the source to the sink is

A 12 **B** 20 **C** 31 **D** 32

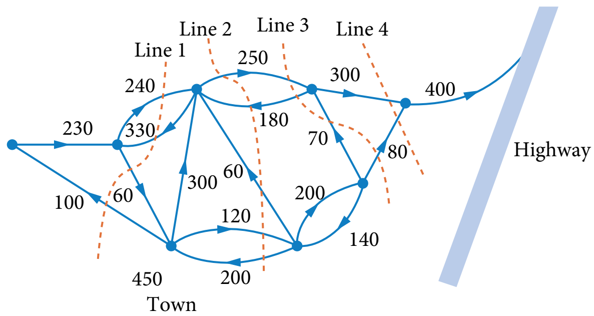
7 (1 mark) The number of vehicles along key roads in a suburb is represented in the directed graph. If there are no non-zero weighted arcs what is the capacity of the cut shown?



8 (2 marks) Vehicles from a town can drive onto a highway using one-way and two-way roads, as shown in the network diagram.

The maximum number of vehicles per hour that can travel along each road is shown. The arrows represent the permitted direction of travel.

Four possible cuts are shown, and one of them is the minimum cut for this network.



Determine the maximum number of vehicles per hour that can travel through this network from the town onto the highway.

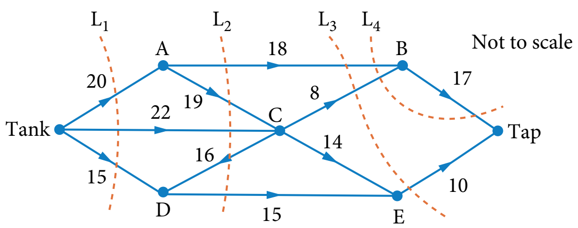
9 (5 marks) A widget manufacturer transports widgets from the factory (source) to a distribution centre (sink) via several handling centres numbered 1 to 4. The capacity to transfer widgets from the source to the sink via the nodes is shown in the table.

From	To	Capacity (widgets)
S	1	63
1	T	33
1	2	33
2	3	58
S	2	51
S	4	53
4	2	16
4	3	15
4	T	26
3	T	76

a Represent the information in the table on a flow network diagram. *[2 marks]*

b Determine the maximum flow for the network. *[3 marks]*

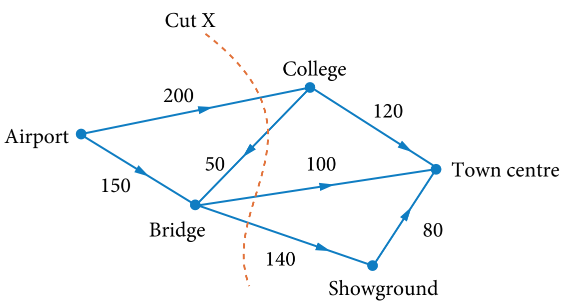
10 ©QCAA 2021 1Q23 (4 marks) The network diagram shows the flow of water from the tank (source) to the kitchen tap (sink).



a Explain which dashed line (L_1, L_2, L_3 or L_4) is not a valid cut. *[1 mark]*

b Calculate the capacity of each dashed line that is a valid cut. *[3 marks]*

11 ©QCAA 2023 1Q21 (4 marks) The road network shows the number of vehicles per hour travelling from the airport to the town centre.

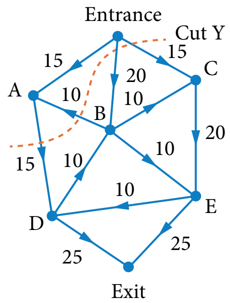


a Determine the capacity of cut X. *[1 mark]*

b Determine the maximum flow from the airport to the town centre. *[2 marks]*

c During a weather emergency, all roads to and from the bridge are closed to vehicles. Determine the maximum flow from the airport to the town centre during this time. *[1 mark]*

12 (5 marks) A museum exhibition has 5 key attractions (A, B, C, D and E), which are connected by directional walkways. The network diagram shows the walkways between each key attraction and the number of visitors who can access each walkway at any one time.

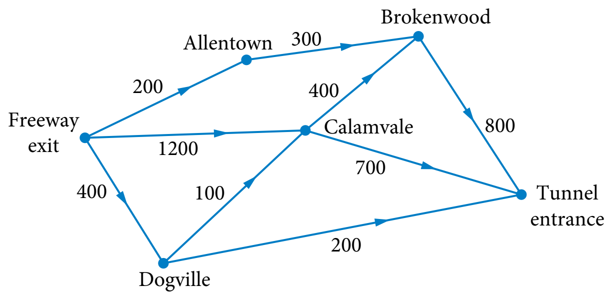


a Determine the flow capacity of cut Y. *[1 mark]*

b By first identifying an alternate cut, explain why the network diagram's current maximum flow capacity is less than 50 visitors. *[2 marks]*

c The number of visitors who can access one walkway is to be increased so that the maximum flow of the network will be 50 at any one time. Identify which walkway could be increased and by how much. *[2 marks]*

13 (6 marks) CF The capacity of a busy road network to move traffic from a freeway exit to a tunnel entrance through 4 towns is shown in the diagram. The flow rate in vehicles per minute is shown on each edge, as is the direction of the traffic.



The Department of Main Roads determines that the traffic flow through Calamvale needs to be reduced to a maximum of 1000 vehicles per minute. It is also determined that the traffic flow on only one other road is to be increased.

If the greatest traffic flow from the freeway exit to the tunnel entrance is to be maintained, which road should be chosen, and by how much must its capacity be increased?

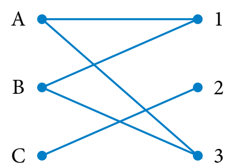
10.3 Bipartite graphs

In Chapter 7, *Graphs and networks*, you learnt that a **bipartite graph** is a graph in which the vertices can be divided into 2 distinct sets, so that each edge of the graph connects a vertex from one subset to the other.

Bipartite graphs can be directed or undirected.

The bipartite graph shown on the right represents 3 tasks {1, 2 and 3} that can be completed by 3 employees {A, B and C}. The graph shows that:

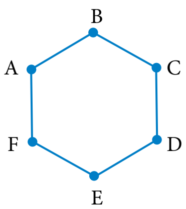
- A can complete task 1 and 3
- B can complete task 1 and 3
- C can only complete task 2.



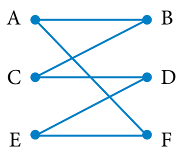
It is not always obvious if a graph is bipartite. For example, consider the graph below on the left. To determine if the graph is bipartite, you could:

- redraw a graph, if possible, so that links are preserved but the vertices are in 2 distinct sets (or groups).
- use the vertex two-colouring technique covered in Chapter 7.

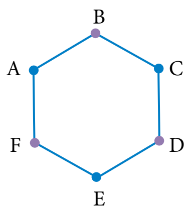
These 2 methods are shown below.



Original graph



Redrawn graph



Vertex two-colouring technique

HINT: When using the vertex two-colouring technique, simply colour the vertices in 2 distinct colours and check that no adjacent vertices are of the same colour.

Bipartite graphs are useful when you are trying to find the best way to match the elements of 2 distinct groups.

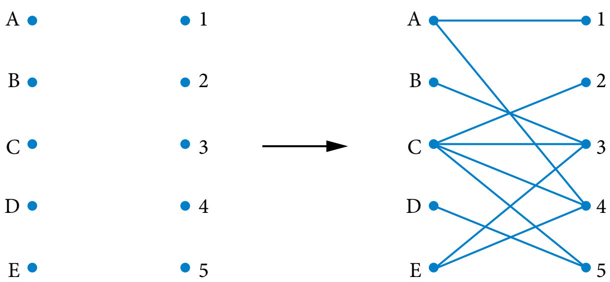
WORKED EXAMPLE 8 Drawing a bipartite graph from a table

A delivery service has 5 drivers (A, B, C, D and E) and various deliveries (1, 2, 3, 4 and 5) that each is suitably located to do. This information is summarised in the table.

Driver	Suitable deliveries
A	1, 4
B	3
C	2, 3, 4, 5
D	5
E	3, 4

a **Sketch** this information as a bipartite graph.

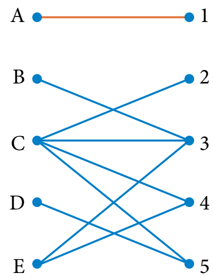
b Use the graph to **determine** which driver should be allocated each delivery.

Steps	Working
a Represent the 2 sets (Drivers and Deliveries) as the vertices of the graph. Then add edges to represent the suitability of the deliveries.	

b **1** Identify deliveries with the smallest number of links.

Driver A is the only person who can do delivery 1.

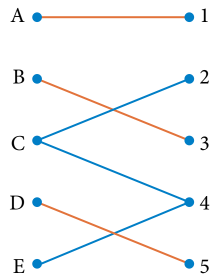
2 Allocate Driver A delivery 1, then eliminate all other deliveries Driver A could do.



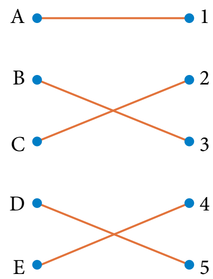
Identify the next smallest link(s).

Driver B can only do delivery 3 and Driver D can only do delivery 5.

Allocate Driver B and D deliveries 3 and 5 and eliminate all other deliveries Drivers B and D could do.



3 Driver E can only do delivery 4 so allocate them this job. This means Driver C must do delivery 2.



4 State the allocations.

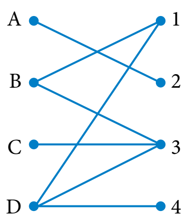
The allocation of deliveries to drivers is A–1, B–3, C–2, D–5 and E–4.

Worked example 8 is an **allocation** (or **assignment**) problem. These types of problems involve finding the best way to match elements in 2 sets based on specific conditions or circumstances.

Adjacency matrices and bipartite graphs

Bipartite graphs can be represented by an **adjacency matrix**. In an adjacency matrix, a 1 represents a connection and a 0 indicate that there is no connection.

For example, the bipartite graph below can be represented by the matrix on its right. Vertex A is only connected to Vertex 2, so in the matrix this connection is represented by a 1.



Bipartite graph

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Adjacency matrix

In the matrix, each row and column is labelled as a vertex, where the rows are the ‘from’ vertices and the columns are the ‘to’ vertices.

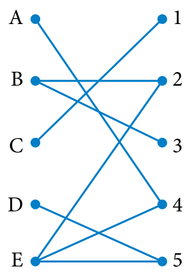
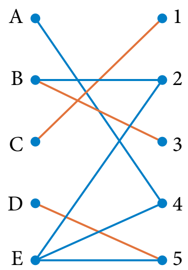
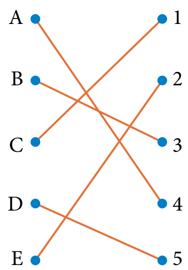
WORKED EXAMPLE 9 Drawing a bipartite graph from an adjacency matrix

Five employees (A, B, C, D and E) are to be allocated a task (1, 2, 3, 4 and 5) to complete on their shift. The employees are only trained to complete certain tasks, as shown in the matrix.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

a **Sketch** the adjacency matrix as a bipartite graph.

b Use the graph to **determine** a valid allocation of tasks for this shift.

Steps	Working
a Draw a connection for every '1' in the matrix. A–4 B–2, B–3 C–1 D–5 E–2, E–4, E–5	
b **1** Identify the tasks with the smallest number of connections. Allocate C to task 1. Allocate B to task 3 and remove other connections from B. Allocate D to task 5 and remove other connections to task 5.	
2 Allocate A to task 4 as they cannot do any other tasks. Remove other connections to task 4. Allocate E to task 2.	
3 State the allocations.	The allocation of tasks is A–4, B–3, C–1, D–5 and E–2.

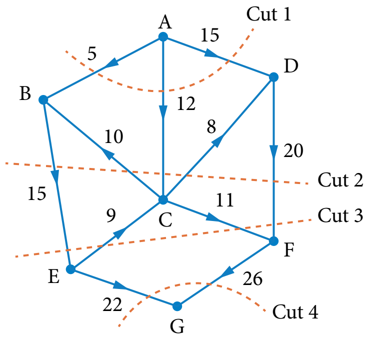
EXERCISE 10.3 Bipartite graphs

ANSWERS p. 468

Recap

Refer to the directed graph for Questions 1 and 2.

In the graph, the values on the edges give the maximum flow between vertices in the direction of the arrows.



1 The source is

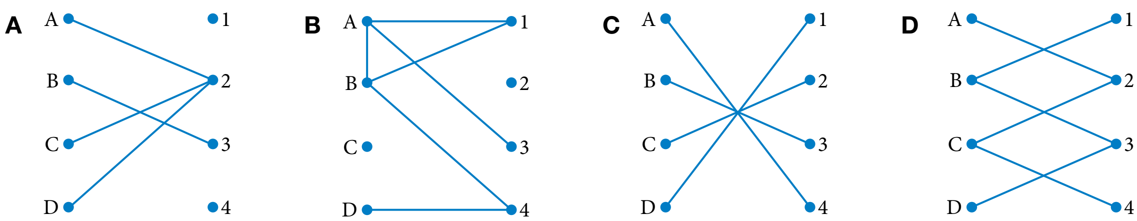
A A **B** C **C** D **D** G

2 The capacity of Cut 3 is

A 31 **B** 35 **C** 46 **D** 55

Mastery

3 Which of these is not a bipartite graph?

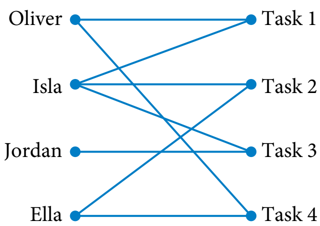


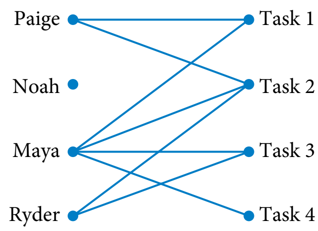
4 WORKED EXAMPLE 8 Five trainees, Chloe, Sam, Harry, Ryan and Zara, work on the production line of a car manufacturer. Their training covers the areas of instrumentation (I), brake systems (B), lighting (L), glasswork (G) and exterior finishing (E). Each apprentice is at a different stage of training and requires experience in the tasks, as shown in the table.

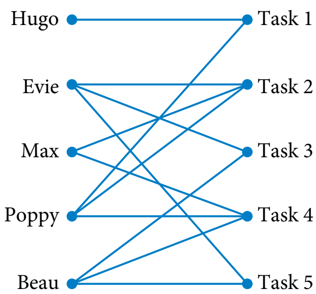
Sketch a bipartite graph to represent this information.

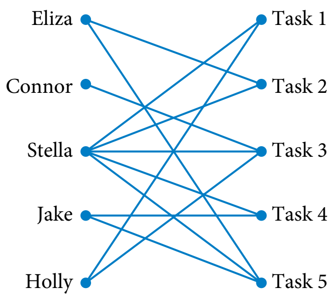
Trainee	Task
Chloe	I, L, E
Sam	B, L, G
Harry	I, L
Ryan	E
Zara	G, B

5 Determine which tasks could be allocated to each person according to the following bipartite graphs.

a


b


c


d


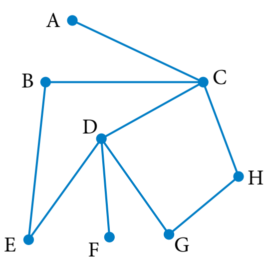
6 A dispatcher for a taxi company has 6 drivers and various jobs that each is suitably located to do. This information is shown in the table.

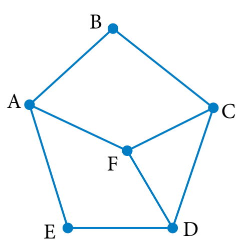
Driver	Job
A	2, 4
B	1, 6
C	4, 6
D	6
E	4, 5
F	1, 3, 5

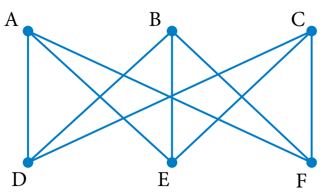
a Draw a bipartite graph to represent this information.

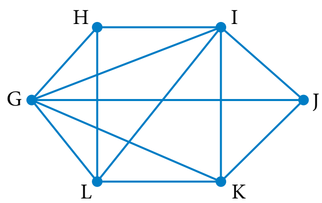
b Determine a valid allocation of jobs to the drivers.

7 Determine whether each graph is bipartite either by inspection, using the vertex two-colouring technique or redrawing the graphs.
If a graph is bipartite, state the vertices in each set.

a


b


c


d


8 WORKED EXAMPLE 9 Four workers (A, B, C and D) are each to be assigned one job from jobs 1, 2, 3 and 4. The matrix representing workers that are qualified to perform the jobs is shown in the matrix. Draw a bipartite graph and hence determine a valid allocation.

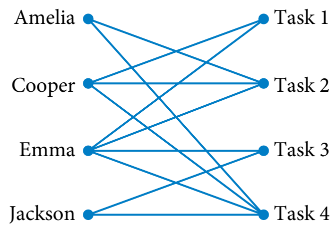
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Exam practice

9 Four trainees are at different stages of their training. The tasks that each trainee is qualified to perform in a production process are shown in the graph.

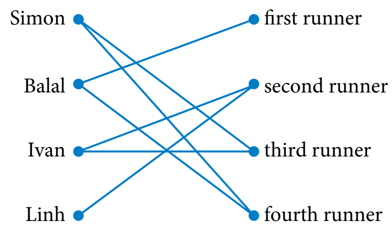
Their supervisor wants to assign each trainee a task for the next production run.

Which of the following tables show an allocation that is not valid?



A	Amelia	Task 2	**B**	Amelia	Task 4	**C**	Amelia	Task 2	**D**	Amelia	Task 2
	Cooper	Task 1		Cooper	Task 2		Cooper	Task 4		Cooper	Task 3
	Emma	Task 3		Emma	Task 1		Emma	Task 1		Emma	Task 1
	Jackson	Task 4		Jackson	Task 3		Jackson	Task 3		Jackson	Task 4

10 ©QCAA 2021 1Q5 90% The coach of a four-person relay team is deciding on the order of the runners. Use the bipartite graph to determine which statement is correct.



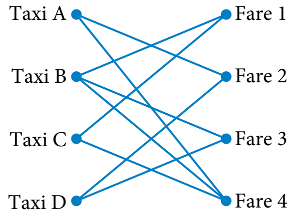
A Linh should be the first runner.

B Simon should be the second runner.

C Ivan should be the third runner.

D Balal should be the fourth runner.

11 A taxi dispatcher has 4 taxis available to pick up passengers in 4 different locations. Company policy states that a taxi must reach a fare within 10 minutes. The bipartite graph shows the fare locations each taxi can reach within 10 minutes.



If each taxi is to collect one passenger, a valid allocation is

A

Taxi A	Fare 2
Taxi B	Fare 3
Taxi C	Fare 1
Taxi D	Fare 4

B

Taxi A	Fare 2
Taxi B	Fare 4
Taxi C	Fare 1
Taxi D	Fare 3

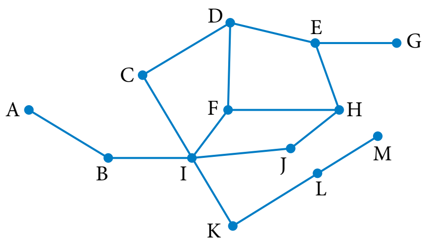
C

Taxi A	Fare 4
Taxi B	Fare 2
Taxi C	Fare 1
Taxi D	Fare 3

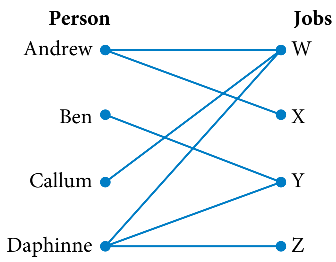
D

Taxi A	Fare 2
Taxi B	Fare 4
Taxi C	Fare 3
Taxi D	Fare 1

12 (3 marks) Determine if this graph is bipartite, and justify your answer. If it is, identify the vertices in each set.



13 (2 marks) The bipartite graph shows the jobs that 4 people can undertake. Represent this as an adjacency matrix.

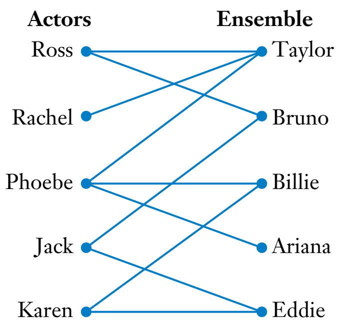


14 (3 marks) Ross, Rachel, Phoebe, Jack and Karen are 5 actors who have all auditioned for various roles in a new sitcom. Each actor can only play one part in the ensemble cast.

The bipartite graph illustrates the roles that each actor has been shortlisted for.

a Which actor must play the role of Bruno? *[1 mark]*

b Which roles must Rachel, Phoebe and Karen fill? *[2 marks]*



15 (8 marks) CF There are 6 applicants for 6 positions advertised by a local employer. The table indicates the positions for which each applicant is qualified. Each position can only accept one applicant, and each applicant can only be appointed to one job.

Applicant	Position
A	2, 3
B	None
C	1, 4
D	3
E	3, 4
F	6

Use a bipartite graph to work out an assignment of jobs to applicants that result in the greatest number of applicants being placed in positions.

10.4

10.4 Assignment problems and the Hungarian algorithm

Worksheet The Hungarian algorithm

The assignment problem and optimum allocations

The goal of an **assignment problem** is to find the **optimum allocation** (best assignment) to meet a specific objective, such as minimising time or cost, or maximising profits. When solving assignment problems, you can only have one-to-one pairings: each person can only be assigned to one job and each job can only be assigned to one person.

In simple cases, assignment problems can often be solved by inspection.

WORKED EXAMPLE 10 The optimum allocation by inspection

A company uses 3 contactors to complete different maintenance tasks. The costs for each task to be performed by the different contractors are shown below.

Contractor A charges \$40 to perform task 1, \$70 for task 2 and \$30 for task 3.

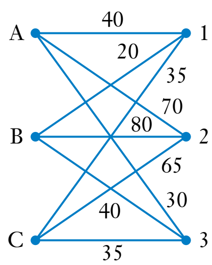
Contractor B charges \$20 to perform task 1, \$80 for task 2 and \$40 for task 3.

Contractor C charges \$35 to perform task 1, \$65 for task 2 and \$35 for task 3.

a **Construct** both a graph and a table to represent this information.

b **Determine** an allocation of tasks to contractors that minimises the cost.

c **Calculate** the minimum cost to complete the 3 tasks.

Steps	**Working**
a 1 Represent the information as a bipartite graph. This is a weighted bipartite graph.	

	2 Represent the information using a 4 × 4 table.	

	Task 1	Task 2	Task 3
A	40	70	30
B	20	80	40
C	35	65	35

The table of weights could be further simplified by converting it to a 3 × 3 matrix.

$$\begin{bmatrix} 40 & 70 & 30 \\ 20 & 80 & 40 \\ 35 & 65 & 35 \end{bmatrix}$$

This type of matrix is often referred to as a **cost matrix**.

b **1** Identify the cheapest contractor to complete task 1.

Contractor B completes task 1.

If you were looking for the maximum cost, you would identify the largest number rather than the smallest.

2 Looking at the remaining contractors (A and C) identify who is the cheapest to complete task 2.

Contractor C completes task 2.

3 A is the only remaining contractor so they will complete task 3.

Contractor A completes task 3.

c **1** Add the costs for all 3 tasks.

$$\begin{aligned} \text{Total cost} &= 20 + 65 + 30 \\ &= 115 \end{aligned}$$

2 State the result.

The total cost to complete the 3 tasks is \$115.

A more reliable method for identifying the optimum allocation is to use matrices.

The Hungarian algorithm – Stage 1 (Row and column reduction)

The **Hungarian algorithm** is used to find the optimum allocation and is completed in 2 stages. Stage 1 involves representing the information as a matrix (the **cost matrix**) and performing a **row and column reduction**. In many situations, this is all that is needed to determine the optimum allocation.

The Hungarian algorithm – Stage 1 (Row and column reduction)

1. Choose the smallest number in each row and subtract it from every element in the same row.
2. For every column that does not have a zero value, choose the smallest number in the column and subtract it from every element in the same column.
3. Cover all the zeros using the smallest number of lines (horizontal or vertical). If the number of lines is equal to the number of rows, then the row and column reduction process is complete. *The zeros in the reduced matrix represent the allocations.*
4. Draw a bipartite graph showing all the allocations.
5. Identify the allocations that produce the optimum total.

WORKED EXAMPLE 11 Stage 1 of the Hungarian algorithm

Four trainees A, B, C and D are available to perform 4 different tasks. Each task is in sequence, so task 1 must be completed before task 2 can commence and so on. Each trainee can only perform one task. The trainees have different skill sets and this means that each can perform the tasks in different periods of time (minutes) as shown in the table below.

	Task 1	Task 2	Task 3	Task 4
A	17	24	27	20
B	29	28	23	26
C	21	18	17	14
D	28	26	22	19

a Convert the table to a matrix and apply stage 1 of the Hungarian algorithm.

b **Determine** the optimum allocation and show it as a bipartite graph.

c **Calculate** the minimum time required for the trainees to perform the tasks.

Steps	Working

a **1** Convert the table to a matrix.

$$\begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 17 & 24 & 27 & 20 \\ 29 & 28 & 23 & 26 \\ 21 & 18 & 17 & 14 \\ 28 & 26 & 22 & 19 \end{bmatrix} \end{array}$$

2 Complete a row reduction. Identify the smallest number in each row and subtract it from all other numbers in the same row.

Row A subtract 17.

Row B subtract 23.

Row C subtract 14.

Row D subtract 19.

$$\begin{array}{ccc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} & \text{Row reduction} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 17 & 24 & 27 & 20 \\ 29 & 28 & 23 & 26 \\ 21 & 18 & 17 & 14 \\ 28 & 26 & 22 & 19 \end{bmatrix} & \begin{array}{c} -17 \\ -23 \\ -14 \\ -19 \end{array} \end{array}$$

$$= \begin{bmatrix} 0 & 7 & 10 & 3 \\ 6 & 5 & 0 & 3 \\ 7 & 4 & 3 & 0 \\ 9 & 7 & 3 & 0 \end{bmatrix}$$

3 For every column that does not have a zero value, complete a column reduction. Identify the smallest number in the column and subtract it from every number in the same column.

Only column 2 requires reduction.

Column 2 subtract 4.

$$\begin{array}{cc} \text{Column reduction} & \begin{array}{cccc} 0 & -4 & 0 & 0 \end{array} \\ = & \begin{bmatrix} 0 & 3 & 10 & 3 \\ 6 & 1 & 0 & 3 \\ 7 & 0 & 3 & 0 \\ 9 & 3 & 3 & 0 \end{bmatrix} \end{array}$$

4 Cover all the zeros with the smallest number of lines (horizontal or vertical).

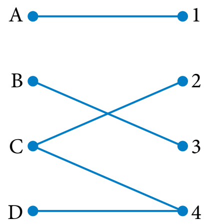
The number of lines is equal to the number of rows, so the reduction process is complete.

$$\begin{bmatrix} 0 & 3 & 10 & 3 \\ 6 & 1 & 0 & 3 \\ 7 & 0 & 3 & 0 \\ 9 & 3 & 3 & 0 \end{bmatrix}$$

The placement of the lines does not matter as long as the zeroes are covered by the minimum number of lines.

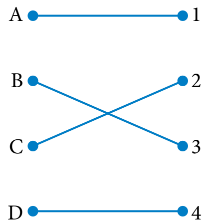
b **1** The zeroes indicate the connections in the bipartite graph.

Draw a graph connecting A–1, B–3, C–2, C–4 and D–4.



2 Look at all the tasks and connections. Where there are multiple connections, identify an allocation.

All tasks have single connections except task 4. Task 4 is connected to trainees C and D, but D is only connected to task 4, so eliminate the connection between C and task 4.



3 State the optimum allocation.

Trainee A must complete task 1.

Trainee B must complete task 3.

Trainee C must complete task 2.

Trainee D must complete task 4.

c **1** Calculate the time taken.

$$\begin{aligned}\text{time taken} &= 17 + 23 + 18 + 19 \\ &= 77\end{aligned}$$

2 State the result.

The minimum time required for the trainees to perform the tasks is 77 minutes.

The Hungarian algorithm – Stage 2

Stage 2 of the Hungarian algorithm is used when the row and column reduction (Stage 1) *does not* produce the optimum solution.

The Hungarian algorithm – Stage 2

1. Complete a row and column reduction and cover all the zeros with the smallest number of horizontal or vertical lines possible. If the number of lines is equal to the number of rows, a solution has been found (Stage 1).
2. If the number of lines is less than the number of rows, identify the smallest uncovered number and add it to every covered number. If a number is covered twice, add this number to it twice.
3. Identify the smallest number in the matrix and subtract it from all the numbers in the matrix.
4. Cover all the zeros using the smallest number of lines (horizontal or vertical). If the number of lines is equal to the number of rows, then the process is complete. If this is *not* the case, repeat steps 2, 3 and 4 until a solution is found.
5. Draw a bipartite graph showing all the allocations and use it to determine the optimum allocation.

10.4

WORKED EXAMPLE 12 Stages 1 and 2 of the Hungarian algorithm

Five couriers are available to pick up parcels at 5 different locations. The distance (in km) each courier is located from the parcels is given in the table.

