ACADEMIC TASK FORCE

MATHEMATICS SPECIALIST

ATAR COURSE TEXT BOOK

YEAR 11 UNITS 1 & 2

O. T. LEE

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SECOND EDITION

A fully worked out Solution Manual for this book is available as a resource to Teachers at www.academicgroup.com.au

O. T. LEE

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About the Author
Dr O. T. Lee is an author of many books which are used extensively in WA schools. Dr Lee is an exceptional, insightful teacher with wide-ranging experience as a WACE marker.

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Contents

Preface

This book addresses the syllabus requirements of Units 1 and 2 of the Mathematics Specialist course of Western Australia.

The use of CAS/graphic calculators is seamlessly integrated into the teaching and learning process. Questions have become more explicit in terms of the required methods and techniques. Knowledge of CAS/graphic calculator techniques empower students to appreciate the relative efficiencies (and accuracies) of *machine based* techniques against traditional pencil and paper techniques. However, the traditional pencil and paper techniques are the ones that convey the actual mathematical concepts and processes and form the backbone of this book. Machine based techniques are at best interpretative techniques.

The use of Hands-on-Tasks is continued in this book. These tasks allow students to conceptualise mathematical concepts on their own without being explicitly "taught". This promotes relational understanding rather than factual knowledge of mathematical concepts and ideas.

A fully worked out ***Solution Manual*** is available as a resource for teachers.

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01 Combinatorics I

1.1 The Factorial Notation

- Consider the product of all integers from 1 to n:
  $n \times (n-1) \times (n-2) \times (n-3) \times \ldots \times 3 \times 2 \times 1$
  - This product may be abbreviated as $n!$, read as n factorial.
  - For example: $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$.
  - $0!$ is defined as numerically equal to 1.
- The value of $n!$ can be found without evaluating the product by using the *factorial button/routine* on your scientific/CAS/Graphics calculator.
- An interesting feature of the factorial notation is that it allows "telescoping" of the product. Hence: $n! = n \times (n-1)!$ or $n(n-1) \times (n-2)!$ or .....

Example 1.1

Without the use of a calculator, evaluate: (a) $\dfrac{5!}{2!}$ (b) $\dfrac{10!}{7!\,3!}$

Solution:

(a) $\dfrac{5!}{2!} = \dfrac{5 \times 4 \times 3 \times 2!}{2!} = 60$

(b) $\dfrac{10!}{7!\,3!} = \dfrac{10 \times 9 \times 8 \times 7!}{7! \times (3 \times 2 \times 1)} = 120$

Exercise 1.1

1. Without the use of a calculator, evaluate:
   (a) $\dfrac{6!}{4!}$ (b) $\dfrac{12!}{10!}$ (c) $\dfrac{8!}{2!6!}$ (d) $\dfrac{9!}{2!3!4!}$
2. Rewrite using factorial notation:
   (a) $5 \times 4 \times 3$ (b) $39 \times 38 \times 37 \times 36 \times 35$
   (c) $n(n-1)(n-2)(n-3)$ (d) $(n+5)(n+4)(n+3)(n+2)(n+1)$
   (e) $\dfrac{14 \times 13}{3 \times 2 \times 1}$ (f) $\dfrac{8 \times 7 \times 6}{6 \times 5 \times 4}$
3. Factorise:
   (a) $8! + 9!$ (b) $7! - 2 \times 5!$
   (c) $3(10!) + 4(8!)$ (d) $n! + (n-1)!$

1.2 The Multiplication Rule

- Two tasks are *mutually exclusive* or *disjoint* if the set of outcomes of the first task and the set of outcomes of the second task have no common element.
- If the tasks A, B, C, D, ... , each *mutually exclusive* or *disjoint* from the other, can each respectively be completed in $n(A)$, $n(B)$, $n(C)$, $n(D)$, ... ways, then:
  tasks A *and* B *and* C *and* D *and* ..... can be completed in
  $n(A) \times n(B) \times n(C) \times n(D) \times \ldots$ ways.
- This rule is known as the *multiplication rule*.
  Note that this rule applies only to tasks that are mutually exclusive.

1.3 Arrangements/Permutations

- Consider the letters A, B, C, D and E.
  We wish to determine the number of ways these letters may be arranged in a line with no letter being used more than once.
  - That is, we wish to determine the number of ways, the five empty spaces in the table below may be "filled" with these letters.

  - The 1st space may be filled in 5 ways as there are five available letters.
    The 2nd space may be filled in 4 ways as there are four remaining letters.
    The 3rd space may be filled in 3 ways as there are three remaining letters.
    The 4th space may be filled in 2 ways as there are two remaining letters.
    The 5th space may be filled in 1 way as there is only one remaining letter.
  - The tasks of filling these spaces are mutually exclusive.
  - Hence, by the multiplication rule,
    the number of ways of filling these spaces is $5 \times 4 \times 3 \times 2 \times 1 = 5!$.

- Consider again the letters A, B, C, D and E.
  We wish to determine the number of ways three of these letters may be arranged in a line with no letter being used more than once.
  - That is, we wish to determine the number of ways, the three empty spaces in the table below may be "filled" with these letters.

  - The 1st space may be filled in 5 ways as there are five available letters.
    The 2nd space may be filled in 4 ways as there are four remaining letters.
    The 3rd space may be filled in 3 ways as there are three remaining letters.
  - The tasks of filling these spaces are mutually exclusive.
  - Hence, by the multiplication rule,
    the number of ways of filling these spaces is $5 \times 4 \times 3 = \frac{5!}{2!}$.

- In general, n objects all different, may be arranged in a line, with no object being used more than once, in $n!$ ways.

- Also, r objects out of n objects all different ($r \le n$) may be arranged in a line, with no object used more than once in $\dfrac{n!}{(n-r)!} = \underbrace{n\times(n-1)\times(n-2)\times.....(n-r+1)}_{r\text{ terms}}$ ways.

- The mathematical expression $\dfrac{n!}{(n-r)!}$ is usually denoted ${}^{n}\mathrm{P}_{r}$ and may be evaluated directly using a scientific/CAS/Graphic Calculator.

- That is, r objects out of n objects all different ($r \le n$) may be arranged in a line, with no object used more than once in

$${}^{n}\mathrm{P}_{r} = \frac{n!}{(n-r)!} = \underbrace{n\times(n-1)\times(n-2)\times.....(n-r+1)}_{r\text{ terms}} \text{ ways.}$$

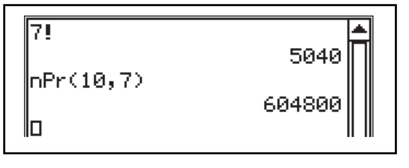
Example 1.2

How many ways are there of arranging: (a) seven students in a line
(b) seven students from a group of ten students in a line?

Solution:

(a) Total number of ways = 7! = 5040.

(b) Total number of ways = ${}^{10}\mathrm{P}_{7}$ = 604 800



Example 1.3

Passwords consisting of n letters are to be made using the letters of the alphabet with no letter being used more than once. Without the use of a calculator, write an expression for the total possible number of passwords if: (a) $n = 10$ (b) $n = 20$.

Solution:

(a) Total number = ${}^{26}\mathrm{P}_{10} = \dfrac{26!}{16!}$.

(b) Total number = ${}^{26}\mathrm{P}_{20} = \dfrac{26!}{6!}$.

Example 1.4

How many 4 digit numbers greater than 4 000 can be formed with the digits 0 to 8 inclusive if: (a) repetition is not allowed (b) repetition is allowed.

Solution:

(a) *For the number to exceed 4 000, the thousands digit must either be a 4, 5, 6, 7, or 8. This condition represents the principal property of the numbers to be formed and must be catered for first.*
The thousands digit can be chosen in 5 ways.
Since, repetition is not allowed, there is a choice of 8 remaining digits for the hundreds digit.
Similarly, there are 7 and 6 ways respectively for choosing the tens and the units digit.

5	8	7	6
Thousands	Hundreds	Tens	Units

Using the multiplication rule, the required numbers can be formed in
$5 \times 8 \times 7 \times 6 = 1\,680$ ways.

(b) *The thousands digit can be chosen in 5 ways (from the digits 4, 5, 6, 7, or 8).*
Since, repetition is allowed, each of the remaining digits can be chosen in 8 ways each.

5	8	8	8
Thousands	Hundreds	Tens	Units

Using the multiplication rule, these numbers can be formed in
$5 \times 8 \times 8 \times 8 = 2\,560$ ways.
However, this includes the number 4 000.
Hence, the required numbers (> 4 000) can be formed in $2\,560 - 1 = 2\,559$ ways

Example 1.5

Three teachers and four students are to be arranged in a line for a photo-shoot. In how many ways can this be done:
(a) if a teacher must be placed at each end of the arrangement
(b) if a teacher must be placed at each end of the arrangement and a teacher must be at the centre of each arrangement?

Solution:

(a) Total number of ways = $3 \times 5! \times 2 = 720$.

(b) Total number of ways = $3! \times 4! = 144$.

3	5	4	3	2	1	2
T						T

3	4	3	2	2	1	1
T	S	S	T	S	S	T

Exercise 1.2

1. In how many ways can 5 different books be arranged in a book rack?

2. A toddler has 8 different coloured cubes. In how many ways can 6 of these cubes be arranged in a straight line?

3. In how many ways can the letters of the word COMPUTER be arranged if
   (a) no letter may be used more than once (b) any letter can be used more than once?

4. In how many ways can the letters of the word JUNGLE be arranged if each letter can only be used once and the first letter must be a vowel?

5. How many 5 character passwords can be formed with the characters A, B, C, 0, 1 and 2 if no character may be used more than once and if the first and last character must be an letter and a digit respectively?

6. How many 5 digit numbers greater than 60 000 can be formed from the digits 0 to 9 inclusive if :
   (a) no digit may be used more than once (b) any digit may be used more than once?

7. The telephone numbers assigned to a certain suburb start with the sequence 9227 followed by 4 other digits. There is no restriction on the number of times a digit may be used. How many telephone numbers are available to this suburb?

8. The telephone numbers assigned to a certain region consists of a nine digit number. The first digit cannot be a 0 and there are no other restrictions. How many telephone numbers are available using this system?

9. Four teachers and five students are to be arranged in a line for a photo-shoot. In how many ways can this be done:
   (a) if a teacher must be placed at each end of the arrangement
   (b) if a teacher must be placed at each end of the arrangement and a teacher must be at the centre of each arrangement?

10. The licence plates of vehicles in certain state consist of 3 letters of the alphabet followed by 4 digits and then another letter. There are no restrictions on the number of times a character can be used. How many licence plates are possible using this system?

11. A certain state licences her vehicles using the following system: licence plates start with 2 letters of the alphabet followed by 4 digits and ends with another letter of the alphabet. The letters O and I are not used at all. Licence plates that start with the letter Q are set aside for government owned vehicles. Licence plates that start with X and Y are reserved for trucks, cranes and other heavy vehicles. There is no restriction on the number of times a character can be used. How many plates are reserved for:
    (a) government owned vehicles (b) trucks, cranes and other heavy vehicles
    (c) non-government owned vehicles that are not trucks, cranes or heavy vehicles?

12. Passwords consisting of n letters are to be made using the letters of the alphabet. Without the use of a calculator, write an expression for the total possible number of passwords if: (a) $n = 5$ with no letter being used more than once (b) $n = 10$.

13. Passwords consisting of n digits are to be made using the digits 0 to 9 inclusive. Without the use of a calculator, write an expression for the total possible number of passwords if: (a) $n = 5$ (b) $n = 8$ with no letter being used more than once.

14. Case sensitive passwords consisting of n characters are to be made using the letters of the alphabet. Without the use of a calculator, write an expression for the total number of passwords if: (a) $n = 10$ with no letter being used more than once (b) $n = 20$.

15. Case sensitive passwords consisting of n characters are to be made using the letters of the alphabet and the digits 0 to 9 inclusive with no character being used more than once. Without the use of a calculator, write an expression for the total possible number of passwords if: (a) $n = 12$ (b) $n = 15$ with the first and last character being digits.

16. Five digit PINs are formed using the digits 0 to 9 inclusive with no digit being used more than once. How many such PINs have digits in consecutive ascending order?

17. Six letter passwords are formed using the letters of the alphabet with no letter used more than once. How many such passwords have letters in consecutive descending order?

1.4 The Grouping Technique and the Complement Rule

1.4.1 The Grouping Technique

- A special technique called the "Grouping Technique" is used to determine the number of arrangements possible when two or more objects are to be together.
- Given n objects all different, all n objects can be arranged in a line with r specific objects adjacent to each other in $(n - r + 1)! \times r!$ ways.

Example 1.6

Determine the number of ways the letters of the word NUMBER may be rearranged with the letters: (a) U and M together (b) N, U and M together.

Solution:

(a) *Consider the letters U and M as bound together.*
Hence, we now have five units to arrange: the "twinned U–M unit" and the letters N, B E and R.
These five units may be arranged in 5! ways.
But within each of these 5! arrangements, the letters U and M may be arranged in 2! ways.

Hence, by the multiplication rule, Number of arrangements = $5! \times 2! = 240$.

(b) *Consider the letters N, U and M as bound together.*
Hence, we now have four units to arrange: the "triplex N–U–M unit" and the letters B E and R.
These four units may be arranged in 4! ways.
But within each of these 4! arrangements, the letters N, U and M may be arranged in 3! ways.

Hence, by the multiplication rule, Number of arrangements = $4! \times 3! = 144$.

1.4.2 The Complement Rule

- The complement rule is used when we wish *two* objects to be arranged apart from each other.
- Given n objects all different, all n objects can be arranged in a line with 2 specific objects apart from each other in $n! - (n-1)! \times 2!$ ways.

Example 1.7

Determine the number of ways the letters of the word COUNTER may be rearranged with:
(a) the letters U and N together (b) the letters U and N apart
(c) no two consonants together.

Solution:

(a) Number of arrangements = $6! \times 2! = 1\ 440$.

(b) Total number of possible arrangements = $7! = 5\ 040$.
The complement of U and N apart is U and N together.
Hence, number of arrangements with U and N apart
= n(total) − n(U and N together)
= 5 040 − 1 440 = 3 600

(c) For the consonants to be apart, the letters must be arranged in the order: CVCVCVC.
Hence, number of arrangements
= $4! \times 3! = 144$

4	3	3	2	2	1	1
C	V	C	V	C	V	C

C: consonant V: vowel

Notes:

- *The complement rule is applied in (b) but can no longer be applied in (c) where more than two items have to be arranged apart.*

Exercise 1.3

1. Determine the number of ways the letters of the word ENGLISH can be arranged if:
   (a) there are no restrictions imposed
   (b) the letters S and H must be together in the order SH
   (c) the letters G and L must be together (d) the letters E and N must be apart
   (e) the letters G and L must be together and E and N must be together.

2. Determine the number of ways the letters of the word CATERING can be arranged if:
   (a) the letters I and N must be together (b) the letters A and T must be apart
   (c) the letters I and N must be together and the letters A and T must be apart.

3. Determine the number of ways the letters of the word FORMULA can be arranged if:
   (a) the letters O and R must be together
   (b) the letters O and R must be apart
   (c) the consonants must be together
   (d) no two consonants are together.

4. Determine the number of ways the letters of the word FRIGHTENS can be arranged if:
   (a) the letters I and E must be apart
   (b) the letters R, I, G, H, T must be together
   (c) no two letters from the set {R, I, G, H, T} must be together.

5. Five different Physics books and four different Chemistry books are to be arranged on a shelf. Determine the number of ways these books may be arranged if:
   (a) the Physics books must be together
   (b) the books of the same subject must be together
   (c) no two Physics books must be together.

6. Six different History books and five different Geography books are to be arranged on a shelf. Determine the number of ways these books may be arranged if:
   (a) the Geography books must be together
   (b) the books of the same subject must be together
   (c) no two History books must be together.

7. Determine the number of ways a family of 3 girls, a boy, mum and dad can be arranged in a straight line if: (a) mum and dad must be together
   (b) mum and dad must be together and none of the girls are next to each other.

8. Eleven persons are required to pose for a group photograph. Five of the eleven are required to pose in a squatting position in front of the remaining six who will pose standing. Determine the number of ways the group can pose for the group photograph if:
   (a) no restrictions are imposed
   (b) the youngest five must pose in a squatting position, assuming that they are all of different ages.

9. Two different Mathematics books and three different Biology books are to be arranged on a shelf. Determine the number of ways these books may be arranged if no two:
   (a) Biology books must be together
   *(b) Mathematics books must be together.

10. Three girls and four boys are to be arranged in a line for a photo-shoot. How many arrangements are there if:
   (a) the girls are to be together
   *(b) no two girls are to be together.

02 Combinatorics II

2.1 The Inclusion-Exclusion Principle for Two Sets/Events

- The inclusion-exclusion principle is used to determine the number of elements in the union of two or more sets. It is also known as the addition rule.
  For two sets:
  - $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.
- If however, the sets A and B are mutually exclusive, then $n(A \cap B) = 0$ and
  - $n(A \cup B) = n(A) + n(B)$.
- Alternatively, if the tasks A and B can each respectively be completed in $n(A)$ and $n(B)$ways, then tasks A <u>*or*</u> B can be completed in:
  - $n(A \cup B) = n(A) + n(B) - n(A \cap B)$ ways.
- If however, the tasks A and B are mutually exclusive, then $n(A \cap B) = 0$ and tasks A *or* B can be completed in $n(A \cup B) = n(A) + n(B)$ ways.

Hands On Task 2.1

In this task, we will explore the proof of the inclusion-exclusion principle for two events.

Consider the accompanying two-way table. A and B are two tasks. The entries in the table indicate the number of ways, the various combinations of tasks may be performed.

	A	$\overline{A}$	Total
B	a	b	$a + b$
$\overline{B}$	c	d	$c + d$
Total	$a + c$	$b + d$	$a + b + c + d$

Use the two-way table to prove:
(a) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
(b) $n(A \cup B) = n(\text{Total}) - n(\overline{A} \cap \overline{B})$.

- The first result is known as the inclusion-exclusion principle for two sets. It is based on the idea that when we include all the elements in A and B, we "double count" the elements in $A \cap B$, and hence, to obtain the number of elements in the union $A \cup B$, we need to exclude "one count" of the elements in $A \cap B$.
- The second result is clearly an application of the complement rule.

Example 2.1

A group of teachers and students are to be arranged in a line for a photo-shoot. There are 240 arrangements with a teacher on the extreme left, 480 arrangements with a student on the extreme right and 288 arrangements with a teacher on the extreme left and a student on the extreme right. How many of these arrangements have either a teacher on the extreme left or a student on the extreme right?

Solution:

n(teacher on left or student on right)
= *n*(teacher on left) + *n*(student on right) – *n*(teacher on left and student on right)
= 240 + 480 – 288 = 432

Example 2.2

Five character passwords are to be formed using the digits 0 to 9 inclusive and the letters of the alphabet with no character being used more than once. How many such passwords:
(a) start with a letter
(b) end with a digit
(c) start with a letter and end with a digit
(d) start with a letter or end with a digit?

Solution:

(a) *n*(start with a letter) = $26 \times 35 \times 34 \times 33 \times 32$ = 32 672 640

(b) *n*(end with a digit) = $35 \times 34 \times 33 \times 32 \times 10$ = 12 566 400

(c) *n*(start with a letter $\cap$ end with a digit) = $26 \times 34 \times 33 \times 32 \times 10$ = 9 335 040

(d) *n*(start with a letter or end with a digit)
= *n*(start with a letter) + *n*(end with a digit) – *n*(start with a letter $\cap$ end with a digit)
= 32 672 640 + 12 566 400 – 9 335 040 = 35 904 000

Example 2.3

Five character passwords are to be formed using the digits 0 to 9 inclusive and the letters of the alphabet with no character being used more than once. How many such passwords:
(a) are possible
(b) start with a digit and end with a letter
(c) start with a letter or end with a digit?

Solution:

(a) *n*(possible passwords) = $36 \times 35 \times 34 \times 33 \times 32$ = 45 239 040

(b) *n*(start with a digit $\cap$ end with a letter) = $10 \times 34 \times 33 \times 32 \times 26$ = 9 335 040

(c) The event "start with a letter or end with a digit" is the complement of the event " start with a digit $\cap$ end with a letter".
Hence, *n*(start with a letter or end with a digit)
= *n*(possible passwords) – *n*(start with a digit $\cap$ end with a letter)
= 45 239 040 – 9 335 040 = 35 904 000

Example 2.4

Consider the set of integers between 1 and 1000 inclusive. How many of these integers are divisible by 3 or 5?

Solution:

Number of integers divisible by 3 = $\frac{999}{3}$ = 333.

Number of integers divisible by 5 = $\frac{1000}{5}$ = 200

Number of integers divisible by 3 and 5 = $\frac{990}{15}$ = 66.

Hence, number of integers divisible by 3 or 5 = 333 + 200 − 66
= 467

Example 2.5

How many code-words each at least 2 characters long can be made with the first five letters of the alphabet if repetition of letters are not allowed?

Solution:

The code-words can be either two, three, four or five characters long.
n(two character code-words) = $5 \times 4 = 20$
n(three character code words) = $5 \times 4 \times 3 = 60$
n(four character code-words) = $5 \times 4 \times 3 \times 2 = 120$
n(five character code-words) = $5! = 120$

As these tasks are jointly mutually exclusive, using the inclusion-exclusion principle:
n(number of code-words) = 20 + 60 + 120 + 120 = 320.

Example 2.6

Four digit numbers are to be formed using the digits 0 to 8 inclusive with no digit being used more than once. How many four digit even numbers greater than 4 000 can be formed?

Solution:

The two principal properties of the numbers required are:

- *the numbers must be greater than 4 000; that is, the first digit must either be 4, 5, 6, 7 or 8*
- *the numbers must be even; that is the last digit must be either 0, 2, 4, 6 or 8.*

Clearly, the task of choosing the first digit and the task of choosing the last digit are not mutually exclusive.

Hence, we need to partition the possible numbers into the following disjoint (mutually exclusive) subsets:

- *the first digit is either a 5 or 7 and the last digit is either a 0, 2, 4, 6 or 8*
- *the first digit is 4, 6 or 8 and the last digit is restricted to a choice of the remaining 4 even digits*

Subset 1: The first digit is 5 or 7 and the last digit is either 0, 2, 4, 6 or 8.

Thousands	Hundreds	Tens	Units
2	7	6	5

There are 2 ways to choose the first digit and 5 ways to choose the last digit.
These 2 tasks must be carried out ahead of the other tasks.
The subsequent digits can be chosen in 7 and 6 ways respectively.

The number of ways numbers in the subset can be formed = $2 \times 7 \times 6 \times 5 = 420$

Subset 2: The first digit is 4, 6 or 8 and the last digit is one of the remaining 4 even digits

Thousands	Hundreds	Tens	Units
3	7	6	4

There are 3 ways to choose the first digit and 4 ways to choose the last digit.
These 2 tasks must be carried out ahead of the other tasks.
The subsequent digits can be chosen in 7 and 6 ways respectively.

The number of ways numbers in the subset can be formed = $3 \times 7 \times 6 \times 4 = 504$

Hence, the required numbers can be formed in 420 + 504 = 924 ways.

Exercise 2.1

1. In a group of 30 spectators; 6 were members of the West Coast Eagles Football Club only, 15 were members of the Fremantle Dockers Football Club only while 2 were not members of any of these two clubs. Find how many of these spectators were members of both these clubs.

2. In a group of 40 children, 20 suffered from asthma and 30 had some form of eczema. 5 of these children had no asthma or eczema. How many of these children had:
   (a) both asthma and eczema (b) eczema only?

3. In a group of 60 Turkish Anggora cats, 45 had different coloured eyes and of these, 35 were deaf. There were 10 cats that were neither deaf nor had different coloured eyes. How many cats: (a) had different coloured eyes but were not deaf (b) were deaf ?

4. A group of boys and girls are to be arranged in a line for a photo-shoot. There are 20 160 arrangements with a girl on the extreme left, 20 160 arrangements with a boy on the extreme right and 11 520 arrangements with a girl on the extreme left and a boy on the extreme right. How many of these arrangements have either a girl on the extreme left or a boy on the extreme right?

5. A toy box contains several soft-toys and toy cars. These toys are all different. The toys are drawn from the box. There are 15 120 permutations where a soft-toy is the first toy drawn, 25 200 permutations where a toy car is the second toy drawn and 10 800 permutations where a soft-toy and a toy car are respectively the first two toys drawn. How many permutations have either a soft-toy drawn first or a toy car drawn second?

6. Six character passwords are to be formed using the digits 0 to 9 inclusive and the letters of the alphabet with no character being used more than once. How many passwords:
   (a) start with a letter (b) end with a digit
   (c) start with a letter and end with a digit (d) start with a letter or end with a digit.

7. Eight character passwords are to be formed using the digits 0 to 9 inclusive and the letters of the alphabet with no character being used more than once. Write expressions for the number of passwords that: (a) start with a digit (b) end with a letter
   (c) start with a digit and end with a letter (d) start with a digit or end with a letter?

8. Nine character passwords are to be formed using the letters of the alphabet with no character being used more than once. Write expressions for the number of passwords that: (a) are possible (b) start with a vowel and end with a vowel
   (c) start with a consonant or end with a consonant?

9. Ten character passwords are to be formed using the digits 0 to 9 inclusive and the letters of the alphabet with no character being used more than once. Write expressions for the number of passwords that: (a) are possible
   (b) start with an consonant and end with an odd digit
   (c) start with a vowel or end with an even digit?

10. A group of 8 boys and 8 girls are to be arranged in a line for a photo-shoot. How many of these arrangements have either a girl on the extreme left or a boy on the extreme right?

11. A toy box contains ten soft-toys and six toy cars. These toys are all different. The toys are drawn from the box. In how many ways can these toys be drawn with either a soft-toy drawn first or a toy car drawn second?

12. How many integers between 1 and 10 000 inclusive are: (a) divisible by 2 or 5
    (b) divisible by 2 or 5 but not both (c) not divisible by 2 or not divisible by 5?

13. How many integers between 1 and 100 000 inclusive are: (a) divisible by 4 or 20
    (b) divisible by 4 or 20 but not both (c) not divisible by 4 or not divisible by 20?

14. How many code-words each at least 5 characters long can be made with the first ten letters of the alphabet if repetition of letters are not allowed?

15. How many PINs each between 4 digits and 8 digits long (inclusive) can be made with the digits 0 to 9 inclusive if:
    (a) repetition of digits are not allowed (b) repetition of digits is allowed?

16. How many six digit even numbers can be formed from the digits 0 to 9 inclusive if
    (a) repetition of digits are not allowed (b) repetition of digits is allowed?

17. How many six digit even numbers greater than 600 000 can be formed from the digits 0 to 9 inclusive if no digit may be used more than once.

18. How many four or five digit even numbers less than 50 000 can be formed from the digits 0 to 9 if repetition of digits is: (a) not allowed (b) allowed?

2.2 The Inclusion-Exclusion Principle for Three Sets/Events

- For three sets:
  $n(A \cup B \cup C) = n(A) + n(B) + n(C)$
  $- n(A \cap B) - n(A \cap C) - n(B \cap C)$
  $+ n(A \cap B \cap C)$.
- If however, the sets A, B and C are mutually exclusive, then:
  - $n(A \cup B \cup C) = n(A) + n(B) + n(C)$.
- Alternatively, if the tasks A, B and C can each respectively be completed in $n(A)$, $n(B)$ and $n(C)$ways, then tasks A *or* B *or* C can be completed in:
  $n(A \cup B \cup C) = n(A) + n(B) + n(C)$
  $- n(A \cap B) - n(A \cap C) - n(B \cap C)$
  $+ n(A \cap B \cap C)$ ways.
- If however, tasks A, B and C are all mutually exclusive with each other, then tasks A *or* B *or* C can be completed in $n(A \cup B \cup C) = n(A) + n(B) + n(C)$ ways.

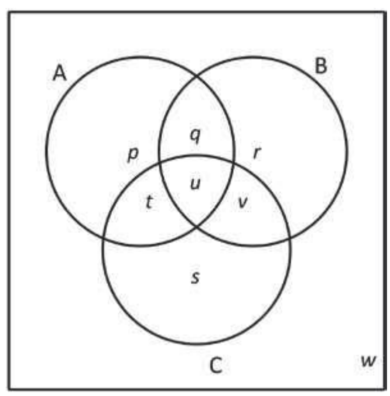
Hands On Task 2.2

In this task, we will explore the proof of the inclusion-exclusion principle for three events.

Consider the accompanying Venn diagram. A, B and C are three sets. The entries in the diagram indicate the number of elements in each region.
Use the Venn diagram to prove:

(a) $n(A \cup B \cup C)$
$= n(A) + n(B) + n(C)$
$- n(A \cap B) - n(A \cap C) - n(B \cap C)$
$+ n(A \cap B \cap C)$

(b) $n(A \cup B \cup C) = n(U) - n(\overline{A} \cap \overline{B} \cap \overline{C})$
where U represents the universal set.

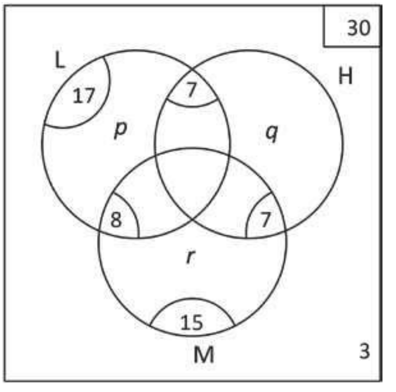


- The first result is known as the inclusion-exclusion principle for three sets. It is based on the idea that when we include all the elements in A and B and C, we "double count" the elements in $A \cap B$, $A \cap C$ and $B \cap C$ hence, to obtain the number of elements in the union $A \cup B \cup C$, we need to exclude "one count" of the elements in the pairwise intersection. However, in so doing, we have over-subtracted one count of the elements in $A \cap B \cap C$ and so it must be added back to obtain the correct value for $n(A \cup B \cup C)$.
- The second result is clearly an application of the complement rule.

Example 2.7

In a group of 30 students, 17 students are enrolled in English Literature, 12 in History, 15 in Specialist Mathematics and 3 students are not enrolled in either English Literature or History or Specialist Mathematics. In addition, 7 of these students are enrolled in English Literature and History, 7 are enrolled in History and Specialist Mathematics and 8 are enrolled in Specialist Mathematics and English Literature. Find how many students are enrolled in:
(a) English Literature and History and Specialist Mathematics
(b) exactly one of the three subjects, English Literature, History and Specialist Mathematics.

Solution:

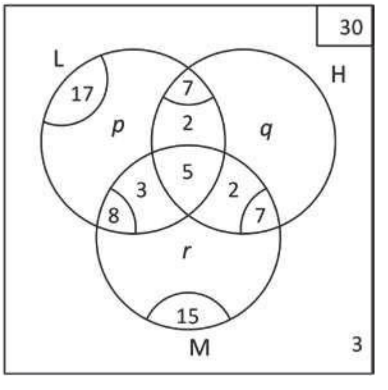


(a) Let the sets L, H and M represent students enrolled in English Literature, History and Specialist Mathematics respectively.
Clearly, $n(\text{L} \cup \text{H} \cup \text{M}) = 30 - 3 = 27.$
Using the inclusion-exclusion principle:

$$n(\text{L} \cup \text{H} \cup \text{M}) = n(\text{L}) + n(\text{H}) + n(\text{M}) - n(\text{L} \cap \text{H}) - n(\text{L} \cap \text{M}) - n(\text{H} \cap \text{M}) + n(\text{L} \cap \text{H} \cap \text{M})$$

$$27 = 17 + 12 + 15 - 7 - 7 - 8 + n(\text{L} \cap \text{H} \cap \text{M})$$

$$\Rightarrow \quad n(\text{L} \cap \text{H} \cap \text{M}) = 5$$



(b) Required number $= p + q + r$.

$$= (17 - 10) + (12 - 9) + (15 - 10)$$
$$= 15.$$

Example 2.8

Two teachers, three boys and four girls are to be arranged in a line for a photo-shoot. Determine the number of arrangements with either a boy on the extreme left or a teacher exactly in the middle or a girl on the extreme right.

Solution:

Let event B: boy on extreme left T: teacher exactly in the middle
G: girl on extreme right

$n(\text{B}) = 3 \times 8!$ $n(\text{T}) = 2 \times 8!$ $n(\text{G}) = 4 \times 8!$
$n(\text{B} \cap \text{T}) = 3 \times 2 \times 7!$ $n(\text{B} \cap \text{G}) = 3 \times 4 \times 7!$ $n(\text{T} \cap \text{G}) = 2 \times 4 \times 7!$
$n(\text{B} \cap \text{T} \cap \text{G}) = 3 \times 2 \times 4 \times 6!$
Using the inclusion-exclusion principle:

$$n(\text{B} \cup \text{T} \cup \text{G}) = (3 + 2 + 4) \times 8! - (3 \times 2 + 3 \times 4 + 2 \times 4) \times 7! + 3 \times 2 \times 4 \times 6!$$
$$= 9! - 26 \times 7! + 24 \times 6!$$
$$= 249\ 120.$$

Example 2.9

How many integers between 1 and 10 000 inclusive are divisible by 2, 3 or 5?

Solution:

Let set A: integers divisible by 2 B: integers divisible by 3 C: integers divisible by 5

Int[*x*] refers to the integer part of *x*.

$n(\text{A}) = \textbf{\textit{Int}}\left[\frac{10000}{2}\right] = 5000 \quad n(\text{B}) = \textbf{\textit{Int}}\left[\frac{10000}{3}\right] = 3333 \quad n(\text{C}) = \textbf{\textit{Int}}\left[\frac{10000}{5}\right] = 2000$

$n(\text{A} \cap \text{B}) = \textbf{\textit{Int}}\left[\frac{10000}{6}\right] = 1666 \qquad n(\text{A} \cap \text{C}) = \textbf{\textit{Int}}\left[\frac{10000}{10}\right] = 1000$

$n(\text{B} \cap \text{C}) = \textbf{\textit{Int}}\left[\frac{10000}{15}\right] = 666 \qquad n(\text{A} \cap \text{B} \cap \text{C}) = \textbf{\textit{Int}}\left[\frac{10000}{30}\right] = 333$

Using the inclusion-exclusion principle:

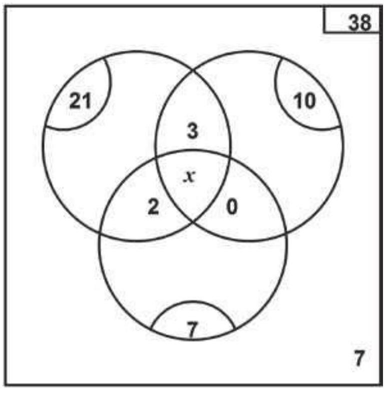
$n(\text{A} \cup \text{B} \cup \text{C}) = 5\,000 + 3\,333 + 2\,000 - 1\,666 - 1\,000 - 666 + 333$

$= 7\,334$

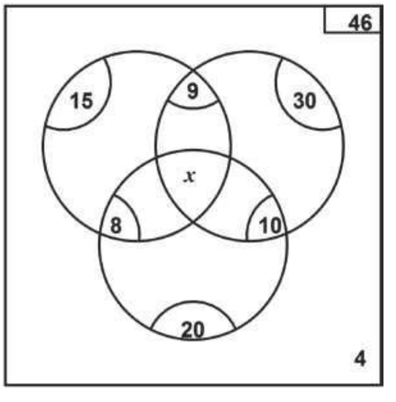
Exercise 2.2

1. Determine the value of *x* in each of the following Venn diagrams:

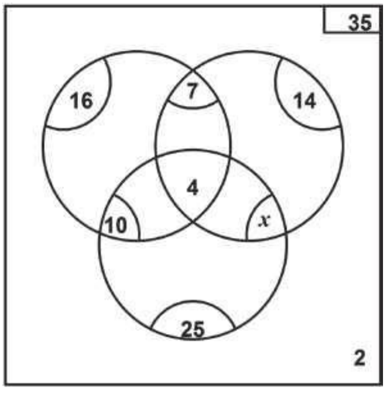
(a)



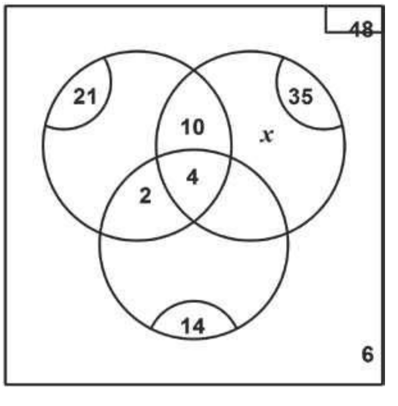
(b)



(c)



(d)



2. In a group of 28 students, 13 students are enrolled in Chemistry, 13 in Physics and 15 in Biology. 4 students are enrolled in Chemistry and Physics, 5 in Physics and Biology and 4 in Chemistry and Biology but not Physics. 3 students are enrolled in all 3 subjects. Find the number of students that are enrolled in:
   (a) at least one of the 3 subjects (b) none of the 3 subjects
   (c) exactly 2 subjects (d) Biology only

3. In a survey conducted by a newspaper, year 10 students were asked to correctly locate on a map of Australia the capital cities of Perth, Sydney and Melbourne. 48 students correctly located all 3 cities, 52 could correctly locate only Perth, 21 students could correctly locate only Sydney and 20 could correctly locate only Melbourne. 68 could correctly locate Perth and Sydney, 80 could correctly locate Melbourne and Sydney and 88 could correctly locate Perth and Melbourne. 5 students were unable to correctly locate any of these cities. Find the number of students who could correctly locate:
   (a) at least one city (b) Perth and Sydney but not Melbourne
   (c) Sydney and Melbourne but not Perth (d) Perth

4. 50 students were asked what sports they would like to play. 31 wanted to play hockey, 26 volleyball, 20 tennis, 9 hockey and volleyball but not tennis, 6 tennis and hockey but not volleyball and 8 volleyball and tennis but not hockey. All chose at least one sport. Find the number of students that chose:
   (a) all three sports (b) volleyball only (c) exactly one sport only.

5. In a survey conducted by a financial magazine on the types of savings instruments used by a group of 60 couples between the ages of 40 to 45 years; the number of couples using share investments, superannuation and real estate investments were respectively 28, 20 and 25. There were 8 couples who invested in shares and had superannuation. 9 couples had share and real estate investments. There were 3 couples with all three forms of savings and 8 couples with none of these instruments of savings. Find the number of couples with:
   (a) only superannuation (b) exactly one of these 3 forms of instruments of savings.

6. In a survey of 35 year 12 students, there were 7 students who studied English, 13 who studied English Literature and 20 who studied Specialist Mathematics. No student studied both English and English Literature. 5 studied none of these subjects. 3 students studied English and Specialist Mathematics. Find the number of students who:
   (a) studied Specialist Mathematics only (b) exactly one of these subjects.

7. In a group of 1 300 students:
   - 901 studied exactly one of Mathematics, History and Geography
   - the number of students studying Geography is 250 less than the number of students studying Mathematics and 300 less than the number of students studying History.
   - 85 students study Mathematics and History and of these 20 do not study Geography
   - One quarter of Geography students also study History and 25 of these do not study Mathematics.

   (a) How many students study Mathematics only?
   (b) How many students study none of these subjects?
   (c) How many students study exactly two subjects?

8. In a health study involving 67 primary school children who had contracted at least one of the diseases chicken pox, measles or mumps; the number of children who had previously contracted chicken pox, measles and mumps were respectively, 37, 28 and 32. There were as many students who had contracted all 3 diseases as those that had contracted chicken pox and measles but not mumps. There were also as many students that had contracted all 3 diseases as those that had contracted measles and mumps but not chicken pox. There were twice as many who had contracted chicken pox and mumps but not measles as those that had contracted all 3 diseases.
   (a) Find the number of students that had contracted exactly one of the 3 diseases.
   (b) An additional 14 students were included in the study so that there were now equal numbers of students who had chicken pox as those who had measles as those who had mumps. Assume that each of these 14 students contracted exactly one of these diseases. Find the number of students who had mumps.

*9. At a relief centre for victims of a flood, 20 parcels containing canned food, 25 parcels containing clothes and 25 parcels containing toys were received. There were 4 parcels containing only canned food, 7 parcels containing only clothes and 8 parcels containing only toys. There were 7 parcels that had canned food, clothes and toys. Assume that the parcels received contained at least one of the items listed.
   (a) Find the total number of parcels received at this centre.
   (b) Find the proportion of parcels with toys from those parcels with clothes.

10. One teacher, five boys and three girls are to be arranged in a line for a photo-shoot. Determine the number of arrangements with:
   (a) a boy on the extreme left and a girl on the extreme right
   (b) a boy on the extreme left, a teacher exactly in the middle and a girl on the extreme right
   (c) either a boy on the extreme left or a teacher exactly in the middle or a girl on the extreme right.

11. Three Australian, two Chinese, two American swimmers and an English swimmer compete in a 100 metres freestyle race. Assume that these swimmers are of the same ability and that there are no ties. Determine the number of arrangements of results where:
   (a) an Australian swimmer will finish first and a Chinese swimmer will finish second
   (b) an Australian swimmer will finish first and an American swimmer will finish third
   (c) an Australian swimmer will finish first or a Chinese swimmer will finish second or an American swimmer will finish third.

12. In a primary school sports carnival, 2 runners each from the Red, Blue, Green and Gold factions compete in the 50 metre sprint race. Assume that these runners are of the same ability and that there are no ties. Determine the number of arrangements of results where:
   (a) the winner is either from the Red, Blue or Green faction
   (b) the winner is from the Gold faction and the runner-up is from the Green faction
   (c) the winner is from the Gold faction or the first runner-up is from the Green faction or the second runner up is from the Blue faction.

13. Amy, Beth, Cate, Dan and Eve compete in a race. Assume that they are of the same ability and there are no ties. Determine the number of arrangements of results where:
    (a) the first three positions are occupied by Amy, Beth and Cate in this order
    (b) the first three positions are occupied by Amy, Beth or Cate
    (c) Amy is first or Beth is second or Cate is third.

14. A Physics book, a Chemistry book, a Biology book, a Mathematics book and a dictionary are randomly arranged in a book shelf. Determine the number of arrangements with:
    (a) either a Physics or Chemistry or Biology book on the extreme left
    (b) the first three books from the left are a Physics, Chemistry or Biology book
    (c) the Physics book on the extreme left or the Chemistry book second from the left or the Biology book third from the left.

15. How many integers between 1 and 100 000 inclusive are divisible by:
    (a) 2, 3 or 7 (b) 3, 5 or 7 (c) 2, 4 or 5?

16. How many integers between 1 and 100 000 inclusive are divisible by:
    (a) 3, 5 or 9 (b) 2, 4 or 8 (c) 2, 5 or 10?

2.3 The Pigeon Hole Principle (Dirichlet's principle)

If m items (pigeons) are placed in n containers (pigeon-holes), where $m > n$, then, at least one container (pigeon-hole) will contain more than one item (pigeon).

- Consider Scenario X where three balls are to be placed into boxes A and B. The table below shows the number of balls each box may have.

Box A	3	2	1	0
Box B	0	1	2	3

  Note that there is at least one box with more than one ball.

- Consider Scenario Y where five balls being placed into boxes A, B and C. The table below shows the number of balls each box may have.

Box A	5	4	4	3	3	3	2	2	2	2	1	1	1	1	1
Box B	0	1	0	2	0	1	3	0	2	1	4	0	3	1	2
Box C	0	0	1	0	2	1	0	3	1	2	0	4	1	3	2

  Note that there is at least one box with more than one ball.

2.3.1 Extending the Pigeon Hole Principle

- Extension A

  If n items are distributed into m containers ($n > m$) there will be at least:
  - one container with at least ***ceiling*** $\left(\frac{n}{m}\right) \equiv \textbf{Int}\left(\frac{n}{m}\right) + 1$ items.
  - one container with no more than ***floor*** $\left(\frac{n}{m}\right) \equiv \textbf{Int}\left(\frac{n}{m}\right)$ items.

- In Scenario X, note that there is at least one box with:
  - at least ***ceiling*** $\left(\frac{3}{2}\right) = 2$ balls
  - with no more than ***floor*** $\left(\frac{3}{2}\right) = 1$ ball.
- In Scenario Y, note that there is at least one box with:
  - at least ***ceiling*** $\left(\frac{5}{3}\right) = 2$ balls
  - with no more than ***floor*** $\left(\frac{5}{3}\right) = 1$ ball.

- Extension B

  Given n containers, for at least one container to have at least 2 items, the minimum number of items required would be $n + 1$.

  In general, given n containers, for at least one container to have at least k items ($k \geq 2$), the minimum number of items required would be $(k - 1) \times n + 1$.

Example 2.10

A laundry basket has a jumble of 5 pairs of different sized socks. Socks are randomly chosen from the basket.
(a) How many socks need to be chosen so that there *could* be one matching pair?
(b) Find the minimum number of socks required to ensure that there:
(i) is at least one matching pair? (ii) are at least two matching pairs?

Solution:

(a) Two socks are required. But there is no guarantee that they form a matching pair.

(b) (i) The socks are distributed across five sizes.
For a matching pair, one of the sizes must have two socks.
Hence, a minimum of 5 + 1 = 6 socks would be needed
(ii) A minimum of 5 + 2 = 7 socks would be needed to ensure at least two matching pairs.

Example 2.11

A paragraph consists of 80 words. How many words need to be randomly chosen to ensure that there are at least: (a) two words that begin with the same letter of the alphabet?
(b) three words that begin with the same letter of the alphabet?

Solution:

(a) There are twenty six letters in the alphabet. Hence, 26 + 1 = 27 words would ensure that there are at least 2 words that begin with the same letter.

(b) Minimum number of words = (3 − 1) × 26 + 1 = 53.

Alternatively

In the worst case scenario, there are no common first letters among the first 26 words chosen and no common first letters among the second set of 26 words chosen. Within these two sets, there is a maximum of 26 pairs of words with a common first letter. Hence, the next word chosen would yield three words with a common first letter. Hence, minimum number of words = 26 + 26 + 1 = 53.

Example 2.12

Twenty balls are distributed among 3 boxes. *Explain* why there must be at least one box:
(a) with no more than six balls (b) with at least 7 balls.

Solution:

(a) Assume that all boxes have more than six balls.
If this is possible, then the total number of balls must be at least 21 which is more than the 20 balls available. Hence, at least one box has no more than six balls.

(b) Assume that there are no boxes with at least 7 balls.
If this is the case, then the total number of balls cannot exceed 18 which is less than the 20 balls available. Hence, there must be at least one box with at least 7 balls.

Example 2.13

A bag has 3 red balls, 5 green balls and 6 white balls. What is the minimum number of balls that need to be randomly drawn to ensure: (a) 2 balls of the same colour?
(b) 3 balls of the same colour? (c) 5 balls of the same colour? (d) 2 red balls?

Solution:

(a) Minimum number of balls = 1 red + 1 green + 1 white + 1 other = 4 balls.

(b) Minimum number of balls = 2 red + 2 green + 2 white + 1 other = 7 balls.

(c) Minimum number of balls = 3 red + 4 green + 4 white+ 1 green or white = 12 balls.

(d) Minimum number of balls = 5 green + 6 white + 2 red = 13 balls.

Exercise 2.3

1. A drawer has a jumble of 6 pairs of different coloured socks.
   What is the minimum number of socks required to ensure that there:
   (a) is at least one matching pair? (b) are at least two matching pairs?

2. There are 12 pairs of different shoes left in a jumble at the front entrance of a house.
   (a) What is the minimum number of shoes required to be chosen to ensure that there:
   (i) is at least one matching pair? (ii) are at least three matching pairs?
   (b) Ten shoes are chosen. What is the minimum and maximum possible number of matching pairs?

3. A mathematics quiz consists of 5 questions, with each correct answer worth one mark.
   (a) What is the minimum number of students required to ensure that there are at least
   (i) two students with the same score (ii) three students with the same score
   (b) In a class of 30 students, what is the minimum and maximum number of students with the same score.

4. A geography quiz consists of 10 questions, with each correct answer worth one mark.
   (a) What is the minimum number of students required to ensure that there are at least
   (i) two students with the same score (ii) six students with the same score
   (b) In a form of 130 students, what is the minimum and maximum number of students with the same score.

5. A paragraph consists of 100 words. How many words need to be chosen to ensure that:
   (a) there are at least two words that begin with the same letter?
   (b) there are at least three words that begin with the same letter?

6. A essay consists of 1 000 words. How many words need to be chosen to ensure that:
   (a) there are at least two words that begin with the same letter?
   (b) there are at least ten words that begin with the same letter?

7. At an international conference, there were five delegates from each of the 40 different countries present. How many delegates need to be chosen so that there are at least:
   (a) 2 delegates from the same country (b) 5 delegates from the same country.

8. Assume that a year consists of 365 days. How many students need to be chosen to ensure that there are at least two students that are born on the same:
   (a) day of the week? (b) calendar month? (c) day of the year?

9. Ten balls are distributed among 3 boxes. Explain why there must be at least one box:
   (a) with no more than three balls (b) with at least four balls.

10. Fifty balls are distributed among 6 boxes. Explain why there must be at least one box:
    (a) with no more than eight balls (b) with at least nine balls.

11. 25 balls are distributed among 4 boxes. There must be at least one box with no more than m balls and there must be at least one box with at least n balls. Find m and n.

12. 100 balls are distributed among 9 boxes. There must be at least one box with no more than m balls and there must be at least one box with at least n balls. Find m and n.

13. A box has 4 red balls, 5 green balls and 5 blue balls. What is the minimum number of balls that need to be drawn to ensure at least:
    (a) 2 balls of the same colour
    (b) 3 balls of the same colour
    (c) 4 balls of the same colour?
    (d) 5 balls of the same colour

14. A bag has 4 red coloured shirts, 2 black coloured shirts, 5 blue coloured shirts. What is the minimum number of shirts that need to be drawn to ensure at least:
    (a) 2 shirts of the same colour?
    (b) 3 shirts of the same colour?
    (c) 2 red shirts?
    (d) 3 blue shirts?

15. A bus carries 12 year twelve students, 11 year eleven students, 10 year ten students and 9 year nine students. What is the minimum number of students that need to be drawn to ensure at least:
    (a) 9 students from the same year group?
    (b) 10 students from the same year group?
    (c) 5 year twelve students?
    (d) 5 year nine students?

03 Combinatorics III

3.1 The Combinatorial Notation

- Consider the expression $\dfrac{n!}{r!(n-r)!}$;
  where n is a whole number and r is a positive integer less than or equal to n.
- This expression is represented by the notation ${}^nC_r$, read as *n combinatorial r* or more simply as *n "C" r*. It is also often represented by a more generalised notation $\binom{n}{r}$, read as *n "down" r*.
- It can be shown that (See Exercise 3.1 and Section 3.5):
  - ${}^nC_r \equiv \dfrac{n!}{r!(n-r)!} = \dfrac{\overbrace{n\times(n-1)\times(n-2)\times......\times(n-r+1)}^{r\ terms}}{r!}$.
  - ${}^nC_r = {}^nC_{n-r}$
  - ${}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$
  - ${}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n = 2^n$.
- Note that ${}^nC_0 = {}^nC_n = 1$ and ${}^nC_1 = {}^nC_{n-1} = n$.
- The values of ${}^nC_r$ can be evaluated directly from your CAS/graphic/scientific calculator using the combinatorial routine.

Example 3.1

Evaluate each of the following without the use of a calculator: (a) ${}^5C_2$ (b) $\binom{20}{18}$.

Solution:

(a) $${}^5C_2 = \frac{5!}{2!\,3!} = \frac{5\times4\times3!}{2!\,3!} = 10$$

(b) $$\binom{20}{18} = \binom{20}{2} = \frac{20\times19}{2\times1} = 190$$

Exercise 3.1

1. Evaluate without the use of a calculator:
   (a) ${}^{15}C_1$ (b) ${}^{20}C_{19}$ (c) ${}^{9}C_3$ (d) ${}^{10}C_7$
   (e) ${}^{100}C_2$ (f) ${}^{50}C_{48}$ (g) ${}^{8}C_4 + {}^{8}C_5$ (h) ${}^{9}C_5 + {}^{9}C_6$.

2. State the value(s) of n and r if: (a) ${}^{n}C_r = \frac{10\times 9\times 8}{3\times 2\times 1}$ (b) ${}^{n}C_r = \frac{15\times 14\times 13\times 12}{1\times 2\times 3\times 4}$.

3. Find the value(s) of r if: (a) ${}^{20}C_r = {}^{20}C_{r+2}$ (b) ${}^{25}C_r = {}^{25}C_{r-3}$

4. Find the value(s) of n if: (a) ${}^{n}C_5 = {}^{n}C_8$ (b) ${}^{n}C_3 = {}^{n}C_{45}$

5. Verify that: (a) ${}^{n}C_r = \frac{\overbrace{n\times(n-1)\times(n-2)\times\ldots\ldots\times(n-r+1)}^{r\text{ terms}}}{r!}$ (b) ${}^{n}C_r = {}^{n}C_{n-r}$.

6. Prove that ${}^{n}C_r + {}^{n}C_{r+1} = {}^{n+1}C_{r+1}$.

3.2 Combinations or Selections

- The total possible number of combinations of *n unlike objects taken r at a time* is given by ${}^{n}C_r \equiv \binom{n}{r}$, where $r \le n$, r and n are positive whole numbers.
- *r objects can be chosen from n unlike objects without replacement* in ${}^{n}C_r \equiv \binom{n}{r}$ ways, where $r \le n$, r and n are positive whole numbers.
- In counting the different combinations, the order in which the objects appear or the order in which the objects are chosen is not important.

Example 3.2

Without the use of a calculator, determine the number of ways of choosing:
(a) 8 students from 11 students. (b) 8 or 9 students from 11 students
(c) at least one student from 11 students.

Solution:

(a) $N = {}^{11}C_8 = {}^{11}C_3 = \frac{11\times 10\times 9}{3\times 2\times 1} = 165.$

(b) $N = {}^{11}C_8 + {}^{11}C_9 = {}^{12}C_9 = \frac{12\times 11\times 10}{3\times 2\times 1} = 220$

(c) $N = {}^{11}C_1 + {}^{11}C_2 + \ldots + {}^{11}C_{11} = 2^{11} - 1 = 2\,047.$

Example 3.3

A committee consisting of three year 11 students and four year 12 students is to be formed from eleven year 11 students and twelve year 12 students. How many different committees can be formed?

Solution:

The three year 11 students can be selected from eleven in $\binom{11}{3}$ = 165 ways.

The four year 12 students can be selected from twelve in $\binom{12}{4}$ = 495 ways.

Using the multiplication rule, the three year 11 students **and** the four year 12 students can be selected in $165 \times 495 = 81\ 675$ ways.

Example 3.4

A committee of four is to be formed from a nominated list of five men and six women. How many committees can be formed where there are more women than men.

Solution:

*For the committees to have more women than men, the committees must either have 3 women and 1 man **or** 4 women and no men. The universal set of "committees with more women than men" is partitioned into 2 distinct disjoint sets: "committees with 3 women and 1 man" and "committees with 4 women".*

3 women can be chosen from 6 **and** 1 man can chosen from 5 in $\binom{6}{3} \times \binom{5}{1}$ = 100 ways.

4 women can be chosen from 6 in $\binom{6}{4}$ = 15 ways.

Hence, using the addition rule (for disjoint sets),
committees of 3 women and 1 man ***or*** committees of 4 women only can be formed
in 100 + 15 = 115 ways.

Exercise 3.2

1. Without the use of a calculator, from a group of 12 teachers, determine the number of ways of selecting:
   (a) 8 teachers
   (b) 8 or 9 teachers
   (c) between 8 and 10 teachers inclusive

2. Without the use of a calculator, from a collection of 15 different books, determine the number of ways of selecting:
   (a) 12 books
   (b) 12 or 13 books
   (c) between 12 and 14 books inclusive

3. Without the use of a calculator, from a group of 20 students, write mathematical expressions in its simplest form for the number of ways of selecting:
   (a) between 15 and 17 students inclusive
   (b) at least 1 student.

4. Without the use of a calculator, from a class of 20 students, write mathematical expressions for the number of ways of selecting:
   (a) at least 1 but no more than 2 students
   (b) at least 2 students
   (c) no more than 18 students.

5. Consider the set of 26 lower case letters and the set of 10 digits. Determine the number of ways of choosing (without replacement):
   (a) 5 lower case letters
   (b) no more than 5 lower case letters
   (c) 5 digits
   (d) 5 lower case letters and 5 digits

6. Consider the set of 10 digits and a set of 30 symbols. Determine the number of ways of choosing (without replacement):
   (a) no more than 6 symbols
   (b) at least 6 digits
   (c) no more than 6 symbols and at least 6 digits

7. A committee comprising 6 teachers and 2 students is to be formed from a list of 20 teachers and 50 students. Determine the number of ways the committee can be formed:
   (a) if Ms Whelan (a teacher) and Erin (the school captain) must be included in the committee, both names are in the list.
   (b) if Mr Hayes (a teacher) refuses to serve in the same committee as Ja'mie (a student), both names are in the list.

8. A committee comprising 4 teachers and 2 academics is to be formed from a list of 35 teachers and 10 academics. Determine the number of ways the committee can be formed:
   (a) if Ms Gillard and Ms Killard (both teachers) refuse to serve in the same committee, both names are in the list.
   (b) if Mr Rudd (a teacher) will not serve unless Prof Thudd (an academic) is also selected, both names are in the nominated list.

9. A delegation of ten persons is to be chosen from 20 men and 20 women. How many possible delegations are there if there are more women than men in each delegation?

10. Twelve characters are to be chosen from the set of 26 letters and 10 digits. How many selections are possible if there are more digits than letters with no character being chosen more than once?

11. Set A has 10 distinct elements and Set B has 8 distinct elements all different from those in A. Set C consisting of 4 elements is formed by choosing at least one element from each of the two sets with no element being chosen more than once. Find the total possible number of such sets.

*12. Set A has 4 distinct elements and Set B has 5 distinct elements all different from those in A. Set C is formed by choosing elements from each of the two sets such that there are always more elements from set B than set A with no element being chosen more than once.. Find the total possible number of such sets.

3.3 Combinations and the inclusion-exclusion principle

- If the tasks A and B can each respectively be completed in $n(A)$ and $n(B)$ways, then tasks A *or* B can be completed in:
  $$n(A \cup B) = n(A) + n(B) - n(A \cap B) \text{ ways.}$$
- If the tasks A, B and C can each respectively be completed in $n(A)$, $n(B)$ and $n(C)$ways, then tasks A *or* B *or* C can be completed in:
  $$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C) \text{ ways.}$$

Example 3.5

A committee consisting of seven students is to be formed from ten year 10, eleven year 11 students and twelve year 12 students. How many different committees can be formed if each:
(a) committee must have three year 11 students
(b) committee must have four year 12 students
(c) committee must have three year 11 students and four year 12 students
(d) committee must have three year 11 students *or* four year 12 students.

Solution:

(a) The three year 11 students can be chosen in ${}^{11}C_3$ ways.

The remaining four students can be chosen in ${}^{22}C_4$ ways.

Hence, number of committees with three year 11 students = ${}^{11}C_3 \times {}^{22}C_4$
= 1 206 975.

(b) Number of committees with four year 12 students = ${}^{12}C_4 \times {}^{21}C_3$
= 658 350.

(c) Number of committees with three year 11 and four year 12 students = ${}^{11}C_3 \times {}^{12}C_4$
= 81 675.

(d) Using the inclusion-exclusion principle:
Number of committees with three year 11 or four year 12 students
= 1 206 975 + 658 350 – 81 675
= 1 783 650.

Example 3.6

A committee consisting of nine students is to be formed from ten year 10, eleven year 11 students and twelve year 12 students. How many different committees can be formed if each committee must have two year 10 *or* three year 11 students *or* four year 12 students.

Solution:

Number of committees with two year 10 students = ${}^{10}C_2 \times {}^{23}C_7$ = 11 032 065.

Number of committees with three year 11 students = ${}^{11}C_3 \times {}^{22}C_6$ = 12 311 145.

Number of committees with four year 12 students = ${}^{12}C_4 \times {}^{21}C_5$ = 10 072 755.

Number of committees with two year 10 and three year 11 students = ${}^{10}C_2 \times {}^{11}C_3 \times {}^{12}C_4$
= 3 675 375.

Number of committees with two year 10 and four year 12 students = ${}^{10}C_2 \times {}^{12}C_4 \times {}^{11}C_3$
= 3 675 375.

Number of committees with three year 11 and four year 12 students = ${}^{11}C_3 \times {}^{12}C_4 \times {}^{10}C_2$
= 3 675 375.

Number of committees with two year 10 and three year 11 and four year 12 students
= ${}^{10}C_2 \times {}^{12}C_4 \times {}^{11}C_3$ = 3 675 375.

Hence, using the inclusion-exclusion principle:

Number of committees with two year 10 *or* three year 11 students *or* four year 12 students
= 11 032 065 + 12 311 145 + 10 072 755 − 3(3 675 375) + 3 675 375
= 26 065 215

Alternatively:

Since committee is to have nine members, having two year 10s and three year 11s automatically implies that there will be four year 12s.
Hence,
Number of committees with two year 10 *or* three year 11 students *or* four year 12 students
= 11 032 065 + 12 311 145 + 10 072 755 − 2(3 675 375)
= 26 065 215

Exercise 3.3

1. A committee consisting of 5 students is to be formed from eight year 8 and nine year 9 students. How many different committees can be formed if each committee must have:
   (a) two year 8 students?
   (b) three year 9 students?
   (c) two year 8 and three year 9 students?
   (d) two year 8 *or* three year 9 students?

2. A committee consisting of 5 students is to be formed from 8 year 8, 9 year 9 students and 10 year 10 students. How many different committees can be formed if each committee must have:
   (a) two year 8 students?
   (b) three year 9 students?
   (c) two year 8 and three year 9 students?
   (d) two year 8 *or* three year 9 students?

3. Ten characters are to be chosen from the set of lower case letters, the set of upper case letters and the set of digits 0 to 9 inclusive. How many different combinations (no character is chosen more than once) will have:
   (a) six digits and four lower case letters?
   (b) six digits or four lower case letters?

4. Ten balls are to be chosen from 5 blue balls, 6 green balls and 7 red balls. How many different selections are there if each selection must have:
   (a) seven red balls and three blue balls?
   (b) seven red balls or three blue balls?

5. A delegation of six students is to be formed from three year 7, four year 8 and five year 9 students. Without the use of a calculator, determine how many delegations can be formed if each delegation must have:
   (a) three year 7 students
   (b) three year 8 students
   (c) three year 7 and three year 8 students
   (d) three year 7 *or* three year 8 students.

6. A focus group consisting of ten persons is to be formed from eight persons from the 18 to 25 age group, seven persons from the 26 to 40 age group and one person from the 41 to 60 age group. Without the use of a calculator, determine how many different focus groups can be formed if each group must have:
   (a) five persons each from the 18 to 25 and 26 to 40 age group
   (b) five persons from the 18 to 25 or five persons from the 26 to 40 age group.

7. A selection of 7 athletes is to be formed from 10 Australian, 12 American and 8 European athletes. How many different selections can be formed if each selection:
   (a) must have four athletes from Australia and three athletes from Europe?
   (b) must have four athletes from Australia or three athletes from Europe but not both?

8. Six characters are to be chosen (no repeats) from the lower and upper case letters of the English alphabet and the digits 0 to 9 inclusive. How many different combinations:
   (a) will have four digits and two upper case letters?
   (b) will have four digits or two upper case letters but not both?

9. A committee consisting of 5 students is to be formed from 10 year 10, 11 year 11 students and 12 year 12 students. How many different committees can be formed if each committee must have 1 year 10 *or* 2 year 11 students *or* 2 year 12 students.

10. Twelve characters are to be chosen (no repeats) from the set of lower case letters, the set of upper case letters and the set of digits 0 to 9 inclusive. How many different combinations will have five digits or four upper case letters or three lower case letters?

11. Six balls are to be chosen from 2 blue balls, 2 green balls, 2 yellow balls and 2 red balls. Determine how many different selections are there if each selection must have either two blue balls or two green balls or two red balls?

12. A committee consisting of seven students is to be formed from nine year 9, ten year 10, eleven year 11 students and twelve year 12 students. How many different committees can be formed if each committee must have one year 10 *or* two year 11 students *or* three year 12 students?

13. A selection of eight swimmers is to be formed from six Australian, five American, five Chinese and eight European swimmers. How many different selections can be formed if each selection must have two swimmers each either from Australia, America or Europe?

14. Twelve characters are to be chosen (no repeats) from the set of lower case letters, the set of upper case letters and the set of digits 0 to 9 inclusive. How many different combinations will have four digits or four upper case letters or four lower case letters but not all three?

15. Six balls are to be chosen from 3 blue balls, 3 green balls, 3 yellow balls and 3 red balls. Determine how many different selections are there if each selection must have either two blue balls or two green balls or two red balls but not all three?

3.4 Combinations and Arrangements

- *To arrange r objects out of n unlike objects in a straight line,*
    - *we first choose r objects out of n*
    - *and then arrange the chosen objects.*

  Hence, the number of ways r objects out of n can be arranged in a straight line is $^{n}C_{r} \times r!$ ways.

- But $^{n}C_{r} \times r! = \dfrac{n!}{r!(n-r)!} \times r! = \dfrac{n!}{(n-r)!}$.

  From Chapter 1, $\dfrac{n!}{(n-r)!} = {}^{n}\mathrm{P}_{r}$.

- Therefore, the number of ways r objects out of n can be arranged in a straight line is $^{n}\mathrm{P}_{r} \equiv {}^{n}C_{r} \times r!$ ways.

Example 3.7

In how many ways can 5 students out of a group of 8 students be seated in a straight line?

Solution:

5 students can be chosen out of 8 students in $^8C_5$ = 56 ways.
The 5 chosen students can be arranged in a straight line in 5! = 120 ways
Hence, 5 students out of 8 can be seated in a straight line in 56 × 120 = 6 720 ways.

OR

5 students out of 8 can be seated in a straight line in $^8P_5$ = 6 720 ways.

Example 3.8

Three letters are chosen from the alphabet set (not case sensitive) and three digits are chosen from the digits 0 to 9 inclusive. The chosen letters and digits are then arranged to form six character passwords. No letter or digit is used more than once. Find the total number of passwords that can be formed.

Solution:

The four letters can be chosen in $^{26}C_3$ ways.

The four digits can be chosen in $^{10}C_3$ ways .

The eight chosen characters can be arranged in 6! ways.

Hence, the total number of possible passwords = $^{26}C_3 \times {^{10}C_3} \times 6!$ = 224 640 000.

***Note**: This example introduces the "choose, choose, pool and arrange" technique!*

Exercise 3.4

1. Seven character passwords are created using the letters of the word GRAPHIC and the digits 0, 1, 2, 3, 4, and 5. No letter or digit is used more than once. Find the total number of passwords that can be formed if the passwords contain:
   (a) four letters and three digits
   (b) five letters and two digits
   (c) more letters than digits

2. Eight character passwords are to be formed using the letters of the alphabet (not case sensitive) and the digits 0 to 9 inclusive. No character is used more than once. Find the total number of passwords that can be formed if the passwords contain:
   (a) four letters and four digits
   (b) four letters and four digits with the digits adjacent to each other
   (c) four letters and four digits with the digits adjacent in consecutive ascending order
   (d) four letters and four digits with the digits adding up to 10.

3. Nine character passwords are to be formed using the letters of the alphabet (not case sensitive) and the digits 0 to 9 inclusive. No character is used more than once. Without the use of a calculator, write expressions for the total number of passwords that can be formed if the passwords contain: (a) five letters and four digits
   (b) five letters and four digits with the digits adjacent to each other
   (c) five letters and four digits with all the letters together and all the digits together
   (d) five letters and four digits with no letters adjacent to each other.

4. A bookshelf has four shelves. Shelf A has 5 books, shelf B has 6 books and shelf C has 4 books. Shelf D is empty. The books are all different. Without the use of a calculator, write expressions for the number of ways:
   (a) all books in shelves A, B and C can be rearranged in shelf D,
   (b) two books can be chosen from shelf B and two books can be chosen from shelf C and then arranged in shelf D?
   (c) four books each can be chosen from shelf A, shelf B and shelf C and arranged in shelf D.

5. Box A has 6 books, box B has 5 books and box C has 4 books. The books are all different. Three books are chosen from each of the boxes and arranged on a shelf.
   (a) How many of these arrangements are possible?
   (b) How many arrangements will have the books from the same box together?
   (c) How many arrangements will have a book from box A at each end of the arrangement?
   (d) How many arrangements will have a book from the same box at each end of the arrangement?

6. From a group of 7 cyclists, 8 gymnasts and 6 athletes, two representatives from each sport are to be chosen and arranged in a line for a photo-shoot. How many possible arrangements are there if:
   (a) the representatives of each sport must be together
   (b) the cyclists must be together and the gymnasts must be together
   (c) the cyclists must be together
   *(d) the representatives from at least one sport are to be together?

7. A media player has access to three folders A, B and C with 10, 12 and 9 tracks respectively.
   (a) In the normal mode, the player plays every track in each folder once through in order, starting from folder A, then B and finally C. In how many ways can the tracks be played exactly once in the normal mode?
   (b) The random mode facility allows the player to play any track from any one of the 3 folders exactly once in a random sequence. How many such random sequences are there?
   (c) The programme mode facility allows the listener to programme the sequence with which the tracks are played. Elle sets up a programme consisting of 5 tracks (all different) from each folder. How many such programmes are there if all the tracks from any given folder must be played before tracks from the next folder are played?

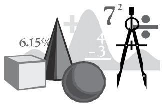
8. A compact disc juke box can be programmed to play any track from any of the 50 discs it holds. Each disc has 10 tracks.
   (a) A disc jockey sets up a programme to play two separate tracks from the discs in the juke box. How many such programmes are there if
   (i) there are no restrictions (ii) the tracks must be from the same disc
   (iii) the tracks cannot be from the same disc?
   (b) In how many ways can a disc jockey play exactly one track from each disc?
   (c) In how many ways can a disc jockey play exactly two tracks from each disc?

9. One of Angeline's document files is protected by a 4 character password. The characters are chosen from the 26 letters of the alphabet (not case sensitive) and the digits 0 to 9 inclusive. How many different passwords are there if
   (a) only letters of the alphabet are used (b) letters of the alphabet and digits are used
   (c) two letters and two digits are used, no character being used more than once
   (d) more letters than digits are used, no character being used more than once
   (e) there must be exactly two letters and the letters must be consecutive and adjacent and
   in ascending order, no character being used more than once?

10. WestNet an Internet service provider allows its customers to choose their own access passwords. Each password is between 4 and 8 characters long inclusive. The characters can either be letters of the alphabet (case sensitive, that is A is different from a) or the digits 0 to 9 inclusive or a space generated by the space-bar.
    (a) How many possible passwords are there if no other restrictions apply?
    (b) How many six different character passwords have exactly two digits?
    (c) How many four different character passwords have exactly 2 letters of the alphabet with the letters adjacent and in consecutive ascending order of the same case type?
    (d) How many four different character passwords have exactly 2 letters of the alphabet with the letters adjacent and in consecutive ascending order of the same case type, and exactly two digits with the digits adjacent and in consecutive descending order?

11. The eight digit telephone number of a city begins with the digits 97. To help its customers remember its telephone number, a request is submitted by a local government agency to the telephone service provider to "customise" its telephone number. How many possible numbers are there if the government agency requests that
    (a) all the digits following the first two digits be the same (for example 97 333 333)
    (b) the digits following the first two digits is a group of 3 repeated digits followed by a second group of 3 repeated digits, the digits in the second group being different from those in the first group (for example 97 333 111)
    (c) the digits in the two groups described in (b) must be consecutive even digits in ascending order (for example 97 222 444)
    (d) the digits following the first two digits consist of three pairs of repeated numbers, the numbers differing from pair to pair (for example 97 22 55 00)
    (e) the digits following the first two digits must either be in consecutive ascending pairs or consecutive descending pairs (for example 97 44 55 66)?

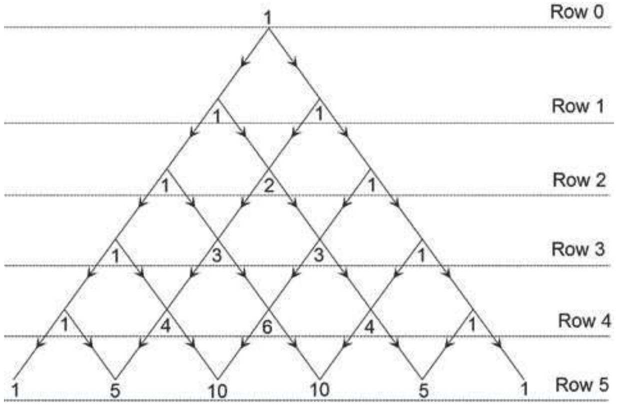
3.5 Pascal's Triangle



Hands On Task 3.1

In this task we will explore some of the properties of Pascal's triangle.

The diagram below displays the first five rows of Pascal's triangle and the expansions for $(x + y)^n$ for $n = 0, 1, 2, 3, 4$ and 5.



$(x + y)^0 = 1$

$(x + y)^1 = x + y$

$(x + y)^2 = x^2 + 2xy + y^2$

$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

$(x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$

1. Extend Pascal's triangle to the sixth row.
2. Express the terms in rows 4 and 5 of Pascal's triangle in terms of the binomial coefficients $\binom{4}{k}$ and $\binom{5}{k}$ respectively.
   (a) Use this observation to write row 10 and row 12 of Pascal's triangle.
   (b) For $n = 5$, Verify that $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ for $k = 1, 2, 3, \dots, n$.
   (c) Prove algebraically that $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ for $k = 1, 2, 3, \dots, n$.
3. Compare the terms in Pascal's triangle with the coefficients of the expansions listed. Hence, state the expansion for: (a) $(x + y)^6$ (b) $(x + y)^8$ (c) $(x - y)^8$ (d) $(1 + x)^6$
4. (a) Add all the terms in row 5 of Pascal's triangle.
   (b) Substitute $x = 1$ and $y = 1$ into the expansion for $(x + y)^5$.
   Compare the value of $(x + y)^5$ with the sum of all terms in row 5 of Pascal's triangle.
   (c) In Pascal's triangle, find the sum of terms in: (a) row 11 (b) row 15.

5. Express each of the following as a sum of the binomial coefficients $\binom{n}{k}$ for k = 0, 1, 2, …., n : (a) 2^8 (b) 2^{10} (c) 2^m where m is a positive integer.

6. (a) Square each element in row 5. Determine the sum of these squares.
   (b) Determine the value of $\binom{10}{5}$.
   (c) Use your observations in parts (a) and (b) to determine the sum of the squares of the terms in row: (a) six (b) row 10.

7. 1, 3, 6, 10, 15, 21, …. are the terms that form the third diagonal. The terms of this sequence of numbers are called triangular numbers.
   (a) Express the first six terms of the sequence of triangular numbers using the binomial coefficients $\binom{n}{k}$.
   (b) Express the 10th and 20th triangular number as binomial coefficients.
   (c) Use binomial coefficients to express the nth triangular number.
   (d) Hence, express the nth triangular number in terms of n.

8. 1, 4, 10, 20, 35, 56, …. are the terms that form the fourth diagonal. The terms of this sequence of numbers are called tetrahedral numbers.
   (a) Express the first six terms of the sequence of tetrahedral numbers using the binomial coefficients $\binom{n}{k}$.
   (b) Express the 10th and 20th tetrahedral number as binomial coefficients.
   (c) Use binomial coefficients to express the nth tetrahedral number.
   (d) Hence, express the nth tetrahedral number in terms of n.

9. 1, 5, 15, 35, 70, 126, …. are the terms that form the fifth diagonal. The terms of this sequence of numbers are called pentatope numbers.
   (a) Express the first six terms of the sequence of pentatope numbers using the binomial coefficients $\binom{n}{k}$.
   (b) Express the 10th and 20th pentatope number as binomial coefficients.
   (c) Use binomial coefficients to express the nth pentatope number.
   (d) Hence, express the nth pentatope number in terms of n.

3.6 More on Arrangements

3.6.1 Arrangements of objects all different with no repeats

- Consider five objects all different: A, B, C, D and E. These objects are to be arranged in a line with no object used more than once.
  - There are five empty spaces to be filled.

  - The first space can be filled in $\binom{5}{1}$ ways as there are five choices.
  - Subsequently the remaining four spaces can be filled in $\binom{4}{1}$, $\binom{3}{1}$, $\binom{2}{1}$ and $\binom{1}{1}$ respectively.
  - Hence, the total number of ways of filling these empty spaces is $\binom{5}{1} \times \binom{4}{1} \times \binom{3}{1} \times \binom{2}{1} \times \binom{1}{1} \equiv 5 \times 4 \times 3 \times 2 \times 1 \equiv 5!$ ways.
  - That is, five unlike objects can be arranged in a line with no object used more than once in 5! ways.

- In general, n unlike objects can be arranged in a line with no object used more than once in $n!$ ways.

3.6.2 Arrangements of objects not all different with no repeats

- Consider five objects: A, A, B, C and D. Note that two of the objects are alike. These objects are to be arranged in a line with no object used more than once.
  - There are five empty spaces to be filled.

  - We need to select two spaces to place the A s.
    This can be done in $\binom{5}{2}$ ways.
  - Having placed the two A s, the remaining three objects can be arranged in 3! ways.
  - Hence, the total number of ways of filling these empty spaces is $\binom{5}{2} \times 3! \equiv \frac{5!}{3!\,2!} \times 3! = \frac{5!}{2!}$ ways.

- Consider now ten objects: A, A, A, A, B, C, D, E, F, G to be arranged in a line with no object used more than once.. Note that four of the objects are alike.
  - We need to select four of ten spaces to place the A s.
    This can be done in $\binom{10}{4}$ ways.
  - Having placed the four A s, the remaining six objects can be arranged in 6! ways.
  - Hence, the total number of ways of filling these empty spaces is
    $\binom{10}{4} \times 6! \equiv \dfrac{10!}{4!\,6!} \times 6! = \dfrac{10!}{4!}$ ways.

- Consider now ten objects: A, A, A, A, B, B, E, F, G, H to be arranged in a line with no object used more than once. Note that four of the objects are alike with another two alike different from the rest.
  - We need to select four of ten spaces to place the A s.
    This can be done in $\binom{10}{4}$ ways.
  - Having placed the four A s, we need to select two of the remaining six spaces to place the B s. This can be done in $\binom{6}{2}$ ways.
  - Having placed the four As and two Bs, the remaining four objects can be arranged in 4! ways.
  - Hence, the total number of ways of filling these empty spaces is
    $\binom{10}{4} \times \binom{6}{2} \times 4! \equiv \dfrac{10!}{4!\,6!} \times \dfrac{6!}{2!\,4!} \times 4! = \dfrac{10!}{4!\,2!}$ ways.

- In general, given a total of *n* objects where there are *a* type A objects, *b* type B objects, *c* type C objects, with the rest all different, the number of ways of arranging these objects with no object used more than once is:

$$N = \binom{n}{a} \times \binom{n-a}{b} \times \binom{n-a-b}{c} \times (n-a-b-c)! \equiv \frac{n!}{a!\,b!\,c!}.$$

Number of type A objects (→ $a!$)

Number of type B objects (→ $b!$)

Number of type C objects (→ $c!$)

3.6.3 Arrangements of objects not all different with repeats

- Consider ten letters: A, A, A, A, B, B, E, F, G, H. Four of the letters are alike with another two alike different from the rest. Any of the ten letters may be used more than once to form a sequence of twelve letters set in a line.
  - There are only six distinct types of letters.
  - Each of the twelve spaces may be "filled" in six ways.
  - Hence, the number of different arrangements is 6^{12}.

- In general, given a total of *n* objects where there are *a* type A objects, *b* type B objects, *c* type C objects, with the rest all different, the number of ways of forming a sequence of *m* objects where an object may be used more than once is:

$$N = (n - a - b - c)^{m} .$$

Number of distinct objects

Example 3.9

Write mathematical expressions for the number of ways of *rearranging* the letters of the following words: (a) SCHOOL (b) EXCELLENCE.

Solution:

(a) There are two Os and four other letters all different.

Hence, number of ways $N = \binom{6}{2} \times 4! = \frac{6!}{2!}$.

(b) There are four E s, two C s, two L s and two other letters all different.

Hence, number of ways $N = \binom{10}{4} \times \binom{6}{2} \times \binom{4}{2} \times 2!$

$$= \frac{10!}{4!\,2!\,2!} .$$

Example 3.10

Write a mathematical expression for the number of different 12 letter length passwords that can be formed from the letters of the word DIFFICULT if any letter can be used more than once.

Solution:

There are seven different letters.

Hence, number of ways is 7^{12}.

Exercise 3.5

1. Write mathematical expressions for the number of ways of *rearranging* the letters of the following words:
   (a) EXERCISES (b) CALCULATOR (c) MATHEMATICS
   (d) SENSELESSNESS (e) EFFERVESCENCE (f) ARRANGEMENTS

2. Write mathematical expressions for the number of ways of *rearranging* the letters of the word CALCULUS if the first letter must be:
   (a) the letter A (b) the letter U (c) the letter C.

3. Write mathematical expressions for the number of ways of *rearranging* the letters of the word DIFFERENTIATION if the last letter must be:
   (a) the letter A (b) the letter F (c) the letter I.

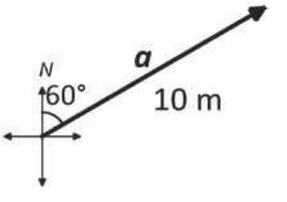
4. Write mathematical expressions for the number of ways of *rearranging* the letters of the word PERMUTATIONS if the first letter must be:
   (a) a vowel (b) a consonant.

5. Write a mathematical expression for the number of different ten letter length passwords that can be formed from the letters of each of the following words if any letter can be used more than once.
   (a) ELEGANT (b) OVERJOYED (c) EXAGERATION.

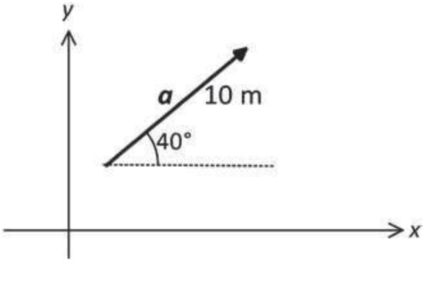
04 Introduction to Vectors

4.1 Arrow Representation of Vectors

- A vector is a quantity with both magnitude and direction whereas a scalar is a quantity with only magnitude.
- Displacement is a vector as it possesses both direction and magnitude, but distance is a scalar as it possesses only magnitude but not direction. Velocity is a vector as it possesses both direction and magnitude, but speed is a scalar as it possesses only magnitude but not direction.
- Diagrammatically, a vector may be represented by an arrow. The length of the arrow corresponds to its magnitude and the direction of the arrow conveys the direction of the vector relative to a set of bearings or the *x*-*y* axes.
- In the accompanying diagram, the arrow shown represents a displacement vector ***a*** of magnitude 10 m acting in the direction 060°.
  - The magnitude of ***a*** is denoted $|\boldsymbol{a}|$. Hence, in this case, $|\boldsymbol{a}| = 10$



- In the accompanying diagram, the arrow shown represents a displacement vector ***b*** of magnitude 10 m in the direction inclined at 40° with the positive *x*-axis.

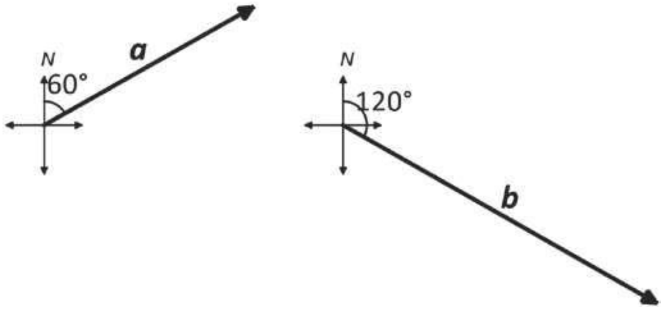


4.1.1 Scalar multiple of a vector

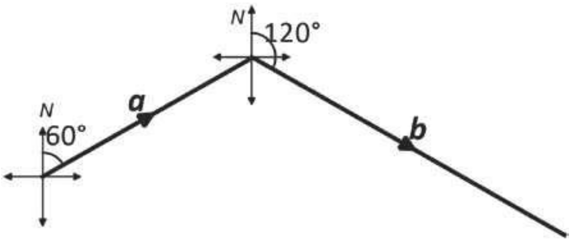
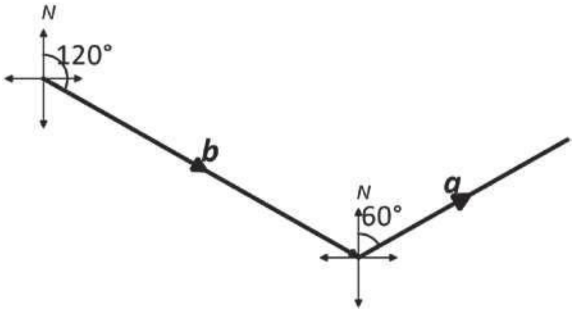
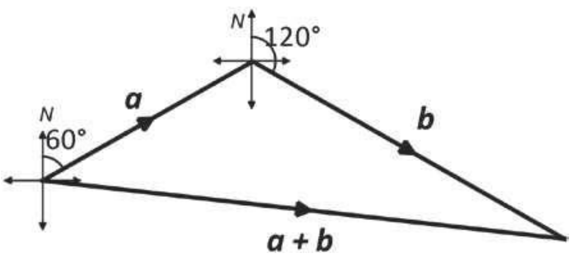
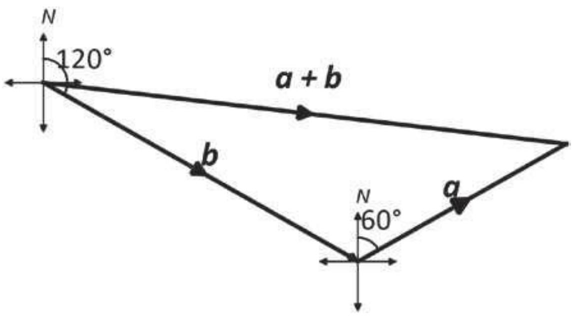
- The vector *k**a*** where *k* is a constant, is a vector parallel to ***a*** but with a magnitude that is *k* times that of ***a***.
  - That is $|k\boldsymbol{a}| = |k| \times |\boldsymbol{a}|$
  - If $k > 0$, then *k**a*** and ***a*** are in the same direction.
  - If $k < 0$, then *k**a*** and ***a*** are parallel but in opposing directions.

4.1.2 Addition of Vectors (arrow representation)

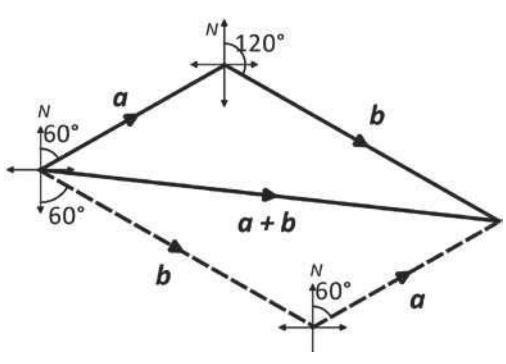
- The addition of vectors ***a*** and ***b*** follow the parallelogram rule.



- To draw ***a*** + ***b***, start with either vector ***a*** or ***b***.

Method 1	*Method 2*
1. *If you start with* ***a****, then place the tail of* ***b*** *on the tip of* ***a****.*  	1. *If you start with* ***b****, then place the tail of* ***a*** *on the tip of* ***b****.* 
2. *Vector* ***a*** + ***b*** *is formed by drawing an arrow from the tail of* ***a*** *to the tip of* ***b****.*  	2. *Vector* ***a*** + ***b*** *is formed by drawing an arrow from the tail of* ***b*** *to the tip of* ***a****.* 

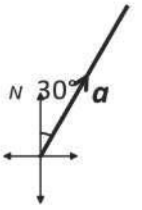
- Notice that the sum ***a*** + ***b*** is the longer diagonal of the parallelogram formed by the vectors ***a*** and ***b***.
- The sum of two or more vectors is also known as the *resultant vector*.

