NEW SENIOR MATHEMATICS

FOR YEARS 11 & 12
SECOND EDITION

J. B. FITZPATRICK
BOB AUS

ALWAYS LEARNING
PEARSON

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FOR YEARS 11 & 12
SECOND EDITION

J. B. Fitzpatrick
Bob Aus

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INTRODUCTION AND DEDICATION

It is interesting to wonder whether J. B. Fitzpatrick ('Bernie') realised in 1983 just how popular his book *New Senior Mathematics* would be. That first edition of *New Senior Mathematics* was to remain in print for almost 30 years. It has stood the test of time thanks to the quality, rigour and variety of its questions, its accuracy and its high mathematical standards.

As Fitzpatrick wrote in 1983: 'Mathematics, like many other things, is best learnt by doing. A student begins to appreciate the power of mathematics when he or she has achieved a mastery of basic techniques, not after reading lengthy explanations … The emphasis throughout the book is on the understanding of mathematical concepts' (Introduction, *New Senior Mathematics* 1984).

Fitzpatrick was only willing to entrust the writing of a second edition to someone who would understand what makes this series so important. Throughout the initial revision process for this book, Fitzpatrick collaborated with respected mathematics author and teacher Bob Aus.

In his revision, Bob Aus has updated and improved aspects of *New Senior Mathematics* to make it better suited to the modern-day classroom. While maintaining the challenging and demanding questions, he has improved the question grading to better scaffold students' skill development. New revision sections at the end of each chapter and five sample exams also provide more opportunities for students to consolidate skills.

We hope all loyal users of *New Senior Mathematics*, or 'Fitzy's book', as it is fondly known, enjoy all the new improvements.

Vebica Evans

Pearson

J. B. Fitzpatrick passed away in 2008 after handing in his final review. Fitzpatrick was a respected author, teacher and figurehead of mathematics education.

NEW SENIOR MATHEMATICS

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New Senior Mathematics for Years 11 & 12 is a new edition of the well-known mathematics series for New South Wales. We've completely updated the series for today's classrooms, continuing the much-loved approach to deliver mathematical rigour with challenging student questions.

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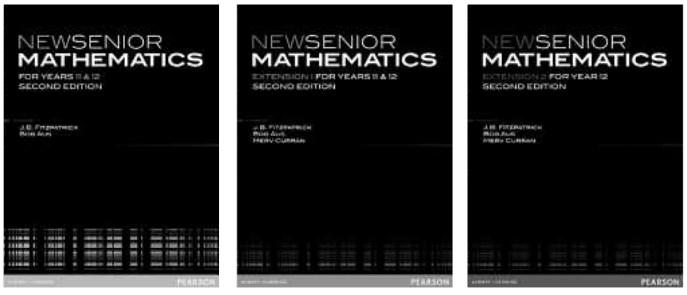

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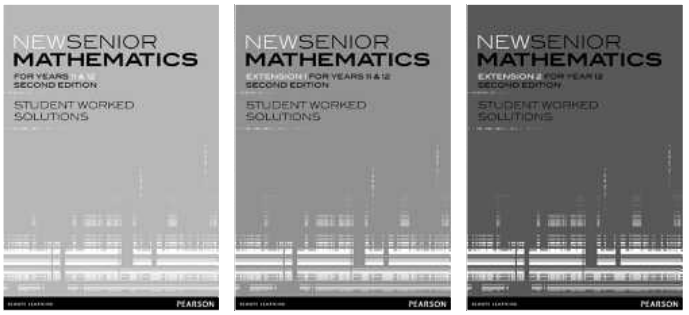
The *New Senior Mathematics for Years 11 & 12 Student Worked Solutions* contain the fully worked solutions for every second question in *New Senior Mathematics for Years 11 & 12*.



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HSC

CHAPTER 1
ARITHMETIC AND SURDS

Outcomes covered:

P1 demonstrates confidence in using mathematics to obtain realistic solutions to problems

P3 performs routine arithmetic and algebraic manipulation involving surds, simple rational expressions and trigonometric identities

P4 chooses and applies appropriate arithmetic, algebraic, graphical, trigonometric and geometric techniques

1.1 REVIEW OF BASICS

Converting between fractions, decimals and percentages without using a calculator is an important skill. A calculator cannot always simplify algebraic fractions, so you are expected to be proficient at this skill.

EXERCISE 1.1 REVIEW OF BASICS

1 Express each fraction as **(i)** a decimal and **(ii)** a percentage.

(a) $\frac{3}{5}$ **(b)** $\frac{7}{8}$ **(c)** $\frac{18}{25}$ **(d)** $\frac{1}{40}$

(e) $\frac{5}{16}$ **(f)** $\frac{13}{20}$ **(g)** $\frac{18}{5}$ **(h)** $\frac{17}{4}$

2 Express each decimal as **(i)** a common fraction in simplest form and **(ii)** a percentage.

(a) 0.25 **(b)** 1.85 **(c)** 0.375 **(d)** 0.23

(e) 1.75 **(f)** 0.025 **(g)** 0.6125 **(h)** 1.82

3 Express each percentage as **(i)** a decimal, **(ii)** a common fraction in simplest form.

(a) 40% **(b)** 1% **(c)** 10% **(d)** 32.5%

(e) 65% **(f)** $7\frac{1}{2}\%$ **(g)** 116% **(h)** $112\frac{1}{2}\%$

4 Select the incorrect statement:

A $\frac{3}{20} = 0.15 = 15\%$ **B** $\frac{3}{10} = 0.03 = 3\%$

C $\frac{15}{16} = 0.9375 = 93.75\%$ **D** $\frac{3}{4} = 0.75 = 75\%$

1.2 REPEATING DECIMALS

Rational numbers

All common fractions $\frac{a}{b}$, where a and b are integers and $b \neq 0$, are called **rational numbers**. They can be expressed as decimals by dividing the numerator by the denominator. For example:

$\frac{3}{4} = 0.75$ $\frac{1}{8} = 0.125$ $\frac{3}{50} = 0.06$ $\frac{1}{3} = 0.3333\ldots$ $\frac{1}{6} = 0.1666\ldots$ $\frac{5}{11} = 0.4545\ldots$

The first three decimals terminate. The last three decimals have an infinite number of digits to the right of the decimal point and one or more of the digits repeat according to a pattern.

0.3333… (3 is repeated) 0.1666… (6 is repeated) 0.4545… (45 is repeated)

These decimals are called repeating (or 'recurring') decimals and can be represented by placing dots over the repeated digits. For example:

$0.3333\ldots = 0.\dot{3}$ $0.1666\ldots = 0.1\dot{6}$ $0.4545\ldots = 0.\dot{4}\dot{5}$

Where more than two digits are repeated, the dot is placed over only the first and last repeating digits:

$\frac{1}{7} = 0.142857142857\ldots = 0.\dot{1}42857\dot{7}$

All decimals that either terminate or repeat are rational numbers.

For common fractions that become terminating decimals, the only factors of the denominator will be 2 and 5. Common fractions with any other factors in the denominator will repeat.

Irrational numbers

Some decimals neither terminate nor repeat. For example:

$\sqrt{2} = 1.414\,213\,562\ldots \qquad \pi = 3.141\,592\,653\ldots$

These numbers are not rational as they cannot be written in the form $\frac{a}{b}$, where a and b are integers, $b \neq 0$.

They are called **irrational numbers**. Their decimal expressions do not terminate or have a pattern that repeats.

Example 1

Express each decimal as a common fraction.

(a) $0.\dot{2}$ **(b)** $0.4\dot{6}$ **(c)** $2.\dot{5}\dot{3}$ **(d)** $4.2\dot{5}\dot{3}$

Solution

(a) Multiply both sides by 10.
Subtract (1) from (2) to eliminate the repeating decimals.

$0.\dot{2} = 0.2222\ldots$ (1)

$10 \times 0.\dot{2} = 2.2222\ldots$ (2)

$9 \times 0.\dot{2} = 2.2222\ldots - 0.2222\ldots$ (2) − (1)

$9 \times 0.\dot{2} = 2$

$0.\dot{2} = \frac{2}{9}$

(b) Multiply both sides by 10.
Subtract (1) from (2) to eliminate the repeating decimals.

$0.4\dot{6} = 0.46666\ldots$ (1)

$10 \times 0.4\dot{6} = 4.6666\ldots$ (2)

$9 \times 0.4\dot{6} = 4.6666\ldots - 0.46666\ldots$ (2) − (1)

$9 \times 0.4\dot{6} = 4.2$

$0.4\dot{6} = \frac{4.2}{9} = \frac{42}{90} = \frac{7}{15}$

(c) Multiply both sides by 100.
Subtract (1) from (2) to eliminate the repeating decimals.

$2.\dot{5}\dot{3} = 2.535353\ldots$ (1)

$100 \times 2.\dot{5}\dot{3} = 253.5353\ldots$ (2)

$99 \times 2.\dot{5}\dot{3} = 253.\dot{5}\dot{3}\ldots - 2.\dot{5}\dot{3}\ldots$ (2) − (1)

$99 \times 2.\dot{5}\dot{3} = 251$

$2.\dot{5}\dot{3} = \frac{251}{99}$

(d) Multiply both sides by 100.
Subtract (1) from (2) to eliminate the repeating decimals.

$4.2\dot{5}\dot{3} = 4.253535\ldots$ (1)

$100 \times 4.2\dot{5}\dot{3} = 425.3\dot{5}\dot{3}$ (2)

$99 \times 4.2\dot{5}\dot{3} = 425.3\dot{5}\dot{3} - 4.2\dot{5}\dot{3}$ (2) − (1)

$99 \times 4.2\dot{5}\dot{3} = 425.3 - 4.2$

$99 \times 4.2\dot{5}\dot{3} = 421.1$

$4.2\dot{5}\dot{3} = \frac{421.1}{99} = 4\frac{25.1}{99} = 4\frac{251}{990}$

OR consider the decimal part alone. Add 4 again at the end:

$$0.2\dot{5}\dot{3} = 0.253535\ldots \quad (1)$$
$$100 \times 0.2\dot{5}\dot{3} = 25.3\dot{5}\dot{3} \quad (2)$$
$$99 \times 0.2\dot{5}\dot{3} = 25.3\dot{5}\dot{3} - 0.2\dot{5}\dot{3} \quad (2) - (1)$$
$$99 \times 0.2\dot{5}\dot{3} = 25.1$$
$$0.2\dot{5}\dot{3} = \frac{25.1}{99} = \frac{251}{990}$$
$$4.2\dot{5}\dot{3} = 4\frac{251}{990}$$

In each of the above examples, we multiplied by the power of ten corresponding to the number of repeated digits. In **(a)** and **(b)** only 1 digit was repeated, so we multiplied by 10^1 (i.e. by 10). In **(c)** and **(d)** 2 digits were repeated, so we multiplied by 10^2 (i.e. by 100). If there had been 3 digits in the repeating set, we would have multiplied by 1000.

EXERCISE 1.2 REPEATING DECIMALS

1 Which of the following fractions can be expressed as repeating decimals?
$\frac{1}{2}, \frac{5}{9}, \frac{7}{16}, \frac{2}{3}, \frac{7}{12}, \frac{4}{11}, \frac{3}{10}, \frac{2}{13}, \frac{9}{25}, \frac{4}{15}$

2 Express each fraction as a repeating decimal.

(a) $\frac{2}{3}$ **(b)** $\frac{7}{9}$ **(c)** $\frac{4}{11}$ **(d)** $\frac{2}{15}$
(e) $\frac{5}{6}$ **(f)** $\frac{9}{13}$ **(g)** $\frac{5}{12}$ **(h)** $\frac{11}{18}$

3 Express each repeating decimal as a common fraction.

(a) $0.\dot{5}$ **(b)** $0.1\dot{7}$ **(c)** $0.\dot{6}\dot{8}$ **(d)** $0.2\dot{1}$
(e) $0.5\dot{1}$ **(f)** $0.58\dot{3}$ **(g)** $1.\dot{3}\dot{7}$ **(h)** $3.5\dot{8}$

4 You have seen that $\frac{1}{7} = 0.142\,857\,142\,857\ldots = 0.\dot{1}42\,85\dot{7}$. Find $\frac{2}{7}$ and $\frac{3}{7}$ as recurring decimals by dividing the numerator by the denominator. By observing the pattern obtained for these three fractions, write the repeating decimals for $\frac{4}{7}$, $\frac{5}{7}$ and $\frac{6}{7}$.

5 Show by division that $\frac{1}{13} = 0.\dot{0}76\,92\dot{3}$ and write the repeating decimals for $\frac{2}{13}, \frac{3}{13}, \frac{4}{13}, \ldots \frac{12}{13}$. Comment on the pattern in the repeating digits.

6 Express $\frac{1}{17}, \frac{2}{17}, \frac{3}{17}, \ldots \frac{16}{17}$ as repeating decimals. What do you observe?

7 The repeating decimal for $\frac{3}{11}$ is:

A $0.3\dot{6}$ **B** $0.0\dot{2}\dot{7}$ **C** $0.2\dot{7}$ **D** $0.\dot{2}0\dot{7}$

1.3 SCIENTIFIC NOTATION (STANDARD FORM)

Scientific notation, also known as 'standard form', is a neat way to write large and small numbers.

A number in scientific notation is expressed as a product of:
- a number between 1 and 10, and
- a power of 10.

The power of ten is determined by the change in the place value of the digits. This is the number of digits that must be moved across the decimal point to obtain the new number (between 1 and 10) from the original number.

Example 2

Write each number in scientific notation.

(a) 540 000 **(b)** 4267.32 **(c)** 0.000 006 8 **(d)** 0.007 562

Solution

(a) Write the number with a decimal point after the first non-zero digit.
Count the number of places from the original position of the decimal point.
Because the original position of the decimal point is 5 places to the right, multiply by 10^5.

$5.40\,000$
540000 (5 places)
$540\,000 = 5.4 \times 10^5$

(b) Write the number with a decimal point after the first non-zero digit.
Count the number of places from the original position of the decimal point.
Because the original position of the decimal point is 3 places to the right, multiply by 10^3.

$4.267\,32$
4267.32 (3 places)
$4267.32 = 4.267\,32 \times 10^3$

(c) Write the number with a decimal point after the first non-zero digit.
Count the number of places from the original position of the decimal point.
Because the original position of the decimal point is 6 places to the left, multiply by 10^{-6}.

6.8
0.0000068 (6 places)
$0.000\,006\,8 = 6.8 \times 10^{-6}$

(d) Write the number with a decimal point after the first non-zero digit.
Count the number of places from the original position of the decimal point.
Because the original position of the decimal point is 3 places to the left, multiply by 10^{-3}.

7.562
0.007562 (3 places)
$0.007\,562 = 7.562 \times 10^{-3}$

Example 3

Perform each calculation, giving your answer in scientific notation.

(a) $8 \times 10^5 \times 3 \times 10^{-2}$ **(b)** $\dfrac{1.5 \times 10^6}{5 \times 10^{-4}}$

Solution

(a) Multiply the numbers and add the indices.
Write using scientific notation.

$8 \times 10^5 \times 3 \times 10^{-2}$
$= 24 \times 10^3$
$= 2.4 \times 10^4$

(b) Divide the numbers and subtract the indices.
Write using scientific notation.

$\dfrac{1.5 \times 10^6}{5 \times 10^{-4}}$
$= 0.3 \times 10^{10}$
$= 3.0 \times 10^9$

EXERCISE 1.3 SCIENTIFIC NOTATION (STANDARD FORM)

1 Write each number in scientific notation.

(a) 3000 **(b)** 468.2 **(c)** 12 million **(d)** 42×10^5
(e) 0.0001 **(f)** 0.0041 **(g)** 0.000 000 22 **(h)** 865×10^{-7}
(i) $862 \times 10^4 \times 250$ **(j)** $0.62 \times 10^{-4} \times 15$ **(k)** $\dfrac{1.44 \times 10^8}{6.4 \times 10^{-2}}$ **(l)** $\dfrac{7.5 \times 10^{-5}}{1.5 \times 10^3}$

2 The product of 3.2×10^6 and 4×10^{-2} expressed in standard form is:

A 12.8×10^4 **B** 1.28×10^5 **C** 0.8×10^8 **D** 8×10^7

3 Simplify:

(a) $4.82 \times 10^3 \times 5 \times 10^5$ **(b)** $1.26 \times 10^8 \div (7 \times 10^2)$ **(c)** $7.5 \times 10^{-6} \div 1500$

(d) $5.5 \times 10^3 \times (4.0 \times 10^3)^3$ **(e)** $7217 \div 700\,000$ **(f)** $1.4 \times 10^{-3} \times 8 \times 10^{-5}$

4 Express in standard form:

(a) the number of cm in 40 km

(b) the number of grams in 3 tonnes

(c) the distance from the Sun to Earth, if light from the Sun travelling at 300 000 km/s takes 8 minutes to reach Earth

(d) the number of cm^2 in 5 hectares, given that 1 hectare = $10^4\,m^2$.

5 The mass of a molecule of hydrogen gas is about 3.34×10^{-24} grams. The mass of an oxygen molecule, which is 16 times that of a hydrogen molecule, is about:

A 5.344×10^{-24} **B** 5.344×10^{-23} **C** 5.344×10^{-22} **D** 5.344×10^{-21}

6 A light-year is the distance light travels in a year (where 1 year = 365.25 days). The speed of light is 3×10^5 km/s. Find the distance from the Earth to the star Proxima Centauri, 4.25 light-years away, in kilometres.

7 The Earth is 9.3×10^7 miles from the Sun. Express this distance in metres, given that 1 mile = 1.6×10^3 metres.

8 How much time will it take for a rocket travelling at a constant speed of 1.5×10^4 km/h to reach the Moon, if the distance to the Moon is 3.84×10^5 km?

1.4 SIGNIFICANT FIGURES AND DECIMAL PLACES

When writing answers, the precision required will not always involve all the digits shown on your calculator. The precision of the information can be defined by the number of significant figures or decimal places used in the answer.

Significant figures

Significant figures are the non-zero digits used in a number. Zeros are only considered significant if they occur between non-zero digits. For example, the number 2874 is correct to four significant figures. The number 301 has three significant figures, because the zero is between the 3 and the 1. The number 30 100 also has three significant figures, because the final zeros are not between non-zero digits.

However, zeros after a decimal point are significant when they represent greater precision in an answer. For example, the number 28 has two significant figures but the number 28.00 has four significant figures. This makes more sense if we consider the numbers as measurements. If 28 m is a measurement rounded to the nearest metre, then the answer has two significant figures. If 28.00 m is a measurement rounded to the nearest centimetre, then the zeros after the decimal point are significant digits of the measurement, so we say it has four significant figures.

To write an answer correct to three significant figures, look at the fourth digit and round it off: if is less than 5, replace it and all following digits with zeros; if it is 5 or more, increase the third digit by one and replace all the following digits with zeros. For example, 2874 correct to three significant figures becomes 2870. 28.00 correct to three significant figures becomes 28.0.

Similarly, to write 2874 correct to two significant figures, round off the third digit. Because the third digit is 7, which is more than five, it rounds up to make 2900.

Decimal places

The π key on your calculator might give the value of π as 3.141 592 654, which is more digits than you really need. Writing π as 3.14 is correct to two decimal places, because there are two non-zero digits after the decimal point.

(3.14 is also correct to three significant figures.) Writing π as 3.142 is correct to three decimal places. In these cases, all digits after the required number of decimal places are rounded off. π correct to four decimal places would be 3.1416, because the fifth digit is 9 and so the fourth digit must be rounded up.

Example 4

(a) Write 58.377 correct to: **(i)** two significant figures (2 s.f.) **(ii)** two decimal places (2 d.p.)

(b) Write 4.061 correct to: **(i)** two significant figures (2 s.f.) **(ii)** two decimal places (2 d.p.)

(c) Write 3.1499 correct to: **(i)** three significant figures (3 s.f.) **(ii)** three decimal places (3 d.p.)

Solution

(a) 58.377

(i) The third digit, 3, is less than 5, so remove all digits after the first two digits. — 58

(ii) The digit in the third decimal place, 7, is more than 5, so remove it and round up the digit in the second decimal place. — 58.38

(b) 4.061

(i) The third digit, 6, is more than 5, so remove it and round up the second digit. — 4.1

(ii) The digit in the third decimal place, 1, is less than 5, so remove it. — 4.06

(c) 3.1499

(i) The fourth digit, 9, is more than 5, so remove it and round up the third digit. — 3.15

(ii) The digit in the fourth decimal place, 9, is more than 5, so remove it and round up the digit in the third decimal place. This also rounds up the digit in the second decimal place, as 49 becomes 50. — 3.150
Note that the final zero must be written to show the precision: 3.15 would be correct to only 2 d.p.

EXERCISE 1.4 SIGNIFICANT FIGURES AND DECIMAL PLACES

1 Write each number correct to three significant figures.

(a) 9726 **(b)** 86.433 **(c)** 18.077 **(d)** 0.37256

2 Write each number correct to three decimal places.

(a) 0.9726 **(b)** 0.86433 **(c)** 18.0707 **(d)** 0.37025

3 Which answer is **not** written correct to two significant figures?

A 2300 **B** 1.73 **C** 0.054 **D** 2.1×10^4

4 How many significant figures are used in each of the following statements?

(a) The distance from Melbourne to Sydney is 900 km.
(b) The time to run a 100 m race was 10.4 s.
(c) The length of the line was measured to be 15.00 cm.
(d) 1 hectare $\approx$ 2.47 acres.

5 Evaluate each expression, writing your answer correct to two significant figures.

(a) $7.89 \times 0.496 \times 15.7$ **(b)** $\dfrac{21.4 \times 0.397}{6.85}$ **(c)** $(2.73 + 35.2) \times 0.762$

6 Evaluate each expression, writing your answer correct to two decimal places.

(a) $\sqrt{\dfrac{37.185}{6.1 \times 3.87}}$ **(b)** $5 \times 10^{-13} \times 17 \times 10^{10}$ **(c)** $\sqrt{\pi^2 + 3}$

7 The distance from the Sun to the planet Mars is about 2.28×10^8 km and the distance from the Sun to the planet Saturn is about 1.427×10^9 km.

(a) Calculate the closest possible distance between Saturn and Mars, giving your answer correct to 3 significant figures.

(b) What is the greatest possible distance between Saturn and Mars? How many significant figures would you give in your answer? Why?