	Parcel 1	Parcel 2	Parcel 3	Parcel 4	Parcel 5
A	9	17	10	4	10
B	7	22	12	7	15
C	10	17	17	9	11
D	12	20	14	4	13
E	8	18	11	9	5

a **Determine** the allocation of couriers to parcels that minimises the distance travelled.

b **Calculate** the minimum distance that needs to be travelled for all 5 parcels to be picked-up.

Steps	Working

a 1 Convert the table to a matrix and complete a row reduction.

Row A subtract 4.

Row B subtract 7.

Row C subtract 9.

Row D subtract 4.

Row E subtract 5.

$$\begin{array}{c c c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} & \text{Row reduction} \\ \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} & \begin{bmatrix} 9 & 17 & 10 & 4 & 10 \\ 7 & 22 & 12 & 7 & 15 \\ 10 & 17 & 17 & 9 & 11 \\ 12 & 20 & 14 & 4 & 13 \\ 8 & 18 & 11 & 9 & 5 \end{bmatrix} & \begin{array}{c} -4 \\ -7 \\ -9 \\ -4 \\ -5 \end{array} \end{array}$$

$$= \begin{bmatrix} 5 & 13 & 6 & 0 & 6 \\ 0 & 15 & 5 & 0 & 8 \\ 1 & 8 & 8 & 0 & 2 \\ 8 & 16 & 10 & 0 & 9 \\ 3 & 13 & 6 & 4 & 0 \end{bmatrix}$$

2 Complete a column reduction.

Column 2 subtract 8.

Column 3 subtract 5.

$$\text{Column reduction} \quad \begin{array}{c} \begin{array}{ccccc} 0 & -8 & -5 & 0 & 0 \end{array} \\ = \begin{bmatrix} 5 & 5 & 1 & 0 & 6 \\ 0 & 7 & 0 & 0 & 8 \\ 1 & 0 & 3 & 0 & 2 \\ 8 & 8 & 5 & 0 & 9 \\ 3 & 5 & 1 & 4 & 0 \end{bmatrix} \end{array}$$

3 Cover all the zeros with the smallest number of lines (horizontal or vertical).

The number of lines (4) is less than the number of rows (5), stage 2 of the Hungarian algorithm must be used.

$$= \begin{bmatrix} 5 & 5 & 1 & 0 & 6 \\ 0 & 7 & 0 & 0 & 8 \\ 1 & 0 & 3 & 0 & 2 \\ 8 & 8 & 5 & 0 & 9 \\ 3 & 5 & 1 & 4 & 0 \end{bmatrix}$$

4 Identify the smallest uncovered number and add this to all covered numbers, adding it twice if a number is covered twice.

The smallest uncovered number is 1. Add 1 to all covered numbers but a 2 to numbers covered twice.

$$= \begin{bmatrix} 5 & 5 & 1 & 1 & 6 \\ 1 & 8 & 1 & 2 & 9 \\ 2 & 1 & 4 & 2 & 3 \\ 8 & 8 & 5 & 1 & 9 \\ 4 & 6 & 2 & 6 & 1 \end{bmatrix}$$

5 Identify the smallest number in the matrix and subtract it from all elements in the matrix.

The smallest number is 1. Subtract 1 from all elements.

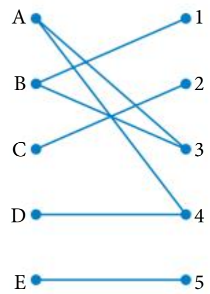
$$= \begin{bmatrix} 4 & 4 & 0 & 0 & 5 \\ 0 & 7 & 0 & 1 & 8 \\ 1 & 0 & 3 & 1 & 2 \\ 7 & 7 & 4 & 0 & 8 \\ 3 & 5 & 1 & 5 & 0 \end{bmatrix}$$

6 Cover all the zeros with the smallest number of lines (horizontal or vertical).

There are 5 lines and 5 rows, the allocation is complete.

$$= \begin{bmatrix} 4 & 4 & 0 & 0 & 5 \\ 0 & 7 & 0 & 1 & 8 \\ 1 & 0 & 3 & 1 & 2 \\ 7 & 7 & 4 & 0 & 8 \\ 3 & 5 & 1 & 5 & 0 \end{bmatrix}$$

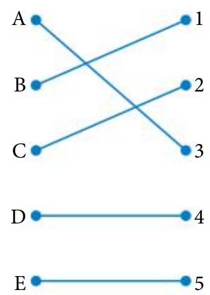
7 Draw the bipartite graph. The zeroes in the matrix indicate the allocations.



8 Determine the allocation.

Parcels 1, 2 and 5 are connected to single workers.
Courier B collects parcel 1.
Courier C collects parcel 2.
Courier E collects parcel 5.
Courier D is only connected to parcel 4.
Courier D collects parcel 4.

This leaves
Courier A collects parcel 3.



9 State the optimum allocations.

The optimum allocations are A–3, B–1, C–2, D–4 and E–5.

b **1** Calculate the distance travelled using the values from the original matrix.

$$\text{distance} = 10 + 7 + 17 + 4 + 5$$
$$= 43$$

2 State the result.

The minimum distance travelled for the couriers to collect the parcels is 43 km.

The Hungarian algorithm – Finding the maximum optimum allocation

The Hungarian algorithm is designed to find the minimal optimal solution. It can easily be adapted to determine the maximum optimal allocation. Assignment problems are transformed into a minimisation problem by subtracting every element from the largest value in the table or matrix.

The Hungarian algorithm for maximisation

1. Identify the largest number in the matrix and subtract each element in the matrix from the largest number.
2. Use the Hungarian algorithm on the resulting matrix – Stage 1 and 2 (if needed).

WORKED EXAMPLE 13 Finding the maximum optimum allocation

A call centre has 4 staff (A, B, C and D) who are each assigned to one of 4 response areas (1, 2, 3 and 4) for a shift. The average number of calls a staff member takes per shift in each response area (based on their past shifts) is given in the table.

	Area 1	Area 2	Area 3	Area 4
A	104	107	109	84
B	125	100	134	106
C	102	111	104	80
D	128	124	132	115

Determine how staff should be allocated to maximise the number of calls answered in a shift and the anticipated number of calls in total.

Steps | **Working**

1 Convert the table to a matrix and identify the largest number in the matrix.

$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 104 & 107 & 109 & 84 \\ 125 & 100 & 134 & 106 \\ 102 & 111 & 104 & 80 \\ 128 & 124 & 132 & 115 \end{bmatrix} \end{array}$$

The largest number is 134.

2 Reduce the matrix using the largest number.

Subtract all elements from the largest number.

$$= \begin{bmatrix} 134-104 & 134-107 & 134-109 & 134-84 \\ 134-125 & 134-100 & 134-134 & 134-106 \\ 134-102 & 134-111 & 134-104 & 134-80 \\ 134-128 & 134-124 & 134-132 & 134-115 \end{bmatrix}$$

$$= \begin{bmatrix} 30 & 27 & 25 & 50 \\ 9 & 34 & 0 & 28 \\ 32 & 23 & 30 & 54 \\ 6 & 10 & 2 & 19 \end{bmatrix}$$

3 Complete a row reduction.

Row A subtract 25.

Row C subtract 23.

Row D subtract 2.

Row reduction

$$\begin{bmatrix} 30 & 27 & 25 & 50 \\ 9 & 34 & 0 & 28 \\ 32 & 23 & 30 & 54 \\ 6 & 10 & 2 & 19 \end{bmatrix} \begin{array}{c} -25 \\ -0 \\ -23 \\ -2 \end{array}$$

$$= \begin{bmatrix} 5 & 2 & 0 & 25 \\ 9 & 34 & 0 & 28 \\ 9 & 0 & 7 & 31 \\ 4 & 8 & 0 & 17 \end{bmatrix}$$

4 Complete a column reduction.

Column 1 subtract 4.

Column 4 subtract 17.

Column reduction $-4 \quad 0 \quad 0 \quad -17$

$$\begin{bmatrix} 5 & 2 & 0 & 25 \\ 9 & 34 & 0 & 28 \\ 9 & 0 & 7 & 31 \\ 4 & 8 & 0 & 17 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 0 & 8 \\ 5 & 34 & 0 & 11 \\ 5 & 0 & 7 & 14 \\ 0 & 8 & 0 & 0 \end{bmatrix}$$

5 Cover all the zeros with lines (horizontal or vertical).

The number of lines (3) is less than the number of rows (4).

$$= \begin{bmatrix} 1 & 2 & 0 & 8 \\ 5 & 34 & 0 & 11 \\ 5 & 0 & 7 & 14 \\ 0 & 8 & 0 & 0 \end{bmatrix}$$

6 The smallest uncovered number is 1. Add 1 to all covered numbers but a 2 to numbers covered twice.

$$= \begin{bmatrix} 1 & 2 & 1 & 8 \\ 5 & 34 & 1 & 11 \\ 6 & 1 & 9 & 15 \\ 1 & 9 & 2 & 1 \end{bmatrix}$$

7 The smallest number is 1. Subtract 1 from all elements.

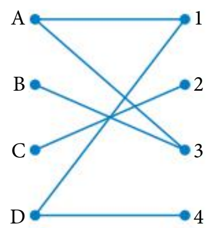
$$= \begin{bmatrix} 0 & 1 & 0 & 7 \\ 4 & 33 & 0 & 10 \\ 5 & 0 & 8 & 14 \\ 0 & 8 & 1 & 0 \end{bmatrix}$$

8 Cover all the zeros with lines (horizontal or vertical).

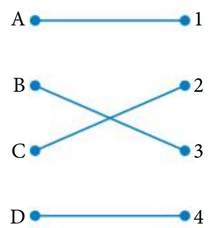
The number of lines (4) equals the number of rows (4), the allocation is complete.

$$= \begin{bmatrix} 0 & 1 & 0 & 7 \\ 4 & 33 & 0 & 10 \\ 5 & 0 & 8 & 14 \\ 0 & 8 & 1 & 0 \end{bmatrix}$$

9 Draw the bipartite graph. The zeroes in the matrix indicate the allocations.



10 Determine the allocation.
Area 2 and 4 are connected to single staff members.
C covers area 2.
D covers area 4.
A is now the only connection to area 1.
A covers area 1.
This leaves B covers area 3.



11 Calculate the maximum number of calls using the values from the original matrix.

Calls = 104 + 134 + 111 + 115
= 464

12 State the optimum allocations and approximate number of calls.

The optimum allocations are:
A–1, B–3, C–2 and D–4.
The anticipated number of calls is 464.

The Hungarian algorithm - Unbalanced allocations

Sometimes situations will arise where a cost matrix is not a square matrix. In this case, the assignment problem is called an **unbalanced assignment problem**. To transform the unbalanced cost matrix into a square matrix, either a **dummy** row or column is added. A dummy row or column has zeroes as all elements.

Unbalanced allocations

1. Transform the cost matrix into a square matrix by adding dummy rows or columns.
2. Use the Hungarian algorithm on the resulting matrix – Stage 1 and 2 (if needed).

WORKED EXAMPLE 14 Unbalanced allocations

The time taken for 4 machines (A, B, C and D) to complete 3 tasks (1, 2 and 3) is shown in the table.
If each machine can complete only one task, **determine** the allocation that minimises the total time taken.

	Task 1	Task 2	Task 3
A	10	14	8
B	8	8	11
C	10	13	9
D	7	7	13

Steps | **Working**

1 Convert the table to a matrix and add a dummy column to transform the matrix into a square 4 × 4 matrix.

$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 10 & 14 & 8 & 0 \\ 8 & 8 & 11 & 0 \\ 10 & 13 & 9 & 0 \\ 7 & 7 & 13 & 0 \end{bmatrix} \end{array}$$

2 Perform a column reduction as a row reduction is not needed as each row contains a zero.

Column 1 subtract 7.

Column 2 subtract 7.

Column 3 subtract 8.

$$\begin{array}{c} \text{Column reduction} \; \begin{array}{cccc} -7 & -7 & -8 & 0 \end{array} \\ \begin{bmatrix} 10 & 14 & 8 & 0 \\ 8 & 8 & 11 & 0 \\ 10 & 13 & 9 & 0 \\ 7 & 7 & 13 & 0 \end{bmatrix} \end{array}$$

$$= \begin{bmatrix} 3 & 7 & 0 & 0 \\ 1 & 1 & 3 & 0 \\ 3 & 6 & 1 & 0 \\ 0 & 0 & 5 & 0 \end{bmatrix}$$

3 Cover all the zeros with lines (horizontal or vertical).

The number of lines (3) is less than the number of rows (4).

$$= \begin{bmatrix} 3 & 7 & 0 & 0 \\ 1 & 1 & 3 & 0 \\ 3 & 6 & 1 & 0 \\ 0 & 0 & 5 & 0 \end{bmatrix}$$

4 The smallest uncovered number is 1. Add 1 to all covered numbers but a 2 to numbers covered twice.

$$= \begin{bmatrix} 4 & 8 & 1 & 2 \\ 1 & 1 & 3 & 1 \\ 3 & 6 & 1 & 1 \\ 1 & 1 & 6 & 2 \end{bmatrix}$$

5 The smallest number is 1.
Subtract 1 from all elements.

$$= \begin{bmatrix} 3 & 7 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 2 & 5 & 0 & 0 \\ 0 & 0 & 5 & 1 \end{bmatrix}$$

6 Cover all the zeros with lines (horizontal or vertical).

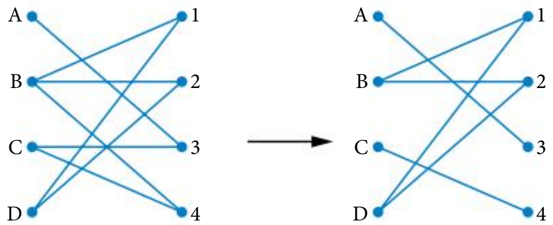
The number of lines (4) equals the number of rows (4), so the allocation is complete.

$$= \begin{bmatrix} 3 & 7 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 2 & 5 & 0 & 0 \\ 0 & 0 & 5 & 1 \end{bmatrix}$$

7 Draw the bipartite graph and determine the allocations.

The possible allocations are A–3, B–1, D–2 or A–3, B–2, D–1 or C–3, B–1, D–2 or C–3, B–2, D–1.

Task 4 is not listed as part of the allocations as it is a dummy task. This means Machine C, say, is not part of the allocation.



8 State the result.

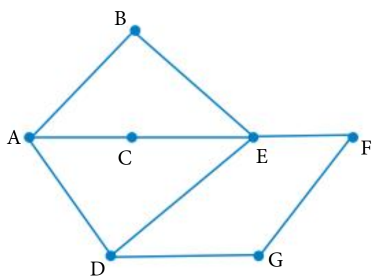
There are 2 possible allocations:
A–3, B–1 and D–2 or
A–3, B–2 and D–1.
Machine C is not allocated a task.

EXERCISE 10.4 Assignment problems and the Hungarian algorithm

ANSWERS p. 468

Recap

1 Determine if this graph is bipartite, justify your answer. If it is, identify the vertices in each set.



2 Five students have been selected by various teachers as subject ambassadors. By first drawing a bipartite graph to represent the information, determine a valid allocation of one student ambassador per subject.

Student	Favourite subjects
Hannah	Maths, Chemistry, Technology
Nate	Maths, Economics
Ella	Legal Studies, Chemistry
Riley	Economics
Mila	Maths, Chemistry

Mastery

3 WORKED EXAMPLE 10 A supervisor needs to allocate 3 different tasks to 3 workers. From experience, the supervisor knows the time taken for the workers to complete the various tasks.

- Emily takes 11 min to perform task 1, 8 min for task 2 and 12 min for task 3.
- Troy takes 13 min to perform task 1, 12 min for task 2 and 10 min for task 3.
- Ruby takes 10 min to perform task 1, 13 min for task 2 and 11 min for task 3.

a Show this information as a table.

b Match the tasks to the workers to minimise the time.

c Find the minimum time to complete the 3 tasks.

4 Each matrix shows the times (in minutes) taken for 4 workers to complete 4 different tasks. By inspection, determine

a the minimum time required to complete the 4 tasks for this matrix.

$$\begin{bmatrix} 12 & 15 & 20 & 18 \\ 22 & 17 & 14 & 20 \\ 19 & 22 & 27 & 15 \\ 35 & 28 & 12 & 18 \end{bmatrix}$$

b the maximum time required to complete the 4 tasks for this matrix.

$$\begin{bmatrix} 41 & 39 & 27 & 33 \\ 52 & 47 & 40 & 46 \\ 30 & 28 & 25 & 29 \\ 44 & 38 & 48 & 39 \end{bmatrix}$$

5 The personnel manager of a company with branches across Queensland must assign 4 new graduates to 4 regional branches. The graduates are equally well qualified, so the decision is based on the costs of relocating the graduates to the regions. The relocation costs (in thousands of dollars) for each graduate are shown in the table.

	Cairns	Mackay	Toowoomba	Ipswich
A	9	10	7	7
B	5	8	7	8
C	9	8	11	10
D	10	12	9	7

a Match the graduates to the regions to minimise relocation costs.

b Find the total cost of relocating the graduates.

6 Given the matrix shown, the total value of the optimal allocation is

A 15

B 16

C 17

D 18

$$\begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 8 & 6 & 6 \\ 4 & 5 & 7 \\ 9 & 5 & 8 \end{bmatrix} \end{array}$$

7 WORKED EXAMPLE 11 A worker for a company needs to purchase 4 different pieces of equipment. There are only 4 suppliers who offer all of the pieces of equipment. Company policy states that only one piece of equipment may be purchased from any supplier. The cost of each piece of equipment, in dollars, from the suppliers is shown in the table.

	Item A	Item B	Item C	Item D
Supplier 1	11	8	13	14
Supplier 2	12	18	9	18
Supplier 3	14	15	13	8
Supplier 4	15	16	5	7

a Convert the table into a matrix and apply stage 1 of the Hungarian algorithm.

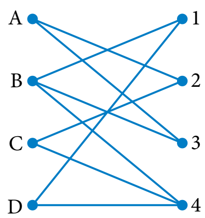
b Find the optimum allocation and show it as a bipartite graph.

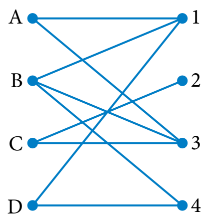
c Find the minimum cost of the equipment.

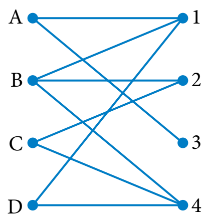
8 Four trainees are required to complete 4 tasks. The times in which each trainee can complete each task are recorded in a table and then a row and column reduction is completed (Stage 1). This process gives the matrix shown.

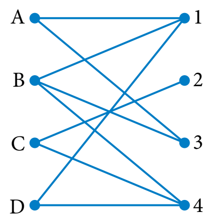
$$\begin{array}{r c}
 & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\
\begin{array}{r} \text{Trainee A} \\ \text{Trainee B} \\ \text{Trainee C} \\ \text{Trainee D} \end{array} & \begin{bmatrix} 0 & 3 & 0 & 2 \\ 0 & 5 & 0 & 0 \\ 6 & 0 & 4 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}
\end{array}$$

The bipartite graph for this network is

A


B


C


D


9 WORKED EXAMPLE 12 Four taxis are available to pick up passengers at 4 different locations. The distance (km) that each taxi must travel to reach each passenger is given in the table.

	Fare 1	Fare 2	Fare 3	Fare 4
Taxi A	20	27	22	21
Taxi B	26	19	23	44
Taxi C	22	13	16	19
Taxi D	16	31	24	33

a Use the Hungarian algorithm to determine the allocation of taxis to passengers that minimises the distance travelled.

b Find the minimum distance that needs to be travelled to pick up all 4 fares.

10 The following tables show the times taken, in minutes, for workers A, B, C and D to perform tasks 1, 2, 3 and 4. Using inspection or otherwise, find the optimum allocation of tasks and the minimum time taken in each case.

a

	Task 1	Task 2	Task 3
A	6	8	9
B	7	9	6
C	10	7	11

b

	Task 1	Task 2	Task 3	Task 4
A	15	10	12	11
B	10	14	9	11
C	10	9	13	9
D	11	12	13	14

11 WORKED EXAMPLE 13 The following tables show the profits, in thousands of dollars ($'000s), for a company's employees A, B, C and D to complete jobs 1, 2, 3 and 4. Use the Hungarian algorithm to find the optimum allocation of tasks and the maximum profit to complete all 4 jobs in each case.

a

	Job 1	Job 2	Job 3	Job 4
A	16	12	15	18
B	16	17	11	16
C	16	13	14	17
D	15	14	13	15

b

	Job 1	Job 2	Job 3	Job 4
A	20	28	45	24
B	28	21	24	23
C	32	17	34	25
D	18	23	20	17

12 WORKED EXAMPLE 14 The times taken in hours for 3 employees (A, B and C) to complete 4 different tasks (1, 2, 3 and 4) are summarised in the table. Determine the allocation that minimises the total time taken if each employee can only be assigned to one task.

	Task 1	Task 2	Task 3	Task 4
A	17	25	26	20
B	28	27	23	25
C	20	18	17	14

Exam practice

13 Four taxis (A, B, C and D) must be assigned to pick up 4 different passengers. The distance (km) that each taxi is from each of the customers is shown in the matrix.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 10 & 4 & 9 & 5 \\ 8 & 11 & 10 & 7 \\ 9 & 8 & 6 & 7 \\ 6 & 11 & 8 & 8 \end{bmatrix} \end{array}$$

This data is converted into a matrix and row and column reductions are performed. The resulting matrix is

A $\begin{bmatrix} 4 & 0 & 3 & 0 \\ 2 & 7 & 4 & 2 \\ 3 & 4 & 0 & 2 \\ 0 & 7 & 2 & 3 \end{bmatrix}$

B $\begin{bmatrix} 6 & 3 & 0 & 1 \\ 1 & 4 & 0 & 3 \\ 3 & 0 & 1 & 2 \\ 2 & 5 & 0 & 2 \end{bmatrix}$

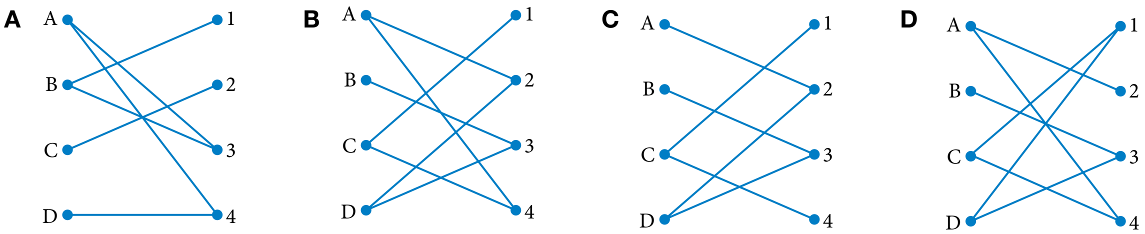
C $\begin{bmatrix} 6 & 0 & 5 & 1 \\ 1 & 4 & 3 & 0 \\ 3 & 2 & 0 & 1 \\ 0 & 5 & 2 & 2 \end{bmatrix}$

D $\begin{bmatrix} 5 & -1 & 4 & 0 \\ 0 & 3 & 2 & -1 \\ 2 & 1 & -1 & 0 \\ -2 & 3 & 0 & 0 \end{bmatrix}$

14 Four employees (A, B, C and D) must be assigned one of 4 different tasks (1, 2, 3 and 4). The time taken for each employee to perform each of the tasks is tabulated and converted into a matrix. The matrix, following row and column reduction, is shown.

$$\begin{array}{c c} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \end{matrix} & \begin{bmatrix} 4 & 0 & 2 & 0 \\ 3 & 1 & 0 & 5 \\ 0 & 6 & 2 & 0 \\ 4 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

The bipartite graph representing the allocation of employees to tasks is



15 ©QCAA 2022 1Q20 (4 marks) The table summarises the distances in kilometres (km) between three flower stores and three delivery locations: A, B and C.

Use the Hungarian algorithm to determine the minimum total distance needed to deliver flowers to all locations if each store delivers flowers to only one location.

	A	B	C
Store 1	19	17	24
Store 2	15	14	22
Store 3	23	16	40

16 (5 marks) Three equally qualified workers are chosen to complete different tasks in a manufacturing process.

- Worker A takes 25 mins to perform task 1, 20 mins for task 2 and 22 mins for task 3
- Worker B takes 18 mins to perform task 1, 17 mins for task 2 and 15 mins for task 3
- Worker C takes 16 mins to perform task 1, 18 mins for task 2 and 17 mins for task 3

Determine the minimum time to complete the 3 tasks if each worker completes only one task.

17 (4 marks) An RACQ dispatcher has 4 service vehicles available to respond to 4 separate calls for assistance in a Southside service area. The times, in minutes, for each of the service vehicles to reach the callers is shown in the table.

	Caller 1	Caller 2	Caller 3	Caller 4
Service vehicle 1	42	36	52	26
Service vehicle 2	36	30	48	35
Service vehicle 3	31	28	50	21
Service vehicle 4	44	38	49	39

Use the Hungarian algorithm to determine minimum time needed for the service vehicles to reach all the callers if each service vehicle can only go to a single location.

18 (8 marks) A factory produces security screen doors for houses. Today, 4 workers are rostered on who are each trained to operate the 3 different machines that make the doors. Each machine will have one worker assigned to it for the whole day. The table below summarises the number of security screen doors that can be produced in a day by each machine operator.

	Machine 1	Machine 2	Machine 3
Anna	35	30	32
Lani	34	46	37
Stephen	24	29	17
Tracey	36	46	33

a Draw the weighted bipartite graph showing the possible allocations for each of the workers. *[2 marks]*

The shift manager, Robert, wants to allocate the workers to the machines so that the production for the day is at a maximum. He decides to use the Hungarian algorithm to determine the allocation. His first step is to rewrite the table in matrix form, adding in a column containing all zeros.

$$\begin{bmatrix} 35 & 30 & 32 & 0 \\ 34 & 46 & 37 & 0 \\ 24 & 29 & 17 & 0 \\ 36 & 46 & 33 & 0 \end{bmatrix}$$

b Why has Robert added the column of zeros? *[1 mark]*

c Use the Hungarian algorithm to determine the optimum allocation of workers to machines. State the maximum total number of doors that can be produced on this day. *[5 marks]*

19 (5 marks) CF The delivery times, in minutes, for 4 couriers (A, B, C and D) to deliver parcels to 4 different locations (1, 2, 3 and 4) is shown in the table below.

	Parcel 1	Parcel 2	Parcel 3	Parcel 4
Courier A	30	22	24	25
Courier B	26	22	36	23
Courier C	43	21	23	39
Courier D	40	22	23	39

Use the Hungarian algorithm to determine who should deliver each parcel to minimise the time taken and how long it will take for all deliveries to be made.

20 ©QCAA 2020S 2Q5 (8 marks) CF Kate, Luca and Marcel are carpenters. Each carpenter needs to build one piece of furniture: a bookcase, a chair or a desk.

As each carpenter works at a different rate, the table shows the total cost for each carpenter to build each piece of furniture.

	Bookcase	Chair	Desk
Kate	\$196	\$62	\$203
Luca	\$150	\$60	\$147
Marcel	\$127	\$77	\$111

a Use the Hungarian algorithm to determine who should build each piece of furniture to minimise the total cost. *[5 marks]*

b Evaluate the reasonableness of your solution to part **a**. *[3 marks]*

21 (6 marks) CF A sports club is selecting a mixed relay team for competition involving 5 events: run (R), swim (S), cycle (C), throw (T) and jump (J). Each athlete's performance in the event is converted into points.

In the relay competition, each athletes' points for their event are added together to give the team's total score. The table below shows the number of points each athlete (Lenny (L), Isabelle (I), Oliver (O), Natalie (N) and Zane (Z)) is anticipated to get in the 5 events based on their past performances.

	R	S	C	T	J
L	125	152	167	112	104
I	172	130	183	150	139
O	136	144	130	109	112
N	178	173	189	172	160
Z	114	102	125	167	184

Use the Hungarian algorithm to predict the maximum score the team could get if assigning each athlete to completing one section.

22 ©QCAA 2020 2Q5 (5 marks) CU A company has three tasks to allocate to three contractors. Each of the contractors has a quote recorded for each task, shown in the table. The quotes are in thousands of dollars ($'000s).

Use a matrix method to determine the minimum cost if each contractor is allocated one task.

Contractor	Task 1	Task 2	Task 3
A	3	3	1
B	4	7	2
C	4	4	1

23 (7 marks) CU A display home village will have 5 different display homes. Five builders are approached and asked to submit quotations to complete the build for each of the homes. The quotes in thousands of dollars ($'000s) submitted by each builder are shown in the table.

	Endeavour	Tropicana	Beachside	Forrest Grove	Alpine
Masters Homes	220	310	460	320	400
Custom Build	280	330	420	450	380
Savannah Homes	280	370	510	440	290
Deakin Homes	460	440	560	470	380
Stylemaster	340	480	390	280	330

The builders are equally qualified and have quoted on the same specifications for each house. The developers who are planning the display home village want to complete the project for the minimum cost.

Find the optimum allocation of builders to construct the houses and the total cost of construction.

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24 (7 marks) CU A business analyst is looking at how a company can maximise its sales. The company has 5 salespeople servicing 5 regions across Queensland. Each salesperson's average sales in thousands of dollars ($'000s) for each region is summarised in the table.

10.4

		Regions				
		1	2	3	4	5
Salesperson	A	34	43	58	44	49
	B	40	45	54	57	50
	C	40	49	63	56	41
	D	58	56	68	59	50
	E	46	60	51	40	45

Use the Hungarian algorithm to predict the maximum average sales if the business analyst assigns each salesperson to one region.