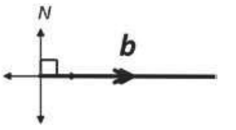


Example 4.1

Vectors ***a*** and ***b*** are as shown in the accompanying diagram. Sketch on separate diagrams:

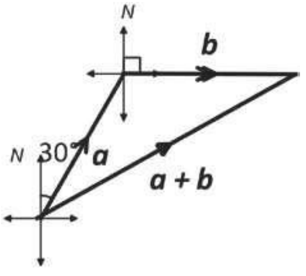
(a) ***a*** + ***b*** (b) 2***a*** + 3***b*** (c) 0.5***a*** + ***b***





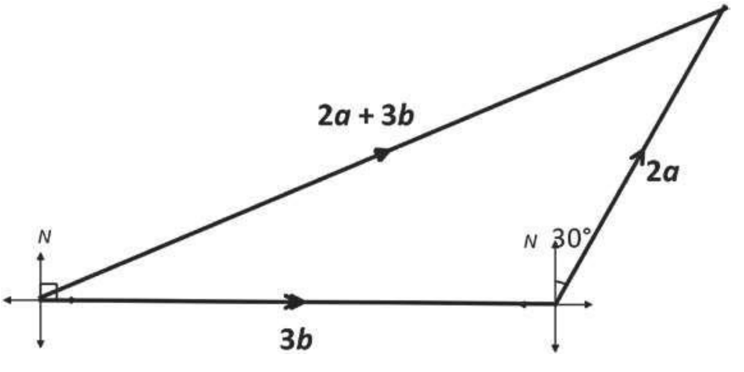
Solution:

(a)

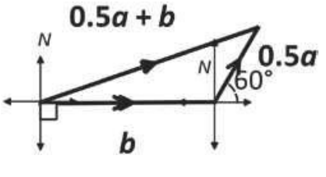


> *Note:*
> *In vector addition, the arrow that completes the triangle always points from the tail of the first vector to the tip of the second vector.*

(b)

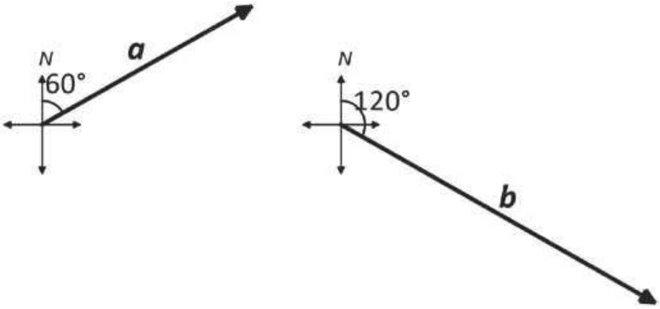


(c)

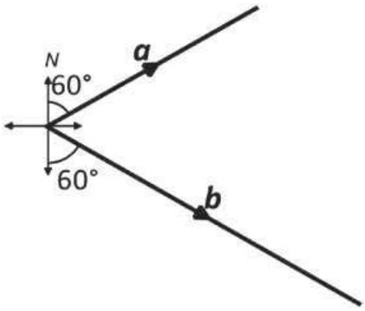
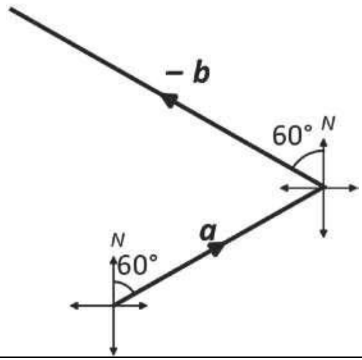
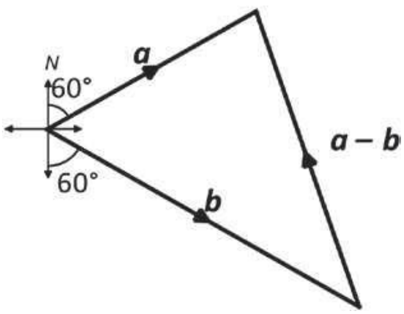
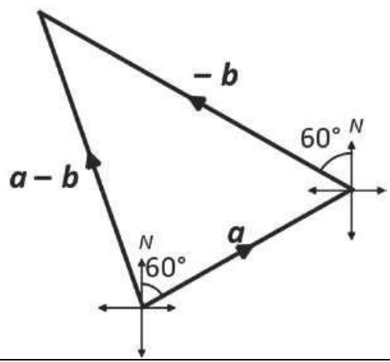


4.1.3 Subtraction of Vectors (arrow representation)

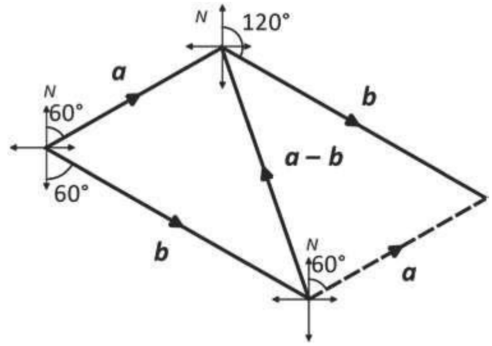
- The subtraction of vectors ***a*** and ***b*** also follow the parallelogram rule.



- To draw ***a*** − ***b***, start with vector ***a*** or −***b***.

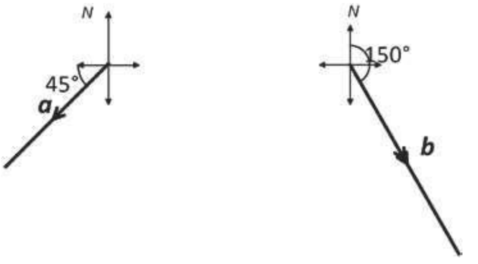
Method 1	*Method 2*
*1. Place the tail of **b** on the tail of **a**.* 	*1. Place the tail of −**b** on the tip of **a**.* 
*2. Vector **a** −**b** is formed by drawing an arrow from the tip of **b** to the tip of **a**.* 	*2. Vector **a** −**b** is formed by drawing an arrow from the tail of **a** to the tip of −**b**.* 

- Notice that the vector ***a*** − ***b*** is the shorter diagonal of the parallelogram formed by the vectors ***a*** and ***b***.



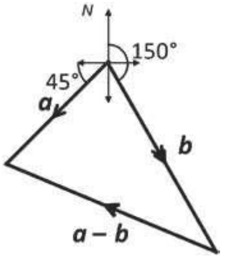
Example 4.2

Vectors ***a*** and ***b*** are as shown in the accompanying diagram. Sketch on separate diagrams:
(a) ***a*** − ***b*** (b) 2***b*** − ***a*** (c) 2***a*** − 0.5***b***



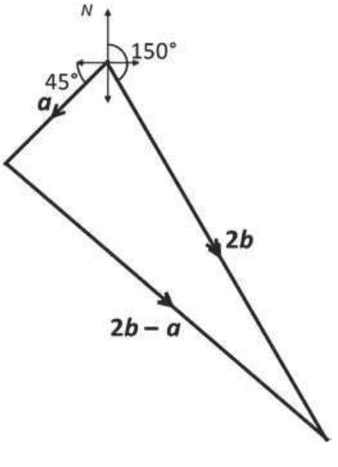
Solution:

(a)



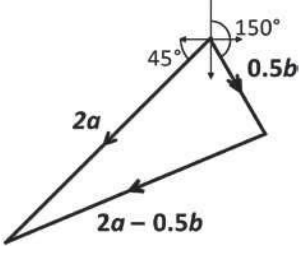
> *Note:*
> *In vector subtraction, the arrow that completes the triangle always points to the first vector, which in this case is **a**.*

(b)



> *Note:*
> *The arrow that completes the triangle points to the first vector, which in this case is 2**b**.*

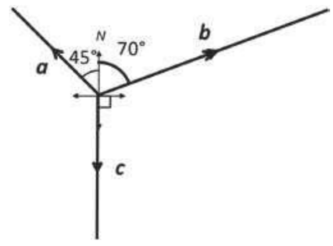
(c)



> *Note:*
> *The vector 2**a** may be considered as the sum of the vectors 0.5**b** and (2a − 0.5**b**).*

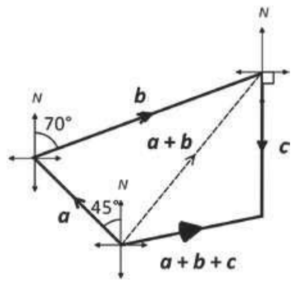
Example 4.3

Vectors ***a***, ***b***, and ***c*** are as shown in the accompanying diagram.
Sketch on separate diagrams:
(a) ***a*** + ***b*** + ***c*** (b) ***b*** + ***a*** – ***c*** (c) 2(***a*** + ***b***) + ***c***



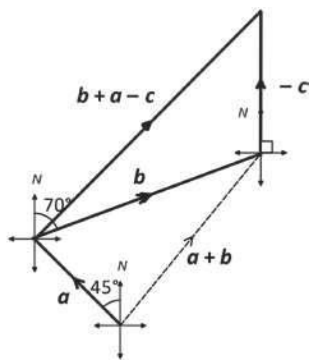
Solution:

(a)

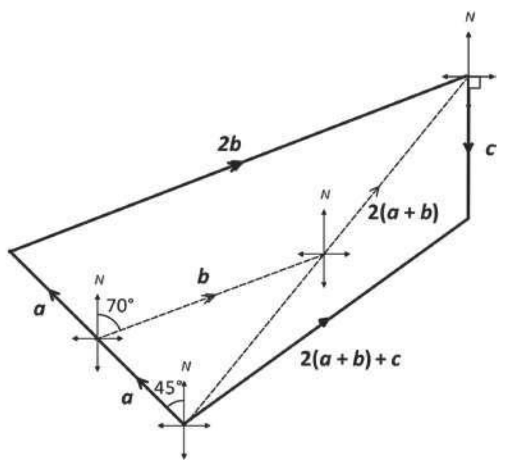


> *Note:*
> *The vector **a** + **b** + **c** is formed from the tail of the first vector **a** to the tip of the last vector **c**. It is not necessary to draw in the intermediate vector **a** + **b**.*

(b)



(c)



> *Note:*
> *The vector 2(**a** + **b)** + **c** can be formed in two ways: either as **a** + **b** + (**a** + **b**) + c or as 2**a** + 2**b** + **c**.*

Exercise 4.1

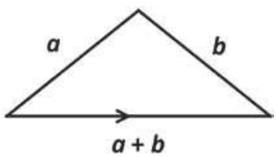
1. Vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are respectively parallel to the positive-x and positive-y axes. $|\boldsymbol{a}|$ = 2 cm and $|\boldsymbol{b}|$ = 3 cm. Use a CAS Geometry App or otherwise to make an accurate scaled drawing of each of the following vectors.

   (a) $\boldsymbol{a} + \boldsymbol{b}$ (b) $\boldsymbol{a} - \boldsymbol{b}$ (c) $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{b}$ (d) $\boldsymbol{a} + 2\boldsymbol{b}$ (e) $2(\boldsymbol{a} + \boldsymbol{b})$

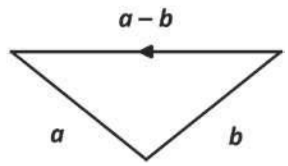
   (f) $2\boldsymbol{a} + 2\boldsymbol{b}$ (g) $\boldsymbol{b} - 2\boldsymbol{a}$ (h) $-2\boldsymbol{a} + \boldsymbol{b}$ (i) $-(\boldsymbol{a} - \boldsymbol{b})$ (j) $-\boldsymbol{a} + \boldsymbol{b}$

2. Complete each of the accompanying diagrams by indicating the directions of :

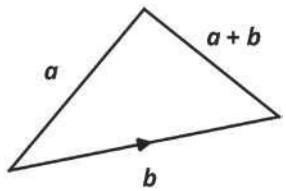
   (a) $\boldsymbol{a}$ and $\boldsymbol{b}$



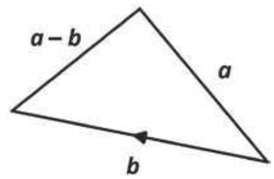
   (b) $\boldsymbol{a}$ and $\boldsymbol{b}$



   (c) $\boldsymbol{a}$ and $\boldsymbol{b}$



   (d) $\boldsymbol{a}$ and $\boldsymbol{a} - \boldsymbol{b}$

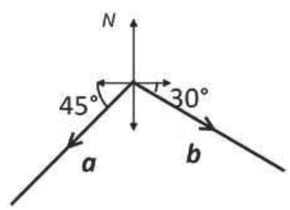


3. Vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are as shown in the accompanying diagram. Sketch on separate diagrams:

   (a) $\boldsymbol{a} - \boldsymbol{b}$

   (b) $-2(\boldsymbol{b} - \boldsymbol{a})$

   (c) $0.5(\boldsymbol{a} - 2\boldsymbol{b})$

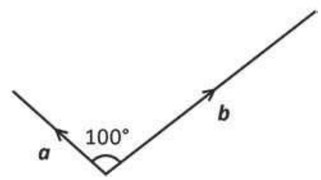


4. Vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are as shown in the accompanying diagram. Sketch on separate diagrams:

   (a) $\boldsymbol{a} + 0.5\boldsymbol{b}$

   (b) $-2(\boldsymbol{b} + \boldsymbol{a})$

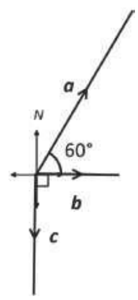
   (c) $0.5(\boldsymbol{a} - 2\boldsymbol{b})$



5. Vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ are as shown in the accompanying diagram. Sketch on separate diagrams:

   (a) $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}$

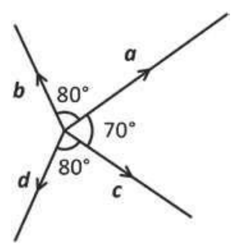
   (b) $\boldsymbol{a} - (\boldsymbol{b} + \boldsymbol{c})$



6. Vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ are as shown in the accompanying diagram. Sketch on separate diagrams:

   (a) $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} + \boldsymbol{d}$

   (b) $(\boldsymbol{a} - \boldsymbol{b}) + (\boldsymbol{c} - \boldsymbol{d})$



7. The angle between vectors ***a*** and ***b*** is 60°. Also, $|\boldsymbol{a}| = 5$ and $|\boldsymbol{b}| = 10$.
   (a) Use a CAS Geometry App or otherwise to make an accurate scaled drawing of ***a*** + ***b***.
   (b) Use your scaled drawing to find the $|\boldsymbol{a} + \boldsymbol{b}|$ and the angle between ***a*** and ***a*** + ***b***.

8. The angle between vectors ***m*** and ***n*** is 150°. Also, $|\boldsymbol{m}| = 10$ and $|\boldsymbol{n}| = 15$.
   (a) Use a CAS Geometry App or otherwise to make an accurate scaled drawing of ***m*** + ***n***.
   (b) Use your scaled drawing to find $|\boldsymbol{m} + \boldsymbol{n}|$ and the angle between ***m*** and ***m*** + ***n***.

9. The angle between the vectors ***u*** and ***v*** is 135°. Given that $|\boldsymbol{u}| = 8$ and $|\boldsymbol{v}| = 10$, use an accurate scaled drawing on paper to find $|\boldsymbol{u} - \boldsymbol{v}|$ and the angle between ***v*** and ***u*** – ***v***.

10. The angle between the vectors ***s*** and ***s*** + ***t*** is 30°. Given that $|\boldsymbol{s}| = 250$ and $|\boldsymbol{s} + \boldsymbol{t}| = 400$, use an accurate scaled drawing on paper to find $|\boldsymbol{t}|$ and the angle between ***s*** and ***t***.

4.1.4 Vectors and Triangle Trigonometry

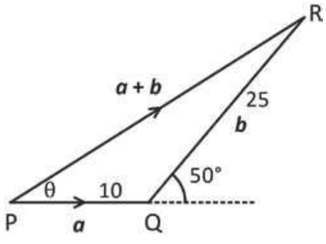
- In questions 7 to 10 of the previous exercise, accurate scale drawings were used to determine the magnitude and direction of certain vectors.
- In this section, the use of accurate scale drawings will be replaced with the use of triangle trigonometry; namely, Pythagoras' Theorem, the sine, cosine and tangent ratios, the sine rule and the cosine rule.

Example 4.4

The angle between the vectors ***a*** and ***b*** is 50°. Given that $|\boldsymbol{a}| = 10$ metres and $|\boldsymbol{b}| = 25$ metres, use the rules of trigonometry to find $|\boldsymbol{a} + \boldsymbol{b}|$ and the angle between ***a*** + ***b*** and ***a***.

Solution:

*A sketch of the situation is required and is shown in the accompanying diagram. The magnitude of **a** + **b** is represented by the length of the line segment PR.*



In Δ PQR, $\angle$ PQR = $180^{\circ} - 50^{\circ} = 130^{\circ}$.

Using the cosine rule:

$$PR^2 = QP^2 + QR^2 - 2 \times QP \times QR \times \cos P\hat{Q}R$$
$$= 10^2 + 25^2 - 2 \times 10 \times 25 \times \cos 130^{\circ} = 1\,046.393805$$
$$PR = 32.3480$$

Hence, $|\boldsymbol{a} + \boldsymbol{b}| = 32.35$.

Let the angle between ***a*** + ***b*** and ***a*** be θ. *(That is $\angle$RPQ in Δ PQR.)*

Using the sine rule: $\dfrac{\sin\theta}{QR} = \dfrac{\sin P\hat{Q}R}{PR} \Rightarrow \dfrac{\sin\theta}{25} = \dfrac{\sin 130^{\circ}}{32.3480}$

$\sin\theta = 0.5920 \Rightarrow \theta = 36.3015^{\circ}$

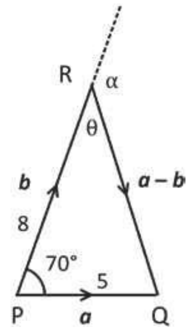
Hence, the angle between ***a*** + ***b*** and ***a*** is 36.3°.

Example 4.5

The angle between the vectors ***a*** and ***b*** is 70°. Given that $|\boldsymbol{a}|$ = 5 metres and $|\boldsymbol{b}|$ = 8 metres, use the rules of trigonometry to find $|\boldsymbol{a} - \boldsymbol{b}|$ and the angle between ***a*** – ***b*** and ***b***.

Solution:

A sketch of the situation is shown in the accompanying diagram.
*The magnitude of **a** – **b** is represented by the length of the line segment RQ.*



In Δ PQR, using the cosine rule:

$$RQ^2 = PR^2 + PQ^2 - 2 \times PR \times PQ \times \cos R\hat{P}Q$$
$$= 8^2 + 5^2 - 2 \times 8 \times 5 \times \cos 70^\circ = 61.638389$$
$$RQ = 7.8510$$

Hence, $|\boldsymbol{a} - \boldsymbol{b}| = 7.85$

Let the angle between ***a*** – ***b*** and ***b*** be α.
Clearly $\alpha = 180 - \theta$ where $\theta = \angle PRQ$ in Δ PQR.

Using the sine rule: $\dfrac{\sin\theta}{PQ} = \dfrac{\sin R\hat{P}Q}{RQ} \Rightarrow \dfrac{\sin\theta}{5} = \dfrac{\sin 70^\circ}{7.8510}$

$\sin\theta = 0.598454 \Rightarrow \theta = 36.7593^\circ \Rightarrow \alpha = 143.2407$

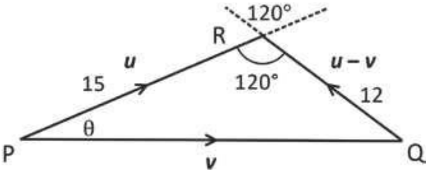
Hence, the angle between ***a*** – ***b*** and ***b*** is 143.2°.

Example 4.6

The angle between the vectors ***u*** and ***u*** – ***v*** is 120°. Given that $|\boldsymbol{u}| = 15\ ms^{-1}$ and $|\boldsymbol{u} - \boldsymbol{v}| = 12\ ms^{-1}$, use the rules of trigonometry to find $|\boldsymbol{v}|$ and the angle between ***u*** and ***v***.

Solution:

A sketch of the situation is shown in the accompanying diagram.
*The magnitude of **v** is represented by the length of the line segment PQ.*



In Δ PQR, Using the cosine rule:

$$PQ^2 = RP^2 + RQ^2 - 2 \times RP \times RQ \times \cos P\hat{R}Q$$
$$= 15^2 + 12^2 - 2 \times 15 \times 12 \times \cos 120^\circ = 549$$
$$PQ = 23.4307$$

Hence, $|\boldsymbol{v}| = 23.43\ ms^{-1}$

The angle between ***u*** and ***v*** = θ. *(Which is ∠RPQ in Δ PQR.)*

Using the sine rule: $\dfrac{\sin\theta}{RQ} = \dfrac{\sin P\hat{R}Q}{PQ} \Rightarrow \dfrac{\sin\theta}{12} = \dfrac{\sin 120^\circ}{23.4307}$

$\sin\theta = 0.443534 \Rightarrow \theta = 26.3296^\circ$

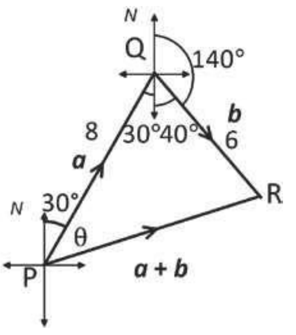
Hence, the angle between ***u*** and ***v*** is 26.3°.

Example 4.7

Vector ***a*** has magnitude 8 m and acts in the direction 030°. Vector ***b*** has magnitude 6 m and acts along 140°. Use the rules of trigonometry to find the magnitude and direction of ***a*** + ***b***

Solution:

*A sketch of the situation is shown in the accompanying diagram. The magnitude of **a** + **b** is represented by the length of the line segment PR.*



In ΔPQR, $\angle PQR = 30^{\circ} + 40^{\circ} = 70^{\circ}$.

Using the cosine rule:

$$PR^2 = QP^2 + QR^2 - 2 \times QP \times QR \times \cos P\hat{Q}R$$

$$= 8^2 + 6^2 - 2 \times 8 \times 6 \times \cos 70^{\circ} = 67.166066$$

$$PR = 8.1955$$

Hence, $|\boldsymbol{a} + \boldsymbol{b}| = 8.20$ m

Using the sine rule: $\dfrac{\sin\theta}{QR} = \dfrac{\sin P\hat{Q}R}{PR} \Rightarrow \dfrac{\sin\theta}{6} = \dfrac{\sin 70^{\circ}}{8.1955}$

$\sin\theta = 0.687958 \Rightarrow \theta = 43.4686^{\circ}$

Hence, the direction of ***a*** + ***b*** is $30^{\circ} + 43.5^{\circ} = 073.5^{\circ}$.

Exercise 4.2

1. The angle between the vectors ***a*** and ***b*** is 150°. Given that $|\boldsymbol{a}| = 10$ metres and $|\boldsymbol{b}| = 25$ metres, use the rules of trigonometry to find $|\boldsymbol{a} + \boldsymbol{b}|$ and the angle between ***a*** + ***b*** and ***a***.

2. The angle between the vectors ***c*** and ***d*** is 20°. Given that $|\boldsymbol{c}| = 3$ metres and $|\boldsymbol{d}| = 1$ metres, use the rules of trigonometry to find $|\boldsymbol{c} + \boldsymbol{d}|$ and the angle between ***c*** + ***d*** and ***d***.

3. The angle between the vectors ***a*** and ***b*** is 140°. Given that $|\boldsymbol{a}| = 15$ m and $|\boldsymbol{b}| = 6$ m, use the rules of trigonometry to find $|\boldsymbol{a} - \boldsymbol{b}|$ and the angle between ***a*** – ***b*** and ***b***.

4. The angle between the vectors ***c*** and ***d*** is 50°. Given that $|\boldsymbol{c}| = 25$ m and $|\boldsymbol{d}| = 20$ m, use the rules of trigonometry to find $|\boldsymbol{c} - \boldsymbol{d}|$ and the angle between ***c*** – ***d*** and ***d***.

5. The angle between the vectors ***a*** and ***a*** + ***b*** is 35°. Given that $|\boldsymbol{a}| = 60$ N and $|\boldsymbol{a} + \boldsymbol{b}| = 100$ N, use the rules of trigonometry to find $|\boldsymbol{b}|$ and the angle between ***a*** and ***b***.

6. The angle between the vectors ***u*** and ***u*** + ***v*** is 110°. Given that $|\boldsymbol{v}| = 30$ N and $|\boldsymbol{u} + \boldsymbol{v}| = 10$ N, use the rules of trigonometry to find $|\boldsymbol{u}|$ and the angle between ***u*** and ***v***.

7. The angle between the vectors ***a*** and ***a*** – ***b*** is 75°. Given that $|\boldsymbol{a}| = 6$ ms^{-1} and $|\boldsymbol{a} - \boldsymbol{b}| = 11$ ms^{-1}, use the rules of trigonometry to find $|\boldsymbol{b}|$ and the angle between ***a*** – ***b*** and ***b***.

8. The angle between the vectors $\boldsymbol{u}$ and $\boldsymbol{v} - \boldsymbol{u}$ is 40°. Given that $|\boldsymbol{v}| = 12$ km and $|\boldsymbol{v} - \boldsymbol{u}| = 5$ km, use the rules of trigonometry to find $|\boldsymbol{u}|$ and the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$.

*9. The angle between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ and $\boldsymbol{b}$ and $\boldsymbol{c}$ are 40° and 60° respectively. Given that $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ have magnitudes 10 N, 15 N and 20 N respectively and the angle between $\boldsymbol{a}$ and $\boldsymbol{c}$ is obtuse, use the rules of trigonometry to find $|\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}|$ and the angle between
$\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}$ and $\boldsymbol{a}$.

10. The angle between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ and $\boldsymbol{b}$ and $\boldsymbol{c}$ are 20° and 100° respectively. Given that $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ have magnitudes 10 N, 8 N and 5 N respectively and the angle between $\boldsymbol{a}$ and $\boldsymbol{c}$ is obtuse, use the rules of trigonometry to find $|\boldsymbol{a} - \boldsymbol{b} - \boldsymbol{c}|$ and the angle between $\boldsymbol{a} - \boldsymbol{b} - \boldsymbol{c}$ and $\boldsymbol{a}$.

11. Vector $\boldsymbol{a}$ has magnitude 20 ms^{-1} and acts in the direction 040°. Vector $\boldsymbol{b}$ has magnitude 30 ms^{-1} and acts in the direction 140°. Use the rules of trigonometry to find the magnitude and direction of $\boldsymbol{a} + \boldsymbol{b}$.

12. Vector $\boldsymbol{u}$ has magnitude 150 N and acts in the direction 060°. Vector $\boldsymbol{v}$ has magnitude 100 N and acts in the direction 320°. Use the rules of trigonometry to find the magnitude and direction of $\boldsymbol{u} - \boldsymbol{v}$.

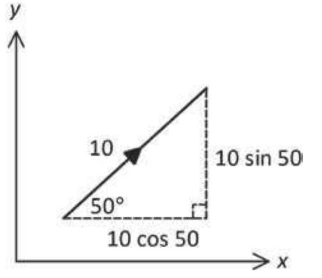
13. Vector $\boldsymbol{c}$ has magnitude 20 kmh^{-1} and acts in the direction 120°. Vector $\boldsymbol{c} + \boldsymbol{d}$ has magnitude 50 kmh^{-1} and acts in the direction 240°. Use the rules of trigonometry to find the magnitude and direction of $\boldsymbol{d}$.

14. Vector $\boldsymbol{r}$ has magnitude 160 km and acts in the direction 250°. Vector $\boldsymbol{s} - \boldsymbol{r}$ has magnitude 200 km and acts in the direction 100°. Use the rules of trigonometry to find: the magnitude and direction of $\boldsymbol{s}$.

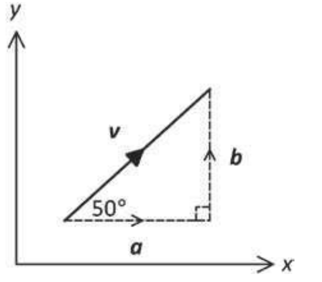
4.2 Component Representation of Vectors

4.2.1 Unit vectors and Basis vectors

- Any vector of magnitude one is termed a *unit vector*.
- A unit vector in the direction of the positive *x*-axis is represented by $\boldsymbol{i}$.
  Hence, 5 $\boldsymbol{i}$ is a vector in the direction of the positive *x*-axis of magnitude 5.
- A unit vector in the direction of the positive *y*-axis is represented by $\boldsymbol{j}$.
  Hence, –8 $\boldsymbol{j}$ is a vector in the direction of the negative *y*-axis of magnitude 8.



- Consider a vector ***v*** of magnitude 10 acting in the direction inclined at an angle of 50° with the positive *x*-axis.
- From the arrow representation of ***v***, the "arrow" can be viewed as the hypotenuse of a right angled triangle with sides parallel to the *x* and *y* axes.
- The side parallel to the *x*-axis has length 10 cos 50°. The side parallel to the *y*-axis has length 10 sin 50°.



- Consider now the same vector ***v*** written as ***v*** = ***a*** + ***b*** where ***a*** is parallel to the *x*-axis and ***b*** is parallel to the *y*-axis.
- Clearly the magnitude of ***a*** is 10 cos 50°.

  $\Rightarrow$ ***a*** = (10 cos 50°) ***i***

  where ***i*** is the unit vector in direction of the positive *x*-axis.
- The magnitude of ***b*** is 10 sin 50°.

  $\Rightarrow$ ***b*** = (10 sin 50°) ***j***

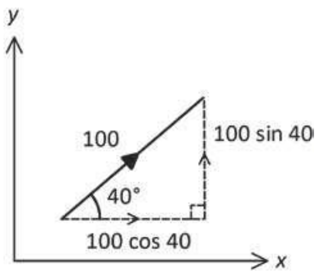
  where ***j*** is the unit vector in direction of the positive *x*-axis.
- Therefore ***v*** = ***a*** + ***b*** can now be written as ***v*** = (10 cos 50°) ***i*** + (10 sin 50°) ***j*** .
- That is the vector ***v*** is now written in terms of the unit vectors ***i*** and ***j***.

- Similarly, any vector in the *x*-*y* plane can be written in terms of the unit vectors ***i*** and ***j***. The vectors ***i*** and ***j*** are called the *basis vectors* for vectors in the *x*-*y* plane.
- In general, if vector ***v*** has magnitude *v* acting in the direction inclined at an angle of θ with the positive *x*-axis, then we can rewrite ***v*** as ***v*** = (v cos θ) ***i*** + (v sin θ) ***j***.
- (v cos θ) ***i*** and (v sin θ) ***j*** are referred to as the ***i*** and ***j*** components of ***v***, or the *x* and *y* components of ***v*** or the horizontal and vertical components of ***v***.

Example 4.8

The vector ***u*** has magnitude 100 m and is inclined at an angle of 40° with the positive *x*-axis. The vector ***v*** has magnitude 60 m and is inclined at an angle of -120° with the positive *x*-axis. Express each of the vectors ***u*** and ***v*** in the form *a* ***i*** + *b* ***j***. Give the values of *a* and *b* correct to 2 decimal places.

Solution:



For vector ***u***:

x–component has magnitude = 100 × cos 40° = 76.60

y–component has magnitude = 100 × sin 40° = 64.28

Hence, ***u*** = 76.60 ***i*** + 64.28 ***j***

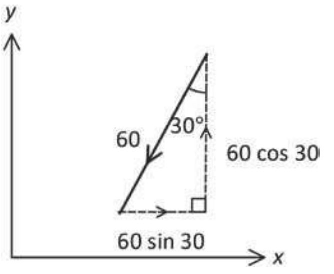
For vector ***v***:

x–component has magnitude = 60 × sin 30° = 30

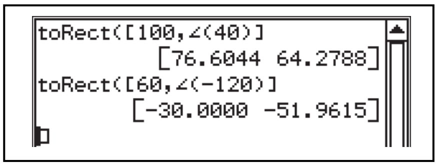
y–component has magnitude = 60 × cos 30° = 51.96

Hence, ***v*** = –30 ***i*** + (–51.96) ***j*** = –30 ***i*** – 51.96 ***j***

*(The x and y components are each in directions opposite to that of **i** and **j**.)*

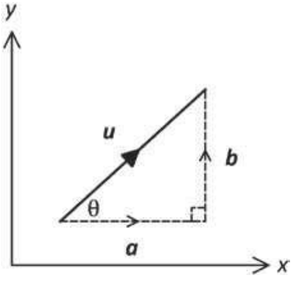


Alternatively:



4.2.2 Magnitude and direction of a vector

- Let ***u*** = *a* ***i*** + *b* ***j***.
  Using Pythagoras' Theorem, the magnitude of ***u***,
  $$|\boldsymbol{u}| = \sqrt{a^2 + b^2}$$
  Using right triangle trigonometry, the direction ***u*** makes with the positive *x*-axis, is given by $\tan\theta = \dfrac{b}{a}$



Example 4.9

Given that ***u*** = –3 ***i*** + 7 ***j***, find $|\boldsymbol{u}|$ and the angle it makes with the positive *x*-axis.

Solution:

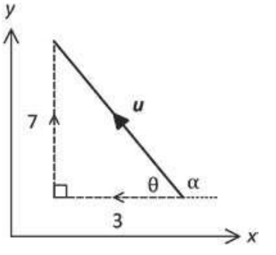
Using Pythagoras' Theorem:

$$|\boldsymbol{u}| = \sqrt{3^2 + 7^2} = \sqrt{58} = 7.62$$

Using right triangle trigonometry:

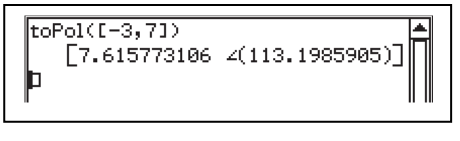
$$\tan\theta = \frac{7}{3} \qquad \Rightarrow \theta = 66.8^{\circ}$$

Hence, the angle it makes with the positive *x*-axis, $\alpha = 180 - \theta = 113.2^{\circ}$



Note: *In determining the direction of a given vector, it is best to make a sketch of the vector.*

Alternatively:



4.2.3 Scalar Multiplication and Parallel Vectors

- Given $\boldsymbol{u} = a\,\boldsymbol{i} + b\,\boldsymbol{j}$, then $\lambda\,\boldsymbol{u} = \lambda(\,a\,\boldsymbol{i} + b\,\boldsymbol{j}\,) = \lambda a\,\boldsymbol{i} + \lambda b\,\boldsymbol{j}$ where λ is a constant.
  Hence, $|\lambda\boldsymbol{u}| = \lambda|\boldsymbol{u}|$. That is, $\lambda\boldsymbol{u}$ has a magnitude λ times the magnitude of $\boldsymbol{u}$.
- Given the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$:
  - if $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel, then $\boldsymbol{u}$ is a scalar multiple of $\boldsymbol{v}$, that is, $\boldsymbol{u} = \lambda\boldsymbol{v}$ where λ is a constant
  - if $\boldsymbol{u} = \lambda\boldsymbol{v}$ where λ is a constant, then $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel.

Example 4.10

Given that $\boldsymbol{u} = 3\,\boldsymbol{i} + 4\,\boldsymbol{j}$, find $|\boldsymbol{u}|$ and find a vector parallel to $\boldsymbol{u}$ with magnitude 50.

Solution:

$|\boldsymbol{u}| = \sqrt{3^2 + 4^2} = 5.$

Since, $|\boldsymbol{u}| = 5$, the vector $10\boldsymbol{u}$ will have a magnitude of 50.

Hence, $10(3\,\boldsymbol{i} + 4\,\boldsymbol{j})$ is parallel to $\boldsymbol{u} = 3\,\boldsymbol{i} + 4\,\boldsymbol{j}$ and has a magnitude of 50.

Example 4.11

Given that $\boldsymbol{v} = 4\,\boldsymbol{i} - 9\,\boldsymbol{j}$, find λ in exact form if $\lambda\boldsymbol{v}$ has a magnitude of 1.
Hence, find in exact form a vector parallel to $\boldsymbol{v}$ with magnitude 5.

Solution:

$|\boldsymbol{v}| = \sqrt{4^2 + (-9)^2} = \sqrt{97}.$

For $\lambda|\boldsymbol{v}| = 1 \Rightarrow \lambda \times \sqrt{97} = 1$ Hence, $\lambda = \dfrac{1}{\sqrt{97}}$ or $\dfrac{\sqrt{97}}{97}$.

Therefore, a vector parallel to $\boldsymbol{v}$ with magnitude 5 is $5 \times \dfrac{1}{\sqrt{97}}\boldsymbol{v} = \dfrac{5}{\sqrt{97}}\boldsymbol{v}$ or $\dfrac{5\sqrt{97}}{97}\boldsymbol{v}$.

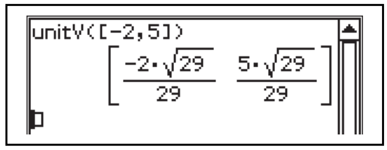
4.2.4 Unit Vectors

- A unit vector in the direction of $\boldsymbol{u}$, denoted $\hat{\boldsymbol{u}}$, is a vector parallel to $\boldsymbol{u}$ with magnitude 1. Clearly; $\hat{\boldsymbol{u}} = \dfrac{1}{|\boldsymbol{u}|}\boldsymbol{u}$.

Example 4.12

Find in exact form a unit vector parallel to $\boldsymbol{v} = -2\,\boldsymbol{i} + 5\,\boldsymbol{j}$. Hence, find in exact form a vector parallel but in the opposite direction to $\boldsymbol{v}$ with magnitude 7.

Solution:



$|\boldsymbol{v}| = \sqrt{(-2)^2 + 5^2} = \sqrt{29}$.

Since, $|\boldsymbol{v}| = \sqrt{29}$, $\hat{\boldsymbol{v}} = \dfrac{1}{\sqrt{29}}\boldsymbol{v}$.

Hence, a vector of magnitude 7 in a parallel but opposite direction to $\boldsymbol{v}$ is

$$7 \times \frac{1}{\sqrt{29}}(-\boldsymbol{v}) = \frac{-7}{\sqrt{29}}\boldsymbol{v} \text{ or } \frac{-7\sqrt{29}}{29}(-2\,\boldsymbol{i} + 5\,\boldsymbol{j}).$$

4.2.5 Alternative notation for vector in component form

- The vector $\boldsymbol{u} = a\,\boldsymbol{i} + b\,\boldsymbol{j}$ can be expressed:
  - in column matrix form as $\boldsymbol{u} = \begin{pmatrix} a \\ b \end{pmatrix}$
  - in ordered pair form as $< a, b >$.

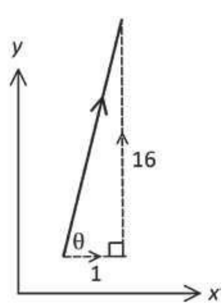
4.2.6 Addition and Subtraction of Vectors

- Given that $\boldsymbol{u} = a\,\boldsymbol{i} + b\,\boldsymbol{j}$ and $\boldsymbol{v} = c\,\boldsymbol{i} + d\,\boldsymbol{j}$, then:
  - $\boldsymbol{u} \pm \boldsymbol{v} = (a \pm c)\,\boldsymbol{i} + (b \pm d)\,\boldsymbol{j}$
  - $\begin{pmatrix} a \\ b \end{pmatrix} \pm \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a \pm c \\ b \pm d \end{pmatrix}$
  - $< a, b > \pm < c, d > = < a \pm c, b \pm d >$

Example 4.13

Given that $\boldsymbol{u} = 2\,\boldsymbol{i} + 5\,\boldsymbol{j}$ and $\boldsymbol{v} = -\,\boldsymbol{i} + 2\,\boldsymbol{j}$, find $|2\boldsymbol{u} + 3\boldsymbol{v}|$ and its direction with respect to the positive x-axis.

Solution:

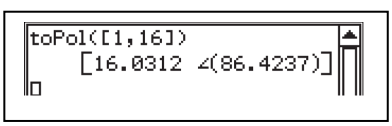


$2\boldsymbol{u} + 3\boldsymbol{v} = 2(2\,\boldsymbol{i} + 5\,\boldsymbol{j}) + 3(-\,\boldsymbol{i} + 2\,\boldsymbol{j}) = \boldsymbol{i} + 16\,\boldsymbol{j}$

Hence, $|2\boldsymbol{u} + 3\boldsymbol{v}| = \sqrt{1^2 + 16^2} = \sqrt{257}$.

Using right triangle trigonometry, $\tan\theta = \dfrac{16}{1} \Rightarrow \theta = 86.4^{\circ}$

Hence, $2\boldsymbol{u} + 3\boldsymbol{v}$ is inclined at an angle of 86.4° to the positive x-axis.

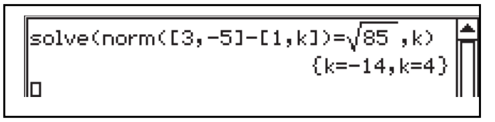


Example 4.14

Given that $\boldsymbol{a} = 3\,\boldsymbol{i} - 5\,\boldsymbol{j}$ and $\boldsymbol{b} = \boldsymbol{i} + k\,\boldsymbol{j}$, find k if $|\boldsymbol{a} - \boldsymbol{b}| = \sqrt{85}$.

Solution:

$\boldsymbol{a} - \boldsymbol{b} = (3\,\boldsymbol{i} - 5\,\boldsymbol{j}) - (\boldsymbol{i} + k\,\boldsymbol{j}) = 2\,\boldsymbol{i} + (-5 - k)\,\boldsymbol{j}$



Since $|\boldsymbol{a} - \boldsymbol{b}| = \sqrt{85}$,

$$\sqrt{2^2 + (-5-k)^2} = \sqrt{85} \Rightarrow 4 + (5+k)^2 = 85$$

$$(5+k)^2 = 81 \Rightarrow (5+k) = \pm 9$$

Hence, $k = -14$ or 4

Example 4.15

Given $\boldsymbol{a} = \begin{pmatrix} -10 \\ 5 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} 2 \\ -8 \end{pmatrix}$ and $\boldsymbol{c} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$, find: (a) $|\boldsymbol{c}|$ (b) $|\boldsymbol{a} + 2\boldsymbol{b} - \boldsymbol{c}|$

(c) a vector parallel to $\boldsymbol{a} + 2\boldsymbol{b} - \boldsymbol{c}$ but with the same magnitude as $\boldsymbol{c}$.

Solution:

(a) $|\boldsymbol{c}| = \sqrt{3^2 + (-1)^2} = \sqrt{10}$

(b) $\boldsymbol{a} + 2\boldsymbol{b} - \boldsymbol{c} = \begin{pmatrix} -10 \\ 5 \end{pmatrix} + 2\begin{pmatrix} 2 \\ -8 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

$$= \begin{pmatrix} -10+4-3 \\ 5-16+1 \end{pmatrix} = \begin{pmatrix} -9 \\ -10 \end{pmatrix}$$

Hence, $|\boldsymbol{a} + 2\boldsymbol{b} - \boldsymbol{c}| = \sqrt{(-9)^2 + (-10)^2} = \sqrt{181}$

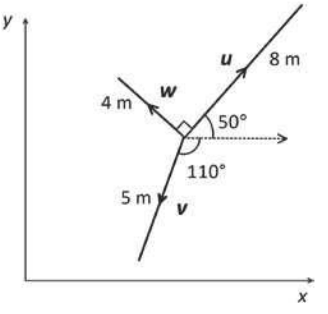
(c) A unit vector parallel to $\boldsymbol{a} + 2\boldsymbol{b} - \boldsymbol{c} = \dfrac{1}{\sqrt{181}}\begin{pmatrix} -9 \\ -10 \end{pmatrix}$

Required vector has a magnitude of $\sqrt{10}$.

Hence, required vector $= \sqrt{10} \times \dfrac{1}{\sqrt{181}}\begin{pmatrix} -9 \\ -10 \end{pmatrix} = \dfrac{\sqrt{10}}{\sqrt{181}}\begin{pmatrix} -9 \\ -10 \end{pmatrix}$.

Exercise 4.3

1. Express the vectors $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ as shown in the accompanying diagram in the form $a\,\boldsymbol{i} + b\,\boldsymbol{j}$ giving the values of a and b correct to two decimal places.



2. Express the following vectors in the form $a\,\boldsymbol{i} + b\,\boldsymbol{j}$ giving the values of a and b correct to two decimal places.
   (a) $|\boldsymbol{r}|$ = 100 ms^{-1} and $\boldsymbol{r}$ is inclined at an angle of 75° with the positive x-axis.
   (b) $|\boldsymbol{s}|$ = 25 ms^{-1} and $\boldsymbol{s}$ is inclined at an angle of –150° with the positive x-axis.
   (c) $|\boldsymbol{t}|$ = 80 ms^{-1} and $\boldsymbol{t}$ is inclined at an angle of 320° with the positive x-axis.
   (d) $|\boldsymbol{u}|$ = 100 ms^{-1} and $\boldsymbol{u}$ is inclined at an angle of 140° with the positive x-axis.

3. Without the use of a calculator, express the following vectors in the form $x\,\boldsymbol{i} + y\,\boldsymbol{j}$.
   (a) $|\boldsymbol{a}|$ = 60 N and $\boldsymbol{a}$ is inclined at an angle of 45° with the positive x-axis.
   (b) $|\boldsymbol{b}|$ = 150 N and $\boldsymbol{b}$ is inclined at an angle of 150° with the positive x-axis.
   (c) $|\boldsymbol{c}|$ = 2000 N and $\boldsymbol{c}$ is inclined at an angle of –60° with the positive x-axis.
   (d) $|\boldsymbol{d}|$ = 400 N and $\boldsymbol{d}$ is inclined at an angle of –135° with the positive x-axis.

4. For each of the given vectors, find its magnitude and the angle it makes with the positive x-axis (in degrees) correct to two and one decimal places respectively.
   (a) $\boldsymbol{a} = 3\,\boldsymbol{i} + 5\,\boldsymbol{j}$ (b) $\boldsymbol{b} = -2\,\boldsymbol{i} + 8\,\boldsymbol{j}$ (c) $\boldsymbol{c} = 10\,\boldsymbol{i} - 25\,\boldsymbol{j}$ (d) $\mathrm{d} = -4\,\boldsymbol{i} - 7\,\boldsymbol{j}$

5. For each of the given vectors, without the use of a calculator, find its magnitude and the angle it makes with the positive x-axis (in radians)
   (a) $\boldsymbol{a} = 9\,\boldsymbol{i} + 9\,\boldsymbol{j}$ (b) $\boldsymbol{b} = -2\,\boldsymbol{i} + 2\,\boldsymbol{j}$ (c) $\boldsymbol{c} = 2\sqrt{3}\,\boldsymbol{i} - 2\,\boldsymbol{j}$ (d) $\mathrm{d} = -6\,\boldsymbol{i} - 6\sqrt{3}\,\boldsymbol{j}$

6. (a) Given that $\boldsymbol{u} = \langle 5, 12 \rangle$, find $|\boldsymbol{u}|$ and find a vector parallel to $\boldsymbol{u}$ with magnitude 39.
   (b) Given that $\boldsymbol{w} = \langle -24, -7 \rangle$, find $|\boldsymbol{w}|$ and a vector parallel to $\boldsymbol{w}$ with magnitude 100.

7. (a) Given that $\boldsymbol{a} = 4\,\boldsymbol{i} + k\,\boldsymbol{j}$, find k in exact form if $|\boldsymbol{a}|$ = 10.
   (b) Given that $\boldsymbol{b} = k\,\boldsymbol{i} - 9\,\boldsymbol{j}$, find k in exact form if $|\boldsymbol{b}|$ = 12.

8. (a) Given that $\boldsymbol{v} = 4\,\boldsymbol{i} + 8\,\boldsymbol{j}$, find λ in exact form if $\lambda\boldsymbol{v}$ has a magnitude of 1.
   Hence, find in exact form a vector parallel to $\boldsymbol{v}$ with magnitude 10.
   (b) Given that $\boldsymbol{w} = \sqrt{2}\,\boldsymbol{i} + \sqrt{3}\,\boldsymbol{j}$, find λ in exact form if $\lambda\boldsymbol{w}$ has a magnitude of 1.
   Hence, find in exact form a vector parallel to $\boldsymbol{w}$ with magnitude 5.

9. Find in exact form a unit vector parallel to $\boldsymbol{v} = \langle 6, -2 \rangle$. Hence, find in exact form a vector parallel but in the opposite direction to $\boldsymbol{v}$ with magnitude 10.

10. Find in exact form a unit vector parallel to $\boldsymbol{u} = \langle 2k, k \rangle$, where $k \geq 0$.
    Hence, find in exact form a vector
    (a) parallel and in the same direction as $\boldsymbol{u}$ with magnitude 5
    (b) parallel but in the opposite direction to $\boldsymbol{u}$ with magnitude 10

11. Given that $\boldsymbol{u} = -5\,\boldsymbol{i} + 2\,\boldsymbol{j}$ and $\boldsymbol{v} = 5\,\boldsymbol{i} - 3\,\boldsymbol{j}$, find $|-\boldsymbol{u} + 2\boldsymbol{v}|$ and its direction with respect to the positive x-axis.

12. Given that $\boldsymbol{a} = \sqrt{2}\,\boldsymbol{i} + \sqrt{3}\,\boldsymbol{j}$ and $\boldsymbol{b} = \sqrt{8}\,\boldsymbol{i} - \sqrt{27}\,\boldsymbol{j}$, find $|5\boldsymbol{a} - 3\boldsymbol{b}|$ and its direction with respect to the positive x-axis.

13. Given that $\boldsymbol{a} = 7\,\boldsymbol{i} - 9\,\boldsymbol{j}$ and $\boldsymbol{b} = -\,\boldsymbol{i} + 4\,\boldsymbol{j}$, find two vectors parallel to $-\boldsymbol{a} - 2\boldsymbol{b}$ with magnitude equal to that of $\boldsymbol{b}$.

14. Given that $\boldsymbol{a} = 2\,\boldsymbol{i} + 5\,\boldsymbol{j}$ and $\boldsymbol{b} = -3\,\boldsymbol{i} + k\,\boldsymbol{j}$, find k if $|\boldsymbol{a} + \boldsymbol{b}| = \sqrt{50}$.

15. Given that $\boldsymbol{r} = k\,\boldsymbol{i} - 8\,\boldsymbol{j}$ and $\boldsymbol{s} = 4\,\boldsymbol{i} + k\,\boldsymbol{j}$, find k if $|\boldsymbol{r} - \boldsymbol{s}| = \sqrt{90}$.

16. Given $\boldsymbol{a} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ and $\boldsymbol{c} = \begin{pmatrix} 5 \\ -14 \end{pmatrix}$, find:

    (a) $|\boldsymbol{a}|$ (b) $|-\boldsymbol{a} + \boldsymbol{b} - 2\boldsymbol{c}|$

    (c) a vector parallel to $\boldsymbol{a} + \boldsymbol{b} - 2\boldsymbol{c}$ but with the same magnitude as $\boldsymbol{a}$.

17. Given $\boldsymbol{p} = \langle 2, 0 \rangle$, $\boldsymbol{q} = \langle 0, -9 \rangle$ and $\boldsymbol{r} = \langle -1, 2 \rangle$, find:

    (a) $|\boldsymbol{p} + \boldsymbol{q}|$ (b) $|\boldsymbol{p} - 2\boldsymbol{q} - \boldsymbol{r}|$

    (c) a vector parallel to $\boldsymbol{p} - 2\boldsymbol{q} - \boldsymbol{r}$ but with the same magnitude as $\boldsymbol{p} + \boldsymbol{q}$.

18. Given $\boldsymbol{u} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix} -7 \\ 0 \end{pmatrix}$ and $\boldsymbol{w} = \begin{pmatrix} 0 \\ 11 \end{pmatrix}$, find the vector in the same direction as $3\boldsymbol{u} + \boldsymbol{v} - 2\boldsymbol{w}$ with the same magnitude as $\boldsymbol{u} - \boldsymbol{w}$.

19. Given $\boldsymbol{u} = \langle 8, -3 \rangle$, $\boldsymbol{v} = \langle k, -k \rangle$ and $\boldsymbol{w} = \langle 0, -k \rangle$, find the exact value(s) of k if $|-\boldsymbol{u} + 2\boldsymbol{v} - \boldsymbol{w}| = |\boldsymbol{u}|$.

20. Given $\boldsymbol{u} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $\boldsymbol{w} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, find α and β given that:

    (a) $\alpha\,\boldsymbol{u} + \beta\,\boldsymbol{v} = \boldsymbol{w}$ (b) $\alpha\,\boldsymbol{v} + \beta\,\boldsymbol{w} = \boldsymbol{u}$

21. Prove that the vectors $\langle 4, -9 \rangle$ and $\langle -40, 90 \rangle$ are parallel.

22. Find k given that $5\,\boldsymbol{i} + k\,\boldsymbol{j}$ and $-21\,\boldsymbol{i} + 29.4\,\boldsymbol{j}$ are parallel.

23. Determine with reasons which of the following vectors are parallel:

$$\begin{pmatrix} 2 \\ 5 \end{pmatrix}, \begin{pmatrix} -44 \\ 55 \end{pmatrix}, \begin{pmatrix} 0.1 \\ 0.25 \end{pmatrix} \text{ and } \begin{pmatrix} \sqrt{20} \\ 5\sqrt{5} \end{pmatrix}.$$

24. Find the relationship between α and β if that the vectors $\langle \alpha, 10 \rangle$, $\langle -4, \beta \rangle$ are parallel.

25. Given that $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}$ is parallel to $\boldsymbol{b}$, prove that $\boldsymbol{a} + \boldsymbol{c}$ must also be parallel to $\boldsymbol{b}$.

4.2.7 Working with Vectors in Component Form

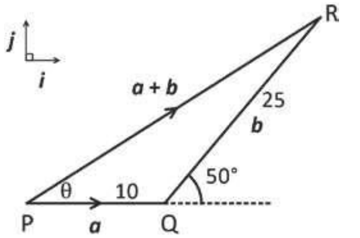
- In Exercise 4.1, accurate scale drawings were used to determine the magnitude and direction of certain vectors. In Exercise 4.2, the use of accurate scale drawings was replaced with the use of triangle trigonometry, namely Pythagoras' Theorem, the sine, cosine and tangent ratios, the sine rule and the cosine rule.
- In this section, we will rework the problems in Exercise 4.2 using vectors in component form, effectively bypassing the need to use the sine and cosine rules.

Example 4.16 (See Example 4.4)

The angle between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is 50°. Given that $|\boldsymbol{a}| = 10$ metres and $|\boldsymbol{b}| = 25$ metres, use vector components to find $|\boldsymbol{a} + \boldsymbol{b}|$ and the angle between $\boldsymbol{a} + \boldsymbol{b}$ and $\boldsymbol{a}$.

Solution:

*Let the unit vector in the direction of **a** be **i** and the unit vector perpendicular to **a** be **j** as shown.*



Let $\boldsymbol{a} = 10\,\boldsymbol{i}$ and $\boldsymbol{b} = 25\cos 50^\circ\,\boldsymbol{i} + 25\sin 50^\circ\,\boldsymbol{j}$.

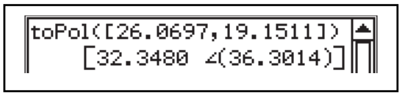
$\Rightarrow\ \boldsymbol{a} + \boldsymbol{b} = (10 + 25\cos 50^\circ)\,\boldsymbol{i} + 25\sin 50^\circ\,\boldsymbol{j}$.

$= 26.0697\,\boldsymbol{i} + 19.1511\,\boldsymbol{j}$

$|\boldsymbol{a} + \boldsymbol{b}| = \sqrt{26.0697^2 + 19.1511^2} = 32.3480 = 32.35$

Let the angle between $\boldsymbol{a} + \boldsymbol{b}$ and $\boldsymbol{a}$ be θ

$\tan\theta = \dfrac{19.1511}{26.0697} \Rightarrow \theta = 36.3015^\circ$



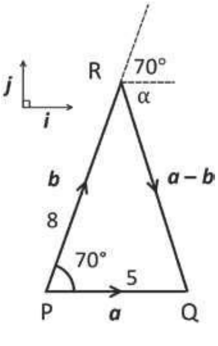
Hence, the angle between $\boldsymbol{a} + \boldsymbol{b}$ and $\boldsymbol{a}$ is 36.3°.

Example 4.17 (See Example 4.5)

The angle between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is 70°. Given that $|\boldsymbol{a}| = 5$ metres and $|\boldsymbol{b}| = 8$ metres, use vector components to find $|\boldsymbol{a} - \boldsymbol{b}|$ and the angle between $\boldsymbol{a} - \boldsymbol{b}$ and $\boldsymbol{b}$.

Solution:

*Let the unit vector in the direction of **a** be **i** and the unit vector perpendicular to **a** be **j** as shown.*



Let $\boldsymbol{a} = 5\,\boldsymbol{i}$ and $\boldsymbol{b} = 8\cos 70^\circ\,\boldsymbol{i} + 8\sin 70^\circ\,\boldsymbol{j}$

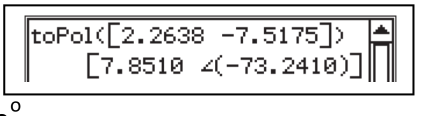
$\Rightarrow\ \boldsymbol{a} - \boldsymbol{b} = (5 - 8\cos 70^\circ)\,\boldsymbol{i} - 8\sin 70^\circ\,\boldsymbol{j}$

$= 2.2638\,\boldsymbol{i} - 7.5175\,\boldsymbol{j}$

$|\boldsymbol{a} - \boldsymbol{b}| = \sqrt{2.2638^2 + (-7.5175)^2} = 7.8510 = 7.85$

The angle between $\boldsymbol{a} - \boldsymbol{b}$ and $\boldsymbol{b} = 70^\circ + \alpha$.

$\tan\alpha = \dfrac{7.5175}{2.2638} \Rightarrow \alpha = 73.2410 = 73.2^\circ$



Hence, the angle between $\boldsymbol{a} - \boldsymbol{b}$ and $\boldsymbol{b}$ is $70^\circ + 73.2^\circ = 143.2^\circ$.

Example 4.18 (See Example 4.6)

The angle between the vectors $\boldsymbol{u}$ and $\boldsymbol{u} - \boldsymbol{v}$ is 120°. Given that $|\boldsymbol{u}|$ = 15 ms^{-1} and $|\boldsymbol{u} - \boldsymbol{v}|$ = 12 ms^{-1}, use vector components to find $|\boldsymbol{v}|$ and the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$.

Solution:

*Let the unit vector in the direction of **u** be **i** and the unit vector perpendicular to **u** be **j** as shown.*

Let $\boldsymbol{u} = 15\,\boldsymbol{i}$

and $\boldsymbol{u} - \boldsymbol{v} = -12\cos 60^{\circ}\,\boldsymbol{i} + 12\sin 60^{\circ}\,\boldsymbol{j}$

Since $\boldsymbol{u} - (\boldsymbol{u} - \boldsymbol{v}) = \boldsymbol{v}$,

$\boldsymbol{v} = 15\,\boldsymbol{i} - (-12\cos 60^{\circ}\,\boldsymbol{i} + 12\sin 60^{\circ}\,\boldsymbol{j})$

$= 21\,\boldsymbol{i} - 10.3923\,\boldsymbol{j}$

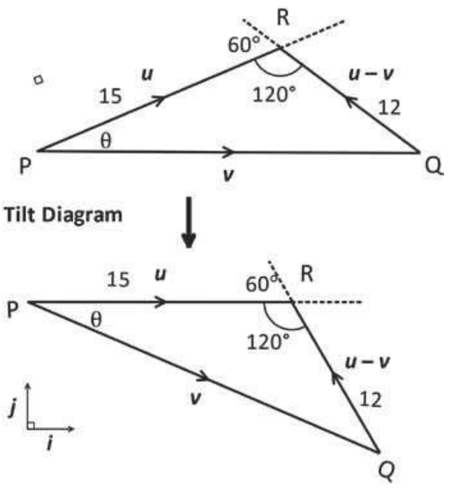
$\Rightarrow\ |\boldsymbol{v}| = \sqrt{21^2 + (-10.3923)^2} = 23.4307$

$= 23.43$ ms^{-1}

Let the angle between $\boldsymbol{u}$ and $\boldsymbol{v} = \theta$.

$$\tan\theta = \frac{10.3923}{21} \quad \Rightarrow \quad \theta = 26.3295 = 26.3^{\circ}$$

Hence, the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$ is 26.3°.




```
toPol([21,-10.3923])
   [23.4307 ∠(-26.3295)]
```

Example 4.19 (See Example 4.7)

Vector $\boldsymbol{a}$ has magnitude 8 m and acts along 030°. Vector $\boldsymbol{b}$ has magnitude 6 m and acts along 140°. Use vector components to find the magnitude and direction of $\boldsymbol{a} + \boldsymbol{b}$

Solution:

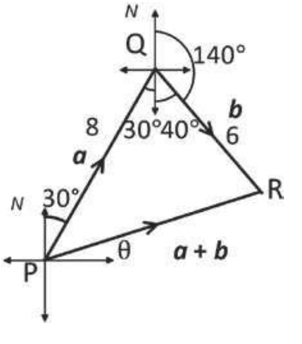
*Let the **i** and **j** be the unit vectors in the Northerly and Easterly directions respectively.*

Let $\boldsymbol{a} = 8\sin 30^{\circ}\,\boldsymbol{i} + 8\cos 30^{\circ}\,\boldsymbol{j}$ and

$\boldsymbol{b} = 6\sin 40^{\circ}\,\boldsymbol{i} - 6\cos 40^{\circ}\,\boldsymbol{j}$

$\Rightarrow\ \boldsymbol{a} + \boldsymbol{b} = 7.8567\,\boldsymbol{i} + 2.3319\,\boldsymbol{j}$

$|\boldsymbol{a} + \boldsymbol{b}| = \sqrt{7.8567^2 + 2.3319^2} = 8.1955 = 8.20$ m.



$$\tan\theta = \frac{2.3319}{7.8567} \quad \Rightarrow \quad \theta = 16.5311 = 16.5^{\circ}$$

Hence, the direction of $\boldsymbol{a} + \boldsymbol{b}$ is $90^{\circ} - 16.5^{\circ} = 073.5^{\circ}$.

```
toPol([7.8567,2.3319])
   [8.1955 ∠(16.5311)]
```

Exercise 4.4 (See Exercise 4.2)

1. The angle between the vectors ***a*** and ***b*** is 150°. Given that |***a***| = 10 metres and |***b***| = 25 metres, use vector components to find |***a*** + ***b***| and the angle between ***a*** + ***b*** and ***a***.
*(Hint: Let **i** and **j** respectively be unit vectors parallel and perpendicular to **a**.)*

2. The angle between the vectors ***c*** and ***d*** is 20°. Given that |***c***| = 3 metres and |***d***| = 1 metres, use vector components to find |***c*** + ***d***| and the angle between ***c*** + ***d*** and ***d***.
*(Hint: Let i and j respectively be unit vectors parallel and perpendicular to **d**.)*

3. The angle between the vectors ***a*** and ***b*** is 140°. Given that |***a***| = 15 metres and |***b***| = 6 metres, use vector components to find |***a*** – ***b***| and the angle between ***a*** – ***b*** and ***b***.
*(Hint: Let **i** and **j** respectively be unit vectors parallel and perpendicular to **b**.)*

4. The angle between the vectors ***c*** and ***d*** is 50°. Given that |***c***| = 25 metres and |***d***| = 20 metres, use vector components to find |***c*** – ***d***| and the angle between ***c*** – ***d*** and ***d***.
*(Hint: Let **i** and **j** respectively be unit vectors parallel and perpendicular to **d**.)*

5. The angle between the vectors ***a*** and ***a*** + ***b*** is 35°. Given that |***a***| = 60 N and |***a*** + ***b***| = 100 N, use vector components to find |***b***| and the angle between ***a*** and ***b***.
*(Hint: Let **i** and **j** respectively be unit vectors parallel and perpendicular to **a**.)*

6. The angle between the vectors ***u*** and ***u*** + ***v*** is 110°. Given that |***v***| = 30 N and |***u*** + ***v***| = 10 N, use vector components to find |***u***| and the angle between ***u*** and ***v***.
*(Hint: Let **i** and **j** respectively be unit vectors parallel and perpendicular to **u**.)*

7. The angle between the vectors ***a*** and ***a*** – ***b*** is 75°. Given that |***a***| = 6 ms^{-1} and |***a*** –***b***| = 11 ms^{-1}, use vector components to find |***b***| and the angle between ***a*** – ***b*** and ***b***.
*(Hint: Let **i** and **j** respectively be unit vectors parallel and perpendicular to **a**.)*

8. The angle between the vectors ***u*** and ***v*** – ***u*** is 40°. Given that |***v***| = 12 km and |***v*** – ***u***| = 5 km, use vector components to find |***u***| and the angle between ***u*** and ***v***.
*(Hint: Let **i** and **j** respectively be unit vectors parallel and perpendicular to **u**.)*

9. Vector ***a*** has magnitude 20 ms^{-1} and acts in the direction 040°. Vector ***b*** has magnitude 30 ms^{-1} and acts in the direction 140°. Use vector components to find the magnitude and direction of ***a*** + ***b*** .

10. Vector ***u*** has magnitude 150 N and acts in the direction 060°. Vector ***v*** has magnitude 100 N and acts in the direction 320°. Use vector components to find the magnitude and direction of ***u*** – ***v*** .

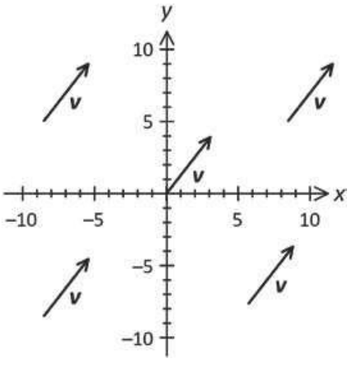
11. Vector ***c*** has magnitude 20 kmh^{-1} and acts in the direction 120°. Vector ***c*** + ***d*** has magnitude 50 kmh^{-1} and acts in the direction 240°. Use vector components to find: the magnitude and direction of ***d*** .

12. Vector ***r*** has magnitude 160 km and acts in the direction 250°. Vector ***s*** – ***r*** has magnitude 200 km and acts in the direction 100°. Use vector components to find: the magnitude and direction of ***s*** .

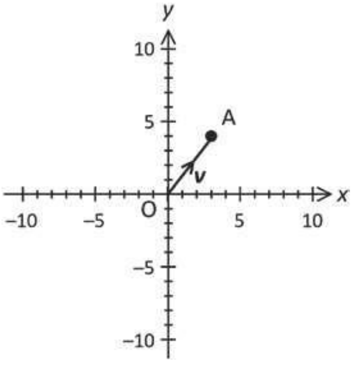
05 Position Vectors

5.1 Free Vectors and Position Vectors

- Consider the vector $\boldsymbol{v} = 3\,\boldsymbol{i} + 4\,\boldsymbol{j}$.
  Vector $\boldsymbol{v}$ is considered a *free vector* as it can be "floated" anywhere within the *x*-*y* plane.



- Let O be the origin.
  Consider the point A with coordinates (3, 4).
  The vector **OA** = $3\,\boldsymbol{i} + 4\,\boldsymbol{j}$.
  OA is considered a *fixed vector* as it is "anchored" between O and A in the direction O to A.

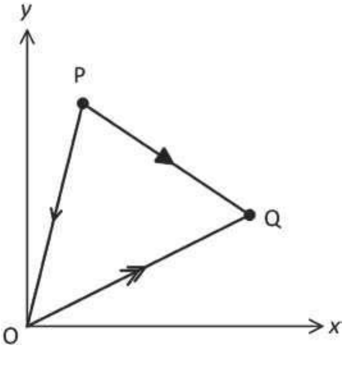


- If O is the origin and A is a point in the *x*-*y* plane, then the *position vector* of A is **OA.** If A has coordinates (h, k), then the Cartesian representation of the position vector of A is $h\,\boldsymbol{i} + k\,\boldsymbol{j}$. That is, in Cartesian form, **OA** = $h\,\boldsymbol{i} + k\,\boldsymbol{j}$.

- If P and Q are points in the *x*-*y* plane with origin O, then:

  PQ = **PO** + **OQ**

  $\Rightarrow$ **PQ** = **OQ** – **OP**.



Note:

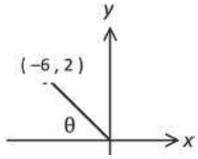
- *Free vectors are represented by single lower case letters in bold (e.g.* ***a****),*
  while fixed vectors are represented by dual upper case letters in bold (e.g. ***AB****)*

Example 5.1

The points A and B have coordinates (2, 3) and (–4, 5) respectively.
(a) Find in Cartesian form, **AB**. (b) Find in simplified exact form, $|\mathbf{AB}|$.
(c) Find the direction of **AB**.

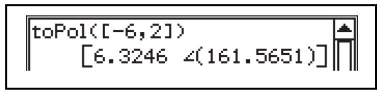
Solution:

(a) $\mathbf{AB} = \mathbf{OB} - \mathbf{OA} = (-4\,\boldsymbol{i} + 5\,\boldsymbol{j}) - (2\,\boldsymbol{i} + 3\,\boldsymbol{j}) = -6\,\boldsymbol{i} + 2\,\boldsymbol{j}$.



(b) $|\mathbf{AB}| = \sqrt{(-6)^2 + (2)^2} = \sqrt{40} = 2\sqrt{10}$.

(c) $\tan\theta = \dfrac{2}{6} \;\Rightarrow\; \theta = 18.4^\circ$.



Hence **AB** acts in the direction 161.6° with the positive *x*-axis.

Example 5.2

Given that **AB** = < 6, 5 > and **AC** = < 11, 1 >, find the position vector of B if the position vector of C is < 10, 4 >.

Solution:

$\mathbf{AC} = \mathbf{OC} - \mathbf{OA} \quad \Rightarrow \quad <11, 1> = <10, 4> - \mathbf{OA}$

$\Rightarrow \quad \mathbf{OA} = <10, 4> - <11, 1> = <-1, 3>$

$\mathbf{AB} = \mathbf{OB} - \mathbf{OA} \quad \Rightarrow \quad <6, 5> = \mathbf{OB} - <-1, 3>$

$\Rightarrow \quad \mathbf{OB} = <6, 5> + <-1, 3> = <5, 8>$

That is, the position vector of B is < 5, 8 >.

Example 5.3

The points A, B and C have position vectors $-2\,\boldsymbol{i} + 4\,\boldsymbol{j}$, $3\,\boldsymbol{i} - 8\,\boldsymbol{j}$ and $3\,\boldsymbol{i} + 2\,\boldsymbol{j}$ respectively.
Find a vector that is parallel to AB but with the same magnitude as AC.

Solution:

$$\mathbf{AB} = \mathbf{OB} - \mathbf{OA} = \begin{pmatrix} 3 \\ -8 \end{pmatrix} - \begin{pmatrix} -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ -12 \end{pmatrix} \quad \Rightarrow \quad |\mathbf{AB}| = \sqrt{(5)^2 + (-12)^2} = 13$$

$$\mathbf{AC} = \mathbf{OC} - \mathbf{OA} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} - \begin{pmatrix} -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \end{pmatrix} \quad \Rightarrow \quad |\mathbf{AC}| = \sqrt{(5)^2 + (2)^2} = \sqrt{29}.$$

Unit vector in the direction of **AB** = $\dfrac{1}{13}\begin{pmatrix} 5 \\ -12 \end{pmatrix}$.

Hence, required vector = $\sqrt{29} \times \dfrac{1}{13}\begin{pmatrix} 5 \\ -12 \end{pmatrix} = \dfrac{\sqrt{29}}{13}\begin{pmatrix} 5 \\ -12 \end{pmatrix}$.

Example 5.4

Use vectors to find k if that the points P (5, 10), Q (3, 8) and R(−1, k) are collinear.

Solution:

$$\mathbf{PQ} = \mathbf{OQ} - \mathbf{OP} = \begin{pmatrix} 3 \\ 8 \end{pmatrix} - \begin{pmatrix} 5 \\ 10 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \end{pmatrix} \text{ and } \mathbf{PR} = \mathbf{OR} - \mathbf{OP} = \begin{pmatrix} -1 \\ k \end{pmatrix} - \begin{pmatrix} 5 \\ 10 \end{pmatrix} = \begin{pmatrix} -6 \\ k-10 \end{pmatrix}.$$

Since, P, Q and R are collinear, then the line segments PQ and PR must be parallel.

That is $\mathbf{PQ} = \lambda\mathbf{PR}$. Hence $\Rightarrow \begin{pmatrix} -2 \\ -2 \end{pmatrix} = \lambda\begin{pmatrix} -6 \\ k-10 \end{pmatrix}$.

Comparing $\boldsymbol{i}$ and $\boldsymbol{j}$ components: $-2 = -6\lambda$ and $-2 = \lambda(k - 10)$

$\Rightarrow \quad \lambda = \frac{1}{3}$ and $\frac{1}{3}(k - 10) = -2$

$\Rightarrow \quad k = 4$

Example 5.5

Let $\boldsymbol{a} = 2\,\boldsymbol{i} + \boldsymbol{j}$ and $\boldsymbol{b} = -\boldsymbol{i} + 2\,\boldsymbol{j}$. The points P and Q have coordinates (3, 2) and (7, 4) respectively. (a) Show that $\boldsymbol{i} = \frac{2}{5}\boldsymbol{a} - \frac{1}{5}\boldsymbol{b}$. (b) Find the unit vector $\boldsymbol{j}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.
(c) Find **PQ** in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

Solution:

(a) Consider $\frac{2}{5}\boldsymbol{a} - \frac{1}{5}\boldsymbol{b}$. $\Rightarrow \frac{2}{5}\boldsymbol{a} - \frac{1}{5}\boldsymbol{b} = \frac{2}{5}(2\,\boldsymbol{i} + \boldsymbol{j}) - \frac{1}{5}(-\boldsymbol{i} + 2\,\boldsymbol{j})$

$$= \frac{4}{5}\boldsymbol{i} + \frac{2}{5}\boldsymbol{j} + \frac{1}{5}\boldsymbol{i} - \frac{2}{5}\boldsymbol{j} = \boldsymbol{i}.$$

(b) Let $\boldsymbol{j} = \alpha\,\boldsymbol{a} + \beta\,\boldsymbol{b}$. $\Rightarrow \boldsymbol{j} = \alpha\,(2\,\boldsymbol{i} + \boldsymbol{j}) + \beta\,(-\boldsymbol{i} + 2\,\boldsymbol{j})$

$$= (2\alpha - \beta)\,\boldsymbol{i} + (\alpha + 2\beta)\,\boldsymbol{j}$$

Comparing $\boldsymbol{i}$ and $\boldsymbol{j}$ coefficients: $2\alpha - \beta = 0$ and $\alpha + 2\beta = 1$.

Solving simultaneously: $\alpha = \frac{1}{5}$ and $\beta = \frac{2}{5}$

Hence, $\boldsymbol{j} = \frac{1}{5}\boldsymbol{a} + \frac{2}{5}\boldsymbol{b}$.

(c) $\mathbf{PQ} = (7\,\boldsymbol{i} + 4\,\boldsymbol{j}) - (3\,\boldsymbol{i} + 2\,\boldsymbol{j}) = 4\,\boldsymbol{i} + 2\,\boldsymbol{j}$.

Hence **PQ** in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$ $= 4 \times \left[\frac{2}{5}\boldsymbol{a} - \frac{1}{5}\boldsymbol{b}\right] + 2 \times \left[\frac{1}{5}\boldsymbol{a} + \frac{2}{5}\boldsymbol{b}\right]$

$$= 2\,\boldsymbol{a}.$$

Alternatively: *As* $\mathbf{PQ} = 4\,\boldsymbol{i} + 2\,\boldsymbol{j}$ *and* $\boldsymbol{a} = 2\,\boldsymbol{i} + \boldsymbol{j}$, $\mathbf{PQ} = 2\,\boldsymbol{a}$.

Note:

- *In this example, the vector* ***PQ*** *has been expressed using two sets of "basis vectors".* ***PQ*** *written in terms of* ***a*** *and* ***b*** *is in a simpler form than when written in terms of* ***i*** *and* ***j****.*

Exercise 5.1

1. The points A and B have coordinates $(-5, -10)$ and $(2, 8)$ respectively.
   (a) Find in Cartesian form, **AB**. (b) Find in simplified exact form, $|\mathbf{AB}|$.
   (c) Find the direction of **AB**.

2. The points A and B have position vectors $< 3, -8 >$ and $< -2, 6 >$ respectively.
   (a) Find in simplified exact form, $|\mathbf{AB}|$. (b) Find the direction of **AB**.

3. Point A has position vector $6\boldsymbol{i} + 2\boldsymbol{j}$. $\mathbf{AB} = -2\boldsymbol{i} + 5\boldsymbol{j}$. Find the position vector of:
   (a) the point B. (b) the point C given that $\mathbf{AC} = 2\mathbf{BA}$.

4. The position vectors of P and Q are $< 6, k >$ and $< 4, 8 >$ respectively.
   (a) Find the value(s) of k if $|\mathbf{OP}| = 10$. (b) Find the value(s) of k if $|\mathbf{PQ}| = 10$.

5. The position vectors of M and N are $\alpha\boldsymbol{i} + 3\boldsymbol{j}$ and $5\boldsymbol{i} + \alpha\boldsymbol{j}$ respectively.
   (a) Find α if $|\mathbf{OM}| = 5$. (b) Find α if $|\mathbf{MN}| = 10$.

6. Given that $\mathbf{PQ} = < 10, 2 >$ and $\mathbf{PR} = < 1, -7 >$, find the position vector of Q if the position vector of R is $< -3, 6 >$.

7. Given that $\mathbf{ST} = -\boldsymbol{i} + 2\boldsymbol{j}$ and $\mathbf{TR} = 5\boldsymbol{i} - 12\boldsymbol{j}$, find the position vector of:
   (a) S if the position vector of R is $8\boldsymbol{i} - 6\boldsymbol{j}$ (b) R if $\mathbf{OS} = 2\boldsymbol{i} + 3\boldsymbol{j}$.

8. The points S and T have position vectors $< -3, 6 >$ and $< 5, 4 >$.
   (a) Find a unit vector parallel to ST.
   (b) Find a vector in the same direction as TS but with a magnitude of 5.

9. The points A, B and C have coordinates $(-2, 4)$, $(3, -8)$ and $(3, 2)$ respectively.
   (a) Find a vector that is parallel to **AB** but with the same magnitude as **AC**.
   (b) Find a vector that is parallel to **AC** with magnitude twice that of **BC**.

10. The points E, F and G have position vectors $< 1, 5 >$, $< -3, 5 >$ and $< 9, 2 >$ respectively.
    (a) Find a vector that is the same direction as **EF** but with the same magnitude as **FG**.
    (b) Find a vector that acts in a direction opposing **EG** and with a magnitude of 5.

11. The position vectors of U, V and W are $\alpha\boldsymbol{i} - 4\boldsymbol{j}$, $\beta\boldsymbol{i} + 2\boldsymbol{j}$ and $6\boldsymbol{i} + 6\boldsymbol{j}$ respectively.
    (a) Find α if **OU** is parallel to **OW**.
    (b) Find the relationship between α and β if **UV** is parallel to **UW**.

12. The position vectors of K, L and M are $< 4, -12 >$, $< \alpha, 6 >$ and $< -3, \beta >$ respectively.
    (a) Find α if **KL** is parallel to **OK**.
    (b) Find the relationship between α and β if **LM** is parallel to **OK**.

13. Use vectors to prove that the points with position vectors $< -2, 2 >$, $< 10, -30 >$ and $< -8, 18 >$ are collinear.

14. Use vectors to find k if that the points P $(-5, -50)$, Q $(11, k)$ and R$(3, -10)$ are collinear.

15. Use vectors to find the algebraic relationship between h and k if that the points with position vectors $h\boldsymbol{i} + 2\boldsymbol{j}$, $k\boldsymbol{i} + 7\boldsymbol{j}$ and $7\boldsymbol{i} + 37\boldsymbol{j}$ are collinear.

16. Let $\boldsymbol{a} = \boldsymbol{i} + \boldsymbol{j}$ and $\boldsymbol{b} = -\boldsymbol{i} + \boldsymbol{j}$. The points P and Q have coordinates (2, 4) and (3, 1) respectively. (a) Show that $2\boldsymbol{j} = \boldsymbol{a} + \boldsymbol{b}$. (b) Find the unit vector $\boldsymbol{i}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$. (c) Find **PQ** in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

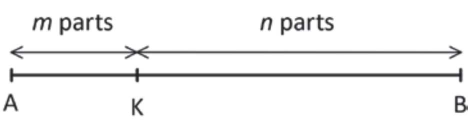
17. Let $\boldsymbol{a} = \frac{\sqrt{2}}{2}(\boldsymbol{i} - \boldsymbol{j})$ and $\boldsymbol{b} = \frac{\sqrt{2}}{2}(-\boldsymbol{i} - \boldsymbol{j})$. The points P and Q have coordinates (7, 8) and (3, 6) respectively. (a) Find the unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$. (b) Find **PQ** in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

18. The position vectors of the points A and B are < 10, 20 > and < 60, 80 > respectively. Find the position vector of the point K if: (a) $\frac{AK}{KB} = \frac{3}{2}$ (b) $\frac{AK}{KB} = -\frac{3}{2}$

5.2 Internal and External Division of a Line Segment

- Consider the point K which divides the line segment AB in the ratio $m : n$.
- If both m and n are positive, then the point K divides the line segment AB internally in the ratio $m : n$. K is located between A and B such that:
  - $\frac{AK}{KB} = \frac{m}{n}$
  - $\mathbf{AK} = \left(\frac{m}{n}\right)\mathbf{KB}$

    $n\,\mathbf{AK} = m\,\mathbf{KB}$
  - $\mathbf{AK} = \left(\frac{m}{m+n}\right)\mathbf{AB}$

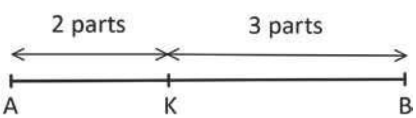


- If either m or n is negative, then the point K divides the line segment AB externally in the ratio $|m|:|n|$.

Example 5.6

The points A and B have position vectors < 5, 10 > and < 10, 20 > respectively.
Find the position vector of the point K if K divides the line segment AB in the ratio 2:3.

Solution:



From the given sketch:

$\mathbf{AK} = \frac{2}{5}\mathbf{AB} \Rightarrow 5\,\mathbf{AK} = 2\,\mathbf{AB}$

$5(\mathbf{OK} - \mathbf{OA}) = 2(\mathbf{OB} - \mathbf{OA})$

$\Rightarrow \mathbf{OK} = \frac{1}{5}(3\mathbf{OA} + 2\mathbf{OB})$

$\mathbf{OK} = \frac{1}{5}$ [< 15, 30 > + < 20, 40 >] = < 7, 14 >

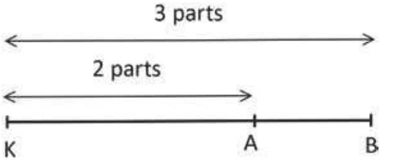
Note:

- *Instead of starting with* $\mathbf{AK} = \frac{2}{5}\mathbf{AB}$, *we could have started with* $\mathbf{AK} = \frac{2}{3}\mathbf{KB}$.

Example 5.7

The points A and B have position vectors < 4, –10 > and < 10, 20 > respectively. Find the position vector of the point K if K divides the line segment AB externally in the ratio 2:3.

Solution:



From the given sketch: $\mathbf{KA} = \frac{2}{3}\mathbf{KB} \Rightarrow 3\,\mathbf{KA} = 2\,\mathbf{KB}$

$$3(\mathbf{OA} - \mathbf{OK}) = 2(\mathbf{OB} - \mathbf{OK})$$

$$\mathbf{OK} = 3\mathbf{OA} - 2\mathbf{OB}$$

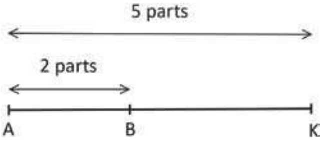
$$= 3 < 4, -10 > - \ 2 < 10, 20 > = < -8, -70 >$$

Example 5.8

The points A and B have position vectors < 4, 4 > and < –2, 7 > respectively. Find the position vector of the point K if K divides the line segment AB in the ratio –5:3.

Solution:

The ratio –5:3, indicates that K divides AB externally in the ratio 5:3. From the given sketch:



$$\mathbf{KA} = \frac{5}{3}\mathbf{KB} \Rightarrow 3\,\mathbf{KA} = 5\,\mathbf{KB}$$

$$3(\mathbf{OA} - \mathbf{OK}) = 5(\mathbf{OB} - \mathbf{OK})$$

$$\mathbf{OK} = \frac{1}{2}(5\mathbf{OB} - 3\mathbf{OA})$$

$$= \frac{1}{2}\,[< -10, 35 > - < 12, 12 >] = < -11, \frac{23}{2} >$$

Exercise 5.2

1. Find the position vector of K if it divides the line segment joining the points with the following position vectors in the stated ratio:
   (a) < –10, 20 > and < 5, 8 >; 1:2 (b) < 24, –12 > and < 10, 9 >; 4:3
   (c) $\begin{pmatrix} -2 \\ 9 \end{pmatrix}$ and $\begin{pmatrix} 12 \\ 18 \end{pmatrix}$, 5:3 externally (d) < 5, 2 > and < –4, 20 >; –4:5

2. The point with position vector < 3, 10 > divides the line segment AB internally in the ratio 3:5. Find the position vector of B if the position vector of A is < –12, 18 >.

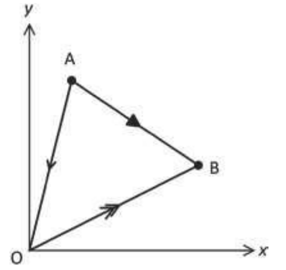
3. The point with position vector < –4, –12 > divides the line segment PQ externally in the ratio 5:2. Find the position vector of Q if the position vector of P is < 17, 23 >.

4. Prove that if the point K divides the line segment AB internally in the ratio $m : n$, then the position vector of K is given by $\mathbf{OK} = \left(\frac{1}{m+n}\right)(n\,\mathbf{OA} + m\,\mathbf{OB})$.

5. Prove that if the point K divides the line segment AB externally in the ratio $m : n$, where $m > n$ then the position vector of K is given by $\mathbf{OK} = \left(\frac{1}{m-n}\right)(-n\,\mathbf{OA} + m\,\mathbf{OB}]$.

06 Relative Vectors

6.1 Relative Position/Displacement Vectors



- Position vectors describe the position or displacement of points with respect (relative) to the origin O.
- Consider the points A and B on the x–y plane. The position vector of A with respect to the origin, $\boldsymbol{r}_{\mathrm{A}}$ = **OA**. The position vector of B with respect to the origin, $\boldsymbol{r}_{\mathrm{B}}$ = **OB**.
- If, however, other points are used as reference points, such vectors are called relative displacement vectors.
- The displacement vector of B relative to A describes the direction and distance of B as viewed from A. To emphasise that B is viewed with A as the reference point, this is written as ${}_{\mathrm{B}}\boldsymbol{r}_{\mathrm{A}}$. That is, ${}_{\mathrm{B}}\boldsymbol{r}_{\mathrm{A}}$ = **AB** = **OB** – **OA** = $\boldsymbol{r}_{\mathrm{B}} - \boldsymbol{r}_{\mathrm{A}}$
- In summary, the position vector or displacement vector of P relative to Q is given by:

$${}_{\mathrm{P}}\boldsymbol{r}_{\mathrm{Q}} = \mathbf{OP} - \mathbf{OQ} = \boldsymbol{r}_{\mathrm{P}} - \boldsymbol{r}_{\mathrm{Q}}$$

Example 6.1

The points A, B and C have position vectors < 3, 4 >, < –2, –7 > and < 5, –6 > respectively.
(a) Find the position vector of A relative to B.
(b) Find the displacement vector of C relative to B. (c) Find ${}_{C}\boldsymbol{r}_{A}$ and hence $|{}_{C}\boldsymbol{r}_{A}|$.

Solution:

(a) ${}_{A}\boldsymbol{r}_{B}$ = **OA** – **OB** = < 3, 4 > – < –2, –7 >
= < 5, 11 > .

(b) ${}_{C}\boldsymbol{r}_{B}$ = **OC** – **OB** = < 5, –6 > – < –2, –7 >
= < 7, 1 > .

(c) ${}_{C}\boldsymbol{r}_{A}$ = **OC** – **OA** = < 5, –6 > – < 3, 4 >
= < 2, –10 > .

Hence $|{}_{C}\boldsymbol{r}_{A}| = \sqrt{2^2 + (-10)^2} = \sqrt{104} = 2\sqrt{26}$.

Example 6.2

The position vector of A is < 3, 4 >. The position vector of B relative to A and C are respectively < –8, 2 > and < 5, 10 >. Find the position vector of C.

Solution:

$_B\mathbf{r}_C$ = < 5, 10 > $\Rightarrow$ **OB** – **OC** = < 5, 10 >

$\Rightarrow$ **OC** = **OB** – < 5, 10 >.

$_B\mathbf{r}_A$ = < –8, 2 > $\Rightarrow$ **OB** – **OA** = < –8, 2 >

$\Rightarrow$ **OB** = < –8, 2 > + **OA**

But, **OA** = < 3, 4 > $\Rightarrow$ **OB** = < –8, 2 > + < 3, 4 >

= < –5, 6 >

Hence, **OC** = < –5, 6 > – < 5, 10 > = < –10, –4 >.

Exercise 6.1

1. The points A, B, C and D have position vectors < 1, 0 >, < 2, 4 >, < –3, –8 > and < 10, 1 > respectively. Find:
   (a) the displacement vector of C relative to B (b) the position vector of A relative to B
   (c) $_C\mathbf{r}_A$ (d) $|\,_D\mathbf{r}_B\,|$ (e) $_B\mathbf{r}_A - {}_A\mathbf{r}_C$.

2. Given that $_C\mathbf{r}_A$ = < 4, –5 > and the position vector of C is < 5, 6 >, find how far and in what direction is A from O.