8 Find the exact value of each expression.

(a) $\sqrt{1.21}$ **(b)** $\sqrt{30.25}$ **(c)** $\sqrt{2\frac{14}{25}}$ **(d)** $\sqrt{0.04}$ **(e)** $\left(\frac{3}{4}\right)^2$ **(f)** $\left(2\frac{3}{4}\right)^2$

(g) $(3.1)^2$ **(h)** $(0.07)^2$ **(i)** 1.1^3 **(j)** $\left(3\frac{1}{2}\right)^3$ **(k)** $\sqrt[3]{125}$ **(l)** $\sqrt[3]{15\frac{5}{8}}$

9 Evaluate, giving your answer correct to two decimal places where necessary:

(a) $\sqrt{5^2+6^2}$ **(b)** $\sqrt{1.2^2+2.1^2}$ **(c)** 5.2^3 **(d)** $\left(2\frac{1}{3}\right)^3$

(e) $\sqrt[3]{99}$ **(f)** $\sqrt[3]{0.526}$ **(g)** $100^{\frac{1}{3}}$ **(h)** $0.1^{\frac{1}{2}}$

1.5 APPROXIMATIONS

When using a calculator, it is useful to be able to do a rough calculation to see if an answer is approximately correct. You can use this technique to check the size of an answer and the position of the decimal point.

Example 5

By writing each number correct to one significant figure, find the approximate value of each expression.

(a) 28.2×0.61 **(b)** $\frac{4327}{21.2}$ **(c)** $\frac{716 \times 18}{0.042}$

Solution

(a) $28.2 \times 0.61 \approx 30 \times 0.6 \approx 18$ (calculator gives 17.202)

(b) $\frac{4327}{21.2} \approx \frac{4000}{20} \approx 200$ (calculator gives 204.10377…)

(c) $\frac{716 \times 18}{0.042} \approx \frac{700 \times 20}{0.04} \approx \frac{700 \times 20 \times 100}{4} \approx 350\,000$ (calculator gives 306 857.1429…)

EXERCISE 1.5 APPROXIMATIONS

1 Express each of the following correct to one significant figure.

(a) 417.2 **(b)** 0.685 **(c)** 2748 **(d)** 8.888

(e) 50 010 **(f)** 0.0056 **(g)** 0.0203 **(h)** 0.155

2 Which answer is written correct to one significant figure?

A 2001 **B** 1.703 **C** 0.05 **D** 2.1×10^4

3 By writing each number correct to one significant figure, find the approximate value of each expression.

(a) 78×6 **(b)** 66×31 **(c)** 196×0.42 **(d)** 48.2×13.9

(e) $\frac{73.6}{4.8}$ **(f)** $\frac{18.23}{49}$ **(g)** $\frac{56.2}{0.006}$ **(h)** $\frac{17 \times 61}{3.2}$

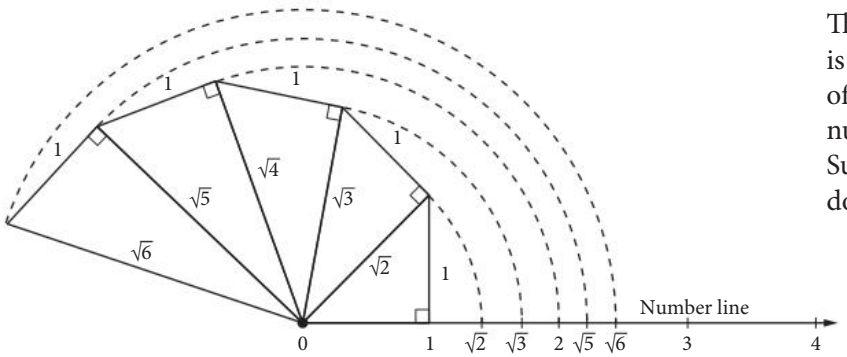
(i) $\frac{117.6 \times 0.002}{0.048}$ **(j)** $\frac{88.9 \times 316}{5.76}$ **(k)** $\frac{39.02 \times 5.9}{4.02 \times 55.1}$ **(l)** $\frac{69.7 \times 0.09}{0.85 \times 7.001}$

1.6 REAL NUMBERS AND SURDS

Rational numbers may be expressed in the form $\frac{a}{b}$ where a and b are integers and $b \neq 0$. Rational numbers include fractions, terminating decimals and recurring decimals. Integers are rational numbers with a denominator of one, for example $5 = \frac{5}{1}$.

Irrational numbers are numbers that cannot be represented as a fraction, such as $\sqrt{2}$, $\sqrt[3]{15}$, $2\sqrt{7}$ or π. They are not rational. Irrational numbers that are roots, such as $\sqrt{2}$, $\sqrt[3]{15}$ and $2\sqrt{7}$, are called surds. Some other irrational numbers, such as π, are called transcendental numbers.

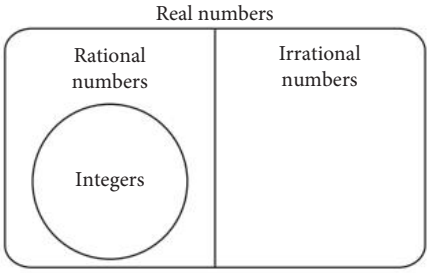
The following diagram shows how to construct exact lengths for $\sqrt{2}$, $\sqrt{3}$, $\sqrt{4}$, $\sqrt{5}$ and $\sqrt{6}$, starting with a right-angled triangle with side lengths of one unit. Each triangle's hypotenuse is an exact value, written as a surd.



The diagram shows that $\sqrt{4} = 2$, which is a rational number. The square root of a perfect square is always a rational number. Similarly, $\sqrt[3]{8} = 2$ is rational. Surds that simplify to a rational number do not represent irrational numbers.

The sets of all rational and irrational numbers together form a larger set called the set of **real numbers**.

The real numbers can be represented by all the points on the real number line.



Operations with surds—basic rules

Surds have their own set of simplifying rules for the four operations of arithmetic.

If a and b are positive numbers, then:

(a) $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$ (b) $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$ (c) $\sqrt{a^2} = a$

(d) $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ (e) $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$ (f) $\sqrt{a} \times \sqrt{a} = a$

(g) $\sqrt{a} \div \sqrt{b} = \frac{\sqrt{a}}{\sqrt{b}} = \frac{\sqrt{a} \times \sqrt{b}}{\sqrt{b} \times \sqrt{b}} = \frac{\sqrt{ab}}{b}$ to write the expression with a rational denominator.

The rules above are used to simplify expressions involving surds. Surds can also sometimes be simplified by identifying a factor in the surd that is a perfect square and taking that factor's square root. (For example, see part **(a)** in Example **6**.) You might also multiply two surds and find that the answer has a perfect square factor.

You should know the perfect squares so that you can look for them as factors in surds. When a surd contains a perfect square factor, you can use rules **(a)**, **(c)** and **(g)** above to simplify the expression.

Example 6

Simplify each expression.

(a) $\sqrt{12}$ (b) $\sqrt{80}$ (c) $3\sqrt{28}$ (d) $\frac{\sqrt{175}}{5}$

(e) $\sqrt{6}\times\sqrt{10}$ (f) $\frac{\sqrt{80}}{\sqrt{5}}$ (g) $\sqrt{3}\times\sqrt{12}$ (h) $4\sqrt{2}\times\sqrt{24}$

Solution

(a) Look for a factor of 12 that is a perfect square.

$12 = 4\times 3$

$\therefore \sqrt{12} = \sqrt{4\times 3} = \sqrt{4}\times\sqrt{3} = 2\sqrt{3}$

(b) Look for a factor of 80 that is a perfect square.

$80 = 16\times 5$

$\therefore \sqrt{80} = \sqrt{16\times 5} = 4\sqrt{5}$

(c) $3\sqrt{28} = 3\sqrt{4}\times\sqrt{7} = 6\sqrt{7}$

(d) $\frac{\sqrt{175}}{5} = \frac{\sqrt{25\times 7}}{5} = \frac{5\sqrt{7}}{5} = \sqrt{7}$

(e) $\sqrt{6}\times\sqrt{10} = \sqrt{60} = \sqrt{4\times 15} = 2\sqrt{15}$

(f) $\frac{\sqrt{80}}{\sqrt{5}} = \sqrt{\frac{80}{5}} = \sqrt{16} = 4$

(g) $\sqrt{3}\times\sqrt{12} = \sqrt{3}\times 2\sqrt{3} = 2\times 3 = 6$

(h) $4\sqrt{2}\times\sqrt{24} = 4\sqrt{2}\times 2\sqrt{6} = 8\sqrt{12} = 16\sqrt{3}$

Example 7

Simplify each expression, writing your answer with a rational denominator (or no denominator).

(a) $\frac{6}{\sqrt{3}}$ (b) $\frac{\sqrt{60}}{3\sqrt{6}}$ (c) $\frac{3\sqrt{5}}{\sqrt{15}}$

Solution

(a) $\frac{6}{\sqrt{3}} = \frac{6}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}$

$= \frac{6\sqrt{3}}{3}$

$= 2\sqrt{3}$

(b) $\frac{\sqrt{60}}{3\sqrt{6}} = \frac{\sqrt{6}\times\sqrt{10}}{3\sqrt{6}}$

$= \frac{\sqrt{10}}{3}$

(c) $\frac{3\sqrt{5}}{\sqrt{15}} = \frac{3\sqrt{5}}{\sqrt{3}\times\sqrt{5}}$

$= \frac{3}{\sqrt{3}}$

$= \frac{\sqrt{3}\times\sqrt{3}}{\sqrt{3}}$

$= \sqrt{3}$

EXERCISE 1.6 REAL NUMBERS AND SURDS

Simplify all the expressions in this exercise, writing each answer with a rational denominator (or no denominator).

1 $\sqrt{8}$ **2** $\sqrt{20}$ **3** $\sqrt{27}$ **4** $\sqrt{32}$

5 When simplified, $\sqrt{40}$ equals:

A $4\sqrt{5}$ **B** $\sqrt{10}$ **C** $2\sqrt{10}$ **D** 20

6 $\sqrt{45}$ **7** $\sqrt{72}$ **8** $\sqrt{84}$ **9** $\sqrt{98}$ **10** $\sqrt{108}$ **11** $\sqrt{125}$

12 $\sqrt{162}$ **13** $\sqrt{200}$ **14** $5\sqrt{128}$ **15** $4\sqrt{800}$ **16** $2\sqrt{150}$ **17** $3\sqrt{52}$

18 When simplified, $7\sqrt{245}$ equals:

A $7\sqrt{5}$ **B** $35\sqrt{7}$ **C** $35\sqrt{5}$ **D** $49\sqrt{5}$

19 $\frac{\sqrt{320}}{2}$ **20** $\frac{\sqrt{175}}{5}$ **21** $\frac{4\sqrt{243}}{3}$ **22** $\frac{\sqrt{90}}{9}$

23 When simplified, $\frac{\sqrt{50}}{\sqrt{5}}$ equals:

A $\sqrt{2}$ **B** $\sqrt{10}$ **C** $5\sqrt{2}$ **D** 10

24 Simplify $\frac{2\sqrt{7}}{\sqrt{35}}$. Indicate whether each answer below is correct or incorrect.

(a) $\frac{2}{\sqrt{5}}$ (b) $\frac{14}{\sqrt{5}}$ (c) $\frac{2\sqrt{5}}{5}$ (d) 10

25 $\sqrt{3}\times\sqrt{5}$ **26** $\sqrt{8}\times\sqrt{2}$ **27** $\sqrt{6}\times\sqrt{2}$ **28** $\sqrt{6}\times\sqrt{10}$ **29** $2\sqrt{3}\times4\sqrt{2}$

30 $\sqrt{8}\times2\sqrt{2}$ **31** $2\sqrt{5}\times5\sqrt{2}$ **32** $4\sqrt{6}\times2\sqrt{3}$ **33** $2\sqrt{5}\times3\sqrt{7}$ **34** $4\sqrt{3}\times\sqrt{18}$

35 $2\sqrt{8}\times\sqrt{12}$ **36** $4\sqrt{5}\times\sqrt{20}$ **37** $\frac{\sqrt{3}}{\sqrt{2}}$ **38** $\frac{2}{\sqrt{3}}$ **39** $\frac{\sqrt{28}}{\sqrt{7}}$

40 $\frac{7}{\sqrt{7}}$ **41** $\frac{\sqrt{80}}{\sqrt{5}}$ **42** $\frac{3\sqrt{2}}{\sqrt{6}}$ **43** $\frac{7\sqrt{2}}{\sqrt{98}}$ **44** $\frac{2\sqrt{18}}{\sqrt{8}}$

1.7 ADDING AND SUBTRACTING SURDS

Surds of the same kind can be added and subtracted just like pronumerals, using the distributive law to collect like terms: $ab + ac = a(b + c)$

You can only add and subtract like algebraic terms, such as $2a + 3a - a = 4a$. Similarly, you can only add and subtract like surds, such as $2\sqrt{2} + 3\sqrt{2} = 5\sqrt{2}$. It may be necessary to first simplify the surd terms by removing perfect square factors.

Example 8

Simplify each expression by collecting like terms.

(a) $2\sqrt{3}+5\sqrt{3}$ (b) $4\sqrt{10}-\sqrt{10}$ (c) $3\sqrt{6}+4\sqrt{6}-\sqrt{5}$ (d) $3\sqrt{5}+4\sqrt{7}+2\sqrt{5}-6\sqrt{7}$

Solution

(a) $2\sqrt{3}+5\sqrt{3}=(2+5)\sqrt{3}=7\sqrt{3}$

(b) $4\sqrt{10}-\sqrt{10}=(4-1)\sqrt{10}=3\sqrt{10}$

(c) $3\sqrt{6}+4\sqrt{6}-\sqrt{5}=7\sqrt{6}-\sqrt{5}$

(d) $3\sqrt{5}+4\sqrt{7}+2\sqrt{5}-6\sqrt{7}=5\sqrt{5}-2\sqrt{7}$

In parts **(c)** and **(d)** the different surds cannot be combined into a single term.

Example 9

Simplify:

(a) $\sqrt{8}-\sqrt{18}+\sqrt{50}$ (b) $5\sqrt{3}+\sqrt{20}-2\sqrt{12}+\sqrt{45}$

Solution

Where possible, simplify each term before attempting to add or subtract the surds.

(a) $\sqrt{8}-\sqrt{18}+\sqrt{50}$
$=\sqrt{4\times2}-\sqrt{9\times2}+\sqrt{25\times2}$
$=2\sqrt{2}-3\sqrt{2}+5\sqrt{2}$
$=4\sqrt{2}$

(b) $5\sqrt{3}+\sqrt{20}-2\sqrt{12}+\sqrt{45}$
$=5\sqrt{3}+2\sqrt{5}-4\sqrt{3}+3\sqrt{5}$
$=\sqrt{3}+5\sqrt{5}$

EXERCISE 1.7 ADDING AND SUBTRACTING SURDS

Simplify the expressions in this exercise.

1 $\sqrt{3}+2\sqrt{3}+4\sqrt{3}$ **2** $5\sqrt{7}-2\sqrt{7}+4\sqrt{7}$ **3** $3\sqrt{5}+5\sqrt{5}-2\sqrt{5}$

4 $4\sqrt{2}-\sqrt{3}+4\sqrt{3}-\sqrt{2}$ **5** $\sqrt{5}+\sqrt{2}+3\sqrt{5}-6\sqrt{2}$ **6** $\sqrt{8}-\sqrt{2}$

7 When $\sqrt{27}-\sqrt{18}+\sqrt{3}$ is simplified, the answer is:

A $4\sqrt{3}-3\sqrt{2}$ B $\sqrt{3}$ C $3+\sqrt{3}$ D $2\sqrt{3}$

8 $\sqrt{20}+\sqrt{5}$ **9** $\sqrt{18}+\sqrt{32}-\sqrt{2}$ **10** $\sqrt{27}+2\sqrt{48}-4\sqrt{3}$

11 $\sqrt{12}+\sqrt{3}+\sqrt{48}$ **12** $2\sqrt{50}-3\sqrt{18}$ **13** $\sqrt{7}+\sqrt{28}-\sqrt{63}$

14 Simplify $\sqrt{5}+\sqrt{2}-\sqrt{45}+\sqrt{8}$. Some steps in this simplification are given below. Indicate whether each statement is a correct or incorrect step.

(a) $\sqrt{10}-\sqrt{45}$ (b) $-\sqrt{35}$ (c) $\sqrt{5}+\sqrt{2}-3\sqrt{5}+2\sqrt{2}$ (d) $3\sqrt{2}-2\sqrt{5}$

15 $6\sqrt{5}+4\sqrt{7}-2\sqrt{5}$ **16** $5\sqrt{3}+\sqrt{27}-\sqrt{45}$ **17** $\sqrt{20}+\sqrt{5}+\sqrt{18}$

18 $3\sqrt{15}+\sqrt{60}-\sqrt{40}$ **19** $4\sqrt{7}-\sqrt{28}+\sqrt{63}$ **20** $2\sqrt{50}-3\sqrt{18}+\sqrt{3}$

21 $\sqrt{3}+3\sqrt{75}-\sqrt{48}$ **22** $\sqrt{150}-\sqrt{200}$ **23** $\sqrt{6}+\sqrt{24}+\sqrt{54}$

24 $5\sqrt{7}+3\sqrt{5}-2\sqrt{28}$ **25** $5\sqrt{45}-2\sqrt{32}$ **26** $\sqrt{98}-2\sqrt{20}-\sqrt{12}$

27 $\sqrt{125}-5\sqrt{2}+\sqrt{50}$ **28** $7\sqrt{3}-2\sqrt{2}+\sqrt{12}+\sqrt{8}$ **29** $\sqrt{150}-\sqrt{96}-\sqrt{24}$

30 $6\sqrt{2}+\sqrt{12}-2\sqrt{3}-3\sqrt{8}$

1.8 THE DISTRIBUTIVE LAW

The distributive law can be used with surds to expand expressions with a binomial factor: $a(b+c)=ab+ac$

Example 10

Expand and simplify:

(a) $\sqrt{6}\left(\sqrt{2}+2\sqrt{3}\right)$ (b) $\left(\sqrt{3}+\sqrt{2}\right)\left(\sqrt{5}+\sqrt{6}\right)$ (c) $\left(\sqrt{5}-\sqrt{3}\right)\left(\sqrt{5}+\sqrt{3}\right)$ (d) $\left(\sqrt{5}+\sqrt{3}\right)\left(\sqrt{5}+\sqrt{3}\right)$

Solution

(a) $\sqrt{6}\left(\sqrt{2}+2\sqrt{3}\right)$
$=\sqrt{6}\times\sqrt{2}+\sqrt{6}\times 2\sqrt{3}$
$=\sqrt{12}+2\sqrt{18}$
$=2\sqrt{3}+6\sqrt{2}$

(b) $\left(\sqrt{3}+\sqrt{2}\right)\left(\sqrt{5}+\sqrt{6}\right)$
$=\sqrt{3}\left(\sqrt{5}+\sqrt{6}\right)+\sqrt{2}\left(\sqrt{5}+\sqrt{6}\right)$
$=\sqrt{15}+3\sqrt{2}+\sqrt{10}+2\sqrt{3}$

(c) $\left(\sqrt{5}-\sqrt{3}\right)\left(\sqrt{5}+\sqrt{3}\right)$
$=\left(\sqrt{5}\right)^2-\left(\sqrt{3}\right)^2$
$=5-3$
$=2$
This is similar to $(a-b)(a+b)=a^2-b^2$.

(d) $\left(\sqrt{5}+\sqrt{3}\right)\left(\sqrt{5}+\sqrt{3}\right)$
$=\sqrt{5}\left(\sqrt{5}+\sqrt{3}\right)+\sqrt{3}\left(\sqrt{5}+\sqrt{3}\right)$
$=5+2\sqrt{15}+3$
$=8+2\sqrt{15}$
This is similar to $(a+b)^2=a^2+2ab+b^2$.

EXERCISE 1.8 THE DISTRIBUTIVE LAW

Expand and simplify the expressions in this exercise.

1 $\sqrt{5}\left(\sqrt{2}+\sqrt{3}\right)$ **2** $\sqrt{5}\left(\sqrt{5}+\sqrt{2}\right)$ **3** $\sqrt{2}\left(\sqrt{2}+\sqrt{8}\right)$

4 $\sqrt{3}\left(\sqrt{2}-\sqrt{6}\right)$ **5** $\sqrt{6}\left(\sqrt{3}-2\right)$ **6** $7\left(2\sqrt{5}-1\right)$

7 When simplified, $\sqrt{2}\left(\sqrt{32}-\sqrt{8}\right)$ equals:

A $8\sqrt{3}$ **B** 4 **C** $2\sqrt{2}$ **D** $8-2\sqrt{2}$

8 $3\sqrt{2}\left(2\sqrt{6}-\sqrt{5}\right)$ **9** $\sqrt{a}\left(\sqrt{a}+\sqrt{b}\right)$ **10** $\sqrt{x}\left(\sqrt{x}-\sqrt{y}\right)$

11 $\left(\sqrt{5}+\sqrt{3}\right)\left(\sqrt{7}-\sqrt{2}\right)$ **12** $\left(\sqrt{2}+\sqrt{7}\right)\left(\sqrt{3}+2\sqrt{2}\right)$ **13** $\left(\sqrt{3}-1\right)\left(\sqrt{2}+3\right)$

14 $\left(\sqrt{5}+2\right)\left(2\sqrt{5}+3\right)$ **15** $\left(2\sqrt{3}-5\right)\left(2\sqrt{3}+3\right)$ **16** $\left(\sqrt{5}+\sqrt{3}\right)\left(\sqrt{6}+1\right)$

17 $\left(\sqrt{7}+1\right)\left(2\sqrt{7}-5\right)$ **18** $\left(\sqrt{3}-\sqrt{2}\left(2\sqrt{3}-\sqrt{2}\right)\right)$ **19** $\left(2\sqrt{5}-\sqrt{2}\right)\left(2\sqrt{5}+3\right)$

20 $\left(2\sqrt{2}-\sqrt{6}\right)\left(2\sqrt{3}-1\right)$ **21** $\left(\sqrt{3}+1\right)^2$ **22** $\left(\sqrt{5}-\sqrt{2}\right)^2$

23 $\left(2\sqrt{6}+\sqrt{3}\right)^2$ **24** $\left(\sqrt{3}+6\right)^2$ **25** $\left(2+\sqrt{3}\right)\left(2-\sqrt{3}\right)$

26 $\left(\sqrt{3}-\sqrt{2}\right)\left(\sqrt{3}+\sqrt{2}\right)$ **27** $\left(\sqrt{7}-\sqrt{5}\right)\left(\sqrt{7}+\sqrt{5}\right)$ **28** $\left(2\sqrt{2}-1\right)\left(2\sqrt{2}+1\right)$

29 $\left(2\sqrt{6}-\sqrt{3}\right)\left(2\sqrt{6}+\sqrt{3}\right)$ **30** $\left(\sqrt{11}-\sqrt{7}\right)\left(\sqrt{11}+\sqrt{7}\right)$ **31** $\left(\sqrt{7}-2\right)\left(\sqrt{7}+2\right)$

32 When simplified, $\left(3\sqrt{7}-2\right)^2$ equals:

A 59 **B** $67+12\sqrt{7}$ **C** $23-12\sqrt{7}$ **D** $67-12\sqrt{7}$

33 $\left(\sqrt{5}-\sqrt{3}\right)^2$ **34** $\left(\sqrt{11}+\sqrt{7}\right)^2$ **35** $\left(2\sqrt{3}-1\right)\left(\sqrt{3}+2\right)$

36 $\left(\sqrt{11}-\sqrt{10}\right)\left(\sqrt{11}+\sqrt{10}\right)$ **37** $\left(\sqrt{6}-\sqrt{5}\right)\left(\sqrt{6}+\sqrt{5}\right)$ **38** $\left(2\sqrt{2}+\sqrt{3}\right)^2$

39 $\left(3\sqrt{5}-2\sqrt{2}\right)\left(3\sqrt{5}+2\sqrt{2}\right)$ **40** $\left(\sqrt{5}+2\sqrt{2}\right)\left(\sqrt{6}-1\right)$ **41** $\left(2\sqrt{6}-\sqrt{3}\right)\left(\sqrt{6}+3\sqrt{3}\right)$

42 Expand and simplify $\left(4\sqrt{3}+1\right)\left(2\sqrt{3}-3\right)$. Some steps in this simplification are given below. Indicate whether each statement is a correct or incorrect step.