25 (5 marks) CU An e-bike company has 4 small workshops that each produce 4 types of e-bikes. The annual cost of production of the e-bikes at each workshop is shown in the table, with all values in thousands of dollars ($'000).

	e-bike 1	e-bike 2	e-bike 3	e-bike 4
Workshop A	33	51	58	47
Workshop B	41	39	64	47
Workshop C	36	55	67	46
Workshop D	44	40	64	49

The revenue matrix, in thousands of dollars ($'000), for the sale of the e-bikes produced annually at each workshop is given by

$$\begin{bmatrix} 44 & 68 & 67 & 60 \\ 52 & 59 & 80 & 57 \\ 45 & 72 & 82 & 62 \\ 51 & 61 & 83 & 52 \end{bmatrix}$$

Given that profit = revenue − cost, complete the profit matrix to determine the appropriate allocation of workshops to e-bike type that will produce the maximum annual profit and state the annual profit.

26 ©QCAA 2024 2Q5 (6 marks) CU A flying doctor coordinator allocates a plane from each of three airbases, A, B and C, to fly to one of three sites, P, Q and R, to provide medical care. Distances (km) are shown in the table.

	P (28° S 136° E)	Q	R (20° S 147° E)
A (20° S 136° E)	x	600	y
B	445	485	340
C	980	1170	770

Determine the optimal allocation for each plane and the minimum total distance flown.

Video
Exam question analysis: Flow and assignment problems

EXAM QUESTION ANALYSIS

©QCAA 2023 2Q1 (5 marks)

A triathlon relay has three sections: swim (S), cycle (C) and run (R). The matrix shows the average number of minutes for three athletes, Jane (J), Knox (K) and Levi (L), to complete each section.

$$\begin{array}{c} \\ J \\ K \\ L \end{array}\begin{array}{c} \begin{array}{ccc} S & C & R \end{array} \\ \begin{bmatrix} 40 & 56 & 66 \\ 36 & 60 & 72 \\ 25 & 48 & 78 \end{bmatrix} \end{array}$$

Use the Hungarian algorithm to predict the minimum total relay time if assigning each athlete to completing one section.

Reading the question

- The question is asking for the minimum total relay time.
- The question is explicitly stating the Hungarian algorithm must be used.
- There are 3 athletes and 3 sections to be allocated.
- A matrix summarising the average times of athletes for each section has been given.

Thinking about the question

- You need to use the Hungarian algorithm.
- First a row and column reduction must be done (Stage 1).
- Stage 2 may need to be completed depending on the row and column reduction result.
- A bipartite graph of the possible allocations should be drawn.
- The optimum allocation must be identified.
- The minimum average total time based on the optimum allocation must be found.

Worked solution (✓ = 1 mark)

Complete a row and column reduction.

$$\begin{array}{c} \\ J \\ K \\ L \end{array}\begin{array}{c} \begin{array}{ccc} S & C & R \end{array} \\ \begin{bmatrix} 40 & 56 & 66 \\ 36 & 60 & 72 \\ 25 & 48 & 78 \end{bmatrix} \end{array} \quad \begin{array}{c} \text{Row reduction} \\ -40 \\ -36 \\ -25 \end{array}$$

$$= \begin{bmatrix} 0 & 16 & 26 \\ 0 & 24 & 36 \\ 0 & 23 & 53 \end{bmatrix} \quad ✓$$

Column reduction $\quad 0 \quad -16 \quad -26$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 8 & 10 \\ 0 & 7 & 27 \end{bmatrix} \quad ✓$$

As the number of lines (2) is less than the number of rows (3), continue with Stage 2 of the algorithm.

The smallest uncovered number is 7. Add 7 to all covered numbers and 14 to numbers covered twice.

$$= \begin{bmatrix} 14 & 7 & 7 \\ 7 & 8 & 10 \\ 7 & 7 & 27 \end{bmatrix}$$

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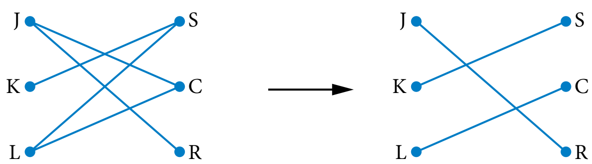
The smallest number in the matrix is 7 so subtract it from all elements in the matrix.

$$= \begin{bmatrix} 7 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 20 \end{bmatrix} \quad \checkmark$$

The number of lines (3) is equal to the number of rows (3), so the process is complete.

Allocate an athlete to each section.



The optimal allocation is **Jane – run, Knox – swim and Levi – cycle.** ✓

Determine the predicted minimum total relay time based on allocation.

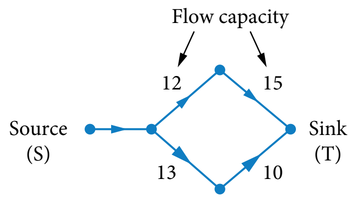
Predicted minimum total relay time = 66 + 36 + 48

= **150 minutes** ✓

10 Chapter summary

Flow networks and capacity

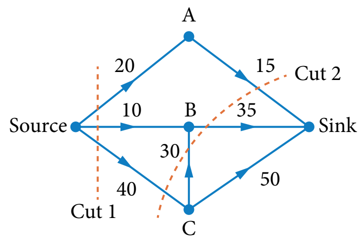
- A **flow network** is a directed graph where each arc has a **capacity** or **flow capacity**.
- The **source (S)** is the first node in the network and is where all flows commence.
- The **sink (T)** is the last node in the network towards which all flows move.
- The **inflow capacity** is the total flow capacity leading into the node and the **outflow capacity** is the total flow capacity leaving the node.



- The **maximum flow** through a node is the smaller of the total inflow capacity and the total outflow capacity of the node.

Maximum flow – minimum cut

- A network **cut** is a line that separates the source from the sink. A **valid** cut stops *all* flow from the source to the sink.
- The **capacity of a cut** is the sum of the flows that it passes across. Only flows that pass in the direction from the source to the sink of a cut are included in the calculation of the capacity of the cut.
- The **maximum flow** from source to sink can be determined by finding the **minimum cut**, which is the cut with the minimum capacity.

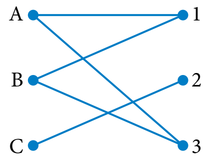


To determine the maximum flow for a network:

1. Identify cuts through the network.
2. Find the capacity of each cut.
3. Maximum flow = capacity of the minimum cut.

Bipartite graphs

- A **bipartite graph** has its vertices in 2 distinct sets. Edges join elements in the first set to elements in the second set.
- Bipartite graphs can be represented by an **adjacency matrix**, where a 1 represents a connection and a 0 indicates there is no connection.



$$\begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

The Hungarian algorithm – Stage 1 (Row and column reduction)

1. Choose the smallest number in each row and subtract it from every element in the same row.
2. For every column that does not have a zero value, choose the smallest number in the column and subtract it from every element in the same column.
3. Cover all the zeros using the smallest number of lines (horizontal or vertical). If the number of lines is equal to the number of rows, then the row and column reduction process is complete. *The zeros in the reduced matrix represent the allocations.*

The Hungarian algorithm – Stage 2

1. Complete a row and column reduction and cover all the zeros with the smallest number of horizontal or vertical lines possible.
   If the number of lines is equal to the number of rows, a solution has been found (Stage 1).
2. If the number of lines is less than the number of rows, identify the smallest uncovered number and add it to every covered number.
   If a number is covered twice add this number to it twice.
3. Identify the smallest number in the matrix and subtract it from all of the numbers in the matrix.
4. Cover all the zeros using the smallest number of lines (horizontal or vertical). If the number of lines is equal to the number of rows, then the process is complete. If this is *not* the case, repeat steps 2, 3 and 4 until a solution is found.
5. Draw a bipartite graph showing all the allocations and use it to determine the optimum allocation.

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The Hungarian algorithm – Maximisation

1. Identify the largest number in the matrix and subtract each element in the matrix from the largest number.
2. Use the Hungarian algorithm on the resulting matrix.

The Hungarian algorithm – Unbalanced allocations

1. Transform the cost matrix into a square matrix by adding dummy rows or columns.
2. Use the Hungarian algorithm on the resulting matrix.

Cumulative examination 1

Worksheet
General Maths formula sheet

Simple familiar

Perusal time: 2 minutes Working time: 30 minutes

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

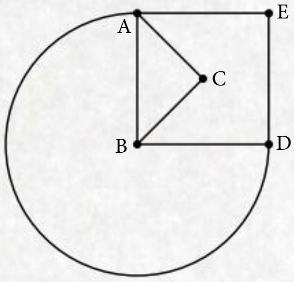
Section 1 (5 marks): 5 multiple choice questions

Section 2 (13 marks): 3 short response questions

Total: 18 marks

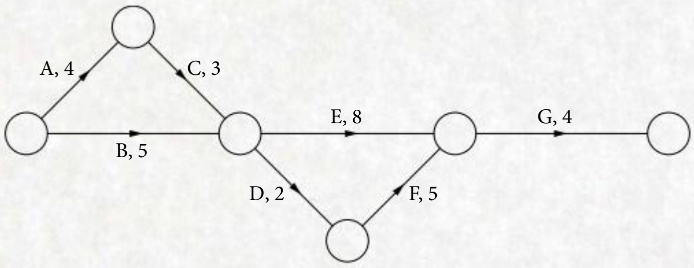
Section 1 5 multiple choice questions 5 marks

1 © QCAA 2021 1Q2 How many faces does this planar graph have?



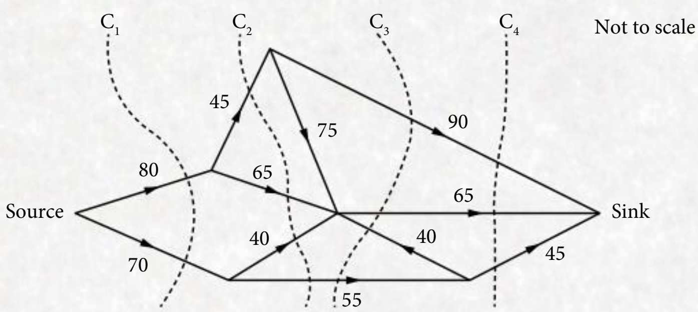
A 3 **B** 4 **C** 5 **D** 7

2 © QCAA 2021 1Q13 Based on this project network, what is the minimum number of days required to complete the project?



A 16 **B** 17 **C** 19 **D** 31

3 © QCAA 2021 1Q7 Which cut (C_1, C_2, C_3 or C_4) could be used to determine the maximum flow from the source to the sink in this network?



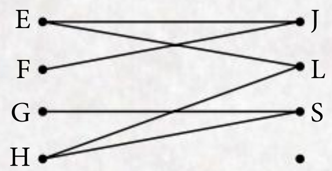
A C_1 **B** C_2 **C** C_3 **D** C_4

4 © QCAA 2023 1Q13 Four athletes, Eoin (E), Fedir (F), Gede (G) and Hana (H), compete in three events: javelin (J), long jump (L) and sprints (S).

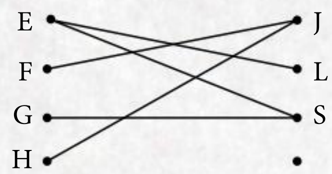
Athlete	Events
Eoin	L, S
Fedir	J
Gede	S
Hana	J, L

Which bipartite graph represents this information?

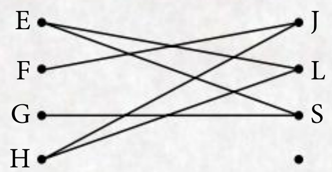
A



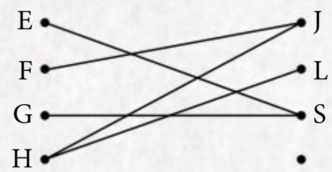
B



C



D



5 © QCAA 2022 1Q7 This matrix was obtained after applying the Hungarian algorithm to determine the optimal allocation of three people, Elandra (E), Farid (F) and Grace (G), to three tasks: legal (L), monitoring (M) and verification (V).

$$\begin{array}{c} \\ E \\ F \\ G \end{array}\begin{array}{c} \begin{array}{ccc} L & M & V \end{array} \\ \begin{bmatrix} 0 & 0 & 4 \\ 0 & 3 & 8 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

The optimal allocation is

A E to V, F to M and G to L.

B E to V, F to L and G to M.

C E to M, F to L and G to V.

D E to M, F to V and G to L.

Section 2 3 short response questions

13 marks

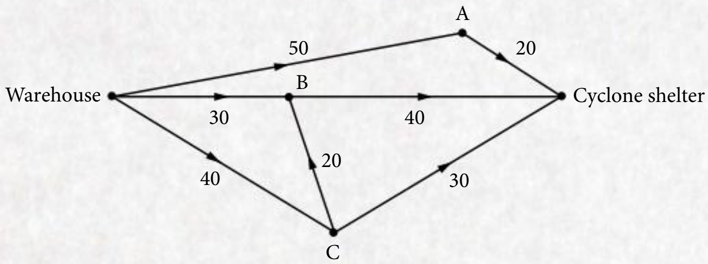
6 © QCAA 2023 1Q19 (4 marks) Ngarra compares two investment options and decides option A will provide the better return.

Option A: 5.60% p.a. compounding monthly

Option B: 5.62% p.a. compounding quarterly

Use the effective annual rate of interest formula to evaluate the reasonableness of Ngarra's decision.

7 © QCAA 2020 1Q21 MODIFIED (5 marks) This network shows the maximum number of supplies (in tonnes) that can be transported from a warehouse to a cyclone shelter along each road each day during an emergency.



a Determine the maximum amount of supplies that can reach the cyclone shelter each day. *[3 marks]*

b During a cyclone, the intersection at vertex A is damaged and no longer allows for any supplies to pass through it. What is the new maximum amount of supplies each day that can reach the cyclone shelter? *[2 marks]*

8 © QCAA 2020 1Q22 (4 marks) A store asked its junior and senior staff whether or not they would like to change the store uniform. The results are in the frequency table.

	Change uniform	Do not change uniform
Junior staff	92	28
Senior staff	23	67

a Convert the two-way table into a percentaged two-way frequency table using column totals. *[2 marks]*

b Explain whether there is an association between staff groups and a desire to change the store uniform. *[2 marks]*

Cumulative examination 2

Complex familiar and unfamiliar

Perusal time: 2 minutes Working time: 30 minutes

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

2 short response questions

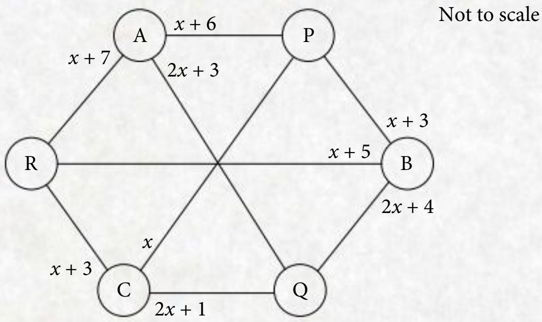
12 marks

1 © QCAA 2021 2Q1 (4 marks) A sailor anchors her yacht near Rocky Island at 14°52′ S, 145°29′ E. Her yacht is at the same latitude as her home, but the sun rises exactly 1 hour and 13 minutes later at home. What are the coordinates of her home?

Worksheet General Maths formula sheet

2 © QCAA 2021 2Q6 MODIFIED (8 marks) A tow truck company has three tow trucks (A, B and C) and receives calls from three motorists (P, Q and R), who have broken down.

The network shows the distances in kilometres (km) from each of the tow trucks to each of the motorists, where x represents the distance between Tow Truck C and Motorist P.



The minimum total distance travelled by the three tow trucks in order for each tow truck to visit exactly one motorist is 32 km. Use the Hungarian algorithm to determine the distance between Tow Truck B and Motorist R.

Answers

CHAPTER 1

EXERCISE 1.1

1 a

	Surf	Swim	Lie on beach	Total
Staying locally	12	15	4	31
Visiting	6	7	6	19
Total	18	22	10	50

b 7

2

	Logging support		
	Yes	No	Total
NSW	20	56	76
Victoria	18	63	81
Queensland	25	43	68
NT	8	23	31
SA	12	48	60
WA	24	39	63
Tasmania	15	32	47
ACT	2	25	27
Total	124	329	453

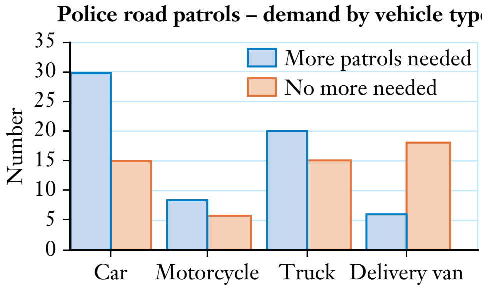
3

		Flat	Unit	House	Total
Refrigerators and freezers	1	40	35	40	115
	2	10	20	70	100
	3	5	5	30	40
	4	0	0	20	20
	> 4	0	0	5	5
Total		55	60	165	280

4 a 23 **b** 47

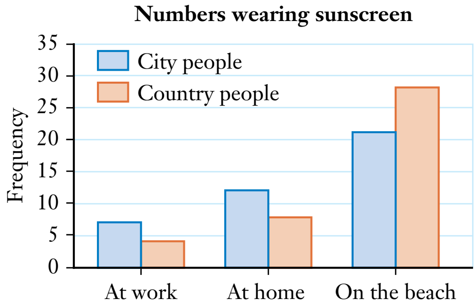
5

Police road patrols – demand by vehicle type



6

Numbers wearing sunscreen



7

		Age		
		≤ 20 (student)	> 20 (teacher)	Total
Music type	C	5	6	11
	J	5	7	12
	E	9	5	14
	CW	5	6	11
	R	12	8	20
Total		36	32	68

It looks like more students have a greater preference for easy listening and rock than teachers.

8 B **9** D

10 a 465 **b** 120

c Under 40s are more in favour of the expansion.
40 year olds and over are split in their opinion. Overall, more customers were in favour of the expansion.

11

	Positive result	Negative result	Total
Has COVID-19	505	20	525
Does not have COVID-19	15	1460	1475
Total	520	1480	2000

12

	Permission	No permission	Total
Ten-pin bowling	68	17	85
Ice-skating	72	18	90
Basketball	36	9	45
Total	176	44	220

176 students will be allowed to travel to the sports venues.

EXERCISE 1.2

1 D

2

		Gender		
		Female	Male	Total
Preference	Music concert	155	100	255
	Comedy show	85	80	165
Total		240	180	420

3 a 66.28% **b** 6.61% **c** 0.09%
d 27.83% **e** 65.02% (drink/drug driving)

4 a 26% **b** 22% **c** 15%
d 5% **e** 15%

5 **a** 27% **b** 15% **c** 26%
d 26% **e** 12%

6 **a**

	Republic			
	Yes	Don't care	No	Total
Labor	28	4	17	49
LNP	25	3	27	55
Minor parties	8	7	6	21
Total	61	14	50	125

b

	Republic			
	Yes	Don't care	No	Total
Labor	57.1%	8.2%	34.7%	100%
LNP	45.5%	5.5%	49.1%	100%
Minor parties	38.1%	33.3%	28.6%	100%
Total	48.8%	11.2%	40%	100%

Labor voters favour a republic, but Liberal/National voters oppose it. Minor party voters are equivocal.

7 Table will vary depending on state groupings.

	Oil exploration support		
	Yes	No	Total
NSW, Qld	35%	65%	100%
ACT, NT, Vic, SA	27.6%	72.4%	100%
Tas, WA	35.7%	64.3%	100%
Total	31.7%	68.3%	100%

Australians are generally opposed to oil exploration on the reef.

8 A **9** B

10 **a** Rugby League **b** AFL

c Rugby League (88%) and AFL (73%) had the higher view rates of both the female and male audiences, so these should be broadcast.

11 **a**

	Antibody result		
	Positive	Negative	Total
Vaccinated	78	7	85
Non-vaccinated	5	110	115
Total	83	117	200

b

	Antibody result	
	Positive	Negative
Vaccinated	94%	6%
Non-vaccinated	6%	94%
Total	100%	100%

c The false positive error rate was approximately 6% and false negative was approximately 6%. The total error rate out of the 200 tests was approximately 6%.

12

		Age group			
		25 or less	26 to 35	36+	Total
Siblings	2 or less	9	8	5	22
	More than 2	4	4	3	11
Total		13	12	8	33

		Age group		
		25 or less	26 to 35	36+
Siblings	2 or less	69%	67%	63%
	More than 2	31%	33%	38%
Total		100%	100%	100%

There is a tendency for those who are older to have more siblings, but this is only slight and could be a random variation, as some of the group sizes are less than 5. The results may suggest that families were larger in the past.

13 63.3%

EXERCISE 1.3

1 C **2** by column

3 **a** temperature, evaporation rate
b time, mass of ice melted
c mass of flour, cost
d number of hens, number of eggs laid
e number of king tides, erosion of beach

4 **a** amount of rainfall, flow rate of river
b number of birds, number of seedlings eaten
c time spent training, time taken to run 100 m
d number of logs milled, amount of sawdust
e amount of sawdust, amount of time taken to sweep

5 **a** incorrectly drawn; length of foot is the explanatory variable so should be shown on the horizontal axis
b correct
c correct
d incorrectly drawn; size of milking herd is the explanatory variable so should be shown on the horizontal axis

6 Sample answers:
age of student, height of student
mass of vegetable, cost of vegetable
sail area of yacht, speed of yacht
number of tiles available, floor area tiled
size of army, cost of army
length of yacht, sail area of yacht

7 The following are not numerical, so cannot be explanatory or response: type of vegetable; hair colour of student.

8 D **9** C

10 heart rate

11 Yes, as ice-cream sales is the response variable as it is dependent on the temperature, the temperature (explanatory variable) is to be shown on the horizontal axis.

12 Check your responses with your teacher. Examples: temperature (EV) and plant height (RV), rainfall (EV) and plant height (RV).

EXERCISE 1.4

1 **a** academic results **b** hearing loss

2 430

3 **a** moderate negative linear association

b non-linear association

c strong positive linear association

d weak negative linear association

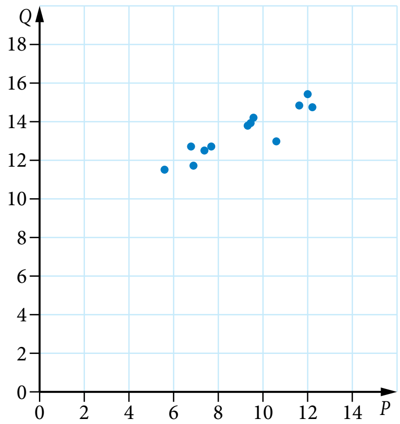
e no apparent association

f strong negative linear association

g weak positive linear association

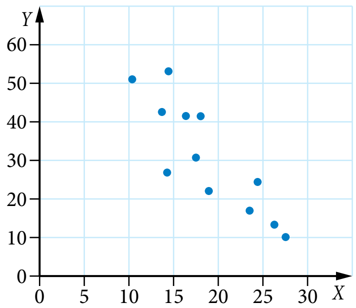
h moderate positive linear association

4 **a**



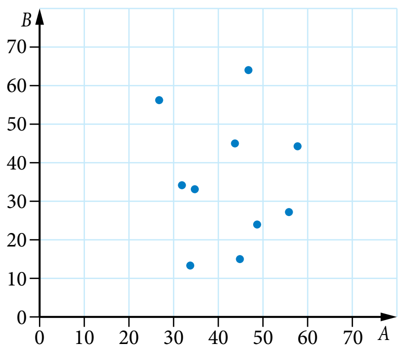
b strong positive linear

5 **a**



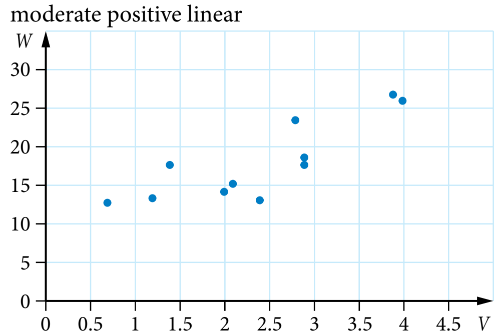
b moderate negative linear

6 **a**

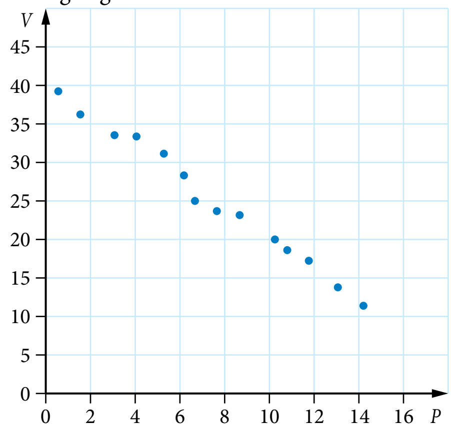


b no association

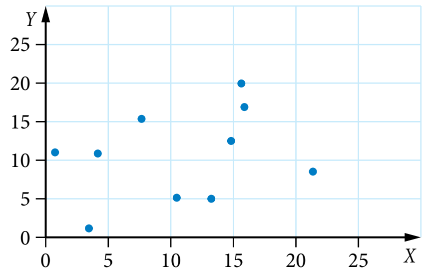
7 **a** moderate positive linear



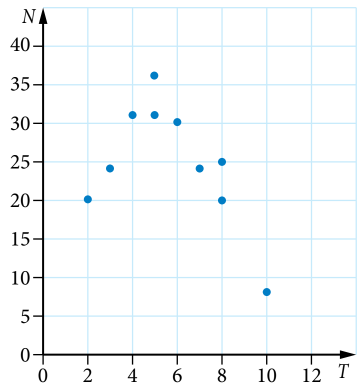
b strong negative linear



c no association



d moderate non-linear association

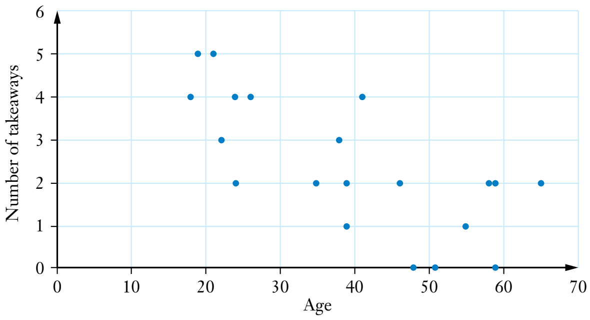


8 A **9** B **10** D

11 amount of “zero sugar” drinks (EV), weight gain (RV)

12 moderate non-linear association

13



There is a weak negative linear association between the age and the number of takeaway meals eaten. This would suggest that younger age groups are more likely to eat takeaway food than older groups.

14 B

EXERCISE 1.5

1 C

2 temperature

3 **a** weak negative **b** perfect positive
c moderate positive **d** strong negative
e weak positive **f** perfect negative
g strong positive **h** moderate positive

4 −0.88, a strong negative association

5 0.33, a weak positive association

6 0.62, a moderate positive association

7 0.84, a strong positive association

8 **a** 0.16 **b** 0.0243 **c** 0.7225
d 1 **e** 0.5776 **f** 0.1521
g 0.3025 **h** 0.3795

9 **a** 0.25 **b** −0.91 **c** −0.66
d 0.85 **e** 0.56 **f** −0.145

10 $r = 0.98$, a very strong positive association. So, the mouse population is increasing over time.

11 A **12** B **13** A

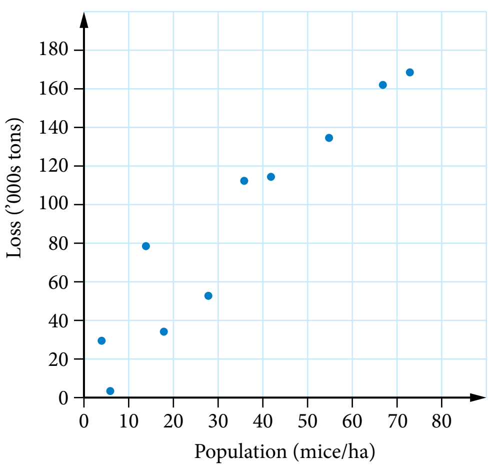
14 0.5449…

15 **a** −0.55

b The data shows a moderate negative linear association. The coefficient of determination suggests that 30.25% of the response variable is explained by the explanatory variable.

16 mice population (EV), loss (RV); the two rows should be swapped.

Impact of Mouse Plagues on Wheat Production



Correlation coefficient $r = 0.947$ ...

Coefficient of determination $R^2 = 0.8971$ ...

$\approx 89.71\%$

The statement by the Department of Primary Industries is not correct. According to the data, the correlation coefficient is 0.947, which is a strong positive linear association. The coefficient of determination suggests that the mice population has an 89.71% impact on wheat yield.