3. Given $_P\mathbf{r}_Q$ = < 2, 10 > km, $_Q\mathbf{r}_R$ = < –11, 9 > km and **OR** = < 8, 2 > km, find how far and in what direction P is from O.

4. The position vector of T is < 3, 4 >. The position vector of S relative to T and R are respectively < –8, 2 > and < 5, 10 >. Find the position vector of R.

5. Given that **OA** = < 4, 2 >, |**OB**| = $\sqrt{40}$ and $|\,_A\mathbf{r}_B\,|$ = 10, find **OB**.

6. Given that **OQ** = < –2, 3 >, |**OP**| = 2 and $|\,_P\mathbf{r}_Q\,|$ = 5, find **OP**.

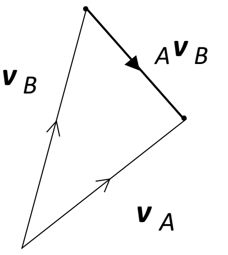
7. Prove that $_A\mathbf{r}_B = -\,_B\mathbf{r}_A$.

8. Prove that $_A\mathbf{r}_B + {}_B\mathbf{r}_C + {}_C\mathbf{r}_D + {}_D\mathbf{r}_E = {}_A\mathbf{r}_E$.

*9. If $|\,_A\mathbf{r}_B\,| = |\,_C\mathbf{r}_B\,|$ = 10, prove that $0 \le |\,_A\mathbf{r}_C\,| \le 20$.

*10. Prove that if $_A\mathbf{r}_B = \lambda\,_B\mathbf{r}_C$, where λ is a constant, then A, B and C are collinear.

6.2 Relative Velocity



- Consider two moving objects A and B.
- The velocity of A relative to B, is the direction and rate at which body A is moving away or towards B as viewed by an observer travelling with object B. This is denoted $_A\boldsymbol{v}_B$.
- In vector terms, $_A\boldsymbol{v}_B = \boldsymbol{v}_A - \boldsymbol{v}_B$.
  - $_A\boldsymbol{v}_B$ is the velocity of A relative to B.
  - $\boldsymbol{v}_A$ and $\boldsymbol{v}_B$ are respectively the *true velocities* of A and B using the *earth* as its frame of reference.
- In summary, the velocity vector of P relative to Q is given by:

$$_P\boldsymbol{v}_Q = \boldsymbol{v}_P - \boldsymbol{v}_Q$$

Example 6.3

The velocities (ms^{-1}) of three moving objects A, B and C are < –1, 2 >, < 1, –3 > and < –4, 2 > respectively.

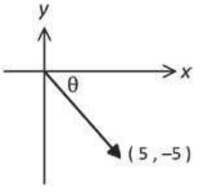
(a) Find the velocity of A relative to B.

(b) In what direction and with what speed is B moving relative to C?

Solution:

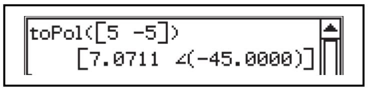
(a) Velocity of A relative to B, $_A\boldsymbol{v}_B$ = < –1, 2 > – < 1, –3 >
= < –2, 5 > ms^{-1}.

(b) Velocity of B relative to C, $_B\boldsymbol{v}_C$ = < 1, –3 > – < –4, 2 >
= < 5, –5 > ms^{-1}.



Therefore, $|< 5, -5 >| = 5\sqrt{2}$ and $\theta = 45^{\circ}$.

Hence, B is moving relative to C at a rate of $5\sqrt{2}$ ms^{-1} in the direction bearing 135°.



Example 6.4

Given that P is moving with velocity < 10, 2 > kmh^{-1}, Q is moving with velocity < *a*, *b* > where *a* > 0 and *b* > 0, with a speed of 5 kmh^{-1}. Given that the speed of P relative to Q is 10 kmh^{-1}, find the direction of Q relative to P.

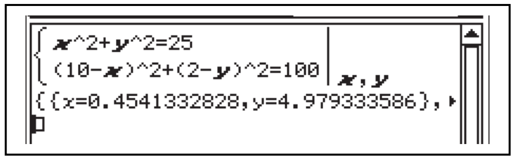
Solution:

Given: $\boldsymbol{v}_P$ = < 10, 2 >, $|\boldsymbol{v}_Q|$ = 5 and $|{}_P\boldsymbol{v}_Q|$ = 10.

Let $\boldsymbol{v}_Q$ = < *a*, *b* >. $\Rightarrow$ $a^2 + b^2 = 25$ I

Also, 0, 2 > – < *a*, *b* > | = 10

$(10 - a)^2 + (2 - b)^2 = 100$ II



Solve I and II simultaneously,

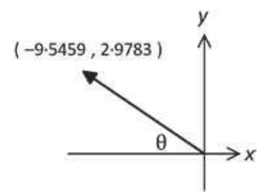
a = 0.4541, *b* = 4.9793 (*a* > 0, *b* > 0)

Hence, $\boldsymbol{v}_Q$ = < 0.4541, 4.9783 >

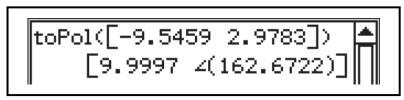
$\Rightarrow$ ${}_Q\boldsymbol{v}_P$ = < 0.4541, 4.9783 > – < 10, 2 >

= < –9.5459, 2.9783 >

$\Rightarrow$ $\theta = 17.3^{o}$.



Therefore, Q is moving in the direction 287.3^{o} bearing relative to P.



Exercise 6.2

1. The velocities (ms^{-1}) of three moving objects P, Q and R are < 5, 4 >, < –1, 7 > and < 9, –2 > respectively.
   (a) Find the velocity of P relative to Q. (b) Find the velocity of R relative to P.
   (c) In what direction and with what speed is Q moving relative to R?

2. Given that ${}_D\boldsymbol{v}_E$ = < 0, –3 > ms^{-1} and the velocity of E is < 3, 0 >, find the velocity of D.

3. Given ${}_A\boldsymbol{v}_B$ = < –1, 6 > cms^{-1}, ${}_B\boldsymbol{v}_C$ = < 3, 10 > cms^{-1} and $\boldsymbol{v}_C$ = < –4, 7 > cms^{-1}, find the true velocity of A.

4. The true velocity of T is < –1, –6 > ms^{-1}. The velocity of S relative to T and R are respectively < 10, –2 > ms^{-1} and < 7, 8 > ms^{-1}. Find the true velocity of R.

5. Given that $\boldsymbol{v}_A$ = < –5, 6 >, $|\boldsymbol{v}_B| = \sqrt{10}$ and $|{}_A\boldsymbol{v}_B| = \sqrt{89}$, find $\boldsymbol{v}_B$.

07 Vector Applications I

7.1 Vectors using Arrow Representation

- In this section we will use the arrow representation of vectors and the rules of triangular trigonometry to model some mathematical situations.
- Note that the term *resultant vector* refers to the sum of two or more vectors.

Example 7.1

A and B are two points directly across each other on opposite banks of a river. The river is 250 m wide from A to B and flows Eastwards at a steady rate of 2 kmh^{-1}. Dianne can row at a steady rate of 5 kmh^{-1} in still waters. Dianne rows Northwards from A towards B and she finds herself landing on the opposite bank at C instead of at B.
(a) Find the bearing of C from A and hence the distance between B and C.
(b) Find the time taken for Dianne to row across the river.

Solution:

*Diagram 1 shows the displacement vector **AB** with magnitude 0.25 km.*
*Diagram 2 shows the velocity vectors **v** (Dianne) and **c** (river current) and the resultant vector **r**.*
*Diagram 3 shows the displacement vectors **AB** and **AC**.*

Diagram 1

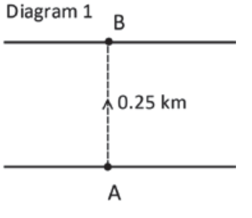


Diagram 2

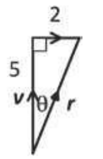
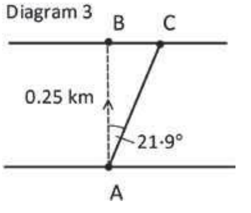


Diagram 3



(a) In Diagram 2, $\tan\theta = \dfrac{2}{5} \Rightarrow \theta = 21.8^{o}$.

Hence, In Diagram 3, Bearing of C from A is N21.8^{o}E.
In Diagram 3, BC = 0.25 tan 21.8^{o} = 0.1 km.
Hence, Dianne lands at C, 100 m downstream from B.

(b) In Diagram 2, $|\boldsymbol{r}|^2 = |\boldsymbol{v}|^2 + |\boldsymbol{c}|^2 = 5^2 + 2^2 = 29$

Hence, resultant speed = $\sqrt{29}$ kmh^{-1}.

In Diagram 3, AC = $\dfrac{0.25}{\cos 21.8^{o}}$ = 0.2693 km.

Hence, time taken = $\dfrac{0.2693}{\sqrt{29}}$ = 0.05 hours = 3 minutes.

Notes:
- *Diagrams 1 and 3 are diagrams showing displacement vectors.*
- *Diagram 2 is a velocity vector diagram.*
- *Care must be taken to distinguish between displacement vectors from the velocity vectors.*

Example 7.2

A and B are two points directly across each other on opposite banks of a river where B is North of A. The river is 250 m wide from A to B and flows Eastwards at a steady rate of 2 kmh^{-1}. Frances can row at a steady rate of 5 kmh^{-1} in still waters.

(a) Find the direction Frances should row her boat to get from A to B.

(b) Find the time taken for Frances to row across the river.

Solution:

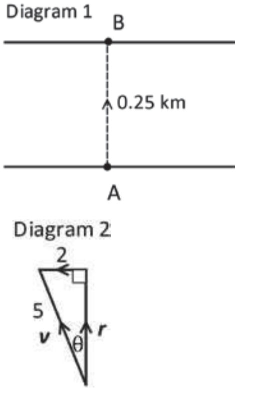


*Diagram 1 shows the displacement vector **AB** with magnitude 0.25 km.*
*Diagram 2 shows the velocity vectors **v** (Frances) and **c** (river current) and the resultant vector **r**. The resultant vector **r** is now parallel to **AB**.*

(a) In Diagram 2, $\sin\theta = \dfrac{2}{5} \Rightarrow \theta = 23.6^\circ$.

Hence, Frances should row in the direction N23.6°W.

(b) In Diagram 2, $|\boldsymbol{r}|^2 = |\boldsymbol{v}|^2 - |\boldsymbol{c}|^2 = 5^2 - 2^2 = 21$

Hence, resultant speed = $\sqrt{21}$ kmh^{-1}.

Hence, time taken = $\dfrac{0.25}{\sqrt{21}}$ = 0.0546 hours = 3.3 minutes.

Example 7.3

Steven wishes to fly his ultra-light plane from A to B where B is 150 km away along 050°. A wind is blowing consistently with velocity 20 kmh^{-1} from 300°. The ultra-light plane can fly at a speed of 100 kmh^{-1}.

(a) Find the direction Steven should fly his plane in order to reach B directly.

(b) Find the flying time from A to B.

Solution:

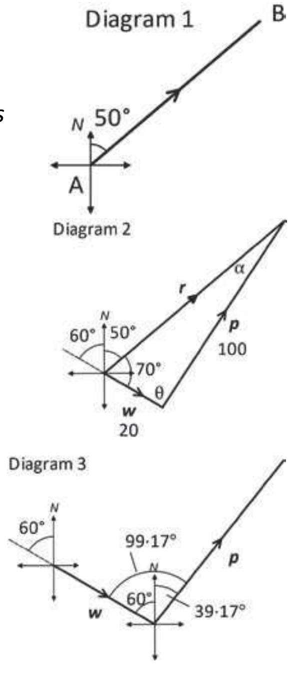


*Diagram 1 shows the displacement vector **AB**. Diagram 2 shows the velocity vectors **w** (wind), **p** (plane) and **r** (resultant vector). **r** is parallel to **AB**.*

(a) In Diagram 2: $\dfrac{\sin\alpha}{20} = \dfrac{\sin 70^\circ}{100} \Rightarrow \sin\theta = 0.1879$

$\alpha = 10.83^\circ$ (Reject 169.17°)

Hence $\theta = 99.17^\circ$.

The direction Steven should fly his plane is $99.17^\circ - 60^\circ = 039.2^\circ$. (see Diagram 3)

(b) In Diagram 2: $r^2 = 100^2 + 20^2 - 2 \times 100 \times 20 \times \cos 99.17^\circ$

$r = 105.06$

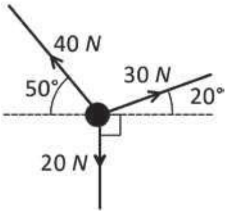
Hence, resultant speed is 105.06 kmh^{-1}.

Therefore, time taken = $\dfrac{150}{105.06}$ = 1.43 hours

= 1 hour 26 minutes.

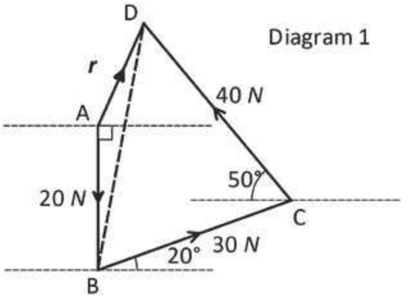
Example 7.4

An object is acted on by three forces as shown in the accompanying diagram. The 20 N force acts vertically downwards. Find the magnitude and direction of the resultant force.



Solution:

Diagram 1 is a vector diagram showing the forces acting on the object where ***r*** *is the resultant vector.*



In ΔBCD, $\angle BCD = 70^\circ$.

$BD^2 = 30^2 + 40^2 - 2 \times 30 \times 40 \times \cos 70^\circ$

$BD = 40.9775$

$$\cos \angle DBC = \frac{30^2 + 40.9775^2 - 40^2}{2 \times 30 \times 40.9775} = 0.3982$$

$$\Rightarrow \angle DBC = 66.53^\circ.$$

Hence, in ΔABD, $\angle ABD = 70^\circ - 66.53^\circ = 3.47^\circ$

$AD^2 = 40.9775^2 + 20^2 - 2 \times 40.9775 \times 20 \times \cos 3.47^\circ$

Hence, magnitude of resultant force $AD = 21.0490 \approx 21.0$ N.

$$\cos \angle BAD = \frac{20^2 + 21.0490^2 - 40.9775^2}{2 \times 20 \times 21.0490} = -0.9930 \Rightarrow \angle BAD = 173.23^\circ.$$

Hence, the resultant force acts in the direction $180^\circ - 173.23^\circ = 6.8^\circ$ to the vertical.

Exercise 7.1

1. A and B are two points directly across each other on opposite banks of a river. The river is 300 m wide from A to B and flows Southwards at a steady rate of 3 ms^{-1}. Mike can row at a steady rate of 2.5 ms^{-1} in still waters. Mike rows Westwards from A towards B and he finds himself landing on the opposite bank at C instead of at B.
   (a) Find the bearing of C from A and hence the distance between B and C.
   (b) Find the time taken for Mike to row across the river.

2. A and B are two points directly across each other on opposite banks of a river. The river is 200 m wide from A to B and flows Westwards at a steady rate of x kmh^{-1}. Sue can row at a steady rate of 4 kmh^{-1} in still waters. Sue rows Northwards from A towards B and she finds herself landing on the opposite bank at C, 400 m West of B.
   (a) Find the value of x. (b) Find the time taken for Sue to row across the river.

*3. P and Q are two points on opposite banks of a river. The river flows Eastwards at a steady rate of 1 ms^{-1}. Lee can row at a steady rate of x ms^{-1} in still waters. Lee rows from P in a Northerly direction and he finds himself landing on the opposite bank at Q after 5 minutes where the bearing of Q from P is 070°. Assume that the width of the river is constant between P and Q.
   (a) Find the value of x. (b) Find the width of the river.

4. A and B are two points directly across each other on opposite banks of a river. The river is 100 m wide from A to B and flows Eastwards at a steady rate of 0.5 ms^{-1}. Greg can swim at a steady rate of 1 ms^{-1} in still waters.
   (a) Find the direction Greg should swim to get from A to B directly.
   (b) Find the time taken for Greg to swim across the river.

5. M and N are two points directly across each other on opposite banks of a river. The river is 100 m wide from M to N and flows Southwards at a steady rate of x kmh^{-1}. Tony can swim at a steady rate of 2.5 kmh^{-1} in still waters. To swim across the river from M to N directly, Tony has to swim in a direction 65^{o} upstream with the river bank.
   (a) Find x. (b) Find the time taken for Tony to swim across the river.

*6. S and T are two points directly across each other on opposite banks of a river. The river flows Southwards at a steady rate of x ms^{-1}. Ian can swim at a steady rate of y ms^{-1} in still waters. Ian manages to swim across the river from S to T, a distance of 250 m, in 4 minutes by swimming in a direction 70^{o} upstream with the river bank. Find x and y.

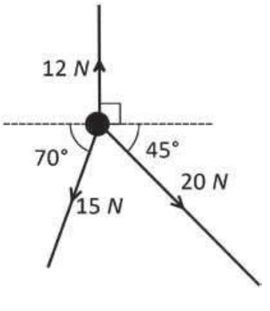
7. Richard wishes to fly his plane from A to B where B is 600 km away along 130^{o}. A wind is blowing consistently with velocity 30 kmh^{-1} from 240^{o}. The plane can fly with a maximum speed of 200 kmh^{-1}.
   (a) Find the direction Richard should fly his plane in order to reach B directly.
   (b) Find the flying time from A to B.

8. Amurtha wishes to fly her plane from P to Q where Q is 540 km away along 245^{o}. A wind is blowing consistently with velocity x kmh^{-1} along bearing θ^{o}. The plane can fly with a maximum speed of 200 kmh^{-1}. Given that if Amurtha flies in the direction 250^{o}, she will reach Q in a minimum time of 3 hours, find x and θ.

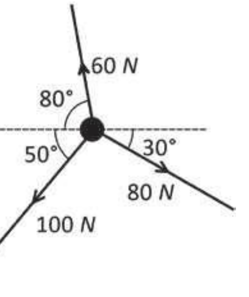
9. Barry wishes to fly his helicopter from A to B, where B is 300 km from A along bearing θ^{o}. The helicopter has a maximum speed of 200 kmh^{-1}. To arrive at B in minimum time, if the wind blows at 20 kmh^{-1} along 100^{o}, then Barry must head along 030^{o}. Find θ and the minimum flight time.

10. A body is acted on by the forces as shown in each of the accompanying diagrams. Find the magnitude and direction of the resultant force.

(a)



(b)



7.2 Vectors in Component Form

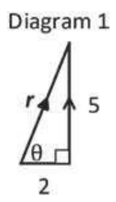
- In this section we will rework some of the questions in the previous section using vectors in component form.

Example 7.5 (See Example 7.1)

A and B are two points directly across each other on opposite banks of a river. The river is 250 m wide from A to B and flows Eastwards at a steady rate of 2 kmh^{-1}. Dianne can row at a steady rate of 5 kmh^{-1} in still waters. Dianne rows Northwards from A towards B and she finds herself landing on the opposite bank at C instead of at B. Use vector components to find: (a) the bearing of C from A and hence the distance between B and C.
(b) the time taken for Dianne to row across the river.

Solution:

*Let **i**: unit vector Eastwards and **j**: unit vector Northwards.*

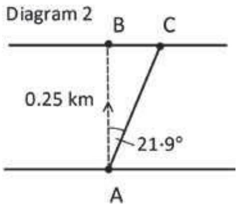


(a) Current $\boldsymbol{c} = 2\,\boldsymbol{i}$, Dianne $\boldsymbol{v} = 5\,\boldsymbol{j}$: $\Rightarrow$ resultant $\boldsymbol{r} = \boldsymbol{c} + \boldsymbol{v} = 2\,\boldsymbol{i} + 5\,\boldsymbol{j}$.

$$\tan\theta = \frac{5}{2} \Rightarrow \theta = 68.2^{\circ}.$$

Hence, bearing of C from A is N21.8°E.

In Diagram 2, in ΔABC, BC = 0.25 tan 21.8° = 0.10 km.
Hence, Dianne lands at C, 100 m downstream from B.



(b) In Diagram 1, $|\,\boldsymbol{r}\,| = \sqrt{2^2 + 5^2} = \sqrt{29}$

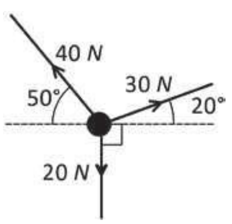
Hence, resultant speed = $\sqrt{29}$ kmh^{-1}.

In Diagram 2, $AC = \dfrac{0.25}{\cos 21.8^{\circ}} = 0.2693$ km.

Hence, time taken = $\dfrac{0.2693}{\sqrt{29}}$ = 0.05 hours = 3 minutes.

Example 7.6 (See Example 7.4)

A body is acted on by three forces as shown in the accompanying diagram. The 20 N force acts vertically downwards. Find the magnitude and direction of the resultant force.



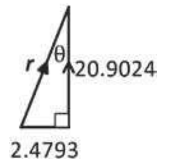
Solution:

*Let **i**: unit vector to the right and **j**: unit vector upwards. Rewrite each of the three forces in component form.*

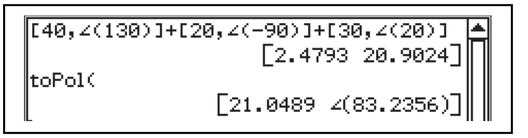
$$\text{Resultant force } \boldsymbol{r} = \begin{pmatrix} -40\cos 50^{\circ} \\ 40\sin 50^{\circ} \end{pmatrix} + \begin{pmatrix} 30\cos 20^{\circ} \\ 30\sin 20^{\circ} \end{pmatrix} + \begin{pmatrix} 0 \\ -20 \end{pmatrix}$$

$$= \begin{pmatrix} 2.4793 \\ 20.9024 \end{pmatrix}$$

$$\Rightarrow \quad |\boldsymbol{r}| = \sqrt{2.4793^2 + 20.9024^2}$$
$$= 21.0489 \text{ N}$$
$$\tan\theta = \frac{2.4793}{20.9024} \quad \Rightarrow \quad \theta = 6.76^{\circ}.$$



Hence, the resultant force has magnitude 21.0 N acting in the direction 6.8° to the vertical.



Example 7.7

Ella wishes to fly her plane from P to Q where Q is 600 km away along 340°. A wind is blowing consistently with speed x kmh^{-1} along bearing θ°. The plane can fly in still air with a maximum speed of 300 kmh^{-1}. Given that if Ella flies in the direction 350°, she will reach Q in a minimum time of 150 minutes, find x and θ.

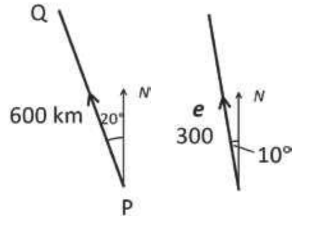
Solution:

*Let **i**: unit vector Eastwards and **j**: unit vector Northwards.*

For Ella to fly directly from P to Q, the resultant velocity vector $\boldsymbol{v}$ must be parallel to the destination vector **PQ**. $\Rightarrow$ **PQ** = $t\,\boldsymbol{v}$ where t is the time of flight.

Let **PQ** = $\begin{pmatrix} -600\sin 20^{\circ} \\ 600\cos 20^{\circ} \end{pmatrix}$, Wind $\boldsymbol{w}$ and Ella $\boldsymbol{e}$ = $\begin{pmatrix} -300\sin 10^{\circ} \\ 300\cos 10^{\circ} \end{pmatrix}$.

Resultant velocity $\boldsymbol{v} = \boldsymbol{w} + \boldsymbol{e}$



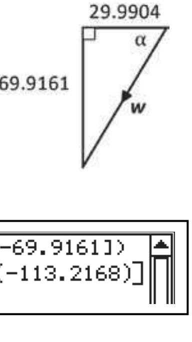
Destination vector **PQ** = time of flight × Resultant velocity.

$$\begin{pmatrix} -600\sin 20^{\circ} \\ 600\cos 20^{\circ} \end{pmatrix} = t \times (\boldsymbol{w} + \boldsymbol{e})$$

$$\begin{pmatrix} -600\sin 20^{\circ} \\ 600\cos 20^{\circ} \end{pmatrix} = 2.5 \times \left(\boldsymbol{w} + \begin{pmatrix} -300\sin 10^{\circ} \\ 300\cos 10^{\circ} \end{pmatrix}\right)$$

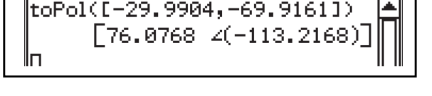
$$\Rightarrow \quad \boldsymbol{w} + \begin{pmatrix} -300\sin 10^{\circ} \\ 300\cos 10^{\circ} \end{pmatrix} = 0.4 \times \begin{pmatrix} -600\sin 20^{\circ} \\ 600\cos 20^{\circ} \end{pmatrix}$$

$$\boldsymbol{w} = \begin{pmatrix} -29.9904 \\ -69.9161 \end{pmatrix}$$



$$|\boldsymbol{w}| = \sqrt{(-29.9904)^2 + (-69.9161)^2} = 76.0768$$

$$\tan\alpha = \frac{69.9161}{29.9904} \quad \Rightarrow \quad \alpha = 66.78^{\circ}.$$



Hence $x = |\boldsymbol{w}|$ = 76.1 kmh^{-1} and θ = 203.2°.

Example 7.8 (See Example 7.2)

A and B are two points directly across each other on opposite banks of a river where B is North of A. The river is 250 m wide from A to B and flows Eastwards at a steady rate of 2 kmh^{-1}. Frances can row at a steady rate of 5 kmh^{-1} in still waters.

(a) Find the direction Frances should row her boat to get from A to B.
(b) Find and the time taken for Frances to row across the river.
(c) In what direction should Frances row her boat from B to return to A.

Solution:

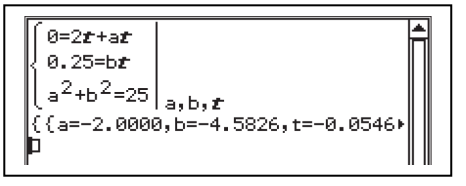
*Let **i**: unit vector Eastwards and **j**: unit vector Northwards.*
*For Frances to row directly from A to B, the resultant velocity vector **v** must be parallel to **AB**. ⇒ **AB** = t **v**.*

(a) Let **AB** = $\begin{pmatrix} 0 \\ 0.25 \end{pmatrix}$, current $\boldsymbol{c} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and Frances $\boldsymbol{f} = \begin{pmatrix} a \\ b \end{pmatrix}$:

As resultant velocity $\boldsymbol{v} = \boldsymbol{c} + \boldsymbol{f} \Rightarrow \mathbf{AB} = t \times (\boldsymbol{c} + \boldsymbol{f})$

$$\Rightarrow \begin{pmatrix} 0 \\ 0.25 \end{pmatrix} = t \left(\begin{pmatrix} 2 \\ 0 \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} \right)$$

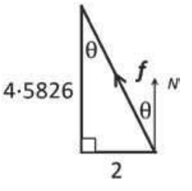
Equate ***i*** components: $0 = 2t + at$ I
Equate ***j*** components: $0.25 = bt$ II
Also $\sqrt{a^2 + b^2} = 5$ III



Solve simultaneously: $a = -2$, $b = -4.5826$, $t = -0.0546$
or $a = -2$, $b = 4.5826$, $t = 0.0546$

But $t > 0$. Hence, $\boldsymbol{f} = \begin{pmatrix} -2 \\ 4.5826 \end{pmatrix}$

$\tan\theta = \dfrac{2}{4.5826} \Rightarrow \theta = 23.57^\circ$.

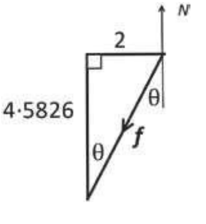


Hence, Frances should row in the direction N23.6°W.

(b) Hence, time taken $t = 0.0546$ hours = 3.3 minutes.

(c) For the return trip, $\mathbf{BA} = t_{return} \times (\boldsymbol{c} + \boldsymbol{f})$ where $t_{return} > 0$.
This can be rewritten as $\mathbf{AB} = -t_{return} \times (\boldsymbol{c} + \boldsymbol{f})$ where $t_{return} > 0$
Let $-t_{return} = t$ where $t < 0$.
Hence, $\mathbf{AB} = t \times (\boldsymbol{c} + \boldsymbol{f})$ where $t < 0$.
From part (a), $a = -2$, $b = -4.5826$, $t = -0.0546$

That is, for the return journey, $\boldsymbol{f} = \begin{pmatrix} -2 \\ -4.5826 \end{pmatrix}$



$\tan\theta = \dfrac{2}{4.5826} \Rightarrow \theta = 23.57^\circ$.

Hence, Frances should row in the direction S23.6°W.

Note:

- *Equation I yields a positive solution for t as well as a negative solution for t.*
  The absolute value of the negative solution corresponds to the time of flight for the return journey.

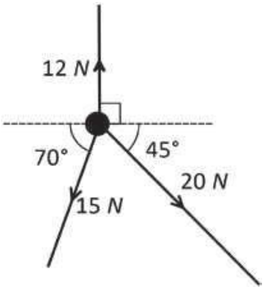
Exercise 7.2 (See Exercise 7.1)

1. A and B are two points directly across each other on opposite banks of a river. The river is 300 m wide from A to B and flows Southwards at a steady rate of 3 kmh^{-1}. Mike can row at a steady rate of 2.5 ms^{-1} in still waters. Mike rows Westwards from A towards B and he finds himself landing on the opposite bank at C instead of at B. Use vector components to find:
   (a) the bearing of C from A and hence the distance between B and C
   (b) the time taken for Mike to row across the river.

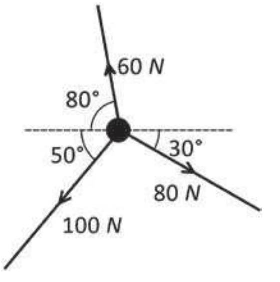
2. A and B are two points directly across each other on opposite banks of a river. The river is 200 m wide from A to B and flows Westwards at a steady rate of *x* kmh^{-1}. Sue can row at a steady rate of 4 kmh^{-1} in still waters. Sue rows Northwards from A towards B and she finds himself landing on the opposite bank at C, 400 m West of B. Use vector components to find:
   (a) the value of *x* (b) the time taken for Mike to row across the river.

3. A body is acted on by the forces as shown in each of the accompanying diagrams. Use vector components to find the magnitude and direction of the resultant force.

(a)



(b)



4. P and Q are two points on opposite banks of a river. The river flows Eastwards at a steady rate of 1ms^{-1}. Lee can row at a steady rate of *x* ms^{-1} in still waters. Lee rows from P in a Northerly direction and he finds himself landing on the opposite bank at Q after 5 minutes where the bearing of Q from P is 070^{o}. Assume that the width of the river is constant between P and Q. Use vector components to find:
   (a) the value of *x* (b) the width of the river.

5. Amurtha wishes to fly her plane from P to Q where Q is 540 km away along 245^{o}. A wind is blowing consistently with velocity *x* kmh^{-1} along bearing θ^{o}. The plane can fly with a maximum speed of 200 kmh^{-1}. Given that if Amurtha flies in the direction 250^{o}, she will reach Q in a minimum time of 3 hours, use vector components to find *x* and θ.

6. S and T are two points directly across each other on opposite banks of a river. The river flows Southwards at a steady rate of *x* ms^{-1}. Ian can swim at a steady rate of *y* ms^{-1} in still waters. Ian manages to swim across the river from S to T, a distance of 250 m, in 4 minutes by swimming in a direction 70^{o} upstream with the river bank. Use vector components to fnd *x* and *y*.

7. A and B are two points directly across each other on opposite banks of a river. The river is 100 m wide from A to B and flows Eastwards at a steady rate of 0.5 ms^{-1}. Greg can swim at a steady rate of 1 ms^{-1} in still waters. Use vector components to find:
   (a) the direction Greg should swim to get from A to B directly.
   (b) the time taken for Greg to swim across the river
   (c) the direction Greg should swim to return directly from B to A.

8. Richard wishes to fly his plane from A to B where B is 600 km away along 130^o. A wind is blowing consistently with velocity 30 kmh^{-1} from 240^o. The plane can fly with a maximum speed of 200 kmh^{-1}. Use vector components to find:
   (a) the direction Richard should fly his plane in order to reach B directly.
   (b) the flying time from A to B
   (c) the direction Richard should fly his plane in return directly from B to A.

9. M and N are two points directly across each other on opposite banks of a river. The river is 100 m wide from M to N and flows Southwards at a steady rate of x kmh^{-1}. Tony can swim at a steady rate of 2.5 kmh^{-1} in still waters. To swim across the river from M to N directly, Tony has to swim in a direction 65^o upstream with the river bank. Use vector components to find the value of x and the time taken for Tony to swim across the river.

10. Barry wishes to fly his helicopter from A to B, where B is 300 km from A along bearing θ^o. The helicopter has a maximum speed of 200 kmh^{-1}. To arrive at B in minimum time, if the wind blows at 20 kmh^{-1} along 100^o, then Barry must head along 030^o. Use vector components to find θ and the minimum flight time.

7.3 Position Vectors

Example 7.9

At 0800 hours, particle P starts moving with constant velocity $< -2, -5 >$ ms^{-1} from the point with position vector $< 3, 10 >$ metres. Find: (a) the position vector of P after 10 seconds (b) when P is 50 metres from the origin.

Solution:

(a) Position vector of P after 10 seconds $\mathbf{OP}(10) = < 3, 10 > + 10 \times < -2, -5 >$
$= < -17, -40 >$

(b) Position vector of P after t seconds $\mathbf{OP}(t) = < 3, 10 > + t < -2, -5 > = < 3 - 2t, 10 - 5t >$

Hence, Distance to the origin $|\mathbf{OP}(t)| = \sqrt{(3-2t)^2 + (10-5t)^2}$.

When distance = 50 m, $\sqrt{(3-2t)^2 + (10-5t)^2} = 50$

$(3 - 2t)^2 + (10 - 5t)^2 = 2500$

Use the "Solve" command: $\Rightarrow$ $t = 11.21$ (reject -7.35)

Hence, P is 50 metres away from the origin 11.2 seconds after 0800 hours.

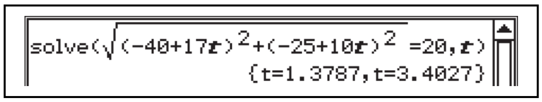
Example 7.10

At 0800 hours, the position vectors of vessels A and B are < 10, 20 > km and < –30, –5 > km respectively. Vessels A and B travel with constant velocity < –5, –5 > and < 12, 5 > kmh^{-1} respectively. (a) Find the position vectors of A and B t hours after 0800 hours.
(b) Find when A and B are 20 km apart.
(c) Find the minimum distance between A and B and state when this occurs.

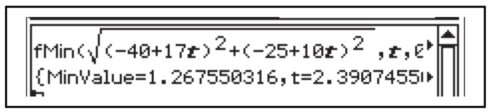
Solution:

(a) $\mathbf{OA}(t) = <10, 20> + t \times <-5, -5> = <10 - 5t, 20 - 5t>$
$\mathbf{OB}(t) = <-30, -5> + t \times <12, 5> = <-30 + 12t, -5 + 5t>$.

(b) Displacement vector between A and B after t hours
$\mathbf{AB}(t) = \mathbf{OB}(t) - \mathbf{OA}(t) = <-30 + 12t, -5 + 5t> - <10 - 5t, 20 - 5t>$
$= <-40 + 17t, -25 + 10t>$
Distance between A and B after t hours = $|\mathbf{AB}(t)|$.
When distance between A and B = 20 km:
$|\mathbf{AB}(t)| = 20 \Rightarrow \sqrt{(-40+17t)^2 + (-25+10t)^2} = 20 \Rightarrow t = 1.3787, 3.4027$ hours.
Hence, A and B are 20 km apart
at 0923 hours and 1124 hours.



(c) Distance $d = \sqrt{(-40+17t)^2 + (-25+10t)^2}$
$\Rightarrow$ Minimum value for AB = 1.27 km
when t = 2.3907 hours i.e. 1023 hours.



Example 7.11

At 6.00 am, the position and velocity vectors of vessels A, B and C are respectively < 5, 10 > km , < 2, 4 > kmh^{-1}; < –1, 1 > km , < 4, 7 > kmh^{-1} and < 2, 5 > km and < 3, 5 > kmh^{-1}.
If the respective velocities are maintained, show that:
(a) A and B will collide, stating when this will occur,(b) A and C will not collide.

Solution:

$\mathbf{OA}(t) = <5, 10> + t \times <2, 4> = <5 + 2t, 10 + 4t>$.
$\mathbf{OB}(t) = <-1, 1> + t \times <4, 7> = <-1 + 4t, 1 + 7t>$.
$\mathbf{OC}(t) = <2, 5> + t \times <3, 5> = <2 + 3t, 5 + 5t>$.

(a) For A and B to collide, $\mathbf{OA}(t) = \mathbf{OB}(t)$; $\Rightarrow$ $<5 + 2t, 10 + 4t> = <-1 + 4t, 1 + 7t>$.
Comparing ***i*** components: $5 + 2t = -1 + 4t \Rightarrow t = 3$
Comparing ***j*** components: $10 + 4t = 1 + 7t \Rightarrow t = 3$
Since, the ***i*** and ***j*** components of **OA** and **OB** are identical for $t = 3$,
A and B collide at 9.00 am.

(b) For A and B to collide, $\mathbf{OA}(t) = \mathbf{OC}(t)$; $\Rightarrow$ $<5 + 2t, 10 + 4t> = <2 + 3t, 5 + 5t>$.
Comparing ***i*** components: $5 + 2t = 2 + 3t \Rightarrow t = 3$
Comparing ***j*** components: $10 + 4t = 5 + 5t \Rightarrow t = 5$
Since, there is no common value of t for the ***i*** and ***j*** components of **OA** and **OC** to be identical, A and C do not collide.

Exercise 7.3

1. At 0600 hours, A starts moving with constant velocity < 1, 3 > m/s from the point with position vector < 3, 10 > metres. Find:
   (a) the position vector of A after 5 seconds (b) when A is 20 metres from the origin
   (c) when A is 10 metres from the point B with position vector < 4, 13 >.

2. At 1000 hours, particle B leaves the point P < 2, 5 > metres with velocity < –1, 5 > m/s.
   Find: (a) the position vector of B after 5 seconds (b) when B is 50 metres from P
   (c) when B is 10 metres from the point Q with position vector < 4, 13 >.

3. Particle C travels with a constant velocity of < 4, –1 > km/h. At 10 am, particle C passes the point Q < 2, 4 > km. Find: (a) the position vector of C at 8 am (same day)
   (b) when C is 20 km from Q (c) when C crosses the x-axis.

4. Particle D travels with a constant velocity of < a, b > km/h and passes the points P < –10, 40 > km and Q < 10, 30 > km at 8 am and 10 am respectively. Find:
   (a) a and b (b) when and where D crosses the y-axis.

5. At 9 am, the position vectors of vessels A and B are < –20, 15 > km and < 50, –25 > km respectively. Vessels A and B travel with constant velocity < 10, –5 > and < –5, 10 > km/h respectively. (a) Find the position vectors of A and B t hours after 9 am.
   (b) Find when A and B are 50 km apart.
   (c) Find the minimum distance between A and B and state when this occurs.

6. At 1 pm, object H travelling with constant velocity < 100, –100 > km/h is sighted at the point with position vector < 100, 120 > km. At 1.30 pm object J travelling with constant velocity < 120, 90 > km/h is sighted at the point with position vector < –20, 40 > km. Find the minimum distance between H and J and state when this occurs.

*7. At 1 pm, the position vectors of yachts Y and Z are < x, y > km and < –10, –20 > km respectively. Yachts Y and Z travel with constant velocity < –2, –8 > and < 3, –6 > km/h respectively. Yachts Y and Z are 5 km apart at 3 pm (same day) and $\sqrt{8}$ km apart at 4 pm (same day). Use vector methods to find x and y.

8. At 1500 hours, the position and velocity vectors of particles A and B are respectively < –15, 6 > m, < –6, 21 > m/s and < –10, 21 > m, < –8, 15 > m/s. Show that A and B will collide stating when and where the collision will occur.

9. At 7.00 am, the position and velocity vectors of particles A, B and C are respectively < 8, 5 > km , < 4, 6 > km/h; < –10, 3 > km , < 7.5, 5 > km/h and < 4, –1 > km and < 4, 6 > km/h. If these velocities were maintained, use vector methods to determine which of these particles will collide. State when and where the collision will occur.

10. Ship J leaves port at < 20, 40 > km at 0700 hours and travels with constant velocity < 10, –5 > km/h. After travelling for 4 hours, the ship J hits a submerged reef and starts sinking. The captain of the ship J sends out a distress call indicating its position and that the ship will be completely submerged in an hour. Ship S is located at < 120, 40 > when it receives the distress call from the ship J and immediately heads towards the J. Find the velocity S should sail in order to reach J before it becomes completely submerged.

08 Vector Applications II

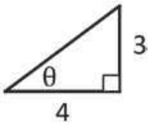
8.1 Relative Displacement and Velocity

- In this section ideas of relative displacement and velocity are applied in several contexts.

Example 8.1

Students A, B and C, are located at points (3, 10) m and (–5, –6) m and (15, 3) m. C starts to run with constant velocity < –4, 3 > ms^{-1} while A and B remain stationary. Find:
(a) how fast and in what direction C is running relative to A
(b) using a relative vector method, the closest C gets to B.

Solution:



(a) Velocity of C relative to A, ${}_C\boldsymbol{v}_A$ = < 4, 3 > – < 0, 0 > = < 4, 3 >.

$|{}_C\boldsymbol{v}_A| = \sqrt{4^2+3^2} = 5 \qquad \theta = \tan^{-1}\left(\frac{3}{4}\right) = 36.9^{\circ}$

Hence, relative to A, C is running at 5 ms^{-1} in the direction 053.1° bearing.

(b) The position vector of B and C after t seconds:

$\boldsymbol{r}_B(t)$ = < –5, –6 > + t < 0, 0 > = < –5, –6 >

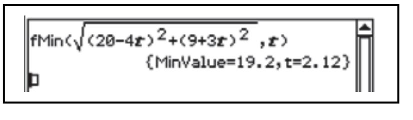
$\boldsymbol{r}_C(t)$ = < 15, 3 > + t < –4, 3 > = < 15 – 4t, 3 + 3t >

Displacement of C relative to B after t seconds:

${}_C\boldsymbol{r}_B(t) = \boldsymbol{r}_C(t) - \boldsymbol{r}_B(t)$
= < 15 – 4t, 3 + 3t > – < –5, –6 >
= < 20 – 4t, 9 + 3t >.

Hence, distance between C and B after t seconds

$= |{}_C\boldsymbol{r}_B(t)| = \sqrt{(20-4t)^2+(9+3t)^2}$



Minimum distance between C and B = 19.2 m.

8.2 Interception & Collision Problems using Relative Vectors

- In Example 7.11, a procedure which equated the position vectors of two moving objects was used to examine the conditions under which two objects will collide. In this section, a method using relative velocities will be used to examine the conditions under which two moving objects will collide.

- Consider objects A and B moving with constant velocities $\boldsymbol{v}_A$ and $\boldsymbol{v}_B$ respectively. Let the position vectors of A and B at time $t = 0$ be $\boldsymbol{r}_A(0)$ and $\boldsymbol{r}_B(0)$ respectively.

- The position vector of A after t units of time, $\boldsymbol{r}_A(t) = \boldsymbol{r}_A(0) + t \times \boldsymbol{v}_A$.
  The position vector of B after t units of time, $\boldsymbol{r}_B(t) = \boldsymbol{r}_B(0) + t \times \boldsymbol{v}_B$.

- For A and B to collide after t units of time, $\boldsymbol{r}_A(t) = \boldsymbol{r}_B(t)$.

$$\Rightarrow \quad \boldsymbol{r}_A(0) + t \times \boldsymbol{v}_A = \boldsymbol{r}_B(0) + t \times \boldsymbol{v}_B$$

$$\boldsymbol{r}_A(0) - \boldsymbol{r}_B(0) = t \times \boldsymbol{v}_B - t \times \boldsymbol{v}_A = t \times (\boldsymbol{v}_B - \boldsymbol{v}_A)$$

Hence $${}_A\boldsymbol{r}_B(0) = t \times {}_B\boldsymbol{v}_A$$

- In summary, for A and B to collide (interception to occur):

$$\boxed{{}_A\boldsymbol{r}_B(0) = t \times {}_B\boldsymbol{v}_A}$$

Initial displacement of A relative to B (${}_A\boldsymbol{r}_B(0)$)

Velocity of B relative to A (${}_B\boldsymbol{v}_A$)

Example 8.2

At 8 am, ships A and B leave their ports located at (10, 0) km and (50, 80) km with constant velocities < 5, 5 > kmh^{-1} and < –5, –15> kmh^{-1} respectively. Use a relative velocity method to show that if these velocities are maintained, A and B will collide. State where and when this collision will occur.

Solution:

Designate $t = 0$ at 8 am.

Initial relative displacement between A and B,

$${}_A\boldsymbol{r}_B(0) = \langle 10, 0 \rangle - \langle 50, 80 \rangle$$
$$= \langle -40, -80 \rangle.$$

Velocity of B relative to A, $${}_B\boldsymbol{v}_A = \langle -5, -15 \rangle - \langle 5, 5 \rangle$$
$$= \langle -10, -20 \rangle.$$

For collision to occur, $${}_A\boldsymbol{r}_B(0) = t \times {}_B\boldsymbol{v}_A$$
$$\langle -40, -80 \rangle = t \times \langle -10, -20 \rangle$$
$$\Rightarrow \quad t = 4$$

When $t = 4$, position vector of A, $\boldsymbol{r}_A(4) = \langle 10, 0 \rangle + 4 \times \langle 5, 5 \rangle = \langle 30, 20 \rangle$.

Hence, A and B will collide at 12.00 noon (4 hours after 8 am) at < 30, 20 > km.

Example 8.3

At 8 am, yacht A and motor vessel B are located at (40, 80) km and (–60, –40) km. Yacht A leaves its position at 8 am and sails with constant velocity < 6, 8 > kmh^{-1}. Motor vessel B has a maximum speed of 50 kmh^{-1}. Use a relative velocity method to determine the velocity of B for interception to occur in minimum time. State where and when the interception occurs should B leave its initial position at 8 am.

Solution:

Designate $t = 0$ at 8 am.

Initial relative displacement between A and B,

$${}_A\boldsymbol{r}_B(0) = \langle 40, 80 \rangle - \langle -60, -40 \rangle = \langle 100, 120 \rangle.$$

Let velocity of B be $\boldsymbol{v}_B$. Velocity of B relative to A, ${}_B\boldsymbol{v}_A = \boldsymbol{v}_B - \langle 6, 8 \rangle$.

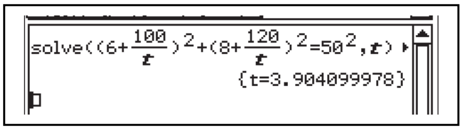
For collision to occur, ${}_A\boldsymbol{r}_B(0) = t \times {}_B\boldsymbol{v}_A$

$$\langle 100, 120 \rangle = t \times [\boldsymbol{v}_B - \langle 6, 8 \rangle]$$

$$\boldsymbol{v}_B - \langle 6, 8 \rangle = \frac{1}{t}\langle 100, 120 \rangle$$

$$\boldsymbol{v}_B = \left\langle 6 + \frac{100}{t}, 8 + \frac{120}{t} \right\rangle.$$

Since, $|\boldsymbol{v}_B| = 50$, $\left(6+\frac{100}{t}\right)^2 + \left(8+\frac{120}{t}\right)^2 = 50^2$



$$\Rightarrow \quad t = 3.9041$$

When $t = 3.9041$, $\quad \boldsymbol{v}_B = \langle 31.6, 38.7 \rangle$

Also, position vector of A, $\boldsymbol{r}_A(3.9041) = \langle 40, 80 \rangle + 3.9041 \times \langle 6, 8 \rangle = \langle 63.4, 111.2 \rangle$.

Hence, B should sail with velocity < 31.6, 38.7 > kmh^{-1} to intercept A at 11.54 am (3 hours 54 min after 8 am) at < 63.4, 111.2 > km.

Exercise 8.1

1. Students A, B and C, are located at points with position vectors < 1, 2 > m and < 3, 4 > m and < –5, 6 > m respectively. A starts to run with constant velocity < 1, 4 > ms^{-1} while B and C remain stationary. Find how fast and in what direction is A running:
   (a) relative to B (b) relative to C.

2. Students P, Q and R, are located at points with position vectors < –8, –4 > m and < 1, 0 > m and < 6, 9 > m respectively. P and Q start to run with constant velocities < –2, 1 > ms^{-1} and < 1, –3 > ms^{-1} respectively, while R remains stationary. Find how fast and in what direction is:
   (a) P running relative to R (b) Q running relative to R
   (c) Q running relative to P.

3. Ella remains stationary at (5, 0) m while her child Ben starts to run from < 20, –10 > m with constant velocity < –0.2, 0.3 > ms^{-1}.
   (a) Find the relative displacement between Ella and Ben after *t* seconds.
   (b) Use your answer in (a) to find the closest distance Ben gets to Ella.

4. Matt and Mark each start running at the same time from (–4, 0) m and (10, 10) m with constant velocities < 2, 1> ms^{-1} and < –2, –1 > ms^{-1} respectively.
   (a) Find the relative displacement between Matt and Mark after *t* seconds.
   (b) Use your answer in (a) to find the closest distance Matt gets to Mark.

5. The position vectors of sailing boats P and Q at time *t* seconds are given by $\boldsymbol{r}_P$ = < 2, 3 > + *t* < 1, 1 > ms^{-1} and $\boldsymbol{r}_Q$ = < –1, 1> + *t* < 2, 1 > ms^{-1} respectively.
   (a) Find the velocity of Q relative to P.
   (b) Find the displacement vector of P relative to Q after *t* seconds.
   (c) Use your answer in (b) to find the initial distance between P and Q.
   (d) Use your answer in (b) to find the shortest distance between P and Q.

6. A ship is sailing at 20 knots in the direction 060^{o}. To an observer on board the ship, the wind appears to be blowing from 300^{o} at 5 knots and a bird appears to be flying in the direction 240^{o} at 4 knots. Find the true velocity of the: (a) wind (b) bird.

*7. Kate, Cathy and James are jogging in an open field. Kate jogs with constant velocity $\boldsymbol{v}_K$. Cathy is jogging with a constant speed of 6 kmh^{-1} in the direction 030^{o}. But to Kate, she is jogging with a speed of *a* kmh^{-1} from the South. The velocity of James relative to Cathy is < 4, 5 > kmh^{-1} and to Kate, James appears to be jogging with a speed of *b* kmh^{-1} from the West. Find $\boldsymbol{v}_K$.

8. The position vectors of sailing boats R and S at time *t* seconds are given by $\boldsymbol{r}_R$ = < –1, 5 > + *t* < 1, –1 > ms^{-1} and $\boldsymbol{r}_S$ = < 0, 5> + *t* < 2, –3 > ms^{-1} respectively.
   (a) Find the velocity of R relative to S.
   (b) Find the initial displacement of S relative to R.
   (c) Use your answers in (a) and (b) to determine if R and S will collide.

9. The position vectors of soccer players L and M at time *t* seconds are given by $\boldsymbol{r}_L$ = < 0, 3 > + *t* < 2, –3 > ms^{-1} and $\boldsymbol{r}_M$ = < 3, –6 > + *t* < 1, 0 > ms^{-1} respectively. Use a relative velocity method to determine if L and M will collide.

10. At 6 am, ships A and B leave their ports located at < –20, 0 > km and < 100, 40 > with constant velocities < 20, 20 > km/h and < –28, 4> km/h respectively. Use a relative velocity method to show that if these velocities are maintained, A and B will collide. State where and when this collision will occur.

11. At 2 pm, ship P leaves a port located at < –20, 70 > nautical miles with constant velocity < 30, 10 > knots. One hour later, ship Q leaves a port located at < 19, 62 > nautical miles with constant velocity < 25, 20 > knots. Use a relative velocity method to show that if these velocities are maintained, Q will intercept P. State where and when this interception will occur.

12. Yachts A and B collide 90 minutes after leaving their respective pens at < *x*, *y* > and < –5, 72 > nautical miles. The yachts were travelling at constant velocities of < 8, 16 > and < 10, 12 > knots respectively. Use a relative velocity method to find *x* and *y*.

13. The position vectors of sailing boats R and S at time *t* seconds are given by $\mathbf{r}_R$ = < 3, 2 > + *t* < 0, 5 > ms^{-1} and $\mathbf{r}_S$ = < –3, 4> + *t* < *x*, *y* > ms^{-1} respectively.
    (a) Find in terms of *x* and *y*, the velocity of R relative to S.
    (b) Find the initial displacement of S relative to R.
    (c) Given that |< *x*, *y* >| = 5, and all velocities are maintained with the two boats eventually colliding, use your answers in (a) and (b) to find *x* and *y*.

14. The position vectors of hockey players A and B at time *t* seconds are given by $\mathbf{r}_A$ = < 0, 3 > + *t* < *x*, *y* > m/s and $\mathbf{r}_B$ = < 3, 8 > + *t* < 1, –3 > m/s respectively.
    Given that |< *x*, *y* >| = √20, and all velocities are maintained with the two players eventually colliding, use a relative velocity method to determine *x* and *y*.

15. At 10 am, sailboats A and B set sail from their respective pens located at < –10, –50> km and < 10, 50 > km respectively. Sailboat A sails with constant velocity < 4, 5 > km/h. Given that sailboat B has a maximum speed of 15 km/h, use a relative velocity method to determine the velocity of B if B meets up with A in minimum time. State where and when this occurs.

16. At 0930 hours, a solo yachtsman is spotted at location < 100, 20 > nautical miles drifting with velocity < 1, 2 > knots. At 1015 hours, a rescue craft with a maximum speed of 50 knots sets sail from < 30, 10 > to give assistance to the yachtsman. Use a relative velocity method to determine the minimum time the rescue craft would take to reach the yachtsman.

09 Scalar Product

9.1 Scalar Product (Dot Product)

- Consider the vectors $a\,\boldsymbol{i} + b\,\boldsymbol{j}$ and $p\,\boldsymbol{i} + q\,\boldsymbol{j}$.
- The scalar product between these two vectors is:

$$(a\,\boldsymbol{i} + b\,\boldsymbol{j}) \cdot (p\,\boldsymbol{i} + q\,\boldsymbol{j}) = a\,p + b\,q\,.$$

- That is $\langle a, b \rangle \cdot \langle p, q \rangle = a\,p + b\,q$.

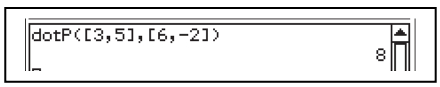
  Alternatively, $\begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} p \\ q \end{pmatrix} = a\,p + b\,q$.

- Note that the scalar product between two vectors is a scalar (a quantity with magnitude but with no direction).

Example 9.1

Given $\boldsymbol{u} = 3\boldsymbol{i} + 5\boldsymbol{j}$ and $\boldsymbol{v} = 6\boldsymbol{i} - 2\boldsymbol{j}$, find (a) $\boldsymbol{u} \cdot \boldsymbol{v}$ (b) $5\boldsymbol{u} \cdot 6\boldsymbol{v}$.

Solution:



(a) $\boldsymbol{u} \cdot \boldsymbol{v} = (3\boldsymbol{i} + 5\boldsymbol{j}) \cdot (6\boldsymbol{i} - 2\boldsymbol{j})$
$= 18 + (-10) = 8$

(b) $5\boldsymbol{u} \cdot 6\boldsymbol{v} = 5(3\boldsymbol{i} + 5\boldsymbol{j}) \cdot 6(6\boldsymbol{i} - 2\boldsymbol{j})$
$= (15\boldsymbol{i} + 25\boldsymbol{j}) \cdot (36\boldsymbol{i} - 12\boldsymbol{j})$
$= 540 + (-300) = 240$

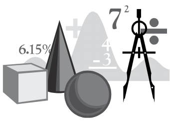
Example 9.2

Given $\boldsymbol{u} = -2\boldsymbol{i} + \alpha\boldsymbol{j}$ and $\boldsymbol{v} = 3\boldsymbol{i} + 4\boldsymbol{j}$, find α if $\boldsymbol{u} \cdot \boldsymbol{v} = 10$.

Solution:

$\langle -2, \alpha \rangle \cdot \langle 3, 4 \rangle = 10$
$\Rightarrow \quad -6 + 4\alpha = 10 \;\Rightarrow\; \alpha = 4$

9.1.1 Properties of the Scalar Product



Hands On Task 9.1

In this task, we will explore the algebraic properties of the scalar product.

Let $\boldsymbol{u} = a\,\boldsymbol{i} + b\,\boldsymbol{j}$, $\boldsymbol{v} = c\,\boldsymbol{i} + d\,\boldsymbol{j}$ and $\boldsymbol{w} = m\,\boldsymbol{i} + n\,\boldsymbol{j}$. Let λ and μ be constants.

1. Find $\boldsymbol{u} \bullet \boldsymbol{u}$. Hence, show that $\boldsymbol{u} \bullet \boldsymbol{u} = |\boldsymbol{u}|^2$.
2. Find $\boldsymbol{u} \bullet \boldsymbol{v}$ and $\boldsymbol{v} \bullet \boldsymbol{u}$. Hence, show that $\boldsymbol{u} \bullet \boldsymbol{v} = \boldsymbol{v} \bullet \boldsymbol{u}$.
3. Find $\lambda\boldsymbol{u} \bullet \mu\boldsymbol{v}$ and $\lambda\mu(\boldsymbol{v} \bullet \boldsymbol{u})$. Hence, show that $\lambda\boldsymbol{u} \bullet \mu\boldsymbol{v} = \lambda\mu(\boldsymbol{v} \bullet \boldsymbol{u})$.
4. Find $(\boldsymbol{u} + \boldsymbol{v}) \bullet \boldsymbol{w}$ and $\boldsymbol{u} \bullet \boldsymbol{w} + \boldsymbol{v} \bullet \boldsymbol{w}$. Hence, show that $(\boldsymbol{u} + \boldsymbol{v}) \bullet \boldsymbol{w} = \boldsymbol{u} \bullet \boldsymbol{w} + \boldsymbol{v} \bullet \boldsymbol{w}$.
5. Find $\boldsymbol{w} \bullet (\boldsymbol{u} + \boldsymbol{v})$ and $\boldsymbol{w} \bullet \boldsymbol{u} + \boldsymbol{w} \bullet \boldsymbol{v}$. Hence, show that $\boldsymbol{w} \bullet (\boldsymbol{u} + \boldsymbol{v}) = \boldsymbol{w} \bullet \boldsymbol{u} + \boldsymbol{w} \bullet \boldsymbol{v}$.
6. For vectors $\boldsymbol{e}$, $\boldsymbol{f}$, $\boldsymbol{g}$ and $\boldsymbol{h}$, show that $(\boldsymbol{e} + \boldsymbol{f}) \bullet (\boldsymbol{g} + \boldsymbol{h}) = \boldsymbol{e} \bullet \boldsymbol{g} + \boldsymbol{e} \bullet \boldsymbol{h} + \boldsymbol{f} \bullet \boldsymbol{g} + \boldsymbol{f} \bullet \boldsymbol{h}$

Summary

- $\boldsymbol{u} \bullet \boldsymbol{u} = |\boldsymbol{u}|^2$
- $\boldsymbol{u} \bullet \boldsymbol{v} = \boldsymbol{v} \bullet \boldsymbol{u}$
- $\lambda\boldsymbol{u} \bullet \mu\boldsymbol{v} = \lambda\mu(\boldsymbol{v} \bullet \boldsymbol{u})$
- $(\boldsymbol{u} + \boldsymbol{v}) \bullet \boldsymbol{w} = \boldsymbol{u} \bullet \boldsymbol{w} + \boldsymbol{v} \bullet \boldsymbol{w}$
- $\boldsymbol{w} \bullet (\boldsymbol{u} + \boldsymbol{v}) = \boldsymbol{w} \bullet \boldsymbol{u} + \boldsymbol{w} \bullet \boldsymbol{v}$
- $(\boldsymbol{e} + \boldsymbol{f}) \bullet (\boldsymbol{g} + \boldsymbol{h}) = \boldsymbol{e} \bullet \boldsymbol{g} + \boldsymbol{e} \bullet \boldsymbol{h} + \boldsymbol{f} \bullet \boldsymbol{g} + \boldsymbol{f} \bullet \boldsymbol{h}$

Exercise 9.1

1. Given $\boldsymbol{u} = 3\boldsymbol{i} + 5\boldsymbol{j}$, $\boldsymbol{v} = 6\boldsymbol{i} - 2\boldsymbol{j}$ and $\boldsymbol{w} = -3\boldsymbol{i} + 4\boldsymbol{j}$ find:
   (a) $\boldsymbol{u} \bullet \boldsymbol{v}$ (b) $\boldsymbol{u} \bullet \boldsymbol{w}$ (c) $(\boldsymbol{v} + \boldsymbol{w}) \bullet \boldsymbol{u}$ (d) $\boldsymbol{u} \bullet (\boldsymbol{v} + \boldsymbol{w})$ (e) $(\boldsymbol{w} \bullet \boldsymbol{w})\,\boldsymbol{w}$

2. Given $\boldsymbol{a} = \langle 10, 3 \rangle$, $\boldsymbol{b} = \langle -4, 0 \rangle$ and $\boldsymbol{c} = \langle 3, 9 \rangle$ find:
   (a) $\boldsymbol{a} \bullet \boldsymbol{b}$ (b) $\boldsymbol{b} \bullet \boldsymbol{c}$ (c) $(\boldsymbol{a} - \boldsymbol{c}) \bullet \boldsymbol{b}$ (d) $\boldsymbol{b} \bullet (\boldsymbol{c} - \boldsymbol{a})$ (e) $(\boldsymbol{a} \bullet \boldsymbol{b})\,\boldsymbol{c}$

3. Given $\boldsymbol{a} = \langle 2, -2 \rangle$, $\boldsymbol{b} = \langle -1, 1 \rangle$, $\boldsymbol{c} = \langle 2, 1 \rangle$ and $\boldsymbol{d} = \langle 4, 4 \rangle$, find:
   (a) $(\boldsymbol{a} + \boldsymbol{b}) \bullet (\boldsymbol{b} + \boldsymbol{a})$ (b) $(\boldsymbol{a} + \boldsymbol{b}) \bullet (\boldsymbol{c} + \boldsymbol{d})$ (c) $(\boldsymbol{a} - \boldsymbol{c}) \bullet (\boldsymbol{b} - \boldsymbol{d})$ (d) $(\boldsymbol{a} + \boldsymbol{b}) \bullet (\boldsymbol{c} - \boldsymbol{d})$

4. Given $\boldsymbol{u} = \langle 6, -5 \rangle$, $\boldsymbol{v} = \langle -2, 3 \rangle$ and $\boldsymbol{w} = \langle 0, -5 \rangle$ find:
   (a) $\boldsymbol{u} \bullet \boldsymbol{v}$ (b) $-2\boldsymbol{u} \bullet 3\boldsymbol{v}$ (c) $(2\boldsymbol{u} - \boldsymbol{w}) \bullet 3\boldsymbol{v}$ (d) $3\boldsymbol{v} \bullet (\boldsymbol{w} - 2\boldsymbol{u})$

5. Prove that $(\boldsymbol{a} + \boldsymbol{b}) \bullet (\boldsymbol{a} - \boldsymbol{b}) = |\boldsymbol{a}|^2 - |\boldsymbol{b}|^2$.

6. Given $\boldsymbol{a} = 5\boldsymbol{i} + \alpha\boldsymbol{j}$ and $\boldsymbol{b} = 6\boldsymbol{i} + 3\boldsymbol{j}$, find α if $\boldsymbol{u} \bullet \boldsymbol{v} = -20$.

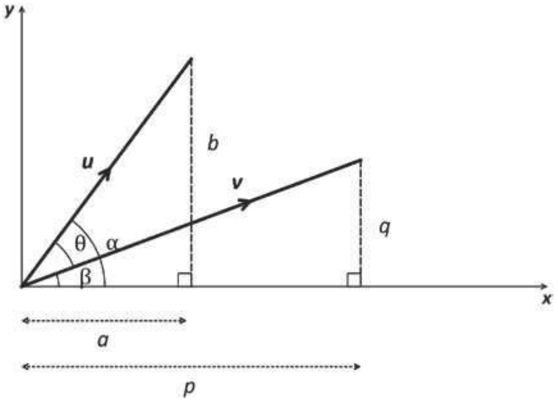
7. Given $\boldsymbol{u} = \langle \alpha, \alpha \rangle$ and $\boldsymbol{v} = \langle -1, \alpha \rangle$, find α if $\boldsymbol{u} \bullet \boldsymbol{v} = 6$.

8. Given $\langle \alpha, 1 \rangle \bullet \langle 1, \beta \rangle = 9$ and $\langle \alpha, 1 \rangle \bullet \langle \beta, 1 \rangle = 21$, find α and β.

9. Given $\boldsymbol{a} = \langle 2, 1 \rangle$, find $\boldsymbol{b}$ if $\boldsymbol{a} \bullet \boldsymbol{b} = 0$ and $\boldsymbol{b} \bullet \boldsymbol{b} = 5$.

10. Given $\boldsymbol{a} = \langle -1, 3 \rangle$, find $\boldsymbol{b}$ if $\boldsymbol{a} \bullet \boldsymbol{b} = 0$ and $\boldsymbol{b} \bullet \boldsymbol{b} = 40$.

9.2 Geometric Interpretation of the Scalar Product



- Consider two vectors ***u*** and ***v*** inclined at an angle of θ to each other.
  Let the angle between ***u*** and ***v*** with the ***i*** vector be α and β respectively.
  Also, let $\boldsymbol{u} = a\,\boldsymbol{i} + b\,\boldsymbol{j}$ and $\boldsymbol{v} = p\,\boldsymbol{i} + q\,\boldsymbol{j}$.

- Hence, $|\boldsymbol{u}| = \sqrt{a^2+b^2}$, $|\boldsymbol{v}| = \sqrt{p^2+q^2}$ and $\boldsymbol{u} \cdot \boldsymbol{v} = ap + bq$.

- From the diagram above,
  $\cos\alpha = \dfrac{a}{\sqrt{a^2+b^2}}$, $\sin\alpha = \dfrac{b}{\sqrt{a^2+b^2}}$, $\cos\beta = \dfrac{p}{\sqrt{p^2+q^2}}$ and $\sin\beta = \dfrac{q}{\sqrt{p^2+q^2}}$

- Clearly $\theta = \alpha - \beta$. Also $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$.

  Hence:
  $$\begin{aligned}\cos\theta &= \frac{a}{\sqrt{a^2+b^2}} \times \frac{p}{\sqrt{p^2+q^2}} + \frac{b}{\sqrt{a^2+b^2}} \times \frac{q}{\sqrt{p^2+q^2}} \\ &= \frac{ap+bq}{\sqrt{a^2+b^2}\sqrt{p^2+q^2}} \\ &= \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{|\boldsymbol{u}| \times |\boldsymbol{v}|}\end{aligned}$$

- Therefore,

  $$\boxed{\boldsymbol{u} \cdot \boldsymbol{v} = |\boldsymbol{u}| \times |\boldsymbol{v}| \cos\theta}$$

  Scalar Product between 2 vectors → $\boldsymbol{u} \cdot \boldsymbol{v}$
  Magnitude of first vector → $|\boldsymbol{u}|$
  Magnitude of second vector → $|\boldsymbol{v}|$
  Angle between the 2 vectors → θ

- The scalar product can hence be used to determine the angle between two vectors.

- Clearly, if $\theta = 90^\circ$, then $\boldsymbol{u} \cdot \boldsymbol{v} = 0$.
  That is,

  $$\boxed{\text{if } \boldsymbol{u} \text{ and } \boldsymbol{v} \text{ are perpendicular} \Leftrightarrow \boldsymbol{u} \cdot \boldsymbol{v} = 0.}$$

- Note that if ***u*** and ***v*** are parallel, then $\boldsymbol{u} = \lambda\,\boldsymbol{v}$.

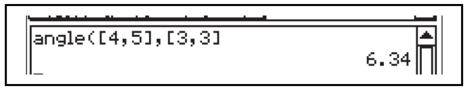
Example 9.3

Find the acute angle between the vectors < 4, 5 > and < 3, 3 >.

Solution:

$< 4, 5 > \cdot < 3, 3 > = 27$, $|< 4, 5 >| = \sqrt{41}$ and $|< 3, 3 >| = 3\sqrt{2}$

Hence, $\cos\theta = \dfrac{27}{(\sqrt{41})(3\sqrt{2})} \Rightarrow \theta = 6.3^{\circ}$.



Example 9.4

Without the use of a calculator, determine if the following pairs of vectors are perpendicular, parallel in the same direction, parallel in opposing direction or otherwise.
(a) < 4, 5 > and < –5, 4 > (b) < 2, 5 > and <4, 10 > (c) < –3, 6 > and < 6, –12 >.

Solution:

(a) $< 4, 5 > \cdot < -5, 4 > = -20 + 20 = 0$. $\Rightarrow$ The two given vectors are perpendicular.

(b) $< 4, 10 > = 2 \times < 2, 5 >$. $\Rightarrow$ The two vectors are parallel in the same direction.

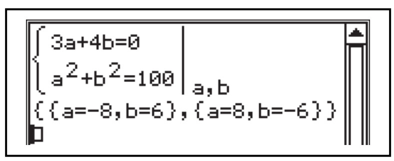
(c) $< 6, -12 > = -2 \times < -3, 6 >$. $\Rightarrow$ The two vectors are parallel in opposing directions.

Example 9.5

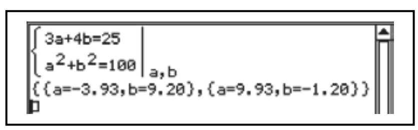
Find a vector of magnitude 10 and :
(a) perpendicular to < 3, 4 > (b) inclined at an angle of 60° with < 3, 4 >

Solution:

(a) Let required vector be < *a*, *b* >.
Then, $< a, b > \cdot < 3, 4 > = 0 \Rightarrow 3a + 4b = 0$
and $|< a, b >| = 10 \Rightarrow a^2 + b^2 = 100$.
Hence, $a = -8$, $b = 6$ or $a = 8$, $b = -6$.
Therefore, required vector is < –8, 6 > or < 8, –6 > .



(b) Let required vector be < *a*, *b* >.
Then, $< a, b > \cdot < 3, 4 > = |< a, b >| \times |< 3, 4 >| \times \cos 60^{\circ}$
$\Rightarrow 3a + 4b = 10 \times 5 \times 0.5 = 25$
and $|< a, b >| = 10 \Rightarrow a^2 + b^2 = 100$.
Hence, $a = -3.93$, $b = 9.20$ or $a = 9.93$, $b = -1.20$.
$\Rightarrow$ Required vector is < –3.93, 9.20 > or <9.93, –1.20>.



Exercise 9.2

1. Use scalar products (where appropriate) to find the acute angle between the vectors:
   (a) < 0, 3 >, < 3, 0 > (b) < 4, 8 >, < –1, – 2 > (c) < 5, 10 >, < 3, –6 >
   (d) < 1, 1>, < 2, 5 > (e) < 5, –1 >, < 1, 2 > (f) < 2, 3 >, < –2, 1 >

2. Determine if the following pairs of vectors are perpendicular, parallel in the same direction, parallel in opposing direction or otherwise.
   (a) < –2, –7 >, < –7, 2 > (b) < 3, 1 >, < –6, –2 > (c) < 5, –4 >, < –4, 5 >.

3. Use scalar products (where appropriate) to find a vector:
   (a) of magnitude 5 and perpendicular to < 0, –2 >
   (b) of magnitude 100 and perpendicular to < 5, 10 >
   (c) of magnitude $\sqrt{10}$ and perpendicular to < –4, –8 >
   (d) of magnitude $\sqrt{30}$ and perpendicular to < 6, 3 >.

4. Use scalar products (where appropriate) to find a vector:
   (a) of magnitude 10 inclined at an angle of 30° with < 3, –4 >
   (b) of magnitude 5 inclined at an angle of 45° with < –6, 2 >
   (c) of magnitude 10 inclined at an angle of 60° with < 1, 5 >
   (d) of magnitude 1 inclined at an angle of $\frac{3\pi}{4}$ with < –10, 10 >.

5. Let ***u*** = < 4, a > and ***v*** = < b, 5 >. Find a and b if $|\boldsymbol{u}| = |\boldsymbol{v}|$ and ***u*** and ***v*** are parallel.

6. Let ***u*** = < a, 10 > and ***v*** = < 40, b >. Find a and b if $|\boldsymbol{v}| = 4|\boldsymbol{u}|$ and ***u*** and ***v*** are:
   (a) parallel (b) perpendicular.

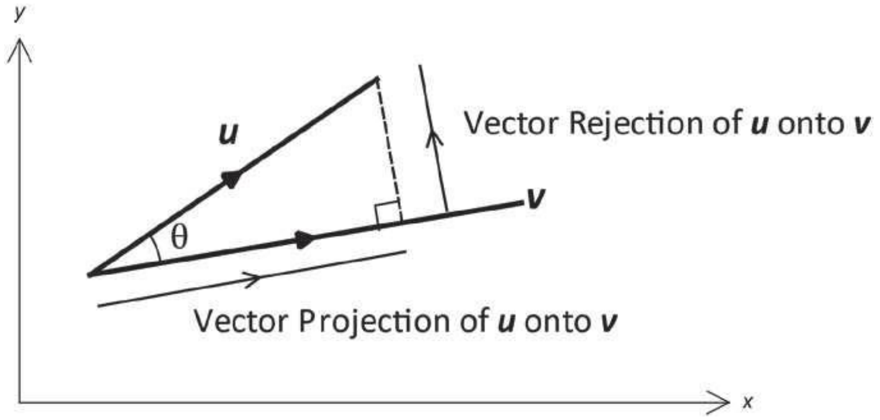
*7. Use a method involving scalar products to find a and b if the acute angle between < 2, a > and < b , 0 > is 60°.

8. The points A, B and C have position vectors < 3, 4 >, < 6, 1 > and < 5, –1 > respectively.
   (a) Use a method involving a scalar product to find $\angle$ ABC.
   (b) Hence, find the area of ΔABC.

9. The points A, B, C and D have position vectors < –1, –5 >, < 0, 5 >, < 1, 0 > and < 2, –10 > respectively.
   (a) Use a method involving scalar products to find $\angle$ ABC and $\angle$ ADC.
   (b) Hence, find the area of the quadrilateral ABCD.

10. Use a method involving scalar products to prove that the triangle with vertices having position vectors < –6, –5 >, < 4, 5 > and < 6, 3 > respectively is a right angled triangle.

9.3 Vector Projection



- The *vector projection* of $\boldsymbol{u}$ onto $\boldsymbol{v}$ is the vector component of $\boldsymbol{u}$ that is parallel to $\boldsymbol{v}$.
- Consider the case where θ the angle between vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is acute.
- Using simple right triangle trigonometry, the *scalar projection* of $\boldsymbol{u}$ onto $\boldsymbol{v}$ is given by $|\boldsymbol{u}|\cos\theta$.
- Since $\cos\theta = \dfrac{\boldsymbol{u} \cdot \boldsymbol{v}}{|\boldsymbol{u}| \times |\boldsymbol{v}|}$,

  the scalar projection of $\boldsymbol{u}$ onto $\boldsymbol{v}$ = $|\boldsymbol{u}| \times \dfrac{\boldsymbol{u} \cdot \boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|} = \boldsymbol{u} \cdot \hat{\boldsymbol{v}}$.
- Hence, the vector projection of $\boldsymbol{u}$ onto $\boldsymbol{v}$, $\text{proj}_{\boldsymbol{v}}\,\boldsymbol{u} = (|\boldsymbol{u}|\cos\theta)\ \hat{\boldsymbol{v}} = (\boldsymbol{u} \cdot \hat{\boldsymbol{v}})\,\hat{\boldsymbol{v}}$.
- In summary, the vector projection of $\boldsymbol{u}$ onto $\boldsymbol{v}$,

$$\boxed{\text{proj}_{\boldsymbol{v}}\,\boldsymbol{u} = (\boldsymbol{u} \cdot \hat{\boldsymbol{v}})\,\hat{\boldsymbol{v}}}$$

Scalar Projection | *Unit vector in direction of* $\boldsymbol{v}$

- The component of vector $\boldsymbol{u}$ that is perpendicular to $\boldsymbol{v}$ is known as the vector rejection of $\boldsymbol{u}$ onto $\boldsymbol{v}$.
  Using the resultant rule for vectors, the rejection of $\boldsymbol{u}$ onto $\boldsymbol{v}$,

$$\begin{aligned}\text{rej}_{\boldsymbol{v}}\,\boldsymbol{u} &= \boldsymbol{u} - \text{proj}_{\boldsymbol{v}}\,\boldsymbol{u} \\ &= \boldsymbol{u} - (\boldsymbol{u} \cdot \hat{\boldsymbol{v}})\,\hat{\boldsymbol{v}}\end{aligned}$$

- If θ, the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$ is obtuse, then the scalar projection will be negative and consequently the vector projection of $\boldsymbol{u}$ onto $\boldsymbol{v}$ will point in the direction opposing $\boldsymbol{v}$.

Example 9.6

The vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ have magnitudes 10 and 8 respectively. The angle between $\boldsymbol{u}$ and $\boldsymbol{v}$ is 60°. Without the use of a calculator, find in terms of the vectors $\boldsymbol{u}$ and/or $\boldsymbol{v}$:

(a) the scalar projection of $\boldsymbol{u}$ onto $\boldsymbol{v}$. (b) the vector projection of $\boldsymbol{u}$ onto $\boldsymbol{v}$.

(c) the vector rejection of $\boldsymbol{u}$ onto $\boldsymbol{v}$.

Solution:

(a) Scalar projection of $\boldsymbol{u}$ onto $\boldsymbol{v}$ = $10 \times \cos 60 = 5$

(b) $\text{proj}_{\boldsymbol{v}}\, \boldsymbol{u} = 5\hat{\boldsymbol{v}} = 5 \times \frac{1}{8}\boldsymbol{v} = \frac{5}{8}\boldsymbol{v}$.

(c) $\text{rej}_{\boldsymbol{v}}\, \boldsymbol{u} = \boldsymbol{u} - \frac{5}{8}\boldsymbol{v}$.

Example 9.7

Given $\boldsymbol{u} = \langle 3, 2 \rangle$ and $\boldsymbol{v} = \langle 5, 12 \rangle$.

(a) Find the scalar projection of $\boldsymbol{u}$ onto $\boldsymbol{v}$. (b) Find the vector projection of $\boldsymbol{u}$ onto $\boldsymbol{v}$.

(c) Find the vector rejection of $\boldsymbol{u}$ onto $\boldsymbol{v}$.

Solution:

(a) Since, $\boldsymbol{v} = \langle 5, 12 \rangle$, $\hat{\boldsymbol{v}} = \frac{1}{13} \langle 5, 12 \rangle$.

$\Rightarrow$ scalar projection of $\boldsymbol{u}$ onto $\boldsymbol{v}$ = $abs(\boldsymbol{u} \bullet \hat{\boldsymbol{v}}) = abs(\langle 3, 2 \rangle \bullet \frac{1}{13} \langle 5, 12 \rangle) = 3$

(b) $\text{proj}_{\boldsymbol{v}}\, \boldsymbol{u} = (\boldsymbol{u} \bullet \hat{\boldsymbol{v}})\hat{\boldsymbol{v}} = 3 \times \frac{1}{13} \langle 5, 12 \rangle = \frac{3}{13} \langle 5, 12 \rangle$.

(c) $\text{rej}_{\boldsymbol{v}}\, \boldsymbol{u} = \langle 3, 2 \rangle - \frac{3}{13} \langle 5, 12 \rangle = \frac{2}{13} \langle 12, -5 \rangle$.

Alternative Solution:

The vector $\langle 12, -5 \rangle$ is perpendicular to $\boldsymbol{v} = \langle 5, 12 \rangle$.
Hence, the components parallel and perpendicular to $\boldsymbol{v} = \langle 12, 5 \rangle$ must be of the form $\lambda \langle 5, 12 \rangle$ and $\mu \langle 12, -5 \rangle$.

Therefore, $\boldsymbol{u} = \langle 3, 2 \rangle = \lambda \langle 5, 12 \rangle + \mu \langle 12, -5 \rangle$.

$\Rightarrow$ $5\lambda + 12\mu = 3$

$12\lambda - 5\mu = 2$

Solving simultaneously: $\lambda = \frac{3}{13}$ and $\mu = \frac{2}{13}$

(b) $\text{proj}_{\boldsymbol{v}}\, \boldsymbol{u} = \frac{3}{13} \langle 5, 12 \rangle$.

(c) $\text{rej}_{\boldsymbol{v}}\, \boldsymbol{u} = \frac{2}{13} \langle 12, -5 \rangle$.

Exercise 9.3

1. The vectors ***u*** and ***v*** have magnitudes 20 and 10 respectively. The angle between ***u*** and ***v*** is 60°. Without the use of a calculator, find in terms of the vectors ***u*** and/or ***v***:
   (a) the scalar projection of ***u*** onto ***v***. (b) the vector projection of ***u*** onto ***v***.
   (c) the vector rejection of ***u*** onto ***v***.

2. The vectors ***u*** and ***v*** have magnitudes 12 and 20 respectively. The angle between ***u*** and ***v*** is 120°. Without the use of a calculator, find in terms of the vectors ***u*** and/or ***v***:
   (a) the scalar projection of ***v*** onto ***u***. (b) the vector projection of ***v*** onto ***u***.
   (c) the vector rejection of ***v*** onto ***u***.

3. The vectors ***u*** and ***v*** have magnitudes $\sqrt{12}$ and 4 respectively. The angle between ***u*** and ***v*** is 30°. Without the use of a calculator, find in terms of the vectors ***u*** and/or ***v***:
   (a) the vector projection of ***u*** onto ***v***. (b) the vector rejection of ***u*** onto ***v***.

4. The vectors ***u*** and ***v*** have magnitudes $\sqrt{12}$ and 24 respectively. The angle between ***u*** and ***v*** is 150°. Without the use of a calculator, find in terms of the vectors ***u*** and/or ***v***:
   (a) the vector projection of ***u*** onto ***v***. (b) the vector rejection of ***u*** onto ***v***.

5. Given ***u*** = < 1, 1 > and ***v*** = < 1, 2 >. Without the use of a calculator, find the vector projection of: (a) ***u*** onto ***v*** (b) ***v*** onto ***u***

6. Given ***u*** = < 1, 2 > and ***v*** = < –1, –4 >. Without the use of a calculator, find the vector projection of: (a) ***u*** onto ***v*** (b) ***v*** onto ***u***

7. Given ***u*** = < 2, 1 > and ***v*** = < –1, 1 >. Without the use of a calculator, find the vector rejection of: (a) ***u*** onto ***v*** (b) ***v*** onto ***u***

8. Given ***u*** = < –4, –3 > and ***v*** = < –1, 3 >. Without the use of a calculator, find the vector rejection of: (a) ***u*** onto ***v*** (b) ***v*** onto ***u***

9. Find the vector component of ***u*** (i) parallel to ***v*** (ii) perpendicular to ***v***.
   (a) ***u*** = < 1, 0 > and ***v*** = < 1, 1 > (b) ***u*** = < 1, 0 > and ***v*** = < 3, 4 >.
   (c) ***u*** = < 1, 1 > and ***v*** = < 3, 4 > (d) ***u*** = < 2, 1 > and ***v*** = < 1, 2 >.

10. Given that ***u*** = < 1, 2 > + < –2, 1 >, find the vector projection and rejection of ***u*** onto:
    (a) < 2, 4 > (b) < –10, 5 >

11. Given that ***u*** = < 5, –2 > + < 4, 10 >, find the vector projection and rejection of ***u*** onto:
    (a) < 2, 5 > (b) < –20, 8 >

12. Given that the vector projection and rejection of ***u*** onto ***v*** are < 3, 5 > and < –15, 9 > respectively, find the cosine of the angle between ***u*** and ***v***.

13. Given that the vector projection and rejection of ***u*** onto ***v*** are < 10, –20 > and < –2, –1 > respectively, find the cosine of the acute between ***u*** and ***v***.

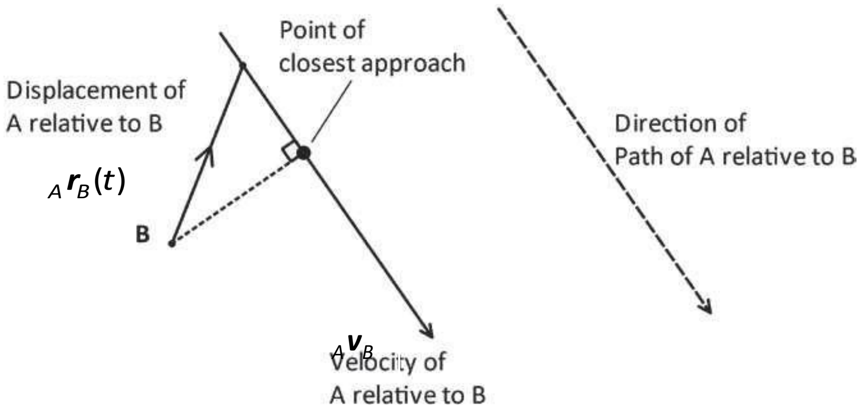
14. The vector components of ***u*** parallel and perpendicular to ***v*** are < 1, 2 > and < 2, –1 > respectively. Find the cosine of the angle between ***u*** and ***v***.

10 Vector Applications III

10.1 Closest Distance

- In this section, the ideas of relative vectors and scalar products are combined to determine the condition for the closest approach between two objects.

10.1.1 Closest distance between two moving objects



- Consider objects A and B moving with constant velocities $\boldsymbol{v}_A$ and $\boldsymbol{v}_B$ respectively. Let the closest approach occur at time t.
  The position vectors of A and B at time t are $\boldsymbol{r}_A(t)$ and $\boldsymbol{r}_B(t)$ respectively.
  The position vector of A relative to B at time t is $_A\boldsymbol{r}_B(t)$.
- Consider the motion of A relative to B.
  The velocity of A relative to B is $_A\boldsymbol{v}_B$.
- hat is, relative to B, at time t, A is $_A\boldsymbol{r}_B(t)$ away, travelling with velocity $_A\boldsymbol{v}_B$.
  Hence, if A and B are closest at time t, $_A\boldsymbol{r}_B(t) \cdot {}_A\boldsymbol{v}_B = 0$
- In summary, A and B are closest together when

$$_A\boldsymbol{r}_B(t) \cdot {}_A\boldsymbol{v}_B = 0$$

relative displacement of A to B at time of closest approach ($_A\boldsymbol{r}_B(t)$)

velocity of A relative to B ($_A\boldsymbol{v}_B$)

Example 10.1

An object P starts moving from < 10, 20 > m with constant velocity < 5, –5 > ms^{-1}. Object Q is stationary and located at < 30, –10 >. Use a scalar product method to find the closest distance between P and Q. State where and when this occurs.

Solution:

Let P be closest to Q at time t.

Position vector of P at time t is $\boldsymbol{r}_P(t) = <10, 20> + t<5, -5> = <10+5t, 20-5t>$.

Relative displacement between Q and P at time t,

$${}_P\boldsymbol{r}_Q(t) = <10+5t, 20-5t> - <30, -10>.$$
$$= <-20+5t, 30-5t>.$$

At closest approach, ${}_P\boldsymbol{r}_Q(t) \cdot {}_P\boldsymbol{v}_Q = 0$.

$\Rightarrow <-20+5t, 30-5t> \cdot <5, -5> = 0$

$$250 - 50t = 0 \Rightarrow t = 5$$

Closest distance $= |{}_P\boldsymbol{r}_Q(5)|$

$= |<-20+5(5), 30-5(5)>|$

$= \sqrt{5^2 + (-5)^2} = 5\sqrt{2}$ m.

Hence, closest distance between P and Q is $5\sqrt{2}$ m.

This occurs 5 seconds after P starts moving

when P is at < 10 + 5(5), 20 – 5(5) > = < 35, –5 > m.

Example 10.2

Objects P and Q respectively, start moving from points with position vectors < –5, –15 > m and < 10, 20 > m with constant velocities < 2, 4 > ms^{-1} and < 5, –5 > ms^{-1}. Use a scalar product method to find the closest distance between P and Q. State where and when this occurs.

Solution:

Let P be closest to Q at time t.

Position vector of P at time t is $\boldsymbol{r}_P(t) = <-5, -15> + t<2, 4> = <-5+2t, -15+4t>$.

Position vector of Q at time t is $\boldsymbol{r}_Q(t) = <10, 20> + t<5, -5> = <10+5t, 20-5t>$.

Displacement of P relative to Q at time t,

$${}_P\boldsymbol{r}_Q(t) = <-5+2t, -15+4t> - <10+5t, 20-5t> = <-15-3t, -35+9t>.$$

Velocity of P relative to Q, ${}_P\boldsymbol{v}_Q = <2, 4> - <5, -5> = <-3, 9>$

At closest approach, ${}_P\boldsymbol{r}_Q(t) \cdot {}_P\boldsymbol{v}_Q = 0$.