(a) $72-12\sqrt{3}+2\sqrt{3}-3$ **(b)** $24-12\sqrt{3}+2\sqrt{3}-3$ **(c)** $21-10\sqrt{3}$ **(d)** $27-10\sqrt{3}$

43 $\left(5\sqrt{2}-4\right)\left(5\sqrt{2}+4\right)$ **44** $\left(2\sqrt{7}+3\sqrt{6}\right)^2$ **45** $\left(2\sqrt{15}+\sqrt{5}\right)\left(\sqrt{15}-3\sqrt{5}\right)$

46 $\left(2\sqrt{2}+3\sqrt{3}\right)^2$ **47** $\left(2\sqrt{3}-1\right)\left(2\sqrt{3}+1\right)$ **48** $\left(2\sqrt{3}-1\right)^2$

1.9 RATIONALISING DENOMINATORS

The expressions $\sqrt{a}\times\sqrt{a}$ and $\left(\sqrt{a}-\sqrt{b}\right)\left(\sqrt{a}+\sqrt{b}\right)$ both have rational answers (that is, answers that do not involve surds): $\sqrt{a}\times\sqrt{a}=a$ and $\left(\sqrt{a}-\sqrt{b}\right)\left(\sqrt{a}+\sqrt{b}\right)=\left(\sqrt{a}\right)^2-\left(\sqrt{b}\right)^2=a-b$. For example:

- $\sqrt{3}\times\sqrt{3}=3$
- $\left(\sqrt{7}-\sqrt{2}\right)\left(\sqrt{7}+\sqrt{2}\right)=7-2=5$

The expansion $\left(\sqrt{a}-\sqrt{b}\right)\left(\sqrt{a}+\sqrt{b}\right)=a-b$ is known as the 'difference of two squares'. You have seen this in previous years as $(x-y)(x+y)=x^2-y^2$. By letting $x=\sqrt{a}$ and $y=\sqrt{b}$, we have obtained a process to convert any binomial surd into a rational number.

When you have a surd expression in the denominator of a fraction, it is normal to make the denominator into a rational number. This process is called **rationalising the denominator**.

Remember: to change a fraction without changing its value, multiply the numerator and the denominator by the same amount.

The expressions $\sqrt{a}-\sqrt{b}$ and $\sqrt{a}+\sqrt{b}$ are called **conjugate** surds, with each expression being the conjugate of the other. To convert any surd into a rational number, multiply the surd by its conjugate.

Example 11

Express with a rational denominator:

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{\sqrt{7}}{\sqrt{3}}$ (c) $\frac{1}{\sqrt{3}+\sqrt{2}}$ (d) $\frac{\sqrt{3}}{2\sqrt{7}-\sqrt{3}}$

Solution

(a) $\frac{1}{\sqrt{2}}$: Multiply by $\frac{\sqrt{2}}{\sqrt{2}}$

$$=\frac{1}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}$$
$$=\frac{\sqrt{2}}{2}$$

(b) $\frac{\sqrt{7}}{\sqrt{3}}$: Multiply by $\frac{\sqrt{3}}{\sqrt{3}}$

$$=\frac{\sqrt{7}}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}$$
$$=\frac{\sqrt{21}}{3}$$

In parts **(c)** and **(d)** the denominator is a binomial surd, so you have to multiply both numerator and denominator by the conjugate of the denominator.

(c) $\frac{1}{\sqrt{3}+\sqrt{2}}=\frac{1}{\sqrt{3}+\sqrt{2}}\times\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}}$

$$=\frac{\sqrt{3}-\sqrt{2}}{3-2}$$
$$=\sqrt{3}-\sqrt{2}$$

(d) $\frac{\sqrt{3}}{2\sqrt{7}-\sqrt{3}}=\frac{\sqrt{3}}{2\sqrt{7}-\sqrt{3}}\times\frac{2\sqrt{7}+\sqrt{3}}{2\sqrt{7}+\sqrt{3}}$

$$=\frac{2\sqrt{21}+3}{4\times7-3}$$
$$=\frac{2\sqrt{21}+3}{25}$$

Example 12

Express $\frac{\sqrt{2}}{2\sqrt{2}+1}+\frac{2}{\sqrt{3}+1}$ as a single fraction with a rational denominator.

Solution

You can add the two fractions by finding a common denominator and then rationalising the result, or by first rationalising each denominator and then adding the resulting fractions. The latter is usually the easier method, unless the denominators happen to be conjugates.

$$\frac{\sqrt{2}}{2\sqrt{2}+1}+\frac{2}{\sqrt{3}+1}=\frac{\sqrt{2}}{2\sqrt{2}+1}\times\frac{2\sqrt{2}-1}{2\sqrt{2}-1}+\frac{2}{\sqrt{3}+1}\times\frac{\sqrt{3}-1}{\sqrt{3}-1}$$

$$=\frac{4-\sqrt{2}}{4\times 2-1}+\frac{2\left(\sqrt{3}-1\right)}{3-1}$$

$$=\frac{4-\sqrt{2}}{7}+\frac{2\left(\sqrt{3}-1\right)}{2}$$

$$=\frac{4-\sqrt{2}}{7}+\frac{\sqrt{3}-1}{1}$$

$$=\frac{4-\sqrt{2}+7\left(\sqrt{3}-1\right)}{7}$$

$$=\frac{7\sqrt{3}-\sqrt{2}-3}{7}$$

EXERCISE 1.9 RATIONALISING DENOMINATORS

For questions **1** to **27**, express each fraction with a rational denominator.

1 $\frac{2}{\sqrt{3}}$ **2** $\frac{\sqrt{5}}{\sqrt{3}}$ **3** $\frac{3\sqrt{5}}{\sqrt{15}}$ **4** $\frac{1}{\sqrt{3}-\sqrt{2}}$

5 $\frac{1}{2\sqrt{7}+\sqrt{6}}$ **6** $\frac{1}{\sqrt{5}+2}$ **7** $\frac{1}{2\sqrt{5}-3\sqrt{2}}$ **8** $\frac{3\sqrt{2}}{\sqrt{5}-\sqrt{3}}$

9 $\frac{\sqrt{6}}{2\sqrt{5}-3\sqrt{2}}$ **10** $\frac{2\sqrt{5}}{3\sqrt{11}-2\sqrt{8}}$ **11** $\frac{\sqrt{3}+2}{\sqrt{3}-2}$ **12** $\frac{4\sqrt{2}+3\sqrt{5}}{2\sqrt{5}-\sqrt{2}}$

13 $\frac{\sqrt{7}-2\sqrt{5}}{3\sqrt{5}-2\sqrt{2}}$ **14** $\frac{3\sqrt{2}+2\sqrt{3}}{3\sqrt{2}-2\sqrt{3}}$ **15** $\frac{5\sqrt{3}+3\sqrt{5}}{5\sqrt{5}-3\sqrt{3}}$ **16** $\frac{2\sqrt{3}}{3\sqrt{3}-2}$

17 $\frac{3\sqrt{6}}{2\sqrt{2}+\sqrt{3}}$ **18** $\frac{5\sqrt{11}+3}{3\sqrt{11}-2}$ **19** $\frac{\sqrt{2}}{2\sqrt{2}-2\sqrt{3}}$ **20** $\frac{3\sqrt{3}}{2\sqrt{3}+\sqrt{2}}$

21 $\frac{\sqrt{3}}{\sqrt{24}-\sqrt{48}}$ **22** $\frac{\sqrt{2}-1}{\sqrt{2}+1}$ **23** $\frac{2\sqrt{5}-\sqrt{2}}{2\sqrt{5}+\sqrt{2}}$ **24** $\frac{\sqrt{6}+2\sqrt{3}}{2\sqrt{6}-\sqrt{3}}$

25 $\frac{2\sqrt{7}-3\sqrt{2}}{2\sqrt{7}-\sqrt{2}}$ **26** $\frac{\sqrt{5}+\sqrt{3}}{2\sqrt{10}-\sqrt{6}}$ **27** $\frac{2\sqrt{2}-5\sqrt{3}}{3\sqrt{6}-\sqrt{15}}$

28 If $x=\frac{3-2\sqrt{2}}{3+2\sqrt{2}}$ then the value of $x+\frac{1}{x}$ is:

A 34 **B** $24\sqrt{2}$ **C** $34-24\sqrt{2}$ **D** 2

29 If $x=\sqrt{3}+1$, find the value of $x^2-\frac{1}{x^2}$.

30 If $x=3\sqrt{2}+1$, find the value of $\frac{x^2-2x}{x-1}$.

31 If $x=\sqrt{5}-2$, find the value of $\frac{x^2+2x}{x+3}$.

32 Show that $x=2\sqrt{2}-3$ is one solution of the equation $x^2+6x+1=0$.

33 Which values of x satisfy $x^2+4x+2=0$? Indicate whether each answer is correct or incorrect.

(a) $x=2-\sqrt{2}$ **(b)** $x=\sqrt{2}-2$ **(c)** $x=\sqrt{2}+2$ **(d)** $x=-2-\sqrt{2}$

34 Show that $x = \sqrt{5} - 1$ is one solution of the equation $x^3 + 3x^2 - 2x - 4 = 0$.

35 Show that $x = \dfrac{1}{\sqrt{3} - 1}$ is one solution of the equation $2x^2 - 2x - 1 = 0$.

For questions **36** to **44**, express as a single fraction with a rational denominator:

36 $\dfrac{1}{2\sqrt{3} - 1} + \dfrac{3}{\sqrt{3} + 1}$

37 $\dfrac{\sqrt{5} + \sqrt{2}}{\sqrt{5} - \sqrt{2}} - \dfrac{\sqrt{5} - \sqrt{2}}{\sqrt{5} + \sqrt{2}}$

38 $\dfrac{\sqrt{3} - \sqrt{2}}{\sqrt{3} + \sqrt{2}} \times \dfrac{2\sqrt{2} - \sqrt{3}}{2\sqrt{2} + \sqrt{3}}$

39 $\dfrac{\sqrt{5} + \sqrt{2}}{\sqrt{5} - \sqrt{2}} \times \dfrac{2\sqrt{5} - 3\sqrt{2}}{\sqrt{5} + 2\sqrt{2}}$

40 $\dfrac{2\sqrt{5} + 1}{2\sqrt{5} - 1} - \dfrac{\sqrt{5} - 1}{2\sqrt{5} - 3}$

41 $\dfrac{2\sqrt{3}}{\sqrt{6} - \sqrt{3}} - \dfrac{\sqrt{3}}{2\sqrt{6} + 3\sqrt{3}}$

42 $\dfrac{\sqrt{3} - 1}{\sqrt{3} + 2} - \dfrac{\sqrt{5} - \sqrt{3}}{2\sqrt{5} + \sqrt{3}}$

43 $\dfrac{2\sqrt{5}}{\sqrt{10} - \sqrt{15}} - \dfrac{3\sqrt{7}}{\sqrt{35} - \sqrt{14}}$

44 $\dfrac{1}{x - 1} + \dfrac{1}{x + 1} - \dfrac{2}{x^2 - 1}$, where $x = 2\sqrt{3} + 1$.

CHAPTER REVIEW 1

1 Simplify $0.2 \times 0.01 \times 0.0003$, writing your answer in standard form.

2 Write each expression as a single fraction.

(a) $\dfrac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2} \times \frac{1}{3}}$ (b) $\dfrac{4}{5} \times \dfrac{5}{9} + \dfrac{1}{5} \times \dfrac{4}{9}$ (c) $\left(2\dfrac{1}{2} - \dfrac{3}{4}\right) \div 2\dfrac{5}{8}$

3 Express as a common fraction:

(a) $0.\dot{6}$ (b) $1.\dot{2}0\dot{3}$ (c) $0.0\dot{7}15\dot{6}$

4 Evaluate, giving your answer correct to two decimal places:

(a) $\sqrt{\dfrac{21.62 \times 4.87}{17.06}}$ (b) $3 \times 10^4 \div (8 \times 10^5)$ (c) $\sqrt{\dfrac{\pi^2}{\pi + 4}}$

5 Simplify:

(a) $\sqrt{12} + 4\sqrt{3} + \sqrt{27} - 3\sqrt{54}$ (b) $\dfrac{2}{\sqrt{5}} + \sqrt{20} + \dfrac{8}{\sqrt{80}}$

(c) $\dfrac{\sqrt{5} - \sqrt{2}}{\sqrt{5} + \sqrt{2}}$ (d) $\dfrac{3^{\frac{5}{4}} \times 15^{\frac{3}{4}}}{5^{-\frac{3}{4}} \times 45^{\frac{1}{2}}}$ (A challenge!)

6 Expand and simplify $\left(3\sqrt{5} - 2\sqrt{3}\right)^2$.

7 The correct expansion of $\left(\sqrt{5} + 2\sqrt{2}\right)\left(\sqrt{6} - \sqrt{5}\right)$ is:

A $\sqrt{30} + 2\sqrt{2} - 5$ B $\sqrt{6} + 2\sqrt{12} - 2\sqrt{10}$ C $4\sqrt{3} - \sqrt{10} - 5$ D $\sqrt{30} + 4\sqrt{3} - 2\sqrt{10} - 5$

8 Express $\dfrac{\sqrt{2} - 1}{2\sqrt{2} - 1}$ with a rational denominator.

9 Find the exact value of $x^4 - 2x^2 + 1$ when $x = 3\sqrt{2}$.

10 Show by substitution that $x = \dfrac{\sqrt{5} - 1}{2}$ is a root of the equation $x^3 + 3x^2 + x - 2 = 0$.

CHAPTER 2
ALGEBRA

Outcomes covered:

P1 demonstrates confidence in using mathematics to obtain realistic solutions to problems

P3 performs routine arithmetic and algebraic manipulation involving surds, simple rational expressions and trigonometric identities

P4 chooses and applies appropriate arithmetic, algebraic, graphical, trigonometric and geometric techniques

2.1 SIMPLIFYING ALGEBRAIC EXPRESSIONS

When adding and subtracting algebraic expressions, you can only combine like terms (that is, algebraic parts that have the same pronumerals).

Example 1

Simplify each expression by collecting the like terms.

(a) $3x + 2y + 5x - 6y$ **(b)** $x^2 + 2x - x + 3x^2$ **(c)** $2(3a - 4b) - 3(a - 5b)$

Solution

(a) $3x + 2y + 5x - 6y$
$= 3x + 5x + 2y - 6y$
$= 8x - 4y$

(b) $x^2 + 2x - x + 3x^2$
$= x^2 + 3x^2 + 2x - x$
$= 4x^2 + x$

(c) $2(3a - 4b) - 3(a - 5b)$
$= 6a - 8b - 3a + 15b$
$= 6a - 3a - 8b + 15b$
$= 3a + 7b$

EXERCISE 2.1 SIMPLIFYING ALGEBRAIC EXPRESSIONS

Simplify each expression by collecting the like terms.

1 $3x + 5 + 7x + 10$ **2** $7x - 3 + 3x - 2$ **3** $4a + b - a - 4b$

4 $6ab + 3ab + 5a + 4a$ **5** $3xy + 2xy - yx$ **6** $mn + 8mn - 3nm$

7 $3a^2b - 3ab^2 + 2a^2b$ **8** $5xy^2 + 3xy^2 - 2xy$ **9** $2x^2y + 3x^2y^2 - x^2y + 3x^2y^2$

10 $3abc + 5bca - 2cba$ **11** $3p^2q - 5pq - 2p^2q$ **12** $12mn + 3m - 6mn - m$

13 $x^2 - 3x + 2x + 4x^2$ **14** $a^2 + 5a^2 - 3a - 5a$ **15** $2x^2 + 5y^2 - 4x^2$

16 $9x^2 - 3xy + 5yx - 6x^2$

Simplify each expression by expanding the brackets and then collecting the like terms.

17 $5a - 3(a + b)$ **18** $4(2x - y) - 6x$ **19** $8m - 5(2m - 3n)$

20 $3(2x + 5y) + 4(x - y)$ **21** $5(2x + 3) - 5(x + 7)$ **22** $6(2a + 3b) + 3(a - b)$

23 $7(2k + 1) - 3(k + 2)$ **24** $5a(a + 2) - 3a(a + 1)$ **25** $5x(x - 2y) + 3x(2x - y)$

26 $4a(2a + b) - a(a + 2b)$ **27** $2a + 3b - (a - b)$ **28** $x + 5y - (3x + 2y)$

29 $3x(x-y)+2x(x-y)$ **30** $5x(2x+1)-(x^2+x)$ **31** $m(3m-5n)+3m(2m-n)$

32 $15(x-2)+4(3x-3)$ **33** $2(x^2-x-6)-3(x^2+2x)$ **34** $3x(x-2)-4(x-1)$

35 $3(x^2+5x-1)-(2x^2+x-2)$ **36** $5x+2y-3-(x-7y+9)$

37 The expression $a(a+1)-3(2a+1)$ simplifies to:

A a^2-7a-3 **B** a^2-5a+3 **C** a^2-5a-3 **D** a^2-7a+3

38 The expression $3(m^2-m)-2(m^2+2m+5)$ simplifies to:

A $5m^2-7m-10$ **B** $m^2-7m-10$ **C** m^2+m-10 **D** $m^2-7m+10$

2.2 SUBSTITUTION IN FORMULAE

Example 2

If $V=\pi r^2h$, find:

(a) V when $r=3.5$, $h=5$ **(b)** r when $V=275$, $h=14$

Solution

(a) $V=\pi\times 3.5^2\times 5$

$=61.25\pi$ (exact value)

$=192.4$ correct to one decimal place

(b) $r^2=\dfrac{V}{\pi h}$

$r^2=\dfrac{275}{14\times\pi}$

$r=\sqrt{\dfrac{275}{14\pi}}$ (exact value)

$r=2.5$ correct to one decimal place

EXERCISE 2.2 SUBSTITUTION IN FORMULAE

Use the value of π on your calculator. Give your answer correct to one decimal place when needed.

1 If $P=2(l+b)$, find the value of P when $l=20$, $b=12$.

2 If $E=IR$, find E when $I=2.4$, $R=40$.

3 If $F=ma$, find F when $m=50$, $a=0.2$.

4 If $F=\dfrac{9C}{5}+32$, find: **(a)** F when $C=60$ **(b)** C when $F=41$.

5 If $A=\pi r^2$, find A when $r=3.5$.

6 If $V=\pi r^2h$, find V when $r=4.2$, $h=10$.

7 If $E=mc^2$, find: **(a)** E when $m=10$, $c=1.6$ **(b)** c when $E=13.5$, $m=1.5$.

8 If $v=u+at$, find v when $u=20$, $a=1.8$, $t=10$.

9 If $s=ut+\frac{1}{2}at^2$, find: **(a)** s when $u=5$, $a=6$, $t=2.4$ **(b)** a when $s=50$, $t=2.5$, $u=10$.

10 If $v^2=u^2+2as$, find v when $u=12$, $a=2$, $s=20.25$.

11 If $s=\frac{1}{2}(u+v)t$, find s when $u=2.6$, $v=3.2$, $t=2.5$.

12 If $S=2\pi rh$, find S when $r=2.5$, $h=3.5$.

13 If $r = \sqrt{\frac{A}{\pi}}$, find: **(a)** r when $A = 154$ **(b)** A when $r = 1.75$.

14 If $E = \frac{m}{2}\left(v^2 - u^2\right)$, find E when $m = 4$, $v = 4$, $u = 2$.

15 If $t = a + (n-1)d$, find: **(a)** t when $a = 3.8$, $n = 20$, $d = 0.2$ **(b)** n when $a = 5.6$, $d = 5$, $t = 25.6$.

16 If $F = \frac{m(v-u)}{t}$, find F when $m = 20$, $v = 4$, $u = 2$, $t = 6$.

17 If $t = ar^5$, find t when $a = 64$, $r = 0.5$.

18 If $S = \frac{a\left(r^3 - 1\right)}{r-1}$, find S when $a = 5$, $r = 3$.

19 If $A = \pi(R^2 - r^2)$, find A when $R = 5.6$, $r = 1.4$.

20 If $V = \pi(R^2 - r^2)h$, find V when $R = 0.9$, $r = 0.2$, $h = 1.5$.

21 If $V = \frac{1}{3}\pi r^2 h$, find V when $r = 3$, $h = 3.5$.

22 If $P = \sqrt{\frac{2R - V}{5}}$, find: **(a)** P when $R = 50$, $V = 20$ **(b)** V when $P = 0.2$, $R = 20$.

23 If $W = \frac{1}{3}d\pi r^2 h$, find W when $d = 3$, $r = \frac{7}{11}$, $h = \frac{11}{14}$.

24 If $f = \frac{vu}{v+u}$, find: **(a)** f when $v = 20$, $u = 25$ **(b)** v when $f = 20$, $u = 25$.

25 If $A = P\left(1 + \frac{r}{100}\right)^n$, find A when $P = 1000$, $r = 10$, $n = 2$.

2.3 BASIC POLYNOMIALS

Common terms

A **monomial** is an expression that contains only **one** term, e.g. $5x$, x^2, $2ab$, $5a^2b^3$.

A **binomial** is an expression that contains **two** terms added or subtracted, e.g. $x + y$, $3a - 2b$, $x^2 + 1$, $3y - 4$.

A **trinomial** is an expression that contains **three** terms added or subtracted, e.g. $x^2 - 5x + 6$, $x + y - 4$, $4x^2 - 2xy + y^2$, $m + n - p$.

A **quadratic trinomial** is a trinomial of the form $ax^2 + bx + c$ (where $a \neq 0$, $b \neq 0$, $c \neq 0$); a is the coefficient of x^2, b is the coefficient of x, and c is the constant term.

Standard results

$(x + m)(x + n) = x^2 + (m + n)x + mn$

$(a + b)^2 = a^2 + 2ab + b^2$

$(a - b)^2 = a^2 - 2ab + b^2$

$(a - b)(a + b) = a^2 - b^2$

In each of these results, the expression on the left-hand side has been **expanded** to obtain the expression on the right.