17 B

CUMULATIVE EXAMINATION 1

1 B **2** B **3** C **4** C **5** B

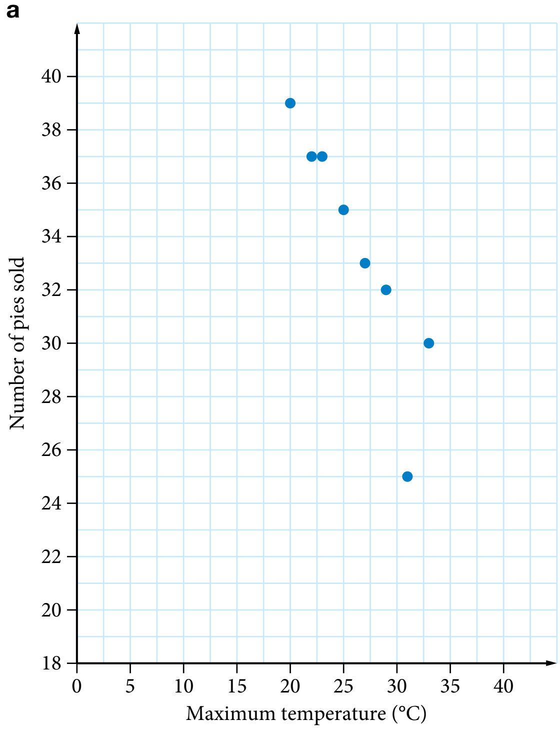
6 **a** −0.558

b The correlation coefficient indicates a moderate negative linear association between the data. This result suggests that plants do grow taller over time.

7 Answers will vary:

number of hours worked, job level, years of service.

8 **a**



b strong, negative, linear association

9 **a** YouTok

b Tikgram

c The most popular social media platforms for surveyed people under 16s is YouTok, closely followed by Tikgram. The government should work with all platforms, as the data does not show the type of content being watched or the length of time people are using the platforms.

ANSWERS

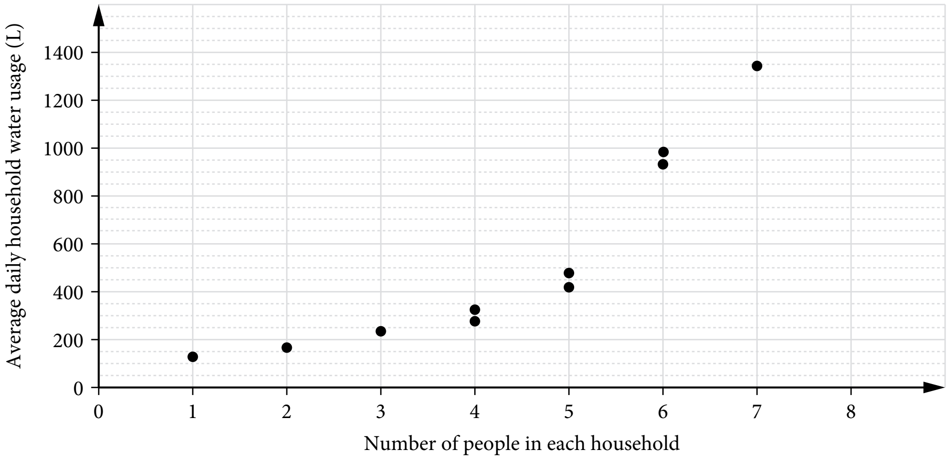
CUMULATIVE EXAMINATION 2

1

	New uniform	Alter existing uniform	No change	Total
Junior students	28.5%	15.7%	55.7%	100%
Senior students	33.7%	41.2%	25.0%	100%
Total	31.3%	29.3%	39.3%	100%

There does appear to be an association between the student groups and wanting to change the uniform. Junior school students seem happy with the uniform, while seniors want to change to a new uniform or alter the existing one.
The school should look into potential uniform changes and based on these results they could look at leaving the junior uniform as is but making some changes to the existing uniform for the senior students.

2



From the calculator $r = 0.886$.
A correlation coefficient of 0.886 indicates that the relationship is a very strong positive relationship.
However, this relationship as shown in the scatterplot does not appear to be linear, therefore the correlation coefficient should not be used.

CHAPTER 2

EXERCISE 2.1

1 **a** **i** yes, $d = 1.2$ **ii** 9.6
b no
c no
d **i** yes, $d = 2$ **ii** −6
e no
f **i** yes, $d = -0.7$ **ii** 4.1

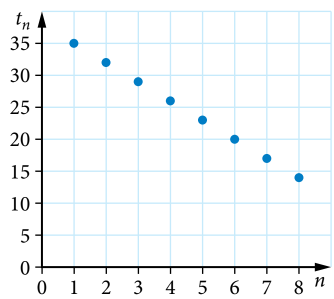
2 **a** $t_{n+1} = t_n + 3, t_1 = 11$ **b** $t_{n+1} = t_n - 3, t_1 = 98$
c $t_{n+1} = t_n + \frac{1}{4}, t_1 = 5$ **d** $t_{n+1} = t_n + 12, t_1 = 34$
e $t_{n+1} = t_n - 0.4, t_1 = 9$ **f** $t_{n+1} = t_n - 0.3, t_1 = 1.7$

3 **a** 75, 67, 59, 51, 43 **b** 11, 14, 17, 20, 23
c −2, −0.5, 1, 2.5, 4 **d** 110, 75, 40, 5, −30
e 0.3, 2.4, 4.5, 6.6, 8.7 **f** $4, 3\frac{1}{3}, 2\frac{2}{3}, 2, 1\frac{1}{3}$

4 **a** **i**

n	1	2	3	4	5	6	7	8
t_n	35	32	29	26	23	20	17	14

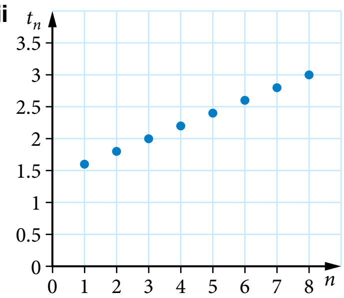
ii



b **i**

n	1	2	3	4	5	6	7	8
t_n	1.6	1.8	2.0	2.2	2.4	2.6	2.8	3.0

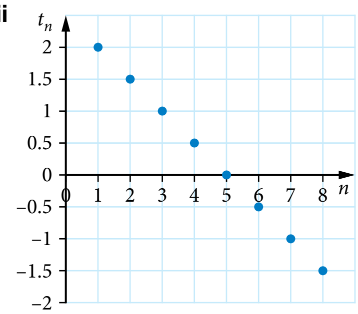
ii



c **i**

n	1	2	3	4	5	6	7	8
t_n	2	1.5	1	0.5	0	−0.5	−1	−1.5

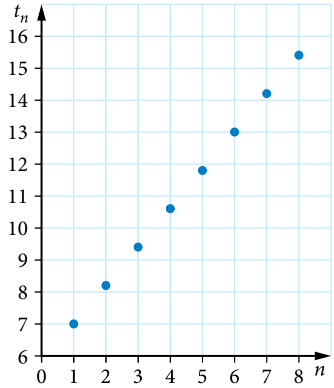
ii



d **i**

n	1	2	3	4	5	6	7	8
t_n	7	8.2	9.4	10.6	11.8	13.0	14.2	15.4

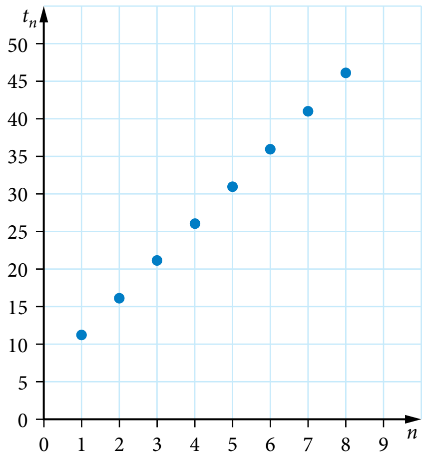
ii



e **i**

n	1	2	3	4	5	6	7	8
t_n	11	16	21	26	31	36	41	46

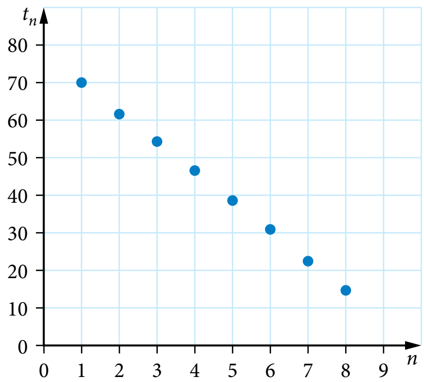
ii



f **i**

n	1	2	3	4	5	6	7	8
t_n	70	62	54	46	38	30	22	14

ii



5 25 weeks

6 **a**

Year	1	2	3	4	5	6	7	8
Diameter	4.26	4.22	4.18	4.14	4.10	4.06	4.02	3.98

b 10 years

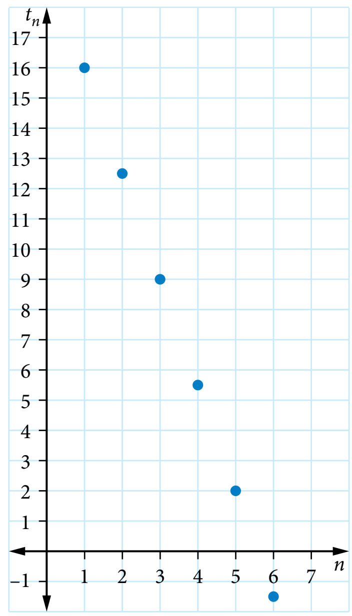
7 C

8 B

9 B

10 11, 18, 25

11



12 **a** $d = 116$ **b** 1063

EXERCISE 2.2

1 C **2** 25, 8, −9, −26, −43, −60

3 **a** −25 **b** 43.6 **c** 142.5 **d** 4.24

4 **a** 64 **b** −208 **c** 31.9 **d** $4\frac{1}{3}$

5 **a** $t_n = 9 + (n - 1) \times 0.6$ **b** 15.6

6 −9.6

7 58th term ($t_{58} = 402$)

8 **a** 20th term ($t_{20} = 27$) **b** 87th term ($t_{87} = 42.2$)

c 16th term ($t_{16} = 0.25$) **d** 18th term ($t_{18} = 9$)

9 16 terms ($t_{16} = 51.5$) **10** 39th term ($t_{39} = 14.4$)

11 4 terms ($t_4 = 0.9$) **12** $a = -2.4, d = 1.8$

13 $a = 46, d = -2$ **14** $a = 44.4, d = -1.1$

15 D **16** C

17 37th term ($t_{37} = -168$)

18 56 minutes

19 **a** 33

b

Row	1	2	3	4
Number of seats	25	33	41	49

Total =148 seats

EXERCISE 2.3

1 −13

2 **a** 78th term **b** 302.5

3 **a** $121.90 **b** $(2300 + 121.9$n$)

c $3153.30

4 **a** $(75 000 + 3375$n$) **b** $95 250

c 8 years

5 **a** $3478.95 **b** $429.50

c $12 498.45 **d** $347.18

e $(12 498.45 − 347.18$n$) **f** $4166.13

6 **a** $(3350 − 402$n$) **b** $2144

7 **a** $(42 500 + 6375$n$) **b** $87 125

c 19 years

8 **a** $(6.30 + 2.17$n$) **b** $62.72

9 **a** $\$(120 + 75n)$ **b** \$645

10 28 weeks

11 6:36 pm, assuming the rain falls at the same rate the entire time and no water is taken out.

12 $\$(10\,735.11 - 223.65n)$; \$2683.71

13 **a** **i** \$900 **ii** $\$(3600 - 900n)$

b **i** \$300 **ii** $\$(3600 - 0.03n)$

14 \$36 000

15 D **16** A

17 **a** $90 - 6n$ **b** 48 L **c** 15 minutes

18 $\$(53\,000 - 4200n)$, \$19 400

EXERCISE 2.4

1 **a** $\$(5.8 + 2.36n)$ **b** \$48.28

2 **a** $\$(65\,000 - 4700n)$ **b** \$36 800

3 **a** yes, $r = 2$ **b** no **c** yes, $r = -\frac{1}{2}$

d no **e** yes, $r = 2.1$ **f** yes, $r = -\frac{1}{3}$

4 ratio $= \frac{2}{3}$; terms: 8, $5\frac{1}{3}$, $3\frac{5}{9}$

5 $t_{n+1} = t_n \times 3$, $t_1 = 5$

6 **a** yes, $t_{n+1} = -5 \times t_n$, $t_1 = -5$

b no

c no

d yes, $t_{n+1} = -1.5 \times t_n$, $t_1 = 6$

e yes, $t_{n+1} = \frac{1}{4} \times t_n$, $t_1 = 16$

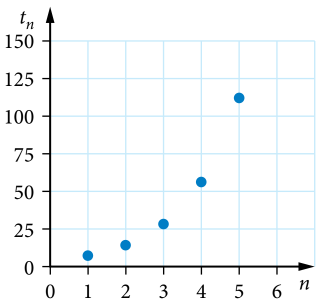
f yes, $t_{n+1} = 3 \times t_n$, $t_1 = 1.8$

7 8, −4, 2, −1, $\frac{1}{2}$

8 **a**

n	1	2	3	4	5
t_n	7	14	28	56	112

b

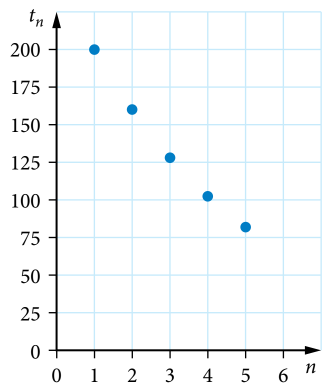


9 **a** $t_{n+1} = t_n \times 0.8$, $t_1 = 200$

b

n	1	2	3	4	5
t_n	200	160	128	102.4	81.92

c



10 **a** $r = 1.15$

b $t_{n+1} = t_n \times 1.15$, $t_1 = 75\,000$

c \$114 065.63

11 **a** Has a common ratio, so it is a geometric sequence.

b $r = 0.8$

c $t_{n+1} = t_n \times 0.8$, $t_1 = 13\,500$

d 4424 penguins

12 C **13** D **14** B

15 7.5, 11.25, 16.875

16 D

EXERCISE 2.5

1 4, 10, 25, 62.5, 156.25

2 **a** It has a common ratio, so it is a geometric sequence.

b $r = 1.08$

c $t_{n+1} = t_n \times 1.08$, $t_1 = 8100$

3 **a** 1458 **b** −640 **c** −0.0486

d $-\frac{2}{27}$ **e** 638.140 7813 **f** $-\frac{7}{512}$

4 **a** 6144 **b** $-\frac{1}{225}$ **c** 562.5

d $\frac{32}{81}$ **e** 4.096 **f** 0.2

5 **a** 768 **b** 121.5 **c** $\frac{3}{32}$

d 1250 **e** 12 **f** 933.12

6 **a** $20 = t_1 \times r^2$ **b** $160 = t_1 \times r^5$ **c** $t_1 = 5, r = 2$

7 $t_1 = 0.08, r = 5$

8 $t_1 = 9, r = \pm\frac{1}{2}$. So, 9, $4\frac{1}{2}$, $2\frac{1}{4}$, or 9, $-4\frac{1}{2}$, $2\frac{1}{4}$

9 12th term as $t_{12} \approx 307.9$

10 6th term as $t_6 = 1458$

11 8th term as $t_8 = \frac{5}{8}$

12 **a** **i** $t_1 = 15$

ii $r = 1.2$

iii $t_n = 15 \times 1.2^{(n-1)}$

b 6 weeks

13 Another 8 weeks as $t_{11} \approx 111$ mice/hectare

14 B **15** A

16 **a** $t_n = 72\,000 \times 1.031^{(n-1)}$

b January 2036

17 C

EXERCISE 2.6

1 2.53125

2 5th term as $t_5 = 2560$

3 **a** **i** $t_1 = 14\,000$ **ii** $r = 0.92$

b

Number of years	1	2	3	4
Population	14 000	12 880	11 849.6	10 901.632

c $t_n = 14\,000 \times 0.92^{(n-1)}$

d 7810

e 33 years as $t_{33} \approx 971$

4 **a** $t_1 = 8500, r = 0.7$

b $t_n = 8500 \times 0.7^{(n-1)}$

c \$2040.85

5 **a** **i** $t_1 = 2$ **ii** $r = 1.15$

b $t_n = 2 \times 1.15^{(n-1)}$

c 4.0227… km

d Using same rate, $t_{20} = 28.46\ldots$ km. This is quite a long way and it's not reasonable to assume she will continue to increase at this rate. At some point it will be too far to be able to walk!

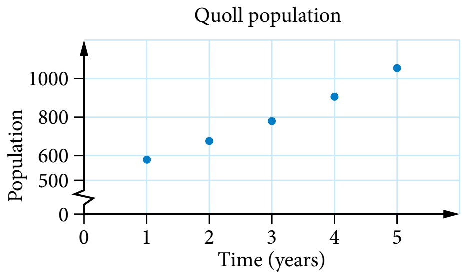
6 **a** $t_1 = 45\,000, r = 1.08$

b $t_n = 45\,000 \times 1.08^{(n-1)}$

c \$89.955.21

7 **a**

Number of years	1	2	3	4	5
Population	580	673	780	905	1050



b increasing exponentially

c 4633

8 **a** $t_n = 124\,000 - (n-1) \times 12\,300$
or $t_n = 136\,300 - 12\,300n$

b $t_n = 124\,000 \times 0.86^{(n-1)}$

c Option 2 as it gives him \$6971 more depreciation.

9 **a** Increasing by a common ratio of 1.023, so geometric sequence.

b $t_n = \$4.50 \times 1.023^{(n-1)}$

c \$5.04

10 **a** $t_{n+1} = t_n \times \frac{1}{2}, t_1 = 11$

b 5th bounce as $t_5 = 0.6875$

11 **a** 8.5%

b 11 years as $t_{11} = 104\,007$

12 **a** **i** $t_1 = 1$ **ii** $r = 1.03$
iii $t_n = 1 \times 1.03^{(n-1)}$

b increase of 75.35%

13 12 cycles; Using guess and check on $t_n = 35 \times \left(\frac{5}{7}\right)^{n-1}$

14 B **15** D

16 **a** $t_{n+1} = t_n \times 1.04, t_1 = 170$

b $t_n = 170 \times 1.04^{(n-1)}$

c 31st month after beginning of 2025.

17 $t_{13} \approx= 633.5$ L, $t_{14} \approx = 475.1$ L. So, between 3 hours and 3 h 15 m.

18 Predicted income = \$1210.97. At least \$1000 is a reasonable prediction if plays continue as a geometric progression.

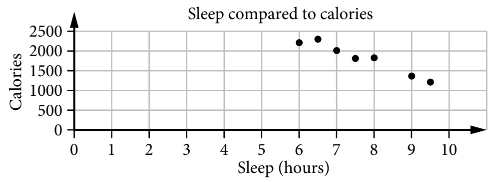
19 $c = m_1 = \$60$, $a_1 = \$120$, giving 2025 total for 7 days of \$1733.83, so \$1500–\$2000 is a reasonable estimate.

CUMULATIVE EXAMINATION 1

1 B **2** D **3** B **4** B **5** D

6 3015 songs

7 **a**



b negative association

8 **a** $t_n = 483 \times 0.83^{(n-1)}$

b 190 birds

CUMULATIVE EXAMINATION 2

1 Two-way table

Hours	Year 7	Year 12
≤ 2	15	2
2 to ≤ 4	53	11
> 4	4	4
Total	72	17

Percentaged two-way table

Hours	Year 7	Year 12
≤ 2	20.8%	11.8%
2 to ≤ 4	73.6%	64.7%
> 4	5.6%	23.5%
Total	100%	100%

The data suggests that there is an association between the hours spent practising the musical instrument each week and the student age group.

Older students practise more overall than younger students.

94% of Year 7 students practise less than 4 hours each week compared to 76% of Year 12 students.

88% of Year 12 students practise more than 2 hours each week compared to only 79% of Year 7 students.

2 To collect from school and 44 houses means $n = 45$. $t_{45} = 1442$ kg, which is more than 1400 kg. So, the truck will not be able to collect garbage at the 44th house.

CHAPTER 3

EXERCISE 3.1

1 D

2 **a** **i** likely

ii An increase of visitors/tourists to far north Queensland may lead to habitat damage or the increase of vehicle strikes.

b **i** likely

ii Increase in visitors and campers to the island may lead to an increase in attacks as the dingoes are attracted to the food in camps.

c **i** unlikely

ii Bilbies may be thriving in the area due to environmental protections. The RFDS will need to increase services as the regional populations grow.

d **i** unlikely

ii Cruise ship popularity and visitors to the reef may have increased. Shortage of rentals due to population growth or lack of building may cause prices to increase.

e **i** likely

ii destruction of habitat

f **i** unlikely

ii Coal mining is dangerous but cannot be the cause of all industrial accidents. Other industries will be involved in the numbers.

3 **a** car accidents

b new students may be great volleyball players

c improvements in health care

d coincidence or marketing and increased population campaigns promoting Queensland and its products

e increase in tourism

f increase in the number of people living at the Gold Coast and travelling to Brisbane for work

4 **a** moderate

b Teacher to check.

Sample answers:

Unlikely, as storms are seasonal and do not regularly occur during the year.

Weak likely, as more storms would encourage more indoor activities, including going to the cinema.

c unlikely as it may be due to coincidence or a confounding variable such as rainfall.

5 **a** rainfall

b moderate – weak

c Rainfall may promote the reproduction of cockroaches and force them inside buildings.

6 **a** unlikely, possibly a coincidence.

b An increase in population may cause an increase of both variables.

7 B **8** C

9 Check your example with your teacher. E.g. Minutes of cardio exercise and heart rate. The association is the relationship between the explanatory variable, exercise in minutes, and the response variable, heart rate. The causation can be explained by an increase in the number of minutes of cardio exercise leads to an increase in heart rate.

10 While the correlation coefficient suggests a strong correlation, it is unlikely that there is a causal relationship between the number of pets and influenza cases. QLD Health should not use this data for forecasting future illnesses. As pets do not cause influenza, this data is not reliable.

EXERCISE 3.2

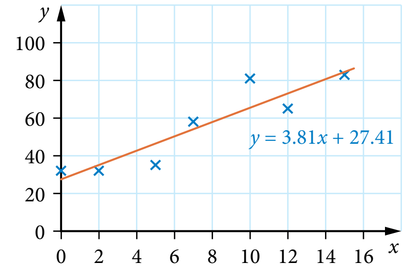
1 D **2** B

3 **a** $y = 0.49x + 6.08$ **b** $y = -2.07x + 38.16$

c $y = 1.90x + 32.41$ **d** $y = 0.29x + 139.96$

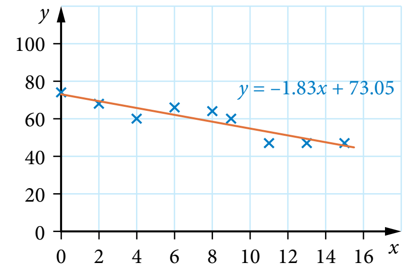
4 **a** $y = 3.81x + 27.41$

b



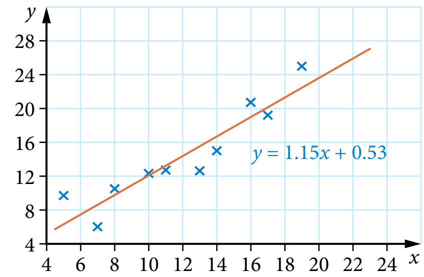
5 **a** $y = -1.83x + 73.05$

b



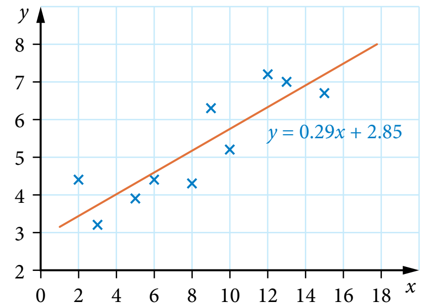
6 **a** $y = 1.15x + 0.53$

b



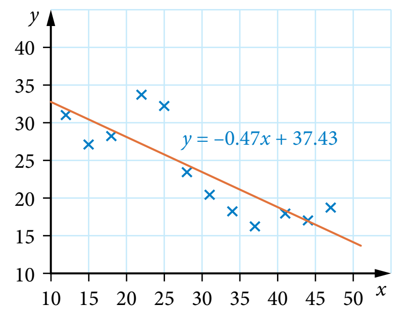
7 **a** $y = 0.29x + 2.85$

b



8 **a** $y = -0.47x + 37.43$

b



9 **a** 75 BPM **b** 90 BPM

c 102 BPM **d** 116 BPM

10 **a** **i** −47 ∴ zero complaints

ii 12 complaints

iii 37 complaints

iv 55 complaints

b It is impossible to have negative complaints. Even with a small crowd, there may still be a complaint. The line suggests the larger the crowd, the greater the number of the complaints. One concert was a significant outlier. This may be due to the type of music played and the composition of the crowd.

11 D **12** C **13** B

14 $m = 36.37$, $c = -748.32$

15 **a** $y = 0.21x + 0.90$

b \$6192

16 **a** 3.25 hours

b interpolation

c The line suggests an upwards trend as age increases. However, it is only moderate association and does not go outside the age range 10-30. 90 year-olds may not follow the trend.

17 $\bar{y} = 3.99$ ∴ the number of hours 10–18-year-olds spend playing sport is approximately 4 hours.

18 **a** 0.986

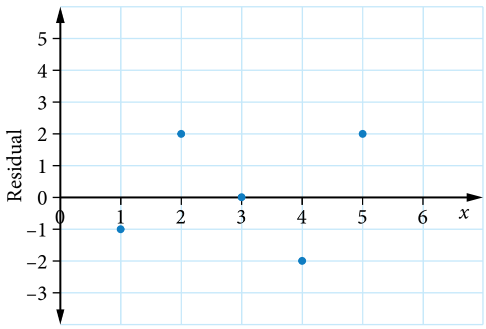
b A third variable that affects the response variable similarly to the explanatory variable

c The number of vehicles parked and temperature are both lower at the start and end of he day, and higher in the middle of the day when commuters are at work.

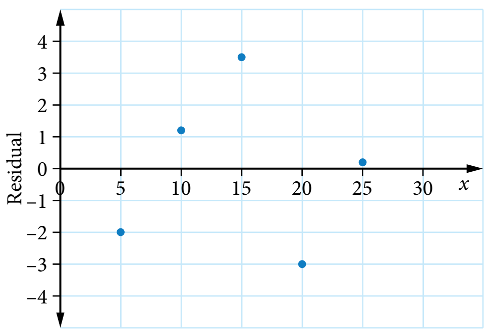
EXERCISE 3.3

1 B **2** D

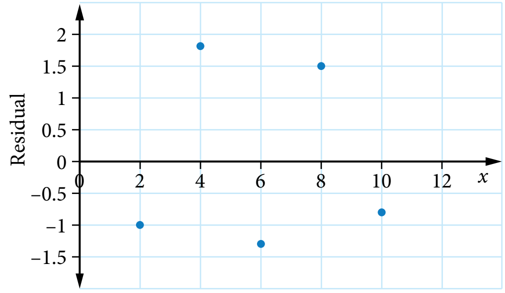
3 **a**



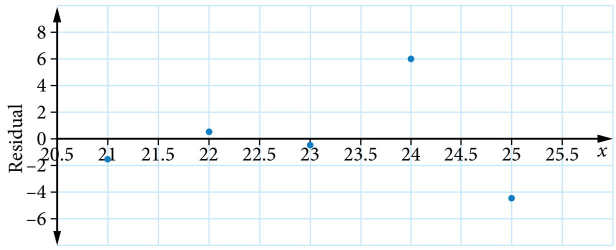
b



c



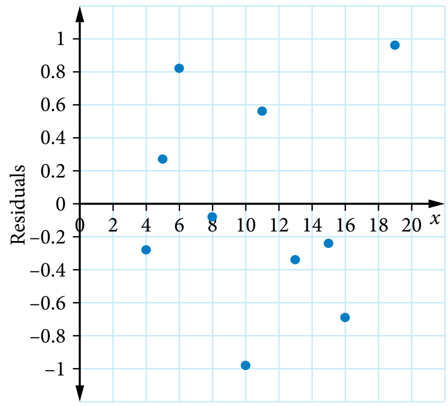
d



4 **a**

x	4	5	6	8	10	11	13	15	16	19
Residuals	−0.28	0.27	0.82	−0.08	−0.98	0.56	−0.34	−0.24	−0.69	0.96

b

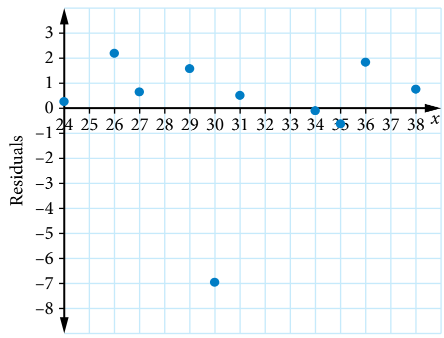


c The residuals are evenly scattered on both sides of the axis. It suggests a linear model is appropriate.

5 **a**

x	24	26	27	29	30	31	34	35	36	38
Residuals	0.25	2.18	0.64	1.57	−6.96	0.5	−0.11	−0.64	1.82	0.75

b

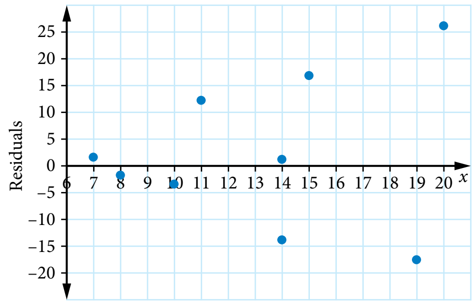


c The residuals have an outlier at $x = 30$. A linear model could be appropriate only if the outlier is invalid.