$\Rightarrow <-15-3t, -35+9t> \cdot <-3, 9> = 0$

$$-270 + 90t = 0 \Rightarrow t = 3.$$

Closest distance $= |{}_P\boldsymbol{r}_Q(3)| = |<-15-3(3), -35+9(3)>| = 8\sqrt{10}$ m.

This occurs 3 seconds after P and Q start moving

when P is at < –5 + 2(3), –15 + 4(3) > = < 1, –3> m

and Q is at < 10 + 5(3), 20 – 5(3) > = < 25, 5 > m.

Exercise 10.1

1. Find the shortest distance between the objects A and B as described below. State when the closest approach occurs.
   (a) $\boldsymbol{r}_A(0) = < 0, 0 >$ m, $\boldsymbol{v}_A = < 0, 0 >$ m/s; $\boldsymbol{r}_B(0) = < -10, -10 >$ m and $\boldsymbol{v}_B = < 1, 2 >$ m/s
   (b) $\boldsymbol{r}_A(0) = < 8, 9 >$ m, $\boldsymbol{v}_A = < 0, 0 >$ m/s; $\boldsymbol{r}_B(0) = < 0, 0 >$ m and $\boldsymbol{v}_B = < 2, 1 >$ m/s
   (c) $\boldsymbol{r}_A(0) = < -5, -10 >$ m, $\boldsymbol{v}_A = < 0, 0 >$ m/s; $\boldsymbol{r}_B(0) = < 20, 10 >$ m and $\boldsymbol{v}_B = < -2, 1 >$ m/s
   (d) $\boldsymbol{r}_A(0) = < 50, -80 >$ m, $\boldsymbol{v}_A = < -1, 2 >$ m/s; $\boldsymbol{r}_B(0) = < -10, -10 >$ m and $\boldsymbol{v}_B = < 0, 0 >$ m/s

2. Find the shortest distance between the objects A and B as described below. State when the closest approach occurs.
   (a) $\boldsymbol{r}_A(0) = < 0, 0 >$ m, $\boldsymbol{v}_A = < -5, 0 >$ m/s; $\boldsymbol{r}_B(0) = < -10, -10 >$ m and $\boldsymbol{v}_B = < 0, 2 >$ m/s
   (b) $\boldsymbol{r}_A(0) = < 0, 5 >$ m, $\boldsymbol{v}_A = < 2, 4 >$ m/s; $\boldsymbol{r}_B(0) = < 10, 5 >$ m and $\boldsymbol{v}_B = < -2, 1 >$ m/s
   (c) $\boldsymbol{r}_A(0) = < -10, -5 >$ m, $\boldsymbol{v}_A = < 1, 4 >$ m/s; $\boldsymbol{r}_B(0) = < -10, 10 >$ m and $\boldsymbol{v}_B = < 2, -2 >$ m/s
   (d) $\boldsymbol{r}_A(0) = < 20, 30 >$ m, $\boldsymbol{v}_A = < -1, -2 >$ m/s; $\boldsymbol{r}_B(0) = < -10, -10 >$ m and $\boldsymbol{v}_B = < 1, 3 >$ m/s

3. At 8 am, a yacht P starts sailing from $< 20, 10 >$ km with constant velocity $< 5, -5 >$ kmh^{-1}. A light house Q is located at $< 50, -30 >$. Use a scalar product method to find the closest P gets to the lighthouse. State when this occurs.

4. A hockey ball is hit from $< 10, -20 >$ m and travels with constant velocity $< -10, 5 >$ m/s for the first 5 seconds. A flagpole is located at $< 5, 5 >$. Use a scalar product method to determine the closest the ball gets to the flagpole. State where and when this occurs.

5. At 8 am, the position vectors of vessels A and B are $< 100, 150 >$ km and $< 50, 25 >$ km respectively. Vessels A and B travel with constant velocity $< 10, -5 >$ and $< -5, 20 >$ kmh^{-1} respectively. Use a scalar product method to determine the minimum distance between A and B. State where and when this occurs.

6. At 1 pm, object H travelling with constant velocity $< 200, 10 >$ kmh^{-1} is sighted at the point with position vector $< -90, -100 >$ km. At 2 pm object J travelling with constant velocity $< 100, -100 >$ kmh^{-1} is sighted at the point with position vector $< 20, -120 >$ km. Use a scalar product method to determine the minimum distance between H and J. State when this occurs.

*7. At 8 am, the position vectors of yachts A and B are $< 0, 0 >$ nautical miles and $\boldsymbol{b} = < x, y >$ nm respectively. Yachts A and B sail with constant velocity $< 10, 5 >$ and $< 0, 15 >$ knots respectively. Use a scalar product method to determine $\boldsymbol{b}$ given that the two yachts were closest together at a distance of $5\sqrt{2}$ nm at 10 am.

*8. At 10 am, object M travelling with constant velocity $< 20, 10 >$ knots is sighted at the point with position vector $< -100, -100 >$ nautical miles (nm). At 11 am object N travelling with constant velocity $\boldsymbol{v} = < x, y >$ knots is sighted at the point with position vector $< -20, -50 >$ nm respectively. Use a scalar product method to determine $\boldsymbol{v}$ given that the two objects were closest together at a distance of 40 nm at 4 pm

10.2 Vector Components/Projections

- The work done by a force of ***F*** Newtons through a distance of *s* metres is given by:
  Work done = Scalar Projection of ***F*** along direction of motion × *s* Joules.
- The work done by a force ***F*** Newtons in moving a body over a displacement ***d*** metres is given by: Work done = ***F* • *d*** Joules.

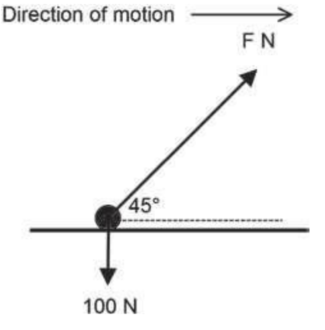
Example 10.3

The accompanying diagram shows an object of weight 100 Newtons being pulled along by a force of magnitude *F* Newtons inclined at an angle of 45^{o} to the surface. The motion of the object is opposed by a horizontal force of magnitude 20 N.

(a) Find *F*, the magnitude of the pulling force.

(b) Find the magnitude of the resultant force in the direction of motion.

(c) Find the work done by the resultant force in moving the object 10 m in its direction of motion.



Solution:

(a) Since the object is not moving in the vertical plane:

Component of F perpendicular to surface = Weight of object

$$F \sin 45 = 100$$

$$F = 100\sqrt{2} \text{ Newtons}$$

(b) Magnitude of Resultant Force in direction of motion

= Scalar projection of Pulling Force along surface − Opposing Force

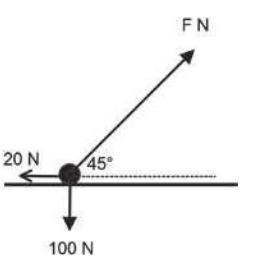
$= F \cos 45 - 20$

$= 100\sqrt{2} \times \frac{1}{\sqrt{2}} - 20 = 80$ Newtons

(c) Work done by resultant force = 80 × 10 = 800 Joules

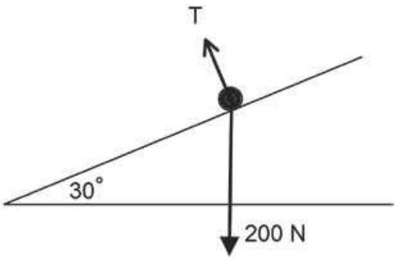
Notes:

- *The accompanying diagram shows the different forces acting on the object.*
- *The object is not moving vertically to the plane. Hence, the vector component of the pulling force (rejection vector) must be exactly balanced by the weight of the object.*
- *The object is moving "to the right". Hence, there must be a resultant force acting in this direction. The resultant force = Vector projection of pulling force in the direction of motion − Opposing force.*
- *The resultant force of 80 N is applied over a distance of 10 m. Hence the work done is 80 × 10 = 800 J.*



Example 10.4

The accompanying diagram shows a body of weight 200 Newtons rolling down a plane inclined at an angle of 30° with the horizontal. The body experiences a reaction force ***T*** perpendicular to the inclined plane. There is a force of magnitude 50 N opposing the motion of the body.



(a) Find magnitude of ***T***.
(b) Find the magnitude of the resultant force in the direction of the motion of the body.
(c) Find the work done by the resultant force when the body:
  (i) has moved 2 m along the inclined plane.
  (ii) has descended a vertical distance of 2 m.

Solution:

(a) Since the object is not moving perpendicular to the inclined plane:

$$\text{Component of weight perpendicular to plane} = |\boldsymbol{T}|$$
$$200 \cos 30 = |\boldsymbol{T}|$$
$$|\boldsymbol{T}| = 100\sqrt{3} \text{ Newtons}$$

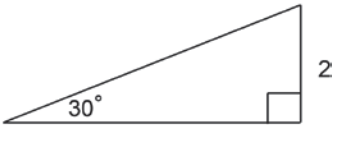
(b) Magnitude of Resultant force in direction of motion
= Scalar projection of Weight along plane − Opposing Force
= 200 sin 30 − 50
= 150 Newtons

(c) (i) Work done by resultant force = 150 × 2 = 300 Joules.

(ii) When the body has descended a vertical distance of 2 m,

$$\text{distance moved along the plane} = \frac{2}{\sin 30}$$
$$= 4 \text{ metres.}$$

Hence, work done = 150 × 4 = 600 Joules



Notes:

- *The force "causing" the body to roll down is the component of the gravitational force parallel to the inclined plane. This is the projection of the gravitational force along the inclined plane.*
- *As the body does not experience any motion perpendicular to the inclined plane, the component of the gravitational force perpendicular to the plane is exactly balanced by the reaction force.*
- *In part (d), when the body has descended a distance of 2 m, the distance travelled in the direction of resultant force is 4 metres.*

Example 10.5

A body undergoes a displacement of 2***i*** + ***j*** metres under the application of a constant force 3***i*** + 4***j*** Newtons.
(a) Find the scalar projection of the force onto the displacement vector.
(b) Find the vector projection of the force onto the displacement vector.
(c) Find the work done by the force.

Solution:

(a) Unit vector in the direction of the displacement vector = $\frac{1}{\sqrt{5}} < 2, 1 >$

Hence, scalar projection of force onto displacement vector

$$= \left| < 3, 4 > \bullet \frac{1}{\sqrt{5}} < 2, 1 > \right|$$

$$= \frac{10}{\sqrt{5}} = 2\sqrt{5}.$$

(b) Vector projection of force onto displacement vector $= 2\sqrt{5} \times \frac{1}{\sqrt{5}} < 2, 1 >$

$= 2 < 2, 1 >$ Newtons

(c) Work done = scalar projection of force onto displacement vector × distance moved

$$= 2\sqrt{5} \times |< 2, 1 >|$$

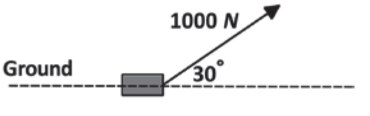
$$= 2\sqrt{5} \times \sqrt{5} = 10 \text{ Joules.}$$

Alternatively:

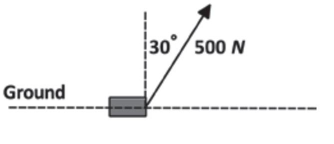
Work done = $< 3, 4 > \bullet < 2, 1 >$
= 10 Joules

Exercise 10.2

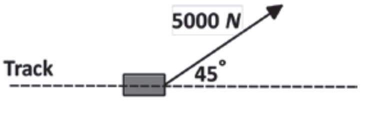
1. A 1000 N force is applied at an angle of 30^{o} to the ground to move an object a horizontal distance of 2 metres. Calculate the work done.



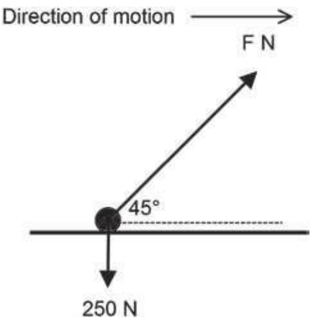
2. A 500 N force is applied at an angle of 30^{o} to the vertical to move an object a horizontal distance of 10 metres. Calculate the work done.



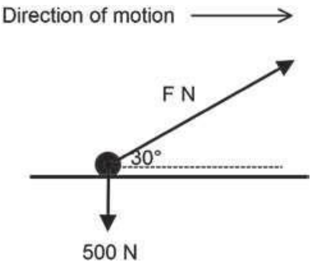
3. A 5000 N force is applied at an angle of 45^{o} to a straight rail track to move a cart 50 metres along the track. Calculate the work done in kJ.



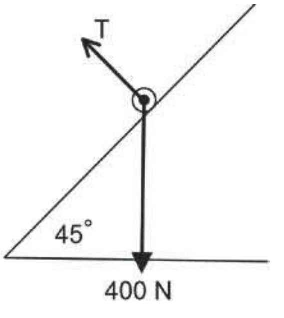
4. An object of weight 250 Newtons is being pulled along by a force of magnitude *F* Newtons inclined at an angle of 45^{o} to the surface. The motion of the object is opposed by a force of magnitude 50 N.
   (a) Find *F*, the magnitude of the pulling force.
   (b) Find the magnitude of the resultant force in the direction of motion.
   (c) Find the work done by the resultant force in (b) in moving the object 20 m in its direction of motion.



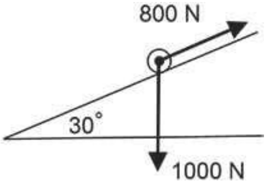
5. An object of weight 500 Newtons is being pulled along by a force of magnitude *F* Newtons inclined at an angle of 30^{o} to the surface. The motion of the object is opposed by a force of magnitude 100 N.
   (a) Find *F*, the magnitude of the pulling force.
   (b) Find the magnitude of the resultant force in the direction of motion.
   (c) Find the work done (kJ) by the resultant force in (b) in moving the object 100 m in its direction of motion.



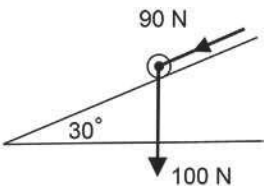
6. A body of weight 400 Newtons is rolling down a plane inclined at an angle of 45^{o} with the horizontal. The body experiences a reaction force ***T*** perpendicular to the inclined plane. There is a force of magnitude 50 N opposing the motion of the body.
   (a) Find the magnitude of ***T***.
   (b) Find the magnitude of the resultant force in the direction of the motion of the body.
   (c) Find the work done by the resultant force in (b) when the:
      (i) body has moved 10 m along the inclined plane.
      (ii) body has descended a vertical distance of 5 m.



7. A body of weight 1000 Newtons is pulled up along a plane inclined at an angle of 30^{o} with the horizontal by a force of magnitude 800 N.
   (a) Find the scalar projection of the gravitational force on the body onto the inclined plane.
   (b) Find the magnitude of the resultant force in the direction of the motion of the body.
   (c) Find the work done by the resultant force in (b) when the body:
      (i) has moved 10 m along the inclined plane.
      (ii) has ascended a vertical distance of 10 m.

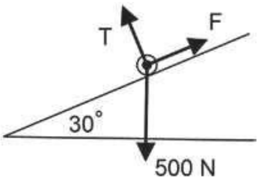


8. A body of weight 100 Newtons is pushed down a plane inclined at an angle of 30° with the horizontal by a force of magnitude 90 N. There is a force of magnitude 100 N opposing the motion.



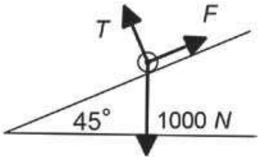
(a) Find the scalar projection of the gravitational force on the body onto the inclined plane.
(b) Find the magnitude of the resultant force in the direction of the motion of the body.
(c) Find the work done by the resultant force in (b) when the body has:
(i) moved 5 m along the inclined plane. (ii) descended a vertical distance of 5 m.

9. A body of weight 500 Newtons is at rest on a plane inclined at an angle of 30° with the horizontal. Frictional forces ***F*** just prevent the body from slipping down the plane. The body experiences a reaction force ***T*** perpendicular to the inclined plane. (a) Find the magnitude of ***T*** and ***F***.



(b) A force of magnitude 100 N parallel to the inclined plane is applied to the body and causes the body to move down along the inclined plane. Find the work done by the resultant force along the inclined plane when the body has:
(i) moved 10 m along the inclined plane. (ii) descended a vertical distance of 10 m.

10. A body of weight 1000 Newtons is at rest on a plane inclined at an angle of 45° with the horizontal. Frictional forces ***F*** just prevent the body from slipping down the plane. The body experiences a reaction force ***T*** perpendicular to the inclined plane. (a) Find the magnitude of ***T*** and ***F***.



(b) A force of magnitude 200 N parallel to the inclined plane is applied to the body and causes the body to move up along the inclined plane. Find the work done by the resultant force along the inclined plane when the body has:
(i) moved 20 m along the inclined plane (ii) ascended a vertical distance of 10 m.

11. A body undergoes a displacement of $3\boldsymbol{i} + 4\boldsymbol{j}$ m under the application of a constant force $12\boldsymbol{i} + 5\boldsymbol{j}$ N. (a) Find the scalar projection of the force onto the displacement vector.
(b) Find the vector projection of the force onto the displacement vector.
(c) Find the work done by the force.

12. A body undergoes a displacement of $-\boldsymbol{i} + 4\boldsymbol{j}$ m under the application of a constant force $-2\boldsymbol{i} + 2\boldsymbol{j}$ N. (a) Find the scalar projection of this force onto the displacement vector.
(b) Find the vector projection of this force onto the displacement vector.
(c) Find the work done by this force.

13. A body is moved 5 metres by a force $6\boldsymbol{i} + 2\boldsymbol{j}$ Newtons. The work done by this force is 30 Joules. Find the displacement vector for the body.

14. A body undergoes a displacement of $5\boldsymbol{i} - 6\boldsymbol{j}$ metres by a force of magnitude 10 N. The work done by the force is 78 Joules. Find the force acting on the body.

15. A body is moved $\sqrt{10}$ metres in the direction of $\boldsymbol{i} + 2\boldsymbol{j}$ by a force of magnitude 20 N. The work done by the force is $40\sqrt{2}$ Joules. Find the force acting on the body.

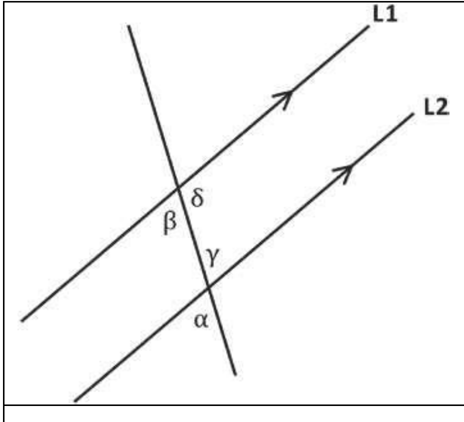
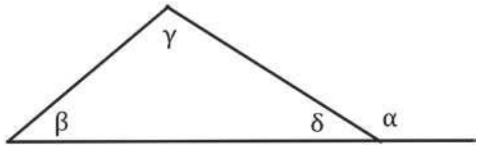
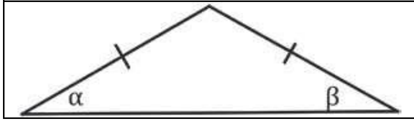
11 Geometric Proofs & Circle Properties

11.1 Geometric Proofs

- In this chapter, we will discuss geometric proofs involving similar and congruent triangles, properties of angles, parallel lines and circle properties.
- Within the context of geometric proofs we will begin to consider the terms used in formal mathematical proofs. A more formal treatment of mathematical proofs is covered in Chapter 26.

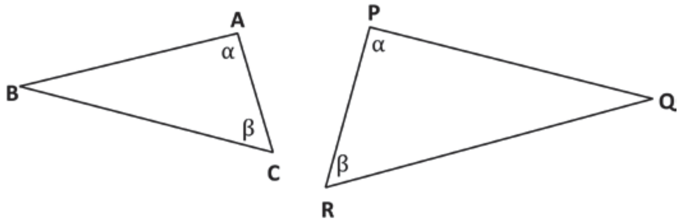
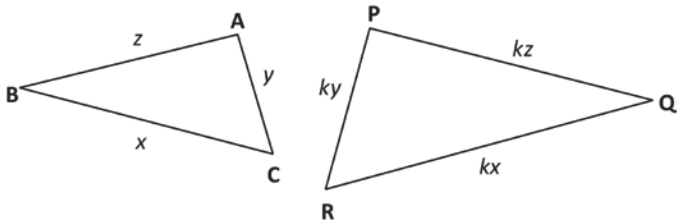
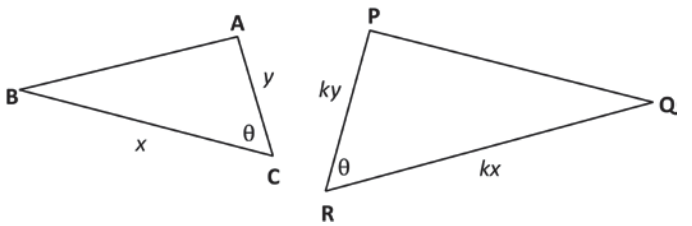
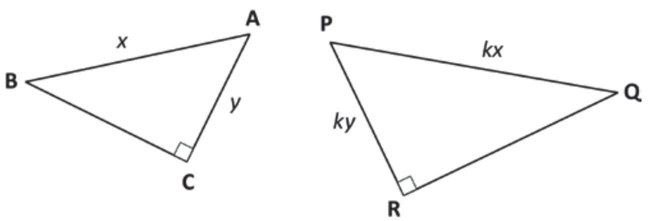
11.2 Properties of angles, parallel lines and triangles

- The table below lists the properties of angles, parallel lines and triangles

	$\alpha = \gamma$ Vertically Opposite angles. $\alpha = \beta$ Corresponding angles L1 // L2. $\beta = \gamma$ Alternate angles L1 // L2. $\delta + \gamma = 180^o$ Co-interior angles L1 // L2. $\beta = 180^o - \delta$ Supplementary angle.
	$\beta + \gamma + \delta = 180$ Sum of angles in a triangle is 180^o. $\alpha = \beta + \gamma$ Exterior angle of a triangle is equal to the sum of the interior opposite angles.
	$\alpha = \beta$ Isosceles triangle.

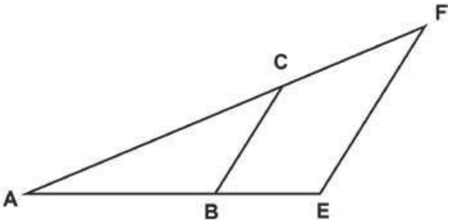
11.3 Similar Triangles

- The table below lists the tests used to determine if two triangles are similar.

AAA or AA Similarity Test

Two angles of the first triangle are equal to two angles of the second triangle.
∠BAC = ∠QPR and ∠ACB = ∠PRQ. Hence $\Delta\begin{matrix}\text{ABC}\\\text{PQR}\end{matrix}$ are similar. (AA)
SSS Similarity Test

Ratios of corresponding side lengths are equal.
$\frac{AC}{PR}=\frac{AB}{PQ}=\frac{BC}{QR}=\frac{1}{k}$. Hence $\Delta\begin{matrix}\text{ABC}\\\text{PQR}\end{matrix}$ are similar. (SSS)
SAS Similarity Test

Ratios of the two corresponding side lengths are equal and included angles are equal.
$\frac{AC}{PR}=\frac{BC}{QR}=\frac{1}{k}$ and ∠ACB = ∠PRQ. Hence $\Delta\begin{matrix}\text{ABC}\\\text{PQR}\end{matrix}$ are similar. (SAS)
RHS Similarity Test

Right Triangles with ratios of hypotenuse and one other corresponding side equal.
∠BCA = ∠QRP = 90° and $\frac{AC}{PR}=\frac{AB}{PQ}=\frac{1}{k}$. Hence $\Delta\begin{matrix}\text{ABC}\\\text{PQR}\end{matrix}$ are similar. (RHS)

Example 11.1

In the diagram given, the points B and C divide the lines AE and AF respectively in the ratio 2:1. Prove that ΔABC and ΔAEF are similar.



Solution:

The included angle ∠CAB is common.

B divides AE in the ratio 2:1. $\Rightarrow \frac{AB}{AE}=\frac{2}{3}$. C divides AF in the ratio 2: 1. $\Rightarrow \frac{AC}{AF}=\frac{2}{3}$.

Hence: $\frac{AB}{AE}=\frac{AC}{AF}=\frac{2}{3}$.

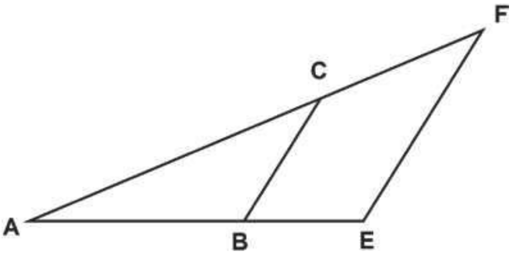
Hence, $\Delta\begin{matrix}ABC\\AEF\end{matrix}$ are similar. (SAS)

Example 11.2

ΔABC and ΔAEF are as shown in the accompanying diagram.

(a) Given that BC and EF are parallel, prove that ΔABC and ΔAEF are similar.

(b) Conversely, if ΔABC and ΔAEF are similar, prove that BC and EF must be parallel.



Solution:

(a) ∠ABC = ∠AEF Reason: Corresponding angles BC parallel to EF.
∠BCA = ∠EFA Reason: Corresponding angles BC parallel to EF.

Hence, $\Delta\begin{matrix}ABC\\AEF\end{matrix}$ are similar. (AA)

(b) If $\Delta\begin{matrix}ABC\\AEF\end{matrix}$ are similar, then ∠ABC = ∠AEF.

Hence, BC must be parallel to EF, ∠ABC and ∠AEF being corresponding angles.

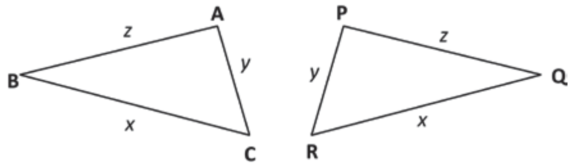
Notes: Converses

- A *premise* or *conjecture* is a mathematical result that needs to be proven.
- Consider the premise "If P then Q", where P and Q represent two separate statements. The *converse* of the premise "If P then Q" is "If Q then P".
- The premise in Example 11.2 (a) may be written as:
  If BE is parallel to EF (statement P), then ΔABC and ΔAEF are similar (statement Q).
  The premise in Example 11.2 (b) reads:
  If ΔABC and ΔAEF are similar (statement Q), then BE is parallel to EF (statement P).
  Hence, the premise in Example 11.2 (b) is the converse of the premise in Example 11.2 (a) and vice-versa.

11.4 Congruent Triangles

- The table below lists the tests used to determine if two triangles are congruent.

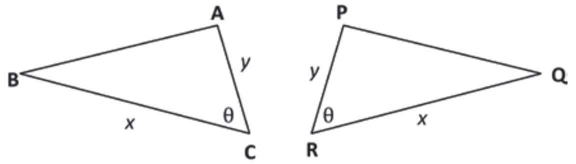
SSS Congruence Test



All three corresponding sides of the two triangles have the same side length.

AB = PQ, AC = PR and BC = QR. Hence $\Delta\begin{matrix}\text{ABC}\\\text{PQR}\end{matrix}$ are congruent. (SSS)

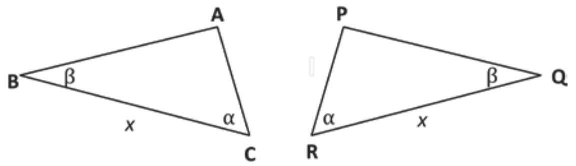
SAS Congruence Test



Two sides and the included angle of both triangles are the same.

CB = RQ, CA = RP and the included angles ∠ACB = ∠PRQ. Hence $\Delta\begin{matrix}\text{ABC}\\\text{PQR}\end{matrix}$ are congruent. (SAS)

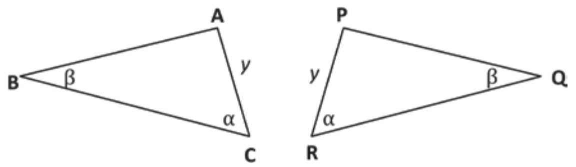
ASA Congruence Test



Two angles and the included side of both triangles are equal.

∠ABC = ∠PQR, ∠ACB = ∠PRQ and the included side BC = QR. Hence $\Delta\begin{matrix}\text{ABC}\\\text{PQR}\end{matrix}$ are congruent. (ASA)

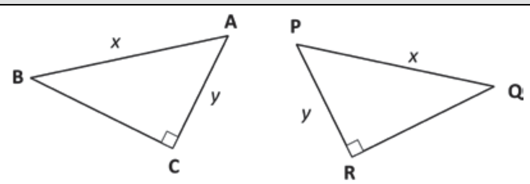
AAS Congruence Test



Two angles and a corresponding non-included side of both triangles are equal.

∠ABC = ∠PQR, ∠ACB = ∠PRQ and the non-included side AC = PR. Hence $\Delta\begin{matrix}\text{ABC}\\\text{PQR}\end{matrix}$ are congruent. (AAS)

RHS Congruence Test



Right Triangles with hypotenuse and one corresponding side equal.

∠BCA = ∠QRP = 90° and hypotenuse AB = PQ and AC = PR. Hence $\Delta\begin{matrix}\text{ABC}\\\text{PQR}\end{matrix}$ are congruent. (RHS)

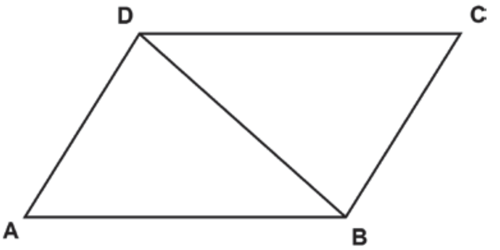
Example 11.3

Consider the quadrilateral ABCD.

(a) Prove that if ABCD is a parallelogram, then ΔABD and ΔCDB are congruent.

(b) Prove that if that ΔABD and ΔCDB are congruent, then ABCD is a parallelogram.

(c) Prove that if ΔABD and ΔCDB are not congruent, then ABCD is not a parallelogram.



Solution:

(a) $\angle$ABD = $\angle$CDB [Alternate angles AB parallel to DC.]

DB is common.

AB = CD [Opposite sides of Parallelogram ABCD.]

Hence, $\Delta\begin{matrix}\text{ABD}\\\text{CDB}\end{matrix}$ are congruent. (SAS)

(b) If $\Delta\begin{matrix}\text{ABD}\\\text{CDB}\end{matrix}$ are congruent, then $\angle$ABD = $\angle$CDB.

Hence, AB is parallel to CD, $\angle$ABD and $\angle$CDB being alternate angles.

Further, AB = CD.

Hence ABCD must be a parallelogram. [A pair of congruent and parallel sides.]

(c) If triangles ABD and CDB are not congruent, then $\angle$ABD ≠ $\angle$CDB.

Hence, AB cannot be parallel to CD.

Therefore, ABCD cannot be a parallelogram as two opposing sides of the quadrilateral are not parallel.

Notes: Converses and Contrapositives

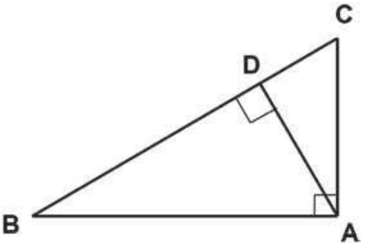
- Consider the premise: *If P then Q*.
  The *converse* of this premise is: *If Q then P*.
  The *negation* of a statement P is: *Not P*.
  The *contrapositive* of this premise is: *If not Q then not P*.

- Consider the premise in Example 11.3 (a):
  If ABCD is a parallelogram then ΔABD and ΔCDB are congruent.
  - The converse of this premise found in Example 11.3 (b) and reads:
    If ΔABD and ΔCDB are congruent, then ABCD is a parallelogram.
  - The contrapositive of this premise is found in Example 11.3 (c) and reads:
    If ΔABD and ΔCDB are not congruent, then ABCD is not a parallelogram.

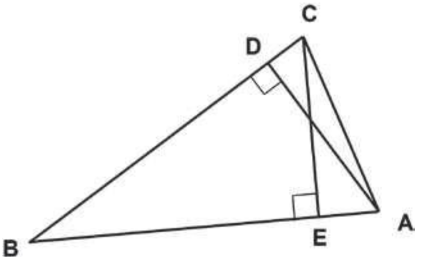
- In general, the converse of a true premise need not always be true but the contrapositive of a true premise is always true.

Exercise 11.1

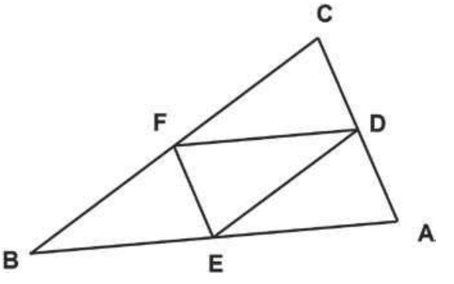
1. Triangles ABC and DBA are right triangles as shown in the accompanying diagram.
   (a) Prove that triangles ABC and DBA are similar.
   (b) Prove that DB × CB = AB^2.



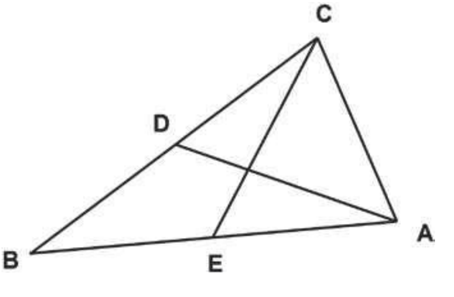
2. AD and CE are the altitudes of ΔABC, as shown in the accompanying diagram.
   (a) Prove that ΔADB and ΔCEB are similar.
   (b) Prove that the AD × BC = CE × AB.
   (c) If in addition the altitudes AD = CE, prove that ΔABC is isosceles with AB = BC.
   (d) Prove that the converse of (c) is true. That is, prove that if ΔABC is isosceles, with AB = CB, then the altitudes AD and CE are congruent.



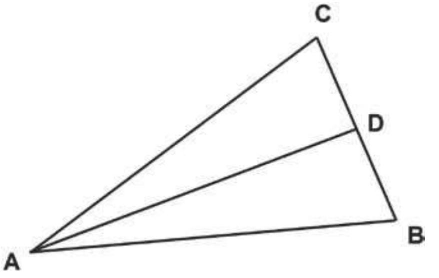
3. In ΔABC, D, E and F are respectively the midpoints of the sides AC, AB and BC.
   (a) Prove that FD is parallel to BA.
   (b) Prove that BA = 2 × FD.
   (c) Prove that ΔFDE and ΔAED are congruent.
   (d) Prove that AEFD is a parallelogram.



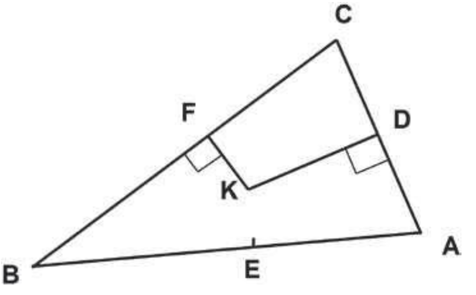
4. AD and CE are medians of ΔABC (i.e. E and D are midpoints of AB and BC respectively).
   (a) Prove that ΔCBE and ΔABD are similar.
   (b) If the medians AD = CE, prove that ΔABC is isosceles with CB = AB.
   (c) Prove that the converse of (b) is true. That is, prove that if ΔABC is isosceles with CB = AB, then the medians AD and CE are congruent.
   (d) If the median BF is such that BF = AD = CE, prove that ΔABC is equilateral.



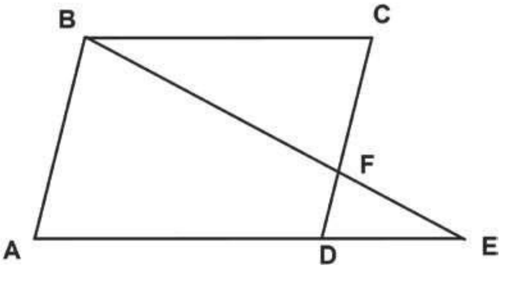
5. Consider ΔABC with AB = AC. AD is the perpendicular from A to BC.
   (a) Prove that if the perpendicular AD bisects BC then AD is the angle bisector of ∠CAB.
   (b) Prove that the converse of (a) is true. That is, prove that if the perpendicular AD is the angle bisector of ∠CAB, then AD bisects BC.
   (c) Prove that the contrapositive of (a) is true. That is, prove that if the perpendicular AD is not the angle bisector of ∠CAB, then AD does not bisect BC.



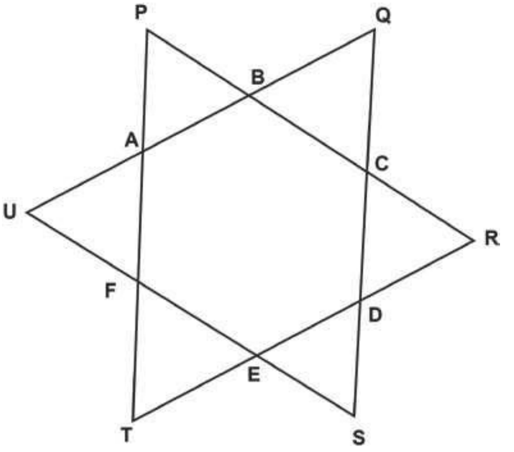
6. In ΔABC, FK is the perpendicular bisector of the side BC and DK is the perpendicular bisector of the side AC. E is the midpoint of AB.
   (a) Prove that ΔBKE and AKE are congruent.
   (b) Prove that the perpendicular bisectors of a triangle meet at a common point.
   Hint: prove that KE is the perpendicular bisector of AB.



7. ABCD is a parallelogram. The point F divides the sided CD in the ratio *m* : *n*. AD extended and BF extended meet at E.
   (a) Prove that ΔFDE and ΔFCB are similar.
   (b) Prove that F divides BE in the ratio *m* : *n*.

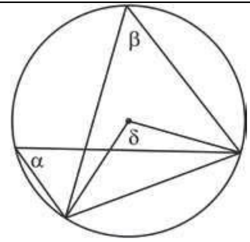
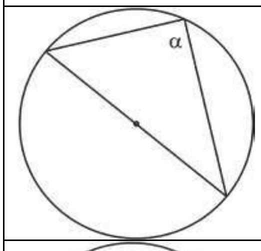
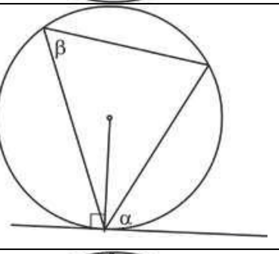
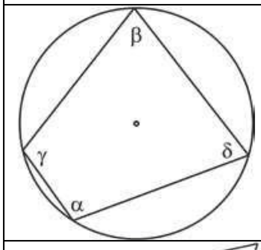
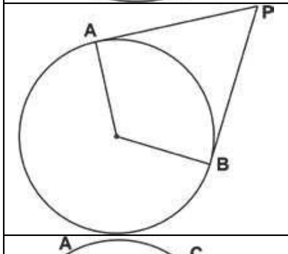
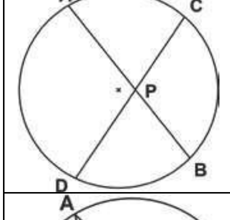
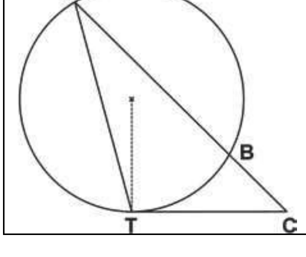


8. ABCDEF is a regular hexagon. Each of the sides of the hexagon is extended at both ends so that the extended sides meet at P, Q R, S, T and U.
   (a) Prove that PQRSTU is a regular hexagon.
   (b) Determine the ratio of the perimeter of PQRSTU to ABCDEF.

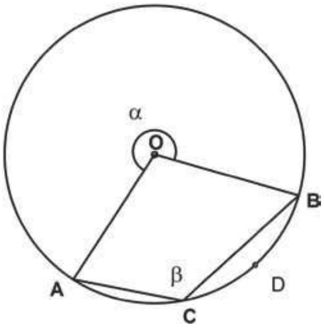


11.5 Circle Properties

- The table below lists some circle properties. The proofs of these properties will be explored in the worked examples and exercise that follow:

	$\alpha = \beta$ Angles in the same segment $\delta = 2\beta$ Angle at centre is twice angle at the circumference
	$\alpha = 90^{\circ}$ Angle in a semi-circle = 90°.
	Angle between radius and tangent = 90°. $\alpha = \beta$ Angle between a tangent and a chord is equal to the angle in the alternate segment.
	$\alpha + \beta = 180^{\circ}$ $\gamma + \delta = 180^{\circ}$ Opposite angles in a cyclic quadrilateral are supplementary.
	PA = PB Length of tangents to a circle from a common external point has the same length.
	AP × PB = CP × PD
	$AC \times BC = TC^2$

Example 11.4



In the accompanying diagram, A, B, C and D are points on the circumference of the circle centre O. Reflex $\angle AOB = \alpha$ and $\angle ACB = \beta$.

(a) Prove that $\alpha = 2\beta$.

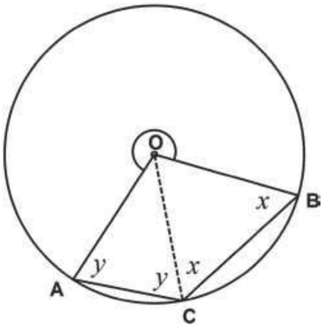
[Prove that the angle at the centre subtended by an arc is twice the angle at the circumference subtended by the same arc.]

(b) Prove that $\angle ADB = \angle ACB = \beta$.

[Prove that angles at the circumference subtended by the same arc are equal, or angles in the same segment are equal.]

Solution:

(a) Construct OC.



Let $\angle OCB = x$ and $\angle OCA = y$.

Δ OCB is isosceles as OC = OB = Radius of circle.

Hence, $\angle OCB = \angle OBC = x$. $\Rightarrow$ $\angle COB = 180^{o} - 2x$.

Δ OAC is isosceles as OA = OC = Radius of circle.

Hence, $\angle OAC = \angle OCA = y$. $\Rightarrow$ $\angle COA = 180^{o} - 2y$.

Clearly $\angle ACB = x + y$.

Also, reflex $\angle AOB = 360^{o} - (180^{o} - 2x) - (180^{o} - 2x)$

$= 2(x + y)$.

Therefore, reflex $\angle AOB = 2 \times \angle ACB$

$\alpha = 2\beta$. Proved.

(b) Since $\angle ADB$ is also an angle at the circumference subtended by the arc AB,

$$\angle ADB = \frac{1}{2} \text{reflex } \angle AOB$$

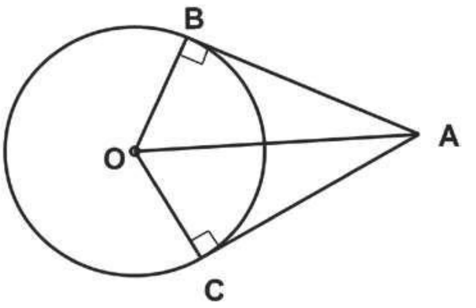
$$= \beta.$$

Hence, $\angle ADB = \angle ACB = \beta$

Notes:

- *The Central Angle Theorem is proven in (a) and is the primary theorem used to prove several other circle properties.*
  - *The right angle in a semicircle property (if $\alpha = 180^{o}$, then $\beta = 90^{o}$)*
  - *The angles in the same segment property.*
  - *The alternate segment property.*
  - *The supplementary property of opposite angles in a cyclic quadrilateral.*

Example 11.5



B and C are points on the circle centre O. A is a point exterior to the circle. Prove that if AB and AC are tangents to the circle then AB = AC.

Solution:

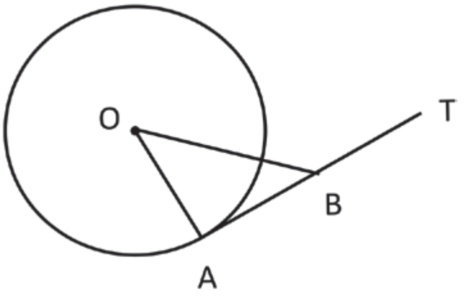
AB and AC are tangents to the circle.
Hence, $\angle OBA = \angle OCA = 90^{o}$.

OA is common and OB = OC = Radius of circle.

Hence, $\Delta\begin{matrix}OBA\\OCA\end{matrix}$ are congruent. (RHS)

Therefore AB = AC.

Example 11.6



A is a point on the circumference of a circle centre O.
AT is the tangent to the circle at A. Prove that AT is perpendicular to OA.

Solution:

Assume that OA is not perpendicular to AT
Hence, there must be a point B on AT such that OB is perpendicular to AT.

In ΔOAB, since $\angle OBA$ is 90^{o}, $\angle OAB$ must be less than 90^{o}.
That is, $\angle OAB < \angle OBA$.

In a triangle, as the longer side is always opposite the larger angle, OB < OA.

This means that OB is shorter than the radius of the circle.
Which is impossible!

Hence, the premise that OA is not perpendicular to AT must be false.
Therefore, OA must be perpendicular to AT.

Note:

- *The proof employed in Example 11.6 uses the method of contradiction which will be discussed in greater detail in Chapter 21.*

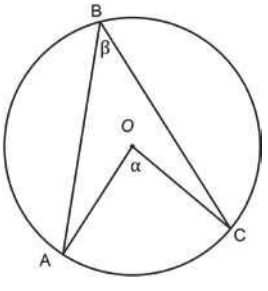
The implication and *iff* symbols

- The symbol $\Rightarrow$ is the *implication* symbol. This is used to convey that the statement following the symbol is a logical conclusion of the statement before the symbol.
  For example: $2x = 100^\circ \Rightarrow x = 50^\circ$.
- Consider the case when a premise and its converse it true.
  If P then Q $\qquad$ $P \Rightarrow Q$
  If Q then P $\qquad$ $Q \Rightarrow P$
  In this instance, we say that the statements P and Q are equivalent.
  This is written symbolically as $P \Leftrightarrow Q$.
- The phrase "*if and only if*" shortened as "*iff*" is used to indicate that the statement before the phrase and the statement after the phrase are equivalent.
  Hence, the phrase "*iff*" is also denoted by the symbol $\Leftrightarrow$.
  When used in a proof statement, both the initial premise and its converse must be proven as shown in Example 11.5.

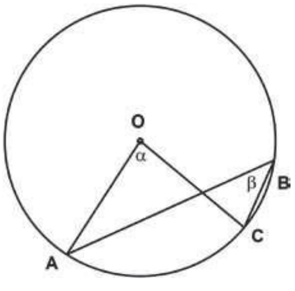
Exercise 11.2

1. In each of the diagrams below, A, B and C are points on the circle centre O. $\angle AOC = \alpha$ and $\angle ABC = \beta$. In each case, prove that $\alpha = 2\beta$. [The Central Angle Theorem.]

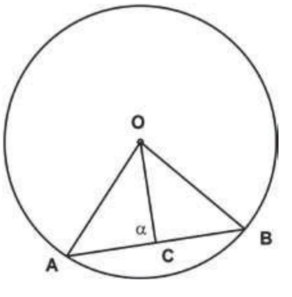
(a)



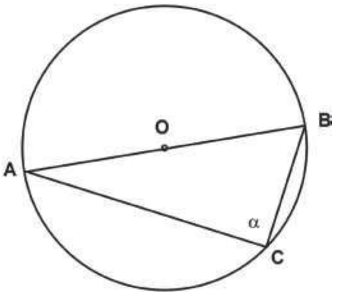
(b)



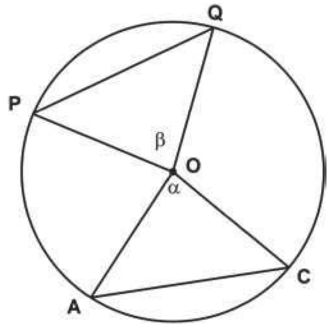
2. In the accompanying diagram, A and B are points on the circle with centre O. $\angle OCA = \alpha$.
   Prove that $\alpha = 90^\circ$ *if and only if* C is the midpoint of AB.



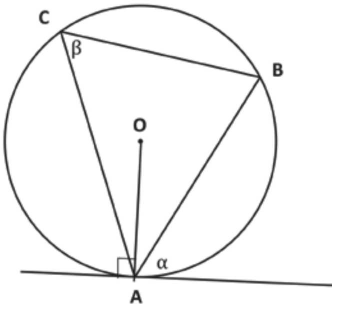
3. In the accompanying diagram, A, B and C are points on the circle centre O. $\angle ACB = \alpha$.
   Prove that $\alpha = 90^\circ$ *if and only if* AB is a diameter of the circle.



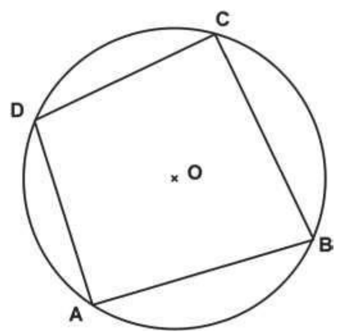
4. AC and PQ are chords of the circle centre O.
   (a) Prove that $\alpha = \beta$ *if and only if* AC = PQ.
   (b) In addition to $\alpha = \beta$, if P, O and C are collinear and Q, O and A are collinear, prove that PQ is parallel to AC



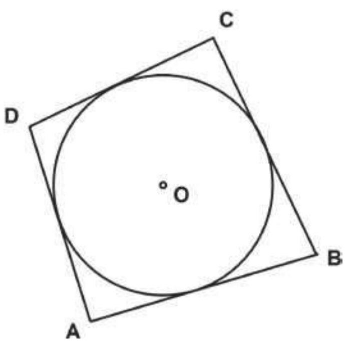
5. The tangent to a circle centre O meets the circle at A. B and C are two other points on the circumference of the circle as shown in the accompanying diagram. $\angle ACB = \beta$. The angle between the chord AB and the tangent is α. Prove that $\alpha = \beta$. [You may assume that the tangent to a circle is always perpendicular to its radius vector.]



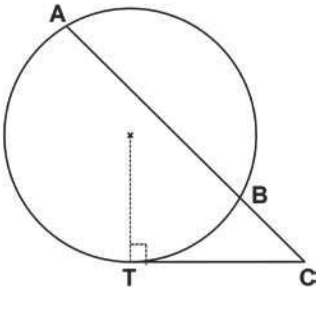
6. ABCD is a cyclic quadrilateral within the circle centre O.
   (a) Prove that the opposite angles are supplementary.
   (b) In addition, if AD is parallel to BC, then AB = DC.



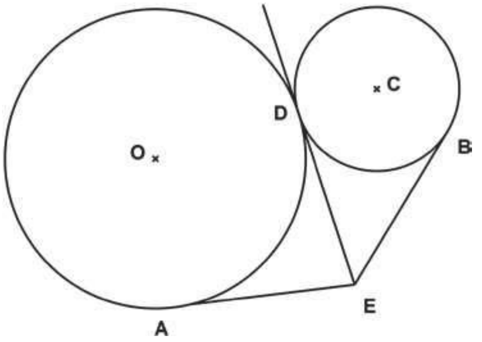
7. The sides of the quadrilateral ABCD are each tangents to the circle centre O. Prove that the sums of the lengths of opposite sides of the quadrilateral are equal.



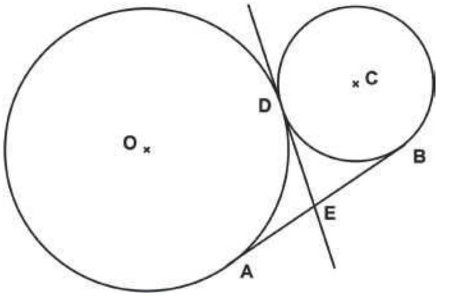
8. TC is a tangent to a circle. AC is a secant to the same circle. AC meets the circle at B.
   Prove that $AC \times BC = TC^2$.



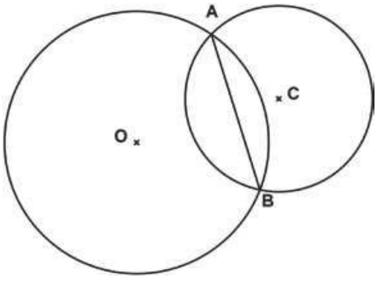
9. The circle with centre at O and the circle with centre at C meet externally at point D. The common tangent at D meets the tangent at A and the tangent at B at E.
   Prove that EA = EB.



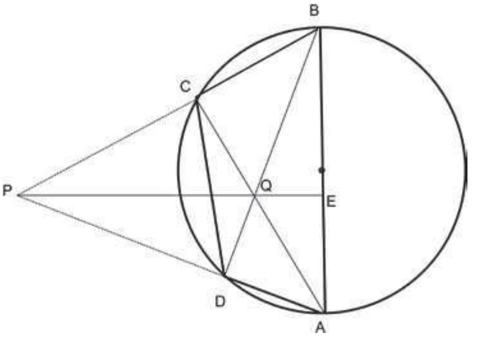
10. The circle with centre at O and the circle with centre at C meet externally at point D.
    (a) Prove that O, D and C are collinear.
    (b) In addition, the line segment AB is tangential to the circles at A and B.
        (i) Prove that the common tangent at D bisects the line AB.
        (ii) Prove that $\angle ADB = 90^{\circ}$.



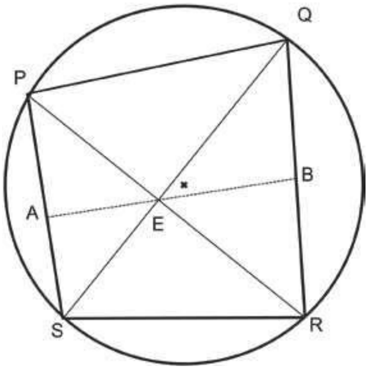
11. The circle with centre at O intersects the circle with centre at C at the points A and B. Prove that OC is the perpendicular bisector of the common chord AB.



12. ABCD is a cyclic quadrilateral. The side AB is also a diameter of the circle. The diagonals of the quadrilateral meet at the point Q. The sides BC and AD are extended to meet at the point P. PQ extended meets AB at E.
    (a) Prove that CQDP is a cyclic quadrilateral
    (b) Prove that PE is perpendicular to AB.



13. PQRS is a cyclic quadrilateral with perpendicular diagonals. The diagonals intersect at E. The line AB is perpendicular to the side PS and passes through E.
    (a) Prove that ΔEQB is isosceles.
    (b) Prove that the line AB bisects QR.



12 Geometric Proofs using Vectors

12.1 Geometric Proofs involving Vectors I

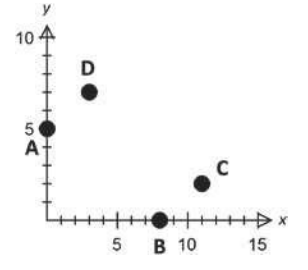
- In this section, the concepts of parallel vectors and null (zero) vectors are used to prove some well-known geometrical properties of some planar figures (triangles, parallelograms etc.). The main ideas are listed below:

> - $\alpha \boldsymbol{u} + \beta \boldsymbol{v} = 0 \quad \Leftrightarrow \quad \alpha = 0$ and $\beta = 0$
> - $\boldsymbol{u} = \boldsymbol{v} \quad \Leftrightarrow \quad$ $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel in the same direction and $|\boldsymbol{u}| = |\boldsymbol{v}|$
> - $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel $\quad \Leftrightarrow \quad \boldsymbol{u} = \lambda \boldsymbol{v}$

Example 12.1

Use a vector method to prove that the points A, B, C and D with position vectors < 0, 5 >, < 8, 0 >, < 11, 2 > and < 3, 7 > respectively, form a parallelogram.

Solution:



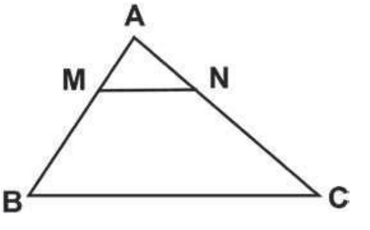
AB = < 8, 0 > – < 0, 5 > = < 8, –5 >
DC = < 11, 2 > – < 3, 7 > = < 8, –5 >.
Hence, **AB** = **DC**. $\Rightarrow$ AB and DC are parallel and congruent.
Therefore, ABCD is a parallelogram.

Note:

- *To prove that a quadrilateral is a parallelogram, it is sufficient to prove that there is a pair of parallel and congruent sides.*

Example 12.2

In ΔABC, the points M and N divide the sides AB and AC respectively in the ratio 1 : 3. Let AB = $\boldsymbol{u}$ and AC = $\boldsymbol{v}$.
Find **BC** and **MN** in terms of $\boldsymbol{u}$ and $\boldsymbol{v}$.
Hence prove that **BC** = 4**MN**.



Solution:

$\mathbf{BC} = \mathbf{BA} + \mathbf{AC} = -\boldsymbol{u} + \boldsymbol{v}$.

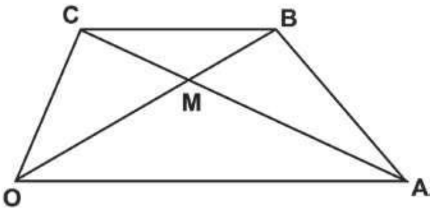
Since AM:MB = 1:3, $\Rightarrow \quad \mathbf{AM} = \frac{1}{4}\mathbf{AB} = \frac{1}{4}\boldsymbol{u}$.

Also, AN:NC = 1:3, $\quad \Rightarrow \quad \mathbf{AN} = \frac{1}{4}\mathbf{AC} = \frac{1}{4}\boldsymbol{v}$.

$\mathbf{MN} = \mathbf{MA} + \mathbf{AN} \qquad \Rightarrow \quad \mathbf{MN} = -\frac{1}{4}\boldsymbol{u} + \frac{1}{4}\boldsymbol{v} = \frac{1}{4}(\boldsymbol{v} - \boldsymbol{u}) = \frac{1}{4}\mathbf{BC}$

Hence, **BC** = 4**MN**.

Example 12.3



OABC is a trapezium with **OA** = ***a*** and **OC** = ***c***. The diagonals OB and AC intersect at M such that AM : MC = 2 : 1 . Let **OM** = α**OB** and **CB** = β**OA** ,
(a) Find **OM** and **OB** in terms of ***a*** and ***c***.
(b) Hence prove that M divides OB in ratio 2 : 1.

Solution:

(a) **OM** = **OA** + **AM**

But AM : MC = 2 : 1 $\Rightarrow$ $\mathbf{AM} = \frac{2}{3}\mathbf{AC} = \frac{2}{3}(\mathbf{AO} + \mathbf{OC})$

$= \frac{2}{3}(\boldsymbol{c} - \boldsymbol{a})$.

$\Rightarrow$ $\mathbf{OM} = \boldsymbol{a} + \frac{2}{3}(\boldsymbol{c} - \boldsymbol{a}) = \frac{1}{3}(2\boldsymbol{c} + \boldsymbol{a})$

OB = **OC** + **CB** $\Rightarrow$ $\mathbf{OB} = \boldsymbol{c} + \beta\boldsymbol{a}$

(b) Since **OM** = α**OB** $\Rightarrow$ $\frac{1}{3}(2\boldsymbol{c} + \boldsymbol{a}) = \alpha(\boldsymbol{c} + \beta\boldsymbol{a})$

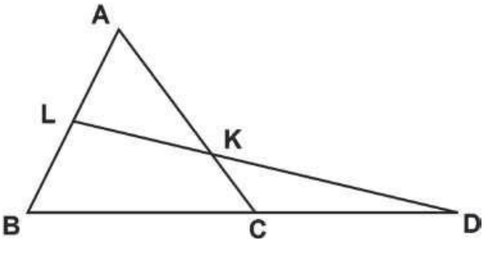
$\Rightarrow$ $\left(\frac{2}{3} - \alpha\right)\boldsymbol{c} + \left(\frac{1}{3} - \alpha\beta\right)\boldsymbol{a} = \mathbf{0}$

$\Rightarrow$ $\alpha = \frac{2}{3}$ and $\alpha\beta = \frac{1}{3}$

$\Rightarrow$ $\alpha = \frac{2}{3}$ and $\beta = \frac{1}{2}$.

Hence, $\mathbf{OM} = \frac{2}{3}\mathbf{OB}$. That is, M divides OB in the ratio 2 : 1.

Example 12.4



In the accompanying diagram, **AB** = ***a*** and **AC** = ***c***. L is the midpoint of AB and C is the midpoint of BD. Let **AK** = α**AC** and **KD** = β**LD**.
(a) Find **BD**, **BK** and **KD** in terms of ***a*** and ***c***.
(b) Prove that K divides DL and AC in the same ratio.

Solution:

(a) **BD** = 2**BC**
= 2(**BA** + **AC**)
= 2(−***a*** + ***c***).

BK = **BA** + **AK**
= −***a*** + α**AC**
= −***a*** + α***c***

KD = β**LD**
= β(**LB** + **BD**)
$= \beta[\frac{1}{2}\boldsymbol{a} + 2(-\boldsymbol{a} + \boldsymbol{c})]$
$= \beta(-\frac{3}{2}\boldsymbol{a} + 2\boldsymbol{c})$

(b) **BD** = **BK** + **KD** $\Rightarrow$ $2(-\boldsymbol{a} + \boldsymbol{c}) = (-\boldsymbol{a} + \alpha\boldsymbol{c}) + \beta(-\frac{3}{2}\boldsymbol{a} + 2\boldsymbol{c})$

$$(-1 + \frac{3}{2}\beta)\boldsymbol{a} + (2 - \alpha - 2\beta)\boldsymbol{c} = 0$$

$$\beta = \frac{2}{3} \text{ and } \alpha + 2\beta = 2 \Rightarrow \beta = \frac{2}{3} \text{ and } \alpha = \frac{2}{3}$$

Hence, **AK** = $\frac{2}{3}$**AC** and **DK** = $\frac{2}{3}$**DL**.

That is, K divides AC and DL in the ratio 2 : 1.

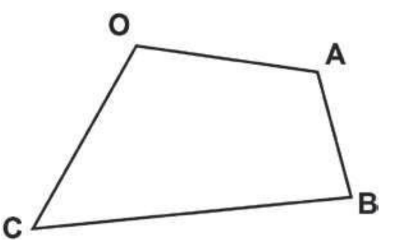
Exercise 12.1

1. Use a vector method to prove that the quadrilateral formed by the points (3,0), (5, 16), (–3, 2) and (–5, –14) is a parallelogram.

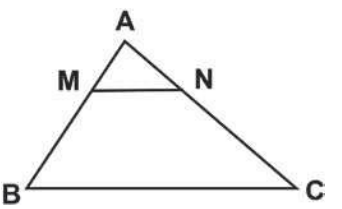
2. Use a vector method to prove that the quadrilateral formed by the points A, B, C and D with position vectors < –4, –7 >, < 4,–1 >, < 4, 9 > and < –4, 3 > respectively is a rhombus.

3. Use a vector method to prove that the quadrilateral formed by the points (2, –3), (5, 3), (–1, 6) and (–4, 0) is a square.

4. OABC is a quadrilateral. E, F, G and H are respectively the midpoints of the sides OA, AB, BC and OC. Let the position vectors of the vertices A, B and C with respect to O be respectively ***a***, ***b*** and ***c***.
   (a) Find **EF** and **HG** in terms of ***a*** and/or ***b*** and/or ***c***.
   (b) Prove that EFGH is a parallelogram, that is, *the midpoints of a trapezium forms a parallelogram.*

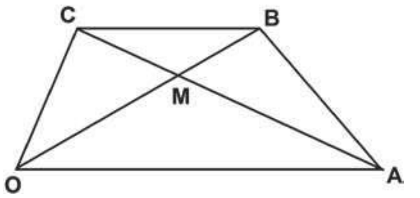


5. In ΔABC, the points M and N divide the sides AB and AC respectively in the ratio 1 : 4. Let AB = ***u*** and AC = ***v***.
   Find **BC** and **MN** in terms of ***u*** and ***v***.
   Hence prove that **BC** = 5**MN**.



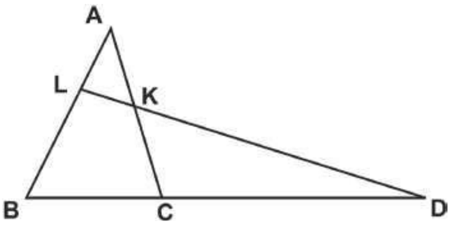
*6. In ΔABC, the points M and N divide the sides AB and AC respectively in the ratio *m* : *n*. Let AB = ***u*** and AC = ***v***. Find **BC** and **MN** in terms of ***u*** and ***v***.
   Hence prove that MN : BC = *m* : *m* + *n*.

7. OABC is a trapezium with **OA** = ***a*** and **OC** = ***c***.
   The diagonals OB and AC intersect at M such that AM : MC = 3 : 1 . Let **OM** = α**OB** and **CB** = β**OA** ,
   (a) Find **OB** and **OM** in terms of ***a*** and ***c***.
   (b) Hence prove that: (i) OA = 3CB
   (ii) M divides OB in the ratio 3 : 1.



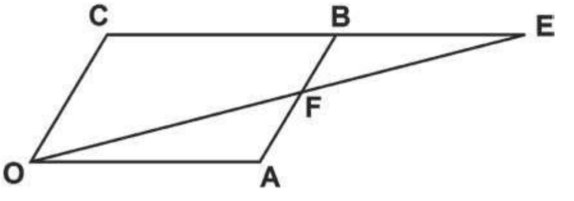
*8. OABC is a trapezium with **OA** = ***a*** and **OC** = ***c***. The diagonals OB and AC intersect at M such that AM : MC = $m : n$. Let **OM** = α**OB** and **CB** = β**OA** . Prove that:
(a) M divides OB in ratio $m : n$ (b) n OA = m CB .

9. In the accompanying diagram, **AB** = ***a*** and **AC** = ***c***. L and C divide the lines AB and DB in the ratio 1 : 2 respectively. Let **AK** = α**AC** and **DK** = β**DL**.
(a) Find **BD**, **BK** and **KD** in terms of ***a*** and ***c***.
(b) Prove that K divides DL and AC in the same ratio.

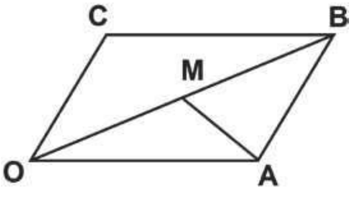


*10. For the diagram in Question 9, **AB** = ***a*** and **AC** = ***c***. L and C divide the lines AB and DB in the ratio $m : n$ respectively. Let **AK** = α**AC** and **DK** = β**DL**. Prove that K divides DL and AC in the same ratio.

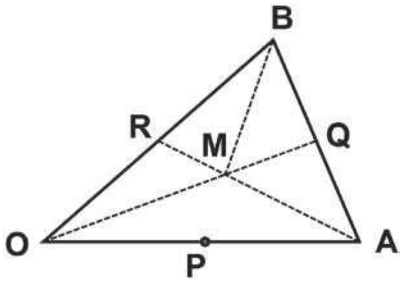
*11. OABC is a parallelogram with **OA** = ***a*** and **OC** = ***c***. The point F divides AB in the ratio $m : n$. OF extended meets the CB extended at E. Prove that:
(a) F divides the line OE in the ratio $m : n$
(b) B divides the line CE in the ratio $m : n$.



12. OABC is a parallelogram with **OA** = ***a*** and **OC** = ***c***. M is the midpoint of the diagonal OB.
(a) Find **OB**, **AC** and **AM** and **MC** in terms of ***a*** and/or ***c***.
(b) Hence show that M is also the midpoint of AC; that is, *the diagonals of a parallelogram bisect each other.*



13. OQ, AR are respectively the medians of ΔOAB; Q and R are the midpoints of AB and OB. Let the medians intersect at M. Also, let **OA** = ***a***, **OB** = ***b***, **OM** = α**OQ** and **AM** = β**AR**.
(a) Find **OQ** and **AR** in terms of ***a*** and/or ***b***.
(b) Find **OM** and **AM** and in terms of ***a*** and/or ***b***.
(c) Prove that $\alpha = \beta = \frac{2}{3}$.
(d) Let P be the midpoint of OA. Find **BM** and **MP** in terms of ***a*** and/or ***b***.
(e) Hence, prove that B, M and P are collinear with BM = $\frac{2}{3}$BP, that is,
the medians of a triangle are coincident and divide each other in the ratio 2 : 1.



12.2 Geometric Proofs involving Vectors II

- In this section, scalar products are introduced to prove the geometrical properties of some planar shapes.

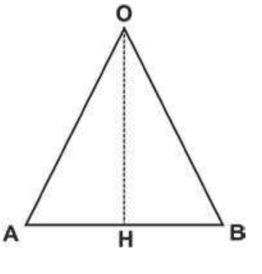
- $\alpha \boldsymbol{u} + \beta \boldsymbol{v} = 0 \iff \alpha = 0$ and $\beta = 0$
- $\boldsymbol{u} = \boldsymbol{v} \iff \boldsymbol{u}$ and $\boldsymbol{v}$ are parallel in the same direction and $|\boldsymbol{u}| = |\boldsymbol{v}|$
- $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel $\iff \boldsymbol{u} = \lambda \boldsymbol{v}$
- $\boldsymbol{u}$ and $\boldsymbol{v}$ are perpendicular $\iff \boldsymbol{u} \cdot \boldsymbol{v} = 0$
- The angle between $\boldsymbol{u}$ and $\boldsymbol{v}$ is given by $\cos\theta = \dfrac{\boldsymbol{u} \cdot \boldsymbol{v}}{|\boldsymbol{u}| \times |\boldsymbol{v}|}$

Example 12.5

OAB is an isosceles triangle with OA = OB and with **OA** = ***a*** and **OB** = ***b***. OH is the angle bisector of ∠AOB and meets AB at H. Let **OH** = ***h***, ∠AOH = α and ∠BOH = β.

(a) Find $\cos\alpha$ and $\cos\beta$ in terms of ***a***, ***b*** and ***h***.

(b) Prove that the angle bisector of ∠AOB is perpendicular to AB.



Solution:

(a) α and β are respectively the angles between the vectors **OA** and **OH** and **OB** and **OH**.

$$\Rightarrow \cos\alpha = \frac{\boldsymbol{a} \cdot \boldsymbol{h}}{|\boldsymbol{a}| \times |\boldsymbol{h}|}$$

$$\text{and} \quad \cos\beta = \frac{\boldsymbol{b} \cdot \boldsymbol{h}}{|\boldsymbol{b}| \times |\boldsymbol{h}|}.$$

(b) Since, $\alpha = \beta$, $\dfrac{\boldsymbol{a} \cdot \boldsymbol{h}}{|\boldsymbol{a}| \times |\boldsymbol{h}|} = \dfrac{\boldsymbol{b} \cdot \boldsymbol{h}}{|\boldsymbol{b}| \times |\boldsymbol{h}|}$

$$\Rightarrow \boldsymbol{a} \cdot \boldsymbol{h} = \boldsymbol{b} \cdot \boldsymbol{h} \quad \text{as } |\boldsymbol{a}| = |\boldsymbol{b}|.$$

Hence, $(\boldsymbol{b} - \boldsymbol{a}) \cdot \boldsymbol{h} = 0$.

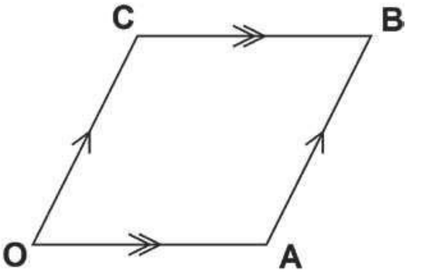
$\Rightarrow$ **AB** $\cdot$ **OH** = 0.

Hence, OH is perpendicular to AB.

Note:

- *This proves that the angle bisector of the apex angle of an isosceles triangle is perpendicular to its base. The converse of Example 12.5b will be proved in Exercise 12.2 Question 1.*

Example 12.6



OABC is a rhombus with **OA** = ***a*** and **OC** = ***c***.
Let ∠AOB = α and ∠COB = β.
(a) Find cos α and cos β in terms of ***a*** and ***c***.
(b) Hence prove that the diagonal OB bisects ∠COA.

Solution:

(a) Clearly **OB** = ***a*** + ***c***.
α and β are respectively the angles between the vectors **OB** and **OA**, and **OB** and **OC**.

$$\Rightarrow \cos\alpha = \frac{(\boldsymbol{a}+\boldsymbol{c}).\boldsymbol{a}}{|(\boldsymbol{a}+\boldsymbol{c})|\times|\boldsymbol{a}|}$$

$$= \frac{\boldsymbol{a}.\boldsymbol{a}+\boldsymbol{a}.\boldsymbol{c}}{|(\boldsymbol{a}+\boldsymbol{c})|\times|\boldsymbol{a}|} = \frac{|\boldsymbol{a}|^2+\boldsymbol{a}.\boldsymbol{c}}{|(\boldsymbol{a}+\boldsymbol{c})|\times|\boldsymbol{a}|}$$

$$\Rightarrow \cos\beta = \frac{(\boldsymbol{a}+\boldsymbol{c}).\boldsymbol{c}}{|(\boldsymbol{a}+\boldsymbol{c})|\times|\boldsymbol{c}|}$$

$$= \frac{\boldsymbol{c}.\boldsymbol{c}+\boldsymbol{a}.\boldsymbol{c}}{|(\boldsymbol{a}+\boldsymbol{c})|\times|\boldsymbol{c}|} = \frac{|\boldsymbol{c}|^2+\boldsymbol{a}.\boldsymbol{c}}{|(\boldsymbol{a}+\boldsymbol{c})|\times|\boldsymbol{c}|}.$$

(b) Since OABC is a rhombus $|\boldsymbol{c}| = |\boldsymbol{a}|$.

$$\Rightarrow \cos\beta = \frac{|\boldsymbol{a}|^2+\boldsymbol{a}.\boldsymbol{c}}{|(\boldsymbol{a}+\boldsymbol{c})|\times|\boldsymbol{a}|}.$$

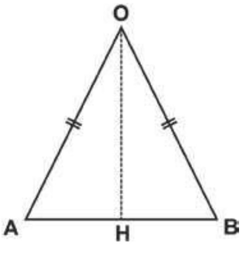
Therefore, cos α = cos β. ⇒ α = β.
Hence, OB bisects ∠COA.

Note:

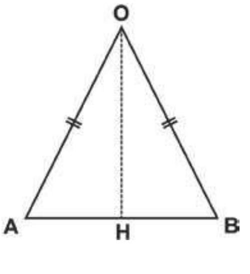
- *By applying the same procedure, it can be shown that the diagonal AC bisects ∠OAB. Hence, the diagonals of a rhombus bisect the respective angles.*

Exercise 12.2

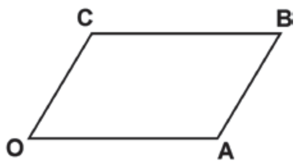
1. OAB is an isosceles triangle with OA = OB and with **OA** = ***a*** and **OB** = ***b***. OH is perpendicular to AB. Let ∠AOH = α and ∠BOH = β and OH = ***h***.
   (a) Find **AB**, cos α and cos β in terms of ***a*** and ***b*** and/or ***h***.
   (b) Prove that OH bisects ∠AOB, that is, *the perpendicular from the apex of an isosceles triangle bisects its apex angle.*
   This is the converse of the premise in Example 12.5b.



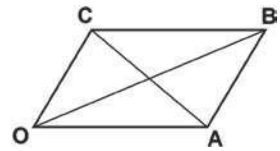
2. OAB is an isosceles triangle with OA = OB and with **OA** = ***a*** and **OB** = ***b***. H is the midpoint of AB. Let OH = ***h***.
   (a) Find **AB** and **OH** in terms of ***a*** and/or ***b***.
   (b) Prove that OH is perpendicular to AB, that is, *the bisector of the side opposite the apex of an isosceles triangle is perpendicular to the said side.*



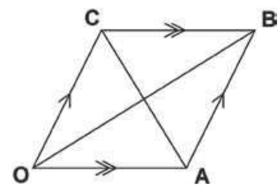
3. OABC is a parallelogram with **OA** = ***a*** and **OC** = ***c***.
   Let ∠AOC = α and ∠CBA = β. Use vector methods to prove that $\alpha = \beta$: that is, *the opposite angles of a parallelogram are congruent.*



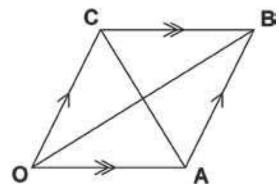
4. OABC is a parallelogram with **OA** = ***a*** and **OC** = ***c***.
   (a) Find $|\mathbf{OB}|$ and $|\mathbf{AC}|$.
   (b) Prove that $|\mathbf{OB}|^2 + |\mathbf{AC}|^2 = 2|\boldsymbol{a}|^2 + 2|\boldsymbol{c}|^2$: that is, *the sum of the squares of the lengths of the diagonals of a parallelogram is equal to the sum of the squares of the lengths of its sides.*



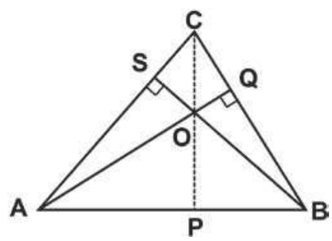
5. OABC is a parallelogram with **OA** = ***a*** and **OC** = ***c***.
   (a) Find OB and AC in terms of ***a*** and/or ***c***.
   (b) Prove that if $|\mathbf{OA}| = |\mathbf{OC}|$, then **AC** is perpendicular to **OB**: that is, *if OABC is a rhombus then its diagonals are perpendicular to each other.*



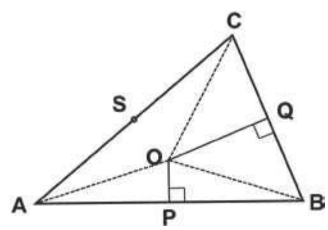
6. OABC is a parallelogram with **OA** = ***a*** and **OC** = ***c***. Given that the diagonals AC and OB are perpendicular, prove that OABC must be a rhombus: that is, *if the diagonals of a parallelogram are perpendicular, then the parallelogram must be a rhombus. This is the converse of the premise in Question 5b.*



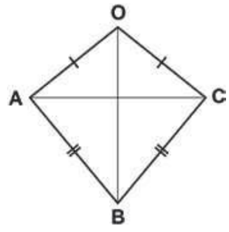
*7. AQ and BS are respectively the altitudes to the sides BC and AC of ΔABC. O is the intersection of the altitudes AQ and BS. P is the foot of the altitude from C to AB.
   Let **OA** = ***a***, **OB** = ***b*** and **OC** = ***c***.
   (a) Find **AC**, **BC** and **AB** in terms of ***a***, ***b*** and/or ***c***.
   (b) By considering the altitudes QA and SB show that $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{c}$ and $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{a} \cdot \boldsymbol{c}$.
   (c) Show that OC is perpendicular to AB and P, O and C are collinear; that is, *the three altitudes of a triangle are coincident (meet at the same point)*.



8. OP and OQ are respectively the perpendicular bisectors of the sides AB and BC of ΔABC. S is the midpoint of AC. Let **OA** = ***a***, **OB** = ***b*** and **OC** = ***c***.
   (a) Find **AC**, **BC**, **AB**, **OP**, **OQ** and **OS** in terms of ***a***, ***b*** and/or ***c***.
   (b) Use the perpendicular bisectors OP and OQ, to show that $|\boldsymbol{a}|^2 = |\boldsymbol{b}|^2$ and $|\boldsymbol{b}|^2 = |\boldsymbol{c}|^2$.
   (c) Hence, show that OS is perpendicular to AC; that is, *the three perpendicular bisectors of a triangle are coincident.*



9. OABC is a kite with OA = OC and BA = BC and ∠OAB = ∠OCB. Let **OA** = ***a***, **OB** = ***b*** and **OC** = ***c***. Let ∠OAB = α and ∠OCB = β.
   (a) Find **AB**, **CB** and **AC** in terms of ***a***, ***b***, and/or ***c***.
   (b) Prove that OB is perpendicular to AC; that is, *the diagonals of a kite are perpendicular.*

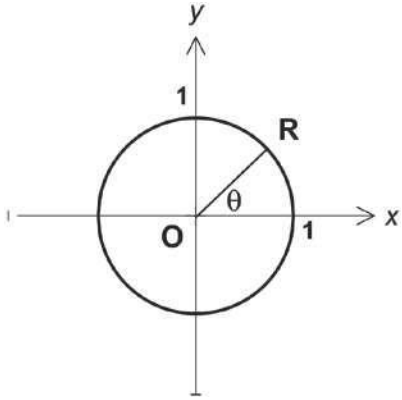


13 Trigonometric Equations

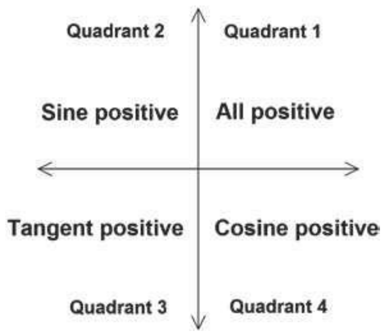
13.1 Review of Basic Concepts

- The reference angle α associated with a given angle θ, is the acute angle which the defining ray makes with either the positive or the negative *x*-axis.
  - If θ is in Quadrant 1: $\alpha = \theta - 360n^\circ$ or $\theta - 2n\pi$.
  - If θ is in Quadrant 2: $\alpha = (2n + 1)180^\circ - \theta$ or $(2n + 1)\pi - \theta$.
  - If θ is in Quadrant 3: $\alpha = \theta - (2n + 1)180^\circ$ or $\theta - (2n + 1)\pi$.
  - If θ is in Quadrant 4: $\alpha = 360n^\circ - \theta$ or $2n\pi - \theta$.

- In general, trigonometric ratios may be defined as circular functions.
  - Consider the point R on a circle with equation $x^2 + y^2 = 1$, where O is the centre of the circle. Let $\angle ROX = \theta$.
    - The *sine* function is defined as: sin (θ) = *y-coordinate of R*
    - The *cosine* function is defined as: cos (θ) = *x-coordinate of R*
    - The *tangent* function is defined as: $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$



- By noting the numerical sign of the coordinates of R, the following results regarding the numerical signs of the sine, cosine and tangent functions may be drawn.



- If α is the reference angle for θ,

$$\sin\theta = \begin{cases} \sin(\alpha) & \text{if } \theta \text{ is in Q1\&Q2} \\ -\sin(\alpha) & \text{if } \theta \text{ is in Q3\&Q4} \end{cases}$$

$$\cos\theta = \begin{cases} \cos(\alpha) & \text{if } \theta \text{ is in Q1\&Q4} \\ -\cos(\alpha) & \text{if } \theta \text{ is in Q2\&Q3} \end{cases}$$

$$\tan\theta = \begin{cases} \tan(\alpha) & \text{if } \theta \text{ is in Q1\&Q3} \\ -\tan(\alpha) & \text{if } \theta \text{ is in Q2\&Q4} \end{cases}$$

- Negative angles are measured in a clockwise direction from the positive *x*-axis. Using reference angles it can be shown that:
  - $\sin(-\theta) = -\sin\theta$ • $\cos(-\theta) = \cos(\theta)$ • $\tan(-\theta) = -\tan\theta$

- A table of exact values for θ in degrees and radians is reproduced below.

θ^{o}	θ radians	$\sin\theta$	$\cos\theta$	$\tan\theta$
0^{o}	0	0	1	0
30^{o}	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
45^{o}	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60^{o}	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90^{o}	$\frac{\pi}{2}$	1	0	∞

- If $\sin\theta = k$, then $\theta = \sin^{-1} k$.
  $\sin^{-1}$ is the inverse sine function and is commonly read as "the arc sine of".

- The values given by using the $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$ buttons on most scientific and CAS/graphic calculators are the *principal values* for the required angle.
  - The principal values of $\cos^{-1} k$, $\sin^{-1} k$ and $\tan^{-1} k$ are:

    $0 \le \cos^{-1} k \le \pi$ or $0^{o} \le \cos^{-1} k \le 180^{o}$ [Q1 & Q2]

    $-\frac{\pi}{2} \le \sin^{-1} k \le \frac{\pi}{2}$ or $-90^{o} \le \sin^{-1} k \le 90^{o}$ [Q1 & Q4]

    $-\frac{\pi}{2} < \tan^{-1} k < \frac{\pi}{2}$ or $-90^{o} < \tan^{-1} k < 90^{o}$ [Q1 & Q4]

13.2 Solving Trigonometric Equations

- In this section, we shall explore the various techniques of solving trigonometric equations using reference angles and principal values.

Example 13.1

Without the use of a calculator find the principal value (in degrees) of:

(a) $\sin^{-1}(-0.5)$ (b) $\cos^{-1}(-0.5)$ (c) $\tan^{-1}(-1)$.

Solution:

(a) $\sin^{-1}(-0.5)$ is the angle in the domain $-90^{o} \le \sin^{-1} k \le 90^{o}$ with a sine value of -0.5.
Hence, $\sin^{-1}(-0.5) = -30^{o}$.

(b) $\cos^{-1}(-0.5)$ is the angle in the domain $0^{o} \le \cos^{-1} k \le 180^{o}$ with a cosine value of -0.5.
Hence, $\cos^{-1}(-0.5) = 120^{o}$.

(c) $\tan^{-1}(-1)$ is the angle in the domain $-90^{o} < \tan^{-1} k < 90^{o}$ with a tangent value of -1.
Hence, $\tan^{-1}(-1) = -45^{o}$.

Example 13.2

Without the use of a calculator, solve for *x* given that $\sin 2(x + 10^\circ) = -\frac{\sqrt{3}}{2}$ $\quad 0 \le x \le 360^\circ$.

Solution:

Adjusted domain for $2(x + 10^\circ)$ is $20^\circ \le x \le 740^\circ$

Reference angle for $2(x + 10^\circ) = \sin^{-1}\left|-\frac{\sqrt{3}}{2}\right| = 60^\circ$

Since, $\sin 2(x + 10^\circ) < 0$, $2(x + 10^\circ)$ must reside in Q3 and/or Q4.

Hence, $2(x + 10^\circ) = 180^\circ + 60^\circ$ [Q3]

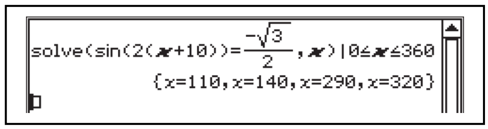
or $360^\circ - 60^\circ$ [Q4]

or $540^\circ + 60^\circ$ [Q3 + 1 complete revolution]

or $720^\circ - 60^\circ$ [Q4 + 1 complete revolution]

$x + 10^\circ = 120^\circ, 150^\circ, 300^\circ, 330^\circ$

$x = 110^\circ, 140^\circ, 290^\circ, 320^\circ$



Note:

- *In this case, the unknown angles have been transformed.*
  Hence, the domain needs to be adjusted to accommodate these changes.
  If we fail to adjust the domain, we may miss relevant solutions.
- *The reference angle for the unknown angle is the principal value of* $\sin^{-1}|k|$.

Example 13.3

Without the use of a calculator, solve for *x* given that $\cos\left(\frac{x}{2} + \frac{\pi}{4}\right) = \frac{1}{2}$ $\quad -\pi \le x \le \pi$

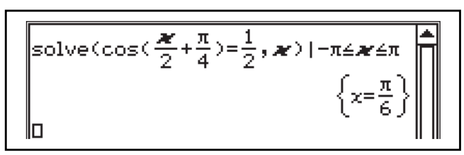
Solution:

Adjusted domain for $\left(\frac{x}{2} + \frac{\pi}{4}\right) \equiv \frac{1}{2}\left(x + \frac{\pi}{2}\right)$ is $-\frac{\pi}{4} \le x \le \frac{3\pi}{4}$.

Reference angle for $\left(\frac{x}{2} + \frac{\pi}{4}\right) = \cos^{-1}\left|\frac{1}{2}\right| = \frac{\pi}{3}$.

Since $\cos\left(\frac{x}{2} + \frac{\pi}{4}\right) > 0$, $\left(\frac{x}{2} + \frac{\pi}{4}\right)$ must reside in Q1 and/or Q4.

Hence, $\left(\frac{x}{2} + \frac{\pi}{4}\right) = \frac{\pi}{3}, -\frac{\pi}{3}$



$\frac{x}{2} = \frac{\pi}{12}, -\frac{7\pi}{12}$

$x = \frac{\pi}{6}$. (Reject $-\frac{7\pi}{6}$ as this solution is outside the domain.)

Example 13.4

Without the use of a calculator, find all solutions for x:

(a) $\cos x = \frac{1}{2}$ (b) $\cos x = -\frac{1}{2}$

Solution:

(a) Reference angle for x is $\cos^{-1}\left|\frac{1}{2}\right| = \frac{\pi}{3}$.

Since $\cos x > 0$, x resides in Q1 or Q4.