If we start with the expression on the right-hand side, then we can **factorise** it to obtain the (usually) shorter form on the left.

Example 3

Expand and simplify each expression.

(a) $(x + 2)(x + 3)$ **(b)** $(3x - 2)(2x + 3)$ **(c)** $(2y + 5)^2$

(d) $(3x - 4)(3x + 4)$ **(e)** $(x + 2)(x^2 - 5x + 6)$ **(f)** $(x - 1)(x + 2)(x + 3)$

Solution

(a) $(x+2)(x+3)$
$= x^2 + 3x + 2x + 6$
$= x^2 + 5x + 6$

(b) $(3x-2)(2x+3)$
$= 6x^2 + 9x - 4x - 6$
$= 6x^2 + 5x - 6$

(c) $(2y+5)^2$
$= 4y^2 + 20y + 25$

(d) $(3x-4)(3x+4)$
$= 9x^2 - 16$

(e) $(x+2)(x^2-5x+6)$
$= x(x^2-5x+6) + 2(x^2-5x+6)$
$= x^3 - 5x^2 + 6x + 2x^2 - 10x + 12$
$= x^3 - 3x^2 - 4x + 12$

(f) $(x-1)(x+2)(x+3)$
$= (x-1)(x^2+5x+6)$
$= x^3 + 5x^2 + 6x - x^2 - 5x - 6$
$= x^3 + 4x^2 + x - 6$

EXERCISE 2.3 BASIC POLYNOMIALS

Write the expansion of the following.

1 $(x+5)(x+1)$ **2** $(x-2)(x-3)$ **3** $(a-3)(a+4)$

4 $(a+5)(a-4)$ **5** $(x-2)^2$ **6** $(y+7)^2$

7 $(2x+3)(x+5)$ **8** $(3x-4)(x-2)$ **9** $(3m+7)(2m-1)$

10 $(5x-3)(7x+2)$ **11** $(3x+2)(3x+2)$ **12** $(2p-9)(2p+9)$

13 $(3x+2)(2x+3)$ **14** $(x+1)(2x-1)$ **15** $(4p-5)^2$

16 $(3x+4)^2$ **17** $(x-3)(2x^2+3x+1)$ **18** $(3x^2-5x+2)(2x-4)$

19 $x(x-2)(x+2)$ **20** $(x-1)(x-1)(x-2)$ **21** $2(x-1)(x-2)(x-3)$

22 $(x^2+5)(x^2-2x-3)$ **23** $(x-2)(x+2)(x+2)$ **24** $(x^2-y^2)^3$

25 The correct expansion of $\left(\sqrt{x}+\sqrt{y}\right)^2$ is:

A $x^2+2xy+y^2$ **B** $x+y+2xy$ **C** $x+y+2\sqrt{xy}$ **D** $x^2+y^2+2\sqrt{xy}$

26 Indicate whether each answer is a correct or incorrect factorisation of $x^2-4xy+4y^2$.

(a) $(x+2y)^2$ **(b)** $(2y-x)^2$ **(c)** $(2x-y)^2$ **(d)** $(x-2y)^2$

2.4 FACTORISING BY GROUPING IN PAIRS

This method is used when there are four terms in the expression.

Example 4

Factorise:

(a) $bx+by+cx+cy$ **(b)** $m^2-mn-2m+2n$

Solution

Group in pairs:
(a) $\underbrace{bx+by}+\underbrace{cx+cy}$
(b) $\underbrace{m^2-mn}-\underbrace{2m+2n}$

Take out common factor:
(a) $= b(x+y)+c(x+y)$
(b) $= m(m-n)-2(m-n)$

Take out common factor:
(a) $= (x+y)(b+c)$
(b) $= (m-2)(m-n)$

EXERCISE 2.4 FACTORISING BY GROUPING IN PAIRS

Factorise:

1 $a(x+2)+b(x+2)$ **2** $3a(2b-3c)-m(2b-3c)$ **3** $p(a+b)+q(a+b)-r(a+b)$

4 $x^2(2x-1)+4(2x-1)$ **5** $ax+4a+bx+4b$ **6** $x^2-xy+xz-yz$

7 $2xy+2xz+y+z$ **8** $a^2-ab-ac+bc$ **9** $10y-25y^2+4x-10xy$

10 $a^3+3a^2b+ab^2+3b^3$ **11** $ac-2bc-2ad+4bd$ **12** $3xy-6y+7x-14$

13 $x^2-2xy-xz+2yz$ **14** $a^3-a^2b-ab+b^2$ **15** $2mn+2mp+pn^2+p^2n$

16 $x^3+3x^2+4x+12$ **17** $p^2q-pq^2+5p-5q$ **18** m^2p+m^2+np+n

19 x^2y+x^2+y+1 **20** $ab-3a-4b+12$ **21** $2x-6y-xy+3y^2$

22 When $3m^2-3mn-m+n$ is factorised, the answer is:

A $(3m-1)(m-n)$ **B** $(3m-n)(m-1)$ **C** $(3m-1)(m+n)$ **D** $(3m+1)(m-n)$

23 Indicate whether each answer is a correct or incorrect factorisation of $2x^3-2x^2-2x+2$.

(a) $2(x+1)(x+1)(x-1)$ **(b)** $2(x+1)(x-1)^2$ **(c)** $2(x+1)(x-1)(x-1)$ **(d)** $2(x-1)(x+1)^2$

2.5 FACTORISING USING THE DIFFERENCE OF TWO SQUARES

Remember the difference of two squares: $a^2-b^2=(a-b)(a+b)$

Example 5

Factorise:

(a) a^2-25 **(b)** $9x^2-49$ **(c)** $(x+1)^2-(y-1)^2$ **(d)** $a^3-a^2b-ab^2+b^3$

Solution

(a) $a^2-25=a^2-5^2$
$=(a-5)(a+5)$

(b) $9x^2-49=(3x)^2-7^2$
$=(3x-7)(3x+7)$

(c) $(x+1)^2-(y-1)^2$
$=[(x+1)-(y-1)][(x+1)+(y-1)]$
$=(x-y+2)(x+y)$

(d) $a^3-a^2b-ab^2+b^3$
$=a^2(a-b)-b^2(a-b)$
$=(a^2-b^2)(a-b)$
$=(a-b)(a+b)(a-b)$
$=(a-b)^2(a+b)$

EXERCISE 2.5 FACTORISING USING THE DIFFERENCE OF TWO SQUARES

Factorise:

1 m^2-1 **2** x^2-16 **3** $64-m^2$ **4** $9a^2-25$

5 $x^2-0.36$ **6** $a^2b^2-c^2$ **7** $9x^2-4y^2$ **8** $(x+1)^2-9$

9 $x^2-y^2z^2$ **10** $\frac{a^2}{25}-1$ **11** $p^2-\frac{1}{4}$ **12** $\frac{x^2}{4}-\frac{1}{9}$

13 $(a+2)^2-4$ **14** $x^2-(y+z)^2$ **15** 99^2-1 **16** 523^2-477^2

17 a^3b-ab^3 **18** $12a^3-3ab^2$ **19** $3x^2y-27y$ **20** $(x+y)^2-4$

21 $a^2-(a-b)^2$ **22** $x^3-x^2y-9x+9y$ **23** $x^3+3x^2-4x-12$ **24** $p^2q-p^2-16q+16$

25 a^2x-x **26** $48a^2-75b^2$ **27** $(1+h)^2-1$ **28** $\frac{x^2}{25}-y^2$

29 When $(p+2)^2-(p-2)^2$ is factorised, the answer is:

A $2p^2+8$ **B** $-8p$ **C** $2p^2-8$ **D** $8p$

30 Indicate whether each answer is a correct or incorrect factorisation of $\frac{a^2}{b^2}-\frac{b^2}{a^2}$.

(a) $\left(\frac{a}{b}-\frac{b}{a}\right)\left(\frac{a}{b}+\frac{b}{a}\right)$ **(b)** $\left(\frac{a}{b}-\frac{b}{a}\right)\left(\frac{b}{a}+\frac{a}{b}\right)$ **(c)** $\left(\frac{a}{b}-\frac{b}{a}\right)\left(\frac{a}{b}-\frac{b}{a}\right)$ **(d)** $\frac{(a-b)(a+b)(a^2+b^2)}{a^2b^2}$

2.6 SUM AND DIFFERENCE OF TWO CUBES

Two important identities are: $a^3+b^3=(a+b)(a^2-ab+b^2)$

$a^3-b^3=(a-b)(a^2+ab+b^2)$

The identities can be verified by expanding the right-hand side.

Example 6

Factorise:

(a) x^3-8 **(b)** $27y^3+64x^3$ **(c)** $(x+2)^3+y^3$ **(d)** $x^2y^3-z^2y^3-x^2w^3+z^2w^3$

Solution

(a) $x^3-8=x^3-2^3$

$=(x-2)(x^2+2x+4)$

(b) $27y^3+64x^3=(3y)^3+(4x)^3$

$=(3y+4x)(9y^2-12xy+16x^2)$

(c) $(x+2)^3+y^3$

$=(x+2+y)[(x+2)^2-(x+2)y+y^2]$

$=(x+2+y)(x^2+4x+4-xy-2y+y^2)$

(d) $x^2y^3-z^2y^3-x^2w^3+z^2w^3$

$=y^3(x^2-z^2)-w^3(x^2-z^2)$

$=(x^2-z^2)(y^3-w^3)$

$=(x-z)(x+z)(y-w)(y^2+yw+w^2)$

EXERCISE 2.6 SUM AND DIFFERENCE OF TWO CUBES

Factorise:

1 y^3-125 **2** z^3+1 **3** $8p^3+27$ **4** $216-a^3$

5 $(x+5)^3+(x-2)^3$ **6** $(2x+3)^3-(x-4)^3$ **7** b^6-a^6 **8** $64a^3+8b^3$

9 $\frac{4}{3}\pi R^3-\frac{4}{3}\pi r^3$ **10** $p^7x^4-p^4x^7$ **11** x^6+y^6 **12** $\frac{8}{a^3}-\frac{27}{b^3}$

13 $a^3m^3+a^3n^3-b^3n^3-b^3m^3$ **14** $4x^5-9x^3-4x^2+9$ **15** $(x+h)^3-x^3$

16 $a^3+(a-b)^3$ **17** $(a+b)^3-(a-b)^3$ **18** $(2x+1)^3-(2x-1)^3$

19 $8-(2-x)^3$ **20** $a^5b^4-a^2b$ **21** $2(x-y)^3+54$

22 When $1000p^3 - q^6$ is factorised, the answer is:

A $(10p - q)(100p^2 + 10pq + q^2)$ **B** $(10p - q^2)(100p^2 + 10pq^2 + q^4)$

C $(10p + q)(100p^2 - 10pq + q^2)$ **D** $(10p + q^2)(100p^2 - 10pq^2 + q^4)$

23 When $(2x + 1)^3 + (2x - 1)^3$ is factorised, the answer is:

A $2(12x^2 + 1)$ **B** $4x(12x^2 + 1)$ **C** $2(4x^2 + 3)$ **D** $4x(4x^2 + 3)$

2.7 FACTORISING QUADRATIC TRINOMIALS

To factorise quadratic trinomials, you must remember how to expand binomial products and then work backwards. We know the following:

- $(x + m)(x + n) = x^2 + (m + n)x + mn = x^2 + (\text{sum of } m \text{ and } n)x + (\text{the product of } m \text{ and } n)$
- $(x - m)(x - n) = x^2 - (m + n)x + mn = x^2 + (\text{sum of } -m \text{ and } -n)x + (\text{the product of } -m \text{ and } -n)$
- $(x + m)(x - n) = x^2 + (m - n)x - mn = x^2 + (\text{sum of } m \text{ and } -n)x + (\text{the product of } m \text{ and } -n)$

To factorise $x^2 + 5x + 6$ you must write it in the form $(x + m)(x + n)$, where $m + n = 5$ and $mn = 6$. This means you must find two numbers whose sum is 5 and whose product is 6.

Example 7

Factorise:

(a) $x^2 + 5x + 6$ **(b)** $x^2 - 7x + 10$ **(c)** $x^2 + x - 12$ **(d)** $x^2 - 6x + 9$ **(e)** $x^2 - 5x - 24$

Solution

(a) $x^2 + 5x + 6$

Write: $x^2 + 5x + 6 = (x + m)(x + n)$

Look for numbers m and n whose sum is 5 and whose product is 6.

List possible factors of 6 and check the sum: $6 \times 1 = 6$ $6 + 1 = 7$

$3 \times 2 = 6$ $3 + 2 = 5$

Hence: $x^2 + 5x + 6 = (x + 3)(x + 2)$

The information could also be set out using the cross method:

x $\times$ 6 3

x 1 2

The correct pair will give $5x$ when multiplied across.

(b) $x^2 - 7x + 10$

Write: $x^2 - 7x + 10 = (x + m)(x + n)$

Look for numbers m and n whose sum is -7 and whose product is 10.

Since the sum is negative and the product is positive, both the numbers are negative.

List possible factors of 10 and check the sum: $-10 \times (-1) = 10$ $-10 + (-1) = -11$

$-5 \times (-2) = 10$ $-5 + (-2) = -7$

Hence: $x^2 - 7x + 10 = (x - 5)(x - 2)$

(c) $x^2 + x - 12$

Write: $x^2 + x - 12 = (x + m)(x + n)$

Look for numbers m and n whose sum is 1 and whose product is -12.

Since the sum is positive and the product is negative, the numbers have different signs and the larger number is positive.

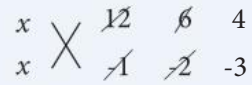
List possible factors of -12 and check the sum: $12 \times (-1) = -12$ $12 + (-1) = 11$

$6 \times (-2) = -12$ $6 + (-2) = 4$

$4 \times (-3) = -12$ $4 + (-3) = 1$

Hence: $x^2 + x - 12 = (x + 4)(x - 3)$

Using the cross method:



The correct pair will give x when multiplied across.

(d) $x^2 - 6x + 9$

Write: $x^2 - 6x + 9 = (x + m)(x + n)$

Look for numbers m and n whose sum is -6 and whose product is 9.

Since the sum is negative and the product is positive, both the numbers are negative.

List possible factors of 9 and check the sum: $-9 \times (-1) = 9$ $-9 + (-1) = -10$

$-3 \times (-3) = 9$ $-3 + (-3) = -6$

Hence: $x^2 - 6x + 9 = (x - 3)(x - 3) = (x - 3)^2$

(e) $x^2 - 5x - 24$

Write: $x^2 - 5x - 24 = (x + m)(x + n)$

Look for numbers m and n whose sum is -5 and whose product is -24.

Since the sum is negative and the product is negative, the numbers have different signs and the smaller number is positive.

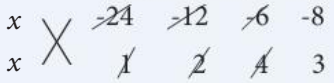
List possible factors of -24 and check the sum: $-24 \times 1 = -24$ $-24 + 1 = -23$

$-12 \times 2 = -24$ $-12 + 2 = -10$

$-6 \times 4 = -24$ $-6 + 4 = -2$

$-8 \times 3 = -24$ $-8 + 3 = -5$

Hence: $x^2 - 5x - 24 = (x - 8)(x + 3)$



The correct pair will give $-5x$ when multiplied across.

With practice, you will be able to write the factors simply by looking at the sum and product.

TECHNOLOGY EXPLORATION

Factorising trinomials

In GeoGebra, enter *y=<trinomial>* in the input bar (where *<trinomial>* is the quadratic trinomial to be graphed, for example $x^2 + 2x - 9$). The resulting graph will show the zeros of the trinomial function and can be used to check the factorisation.

EXERCISE 2.7 FACTORISING QUADRATIC TRINOMIALS

Factorise the following quadratic trinomials:

1 $x^2 + 4x + 3$ **2** $x^2 + 10x + 21$ **3** $x^2 + 11x + 24$ **4** $x^2 + 6x + 5$

5 $a^2 + 7a + 6$ **6** $a^2 + 12a + 32$ **7** $m^2 + 9m + 20$ **8** $p^2 + 9p + 18$

9 $p^2 + 11p + 18$ **10** $p^2 + 19p + 18$ **11** $x^2 + 7x + 12$ **12** $x^2 + 13x + 12$

13 $x^2 + 8x + 12$ **14** $x^2 - 7x + 12$ **15** $x^2 - 13x + 12$ **16** $x^2 - 8x + 12$

17 $m^2 - 9m + 20$ **18** $m^2 - 12m + 20$ **19** $m^2 - 21m + 20$ **20** $x^2 - 14x + 13$

21 $p^2 + 2p - 15$ **22** $p^2 + 14p - 15$ **23** $p^2 - 2p - 15$ **24** $p^2 - 14p - 15$

25 $x^2 - 2x - 35$ **26** $x^2 - 3x - 10$ **27** $x^2 + 17x + 72$ **28** $a^2 - 4a - 12$

29 $x^2 - 7x + 6$ **30** $x^2 - x - 72$ **31** $x^2 + 6x - 72$ **32** $x^2 - 21x - 72$

33 $a^2 + 13a + 30$ **34** $x^2 - x - 42$ **35** $x^2 - 19x - 42$ **36** $x^2 + 19x - 42$

37 The factors of $x^2 - 11x - 42$ are:

A $(x + 14)(x - 3)$ **B** $(x - 7)(x + 6)$ **C** $(x - 6)(x + 7)$ **D** $(x - 14)(x + 3)$

38 Indicate whether each answer is a correct or incorrect factorisation of $x^2 - 8x + 7$.

(a) $(x + 1)(x - 7)$ **(b)** $(1 - x)(7 - x)$ **(c)** $(x - 1)(x + 7)$ **(d)** $(x - 1)(x - 7)$

2.8 FACTORISING NON-MONIC TRINOMIALS

When the x^2 term in the quadratic has a coefficient other than 1, finding the factors becomes more difficult because there are more possibilities. You must find factors of the coefficient of x^2 as well as the factors of the constant term and get the correct pairs together. You could use trial and error, but the cross method is easier because it keeps the information more organised.

Example 8

Factorise $6x^2 + 19x + 10$.

Solution

Write: $6x^2 + 19x + 10 = (ax + m)(bx + n) = abx^2 + (an + bm)x + mn$

This gives: $ab = 6$, $an + bm = 19$, $mn = 10$

List the factors of 6: 6, 1 or 3, 2

List the factors of 10: 10, 1 or 5, 2

List possible binomial factors:

$(6x + 10)(x + 1)$ $(6x + 1)(x + 10)$ $(6x + 5)(x + 2)$ $(6x + 2)(x + 5)$

$(3x + 10)(2x + 1)$ $(3x + 1)(2x + 10)$ $(3x + 5)(2x + 2)$ $(3x + 2)(2x + 5)$

You can expand the binomial factors to see which answer gives the original quadratic trinomial.

Before doing this, you can eliminate any possibility that has a common factor, because there is no common factor in the original quadratic trinomial. This means you can eliminate the answers containing $(6x + 10)$, $(6x + 2)$, $(2x + 10)$ and $(2x + 2)$, because they all have a common factor of 2 (and the original quadratic trinomial does not).

Expand the others:

$(6x + 1)(x + 10) = 6x^2 + 61x + 10$ Not correct

$(6x + 5)(x + 2) = 6x^2 + 17x + 10$ Not correct

$(3x + 10)(2x + 1) = 6x^2 + 23x + 10$ Not correct

$(3x + 2)(2x + 5) = 6x^2 + 19x + 10$ Correct

Hence: $6x^2 + 19x + 10 = (3x + 2)(2x + 5)$

Alternatively, using the cross method:

$6x$	10	1	5	2	or	$3x$	10	1	5	2
x	1	10	2	5		$2x$	1	10	2	5

The correct pair will give $19x$ when multiplied across.

Hence: $6x^2 + 19x + 10 = (3x + 2)(2x + 5)$

Example 9

Factorise $4x^2 - 4x - 15$.

Solution

The factors of $4x^2$ are either $4x$ and x, or $2x$ and $2x$.

The factors of -15 are -15 and 1, 15 and -1, -5 and 3, or 5 and -3.

Set up the cross method:

$4x$		-15	15	-5	5	or	$4x$	1	-1	3	-3
x		1	-1	3	-3		x	-15	15	-5	5

or

$2x$	-15	15	-5	5	or	$2x$	1	-1	3	-3
$2x$	1	-1	3	-3		$2x$	-15	15	-5	5

The only combination that gives $-4x$ is:

$2x$		-5
$2x$		3

Hence: $4x^2 - 4x - 15 = (2x - 5)(2x + 3)$

Example 10

Factorise $3x^2 + 8x - 16$.

Solution

The factors of $3x^2$ are $3x$ and x.

The factors of -16 are -16 and 1; 16 and -1; -8 and 2; 8 and -2; -4 and 4; or 4 and -4.

Set up the cross method:

$3x$		-16	1	16	-1	-8	2	-2	8	-4	4
x		1	-16	-1	16	2	-8	8	-2	4	-4

The only combination that gives $8x$ is:

$3x$		-4
x		4

Hence: $3x^2 + 8x - 16 = (3x - 4)(x + 4)$

EXERCISE 2.8 FACTORISING NON-MONIC TRINOMIALS

Factorise:

1 $2x^2 + 3x + 1$	**2** $3x^2 + 11x - 4$	**3** $2x^2 + 7x + 6$	**4** $4a^2 + 13a + 3$
5 $4x^2 + 5x + 1$	**6** $3a^2 - 5a + 2$	**7** $8x^2 - 14x + 3$	**8** $2x^2 - 9x - 5$
9 $13c^2 - 7c - 6$	**10** $8x^2 + 14x + 5$	**11** $6x^2 + 17x + 12$	**12** $3x^2 - 13x + 4$
13 $3x^2 - 17x + 10$	**14** $6a^2 - 13a - 63$	**15** $3x^2 - 11x - 4$	**16** $10x^2 - 11x - 8$
17 $2x^2 + 3x - 2$	**18** $4x^2 - 12x + 9$	**19** $9x^2 - 12x + 4$	**20** $2x^2 - 9x + 10$
21 $6x^2 - 85x + 14$	**22** $15m^2 + 17m - 18$	**23** $10a^2 - 41a - 18$	**24** $y^2 - 2y - 3$
25 $12y^2 + 14y - 6$	**26** $6x^2 - 25x + 14$	**27** $6x^2 - 29x + 28$	**28** $6x^2 - 19x + 14$
29 $6x^2 - 20x + 14$	**30** $8x^2 + 2x - 3$	**31** $6p^2 + 25p + 21$	**32** $10a^2 - 11a - 6$
33 $12y^2 + 28y - 5$	**34** $24x^2 - 59x + 36$	**35** $15x^2 - 19x + 6$	**36** $3x^2 - 2x - 1$
37 $9x^2 + 9x - 10$	**38** $2x^2 - 9x + 4$	**39** $10p^2 + 11p + 3$	**40** $5x^2 - 2x - 3$

41 The factors of $8x^2 - 6x - 9$ are:

A $(4x-3)(2x+3)$ **B** $(8x-9)(x+1)$ **C** $(4x+3)(2x-3)$ **D** $(4x-9)(2x+1)$

42 Indicate whether each answer is a correct or incorrect factorisation of $4x^2 + 12x + 9$.

(a) $(2x+3)(2x+3)$ **(b)** $(3+2x)(3+2x)$ **(c)** $(2x+3)^2$ **(d)** $(2x-3)^2$

2.9 MIXED FACTORISATIONS

To factorise the expressions in the next exercises, you will need to use **one or more** of the techniques learned so far:

- remove a common factor
- group in pairs and remove common factors
- difference of two squares
- quadratic trinomials.