6 **a**

x	7	8	10	11	14	14	15	17	19	20
Residuals	1.54	−1.81	−3.49	12.16	−13.87	1.13	16.78	−20.91	−17.59	26.06

b

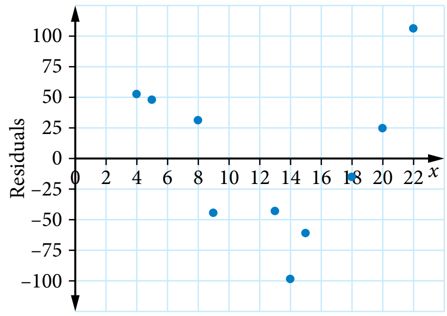


c The residuals spread out to the right. A linear model is not appropriate.

7 a

x	4	5	8	9	13	14	15	18	20	22
Residuals	52.54	47.9	31.07	−44.55	−43.03	−98.7	−61.3	−15.12	24.64	106.4

b



c The residuals start as positive, go negative and end as positive again; possible curve.

8 a inappropriate, outliers

b inappropriate, spread to the right

c appropriate, evenly spread to both sides with no pattern

d inappropriate, has a pattern that could be a curve

e appropriate, evenly spread to both sides with no pattern

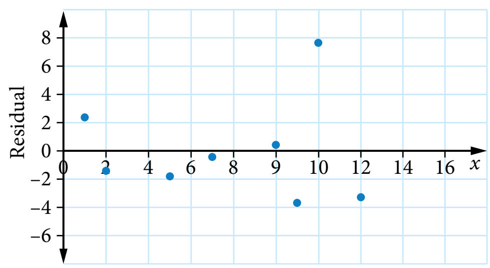
f inappropriate, outliers

9 $a = 4$, $b = 8.5$, $c = 6$, $d = -1$, $e = 10$, $f = 14.5$

10 C **11** C

12 a $y = -0.831 + 23.627$

b



A linear model may be appropriate if the outlier is invalid.

13 33°C is the highest temperature he can operate at and still break even.

EXERCISE 3.4

1 D

2 The scatterplot displays a strong non-linear form. The residual shows randomly placed points close to the line, which suggest a linear association.

3 a Males: 9.7%, Females: 12.1%; Females are 24.7% times more likely to have asthma than males.

b As females are approximately 25% more likely than males to have asthma, education programs should be focused towards them. Health funding for asthma is important but more of the respondents did not have asthma, meaning funding may be needed elsewhere.

4 Overweight males: 60.2%, Overweight females: 52%; Males are 15.8% times more likely to be overweight or obese than females. This may be due to lifestyle choices between the genders.

5 a ≈ 690 000 people

b The institute predicts an additional 81 643 people will have dementia. This may be due to a different method of calculating prediction values or through the inclusion of confounding variables as part of the data collection process.

6 a The table can be written as percentages horizontally and vertically.

	Blue eyes	Dark eyes
Fair hair	66.3%	39.0%
Dark hair	33.7%	61.0%
Total	100%	100%

Or

	Blue eyes	Dark eyes	Total
Fair hair	53.4%	46.6%	100%
Dark hair	27.1%	72.9%	100%

b Someone with blue eyes is nearly twice as likely to have fair hair as dark hair, while those with fair hair are only a little more likely to have blue eyes than dark eyes.

Students with dark eyes are about 50% more likely to have dark hair than light hair, while those with dark hair are about 2.7 times as likely to have dark eyes than blue eyes.

The results suggest fair hair goes with blue eyes, and dark eyes go with dark hair, however students' natural hair colour may be covered by dye, making this a confounding variable.

7 eye colour and type of metal; both are categorical data types and cannot be compared in the same manner as numerical data values.

8 Answers will vary. Sample responses:

- Association may not have a cause.
- Association may not be strong.
- The variables may have a non-linear relationship.
- The trend may only be for a specific period of time.
- The trend may not be logical over time, e.g., negative rainfall may be suggested by a trend, but cannot occur.

9 a 3°C **b** 13°C

c The answers are not completely reasonable.
The answer to part **a** seems unlikely as even during winter, it is more reasonable for the temperature to be higher than this in a Queensland town.
The answer to part **b** is reasonable.

10 The statement does not sound reasonable as the prediction of the 75-year-old group is 12%. This is the same as the 45-year-old group and just less than the 20-year-old group. While it is reasonable that there may be fewer 75-year-olds with phones, the percentage is in line with younger age groups.

11 The correlation coefficient is 0.775, suggesting a strong correlation. The coefficient of determination suggests 60% of the change is common to both variables. As logically there should not be a causal relationship between the two variables, it is likely that the catch is decreasing over time and the amount of gambling is increasing, and the variables are confounded by time.

12 The scatterplot shows a moderate negative linear trend: $M \approx 128\,578 - 77.961C$, with $R^2 \approx 0.4979$. The data shows a moderate negative linear trend, with a correlation coefficient of $r = -0.7056$. The correlation coefficient suggests venue costs affect the number of marriages 50% of the time. In 2021 the lower value may be due to increasing costs, but would more likely be due to Covid restrictions on weddings at the time.

13 $y = 6.7 - 0.227x$. For $x = 43$, $y = 6.939 \approx 7$, high, so the phone app notification is reasonable.

CUMULATIVE EXAMINATION 1

1 C **2** D **3** A **4** B **5** B

6 **a** common difference = 43
b 568 people

7 **a** 59 fish
b The correlation coefficient of 0.688 suggests a moderate association, which means that as the hours spent fishing increase so do the number of fish caught. A coefficient of determination of 0.473 means that 47% of the variation in results can be explained by the variation of hours spent fishing. Therefore, the prediction of catching 59 fish after fishing for 12 hours may be valid, however other factors will also come into play.

8 $p = 4n + 12$

9 **a** $r = 0.65$, therefore each term decreases by 35%.
b $t_4 = 32.955$

CUMULATIVE EXAMINATION 2

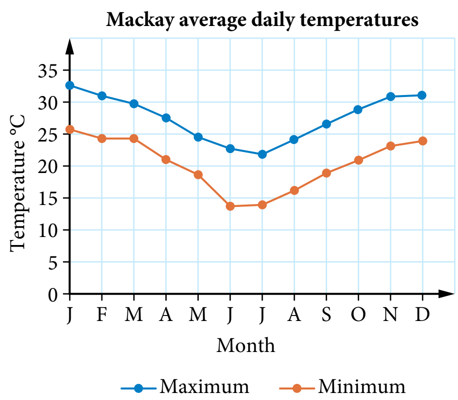
1 $p = 12, q = 15$

2 $b(m) = 992.878$ $a(c) = 4199.17$
Predicted profit in the 11th year is \$15 121.

CHAPTER 4

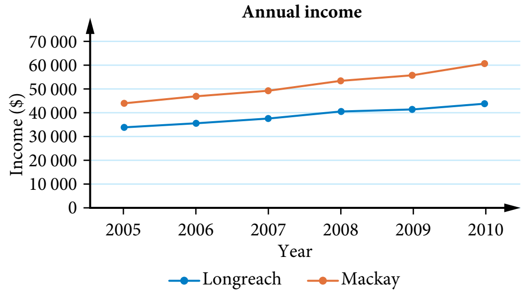
EXERCISE 4.1

1 **a**



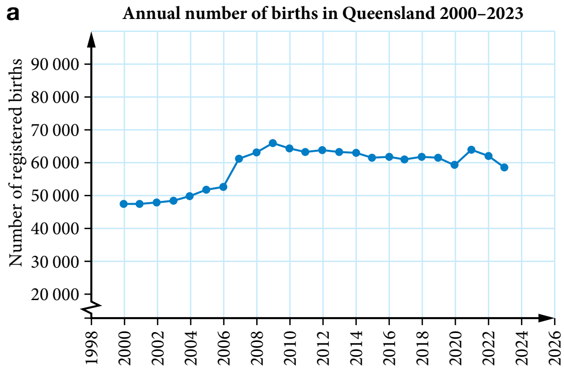
b The overall average temperatures are lowest in winter. The difference between the average maximum and minimum temperatures for each month also occurs in winter.

2 **a**



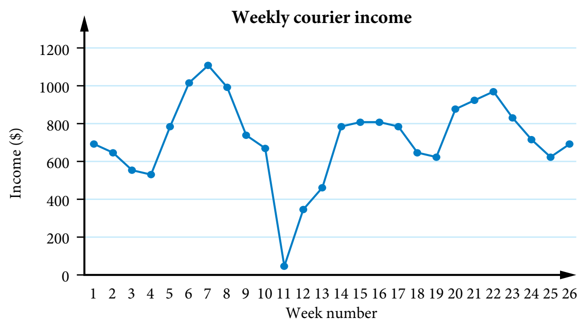
b There is a steady increase of income, but the incomes in Mackay are higher than those in Longreach, with the pay difference getting wider in later years.

3 **a**



b There was a marked increase in births between 2006–2010, and then it stabilised at the higher level, but with a slight decreasing trend. There was an increase in 2021 which may be due to the 2020 COVID-19 lockdowns.

4 **a**



b The income fluctuates over the weeks, with a sudden drop around week 11. This may be due to a decrease in working hours because of time off work.

5 **a** increasing trend **b** seasonal
c decreasing trend **d** seasonal
e cyclic **f** seasonal

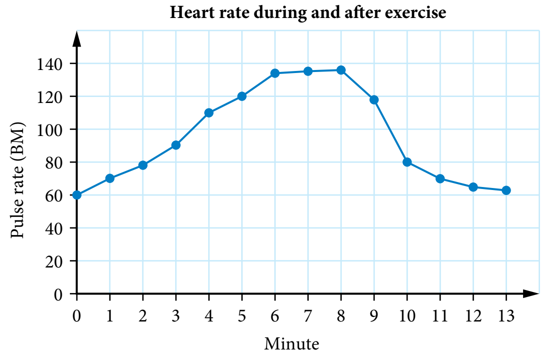
6 The plot fluctuates with a decreasing trend from 2000 to 2010, before showing an increasing trend with growing victim numbers until 2023.

7 **a** Towards the end of week 3, the sales appear to drop. As they quickly returned to the previous level, this suggests the presence of an outlier.
b This may have been caused by a fault with the pumps, interruption of power to the station, or running out of fuel.

8 General increasing trend with outliers in 2009 and 2015.

9 C **10** D **11** A

12 **a**



b Delia's heart rate goes up during exercise and recovers about 5 minutes after exercising. This is displayed by an increase in the trend during exercise and a decrease during rest.

13 General increasing trend across the plot. Outliers are evident in May 2020 and August 2021.

14 An increasing trend is evident across the graph. A cyclic pattern is evident. The cycles are irregular at the beginning of the plot but become more regular towards mid-2022.

EXERCISE 4.2

1 D **2** C

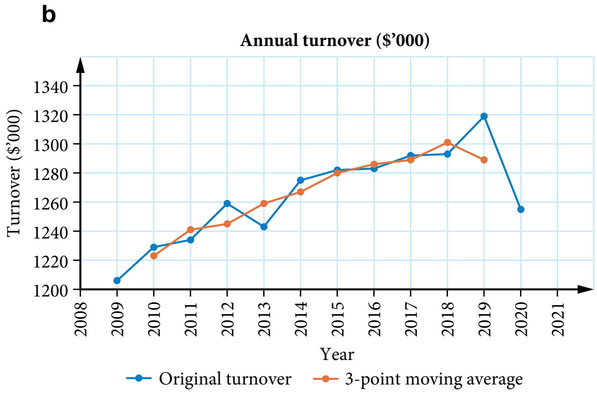
3 **a** 151 **b** 185 **c** 258.67

d 177 **e** 216.4 **f** 219.86

4 **a**

Year	Turnover $'000	3-point moving average
2009	1206	
2010	1229	1223
2011	1234	1241
2012	1259	1245
2013	1243	1259
2014	1275	1267
2015	1282	1280
2016	1283	1286
2017	1292	1289
2018	1293	1301
2019	1319	1289
2020	1255	

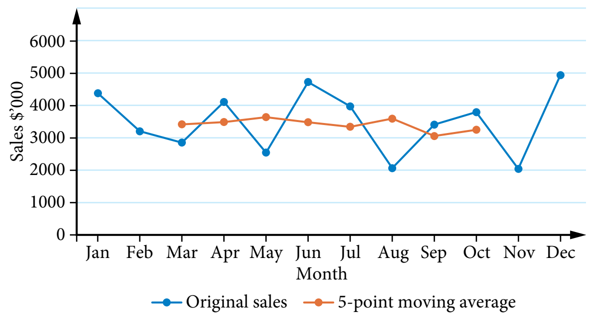
b



5 **a**

Month	Sales $'000	5-point moving average
Jan	4375	
Feb	3200	
Mar	2850	3414
Apr	4105	3483
May	2540	3637
Jun	4720	3479
Jul	3970	3339
Aug	2060	3589
Sep	3405	3053
Oct	3790	3245
Nov	2040	
Dec	4930	

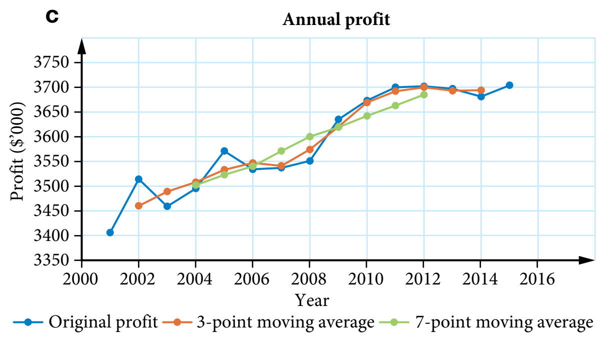
b



6 **a–b**

Year	Profit $'000	3-point moving average	7-point moving average
2001	3406		
2002	3514	3460	
2003	3459	3489	
2004	3495	3508	3502
2005	3571	3533	3523
2006	3534	3547	3540
2007	3537	3541	3571
2008	3551	3574	3600
2009	3635	3620	3619
2010	3673	3669	3642
2011	3700	3692	3663
2012	3702	3700	3685
2013	3697	3693	
2014	3681	3694	
2015	3704		

c



d The smoothed averages straighten the line. The 7-point moving average has fewer point and smooths the line more than the 3-point moving average.

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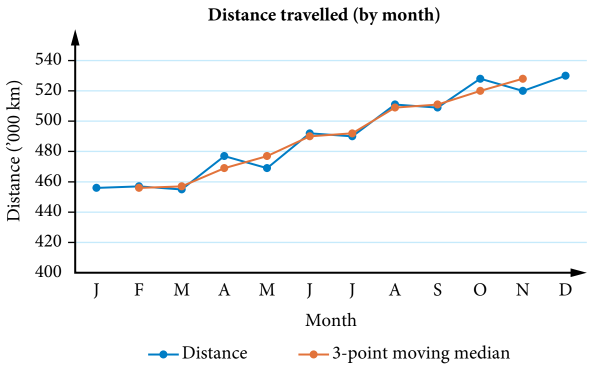
7 a

Month	Distance	3-point moving median
J	456	
F	457	456
M	455	457
A	477	469
M	469	477
J	492	490
J	490	492
A	511	509
S	509	511
O	528	520
N	520	528
D	530	

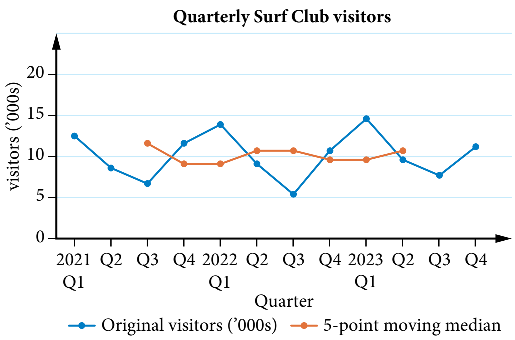
b



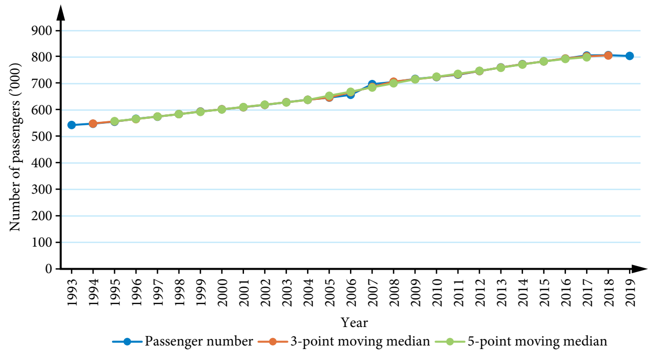
8 a

Quarter	Original visitors ('000s)	5-point moving median
2021 Q1	12.5	
Q2	8.6	
Q3	6.7	11.6
Q4	11.6	9.1
2022 Q1	13.9	9.1
Q2	9.1	10.7
Q3	5.4	10.7
Q4	10.7	9.6
2023 Q1	14.6	9.6
Q2	9.6	10.7
Q3	7.7	
Q4	11.2	

b



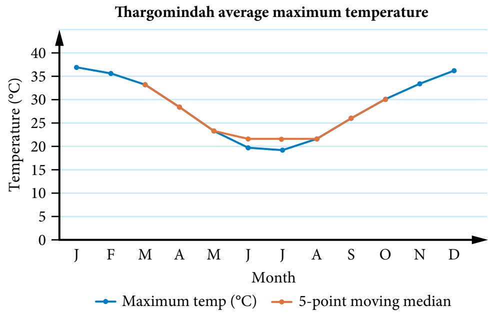
9



10 B **11** A **12** D

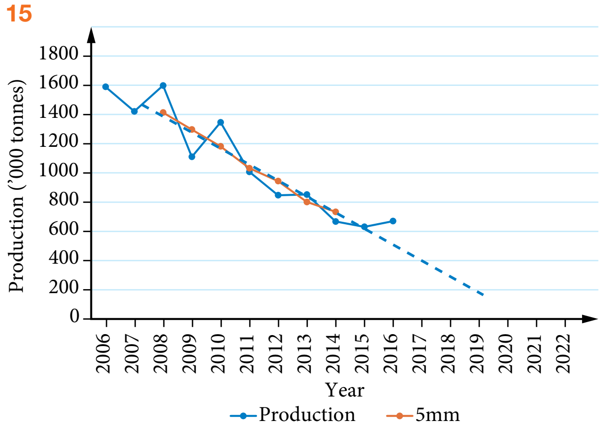
13 Higher value moving means will be more smoothed but will also have fewer points on their line. This may be better when smoothing out large fluctuations, but the lack of points may make the graph less useful.

14 a and **b**



c Because the original data followed a smoothed trend, the 5-point moving average made little difference.

15



Approximately 2019, by extending the smoothed line. The prediction could be wrong if production methods changed or more coal was discovered in the mine.

EXERCISE 4.3

1 C **2** A

3 a

Quarter	Q1	Q2	Q3	Q4
Index	0.868	1.060	1.100	0.972

b

Quarter	Q1	Q2	Q3	Q4
Index	1.04	0.994	0.943	1.023

c

Month	J	F	M	A	M	J	J	A	S	O	N	D
Index	1.004	0.918	0.944	0.888	0.952	0.974	0.926	1.025	1.064	1.085	1.128	1.094

d

Month	J	F	M	A	M	J	J	A	S	O	N	D
Index	0.837	0.857	0.971	0.991	1.105	1.172	1.092	1.138	1.031	1.011	0.871	0.924

4 a 0.915 **b** 0.898 **c** 0.77

5 a

Quarter	Q1	Q2	Q3	Q4
Index	1.053	0.972	0.934	1.041

b

Season	Summer	Autumn	Winter	Spring
Index	1.057	0.980	0.955	1.008

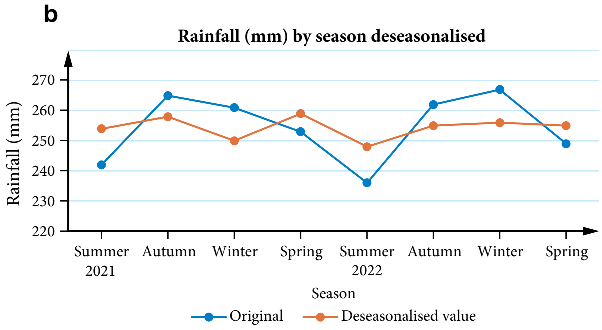
6

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Index	1.037	1.126	1.091	1.013	0.951	0.956	0.960	0.900	0.944	0.967	1.002	1.054

7 a

Season	Original	Deseasonalised value
Summer 2021	242	254
Autumn	265	258
Winter	261	250
Spring	253	259
Summer 2022	236	248
Autumn	262	255
Winter	267	256
Spring	249	255

b

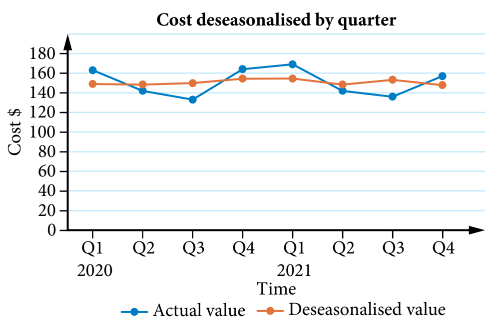


c ≈ 291 mm

8 a

Quarter	Actual value	Deseasonalised value
Q1 2020	163	149.0
Q2	142	148.4
Q3	133	149.8
Q4	164	154.4
Q1 2021	169	154.5
Q2	142	148.4
Q3	136	153.2
Q4	157	147.8

b

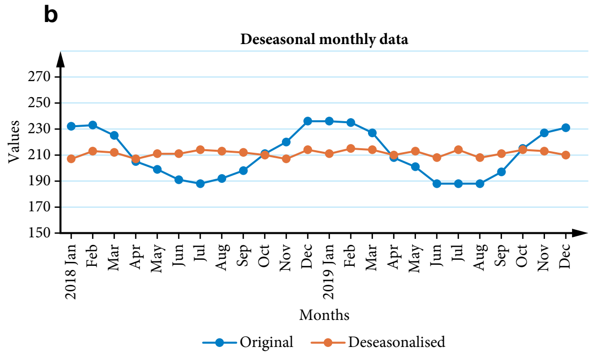


c The smoothed series reduces the impact of the quarterly peaks and troughs.

9

Month	Original	Deseasonalised value
2023 Jan	232	207
Feb	233	213
Mar	225	212
Apr	205	207
May	199	211
Jun	191	211
Jul	188	214
Aug	192	213
Sep	198	212
Oct	211	210
Nov	220	207
Dec	236	214
2024 Jan	236	211
Feb	235	215
Mar	227	214
Apr	208	210
May	201	213
Jun	188	208
Jul	188	214
Aug	188	208
Sep	197	211
Oct	215	214
Nov	227	213
Dec	231	210

b



c Deseasonalising the original data has smoothed out the differences in the summer and winter months.

10 D **11** D **12** B **13** 0.97

14 April 0.97, May 0.98, June 0.99

15 0.983

16 $78 493.29

EXERCISE 4.4

1 C **2** ≈ 29.43

3 a negative gradient, meaning the values should decrease over time

b $\approx$ 7 complaints

c i −4 complaints

ii As it is impossible to have a negative number of complaints, using the trend is not reliable.

d 18th month

4 a i 13.4 min **ii** 76.7 min

iii 108.35 min **iv** 319.35 min

b The predictions are reasonable for the first 7 days. At 10 days, nearly 2 hours of exercise seems unlikely. At 30 days, the predictions are unreasonable as Althea would be exercising for at least 5 hours in a day. It is unlikely that she starts at 13 min on day one and by the end of the month is exercising for 5 hours.

5 a $V = 4.1Y + 102.7$ **b** 135.5

c The data is over 5 years and this is 3 years later, so it is probably accurate.

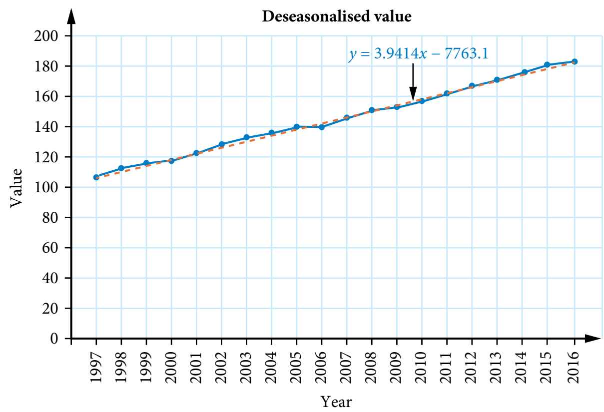
6 a $C = 6.786q + 180.964$ **b** $316.684

c The data is over 8 quarters, but this is 12 quarters later, so it may not be accurate.

7 a $d = -0.566 \times month + 11.515$ **b** $\approx$ 3 days

c As the data goes to the 12th month prediction is likely to be accurate. As time goes on, longer-term predictions will be less accurate or reliable. There are also external factors that may affect the number of days that tennis can be played.

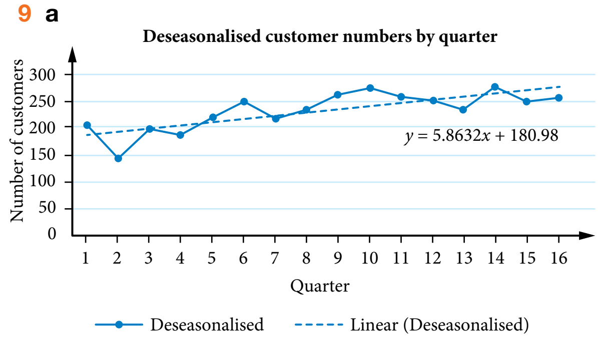
8 a



b $y = 3.9414x - 7763.1$

c ≈ 198

9 a



b $y = 5.8632x + 180.98$

c Q4 2024 = 20th quarter. Actual number of customers = 341788

10 B **11** D **12** C

13 **a** ≈ 344 **b** ≈ 313 **c** ≈ 350 **d** ≈ 381.5

14 ≈ 256

CUMULATIVE EXAMINATION 1

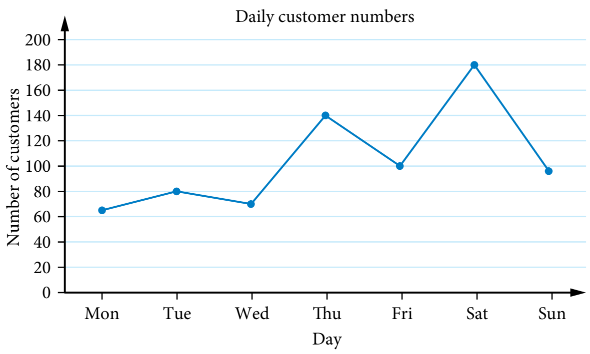
1 D **2** A **3** C **4** C **5** D

6 **a** $n = 42.6t + 55.4$ **b** 950

7 **a** $d = 43$ **b** 568 people

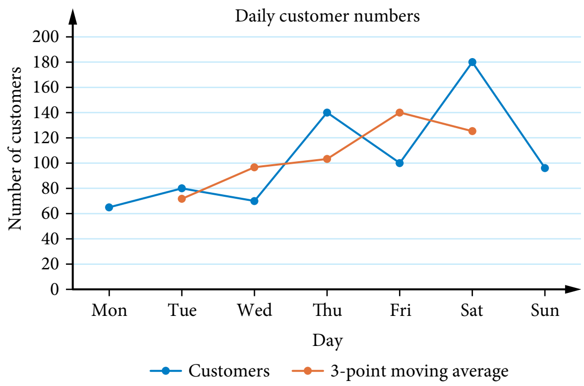
8 non-linear form, seasonal cycle, positive long-term trend

9 **a**



b

Day	Number of customers	3-point moving average
Mon	65	
Tue	80	71.7
Wed	70	96.7
Thu	140	103.3
Fri	100	140.0
Sat	180	125.3
Sun	96	



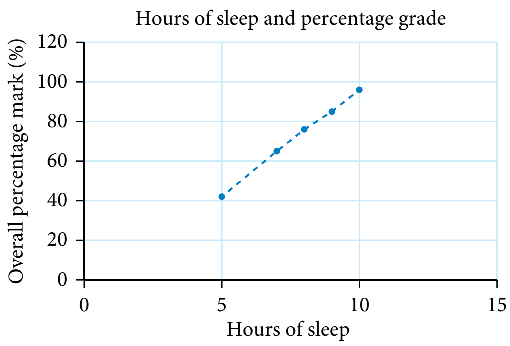
CUMULATIVE EXAMINATION 2

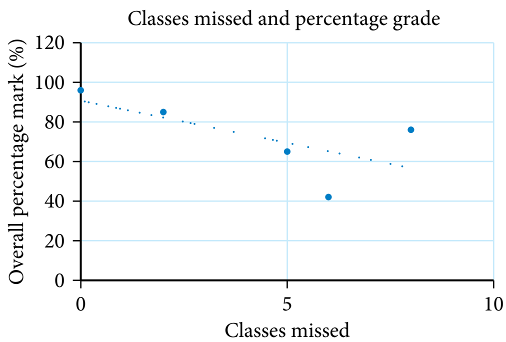
1 Actual swimsuit sales are $30 940.

2

Year	Season	Number of skin wounds	Yearly average	$\frac{\text{number}}{\text{yearly average}}$	Seasonal indices	Deseasonalised number
2021	Autumn	285	242	1.1776…	1.2307…	232
	Winter	28		0.1157…	0.1090…	257
	Spring	195		0.8057…	0.7982…	244
	Summer	460		1.9008…	1.8620…	247
2022	Autumn	276	215	1.2837…	1.2307…	224
	Winter	22		0.1023…	0.1090…	202
	Spring	170		0.7906…	0.7982…	213
	Summer	382		1.8232…	1.8620…	211

3





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From the graphs, it can be observed that:

- there is a very strong positive correlation between hours of sleep and overall percentage grades.
- there is a strong negative correlation between the number of times a student is absent from class and overall percentage grades.