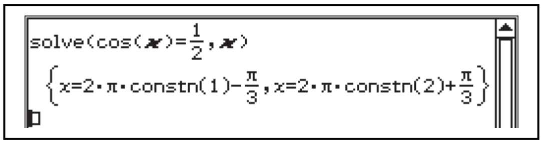
In Q1: $x = \frac{\pi}{3}, \frac{\pi}{3} + 2\pi, \frac{\pi}{3} + 4\pi, \frac{\pi}{3} + 6\pi, \ldots$

$= \frac{\pi}{3} + 2n\pi$

In Q4 $x = -\frac{\pi}{3} + 2\pi, -\frac{\pi}{3} + 4\pi, -\frac{\pi}{3} + 6\pi, -\frac{\pi}{3} + 8\pi, \ldots$

$= -\frac{\pi}{3} + 2n\pi$

Hence, $x = 2n\pi \pm \frac{\pi}{3}$ for integer n.



(b) Reference angle for x is $\cos^{-1}\left|-\frac{1}{2}\right| = \frac{\pi}{3}$.

Since $\cos x < 0$, x resides in Q2 or Q3.

In Q2: $x = \pi - \frac{\pi}{3}, 3\pi - \frac{\pi}{3}, 5\pi - \frac{\pi}{3}, 7\pi - \frac{\pi}{3}, \ldots$

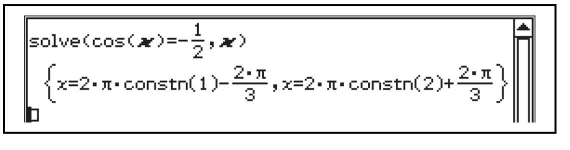
$= (2n + 1)\pi - \frac{\pi}{3} = 2n\pi + \frac{2\pi}{3}$

In Q3 $x = \pi + \frac{\pi}{3}, 3\pi + \frac{\pi}{3}, 5\pi + \frac{\pi}{3}, 7\pi + \frac{\pi}{3}, \ldots$

$= (2n + 1)\pi + \frac{\pi}{3} = 2n\pi + \frac{4\pi}{3}$

$= 2n\pi - \frac{2\pi}{3}$

Hence, $x = 2n\pi \pm \frac{2\pi}{3}$ for integer n.



Notes:

- *The results in (a) and (b) are valid for both positive and negative integers n including 0.*
- *In (a), the principal value for x is* $\cos^{-1}\frac{1}{2} = \frac{\pi}{3}$ *and the solution is* $x = 2n\pi \pm \frac{\pi}{3}$ *for integer n.*
- *In (b), the principal value for x is* $\cos^{-1}-\frac{1}{2} = \frac{2\pi}{3}$ *and the solution is* $x = 2n\pi \pm \frac{2\pi}{3}$ *for integer n.*
- *In general, the solution to cos x = k is* $x = 2n\pi \pm \cos^{-1}k$ *where* $\cos^{-1}k$ *is the principal value of x for integer n.*

Example 13.5

Without the use of a calculator, find all solutions for x:

(a) $\tan x = 1$ (b) $\tan x = -1$.

Solution:

(a) Reference angle for x is $\tan^{-1}\left|1\right| = \dfrac{\pi}{4}$. Since $\tan x > 0$, x resides in Q1 or Q3.

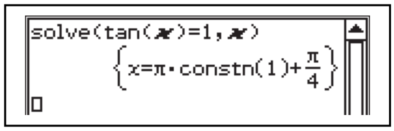
In Q1: $x = \dfrac{\pi}{4}, \dfrac{\pi}{4} + 2\pi, \dfrac{\pi}{4} + 4\pi, \dfrac{\pi}{4} + 6\pi, \ldots$

$= \dfrac{\pi}{4} + 2n\pi$

In Q3 $x = \dfrac{\pi}{4} + \pi, \dfrac{\pi}{4} + 3\pi, \dfrac{\pi}{4} + 5\pi, \dfrac{\pi}{4} + 7\pi, \ldots$

$= \dfrac{\pi}{4} + (2n + 1)\pi$

Hence, $x = \dfrac{\pi}{4} + n\pi$ for integer n.



(b) Reference angle for x is $\tan^{-1}\left|-1\right| = \dfrac{\pi}{4}$. Since $\tan x < 0$, x resides in Q2 or Q4.

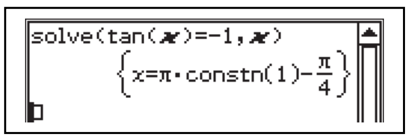
In Q2: $x = \pi - \dfrac{\pi}{4}, 3\pi - \dfrac{\pi}{4}, 5\pi - \dfrac{\pi}{4}, 7\pi - \dfrac{\pi}{4}, \ldots$

$= (2n + 1)\pi - \dfrac{\pi}{4}$

In Q4 $x = 2\pi - \dfrac{\pi}{4}, 2(2\pi) - \dfrac{\pi}{4}, 3(2\pi) - \dfrac{\pi}{4}, 4(2\pi) - \dfrac{\pi}{4}, \ldots$

$= 2n\pi - \dfrac{\pi}{4}$

Hence, $x = -\dfrac{\pi}{4} + n\pi$ for integer n.



Notes:

- *The results in (a) and (b) are valid for both positive and negative integers n including 0.*
- *In (a), the principal value for x is* $\tan^{-1} 1 = \dfrac{\pi}{4}$ *and the solution is* $x = \dfrac{\pi}{4} + n\pi$ *for integer n.*
- *In (b), the principal value for x is* $\tan^{-1} -1 = -\dfrac{\pi}{4}$ *and the solution is* $x = -\dfrac{\pi}{4} + n\pi$ *for integer n.*
- *In general, the solution to* $\tan x = k$ *is* $x = \tan^{-1} k + n\pi$ *where* $\tan^{-1} k$ *is the principal value of x.*

Example 13.6

Without the use of a calculator, find all solutions for x given that:

(a) $\sin x^{\circ} = \frac{1}{2}$ (b) $\sin x^{\circ} = -\frac{1}{2}$

Solution:

(a) Reference angle for x is $\sin^{-1}\left|\frac{1}{2}\right| = 30^{\circ}$.

Since $\sin x^{\circ} > 0$, x° must reside in Q1 or Q2.

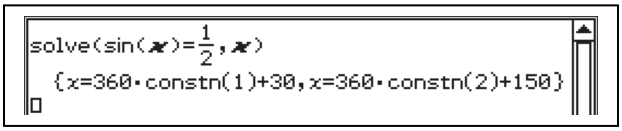
In Q1: $x = 30^{\circ}, 30^{\circ} + 360^{\circ}, 30^{\circ} + 2(360^{\circ}), 30^{\circ} + 3(360^{\circ}), \ldots$

That is: $x = 30^{\circ} + 360n^{\circ}$ for integer n.

In Q2: $x = 150^{\circ}, 150^{\circ} + 360^{\circ}, 150^{\circ} + 2(360^{\circ}), 150^{\circ} + 3(360^{\circ}), \ldots$

That is: $x = 150^{\circ} + 360n^{\circ}$ for integer n.

Hence, $x = 30^{\circ} + 360n^{\circ}$
or $150^{\circ} + 360n^{\circ}$
for integer n.



(b) Reference angle for x is $\sin^{-1}\left|-\frac{1}{2}\right| = 30^{\circ}$.

Since $\sin x^{\circ} < 0$, x° must reside in Q3 or Q4.

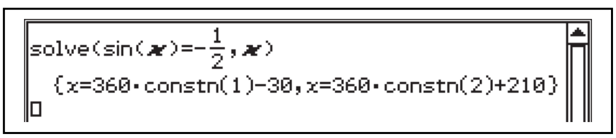
In Q3: $x = 180^{\circ} + 30^{\circ}, 210^{\circ} + 360^{\circ}, 210^{\circ} + 2(360^{\circ}), 210^{\circ} + 3(360^{\circ})\ldots$

That is: $x = 210^{\circ} + 360n^{\circ}$ for integer n.

In Q4: $x = -30^{\circ}, -30^{\circ} + 360^{\circ}, -30^{\circ} + 2(360^{\circ}), -30^{\circ} + 3(360^{\circ}), \ldots$

That is: $x = -30^{\circ} + 360n^{\circ}$ for integer n.

Hence, $x = 210^{\circ} + 360n^{\circ}$
or $-30^{\circ} + 360n^{\circ}$
for integer n.



Notes:

- *The results in (a) and (b) are valid for both positive and negative integers n including 0.*
- *In (a), the principal value for x is* $\sin^{-1} 1 = 30^{o}$ *and the solution is* $x = 30^{o} \pm 360n^{o}$ *or* $150^{o} \pm 360n^{o}$ *for integer n.*
- *In (b), the principal value for x is* $\sin^{-1} -1 = -30^{o}$ *and the solution is* $x = 210^{o} \pm 360n^{o}$ *or* $-30^{o} \pm 360n^{o}$ *for integer n.*
- *In general, the solution to* $\sin x = k$ *is* $x = \sin^{-1} k + 2n\pi$ *or* $x = -\sin^{-1} k + (2n + 1)\pi$ *where* $\sin^{-1} k$ *is the principal value of x.*
- *This can further be condensed into* $x = (-1)^{n} \sin^{-1} k + n\pi$ *for integer n.*

13.2.1 General Solutions

- As seen in the previous three examples, the solutions to each of the trigonometric equations may be expressed in general form. These are known as the general solutions to the trigonometric equation concerned.
- The table below lists the general solutions to the various trigonometric ratios. Note that $\mathbb{Z}$ represents the set of all integers.

Equation	General Solution
$\sin x = k$	$x = (-1)^n \sin^{-1} k + n\pi \quad n \in \mathbb{Z}$. Alternatively: $x = \sin^{-1} k + 2n\pi, -\sin^{-1} k + (2n+1)\pi \quad n \in \mathbb{Z}$
$\cos x = k$	$x = 2n\pi \pm \cos^{-1} k \quad n \in \mathbb{Z}$.
$\tan x = k$	$x = \tan^{-1} k + n\pi \quad n \in \mathbb{Z}$

Example 13.7

Without the use of a calculator, find all solutions for x given that:

(a) $\cos\left(\frac{\pi}{5} - 2x\right) = -\frac{\sqrt{3}}{2}$ (b) $\sin\left(\frac{x}{2} - \frac{\pi}{4}\right) = \frac{1}{2}$

Solution:

(a) Principal value for $\left(\frac{\pi}{5} - 2x\right)$ is $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$

Hence, $\left(\frac{\pi}{5} - 2x\right) = 2n\pi \pm \frac{5\pi}{6}$

$$2x = \frac{\pi}{5} \pm \frac{5\pi}{6} - 2n\pi$$

$$x = \frac{\pi}{10} \pm \frac{5\pi}{12} - n\pi \quad n \in \mathbb{Z}.$$

(b) Principal value for $\left(\frac{x}{2} - \frac{\pi}{4}\right)$ is $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

Hence, $\left(\frac{x}{2} - \frac{\pi}{4}\right) = (-1)^n \times \frac{\pi}{6} + n\pi$

$$\frac{x}{2} = (-1)^n \times \frac{\pi}{6} + n\pi + \frac{\pi}{4}$$

$$x = (-1)^n \times \frac{\pi}{3} + 2n\pi + \frac{\pi}{2} \quad n \in \mathbb{Z}.$$

Example 13.8

Without the use of a calculator, solve for all values of x:

(a) $\sin x = \sqrt{3}\cos x$ (b) $\tan^2 x + \tan x - 6 = 0$

Solution:

(a) $\sin x = \sqrt{3}\cos x \Rightarrow \tan x = \sqrt{3}$

Principal value of $x = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$

$$x = \frac{\pi}{3} + n\pi \quad n \in \mathbb{Z}.$$

(b) $\tan^2 x + \tan x - 6 = 0 \Rightarrow (\tan x + 3)(\tan x - 2) = 0$

$\tan x = -3$ or 2

Hence: $x = \tan^{-1}(-3) + n\pi$

or $\tan^{-1} 2 + n\pi \quad n \in \mathbb{Z}$.

Example 13.9

Without the use of a calculator, solve for all values of x (in degrees):

(a) $2\cos^2 x - 3\cos x - 2 = 0$ (b) $\sin^2 x - \sin x \cos x - 2\cos^2 x = 0$

Solution:

(a) $2\cos^2 x - 3\cos x - 2 = 0 \Rightarrow (2\cos x + 1)(\cos x - 2) = 0$

$\Rightarrow \cos x = -\frac{1}{2}$ or $\cos x = 2$ (reject as $-1 \le \cos x \le 1$)

For $\cos x = -\frac{1}{2}$, principal value for $x = 120^{\circ}$.

$\Rightarrow x = 360n^{\circ} \pm 120^{\circ} \quad n \in \mathbb{Z}$.

(b) $\sin^2 x - \sin x \cos x - 2\cos^2 x = 0 \Rightarrow (\sin x - 2\cos x)(\sin x + \cos x) = 0$

For $\sin x - 2\cos x = 0$, rewrite as $\sin x = 2\cos x$.

$\tan x = 2$

$x = \tan^{-1} 2 + 180n^{\circ} \quad n \in \mathbb{Z}$.

For $\sin x + \cos x = 0$, rewrite as $\sin x = -\cos x$.

$\tan x = -1$

For $\tan x = -1$, principal value for $x = \tan^{-1}(-1) = -45^{\circ}$

$\Rightarrow x = -45^{\circ} + 180n^{\circ} \quad n \in \mathbb{Z}$.

Therefore, $x = \tan^{-1} 2 + 180n^{\circ}$ or $-45^{\circ} + 180n^{\circ} \quad n \in \mathbb{Z}$.

Exercise 13.1

1. Without the use of a calculator, solve for x within the given domain:
   (a) $2 \sin 2x = 1 \quad 0 \le x \le 360^\circ$
   (b) $2 \cos (2x + 10^\circ) = \sqrt{3} \quad 0 \le x \le 360^\circ$
   (c) $\sqrt{3} \tan (10 - 2x) = -1 \quad 0 \le x \le 360^\circ$
   (d) $\cos 3x = -1 \quad 0 \le x \le 360^\circ$

2. Without the use of a calculator, solve for x within the given domain:
   (a) $\tan \left(\frac{x}{2} + \frac{\pi}{6}\right) = -1 \quad 0 \le x \le 2\pi$
   (b) $\sin \frac{1}{2}\left(x + \frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \quad 0 \le x \le 2\pi$
   (c) $\cos 2\left(x - \frac{\pi}{12}\right) = 0 \quad 0 \le x \le 2\pi$
   (d) $1 + 2 \sin \frac{x}{3} = 0 \quad -2\pi \le x \le 2\pi$

3. Without the use of a calculator, solve for all values of θ (radians).
   (a) $(\sin \theta + 1)(\cos \theta - 2) = 0$
   (b) $\cos \theta (2 \cos \theta - 1) = 0$
   (c) $2\sin^2\theta + \sin \theta = 0$
   (d) $\tan^2\theta = \tan \theta$

4. Without the use of a calculator, solve for all values of θ (radians).
   (a) $\cos^2\theta + \cos \theta = 0$
   (b) $2 \sin^2\theta - \sqrt{3} \sin \theta = 0$
   (c) $\sin \theta + \cos \theta = 0$
   (d) $\tan^2\theta - 3 = 0$

5. Solve for all values of x (degrees):
   (a) $\tan^2 x - 4 \tan x + 3 = 0$
   (b) $2 \sin^2 x = 3 \sin x + 2$
   (c) $2 \cos^2 x + 5 \cos x = 1$
   (d) $\cos^2 x + \cos x - 3 = 0$

6. Without the use of a calculator, solve for all values of t (radians):
   (a) $(\sin t - 1)(\sin 2t + 1) = 0$
   (b) $\cos^2 2t - 2 \cos 2t = -1$
   (c) $3 \sin^2 t - \cos^2 t = 0$
   (d) $4 \sin^2 2t - 1 = 0$

7. Without the use of a calculator, solve for all values of θ (radians):
   (a) $\sqrt{3} \sin \theta \cos \theta - \cos \theta = 0$
   (b) $(\tan \theta + 3)(\tan^2\theta - 4) = 0$
   (c) $\sin^2\theta - 3 \sin \theta \cos \theta + 2 \cos^2 \theta = 0$
   (d) $2 \cos^3\theta + 7 \cos^2 \theta - 4 \cos \theta = 0$

13.3 Reciprocal Trigonometric Functions

- The *reciprocal* trigonometric functions are defined as:
  - Secant (θ) = $\frac{1}{\cos\theta}$, denoted sec (θ);
  - Cosecant (θ) = $\frac{1}{\sin\theta}$, denoted cosec (θ);
  - Cotangent (θ) = $\frac{1}{\tan\theta}$, abbreviated cot (θ).
- There are no calculator dedicated "buttons" for these functions.
  Hence, to calculate the values of these functions, we need to return to the basic trigonometric functions of sine, cosine and tangent.

Example 13.10

Without the use of a calculator, determine the values of:

(a) $\sec 45^{\circ}$ (b) $\operatorname{cosec} 90^{\circ}$ (c) $\cot \frac{\pi}{2}$

Solution:

(a) $\sec 45^{\circ} = \frac{1}{\cos 45^{\circ}} = \sqrt{2}$.

(b) $\operatorname{cosec} 0^{\circ} = \frac{1}{\sin 90^{\circ}} = 1$

(c) $\cot \frac{\pi}{2} = \frac{1}{\tan\frac{\pi}{2}} \to 0$.

Example 13.11

Without the use of a calculator, solve for x given that cosec $x = 2$, where $0 \le x \le 360^{\circ}$.

Solution:

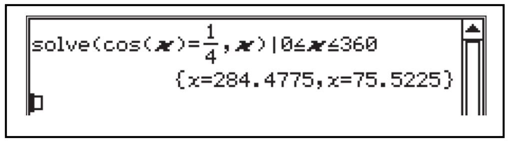
$\operatorname{cosec} x = 2 \Rightarrow \sin x = \frac{1}{2}$

$x = 30^{\circ}, 150^{\circ}$.

Example 13.12

Solve for x given that sec x = 4, where $0 \le x \le 360^{\circ}$.

Solution:



$$\sec x = 4 \;\Rightarrow\; \cos x = \frac{1}{4}$$

$$x = 284.5^{\circ},\ 75.5^{\circ}.$$

Example 13.13

Without the use of calculator solve for all values of x given that $\cot\left(2x+\frac{\pi}{4}\right) = \sqrt{3}$.

Solution:

$$\cot\left(2x+\frac{\pi}{4}\right) = \sqrt{3} \;\Rightarrow\; \tan\left(2x+\frac{\pi}{4}\right) = \frac{1}{\sqrt{3}}$$

$$2x+\frac{\pi}{4} = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) + n\pi$$

$$2x = \frac{\pi}{6}-\frac{\pi}{4} + n\pi$$

$$x = -\frac{\pi}{24} + n\pi \quad \text{for integer } n.$$

Exercise 13.2

1. Without the use of a calculator, determine the values of:

   (a) $\sec 30^{\circ}$ (b) $\operatorname{cosec} 45^{\circ}$ (c) $\cot 120^{\circ}$ (d) $\operatorname{cosec} 0^{\circ}$

   (e) $\cot \frac{3\pi}{4}$ (f) $\sec \frac{7\pi}{6}$ (g) $\operatorname{cosec} -\frac{5\pi}{6}$ (h) $\frac{1}{\sec\pi}$.

2. Without the use of a calculator, solve for x, within the indicated domain.

   (a) $\cot x = -1 \quad 0 \le x \le 360^{\circ}$ (b) $\sqrt{3}\sec x = 2 \quad 0 \le x \le 360^{\circ}$

   (c) $\operatorname{cosec}(2x) = -2 \quad 0 \le x \le 360^{\circ}$ (d) $\cot \frac{x}{2} = \sqrt{3} \quad 0 \le x \le 360^{\circ}$

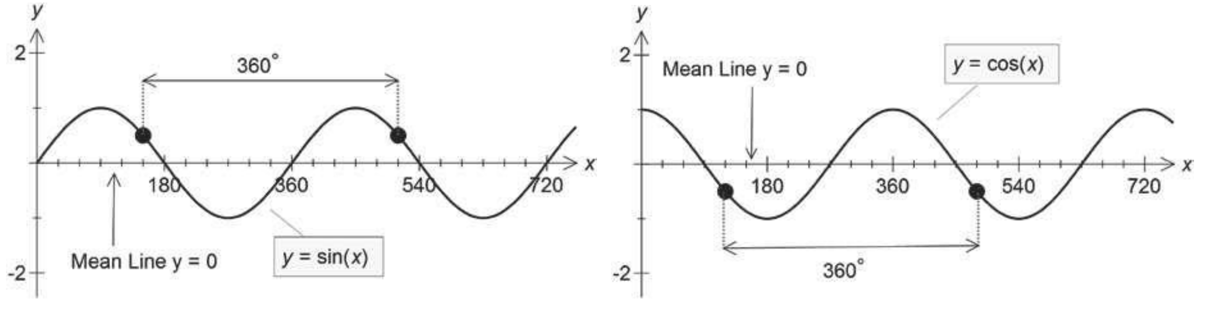
3. Without the use of a calculator, solve for all values of x (radians)>

   (a) $\sec(2x) = -2$ (b) $\cot(2x) = -1$ (c) $\operatorname{cosec}\left(\frac{x}{2}\right) = \frac{2\sqrt{3}}{3}$.

14 Trigonometric Graphs

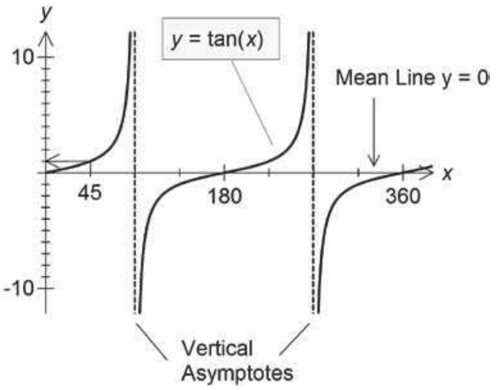
14.1 Graphs of basic trigonometric functions

- In this section we will examine the graphs of trigonometric functions.
- Consider the graphs of $y = \sin x$ and $y = \cos x$ for $0 \le x \le 360^{o}$.



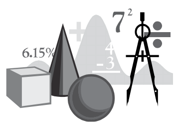
- The graphs of $y = \sin x$ and $y = \cos x$ consist of a repeated regular waveform. Such functions are referred to as *periodic functions*.
- In each case, the pattern covers a horizontal distance of 360^{o} before the pattern repeats itself. The *period* of the sin x and cos x function is 360^{o}.
- The heights of the curve after every 360^{o} are identical.
  That is: $\sin(x + 360^{o}) = \sin(x)$ and $\cos(x + 360^{o}) = \cos x$.
  In general, $\sin(x + 360^{o}n) = \sin(x)$ and $\cos(x + 360^{o}n) = \cos(x)$,
  where n is an integer.
- The mean line or equilibrium line cuts through the “middle” of the graph. The equation of the mean line is $y = 0$.
- The *amplitude* is the distance from the mean line to the maximum point or minimum point of the graph. The amplitude of the sin x and cos x function is 1.
- Hence for $y = \sin x$ and $y = \cos x$:
  - Equation of mean line is $y = 0$.
  - Period = 360^{o} or 2π radians
  - Amplitude = 1 unit
  - Minimum value -1
  - Maximum value = 1 .
- Notice that the $y = \cos x$ curve can be obtained by translating the $y = \sin x$ curve 90^{o} to the left. Hence: $\sin(x + 90^{o}) = \cos x$.
- Similarly the $y = \sin x$ curve can be obtained by translating the $y = \cos x$ curve 90^{o} to the right. Hence: $\cos(x - 90^{o}) = \sin x$.

- Consider now the graph of $y = \tan x$ for $0 \leq x \leq 360^{\circ}$.



- The period of the tan x function is 180°.
  Hence, $\tan(x + 180^{\circ}n) = \tan(x)$, where n is an integer.
- The equation of the mean line is $y = 0$.
- The function does not have any maximum or minimum points.
  Hence, the concept of amplitude is not applicable.
- The graph has vertical asymptotes at $x = 90^{\circ}$ and $x = 270^{\circ}$.
- One point of interest is $(45^{\circ}, 1)$.

14.2 Graphs of Trigonometric functions of the form $y = a\,f\,(bx + c) + d$



Hands On Task 14.1

Recall that for $y = \sin x$ and $y = \cos x$:

- Equation of mean line is $y = 0$.
- Period = 360° or 2π radians
- Amplitude = 1 unit

1. By applying the appropriate translation to the curve of $y = \sin x$ or $y = \cos x$, determine the equation of the mean line for the curve with equation:

   (a) $y = \sin(x) + 5$ (b) $y = \cos(2x) - 3$ (c) $y = 2 - \sin(x + 30^{\circ})$

2. By applying the appropriate dilation to the curve of $y = \sin x$ or $y = \cos x$, determine the amplitude for the curve with equation:

   (a) $y = 2\sin x$ (b) $y = 3\cos(2x) + 3$ (c) $y = -4\sin(x - \frac{\pi}{6})$.

3. By applying the appropriate dilation to the curve of $y = \sin x$ or $y = \cos x$, determine the period for the curve with equation:

   (a) $y = \sin 2x$ (b) $y = \cos \frac{x}{2} + 3$ (c) $y = 1 - 2\sin(x + 60^\circ)$

4. The graph of $y = \sin(x)$ passes through the point (0, 0). By applying the appropriate translation (and dilation) to the curve of $y = \sin x$, determine the image of this point for:

   (a) $y = \sin(x - 50^\circ)$ (b) $y = \sin(2x + \frac{\pi}{3})$ (c) $y = \sin 2\left(x + \frac{\pi}{3}\right)$.

5. The graph of $y = \cos(x)$ passes through the point (0, 1). By applying the appropriate translation (and dilation) to the curve of $y = \cos x$, determine the image of this point for:

   (a) $y = \cos(x + 50^\circ)$ (b) $y = \cos(2x - \frac{\pi}{3})$ (c) $y = \cos 2\left(x + \frac{\pi}{3}\right)$.

6. By applying the appropriate translation and dilation to the curve of $y = \sin x$ or $y = \cos x$, determine the minimum and maximum value for y for:

   (a) $y = -2\sin(x + 10^\circ)$ (b) $y = 1 + 2\cos 2x$ (c) $y = 3 - 4\sin \frac{x}{2}$.

7. Use the observations made above to determine for the curves with equations $y = a\sin(bx + c) + d$ and $y = a\cos(bx + c) + d$:
   (i) the equation of the mean line
   (ii) the amplitude
   (iii) the period
   (iv) the minimum and maximum values of y.

8. By applying the appropriate dilation to the curve of $y = \tan x$ determine the period for the curve with equation:

   (a) $y = \tan 2x$ (b) $y = -\tan \frac{x}{2}$ (c) $y = 1 + 2\tan(x + 60^\circ)$

9. The graph of $y = \tan(x)$ passes through the point (0, 0). By applying the appropriate translation (and dilation) to the curve of $y = \tan x$ determine the image of this point for:

   (a) $y = \tan(x + 20^\circ)$ (b) $y = \tan(2x - \frac{\pi}{4})$ (c) $y = \tan 2\left(x - \frac{\pi}{4}\right)$.

10. Use the observations made above to determine for the curve with equation $y = a\tan(bx + c) + d$:
    (i) the equation of the mean line
    (ii) the period.

Summary

	Equation of Mean Line	Amplitude	Minimum/ Maximum y-value	Period	Phase shift
$y = \sin x$ $y = \cos x$	$y = 0$	1	Min: -1 Max: 1	360° or 2π	0
$y = a \sin (bx + c) + d$ $y = a \cos (bx + c) + d$	$y = d$	$\lvert a \rvert$	Min: $d - \lvert a \rvert$ Max: $d + \lvert a \rvert$	$\frac{360^\circ}{b}$ or $\frac{2\pi}{b}$	*Shifted* $\frac{c}{b}$ *degrees/ radians to the left*
$y = \tan x$	$y = 0$			180° or π	0
$y = a \tan (bx + c) + d$	$y = d$			$\frac{180^\circ}{b}$ or $\frac{\pi}{b}$	*Shifted* $\frac{c}{b}$ *degrees/ radians to the left*

Notes:

- *The phase shift refers to the magnitude of the horizontal translation applied to the $y = \sin x$, $y = \cos x$ or $y = \tan x$ curve.*

Example 14.1

For the curve $y = 5 \sin (2x + 60^\circ)$, without the use of a calculator, state:
(a) the equation of the mean line, (b) the amplitude, (c) the period (d) the phase shift
(e) the maximum and minimum y-values.

Solution:

(a) Equation of mean line is $y = 0$

(b) Amplitude = 5

(c) Period = $\frac{360^\circ}{2} = 180^\circ$

(d) Phase shift = $\frac{60^\circ}{2} = 30^\circ$ to the left

(e) Minimum y-value: -5 maximum y-value: 5

Example 14.2

For the curve $y = 10 - 4\cos\frac{1}{2}\left(x - \frac{\pi}{6}\right)$, without the use of a calculator, state:

(a) the equation of the mean line, (b) the amplitude, (c) the period (d) the phase shift
(e) the maximum and minimum y-values.

Solution:

(a) Equation of mean line is $y = 10$

(b) Amplitude = 4

(c) Period = $\frac{2\pi}{\frac{1}{2}} = 4\pi$

(d) Phase shift $\frac{\pi}{6}$ to the right

(e) Minimum y-value: 6 maximum y-value: 14

Note:

- *For the trigonometric curve with equation $y = a\,f\,(bx + c) + d$, the phase shift is $-\frac{c}{b}$.*
- *For the trigonometric curve with equation $y = a\,f\,(b(x + c)) + d$, the phase shift is $-\frac{bc}{b} = -c$.*

Example 14.3

A tangent curve $y = f\,(x)$ has period $\frac{\pi}{2}$ with mean line $y = 2$ and phase shift $\frac{\pi}{6}$ to the right.

Given that the curve passes through the point $(\frac{7\pi}{24}, -1)$, find $f\,(x)$.

Solution:

Let $y = a\tan(bx + c) + d$.

Equation of mean line is $y = 2 \Rightarrow d = 2$.

Period is $\frac{\pi}{b} = \frac{\pi}{2} \Rightarrow b = 2$.

Phase shift is $-\frac{c}{b} = \frac{\pi}{6} \Rightarrow c = -\frac{\pi}{6} \times 2 = -\frac{\pi}{3}$.

Hence, $y = a\tan(2x - \frac{\pi}{3}) + 2$.

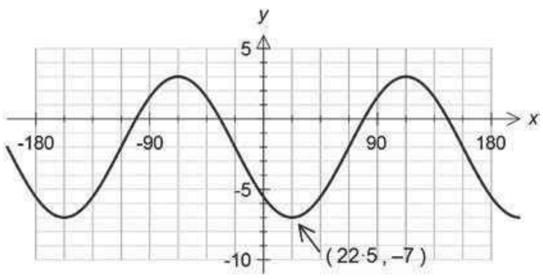
When $x = \frac{7\pi}{24}$, $y = -1$ $\Rightarrow -1 = a \times \tan\left(\frac{7\pi}{12} - \frac{\pi}{3}\right) + 2$

$$-3 = a \times \tan\frac{\pi}{4} \Rightarrow a = -3$$

Therefore, equation of curve is $y = -3\tan(2x - \frac{\pi}{3}) + 2$.

Example 14.4

The sketch of the curve $y = a \cos(bx + c) + d$ is given in the accompanying diagram. Given that $a < 0$ and the curve passes through the point with coordinates (22.5, −7), find the values for a, b, c and d.



Solution:

Minimum y-value = −7 Maximum y-value = 3

Hence, mean y-value = $\frac{-7+3}{2} = -2$.

Therefore, the equation of the mean line is $y = -2$. $\Rightarrow$ $d = -2$

Difference between maximum y-value and mean y-value = $3 - (-2) = 5$
$\Rightarrow$ amplitude = 5 $\Rightarrow$ $|a| = 5$. But, $a < 0$, $\Rightarrow$ $a = -5$.

Period = 180° $\Rightarrow$ $\frac{360^\circ}{b} = 180^\circ$ $\Rightarrow$ $b = 2$

Hence $y = -5 \cos(2x + c) - 2$.

In sketch above, when $x = 22.5^\circ$, $y = -7$ $\Rightarrow$ $-7 = -5\cos(45^\circ + c) - 2$

$\cos(45^\circ + c) = 1$ $\Rightarrow$ $c = -45^\circ$.

Therefore, equation of given curve is $y = -5\cos(2x - 45^\circ) - 2$.

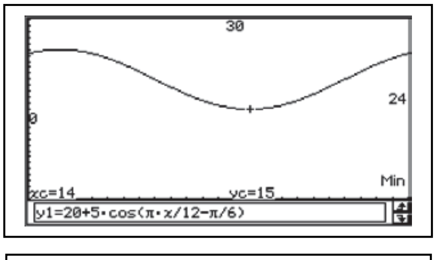
Example 14.5

The temperature θ C inside a warehouse t hours after 12 noon is modelled by the equation $\theta = 20 + 5\cos\left(\frac{\pi t}{12} - \frac{\pi}{6}\right)$. Use a graphical method to determine the:

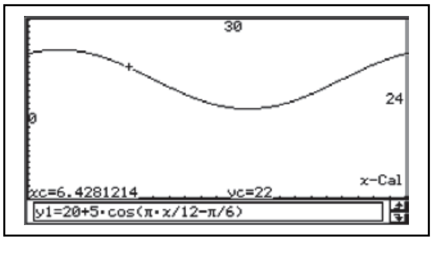
(a) minimum temperature inside the warehouse and the first time after 12 noon this occurs

(b) number of hours in a day when the temperature inside the warehouse is above 22 °C.

Solution:



(a) Use the "Min" command: .
The minimum temperature of 15 C occurs 14 hours after 12 noon, i.e. at 2 am.



(b) Use the "x-cal" command.
$\theta = 22$ for $t = 6.4281$ and 21.5719.

From the graph drawn,
$\theta \geq 22$ for $0 \leq t \leq 6.4281$ and $21.5719 \leq t \leq 24$,
that is for 8.8562 hours a day.
Hence, 8 hours and 51 min.

Exercise 14.1

1. Without the use of a calculator, for each of the following, find x where $|x|$ is acute:

   (a) $\cos 70^{\circ} = \sin x$ (b) $\sin \frac{5\pi}{6} = \cos x$ (c) $\cos 150^{\circ} = \sin x$

2. For each of the given curves, state the: (i) equation of the mean line, (ii) amplitude, (iii) period (iv) phase shift (v) maximum and minimum y-values.

   (a) $y = -3 \sin x$ (b) $y = 0.5 \cos (2x)$ (c) $y = -2 \cos (x + 40^{\circ})$

   (d) $y = -6 - 2 \cos (\pi x)$ (e) $y = 2 + 3 \cos (\pi x + \frac{\pi}{6})$ (f) $y = 2\pi \cos (2\pi x + \frac{\pi}{12})$

   (g) $y = 5 - 8 \sin (2\pi x - \frac{\pi}{8})$ *(h) $y = 4 \cos (x) + 3 \sin (x)$

3. For each of the given curves, state the: (i) equation of the mean line, (ii) period (iii) phase shift (iv) equation of the asymptotes enclosing one cycle of the curve.

   (a) $y = \tan 2x$ (b) $y = 1 + 0.5 \tan x$ (c) $y = \tan (2x - 30^{\circ})$

   (d) $y = 5 + 2 \tan (\pi x)$ (e) $y = 3 - 3 \tan (2\pi x - \frac{\pi}{6})$ (f) $y = \pi \tan (\pi x + \frac{\pi}{6})$

4. A sine curve $y = f(x)$ has period 90° with mean line $y = 2$ and no phase shift. Given that y has a maximum value of 5, find a possible equation of this curve.

5. A sine curve $y = f(x)$ has period 45° with mean line $y = 7$ and a phase shift of -30°. Given that y has a minimum value of 4, find a possible equation of this curve.

6. A cosine curve $y = f(x)$ has period 1 with mean line $y = 3$ and a phase shift of 1/6. Given that y has a minimum value of -2, find a possible equation of this curve.

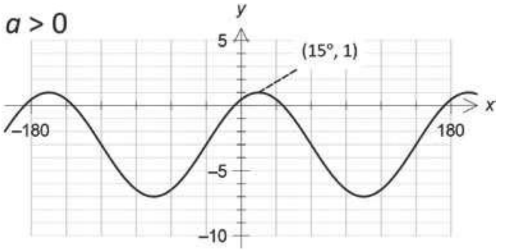
7. A cosine curve $y = f(x)$ has period 2 with mean line $y = -1$ and a phase shift of $-1/8$. Given that y has a maximum value of 5, find a possible equation of this curve.

8. A tangent curve $y = f(x)$ has period 180° with mean line $y = 2$ and phase shift of 45°. Given that the curve passes through the point $(90^{\circ}, 7)$, find $f(x)$.

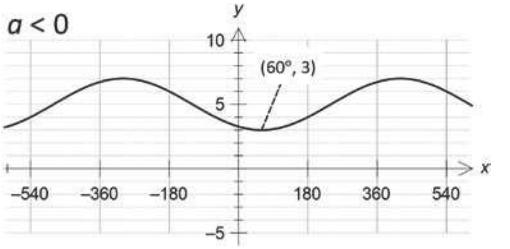
9. A tangent curve $y = f(x)$ has period 360° with mean line $y = -5$ and phase shift of -30°. Given that the curve passes through the point $(60^{\circ}, -4.5)$, find $f(x)$.

10. Find the values for a, b, c and d in the sketch of $y = a \cos (bx + c) + d$ given below.

    (a) $a > 0$

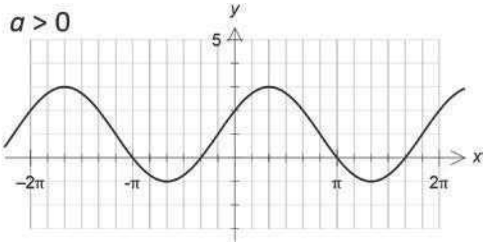


    (b) $a < 0$

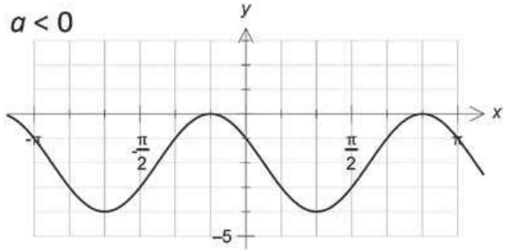


11. The sketch of the curve $y = a \sin(bx + c) + d$ is given in the accompanying diagram. Find the values for a, b, c and d.

(a) $a > 0$



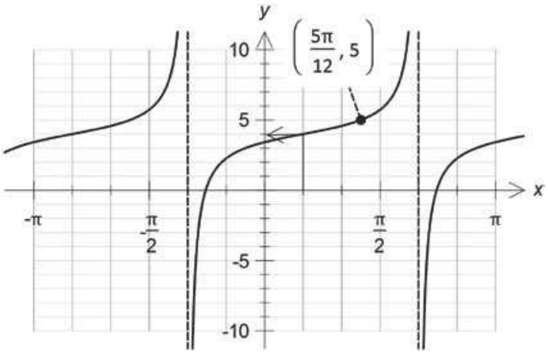
(b) $a < 0$

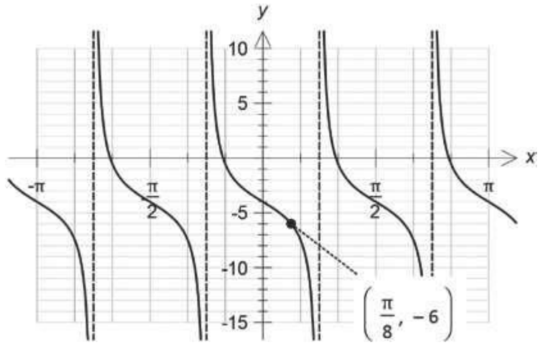


12. Find the values for a, b, c and d in the sketch of $y = a \tan(bx + c) + d$ given below.

(a) $a > 0$

(b) $a < 0$





13. The temperature θ °C inside a garden shed t hours after 6 am is modelled by the equation $\theta = 30 + 15 \sin\left(\frac{\pi t}{12} - \frac{\pi}{4}\right)$. Use a non-graphical method to determine the:

(a) temperature inside the shed at 6 am

(b) maximum temperature inside the shed and the first time after 6 am when this occurs

(c) number of hours in a 24 hour day when the temperature inside the shed is above 40 °C.

14. The water depth, h meters, measured from the bottom of a lake, t hours after 6 am is modelled by the equation $h = 10 - 2\cos\left(\frac{\pi t}{6} - \frac{\pi}{12}\right)$. Use a graphical method to determine the:

(a) water depth at 12 noon

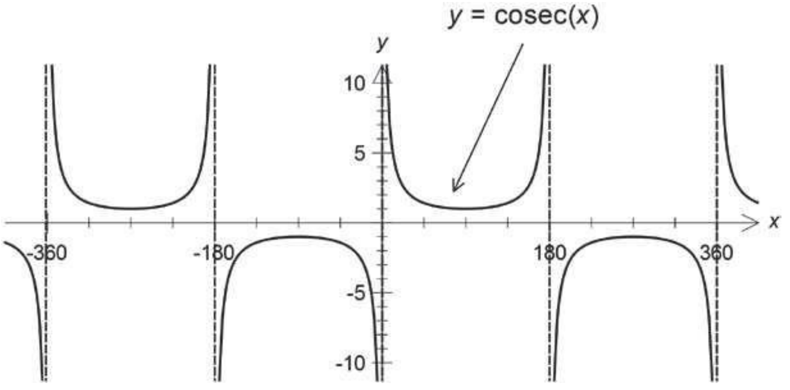
(b) minimum water depth and the first time after 6 am when this occurs

(c) number of hours in one cycle when the water depth is below 9 metres.

*15. The displacement, x cm, of a particle from a fixed point O, t seconds after it is released is modelled by the equation $x = -5\cos 2\pi t$. Use an appropriate method to find the percentage of time over a given cycle when the particle is at least 3 cm away from O. Explain clearly how you obtained your answer.

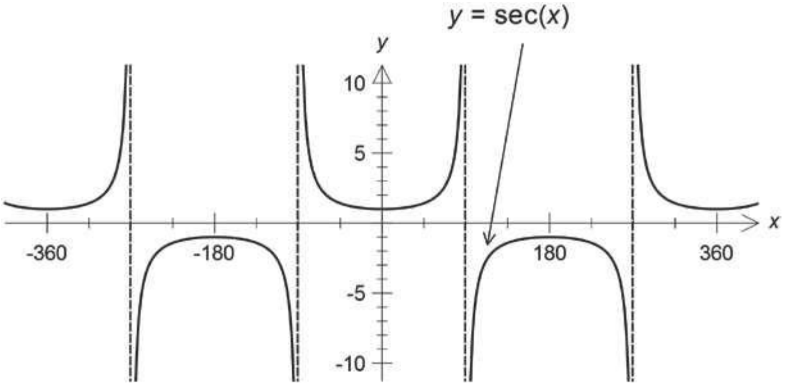
14.3 Graphs of reciprocal trigonometric functions

- Consider the graph of $y = \text{cosec } x$.



  - Equation of mean line is $y = 0$.
  - Period of graph = 360^{o} or 2π.
  - Vertical asymptotes at $x = \pm 180n^{o}$ or $x = \pm n\pi \quad n \in \mathbb{Z}$.
  - Domain = $\{x: x \neq \pm 180n^{o},\ x \in \mathbb{R},\ n \in \mathbb{Z}\}$.
  - Range = $\{y: y \leq -1, y \geq 1, y \in \mathbb{R}\}$.

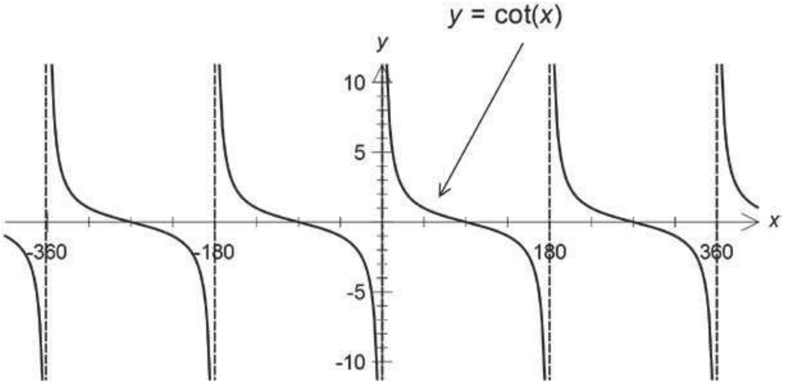
- Consider the graph of $y = \sec x$.



  - Equation of mean line is $y = 0$.
  - Period of graph = 360^{o} or 2π.
  - Vertical asymptotes at $x = \pm 90(2n + 1)^{o}$

    or $x = \pm \dfrac{(2n+1)\pi}{2} \quad n \in \mathbb{Z}$.
  - Domain = $\{x: x \neq 90(2n + 1)^{o},\ x \in \mathbb{R},\ n \in \mathbb{Z}\}$.
  - Range = $\{y: y \leq -1, y \geq 1, y \in \mathbb{R}\}$.

- Consider the graph of $y = \cot x$.



- Equation of mean line is $y = 0$.
- Period of graph = 180° or π.
- Vertical asymptotes at $x = \pm 180n^{\circ}$ or $x = \pm n\pi \quad n \in \mathbb{Z}$
- Domain = $\{x: x \neq 180n^{\circ}, x \in \mathbb{R}, n \in \mathbb{Z}\}$.
- Range = $\mathbb{R}$.

Example 14.6

For the curves (a) $y = -2 \sec x + 4$ (b) $y = 3 \cot (x - 45^{\circ})$ (c) $y = \operatorname{cosec} 2\left(x + \frac{\pi}{6}\right)$,

without the use of a calculator, state:
(i) the equation of the mean line (ii) the period (iii) the phase shift (iv) the range for y
(v) the equations of the vertical asymptotes.

Solution:

(a) (i) Equation of mean line is $y = 4$ (ii) Period = 2π
(iii) No Phase shift (iv) range = $\{y: y \le 2, y \ge 6, y \in \mathbb{R}\}$
(v) vertical asymptotes $x = \pm \dfrac{(2n+1)\pi}{2} \quad n \in \mathbb{Z}$.

(b) (i) Equation of mean line is $y = 0$ (ii) Period = 180°
(iii) Phase shift: 45° to the right (iv) range = **R**
(v) vertical asymptotes $x = 45^{\circ} \pm 180n^{\circ} \quad n \in \mathbb{Z}$.

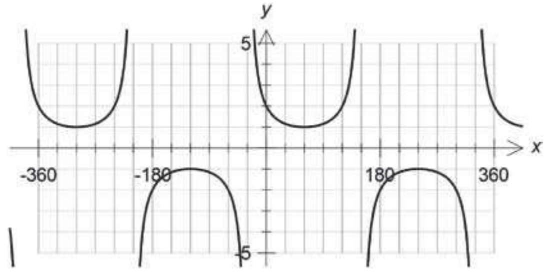
(c) (i) Equation of mean line is $y = 0$ (ii) Period = π
(iii) Phase shift: $\dfrac{\pi}{6}$ to the left (iv) range = $\{y: y \le -1, y \ge 1, y \in \mathbb{R}\}$
(v) vertical asymptotes $x = -\dfrac{\pi}{6} \pm \dfrac{n\pi}{2} \quad n \in \mathbb{Z}$

Exercise 14.2

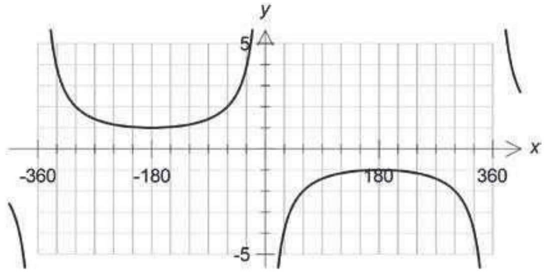
1. Without the use of a calculator, for each of the following, find *x* where $|x|$ is acute:
   (a) $\text{cosec } 50^\circ = \sec x$ (b) $\sec(3\pi/4) = \text{cosec } x$ (c) $\text{cosec } 150^\circ = \sec x$

2. For each of the given curves, state: (i) the equation of the mean line (ii) the period (iii) the phase shift (iv) the range for *y* (v) the equations of the vertical asymptotes.
   (a) $y = -2 \text{ cosec } x$ (b) $y = \cot 3x$ (c) $y = 1 - \sec 2x$
   (d) $y = \sec(2x + 30^\circ)$ (e) $y = \text{cosec}\left(2x + \frac{\pi}{6}\right)$ (f) $y = \cot\left(x - \frac{\pi}{6}\right) + 2.$

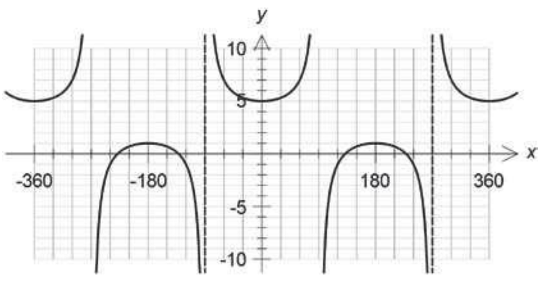
3. Find the values for *a*, *b*, *c* and *d* in the sketch of $y = a \text{ cosec}(bx + c) + d$ given below.
   (a)



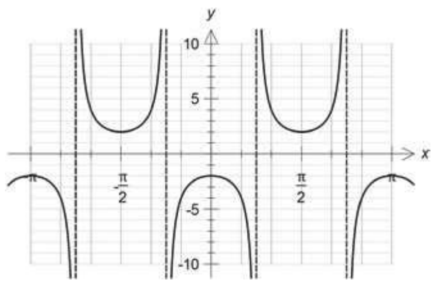
   (b)



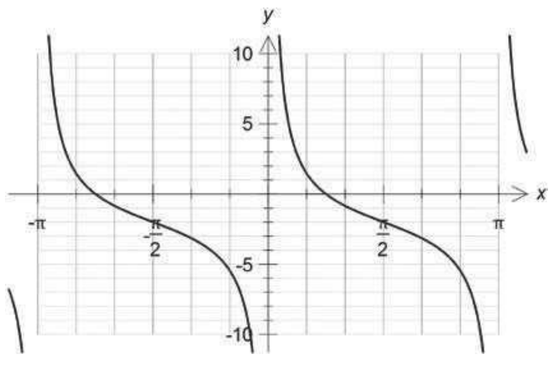
4. Find the values for *a*, *b* (where $b > 0$), *c* and *d* in the sketch of $y = a \sec(bx + c) + d$ given below.
   (a)



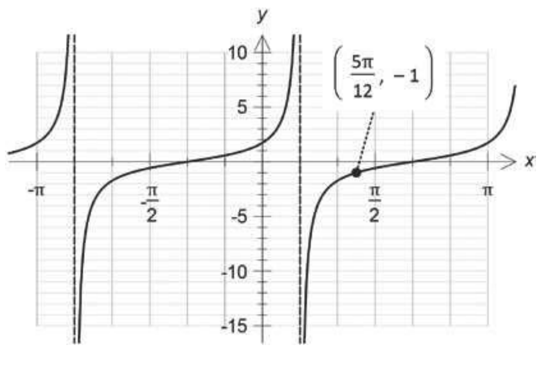
   (b)



5. Find the values for *a*, *b*, *c* and *d* in the sketch of $y = a \cot(bx + c) + d$ given below.
   (a)



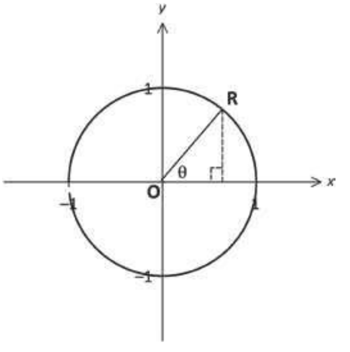
   (b)



15 Trigonometric Identities

15.1 Pythagorean Identities

- Recall the circular function definition for sin θ and cos θ.
- On a unit circle (see accompanying diagram):
  sin (x) = y-coordinate of the radius vector OR
  cos (x) = x-coordinate of the radius vector OR



- Using Pythagoras' Theorem: $x^2 + y^2 = 1$
  Since, $x = \cos\theta$ and $y = \sin\theta$;

  $\Rightarrow \sin^2\theta + \cos^2\theta = 1$
- This result is true for all values of θ and is known as the first of a set of three Pythagorean Identities.
- Consider $\sin^2\theta + \cos^2\theta = 1$.
  - Divide each term with $\cos^2\theta$ where $\theta \neq \dfrac{(2n+1)\pi}{2}$.

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\Rightarrow 1 + \tan^2\theta = \sec^2\theta$$

  - Divide each term with $\sin^2\theta$ where $\theta \neq n\pi$.

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$\Rightarrow 1 + \cot^2\theta = \text{cosec}^2\theta$$

- Hence, the three Pythagorean Identities are:

• $\sin^2\theta + \cos^2\theta = 1$
• $\sec^2\theta = 1 + \tan^2\theta$
• $\text{cosec}^2\theta = 1 + \cot^2\theta$

Example 15.1

Express $3\cos^2\theta + \sin\theta$ in terms of $\sin\theta$.

Solution:

Since $\cos^2\theta = 1 - \sin^2\theta$,

$$3\cos^2\theta + \sin\theta = 3(1 - \sin^2\theta) + \sin\theta$$
$$= -3\sin^2\theta + \sin\theta + 3.$$

Example 15.2

Without the use of a calculator solve for all values of θ (radians) in $2\cos^2\theta - 3\sin\theta = 0$.

Solution:

$$2\cos^2\theta - 3\sin\theta = 0 \quad \Rightarrow \quad 2(1 - \sin^2\theta) - 3\sin\theta = 0$$
$$2\sin^2\theta + 3\sin\theta - 2 = 0$$
$$(2\sin\theta - 1)(\sin\theta + 2) = 0$$
$$\Rightarrow \quad \sin\theta = \frac{1}{2} \text{ or } -2.$$

But, $-1 \le \sin\theta \le 1$, hence reject $\sin\theta = -2$.

Hence, $\sin\theta = \frac{1}{2}$ $\quad \Rightarrow \quad \theta = (-1)^n \sin^{-1}\left(\frac{1}{2}\right) + n\pi$

$$= (-1)^n \frac{\pi}{6} + n\pi \quad n \in \mathbb{Z}.$$

Example 15.3

Without the use of a calculator solve for all values of θ (radians) in $\sec^2\theta - \sqrt{3}\tan\theta - 1 = 0$.

Solution:

$$\sec^2\theta - \sqrt{3}\tan\theta - 1 = 0 \quad \Rightarrow \quad (1 + \tan^2\theta) - \sqrt{3}\tan\theta - 1 = 0$$
$$\tan^2\theta - \sqrt{3}\tan\theta = 0$$
$$\tan\theta(\tan\theta - \sqrt{3}) = 0$$
$$\Rightarrow \quad \tan\theta = 0 \quad \text{or} \quad \tan\theta = \sqrt{3}$$

If $\tan\theta = 0 \quad \Rightarrow \quad \theta = n\pi \quad n \in \mathbb{Z}$.

If $\tan\theta = \sqrt{3} \quad \Rightarrow \quad \theta = \frac{\pi}{3} + n\pi \quad n \in \mathbb{Z}$.

Exercise 15.1

1. Express each of the following expressions in terms of $\sin\theta$.
   (a) $5 - \sin\theta + 5\cos^2\theta$
   (b) $(4\sin\theta + \cos\theta)^2 - 8\sin\theta\cos\theta$
   (c) $\cos\theta\,(\tan\theta + \cos\theta)$
   (d) $\sec^2\theta + 2$

2. Express each of the following expressions in terms of $\cos\theta$.
   (a) $4\sin^2\theta - 2\cos\theta$
   (b) $(1 - \sin\theta)^2 + 2\sin\theta$
   (c) $(1 + \sin^2\theta)^2$
   (d) $\sec^2\theta + \text{cosec}^2\theta$

3. Without the use of a calculator, solve for all values of θ (radians):
   (a) $\cos^2\theta - \sin\theta = 1$
   (b) $2\cos^2\theta + 3\sin\theta - 3 = 0$
   (c) $3\sin^2\theta - 5\cos\theta = 1$
   (d) $6\cos^2\theta - 8\cos\theta\sin\theta + 1 = 0$

4. Without the use of a calculator, solve for all values of θ (radians):
   (a) $2\cos^2\theta - 5\sin\theta - 4 = 0$
   (b) $\cos^2\theta + 3\sin\theta = 3$
   (c) $2\sin^2\theta + 5\cos\theta = 2$ `
   (d) $2\sin^2\theta - 3\sqrt{3}\cos\theta = 5$
   (e) $\sqrt{3}\sin^2\theta - 2\sin\theta\cos\theta - \sqrt{3}\cos^2\theta = 0$
   (f) $\sqrt{3}\,\text{cosec}^2\theta + 2\cot\theta - 2\sqrt{3} = 0$.

15.2 Proving Trigonometric Identities

- To prove a mathematical identity, we need to start with either the expression on the Left Hand Side (LHS) or the Right Hand Side (RHS) and transform it so that it is identical to the expression on the other side.
- As a rule of thumb, start with the side that "looks" more complicated.

Example 15.4

Prove that $(\cos\theta + \sin\theta)^2 = 1 + 2\sin\theta\cos\theta$.

Solution:

$$\begin{aligned} \text{LHS} &= (\cos\theta + \sin\theta)^2 \\ &= \cos^2\theta + 2\sin\theta\cos\theta + \sin^2\theta \\ &= (\cos^2\theta + \sin^2\theta) + 2\sin\theta\cos\theta \\ &= 1 + 2\sin\theta\cos\theta \\ &= \text{RHS} \end{aligned}$$

Proved.

OR

$$\begin{aligned} \text{RHS} &= 1 + 2\sin\theta\cos\theta \\ &= (\cos^2\theta + \sin^2\theta) + 2\sin\theta\cos\theta \\ &= \cos^2\theta + 2\sin\theta\cos\theta + \sin^2\theta \\ &= (\cos\theta + \sin\theta)^2 \\ &= \text{LHS} \end{aligned}$$

Proved.

Note:
- *An often used technique is applied in the alternative proof where "1" is replaced with* $\cos^2\theta + \sin^2\theta$.
- *In this instance it is equally advantageous to start from either side of the identity.*

Example 15.5

Prove each of the following trigonometric identities:

(a) $5 + 4\cos\theta - 4\sin^2\theta = (2\cos\theta + 1)^2$ (b) $(1 - \tan^2\theta)\cos^2\theta = 2\cos^2\theta - 1$

(c) $\dfrac{\cos^2\theta\tan^2\theta}{1-\cos\theta} = 1 + \cos\theta$.

Solution:

(a) To Prove: $5 + 4\cos\theta - 4\sin^2\theta = (2\cos\theta + 1)^2$

LHS $= 5 + 4\cos\theta - 4\sin^2\theta$
$= 5 + 4\cos\theta - 4(1 - \cos^2\theta)$
$= 4\cos^2\theta + 4\cos\theta + 1$
$= (2\cos\theta + 1)^2$
= RHS Proved.

OR

RHS $= (2\cos\theta + 1)^2$
$= 4\cos^2\theta + 4\cos\theta + 1$
$= 4(1 - \sin^2\theta) + 4\cos\theta + 1$
$= 5 + 4\cos\theta - 4\sin^2\theta$
= LHS Proved.

Note: *As in the previous example, it is equally advantageous to start from either side of the identity.*

(b) To Prove: $(1 - \tan^2\theta)\cos^2\theta = 2\cos^2\theta - 1$.

LHS $= (1 - \tan^2\theta)\cos^2\theta$
$= \left(1 - \dfrac{\sin^2\theta}{\cos^2\theta}\right)\cos^2\theta$
$= \cos^2\theta - \sin^2\theta$
$= \cos^2\theta - (1 - \cos^2\theta)$
$= 2\cos^2\theta - 1$ = RHS Proved.

Note: *Unlike the two previous examples, in this case it is more advantageous to start with the LHS. However, this does not imply that the identity cannot be proved starting from the RHS*

(c) To Prove: $\dfrac{\cos^2\theta\tan^2\theta}{1-\cos\theta} = 1 + \cos\theta$

LHS $= \dfrac{\cos^2\theta\tan^2\theta}{1-\cos\theta}$
$= \dfrac{\cos^2\theta \times \dfrac{\sin^2\theta}{\cos^2\theta}}{1-\cos\theta}$
$= \dfrac{\sin^2\theta}{1-\cos\theta}$
$= \dfrac{1-\cos^2\theta}{1-\cos\theta}$
$= \dfrac{(1-\cos\theta)(1+\cos\theta)}{1-\cos\theta}$
$= 1 + \cos\theta$ = RHS Proved.

Note: *Another commonly used technique is to use the expansion for the difference of two squares;*
$a^2 - b^2 = (a - b)(a + b)$.

Example 15.6

Prove each of the following trigonometric identities:

(a) $\sin^2\theta\,(1 + \cot^2\theta) = 1$ (b) $\dfrac{1}{1+\cot\theta} = \dfrac{\tan\theta}{1+\tan\theta}$.

Solution:

(a) To Prove $\sin^2\theta\,(1 + \cot^2\theta) = 1$.

$$\begin{aligned}\text{LHS} &= \sin^2\theta\,(1 + \cot^2\theta)\\ &= \sin^2\theta\,(\text{cosec}^2\theta)\\ &= \sin^2\theta \times \frac{1}{\sin^2\theta}\\ &= 1\\ &= \text{RHS}\end{aligned}$$

Proved.

(b) To Prove $\dfrac{1}{1+\cot\theta} = \dfrac{\tan\theta}{1+\tan\theta}$.

$$\begin{aligned}\text{LHS} &= \frac{1}{1+\cot\theta}\\ &= \frac{1}{\left(1+\dfrac{1}{\tan\theta}\right)}\\ &= \frac{1}{\left(\dfrac{\tan\theta+1}{\tan\theta}\right)}\\ &= \frac{\tan\theta}{1+\tan\theta}\\ &= \text{RHS}\end{aligned}$$

Proved.

OR

$$\begin{aligned}\text{RHS} &= \frac{\tan\theta}{1+\tan\theta}\\ &= \frac{\left(\tan\theta\times\dfrac{1}{\tan\theta}\right)}{\left((1+\tan\theta)\times\dfrac{1}{\tan\theta}\right)}\\ &= \frac{1}{\left(\dfrac{1}{\tan\theta}+1\right)}\\ &= \frac{1}{\cot\theta+1}\\ &= \text{RHS}\end{aligned}$$

Proved.

Exercise 15.2

1. Prove each of the following identities.

(a) $\cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$

(b) $1 - 2\sin^2\theta = 2\cos^2\theta - 1$

(c) $\cos^2\theta\,(1 + \tan^2\theta) = 1$

(d) $\sin^2\theta\,(1 + \tan^2\theta) = \tan^2\theta$

(e) $\cos^2\theta - \sin^2\theta = \cos^2\theta\,(1 - \tan^2\theta)$

(f) $\cos^4\theta - \sin^4\theta = 2\cos^2\theta - 1$

(g) $(\cos\theta - \sin\theta)^2 = 1 - 2\sin\theta\cos\theta$

(h) $5\cos^2\theta + 4\cos\theta + 4\sin^2\theta = (2 + \cos\theta)^2$

(i) $(\sin\theta - \cos\theta)^2 = 1 - 2\tan\theta + 2\tan\theta\sin^2\theta$

(j) $(\cos\theta - \sin\theta)^2 + (\cos\theta + \sin\theta)^2 = 2$

2. Prove each of the following identities.

(a) $\dfrac{1}{1+\sin x}+\dfrac{1}{1-\sin x} = \dfrac{2}{\cos^2 x}$

(b) $\dfrac{\cos x}{1+\sin x} + \dfrac{1+\sin x}{\cos x} = \dfrac{2}{\cos x}$

(c) $\cos^2 x - \sin^2 x = \dfrac{1-\tan^2 x}{1+\tan^2 x}$

(d) $(1 + \tan x)^2 = \dfrac{1+2\sin x\cos x}{\cos^2 x}$

(e) $\dfrac{1}{\sin x\cos x} = \tan x + \dfrac{1}{\tan x}$

(f) $\dfrac{\cos^2 x}{1-\sin^2 x} - \dfrac{\cos^2 x}{1+\sin^2 x} = \dfrac{2\sin^2 x}{1+\sin^2 x}$

(g) $\dfrac{1-\sin x}{\cos x} = \dfrac{\cos x}{1+\sin x}$

(h) $\tan x = \dfrac{\tan x+\sin x}{1+\cos x}$

(i) $\dfrac{\tan x}{\tan x+\sin x} = \dfrac{1-\cos x}{\sin^2 x}$

(j) $\dfrac{\tan x}{1-\tan x} + \dfrac{\cos x}{\sin x-\cos x} = -1$

3. Prove each of the following identities.

(a) $\operatorname{cosec}^2\alpha - \sec^2\alpha = \cot^2\alpha - \tan^2\alpha$

(b) $\tan^2\alpha(\operatorname{cosec}^2\alpha - 1) = 1$

(c) $(\sec^2\alpha - 1)(\operatorname{cosec}^2\alpha - 1) = 1$

(d) $\sec^4\alpha - \tan^4\alpha = 1 + 2\tan^2\alpha$

(e) $\sec\alpha + \tan\alpha = \dfrac{\cos x}{1-\sin x}$

(f) $\dfrac{1+\sec\alpha}{\sec\alpha-1} = \dfrac{1+\cos\alpha}{1-\cos\alpha}$

(g) $\dfrac{1}{\operatorname{cosec}^2\alpha} + \dfrac{1}{\sec^2\alpha} = 1$

(h) $(\operatorname{cosec}\alpha + \cot\alpha)^2 = \dfrac{1+\cos\alpha}{1-\cos\alpha}$

(i) $\dfrac{1+\cot\alpha}{\operatorname{cosec}\alpha} - \dfrac{\sec\alpha}{\cot\alpha+\tan\alpha} = \cos\alpha$

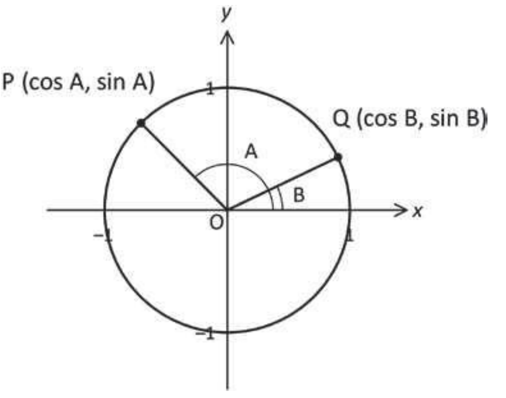
(j) $\dfrac{1+\cos\alpha}{1-\cos\alpha} = 1 + 2\operatorname{cosec}\alpha\cot\alpha + 2\cot^2\alpha$

15.3 Compound Angle Formulae

- The following formulae are known as the compound angle formulae or the sum and difference formulae.

  - $\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$
  - $\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$
  - $\tan(A \pm B) = \dfrac{\tan A\pm\tan B}{1\mp\tan A\tan B}$ $\quad (A \neq \dfrac{\pi}{2}, B \neq \dfrac{\pi}{2})$

- Consider the points P and Q on the unit circle. OP, OQ are inclined to the positive *x*-axis with angles A and B respectively.
  Hence P and Q have coordinates (cos A, sin A) and (cos B, sin B) respectively.



- In ΔOPQ:
  - $\angle$POQ = A – B
    OP = 1 and OQ = 1
- Using the distance formula:

$$PQ^2 = (\cos A - \cos B)^2 + (\sin A - \sin B)^2$$
$$= (\cos^2 A - 2\cos A\cos B + \cos^2 B) + (\sin^2 A - 2\sin A\sin B + \sin^2 B)$$
$$= \cos^2 A + \sin^2 A + \cos^2 B + \sin^2 B - 2\cos A\cos B - 2\sin A\sin B$$
$$= 2 - 2\cos A\cos B - 2\sin A\sin B$$

  - Using the cosine rule:

$$PQ^2 = OP^2 + OQ^2 - 2 \times OP \times OQ \times \cos P\hat{O}Q,$$
$$2 - 2\cos A\cos B - 2\sin A\sin B = 1 + 1 - 2 \times 1 \times 1 \times \cos(A - B)$$
$$2 - 2\cos A\cos B - 2\sin A\sin B = 2 - 2\cos(A - B)$$

$\Rightarrow$ $\cos(A - B) = \cos A\cos B + \sin A\sin B$. Proved.

  - Rewrite cos (A + B) as cos(A – (–B)):

$\cos(-B) = \cos B$ $\sin(-B) = -\sin B$

$$\cos(A + B) = \cos A\cos(-B) + \sin A\sin(-B).$$

$\Rightarrow$ $\cos(A + B) = \cos A\cos B - \sin A\sin B$. Proved.

- Expand $\cos[A + (90^o - B)] = \cos A\cos(90^o - B) - \sin A\sin(90^o - B)$
  $= \cos A\sin B - \sin A\cos B$

$\sin\theta = \cos(90^o - \theta)$

  - But $\cos[A + (90^o - B)] = \cos[(A - B) + 90^o]$
    $= \cos(A - B)\cos 90^o - \sin(A - B)\sin 90^o$
    $= -\sin(A - B)$.
  - Hence, $-\sin(A - B) = \cos A\sin B - \sin A\cos B$

    $\Rightarrow$ $\sin(A - B) = \sin A\cos B - \cos A\sin B$. Proved.

  - Rewrite sin (A + B) as sin (A – (–B)):

$$\sin(A + B) = \sin A\cos(-B) - \cos A\sin(-B)$$

$\Rightarrow$ $\sin(A + B) = \sin A\cos B + \cos A\sin B$. Proved.

- The proof for tan (A ± B) will be established in Exercise 15.3.

Example 15.7

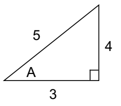
Given that cos A = $\frac{3}{5}$ where $0 \le A \le 90^{\circ}$, and sin B = $\frac{1}{3}$ where $90^{\circ} \le B \le 180^{\circ}$,

find in exact form: (a) sin (A + B) (b) tan (A – B).

Solution:

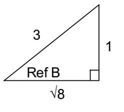
(a) sin (A + B) = sin A cos B + cos A sin B

$= \sin A \times \cos B + \frac{3}{5} \times \frac{1}{3}$.



Since cos A = $\frac{3}{5}$, from the reference triangle, sin A = $\frac{4}{5}$.

Similarly, since sin B = $\frac{1}{3}$, sin (ref ∠ B) = $\frac{1}{3}$ and cos (ref ∠ B) = $\frac{\sqrt{8}}{3}$.



Since B is located in Q2, cos B = $-\frac{\sqrt{8}}{3}$.

Hence, sin (A + B) = $\frac{4}{5} \times -\frac{\sqrt{8}}{3} + \frac{3}{5} \times \frac{1}{3} = \frac{1}{15}(3 - 8\sqrt{2})$.

(b) tan (A – B) = $\frac{\tan A - \tan B}{1 + \tan A \tan B}$

From the reference triangles in part (a), tan A = $\frac{4}{3}$, tan B = $-\frac{1}{\sqrt{8}} = -\frac{\sqrt{2}}{4}$.

Hence, tan (A – B) = $\frac{\frac{4}{3} - \left(\frac{-\sqrt{2}}{4}\right)}{1 + \left(\frac{4}{3}\right) \times \left(\frac{-\sqrt{2}}{4}\right)} = \frac{25\sqrt{2}}{28} + \frac{27}{14}$

Example 15.8

Without the use of a calculator, solve for all values of θ in $\cos\left(\theta + \frac{\pi}{3}\right) = 2\sin\left(\theta - \frac{\pi}{6}\right)$.

Solution:

$$\cos\left(\theta + \frac{\pi}{3}\right) = 2\sin\left(\theta - \frac{\pi}{6}\right)$$

$$\Rightarrow \cos\theta\cos\frac{\pi}{3} - \sin\theta\sin\frac{\pi}{3} = 2\sin\theta\cos\frac{\pi}{6} - 2\cos\theta\sin\frac{\pi}{6}$$

$$\frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta = \sqrt{3}\sin\theta - \cos\theta$$

$$\frac{3\sqrt{3}}{2}\sin\theta = \frac{3}{2}\cos\theta$$

$$\Rightarrow \tan\theta = \frac{1}{\sqrt{3}} \quad \Rightarrow \theta = \frac{\pi}{6} + n\pi \quad n \in \mathbb{Z}.$$

Example 15. 9

Prove that $\cos(A + B)\cos(A - B) = \cos^2 A - \sin^2 B$.

Solution:

$$
\begin{aligned}
\text{LHS} &= \cos(A + B)\cos(A - B)\\
&= (\cos A \cos B - \sin A \sin B)(\cos A \cos B + \sin A \sin B)\\
&= (\cos A \cos B)^2 - (\sin A \sin B)^2\\
&= \cos^2 A\,(1 - \sin^2 B) - (1 - \cos^2 A)\sin^2 B\\
&= \cos^2 A - \sin^2 B = \text{RHS} \qquad \text{Proved.}
\end{aligned}
$$

Exercise 15.3

1. Given that A is an acute angle and B is an obtuse angle with $\sin A = \frac{3}{5}$ and $\sin B = \frac{5}{13}$, find in exact form: (a) $\sin(A - B)$ (b) $\cos(A + B)$ (c) $\tan(A - B)$.

2. Given that $\cos A = \frac{7}{25}$ where $0 \le A \le 90^\circ$, and $\sin B = \frac{8}{17}$ where $90^\circ \le B \le 180^\circ$, find in exact form: (a) $\sin(A - B)$ (b) $\cos(A - B)$ (c) $\tan(A + B)$.

3. Given that $\sin A = \frac{1}{4}$ where $0 \le A \le 90^\circ$, and $\tan B = \frac{4}{3}$ where $180^\circ \le B \le 270^\circ$, find in exact form: (a) $\sin(A - B)$ (b) $\cos(A + B)$ (c) $\tan(A - B)$.

4. Given that A and B are both obtuse angles and $\cos A = -\frac{3}{5}$ and $\tan B = -\frac{12}{35}$, find in exact form: (a) $\sin(A + B)$ (b) $\cos(A - B)$ (c) $\tan(A + B)$.

5. Without the use of a calculator, solve for all values of θ.
   (a) $\cos(\theta + \frac{\pi}{3}) = \cos\theta$ (b) $\cos(\theta - \frac{5\pi}{4}) = -\sqrt{2}\sin\theta$
   (c) $\cos(\theta + \frac{\pi}{6}) = 2\sin(\theta - \frac{\pi}{3})$ (d) $\sin(\theta + \frac{\pi}{4}) = \sqrt{2}\cos(\theta - \frac{\pi}{4})$

*6. Without the use of a calculator, solve for all values of x if $\tan(x - \frac{\pi}{4}) = -\tan(x + \frac{\pi}{2})$.

7. Prove each of the following, where n is an integer:
   (a) $\sin(x - 2n\pi) = \sin x$ (b) $\cos(x - 2n\pi) = \cos x$ (c) $\tan(x - n\pi) = \tan x$

8. Use the compound angle formulae for $\sin(A \pm B)$ and $\cos(A \pm B)$ to prove that:

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \qquad (A \ne \frac{\pi}{2},\ B \ne \frac{\pi}{2}).$$

9. Prove each of the following identities.
   (a) $\sin(x + y) + \sin(x - y) = 2\sin x\cos y$
   (b) $\cos(\alpha - \beta) - \cos(\alpha + \beta) = 2\sin\alpha\sin\beta$
   (c) $\sin(A + B)\sin(A - B) = \sin^2 A - \sin^2 B$
   (d) $\sin(A + B)\cos B - \cos(A + B)\sin B = \sin A$
   (e) $\cos(P + Q)\cos(P - Q) + \sin(P + Q)\sin(P - Q) = \cos^2 Q - \sin^2 Q$
   (f) $\cos(P + Q)\cos(P - Q) = \cos^2 P + \cos^2 Q - 1$
   (g) $\dfrac{\sin(A - B)}{\cos A\cos B} = \tan A - \tan B$
   (h) $\dfrac{\cos(A + B)}{\cos A\cos B} = 1 - \tan A\tan B$
   (i) $\cot(x + y) = \dfrac{\cot x\cot y - 1}{\cot x + \cot y}$

*10. If x, y and z are the three angles within a triangle, prove that:
   (a) $\cos(x + y) = -\cos z$ (b) $\cos(x - y) - \cos z = 2\cos x\cos y$.

15.4 The Double and Half-Angle Formulae

- Consider the compound angle formulae:
  - $\sin(A + B) = \sin A\cos B + \cos A\sin B$
  - $\cos(A + B) = \cos A\cos B - \sin A\sin B$
  - $\tan(A + B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A\ \tan B}$ $(A \neq \frac{\pi}{2}, B \neq \frac{\pi}{2})$

- By letting B = A, we obtain the double angle formulae:
  - $\sin 2A = 2\sin A\cos A$
  - $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$
  - $\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$

- By letting $A = \dfrac{x}{2}$, we obtain the half angle formulae:
  - $\sin x = 2\sin\dfrac{x}{2}\cos\dfrac{x}{2}$
  - $\cos x = \cos^2\dfrac{x}{2} - \sin^2\dfrac{x}{2} = 2\cos^2\dfrac{x}{2} - 1 = 1 - 2\sin^2\dfrac{x}{2}$
  - $\tan x = \dfrac{2\tan\frac{x}{2}}{1 - \tan^2\frac{x}{2}}$

Example 15.10

Without the use of a calculator, solve for all values of x (radians) in $\cos 2x + \cos x + \sin^2 x = 0$.

Solution:

Rewrite the equation $\cos 2x + \cos x + \sin^2 x = 0$ solely in terms of $\cos x$:

$$\Rightarrow \quad (2\cos^2 x - 1) + \cos x + (1 - \cos^2 x) = 0$$
$$\cos^2 x + \cos x = 0$$
$$\cos x\,(\cos x + 1) = 0$$
$$\Rightarrow \quad \cos x = 0 \text{ or } \cos x = -1$$

$\cos x = 0 \quad \Rightarrow \quad x = \dfrac{(2n+1)\pi}{2} \quad n \in \mathbb{Z}$.

$\cos x = -1 \quad \Rightarrow \quad x = (2n+1)\pi \quad n \in \mathbb{Z}$.

Example 15.11

Prove: (a) $\sin 3A = 3\sin A - 4\sin^3 A$ (b) $\sin 2A = \dfrac{2\tan A}{1+\tan^2 A}$.

Solution:

(a) LHS = $\sin 3A$

$= \sin(2A + A)$

$= \sin 2A \cos A + \cos 2A \sin A$

$= 2\sin A \cos^2 A + (1 - 2\sin^2 A)\sin A$

$= 2\sin A\,(1 - \sin^2 A) + \sin A - 2\sin^3 A$

$= 3\sin A - 4\sin^3 A$

= RHS Proved.

(b) RHS = $\dfrac{2\tan A}{1+\tan^2 A}$

$= \dfrac{\left(\dfrac{2\sin A}{\cos A}\right)}{1+\left(\dfrac{\sin^2 A}{\cos^2 A}\right)}$

$= \dfrac{2\sin A}{\cos A} \times \dfrac{1}{\left(\dfrac{\cos^2 A + \sin^2 A}{\cos^2 A}\right)}$

$= \dfrac{2\sin A}{\cos A} \times \dfrac{\cos^2 A}{1}$

$= 2\sin A \cos A$

$= \sin 2A$

= RHS Proved.

Exercise 15.4

1. Without the use of a calculator, solve for all values of x (radians).
   (a) $\sin 2x = \cos x$
   (b) $\sin 2x = \sqrt{3} \sin x$
   (c) $\cos 2x - \cos x = 0$
   (d) $\cos 2x - \sin x = 1$
   (e) $\cos 2x - 6 \sin x + 2 \cos^2 x = -1$
   (f) $\cos 2x + 2\cos x - 2 \sin^2 x = -1$
   (g) $\tan x \tan 2x - 1 = 0$
   *(h) $\cos 2x - \sin 2x + 2 \cos x - 2 \sin x = -1$

2. Without the use of a calculator, solve for all values of x (radians).
   (a) $2 \cos (\theta + \frac{\pi}{12}) = 1$
   (b) $\sin 2\theta + \sqrt{3} \cos \theta = 0$
   (c) $\cos (\theta + \frac{\pi}{4}) - \sqrt{3} \sin (\theta + \frac{\pi}{4}) = 0$
   (d) $\tan^2 \theta - 2 \sec \theta \tan \theta + 1 = 0$
   (e) $\sin 2\theta + \sqrt{3} \cos 2\theta = \sqrt{3}$
   (f) $\cos 2\theta + 2 \cos^2 \theta + 2\sqrt{3} \sin \theta - 3 = 0$

3. Use appropriate trigonometric identities to solve each of the following equations, giving answers correct to one decimal place between 0 and 360°.
   (a) $\cos x + 2 \sin 2x = 0$
   (b) $\sin 4x = \sin 2x$
   (c) $16 \cos x + 3 \cos 2x - 3= 0$
   (d) $\cos 2x + 6 \sin x - 5 = 0$
   (e) $4 \cos^2 x - \cos 2x - \sin x - 2 = 0$
   (f) $\cos 4x + 6 = 7 \cos 2x + \sin^2 2x$
   *(g) $\tan x \tan 2x - \tan 2x = 1$
   *(h) $2 \cos 2x + \sin 2x + 4 \cos x - 8 \sin x - 2 = 0$

4. Using appropriate trigonometric techniques and/or identities, solve each of the following equations, for $-180^\circ \le \theta \le 180^\circ$.
   (a) $\sin (2\theta + 10^\circ) = -0.7$
   (b) $2 \sin 2\theta - 3 \cos \theta = 0$
   (c) $2 \sin 2\theta + \cos 2\theta = 1$
   (d) $3 \cos 2\theta - 14 \cos \theta + 7 = 0$
   (e) $2 \cos^2 \theta + \cos 2\theta - 10 \sin^2 \theta + 3 = 0$
   *(f) $\sqrt{10} \cos \theta - 5 \sin \theta = -\sqrt{10}$.

5. Prove each of the following identities:
   (a) $\cos^2 A = \frac{1+\cos 2A}{2}$
   (b) $\sin^2 A = \frac{1-\cos 2A}{2}$
   (c) $\cos^2 2A = \frac{1+\cos 4A}{2}$
   (d) $\sin^2 2A = \frac{1-\cos 4A}{2}$

6. Prove each of the following trigonometric identities.
   (a) $(\cos \theta - \sin \theta)^2 = 1 - \sin 2\theta$
   (b) $\cos 2\theta = \cos^4 \theta - \sin^4 \theta$
   (c) $2 - \tan \theta \sin 2\theta = 2 \cos^2 \theta$
   (d) $\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$
   (e) $\cos 4\theta = 8 \cos^4 \theta - 8 \cos^2 \theta + 1$
   (f) $\sin 4\theta = 4 \sin \theta \cos^3 \theta - 4 \cos \theta \sin^3 \theta$
   (g) $\cot 2\theta + \text{cosec } 2\theta = \cot \theta$
   (h) $\sec 2\theta = 1 + \tan \theta \tan 2\theta$
   (i) $\text{cosec } 2\theta = \frac{\cot\theta + \tan\theta}{2}$
   (j) $\cot 2\theta = \frac{\cot\theta - \tan\theta}{2}$

7. Prove each of the following trigonometric identities.

(a) $\tan\theta = \dfrac{\sin 2\theta}{1+\cos 2\theta}$

(b) $\cos 2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$

(c) $\dfrac{2\cos 2\theta}{1+\cos 2\theta} = 1 - \tan^2\theta$

(d) $\sin^2\theta = \dfrac{1-\cos 4\theta}{4\,(1+\cos 2\theta)}$

(e) $\dfrac{\sin 2\theta}{1+\cos 2\theta} = \dfrac{1-\cos 2\theta}{\sin 2\theta}$

(f) $\dfrac{\cos 2\theta}{\cos\theta-\sin\theta} = \dfrac{1+\sin 2\theta}{\cos\theta+\sin\theta}$

(g) $\dfrac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta} = \dfrac{\cos 2\theta}{1+\sin 2\theta}$

(h) $\dfrac{\cos 2\theta}{1-\sin 2\theta} = \dfrac{1+\tan\theta}{1-\tan\theta}$

(i) $\tan 3\theta = \dfrac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}$

(j) $\tan 4\theta = \dfrac{4\tan\theta-4\tan^3\theta}{1-6\tan^2\theta+\tan^4\theta}$

8. Prove each of the following:

(a) $\left(\cos\dfrac{\theta}{2}+\sin\dfrac{\theta}{2}\right)^2 = 1 + \sin\theta$

(b) $\tan\dfrac{\theta}{2} = \dfrac{\sin\theta}{1+\cos\theta}$

(c) $1 + \tan\dfrac{\theta}{2}\tan\theta = \sec\theta$

(d) $\dfrac{1-\cos\theta}{\sin\theta} = \tan\dfrac{\theta}{2}$

(e) $\dfrac{\cos\theta}{1+\cos\theta} = \dfrac{1}{2}\left(1-\tan^2\dfrac{\theta}{2}\right)$

(f) $\cot\theta + \operatorname{cosec}\theta = \cot\dfrac{\theta}{2}$

(g) $(\cos 4x - 1)(\cos 4x + 1) = -\sin^2 4x$

(h) $\tan x = \dfrac{\sin 2x-\sin x}{\cos 2x-\cos x+1}$

(i) $\cot\dfrac{\theta}{2} + \tan\dfrac{\theta}{2} = 2\operatorname{cosec}\theta$

(j) $\dfrac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta} = \tan\dfrac{\theta}{2}$

15.5 Product to Sum and Sum to Product Formulae

- The product to sum formulae are:
  - $\sin A\cos B = \dfrac{1}{2}[\sin(A+B) + \sin(A-B)]$
  - $\cos A\cos B = \dfrac{1}{2}[\cos(A+B) + \cos(A-B)]$
  - $\sin A\sin B = -\dfrac{1}{2}[\cos(A+B) - \cos(A-B)]$
- The sum to product formulae are:
  - $\sin A \pm \sin B = 2\sin\left(\dfrac{A\pm B}{2}\right)\cos\left(\dfrac{A\mp B}{2}\right)$
  - $\cos A + \cos B = 2\cos\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A-B}{2}\right)$
  - $\cos A - \cos B = -2\sin\left(\dfrac{A+B}{2}\right)\sin\left(\dfrac{A-B}{2}\right)$

Example 15.12

Prove that: (a) $\sin A \cos B = \frac{1}{2}\,[\sin (A + B) + \sin (A - B)]$

(b) $\sin A + \sin B = 2 \sin\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right)$.

Solution:

(a) $\text{RHS} = \frac{1}{2}\,[\sin (A + B) + \sin (A - B)]$

$= \frac{1}{2}\,[\sin A \cos B + \cos A \sin B + \sin A \cos B - \cos A \sin B]$

$= \sin A \cos B = \text{LHS}$ Proved

(b) $\text{RHS} = 2 \sin\left(\frac{A}{2}+\frac{B}{2}\right) \cos\left(\frac{A}{2}-\frac{B}{2}\right)$

$= 2 \times \frac{1}{2}\left\{\sin\left[\left(\frac{A}{2}+\frac{B}{2}\right)+\left(\frac{A}{2}-\frac{B}{2}\right)\right] + \sin\left[\left(\frac{A}{2}+\frac{B}{2}\right)-\left(\frac{A}{2}-\frac{B}{2}\right)\right]\right\}$

$= 2 \times \frac{1}{2}\{\sin A + \sin B\}$

$= \sin A + \sin B = \text{LHS}$ Proved.

Example 15.13

Without the use of calculator, evaluate: (a) $\cos 75^{o} \cos 15^{o}$ (b) $\sin \frac{3\pi}{8} \sin \frac{\pi}{4} \sin \frac{\pi}{8}$.

Solution:

(a) $\cos 75^{o} \cos 15^{o} = \frac{1}{2}[\cos (75^{o} + 15^{o}) + \cos (75^{o} - 15^{o})]$

$= \frac{1}{2}\,[\cos 90^{o} + \cos 60^{o}] = \frac{1}{4}$

(b) $\sin \frac{3\pi}{8} \sin \frac{\pi}{4} \sin \frac{\pi}{8} = \sin \frac{3\pi}{8} \sin \frac{\pi}{8} \sin \frac{\pi}{4}$

$= -\frac{1}{2}\left[\cos\left(\frac{3\pi}{8}+\frac{\pi}{8}\right) - \cos\left(\frac{3\pi}{8}-\frac{\pi}{8}\right)\right] \sin \frac{\pi}{4}$

$= -\frac{1}{2}\left[\cos\frac{\pi}{2} - \cos \frac{\pi}{4}\right] \sin \frac{\pi}{4}$

$= -\frac{1}{2}\left[0 - \frac{1}{\sqrt{2}}\right] \frac{1}{\sqrt{2}}$

$= \frac{1}{4}$.

Example 15.14

Without the use of calculator, solve for all values of θ (radians) in $\cos 4\theta + \cos 3\theta = 0$.