EXERCISE 2.9 MIXED FACTORISATIONS

Factorise completely:

1 $x^2 - 3x$ **2** $2a^3 - 8a$ **3** $3x^2 + 9x$ **4** $x^2 - 8x$

5 $x^2 - 9$ **6** $x^2 - 8x - 9$ **7** $3x^2y - 12y^3$ **8** $5x^3y - 20xy^3$

9 $1 - (b+c)^2$ **10** $10x^2 + 9x - 1$ **11** $(a+b)^2 - b^2$ **12** $6x^2 - 24$

13 $a^2 - a - 42$ **14** $a(m+n) - b(m+n)$ **15** $2x^3 + 14x^2 - 16x$ **16** $3a^3 + 24a^2 + 21a$

17 $(x+2y)^2 - 4$ **18** $x^2 - 3x - 10$ **19** $a^2 - 1$ **20** $ab^2 + abc + abd$

21 $x^2 - 36y^2$ **22** $x^2 + x$ **23** $x^2 + x - 12$ **24** $x(y-z) + y(y-z)$

25 $4x^2 - 28x - 480$ **26** $bx^2 - 14bxy + 49by^2$ **27** $6y^3 + 3y^2 - 3y$ **28** $6y^3 + 26y^2 + 8y$

29 $15a^2 - 60$ **30** $9mn - 25m^3n^3$ **31** $5a^2x - 125x$ **32** $(x+y)^2 - (x-y)^2$

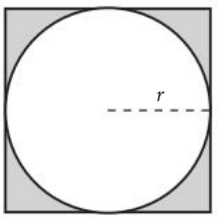
33 $5t^3 + 5t^2 - 360t$ **34** $m^2 - mn + 6m - 6n$ **35** $x^2(x+3) - 4(x+3)$ **36** $mx^2 - xy + ly - mlx$

37 The factors of $4 - (x+1)^2$ are:

A $(3-x)(5+x)$ **B** $(2-x)(2+x)$ **C** $(1+x)(3-x)$ **D** $(1-x)(3+x)$

For questions **38** to **45**, write an algebraic expression in factorised form for the shaded area in each of the figures.

38



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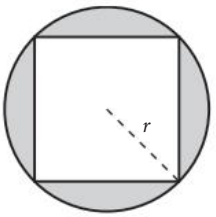

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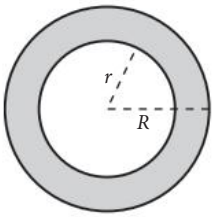

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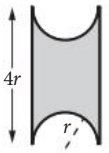

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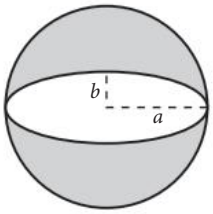

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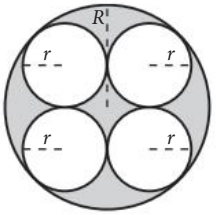

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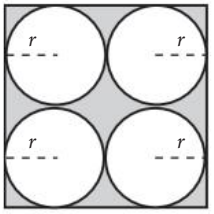

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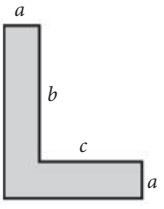


46 Indicate whether each answer is a correct or incorrect expression for the shaded area in question **45**.

(a) $(a+b)(a+c)-bc$ **(b)** $a^2+ab+ac$ **(c)** $(a+b)(a+c)$ **(d)** $a(a+b+c)$

2.10 ALGEBRAIC FRACTIONS

To **simplify** algebraic fractions:

1. Factorise the numerator and denominator.
2. Cancel any common factors.

Example 11

Simplify:

(a) $\dfrac{9x+6}{3x+2}$ **(b)** $\dfrac{15a^2-5ab}{10ab}$ **(c)** $\dfrac{9x^2-y^2}{6xy-2y^2}$

Solution

(a) $\dfrac{9x+6}{3x+2}=\dfrac{3(3x+2)}{3x+2}$

$=3$

(b) $\dfrac{15a^2-5ab}{10ab}=\dfrac{5a(3a-b)}{10ab}$

$=\dfrac{3a-b}{2b}$

(c) $\dfrac{9x^2-y^2}{6xy-2y^2}=\dfrac{(3x-y)(3x+y)}{2y(3x-y)}$

$=\dfrac{3x+y}{2y}$

Example 12

Simplify: $\dfrac{x^2-5x+6}{x^2-9}\times\dfrac{x^2+3x}{x^2-x-2}$

Solution

$\dfrac{x^2-5x+6}{x^2-9}\times\dfrac{x^2+3x}{x^2-x-2}=\dfrac{(x-2)(x-3)}{(x-3)(x+3)}\times\dfrac{x(x+3)}{(x+1)(x-2)}$

$=\dfrac{x}{x+1}$

EXERCISE 2.10 ALGEBRAIC FRACTIONS

Simplify:

1 $\dfrac{8a-4b}{4}$ **2** $\dfrac{15x+10y}{15}$ **3** $\dfrac{14x-7y}{2x-y}$ **4** $\dfrac{8x^2-4xy}{8xy}$

5 $\dfrac{12ab-6b^2}{9ab}$ **6** $\dfrac{8x+2}{4x+1}$ **7** $\dfrac{3a-5b}{3a^2-5ab}$ **8** $\dfrac{m+m^2}{m}$

9 $\dfrac{mn-n^2}{n}$ **10** $\dfrac{p^2q-pq^2}{pq}$ **11** $\dfrac{x^2+xy}{2x}$ **12** $\dfrac{2rs-12r}{r^2+rs}$

13 $\dfrac{x^2-y^2}{(x+y)^2}$ **14** $\dfrac{k^2+k}{k+1}$ **15** $\dfrac{x^2-9}{x^2+3x}$ **16** $\dfrac{15a^2-5ab}{3ab-b^2}$

17 $\dfrac{4x^2-4xy}{x^2-y^2}$ **18** $\dfrac{x^2-7x+6}{x^2-36}$ **19** $\dfrac{a^2+ab}{ab+b^2}$ **20** $\dfrac{a^2-b^2}{a^2+ab}$

21 $\dfrac{x^2-1}{x^2-5x+4}$ **22** $\dfrac{x^2-6x+8}{x^2-x-2}$ **23** $\dfrac{x^2+3x+2}{x^2-4}$ **24** $\dfrac{x^2-5x+6}{x^2+x-12}$

25 $\dfrac{x^2+4x+4}{x^2-3x-10}$ **26** $\dfrac{4x^3y-16xy}{x^2+2x-8}$ **27** $\dfrac{12a+9}{15}\times\dfrac{5}{4a+3}$ **28** $\dfrac{3x^2-xy}{xy}\times\dfrac{x^2y}{3xy-y^2}$

29 $\dfrac{m^2+m-2}{m^2-m}$ simplifies to:

A $\dfrac{(m-2)(m+1)}{m(m-1)}$ B 2 C $\dfrac{m+2}{m}$ D $\dfrac{m-2}{m}$

30 $\dfrac{2a^2-3ab}{ab-b^2}\times\dfrac{2a^2-2ab}{4a-6b}$ **31** $\dfrac{15x^2-5xy}{10xy}\div\dfrac{3x-y}{2y}$ **32** $\dfrac{12x^2-4x}{3x^2-x}\div\dfrac{10x^2y}{5x^2y^2}$

33 $\dfrac{x^2-2x-3}{x^2-4x-5}\times\dfrac{x^2-25}{(x-3)(x+5)}$ **34** $\dfrac{(a+2b)(a-b)}{a^2-4b^2}\times\dfrac{a^2-3ab+2b^2}{ab-b^2}$

35 Simplify $\dfrac{m^2-9}{m^2-m-12}\div\dfrac{m^2-3m}{m^2-9m+20}$. Indicate whether each answer is correct or incorrect.

(a) $\dfrac{m(m-3)^2}{(m-4)(m-5)}$ (b) $\dfrac{3}{4}$ (c) $\dfrac{m+5}{m}$ (d) $\dfrac{m-5}{m}$

Simplify:

36 $\dfrac{x^3+y^3}{x^2-y^2}$ **37** $\dfrac{8x^2+4x+2}{8x^3-1}$ **38** $\dfrac{2x+2y}{x^3-y^3}\times\dfrac{x^2-2xy+y^2}{x^2-y^2}$

39 $\dfrac{(x+h)^3-x^3}{h}$ **40** $\dfrac{x^3-(x-y)^3}{x^2-(x-y)^2}$

2.11 ADDING AND SUBTRACTING ALGEBRAIC FRACTIONS

When adding and subtracting fractions, first rewrite each fraction with the same (common) denominator, then add or subtract the numerators.

Example 13

Express as a single fraction:

(a) $\dfrac{3}{5}-\dfrac{2}{7}$ (b) $\dfrac{3x}{5}-\dfrac{2x+1}{7}$ (c) $\dfrac{3}{5x}+\dfrac{2}{7x}$ (d) $\dfrac{2x-y}{3}-\dfrac{x-y}{6}$

Solution

(a) $\dfrac{3}{5}-\dfrac{2}{7}$
$=\dfrac{21}{35}-\dfrac{10}{35}$
$=\dfrac{11}{35}$

(b) $\dfrac{3x}{5}-\dfrac{2x+1}{7}$
$=\dfrac{21x}{35}-\dfrac{5(2x+1)}{35}$
$=\dfrac{21x-10x-5}{35}$
$=\dfrac{11x-5}{35}$

(c) $\dfrac{3}{5x}+\dfrac{2}{7x}$
$=\dfrac{21}{35x}+\dfrac{10}{35x}$
$=\dfrac{31}{35x}$

(d) $\dfrac{2x-y}{3}-\dfrac{x-y}{6}$
$=\dfrac{2(2x-y)}{6}-\dfrac{(x-y)}{6}$
$=\dfrac{4x-2y-x+y}{6}$
$=\dfrac{3x-y}{6}$

EXERCISE 2.11 ADDING AND SUBTRACTING ALGEBRAIC FRACTIONS

Simplify:

1 $\dfrac{x}{5}-\dfrac{x}{6}$ **2** $\dfrac{3x}{8}+\dfrac{x}{2}$ **3** $\dfrac{a}{3}+\dfrac{4a}{5}-\dfrac{a}{6}$ **4** $\dfrac{y}{2}+\dfrac{2y}{3}-\dfrac{y}{4}$

5 $\frac{a+2}{5}-\frac{a-1}{3}$ **6** $\frac{2x-y}{3}-\frac{x-3y}{6}$ **7** $\frac{3x+2}{6}-\frac{x+1}{4}$ **8** $\frac{3m-2n}{5}+\frac{m+n}{10}$

9 $\frac{x}{2}+\frac{y}{4}-\frac{x+y}{3}$ **10** $\frac{a-2b}{6}-\frac{2a+b}{9}$ **11** $\frac{3(a+b)}{4}-\frac{a-b}{6}$ **12** $\frac{1}{x}-\frac{2}{3x}$

13 $\frac{3}{a}+\frac{1}{a^2}$ **14** $\frac{1}{ab}-\frac{2}{b}$ **15** $\frac{m}{n}-\frac{n}{m}$ **16** $\frac{4}{xy}+\frac{3}{yz}$

17 $\frac{5}{a^2b}-\frac{2}{ab^2}$ **18** $\frac{a+1}{6a}+\frac{a-4}{2a}$ **19** $\frac{1}{x+1}+\frac{2}{3}$ **20** $\frac{1}{x}+\frac{2}{x}-\frac{1}{x^2}$

21 Simplify $\frac{1}{ab}+\frac{a}{bc}$. Which of the following is correct?

A $\frac{1+a}{abc}$ B $\frac{c+a^2}{abc}$ C $\frac{1+a}{ab^2c}$ D $\frac{c+a^2}{ab^2c}$

2.12 HARDER ALGEBRAIC FRACTIONS

More complex algebraic fractions require you to factorise the denominator before you find the common denominator. Write each fraction with the common denominator before you add or subtract the numerators.

Example 14

Express as a single fraction:

(a) $\frac{3}{x^2-4}+\frac{1}{x-2}$ (b) $\frac{1}{x-y}-\frac{1}{x+y}$ (c) $\frac{3}{x^2+2x}-\frac{2}{x^2-4}$ (d) $\frac{1}{x^2-5x+6}-\frac{1}{x^2+2x-8}$

Solution

(a) $\frac{3}{x^2-4}+\frac{1}{x-2}$

$=\frac{3}{(x-2)(x+2)}+\frac{1}{x-2}$

$=\frac{3}{(x-2)(x+2)}+\frac{x+2}{(x-2)(x+2)}$

$=\frac{x+5}{(x-2)(x+2)}$

(b) $\frac{1}{x-y}-\frac{1}{x+y}$

$=\frac{x+y}{(x-y)(x+y)}-\frac{x-y}{(x-y)(x+y)}$

$=\frac{x+y-(x-y)}{(x-y)(x+y)}$

$=\frac{2y}{(x-y)(x+y)}$

(c) $\frac{3}{x^2+2x}-\frac{2}{x^2-4}$

$=\frac{3}{x(x+2)}-\frac{2}{(x-2)(x+2)}$

$=\frac{3(x-2)}{x(x+2)(x-2)}-\frac{2x}{x(x-2)(x+2)}$

$=\frac{3x-6-2x}{x(x+2)(x-2)}$

$=\frac{x-6}{x(x+2)(x-2)}$

(d) $\frac{1}{x^2-5x+6}-\frac{1}{x^2+2x-8}$

$=\frac{1}{(x-2)(x-3)}-\frac{1}{(x-2)(x+4)}$

$=\frac{x+4}{(x-2)(x-3)(x+4)}-\frac{x-3}{(x-2)(x-3)(x+4)}$

$=\frac{x+4-(x-3)}{(x-2)(x-3)(x+4)}$

$=\frac{x+4-x+3}{(x-2)(x-3)(x+4)}$

$=\frac{7}{(x-2)(x-3)(x+4)}$

EXERCISE 2.12 HARDER ALGEBRAIC FRACTIONS

For questions **1** to **9**, write the lowest common multiple (LCM).

1 $(x-3)$ and $(x+3)$ **2** $(x-5)$ and (x^2-25) **3** x and $(x-2)$

4 $(2x-4)$ and $(3x-6)$ **5** (x^2-4x) and $(x-4)$ **6** (x^2-4x) and (x^2-16)

7 $(x+2)$ and (x^2+4x+4) **8** $(x-y)$, $(x+y)$, and (x^2-y^2) **9** (x^2-y^2), (x^2+xy), $(xy-y^2)$

10 The lowest common multiple of x, $x+3$ and x^2-9 is:

A $x(x-3)(x+3)$ **B** $x(x-3)(x^2-9)$ **C** $x(x+3)(x^2-9)$ **D** $x(x-3)^2$

Express each of the following as a single fraction.

11 $\frac{1}{a-b}+\frac{1}{a+b}$ **12** $\frac{3}{x-y}-\frac{2}{x+y}$ **13** $\frac{x}{x-y}+\frac{y}{x-y}$

14 $\frac{x}{x-y}+\frac{y}{x+y}$ **15** $\frac{3a-b}{a^2-b^2}+\frac{1}{a-b}$ **16** $\frac{1}{x^2-4}-\frac{1}{x+2}$

17 $\frac{x}{x^2-y^2}-\frac{y}{x^2-y^2}$ **18** $\frac{3}{(x-2)^2}+\frac{2}{x-2}$ **19** $\frac{1}{x^2-4x+3}-\frac{1}{x^2-1}$

20 $\frac{3}{x-2}+\frac{1}{x+3}$ **21** $\frac{1}{x+y}-\frac{1}{x-y}$ **22** $\frac{5}{2a+6}+\frac{a}{a^2-9}$

23 $\frac{1}{x-5}-\frac{1}{x+5}+\frac{x+10}{x^2-25}$ **24** $\frac{6}{3x-2}-\frac{8}{4x+1}$ **25** $\frac{7a}{3a-4}-\frac{5a}{2a-3}$

26 $\frac{y}{x^2-xy}+\frac{1}{x}$ **27** $\frac{1}{x+4}+\frac{x}{x^2-16}$ **28** $\frac{1}{x+2}+\frac{1}{x-2}+\frac{4}{x^2-4}$

29 $\frac{1}{a+3}+\frac{a+4}{a^2+5a+6}$ **30** $\frac{3}{x^2-4}-\frac{2}{x^2-3x+2}$ **31** $\frac{x+1}{x-1}-\frac{x-1}{x+1}$

32 $\frac{5}{x-2}+\frac{3}{x^2-4}$ **33** $\frac{3x}{x^2-16}-\frac{2}{x+4}$ **34** $\frac{5}{3x}-\frac{2}{x^2-5x}$

35 Simplify $\frac{5}{x+1}-\frac{3}{x-2}$. Indicate whether each answer is correct or incorrect.

(a) $\frac{2x-7}{(x+1)(x-2)}$ **(b)** $\frac{2x-13}{(x+1)(x-2)}$ **(c)** $\frac{2}{(x+1)(x-2)}$ **(d)** $\frac{2x-1}{(x+1)(x-2)}$

CHAPTER REVIEW 2

1 Expand and simplify:

(a) $(2x-5)^2$ **(b)** $(x+3)(x-7)$ **(c)** $(2y+1)(3y+4)$

(d) $(5x-4)(5x+4)$ **(e)** $(2x-y)(x^2-xy+y^2)$

2 Factorise:

(a) $81-4a^2$ **(b)** $10xy-5y$ **(c)** $5x^2y-10xy^2-5xy$ **(d)** $a^2-18a+56$

(e) $16x^2-1$ **(f)** $3x^2+4x-7$ **(g)** $a^2-b^2+2a-2b$ **(h)** $3x+7x^2-6x^3$

(i) $8a^3-27$ **(j)** $8-(x+h)^3$ **(k)** $8y^2-6y-9$ **(l)** x^4-8x

3 If $a=-3$ and $b=4$, evaluate $\sqrt{a^2+b^2}$.

4 Simplify:

(a) $\dfrac{3x^3}{4a^2}\times\dfrac{ay-a}{xy^2}\div\dfrac{3y-3}{4ay^2}$ **(b)** $\dfrac{5x}{3}-\dfrac{2x+3}{4}+\dfrac{x}{6}$ **(c)** $\dfrac{12m^2-4n}{3m^2-n}\div\dfrac{10m^2n}{5m^2n^2}$

(d) $\dfrac{3x+4}{x^2}-\dfrac{5}{x}$ **(e)** $\dfrac{3}{x+1}+\dfrac{1}{x^2+2x+1}$ **(f)** $\dfrac{2x^2-3xy}{xy-y^2}\div\dfrac{4x-6y}{2x^2-2xy}$

(g) $\dfrac{a^3+b^3}{a^2-b^2}$ **(h)** $\dfrac{2}{m^2-4}-\dfrac{1}{m^2-3m+2}$ **(i)** $\dfrac{1}{x^2-9x+20}+\dfrac{1}{x^2-11x+30}$

5 Expand and simplify:

(a) $\left(a^{\frac{1}{3}}-b^{\frac{1}{3}}\right)\left(a^{\frac{2}{3}}+a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}}\right)$ **(b)** $\left(a^{\frac{1}{2}}-b^{-\frac{1}{2}}\right)^2$

6 Find the exact value of $\dfrac{a^4b}{c^4}$ where $a=\left(\frac{2}{3}\right)^2$, $b=\left(\frac{8}{3}\right)^7$, $c=\left(\frac{4}{3}\right)^4$.

7 If $E=\dfrac{m}{2}\left(v^2-u^2\right)$, find:

(a) E when $m=13$, $v=17$, $u=15$ **(b)** v when $E=98$, $m=4$, $u=24$, if $v>0$

CHAPTER 3
EQUATIONS AND INEQUALITIES

Outcomes covered:

P1 demonstrates confidence in using mathematics to obtain realistic solutions to problems

P3 performs routine arithmetic and algebraic manipulation involving surds, simple rational expressions and trigonometric identities

P4 chooses and applies appropriate arithmetic, algebraic, graphical, trigonometric and geometric techniques

3.1 LINEAR EQUATIONS IN ONE VARIABLE

To **solve** a linear equation in one variable is to find the value of the variable that will make the equation true.

Example 1

Solve:

(a) $5x - 3 = 2x + 9$ **(b)** $5(x - 3) - 3(x - 2) = 19$

Solution

(a)
$$\begin{aligned} 5x - 3 &= 2x + 9 \\ 5x - 3 - 2x &= 9 \\ 3x - 3 &= 9 \\ 3x &= 12 \\ x &= 4 \end{aligned}$$

(b)
$$\begin{aligned} 5(x - 3) - 3(x - 2) &= 19 \\ 5x - 15 - 3x + 6 &= 19 \\ 2x - 9 &= 19 \\ 2x &= 28 \\ x &= 14 \end{aligned}$$

TECHNOLOGY EXPLORATION

Solving polynomial equations

In GeoGebra, enter the equation in the input bar (replacing the equation's variable with the letter x). The solutions will be shown in the **Algebra** view as well as in the **Graphics** view.

EXERCISE 3.1 LINEAR EQUATIONS IN ONE VARIABLE

Solve:

1 $5x = 3x + 9$ **2** $4(2x - 7) = 3x - 5$ **3** $3(x + 3) = 3(9 - 3x)$

4 $x - 7 = 3 - x$ **5** $5(2a + 1) = 6(a - 5)$ **6** $3(5x - 1) - 2(4 - x) = 6$

7 $4 - 3x = x + 9$ **8** $3x - 2 = 5x$ **9** $5(2x - 1) = 2(x - 5) - 3$

10 $4(3a + 2) - 6(3 - a) = 8$ **11** $5 - 9y = 10 - 11y$ **12** $4(3m - 1) = 11 + 2m$

13 $2c - (4 - c) = 5 - c$ **14** $m + 8 = 5(m - 1) - 2m$ **15** $4(x + 5) - (x - 1) = 9$

16 $18 - 3(a - 2) = 2(a + 2)$ **17** $3(2x + 5) = 4 - (3x - 2)$ **18** $5(c - 7) = 3(3c + 8)$

19 $6(2x - 3) = 8 - 2(3x + 1)$ **20** $a + 6 = 3 - a$

3.2 LINEAR EQUATIONS INVOLVING FRACTIONS

For equations involving fractions, write the terms with a common denominator and then solve the equation given by the numerators.