The graphs indicate getting plenty of sleep has a stronger impact on improving grades than attending all classes. However, doing both is likely to have the greatest impact.

CHAPTER 5

EXERCISE 5.1

1 a no **b** equator
c South Pole **d** yes
2 a 2513.3 km **b** 31 415.9 km
3 a 19 792 m **b** 23 562 m
4 east
5 a 30° S **b** 45° N **c** 65° N **d** 35° N
e 40° S **f** 10° S **g** 0° **h** 40° N
6 a 80° W **b** 60° W **c** 115° E **d** 0° E
e 30° E **f** 20° W
g 10° E **h** 100° W
7 A = 50° N 20° W, B = 0° 50° W, C = 20° N 70° E, D = 30° S 30° E, E = 30° S 20° W, F = 70° N 70° E, G = 60° S 50° W, H = 20° N 30° E, I = 0° 70° E
8 a 23° 24′ **b** 86° 9′
9 a 102.87° **b** 35.42°
10 a 50° N 14° E **b** 39° N 09° W
c 52° N 0° W **d** 42° N 12° E
11 a Paris **b** Oslo
c Athens **d** Dublin
12 a 27° 30′ S 153° 21′ E **b** 27° 28′ S 153° 21′ E
c 27° 30′ S 153° 16′ E
13 a West Peel Artificial Reef **b** Raby Bay
14 a 25° 21′ S 131° 2′ E **b** 60° 11′ N 24° 56′ E
c 27° 25′ S 153° 2′ E **d** 36° 08′ N 5° 21′ W
e 33° 51′ S 151° 13′ E **f** 41° 54′ N 12° 29′ E
g 43° 5′ N 79° 4′ W **h** 33° 43′ S 143° 03′ E
15 a K'gari
b Table Mountain, Cape Town
c Rottnest Island **d** New York
e The Great Pyramid of Giza **f** Brussels
g Hong Kong **h** Buckingham Palace
16 A **17** A **18** A
19 Mumbai and Mexico City, 19° N
20 B

EXERCISE 5.2

1 C **2** B
3 a 71° **b** 7895.2 km
4 2307 km **5** 8340 km
6 12 232 km **7** 9314 km
8 14 119 km
9 C **10** B **11** 9026 km **12** 9744 km
13 60° N and 60° S, 20 015 km

EXERCISE 5.3

1 3781 km **2** 1149 km
3 latitude = 0.2′, longitude = 1′
4 between latitudes 26° 37.95′ S and 26° 38.2′ S and between longitudes 135° 5.8′ E and 135° 5.95′ E
5 about 7 h and 12 min
6 about 26 days and 7 hours
7 a 20 435.68 km **b** 10 230.62 km
c 34.06 hours and 17.05 hours
8 B **9** B
10 a Marovoay 16.1° S 46.6° E, Iakora 23.1° S 46.6° E
b 778 km
11 19° S 147° E (Townsville)
12 4675 km
13 It is possible as the trip should take about 44 hours.

EXERCISE 5.4

1 32 792 km
2 3395 km
3 a 6 hours **b** 4 hours 56 minutes
c 2 hours 8 minutes **d** 10 hours 12 minutes
4 10 am
5 3:30 pm
6 a 11:00 pm **b** 10:00 am Saturday
c 9:00 pm Tuesday **d** 9:00 pm
7 B **8** B **9** D
10 8 hours **11** 12 pm
12 3.33 hours (3 h 20 min) after it rises in City *Y*.
13 10:15 pm Saturday
14 C

EXERCISE 5.5

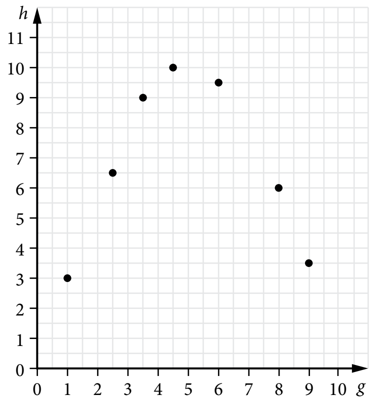
1 C **2** D
3 a 4:00 am the next day **b** 1:20 pm
c 12:45 am the next day **d** 7:15 pm
4 a 8:00 am Wednesday **b** 10:00 pm Tuesday
c 2:00 am Wednesday **d** 8:00 pm Tuesday
5 5 pm Monday
6 a 4:30 pm Saturday **b** 3:30 pm Saturday
7 a 3:05 pm Wednesday **b** 5:05 am Wednesday
8 A
9 7:15 am Wednesday
10 a 105° W **b** 4781.6 km **c** 12 hours
d The South Pole 90° S 0° **e** 23 hours 54 minutes
11 3:45 pm Thursday
12 5° East
13 Marcus left Lima at 11:18 am on Friday.
14 The time and date became 12:01 am, December 31st, 2011.
15 3:50 am Wednesday

CUMULATIVE EXAMINATION 1

1 B 2 A 3 A 4 D 5 D

6 approximately 600 km

7 a



b non-linear and strong

8 a Trend — long term is positive because the amount of rainfall generally increases as time increases. Seasonality — the data is seasonal with a high 4th quarter every year.

b y-intercept — the model predicts that 156.5 mm of rainfall was falling in the 4th quarter of 2015. Slope — on average an additional 1.763 mm of rainfall was precipitated each quarter.

CUMULATIVE EXAMINATION 2

1

Dataset	Correlation coefficient, r
t vs f	0.309...
p vs f	0.806...

The explanatory variable is p. The least-squares line equation is $f = -130 + 11p$.

It is predicted that 420 fish will be caught.

2 7:12 am Friday

CHAPTER 6

EXERCISE 6.1

1 $A_{n+1} = 1.068A_n$, where $A_0 = 10\,000$

2 C

3 a $20 140, $21 348.40, $23 987.06

b $1208.40 c $4987.06

4 C

5 a $14 038.30 b $2038.30

6 $1724.71, $224.71

7 $12 441.79 8 18.92% p.a.

9 B 10 C 11 A

12 Investment A: A = $9338.08, interest = $1838.08

Investment B: A = $9323.31, interest = $1823.31

Investment C: A = $9389.38, interest = $1889.38

Investment C is best.

13 $8101.72 14 4 years

15 B 16 D 17 D

18 a $31 815.56 b $1558.15

19 a $1331.40 b $531.40

20 $1816.09

21 6.79% p.a.

22 The investment with weekly rests is better (Option B) as it returns $29 237.58 while the other investment (Option A) returns $29 115.47.

23 5.79% p.a.

24 The investment compounded annually is better. It will return about 1.4937 times the amount invested, while the other investment only returns about 1.4390 times the amount invested.

25 $3380.79

EXERCISE 6.2

1 a $15 525, $17 815.30

b $525 c $2815.30

2 $8263.97

3 8.54%

4 a 7.76% b 14.34%

c 8.67% d 8.32%

5 7.82%, 7.76%, 7.68%; 7.6% compounded quarterly is best.

6 Option 3, 7.59% p.a. compounded monthly provides the best return.

7 B 8 C 9 0.1%

10 Option 1 (≈0.071 86) is better than Option 2 (≈0.070 16).

11 The insurance bonds are best with an effective rate of 7.98%.

12 The interest rate compounding quarterly is better (6.22% effective).

13 C

EXERCISE 6.3

1 7.23%

2 Option 2, with a 6.27% effective interest rate, provides a better return than Option 1, which has a 5.72% effective interest rate.

3 $9814.47

4 $3986.39

5 a $12 762.32 b $28 704.93

c $38 982.18 d $9835.76

6 a $5638.83 b $11 969.36

c $11 824.14 d $7462.15

7 $30 000.43

8 $762.79 per week

9 C 10 D

11 $7747.10

12 a $8703.96 b $6296.04

13 a $13 299.19 b $14 075.18

14 $55 256.18 15 5.79% p.a.

16 $15 493.47 17 14 years 9 months

EXERCISE 6.4

1 $13 854.47 2 $2541.60

3 a $A_{n+1} = 1.0625A_n - 5200$ where $A_0 = 18\,000$
b $1995.31
c 2.5 years and the final payment would be $113.91

4 a $A_{n+1} = 1.027A_n - 2500$ where $A_0 = 25\,000$
b $17 398.99

5 a $20 000 b $600
c $A_1 = 19\,560, A_2 = 19\,116.46,$
$A_3 = 18\,669.41, A_4 = 18\,218.77 \approx 18\,219$
d 9.6% p.a. e 7 months

6 a 0.007 b 8.4% p.a.
c

Payment number	Opening balance	Payment	Interest	Principal reduction	Closing balance
1	850 000.00	6476.00	5950.00	526.00	849 474.00
2	849 474.00	6476.00	5946.32	529.68	848 944.32
3	848 944.32	6476.00	5942.61	533.39	848 410.93
4	848 410.93	6476.00	5938.88	537.12	847 873.80

d $847 873.80 e $23 777.80

7 a

Payment number	Opening balance	Payment	Interest	Principal reduction	Closing balance
1	25 000.00	1130.00	177.08	952.92	24 047.08
2	24 047.08	1130.00	170.33	959.67	23 087.42
3	23 087.42	1130.00	163.54	966.46	22 120.95
4	22 120.95	1130.00	156.69	973.31	21 147.64

b

Payment number	Opening balance	Payment	Interest	Principal reduction	Closing balance
1	63 000.00	800.00	290.77	509.23	62 490.77
2	62 490.77	800.00	288.42	511.58	61 979.19
3	61 979.19	800.00	286.06	513.94	61 465.25
4	61 465.25	800.00	283.69	516.31	60 948.93
5	60 948.93	800.00	281.30	518.70	60 430.23
6	60 430.23	800.00	278.91	521.09	59 909.14

8 a A = B = $7597.54, C = $86.74, D = $6884.28
b $2115.72 c $887.11
d $9687.11 e $687.11

9 a $21 946.74 b $2053.26 c $946.74

10 C 11 C 12 D

13 A 14 B 15 D

16 a i $493.66 ii $118 027.48
b $A_{n+1} = 1.00215A_n - 750$ where $A_0 = 120\,000$

17 a $124.20 = i \times 5400$
$i = 0.023$
nominal rate $= 0.023 \times 4 = 9.2\%$ p.a.
b $A_{n+1} = 1.023A_n - 715$ where $A_0 = 5400$
c

Payment number	Opening balance	Payment	Interest	Principal reduction	Closing balance
4	3586.52	715.00	82.49	632.51	2954.01

d 8 quarters = 2 years
e $595.11

18 Interest $1555.14, loan amount $28 455.14

19 Amount owing $686.34 , interest paid $17.07

20 a $A_{n+1} = A_n - 250$ b $14 708.99 c $208.99

EXERCISE 6.5

1 a $218 176.03 b $7583.28

2 a

Payment number	Opening balance	Payment	Interest	Principal reduction	Closing balance
1	19 000	251	142.50	108.50	18 891.50
2	18 891.50	251	141.69	109.31	18 782.19
3	18 782.19	251	140.87	110.13	18 672.06
4	18 672.06	251	140.04	110.96	18 561.10
5	18 561.10	251	139.21	111.79	18 449.31
6	18 449.31	251	138.37	112.63	18 336.68

b total interest = $842.68, total repayments = $1506, reduction of the principal = $663.32, loan balance = $18 336.68

3 B

4 a $A_{n+1} = 1.057A_n - 9000$, where $A_1 = 90\,000$
b $68 314.74 c $5106.06 d 10 years

5 a amount inversted is $300 000 and six-monthly repayments are $20 000
b i = 0.0315 or 3.15% p.a. compounded six-monthly
c 6.3% d $278 567.67
e $8774.88 f $11 225.12

6 $14 460.78

7 a $835.55 b $732.54 c $1219.18

8 a $454.04 b $27 242.30 c $7242.30

9 a i $644.74/month, $1473.70
ii $450.43/month, $2215.37
iii $353.73/month, $2979.18
iv $321.65/month, $3369.34
b As the time taken to repay the loan increases, the monthly repayment value decreases but the total amount of interest paid increases.

10 C 11 D

12 a $A_{n+1} = 1.00492A_n - 5000$, where $A_0 = 210\,000$
b 5 months

13 a 13% p.a. b $426.52

14 a 41 389.00 b $4355.85 c $757.32

15 $13 715.82

16 a $150 000
b $A_{n+1} = 1.002A_n - 993.14$, where $A_0 = 150\,000$

17 $8 130 186.88

18 No, the annuity pays an extra $18 479.23 over the 10-year period.

19 Yes, the present value of the periodic payments is $291 339.57, so she is $8660.43 better off with the lump sum.

CUMULATIVE EXAMINATION 1

1 B **2** D **3** B **4** D **5** B

6 Option A ≈ 0.05745. Option B ≈ 0.05739.
Ngarra's decision is reasonable.

7 **a** 11 cm

b The least-squares line provided does suggest that at 29 days, the seedling will be 32 cm high. However, the data values are levelling off at about 25 cm, so extrapolation is unwise.

8 \$4126

9 4:10 pm Mon 7th December

CUMULATIVE EXAMINATION 2

1 Because the interest rates are the same, the one that is compounded more often has the higher effective interest rate.

2 14.87%

3 Leigh = \$139 360
Tony = \$189 980
The difference in account balances for Leigh and Tony is predicted to be \$50 620.

CHAPTER 7

EXERCISE 7.1

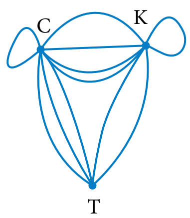
1 **a** 5 vertices, 4 edges **b** 6 vertices, 10 edges

2 **a** M **b** R **c** S
d T and V **e** P or T

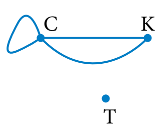
3 True in all cases.

4 **a** There are two loops, one at C and the other at K.
There are 2 multiple edges between K–T
There are 3 multiple edges between C–T.
There are 4 multiple edges between C–K.

b

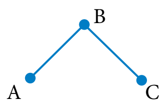


c

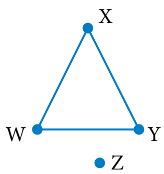


5 Other answers are possible.

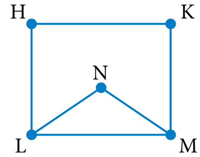
a



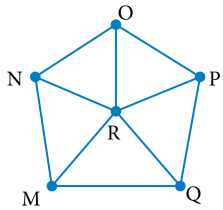
b



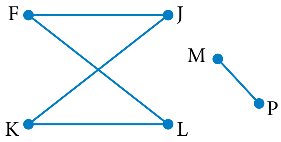
c



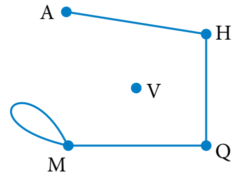
d



e



f



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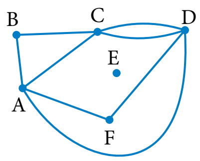

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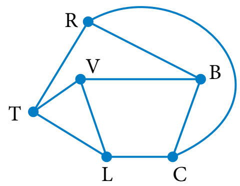

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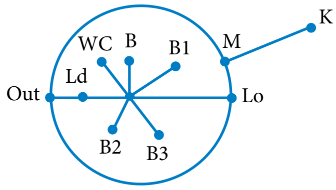

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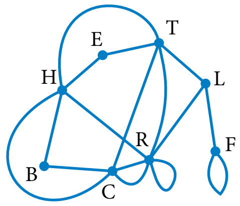


10 D **11** D **12** C

13 **a** A

b D, it only borders NSW so can only have 1 edge attached to it.

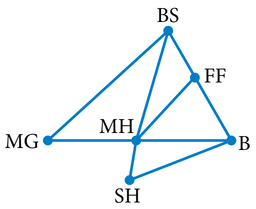
c G, deg (G) = 0

d degree sum = 20. The degree sum of 20 is twice the number of edges (shared borders) (10) between any two states and/or territories in Australia.

14 **a** A driver can leave Dalby and return to Dalby without visiting any other towns.

b A connection is missing between Dalby and Toowoomba. An additional edge must be added between D and T.

15



16 3 colours

EXERCISE 7.2

For answers where a graph needs to be drawn, many different graphs are possible. Any graph that is isomorphic with the answer supplied is correct.

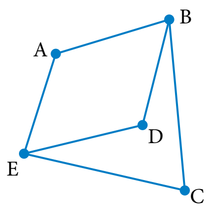
1 D **2** D

3 **a** **i** not simple as it has multiple edges
ii not complete as it is not simple.
b **i** simple
ii complete
c **i** simple
ii not complete
d **i** not simple as it has a loop
ii not complete as it is not simple

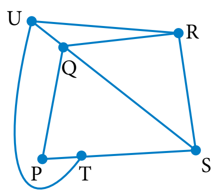
4 D

5 **a** ZW is a bridge.
b 6 bridges: AB, AC, AD, AE, AF and AG
c 2 bridges: FI and IJ
d no bridges

6 **a**



b

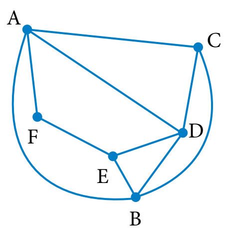


c non-planar

7 B

8 **a** 2 **b** 5 **c** 4

9 **a**

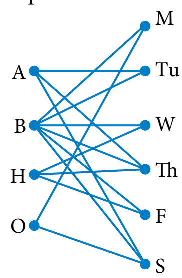


b Holds true as $6 + 6 - 10 = 2$

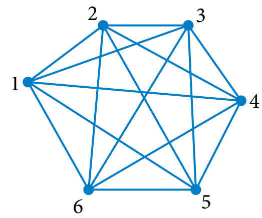
10 C

11 **a** **B**, **D**, **E** and **F**
b no graphs are complete
c **D** **d** **B, C, D** and **F**
e C
f **A** and **C**

12 bipartite



13 **a**



b complete **c** 15 **d** 10

14 **a** yes: C–P–B–S–T–M–H
b M–H–T–B and M–H–S–T–B
c no
d two: B–S–T–M–H and B–C–P–B–S–T–M–H

15 A **16** B

17 D **18** B

19 No, the values do not satisfy Euler's formula.

20 **a** $f = 2$, $v = 7$, $e = 7$; $7 + 2 - 7 = 2$, so the graph is planar.
b They cannot remove Shop entry–B, B–C or C–D.
c a bridge

21 Using Euler's formula: $6 + 5 - 9 = 2$ ✓

22 Using Euler's formula: $6 + f - 10 = 2$, so $f = 6$.

EXERCISE 7.3

For answers where a graph needs to be drawn, many different graphs are possible. Any graph that is isomorphic to the answer supplied is correct.

1 **a** Graph I has 5 vertices and 5 edges.
Graph II has 7 vertices and 6 edges.
Graph III has 5 vertices and 7 edges.
Graph IV has 7 vertices and 9 edges.
b Graphs I, III and IV are connected graphs.
c Graphs I and II are simple graphs.
d Graph III is a directed graph.
e Graph IV is a bipartite graph.

2 B

3 **i** All adjacency matrices are correct.
ii **a** 8 **b** 12 **c** 12 **d** 20
iii For **a** the sum of cells is one less as B has a loop.
The answers are the same for **b-d**.

4 D

5 a

$$\begin{array}{cc} & \begin{array}{ccccc} P & Q & R & S & T \end{array} \\ \begin{array}{c} P \\ Q \\ R \\ S \\ T \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

b

$$\begin{array}{cc} & \begin{array}{ccccc} L & M & N & O & P \end{array} \\ \begin{array}{c} L \\ M \\ N \\ O \\ P \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 2 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

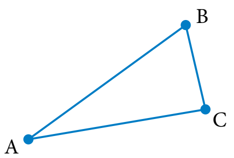
c

$$\begin{array}{cc} & \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \end{array} & \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

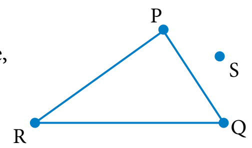
d

$$\begin{array}{cc} & \begin{array}{cccccc} U & V & W & X & Y & Z \end{array} \\ \begin{array}{c} U \\ V \\ W \\ X \\ Y \\ Z \end{array} & \begin{bmatrix} 0 & 1 & 2 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

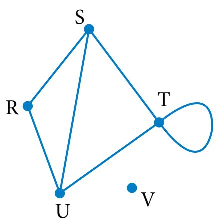
6 a simple,



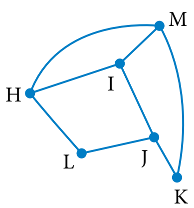
b simple,



c not simple – contains loop,



d simple,



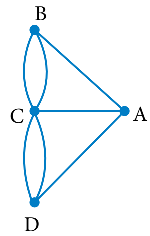
7 B

8 a

$$\begin{array}{cc} & \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} & \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

b The entry in the cell B–B would change from 0 to 1.

9 a



b

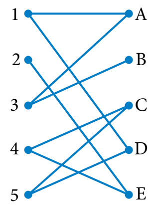
$$\begin{array}{cc} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 2 & 0 \\ 1 & 2 & 0 & 2 \\ 1 & 0 & 2 & 0 \end{bmatrix} \end{array}$$

c Use a pencil to try to draw a path on the graph that passes through each vertex exactly once without lifting the pencil.

d There is no solution.

10 B **11** C **12** D **13** C **14** C

15 a



b It is a simple graph as there are no loops or multiple edges.

16

$$\begin{array}{cc} & \begin{array}{ccccccc} A & B & C & D & E & F & G \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \\ G \end{array} & \begin{bmatrix} 0 & 1 & 1 & 2 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 2 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 3 & 0 \end{bmatrix} \end{array}$$

EXERCISE 7.4

1 B

2

$$\begin{array}{cc} & \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \end{array} & \begin{bmatrix} 0 & 3 & 0 & 1 & 1 & 0 \\ 3 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

For questions where answers involve identifying a route, many different answers are possible. The routes provided are examples.

3 B **4** D **5** C

6 **a** closed walk **b** open path
c cycle **d** cycle
e cycle **f** open trail

7 **a** P–U–Q–V–R–S **b** P–U–Q–V–Q–U
c U–Q–V–R–S–T–U **d** T–V–R–Q–U–P–Q
e V–R–Q–P–U–Q–V

8 A shortest path from A to H is A–F–G–H, which is 11 km long.

9 **a** H–G–F–B–C = 23 **b** D–C–I–J = 29
c A–B–L = 36 **d** G–F–I–K–C–B = 46
e H–A–B = 22

10 **a** P–G = 40 **b** J–I–F–E–B–A = 42
c C–P–N = 39 **d** E–B–A–P–O = 48
e I–J–K–L = 28

11 **a** A–F–H = 20c **b** D–I–B = 16c
c F–D–I = 20c **d** H–F–E = 19c

12 B **13** D **14** C

15 D **16** A

17 A–P–Q–S–B, 200 km

18 49 km

19 No as there is now no longer a path that connects every animal enclosure.

20 $21 + (26 - x) < 44 \Rightarrow x > 3$, also $x < 26$, so $3 < x < 26$. The direct road between B and G can be reduced by between 3 and 26 km.

EXERCISE 7.5

1 D **2** A

3 **a** deg (A) = 4, deg (B) = 3, deg (C) = 2, deg (D) = 3, deg (E) = 2, deg (F) = 2

b Semi-Eulerian, because it has 2 vertices of odd degree and the remaining vertices are of even degree.

4 **a** The graph is semi-Eulerian because it has 2 vertices of odd degree and the remaining vertices are of even degree.

b Either A or F

c One Eulerian trail is F–E–A–C–B–D–C–F–D–B–A, but others exist.

5 **a** The graph is Eulerian because all vertices are of even degree.

b One Eulerian circuit is S–U–T–S–Q–P–V–U–Q–R–S, but others exist.

6 **i** semi-Eulerian: **b**, **c** and **f**
Eulerian: **a**, **d** and **e**

ii **a** A–B–C–D–E–F–G–B–F–A (Eulerian circuit)
b T–P–R–S–T–Q–R (Eulerian trail)
c N–M–S–R–N–O–P–T–Q–R–T (Eulerian trail)
d D–E–F–K–E–C–K–H–G–F–I–J–C–D (Eulerian circuit)
e I–J–K–L–M–N–O–M–K–Q–H–P–O–H–I (Eulerian circuit)
f S–T–U–V–W–X–Y–V–T–Y–Z–X–S–Z–T (Eulerian trail)

7 **a** Eulerian graph

b The most efficient route will be an Eulerian cycle. D–F–B–D–B–C–Ph–G–C–D–G–P–D is an example of the most efficient route because it only travels on each road once.

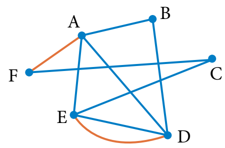
8 C **9** C **10** A

11 **a** The graph is Eulerian because all vertices are of even degree.

b B–A–D–E–C–D–C–B–D–B

12 5 ways: remove A–F, C–D, D–C, D–F or C–F.

13



14 No, it is not possible because the graph is not Eulerian and so cannot have an Eulerian circuit (closed trail).

15 A tour that walks along every path exactly once without repeats is possible, but it will not finish back at the Administration block.

EXERCISE 7.6

For questions where answers involve identifying a route, many different answers are possible. The routes provided are examples.

1 C **2** D

3 **a** B–A–D–G–F–E–C
b V–W–U–Y–X
c R–P–V–U–T–S–Q
d P–L–M–N–O–J–K

4 **a** F–A–C–D–B–E–F
b V–T–U–P–R–S–Q–W–V
c O–L–K–M–N–O
d W–Z–Y–X–S–T–U–V–W

5 **a** G–F–A–B–C–E–D–G
b F–G–E–D–C–B–A
c E–C–D–G–B–A–F–E
d G–F–E–D–C–B–A
e A–F–G–E–D–C–B–A

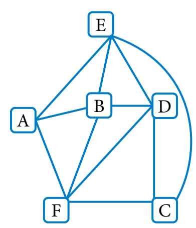
6 **a** a Hamiltonian cycle
b 3–4–9–5–6–7–8–1–2–3
c 2–8, 3–9, 8–9, 9–6 and 7–1

7 D–I–K–H–C–F–A–E–J–B–G–D

8 A–L–K–J–I–H–P–G–F–E–D–C–N–M–B–O–A

9 A **10** D **11** B

12 **a**

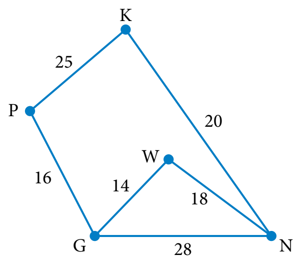


b E–A–B–D–C–F

13 A–D–C–B–E–G–H–F; 36 minutes

14 There 2 combinations of internal doors which can be locked and still allow a Hamiltonian cycle. Doors DB, DC and AB; or BC, BD and BA allow cycles ODABCO, OBADCO and the reverses.

15



Total travel time for WGPKNW = 93 min

CUMULATIVE EXAMINATION 1

1 B **2** B **3** C **4** B **5** A

6 7:20 am on Tuesday in Dubai

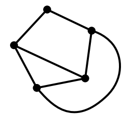
7 **a** 4.5% p.a. compounding monthly

b 6 months

8 **a** open walk **b** cycle

9 **a** $v = 4, f = 3, e = 5; v + f - e = 4 + 3 - 5 = 2$

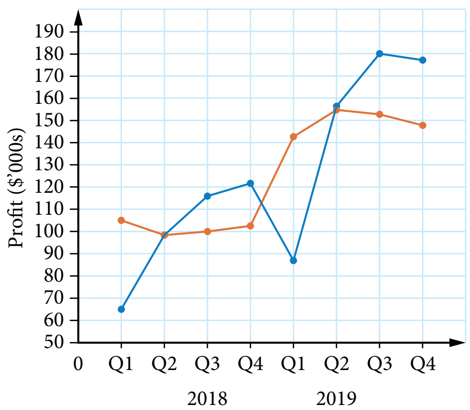
b



CUMULATIVE EXAMINATION 2

1

Year	Quarter	Profit (in $1000s)	Yearly average	Profit/ yearly average	Seasonal indices	Deseasonalised
2018	1	64	100	0.64	0.61	104.92
	2	98		0.98	1.01	97.03
	3	116		1.16	1.18	98.31
	4	122		1.22	1.2	101.67
2019	1	87	150	0.58	0.61	142.62
	2	156		1.04	1.01	154.46
	3	180		1.2	1.18	152.54
	4	177		1.18	1.2	147.5



2 11:39 am, A–G–B–C–D–H–E–F–A

3 **a**

Payment number	Opening balance	Payment	Investment gain	Principal addition	Closing balance
1	8000.00	1200.00	440.00	1640.00	9640.00
2	9640.00	1200.00	530.20	1730.20	11 370.20
3	11 370.20	1200.00	625.36	1825.36	13 195.56
4	13 195.56	1200.00	725.76	1925.76	15 121.32
5	15 121.32	1200.00	831.67	2031.67	17 152.99

b $3152.99 **c** $17 152.99

4 **a** $15 903.00 **b** $456.02

c $700 **d** $418.08

5 $20279.18

6 $32 716.17

7 $65 932.48

8 B **9** C **10** C

11 **a** $A_{n+1} = 1.07(A_n + 2400)$, where $A_0 = 0$

b $14 767.90

12 $22 066.15

13 $1099.80

14 $14 002.38

CHAPTER 8

EXERCISE 8.1

1 **a** $A_{n+1} = 1.005\,75A_n + 650$, where $A_0 = 7500$

b $10 296.50

2 **a** 4.8% p.a. **b** $217 104.83 **c** $17 104.83

EXERCISE 8.2

1 a $11 658.26 b $58.29
c $900 d $277.63

2 a $A_{n+1} = 1.01625A_n + 1960$, where $A_0 = 0$
b $35487.76

3 a $A_{n+1} = 1.006833\,(A_n + 380)$, where $A_1 = 380$
b $1546.14 c $26.14

4 a $146 430.22 b $358.22

5 $1034.34

6 a $320 257.40 b $843.07

7 $950.93

8 D 9 A 10 B

11 $983.64

12 $2722.68

13 $91 392.41

14 $565 800

15 $11 336.36

EXERCISE 8.3

1 $307 041.31 2 $127,492.61

3 $1150 4 $21 600

5 a $1375 b about 12.3% p.a.