Solution:

$$\cos 4\theta + \cos 3\theta = 0 \Rightarrow 2\cos\left(\frac{4\theta+3\theta}{2}\right)\cos\left(\frac{4\theta-3\theta}{2}\right) = 0$$

$$\cos\frac{7\theta}{2} = 0 \quad \text{or} \quad \cos\frac{\theta}{2} = 0$$

For $\cos\frac{7\theta}{2} = 0 \Rightarrow \frac{7\theta}{2} = \frac{(2n+1)\pi}{2}$

$$\theta = \frac{(2n+1)\pi}{7} \quad n \in \mathbb{Z}.$$

For $\cos\frac{\theta}{2} = 0 \Rightarrow \frac{\theta}{2} = \frac{(2n+1)\pi}{2}$

$$\theta = (2n+1)\pi \quad n \in \mathbb{Z}.$$

Example 15.15

Prove that $\frac{\sin 6A + \sin 2A}{\sin 6A - \sin 2A} = \tan 4A \cot 2A$.

Solution:

$$\text{LHS} = \frac{\sin 6A + \sin 2A}{\sin 6A - \sin 2A}$$

$$= \frac{2\sin\left(\frac{6A+2A}{2}\right)\cos\left(\frac{6A-2A}{2}\right)}{2\sin\left(\frac{6A-2A}{2}\right)\cos\left(\frac{6A+2A}{2}\right)}$$

$$= \frac{2\sin 4A\cos 2A}{2\sin 2A\cos 4A}$$

$= \tan 4A \cot 2A$

$= \text{RHS}$ Proved.

Exercise 15.5

1. Prove that: (a) $\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$

   (b) $\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$.

2. Prove that: (a) $\sin A \sin B = -\frac{1}{2}[\cos(A+B) - \cos(A-B)]$

   (b) $\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$.

3. Without the use of calculator, evaluate:

   (a) $\sin 75^{\circ} \sin 15^{\circ}$ (b) $\sin 75^{\circ} \cos 75^{\circ}$ (c) $\cos 15^{\circ} \cos 45^{\circ}$ (d) $\cos 105^{\circ} \sin 15^{\circ}$

   (e) $\cos\frac{3\pi}{8}\cos\frac{\pi}{8}$ (f) $\cos\frac{3\pi}{8}\sin\frac{3\pi}{8}$ (g) $\sin\frac{11\pi}{12}\sin\frac{\pi}{12}$ (h) $\cos\frac{5\pi}{12}\sin\frac{7\pi}{12}$.

4. Without the use of calculator, evaluate:

   (a) $\sin 75^{\circ} \sin 45^{\circ} \sin 15^{\circ}$ (b) $\cos\frac{3\pi}{8}\cos\frac{\pi}{4}\cos\frac{\pi}{8}$

   *(c) $\cos 20^{\circ} \cos 40^{\circ} \cos 80^{\circ}$ *(d) $\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}$.

5. Without the use of a calculator, evaluate:

   (a) $\cos 75^{\circ} - \cos 15^{\circ}$ (b) $\sin 75^{\circ} - \sin 15^{\circ}$ *(c) $\sin\frac{11\pi}{12} + \cos\frac{19\pi}{12}$

6. Without the use of a calculator, solve for all values of θ (radians) in:

   (a) $\sin 4\theta + \sin 3\theta = 0$ (b) $\cos 5\theta + \cos 3\theta = 0$ (c) $\sin 4\theta - \sin\theta = 0$

7. Solve for all values of θ: (a) $\sin\theta + \sin 3\theta + \sin 5\theta = 0$ (b) $\sin\theta + \sin 3\theta = \cos\theta$.

8. Use a sum to product formula to prove that $\sin\theta + \cos\theta = \sqrt{2}$ can be rewritten as $\cos\left(\theta - \frac{\pi}{4}\right) = 1$. Hence, solve for all values of θ (radians).

9. Use a sum to product formula to prove that $\cos 2\theta - \sin 2\theta = \sqrt{2}$ can be rewritten as $\sin\left(2\theta - \frac{\pi}{4}\right) = -1$. Hence, solve for all values of θ (radians).

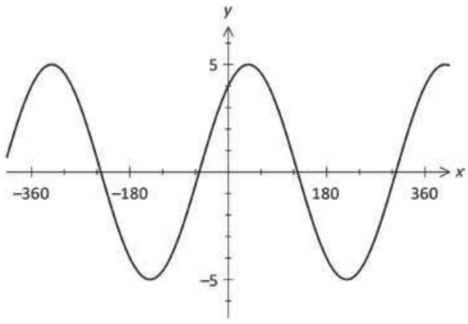
10. Prove that:

    (a) $\frac{\sin x + \sin 3x}{\cos x + \cos 3x} = \tan 2x$ (b) $\frac{\cos x - \cos 3x}{\cos x + \cos 3x} = \tan 2x \tan x$

    (c) $\frac{\sin 8A + \sin 4A}{\sin 8A - \sin 4A} = \tan 6A \cot 2A$ (d) $\frac{\sin A + \sin B}{\cos A + \cos B} = \tan\left(\frac{A+B}{2}\right)$

15.6 Linear combinations of sine and cosine (Auxiliary Angles)

- The accompanying diagram shows the graph of $y = 3 \sin x + 4 \cos x$.



- Note that the graph of $y = 3 \sin x + 4 \cos x$ is sinusoidal (of a sine waveform).
- Hence, we should be able to rewrite $3 \sin x + 4 \cos x$ in the form $R \sin (x + \alpha)$ where α is an acute angle called the *phase shift* or the *auxiliary angle*.
- We need to transform $a \sin x + b \cos x$ into $R \sin (x + \alpha)$ where the constants a, b, and R are all positive and α is acute.
  - $a \sin x + b \cos x \equiv R \sin (x + \alpha)$.
    $\equiv R \sin x \cos \alpha + R \cos x \sin \alpha$
    $\equiv (R \cos \alpha) \sin x + (R \sin \alpha) \cos x$
  - Note that on the RHS of the identity, $R \cos \alpha$ and $R \sin \alpha$ are constants and are respectively the coefficients of the sin x and cos x terms.
  - By comparing the coefficients of the sin x and cos x terms on the two sides of the identity, note that:
    $R \cos \alpha = a$ (I)
    and $R \sin \alpha = b$ (II)
  - $(\text{I})^2 + (\text{II})^2$ $R^2 \cos^2 \alpha + R^2 \sin^2 \alpha = a^2 + b^2$
    $R^2 (\cos^2 \alpha + \sin^2 \alpha) = a^2 + b^2$
    $R^2 = a^2 + b^2$
  - Divide (II) with (I) $\tan \alpha = \dfrac{b}{a}$
  - Hence, $a \sin x + b \cos x \equiv \sqrt{a^2 + b^2} \sin \left[x + \tan^{-1}\left(\dfrac{b}{a}\right)\right]$.

- In general, for $a > 0$, $b > 0$:
  - $a \sin x \pm b \cos x \equiv \sqrt{a^2 + b^2} \sin \left[x \pm \tan^{-1}\left(\dfrac{b}{a}\right)\right]$
  - $a \cos x \pm b \sin x \equiv \sqrt{a^2 + b^2} \cos \left[x \mp \tan^{-1}\left(\dfrac{b}{a}\right)\right]$.

Example 15.16

Without the use of a calculator solve for all values of x (radians) in $\sqrt{3}\,\sin x + \cos x = \sqrt{2}$.

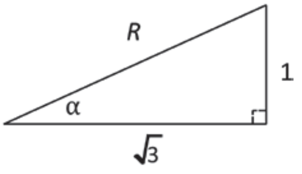
Solution:

Rewrite $\sqrt{3}\,\sin x + \cos x \equiv R\sin(x+\alpha)$

$$\equiv R\sin x\cos\alpha + R\cos x\sin\alpha\,.$$

$\Rightarrow$ $R\cos\alpha = \sqrt{3}$ and $R\sin\alpha = 1$.

$$\cos\alpha = \frac{\sqrt{3}}{R} \quad\text{and}\quad \sin\alpha = \frac{1}{R}.$$



Hence, there exists a reference triangle as drawn.

From the reference triangle: $R^2 = 1^2 + (\sqrt{3})^2 \Rightarrow R = 2$.

$$\tan\alpha = \frac{1}{\sqrt{3}} \qquad \Rightarrow \alpha = \frac{\pi}{6}.$$

Therefore, $\sqrt{3}\,\sin x + \cos x = \sqrt{2}$ can be rewritten as $2\sin\left(x+\frac{\pi}{6}\right) = \sqrt{2}$

Hence: $$\sin\left(x+\frac{\pi}{6}\right) = \frac{\sqrt{2}}{2}$$

$$x + \frac{\pi}{6} = (-1)^n \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) + n\pi$$

$$x = (-1)^n\left(\frac{\pi}{4}\right) - \frac{\pi}{6} + n\pi \quad n \in \mathbb{Z}.$$

Note:

- *In this example, an alternative method of determining R and α is shown.*

Example 15.17

Use an appropriate trigonometric technique to find in exact form the maximum value of $y = 2\cos x + \sqrt{12}\sin x$, $0 \le x \le 2\pi$, and the values of x at which this occurs

Solution:

Rewrite $2\cos x + \sqrt{12}\sin x \equiv R\cos(x-\alpha)$

$$R^2 = 4 + 12 \qquad \Rightarrow R = 4$$

$$\tan\alpha = \frac{\sqrt{12}}{2} = \sqrt{3} \qquad \Rightarrow \alpha = \frac{\pi}{3}$$

Hence, $y = 2\cos x + \sqrt{12}\sin x \equiv 4\cos\left(x-\frac{\pi}{3}\right)$.

Therefore maximum value of $y = 4$ when $\cos\left(x-\frac{\pi}{3}\right) = 1$.

That is, when $x - \frac{\pi}{3} = 2n\pi$

$$x = 2n\pi + \frac{\pi}{3} \quad n \in \mathbb{Z}.$$

Exercise 15.6

1. Express each of the following in the form $\pm R\sin(\theta \pm \alpha^\circ)$, where $R > 0$ and α° is acute:
   (a) $2\sin\theta + 3\cos\theta$ (b) $\sin\theta - 2\cos\theta$
   (c) $5\cos\theta + 2\sin\theta$ (d) $4\cos\theta - \sin\theta$
   (e) $-2\sin\theta + 2\cos\theta$ (f) $-4\cos\theta - 3\sin\theta$

2. Express each of the following in the form $\pm R\cos(\theta \pm \alpha)$, where $R > 0$ and α° is acute:
   (a) $\sqrt{2}\sin\theta + 2\cos\theta$ (b) $\sin\theta - \sqrt{3}\cos\theta$
   (c) $-\cos\theta + \sqrt{5}\sin\theta$ (d) $\frac{1}{2}\cos\theta + \frac{1}{3}\sin\theta$
   (e) $0.5\sin\theta - 0.2\cos\theta$ (f) $-3\sqrt{2}\cos\theta - \sqrt{3}\sin\theta$

3. Without the use of a calculator solve for all values of x (radians) in:
   (a) $\sin x + \cos x = \frac{\sqrt{2}}{2}$ (b) $\cos x - \sin x = -\frac{\sqrt{2}}{2}$
   (c) $\sin x - \sqrt{3}\cos x = \sqrt{3}$ (d) $-\cos x - \sqrt{3}\sin x = \sqrt{3}$
   (e) $\frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x = \frac{1}{\sqrt{2}}$ (f) $3\sin 2x + 3\cos 2x = -\frac{\sqrt{54}}{2}$

4. Use an appropriate trigonometric technique to find in exact form the maximum value of $y = \cos x + \sin x$, $0 \le x \le 2\pi$, and the smallest positive value of x at which this occurs.

5. Use an appropriate trigonometric technique to find in exact form the minimum value of $y = \sqrt{3}\sin x - \cos x$, the values of x (radians) at which this occurs.

6. Use an appropriate trigonometric technique to find in exact form the maximum and minimum values of $A = 3 + 3\cos t + \sqrt{3}\sin t$, and values of t at which these values occur.

7. Use an appropriate trigonometric technique to find in exact form the maximum and minimum values of $A = 10 - 12\cos t - 5\sin t$, and values of t (radians) at which these values occur.

*8. Use trigonometric techniques to solve for all value of x (radians):
   (a) $\sin x + \cos x = \sqrt{2}\sin 2x$ (b) $\sqrt{3}\cos x + \sin x = 2\cos 2x$

16 Matrix Algebra

16.1 Definitions

- A matrix consists of an ordered array of numbers.
  These numbers or entries are referred to as elements of the matrix.

  $\mathbf{A} = \begin{pmatrix} 1 & 4 & -8 \\ 9 & 6 & 8 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 3 & -6 \\ 4 & 8 \\ -6 & 5 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 1 & 4 & 5 \\ -3 & 2 & 0 \\ 5 & 6 & -5 \end{pmatrix}$ are examples of matrices.

  Matrices are usually referred to by using upper case letters in bold.

- The positions of the elements are of the utmost importance.
  Hence the matrix $\begin{pmatrix} 1 & 4 \\ 5 & 6 \end{pmatrix}$ is different from the matrix $\begin{pmatrix} 4 & 1 \\ 5 & 6 \end{pmatrix}$ despite the fact that the elements in the array are the same.

- The dimension or order/size of a matrix is stated as

  "Number of Rows" × " Number of Columns".

  Hence, the dimension of matrix **A** above is 2 × 3. This is written as dim(**A**) = 2 × 3.
  Clearly then, dim(**B**) = 3 × 2 and dim(**C**) = 3 × 3.

- The elements in a matrix are referred to by identifying the row and column number of its position. $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is the commonly used notation to represent a matrix. The element a_{12} refers to the element in row 1 and column 2.

- A matrix with dimension $n \times n$ (equal number of rows and columns) is called a square matrix.

- In this book, we will adopt the convention that the unit matrix is a square matrix where all the elements in the leading or main diagonal are ones and all other entries are zeros. Unit matrices are represented by the symbol **I**. They are often referred to as Identity matrices.

  $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ are respectively the 2 × 2 and 3 × 3 unit matrices.

- A matrix where all the elements are "ones" will be referred to as a "matrix of ones".

  $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ are respectively the 2 × 2 and 3 × 3 matrices of ones.

- A null matrix is a matrix where all the elements are zero. Null matrices whatever the dimension are represented by the symbol **0**.

16.1.1 Equality of Matrices

- Two matrices **A** and **B** are equal if the dimensions of the two matrices are identical and the corresponding elements in each matrix are identical.

Example 16.1

Given that $\begin{pmatrix} 4x & 3+a \\ b-4 & 3y+2 \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ 2 & 4 \end{pmatrix}$, find the values of a, b, x and y.

Solution:

Match the corresponding elements:

$4x = 5 \quad \Rightarrow \quad x = \dfrac{5}{4}$ $\qquad\qquad$ $3 + a = 1 \quad \Rightarrow \quad a = -2$

$b - 4 = 2 \quad \Rightarrow \quad b = 6$ $\qquad\qquad$ $3y + 2 = 4 \quad \Rightarrow \quad y = \dfrac{2}{3}$.

16.2 Operations on Matrices

16.2.1 Matrix Addition and Subtraction

- The operations of addition and subtraction between two matrices can only occur if both matrices have identical dimensions. In such an instance, the two matrices are said to be *conformable for addition and subtraction*.
- In matrix addition/subtraction, the corresponding elements are added/subtracted.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \pm \begin{pmatrix} u & v \\ w & x \end{pmatrix} = \begin{pmatrix} a \pm u & b \pm v \\ c \pm w & d \pm x \end{pmatrix}$$

- For any matrix **M**, **M** + **0** = **0** + **M** = **M**, where **0** is the corresponding null matrix. The null matrix in this context is known as the *additive-identity matrix*.
- Matrix addition is commutative. That is **A** + **B** = **B** + **A** .

  However, matrix subtraction is not commutative as **A** − **B** ≠ **B** − **A** .
- Matrix addition/subtraction is associative. That is (**A** ± **B**) ± **C** = **A** ± (**B** ± **C**) .

16.2.2 Scalar Multiplication

- In scalar multiplication, every element in the matrix is multiplied by the scalar.

$$k\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}$$

Example 16.2

If $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 3 & 2 \\ 7 & -3 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} x & 4 \\ 3+a & y \end{pmatrix}$ find:

(a) $3\mathbf{A} + 2\mathbf{B}$ (b) a, x and y if $\mathbf{A} + \mathbf{B} = \mathbf{C}$

Solution:

(a) $3\mathbf{A} + 2\mathbf{B} = 3 \times \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} + 2 \times \begin{pmatrix} 3 & 2 \\ 7 & -3 \end{pmatrix}$

$$= \begin{pmatrix} 3 & 6 \\ 3 & 9 \end{pmatrix} + \begin{pmatrix} 6 & 4 \\ 14 & -6 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & 10 \\ 17 & 3 \end{pmatrix}$$

(b) $\mathbf{A} + \mathbf{B} = \mathbf{C}$

$$\Rightarrow \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} + \begin{pmatrix} 3 & 2 \\ 7 & -3 \end{pmatrix} = \begin{pmatrix} x & 4 \\ 3+a & y \end{pmatrix}$$

$$\begin{pmatrix} 4 & 4 \\ 8 & 0 \end{pmatrix} = \begin{pmatrix} x & 4 \\ 3+a & y \end{pmatrix}$$

Compare corresponding elements: $x = 4$, $a = 5$ and $y = 0$.

16.2.3 Basic Rules for Manipulation of Matrices

- The rules for manipulating matrices involving the operations of addition, subtraction and scalar multiplication are similar to those for "ordinary algebra".
  - $n\mathbf{A} + m\mathbf{A} = (n + m)\mathbf{A}$
  - If $\mathbf{A} + \mathbf{X} = \mathbf{B}$, then $\mathbf{X} = \mathbf{B} - \mathbf{A}$
  - If $n\mathbf{X} = \mathbf{A}$, then $\mathbf{X} = \frac{1}{n}\mathbf{A}$ where $n \neq 0$

    Note that $\frac{\mathbf{A}}{n}$, the division of a matrix by a scalar is not defined.
  - $k(\mathbf{A} + \mathbf{B}) = k\mathbf{A} + k\mathbf{B}$

Example 16.3

Find matrix **X** in each case:

(a) $2\mathbf{X} + \mathbf{A} = \mathbf{B}$ (b) $3\mathbf{X} + \mathbf{A} = \mathbf{X} + \mathbf{B}$ (c) $2(\mathbf{X} + \mathbf{A}) = \mathbf{X} + \mathbf{C}$ (d) $\frac{1}{2}\mathbf{X} + \begin{pmatrix} 1 & 3 \\ 4a & 5 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ a & 3 \end{pmatrix}$.

Solution:

(a)
$$2\mathbf{X} + \mathbf{A} = \mathbf{B}$$
$$2\mathbf{X} = \mathbf{B} - \mathbf{A} \Rightarrow \mathbf{X} = \tfrac{1}{2}(\mathbf{B} - \mathbf{A})$$

(b)
$$3\mathbf{X} + \mathbf{A} = \mathbf{X} + \mathbf{B}$$
$$3\mathbf{X} - \mathbf{X} = \mathbf{B} - \mathbf{A}$$
$$2\mathbf{X} = \mathbf{B} - \mathbf{A} \Rightarrow \mathbf{X} = \tfrac{1}{2}(\mathbf{B} - \mathbf{A})$$

(c)
$$2(\mathbf{X} + \mathbf{A}) = \mathbf{X} + \mathbf{C}$$
$$2\mathbf{X} + 2\mathbf{A} = \mathbf{X} + \mathbf{C}$$
$$\mathbf{X} = \mathbf{C} - 2\mathbf{A}$$

(d)
$$\frac{1}{2}\mathbf{X} + \begin{pmatrix} 1 & 3 \\ 4a & 5 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ a & 3 \end{pmatrix}$$
$$\frac{1}{2}\mathbf{X} = \begin{pmatrix} 3 & 4 \\ a & 3 \end{pmatrix} - \begin{pmatrix} 1 & 3 \\ 4a & 5 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -3a & -2 \end{pmatrix}$$

Hence,
$$\mathbf{X} = \begin{pmatrix} 4 & 2 \\ -6a & -4 \end{pmatrix}$$

Exercise 16.1

1. For the each of the following matrices:

$$\mathbf{A} = \begin{pmatrix} 3 \\ 2 \end{pmatrix},\ \mathbf{B} = \begin{pmatrix} 6 & -4 & -2 & 0 \end{pmatrix},\ \mathbf{C} = \begin{pmatrix} -1 & 0 & 1 \\ -2 & 5 & 1 \end{pmatrix} \text{ and } \mathbf{D} = \begin{pmatrix} 6 & 3 \\ -1 & 0 \\ 3 & -1 \\ 2 & 0 \\ 1 & 3 \end{pmatrix}$$

(a) state the dimension of each matrix
(b) state the elements a_{12} and a_{21} where they exist
(c) state the value of p if matrix **E** of dimension $p \times 2$ is conformable under addition with matrix **D**
(d) state the value of p if matrix **F** with dimension $1 \times p$ is conformable under subtraction with matrix **B**.

2. Without the use of a CAS calculator, evaluate where possible:

(a) $\begin{pmatrix}1 & 5 & 2\end{pmatrix} - 2\begin{pmatrix}1 & 5 & 2\end{pmatrix}$

(b) $8\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} + 2\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} - 5\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}$

(c) $4\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} + \begin{pmatrix}1\\ -1\end{pmatrix} + \begin{pmatrix}-1\\ 1\end{pmatrix}$

(d) $\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} + 2\begin{pmatrix}x & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix} - 5\begin{pmatrix}-1 & 3 & y\\ 2 & -1 & 6\\ 7 & -4 & 0\end{pmatrix}$

3. Given the following matrices, without the use of a CAS calculator, find where they exist:

$\mathbf{A} = \begin{pmatrix}3 & 5\\ -1 & 3\end{pmatrix}$, $\mathbf{B} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, $\mathbf{C} = \begin{pmatrix}-2 & c\\ 5 & 1\end{pmatrix}$, $\mathbf{D} = \begin{pmatrix}1 & 4\end{pmatrix}$ and $\mathbf{E} = \begin{pmatrix}2\\ 0\end{pmatrix}$

(a) $5\mathbf{A} + 2\mathbf{B}$

(b) $2\mathbf{A} + 4\mathbf{E}$

(c) $3(\mathbf{A} - \mathbf{B})$

(d) $4\mathbf{A} - 2(\mathbf{B} + \mathbf{C})$

(e) $\frac{1}{2}$ (D + B)

(f) $-\frac{1}{2}$ $(\mathbf{A} + \mathbf{B} - \mathbf{C})$

4. Find the value(s) of the unknowns in each of the following:

(a) $\begin{pmatrix}1 & -a & 5 & 2b\end{pmatrix} = \begin{pmatrix}-c & c+1 & d & 4a\end{pmatrix}$

(b) $\begin{pmatrix}2 & a+b\\ 0 & 4\end{pmatrix} = 2\begin{pmatrix}c & 4\\ 0 & a\end{pmatrix}$

(c) $\begin{pmatrix}a^2\\ a\\ b\end{pmatrix} = \begin{pmatrix}4\\ -2\\ b\end{pmatrix}$

(d) $\begin{pmatrix}2a & 4 & -1\\ -b & 0 & 5\\ d & 1 & 1\end{pmatrix} = \begin{pmatrix}-6 & d & -1\\ b & 0 & c-2\\ d & 1 & 1\end{pmatrix}$

5. Given the following matrices, find the unknowns in each of the following cases:

$\mathbf{A} = \begin{pmatrix}1 & 9\\ -3 & 5\end{pmatrix}$, $\mathbf{B} = \begin{pmatrix}0 & 3\\ p & 2\end{pmatrix}$, $\mathbf{C} = \begin{pmatrix}q & 12\\ 0 & 7\end{pmatrix}$, $\mathbf{D} = \begin{pmatrix}p & -9\\ -q & -5\end{pmatrix}$

(a) $\mathbf{A} + \mathbf{B} = \mathbf{C}$

(b) $\mathbf{C} + \mathbf{D} = \mathbf{B}$

6. Find matrix **X** in each case:

(a) $2\mathbf{A} + \mathbf{X} = \mathbf{B}$

(b) $2(\mathbf{X} + \mathbf{A}) = 4\mathbf{X}$

(c) $\frac{1}{2}$ $[\mathbf{X} + 3\mathbf{C}] = \mathbf{A}$

7. Find the matrix **X** in each case:

(a) $\begin{pmatrix}2\\ 5\\ 4\end{pmatrix} + \mathbf{X} = \begin{pmatrix}-1\\ 2a\\ 5\end{pmatrix}$

(b) $3\begin{pmatrix}-1 & 5 & 4\end{pmatrix} - 2\mathbf{X} = \begin{pmatrix}b & 4 & 1\end{pmatrix}$

(c) $\begin{pmatrix}6 & 7\\ 1 & c\end{pmatrix} + 3\mathbf{X} = 2\left[\begin{pmatrix}4 & -1\\ 3 & 4\end{pmatrix} + \mathbf{X}\right]$

(d) $2\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} - \frac{1}{2}\mathbf{X} = \frac{1}{4}\left[\begin{pmatrix}2 & 0 & 6\\ 1 & 0 & 1\\ -2 & -1 & 1\end{pmatrix} + \mathbf{X}\right]$

16.2.4 Matrix Multiplication

- The matrix product **AB** is possible only if:
  the number of *columns* in **A** = the number of *rows* in **B**

- Let $\mathbf{A}_{m \times n}$ denote that matrix **A** has *m* row and *n* columns.
  Let $\mathbf{B}_{p \times q}$ denote that matrix **B** has *p* rows and *q* columns.
  Then the product **AB** exists only if *n* = *p*.

- The product matrix **AB** has dimensions $m \times q$.
  This is schematically shown as: $\mathbf{A}_{m \times n} \times \mathbf{B}_{p \times q} = \mathbf{C}_{m \times q}$
  The product "picks up" the number of rows of matrix **A** and the number of columns of matrix **B**.

- **A** and **B** are then said to be conformable under matrix multiplication.
  A is said to be post-multiplied (right-multiplied) by **B** or **B** is pre-multiplied (left-multiplied) by A.

- The rule for matrix multiplication is illustrated below:

  - $$\begin{pmatrix} a & b & c \end{pmatrix}\begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{pmatrix} ad + be + cf \end{pmatrix}$$

  - $$\begin{pmatrix} a & b & c \end{pmatrix}\begin{pmatrix} d & e \\ f & g \\ h & j \end{pmatrix} = \begin{pmatrix} ad + bf + ch & ae + bg + cj \end{pmatrix}$$

  - $$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} ap + bq + cr \\ dp + eq + fr \\ gp + hq + jr \end{pmatrix}$$

  - $$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}$$

Example 16.4

If dim(**A**) = $3 \times p$ and dim(**B**) = 4×5, find *p* and dim(**AB**) if the product exists.
Explain clearly why **BA** does not exist.

Solution:

Since **AB** exists, *p* = 4. Also, dim(**AB**) = 3×5.
No of columns in **B** $\neq$ No of rows in **A**. Hence **BA** does not exist.

Example 16.5

Given the following matrices, without the use of a calculator, find each of the following products. If a product does not exist, explain why it does not exist.

$\mathbf{A} = \begin{pmatrix} 1 & 2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 5 & 6 \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} -1 & 3 \\ 4 & d \end{pmatrix}$, $\mathbf{E} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$, $\mathbf{F} = \begin{pmatrix} 0 & 4 \\ f & 3 \end{pmatrix}$

(a) **AB** (b) **BC** (c) **AD** (d) **DA**
(e) **(DE)F** (f) **D(E + F)** (g) **(E + F)A** (h) $\mathbf{D}^2$

Solution:

(a) $\mathbf{AB} = \begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1(3)+2(4) \end{pmatrix} = \begin{pmatrix} 11 \end{pmatrix}$

(b) $\mathbf{BC} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}\begin{pmatrix} 5 & 6 \end{pmatrix} = \begin{pmatrix} 3(5) & 3(6) \\ 4(5) & 4(6) \end{pmatrix} = \begin{pmatrix} 15 & 18 \\ 20 & 24 \end{pmatrix}$

(c) $\mathbf{AD} = \begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 4 & d \end{pmatrix} = \begin{pmatrix} 1(-1)+2(4) & 1(3)+2(d) \end{pmatrix} = \begin{pmatrix} 7 & 3+2d \end{pmatrix}$

(d) $\mathbf{DA} = \begin{pmatrix} -1 & 3 \\ 4 & d \end{pmatrix}\begin{pmatrix} 1 & 2 \end{pmatrix}$. The product does not exist as the number of columns in **D** is not equal to the number of rows in **A**.

(e) $(\mathbf{DE})\mathbf{F} = \left[\begin{pmatrix} -1 & 3 \\ 4 & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}\right]\begin{pmatrix} 0 & 4 \\ f & 3 \end{pmatrix} = \left[\begin{pmatrix} -1 & 12 \\ 4 & 4d \end{pmatrix}\right]\begin{pmatrix} 0 & 4 \\ f & 3 \end{pmatrix} = \begin{pmatrix} 12f & 32 \\ 4df & 16+12d \end{pmatrix}$

(f) $\mathbf{D}(\mathbf{E}+\mathbf{F}) = \begin{pmatrix} -1 & 3 \\ 4 & d \end{pmatrix}\left[\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}+\begin{pmatrix} 0 & 4 \\ f & 3 \end{pmatrix}\right] = \begin{pmatrix} -1 & 3 \\ 4 & d \end{pmatrix}\left[\begin{pmatrix} 1 & 4 \\ f & 7 \end{pmatrix}\right] = \begin{pmatrix} -1+3f & 17 \\ 4+df & 16+7d \end{pmatrix}$

(g) $(\mathbf{E}+\mathbf{F})\mathbf{A} = \left[\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}+\begin{pmatrix} 0 & 4 \\ f & 3 \end{pmatrix}\right]\begin{pmatrix} 1 & 2 \end{pmatrix}$. The product does not exist as the number of columns in (**E** + **F**) is not equal to the number of rows in **A**.

(h) $\mathbf{D}^2 = \begin{pmatrix} -1 & 3 \\ 4 & d \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 4 & d \end{pmatrix} = \begin{pmatrix} 13 & -3+3d \\ -4+4d & 12+d^2 \end{pmatrix}$

Notes:

- *In part (a) the product **AB** is a 1 × 1 matrix. The answer must be expressed in the form of a matrix. That is, the number 11 must be enclosed by brackets.*
- *Parts (c) and (d) compare the products **AD** and **DA**. Quite clearly, **AD** ≠ **DA**. This is known as the **non-commutative property of matrix multiplication**.*
- *The square of a matrix is obtained by multiplying the matrix by itself.*

16.2.5 More Rules for Manipulating Matrices

- The following rules are similar to those for manipulating algebraic expressions except that it must always be remembered that matrix multiplication is *not commutative*. That is $\mathbf{AB} \neq \mathbf{BA}$.
- Provided that the products exist:
  - $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$
  - $\mathbf{AX} + \mathbf{BX} = (\mathbf{A} + \mathbf{B})\mathbf{X}$
  - $\mathbf{XA} + \mathbf{XB} = \mathbf{X}(\mathbf{A} + \mathbf{B})$

Example 16.6

Find the expansion of $(\mathbf{A} + \mathbf{B})^2$ if: (a) A and B are not commutative under multiplication (b) A and B are commutative under multiplication.

Solution:

(a) $(\mathbf{A} + \mathbf{B})^2 = (\mathbf{A} + \mathbf{B})(\mathbf{A} + \mathbf{B})$

$= \mathbf{A}^2 + \mathbf{AB} + \mathbf{BA} + \mathbf{B}^2$

(b) $(\mathbf{A} + \mathbf{B})^2 = (\mathbf{A} + \mathbf{B})(\mathbf{A} + \mathbf{B})$

$= \mathbf{A}^2 + \mathbf{AB} + \mathbf{BA} + \mathbf{B}^2$

$= \mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2$ (as $\mathbf{AB} = \mathbf{BA}$)

Example 16.7

Given that $\mathbf{A}\begin{pmatrix}1\\3\\-2\end{pmatrix} = \begin{pmatrix}5\\9\\-1\end{pmatrix}$ and $\mathbf{A}\begin{pmatrix}2\\0\\3\end{pmatrix} = \begin{pmatrix}5\\6\\7\end{pmatrix}$ find: (a) $\mathbf{A}\begin{pmatrix}2\\6\\-4\end{pmatrix}$ (b) $\mathbf{A}\begin{pmatrix}3\\3\\1\end{pmatrix}$

Solution:

(a)

$$\mathbf{A}\begin{pmatrix}2\\6\\-4\end{pmatrix} = \mathbf{A} \times 2\begin{pmatrix}1\\3\\-2\end{pmatrix}$$

$$= 2\mathbf{A}\begin{pmatrix}1\\3\\-2\end{pmatrix}$$

$$= 2 \times \begin{pmatrix}5\\9\\-1\end{pmatrix} = \begin{pmatrix}10\\18\\-2\end{pmatrix}$$

(b)

$$\mathbf{A}\begin{pmatrix}3\\3\\1\end{pmatrix} = \mathbf{A}\left[\begin{pmatrix}1\\3\\-2\end{pmatrix} + \begin{pmatrix}2\\0\\3\end{pmatrix}\right]$$

$$= \mathbf{A}\begin{pmatrix}1\\3\\-2\end{pmatrix} + \mathbf{A}\begin{pmatrix}2\\0\\3\end{pmatrix}$$

$$= \begin{pmatrix}5\\9\\-1\end{pmatrix} + \begin{pmatrix}5\\6\\7\end{pmatrix} = \begin{pmatrix}10\\15\\6\end{pmatrix}$$

Exercise 16.2

1. Given that dim(**A**) = 3×3 and dim(**B**) = $p \times 4$, find p and dim(**AB**) if **AB** exists.

2. Given that dim(**A**) = $p \times 4$ and dim(**B**) = 3×2, find p and dim(**BA**) if **BA** exists.

3. Given that dim(**A**) = $m \times n$, find m and n if (a) $\mathbf{A}^2$ exists (b) $\mathbf{A}^2$ does not exist.

4. Given that dim(**A**) = 2×2, find dim($\mathbf{A}^{10}$).

5. Given that dim(**A**) = 2×2, dim(**B**) = 2×2 and dim(**C**) = 2×1, find dim[(**A** + **B**)**C**] .

6. Given the following matrices, without the use of a CAS calculator, find each of the following products. If the product does not exist, explain why it does not exist.

$$\mathbf{A} = \begin{pmatrix} -2 & 0 \\ 3 & 4 \end{pmatrix},\ \mathbf{B} = \begin{pmatrix} 2 & -1 \\ b & 3 \end{pmatrix},\ \mathbf{C} = \begin{pmatrix} 4 & 1 \\ -1 & 3 \end{pmatrix},\ \mathbf{D} = \begin{pmatrix} 2 \\ 5 \end{pmatrix},\ \mathbf{E} = \begin{pmatrix} -1 & e \end{pmatrix}$$

(a) **AD** (b) **DA** (c) **DE** (d) **ED**
(e) **AB** (f) **BA** (g) (**AB**)**C** (h) **A**(**BC**)
(i) **A**(**B** + **C**) (j) (**B** + **C**)**D** (k) $\mathbf{A}^2$ (l) $\mathbf{A}^3$

7. Given the following matrices, find each of the following products. If the product does not exist, explain why it does not exist.

$$\mathrm{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix},\ \mathrm{B} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix},\ \mathrm{C} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & c & 0 \\ 1 & 0 & 0 \end{pmatrix},\ \mathrm{D} = \begin{pmatrix} -1 \\ d \\ -2 \end{pmatrix},\ \mathrm{E} = \begin{pmatrix} -1 & 0 & 1 \end{pmatrix}$$

(a) **AD** (b) **DA** (c) **DE** (d) **ED**
(e) **AB** (f) **BA** (g) (**AB**)**C** (h) **A**(**BC**)
(i) **A**(**B** + **C**) (j) (**B** + **C**)**D** (k) $\mathbf{A}^2$ (l) $\mathbf{A}^3$

8. Without the use of a CAS calculator, find all possible products of exactly two ***different*** matrices from the following matrices: $\begin{pmatrix} 1 & k \\ 0 & -1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$, $\begin{pmatrix} 1 & 3 \end{pmatrix}$, $\begin{pmatrix} 1 \\ k \end{pmatrix}$.

9. If $\mathbf{A} = \begin{pmatrix} p & 1 \\ q & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}$ find the values of p and q if **A** and **B** are commutative under multiplication.

10. Given that $\mathbf{M} = \begin{pmatrix} 2 & 3 \\ -4 & -1 \end{pmatrix}$, show that $\mathbf{M}^2 - \mathbf{M} + 10\mathbf{I} = \mathbf{0}$.

11. Given that $\mathbf{M} = \begin{pmatrix} 2 & -1 \\ -3 & 4 \end{pmatrix}$, find the value of k such that $\mathbf{M}^2 - 6\mathbf{M} + k\mathbf{I} = \mathbf{0}$.

12. Given that $\mathbf{M} = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$, find the value(s) of p and q given that $\mathbf{M}^2 + p\mathbf{M} + q\mathbf{I} = \mathbf{0}$.

13. Given that dim(**M**) = 2×2 and $\mathbf{M}^2 - \mathbf{M} + 2\mathbf{I} = \mathbf{0}$, show that $\mathbf{M}^4 + 3\mathbf{M}^2 + 4\mathbf{I} = \mathbf{0}$.

14. Given that $\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$, verify that $\mathbf{M}^n = \begin{pmatrix} 1 & 0 \\ 3n & 1 \end{pmatrix}$ where n is a positive integer.

15. Given that $\mathbf{M} = \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix}$, verify that $\mathbf{M}^n = \begin{pmatrix} 2n+1 & -n \\ 4n & 1-2n \end{pmatrix}$ where n is a positive integer.

16. Find the expansion for $(\mathbf{A} - \mathbf{B})^2$ given that **A** and **B** are:
    (a) not commutative under multiplication (b) commutative under multiplication.

17. Find the expansion for $(\mathbf{A} - \mathbf{B})(\mathbf{A} + \mathbf{B})$ given that **A** and **B** are:
    (a) not commutative under multiplication (b) commutative under multiplication.

18. Factorise the following matrix expressions:
    (a) $\mathbf{AB} + \mathbf{CB}$ (b) $\mathbf{AC} + \mathbf{AD}$ (c) $\mathbf{A}^3 - \mathbf{A}^2$
    (d) $\mathbf{AB} + \mathbf{A}$ (e) $\mathbf{AB} + \mathbf{B}$ (f) $\mathbf{ABC} + \mathbf{ADC}$

19. Given that $\mathbf{A}\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ -3 \end{pmatrix}$ and $\mathbf{A}\begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \\ 19 \end{pmatrix}$ find:

    (a) $\mathbf{A}\begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix}$ (b) $\mathbf{A}\begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix}$ (c) $\mathbf{A}\begin{pmatrix} -4 \\ 0 \\ 6 \end{pmatrix}$ (d) $\mathbf{A}\begin{pmatrix} -2 \\ 3 \\ 9 \end{pmatrix}$

20. Given that $\mathbf{A}\begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 9 \\ -15 \\ -5 \end{pmatrix}$ and $\mathbf{A}\begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} -9 \\ 3 \\ 1 \end{pmatrix}$ find:

    (a) $\mathbf{A}\begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix}$ (b) $\mathbf{A}\begin{pmatrix} -5 \\ -2 \\ 1 \end{pmatrix}$ (c) $\mathbf{A}\begin{pmatrix} 9 \\ 0 \\ 0 \end{pmatrix}$ (d) $\mathbf{A}\begin{pmatrix} 6 \\ 6 \\ -3 \end{pmatrix}$

*21. Given that $\mathbf{A}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ -5 \end{pmatrix}$ and $\mathbf{A}\begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$ find:

    (a) $\mathbf{A}\begin{pmatrix} 1 & 1 \\ -2 & 4 \\ 1 & -2 \end{pmatrix}$ (b) $\mathbf{A}\begin{pmatrix} 1 & 1 \\ 4 & -2 \\ -2 & 1 \end{pmatrix}$ (c) $\mathbf{A}\begin{pmatrix} 1 & 2 \\ 4 & -4 \\ -2 & 2 \end{pmatrix}$ (d) $\mathbf{A}\begin{pmatrix} 1 & -1 \\ -2 & -4 \\ 1 & 2 \end{pmatrix}$

*22. Given that $\mathbf{A}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}$ and $\mathbf{A}\begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 9 \\ 5 \\ 5 \end{pmatrix}$ find:

    (a) $\mathbf{A}\begin{pmatrix} 2 & 2 \\ -1 & 3 \\ 1 & 2 \end{pmatrix}$ (b) $\mathbf{A}\begin{pmatrix} 0 & 4 \\ -4 & 6 \\ -1 & 4 \end{pmatrix}$ (c) $\mathbf{A}\begin{pmatrix} 2 & -2 & 4 \\ -1 & -3 & 2 \\ 1 & -2 & 3 \end{pmatrix}$

*23. Find **A** if: (a) $\mathbf{A}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\mathbf{A}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$

    (b) $\mathbf{A}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix}$, $\mathbf{A}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$ and $\mathbf{A}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$

16.2.6 Properties of Special Matrices

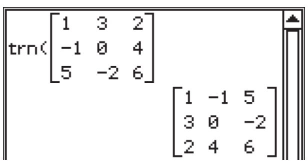
- A diagonal matrix is a square matrix where all the entries except the entries along the main diagonal are zero. For example:

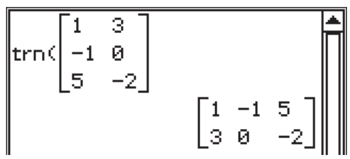
$$\begin{pmatrix} 5 & 0 \\ 0 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{pmatrix} \text{ and } \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

- The transpose of a matrix is obtained by interchanging the rows and the columns. For example:

If $\mathbf{A} = \begin{pmatrix} 1 & 3 & 2 \\ -1 & 0 & 4 \\ 5 & -2 & 6 \end{pmatrix}$, the transpose of **A**, written $\mathbf{A}^\mathbf{T}$, is $\mathbf{A}^\mathbf{T} = \begin{pmatrix} 1 & -1 & 5 \\ 3 & 0 & -2 \\ 2 & 4 & 6 \end{pmatrix}$.

If $\mathbf{A} = \begin{pmatrix} 1 & 3 \\ -1 & 0 \\ 5 & -2 \end{pmatrix}$, the transpose of **A**, written $\mathbf{A}^\mathbf{T}$, is $\mathbf{A}^\mathbf{T} = \begin{pmatrix} 1 & -1 & 5 \\ 3 & 0 & -2 \end{pmatrix}$.





- A square matrix **M** is symmetric if $\mathbf{M} = \mathbf{M}^\mathbf{T}$.
- A square matrix **M** is anti-symmetric (or skew-symmetric) if $\mathbf{M} = -\mathbf{M}^\mathbf{T}$.

Exercise 16.3

1. If $\mathbf{A} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$, find $\mathbf{A}^n$ for n = 1, 2, 3, … , 10.

   Hence make a conjecture with regards $\mathbf{A}^n$ where n is a positive integer.

2. If $\mathbf{A} = \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}$, find $\mathbf{A}^n$ for n = 1, 2, 3, … , 10.

   Hence make a conjecture with regards $\mathbf{A}^n$ where n is a positive integer.

3. If $\mathbf{A} = \begin{pmatrix} 0 & 0 & a \\ 0 & b & 0 \\ c & 0 & 0 \end{pmatrix}$, find $\mathbf{A}^n$ for n = 1, 2, 3, … , 10.

   Hence make a conjecture with regards $\mathbf{A}^n$ where n is a positive integer.

4. Prove or disprove the conjecture: If $\mathbf{A} \times \mathbf{A}^\mathbf{T}$ exists, then **A** must be a square matrix.

5. If **A** and **B** are non-zero square 2 × 2 matrices, prove that $(\mathbf{AB})^{T} = \mathbf{B}^{T}\mathbf{A}^{T}$.

6. If $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $a \neq d$, $b \neq 0$, $c \neq 0$ and $\mathbf{A} \times \mathbf{A}^{\mathbf{T}} = \mathbf{A}^{\mathbf{T}} \times \mathbf{A}$, prove that $b = c$.

7. If $\mathbf{A} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$, prove that $\mathbf{A} = \mathbf{A}^{\mathbf{T}}$ (i.e. **A** is a symmetric matrix).

8. If **A** is a 3 × 3 non-zero matrix and $\mathbf{A} = -\mathbf{A}^{\mathbf{T}}$ (i.e. A is anti-symmetric), prove that *each element in the main diagonal must be zero.*

16.3 The Multiplicative Inverse of a Matrix

- Given that **A** and **B** are square matrices and **I** is the corresponding unit matrix and if **AB** = **BA** = **I,** then:
  - the multiplicative inverse of **A**, written $\mathbf{A}^{-1}$, is **B**
  - the multiplicative inverse of **B**, written $\mathbf{B}^{-1}$, is **A**
  - the unit matrix **I** is known as the multiplicative identity.
- Hence, the statement, $\mathbf{A}\,\mathbf{A}^{-1} = \mathbf{A}^{-1}\,\mathbf{A} = \mathbf{I}$.
- In general, if $\mathbf{AB} = k\mathbf{I}$ where k is a real number

  then, $\mathbf{A}^{-1} = \frac{1}{k}\mathbf{B}$ and $\mathbf{B}^{-1} = \frac{1}{k}\mathbf{A}$

Example 16.8

Verify that the multiplicative inverse of $\begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix}$ is $\begin{pmatrix} 1 & -1 \\ -3 & 4 \end{pmatrix}$.

Solution:

The product of the 2 matrices is:

$$\begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & -1 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence the inverse of $\begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix}$ is $\begin{pmatrix} 1 & -1 \\ -3 & 4 \end{pmatrix}$.

Example 16.9

Given that $\mathbf{A} = \begin{pmatrix} 3 & 6 & 0 \\ 0 & -3 & 3 \\ -3 & 0 & -3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -1 & -2 & -2 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$, find **AB**. Hence, find $\mathbf{A}^{-1}$.

Solution:

The product **AB** is: $\begin{pmatrix} 3 & 6 & 0 \\ 0 & -3 & 3 \\ -3 & 0 & -3 \end{pmatrix}\begin{pmatrix} -1 & -2 & -2 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

$= 3\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Hence: $\mathbf{A}^{-1} = \frac{1}{3}\begin{pmatrix} -1 & -2 & -2 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$

Example 16.10

Disprove the conjecture that if $\mathbf{A} \times \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then $\mathbf{A}^{-1} = \mathbf{B}$.

Solution:

Let $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$.

Clearly $\mathbf{A} \times \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, but **A** does not have an inverse as **A** is not a square matrix.

Exercise 16.4

1. Verify that each of the given pairs of matrices are inverses of each other:

(a) $\begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}$ (b) $\begin{pmatrix} 3 & 7 \\ 2 & 5 \end{pmatrix}, \begin{pmatrix} 5 & -7 \\ -2 & 3 \end{pmatrix}$

(c) $\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix}, \begin{pmatrix} 7 & -4 \\ -5 & 3 \end{pmatrix}$ (d) $\begin{pmatrix} 7 & -25 \\ 2 & -7 \end{pmatrix}, \begin{pmatrix} -7 & 25 \\ -2 & 7 \end{pmatrix}$

2. Determine if the following pairs of matrices are inverses of each other:

(a) $\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{pmatrix}, \begin{pmatrix} -1 & -1 & 0 \\ -1 & 0 & 0 \\ 1 & -1 & -1 \end{pmatrix}$ (b) $\begin{pmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 2 & 3 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 6 & 1 & 20 \\ 1 & -1 & 0 \\ 0 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 3 & -17 & -20 \\ 3 & -18 & -20 \\ -1 & 6 & 7 \end{pmatrix}$ (d) $\begin{pmatrix} 3 & -2 & 4 \\ 2 & 1 & 2 \\ 5 & 3 & 5 \end{pmatrix}, \begin{pmatrix} -1 & 22 & -8 \\ 0 & -5 & 1 \\ 1 & -19 & 7 \end{pmatrix}$

3. Given that the first matrix is **A** and the second matrix is **B**, find **AB** and hence $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$ for each of the following:

(a) $\begin{pmatrix} 3 & -4 \\ -1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 3 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 3 \\ 2 & -1 \end{pmatrix}$

(c) $\begin{pmatrix} 4 & 3 \\ 5 & 3 \end{pmatrix}, \begin{pmatrix} 3 & -3 \\ -5 & 4 \end{pmatrix}$

(d) $\begin{pmatrix} -8 & 6 \\ -2 & -4 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix}$

4. Given that the first matrix is **A** and the second matrix is **B**, find **AB** and hence $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$ for each of the following:

(a) $\begin{pmatrix} 0 & 1 & 2 \\ -1 & 1 & 2 \\ 1 & -2 & -5 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 0 \\ 3 & 2 & 2 \\ -1 & -1 & -1 \end{pmatrix}$

(b) $\begin{pmatrix} 0 & 1 & 2 \\ 2 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 2 & 3 \\ -1 & 2 & 4 \\ 1 & -1 & -2 \end{pmatrix}$

(c) $\begin{pmatrix} -4 & 4 & 4 \\ 7 & 1 & -3 \\ 13 & -5 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & -1 \\ 3 & -2 & 2 \end{pmatrix}$

(d) $\begin{pmatrix} 2 & -3 & 1 \\ 1 & 2 & -2 \\ 1 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 8 & 10 & 4 \\ -5 & 5 & 5 \\ -1 & -5 & 7 \end{pmatrix}$

5. Let $\mathbf{A} = \begin{pmatrix} 5 & 4 \\ 6 & 5 \end{pmatrix}$. Find the value of p if the inverse of **A** has the form $\begin{pmatrix} 5 & p \\ -6 & 5 \end{pmatrix}$.

6. Find the value of p if the multiplicative inverse of $\begin{pmatrix} 2 & 2 \\ 14 & p \end{pmatrix}$ has the form $\frac{1}{2}\begin{pmatrix} 15 & -2 \\ -14 & 2 \end{pmatrix}$.

7. Find the value of p if the multiplicative inverse of $\begin{pmatrix} p & -5 \\ -7 & 4 \end{pmatrix}$ has the form $\begin{pmatrix} 4 & 5 \\ 7 & p \end{pmatrix}$.

8. Find $\mathbf{A}^{-1}$, if $\mathbf{A} \times \begin{pmatrix} p & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

9. Find $\mathbf{P}^{-1}$, if $\begin{pmatrix} x & y \\ -1 & 2 \end{pmatrix} \times \mathbf{P} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

10. Find the values of a and b if the inverse of $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ is of the form $\begin{pmatrix} a & 0 & b \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix}$.

11. Find the values of a and b if the inverse of $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix}$ is of the form $\begin{pmatrix} a & 0 & 0 \\ b & a & -a \\ -b & 0 & a \end{pmatrix}$.

12. Find $\begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 3 & 0 \\ 2 & 4 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} -12 & 0 & 8 & 1 \\ 6 & 0 & -4 & 2 \\ 2 & 0 & 2 & -1 \\ 0 & 10 & 0 & 0 \end{pmatrix}$. Hence, find $\begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 3 & 0 \\ 2 & 4 & 0 & 0 \end{pmatrix}^{-1}$.

16.3.1 Determinant of a 2 × Matrix

- The **determinant** of matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, denoted det(**A**) or $|\mathbf{A}|$ is defined as:

$$\det(\mathbf{A}) = |\mathbf{A}| = ad - bc$$

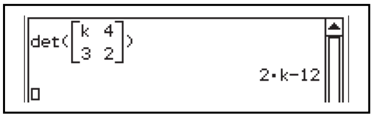
- That is, the determinant of a 2 × 2 matrix is defined as the difference between the product of the elements in the main or leading diagonal with the product of the elements in the minor diagonal.
  The diagonal consisting of the terms a and d is called the main diagonal while the diagonal consisting of the terms b and c is called the minor diagonal.

Example 16.11

Given that $A = \begin{pmatrix} k & 4 \\ 3 & 2 \end{pmatrix}$, find:

(a) $|\mathbf{A}|$ (b) $|2\mathbf{A}|$ (c) k if (i) $|\mathbf{A}| = 0$ (ii) $|\mathbf{A}| \neq 0$ (iii) $|\mathbf{A}| = 4$.

Solution:



(a) $|\mathbf{A}| = 2k - 12$

(b) $|2\mathbf{A}| = \begin{vmatrix} 2k & 8 \\ 6 & 4 \end{vmatrix} = 8k - 48$

(c) (i) Since $|\mathbf{A}| = 0$, $2k - 12 = 0 \Rightarrow k = 6$

(ii) Since $|\mathbf{A}| \neq 0$, $2k - 12 \neq 0 \Rightarrow k \neq 6$

(iii) Since $|\mathbf{A}| = 4$, $2k - 12 = 4 \Rightarrow k = 8$

16.3.2 Formula for the inverse of a 2 × atrix

- Given that $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the inverse of matrix **A** denoted $\mathbf{A}^{-1}$, exists only if $|\mathbf{A}| \neq 0$

  and is given by $$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

- Notice that the elements in the main diagonal have been “exchanged” and the elements in the minor diagonal have had their signs reversed.

- If $\mathbf{A}^{-1}$ does not exist then **A** is *singular* or *non-invertible*: In which case $|\mathbf{A}| = 0$.

- If $\mathbf{A}^{-1}$ exists, then **A** is *non-singular* or *invertible*: In which case $|\mathbf{A}| \neq 0$.

Example 16.12

Without the use of a calculator, find, if it exists, the inverse of:

(a) $\begin{pmatrix} 3 & 2 \\ 4 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 4 & 6 \\ 6 & 9 \end{pmatrix}$

Solution:

(a) The determinant of the matrix is 6 – 8 = –2.

Hence: $\begin{pmatrix} 3 & 2 \\ 4 & 2 \end{pmatrix}^{-1} = -\frac{1}{2}\begin{pmatrix} 2 & -2 \\ -4 & 3 \end{pmatrix}$

(b) The determinant of the matrix is 36 – 36 = 0. Hence, the inverse does not exist.

Example 16.13

Find the value of *k* for which the inverse of **A** = $\begin{pmatrix} k & 4 \\ 1 & k \end{pmatrix}$ exists. Find $\mathbf{A}^{-1}$.

Solution:

For $\mathbf{A}^{-1}$ to exist, $\begin{vmatrix} k & 4 \\ 1 & k \end{vmatrix} \neq 0$

Hence $k^2 - 4 \neq 0 \quad \Rightarrow \quad k \neq \pm 2$

Therefore: $\mathbf{A}^{-1} = \frac{1}{k^2 - 4}\begin{pmatrix} k & -4 \\ -1 & k \end{pmatrix}$ where $k \neq \pm 2$

Exercise 16.5

1. Evaluate the following:

(a) $\begin{vmatrix} 1 & 5 \\ 6 & 3 \end{vmatrix}$ (b) $\begin{vmatrix} -2 & 3 \\ 0 & 1 \end{vmatrix}$ (c) $\begin{vmatrix} k & 3 \\ k & 2 \end{vmatrix}$ (d) $\begin{vmatrix} 4 & k \\ -k & 3 \end{vmatrix}$

2. Find the value of *k* in each of the following:

(a) $\begin{vmatrix} k & 4 \\ 5 & 2 \end{vmatrix} = 4$ (b) $\begin{vmatrix} k+1 & 2 \\ 1 & 2 \end{vmatrix} = 20$

(c) $\begin{vmatrix} k-1 & 2 \\ -1 & k+2 \end{vmatrix} = 0$ (d) $\begin{vmatrix} k+1 & k+1 \\ 3 & k \end{vmatrix} = 0$

3. Given that **A** = $\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$ and **B** = $\begin{pmatrix} 4 & 2 \\ 3 & 2 \end{pmatrix}$:

(a) find **AB** (b) find $|\mathbf{AB}|$ (c) find $|\mathbf{A}||\mathbf{B}|$

(d) comment on your answers in (b) and (c) (e) find $|\mathbf{B}^5|$.

4. Find, when it exists, the inverse of each of the following matrices.

(a) $\begin{pmatrix} 6 & 2 \\ 9 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} 7 & 3 \\ 10 & 4 \end{pmatrix}$ (c) $\begin{pmatrix} 9 & 8 \\ 10 & 12 \end{pmatrix}$ (d) $\begin{pmatrix} -5 & -4 \\ -3 & 2 \end{pmatrix}$

5. Find the value(s) of k for which each of the following matrices are singular:

(a) $\begin{pmatrix} 3 & 4 \\ 6 & k \end{pmatrix}$ (b) $\begin{pmatrix} 25 & k \\ 10 & 2 \end{pmatrix}$ (c) $\begin{pmatrix} -8 & 4 \\ k & 8 \end{pmatrix}$ (d) $\begin{pmatrix} k+1 & 1 \\ 2 & k \end{pmatrix}$

6. Find the value(s) of k for which each of the following matrices are invertible.

(a) $\begin{pmatrix} k & 8 \\ 2 & k \end{pmatrix}$ (b) $\begin{pmatrix} k+1 & k \\ 2 & 2 \end{pmatrix}$ (c) $\begin{pmatrix} k+1 & k+1 \\ -1 & k-3 \end{pmatrix}$ (d) $\begin{pmatrix} k & 1 \\ -1 & k \end{pmatrix}$

7. Given that $\mathbf{A} = \begin{pmatrix} x & 1 \\ 3 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 4 & -1 \\ -3 & y \end{pmatrix}$, find x and y if $\mathbf{A} = \mathbf{B}^{-1}$.

8. Given that $\mathbf{A} = \begin{pmatrix} x & 3 \\ 1 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 4 & -3 \\ y & 2 \end{pmatrix}$, find x, y and k if $\mathbf{AB} = k\mathbf{I}$.

9. Given that $\mathbf{A}(x) = \begin{pmatrix} 1-x & 0 \\ 0 & 1+x \end{pmatrix}$ and $\mathbf{B}(x) = \begin{pmatrix} 1+x & 0 \\ 0 & 1-x \end{pmatrix}$. Find:

(a) $\mathbf{A}(1) + 2\mathbf{B}(2)$

(b) the value of x for which $\mathbf{A}(x) + \mathbf{B}(x) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

(c) $\mathbf{A}(1) \times \mathbf{B}(2)$

(d) the value of x for which $\mathbf{A}(x) .\mathbf{B}(x)$ is non-singular.

10. Given that $\mathbf{A} = \begin{pmatrix} 1 & 5 \\ 2 & 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -1 & 4 \\ -1 & -1 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 3 & 5 \\ 1 & 3 \end{pmatrix}$,

verify that $(\mathbf{ABC})^{-1} = \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}$.

11. Given the non-singular matrices $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$,

prove that $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

12. Given that $\mathbf{A} = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$, verify that $(\mathbf{A}^n)^{-1} = (\mathbf{A}^{-1})^n$ for n = 2, 3 and 4.

13. Given the non-singular matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ prove that $(\mathbf{A}^2)^{-1} = (\mathbf{A}^{-1})^2$.

14. If $\mathbf{A} = \begin{pmatrix} 4 & -2 \\ 6 & -3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 5 & 6 \\ 10 & 12 \end{pmatrix}$, verify that if $\mathbf{AB} = \mathbf{0}$ then either $|\mathbf{A}| = 0$ or $|\mathbf{B}| = 0$.

16.4 Manipulating Matrix Equations

- Consider the matrix equation $\mathbf{AX} = \mathbf{B}$
  Provided **A** is non-singular, we can manipulate the equation and find an expression for **X** by using the result $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.
- Pre-multiply both sides of the equation by the inverse of **A**, $\mathbf{A}^{-1}$

  Hence $\mathbf{A}^{-1}(\mathbf{AX}) = \mathbf{A}^{-1}\mathbf{B}$

  Rewriting $(\mathbf{A}^{-1}\mathbf{A})\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$

  But $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$. $\Rightarrow$ $\mathbf{IX} = \mathbf{A}^{-1}\mathbf{B}$

  But $\mathbf{IX} = \mathbf{X}$. $\Rightarrow$ $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$
- his algebraic procedure is referred to as the procedure of **matrix inversion**.
  X has been isolated by inverting **A** onto the other side of the equation.
  If the matrix to be inverted is a pre-multiplied (or left-multiplied matrix), the inverted matrix must also be a pre-multiplied matrix (left-multiplied matrix).
- Similarly, it can be shown that if $\mathbf{XA} = \mathbf{B} \;\Rightarrow\; \mathbf{X} = \mathbf{BA}^{-1}$.
- Hence:
  - Given that $\mathbf{AX} = \mathbf{B} \;\Rightarrow\; \quad = \mathbf{A}^{-1}\mathbf{B}$.
  - Given that $\mathbf{XA} = \mathbf{B} \;\Rightarrow\; \mathbf{X} = \mathbf{BA}^{-1}$

Example 16.14

Solve for **X**:

(a) $\begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}\mathbf{X} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$ (b) $\mathbf{X}\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 5 & -1 \\ 3 & 2 \end{pmatrix}$

Solution:

(a)
$$\begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}\mathbf{X} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$
$$\Rightarrow \mathbf{X} = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}^{-1}\begin{pmatrix} 5 \\ 4 \end{pmatrix}$$
$$= \begin{pmatrix} 3 & -2 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 7 \\ -8 \end{pmatrix}$$

(b)
$$\mathbf{X}\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 5 & -1 \\ 3 & 2 \end{pmatrix}$$
$$\Rightarrow \mathbf{X} = \begin{pmatrix} 5 & -1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}^{-1}$$
$$= \begin{pmatrix} 5 & -1 \\ 3 & 2 \end{pmatrix}\left[-\frac{1}{10}\begin{pmatrix} 2 & -3 \\ -4 & 1 \end{pmatrix}\right]$$
$$= -\frac{1}{10}\begin{pmatrix} 5 & -1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ -4 & 1 \end{pmatrix} =$$
$$= -\frac{1}{10}\begin{pmatrix} 14 & -16 \\ -2 & -7 \end{pmatrix}$$

Example 16.15

For the non-singular matrices **A**, **B,** given that **AB** = **BA** , show that:

(a) $\mathbf{B} = \mathbf{A}^{-1}\mathbf{BA}$ (b) $\mathbf{B}^2 = \mathbf{A}^{-1}\mathbf{B}^2\mathbf{A}$

Solution:

(a)
$$\mathbf{AB} = \mathbf{BA}$$
$$\mathbf{A}^{-1}\mathbf{AB} = \mathbf{A}^{-1}\mathbf{BA}$$
$$\mathbf{IB} = \mathbf{A}^{-1}\mathbf{BA}$$
Hence,
$$\mathbf{B} = \mathbf{A}^{-1}\mathbf{BA}$$

(b)
$$\mathbf{B} = \mathbf{A}^{-1}\mathbf{BA}$$
$$\begin{aligned}\mathbf{B}^2 &= (\mathbf{A}^{-1}\mathbf{BA}) \times (\mathbf{A}^{-1}\mathbf{BA})\\ &= (\mathbf{A}^{-1}\mathbf{B}) \times (\mathbf{AA}^{-1}) \times (\mathbf{BA})\\ &= (\mathbf{A}^{-1}\mathbf{B}) \times (\mathbf{I}) \times (\mathbf{BA})\\ &= \mathbf{A}^{-1}(\mathbf{B} \times \mathbf{I} \times \mathbf{B})\mathbf{A}\\ &= \mathbf{A}^{-1}\mathbf{B}^2\mathbf{A}\end{aligned}$$

Exercise 16.6

1. Find an expression for matrix X in each of the following:
   (a) $2\mathbf{X} = \mathbf{A}$ (b) $3\mathbf{X} + \mathbf{B} = \mathbf{C}$ (c) $\mathbf{AX} - \mathbf{B} = \mathbf{0}$
   (d) $\mathbf{AX} + \mathbf{B} = \mathbf{C}$ (e) $\mathbf{AX} + \mathbf{BX} = \mathbf{C}$ (f) $\mathbf{XA} + \mathbf{XB} = \mathbf{C}$
   (g) $\mathbf{AX} + \mathbf{C} = \mathbf{BX} + \mathbf{D}$ (h) $\mathbf{AX} + 2\mathbf{X} = \mathbf{B}$ (i) $\mathbf{AX} - \mathbf{BX} = 3\mathbf{X} + \mathbf{D}$

2. Solve for matrix **X** in each of the following:

   (a) $\begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}\mathbf{X} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ (b) $\begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}\mathbf{X} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

   (c) $\mathbf{X}\begin{pmatrix} 10 & 3 \\ 6 & 2 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ (d) $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\mathbf{X} + \mathbf{X}\begin{pmatrix} 25 & 4 \\ 11 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$

   (e) $\begin{pmatrix} 25 & 4 \\ 11 & 2 \end{pmatrix}\mathbf{X} - \mathbf{X}\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ (f) $\begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & -2 & 1 \end{pmatrix}\mathbf{X} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 2 & 3 & 1 \end{pmatrix}$

   (g) $\begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & -2 & 1 \end{pmatrix}\mathbf{X} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & -2 & 0 \end{pmatrix}\mathbf{X} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ (h) $\mathbf{X}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} = 2\mathbf{X}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

   (i) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\mathbf{X} + \mathbf{X}\begin{pmatrix} 1 & 2 & 1 \\ 3 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (j) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}\mathbf{X} - \mathbf{X}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$

3. Given that $\mathbf{PQ} = \begin{pmatrix} 17 & -8 \\ 13 & -6 \end{pmatrix}$, find: (a) **P** if $\mathbf{Q} = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$ (b) **Q** if $\mathbf{P}^{-1} = \begin{pmatrix} 4 & -3 \\ -5 & 4 \end{pmatrix}$.

4. Given that $\mathbf{PQ} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 3 & 0 \\ 4 & 4 & 0 \end{pmatrix}$, find: (a) $\mathbf{Q}$ if $\mathbf{P} = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix}$ (b) $\mathbf{P}$ if $\mathbf{Q}^{-1} = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

5. Given that $\mathbf{M}^2 - 4\mathbf{M} - 3\mathbf{I} = \mathbf{0}$, express $\mathbf{M}^{-1}$ in the form $p\mathbf{M} + q\mathbf{I}$.

6. Given that $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix}$, find p and q such that $\mathbf{A}^2 + p\mathbf{A} + q\mathbf{I} = \mathbf{0}$.

   Hence find $\mathbf{A}^{-1}$ in the form $u\mathbf{A} + v\mathbf{I}$.

7. Given the non-singular matrices $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{BA}$, show that:
   (a) $\mathbf{A}^2 = \mathbf{BA}^2\mathbf{B}^{-1}$ (b) $\mathbf{A}^3 = \mathbf{BA}^3\mathbf{B}^{-1}$

8. If $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are non-singular square matrices, such that $\mathbf{AB} = \mathbf{BC}$, show that
   (a) $\mathbf{A}^2 = \mathbf{BC}^2\mathbf{B}^{-1}$ (b) $\mathbf{A}^3 = \mathbf{BC}^3\mathbf{B}^{-1}$

9. Given the non-singular square matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ such that $\mathbf{ABC} = \mathbf{I}$.
   (a) Show that $\mathbf{C} = \mathbf{B}^{-1}\mathbf{A}^{-1}$. (b) Hence, show that $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

10. Given the non-singular square matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ such that $\mathbf{ABC} = \mathbf{I}$.
    (a) Show that $\mathbf{B} = \mathbf{A}^{-1}\mathbf{C}^{-1}$. (b) Hence, show that $\mathbf{B}^{-1} = (\mathbf{CA})$.

11. If $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{R}$ are non-singular square matrices, such that $\mathbf{PQ} = \mathbf{QR}$,
    use the result $(\mathbf{PQ})^{-1} = \mathbf{Q}^{-1}\mathbf{P}^{-1}$, to show that:
    (a) $\mathbf{P}^{-1} = \mathbf{QR}^{-1}\mathbf{Q}^{-1}$ (b) $(\mathbf{P}^{-1})^2 = \mathbf{Q}(\mathbf{R}^{-1})^2\mathbf{Q}^{-1}$

12. Given that $\mathbf{T} = \begin{pmatrix} 5 & 4 \\ 4 & 3 \end{pmatrix}$, $\mathbf{M} = \begin{pmatrix} -14 & 20 \\ -12 & 17 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$.
    (a) Verify that $\mathbf{M} = \mathbf{TDT}^{-1}$.
    (b) Find $\mathbf{D}^2$, $\mathbf{D}^3$ and $\mathbf{D}^4$. Hence find in terms of n, $\mathbf{D}^n$ for n = 1, 2, 3... .
    Comment on the matrices $\mathbf{D}$ and $\mathbf{D}^n$.
    (c) Use matrix algebra to verify that $\mathbf{M}^2 = \mathbf{TD}^2\mathbf{T}^{-1}$ and $\mathbf{M}^3 = \mathbf{TD}^3\mathbf{T}^{-1}$.
    Hence, find an expression for $\mathbf{M}^n$, for n = 1,2,3... .
    Use this expression to find $\mathbf{M}^{10}$.

13. Given that $\mathbf{A}$ is a 2 × 2 non-singular matrix, prove that $(\mathbf{A}^{\mathrm{T}})^{-1} = (\mathbf{A}^{-1})^{\mathrm{T}}$ where $\mathbf{A}^{\mathrm{T}}$ is the transpose of $\mathbf{A}$.

14. Given that $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $|\mathbf{M}| = 1$ and $a + d = -1$, Prove that $\mathbf{M}^{-1} = \mathbf{M}^2$.

15. Given that $\mathbf{A}$ and $\mathbf{B}$ are non-singular square matrices,
    prove that $\mathbf{A} \times \mathbf{B} = 0 \Rightarrow |\mathbf{A}| = 0$ or $|\mathbf{B}| = 0$.

16. Let $\mathbf{X}$ be a 2 × 1 matrix, $\mathbf{A}$ be a 2 × 2 matrix and λ be a real non-zero constant.
    Given that $\mathbf{A} \times \mathbf{X} = \lambda\mathbf{X}$, prove that $|\mathbf{A} - \lambda\mathbf{I}| = 0$.

17 Systems of Linear Equations

17.1 Systems of Two Linear Equations

- Consider the linear equation $x + y = 10$.
  In this situation, $x = 0$, $y = 10$ is one solution to the equation.
  Another solution could be $x = 10$, $y = 0$.
  Clearly, there are an infinite number of solutions to this equation.
- Consider now the set of linear equations
  $$x + y = 10$$
  $$2x + 2y = 20.$$
  In this situation, the two equations are identical.
  Hence, there are an infinite number of solutions to this set of equations.
- Consider now the set of linear equations
  $$x + y = 10$$
  $$x + y = 20.$$
  In this situation, the two equations are contradictory.
  Hence, there no solutions to this set of equations.
- Consider now the set of linear equations
  $$x + y = 10$$
  $$x - y = 0.$$
  In this situation, $x = 5$, $y = 5$ is the only solution to this set of equations.
- In summary, where two variables x and y are linearly related, two distinct consistent equations are required to determine uniquely the values of x and y.

17.2 The Matrix Inversion Method

- Consider the following set of simultaneous equations.
  $$x + y = 3$$
  $$x - y = 5$$
  - These equations can be rewritten as the matrix equation:
    $$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}.$$
    - $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ is termed the coefficient matrix.
    - $\begin{pmatrix} x \\ y \end{pmatrix}$ is the matrix of variables. $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$ is the matrix of constants.
  - The matrix inversion method introduced in Section 16.4 can then be used to solve for the matrix of variables and hence x and y.

Example 17.1

Without the use of a calculator, use the matrix inversion method to solve for x and y in

$x + 3y = 14$ and $5x - y = 6$.

Solution:

Rewrite the system of linear equations as a matrix equation.

$$\begin{pmatrix} 1 & 3 \\ 5 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 14 \\ 6 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 5 & -1 \end{pmatrix}^{-1}\begin{pmatrix} 14 \\ 6 \end{pmatrix}$$

$$= -\frac{1}{16}\begin{pmatrix} -1 & -3 \\ -5 & 1 \end{pmatrix}\begin{pmatrix} 14 \\ 6 \end{pmatrix}$$

$$= -\frac{1}{16}\begin{pmatrix} -32 \\ -64 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

Hence, $x = 2, y = 4$.

Example 17.2

Without the use of a calculator, use the matrix inversion method to solve for x and y in

$\mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -10 \\ 4 \end{pmatrix}$ if: (a) $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ (b) $\mathbf{A} = \begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix}$.

Solution:

(a) $$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -10 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}^{-1}\begin{pmatrix} -10 \\ 4 \end{pmatrix}$$

$$= -\frac{1}{3}\begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} -10 \\ 4 \end{pmatrix} = \begin{pmatrix} 6 \\ -8 \end{pmatrix}.$$

Hence: $x = 6, y = -8$.

(b) $$\begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -10 \\ 4 \end{pmatrix}.$$

Clearly, the coefficient matrix is singular.

Re-examining the linear equations: $x - 2y = -10$

$-x + 2y = 4$ which is $x - 2y = -4$

These two equations are contradictory.

Hence, there are no solutions to x and y.

Example 17.3

(a) Given $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix}$, find **AB**.

(b) Use result in (a) and an appropriate matrix operation to solve for x and y in:

$$3x + 2y = 6$$
$$-x + 4y = -16$$

Solution:

(a) $\mathbf{AB} = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 6 \\ 0 & 14 \end{pmatrix}$.

(b) Rewrite the system of equations as a matrix equation $\begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ -16 \end{pmatrix}$.

Note that the coefficient matrix is matrix **B**.
Pre-multiply the matrix equation with the matrix **A**:

$$\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 6 \\ -16 \end{pmatrix}$$

But $\mathbf{AB} = \begin{pmatrix} 2 & 6 \\ 0 & 14 \end{pmatrix}$.

Hence: $$\begin{pmatrix} 2 & 6 \\ 0 & 14 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 6 \\ -16 \end{pmatrix}$$

Thus $$\begin{pmatrix} 2x + 6y \\ 14y \end{pmatrix} = \begin{pmatrix} -10 \\ -42 \end{pmatrix}$$

Hence, $$14y = -42 \Rightarrow y = -3$$
$$2x - 18 = -10 \Rightarrow x = 4$$

Notes:

- *This example shows that to solve for an unknown matrix in a matrix equation, it is not always necessary to use the inverse of the coefficient matrix.*

Exercise 17.1

1. Without the use of a calculator, use the matrix inversion method to solve for x and y in:
   (a) $3x + 4y = 15$ and $2x - y = -1$
   (b) $2x - 3y = -11$ and $5x + 4y = -16$
   (c) $2x + 4y = -9$ and $4x + 3y = -8$
   (d) $0.2x + 0.3y = -2$ and $0.7x + 0.5y = 15$
   (e) $\frac{x}{2} + \frac{y}{3} = 4$ and $x - \frac{y}{6} = 3$
   (f) $2x^2 - y^2 = 7$ and $4x^2 + 3y^2 = 19$

2. Use the matrix inversion method to solve where possible for x and y in $\mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{B}$ if:

(a) $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$ (b) $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 4 \\ 8 \end{pmatrix}$

(c) $\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 3 & -9 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 \\ 9 \end{pmatrix}$ (d) $\mathbf{A} = \begin{pmatrix} -1 & 3 \\ 3 & -9 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 \\ 9 \end{pmatrix}$

3. (a) Given $\mathbf{A} = \begin{pmatrix} 2 & 5 \\ 1 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & 5 \\ 3 & -2 \end{pmatrix}$, find **AB**.

   (b) Use result in (a) and an appropriate matrix operation to solve for x and y in:

$$2x + 5y = -17$$
$$3x - 2y = 22$$

4. (a) Given $\mathbf{A} = \begin{pmatrix} 8 & 5 \\ 1 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 5 & 9 \\ -8 & 7 \end{pmatrix}$, find **AB**.

   (b) Use result in (a) and an appropriate matrix operation to solve for x and y in:

$$5x + 9y = 48$$
$$8x - 7y = -73$$

5. (a) Given $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -3 & 4 \\ 5 & -3 \end{pmatrix}$, find **AB**.

   (b) Use result in (a) and an appropriate matrix operation to solve for x and y in:

$$\frac{3}{x} - \frac{4}{y} = -2$$
$$\frac{5}{x} - \frac{3}{y} = 26.$$

6. Solve for x and y in:

(a) $\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0.7 & -0.1 \end{pmatrix}$ (b) $\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 4 & 2 \\ 5 & 3 \end{pmatrix} = \begin{pmatrix} 950 & 530 \end{pmatrix}$.

18 Applications using Matrices

- In Chapter 16 the algebraic groundwork for using matrices was established, while in chapter 17 the use of matrices in solving systems of equations was discussed. This chapter will focus on the use of matrices in problem solving.

Example 18.1

The matrices $\mathbf{A} = \begin{array}{c} \\ \text{Pen} \\ \text{Pencil} \end{array}\begin{array}{c} \begin{array}{cc} \text{Pack A} & \text{Pack B} \end{array} \\ \begin{pmatrix} 5 & 4 \\ 3 & 6 \end{pmatrix} \end{array}$ and $\mathbf{B} = \begin{array}{c} \\ \text{Pen} \\ \text{Pencil} \end{array}\begin{array}{c} \text{Cost (\$)} \\ \begin{pmatrix} 2.80 \\ 1.20 \end{pmatrix} \end{array}$ respectively describe the number of identical pens and identical pencils in Packs A and B, and the cost of each pen and pencil.

Interpret and determine: (a) $\mathbf{A} \times \mathbf{B}$ (b) $\mathbf{A}^T \times \mathbf{B}$ (c) $\mathbf{B}^T \times \mathbf{A}$.

Solution:

(a) $$\mathbf{A} \times \mathbf{B} = \begin{array}{c} \\ \text{Pen} \\ \text{Pencil} \end{array}\begin{array}{c} \begin{array}{cc} \text{Pack A} & \text{Pack B} \end{array} \\ \begin{pmatrix} 5 & 4 \\ 3 & 6 \end{pmatrix} \end{array} \times \begin{array}{c} \\ \text{Pen} \\ \text{Pencil} \end{array}\begin{array}{c} \text{Cost (\$)} \\ \begin{pmatrix} 2.80 \\ 1.20 \end{pmatrix} \end{array}.$$

This product is possible but conveys no possible meaning.

(b) $\mathbf{A}^T$ is the transpose of matrix A (interchange row and columns).

Hence, $$\mathbf{A}^T \times \mathbf{B} = \begin{array}{c} \\ \text{Pack A} \\ \text{Pack B} \end{array}\begin{array}{c} \begin{array}{cc} \text{Pen} & \text{Pencil} \end{array} \\ \begin{pmatrix} 5 & 3 \\ 4 & 6 \end{pmatrix} \end{array} \times \begin{array}{c} \\ \text{Pen} \\ \text{Pencil} \end{array}\begin{array}{c} \text{Cost (\$)} \\ \begin{pmatrix} 2.80 \\ 1.20 \end{pmatrix} \end{array} = \begin{array}{c} \\ \text{Pack A} \\ \text{Pack B} \end{array}\begin{array}{c} \text{Cost (\$)} \\ \begin{pmatrix} 17.60 \\ 18.40 \end{pmatrix} \end{array}.$$

The product matrix describes the total cost of the pens and pencils in each pack.

(c) $$\mathbf{B}^T \times \mathbf{A} = \text{Cost(\$)} \begin{array}{c} \begin{array}{cc} \text{Pen} & \text{Pencil} \end{array} \\ \begin{pmatrix} 2.80 & 1.20 \end{pmatrix} \end{array} \times \begin{array}{c} \\ \text{Pen} \\ \text{Pencil} \end{array}\begin{array}{c} \begin{array}{cc} \text{Pack A} & \text{Pack B} \end{array} \\ \begin{pmatrix} 5 & 4 \\ 3 & 6 \end{pmatrix} \end{array} = \text{Cost(\$)} \begin{array}{c} \begin{array}{cc} \text{Pack A} & \text{Pack B} \end{array} \\ \begin{pmatrix} 17.60 & 18.40 \end{pmatrix} \end{array}$$

The product matrix describes the total cost of the pens and pencils in each pack.

Note:

- *For the product of two compatible matrices to be meaningful, generally, the column labels of the left matrix must match the row labels of the right matrix.*
- *To create a meaningful matrix product, sometimes one of the matrices in the product needs to be transposed.*

Example 18.2

A clothing manufacturer sources its garments from a factory in China and Indonesia. The monthly costs for raw materials and labour for every 1 000 pieces of a particular garment for the factory in China are \$500 and \$2 000 respectively while the corresponding costs for the factory in Indonesia are \$500 and \$3 000 respectively. In March 2012, the Chinese and Indonesian factories manufactured 8 000 and 7 000 pieces of garment respectively.

(a) Express the monthly component costs per 1 000 pieces of garment manufactured in the form of a clearly labelled 2 × 2 matrix **C** for the factories in China and Indonesia.

(b) Write a column matrix **N** to describe the number of garments (in thousands) manufactured in March 2012. Hence, use a matrix method to determine the total costs in each of the categories; raw materials and labour,

(c) The profit on each Chinese made garment and each Indonesian made garment are respectively \$48.00 and \$36.00. Use a matrix method to determine the total profit for March 2012.

(d) In April 2012, the total cost per thousand units in each of the categories, raw materials and labour were respectively \$9 000 and \$44 000. Use a matrix method to determine the number of garments produced in each of the two factories.

Solution:

(a) The required cost matrix **C** =.

$$\begin{array}{c} \\ \text{China} \\ \text{Indon} \end{array} \begin{array}{c} \begin{array}{cc} \text{raw} & \text{labour} \end{array} \\ \begin{pmatrix} 500 & 2000 \\ 500 & 3000 \end{pmatrix} \end{array}$$

(b) $\mathbf{N} = \begin{array}{c} \\ \text{China} \\ \text{Indon} \end{array} \begin{array}{c} \text{No. ('000)} \\ \begin{pmatrix} 8 \\ 7 \end{pmatrix} \end{array}$.

$$\mathbf{C}^T \times \mathbf{N} = \begin{array}{c} \\ \text{raw} \\ \text{labour} \end{array} \begin{array}{c} \begin{array}{cc} \text{China} & \text{Indon} \end{array} \\ \begin{pmatrix} 500 & 500 \\ 2000 & 3000 \end{pmatrix} \end{array} \times \begin{array}{c} \\ \text{China} \\ \text{Indon} \end{array} \begin{array}{c} \text{No. ('000)} \\ \begin{pmatrix} 8 \\ 7 \end{pmatrix} \end{array} = \begin{pmatrix} 7\,500 \\ 37\,000 \end{pmatrix}$$

Hence the cost for raw materials and labour are respectively \$7 500 and \$37 000.

(c) The total profit is obtained as follows:

$$\begin{pmatrix} 8000 & 7000 \end{pmatrix} \begin{pmatrix} 48 \\ 36 \end{pmatrix} = \begin{pmatrix} 636\,000 \end{pmatrix}$$

Hence, the total profit is \$636 000.

(d) Let x: number of garments (in thousands) produced at the Chinese factory
y: number of garments (in thousands) produced at the Indonesian factory

Hence:

For raw materials $500x + 500y = 9\ 000$

For labour $2\ 000x + 3\ 000y = 44\ 000$

Rewrite as an matrix equation (after dividing each equation by 100):

$$\begin{pmatrix} 5 & 5 \\ 20 & 30 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 90 \\ 440 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 & 5 \\ 20 & 30 \end{pmatrix}^{-1}\begin{pmatrix} 90 \\ 440 \end{pmatrix} = \begin{pmatrix} 10 \\ 8 \end{pmatrix}$$

Therefore the production figures for the Chinese and Indonesian factories are respectively, 10 000 and 8 000.

Note:

- *If in part (a) matrix **C** had been written as* $\begin{matrix} & \textit{China} & \textit{Indon} \\ \textit{raw} & 500 & 500 \\ \textit{labour} & 2000 & 3000 \end{matrix}$ *and **N** as written in part (b),*

  *then, there would not have been any need to transpose matrix **C**.*

Exercise 18.1

1. The matrix **A** describes the number of identical pencils and identical writing pads in Packs A and B respectively while the matrix **B** describes the cost of each pencil and writing pad in Packs **A** and **B**:

   $$\mathbf{A} = \begin{matrix} & \text{Pack X} & \text{Pack Y} \\ \text{Pencils} & 2 & 5 \\ \text{Writing Pad} & 3 & 5 \end{matrix}, \quad \mathbf{B} = \begin{matrix} & \text{Cost (\$)} \\ \text{Pencils} & 0.90 \\ \text{Writing Pad} & 2.10 \end{matrix}.$$

   Interpret and determine: (a) $\mathbf{A} \times \mathbf{B}$ (b) $\mathbf{A}^T \times \mathbf{B}$ (c) $\mathbf{B}^T \times \mathbf{A}$.

2. The matrix **A** describes the number of hamburgers and cans of drinks in Meals P and Q respectively while the matrix **B** describes the cost of each hamburger and can of drinks in Meals P and Q:

   $$\mathbf{A} = \begin{matrix} & \text{Burgers} & \text{Drink Cans} \\ \text{MealP} & 4 & 2 \\ \text{MealQ} & 6 & 4 \end{matrix}, \quad \mathbf{B} = \begin{matrix} & \text{Burger} & \text{Drink Cans} \\ \text{Cost (\$)} & 5.90 & 3.50 \end{matrix}.$$

   Interpret and determine: (a) $\mathbf{B} \times \mathbf{A}$ (b) $\mathbf{B} \times \mathbf{A}^T \times$ (c) $\mathbf{A} \times \mathbf{B}^T$

3. The Smiths have 2 children in primary school. Scott (year 5) requires six scrap books and four 2B pencils. Marcia (year 3) requires eight scrap books, and two 2B pencils.
   (a) Describe the stationery requirements for the Smiths children in the form of a clearly labelled matrix **A**.
   (b) If the cost for a scrap book and 2B pencil are respectively, \$1.30 and \$0.50, convey this information as cost matrix **C**. Use matrices **A** and **C** to determine the total cost of stationery for each of the two children.

4. Austar is a giant retail company which operates two chain stores. The gross retail figures (\$million) for each of the two stores in 2012 and 2013 are summarised in the accompanying table.

Year	Q-Mart	AusOne
2012	800	1 200
2013	1 000	1 500

   (a) Display the information above as a clearly labelled matrix **G**.
   (b) If each year, 15% of Q-Mart's gross retail figures translates as profit and the corresponding figures for AusOne is 18%, display this information as a clearly labelled matrix **P**.
   (c) Use matrices **G** and **P** to find the total profit for Austar for each of the given years.
   (d) Use matrices to find the profit for each chain store in each of the given years.

5. A manufacturer produces two models of a certain product at two factories. The accompanying table summarises the production numbers of each model at each of the two factories in April 2012.

	Model 100	Model 200
Factory A	5 000	6 000
Factory B	6 000	4 000

   (a) In April 2012, 25% and 40% respectively of model 100 and model 200 were exported. Use a matrix method to find the number of units of each model that was exported.
   (b) In May 2012, the production numbers for model 100 and model 200 were increased by 10% and 5% respectively. Use a matrix method to find the total number of each model produced in May 2012.
   (c) Each unit of model 100 and model 200 exported brought in a profit of \$20 and \$25 respectively. Use a matrix method to find the total profit for April 2012.

6. ComputerWest assembles and sells two versions of a desktop computer system. Version A requires 3 pieces of component P and 7 pieces of component R. The corresponding numbers for version B are respectively 4 and 16.
   (a) In January 2013, ComputerWest assembled and sold 20 version A and 40 version B machines respectively. At the start of the month there were 280 and 850 pieces of components P and R respectively. Use a matrix method to determine how many pieces of each type of component were left at the end of the month.
   (b) The profit on each version A and B machine were respectively \$170 and \$300. Use a matrix method to find the total profit made in January 2013.
   (c) In February 2013, 790 and 2 510 pieces of component P and R respectively were used. Use a matrix method to find how many of each version of machine was assembled in February 2013.

7. Helen and Frances bought tickets for two separate events. The table below shows the number of tickets bought by each person.

	Helen	Frances
AFL Final	2	5
Concert	5	4

(a) If the total cost for Helen and Frances were \$900 and \$1230 respectively, represent this information in the form of a matrix.

(b) Use a matrix method to find the cost for each of the events.

(c) Use a matrix method to find how much it would cost Naomi to purchase 1 AFL final and 2 concert tickets.

8. A supermarket sells two types of Christmas hampers during the Christmas season. Hamper A has 2 bars of chocolates and twice as many cans of cool drinks as bars of chocolates. Hamper B has 6 cans of cool drink and an equal number of bars of chocolates.

(a) A worker who prepares the hampers makes use of 218 cans of cool drinks and 202 bars of chocolates. Use a matrix method to find the number of each type of hamper that the worker made up.

(b) The cost to the supermarket for each can of cool drink and each bar of chocolate are \$0.12 and \$0.60 respectively. In addition, hampers A and B have other goods worth (to the supermarket) \$15.00 and \$18.50 respectively. Use a matrix method to find the cost price for each of the 2 different types of hampers.

9. A fish of species P consumes 8g of food A and 3g of food C each day.
A fish of species Q consumes 5g of food A and 2g of food C each day.
The environment has 285g of food A and 110g of food C.

(a) Given that there were 10 fish of each species in the environment, use a matrix method to determine how much food of each type would be required each day.

(b) Use a matrix method to find the population size of the two species that will consume exactly all of the available food in (i) one day (ii) five days.