If the equation has only one term on each side of the equals sign, you can multiply each numerator by the other term's denominator to achieve the same result (see Example **2(d)**).

Example 2

Solve:

(a) $\frac{x}{5}+\frac{4x}{3}=23$ **(b)** $\frac{2x+1}{4}-\frac{2x-3}{6}=\frac{7}{3}$ **(c)** $\frac{3}{x-4}=\frac{2}{x-2}$ **(d)** $\frac{3a-2}{2a-1}=\frac{3a+1}{2a+3}$

Solution

(a) $\frac{x}{5}+\frac{4x}{3}=23$

$$\frac{3x}{15}+\frac{5\times 4x}{15}=23$$
$$\frac{23x}{15}=23$$
$$23x=15\times 23$$
$$x=15$$

(b) $\frac{2x+1}{4}-\frac{2x-3}{6}=\frac{7}{3}$

$$\frac{3(2x+1)}{12}-\frac{2(2x-3)}{12}=\frac{28}{12}$$
$$6x+3-4x+6=28$$
$$2x+9=28$$
$$2x=19$$
$$x=9.5$$

(c) $\frac{3}{x-4}=\frac{2}{x-2}$, $x\neq 4, x\neq 2$

$$\frac{3(x-2)}{(x-4)(x-2)}=\frac{2(x-4)}{(x-2)(x-4)}$$
$$3(x-2)=2(x-4)$$
$$3x-6=2x-8$$
$$3x-2x=6-8$$
$$x=-2$$

(d) $\frac{3a-2}{2a-1}=\frac{3a+1}{2a+3}$, $a\neq 0.5, a\neq -1.5$

$$\frac{3a-2}{2a-1}=\frac{3a+1}{2a+3}$$
$$(3a-2)(2a+3)=(3a+1)(2a-1)$$
$$6a^2+5a-6=6a^2-a-1$$
$$6a=5$$
$$a=\tfrac{5}{6}$$

Solutions may have restrictions, which can be easy to miss. In **(c)**, $x\neq 4$ and $x\neq 2$, because these values would make the denominators zero. In **(d)**, $a\neq 0.5$ and $a\neq -1.5$.

EXERCISE 3.2 LINEAR EQUATIONS INVOLVING FRACTIONS

Solve:

1 $\frac{2x}{3}+\frac{3x}{4}=34$ **2** $\frac{3x}{4}-\frac{2x}{5}=14$ **3** $\frac{3x}{5}=\frac{4x}{3}-22$

4 $\frac{2x+5}{3}=\frac{x+2}{7}+4$ **5** $\frac{5x-3}{2}=x+2$ **6** $\frac{x+3}{3}-\frac{2x-1}{6}=x+1$

7 $\frac{2x-1}{8}=\frac{3x+1}{4}$ **8** $\frac{2x-1}{3}-5=\frac{x}{6}$ **9** $\frac{2(2a+1)}{3}=\frac{5(a-2)}{2}$

10 $\frac{5y+1}{4}=6-\frac{2y}{3}$ **11** $\frac{x}{3}+\frac{x-7}{4}=\frac{x+1}{6}$ **12** $\frac{x+4}{2}-\frac{3-4x}{4}=\frac{5-x}{8}$

13 $\frac{3(x-2)}{5}=\frac{2(x-1)}{3}-\frac{2}{5}$ **14** $\frac{x-4}{x+2}=5$ **15** $\frac{5}{x}+\frac{3}{2x}=2$

16 $\frac{7}{x}+2=\frac{3}{x}$ **17** $\frac{3}{p+1}=\frac{2}{2p+1}$ **18** $\frac{3}{x+2}=\frac{5}{2x-1}$

19 $\frac{5}{a+3}=2$ **20** $\frac{y+3}{y+2}=\frac{y+1}{y+4}$ **21** $\frac{x-2}{x+3}=\frac{x+4}{x-5}$

22 Solve $\frac{2x+3}{3x-1}=\frac{2x-5}{3x+4}$. Which of the following is correct?

A $\frac{1}{3}$ B $-\frac{4}{3}$ C $-\frac{7}{34}$ D $-\frac{1}{2}$

Solve:

23 $\frac{2m-5}{m}=\frac{2m}{m+3}$ **24** $\frac{2}{x+1}+\frac{3}{x+9}=0$ **25** $\frac{5}{y}=\frac{2}{y-7}$

26 $\frac{3}{x-2}-\frac{2}{x+2}=\frac{1}{x^2-4}$ **27** $\frac{4}{a}-\frac{3}{a+2}=\frac{1}{a(a+2)}$ **28** $\frac{1}{x-1}+\frac{1}{2x-1}=0$

29 $\frac{1}{x+2}+\frac{1}{x-3}=\frac{1}{(x+2)(x-3)}$ **30** $\frac{1}{x+1}+\frac{1}{x+2}=\frac{1}{x^2+3x+2}$ **31** $\frac{1}{x-3}+\frac{1}{x+3}=\frac{1}{x^2-9}$

32 Solve $\frac{1}{1-t}+\frac{1}{1+t}=\frac{2t}{1-t^2}$. Indicate whether each answer is correct or incorrect.

(a) -1 (b) 0 (c) 1 (d) 2

3.3 SIMPLE LINEAR INEQUALITIES

The solution to an inequality is usually a range of numbers described by another inequality, related to the inequality in the question. The solution is also usually a range of real numbers, unless another set of numbers is specified in the question (e.g. integers).

Rules for inequalities

If both sides of an inequality are multiplied or divided by a negative number, then the direction of the inequality is reversed.

If $a > b$, then: $a + c > b + c$

$a - c > b - c$

$ac > bc$ if $c > 0$

$ac < bc$ if $c < 0$

$\frac{a}{c} > \frac{b}{c}$ if $c > 0$

$\frac{a}{c} < \frac{b}{c}$ if $c < 0$

If $a < b$, then: $a + c < b + c$

$a - c < b - c$

$ac < bc$ if $c > 0$

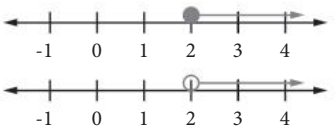
$ac > bc$ if $c < 0$

$\frac{a}{c} < \frac{b}{c}$ if $c > 0$

$\frac{a}{c} > \frac{b}{c}$ if $c < 0$

On number lines:

- $a > b$ means that a is to the right of b.
- $a < b$ means that a is to the left of b.
- $x \geq 2$ is shown by a solid circle over 2 and an arrow to the right.
- $x > 2$ is shown by an empty circle over 2 and an arrow to the right.



This is demonstrated in the following examples.

Example 3

Solve $2x + 3 \geq 9$ and show the solution on a number line.

Solution

$2x + 3 \geq 9$

$2x \geq 6$

$x \geq 3$

Example 4

Solve $35 - 3x > 19 - x$ and show the solution on a number line, for the conditions:

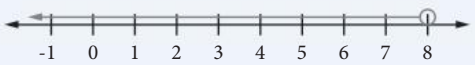
(a) x is a real number **(b)** x is an integer **(c)** x is not negative

Solution

$35 - 3x > 19 - x$

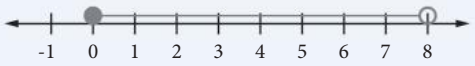
$35 > 19 + 2x$	Add $3x$ to both sides
$16 > 2x$	Subtract 19 from both sides
$8 > x$	Divide both sides by 2
$x < 8$	Rewrite starting with x

(a) For real numbers:
Solution is $x < 8$



(b) For integers:
Solution is $x = 7, 6, 5, \ldots$
(all integers to the left on the number line)

(c) For non-negative numbers, $x \geq 0$:
Solution is $0 \leq x < 8$



Example 5

Solve $\frac{x-1}{5} \leq \frac{x+1}{3}$ and show the solution on a number line.

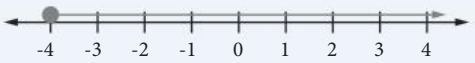
Solution

$$\frac{x-1}{5} \leq \frac{x+1}{3}$$

$15 \times \frac{(x-1)}{5} \leq 15 \times \frac{(x+1)}{3}$ Multiply both sides by 15

$3(x-1) \leq 5(x+1)$

$3x - 3 \leq 5x + 5$	Expand both sides
$-3 \leq 2x + 5$	Subtract $3x$ from both sides
$-8 \leq 2x$	Subtract 5 from both sides
$-4 \leq x$	Divide both sides by 2
$x \geq -4$	Rewrite starting with x



Example 6

Solve $-5 < 2x - 3 \leq 7$ and show the solution on a number line.

Solution

This is really two separate inequalities, $-5 < 2x - 3$ and $2x - 3 \leq 7$, solved simultaneously.

Solve them together by applying the same steps to each part.

		OR		
$-5 < 2x - 3 \leq 7$			$2x - 3 > -5$	$2x - 3 \leq 7$
$-5 + 3 < 2x \leq 7 + 3$	Add 3 to each part		$2x > -5 + 3$	$2x \leq 7 + 3$
$-2 < 2x \leq 10$	Simplify		$2x > -2$	$2x \leq 10$
$-1 < x \leq 5$	Divide each part by 2		$x > -1$	$x \leq 5$

$-1 < x \leq 5$

The solution shows that x is greater than -1 but less than or equal to 5.

EXERCISE 3.3 SIMPLE LINEAR INEQUALITIES

In questions **1** to **15**, solve each inequality and show the solution on a number line.

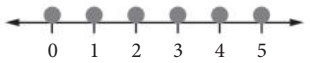
1 $5x - 2 \geq 13$ **2** $2x - 2 < 0$ **3** $2x + 5 < -5$ **4** $6x + 2 \geq -10$

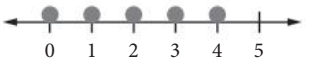
5 $3x > 2x + 12$ **6** $3(x + 1) \geq 9$ **7** $7x < 3(2x + 1)$ **8** $2(x - 6) \geq 8$

9 $\frac{x}{3} - \frac{x}{4} > 1$ **10** $\frac{3x}{2} - \frac{x}{3} < \frac{3}{2}$ **11** $3x - 1 \geq x + 2$ **12** $2x + 7 < 3x + 10$

13 $\frac{x-1}{3} > -1$ **14** $\frac{x+3}{5} < 7$ **15** $x - 1 > 5$, x an integer

16 Solve $4x < x + 15$ for x as a positive integer. The solution on a number line is:

A 

B 

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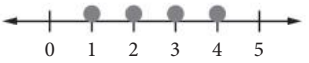
D 

In questions **17** to **23**, solve each inequality and show the solution on a number line.

17 $\frac{3x}{5} - \frac{2x}{3} > -2$ **18** $\frac{7x}{3} < 3 + \frac{4x}{3}$ **19** $\frac{x-5}{2} > \frac{5x-3}{6}$ **20** $\frac{5x-3}{2} < x + 2$

21 $-5 < x + 4 < 1$ **22** $22 \leq 5x - 3 \leq 32$ **23** $-3 \leq \frac{2x-1}{3} < 3$

24 Solve simultaneously $x - 2 > -2$ and $x - 3 \leq 0$. Indicate whether each answer is correct or incorrect.

(a) $0 \leq x \leq 3$ **(b)** $0 < x \leq 3$ **(c)** $0 \leq x \leq 3$ **(d)** $x > 0$ or $x \leq 3$

25 If 5 is subtracted from a certain positive integer, the result is greater than 5 but less than 12. What values can this integer take? (Let the positive integer be x, so that $5 < x - 5 < 12$.)

26 If a certain number is divided by 2, the result is greater than 4 but less than 8. What values can this number take?

27 The sum of two consecutive positive integers is no more than 35. What are the possible values of these integers?

28 A committee consists of 3 more women than men. The total number of committee members is at least 7 but not more than 15. How many women could be on the committee?

29 When a certain number is added to 5 and the sum is divided by 5, the result is not more than if this same number is added to 13 and the sum is divided by 9. What is the largest value this number can take?

30 The base length of an isosceles triangle is an integer (in cm) and is 4 cm less than the sum of the two equal sides. The perimeter is an integer (in cm) less than 80 cm. What are the possible base lengths?

31 The sum of three consecutive integers is greater than 7 and less than 25. Find all possible values of the smallest of these integers.

32 The side lengths of a triangle are 8 cm, 10 cm and x cm. What are the possible values of x?

3.4 QUADRATIC EQUATIONS

An equation of the form $ax^2 + bx + c = 0$, $a \neq 0$ is called a quadratic equation in x. The values of x that make this equation true are called the solutions or the roots of the equation.

A quadratic equation can be solved by factorising the quadratic expression and making the factored expression equal to zero:

If $AB = 0$, then $A = 0$ or $B = 0$ or $A = B = 0$.

Example 7

Solve:

(a) $x(x-2) = 0$ **(b)** $(x+1)(x-4) = 0$

Solution

(a) $x(x-2) = 0$

$x = 0$ or $x - 2 = 0$

$x = 0$ or $x = 2$

(b) $(x+1)(x-4) = 0$

$x + 1 = 0$ or $x - 4 = 0$

$x = -1$ or $x = 4$

EXERCISE 3.4 QUADRATIC EQUATIONS

Solve:

1 $x(x-5) = 0$ **2** $(x-2)(x-3) = 0$ **3** $x(2x+1) = 0$

4 $(x-7)(2x+5) = 0$ **5** $3x(2x-9) = 0$ **6** $-5x(x+1) = 0$

7 $(x-a)(x-b) = 0$ **8** $(x-3a)(x+2b) = 0$ **9** $(x-2)(x+2) = 0$

10 $(2x-11)(2x+11) = 0$ **11** $(x-1)^2 = 0$ **12** $(2x+3)^2 = 0$

3.5 QUADRATIC EQUATIONS WITHOUT A LINEAR TERM

To solve $ax^2 + c = 0$, $c < 0$, remember the difference of two squares: $a^2 - b^2 = (a-b)(a+b)$

Example 8

Solve:

(a) $x^2 - 4 = 0$ **(b)** $2x^2 - 18 = 0$ **(c)** $9x^2 = 25$ **(d)** $(x-2)^2 = 9$

Solution

(a) $x^2 - 4 = 0$

$(x-2)(x+2) = 0$

$x - 2 = 0 \quad \text{or} \quad x + 2 = 0$

$x = 2 \quad \text{or} \quad x = -2$

(b) $2x^2 - 18 = 0$

$2(x^2 - 9) = 0$

$2(x-3)(x+3) = 0$

$x = 3 \quad \text{or} \quad x = -3$

(c) $9x^2 = 25$

$9x^2 - 25 = 0$

$(3x-5)(3x+5) = 0$

$3x - 5 = 0 \quad \text{or} \quad 3x + 5 = 0$

$x = \frac{5}{3} \quad \text{or} \quad x = -\frac{5}{3}$

(d) $(x-2)^2 = 9$

$(x-2)^2 - 9 = 0$

$(x-2-3)(x-2+3) = 0$

$(x-5)(x+1) = 0$

$x = 5 \quad \text{or} \quad x = -1$

Example 9

Solve:

(a) $x^2 - 6 = 0$ **(b)** $3x^2 = 15$

Solution

(a) $x^2 - 6 = 0$

Use $6 = \left(\sqrt{6}\right)^2$

$x^2 - \left(\sqrt{6}\right)^2 = 0$

$\left(x - \sqrt{6}\right)\left(x + \sqrt{6}\right) = 0$

$x = \sqrt{6} \quad \text{or} \quad x = -\sqrt{6}$

OR $x^2 = 6$

$x = \pm\sqrt{6}$

(b) $3x^2 = 15$

$3(x^2 - 5) = 0$

$3\left(x - \sqrt{5}\right)\left(x + \sqrt{5}\right) = 0$

$x = \sqrt{5} \quad \text{or} \quad x = -\sqrt{5}$

OR $x^2 = 5$

$x = \pm\sqrt{5}$

TECHNOLOGY EXPLORATION

Solving quadratic equations

In GeoGebra, enter the equation in the input bar. The resulting vertical lines will show the solutions. Closer approximations can be found using the **Intersect Two Objects** tool  . Use this to click on the vertical line and then on the *x*-axis: the resulting point is shown in the **Algebra** view and the *x*-coordinate will be the solution.

EXERCISE 3.5 QUADRATIC EQUATIONS WITHOUT A LINEAR TERM

Solve:

1 $x^2 - 1 = 0$ **2** $x^2 - 25 = 0$ **3** $x^2 - 49 = 0$ **4** $x^2 = 16$

5 The roots of the equation $x^2 - 0.25 = 0$ are:

A $x = 0.5$ **B** $x = -0.5$ **C** $x = \pm 0.5$ **D** $x = \pm\frac{1}{16}$

6 $4x^2 = 9$ **7** $x^2 - 6\frac{1}{4} = 0$ **8** $4x^2 - 25 = 0$ **9** $x^2 - 2\frac{1}{4} = 0$

10 $5x^2 - 5 = 0$ **11** $16x^2 - 1 = 0$ **12** $16 - x^2 = 0$ **13** $25x^2 = 49$

14 For the equation $9(x-1)^2 - 36 = 0$, indicate whether each statement is correct or incorrect.

(a) $(x-1)^2 = 4$ **(b)** $x = 1 \pm \sqrt{4}$ **(c)** $x = -3, 1$ **(d)** $x = -1, 3$

15 $(x-2)^2 = 16$ **16** $5x^2 - 245 = 0$ **17** $7x^2 = 63$ **18** $(5x-1)^2 = 16$

19 $(x+3)^2 - 4 = 0$ **20** $x^2 - 5 = 0$ **21** $x^2 = 2$ **22** $(x+1)^2 = 8$

3.6 QUADRATIC EQUATIONS WITHOUT A CONSTANT TERM

To solve $ax^2 + bx = 0$: the equation has no constant term, so the common factor is x or a multiple of x.

Example 10

Solve:

(a) $x^2 - 3x = 0$ **(b)** $4x^2 = 8x$

Solution

(a) $x^2 - 3x = 0$

$x(x-3) = 0$

$x = 0$ or $x = 3$

(b) $4x^2 = 8x$

$4x^2 - 8x = 0$

$4x(x-2) = 0$ Divide by 4, do **not** divide by x

$x = 0$ or $x = 2$

EXERCISE 3.6 QUADRATIC EQUATIONS WITHOUT A CONSTANT TERM

Solve:

1 $x^2 - 6x = 0$ **2** $x^2 - 5x = 0$ **3** $x^2 + 5x = 0$ **4** $x^2 + 10x = 0$

5 The solution to $x^2 = 4x$ is:

A $x = 0$ **B** $x = 4$ **C** $x = 0, 2$ **D** $x = 0, 4$

6 $2x^2 - 5x = 0$ **7** $x^2 = 7x$ **8** $3x^2 - 21x = 0$ **9** $2x^2 + 20x = 0$

10 $6x^2 = 24x$ **11** $2x^2 - x = 0$ **12** $5x^2 + x = 0$ **13** $3x^2 = -9x$

14 $12x^2 - 5x = 0$ **15** $15x - x^2 = 0$

3.7 GENERAL QUADRATIC EQUATIONS

To solve $ax^2 + bx + c = 0$, factorise the trinomial if possible.

Example 11

Solve:

(a) $x^2 - 5x + 6 = 0$ **(b)** $2x^2 = x + 3$ **(c)** $x(x-2) = 3$ **(d)** $(3x+4)(x-3) = 16$

Solution

(a) $x^2 - 5x + 6 = 0$

$(x-2)(x-3) = 0$

$x = 2$ or $x = 3$

(b) $2x^2 = x + 3$

$2x^2 - x - 3 = 0$

$(2x-3)(x+1) = 0$

$x = 1.5$ or $x = -1$

(c) $x(x-2)=3$

$x^2-2x-3=0$

$(x-3)(x+1)=0$

$x=3$ or $x=-1$

(d) $(3x+4)(x-3)=16$

$3x^2-5x-12=16$

$3x^2-5x-28=0$

$(3x+7)(x-4)=0$

$x=-2\frac{1}{3}$ or $x=4$

EXERCISE 3.7 GENERAL QUADRATIC EQUATIONS

Solve:

1 $x^2-3x+2=0$ **2** $x^2-6x+5=0$ **3** $x^2-2x-8=0$ **4** $x^2-4x+3=0$

5 $x^2-6x+9=0$ **6** $x^2-5x+4=0$ **7** $x^2+9x+8=0$ **8** $9x^2+4x-5=0$

9 The roots of the equation $5x^2+7x-12=0$ are:

A $x=-1, 2.4$ **B** $x=-2.4, 1$ **C** $x=-1.2, 1$ **D** $x=-0.8, 3$

Solve:

10 $x^2+4x-12=0$ **11** $5x^2-11x+2=0$ **12** $4x^2-12x-7=0$ **13** $2x^2-x-10=0$

14 $x^2+10x+25=0$ **15** $x^2+5x+4=0$ **16** $4x^2-8x-21=0$ **17** $3x^2-28x+25=0$

18 $x^2-8x+16=0$ **19** $5x^2+26x+24=0$ **20** $3x^2-41x+60=0$ **21** $30-7x-x^2=0$

22 $5x^2=8x-3$ **23** $x(2x-11)=6$ **24** $x(x+5)=6$ **25** $x(3x+19)=72$

26 $x^2+15=8x$ **27** $(x-2)(2x+5)=2x+5$ **28** $12-4x-x^2=0$ **29** $(2x+1)^2=4$

30 $(x+1)^2=4x$ **31** $(x+6)^2=x+6$ **32** $6x^2=10-11x$ **33** $7x^2=2(17x-12)$

3.8 COMPLETING THE SQUARE

In the expansion $(x+6)^2=x^2+12x+36$, the constant term 36 is half the coefficient of x squared: $\left(\frac{12}{2}\right)^2=36$.

In the expansion $(x-5)^2=x^2-10x+25$, the constant term 25 is half the coefficient of x squared: $\left(\frac{-10}{2}\right)^2=25$.

This can be shown for the general expansion $(x-a)^2=x^2-2ax+a^2$ as $\left(\frac{-2a}{2}\right)^2=a^2$.

Thus, an expression like x^2+6x can be made into a perfect square by adding $\left(\frac{6}{2}\right)^2=9$ to obtain $x^2+6x+9=(x+3)^2$.

Example 12

What must be added to each expression to complete the square?

(a) x^2+8x **(b)** x^2-3x

Solution

(a) x^2+8x

- Half of 8 is 4
- The square of 4 is 16
- Hence 16 must be added

Check: $x^2+8x+16=(x+4)^2$

(b) x^2-3x

- Half of -3 is $-\frac{3}{2}$
- The square of $-\frac{3}{2}$ is $\frac{9}{4}$
- Hence $\frac{9}{4}$ must be added

Check: $x^2-3x+\frac{9}{4}=\left(x-\frac{3}{2}\right)^2$

EXERCISE 3.8 COMPLETING THE SQUARE

In questions **1**–**4** and **6**–**13**, write the number to be added to complete the square.