6 $A_1 = A_2 = 30\,000$

7 $228,923 8 6.67% 9 B

10 C 11 $2790 12 $490 909.09

13 a $10 350 b about 8.3% p.a.
c $333.67 d about 8.5% p.a.

14 $150 000

15 about $85 073

16 a 1.3% or 0.013 b $75 000

EXERCISE 8.4

1 A

2 a $1100 b 18% p.a.

3 a This is an example of an investment annuity.
b No, she will fall about $10 071 short.
c $538.51

4 a This is an example of an ordinary annuity.
b $138 051.77

5 $40 111.13

6 a $370.03 b $31 082.52 c $6082.52
d $380.92 e $522.72

7 $8553.22 8 B 9 $10 100

10 $526.67 11 $1688.86

12 a $11 520 b 7.5%

13 a $77 784.36 b $1005.16

14 a $171 045.68 b $784.26

15 $31 699.40

16 $3401.37

17 $483 103.97 18 $73 951.50

19 Option 1 = $599 402.78
Option 2 = $629 718.14
Option 2 is a better investment by $30 315.36.

20 $8526.218… ≈ 8526.22

21 There are 26 fortnights in a year, so this is effectively the same as making 13 monthly payments in a year. You can work out the correct fortnightly rate by multiplying the monthly payment by $\frac{6}{13}$.

22 $797.17 23 $415 655.51

24 $84 999.96 25 $4 989 054.86

26 $3276.50

CUMULATIVE EXAMINATION 1

1 A 2 C 3 D 4 D 5 D

6 $2363.23 7 600 km (600.14)

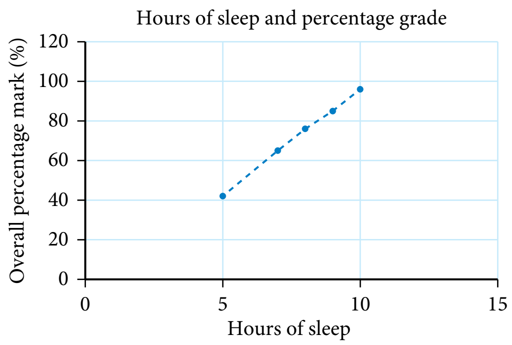
8 a $51 343.85 b $55 285.73 c Option B

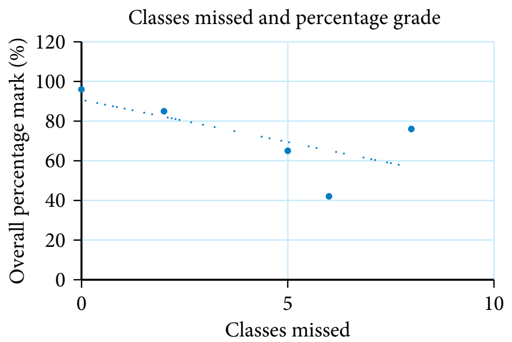
CUMULATIVE EXAMINATION 2

1 $77 383

2 Option 1 (≈ 0.0293) is better than Option 2 (≈ 0.0243)

3





From the graphs, it can be observed that:

- there is a very strong positive correlation between hours of sleep and overall percentage grades.
- there is a strong negative correlation between the number of times a student is absent from class and overall percentage grades.

The graphs indicate getting plenty of sleep has a stronger impact on improving grades than attending all classes. However, doing both is likely to have the greatest impact.

CHAPTER 9

EXERCISE 9.1

1 **a** tree; the graph is connected and contains no loops, cycles or multiple edges

b not a tree; the graph contains a cycle

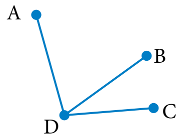
c not a tree; the graph is disconnected

d tree; the graph is connected and contains no loops, cycles or multiple edges

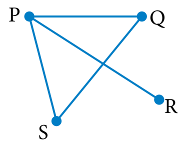
e tree; the graph is connected and contains no loops, cycles or multiple edges

f not a tree; the graph contains a loop

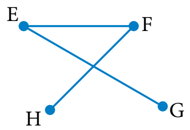
2 **a** tree



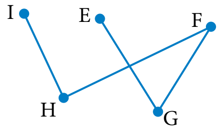
b not a tree



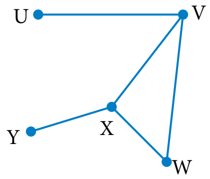
c tree



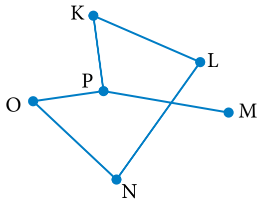
d tree



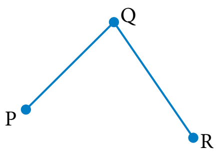
e not a tree

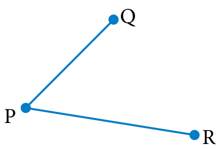


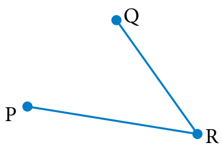
f not a tree



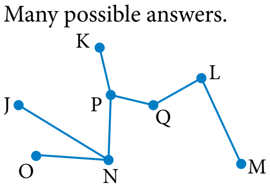
3





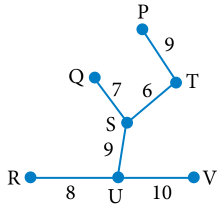


4 **a** Many possible answers.

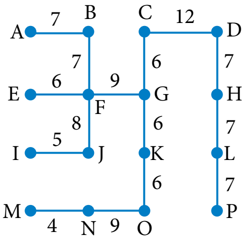


b Answers will vary depending on the spanning tree. JK, JO, NQ, NM, KQ and KL.

5 **a**



b

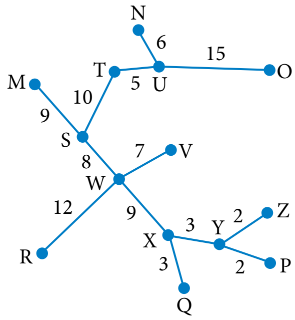


For **b**, you could use BC or JK instead of FG.

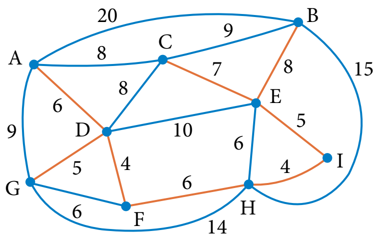
6 **a** The minimum spanning tree is BC, CD, DE, AE, DF.

b total weight of minimum spanning tree = 31

7 **a**

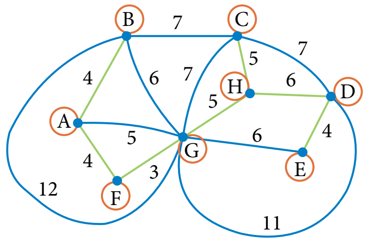


b



8 length = 31 km

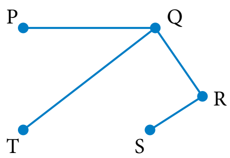
You can use GE instead of HD.



9 D **10** B **11** B **12** B

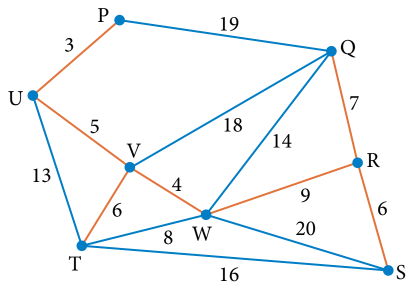
13 AB, BC, FE and ED.

14 tree



 9780170484824

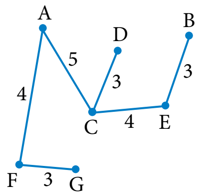
15 total weight = 40



EXERCISE 9.2

1 A

2 **a**



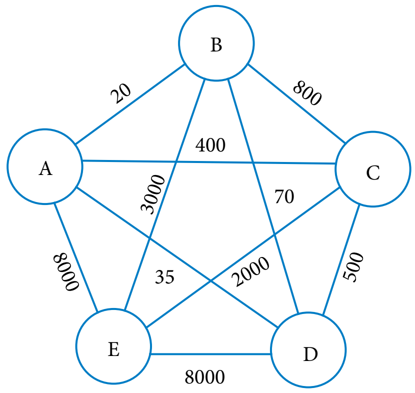
b 22 **c** FB, AB, AD, DB, AG, GB

3 B **4** D **5** $k = 5$ m

6 **a** B–C–E–F–G, 30 minutes

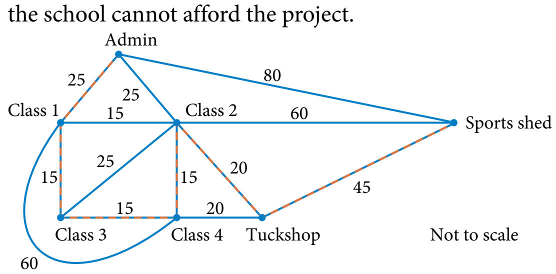
b not a minimum spanning tree; CD is not the shortest edge to join D to the tree

7 **a**



b $6137.50

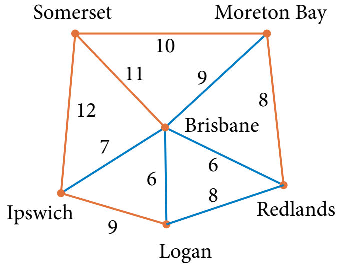
8 Total cost is $162 000. This is greater than the budget so the school cannot afford the project.



9 length of cable C = 2.6 km, minimum total cable length is 15 km

10 U–X–W–Z

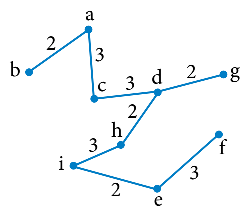
11



EXERCISE 9.3

1 28 km

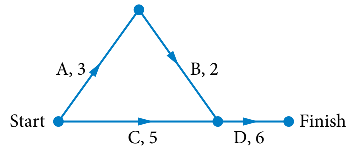
2 **a**



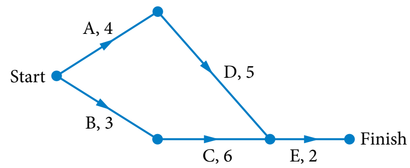
b 20 km

3 B **4** D

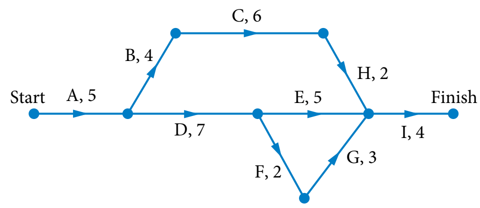
5 **a**



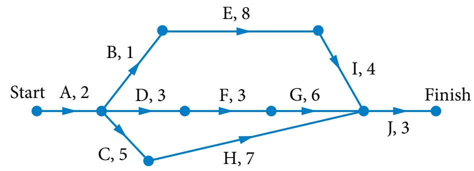
b



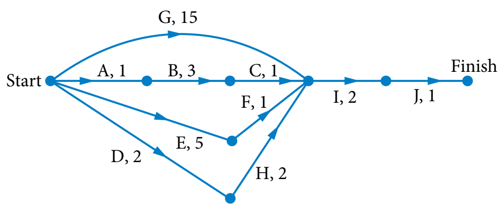
c



d



6



7 **a**

Activity	Time (days)	Prerequisites
A	2	None
B	4	A
C	3	A
D	3	B
E	1	C

b

Activity	Time (days)	Prerequisites
A	1	None
B	4	None
C	6	A
D	4	A
E	7	B, C
F	5	B, C
G	7	F
H	6	D
I	2	G

c

Activity	Time (days)	Prerequisites
E	3	None
F	7	E
G	5	E
H	2	G
I	6	G
J	3	F
K	1	J, H
L	9	F
M	3	K, I

d

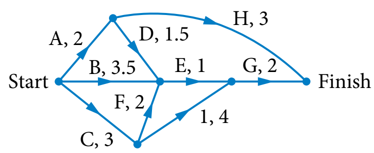
Activity	Time (days)	Prerequisites
P	13	None
Q	14	None
R	15.5	None
S	10	P
T	3.5	S, Q
U	8	R, T
V	1.5	U

8 C

9 C

10 B

11

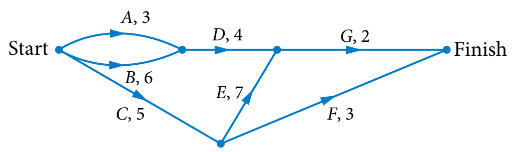


12 Activity G is missing and needs to be included going from the end of activities B, D and H and connecting to the end of activity F and the start of activity I. There is an additional activity N which needs to be deleted.

EXERCISE 9.4

1 C

2



3 C

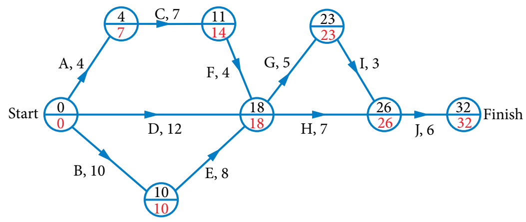
4 B

5 D

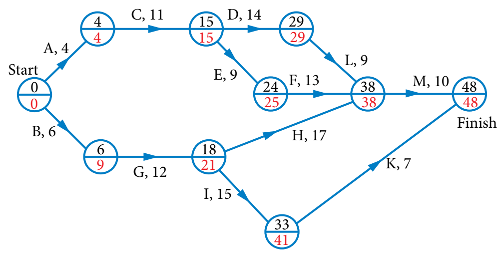
6 C

7 **a** 0 **b** 12 **c** 15 **d** 21

8 **a, b, c**



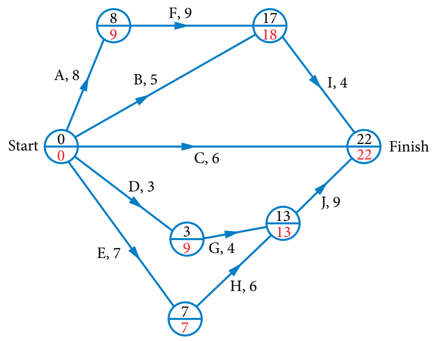
9 **a, b**



c It would have no effect.

d It would have no effect.

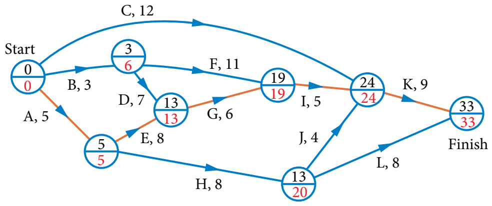
10 **a, b**



c no effect

d It would increase the time by 2 hours.

11 a, b



c 33 weeks

12 a B–E–H–J = 44

b C–F–E–I = 35

c C–G–I–P–Q–R–T–O = 39

d A–D–I–K–M–N–O–Z’ or
A–B–F–J–K–M–N–O–Z’ or
A–B–F–X–Y–Z–O–Z’ or
A–C–E–I–K–M–N–O–Z’ = 40

13 a B–G–J–L

b **i** 1 day **ii** 1 day
iii 8 days **iv** 8 days

14 a A–B–F–H–K–M

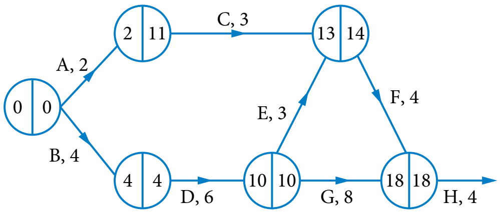
b A–C–E–J–M

15 B **16** C **17** D **18** C

19 A **20** B **21** A **22** A

23 a A – H **b** end of day 12 **c** 5 days

24 a



b B–D–G–H

c 22 days

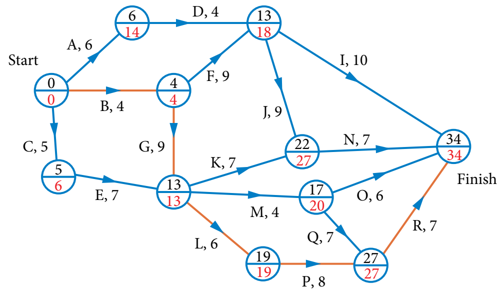
25 Delay of F will extend the completion time by 5 hours. but delay of I by 3 hours has no effect. Extra resources should be devoted to F as it is a critical activity that will reduce completion time by 1 hour. H is non-critical and reducing M will have no effect unless you can also reduce J, N or L.

26 G is critical, but J is not, so the original completion time of 36 days is extended to 38 days

EXERCISE 9.5

1 D

2 a, b



c 34 days

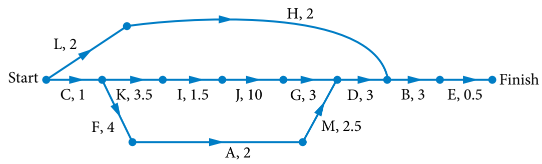
3 a float time for A, B, D, F and L = 3 weeks; Float time for E and J = 5 weeks;
float time for I = 7 weeks

b critical path = A–D–L; Completion time = 20 weeks

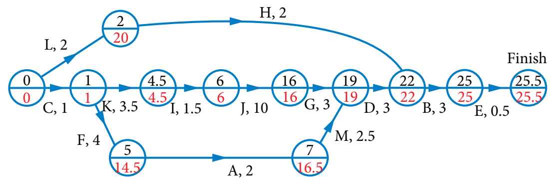
c Critical path = B–E–J–M;
Completion time = 19 weeks

d Reduce the duration of C by 1 week.

4 a Getting rid of old vegetation before getting new soil makes less clutter on site.

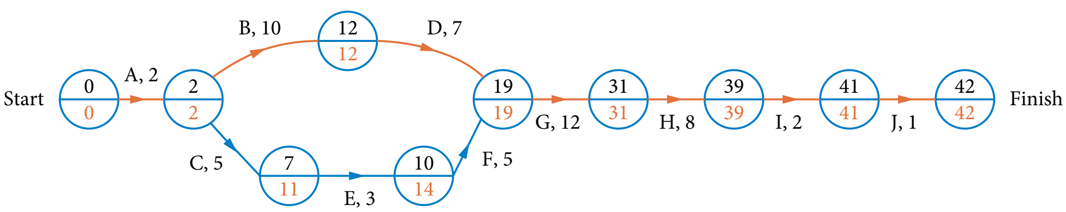


b



c It will have no effect.

5 a

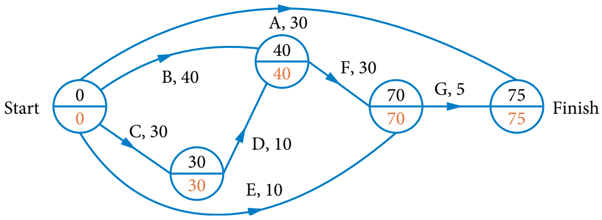


b Project completion time = 42 days

c It will have no effect. The activity is non-critical and has a float time of 4 days.

d It reduces the project completion time to 38 days.

6 **a**



C–D–F–G or B–F–G = 75 days

b no change

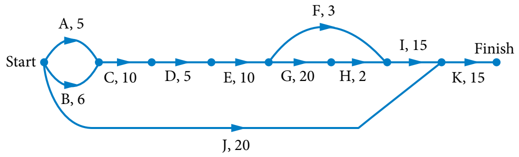
c Critical path B–F–G only and 20 days delay.

d Critical path C–D–F–G only and 20 days delay.

7 **a** pond and café

b Path = Garden shed, Pond, Playground, Butterfly house. 9:01 am

8 **a** In minutes:



b **i** 1 min **ii** 0 min

iii 19 mins **iv** 48 mins

9 The project completion time is 61 days which is more than 8 weeks ($8 \times 7 = 56$ days). They will need 5 more days to complete the project.

10 Activity G should be reduced to 3 hours

CUMULATIVE EXAMINATION 1

1 D **2** C **3** B **4** A **5** D

6 **a** $n = 42.6t + 55.4$ **b** 950

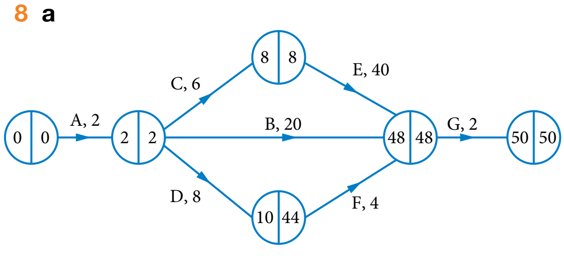
7 **a** R–W–P–D–R

b semi-Eulerian; has two vertices of odd degree

c

$$\begin{array}{c} \\ B \\ C \\ D \\ P \\ R \\ W \end{array}\begin{array}{c} \begin{array}{cccccc} B & C & D & P & R & W \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 2 & 1 & 1 \\ 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 1 & 1 & 1 & 1 & 2 & 0 \end{bmatrix}\end{array}$$

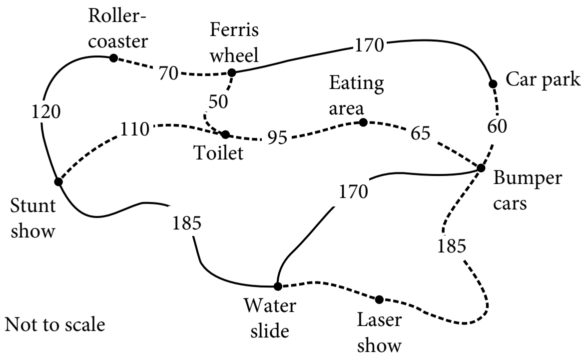
8 **a**



b A, C, E and G; 50 minutes.

CUMULATIVE EXAMINATION 2

1



The manager is correct and they will save \$138 030 annually.

2 The monthly savings were \$1999.29.

CHAPTER 10

EXERCISE 10.1

1 **a** **i** A = 20, B = 35, C = 15, T = 80

ii A = 15, B = 40, C = 25, S = 70

iii A = 15, B = 35, C = 15

b **i** W = 20, X = 55, Y = 50, Z = 45, T = 95

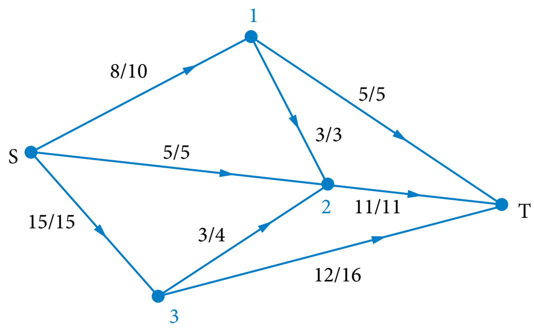
ii W = 40, X = 30, Y = 40, Z = 45, S = 110

iii W = 20, X = 30, Y = 40, Z = 45

2 D **3** D **4** C

5 **a** 50 **b** 35 **c** 35

6



7 maximum flow = 35

8 B **9** D **10** A **11** A

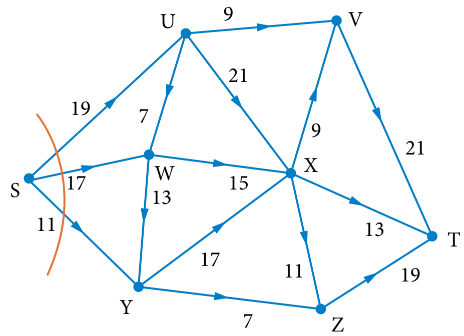
12 **a** **i** Cuts 2 and 3 are not valid.

ii Cut 1 = 27, Cut 4 = 33

b **i** Cut 2 is not valid.

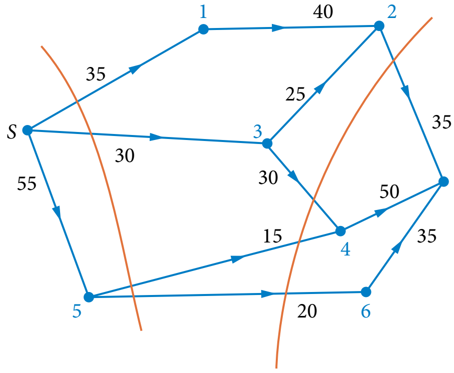
ii Cut 1 = 48, cut 3 = 57, Cut 4 = 56, Cut 5 = 54

13 **a** maximum flow = 47



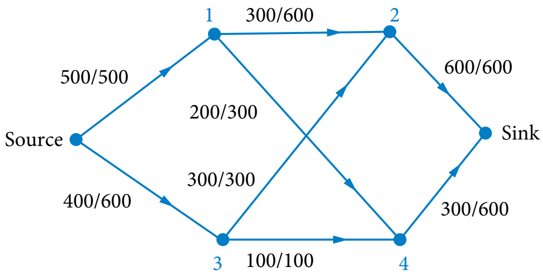
9780170484824

b maximum flow = 100 (Both cuts are correct).



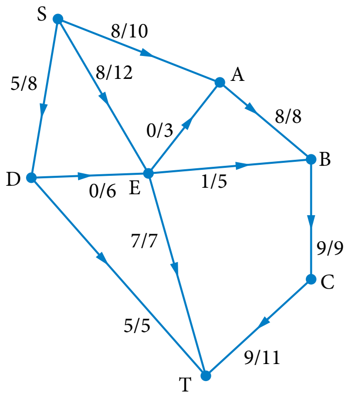
14 C **15** B **16** C **17** A **18** C

19



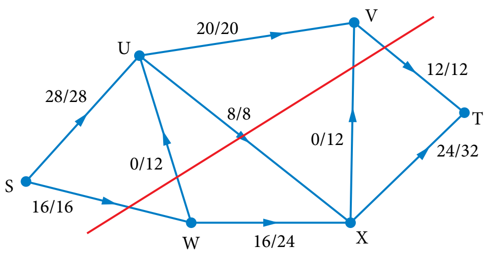
20 **a** **i** 18 L/s **ii** 15 L/s

b



c 21 L/s

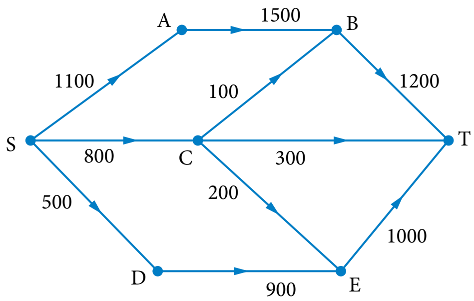
21



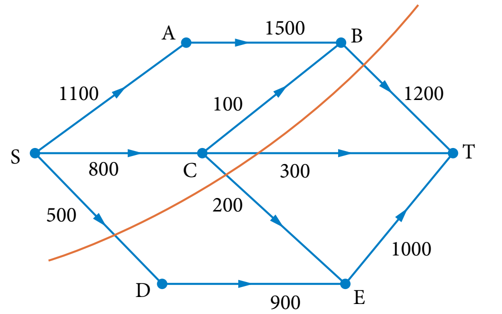
EXERCISE 10.2

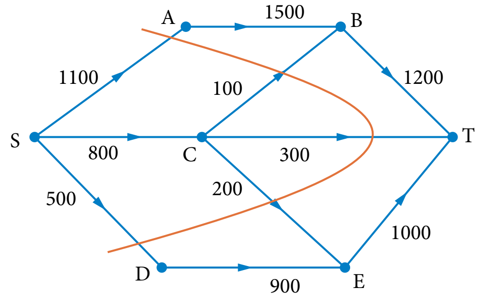
1 C **2** B

3 **a**



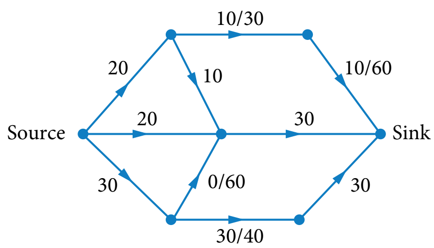
b maximum flow = 2200 MW (Both cuts are correct).





4 **a** maximum flow = 70 L/min

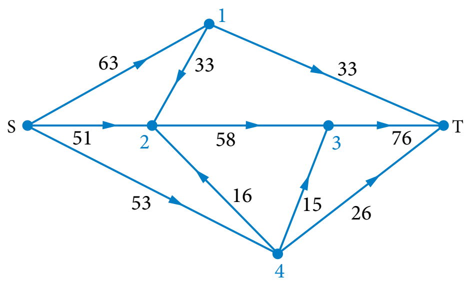
b



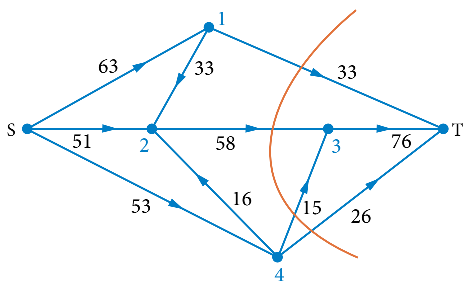
5 C **6** C

7 a + c + d + e **8** 370

9 **a**



b maximum flow =132 widgets



10 **a** L_4 is not valid as the tank and the tap are on the same side of the line.

b $L_1 = 57$, $L_2 = 74$, $L_3 = 36$

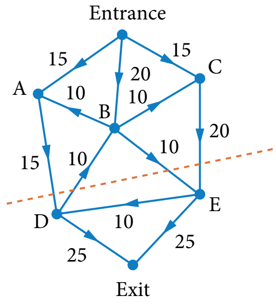
11 **a** 440

b maximum flow = 300 vehicles per hour

c maximum flow during weather emergency = 120 vehicles per hour

12 a 50 visitors

b



For the maximum flow to be at least 50, all cuts must be equal to or greater than 50. The cut illustrates that the maximum flow is 45 visitors.