*10. A midipak consists of 3 chocolate éclairs and 6 lollipops. A maxipak consists of 5 chocolate éclairs and *n* lollipops. Jane had 105 chocolate éclairs and 138 lollipops. Find the value of *n* so that each and every chocolate éclair and lollopop available is used up. The composition of each pack is as stated.

19 Transformation Matrices

19.1 Linear Transformations in the *x*-*y* plane

- A linear transformation maps the point (0, 0) to the point (0, 0) and a set of parallel lines to another set of parallel lines (not necessarily the same).
- The linear transformations considered in this chapter are: dilations parallel to the *x*-axis and *y*-axis; rotations about the origin, reflections about the axes and reflections about lines passing through the origin.
- Translations are not linear transformations as the point (0, 0) is not mapped to (0, 0). Similarly reflections about line $y = mx + c$ where $c \neq 0$ are not linear transformations as the point (0, 0) is not preserved.

19.2 Using Matrices to Represent Linear Transformations

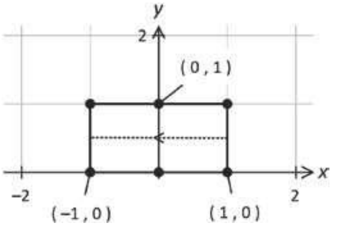
- A linear transformation **T** may be represented by a 2 × 2 matrix **M**.

  Consider the point A with position vector $\begin{pmatrix} a \\ b \end{pmatrix}$.

  Let A' be the image of A under **T**.

  Then the position vector of A' under **T** is given by $\mathbf{M} \times \begin{pmatrix} a \\ b \end{pmatrix}$.

19.2.1 Procedure for determining the matrix representation of a linear transformation



- The vertices of a unit square have position vectors $\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ respectively.
- Let the unit square be reflected about the *y*-axis.
  - The vertices are mapped as follows.

    $$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix} \to \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$
  - Let the 2 × 2 matrix **M** represent a reflection about the *y*-axis.

    Then: $\mathbf{M} \times \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\mathbf{M} \times \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$,

    $\mathbf{M} \times \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\mathbf{M} \times \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

- If however, we considered only **M** × $\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}-1\\0\end{pmatrix}$ and **M** × $\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}$,

  then $\mathbf{M}\times\begin{pmatrix}1&0\\0&1\end{pmatrix}=\begin{pmatrix}-1&0\\0&1\end{pmatrix}$.

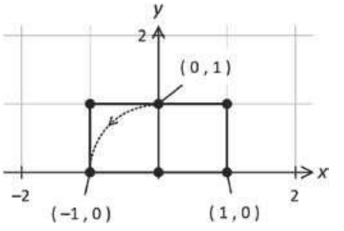
  Hence $\mathbf{M}=\begin{pmatrix}-1&0\\0&1\end{pmatrix}$.

- Hence, a reflection about the *y*-axis can be represented by **M** = $\begin{pmatrix}-1&0\\0&1\end{pmatrix}$.

- Let the unit square be rotated 90° anticlockwise about the origin.
  - The vertices are mapped as follows:

    $\begin{pmatrix}0\\0\end{pmatrix}\to\begin{pmatrix}0\\0\end{pmatrix}$, $\begin{pmatrix}1\\0\end{pmatrix}\to\begin{pmatrix}0\\1\end{pmatrix}$,

    $\begin{pmatrix}1\\1\end{pmatrix}\to\begin{pmatrix}-1\\1\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}\to\begin{pmatrix}-1\\0\end{pmatrix}$.



  - Let the 2 × 2 matrix **M** represent a 90° anti-clockwise rotation about the origin.
    To determine **M**, it is sufficient to consider:

    $\mathbf{M}\times\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}$ and $\mathbf{M}\times\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}-1\\0\end{pmatrix}$.

  - This can be condensed to $\mathbf{M}\times\begin{pmatrix}1&0\\0&1\end{pmatrix}=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$. Hence $\mathbf{M}=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$.

  - Hence, a 90° anti-clockwise rotation about the origin can be represented by $\mathbf{M}=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$.

- Note that in each of transformation matrices above:
  - the first column of the transformation matrix is the image of $\begin{pmatrix}1\\0\end{pmatrix}$
  - the second column is the image of $\begin{pmatrix}0\\1\end{pmatrix}$.

- In summary, given that the 2 × 2 matrix **M** represents a linear transformation **T**:
  - the first column in **M** represents the image of $\begin{pmatrix}1\\0\end{pmatrix}$
  - the second column in **M** represents the image $\begin{pmatrix}0\\1\end{pmatrix}$.

19.2.2 Linear Transformations considered

- The table below lists the matrices for the linear transformations used in this book.

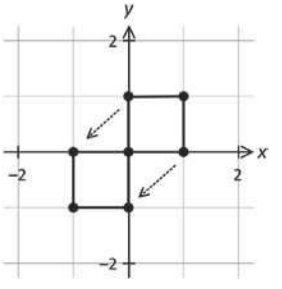
Linear Transformation	**Matrix**
Reflection about the x-axis	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
Reflection about the y-axis	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
Reflection about the line $y = x$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
Reflection about the line $y = -x$	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$
Reflection about the line $y = x \tan \theta$ where $\theta \neq (2n + 1)\pi/2$	$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$
Rotation 90° clockwise about the origin	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
Rotation 90° anti-clockwise about the origin	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
Rotation 180° clockwise about the origin	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$
Rotation θ° anti-clockwise about the origin	$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$
Dilation factor $k > 0$ along the x-axis	$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$
Dilation factor $k > 0$ along the y-axis	$\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$
Enlargement factor $k > 0$ about the origin	$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$

Example 19.1

Describe the transformation **T** represented by the matrix $\mathbf{M} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$.

Solution:

From the information contained in the first two columns of **M**, **T** maps (1, 0) to (0, –1) and (0, 1) to (–1, 0).
The effect of **T** on a unit square is shown in the accompanying diagram.
From the diagram, clearly **T** is a reflection about the line $y = -x$.



Example 19.2

The image of (a, b) under the transformation represented by $\begin{pmatrix} 6 & -3 \\ 2 & 4 \end{pmatrix}$ is (9, 8). Find a and b.

Solution:

$$\begin{pmatrix} 6 & -3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 9 \\ 8 \end{pmatrix}$$

Hence
$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 6 & -3 \\ 2 & 4 \end{pmatrix}^{-1}\begin{pmatrix} 9 \\ 8 \end{pmatrix}$$
$$= \frac{1}{30}\begin{pmatrix} 4 & 3 \\ -2 & 6 \end{pmatrix}\begin{pmatrix} 9 \\ 8 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Hence $a = 2$ and $b = 1$.

Example 19.3

Find the matrix **M** that represents the transformation that maps (1, 1) to (4, 1) and (1, 2) to (7, 2).

Solution:

$$\mathbf{M}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 7 \\ 1 & 2 \end{pmatrix}$$

$$\mathbf{M} = \begin{pmatrix} 4 & 7 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 4 & 7 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

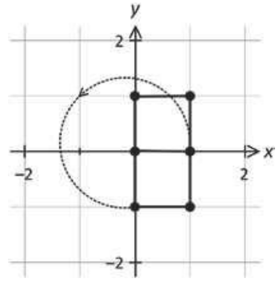
Hence, $\mathbf{M} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$.

Example 19.4

Determine the transformation matrix for a 270° anticlockwise rotation about the origin.

Solution:

From the accompanying sketch, the point (1, 0) is mapped to (0, −1) and the point (0, 1) is mapped to (1, 0).



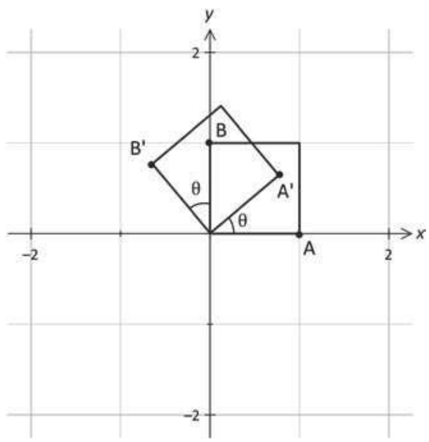
Hence, the required matrix is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Example 19.5

Prove that the transformation matrix for a θ° anti-clockwise rotation about the origin is given by **M** = $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, $0 \le \theta \le \dfrac{\pi}{2}$.

Solution:

Let **OA** = $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and **OB** = $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.



The unit square is rotated θ° anti-clockwise about O. Let A' and B' be the images of A and B respectively.

Hence $\angle AOA' = \angle BOB' = \theta$.

Since, $|\mathbf{OA}| = |\mathbf{OA}'| = 1$,

$$\mathbf{OA}' = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}.$$

Since, $|\mathbf{OB}| = |\mathbf{OB}'| = 1$,

$$\mathbf{OB}' = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}.$$

Let M be the required matrix. Then:

$$\mathbf{M} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\Rightarrow \mathbf{M} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

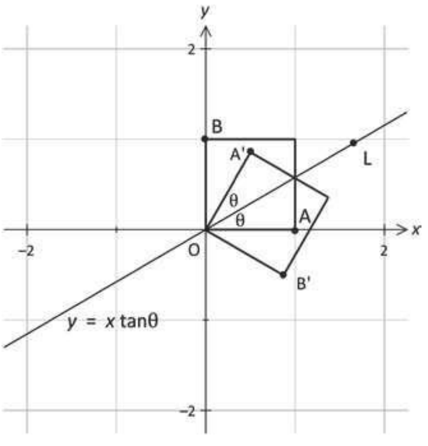
Example 19.6

Prove that the transformation matrix for a reflection about the line $y = x \tan\theta$ where $0 < \theta < \frac{\pi}{2}$ is given by $\mathbf{M} = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$.

Solution:

Let $\mathbf{OA} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{OB} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

OA and **OB** are reflected about the line $y = x\tan\theta$.
Let $\mathbf{OA}'$ and $\mathbf{OB}'$ be the images of **OA** and **OB** respectively.
Let L be a point on the line $y = x\tan\theta$.
Hence $\angle AOL = \angle A'OL = \theta$.



Since, $|\mathbf{OA}| = |\mathbf{OA}'| = 1$,

$$\Rightarrow \quad \mathbf{OA}' = \begin{pmatrix} \cos 2\theta \\ \sin 2\theta \end{pmatrix}.$$

Also, $\angle BOL = \angle B'OL = 90 - \theta$.
Hence, $\angle B'OA = 90 - \theta - \theta = 90 - 2\theta$.

Since, $|\mathbf{OB}| = |\mathbf{OB}'| = 1$,

$$\Rightarrow \quad \mathbf{OB}' = \begin{pmatrix} \cos(90-2\theta) \\ -\sin(90-2\theta) \end{pmatrix} = \begin{pmatrix} \sin 2\theta \\ -\cos 2\theta \end{pmatrix}.$$

Let **M** be the required matrix. Then $\mathbf{M} = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$.

Example 19.7

Find the image of the line with equation $y = 2x - 1$ after it is reflected about the line $y = -x$.

Solution:

The matrix representing a reflection about the line $y = -x$ is $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$.

The position vector of any point on the line $y = 2x - 1$ is $\begin{pmatrix} t \\ 2t-1 \end{pmatrix}$.

Hence, the image of this point has position vector $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} t \\ 2t-1 \end{pmatrix} = \begin{pmatrix} -2t+1 \\ -t \end{pmatrix}$.

Hence, the image of this point has coordinates $x = -2t + 1$, $y = -t$
Substitute $t = -y$ into x: $\Rightarrow$ Equation of image is $x = 2y + 1$ or $x - 2y = 1$.

Exercise 19.1

1. The quadrilateral OABC with coordinates (1,0), (3,0), (2,2) and (0,1) respectively is mapped to O´A´B´C´ by the transformation **T** represented by the matrix **M** = $\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$.
   (a) Find the coordinates of the points O´, A´, B´ and C´.
   (b) Sketch on the same diagram OABC and O´A´B´C´.

2. Repeat Question 1 for the following transformations **T** represented by matrix **M**:
   (a) $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 0 \\ 4 & 2 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & 2 \\ 3 & 0 \end{pmatrix}$

3. Repeat Question 1 for the following transformations **T** represented by matrix **M**:
   (a) $\begin{pmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$ (b) $\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} \\ \frac{1}{2} & \frac{-1}{2} \end{pmatrix}$ (c) $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}$

4. By considering the geometrical effects of the following transformations on the points (1, 0) and (0, 1), write down the matrices representing each of these transformations:
   (a) reflection about the y-axis
   (b) reflection about the line $y = x$
   (c) vertical dilation of factor 5
   (d) horizontal dilation of factor 4
   (e) enlargement about the origin of factor 4
   (f) enlargement about the origin of factor $\frac{1}{2}$
   (g) clockwise rotation of 90^o about the origin
   (h) anti-clockwise rotation of 180^o about the origin

5. (a) For each transformation in Question 4, describe the transformation that reverses the effects of the original transformation.
   (b) Hence or otherwise give the matrix representing each of these "reverse" transformations.
   (c) Comment on the relationship between the matrix for the original transformation and the matrix for the "reverse" transformations.

6. By considering the geometrical effects of the following transformations on the points (1,0) and (0,1), write down the matrices representing each of these reflections:
   (a) about the line $y = x\sqrt{3}$
   (b) about the line $y = -\frac{x\sqrt{3}}{3}$

   For each transformation above, find the matrix representing the reverse transformations.

7. By considering the geometrical effects of the following transformations on the points (1, 0) and (0, 1), write down the matrices representing each of these rotations:

   (a) $\frac{\pi}{6}$ anti-clockwise about the origin (b) $\frac{3\pi}{4}$ clockwise about the origin

   For each transformation above, find the matrix representing the reverse transformations.

8. By considering the geometrical effects on the points (1,0) and (0,1), describe the transformation represented by each of the following matrices:

   (a) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$

   (e) $\begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}$ (f) $\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$ (g) $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ (h) $\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$

9. By considering the geometrical effects on the points (1,0) and (0,1), describe the transformation represented by each of the following matrices:

   (a) $\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$ (b) $\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$ (c) $\begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}$ (d) $\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} \end{pmatrix}$

10. The transformation **T** represented by the matrix **M** = $\begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}$ maps the point A with coordinates (a, b) to the point B. Find a and b if the coordinates of B are:

    (a) (8, 3) (b) (12, –6) (c) (–4, 9)

11. The transformation **T** represented by the matrix **M** = $\begin{pmatrix} 3 & 0 \\ 3 & 3 \end{pmatrix}$ maps the point A with coordinates (a, b) to the point B. Find a and b if the coordinates of B are:

    (a) (6, 12) (b) (12, 9) (c) (–9, –3)

12. The transformation **T** represented by the matrix **M** maps the point (0, 1) to (2, 3) and the point (1, 0) to (1, 0). (a) Find **M**.

    (b) Find the matrix representation of the transformation that reverses the effects of **T** .

13. The transformation **T** represented by the matrix **M** maps the point (2, 1) to (–8, 4) and the point (1, 2) to (–4, 8). (a) Find **M**.

    (b) Find the matrix representation of the transformation that reverses the effects of **T** .

14. The transformation **T** represented by the matrix **M** maps the point (2, –1) to (4, 5) and the point (1, –1) to (2, 1). (a) Find **M**.
    (b) Find the matrix representation of the transformation that reverses the effects of **T** .

15. Show that it is not possible to find a single linear transformation that maps the points (0, 0) to (1, 0) and (1, 1) to (2, 1).

16. Show that it is not possible to find a single linear transformation that maps the points (1, 0) to (1, 0), (1, 1) to (1, 3) and (0, 1) to (0, 2)

17. Show that it is not possible to find a single linear transformation that maps the points (1, 2) to (1, 4), (2, 1) to (2, 2) and (3, 1) to (3, 3).

18. Prove that the transformation matrix for a θ radians clockwise rotation about the origin is given by **M** = $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ where $0 \le \theta \le \frac{\pi}{2}$.

19. Prove that the transformation matrix for a reflection about the line $y = -x \tan\theta$ is given by **M** = $\begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ -\sin 2\theta & -\cos 2\theta \end{pmatrix}$ where $0 \le \theta \le \frac{\pi}{2}$.

20. Prove that all reflections about a line passing through the origin are represented by matrices that are self-inverses.

21. The transformation represented by matrix **M** = $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ maps the line with equation $y = 2x + 1$ to $y = ax + b$. Find a and b.

22. Find the image of the line with equation $y = x + 1$ after it is rotated 45° clockwise about the origin.

23. (a) Find the image of the line with equation $x + y = 2$ after it is mapped by a transformation represented by the matrix **M** = $\begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix}$.
    (b) Explain clearly why the reverse transformation does not exist.

24. (a) Find the image of the line with equation $y = -3x + 1$ after it is mapped by a transformation represented by the matrix **M** = $\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}$.
    (b) Explain clearly why the reverse transformation does not exist.

19.3 Scale Factor for Area

Hands On Task 19.1

In this task, we will consider a geometrical interpretation to the *determinant* of a transformation matrix.

1. The unit square OABC where, O (0, 0), A (1, 0), B (1, 1) and C (0, 1) is mapped by each of the given transformations to O′A′B′C′. Complete the table below.

Transformation	Transformation Matrix **M**	q = Area of OABC	r = Area of O′A′B′C′	Ratio $\frac{r}{q}$	det(**M**)
Reflection about y-axis					
Reflection about the x-axis					
Reflection about $y = x$					
Reflection about $y = -x$					
Vertical dilation factor k					
Horizontal dilation factor k					
Enlargement factor k					
Rotation 90° clockwise about O					
Rotation 90° anticlockwise about O					

2. For each transformation listed in Question 1, compare the value of the ratio $\frac{r}{q}$ with the absolute value (ignore the sign) of the determinant of the transformation matrix. Hence, give a geometrical interpretation for the absolute value of the determinant.

3. The determinant is negative for some transformations. Suggest a geometrical interpretation for the **sign** of the determinant of the transformation matrix.

4. Suggest a geometrical interpretation of a transformation matrix with a **zero determinant**.

Summary

- The absolute value of the determinant of a transformation matrix is defined as the scale factor for area, which is the ratio $\frac{\text{Area of Image}}{\text{Area of Object}}$.
- A negative determinant indicates that the transformation involves a reflection or the sequence of transformations involves an odd number of reflections.

Example19.8

A pentagon OABCD of area 10 square units is mapped to O´A´B´C´D´ by a transformation **T** represented by the matrix $\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$. The vertices of OABCD are arranged and read in an anticlockwise manner.

(a) Determine the area of O´A´B´C´D´.
(b) Comment on the way in which the vertices of O´A´B´C´D´ would be read.

Solution:

(a) The determinant of the transformation matrix is -4.
Hence, area of O´A´B´C´D´ is 4 times the area of OABCD.
Therefore, the area of O´A´B´C´D´ is $4 \times 10 = 40$ square units.

(b) Since the determinant is negative, a reflection has occurred.
Hence, the vertices of O´A´B´C´D´ would be read in a clockwise manner.

19.4 Combining Transformations

- Let the transformations **T** and **R** be represented by matrices **M** and **Q** respectively. The image of the point (a, b) under the sequence of transformations **T** followed by **R** (the transformation sequence is denoted **RT**), is given by **QM**$\begin{pmatrix} a \\ b \end{pmatrix}$.
- The sequence is represented symbolically as **RT**, *the transformation which is applied first, in this case* **T**, *is written on the right***.**

  Hence in the matrix representation, *the matrix* **M** *is closest to* $\begin{pmatrix} a \\ b \end{pmatrix}$ *as it represents the transformation that is applied first***.**
- As matrix multiplication is not generally commutative, the process of combining two or more transformations is also generally *not commutative*.

Example 19.9

The transformations **T** and **R** are represented by the matrixes $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$ respectively. Find the image of the point (1, 1) under:
(a) **T** followed by **R** (b) **R** followed by **T**.

Solution:

(a) The image of (1, 1) in column form = $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Hence, the image of (1, 1) under **T** followed by **R** is (2, 1).

(b) The image of (1, 1) in column form = $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$$

Hence, the image of (1, 1) under **R** followed by **T** is (–4, 1).

Example 19.10

The transformations **A** and **B** are the reflection about the line $y = x$ and the reflection about the y-axis respectively. Determine the matrix that represents the combined transformation:
(a) **AB** (b) **BA**. In each case, name the resulting equivalent single transformation.

Solution:

(a) **AB** = $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Clearly, $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ represents a 90° clockwise rotation about the origin O.

(b) **BA** = $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Clearly, $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ represents a 90° anti-clockwise rotation about the origin O.

Notes:

- *In general, it may be proved that the combined effects of two reflections about lines passing through the origin is equivalent to the effect produced by a single rotation. See Exercise 19.2, Question 21.*

Example 19.11

The transformations **A** and **B** are combined as **AB**. **AB** is represented by $\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$.

Find the one possible set of linear transformations represented by **A** and **B**.

Solution:

Decomposing $\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$ as a product of two 2 × 2 matrices: $\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \equiv \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Hence, one possible set is: **B** is represented by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and **A** is represented by $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

Hence, the combined transformation is made up of the sequence:
A reflection about the *x*-axis followed by an enlargement about the origin of factor 2.

Note:

- $\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$ *could also be written as* $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. *Hence, the sequence, an enlargement about the origin of factor 2 followed by a reflection about the x-axis would also be correct.*

Exercise 19.2

1. The linear transformations **T** and **R** are represented by $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ respectively.
   (a) Find the image of the points O (0, 0), A (2, 0) and B (0, 3) under **T**.
   (b) Plot on a diagram the points O (0, 0), A (2, 0), B (0, 3) and their corresponding images under **T**. Hence, or otherwise, describe the effects of **T**.
   (c) Find the matrix representation of a transformation that will reverse the effects of **T**.
   (d) An object with area 20 cm^2 is mapped by **T** into an object with area *k* cm^2. Find *k*.
   (e) Find the image of the points A (2, 0) and B (0, 3) under the combined transformation **R** followed by **T**.
   (f) Find a single matrix that has the same effect as **R** followed by **T**.

2. The transformations **T** and **R** are represented by $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ respectively.
   (a) Find a single matrix that represents the successive transformations:
   (i) **R** then **T** (ii) **T** then **R** (iii) **T** then **T** (iv) **R** then **R** (v) **T** then **R** then **T.**
   (b) Hence, or otherwise find the image of the point (0, 1) under each of the transformations in part (a).
   (c) Find a single matrix that will reverse each of the sequence of transformations in (a).

3. The transformations **S** and **F** are represented by $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ respectively.
   (a) Find a single matrix that represents the successive transformations:
   (i) **F** then **S** (ii) **S** then **F** (iii) **S** then **F** then **S.**
   (b) Find a single matrix that will reverse each of the sequence of transformations in (a).
   (c) Hence, or otherwise, find the point whose image is (8, 4) under each of the sequence of transformations in (a).

4. The transformations **A** and **B** are a reflection about the line $y = -x$ and the reflection about the y-axis respectively. Determine the matrix that represents the combined transformation: (a) **B** then **A** (b) **A** then **B**.
   In each case, determine the corresponding equivalent single transformation.

5. The transformations **A** and **B** are a reflection about the line $y = \sqrt{3}\,x$ and the reflection about the line $y = \dfrac{-x}{\sqrt{3}}$. Determine the matrix that represents the combined transformation: (a) **B** then **A** (b) **A** then **B**.
   In each case, determine the corresponding equivalent single transformation.

6. The transformations **S** and **T** are represented by $\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}$ respectively.
   A quadrilateral OABC is mapped to O´A´B´C´ by the sequence of transformations:
   (i) **S** (ii) **T** (iii) **T** then **S** (iv) **T** then **T**. Under each of these sequences, find:
   (a) the area of O´A´B´C´ if the area of OABC is 12 square units
   (b) the area of OABC if the area of O´A´B´C´ is 4 square units.

7. The triangle with vertices O (0, 0), A (2, 0) and B (0, 4) is mapped to the triangle O´A´B´ by the transformation represented by the matrix $\begin{pmatrix} -x & y \\ x & 2 \end{pmatrix}$. The area of triangle O´A´B´ is 24 cm^2 and B' has coordinates (0, 8). Find x and y.

8. The triangle with vertices O (0, 0), A (2, 0) and B (0, 3) is mapped to the triangle O´A´B´ by the transformation represented by the matrix $\begin{pmatrix} 4 & x \\ x & y \end{pmatrix}$. The area of triangle O´A´B´ is 36 cm^2 and A´ has coordinates (8,0). Find x and y.

9. The transformation **T** represented by $\begin{pmatrix} 4 & 0 \\ 0 & 5 \end{pmatrix}$ is a combination of 2 other transformations **R** and **F**.
   (a) Suggest matrices to represent **R** and **F** respectively.
   (b) Find the area of the image of an object of area 2 square units under **T**.
   (c) Find the area of the image of an object of area 40 square units under $\mathbf{T}^{-1}$.

10. Let **A** and **B** be each a reflection about a line passing through the origin. The combined effects of **A** and **B** is equivalent to a rotation θ^{o} anti-clockwise about the origin.
Find a possible pair of **A** and **B** if: (a) $\theta = 180^{o}$ (b) $\theta = 90^{o}$ (c) $\theta = 60^{o}$

11. The transformation **T** is formed by a reflection about the *y*-axis followed by a horizontal dilation of factor 4.
(a) Find a single matrix that represents **T**.
(b) Find the image of the point (1, 4) under **T**.
(c) Find the single matrix that undoes the effects of **T**.

12. The transformation **T** is formed by 2 successive reflections about the line $y = -x$.
(a) Find a single matrix that represents **T**.
(b) Comment on your answer in (a).
(c) Find the image of the point (4, 5) under $\mathbf{T}^{8}$.
(d) Find the coordinates of the point whose image under $\mathbf{T}^{8}$ is (5, 6).

13. The transformation **T** is formed by 2 successive applications of the transformation represented by $\mathbf{M} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.
(a) Find a single matrix that represents **T**.
(b) Comment on your answer in (a).
(c) Find the image of the point (10, 0) under $\mathbf{T}^{20}$.
(d) Find the coordinates of the point whose image under $\mathbf{T}^{21}$ is (0, 10).

14. The transformation **F** is represented by the matrix $\begin{pmatrix} 8 & 4 \\ 2 & 1 \end{pmatrix}$.
(a) Find the images of the following points under **F**:
(i) (1, 0) (ii) (0, 1) (iii) (1, 1) (iv) (0, 2) (v) (3, 5)
(b) Find the equation of the line on which the images of the points in (a) lie.
(c) Find (if possible) a single matrix that will reverse the effects of **F**.
(d) Describe geometrically the effects of **F**.

15. The transformation **T** is represented by the matrix **M** with det(**M**) = k.
(a) Find the area of the image of rectangle OABC with an area of 5 square units under **T** given that (i) $k = 0.5$(ii) $k = -2$.
(b) Discuss the orientation of the image in relation to that of the OABC under **T** for
(i) $k = -0.5$ (ii) $k = 2$

16. The transformations **S**, **T** and **U** are represented by the matrices **M**, **N** and **P** respectively, where det(**M**) = 2, det(**N**) = –2 and det(**P**) = 0.
    (a) Find the area of the image of a square of area 5 square units under
        (i) **ST** (ii) **TS** (iii) **UST** (iv) $\mathbf{T}^2$ (v) $\mathbf{T}^3$ (vi) $\mathbf{U}^2$
    (b) Discuss the orientation of the image with respect to the object for each of the sequence of transformations in (a).
    (c) Discuss the existence of a single transformation that undoes the effects of each of the sequence of transformations in (a).

17. Let $\mathbf{T}_\theta$ and $\mathbf{T}_\phi$ be the transformations of an anti-clockwise rotation of θ radians and ϕ radians about the origin respectively. Let the matrices $\mathbf{R}_\theta$ and $\mathbf{R}_\phi$ represent $\mathbf{T}_\theta$ and $\mathbf{T}_\phi$ respectively.
    (a) Explain geometrically why $\mathbf{T}_\theta \mathbf{T}_\phi = \mathbf{T}_\phi \mathbf{T}_\theta$
    (b) Prove algebraically that $\mathbf{R}_\theta \times \mathbf{R}_\phi = \mathbf{R}_\phi \times \mathbf{R}_\theta$.
    (c) Explain geometrically why $\mathbf{T}_\theta \mathbf{T}_\phi = \mathbf{T}_{\theta+\phi}$.
    (d) Use your result in (c) to prove:
        (i) $\cos(\theta + \phi) = \cos\theta \cos\phi - \sin\theta \sin\phi$
        (ii) $\sin(\theta + \phi) = \sin\theta \cos\phi + \cos\theta \sin\phi$.

18. By using two appropriate rotations and their matrix representations, prove:
    (a) $\cos(\theta - \phi) = \cos\theta \cos\phi + \sin\theta \sin\phi$ (b) $\sin(\theta - \phi) = \sin\theta \cos\phi - \cos\theta \sin\phi$.

19. Prove that if **A** and **B** represent reflections about the line $y = x \tan\alpha$ and $y = x \tan\beta$ respectively, then **A** and **B** are not commutative.

20. Prove that if **A** and **B** represent a reflection about the line $y = a \tan\alpha$ ($\alpha \neq 0$) and a rotation of θ° ($\theta \neq 0$) anti-clockwise about the origin respectively, then **A** and **B** are not commutative.

21. Let **A** represent a reflection about the line $y = x \tan\alpha$. Let **B** represent a reflection about the line $y = x \tan\beta$. Use the matrix representations of **A** and **B** to prove that the combined transformation **AB** is equivalent to an anti-clockwise rotation of $2(\alpha - \beta)$ about the origin.

19.5 Transformation Mappings

- In this section we will explore a more general method of using matrices to represent transformations. This will include transformations that are not linear. In particular, we will explore how translations are represented using matrices.

- The transformation **T** that maps the point (x, y) to (x' , y') can written as:

  $$\mathbf{T}:\ (x, y) \rightarrow (x', y').$$

  or alternatively,
  $$\mathbf{T}:\ \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x' \\ y' \end{pmatrix}.$$

- Transformation **T** need not be a linear transformation.

Example 19.12

Express each of the given linear transformations as matrix equations.
In each case Identify the transformation involved.

Solution:

(a) $\mathbf{T}:\ (x, y) \rightarrow (-x, y)$ (b) $\mathbf{T}:\ (x, y) \rightarrow (2x, \frac{y}{2})$ (c) $\mathbf{T}:\ \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} \frac{\sqrt{3}x}{2}+\frac{y}{2} \\ \frac{x}{2}-\frac{y\sqrt{3}}{2} \end{pmatrix}$.

Solution:

(a) $$\mathbf{M}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix} \Rightarrow \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}$$

T is a reflection about the y-axis.

(b) $$\mathbf{M}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ \frac{y}{2} \end{pmatrix} \Rightarrow \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ \frac{y}{2} \end{pmatrix}$$

T is a dilation of factor 2 along the x-axis and a dilation of factor $\frac{1}{2}$ along the y-axis.

(c) $$\mathbf{M}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}x}{2}+\frac{y}{2} \\ \frac{x}{2}-\frac{y\sqrt{3}}{2} \end{pmatrix} \Rightarrow \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}x}{2}+\frac{y}{2} \\ \frac{x}{2}-\frac{y\sqrt{3}}{2} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \cos\frac{\pi}{6} & \sin\frac{\pi}{6} \\ \sin\frac{\pi}{6} & -\cos\frac{\pi}{6} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}x}{2}+\frac{y}{2} \\ \frac{x}{2}-\frac{y\sqrt{3}}{2} \end{pmatrix}$$

T is a reflection about the line with equation $y = x\ \tan\left(\frac{\pi}{12}\right)$.

19.5.1 Translations

- A translation *a* units along the positive *x*-axis and *b* units along the positive *y*-axis can be written using matrices as:

$$\mathbf{T}: (x, y) \to (x + a, y + b) \quad \text{or} \quad \mathbf{T}: \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$$

- Translations are not linear transformations as (0, 0) is not mapped to (0, 0).

Example 19.13

Identify each of the transformations given below:
(a) $\mathbf{T}: (x, y) \to (x + 1, y - 2)$ (b) $\mathbf{T}: (x, y) \to (2 - x, y + 1)$

Solution:

(a) Translation 1 unit along the positive *x*-axis and translation of 2 units along the negative *y*-axis.

(b) $\mathbf{T}: \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 2-x \\ y+1 \end{pmatrix} \Rightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$$= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Hence, **T** is a reflection about the *y*-axis followed by a translation 2 units along the positive x-axis and a translation 1 unit along the positive *y*-axis.

Example 19.14

Find the coordinates of the image of the point (3, 5) under a 90° clockwise rotation followed by translations of 2 units along the positive *x*-axis and 1 unit along the negative *y*-axis.

Solution:

Transformation mapping is: $\mathbf{T}: \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

Hence: $\mathbf{T}: \begin{pmatrix} 3 \\ 5 \end{pmatrix} \to \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ 5 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ -4 \end{pmatrix}$

Therefore image of (3, 5) is (7, –4).

Exercise 19.3

1. Express each of the given linear transformation as a matrix equation and identify the transformation(s) used.
   (a) **T**: $(x, y) \to (-x, -y)$ (b) **T**: $(x, y) \to (2x, 5y)$ *(c) **T**: $(x, y) \to (-2y, 3x)$

2. Identify the linear transformation represented by each of the following mappings:
   (a) **T**: $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} y \\ x \end{pmatrix}$ (b) **T**: $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \frac{-\sqrt{3}x}{2}+\frac{y}{2} \\ \frac{x}{2}+\frac{y\sqrt{3}}{2} \end{pmatrix}$ (c) **T**: $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \frac{-\sqrt{3}x-y}{2} \\ \frac{x-y\sqrt{3}}{2} \end{pmatrix}$

*3. Identify the sequence of linear transformations represented by each of the following mappings:
   (a) **T**: $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 2y \\ 5x \end{pmatrix}$ (b) **T**: $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \frac{-\sqrt{3}x}{2}-\frac{y}{2} \\ \frac{-x}{2}+\frac{y\sqrt{3}}{2} \end{pmatrix}$ (c) **T**: $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \frac{-\sqrt{3}x+y}{2} \\ \frac{-x-y\sqrt{3}}{2} \end{pmatrix}$

4. Write the mapping for each of the following sequence of transformations.
   (a) Translations of 3 units along the positive x-axis and 7 units along the negative y-axis
   (b) Translations of 5 units along the negative x-axis and 2 units along the positive y-axis
   (c) A rotation of 30° clockwise about the origin followed by translations of 2 units along the positive y-axis and 5 units along the negative x-axis.
   (d) Translations of 2 units along the positive y-axis and 5 units along the negative x-axis followed by a rotation of 30° clockwise.

5. Identify each of the transformations given below:
   (a) **T**: $(x, y) \to (x - 1, y + 2)$ (b) **T**: $(x, y) \to (1 - x, -y - 2)$
   (c) **T**: $(x, y) \to (y + 3, -x)$ (d) **T**: $(x, y) \to \left(\frac{-x+y\sqrt{3}+1}{2}, \frac{x\sqrt{3}+y-1}{2}\right)$

6. Let **T** represent the translations of 2 units along the negative x-axis and 4 units along the positive y-axis. Let **M** represent an clockwise rotation of 60° about the origin. Find the coordinates of the image of the point $(-2, 2)$ under each of the following:
   (a) TM (b) MT (c) TMT (d) MTM

7. Let **T** represent the translations of 2 units along the negative x-axis and 4 units along the positive y-axis. Let **M** represent a reflection about the line $y = -x\sqrt{3}$. Find the coordinates of the image of the point $(2, -2)$ under each of the following:
   (a) MT (b) TMT (c) TM (d) MTM

20 Introduction to Complex Numbers

20.1 Introduction to Complex Numbers

- Consider the equation $x^2 = -4$. This equation has *no real solution* as $\sqrt{-4}$ is as yet undefined. However, if we define $i^2 = -1$, then the equation becomes:

$$x^2 = 4i^2$$

This gives $x = \pm 2i$

- Similarly the equation $x^2 - 2x + 2 = 0$ gives:

$$x = \frac{2 \pm \sqrt{-4}}{2} = 1 \pm i$$

- We define a complex number $z = x + yi$ where $i^2 = -1$.

  The real part of z, $\mathrm{Re}(z) = x$

  and the imaginary part of z, $\mathrm{Im}(z) = y$.

20.2 Basic Properties of Complex Numbers (Cartesian Form)

1. Equality of two complex numbers: if $x + yi = a + bi$, then $x = a$, and $y = b$.
2. Addition and subtraction of complex numbers:
   $$(x + yi) \pm (a + bi) = (x \pm a) + (y \pm b)i$$
3. Multiplication of complex numbers: $(x + yi)(a + bi) = (xa - yb) + (xb + ya)i$
4. The complex conjugate of $z = x + yi$ is $\bar{z} = x - yi$. Note that, $z\bar{z} = x^2 + y^2$
5. Properties of conjugates: $\overline{u \pm v} = \bar{u} \pm \bar{v}$ and $\overline{u \times v} = \bar{u} \times \bar{v}$
6. The modulus of $z = x + yi$ denoted by $|x + yi|$ is given by $|x + yi| = \sqrt{x^2 + y^2}$.
7. "Rationalizing" $\dfrac{1}{a+bi}$: $\dfrac{1}{a+bi} \equiv \dfrac{1}{a+bi} \times \dfrac{a-bi}{a-bi} \equiv \dfrac{a-bi}{a^2+b^2}$
8. Dividing complex numbers: $\dfrac{x+yi}{a+bi} \equiv \dfrac{x+yi}{a+bi} \times \dfrac{a-bi}{a-bi} \equiv \dfrac{(x+yi)(a-bi)}{a^2+b^2}$

Example 20.1

Without the use of a calculator, express each of the following in exact Cartesian form. (a is a real number.)

(a) $\dfrac{2}{2+3i}$ (b) $\dfrac{5-3i}{2-5i}$ (c) $\dfrac{a+2i}{2-3i}$ (d) $\dfrac{a+i}{3+4i}$

Solution:

(a) $\dfrac{2}{2+3i} \equiv \dfrac{2}{2+3i} \times \dfrac{2-3i}{2-3i}$

$\equiv \dfrac{4-6i}{2^2+3^2} \equiv \dfrac{4}{13} - \dfrac{6}{13}i$

(b) $\dfrac{5-3i}{2-5i} \equiv \dfrac{5-3i}{2-5i} \times \dfrac{2+5i}{2+5i}$

$\equiv \dfrac{10+25i-6i-15i^2}{2^2+5^2} \equiv \dfrac{25}{29} + \dfrac{19}{29}i$

(c) $(a + 2i)(2 - 3i) = 2a - 3ai + 4i - 6i^2 = (2a + 6) - (3a - 4)i$

(d) $\dfrac{a+3i}{3+4i} \equiv \dfrac{a+3i}{3+4i} \times \dfrac{3-4i}{3-4i}$

$\equiv \dfrac{3a+12-4ai+9i}{5} \equiv \dfrac{3a+12}{25} + \left(\dfrac{9-4a}{25}\right)i$

Example 20.2

Given $z_1 = 3 + 4i$ and $z_2 = 2 - 3i$, express in exact Cartesian form (a) $\overline{z_1 z_2}$ (b) $\overline{z_1}.\overline{z_2}$.

Solution:

(a) $\overline{z_1 z_2} = \overline{(3+4i)(2-3i)} = \overline{(18-i)} = 18 + i$

(b) $\overline{z_1}.\overline{z_2} = (3-4i)(2+3i) = 18 + i$

Example 20.3

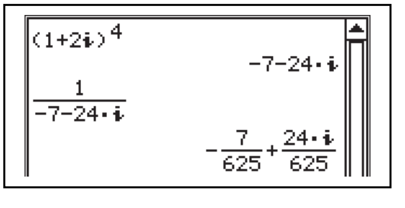
Find $(1 + 2i)^4$. Hence, use this result to find $(1 + 2i)^{-4}$.

Solution:

$(1 + 2i)^4 = [(1 + 2i)^2]^2$

$= (-3 + 4i)^2 = -7 - 24i$

$\Rightarrow (1+2i)^{-4} = \dfrac{1}{-7-24i} = \dfrac{-7}{625} + \dfrac{24}{625}i$



Exercise 20.1

1. Express in terms of i (a and b are real numbers):
   (a) $\sqrt{-25}$ (b) $\sqrt{-81}$ (c) $5-\sqrt{-100}$ (d) $-2+\sqrt{-144}$
   (e) $\sqrt{-(a^2)}$ (f) $4+\sqrt{-(b^2)}$ (g) $a-\sqrt{-(25a^2)}$ (h) $a+\sqrt{-(8b^2)}$

2. State Re(z) and Im(z) for the following complex numbers z (a is a real number):
   (a) $2-3i$ (b) $4+4i$ (c) 0 (d) $-19i$
   (e) $3-\sqrt{-16}$ (f) $\sqrt{-25}+6$ (g) $4+\sqrt{-(9a^2)}$ (h) $\sqrt{a^2}+2i$

3. In each of the following cases, find the complex number z given that:
   (a) $\text{Re}(z)+\text{Im}(z)=10$ and $\text{Im}(z)=3$ (b) $\text{Re}(z)+\text{Im}(z)=12$ and $\text{Re}(z)-\text{Im}(z)=2$.

4. In each of the following cases, find the value(s) of the real numbers a and b:
   (a) $a+2i=3-bi$ (b) $4-2i=a-bi$ (c) $(a+b)+(a-b)i=6+2i$ (d) $ab+b^2i=6+9i$

5. Evaluate without the use of a calculator (a is a real number):
   (a) $(2+3i)+(3-2i)$ (b) $(2-3i)-(3-2i)$ (c) $3i+(3-6i)$
   (d) $(4-i)+(2+7i)-(3-6i)$ (e) $(4+ai)+(2+3i)$ (f) $(a+2i)+(3-2i)-(1+ai)$

6. Evaluate without the use of a calculator:
   (a) $(2+i).(3+2i)$ (b) $(4+3i).(2-3i)$ (c) $(-3-2i).(-1-5i)$
   (d) $(-4+3i)(-2-i)$ (e) $(3-4i)(3+4i)$ (f) $(3-2i)^2$

7. Express in exact Cartesian form (a and b are real numbers):
   (a) $\frac{4}{1+i}$ (b) $\frac{1-i}{1+2i}$ (c) $\frac{3+i}{5-i}$ (d) $\frac{-3-5i}{3-5i}$
   (e) $\frac{(2+4i)(3-2i)}{1-2i}$ (f) $(3+4i)+\frac{25}{3+4i}$ (g) $\frac{a+bi}{5+12i}$ (h) $(2+i)+\frac{a}{6+8i}$

8. Given that $z=12+5i$, determine: (a) $\bar{z}$ (b) $z+\bar{z}$ (c) z^{-1} (d) $\bar{z}+z^{-1}$

9. Given that $w=2-3i$ and $z=4+3i$, express in exact Cartesian form:
   (a) $(w+z)^2$ (b) $\left(\frac{w}{z}\right)^2$ (c) $\overline{wz}$ (d) $\overline{\left(\frac{w}{z}\right)}$ (e) $\frac{w+z}{w-z}$ (f) $w+\frac{\bar{w}}{z}$

10. Expand and simplify: (a) $(1-i)^3$ (b) $(1+2i)^4$ (c) $(2-i)^{-3}$ (d) $(3+4i)^{-4}$

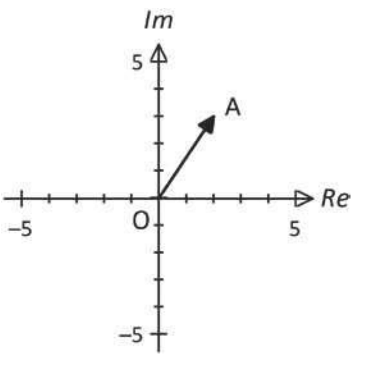
11. Given that $w=a+2i$, $z=5+12i$, express in exact Cartesian form (a is a real number):
    (a) $(w+z)^2$ (b) $\bar{w}z$ (c) $\frac{1}{wz}$ (d) $\frac{z}{iw}$

12. Find in exact Cartesian form $z^2+iz+(1-3i)$ for:
    (a) $z=1+i$ (b) $\bar{z}=2-i$ (c) $\frac{1}{z}=3+4i$ (d) $iz=a-i$

13. Find in exact Cartesian form z^3+iz^2+4i for:
    (a) $\frac{1}{z}=2i$ (b) $z=1+i$ (c) $\bar{z}=2+3i$ (d) $z=\frac{1}{i}$

20.3 Complex Numbers as Ordered Pairs

- We have seen that essentially a complex number consists of two parts, the real part and the imaginary part. The Cartesian representation,
  $$z = \text{(real part)} + \text{(imaginary part)}i$$
  provides some distinct algebraic advantages as observed in the previous section.
- However, rather than using the "+" sign to bring the two parts together, we could rewrite z as;
  (real part, imaginary part).
  In so doing we are representing complex numbers as ordered pairs.
- An Argand diagram represents a complex number $x + yi$ either as an ordered pair (x, y) or as a vector on a set of x-y axes. The x-axis is labelled the Real axis and the y-axis is labelled the Imaginary axis.
- For example the complex number $z = 2 + 3i$ is represented as a point A (2, 3) or as a vector **OA** = < 2, 3 > on an Argand diagram.



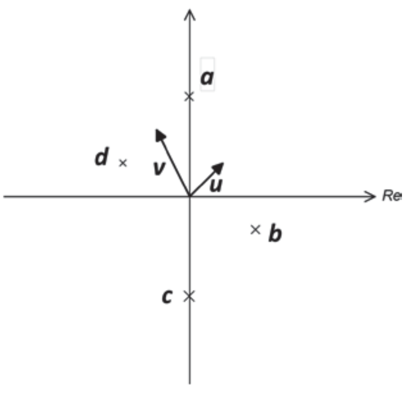
Exercise 20.2

1. For each of the complex numbers z, locate in the Argand plane, the points corresponding to (i) z (ii) $\overline{z}$ (iii) $z + \overline{z}$ (iv) $z - \overline{z}$:
   (a) $-4i$ (b) $-3 - 5i$ (c) $2 + 4i$ (d) $5 - 2i$

2. For each of the complex numbers z, locate in the Argand plane, the points corresponding to (i) z (ii) $\frac{1}{z}$ (iii) $z + \frac{1}{z}$ (iv) $z - \frac{1}{z}$:
   (a) $2i$ (b) $1 + i$ (c) $0.5 + 0.5i$ (d) $-0.25 + 0.25i$

3. For each of the complex numbers w and z, locate in the Argand plane, the points corresponding to (i) $w + z$ (ii) $\overline{w + z}$ (iii) $\overline{w} + \overline{z}$:
   (a) $w = 1 + i$, $z = -2 + 3i$ (b) $w = -3 - 4i$, $z = \sqrt{2} + i\sqrt{2}$

4. In the accompanying diagram, the complex numbers u and v are represented as vectors. Find in terms of u and/or v and/or their conjugates, the complex numbers a, b, c and d.



20.4 Quadratic Equations with Complex Roots

- For the equation $ax^2 + bx + c = 0$, the roots are given by:

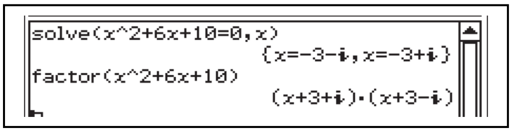
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- When the discriminant $(b^2 - 4ac)$ is negative, then $\sqrt{b^2 - 4ac}$ is imaginary and the roots are complex (and appear as conjugates).

Example 20.4

Without the use of a calculator, solve $x^2 + 6x + 10 = 0$.
Hence express $x^2 + 6x + 10$ as a product of 2 complex factors.

Solution:



Roots are $x = \dfrac{-6 \pm \sqrt{36 - 40}}{2} \Rightarrow x = -3 \pm i$

Hence $x^2 + 6x + 10 \equiv [x - (-3 + i)][x - (-3 - i)] \equiv [x + 3 - i][x + 3 + i]$

Exercise 20.3

Work each question without the use of a CAS calculator.

1. Solve each of the following equations:
   (a) $z^2 + 16 = 0$ (b) $49 + z^2 = 0$ (c) $z^2 + 4z + 8 = 0$
   (d) $4z^2 + 4z + 2 = 0$ (e) $z(z + 2) = -5$ (f) $(z + 1)^2 + 16 = 0$
   (g) $2[(z - 3)^2 + 36] = 0$ (h) $2(z - 2)^2 + 45 = -5$ (i) $z = \dfrac{-2}{z+2}$
   (j) $\dfrac{3-2z}{z+1} = 1 - z$

2. Factorise each of the following completely (include complex factors):
   (a) $z^2 + 25$ (b) $100 + z^2$ (c) $z^2 - 2z + 10$ (d) $(z - 1)^2 + 49$

3. Solve for z, giving all roots in exact form:
   (a) $(z - 1)(z^2 + 2z + 4) = 0$ (b) $(z - 3)(z^2 + 9) = 0$
   (c) $(z + 1)(z^2 + z + 1) = 0$ (d) $(z^2 + 1)(z^2 + 25) = 0$

21 Methods of Proofs

21.1 Conjectures

- A mathematical conjecture is an unproven mathematical statement which appears to be true. Once a conjecture has been proved, it is called a theorem.
- There are various ways of conducting mathematical proofs.
  Geometric proofs were introduced in Chapter 11 and Geometric proofs using vectors utilising the method of deduction was introduced in Chapter 12.
- In this chapter, we will further explore the use of deductive proof techniques and introduce the method of inductive proof.

21.1.1 General Terms and Assumed Results

- The table below shows how "numbers" are represented algebraically.

Description	Algebraic terms
Even numbers	$2n$
Odd numbers	$2n + 1$
Consecutive numbers	$n, n + 1, n + 2, \ldots$..or $n - 1, n, n + 1$
Consecutive even numbers	$2n, 2n + 2, 2n + 4, \ldots$.
Consecutive odd numbers	$2n + 1, 2n + 3, 2n + 5$
Any two even numbers	$2n$ and $2m$
Any two odd numbers	$2n + 1$ and $2m + 1$

- In general, proofs are built on other proven results.
  In this chapter, the following results will be assumed without proof.
  - Even number + even number = even number
  - Odd number + even number = odd number
  - Odd number + odd number = even number
  - Even number × Even number = Even number
  - Even number × Odd number = Even number
  - Product of two consecutive numbers must be even.

    $\Rightarrow \quad n \times (n + 1) = n^2 + n$ must be even

    or $(n - 1) \times n = n^2 - n$ must be even

21.2 Deductive Proofs

- A deductive proof is one of the methods of direct proof used in Mathematics. In this section, the use of the deductive proof method will be applied to prove several interesting facts about whole numbers.

Example 21.1

Prove that:
(a) the square of an odd number is always odd,
(b) the sum of the squares of any two consecutive whole numbers is always odd.

Solution:

(a) $(2k + 1)$ is an odd number.

$(2k + 1)^2 = 4k^2 + 4k + 1$

$= 4(k^2 + k) + 1$ which is an odd number. Proved.

(b) Let the two consecutive numbers be x and $x + 1$.

Sum of squares $= x^2 + (x + 1)^2$

$= 2(x^2 + x) + 1$ which is odd. Proved.

Example 21.2

(a) Prove that the product of any three consecutive whole numbers is a multiple of three.
(b) Hence, prove that $n^3 - n + 3$ is a multiple of three for any whole number n.

Solution:

(a) In any sequence of three consecutive whole numbers, one of these numbers must be a multiple of three.

Hence, the three consecutive numbers must be in one of these forms:

- $3k$, $3k + 1$ and $3k + 2$ giving the product $3k(3k + 1)(3k + 2)$
- $3k - 1$, $3k$, $3k + 1$ giving the product $3k(3k - 1)(3k + 1)$
- $3k - 2$, $3k - 1$, $3k$ giving the product $3k(3k - 1)(3k - 2)$

In each instance, the product is a multiple of three. Proved.

(b) $n^3 - n + 3 = n(n^2 - 1) + 3 = n(n - 1)(n + 1) + 3$

$= (n - 1) \times n \times (n + 1) + 3.$

But $(n - 1) \times n \times (n + 1)$ is the product of three consecutive whole numbers and from the result in (a) must be a multiple of three.

Hence, $n^3 - n + 3 = (n - 1) \times n \times (n + 1) + 3$

$= 3k + 3 = 3(k + 1)$ which is a multiple of three. Proved.

Note:

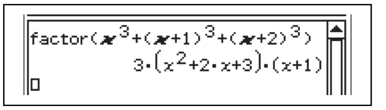
- *The proof of the result in (b) is built upon the proof used in (a). In general, proofs are built on previously proven results.*

Example 21.3

(a) Prove that the sum of the cubes of any 3 consecutive whole numbers is a multiple of 3.
(b) Under what conditions will this sum be a multiple of 12? Prove this conjecture.

Solution:

(a) Let the three consecutive numbers be x, x +1 and x + 2.



Hence, Sum of cubes = $x^3 + (x + 1)^3 + (x + 2)^3$
$= 3(x + 1)(x^2 + 2x + 3)$
which is clearly a multiple of three. Proved.

(b) Exploratory Stage:

Trialling with numbers OR	Algebraic Approach
$1^3 + 2^3 + 3^3 = 36$ multiple of 12 $2^3 + 3^3 + 4^3 = 99$ not multiple of 12 $3^3 + 4^3 + 5^3 = 216$ multiple of 12 $4^3 + 5^3 + 6^3 = 405$ not multiple of 12 $5^3 + 6^3 + 7^3 = 684$ multiple of 12 $6^3 + 7^3 + 8^3 = 1071$ not multiple of 12 $7^3 + 8^3 + 9^3 = 1584$ multiple of 12	*Consider the sum of cubes* $= 3(x + 1)(x^2 + 2x + 3)$. *If x is an odd number, then (x + 1) will be even and* x^2 *will be odd.* *Hence, the factor* $(x^2 + 2x + 3)$ *consists of the sum of two odd numbers with an even number, which will result in an even number.* *Hence, the sum of cubes* *= 3 × an even no. × another even no* *and therefore will be a multiple of twelve.*

Therefore, we make the conjecture:
Starting with an odd number, the sum of cubes of three consecutive numbers will be a multiple of twelve.

Let the three consecutive numbers starting with an odd number be
$2k + 1$, $2k + 2$ and $2k + 3$.
Hence, Sum of cubes = $(2k + 1)^3 + (2k + 2)^3 + (2k + 3)^3$
$= 12(x + 1)(2x^2 + 4x + 3)$
which is a multiple of twelve. Proved.

Note:

- *The stages involved in creating a conjecture involves firstly an exploratory stage where we search for patterns through the use of numerical examples or through algebraic considerations.*
- *The next stage involves writing the conjecture describing the mathematical pattern observed.*

Exercise 21.1

1. (a) Prove that the sum of any 3 consecutive positive integers will be a multiple of three.
   (b) Prove that the sum of any 5 consecutive positive integers will be a multiple of five.
   (c) Prove that the sum of any 7 consecutive positive integers will be a multiple of seven.
   (d) Under what conditions will the sum of any n consecutive positive integers be a multiple of n. Prove this conjecture. [Use 1 + 2 + 3 + + k = $\frac{k(1+K)}{2}$.]

2. (a) Prove that the square of an odd number greater than one when divided by four leaves a remainder of one.
   (b) Prove that the square of an odd number greater than one when divided by eight leaves a remainder of one.
   (c) Under what conditions will the square of an odd number when divided by twenty four leave a remainder of one.

3. (a) Prove that the square of an odd number add three is a multiple of four.
   (b) Prove that the square of an odd number add seven is a multiple of eight.
   (c) Prove that the square of an odd number add eleven is a multiple of four.
   (d) For what values of α, would the square of an odd number add α be a multiple of eight? Prove your conjecture.

4. Prove that the:
   (a) sum of the squares of any two consecutive odd numbers is always even.
   (b) sum of the squares of any three consecutive odd numbers add one is divisible by 12.
   (c) sum of the squares of any four consecutive odd numbers is divisible by 4.
   (d) sum of the squares of any five consecutive odd numbers is divisible by 5.

5. Prove that:
   (a) the product of any two consecutive positive integers must be divisible by 2.
   (b) the product of any three consecutive positive integers is divisible by 3!.
   (c) the product of any four consecutive positive integers is divisible by 4!.
   (d) the product of any five consecutive positive integers is divisible by 5!.
   (e) $x^4 - 2x^3 - x^2 + 2x + 24$ is divisible by 24, for x as a positive integers.

6. (a) Prove that the product of any three consecutive odd numbers is divisible by three.
   (b) Prove that the product of any six consecutive odd numbers is divisible by nine.
   (c) Prove that the product of any $3n$ consecutive odd numbers is divisible by 3^n.
   (d) Prove that $16x^3 + 24x^2 - 4x - 6$ is divisible by six, for x as a whole number.

7. (a) Prove that the product of any two consecutive even numbers is divisible by 8.
   (b) Prove that the product of any three consecutive even numbers is divisible by 48.
   (c) Prove that the product of any four consecutive even numbers is divisible by 384.
   (d) Prove that the product of any n consecutive even numbers is divisible by $2^n \times n!$.

8. (a) Prove that $3^n + 1$ is an even number for positive integer n.
   (b) Prove that $3^n + 3$ is an even number for positive integer n.
   (c) Prove that $3^n + 3$ is a multiple of six for positive integer n.
   (d) Prove that $3^n + 3^m$ is a multiple of six for positive integers m and n.

9. (a) Prove that $x^3 - x$ is divisible by six for positive integer x.
   (b) Prove that $5x^5 - 5x$ is divisible by thirty for positive integer x.
   (c) Prove that $7x^7 - 7x$ is divisible by forty-two for positive integer x.
   (d) Under what conditions will $x^n - x$ be divisible by six for positive integers x and n. Prove your conjecture [Very Challenging!].

10. (a) Prove that the sum of the cubes of any three consecutive positive integers is divisible by 3.
   *(b) Prove that the sum of the cubes of any four consecutive positive integers is divisible by 4.
   (c) Prove that the sum of the cubes of any five consecutive positive integers is divisible by 5.
   *(d) For integer n, where $1 \le n \le 10$, investigate the conjecture that the sum of the cubes of any n consecutive positive integers is divisible by n.

21.3 Counter Examples

- To disprove a mathematical conjecture it is sufficient to produce one example that invalidates the conjecture. This example is referred to as a counter-example.
- Often, conjectures are amended to neutralise counter examples. In this way, counter examples are used to refine and improve conjectures.

Example 21.4

Provide a counter-example to disprove the following conjectures.

(a) $x \le 5 \Rightarrow x^2 \le 25$ (b) $x \le 3 \Rightarrow \frac{3}{x} \ge 1$

Solution:

(a) Consider $x = -6$. Clearly $-6 \le 5$ but $(-6)^2 \not\le 25$.

(b) Consider $x = -1$. Clearly $-1 < 3$ but $\frac{3}{-1} = -3 \not\ge 3$.

Example 21.5

Provide a counter-example to disprove the conjecture that the sum of the squares of any five consecutive odd numbers is divisible by 15. Amend the conjecture to neutralise the counter-example.

Solution:

Using a spread-sheet.

a	b	c	d	e	$a^2+b^2+c^2+d^2+e^2$	sum/15
1	3	5	7	9	165	11
3	5	7	9	11	285	19
5	7	9	11	13	445	29.66667
7	9	11	13	15	645	43
9	11	13	15	17	885	59
11	13	15	17	19	1165	77.66667
13	15	17	19	21	1485	99
15	17	19	21	23	1845	123
17	19	21	23	25	2245	149.6667
19	21	23	25	27	2685	179

From the table above, $5^2 + 7^2 + 9^2 + 11^2 + 13^2 = 445$ which is *not* divisible by 15.

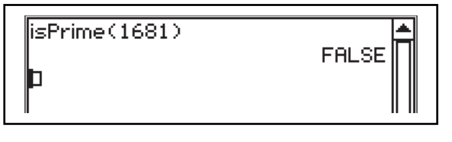
Amend the conjecture to:
The sum of the squares of any five consecutive odd numbers is divisible by 15 provided the first number in the sequence is not of the form $3n - 1$ for $n = 1, 2, 3, \ldots$.

Example 21.6

Provide a counter-example to disprove the conjecture that if n is a positive integer, then $P(n) = n^2 - 79n + 1601$ is a prime number.. Amend the conjecture to neutralise the counter-example.

Solution:

Using a process of trial and error:
$P(80) = 1681$
which is divisible by 41.



Amend the conjecture to:
If n is a positive integer where $n < 80$, then $P(n) = n^2 - 79n + 1601$ is a prime number.

Note:

- *The process of obtaining a counter-example can be quite arduous!*

Exercise 21.2

1. Provide a counter-example to disprove each of the following conjectures. Where possible, amend the conjecture to neutralize the counter-example.
   For real numbers a and b:
   (a) If $a^2 > b^2$, then $a > b$. (b) If $a^2 < b^2$, then $a < b$. (c) If $a < b$, then $ac < bc$.
   (d) If $a > b$, then $\frac{1}{a} < \frac{1}{b}$ (e) If $a^2 > b$, then $a > \sqrt{b}$ (f) $\sqrt{a^2} = a$

2. Provide a counter-example to disprove each of the following conjectures.
   (a) $x \le 5 \Rightarrow \frac{5}{x} \ge 1$ (b) $\frac{10}{x} < 1 \Rightarrow x > 10$
   (c) $x(x-1) > 0 \Rightarrow x > 1$ (d) $(x-2)(x+3) \ge 0 \Rightarrow x \ge 2$.

3. Provide a counter-example to disprove each of the following conjectures.
   For real numbers a and b:
   (a) $a - b \le a + b$ (b) $a + b \le ab$ (c) $ab \ge \frac{a}{b}$ (d) $a^2 + b^2 \ge (a-b)^2$.

4. Provide a counter-example to disprove each of the following conjectures. Where possible, amend the conjecture to neutralise the counter-example.
   (a) $x^3 + x^2 + x$ is divisible by 3 for x as a positive integer.
   (b) $x^4 + x^3 + x^2 + x$ is divisible by 5 for x as a positive integer greater than one.
   (c) $x^5 + x^4 + x^3 + x^2$ is divisible by 5 for x as a positive integer greater than one.

5. Provide a counter-example where possible to disprove each of the following conjectures.
   (a) The square of an even number add one or less one is always a multiple of 5.
   (b) The cube of an even number greater than two less the number itself is divisible by 5.
   (c) The cube of an odd number greater than one add the number itself is divisible by 5.

*6. Provide a counter-example where possible to disprove each of the following conjectures.
   (a) If n is a positive integer, then P(n) is a prime number for:
      (i) $P(n) = n^2 + n + 41$ (ii) $P(n) = 5(n^2 - n) + 1$ for $n > 1$.
   (b) If n is a prime number, then $2^n - 1$ is also a prime number.

*7. Provide a counter-example to disprove each of the following conjectures involving matrices.
   (a) If **AB** = **I**, then $\mathbf{A}^{-1} = \mathbf{B}$. (b) If **AB** = **0**, then **A** = **0** or = **B** = **0**.
   (c) If **AB** = **BC**, then **A** = **C**.
   (d) If a linear transformation is a self-inverse transformation, then it must be a reflection about a line passing through the origin.

21.4 Proof by Contradiction

- In this section, we shall explore yet another method of mathematical proof, referred to as proof by contradiction.
- The method of proof by contradiction involves the following steps.

  - *Negate* the given conjecture.
    That is, assume the given conjecture is *false*.

  - Using this assumption, we then show that this leads to some impossible conclusions: conclusions that contradict our initial assumption or that contradict known facts.

  - It is then concluded that the negation of the conjecture must be incorrect.

  - The conjecture must then be true.

Example 21.7

Prove that if x^2 is even, then x is also even, for x as a whole number.

Solution:

Negate the given conjecture.
That is, assume that if x^2 is even, then x is not even.

If x^2 is even $\Rightarrow$ $\exists$ a whole number m such that $x^2 = 2m$.
Since, x is not even, $x = 2n + 1$ for some whole number n.
Hence, $(2n + 1)^2 = 2m$
$\Rightarrow$ $4n^2 + 4n + 1 = 2m$.

But $2m$ is an even number hence $4n^2 + 4n + 1$ must also be even
and this implies that 1 is an even number which is a contradiction.

Hence, if x^2 is even, then x is must also even.

Note:

- The symbol $\exists$ stands for "there exists".

Example 21.8

Prove that if x^2 is a multiple of 5, then x is also a multiple of 5, for x as a whole number.

Solution:

Negate the given conjecture.

That is, assume that if x^2 is a multiple of 5, then x is not a multiple of 5.

x^2 is a multiple of 5, then $x^2 = 5n$ for some whole number n.

Since, x is not a multiple of 5, $x = 5q + r$ for some whole number q and $r = 1, 2, 3, 4$.

Hence, $(5q + r)^2 = 5n$

$25q^2 + 10qr + r^2 = 5n.$

The RHS is a multiple of 5.

The first two terms on the LHS are each multiples of 5.

For the entire LHS to be a multiple of 5, r^2 must also be a multiple of 5;

which is a contradiction as none of 1^2, 2^2, 3^2 or 4^2 are multiples of 5.

Hence, if x^2 is a multiple of a whole number 5, then x is also a multiple of 5.

Example 21.9

A rational number must be expressible as a fraction of two integers.

Prove that $\sqrt{5}$ is an irrational number.

Solution:

Assume that the conjecture that $\sqrt{5}$ is an irrational number is false.

That is, assume that $\sqrt{5}$ is a rational number.

Hence, $\exists$ integers a and b with no factors in common such that $\sqrt{5} = \frac{a}{b}$

$\Rightarrow \quad 5 = \frac{a^2}{b^2}$

$5b^2 = a^2$

This implies that a^2 is a multiple of 5 and hence a is also a multiple of 5. (Example 21.8)

Hence, $a = 5k$ for some integer k.

$\Rightarrow \quad 5b^2 = (5k)^2 \quad \Rightarrow \quad b^2 = 5k^2$

Which implies that b is also a multiple of 5.

This contradicts the initial assumption that a and b have no factors in common.

Hence, the assumption that "$\sqrt{5}$ is an irrational number is false" must be incorrect.

Hence $\sqrt{5}$ must be an irrational number.

Exercise 21.3

1. Use the method of contradiction to prove that:
   (a) $n^2 - n$ must be an even number for n as a whole number.
   (b) $5^n + 1$ is an even number for positive integer n.
   (c) $3^n > 3n$ for integer $n \geq 2$.
   (d) for n as a whole number, n^2 is even if and only if n is even.
   (e) for n as a whole number, n^2 is odd if and only if n is odd.

*2. If $\log_a y = x$ and $y = a^x$ are equivalent, use the method of contradiction to prove that:
   (a) $\log_2 5$ is irrational. (b) $\log_{10} 5$ is irrational.

3. Use the method of contradiction to prove that:
   (a) $\sqrt{10}$ is an irrational number. (b) $\sqrt{7}$ is an irrational number.
   (c) $\dfrac{1}{\sqrt{2}}$ is an irrational number.
   (d) if a is rational and b is irrational, then $a + b$ must be irrational.
   (e) if a is rational and b is irrational, then $a \times b$ must be irrational.

4. Use the method of contradiction to prove that:
   (a) if vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are perpendicular, then, $\boldsymbol{u} \bullet \boldsymbol{v} = 0$.
   (b) if vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are such that $\boldsymbol{u} \bullet \boldsymbol{v} = 0$, then, $\boldsymbol{u}$ and $\boldsymbol{v}$ are perpendicular.
   (c) if vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are such that $\boldsymbol{u} = \lambda\boldsymbol{v}$ where λ is a constant, then, $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel.
   (d) if vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel, then, $\boldsymbol{u} = \lambda\boldsymbol{v}$ where λ is a constant.

21.5 Mathematical Induction

- In this section, we will explore the use of the inductive method in proving mathematical conjectures.
- The method of proof by induction when applied to a mathematical conjecture which claims to be true for some or all values of n, comprises three stages.
  - Verify that the mathematical conjecture is true for the initial value of n.
  - Assume that the conjecture is true for $n = k$
    and use algebra to show that this assumption leads to the unavoidable conclusion that it must also be true for $n = k + 1$.
  - Merge the two results above to conclude inductively that it must be true for all required values of n.

Example 21.10

Use mathematical induction to prove $1 + 2 + 3 + 4 + 5 + ... + n = \frac{n(n+1)}{2}$ where n is a positive integer.

Solution:

When $n = 1$, LHS of statement = 1,

RHS of statement = $\frac{1(1+1)}{2} = 1$.

Hence, the statement is true for $n = 1$.

Assume that the result is true for $n = k$,

that is, $1 + 2 + 3 + 4 + 5 + ... + k = \frac{k(k+1)}{2}$

When $n = k + 1$

$$\begin{aligned} \text{LHS} &= 1 + 2 + 3 + 4 + 5 + ... + k + (k + 1) \\ &= \{1 + 2 + 3 + 4 + 5 + ... + k\} + (k + 1) \\ &= \frac{k(k+1)}{2} + (k + 1) \\ &= \frac{k(k+1)+2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2}. \end{aligned}$$

But, when $n = k + 1$, RHS = $\frac{(k+1)(k+2)}{2}$

Hence, if the result is assumed to be true for $n = k$, then it must be true for $n = k + 1$.

Having shown in the first instance that it is true for $n = 1$,

using the result just shown, it must then be true for $n = 1 + 1 = 2$.

Since it is true for $n = 2$, it must be true for $n = 2 + 1 = 3$.

Since it is true for $n = 3$, it must be true for $n = 3 + 1 = 4$, and so on.

Hence, the result must be true for all positive integers n.

Example 21.11

Use mathematical induction to prove $1 + r + r^2 + r^3 + r^4 + ... + r^n = \frac{1-r^{n+1}}{1-r}$ where $n \geq 1$ and n is an integer.

Solution:

When $n = 1$, LHS of statement = $1 + r$,

RHS of statement = $\frac{1-r^2}{1-r} = 1 + r$.

Hence, the statement is true for $n = 1$.

Assuming that the result is true for $n = k$,

that is, $1 + r + r^2 + r^3 + r^4 + ... + r^k = \frac{1-r^{k+1}}{1-r}$

When $n = k + 1$

$$\begin{aligned} \text{LHS} &= 1 + r + r^2 + r^3 + r^4 + ... + r^k + r^{k+1} \\ &= \{1 + r + r^2 + r^3 + r^4 + ... + r^k\} + r^{k+1} \\ &= \frac{1-r^{k+1}}{1-r} + r^{k+1} \\ &= \frac{1-r^{k+1} + r^{k+1}(1-r)}{1-r} \\ &= \frac{1-r^{k+2}}{1-r}. \end{aligned}$$

But, when $n = k + 1$, RHS = $\frac{1-r^{k+2}}{1-r}$.

Hence, if the result is assumed to be true for $n = k$, then it must be true for $n = k + 1$.

Having shown in the first instance that it is true for $n = 1$,

using the result just shown, it must then be true for $n = 1 + 1 = 2$.

Since it is true for $n = 2$, it must be true for $n = 2 + 1 = 3$.

Since it is true for $n = 3$, it must be true for $n = 3 + 1 = 3$, and so on.

Hence, the result must be true $\forall$ integers $n \geq 1$.

Note:

- The symbol $\forall$ represents the phrase "for all".

Example 21.12

Use mathematical induction to prove for integer $x \geq 1$,

$$\frac{1}{1(3)} + \frac{1}{2(4)} + \frac{1}{3(5)} + \ldots + \frac{1}{x(x+2)} = \frac{3}{4} - \frac{2x+3}{2(x+1)(x+2)}.$$

Solution:

When $x = 1$, LHS of statement $= \frac{1}{3}$,

RHS of statement $= \frac{3}{4} - \frac{5}{2(2)(3)} = \frac{1}{3}$.

Hence, the statement is true for $x = 1$.

Assume that the result is true for $x = k$, that is,

$$\frac{1}{1(3)} + \frac{1}{2(4)} + \frac{1}{3(5)} + \ldots + \frac{1}{k(k+2)} = \frac{3}{4} - \frac{2k+3}{2(k+1)(k+2)}$$

When $x = k + 1$

$$\text{LHS} = \frac{1}{1(3)} + \frac{1}{2(4)} + \frac{1}{3(5)} + \ldots + \frac{1}{k(k+2)} + \frac{1}{(k+1)(k+1+2)}.$$

$$= \left[\frac{1}{1(3)} + \frac{1}{2(4)} + \frac{1}{3(5)} + \ldots + \frac{1}{k(k+2)}\right] + \frac{1}{(k+1)(k+1+2)}$$

$$= \frac{3}{4} - \frac{2k+3}{2(k+1)(k+2)} + \frac{1}{(k+1)(k+3)}$$

$$= \frac{3}{4} - \left[\frac{(2k+3)(k+3) - 2(k+2)}{2(k+1)(k+2)(k+3)}\right]$$

$$= \frac{3}{4} - \left[\frac{(2k+5)(k+1)}{2(k+1)(k+2)(k+3)}\right]$$

$$= \frac{3}{4} - \left[\frac{(2k+5)}{2(k+2)(k+3)}\right]$$

But, when $x = k + 1$, $\text{RHS} = \frac{3}{4} - \left[\frac{(2k+5)}{2(k+2)(k+3)}\right]$

Hence, if the result is assumed to be true for $x = k$, then it must be true for $x = k + 1$.

Having shown in the first instance that it is true for $x = 1$,

using the result just shown, it must then be true for $x = 1 + 1 = 2$.

Since it is true for $x = 2$, it must be true for $x = 2 + 1 = 3$.

Since it is true for $x = 3$, it must be true for $x = 3 + 1 = 4$, and so on.

Hence, the result must be true $\forall$ integers $x \geq 1$.

Example 21.13

Use mathematical induction to prove that for integer $n \geq 1$, $4^n + 5$ is divisible by 3.

Solution:

When $n = 1$, Expression = 4 + 5 = 9 = 3 × 3 which is divisible by 3.
Hence, the statement is true for $n = 1$.

Assume that the result is true for $n = k$, that is $4^k + 5$ is divisible by 3.
Hence, we can write $4^k + 5 = 3 \times m$ for some integer m.

When $n = k + 1$ Expression = $4^{k+1} + 5$

$= 4 \times 4^k + 20 - 15$

$= 4\,(4^k + 5) - 15$

$= 4\,(3 \times m) - 15$

$= 3\,(4m - 5)$ which is a multiple of 3.

Hence, if the result is assumed to be true for $n = k$, then it must be true for $n = k + 1$.

Having shown in the first instance that it is true for $n = 1$,
using the result just shown, it must then be true for $n = 1 + 1 = 2$.
Since it is true for $n = 2$, it must be true for $n = 2 + 1 = 3$.
Since it is true for $n = 3$, it must be true for $n = 3 + 1 = 4$, and so on.

Hence, the result must be true $\forall$ integers $n \geq 1$.

Note:

- The conjecture refers to a property of a mathematical expression. Hence, there is no RHS as in the previous examples.
- In this instance, we rewrite the property in mathematical form as seen above and show that this property can be inductively conferred on the expression for all required values of n.

Exercise 21.4

1. Use mathematical induction to prove each of the following mathematical conjectures.

   (a) $5 + 10 + 15 + 20 + ... + 5n = \frac{5n(n+1)}{2}$ for integer $n \geq 1$.

   (b) $1 + 3 + 5 + ... + (2n - 1) = n^2$ for integer $n \geq 1$.

   (c) $7 + 10 + 13 + 16 + ... + (3n + 4) = \frac{n(3n+11)}{2}$ for integer $n \geq 1$.

   (d) $46 + 42 + 38 + 34 + ... + (50 - 4n) = 2n(24 - n)$ for integer $n \geq 1$.

2. Use mathematical induction to prove each of the following mathematical conjectures.

   (a) $1 + 4 + 9 + 25 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$ for integer $n \geq 1$.

   (b) $1 + 3 + 6 + 10 + 15 + ... + \frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{6}$ for integer $n \geq 1$.

   (c) $1^2 + 3^2 + 5^2 + 7^2 + ... + (2n - 1)^2 = \frac{n(4n^2-1)}{3}$ for integer $n \geq 1$.

   (d) $3 + 7 + 13 + 21 + ... + (n^2 + n + 1) = \frac{n(n^2+3n+5)}{3}$ for integer $n \geq 1$.

3. Use mathematical induction to prove each of the following mathematical conjectures for integer $n \geq 1$.

   (a) $1^3 + 2^3 + 3^3 + 4^3 + ... + n^3 = \frac{n^2(n+1)^2}{4}$

   (b) $1(2)(3) + 2(3)(4) + 3(4)(5) + ... + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}$

   (c) $1^2(2) + 2^2(3) + 3^2(4) + 4^2(5) + ... + n^2(n + 1) = \frac{n(n+1)(n+2)(3n+1)}{12}$

   (d) $1^4 + 2^4 + 3^4 + 4^4 + ... + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$

4. Use mathematical induction to prove each of the following mathematical conjectures for integer $n \geq 1$.

   (a) $\frac{1}{1(2)} + \frac{1}{2(3)} + \frac{1}{3(4)} + \frac{1}{4(5)} ... + \frac{1}{n(n+1)} = \frac{n}{n+1}$

   (b) $\frac{1}{1(3)} + \frac{1}{3(5)} + \frac{1}{5(7)} + \frac{1}{7(9)} ... + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$

   (c) $\frac{1}{1(2)(3)} + \frac{1}{2(3)(4)} + \frac{1}{3(4)(5)} + \frac{1}{4(5)(6)} + ... + \frac{1}{n(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$

   (d) $\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + ... + \frac{n}{2^n} = 2 - \frac{n+2}{2^n}$

*5. Use mathematical induction to prove the following conjectures for integer $n \geq 1$.

   (a) $1(1!) + 2(2!) + 3(3!) + 4(4!) + ... + n(n!) = (n + 1)! - 1$

   (b) $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} + ... + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$

6. Use mathematical induction to prove each of the following for integer $n \geq 1$.
   (a) $7^n + 2$ is divisible by 3 (b) $8^n + 6$ is divisible by 14
   (c) $3^{2n} - 1$ divisible by 8

*7. Use mathematical induction to prove that for integer n, $n \geq 2$,
   $(x + 1)^n - nx - 1$ is divisible by x^2.

8. Use mathematical induction to prove:
   (a) $3^n > 3n$ for integer $n \geq 2$ (b) $4n^2 \geq (n + 1)^2$ for integer $n \geq 1$
   (c) $n! > n^2$ for integer $n \geq 4$ (d) $n! > 2^n$ for integer $n \geq 4$.

9. Given the non-singular matrices **A** and **B** such that **AB** = **BA**, use mathematical induction to prove that for integer $n \geq 1$, $\mathbf{A}^n = \mathbf{B}\mathbf{A}^n\mathbf{B}^{-1}$.

10. Use mathematical induction to prove:
   (a) $\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ an & 1 \end{pmatrix}$ for integer $n \geq 1$ (b) $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^n = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}$ for integer $n \geq 1$

11. Use mathematical induction to prove that for integer $n \geq 1$,
   (a) $\begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix}^n = \begin{pmatrix} \cos nx & -\sin nx \\ \sin nx & \cos nx \end{pmatrix}$
   (b) $\begin{pmatrix} \cos x - \sin x & -\sqrt{2}\sin x \\ \sqrt{2}\sin x & \cos x + \sin x \end{pmatrix}^n = \begin{pmatrix} \cos nx - \sin nx & -\sqrt{2}\sin nx \\ \sqrt{2}\sin nx & \cos nx + \sin nx \end{pmatrix}$.

Extremely Challenging

12. If $\mathbf{A} = \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}$, find $\mathbf{A}^n$ for n = 1, 2, 3, … , 10. (See Exercise 19.3 No. 2)
   Hence make a conjecture with regards $\mathbf{A}^n$ where n is a positive integer.
   Use mathematical induction to prove your conjecture.

13. If $\mathbf{A} = \begin{pmatrix} 0 & 0 & a \\ 0 & b & 0 \\ c & 0 & 0 \end{pmatrix}$, find $\mathbf{A}^n$ for n = 1, 2, 3, … , 10.
   Hence make a conjecture with regards $\mathbf{A}^n$ where n is a positive integer.
   Use mathematical induction to prove your conjecture.

21.6 Rational numbers as terminating or recurring decimals

- In the previous sections, we have used various methods of proof to establish several properties of numbers.
- In Example 21.1, we used the method of deductive proof to prove that the square of an even number is always even. In Example 21.7, we used the method of contradiction to prove that if the square of a number is even, that number itself must be even.
- In Example 21.9, we introduced the idea of a rational number and used the method of contradiction to prove that $\sqrt{5}$ is an irrational number.
- In Example 21.14, we used the method of proof by mathematical induction to prove that $4^n + 5$ is divisible by 3.
- In this section, we will use the method of deductive proof to prove that terminating decimals (e.g. 0.7458) and recurring decimals (e.g. $2.\overline{1345}$) are rational numbers.

Example 21.14

Prove that each of the following are rational numbers.

(a) 0.6125 (b) $1.\overline{13}$ (c) $5.6\overline{29}$.

Solution:

(a) $0.6125 \equiv 0.6125 \times \dfrac{10\,000}{10\,000}$

$\equiv \dfrac{6\,125}{10\,000}$

Notes:
0.6125 is a terminating decimal with 4 dp.
Eliminate the decimal part by multiplying and dividing 0.6125 with 10^4.

Hence, 0.6125 can be expressed as a fraction of two integers and is therefore rational.

(b) Let $x = 1.\overline{13}$ I

$100x = 113.\overline{13}$ II

II – I $99x = 112$

$\Rightarrow x = \dfrac{112}{99}$

Notes
$1.\overline{13}$ is a recurring decimal with a period of 2.
Multiply $1.\overline{13}$ with 10^2.
We now have two numbers with the same recurring decimals.
By subtracting these two numbers, the recurring decimals are eliminated.
$1.\overline{13}$ can now be written as a fraction of two integers.

Hence, $1.\overline{13}$ can be expressed as a fraction of two integers and is therefore rational.

(c) Let
$$x = 5.6\overline{29}$$
$$10x = 56.\overline{29} \quad \text{I}$$
$$1000x = 5629.\overline{29} \quad \text{II}$$

II − I
$$990x = 5573$$
$$\Rightarrow \quad x = \frac{5573}{990}.$$

Hence, $5.6\overline{29}$ can be expressed as a fraction of two integers and is therefore rational.

Notes
$5.6\overline{29}$ is a recurring decimal of period 2 from the second decimal place on. We need two numbers with the same recurring decimals. The first number is obtained by multiplying $5.6\overline{29}$ by 10^1 since the recurring decimals start from the second decimal place. The second number is obtained by multiplying $5.6\overline{29}$ with $10^1 \times 10^2$. By subtracting these two numbers, the recurring decimals are eliminated. We can now express $5.6\overline{29}$ as a fraction of two integers.

Example 21.15

Prove that the infinite sum $1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \ldots$ is a rational number.

Solution:

The infinite sum is the sum of a geometric sequence with first term $a = 1$ and common ratio $r = \frac{1}{2}$.

As $|r| < 1$, the sum to infinity of the series exists.

Hence:
$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \ldots = \left.\frac{a}{1-r}\right|_{a=1, r=\frac{1}{2}}$$
$$= \frac{1}{\left(\frac{1}{2}\right)} = 2.$$

Therefore, $1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \ldots$ is a rational number.

Example 21.16

Express $0.\overline{37}$ as an infinite sum. Hence, prove that $0.\overline{37}$ is a rational number.

Solution:

$$0.\overline{37} \equiv 0.37 + 0.00\,37 + 0.00\,00\,37 + 0.00\,00\,00\,37 + \ldots$$
$$\equiv \frac{37}{100} + \frac{37}{10\,000} + \frac{37}{1\,000\,000} + \frac{37}{100\,000\,000} + \ldots$$
$$\equiv \frac{37}{100} + \frac{37}{100^2} + \frac{37}{100^3} + \frac{37}{100^4} + \ldots$$

Hence $0.\overline{37}$ is the sum to infinity of a geometric series with first term $a = \frac{37}{100}$ and common ratio $r = \frac{1}{100}$.

Therefore: $$0.\overline{37} = \left.\frac{a}{1-r}\right|_{a=\frac{37}{100},\, r=\frac{1}{100}}$$
$$= \frac{\left(\frac{37}{100}\right)}{1-\left(\frac{1}{100}\right)} = \frac{37}{99}.$$

Hence, $0.\overline{37}$ is a rational number.

> ***Note:***
> *This example uses the concepts of an infinite geometric series to prove that recurring decimals are rational numbers.*

Example 21.17

Prove that $2 \equiv 1.\overline{9}$.

Solution:

Let $\quad x = 1.\overline{9} \quad$ I

$\quad 10x = 19.\overline{9} \quad$ II

II − I $\quad 9x = 18$

$\Rightarrow \quad x = 2$

Hence: $\quad 2 \equiv 1.\overline{9}$

> Note:
> This example shows that all integers can be written as terminating decimals

Exercise 21.5

1. Prove that each of the following numbers are rational numbers:
   (a) 0.12345 (b) 5.5974 (c) −2.145 (d) $\frac{1}{2.1}$
   (e) $1.\overline{8}$ (f) $1.\overline{17}$ (g) $2.\overline{357}$ (h) $8.\overline{9357}$

2. Prove that each of the following numbers are rational numbers:
   (a) $1.1\overline{8}$ (b) $12.3\overline{47}$ (c) $0.15\overline{537}$ (d) $1.34\overline{901}$

3. Prove that each of the following infinite sums are rational numbers:
   (a) $1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + \ldots$ (b) $5 + \frac{5}{4} + \frac{5}{4^2} + \frac{5}{4^3} + \frac{5}{4^4} + \ldots$
   (c) $1 + \frac{2}{7} + \left(\frac{2}{7}\right)^2 + \left(\frac{2}{7}\right)^3 + \left(\frac{2}{7}\right)^4 + \ldots$ (d) $1 - \frac{1}{9} + \frac{1}{9^2} - \frac{1}{9^3} + \frac{1}{9^4} + \ldots$

4. Prove that each of the following numbers are rational numbers by first expressing each of them as infinite sums.
   (a) $5.\overline{2}$ (b) $6.4\overline{6}$ (c) $2.10\overline{04}$ (d) $1.105\overline{107}$

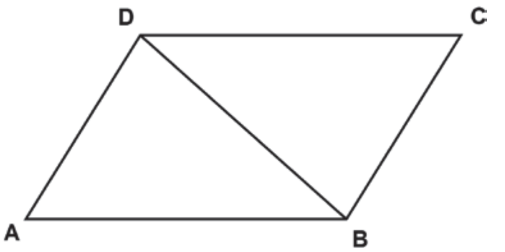
5. Prove that:
   (a) $1 \equiv 0.\overline{9}$ (b) $40 \equiv 39.\overline{9}$ (c) $2.8 \equiv 2.7\overline{9}$ (d) $5.14 \equiv 5.13\overline{9}$

21.7 Formal Considerations

- In Chapter 11 the terms *converse* and *contrapositive* in relation to conjectures were introduced. The idea of the *negative* of a conjecture was introduced in Section 21.4.
- In this section, we will review these concepts and include the concept of the *inverse* of a conjecture or statement.
- The following table summarises the four concepts considered.

Concept	Key Idea	Truth Value
Conjecture	If P then Q.	
Negative	If P then not Q.	
Contrapositive	If not Q then not P.	Always true for a true statement
Converse	If Q then P.	Not necessarily true for a true statement.
Inverse	If not P then not Q.	Not necessarily true for a true statement.

- The negation of a conjecture is an inherent part of the method of proof by contradiction.
  - If the negative of a conjecture is false, then the conjecture is true. This forms the basis of proofs by contradiction.
  - Note that if the negative of a conjecture is true, then the conjecture is false.
- The contrapositive of a true conjecture is always true.
  This was illustrated in Example 11.3.
  Conjecture:
  If ABCD is a parallelogram then ΔABD and ΔCDB are congruent.
  Contrapositive:
  If ΔABD and ΔCDB are not congruent, then ABCD is not a parallelogram.



- If the contrapositive of a conjecture is true, the conjecture must itself be true.
- The converse of the conjecture in Example 11.3 is:
  If ΔABD and ΔCDB are congruent, then ABCD is a parallelogram.
  In this case, the converse is true.
- The inverse of the conjecture in Example 11.3 is:
  If ΔABD and ΔCDB are not congruent, then ABCD is a not parallelogram.
  In this case, the inverse is also true.
- The converse and the inverse of a true conjecture need not always be true.
  - If the converse of a conjecture is false, then the negative of the conjecture must necessarily be false.
  - If the converse of a conjecture is true, then the negative of the conjecture must also be true.

Example 21.18

Consider the conjecture: If x is a factor of 12, then x is a factor of 48 for x as a whole number.

(a) Prove that the conjecture is true.

(b) Prove that both the converse and inverse of the conjecture are false.

Solution:

(a) x is a factor of 12 $\Rightarrow$ $\exists$ a whole number n such that $nx = 12$.
But $48 = 12 \times 4 = 4nx$ $\Rightarrow$ x is also a factor of 48.

(b) Converse of conjecture: If x is a factor of 48, then x is a factor of 12.
Counter-example: $x = 16$ is a factor of 48 but not a factor of 12.
Hence, the converse of the conjecture is false.