1 $x^2 + 4x$ **2** $x^2 - 6x$ **3** $x^2 + 14x$ **4** $x^2 + 2x$

5 The number added to $x^2 - 12x$ to complete the square is:

A 36 **B** 144 **C** -36 **D** -144

6 $x^2 - x$ **7** $x^2 + 5x$ **8** $x^2 + 3x$ **9** $x^2 - 7x$

10 $x^2 + x$ **11** $x^2 + 2ax$ **12** $x^2 - 2bx$ **13** $x^2 + cx$

14 The square is completed for the expression $x^2 - ax$. Indicate whether the following statements would be correct or incorrect.

(a) $x^2 - ax + \frac{a^2}{2}$ **(b)** $x^2 - ax + \frac{a^2}{4}$ **(c)** $\left(x - \frac{a}{2}\right)^2$ **(d)** $x^2 - ax + a^2$

3.9 SOLVING QUADRATIC EQUATIONS BY COMPLETING THE SQUARE

Example 13

Complete the square to solve:

(a) $x^2 + 4x - 5 = 0$ **(b)** $x^2 - 5x + 6 = 0$ **(c)** $x^2 = 8x$

Solution

(a) $x^2 + 4x - 5 = 0$

$x^2 + 4x = 5$ Add 5 to both sides

$x^2 + 4x + 4 = 5 + 4$ Add 4 to both sides to complete the square

$(x + 2)^2 = 9$ Factorise

$x + 2 = \pm 3$ Take the square roots of each side

$x + 2 = 3$ or $x + 2 = -3$

$x = 1$ or $x = -5$

(b) $x^2 - 5x + 6 = 0$

$x^2 - 5x = -6$ Subtract 6 from both sides

$x^2 - 5x + \frac{25}{4} = -6 + \frac{25}{4}$ Add $\frac{25}{4}$ to both sides to complete the square

$\left(x - \frac{5}{2}\right)^2 = \frac{1}{4}$ Factorise

$x - \frac{5}{2} = \pm\frac{1}{2}$ Take the square roots of each side

$x - \frac{5}{2} = \frac{1}{2}$ or $x - \frac{5}{2} = -\frac{1}{2}$

$x = 3$ or $x = 2$

OR, instead of taking the square roots of each side, rewrite the expression as:

$\left(x - \frac{5}{2}\right)^2 - \frac{1}{4} = 0$

Then, factorise using the difference of squares:

$\left(x - \frac{5}{2} - \frac{1}{2}\right)\left(x - \frac{5}{2} + \frac{1}{2}\right) = 0$

$(x - 3)(x - 2) = 0$

$x = 3$ or $x = 2$

(c) $x^2 = 8x$

$x^2 - 8x = 0$	Subtract $8x$ from both sides
$x^2 - 8x + 16 = 16$	Add 16 to both sides to complete the square
$(x-4)^2 = 16$	Factorise
$x - 4 = \pm 4$	Take the square roots of each side
$x - 4 = 4$ or $x - 4 = -4$	
$x = 8$ or $x = 0$	

EXERCISE 3.9 SOLVING QUADRATIC EQUATIONS BY COMPLETING THE SQUARE

Complete the square to solve the following quadratic equations.

1 $x^2 - 6x + 5 = 0$ **2** $x^2 - 2x - 8 = 0$ **3** $x^2 + 4x - 5 = 0$ **4** $x^2 + 4x = 12$

5 When solving $x^2 - 10x + 24 = 0$ by completing the square, the line of working after completing the square could be:

A $x^2 - 10x + 16 = -8$ **B** $x^2 - 10x + 25 = 0$ **C** $x^2 - 10x + 100 = 76$ **D** $x^2 - 10x + 25 = 1$

6 $x^2 - 4x = 21$ **7** $x^2 - 26x + 25 = 0$ **8** $x^2 - 3x + 2 = 0$ **9** $x^2 + x - 12 = 0$

10 $x^2 - 5x + 4 = 0$ **11** $x^2 + 7x = 30$ **12** $x^2 - 11x = 12$ **13** $x^2 - 3x - 10 = 0$

14 $x^2 = 7x - 10$ **15** $x^2 + x = 72$ **16** $x^2 - 10x - 11 = 0$ **17** $x^2 - 10x = 0$

18 $x^2 = 3x$ is solved by completing the square. Indicate whether the following steps in the working are correct or incorrect.

(a) $x^2 - 3x + \frac{9}{4} = \frac{9}{4}$ **(b)** $\left(x - \frac{3}{2}\right)^2 = \frac{3}{2}$ **(c)** $x - \frac{3}{2} = \pm\frac{3}{2}$ **(d)** $x = 0, 3$

3.10 QUADRATIC EQUATIONS WITH NON-RATIONAL SOLUTIONS

All the questions in Exercise 3.9 could have been solved by factorising. Practice in completing the square will also enable you to solve equations that do not have rational factors:

Example 14

Complete the square to solve:

(a) $x^2 + 2x - 5 = 0$ **(b)** $x^2 = 4x + 8$ **(c)** $x^2 - 5x + 2 = 0$

Solution

(a) $x^2 + 2x - 5 = 0$

$x^2 + 2x = 5$	Move constant to RHS
$x^2 + 2x + 1 = 5 + 1$	Add 1^2 to complete the square
$(x+1)^2 = 6$	Factorise
$x + 1 = \pm\sqrt{6}$	Take square roots of both sides
$x = -1 + \sqrt{6}$ or $x = -1 - \sqrt{6}$	Exact answers
$x = 1.45$ or $x = -3.45$	Answers correct to 2 decimal places

(b) $x^2 = 4x + 8$

$x^2 - 4x = 8$	Rewrite with only the constant on RHS
$x^2 - 4x + 4 = 8 + 4$	Add 2^2 to complete the square
$(x-2)^2 = 12$	Factorise
$x - 2 = \pm\sqrt{12}$	Take square roots of both sides
$x = 2 + 2\sqrt{3}$ or $x = 2 - 2\sqrt{3}$	Exact answers
$x = 5.46$ or $x = -1.46$	Answers correct to 2 decimal places

(c) $x^2 - 5x + 2 = 0$

$x^2 - 5x = -2$	Move constant to RHS
$x^2 - 5x + \frac{25}{4} = -2 + \frac{25}{4}$	Add $\left(\frac{5}{2}\right)^2$ to complete the square
$\left(x - \frac{5}{2}\right)^2 = \frac{17}{4}$	Factorise
$x - \frac{5}{2} = \pm\frac{\sqrt{17}}{2}$	Take square roots of both sides
$x = \frac{5}{2} + \frac{\sqrt{17}}{2}$ or $x = \frac{5}{2} - \frac{\sqrt{17}}{2}$	Exact answers
$x = 4.56$ or $x = 0.44$	Answers correct to 2 decimal places

EXERCISE 3.10 QUADRATIC EQUATIONS WITH NON-RATIONAL SOLUTIONS

Complete the square to solve the following quadratic equations, giving answers to even-numbered questions in surd form and answers to odd-numbered questions correct to two decimal places.

1 $x^2 - 2x - 4 = 0$ **2** $x^2 + 4x - 4 = 0$ **3** $x^2 - x - 5 = 0$ **4** $x^2 - 6x + 2 = 0$

5 $x^2 - 5x + 1 = 0$ **6** $x^2 + 2x - 2 = 0$ **7** $x^2 = 6x - 4$ **8** $x^2 + x - 1 = 0$

9 $x^2 - 6x - 5 = 0$ **10** $x^2 + 4x = 1$ **11** $x^2 = 2x + 5$ **12** $x^2 + 3x - 6 = 0$

3.11 COMPLETING THE SQUARE FOR NON-MONIC EQUATIONS

Completing the square is more difficult when the coefficient of x^2 is not 1. You can overcome this problem by first dividing every term by the coefficient of x^2.

Example 15

Complete the square to solve $2x^2 - 3x - 3 = 0$.

Solution

$2x^2 - 3x - 3 = 0$

$2x^2 - 3x = 3$	Move constant to RHS
$x^2 - \frac{3x}{2} = \frac{3}{2}$	Divide by coefficient of x^2
$x^2 - \frac{3x}{2} + \frac{9}{16} = \frac{3}{2} + \frac{9}{16}$	Add $\left(\frac{3}{4}\right)^2$ to complete the square
$\left(x - \frac{3}{4}\right)^2 = \frac{33}{16}$	Factorise
$x - \frac{3}{4} = \pm\frac{\sqrt{33}}{4}$	Take square roots of both sides

$x = \frac{3}{4} + \frac{\sqrt{33}}{4} = \frac{3+\sqrt{33}}{4}$ or $x = \frac{3}{4} - \frac{\sqrt{33}}{4} = \frac{3-\sqrt{33}}{4}$ Exact answers

$x = 2.19$ or $x = -0.69$ Answers correct to two decimal places

EXERCISE 3.11 COMPLETING THE SQUARE FOR NON-MONIC EQUATIONS

Complete the square to solve the following quadratic equations, giving answers to even-numbered questions in surd form and answers to odd-numbered questions correct to two decimal places.

1 $2x^2 - x - 5 = 0$ **2** $2x^2 + 6x - 5 = 0$ **3** $2x^2 + x - 2 = 0$ **4** $2x^2 + 3x - 1 = 0$

5 $3x^2 - 5x - 1 = 0$ **6** $3x^2 + 4x = 5$ **7** $3x^2 - 2x = 4$ **8** $2x^2 - 6x + 1 = 0$

9 $3x^2 = 7x + 3$ **10** $4x^2 + 4x - 5 = 0$ **11** $2x^2 - 5x = 9$ **12** $3x^2 - 2x - 2 = 0$

13 $2x^2 + x = 4$ **14** $3x^2 - 8x + 3 = 0$ **15** $6x^2 - 10x + 3 = 0$

3.12 THE QUADRATIC FORMULA

The equation $ax^2 + bx + c = 0$, $a \neq 0$ is the general quadratic equation.

If we solve this equation by completing the square, we obtain the **quadratic formula**, which is a solution that must be true for all quadratic equations. This formula enables us to solve quadratic equations even when the factors are not obvious.

$ax^2 + bx + c = 0$

$ax^2 + bx = -c$ Move constant to RHS

$x^2 + \frac{b}{a}x = -\frac{c}{a}$ Divide by coefficient of x^2

$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$ Add $\left(\frac{b}{2a}\right)^2$ to complete the square

$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$ Factorise

$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$ Take square roots of both sides

$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ The quadratic formula

Example 16

Use the quadratic formula to solve the following quadratic equations.

(a) $x^2 + 8x + 12 = 0$ **(b)** $x^2 - 3x - 2 = 0$ **(c)** $2x^2 - 4x + 1 = 0$ **(d)** $4x^2 + 5x - 2 = 0$

Solution

(a) $x^2 + 8x + 12 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},\ a = 1,\ b = 8,\ c = 12$$

$$x = \frac{-8 \pm \sqrt{8^2 - 4 \times 1 \times 12}}{2}$$

$$= \frac{-8 \pm \sqrt{64 - 48}}{2}$$

$$= \frac{-8 \pm \sqrt{16}}{2}$$

$$= \frac{-8 + 4}{2} \quad \text{or} \quad \frac{-8 - 4}{2}$$

$$= -2 \quad \text{or} \quad -6$$

It would have been faster to use factors to solve **(a)**: $x^2 + 8x + 12 = (x + 2)(x + 6)$

(b) $x^2 - 3x - 2 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},\ a = 1,\ b = -3,\ c = -2$$

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4 \times 1 \times (-2)}}{2}$$

$$= \frac{3 \pm \sqrt{9 + 8}}{2}$$

$$= \frac{3 \pm \sqrt{17}}{2}$$

$$= \frac{3 + \sqrt{17}}{2} \quad \text{or} \quad \frac{3 - \sqrt{17}}{2} \quad \text{(Exact solution)}$$

$$= 3.56 \quad \text{or} \quad -0.56 \quad \text{(Correct to 2 d.p.)}$$

(c) $2x^2 - 4x + 1 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},\ a = 2,\ b = -4,\ c = 1$$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \times 2 \times 1}}{4}$$

$$= \frac{4 \pm \sqrt{16 - 8}}{4}$$

$$= \frac{4 \pm \sqrt{8}}{4}$$

$$= \frac{4 \pm 2\sqrt{2}}{4}$$

$$= \frac{2 \pm \sqrt{2}}{2} \quad \text{(Exact solution)}$$

$$= 1.71 \quad \text{or} \quad 0.29 \quad \text{(Correct to 2 d.p.)}$$

(d) $4x^2 + 5x - 2 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},\ a = 4,\ b = 5,\ c = -2$$

$$x = \frac{-5 \pm \sqrt{5^2 - 4 \times 4 \times (-2)}}{8}$$

$$= \frac{-5 \pm \sqrt{25 + 32}}{8}$$

$$= \frac{-5 \pm \sqrt{57}}{8}$$

$$= \frac{-5 + \sqrt{57}}{8} \quad \text{or} \quad \frac{-5 - \sqrt{57}}{8} \quad \text{(Exact solution)}$$

$$= 0.32 \quad \text{or} \quad -1.57 \quad \text{(Correct to 2 d.p.)}$$

EXERCISE 3.12 THE QUADRATIC FORMULA

Solve the following quadratic equations, giving the answers to the even-numbered questions in surd form, and to the odd-numbered questions correct to two decimal places (if necessary).

1 $x^2 + 6x + 5 = 0$
2 $x^2 + 2x - 8 = 0$
3 $x^2 - 6x - 7 = 0$
4 $x^2 - 7x + 10 = 0$
5 $x^2 + 2x - 1 = 0$
6 $x^2 - 6x + 4 = 0$
7 $x^2 - 2x - 5 = 0$
8 $x^2 + 5x - 1 = 0$
9 $x^2 - 2x - 9 = 0$
10 $x^2 + 4x + 2 = 0$
11 $x^2 - 15x + 56 = 0$
12 $x^2 + 2x - 15 = 0$
13 $2x^2 + 5x + 1 = 0$
14 $2x^2 - 8x + 3 = 0$
15 $2x^2 + 3x + 1 = 0$
16 $2x^2 - 3x = 0$
17 $3x^2 + 2x - 2 = 0$
18 $2x^2 + 3x - 5 = 0$
19 $x^2 + 6x + 1 = 0$
20 $x^2 - 8x + 16 = 0$
21 $2x^2 - x - 3 = 0$
22 $7x^2 - 7x - 2 = 0$
23 $4x^2 - 9x + 4 = 0$
24 $3x^2 - 11x - 4 = 0$
25 $2x^2 + x = 3$
26 $x(x + 3) = 2$
27 $2x^2 + 6x + 1 = 0$
28 $2x^2 - 6x = 3$
29 $x^2 = 2x + 2$
30 $x^2 = 6x + 2$
31 $2x^2 = 3x + 4$
32 $2x^2 + 10x + 5 = 0$
33 $x^2 + 17x = 60$
34 $3x^2 + 9x + 5 = 0$
35 $3x^2 - 15 = 0$
36 $x(x + 1) = 1$

3.13 PROBLEMS INVOLVING QUADRATIC EQUATIONS

Example 17

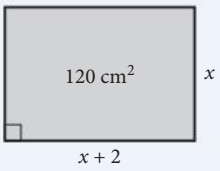
One side of a rectangle is 2 cm longer than the other side. The area of the rectangle is 120 cm^2. What are the dimensions of the rectangle?

Solution

Let one side length be x cm.

The other side length is $(x + 2)$ cm.

Draw a diagram to show this information.



The area of the rectangle is 120 cm^2, so:

$$x(x + 2) = 120$$
$$x^2 + 2x = 120$$
$$x^2 + 2x - 120 = 0$$
$$(x + 12)(x - 10) = 0$$
$$x + 12 = 0 \quad \text{or} \quad x - 10 = 0$$
$$x = -12 \quad \text{or} \quad x = 10$$

Because x represents the side length of a rectangle, $x > 0$. This means the only possible solution is $x = 10$.

Therefore the dimensions of the rectangle are 10 cm by 12 cm.

We should have written the original equation as $x(x + 2) = 120$, $x > 0$, to remember that x must represent a positive length.

Also, don't forget that when solving $AB = 0$, either $A = 0$ or $B = 0$ or $A = B = 0$, but zero is not always a valid solution to the problem. This is one reason why we don't always use all solutions to the quadratic equation in practical problems.

Example 18

The height h metres of a stone, t seconds after being thrown straight up, is given by the equation $h = 30t - 5t^2$. When is the stone at a height of 40 metres?

Solution

$$h = 30t - 5t^2$$

For $h = 40$: $\quad 40 = 30t - 5t^2$

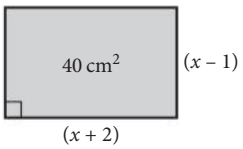
$$5t^2 - 30t + 40 = 0$$
$$5(t^2 - 6t + 8) = 0$$
$$5(t - 2)(t - 4) = 0$$
$$t = 2 \quad \text{or} \quad t = 4$$

On the way up, the stone reaches a height of 40 m after 2 seconds; on the way down, it comes back to a height of 40 m at 4 seconds. In this problem both answers make sense.

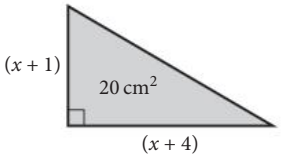
EXERCISE 3.13 PROBLEMS INVOLVING QUADRATIC EQUATIONS

1 In each diagram, all measurements are in centimetres and the area of the shaded region is given. Find the value of x in each case.

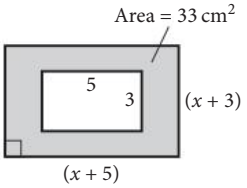
(a)



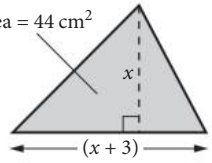
(b)



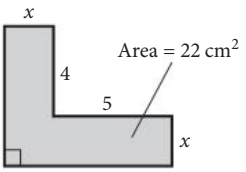
(c)



(d)



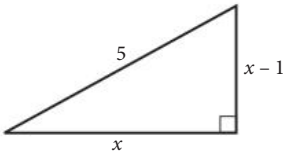
(e)



2 For the information given on the triangle, which statement is correct?

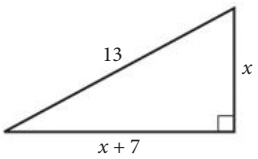
A $x(x-1) = 2.5$ **B** $2x - 1 = 5$

C $x^2 = 12$ **D** $x^2 - x - 12 = 0$

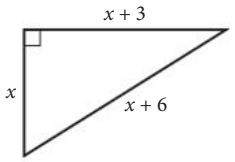


3 Use Pythagoras' theorem to find the value of x, given that all measurements are in centimetres.

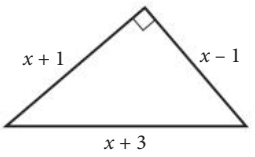
(a)



(b)



(c)



4 The sum of a certain positive number and the square of that number is 12. What is the number?

5 The product of two numbers is 88. If one of the numbers is 3 more than the other, what are the numbers?

6 The product of two consecutive numbers is 72. What are the numbers?

7 The height h metres of a stone, t seconds after being thrown straight up, is given by $h = 40t - 5t^2$. At what times is the stone at a height of: **(a)** 60 m **(b)** 80 m?

8 The sum of the square of a positive number and four times the number is 60. What is the number?

9 A rectangular swimming pool, 12 m by 8 m, is surrounded by a concrete path of uniform width. If the area of the path alone is 224 m^2, find its width.

10 A carpet is placed in a room measuring 6 m by 4 m, leaving an uncarpeted border of uniform width around it. If the area of the carpet is 8 m^2, find the width of the border.

11 A picture on a wall measures 24 cm by 20 cm. It is surrounded by a frame of uniform width whose area, not including the picture, is 416 cm^2. What is the width of the frame?

12 A rectangle is 8 cm longer than its breadth. If the area of the rectangle is 48 cm^2, what are its dimensions?

13 In a right-angled triangle, one of the sides adjacent to the right angle is 4 cm longer than the other side. If the area of the triangle is 96 cm^2, find the length of each of the three sides.

14 The perimeter of a rectangle is 40 cm and its area is 84 cm^2.

(a) If the breadth of the rectangle is x cm, express the length in terms of x.

(b) Write the area of the rectangle in terms of x.

(c) Form a quadratic equation in x and solve it to find the length and breadth.

3.14 SQUARE ROOTS AND ABSOLUTE VALUE

Square roots

If $a \geq 0$, then $\sqrt{a}$ is a non-negative number such that $\left(\sqrt{a}\right)^2 = a$. $\sqrt{a}$ is called the positive square root of a.

An equation like $x^2 = 9$ always has two solutions: the positive and negative square roots of 9. In this case, they are $x = \sqrt{9} = 3$ and $x = -\sqrt{9} = -3$.

The important fact is that the value of $\sqrt{a}$ will always be either positive or zero (non-negative).

What meaning can be given to $\sqrt{x^2}$?

If $x = 2$, then $\sqrt{x^2} = \sqrt{(2)^2} = \sqrt{4} = 2$.

If $x = -2$, then $\sqrt{x^2} = \sqrt{(-2)^2} = \sqrt{4} = 2$.

If $x = 0$, then $\sqrt{x^2} = \sqrt{0^2} = \sqrt{0} = 0$.

This means:

$$\begin{aligned} \sqrt{x^2} &= x && \text{if } x > 0 \\ &= -x && \text{if } x < 0 \\ &= 0 && \text{if } x = 0 \end{aligned}$$

Here $-x$ means the opposite sign of x, hence $-x > 0$.

Absolute value

The absolute value (also called 'modulus') of a real number x is written $|x|$. It is the non-negative number that defines the magnitude of the given number.

Thus $|3| = 3, |-3| = 3$ and $|0| = 0$.

This means:

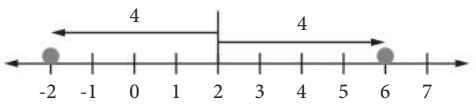
$$\begin{aligned} |x| &= x && \text{if } x > 0 \\ &= -x && \text{if } x < 0 \\ &= 0 && \text{if } x = 0 \end{aligned}$$

This is identical to $\sqrt{x^2}$, so it leads to another definition of absolute value: $|x| = \sqrt{x^2}$.

Because x is a real number, it can be represented by a point on the number line, and $|x|$ is the distance of the point x from the origin. Distance is always positive, so $|x| > 0$ for all $x \neq 0$.

The expression $|x| = 2$ thus describes a distance of 2 from the origin. The two points that are 2 units distant from 0 are $x = 0 + 2 = 2$ and $x = 0 - 2 = -2$.

Similarly, the expression $|x - a| = 4$ is saying that the distance between the points x and a is 4 units. For example, the values of x for which $|x - 2| = 4$ are $x = 2 + 4 = 6$ and $x = 2 - 4 = -2$. This can be shown on a number line:



In general:

$$\begin{aligned} |x - y| &= x - y && \text{if } x > y \\ &= y - x && \text{if } x < y \\ &= 0 && \text{if } x = y \end{aligned}$$

Thus $|5 - 3| = |3 - 5| = 2$.