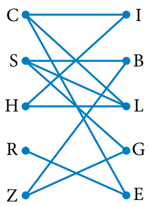
c A–D or B–E or C–E could increase by 5.

13 The road from the freeway exit to Allentown (or the edge from the Dogville to the tunnel entrance) needs to be increased by 100 vehicles/minute.

EXERCISE 10.3

1 A **2** C **3** B

4



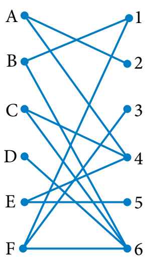
5 a Oliver = Tasks 1, 4; Isla = Tasks 1, 2, 3; Jordan = Task 3; Ella = Tasks 2, 4

b Paige = Tasks 1, 2; Noah = No task; Maya = Tasks 1, 2, 3 and 4; Ryder = Tasks 2, 3

c Hugo = Task 1; Evie = Tasks 2, 3, 5; Max = Tasks 2, 4; Poppy = Tasks 1, 2 and 4; Beau = Tasks 3, 4, 5

d Eliza = Tasks 2, 5; Connor = Task 3; Stella = Tasks 1, 2, 3, 4, 5; Jake = Tasks 4, 5; Holly = Tasks 1, 3

6 a



b A–2, B–1, C–4, D–6, E–5 and F–3.

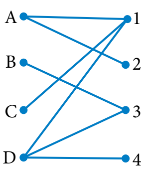
7 a bipartite; Group 1 = {A, B, D, H}, Group 2 = {C, E, F, G}

b not bipartite

c bipartite; Group 1 = {A, B, C}, Group 2 = {D, E, F}

d not bipartite

8



A–2, B–3, C–1 and D–4

9 D **10** C **11** B

12 The graph is bipartite;
Group 1 = {A, I, D, G, H, L},
Group 2 = {B, C, F, E, J, K, M}

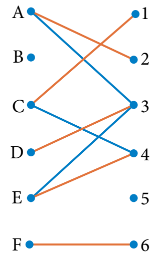
13

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} W & X & Y & Z \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

14 a Ross

b Rachel–Taylor, Phoebe–Ariana and Karen–Billie

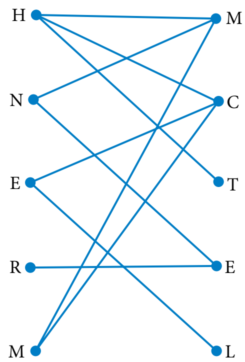
15 The matching shown in orange is the maximum number of applicants placed in positions.



EXERCISE 10.4

1 bipartite; Group 1 = {A, E, G}; Group 2 = {B, C, D, F}

2



Hannah–Technology, Nate–Maths, Ella–Legal Studies, Riley–English, Mila–Chemistry

3 a

	Task 1	Task 2	Task 3
Emily	11	8	12
Troy	13	12	10
Ruby	10	13	11

b Emily = task 2, Troy = task 3, Ruby = task 1

c 28 minutes

4 a A–1, B–2, C–4, D–3; 56 minutes

b A–1, B–2, C–4, D–3, 165 minutes

5 a A = Toowoomba, B = Cairns, C = Mackay, D = Ipswich

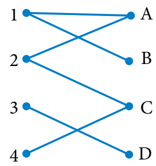
b $27 000

6 A

7 a

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 0 & 5 & 6 \\ 0 & 9 & 0 & 9 \\ 3 & 7 & 5 & 0 \\ 7 & 11 & 0 & 2 \end{bmatrix} \end{array}$$

b Optimum allocation is 2A, 1B, 4C and 3D.



c $33

8 D

9 a A–4, B–2, C–3, D–1 b 72 km

10 a A–1, B–3, C–2; 19 minutes

b A–2, B–3, C–4, D–1; 39 minutes

11 a A3, B2, C4, D1 or A4, B2, C1, D3 or A4, B2, C3, D1; $64 000

b A–3, B–4, C–1, D–2; $123 000

12 A–1, B–3, C–4

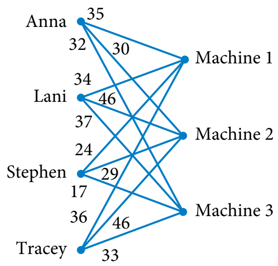
13 C **14** B

15 1–C, 2–A, 3–B; 55 km

16 A = Task 2, B = Task 3, C = Task 1, Minimum time = 51 minutes

17 S1 to C4, S3 to C1, S2 to C2 and S4 to C3.; 136 minutes

18 a



b The number of columns does not equal the number of rows, it is not a square matrix, so a dummy column is added.

c Anna to Machine 1, Lani to Machine 3 and Tracey to Machine 2.; Maximum number of doors = 118 doors

19 A–4, B–1, C–2, D–3; 95 minutes

20 a Kate–Chair, Luka–bookcase, Marcel–desk; $323

b Reasonable because it is the cheapest option.

21 Lenny–cycle, Issabelle–run, Oliver–swim, Natalie–throw, Zane–Jump and 839 points

22 A–2, B–1, C–3; $8000

23 Masters Homes–Endeavour, Custom Build–Beachside, Savannah Homes–Alpine, Deakin Homes–Tropicana, Stylemaster–Forrest Grove.

Total cost = $1 650 000.

24 A–5, B–4, C–3, D–1 and E–2; Sales = $287 000

25 Workshop A–e-bike 1, Workshop B–e-bike 2, Workshop C–e-bike 4, Workshop D–e-bike 3; Annual profit = $ 66 000

26 A-Q, B-P, C-R, 1815 km

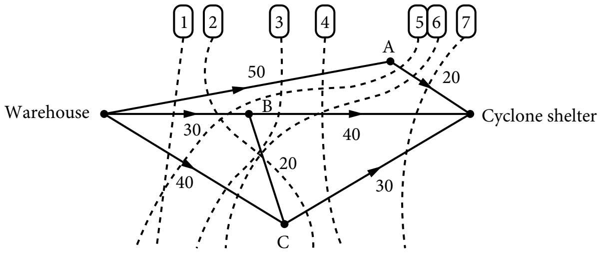
CUMULATIVE EXAMINATION 1

1 B **2** C **3** A

4 C **5** C

6 Option A is reasonable as it has the higher effective interest rate.

7 a The maximum amount is 90.



Source: General Mathematics SEE marking guide External assessment, © State of Queensland (QCAA) 2024.

b The maximum amount is now 70.

8 a

	Change uniform	Do not change uniform
Junior staff	80%	29.5%
Senior staff	20%	70.5%
Total	100%	100%

b There does appear to be an association. The data suggests that junior staff want to change the uniform (80% as opposed to 20% of senior staff) and senior staff do not want to change (70.5% compared with 29.5% of junior staff).

CUMULATIVE EXAMINATION 2

1 14° 52′ S, 127° 14′ E

2 11 km

Glossary and index

accrued value The total value of an investment. (p. 202)

activity A task required to complete a project. (p. 356)

activity table A table that shows the order and estimated completion time of each activity required to complete a project. (p. 353)

actual flow The actual quantity of a commodity that moves through an edge of a network. (p. 390)

actual value *See* **residual value**.

adjacent vertices Vertices in a graph or network that are connected by an edge. (p. 242)

adjacency matrix A matrix that shows how adjacent edges are joined. (p. 263, 405)

allocation A situation where each agent is assigned to one and only one task. (p. 405)

amortisation The process of paying off a loan with regular payments over a period of time. (p. 217)

amortisation table A table showing time, interest, change in principal and balance of a loan or investment. (p. 217)

angular distance On the Earth, the angle subtended at the centre by two points on the surface. In particular, the difference in latitude or longitude. (p. 162) *See also* **latitude**, **longitude**.

annuity An investment that has regular deposits made over a period of time. (p. 225)

arc An edge with a direction (shown by an arrow). (p. 244)

arithmetic sequence, **arithmetic progression (AP)** A sequence whose successive terms differ by the same amount. (p. 40) *See also* **common difference**.

assignment *See* **allocation**.

association A pattern in the ordered pairs of bivariate data that suggests a **relationship**, often seen in a scatterplot. Also called a **correlation**. (p. 18)

Australian Central Standard Time (ACST) UTC +9.5 h; the time zone used in South Australia and the Northern Territory. (p. 183)

Australian Eastern Standard Time (AEST) UTC +10 h; the time zone in Queensland, New South Wales, the Australian Capital Territory, Victoria and Tasmania. (p. 183)

Australian Western Standard Time (AWST) UTC +8 h; the time zone used in Western Australia. (p. 183)

backward scanning A process used to determine the latest starting time for each activity in a network. (p. 360)

bipartite graph A graph in which the vertices can be divided into 2 distinct groups so that every edge of the graph connects a vertex in one group to a vertex in the other. (p. 256)

bivariate Bivariate data has 2 variables. It is collected in pairs. (p. 3)

book value *See* **depreciation**.

bridge An edge that keeps a graph connected. (p. 253)

capacity The maximum quantity of a commodity that moves through an edge of a network. (p. 389)

capacity of a cut *See* **cut**.

categorical Data with values that are names. (p. 100) *See also* **numerical**.

causation Where changes in an explanatory variable produce changes in the response variable. (p. 14)

Central Standard Time *See* **Australian Central Standard Time**.

circle of illumination The division between day and night on the Earth when viewed on a globe. (p. 180)

circuit A trail that returns to its starting point. (p. 272)

closed path A path that starts and finishes at the same vertex. (p. 273)

closed trail A trail that starts and finishes at the same vertex. (p. 272)

closed walk A walk that starts and finishes at the same vertex. (p. 272)

coefficient of determination, R^2 The square of the correlation coefficient of bivariate data that shows the proportion of the response variable that is determined by the least-squares line of best fit. (p. 27)

coincidence A random relationship between variables of bivariate data that depends on the particular sample and does not show a real relationship. (p. 80)

column percentages Percentages in a contingency table based on totals down the column. (p. 8) *See also* **row percentages**, **table percentages**.

common difference The difference between successive terms of an arithmetic sequence. (p. 40)

common ratio The ratio of successive terms of a geometric sequence. (p. 54)

complete graph A graph that has edges connecting each vertex to every other vertex in the graph. (p. 252)

compound interest The interest paid on the original principal and on the accumulated past interest. (p. 199)

compounding period The length of time between interest calculations in compound interest. (p. 199)

concurrent activities Activities that can occur at the same time. (p. 352)

confounding variable A third variable that is the explanatory variable of both variables of bivariate data; a variable that causes the changes in both the variables of the data. (p. 80)

connected graph A graph in which it is possible to reach every vertex of a graph by moving along the edges. (p. 253)

connector problem A problem requiring minimisation of connections between vertices of a graph. (p. 343)

continuous data Data that can have any value between its limits. (p. 4) *See also* **discrete data**.

Coordinated Universal Time Abbreviated as UTC, reference for all time zones. For example, Queensland is in a UTC +10 h time zone, meaning it is 10 hours ahead of the reference. (p. 181)

correlation *See* **association**.

correlation coefficient A number between −1 and 1 that measures how well the least-squares line fits a set of data. It is given by $r = \frac{s_{xy}}{s_x s_y}$, where s_{xy} is the covariance and s_x, s_y are the standard deviations of the variables x and y. A value of 0 means there is no association. −1 and 1 show perfect negative and positive associations respectively. (p. 25)

cost matrix A matrix showing the cost of assignment of each agent to a task, each agent having the costs shown in a row, so each task is shown as a column. (p. 412) *See also* **allocation**.

critical activity An activity that lies on the critical path. (p. 361)

critical path The path with the longest total time in a network. (p. 361)

critical path analysis Analysing a project network, particularly to find minimum time to complete the project by finding **critical path**. (p. 358) *See also* **project network**.

cut A line that separates a flow network source from the sink. The **capacity of a cut** is the sum of the flows in the direction from source to sink that pass through the cut. (p. 396)

cycle A path beginning and ending at the same vertex. (p. 273)

cyclical (time series) A time series with values that rise and fall over a period or cycle (usually more than a year). For example, an economic or business cycle. The cycles may vary in length. (p. 116) *See also* **irregular**, **seasonal**.

daylight saving Adjusting clocks forward in summer to have an extra hour of daylight in the evening. (p. 183)

degree (of a vertex) The number of edges that are directly connected to the vertex, written deg(V). (p. 244)

dependent variable *See* **response variable**.

depreciation Loss of value of an asset (item) over its (working) life. The **book value** is its value after depreciation. The **salvage value (scrap value)** is the value at the end of its life. (p. 50) *See also* **straight-line depreciation, diminishing value depreciation**.

deseasonalisation Smoothing of a time series by dividing the values for each quarter or month by the corresponding value of the seasonal index. (p. 139)

directed graph, directed network, digraph A graph in which each edge has a direction. All the edges are arcs. (p. 257)

disconnected graph A graph in which it is not possible to reach every vertex of a graph by moving along the edges. (p. 253)

discrete data Data that has only some possible values, like integers. (p. 4) *See also* **continuous data**.

diminishing-value depreciation (**declining** or **reducing balance** method) Depreciation by the same proportion or percentage of the previous value each year. Successive values form a **geometric sequence**. (p. 66) *See also* **straight-line depreciation**.

dummy A row or column consisting of zeros inserted into a non-square matrix to make it square, particularly in unbalanced assignment problems. (p. 419)

earliest starting time (EST) The earliest possible time that an activity can begin. (p. 358)

Eastern Standard Time *See* **Australian Eastern Standard Time**.

edge A line that joins 2 vertices. (p. 244) *See also* **arc, graph**.

effective interest rate The interest rate that is actually earned or paid in one year by compound interest with a shorter compounding period. (p. 207)

equator The centremost and longest parallel of latitude that divides the Earth equally into the northern and southern hemispheres. (p. 162)

Euler's formula For any planar graph: vertices + faces − edges = 2 or $v + f - e = 2$. (p. 254)

Eulerian circuit *See* **Eulerian cycle**.

Eulerian cycle A trail that traverses every edge of a graph exactly once. (p. 281)

Eulerian graph Contains an Eulerian circuit. It has all vertices of even degree and a closed trail. (p. 281)

Eulerian path An open trail that traverses each edge only once. (p. 281)

Eulerian trail *See* **Eulerian path**.

explanatory variable, independent variable A statistical variable not changed by another statistical variable. (p. 14) *See also* **response variable**.

exponential decay, exponential growth, exponential relationship Reduction or growth of a sequence of values by the same ratio in each period; relationship with successive values that form a geometric sequence; relationship such that $t_n = t_0 r^n$, where $r > 0$: decay for $r < 1$, growth for $r > 1$. (p. 65)

exponential function A relationship in which the variable is an exponent, such as $t_n = t_1 r^n$, the general term of a geometric sequence. (p. 55)

extrapolation Prediction of values of the response variable of bivariate data that is outside the range of the data. (p. 86) *See also* **interpolation**.

face (of a network) A region of a planar graph enclosed by edges (including the outside as one face). (p. 258)

first term *See* **n^{th} term, sequence**.

fixed-term loan A loan in which the interest rate, instalment value and the frequency of payments remain unchanged. (p. 326)

flexible-term loan A loan in which the interest rate, instalment value or the frequency of payments may vary. (p. 326)

float time The maximum delay possible in starting an activity that does not affect the completion time of a project. $= LST_{End} - EST_{Start}$ − activity time. (p. 362)

flow capacity *See* **capacity**.

flow network A directed graph where each arc has a capacity or flow capacity. (p. 389)

forward scanning A process used to determine the earliest starting time for each activity in a network. (p. 358)

future value The value of an amount such as an annuity, loan or investment at a specific future date, taking interest earned into account. (p. 210)

general term A term in a sequence, usually written as a formula involving the term number, *n*. (p. 45) *See also* **n^{th} term**.

geometric decay, geometric growth Decline or growth where successive values form a geometric sequence. (p. 65)

geometric sequence, geometric progression, GP A sequence whose successive terms are in the same ratio. (p. 54)

gradient *See* **slope**.

graph, network A set of points called vertices (or nodes) connected by lines called edges (or arcs) joining pairs of vertices. (p. 244)

great circles Circles on a sphere (e.g. the Earth) that have their centres at the same place as the sphere's centre. The shortest distance on the surface between 2 points on a sphere is along a great circle. (p. 161)

Greenwich Mean Time Abbreviated as GMT; the predecessor to UTC and the same as UTC. (p. 181)

Hamiltonian cycle A route that includes each vertex in a graph exactly once and starts and finishes at the same vertex. (p. 286)

Hamiltonian graph A graph that contains a Hamiltonian cycle. (p. 290) *See also* **semi-Hamiltonian graph**.

Hamiltonian path A route that includes each vertex in a graph exactly once and starts and finishes at different vertices. (p. 286)

Hungarian algorithm A process for determining optimum allocation. (p. 412)

immediate predecessor An activity that must be completed immediately before another activity can commence. (p. 352)

independent variable *See* **explanatory variable**.

inflow capacity The total of the flow capacities of all edges leading into a node. (p. 389)

instalment One of a number of successive payments (usually equal) made to settle a debt. (p. 217)

intercept, *y*-intercept The value of *y* where a straight-line model of bivariate data crosses the vertical axis, particularly the least-squares line. (p. 84) *See also* **slope**.

interest The price of borrowing or lending money. (pp. 49, 199)

interest rate The ratio of interest to the amount invested or borrowed, normally expressed as a decimal or percentage. (p. 199) *See also* **nominal interest rate, effective interest rate**.

International Date Line The imaginary boundary line on the Earth that separates the current day from the next day, according to standard time zones. (p. 162)

interpolation Prediction of the response variable values for bivariate data that is within the range of the data. (p. 86) *See also* **extrapolation**.

investment annuity An annuity that has an initial balance with regular deposits and no withdrawals. (p. 302)

investment perpetuity *See* **perpetuity**.

irregular (time series) A time series with values that do not follow a pattern. (p. 116) *See also* **cyclical**, **seasonal**.

latest starting time (LST) The latest time it is possible to start an activity without delaying the completion of the project. (p. 360)

latitude Position, in degrees north or south, of the equator. The angle between the position and the equator. (p. 162) *See also* **angular distance, longitude**.

least-squares line Straight-line model of bivariate data with the smallest possible total of the squares of the distances of the points from the line. (p. 84) *See also* **intercept**, **straight-line relationship**.

linear Ordered pairs forming a straight line or nearly straight line (for data). (p. 18)

line of latitude *See* **parallel of latitude**.

line of longitude *See* **meridian of longitude**.

linear decay, linear growth Reduction or growth of a sequence of values by the same amount in each period; successive values for an arithmetic sequence. (p. 42) *See also* **exponential growth**, **exponential decay**.

loan schedule A table of the interest, payments and amount owing for a loan over time. (p. 217)

longitude Position, in degrees east or west, of the prime meridian. The angle between the position and the prime meridian. (p. 162) *See also* **angular distance, latitude, meridian of longitude**.

loop An edge that connects a vertex to itself in a graph or network. (p. 242)

maximum flow (through a node) The smaller of the total inflow capacity and the total outflow capacity of the node. (p. 389)

mean The average of a set of data calculated as the total divided by the number of data items. (p. 124) *See also* **median**.

median The average of a set of data calculated as the central value of the items placed in numeric order. (p. 124) *See also* **mean**.

meridian of longitude A line connecting the north and south poles showing the longitude east or west from the prime meridian. Also called a **line of longitude**. (p. 162) *See also* **prime meridian**.

minimum cut The cut with the minimum capacity. (p. 393)

minimum spanning tree A spanning tree with the minimum total length of its edges. (p. 339)

moderate association A linear association with points that are scattered around a straight line, but mostly close to the line. (p. 18)

moving average The average using the mean or median of the same number of successive values to smooth a time series. (p. 124)

multiple edges Where 2 or more edges connect the same pair of vertices. (p. 242)

negative association A pattern in numeric bivariate data where the low values of a variable have a tendency to occur with the high values of the other. (p. 18)

network *See* **graph**.

node *See* **vertex**.

nominal interest rate The stated annual interest rate for a loan or investment. (p. 207)

non-critical activity An activity that is not on the **critical path**. (p. 359)

non-linear An association that is not linear. (p. 18)

n^{th} term Term number n of a sequence, the general term of a **sequence**, shown as t_n. (p. 40) *See also* **general term**.

numerical Data with values that are numbers. (p. 18) *See also* **categorical**.

open path A path that starts and finishes at different vertices; an open walk with no repeated edges or vertices. (p. 273)

open trail A trail that starts and finishes at different vertices; an open walk with no repeated edges. (p. 272)

open walk A walk that starts and finishes at different vertices. (p. 272)

optimum allocation The most efficient allocation in terms of time, cost, energy consumption, etc. (p. 411)

ordinary annuity An annuity that has an initial balance with regular deposits and regular withdrawals. (p. 225)

outflow capacity The sum of the flow capacities of the edges directed away from a node in a flow network. (p. 389)

outlier A value that is very different from the rest of the values particularly for bivariate data or time series. (p. 93)

parallels of latitude Small circles parallel to the equator with their centres at the axis. (p. 162)

path A trail with no repeated vertices. (p. 272)

Pearson's correlation coefficient *See* **correlation coefficient**.

per annum (p.a.) For each year; usually how interest rates are expressed. (p. 201)

percentaged two-way frequency table A two-way table with frequencies expressed as percentages by row, column, or table totals. (p. 8)

periodic interest rate The annual (nominal) interest rate divided by the number of compounding periods per year. (p. 201)

periodic payment A payment made at regular time intervals (e.g., week, month, quarter) for a loan or investment; the regular amount deposited/withdrawn from an investment annuity, ordinary annuity, superannuation account or perpetuity. (p. 213)

perpetuity An annuity that has regular payments that continue indefinitely. (p. 316)

planar graph A graph that can be drawn without edges that cross. (p. 258)

positive association A pattern in numeric bivariate data where the variables have a tendency to decrease or increase together. (p. 18)

prediction An estimate of the value of a response (dependent) variable for a given value of the explanatory (independent) variable. (p. 86)

prerequisite *See* **immediate predecessor**.

present value The current value of a sum of money that must be invested to achieve a future value, taking interest earned into account. (p. 210)

Prim's algorithm An algorithm used to identify a minimum spanning tree. (p. 341)

prime meridian The line of longitude at 0° used as the reference for time zones. The time zone on the prime meridian is UTC (or GMT). (p. 162)

principal The amount of money borrowed or invested. (p. 199)

project network (diagram) The sequence of activities that make up a project, shown as a directed network (or directed graph). (p. 352)

quarter A 3-month period of the year, particularly one of Q1, Q2, Q3, Q4. (p. 116)

R^2 *See* **coefficient of determination**.

recurrence relation A relationship between successive values of a (financial) sequence such as $A_{n+1} = rA_n - d$ for the values of a reducing balance loan. (pp. 41, 199)

recursion, recursive Returning to the previous term, the definition of a sequence that shows how each term is calculated from the previous term. (pp. 41, 199)

recursive rule The rule for terms of a sequence stated in terms of previous terms. (p. 199)

reducing balance loan A loan in which the interest is calculated at each compounding period. (p. 213)

reducing interest loan *See* **reducing balance loan**.

relationship An association with an equation that models the data, particularly a linear association. (p. 18) *See also* **correlation**.

repayment (payment) *See* **instalment**.

reseasonalisation Restoring the seasonal value of a deseasonalised amount. (p. 139)

residual plot A plot of all the residuals for a straight-line model of bivariate data against the explanatory variable x. (p. 92)

residual value (or **residual**) The vertical distance between the y value of a point on on a scatterplot (the **actual value**) and the predicted value from the line of best fit (least-squares line)

residual value = actual value − predicted value (p. 92)

response variable, dependent variable A statistical variable that could be changed by another statistical variable. (p. 14) *See also* **explanatory variable**.

route A sequence of edges on a graph or network. (p. 276)

row and column reduction A process for determining optimum allocation. (p. 412)

row percentages Percentages in a contingency table based on totals across the rows. (p. 8)
See also **column percentages**, **table percentages**.

salvage value *See* **depreciation.**

scatterplot, scatter graph, scattergram A graph of the ordered pairs of numerical bivariate data. (p. 15)
See also **time series plot**.

scrap value *See* **depreciation**.

seasonal index The values that show the variation of a time series across the quarters or months of a year. (p. 135)

seasonal (time series) A time series with values that rise and fall in the same way each year, often linked to the seasons. (p. 116) *See also* **cyclical, irregular**.

semi-Eulerian graph Contains an Eulerian path. It has only two vertices of odd degree, an open trail or Eulerian path exists and it must start and finish at a vertex of odd degree. (p. 281)

semi-Hamiltonian graph A graph that contains a **Hamiltonian path**. (p. 284) *See also* **Hamiltonian graph**.

sequence A list of numbers in a particular order, usually written as $t_1, t_2, t_3, \ldots$ The first term is written as t_1 and the n^{th} term as t_n. (p. 40) *See also* **arithmetic sequence, geometric sequence**.

shortest path In a weighted network, the path between 2 vertices with the lowest total length. (p. 275)

simple (path or circuit) A path or circuit that does not cross over itself, so does not repeat nodes.

simple graph A graph where the vertices are connected by no more than one edge. (p. 252)

simple interest Interest that is calculated using the formula $I = Pin$. (pp. 49, 199)

sink (T) The node in a flow network towards which all flows move, the end node. (p. 389)

sinking fund An annuity set up to receive periodic payments to pay for an item at some future time. (p. 313)

slack time *See* **float time**.

slope The vertical distance between two points of a least-squares model of bivariate data, divided by the horizontal distance. It measures the change of the response variable per unit change in the explanatory variable. (p. 84)
See also **intercept**.

small circles Circles on the Earth that *do not* have their centre at the centre of the Earth. (p. 161)
See also **great circles**.

smoothing Removal of variation from a time series (to make important trends more obvious). (p. 124)
See also **moving average**.

source(S) The node in a flow network from which all flows commence. (p. 389)

spanning tree A connected graph/network that includes all the vertices of the original. (p. 338)

standard deviation The standard deviation of a variable measures its spread from the mean. It is defined as

$$\sigma_x = \sqrt{\frac{\sum (x-\bar{x})^2}{n}} \text{ for a population of size } n \text{ or}$$

$$s_x = \sqrt{\frac{\sum (x-\bar{x})^2}{n-1}} \text{ for a sample of size } n. \text{ (p. 84)}$$

standard time zones Time zones that are internationally recognised as standard. (p. 181)

straight-line depreciation The method of depreciation that reduces the value of an asset by the same amount each year using the formulas

$$V_{n+1} = V_n - d \text{ and } V_n = V_0 - nd,$$

where V_0 is the initial value, V_n the value after n years and d is the depreciation per year.The **book values** (V_n) form an **arithmetic sequence**. (p. 49)
See also **depreciation**, **salvage value**.

straight-line relationship, straight-line model Model of the relationship between the explanatory variable (x) and response variable (y) of bivariate data of the form $y = mx + c$, $y = ax + b$ or $y = a + bx$. (p. 84) *See also* **least-squares model**.

strong association A linear association with points that are almost all on a straight line. (p. 18)

subgraph A graph that is all or part of another graph without any new edges or vertices. (p. 243)

superannuation Money that is put aside and saved while you're working, so you can have a regular income later in life when you retire. (p. 310)

table percentages Percentages based on the grand total of all frequencies in a contingency table. (p. 8)
See also **column percentages**, **row percentages**.

term Any of the members of a sequence. (p. 40)
See also **n^{th} term**.

time series A set of (numeric) data collected at regular periods of time. (p. 113)

time series plot A graph of a time series with time on the horizontal axis. Points are often joined to make a line graph. (p. 113) *See also* **scatterplot**.

time zone An area of a country or state that has its clocks set to a common time. (p. 180)

trail A walk with no repeated edges. (p. 272)

tree A connected graph/network that does not contain any loops, multiple edges or cycles. (p. 337)

trend A gradual increase or decrease in the values of a time series. (p. 116)

two-way table A table of bivariate data with one variable across columns and the other down rows. (p. 3)
See also **percentaged two-way table**.

unbalanced assignment problem An assignment problem with a non-square cost matrix. (p. 419)

undirected graph A network that has edges with no directions. (p. 257)

univariate Data with only one variable. (p. 3)

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valid (cut) A cut in a flow network that stops all flow from the source to the sink. (p. 396)

vertex, node The points joined by edges in a graph or network. (p. 242)

walk Any route between 2 vertices of a network, perhaps including repeated edges and vertices. (p. 272)

weak association A linear association with points that have a pattern, but do not have a clear straight line. (p. 18)

weight (of an edge of a network) A number given to the edge of a network. It represents a property such as the length, time or cost of the edge. (p. 258)
See also **weighted network**.

weighted graph A graph or network in which a number is given for the 'length' of each edge. (p. 258)

Western Standard Time *See* **Australian Western Standard Time**.

***y*-intercept** *See* **intercept**.