Inverse of conjecture: If x is not a factor of 12, then x is not a factor of 48.
Counter-example: $x = 16$ is a not factor of 12 but is a factor of 48.
Hence, the inverse of the conjecture is false.

Exercise 21.6 This exercise covers the various types of proofs discussed.

1. Consider the conjecture: If x is even, then x^2 is even for x as a whole number.
   (a) Prove that both the conjecture and its contrapositive is true.
   (b) Prove that both the converse and inverse are true.

2. Consider the conjecture: If x is a multiple of 10, then x^2 is a multiple of 10 for x as a whole number. (a) Prove that both the conjecture and its contrapositive is true.
   (b) Prove that both the converse and inverse are true.

3. Consider the conjecture: If x is a factor of 9, then x is a factor of 36 for x as a whole number. (a) Prove that the conjecture is true.
   (b) Prove that both the converse and inverse are false.

4. Consider the conjecture: If x is a factor of 20, then x is a factor of 100 for x as a whole number. (a) Prove that the conjecture is true.
   (b) Prove that both the converse and inverse are false.

5. Consider the true statement: If $x = 10$, then $x^2 = 100$.
   Prove that both the converse and the inverse are false.

6. Consider the true statement: If $x = 2$, then $x^3 = 8$.
   Prove that both the converse and the inverse are true.

7. Consider the true statement: If $\sin \theta = 1$, then $\cos \theta = 0$.
   Prove that both the converse and the inverse are false.

8. Consider the true statement: If $\sin \theta = 0$, then $\tan \theta = 0$.
   Prove that both the converse and the inverse are true.

Answers

Exercise 1.1

1. (a) 30 (b) 132 (c) 28 (d) 1 260
2. (a) $\frac{5!}{2!}$ (b) $\frac{39!}{34!}$ (c) $\frac{n!}{(n-4)!}$
   (d) $\frac{(n+5)!}{n!}$ (e) $\frac{14!}{3!\,12!}$ (f) $\frac{8!\,3!}{5!\,6!}$
3. (a) $10 \times 8!$ (b) $40 \times 5!$
   (c) $274 \times 8!$ (d) $(n-1)!\,(n+1)$

Exercise 1.2

1. 120
2. 20 160
3. (a) 40 320 (b) 16 777 216
4. 240
5. 216
6. (a) 12 096 (b) 39 999
7. 10 000
8. 900 000 000
9. (a) 60 480 (b) 17 280
10. 4 569 760 000
11. (a) 5 760 000 (b) 11 520 000
    (c) 120 960 000
12. (a) $\frac{26!}{21!}$ (b) 26^{10}
13. (a) 10^{5} (b) $\frac{10!}{2!}$
14. (a) $\frac{52!}{42!}$ (b) 52^{20}
15. (a) $\frac{62!}{50!}$ (b) $\frac{10!}{8!} \times \frac{60!}{47!}$
16. 6
17. 21

Exercise 1.3

1. (a) 5040 (b) 720 (c) 1440
   (d) 3600 (e) 480
2. (a) 10 080 (b) 30 240 (c) 7200
3. (a) 1 440 (b) 3 600
   (c) 576 (d) 144
4. (a) 282 240 (b) 14 400 (c) 2 880
5. (a) 14 400 (b) 5760 (c) 2 880
6. (a) 604 800 (b) 172 800 (c) 86 400
7. (a) 240 (b) 24
8. (a) 39 916 800 (b) 86 400
9. (a) 12 (b) 72
10. (a) 720 (b) 1 440

Exercise 2.1

1. 7
2. (a) 15 (b) 15
3. (a) 10 (b) 40
4. 28 800
5. 29 520
6. (a) 1 012 851 840 (b) 389 558 400
   (c) 289 386 240 (d) 1 113 024 000
7. (a) $10 \times {}^{35}P_7$ (b) $26 \times {}^{35}P_7$
   (c) $260 \times {}^{34}P_6$
   (d) $36 \times {}^{35}P_7 - 260 \times {}^{34}P_6$
8. (a) ${}^{26}P_9$ (b) $20 \times {}^{24}P_7$
   (c) ${}^{26}P_9 - 20 \times {}^{24}P_7$
9. (a) ${}^{36}P_{10}$ (b) $105 \times {}^{34}P_8$
   (c) $325 \times {}^{34}P_8$
10. $176 \times 14!$
11. $12 \times 15!$
12. (a) 6 000 (b) 5 000 (c) 9 000
13. (a) 25 000 (b) 20 000 (c) 95 000
14. 9 858 240
15. (a) 2 605 680 (b) 111 110 000
16. (a) 68 880 (b) 450 000
17. 30 240
18. (a) 8 344 (b) 24 500

Exercise 2.2

1. (a) $x = 1$ (b) $x = 4$
   (c) $x = 9$ (d) $x = 13$
2. (a) 28 (b) 0 (c) 7 (d) 6
3. (a) 233 (b) 20 (c) 32 (d) 160
4. (a) 2 (b) 7 (c) 25
5. (a) 8 (b) 34
6. (a) 10 (b) 20
7. (a) 303 (b) 67 (c) 267
8. (a) 42 (b) 37
9. (a) 41 (b) 13/25
10. (a) 75 600 (b) 10 800 (c) 257 760
11. (a) 4 320 (b) 4 320 (c) 25 200
12. (a) 30 240 (b) 2 880 (c) 22 560
13. (a) 2 (b) 12 (c) 56
14. (a) 72 (b) 12 (c) 56
15. (a) 71 429 (b) 54 286 (c) 60 000
16. (a) 46 667 (b) 50 000 (c) 60 000

Exercise 2.3

1. (a) 7 (b) 8
2. (a) (i) 13 (ii) 15
   (b) Min = 0, Max = 5
3. (a) (i) 7 (ii) 13
   (b) Min = 5, Max = 30
4. (a) (i) 12 (ii) 56
   (b) Min = 11, Max = 130
5. (a) At least 27 words. (b) At least 53 words.
6. (a) At least 27 words. (b) At least 235 words.
7. (a) 41 (b) 161
8. (a) 8 (b) 13 (c) 366
11. $m = 6, n = 7$
12. $m = 11, n = 12$
13. (a) 4 (b) 7 (c) 10 (d) 13
14. (a) 4 (b) 7 (c) 9 (d) 9
15. (a) 33 (b) 37 (c) 35 (d) 38

Exercise 3.1

1. (a) 15 (b) 20 (c) 84 (d) 120
   (e) 4950 (f) 1 225 (g) 126 (h) 210
2. (a) $n = 10$; $r = 3$ or 7 (b) $n = 15$; $r = 4$ or 11
3. (a) 9 (b) 14
4. (a) 13 (c) 48

Exercise 3.2

1. (a) 495 (b) 715 (c) 781
2. (a) 455 (b) 560 (c) 575
3. (a) ${}^{21}C_{16} + {}^{20}C_{17}$ (b) $2^{20} - 1$
4. (a) ${}^{21}C_2$ (b) $2^{20} - 21$ (c) $2^{20} - 21$
5. (a) 65 780 (b) 83 682
   (c) 252 (d) 16 576 560
6. (a) 768 212 (b) 386
   (c) 296 529 832
7. (a) 569 772 (b) 46 911 228
8. (a) 2 332 440 (b) 517 616
9. 303 643 256
10. 8 592 675
11. 2 780
12. 256

Exercise 3.3

1. (a) 2 352 (b) 2 352
   (c) 2 352 (d) 2 352
2. (a) 27 132 (b) 12 852
   (c) 2 352 (d) 37 632
3. (a) 3 139 500 (b) 29 173 203 150
4. (a) 10 (b) 17 315
5. (a) 84 (b) 224
   (c) 4 (d) 304
6. (a) 1 176 (b) 4 606
7. (a) 11 760 (b) 625 520
8. (a) 68 250 (b) 19 286 085
9. 188 430
10. 711 287 840 120
11. 28
12. 17 656 275
13. 528 154
14. 922 010 602 500
15. 729

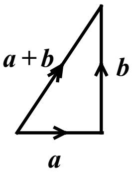
Exercise 3.4

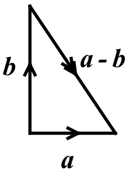
1. (a) 3 528 000 (b) 1 587 600
   (c) 5 332 320
2. (a) 126 584 640 000 (b) 9 041 760 000
   (c) 12 558 000 (d) 3 013 920 000
3. (a) $\binom{26}{5}\binom{10}{4} \times 9!$ (b) $\binom{26}{5}\binom{10}{4} \times 6! \times 4!$
   (c) $\binom{26}{5}\binom{10}{4} \times 2! \times 5! \times 4!$
   (d) $\binom{26}{5}\binom{10}{4} \times 5! \times 4!$
4. (a) 15! (b) $\binom{6}{2}\binom{4}{2} \times 4!$
   (c) $\binom{5}{4}\binom{6}{4}\binom{4}{4} \times 12!$
5. (a) 290 304 000 (b) 1 036 800
   (c) 24 192 000 (d) 72 576 00
6. (a) 423 360 (b) 846 720
   (c) 2 116 800 (d) 4 233 600
7. (a) 1 (b) $31! = 8.222\,8 \times 10^{33}$
   (c) $2.607\,3 \times 10^{14}$
8. (a) (i) 249 500 (ii) 4 500
   (iii) 245 000
   (b) $\binom{10}{1}^{50} \times 50!$ (c) $\binom{10}{2}^{50} \times 100!$
9. (a) 456 976 (b) 1 679 616
   (c) 351 000 (d) 982 800
   (e) 6 750
10. (a) $2.521\,6 \times 10^{14}$ (b) 9 487 530 000
    (c) 16 500 (d) 900
11. (a) 10 (b) 90
    (c) 4 (d) 720
    (e) 16

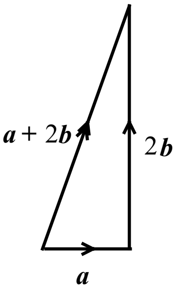
Exercise 3.5

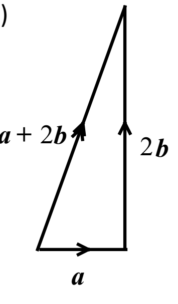
1. (a) $\dfrac{9!}{3!\,2!}$ (b) $\dfrac{10!}{2!\,2!\,2!}$ (c) $\dfrac{11!}{2!\,2!\,2!}$
   (d) $\dfrac{13!}{6!\,4!\,2!}$ (e) $\dfrac{13!}{5!\,2!\,2!}$ (f) $\dfrac{12!}{2!\,2!\,2!\,2!}$
2. (a) $\dfrac{7!}{2!\,2!\,2!}$ (b) $\dfrac{7!}{2!\,2!}$ (c) $\dfrac{7!}{2!\,2!}$
3. (a) $\dfrac{14!}{3!\,(2!)^4}$ (b) $\dfrac{14!}{3!\,(2!)^3}$ (c) $\dfrac{14!}{(2!)^5}$
4. (a) $\dfrac{5 \times 11!}{2!}$ (b) $\dfrac{12!}{2!} - \dfrac{5 \times 11!}{2!}$
5. (a) 6^{10} (b) 7^{10} (c) 9^{10}

Exercise 4.1

1. (a) 

   (b) 

   (c) 

   (d) 

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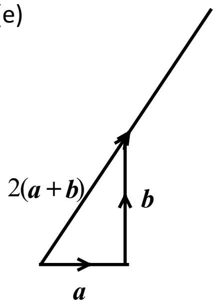

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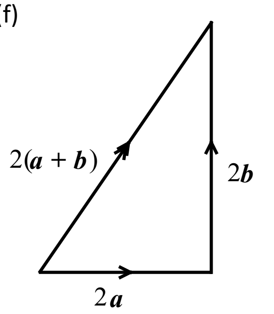
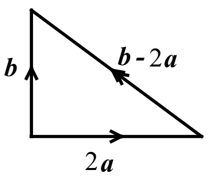


(g)



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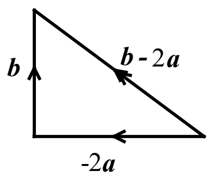
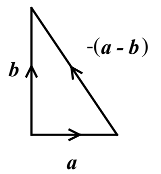


(i)



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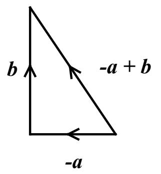

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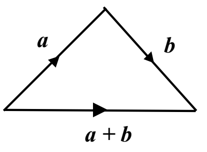

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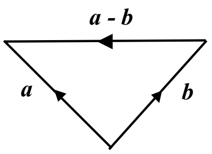

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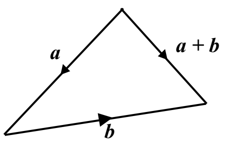

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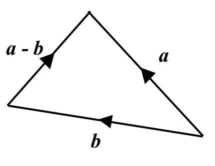

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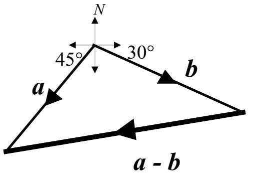
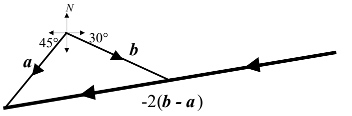
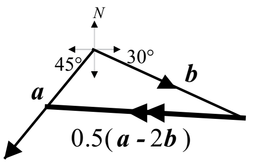


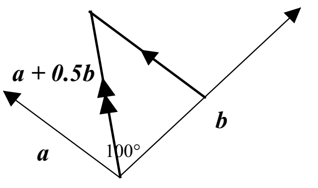
(b)



(c)



4. (a)



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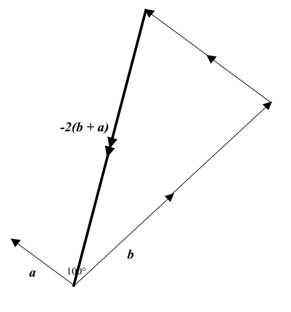
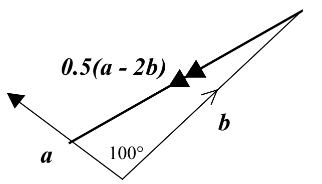
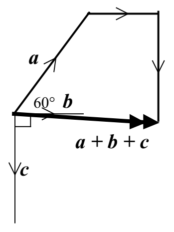


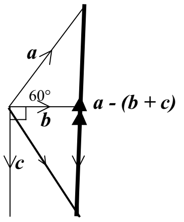
(c)



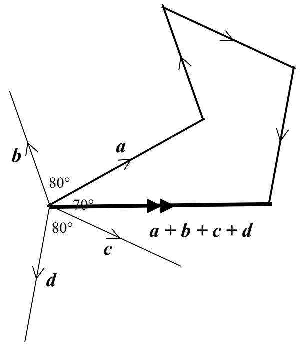
5. (a)



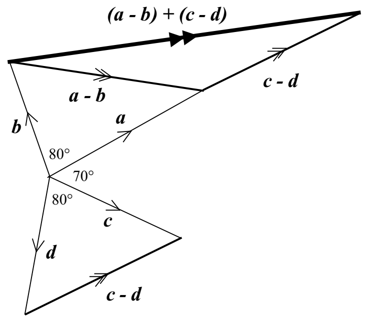
5. (b)



6. (a)



(b)



7. (b) 13.2, 41° 8. (b) 8.1, 112°
9. 16.6, 160° 10. 220, 65°

Exercise 4.2

1. 17.09 m, 133.0° 2. 3.95 m, 15.0°
3. 19.97 m, 151.1° 4. 19.55 m, 101.6°
5. 61.40 N, 69.1° 6. 25.07 N, 161.7°
7. 11.08 ms^{-1}, 148.5° 8. 7.73 km, 15.5°
9. 34.43 N, 58.4° 10. 8.36 N, 66.4°
11. 33.04 ms^{-1}, 103.4° 12. 194.19 N, 90.5°
13. 62.45 kmh^{-1}, 256.1° 14. 100.87 km, 152.5°

Exercise 4.3

1. < 5.14, 6.13 >, < –1.71, –4.70 >, < –3.06, 2.57>
2. (a) < 25.88, 96.59 > (b) < –21.65, –12.50 >, (c) < 61.28, –51.42> (d) < –76.60, 64.28 >
3. (a) $< 30\sqrt{2}, 30\sqrt{2} >$ (b) $< -75\sqrt{3}, 75 >$, (c) $< 1\,000, -1\,000\sqrt{3} >$ (d) $< -200\sqrt{2}, -200\sqrt{2} >$
4. (a) 5.83, 59.0° (b) 8.25, 104.0° (c) 26.93, –68.2° (d) 8.06, –119.7°
5. (a) $9\sqrt{2}, \pi/4$ (b) $2\sqrt{2}, 3\pi/4$ (c) $4, -\pi/6$ (d) $12, -2\pi/3$
6. (a) 13, 3 < 5, 12 > (b) 25, 4 < –24, –7 >
7. (a) $\pm 2\sqrt{21}$ (b) $\pm 3\sqrt{7}$
8. (a) $\frac{\sqrt{5}}{20}, \frac{\sqrt{5}}{2} < 4, 8 >$ (b) $\frac{\sqrt{5}}{5}, \sqrt{5} < \sqrt{2}, \sqrt{3} >$
9. $\frac{1}{2\sqrt{10}} < 6, -2 >, -\frac{\sqrt{10}}{2} < 6, -2)$
10. $\frac{\sqrt{5}}{5} < 2, 1>$ (a) $\sqrt{5} < 2, 1 >$ (b) $-2\sqrt{5} < 2, 1 >$
11. 17, –28.1° 12. 24.29, 93.3°
13. $\pm\sqrt{(17/26)} < -5, 1 >$ 14. –12, 2 15. –5, 1
16. (a) $\sqrt{10}$ (b) $6\sqrt{17}$ (c) $\frac{\sqrt{10}}{6\sqrt{17}} < -6, 24 >$
17. (a) $\sqrt{85}$ (b) $\sqrt{265}$ (c) $\frac{\sqrt{85}}{\sqrt{265}} < 3, 16 >$
18. $\frac{4\sqrt{10}}{5\sqrt{26}} < 5, -25 >$
19. 0, 38/5
20. (a) $\alpha = 5/6, \beta = 7/6$ (b) $\alpha = -7/5, \beta = 6/5$
22. –7
23. < 2, 5 >, < 0.1, 0.25 > and $< \sqrt{20}, 5\sqrt{5} >$
24. $\alpha\beta = -40$

Exercise 4.4

1. 17.09 m, 133.0° 2. 3.95 m, 15.0°
3. 19.97 m, 151.1° 4. 19.55 m, 101.6°
5. 61.40 N, 69.1° 6. 25.07 m, 161.7°
7. 11.08 ms^{-1}, 148.5° 8. 7.73 km, 15.5°
9. 33.04 ms^{-1}, 103.4° 10. 194.19 N, 90.5°
11. 62.45 kmh^{-1}, 256.1° 12. 100.87 km, 152.5°

Exercise 5.1

1. (a) < 7, 18 > (b) $\sqrt{373}$ (c) 68.8°
2. (a) $\sqrt{221}$ (b) 109.7°
3. (a) < 4, 7 > (b) < 10, –8 >
4. (a) ± 8 (b) $8 \pm 4\sqrt{6}$
5. (a) ± 4 (b) –3, 11
6. < 6, 15 > 7. (a) < 4, 4 > (b) < 6, –7 >
8. (a) $\frac{1}{\sqrt{17}} < 4, -1 >$ (b) $\frac{5}{\sqrt{17}} < -4, 1 >$
9. (a) $\frac{\sqrt{29}}{13} < 5, -12 >$ (b) $\frac{20}{\sqrt{29}} < 5, -2 >$
10. (a) $3\sqrt{17} < -1, 0 >$ (b) $\frac{5}{\sqrt{73}} < -8, 3 >$
11. (a) $\alpha = -4$ (b) $5\beta - 2\alpha = 18$
12. (a) $\alpha = -2$ (b) $3\alpha - \beta = -15$
14. $k = 30$ 15. $k = (7 + 6h)/7$, h is real
16. (b) ½ ($\boldsymbol{a} - \boldsymbol{b}$) (c) $-\boldsymbol{a} - 2\boldsymbol{b}$
17. (a) $\boldsymbol{i} = \frac{\sqrt{2}}{2}(\boldsymbol{a} - \boldsymbol{b}), \boldsymbol{j} = -\frac{\sqrt{2}}{2}(\boldsymbol{a} + \boldsymbol{b})$
(b) $-\sqrt{2}\,\boldsymbol{a} + 3\sqrt{2}\,\boldsymbol{b}$
18. (a) < 40, 56 > (b) < 160, 200 >

Exercise 5.2

1. (a) < –5, 16 > (b) < 16, 0 >
   (c) < 33, 63/2 > (d) < 41, –70 >
2. < 28, –10/3 > 3. < 22/5, 2 >

Exercise 6.1

1. (a) ${}_C\boldsymbol{r}_B$ = < –5, –12 > (b) ${}_A\boldsymbol{r}_B$ = < –1, –4 >
   (c) ${}_C\boldsymbol{r}_A$ = < –4, –8 > (d) $\sqrt{73}$
   (e) < –3, –4 >
2. $\sqrt{122}$, direction 005.2°
3. $\sqrt{442}$, direction 357.3°
4. < –10, –4 > 5. < –6, 2 > or < –2, –6 >
6. < 2, 0 > or < –10/13, –24/13 >

Exercise 6.2

1. (a) < 6, –3 > (b) < 4, –6 >
   (c) $\sqrt{181}$ ms^{-1}, direction 312°
2. < 3, –3 > 3. < –2, 23 > 4. < 2, –16 >
5. < 3, 1 > or < –93/61, –169/61 >

Exercise 7.1

1. (a) 219.8°, 360 m (b) 120 sec
2. (a) 8 km/h (b) 3 min
3. (a) 0.36 ms^{-1} (b) 109.19 m
4. (a) 330° (b) 1 min 55 sec
5. (a) 1.06 kmh^{-1} (b) 2 min 39 sec
6. x = 0.38 ms^{-1}, y = 1.11 ms^{-1}
7. (a) 138.1° (b) 2 hr 53 min
8. x = 25.96 kmh^{-1}, θ = 107.18°
9. θ = 35.2°, 1 hr 27 min
10. (a) 18.57 N, 150.97° with the 12N force.
    (b) 55.77 N, 164.6° with the 60N force.

Exercise 7.2

1. (a) 219.8°, 360 m (b) 120 sec
2. (a) 8 km/h (b) 3 min
3. (a) 18.57 N, 150.97° with the 12N force.
   (b) 55.77 N, 164.6° with the 60N force.
4. (a) 0.36 ms^{-1} (b) 109.19 m
5. x = 25.96 kmh^{-1}, θ = 107.18°
6. x = 0.38 ms^{-1}, y = 1.11 ms^{-1}
7. (a) 330° (b) 1 min 55 sec (c) 210°
8. (a) 138.1° (b) 2 hr 53 min (c) 301.9°
9. 1.06 kmh^{-1}; 2 min 39 sec
10. θ = 35.2°, 1 hr 27 min

Exercise 7.3

1. (a) < 8, 25 > (b) 3.02 sec (c) 4.16 sec
2. (a) < –3, 30 > (b) 8.90 sec (c) 3.30 sec
3. (a) < –6, 6 > (b) 2.30 pm (c) 2.00 pm
4. (a) a = 10, b = –5 (b) 9.00 am at < 0, 35 >
5. (a) < –20 + 10t, 15 – 5t >, < 50 – 5t, –25 + 10t >
   (b) 10.32 am, 2.48 pm
   (c) 21.2 km at 12.40 pm
6. 165.93 km at 1.45 pm
7. x = 3, y = –12 or x = 4.93, y = –16.83
8. A & B collide 2.5 sec after 1500 hours
   at < –30, 58.5 > km
9. B & C collide at 11 am at < 20, 23 >
10. 20k < –3, –1 > kmh^{-1} where k > 1.

Exercise 8.1

1. (a) $\sqrt{17}$ ms^{-1}, direction 014°
   (b) $\sqrt{17}$ ms^{-1}, direction 014°
2. (a) $\sqrt{5}$ ms^{-1}, direction 296.6°
   (b) $\sqrt{10}$ ms^{-1}, direction 161.6°
   (c) 5 ms^{-1}, direction 143.1°
3. (a) < –15 + 0.2t, 10 – 0.3t > (b) 6.9 m
4. (a) < –14 + 4t, –10 + 2t > (b) 2.7 m
5. (a) < 1, 0 > (b) < 3 – t, 2 >
   (c) $\sqrt{13}$ m (d) 2 m
6. (a) < (25$\sqrt{3}$)/2, 15/2 > knots
   (b) < 8$\sqrt{3}$, 8 > knots
7. < 3, 5 + 3$\sqrt{3}$ > kmh^{-1}
8. (a) < –1, 2 > (b) < 1, 0 > (c) No collision
9. Collision after 3 seconds
10. < 30, 50 > km at 8.30 am
11. < 64, 98 > nm at 4.48 pm
12. x = –2, y = 66
13. (a) < –x, 5 – y > (b) < –6, 2 > (c) x = 3, y = 4
14. x = 4, y = 2
15. v_B = < 0, –15 >; < 10, –25 > km at 3 pm
16. 1 hour 28 min

Exercise 9.1

1. (a) 8 (b) 11 (c) 19 (d) 19
   (e) 25 < –3, 4 >
2. (a) –40 (b) –12 (c) –28 (d) 28
   (e) –40 < 3, 9 >
3. (a) 2 (b) 1 (c) 9 (d) 1
4. (a) –27 (b) 162 (c) –117 (d) 117
6. –50/3 7. –2, 3
8. α = 4, β = 5 or α = 5, β = 4
9. x = –1, y = 2 or x = 1, y = –2
10. x = –6, y = –2 or x = 6, y = 2

Exercise 9.2

1. (a) 90° (b) 0° (c) 53.1°
   (d) 23.2° (e) 74.7° (f) 82.9°
2. (a) perpendicular (b) parallel and opposite
   (c) neither parallel nor perpendicular

3. (a) < 5, 0 > (b) $20\sqrt{5}$ < −2, 1 >
   (c) $\sqrt{2}$ < −2, 1 > (d) $\sqrt{6}$ < 1, −2 >
4. (a) < 1.1962, −9.9282 > or < 9.1962, −3.9282>
   (b) < $-2\sqrt{5}$, $-\sqrt{5}$ > or < $-\sqrt{5}$, $2\sqrt{5}$ >
   (c) < −7.5115, 6.6013 > or < 9.4727, 3.2045 >
   (d) < 0, −1 > or < 1, 0 >
5. $a = 5$, $b = 4$ or $a = -5$, $b = -4$
6. (a) $a = -10$, $b = -40$ or $a = 10$, $b = 40$
   (b) $a = k$, $b = -4k$ where k is a real number or $a = -k/4$, $b = k$ where k is a real number
7. $a = \pm 2\sqrt{3}$, $b = k$ where k is a real number
8. (a) 108.4° (b) 4.5
9. (a) $\angle ABC = 17.0^\circ$, $\angle ADC = 25.3^\circ$ (b) 20.0

Exercise 9.3

1. (a) 10 (b) $\boldsymbol{v}$ (c) $\boldsymbol{u} - \boldsymbol{v}$
2. (a) −10 (b) $-\frac{5}{6}\boldsymbol{u}$ (c) $\boldsymbol{v} + \frac{5}{6}\boldsymbol{u}$
3. (a) $\frac{3}{4}\boldsymbol{v}$ (b) $\boldsymbol{u} - \frac{3}{4}\boldsymbol{v}$
4. (a) $-\frac{1}{8}\boldsymbol{v}$ (b) $\boldsymbol{u} + \frac{1}{8}\boldsymbol{v}$
5. (a) $\frac{3}{5}\begin{pmatrix}1\\2\end{pmatrix}$ (b) $\frac{3}{2}\begin{pmatrix}1\\1\end{pmatrix}$
6. (a) $\frac{9}{17}\begin{pmatrix}1\\4\end{pmatrix}$ (b) $-\frac{9}{5}\begin{pmatrix}1\\2\end{pmatrix}$
7. (a) $\frac{3}{2}\begin{pmatrix}1\\1\end{pmatrix}$ (b) $\frac{3}{5}\begin{pmatrix}-1\\2\end{pmatrix}$
8. (a) $\frac{3}{2}\begin{pmatrix}-3\\-1\end{pmatrix}$ (b) $\frac{3}{5}\begin{pmatrix}-3\\4\end{pmatrix}$
9. (a) (i) $\frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}$ (ii) $\frac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix}$
   (b) (i) $\frac{3}{25}\begin{pmatrix}3\\4\end{pmatrix}$ (ii) $\frac{4}{25}\begin{pmatrix}4\\-3\end{pmatrix}$
   (c) (i) $\frac{7}{25}\begin{pmatrix}3\\4\end{pmatrix}$ (ii) $\frac{1}{25}\begin{pmatrix}4\\-3\end{pmatrix}$
   (d) (i) $\frac{4}{5}\begin{pmatrix}1\\2\end{pmatrix}$ (ii) $\frac{3}{5}\begin{pmatrix}2\\-1\end{pmatrix}$
10. (a) < 1, 2 >; < −2, 1 > (b) < −2, 1 >; < 1, 2 >
11. (a) < 4, 10 >; < 5, −2 > (b) < 5, −2 >; < 4, 10>
12. $\frac{\sqrt{10}}{10}$ 13. $\frac{10\sqrt{101}}{101}$ 14. $\frac{\sqrt{2}}{2}$

Exercise 10.1

1. (a) $2\sqrt{5}$ m at $t = 6$ s (b) $2\sqrt{5}$ m at $t = 5$ s
   (c) $13\sqrt{5}$ m at $t = 6$ s (d) $10\sqrt{5}$ m at $t = 40$ s
2. (a) $\frac{30\sqrt{29}}{29}$ m at $t = 70/29$ s
   (b) 6 m at $t = 8/5$ s
   (c) $\frac{15\sqrt{37}}{37}$ m at $t = 90/37$ s
   (d) $\frac{70\sqrt{29}}{29}$ m at $t = 260/29$ s
3. $5\sqrt{2}$ km at 1500 hours
4. $9\sqrt{5}$ m after 7/5 sec when ball is at < −4, −13 >
5. 107.2 km at 10.48 am when A is at < 127.9, 136.0 > and B is at < 36.0, 80.9 > km
6. 46.4 km at 1327 hours
7. < 15, – 25 > or < 25, −15 > nautical miles
8. < 8, 10 > or < 200/13, −14/13 > knots

Exercise 10.2

1. $1000\sqrt{3}$ J 2. 2 500 J 3. $125\sqrt{2}$ kJ
4. (a) $250\sqrt{2}$ N (b) 200 N (c) 4 000 J
5. (a) 1 000 N (b) $500\sqrt{3} - 100$ N
   (c) $10(5\sqrt{3} - 1)$ kJ
6. (a) $200\sqrt{2}$ N (b) $200\sqrt{2} - 50$ N
   (c) (i) $500(4\sqrt{2} - 1)$ J (ii) $250(8 - \sqrt{2})$ J
7. (a) 500 N (b) 300 N
   (c) (i) 3 000 J (ii) 6 000 J
8. (a) 50 N (b) 40 N
   (c) (i) 200 J (ii) 400 J
9. (a) $T = 250\sqrt{3}$ N, F = 250 N
   (b) (i) 1 000 J (ii) 2 000 J
10. (a) $T = 500\sqrt{2}$ N, $F = 500\sqrt{2}$ N
    (b) (i) 4 000 J (ii) $2000\sqrt{2}$ J
11. (a) 56/5 (b) $\frac{56}{25}\begin{pmatrix}3\\4\end{pmatrix}$ (c) 56 J
12. (a) $(10\sqrt{17})/17$ (b) $\frac{10}{17}\begin{pmatrix}-1\\4\end{pmatrix}$ (c) 10 J
13. < 4, 3 > or < 5, 0 >
14. < 6, −8 > or < 414/61, −448/61 >
15. < 0, 20 > or < 16, 12 >

Exercise 11.1

Please refer to Solution Manual.

Exercise 11.2

Please refer to Solution Manual.

Exercise 12.1

Please refer to Solution Manual.

Exercise 12.2

Please refer to Solution Manual.

Exercise 13.1

1. (a) $15^\circ, 75^\circ, 195^\circ, 255^\circ$
   (c) $10^\circ, 160^\circ, 190^\circ, 340^\circ$
   (c) $20^\circ, 110^\circ, 200^\circ, 290^\circ$
   (d) $60^\circ, 180^\circ, 300^\circ$
2. (a) $7\pi/6$ (b) $\pi/2, 7\pi/6$
   (c) $\pi/3, 4\pi/3, 5\pi/6, 11\pi/6$ (d) $-\pi/2$
3. (a) $(-1)^n(-\pi/2) + n\pi$
   (b) $2n\pi \pm \pi/2$ or $2n\pi \pm \pi/3$

3. (c) $n\pi$ or $(-1)^n(-\pi/6) + n\pi$
   (d) $n\pi$ or $(\pi/4) + n\pi$
4. (a) $2n\pi \pm \pi/2$ or $(2n + 1)\pi$
   (b) $n\pi$ or $(-1)^n(\pi/3) + n\pi$
   (c) $-(\pi/4) + n\pi$ (d) $(\pm\pi/3) + n\pi$
5. (a) $71.6^\circ + 180^\circ n$ or $45^\circ + 180^\circ n$
   (b) $(-1)^n(30^\circ) + 180^\circ n$
   (c) $360^\circ n \pm 79.3^\circ$ (d) No solution.
6. (a) $(-1)^n(\pi/2) + n\pi$ or $(-1)^n(\pi/4) + n\pi/2$
   (b) $n\pi$ (c) $(\pm\pi/6) + n\pi$
   (d) $(-1)^n(\pm\pi/12) + n\pi/2$
7. (a) $(2n + 1)\pi/2$ or $(-1)^n(\sin^{-1}\frac{1}{\sqrt{3}}) + n\pi$
   (b) $\tan^{-1}(-3) + n\pi$ or $\tan^{-1}(\pm 2) + n\pi$
   (c) $\pi/4 + n\pi$ or $\tan^{-1}(2) + n\pi$
   (d) $(2n + 1)\pi$ or $(\pi/3) \pm 2n\pi$

Exercise 13.2

1. (a) $(2\sqrt{3})/3$ (b) $\sqrt{2}$
   (c) $-(\sqrt{3})/3$ (d) undefined
   (e) -1 (f) $-(2\sqrt{3})/3$
   (g) -2 (h) -1
2. (a) $135^\circ, 315^\circ$ (b) $30^\circ, 330^\circ$,
   (c) $105^\circ, 165^\circ, 285^\circ, 345^\circ$ (d) 60°
3. (a) $n\pi \pm \pi/3$ (b) $n\pi/2 - \pi/8$
   (c) $(-1)^n(2\pi/3) + 2n\pi$

Exercise 14.1

1. (a) 20° (b) $-\pi/3$ or $\pi/3$ (c) -60°
2.

	Mean Line	Amp	Period	Phase Shift	Max y	Min y
(a)	y=0	3	2π	0	3	-3
(b)	y=0	0.5	π	0	0.5	-0.5
(c)	y=0	2	360°	-40°	2	-2
(d)	y=-6	2	2	0	-4	-8
(e)	y=2	3	2	$-1/6$	5	-1
(f)	y=0	2π	1	1/24	2π	-2π
(g)	y=5	8	1	1/16	13	-3
(h)	y=0	5	2π	$-\tan^{-1}(4/3)$	5	-5

3.

	Mean Line	Period	Phase Shift	Asymp.
(a)	$y = 0$	$\pi/2$	0	$x = \pm\pi/4$
(b)	$y = 1$	π	0	$x = \pm\pi/2$
(c)	$y = 0$	90°	$+15^\circ$	$x = -30^\circ$ $x = 60^\circ$
(d)	$y = 5$	1	0	$x = \pm 1/2$
(e)	$y = 3$	1/2	1/12	$x = -1/6$ $x = 1/3$
(f)	$y = 0$	1	$-1/6$	$x = 1/3$ $x = -2/3$

4. $y = 2 + 3\sin(4x)$
5. $y = 7 - 3\sin[8(x + 30^\circ)]$
6. $y = 3 - 5\cos[2\pi(x - 1/6)]$
7. $y = -1 + 6\cos[\pi(x + 1/8)]$
8. $y = 2 + 5\tan(x - 45^\circ)$
9. $y = -5 + 0.5\tan[0.5(x + 30^\circ)]$
10. (a) $y = 4\cos(2x - 30^\circ) - 3$
    (b) $y = -2\cos(x/2 - 30^\circ) + 5$
11. (a) $y = 2\sin(x + \pi/6) + 1$
    (b) $y = -2\sin(2x - \pi/6) - 2$
12. (a) $y = \tan(x - \pi/6) + 4$
    (b) $y = -2\tan(2x) - 4$
13. (a) 19.4° C (b) 45° C at 3 pm
    (c) 6 hours 26 minutes
14. (a) 11.9 m (b) 8 m at 6.30 pm
    (c) 4 hours
15. 60%

Exercise 14.2

1. (a) $\pm 40^\circ$ (b) $-\pi/4$ (c) $\pm 60^\circ$
2. (a) (i) $y = 0$ (ii) 2π
   (iii) 0 (iv) $y \le -2, y \ge 2$
   (v) $x = \pm n\pi$
   (b) (i) $y = 0$ (ii) $\pi/3$ (iii) 0 (iv) $\mathbb{R}$
   (v) $x = \pm n\pi/3$
   (c) (i) $y = 1$ (ii) π
   (iii) 0 (iv) $y \le 0, y \ge 2$
   (v) $x = \pm(2n + 1)\pi/4$
   (d) (i) $y = 0$ (ii) 180°
   (iii) -15° (iv) $y \le -1, y \ge 1$
   (v) $x = -15^\circ \pm (2n + 1)45^\circ$
   (e) (i) $y = 0$ (ii) π
   (iii) $-\pi/12$ (iv) $y \le -1, y \ge 1$
   (v) $x = -\pi/12 \pm n\pi/2$
   (f) (i) $y = 2$ (ii) π (iii) $\pi/6$ (iv) $\mathbb{R}$
   (v) $x = \pi/6 \pm n\pi$
3. (a) $y = \operatorname{cosec}(x + 30^\circ)$ (b) $y = -\operatorname{cosec}(x/2)$
4. (a) $y = 2\sec x + 3$ (b) $y = -2\sec(2x)$
5. (a) $y = 2\cot x - 2$ (b) $y = -\cot(x - \pi/6)$

Exercise 15.1

1. (a) $10 - \sin\theta - 5\sin^2\theta$
   (b) $15\sin^2\theta + 1$
   (c) $1 + \sin\theta - \sin^2\theta$
   (d) $(3 - 2\sin^2\theta)/(1 - \sin^2\theta)$
2. (a) $4 - 2\cos\theta - 4\cos^2\theta$
   (b) $2 - \cos^2\theta$
   (c) $4 - 4\cos^2\theta + \cos^4\theta$
   (d) $1/[(\cos^2\theta)(1 - \cos^2\theta)]$

3. (a) $n\pi$ or $(-1)^n(-\pi/2) + n\pi$
   (b) $(-1)^n(\pi/6) + n\pi$ or $(-1)^n(\pi/2) + n\pi$
   (c) $2n\pi \pm \cos^{-1}(1/3)$
   (d) $n\pi + \pi/4$ or $n\pi + \tan^{-1}(7)$
4. (a) $(-1)^n(-\pi/6) + n\pi$ (b) $(-1)^n(\pi/2) + n\pi$
   (c) $(2n + 1)\pi/2$ (d) $2n\pi \pm 5\pi/6$
   (e) $-\pi/6 + n\pi$ or $\pi/3 + n\pi$
   (f) $n\pi + \pi/3$ or $n\pi - \pi/6$

Exercise 15.2

Please refer to Solution Manual.

Exercise 15.3

1. (a) $-56/65$ (b) $-63/65$ (c) $56/33$
2. (a) $-416/425$ (b) $87/425$ (c) $304/297$
3. (a) $(-3 + 4\sqrt{15})/20$ (b) $(4 - 3\sqrt{15})/20$
   (c) $(-192 + 25\sqrt{15})/119$
4. (a) $-176/185$ (b) $153/185$ (c) $-176/57$
5. (a) $-\pi/6 + n\pi$ (b) $\pi/4 + n\pi$
   (c) $\pi/3 + n\pi$ (d) $-\pi/4 + n\pi$
6. $-\pi/8 + n\pi/2$

8 to 10. Please refer to Solution Manual.

Exercise 15.4

1. (a) $(2n + 1)\pi/2$ or $(-1)^n(\pi/6) + n\pi$
   (b) $n\pi$ or $2n\pi \pm (\pi/6)$
   (c) $2n\pi$ or $2n\pi \pm (2\pi/3)$
   (d) $n\pi$ or $(-1)^n(-\pi/6) + n\pi$
   (e) $(-1)^n(\pi/6) + n\pi$
   (f) $(2n + 1)\pi$ or $2n\pi \pm (\pi/3)$
   (g) $\pm \pi/6 + n\pi$
   (h) $(2n + 1)\pi$ or $\pi/4 + n\pi$
2. (a) $2n\pi \pm (\pi/3) - (\pi/12)$
   (b) $(2n + 1)\pi/2$ or $(-1)^n(-\pi/3) + n\pi$
   (c) $n\pi - \pi/12$(d) $(-1)^n(\pi/6) + n\pi$
   (e) $n\pi$ or $n\pi + \pi/6$
   (f) $n\pi$ or $(-1)^n(\pi/3) + n\pi$
3. (a) $90^\circ, 270^\circ; 194.5^\circ, 345.5^\circ$
   (b) $0^\circ, 90^\circ, 180^\circ, 270^\circ, 360^\circ;$
   $30^\circ; 150^\circ, 210^\circ, 330^\circ$
   (c) $70.5^\circ, 289.5^\circ$ (d) 90°
   (e) $30^\circ, 150^\circ; 270^\circ$ (f) $0^\circ, 180^\circ, 360^\circ$
   (g) $161.6^\circ, 341.6^\circ$ (h) $26.6^\circ, 206.6^\circ$
4. (a) $-27.2^\circ, 152.8^\circ; 107.2^\circ, -72.8^\circ$
   (b) $-90^\circ, 90^\circ; 48.6^\circ, 131.4^\circ$
   (c) $-180^\circ, 0^\circ, 180^\circ; 63.4^\circ, -116.6^\circ$
   (d) $\pm 70.5^\circ$ (e) $\pm 40.9^\circ, \pm 139.1^\circ$
   (f) $\pm 180^\circ, 64.6^\circ$

5 to 8. Please refer to Solution Manual.

Exercise 15.5

Questions 1 & 2, please refer to Solution Manual.

3. (a) $1/4$ (b) $1/4$
   (c) $(1 + \sqrt{3})/4$ (d) $(\sqrt{3}/2 - 1)/2$
   (e) $\sqrt{2}/4$ (f) $\sqrt{2}/4$
   (g) $(1 - \sqrt{3}/2)/2$ (h) $1/4$
4. (a) $\sqrt{2}/8$ (b) $1/4$
   (c) $1/8$ (d) $-1/8$
5. (a) $-\sqrt{2}/2$ (b) $\sqrt{2}/2$
   (c) $\sqrt{2}(\sqrt{3} - 1)/2$
6. (a) $2n\pi/7$ or $(2n + 1)\pi$
   (b) $(2n + 1)\pi/8$ or $(2n + 1)\pi/2$
   (c) $2n\pi/3$ or $(2n + 1)\pi/5$
7. (a) $n\pi \pm \pi/3$ or $n\pi/3$
   (b) $(2n + 1)\pi/2$ or $(-1)^n(\pi/12) + n\pi/2$
8. $2n\pi + \pi/4$
9. $(-1)^n(-\pi/4) + n\pi/2 + \pi/8$
10. Please refer to Solution Manual.

Exercise 15.6

1. (a) $\sqrt{13}\sin(\theta + 56.31^\circ)$
   (b) $\sqrt{5}\sin(\theta - 63.43^\circ)$
   (c) $\sqrt{29}\sin(\theta + 68.2^\circ)$
   (d) $-\sqrt{17}\sin(\theta - 75.96^\circ)$
   (e) $-2\sqrt{2}\sin(\theta - 45^\circ)$
   (f) $-5\sin(\theta + 53.13^\circ)$
2. (a) $\sqrt{6}\cos(\theta - 35.26^\circ)$
   (b) $-2\cos(\theta + 30^\circ)$
   (c) $-\sqrt{6}\cos(\theta + 65.91^\circ)$
   (d) $(\sqrt{13}/6)\cos(\theta - 33.69^\circ)$
   (e) $(-\sqrt{29}/10)\cos(\theta + 68.2^\circ)$
   (f) $-\sqrt{21}\cos(\theta - 22.25^\circ)$
3. (a) $(-1)^n(\pi/6) + n\pi - \pi/4$
   (b) $2n\pi \pm 2\pi/3 - \pi/4$
   (c) $(-1)^n(\pi/3) + n\pi + \pi/3$
   (d) $2n\pi \pm 5\pi/6 + \pi/3$ (e) $2n\pi \pm \pi/4 - \pi/6$
   (f) $(-1)^n(-\pi/6) + n\pi/2 - \pi/8$
4. Max. for $y = \sqrt{2}$ when $x = \pi/4$
5. Min. for $y = -2$ when $(-1)^n(-\pi/2) + n\pi + \pi/6$
6. Max. for $A = 3 + 2\sqrt{3}$ when $t = 2n\pi + \pi/6$
   Min. for $A = 3 - 2\sqrt{3}$ when $t = (2n + 1)\pi + \pi/6$
7. Max. for $A = 23$ when $t = (2n + 1)\pi + 0.3948$
   Min. for $A = -3$ when $t = 2n\pi + 0.3948$
8. (a) $2n\pi + \pi/4$ or $(2n + 1)\pi/3 - \pi/12$
   (b) $2n\pi/3 + \pi/18$ or $2n\pi - \pi/6$

Exercise 16.1

1. (a) dim(**A**) = 2 × 1, dim(**B**) = 1 × 4
   dim(**C**) = 2 ×3, dim(**D**) = 5 × 2
   (b) a_{12} does not exist, $a_{21} = 2$
   (c) $p = 5$ (d) $p = 4$
2. (a) $\begin{pmatrix} -1 & -5 & -2 \end{pmatrix}$ (b) $\begin{pmatrix} 5 & -5 \\ -5 & 5 \end{pmatrix}$
   (c) Does not exist
   (d) $\begin{pmatrix} 2x+6 & -15 & 2-5y \\ -10 & 8 & -30 \\ -33 & 20 & 1 \end{pmatrix}$
3. (a) $\begin{pmatrix} 17 & 25 \\ -5 & 17 \end{pmatrix}$ (b) Does not exist
   (c) $\begin{pmatrix} 6 & 15 \\ -3 & 6 \end{pmatrix}$ (d) $\begin{pmatrix} 14 & 20-2c \\ -14 & 8 \end{pmatrix}$
   (e) Does not exist (f) $\begin{pmatrix} -3 & \frac{c-5}{2} \\ 3 & -\frac{3}{2} \end{pmatrix}$
4. (a) $a = 0$, $b = 0$, $c = -1$, $d = 5$
   (b) $a = 2$, $b = 6$, $c = 1$
   (c) $a = -2$, b = any real number
   (d) $a = -3$, $b = 0$, $c = 7$, $d = 4$
5. (a) $p = 3$, $q = 1$
   (b) $p = k$, $q = -k$ where k is any real number
6. (a) **B** – 2**A** (b) **A** (c) 2**A** – 3**C**
7. (a) $\begin{pmatrix} -3 \\ 2a-5 \\ 1 \end{pmatrix}$ (b) $\frac{1}{2}\begin{pmatrix} -3-b & 11 & 11 \end{pmatrix}$
   (c) $\begin{pmatrix} 2 & -9 \\ 5 & 8-c \end{pmatrix}$ (d) $\frac{1}{3}\begin{pmatrix} 6 & 0 & -6 \\ -1 & 8 & -1 \\ 2 & 1 & 7 \end{pmatrix}$

Exercise 16.2

1. $p = 3$; dim(**AB**) = 3 × 4
2. $p = 2$; dim(**BA**) = 3 × 4
3. (a) $m = n$ = any positive integer
   (b) $m \neq n$, m, n are positive integers
4. 2 × 2 5. 2 × 1
6. (a) $\begin{pmatrix} -4 \\ 26 \end{pmatrix}$ (b) Does not exist
   (c) $\begin{pmatrix} -2 & 2e \\ -5 & 5e \end{pmatrix}$ (d) $(-2 + 5e)$
   (e) $\begin{pmatrix} -4 & 2 \\ 6+4b & 9 \end{pmatrix}$ (f) $\begin{pmatrix} -7 & -4 \\ -2b+9 & 12 \end{pmatrix}$
   (g) $\begin{pmatrix} -18 & 2 \\ 16b+15 & 4b+33 \end{pmatrix}$
   (h) $\begin{pmatrix} -18 & 2 \\ 16b+15 & 4b+33 \end{pmatrix}$
   (i) $\begin{pmatrix} -12 & 0 \\ 4b+14 & 24 \end{pmatrix}$ (j) $\begin{pmatrix} 12 \\ 2b+28 \end{pmatrix}$
   (k) $\begin{pmatrix} 4 & 0 \\ 6 & 16 \end{pmatrix}$ (l) $\begin{pmatrix} -8 & 0 \\ 36 & 64 \end{pmatrix}$
7. (a) $\begin{pmatrix} -1 \\ d \\ d \end{pmatrix}$ (b) Does not exist
   (c) $\begin{pmatrix} 1 & 0 & -1 \\ -d & 0 & d \\ 2 & 0 & -2 \end{pmatrix}$ (d) (-1)
   (e) $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ (f) $\begin{pmatrix} 0 & 3 & 0 \\ 1 & 1 & 0 \\ 0 & 2 & 0 \end{pmatrix}$
   (g) $\begin{pmatrix} 2 & c & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ (h) $\begin{pmatrix} 2 & c & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$
   (i) $\begin{pmatrix} 0 & 1 & 3 \\ 1 & c & 1 \\ 1 & c & 1 \end{pmatrix}$ (j) $\begin{pmatrix} d-6 \\ cd-3 \\ d-3 \end{pmatrix}$
   (k) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ (l) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
8. $\begin{pmatrix} 1+k^2 \\ -k \end{pmatrix}$, $\begin{pmatrix} k \\ 1+2k \end{pmatrix}$, $\begin{pmatrix} k & 1+2k \\ -1 & -2 \end{pmatrix}$, $(3 \quad 7)$
   $\begin{pmatrix} 0 & -1 \\ 1 & k-2 \end{pmatrix}$, $\begin{pmatrix} 1 & 3 \\ k & 3k \end{pmatrix}$, $\begin{pmatrix} 1 & k-3 \end{pmatrix}$, $(1 + 3k)$,
9. $p = 1$, $q = -3$ 11. $k = 5$ 12. $p = -4$, $q = 5$
16. (a) $\mathbf{A}^2 - \mathbf{AB} - \mathbf{BA} + \mathbf{B}^2$
    (b) $\mathbf{A}^2 - 2\mathbf{AB} + \mathbf{B}^2$ or $\mathbf{A}^2 - 2\mathbf{BA} + \mathbf{B}^2$
17. (a) $\mathbf{A}^2 + \mathbf{AB} - \mathbf{BA} - \mathbf{B}^2$ (b) $\mathbf{A}^2 - \mathbf{B}^2$
18. (a) (**A** + **C**)**B** (b) **A**(**C** + **D**)
    (c) $\mathbf{A}^2(\mathbf{A} - \mathbf{I})$ (d) **A**(**B** + **I**)
    (e) (**A** + **I**)**B** (f) **A**(**B** + **D**)**C**
19. (a) $\begin{pmatrix} 6 \\ -8 \\ -6 \end{pmatrix}$ (b) $\begin{pmatrix} 8 \\ 4 \\ 16 \end{pmatrix}$ (c) $\begin{pmatrix} 2 \\ 12 \\ 22 \end{pmatrix}$ (d) $\begin{pmatrix} 13 \\ 12 \\ 35 \end{pmatrix}$
20. (a) $\begin{pmatrix} 0 \\ -12 \\ -4 \end{pmatrix}$ (b) $\begin{pmatrix} 0 \\ 12 \\ 4 \end{pmatrix}$ (c) $\begin{pmatrix} 9 \\ -27 \\ -9 \end{pmatrix}$ (d) $\begin{pmatrix} -9 \\ -9 \\ -3 \end{pmatrix}$
21. (a) $\begin{pmatrix} 2 & 0 \\ -3 & 2 \\ -5 & -1 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 2 \\ 2 & -3 \\ -1 & -5 \end{pmatrix}$
    (c) $\begin{pmatrix} 0 & 4 \\ 2 & -6 \\ -1 & -10 \end{pmatrix}$ (d) $\begin{pmatrix} 2 & 0 \\ -3 & -2 \\ -5 & 1 \end{pmatrix}$

22. (a) $\begin{pmatrix} 1 & 9 \\ 0 & 5 \\ -4 & 5 \end{pmatrix}$ (b) $\begin{pmatrix} -8 & 18 \\ -5 & 10 \\ -9 & 10 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & -9 & 10 \\ 0 & -5 & 5 \\ -4 & -5 & 1 \end{pmatrix}$

23. (a) $\begin{pmatrix} 2 & -3 \\ 3 & 4 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 2 & 2 \\ 1 & 3 & 3 \\ -4 & 0 & 0 \end{pmatrix}$

Exercise 16.3

1. If $\mathbf{A} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$, then $\mathbf{A}^n = \begin{pmatrix} a^n & 0 & 0 \\ 0 & b^n & 0 \\ 0 & 0 & c^n \end{pmatrix}$,

where n is a positive integer.

2. If $\mathbf{A} = \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}$, then, for positive integer n:

$\mathbf{A}^n = \begin{pmatrix} x^{\frac{n}{2}} y^{\frac{n}{2}} & 0 \\ 0 & x^{\frac{n}{2}} y^{\frac{n}{2}} \end{pmatrix}$ if n is even,

$\mathbf{A}^n = \begin{pmatrix} 0 & x^{\frac{n+1}{2}} y^{\frac{n-1}{2}} \\ x^{\frac{n-1}{2}} y^{\frac{n+1}{2}} & 0 \end{pmatrix}$ if n is odd.

3. If $\mathbf{A} = \begin{pmatrix} 0 & 0 & a \\ 0 & b & 0 \\ c & 0 & 0 \end{pmatrix}$, then, for positive integer n:

$\mathbf{A}^n = \begin{pmatrix} a^{\frac{n}{2}} c^{\frac{n}{2}} & 0 & 0 \\ 0 & b^n & 0 \\ 0 & 0 & a^{\frac{n}{2}} c^{\frac{n}{2}} \end{pmatrix}$ if n is even,

$\mathbf{A}^n = \begin{pmatrix} 0 & 0 & a^{\frac{n+1}{2}} c^{\frac{n-1}{2}} \\ 0 & b^n & 0 \\ a^{\frac{n-1}{2}} c^{\frac{n+1}{2}} & 0 & 0 \end{pmatrix}$ if n is odd.

4. Conjecture is false. Use counter-example.

5 to 8. Please refer to Solution Manual.

Exercise 16.4

2. (a) No (b) Yes (c) Yes (d) No

3. (a) $2\mathbf{I}$; $\mathbf{A}^{-1} = \frac{1}{2}\mathbf{B}$, $\mathbf{B}^{-1} = \frac{1}{2}\mathbf{A}$

(b) $6\mathbf{I}$; $\mathbf{A}^{-1} = \frac{1}{6}\mathbf{B}$, $\mathbf{B}^{-1} = \frac{1}{6}\mathbf{A}$

(c) $-3\mathbf{I}$; $\mathbf{A}^{-1} = \frac{-1}{3}\mathbf{B}$, $\mathbf{B}^{-1} = \frac{-1}{3}\mathbf{A}$

(d) $-22\mathbf{I}$,;$\mathbf{A}^{-1} = \frac{-1}{22}\mathbf{B}$, $\mathbf{B}^{-1} = \frac{-1}{22}\mathbf{A}$

4. (a) $\mathbf{I}$; $\mathbf{A}^{-1} = \mathbf{B}$, $\mathbf{B}^{-1} = \mathbf{A}$

(b) $\mathbf{I}$; $\mathbf{A}^{-1} = \mathbf{B}$, $\mathbf{B}^{-1} = \mathbf{A}$

(c) $16\mathbf{I}$; $\mathbf{A}^{-1} = \frac{1}{16}\mathbf{B}$, $\mathbf{B}^{-1} = \frac{1}{16}\mathbf{A}$

(d) $30\mathbf{I}$, $\mathbf{A}^{-1} = \frac{1}{30}\mathbf{B}$, $\mathbf{B}^{-1} = \frac{1}{30}\mathbf{A}$

5. $p = -4$ 6. $p = 15$ 7. $p = 9$

8. $\begin{pmatrix} p & 2 \\ 1 & 3 \end{pmatrix}$ 9. $\begin{pmatrix} x & y \\ -1 & 2 \end{pmatrix}$

10. $a = 1, b = -1$ 11. $a = 1, b = 2$

12. $10\mathbf{I}$; $\frac{1}{10}\begin{bmatrix} -12 & 0 & 8 & 1 \\ 6 & 0 & -4 & 2 \\ 2 & 0 & 2 & -1 \\ 0 & 10 & 0 & 0 \end{bmatrix}$

Exercise 16.5

1. (a) -27 (b) -2 (c) $-k$ (d) $12 + k^2$
2. (a) 12 (b) 10 (c) $-1, 0$ (d) $-1, 3$
3. (a) $\begin{pmatrix} 10 & 6 \\ 23 & 14 \end{pmatrix}$ (b) 2 (c) 2

(d) $|\mathbf{AB}| = |\mathbf{A}| \times |\mathbf{B}|$ (e) 32

4. (a) Does not exist (b) $\frac{-1}{2}\begin{pmatrix} 4 & -3 \\ -10 & 7 \end{pmatrix}$

(c) $\frac{1}{28}\begin{pmatrix} 12 & -8 \\ -10 & 9 \end{pmatrix}$ (d) $\frac{-1}{22}\begin{pmatrix} 2 & 4 \\ 3 & -5 \end{pmatrix}$

5. (a) 8 (b) 5 (c) -16 (d) $-2, 1$
6. (a) $k \neq -4, k \neq 4$ (b) All real values of k

(c) $k \neq -1, k \neq 2$ (d) All real values of k

7. $x = 1, y = 1$ 8. $x = 2, y = -1, k = 5$
9. (a) $\begin{pmatrix} 6 & 0 \\ 0 & 0 \end{pmatrix}$ (b) All real values of x

(c) $\begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}$ (d) $x \neq \pm 1$

10 to 14. Please refer to Solution Manual.

Exercise 16.6

1. (a) $\frac{1}{2}\mathbf{A}$ (b) $\frac{1}{3}[\mathbf{C} - \mathbf{B}]$ (c) $\mathbf{A}^{-1}\mathbf{B}$

(d) $\mathbf{A}^{-1}[\mathbf{C} - \mathbf{B}]$ (e) $(\mathbf{A} + \mathbf{B})^{-1}\mathbf{C}$

(f) $\mathbf{C}(\mathbf{A} + \mathbf{B})^{-1}$ (g) $(\mathbf{A} - \mathbf{B})^{-1}(\mathbf{D} - \mathbf{C})$

(h) $(\mathbf{A} + 2\mathbf{I})^{-1}\mathbf{B}$ (i) $(\mathbf{A} - \mathbf{B} - 3\mathbf{I})^{-1}\mathbf{D}$

2. (a) $\begin{pmatrix} 5 \\ -8 \end{pmatrix}$ (b) $\begin{pmatrix} 4 \\ -7 \end{pmatrix}$ (c) No solution

(d) $\frac{1}{64}\begin{pmatrix} 4 & -4 \\ -15 & 31 \end{pmatrix}$ (e) $\frac{1}{33}\begin{pmatrix} 1 & 2 \\ 11 & -11 \end{pmatrix}$

2. (f) $\frac{1}{5}\begin{pmatrix} 9 & 7 & 5 \\ 3 & -1 & 0 \\ 7 & 6 & 0 \end{pmatrix}$ (g) $\begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}$ (h) No solution

(i) $\frac{1}{8}\begin{pmatrix} -1 & 3 & 1 \\ 3 & -1 & -3 \\ 4 & -4 & 4 \end{pmatrix}$ (j) $\begin{pmatrix} 0 & -1 & 0 \\ -3 & -3 & -4 \\ -1 & -1 & -1 \end{pmatrix}$

3. (a) $\begin{pmatrix} 42 & -109 \\ 32 & -83 \end{pmatrix}$ (b) $\begin{pmatrix} 29 & -14 \\ -33 & 16 \end{pmatrix}$

4. (a) $\frac{1}{3}\begin{pmatrix} 4 & -11 & 4 \\ 4 & 7 & -2 \\ 0 & 9 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

5. $\mathbf{M}^{-1} = \frac{1}{3}\mathbf{M} - \frac{4}{3}\mathbf{I}$

6. $p = -2$, $q = 1$; $\mathbf{A}^{-1} = -\mathbf{A} + 2\mathbf{I}$

7 to 11. Please refer to Solution Manual.

12. (b) $\mathbf{D}^2 = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$, $\mathbf{D}^3 = \begin{pmatrix} 8 & 0 \\ 0 & 1 \end{pmatrix}$,

$\mathbf{D}^4 = \begin{pmatrix} 16 & 0 \\ 0 & 1 \end{pmatrix}$

$\mathbf{D}^n = \begin{pmatrix} 2^n & 0 \\ 0 & 1 \end{pmatrix}$ for n = 1, 2, 3 ...

D and $\mathbf{D}^n$ are diagonal matrices.
The power of a diagonal matrix is obtained by raising the elements in the matrix to the required power.

(c) $\mathbf{M}^n = \mathbf{T}\mathbf{D}^n\mathbf{T}^{-1}$;

$\mathbf{M}^{10} = \begin{pmatrix} -15344 & 20460 \\ -12276 & 16369 \end{pmatrix}$

13 to 16: Please refer to Solution Manual.

Exercise 17.1

1. (a) $x = 1, y = 3$ (b) $x = -4, y = 1$
   (c) $x = -1/2, y = -2$ (c) $x = 50, y = -40$
   (e) $x = 4, y = 6$ (f) $x = \pm 2, y = \pm 1$
2. (a) No solution.
   (b) $x = k, y = (4 - 2k)/3, k \in \mathbf{R}$
   (c) $x = 2, y = -1/3$ (d) No solution.
3. (a) $\begin{pmatrix} 19 & 0 \\ 2 & 5 \end{pmatrix}$ (b) $x = 4, y = -5$
4. (a) $\begin{pmatrix} 0 & 107 \\ 5 & 9 \end{pmatrix}$ (b) $x = -3, y = 7$
5. (a) $\begin{pmatrix} 2 & 1 \\ 11 & 0 \end{pmatrix}$ (b) $x = 1/10, y = 1/8$
6. (a) $x = 0.4, y = -0.5$ (b) $x = 100, y = 110$

Exercise 18.1

1. (a) Meaningless
   (b) Pack X: \$8.10; Pack Y: \$15.00
   (c) Pack X: \$8.10; Pack Y: \$15.00
2. (a) Meaningless
   (b) Meal P: \$30.60; Meal Q: \$49.40
   (c) Meal P: \$30.60; Meal Q: \$49.40
3. (a)

	Scott	Marcia
Scrap Bk	6	8
Pencil	4	2

   (b) Scott: \$9.80; Marcia: \$11.40
4. (a)

	Q-M	A
2012	800	1200
2013	1000	1500

   (b)

	% Profit
Q-M	0.15
A	0.18

   (c)

	Profit \$m
2012	336
2013	420

   (d)

Profit (\$m)	2012	2013
Q-M	120	150
A	216	270

5. (a) Model-100: 2750; Model-200: 4000
   (b) Model-100: 12 100; Model-200: 10 500
   (c) \$155 000
6. (a) P: 60; Q: 70 (b) \$15 400
   (c) A: 130; B: 100
7. (a)

	H	F
Cost (\$)	900	1230

   (b) AFL Final: \$150; Concert \$120
   (c) \$390
8. (a) A: 8; B: 31
   (b) A: \$16.68; B: \$22.82
9. (a) A: 130g; B: 50g
   (b) (i) P: 20; Q: 25 (ii) P: 4; Q: 5
10. n = 2, 4 or 6

Exercise 19.1

1. (a) O′(–1, 0), A′(–3, 0), B′(0, 2), C′(1, 1)
2. (a) O′(3, 0), A′(9, 0), B′(6, 6), C′(0, 3)
   (b) O′(2, 4), A′(6, 12), B′(4, 12), C′(0, 2)
   (c) O′(0, 3), A′(0, 9), B′(4, 6), C′(2, 0)
3. (a) O′($1/2$, $\sqrt{3}/2$), A′($3/2$, $3\sqrt{3}/2$), B′($1 - \sqrt{3}$, $1 + \sqrt{3}$), C′($-\sqrt{3}/2$, $1/2$)
   (b) O′($\sqrt{3}/2$, $1/2$), A′($3\sqrt{3}/2$, $3/2$), B′($2\sqrt{3}$, 0), C′($\sqrt{3}/2$, $-1/2$)
   (c) O′($\sqrt{2}/2$, $\sqrt{2}/2$), A′($3\sqrt{2}/2$, $3\sqrt{2}/2$), B′($2\sqrt{2}$, 0), C′($\sqrt{2}/2$, $-\sqrt{2}/2$)

4. (a) $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$ (d) $\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$

(e) $\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$ (f) $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$

(g) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ (h) $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

5. (a) reflection about the *y*-axis; $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

(b) reflection about $y = x$; $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(c) vertical dilation of factor $\frac{1}{5}$; $\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{5} \end{pmatrix}$

(d) horizontal dilation of factor $\frac{1}{4}$; $\begin{pmatrix} \frac{1}{4} & 0 \\ 0 & 1 \end{pmatrix}$

(e) enlargement about origin factor $\frac{1}{4}$; $\begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$

(f) enlargement about origin factor 2; $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

(g) anticlockwise rotation 90° about O; $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

5. (h) clockwise rotation 180° about O; $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

In each case the matrices are inverses of the matrices in Question 4.

6. (a) $\begin{pmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$, $\begin{pmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$

(b) $\begin{pmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{2} & \frac{-1}{2} \end{pmatrix}$, $\begin{pmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{2} & \frac{-1}{2} \end{pmatrix}$

7. (a) $\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$, $\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$

(b) $\begin{pmatrix} \frac{-\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{-\sqrt{2}}{2} & \frac{-\sqrt{2}}{2} \end{pmatrix}$, $\begin{pmatrix} \frac{-\sqrt{2}}{2} & \frac{-\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{-\sqrt{2}}{2} \end{pmatrix}$

8. (a) Identity transformation or rotation 360° about O.
(b) Reflection about the *y*-axis.
(c) Horizontal dilation factor 3.

8. (d) Vertical dilation of factor 4.
(e) Horizontal dilation of factor 5.
(f) Enlargement about O of factor 5.
(g) Reflection about $y = -x$
(h) enlargement about O of factor 4

9. (a) $\pi/3$ clockwise rotation about origin
(b) $\pi/6$ clockwise rotation about origin
(c) $3\pi/4$ anti-clockwise rotation about origin
(d) Reflection about the line $y = x \tan \pi/6$

10. (a) (2, 1) (b) (3, −2) (c) (−1, 3)

11. (a) (2, 2) (b) (4, −1) (c) (−3, 2)

12. (a) $\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$ (b) $\frac{1}{3}\begin{pmatrix} 3 & -2 \\ 0 & 1 \end{pmatrix}$

13. (a) $\begin{pmatrix} -4 & 0 \\ 0 & 4 \end{pmatrix}$ (b) $\frac{-1}{16}\begin{pmatrix} 4 & 0 \\ 0 & -4 \end{pmatrix}$

14. (a) $\begin{pmatrix} 2 & 0 \\ 4 & 3 \end{pmatrix}$ (b) $\frac{1}{6}\begin{pmatrix} 3 & 0 \\ -4 & 2 \end{pmatrix}$

15 to 20. Please refer to Solution Manual.

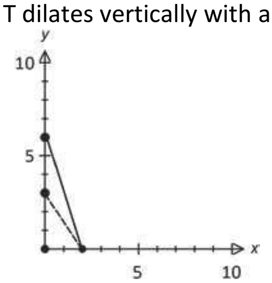
21. $a = 3, b = 1$ 22. $y = (\sqrt{2})/2$

23. (a) $x = 3y$
(b) Transformation matrix is singular.

24. (a) The point (1, 1).
(b) Transformation matrix is singular.

Exercise 19.2

1. (a) O´(0, 0), A´(2, 0), B´(0, 6)
(b) T dilates vertically with a factor of 2.



(c) $\frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ (d) 40 cm^2

(e) A´(−2, 0), B´(0, −6) (f) $\begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$

2. (a) (i) $\begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}$ (ii) $\begin{pmatrix} -1 & 0 \\ -2 & 1 \end{pmatrix}$

(iii) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (iv) $\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$ (v) $\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$

(b) (i) (0, 1) (ii) (0, 1) (iii) (0, 1)
(iv) (0, 1) (v) (0, 1)

(c) (i) $\begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}$ (ii) $\begin{pmatrix} -1 & 0 \\ -2 & 1 \end{pmatrix}$

(iii) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (iv) $\begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}$ (v) $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$

3. (a) (i) $\begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$ (ii) $\begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$

3. (a) (iii) $\begin{pmatrix} -4 & 0 \\ 0 & -1 \end{pmatrix}$

(b) (i) $\frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$ (ii) $\frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$

(iii) $\frac{1}{4}\begin{pmatrix} -1 & 0 \\ 0 & -4 \end{pmatrix}$

(c) (i) (−4, −4) (ii) (−4, −4) (iii) (−2, −4)

4. (a) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$: An anti-clockwise rotation of 90° about the origin.

(b) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$: A clockwise rotation of 90° about the origin.

5. (a) $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$: An anti-clockwise rotation of 180° about the origin.

(b) $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$: An anti-clockwise rotation of 180° about the origin.

6. (a) (i) 12 (ii) 48 (iii) 48 (iv) 192

(b) (i) 4 (ii) 1 (iii) 1 (iv) $\frac{1}{4}$

7. $x = \pm 3$, $y = 0$ 8. $x = 0$, $y = \pm 3$

9. (a) $\mathbf{R} = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$, $\mathbf{F} = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$ $\mathbf{T} = \mathbf{RF}$

(b) 40 square units (c) 2 square units

10. (a) An anti-clockwise rotation of 180° about O: A reflection about the y-axis followed by a reflection about the x-axis.

(b) An anti-clockwise rotation of 90° about O: A reflection about the y-axis followed by a reflection about the line $y = -x$.

(c) An anti-clockwise rotation of 60° about O: A reflection about the line $y = x\sqrt{3}$ followed by a reflection about the line $y = x\sqrt{3}$.

11. (a) $\begin{pmatrix} -4 & 0 \\ 0 & 1 \end{pmatrix}$ (b) (−4, 4) (c) $\frac{-1}{4}\begin{pmatrix} 1 & 0 \\ 0 & -4 \end{pmatrix}$

12. (a) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(b) Reflection about the y-axis is a self-inverse transformation.

(c) (4, 5) (d) (5, 6)

13. (a) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(b) **T** is a self-inverse transformation

(c) (10, 0) (d) (0, −10)

14. (a) (i) (8, 2) (ii) (4, 1) (iii) (12, 3)
(iv) (8, 2) (v) (44, 11)

(b) $y = x/4$ (c) Does not exist

(d) Maps points on the x-y plane to points on the line $y = x/4$.

15. (a) (i) 2.5 square units (ii) 10 square units

(b) (i) Orientation reversed.
(ii) Orientation preserved.

16. (a) (i) 20 (ii) 20 (iii) 0
(iv) 20 (v) 40 (vi) 0

(b) (i) Orientation reversed.
(ii) Orientation reversed.
(iii) Not applicable.
(iv) Orientation preserved.
(v) Orientation reversed.
(vi) Not applicable.

14. (c) Reverse transformation exists if transformation matrix is non-singular. Hence, reverse transformations exists for all cases except (iii) and (vi).

17 to 21. Please refer to Solution Manual.

Exercise 19.3

1. (a) $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}$

Rotation 180° about origin.

(b) $\begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ 5y \end{pmatrix}$

Dilation along the x-axis factor 2 and dilation along the y-axis factor 5.

(c) $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2y \\ 3x \end{pmatrix}$

Rotation 90° anti-clockwise about the origin followed by dilations of factor 2 along the x-axis and factor 3 along the y-axis.

OR

$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2y \\ 3x \end{pmatrix}$

Dilations of factor 3 along the x-axis and factor 2 along the y-axis followed by a rotation 90° anti-clockwise about the origin.

2. (a) **T** is a reflection about the line $y = x$.

(b) **T** is a reflection about the line with equation $y = x\tan\left(\frac{5\pi}{12}\right)$

(c) **T** is a rotation $\frac{5\pi}{6}$ radians anti-clockwise about the origin.

3. (a) Reflection about the line $y = x$ followed by dilations of factor 2 along the x-axis and factor 5 along the y-axis.

   OR

   Dilations of factor 5 along the x-axis and factor 2 along the y-axis followed by a reflection about the line $y = x$.

   (b) **T** is a reflection about the line with equation $y = x\tan\left(\frac{7\pi}{12}\right)$

   (c) Rotation of $\frac{7\pi}{6}$ radians anti-clockwise about the origin.

4. (a) **T**: $\begin{pmatrix}x\\y\end{pmatrix} \to \begin{pmatrix}x'\\y'\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}3\\-7\end{pmatrix}$

   (b) **T**: $\begin{pmatrix}x\\y\end{pmatrix} \to \begin{pmatrix}x'\\y'\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}-5\\2\end{pmatrix}$

   (c) **T**: $\begin{pmatrix}x\\y\end{pmatrix} \to \begin{pmatrix}x'\\y'\end{pmatrix} = \begin{pmatrix}\frac{\sqrt{3}}{2} & \frac{1}{2}\\ \frac{-1}{2} & \frac{\sqrt{3}}{2}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}-5\\2\end{pmatrix}$

   (d) **T**: $\begin{pmatrix}x\\y\end{pmatrix} \to \begin{pmatrix}x'\\y'\end{pmatrix} = \begin{pmatrix}\frac{\sqrt{3}}{2} & \frac{1}{2}\\ \frac{-1}{2} & \frac{\sqrt{3}}{2}\end{pmatrix}\begin{pmatrix}x-5\\y+2\end{pmatrix}$

5. (a) Translations 1 unit along the negative x-axis and 2 units along the positive y-axis

   (b) Rotation 180° about the origin followed by translations of 1 unit in the direction of the positive x-axis and 2 units along the negative y-axis.

   OR

   Translations of 1 unit in the direction of the negative x-axis and 2 units along the positive y-axis followed by a rotation 180° about the origin.

   (c) Rotation 90° clockwise about the origin followed by a translation of 3 units in the direction of the positive x-axis.

   OR

   A translation of 3 units in the direction of the positive y-axis followed by a rotation 90° clockwise about the origin.

   (d) A reflection about the line with equation $y = x\sqrt{3}$ followed by translations of 1 unit along the positive x-axis and 1 unit along the negative y-axis.

6. (a) $\begin{pmatrix}\sqrt{3}-3\\ \sqrt{3}+5\end{pmatrix}$ (b) $\begin{pmatrix}3\sqrt{3}-2\\ 2\sqrt{3}+3\end{pmatrix}$

   (c) $\begin{pmatrix}3\sqrt{3}-4\\ 2\sqrt{3}+7\end{pmatrix}$ (d) $\begin{pmatrix}3\sqrt{3}\\ 2\sqrt{3}+1\end{pmatrix}$

7. (a) $\begin{pmatrix}-\sqrt{3}\\ 1\end{pmatrix}$ (b) $\begin{pmatrix}-\sqrt{3}-2\\ 5\end{pmatrix}$

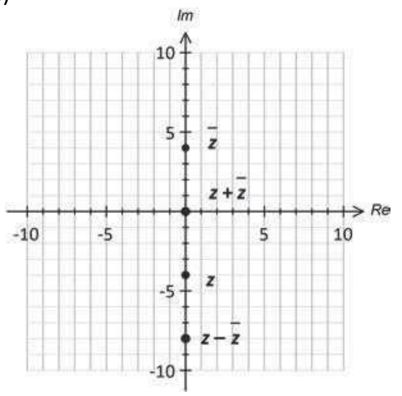
   (c) $\begin{pmatrix}\sqrt{3}-3\\ -\sqrt{3}+3\end{pmatrix}$ (d) $\begin{pmatrix}-2\sqrt{3}+3\\ \sqrt{3}\end{pmatrix}$

Exercise 20.1

1. (a) $5i$ (b) $9i$ (c) $5 - 10i$
   (d) $-2 + 12i$ (e) $|a|i$ (f) $4 + |b|i$
   (g) $a - 5|a|i$ (h) $a + 2\sqrt{2}|b|i$
2. (a) 2, −3 (b) 4, 4 (c) 0, 0
   (d) 0, −19 (e) 3, −4 (f) 6, 5
   (g) $4, 3|a|$ (h) $|a|, 2$
3. (a) $7 + 3i$ (b) $7 + 5i$
4. (a) $a = 3, b = -2$ (b) $a = 4, b = 2$
   (c) $a = 4, b = 2$
   (d) $a = 2, b = 3$; $a = -2, b = -3$
5. (a) $5 + i$ (b) $-1 - i$ (c) $3 - 3i$
   (d) $3 + 12i$ (e) $6 + (a + 3)i$
   (f) $(a + 2) - ai$
6. (a) $4 + 7i$ (b) $17 - 6i$ (c) $-7 + 17i$
   (d) $11 - 2i$ (e) 25 (f) $5 - 12i$
7. (a) $2 - 2i$ (b) $(-1/5) - (3/5)i$
   (c) $(7/13) + (4/13)i$ (d) $(8/17) - (15/17)i$
   (e) $(-2/5) + (36/5)i$ (f) 6
   (g) $(5a + 12b)/169 + [(15b-12a)/169]\,i$
   (h) $(2 + 3a/50) + (1 - 2a/25)i$
8. (a) $12-5i$ (b) 24
   (c) $(12/169) - (5/169)i$
   (d) $(2040/169) - (850/169)i$
9. (a) 36 (b) $(-323/625) + (36/625)i$
   (c) $17 + 6i$
   (d) $(-1/25) + (18/25)i$
   (e) $(-3/10) + (9/10)i$
   (f) $(67/25) - (69/25)i$
10. (a) $-2 - 2i$ (b) $-7 - 24i$
    (c) $(2/125) + (11/125)i$
    (d) $[-527 + 336i]/390625$
11. (a) $[a^2 + 10a - 171] + 28(5 + a)i$
    (b) $(5a + 24) + (12a - 10)i$
    (c) $(5a-24)/[169(a^2+4)] - \{2(6a+5)/[129(a^2+4)]\}i$
    (d) $(12a-10)/(4 + a^2) - [(24 + 5a)/(4 + a^2)]i$
12. (a) 0 (b) $3 + 3i$
    (c) $718/625 - (1824/625)i$
    (d) $(2 + a - a^2) + (2a - 4)i$
13. (a) $(31/8)i$ (b) $-4 + 6i$
    (c) $-34 - 10i$ (d) $4i$

Exercise 20.2

1. (a)



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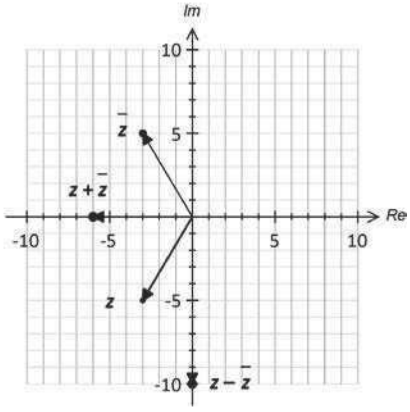
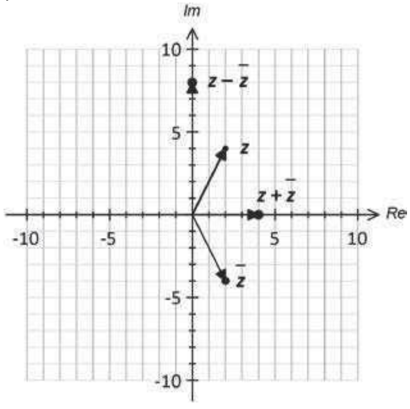
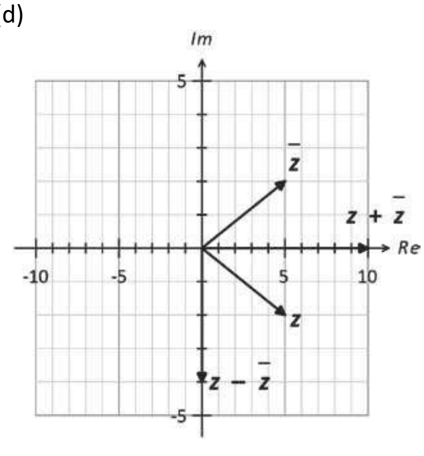


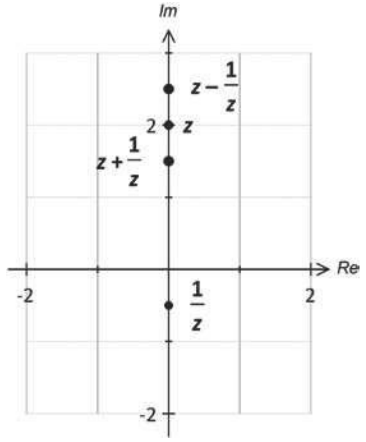
(c)



(d)



2. (a)



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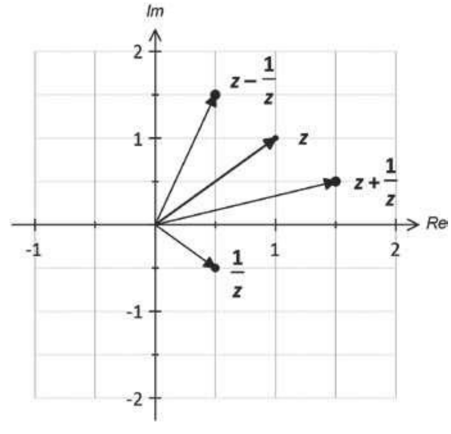

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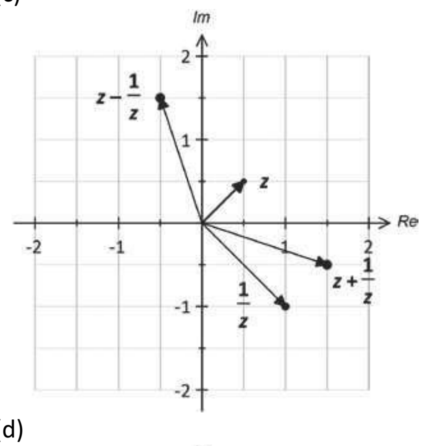
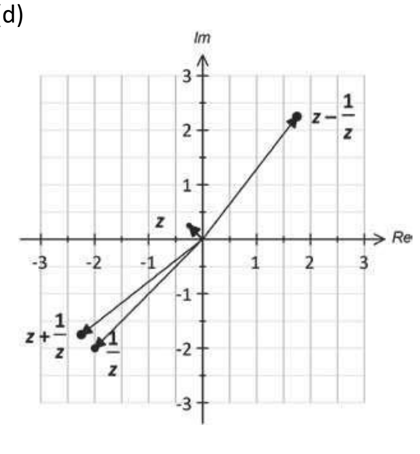
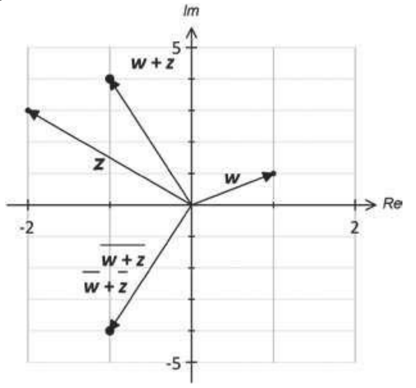


(d)



3. (a)



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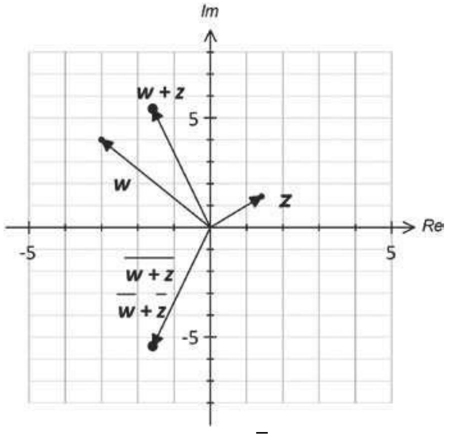

4. ***a*** = ***u*** + ***v***, ***b*** = ***u*** – , ***c*** = $\bar{v}$ – ***v*** , ***d*** = ***v*** – ***u***

Exercise 20.3

1. (a) $\pm 4i$ (b) $\pm 7i$ (c) $-2 \pm 2i$
   (d) $-1/2 \pm (1/2)i$ (e) $-1 \pm 2i$
   (f) $-1 \pm 4i$ (g) $3 \pm 6i$ (h) $2 \pm 5i$
   (i) $-1 \pm i$ (j) $1 \pm i$
2. (a) $(z + 5i)(z - 5i)$ (b) $(z + 10i)(z - 10i)$
   (c) $(z - 1 - 3i)(z - 1 + 3i)$
   (d) $(z - 1 + 7i)(z - 1 - 7i)$
3. (a) $1, -1 \pm \sqrt{3}\, i$ (b) $3, \pm 3i$
   (c) $-1, -1/2 \pm (\sqrt{3})/2\, i$
   (d) $\pm i, \pm 5\, i$

Exercise 21.1

Please refer to Solution Manual.

Exercise 21.2

Please refer to Solution Manual.

Exercise 21.3

Please refer to Solution Manual.

Exercise 21.4

Please refer to Solution Manual.

Exercise 21.5

Please refer to Solution Manual.

Exercise 21.6

Please refer to Solution Manual.

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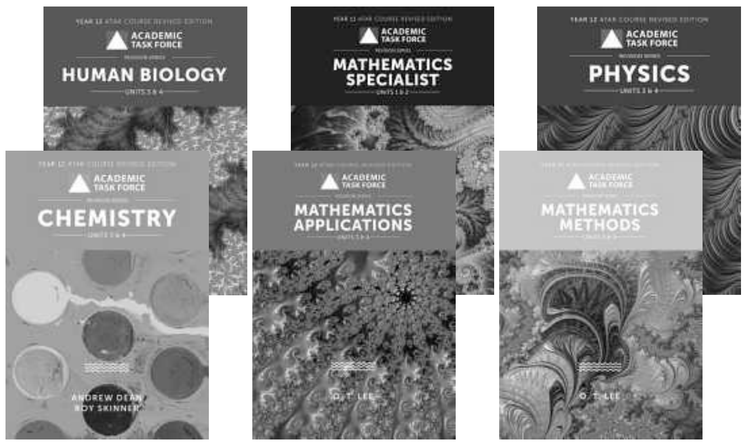

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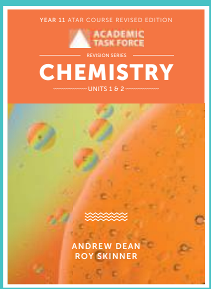

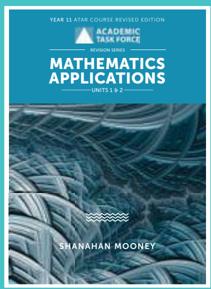

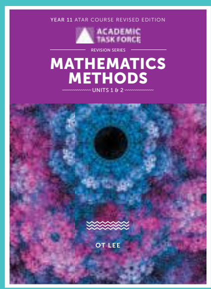

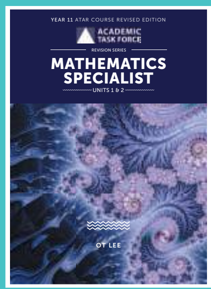

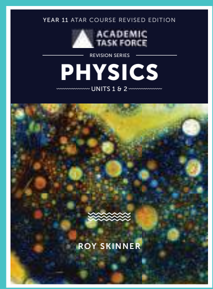

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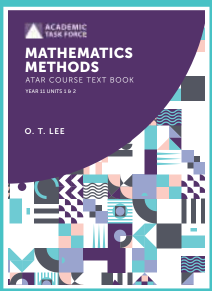

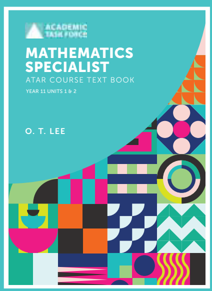

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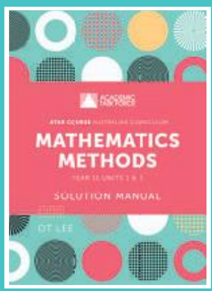

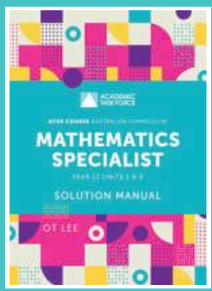

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