Important results

1 $|xy| = |x| \times |y|$

2 $|x + y| \leq |x| + |y|$ (the 'triangle inequality')
and $|x + y| = |x| + |y|$ if and only if x and y are either zero or have the same sign.

These results can be checked with specific values of x and y. Test them for $x = -7$ and $y = 3$.

TECHNOLOGY EXPLORATION

What is meant by $|a|$?

1 In GeoGebra, add a slider using the **Slider** tool  . Leave the name as *a* and the Increment as 0.1.

2 In the input bar, enter *y=abs(a)*. This line will be called *b*.

3 Use the **Move** tool  to move the slider.

The **Algebra** view on the left shows the results clearly. What do you notice about the line? What happens when you make the value negative?

Demonstration of $|a \times c| = |a| \times |c|$

4 Add a second slider. Leave the name as *c* and the Increment as 0.1.

5 In the input bar, enter *y=abs(c)*. This line will be called *d*.

6 Hide lines *b* and *d* by clicking on the circle next to each one in the **Algebra** view.

7 In the input bar, enter *y=abs(a*c)*. A new horizontal line will appear. Right-click on the line and select **Object Properties** to change the colour.

8 In the input bar, enter *y=abs(a)*abs(c)*. A new horizontal line will appear. Right-click on the line and select **Object Properties** to change the colour.

What do you notice about the two lines?

9 Use the **Move** tool  to move the sliders for *a* and *c*.

What do you notice about the two lines now?

Demonstration of $|a + c| \leq |a| + |c|$

10 Hide the lines (by clicking on the circles in **Algebra** view) so that you have a clear working space. Set both sliders to 1.

11 In the input bar, enter *y=abs(a+c)*. A new horizontal line will appear. Right-click on the line and select **Object Properties** to change the colour.

12 In the input bar, enter *y=abs(a)+abs(c)*. A new horizontal line will appear. Right-click on the line and select **Object Properties** to change the colour.

13 Use the **Move** tool  to move the sliders for *a* and *c*.

What do you notice about the two lines? When does the condition become an equality?

Example 19

On a number line, show the values of x for which:

(a) $|x| > 1$ **(b)** $|x| \leq 2$

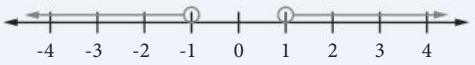
Solution

(a) $|x| > 1$

$x > 1$ or $-x > 1$

$x > 1$ or $x < -1$

$x < -1$ or $x > 1$

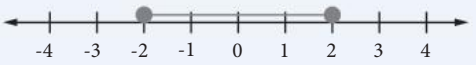


(b) $|x| \leq 2$

$x \leq 2$ or $-x \leq 2$

$x \leq 2$ or $x \geq -2$

$-2 \leq x \leq 2$



When the circle is filled in, the point is included, as it is in part **(b)**.

Example 20

Solve for x:

(a) $|2x-1| = 3$ **(b)** $|3x+2| = 1$ **(c)** $|2x-1| \geq 3$ **(d)** $|3x+2| < 1$

Solution

(a) $|2x-1| = 3$

$2x-1 = 3$ or $-(2x-1) = 3$

$2x-1 = 3$ or $2x-1 = -3$

$2x = 4$ or $2x = -2$

$x = 2$ or $x = -1$

(b) $|3x+2| = 1$

$3x+2 = 1$ or $3x+2 = -1$

$3x = -1$ or $3x = -3$

$x = -\frac{1}{3}$ or $x = -1$

(c) $|2x-1| \geq 3$

$2x-1 \geq 3$ or $-(2x-1) \geq 3$

(d) $|3x+2| < 1$

$3x+2 < 1$ or $-(3x+2) < 1$

Multiplying both sides of an inequality by -1 reverses the direction of the inequality, so:

$2x-1 \geq 3$ or $2x-1 \leq -3$

$2x \geq 4$ or $2x \leq -2$

$x \geq 2$ or $x \leq -1$

$3x+2 < 1$ or $3x+2 > -1$

$3x < -1$ or $3x > -3$

$x < -\frac{1}{3}$ or $x > -1$

$-1 < x < -\frac{1}{3}$

In **(b)** above, the first line of working has been left out. When you are confident solving absolute value equations, you can do this too. However, beware that skipping the first line of working in problems like parts **(c)** and **(d)** could easily lead to wrong inequality signs.

Example 21

Write expressions to give meaning to the following:

(a) $|2x-3|$ (b) $\sqrt{(y-2)^2}$ (c) $\sqrt{(2y-x)^2}$

Solution

(a) $|2x-3| = 2x-3$ if $2x-3>0$, i.e. if $x>\frac{3}{2}$

$= 3-2x$ if $2x-3<0$, i.e. if $x>\frac{3}{2}$

$= 0$ if $2x-3=0$, i.e. if $x=\frac{3}{2}$

OR $|2x-3| = 2x-3$ if $x \geq \frac{3}{2}$

$= 3-2x$ if $x<\frac{3}{2}$

(b) $\sqrt{(y-2)^2} = |y-2|$

$= y-2$ if $y>2$

$= 2-y$ if $y<2$

$= 0$ if $y=2$

(c) $\sqrt{(2y-x)^2} = |2y-x|$

$= 2y-x$ if $y>\frac{x}{2}$

$= x-2y$ if $y<\frac{x}{2}$

$= 0$ if $y=\frac{x}{2}$

Algebraic denominators

A fraction cannot have zero as a denominator. If we write a fraction with an algebraic expression as the denominator, then that fraction is undefined when that algebraic expression is zero.

For example, $\frac{1}{x}$ is undefined when $x=0$, so we say that the expression is defined for all real x, $x \neq 0$.

The expression $\frac{x-1}{x(x-1)}$ is undefined for $x=0$ or 1. The expression is defined for all real x, $x \neq 0, 1$.

When an expression is undefined for some values, you must write the restrictions in the answer.

Example 22

Simplify these expressions:

(a) $\frac{|x|}{x}$ for $x \neq 0$ (b) $\frac{|x-2|}{x^2-4}$ for $x \neq -2, 2$ (c) $\frac{\sqrt{(x-1)^2}}{x-1}$ for x between 0 and 1

Solution

(a) $\frac{|x|}{x}$ for $x \neq 0$

$\frac{|x|}{x} = \frac{x}{x} = 1$ if $x>0$

$\frac{|x|}{x} = \frac{-x}{x} = -1$ if $x<0$

(b) $\frac{|x-2|}{x^2-4}$ for $x \neq -2, 2$

$\frac{|x-2|}{x^2-4} = \frac{x-2}{(x-2)(x+2)} = \frac{1}{x+2}$ if $x>2$

$\frac{|x-2|}{x^2-4} = \frac{-(x-2)}{(x-2)(x+2)} = \frac{-1}{x+2}$ if $x<2, x \neq -2$

(c) $\frac{\sqrt{(x-1)^2}}{x-1}$ for x between 0 and 1

$\frac{\sqrt{(x-1)^2}}{x-1} = \frac{-(x-1)}{x-1} = -1$ because $0<x<1$ means that $(x-1)<0$, but the square root of its square must be positive.

TECHNOLOGY EXPLORATION

Absolute value inequalities: $|ax-b| < c$ and $|ax-b| > c$

1. In GeoGebra, create three sliders *a*, *b* and *c* using the **Slider** tool  . Set each slider to have an Increment of 1.
2. In the input bar, enter *y=abs(a*x-b)*. This will be labelled *f(x)* in the **Algebra** view.
3. In the input bar, enter *y=c*. This will be labelled *d*.
4. Use the **Intersect Two Objects** tool  to find the points of intersection of the horizontal line and the absolute value graph. Because we are solving for *x*, the solution will be related to the *x* values of these two points *A* and *B*.

Example: Solve $|2x + 1| < 3$

This means that we want the graph of *y=abs(2*x+1)* to be less than (or 'under') the graph of *y=3*.

5. Move the sliders so that $a = 2$, $b = -1$ and $c = 3$. You can check this in the **Algebra** view, which should show $f(x) = |2x + 1|$ and $d: y = 3$.
6. The points of intersection show *x* values of -2 and 1. The section of the absolute function under the line $y = 3$ is between these two values: the solution is $-2 < x < 1$.

You can try this with other examples too. As you move the sliders, the points on the right-hand side will sometimes shift to the left-hand side—when this happens, you can just add new points as required.

EXERCISE 3.14 SQUARE ROOTS AND ABSOLUTE VALUE

Write expressions for the following:

1 $\sqrt{81}$ **2** $\sqrt{(-2.5)^2}$ **3** $|6x-4|$ **4** $|\sqrt{3}-2|$

5 $|x+y|$ **6** $|x| + |y|$ **7** $\sqrt{x^2}+x$ **8** $|x-5| + |x+5|$

9 $\sqrt{16}+\sqrt{(-4)^2}$ **10** $\sqrt{(2x+3)^2}$ **11** $\sqrt{9-6x+x^2}$ when $x > 3$

12 Solve for *x*:

(a) $|x-2|=3$ (b) $|x+3|=7$ (c) $|4-x|=5$ (d) $|x+7|=2$

(e) $|x-6|=0$ (f) $|x-5|=1$ (g) $|x+1|=0$ (h) $|10+x|=3$

(i) $|2x+1|=2$ (j) $|2x-5|=3$ (k) $|5x+1|=4$ (l) $|3x-4|=5$

(m) $|3x+1|=0$ (n) $|6x+1|=7$ (o) $|4x-1|=0$ (p) $|2x-9|=13$

13 The solution to $|2+x|=5$ is:

A $x=-3, 7$ B $x=-7, -3$ C $x=-7, 3$ D $x=3, 7$

14 Solve

(a) $|x-1|<3$ (b) $|y+2|>4$ (c) $|t-6|\le 2$ (d) $|x+4|\ge 2$

(e) $|m-5|\ge 0$ (f) $|3-x|\le 5$ (g) $|y+1|\le 0$ (h) $|7+x|<3$

(i) $|2x+1|>3$ (j) $|3z-5|<1$ (k) $|4x+3|\ge 5$ (l) $|3t-2|<5$

(m) $|3y+1|<0$ (n) $|5x+4|>9$ (o) $|1-2x|>0$ (p) $|2x-7|\ge 11$

Solve questions **15** to **32** and show your solution on a number line.

15 $|x+1|>1$ **16** $|y-4|<3$ **17** $|x+2|<4$ **18** $|y-2|>3$

19 $|t-3|\le 2$ **20** $|x+2|\ge 1$ **21** $|3-m|\le 2$ **22** $|3+x|\ge 3$

23 $|3t-1|<4$ **24** $|2x+5|<3$ **25** $|1-2x|>3$ **26** $|2+4x|\geq 6$

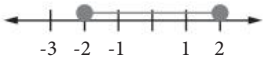
27 $|x-1|<-2$ **28** $|2x-3|\leq 5$ **29** $|3x+2|<2$ **30** $|x^2-1|\leq 4$

31 $|x+3|>1$ and $|2x+5|<3$ **32** $|2x+5|<3$ or $|2+4x|\geq 6$

33 $|x^2-1|\leq 3$ and $|x^2+1|\geq 0$ are solved and shown on a number line. Indicate whether each of the following is a correct or incorrect part of the solution.

(a) $-3\leq x^2-1\leq 3$ and $x^2+1\geq 0$ **(b)** $0\leq x\leq 2$

(c) $-2\leq x\leq 2$ **(d)**



Simplify the following expressions, stating the values of x for which your answers apply.

34 $\dfrac{\sqrt{x^2}}{|x|}$ **35** $\dfrac{\sqrt{x^2}}{x}$ **36** $\sqrt{\dfrac{(x-4)^2}{x^2}}$ **37** $\dfrac{1-x}{|1-x|}$

38 $\sqrt{x^2-10x+25}$ **39** $\dfrac{|x^2-1|}{x-1}$

40 For the following values of x and y, verify that (i) $|xy|=|x|\times|y|$ and (ii) $|x+y|\leq|x|+|y|$.

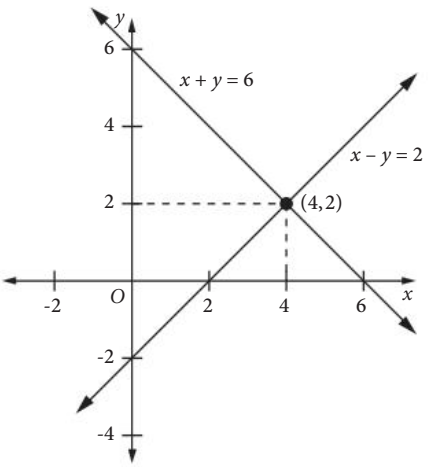
(a) $x=5, y=2$ **(b)** $x=3, y=-2$ **(c)** $x=-6, y=8$ **(d)** $x=-4, y=-3$

3.15 SIMULTANEOUS EQUATIONS

A **linear equation** in two variables x and y is an equation in which both the pronumerals are of the **first degree**, e.g. $x+y=6$. There are an infinite number of values for x and y that satisfy an equation like this, e.g. $(0,6)$, $(1,5)$, $(2,4)$, $(4,2)$, $(-8,14)$. Graphically, the points that satisfy a linear equation all lie on the same straight line.

Consider another equation $x-y=2$, for which the graph is also a straight line containing an infinite number of points, e.g. $(0,-2)$, $(1,-1)$, $(2,0)$, $(4,2)$, $(-10,-12)$.

From the two lists of points, you can see that the ordered pair $(4,2)$ satisfies both equations. Hence $x=4$, $y=2$ is said to be the solution of the simultaneous linear equations $x+y=6$ and $x-y=2$. Graphically, $x=4$, $y=2$ are the coordinates of the point of intersection of the two lines.



TECHNOLOGY EXPLORATION

Intersection points

Use GeoGebra to draw the graphs of $x+y=6$ and $x-y=2$. Find the coordinates of their point of intersection using the **Intersect Two Objects** tool.

Use GeoGebra to check each of the worked examples that follow.

Algebraic solution of simultaneous equations

The most common algebraic methods of solving a simultaneous pair of linear equations in two variables are **(a)** elimination and **(b)** substitution.

In the elimination method, we can eliminate one of the variables x or y by adding or subtracting the equations, as long as the coefficients of one of the variables are equal in both equations.

Solving simultaneous equations by elimination:

- If the coefficients have the same size but opposite sign, eliminate by addition.
- If the coefficients are equal, eliminate by subtraction.

In the substitution method, rewrite one of the equations to make it equal to one of the variables, then substitute this into the other equation.

Example 23

Solve the simultaneous equations $x + y = 6$ and $x - y = 2$.

Solution

$$x + y = 6 \quad (1)$$
$$x - y = 2 \quad (2)$$

Elimination method

The coefficients of y have the same size but opposite sign, so eliminate y by adding the equations (1) and (2).

$(1) + (2)$: $2x = 8$

$x = 4$

Substitute $x = 4$ into (1): $4 + y = 6$

$y = 2$

Check this solution by substituting it into the other equation, (2).

LHS $= 4 - 2 = 2 =$ RHS

$\therefore$ Solution is $x = 4$, $y = 2$.

Substitution method

Rewrite (1) in the form $y = \ldots$ to make a new equation, then substitute into (2).

Rewrite (1): $y = 6 - x$ (3)

Substitute (3) into (2): $x - (6 - x) = 2$

$x - 6 + x = 2$

$2x = 8$

$x = 4$

Substitute $x = 4$ into (3): $y = 6 - 4 = 2$

$\therefore$ Solution is $x = 4$, $y = 2$.

Example 24

Solve the simultaneous equations $3x + 2y = 10$ and $4x + 3y = 13$.

Solution

$$3x + 2y = 10 \quad (1)$$
$$4x + 3y = 13 \quad (2)$$

Elimination method

To use the elimination method, make the coefficients of y both the same by multiplying (1) by three and multiplying (2) by two.

$3 \times (1)$: $9x + 6y = 30$ (3)

$2 \times (2)$: $8x + 6y = 26$ (4)

$(3) - (4)$: $x = 4$

Substitute $x = 4$ into (1): $12 + 2y = 10$

$2y = -2$

$y = -1$

$\therefore$ Solution is $x = 4$, $y = -1$.

Substitution method

Rewrite (1): $y = \dfrac{10 - 3x}{2}$ (3)

Substitute (3) into (2): $4x + \dfrac{3(10 - 3x)}{2} = 13$

$8x + 30 - 9x = 26$

$-x = -4$

$x = 4$

Substitute $x = 4$ into (3): $y = \dfrac{10 - 12}{2} = \dfrac{-2}{2} = -1$

$\therefore$ Solution is $x = 4$, $y = -1$.

Example 25

Solve the simultaneous equations $\frac{x}{6}+\frac{y}{4}=1$ and $\frac{3x}{4}-\frac{x-y}{2}=\frac{7}{4}$.

Solution

First, it is best to get rid of the fractions in each equation by multiplying each term by the LCM (lowest common multiple) of the denominators.

$\frac{x}{6}+\frac{y}{4}=1$, LCM of 6 and 4 is 12: $2x+3y=12$ (1)

$\frac{3x}{4}-\frac{x-y}{2}=\frac{7}{4}$, LCM of 4 and 2 is 4: $3x-2(x-y)=7$

$x+2y=7$ (2)

$2\times(2)$: $2x+4y=14$ (3)

(1) – (3): $-y=-2$

$y=2$

Substitute into (2): $x+4=7$

$x=3$

$\therefore$ Solution is $x=3$, $y=2$.

Check your solutions by substituting them back into the original equations.

TECHNOLOGY EXPLORATION

Simultaneous equations

In GeoGebra, enter each equation in the input bar (replacing the equation's variables with the letters *x* and *y*). To find points of intersection (or solutions for *x* and *y*), click on each line using the **Intersect Two Objects** tool  . Results will be displayed in the **Algebra** view.

EXERCISE 3.15 SIMULTANEOUS EQUATIONS

Solve the simultaneous equations in this exercise. (Try to solve some using GeoGebra.)

1 $x+7y=5$, $x-7y=-9$

2 $x+5y=34$, $x-5y=-6$

3 $4x-5y=30$, $4x-2y=24$

4 $3x-y=5$, $5x+3y=-8$

5 $2m+3n=-4$, $3m+2n=-6$

6 $-2x+7y=4$, $-3x+5y=-5$

7 $x+5y=-13$, $2x-y=7$

8 $5x+2y=9$, $9x-7y=-5$

9 The solution to the simultaneous equations $2a-3b=5$ and $2a-5b=-1$ is:

A $a=-7, b=3$ **B** $a=7, b=-3$ **C** $a=7, b=3$ **D** $a=3, b=7$

10 $2x+5y=16$, $10x-3y=-4$

11 $2x+3y=14$, $4x-5y=17$

12 $5m-6n=12$, $2m+9n=20$

13 $y=4x-2$, $y=-3x+5$

14 $x-4=4(y+2)$, $3(x-2)=2y+20$

15 $3(x-y)+8y=42$, $9x-(x-2y)=78$

16 $2x - \frac{y}{4} = 5$

$x + \frac{3y}{4} = -1$

17 $\frac{3a-2b}{2} = 9$

$\frac{6a-b}{5} = 9$

18 $\frac{x-3}{2} = \frac{2y+1}{3}$

$\frac{3x-1}{5} - \frac{2y+1}{2} = 1$

19 $3(x-y) - 8(x+y) = 7$

$2(x+y) + 5(x-y) = -65$

20 $2(3a-b) = 3(a+b)$

$3(a-4b) + 46 = 5a$

21 $5(2x-y) = 7x+1$

$3(3x+y) = 5(x-y+12)$

3.16 PROBLEM SOLVING WITH SIMULTANEOUS EQUATIONS

Example 26

Three books and five pens cost \$5.55. A book costs 10 cents more than ten pens. Find the cost of a book and a pen.

Solution

Let a book cost x cents and a pen cost y cents.

3 books and 5 pens cost 555 cents:	$3x + 5y = 555$	(1)
1 book costs 10 cents more than 10 pens:	$x = 10y + 10$	(2)
Substitute (2) into (1):	$3(10y + 10) + 5y = 555$	
	$30y + 30 + 5y = 555$	
	$35y = 525$	
	$y = 15$	
Substitute back into (2):	$x = 150 + 10$	
	$x = 160$	

Hence, a book costs \$1.60 and a pen costs 15 cents, so a book and a pen together cost \$1.75.

Example 27

$y = mx + b$ is the equation of a straight line that passes through the points $(1, 8)$ and $(-2, -1)$. Form a pair of simultaneous equations in m and b and solve them to find m and b. Find the equation of the line.

Solution

	$y = mx + b$	
At $(1, 8)$:	$8 = m + b$	(1)
At $(-2, -1)$:	$-1 = -2m + b$	(2)
(1) – (2):	$9 = 3m$	
	$m = 3$	
Substitute into (1):	$8 = 3 + b$	
	$b = 5$	

The equation of the line is $y = 3x + 5$.

EXERCISE 3.16 PROBLEM SOLVING WITH SIMULTANEOUS EQUATIONS

1 There are 450 students at Newton High School. If there are 50 more boys than girls, how many boys and girls are there at the school?

2 A contractor has 8 trucks. Some trucks carry a load of 10 tonnes and the other trucks carry a load of 5 tonnes. When all 8 trucks are filled, they contain a total load of 70 tonnes. How many of each size of truck does the contractor own?

3 A father is 7 times as old as his daughter. In 5 years' time he will be 4 times as old as his daughter. If he is now x years old and his daughter is y years old, which set of equations correctly describes their ages?

A $x - 7y = 0,\ x - 4y + 5 = 0$ **B** $x + 7y = 0,\ x - 4y - 15 = 0$
C $x - 7y = 0,\ x - 4y - 15 = 0$ **D** $x + 7y = 0,\ x - 4y + 5 = 0$

4 John's mother is now 5 times as old as John. Three years ago, she was 9 times his age. What are their ages now?

5 Tickets to a movie cost \$15 for adults and \$12 for children. If 1000 people paid to see a movie and the total money paid was \$13 800, how many adults and how many children were there?

6 The straight line $ax + by = 12$ passes through the points $(2, 2)$ and $(-4, 5)$. Form two equations in a and b, then solve them to find the equation of the line.

7 Let x be the numerator and y be the denominator of a fraction. The denominator is 5 more than the numerator. If 2 is subtracted from both the numerator and the denominator, the denominator is then twice the numerator. What is the fraction?

8 Find two numbers such that if 18 is added to the first number it becomes twice the second number, and if 6 is added to the second number it becomes three times the first number.

9 The weekly wages of 5 carpenters and 3 apprentices total \$5480 while the wages of 3 carpenters and 5 apprentices total \$4440. Find the weekly wages of a carpenter and of an apprentice.

10 The equation of a straight line is given as $y = mx + b$. By forming a pair of simultaneous equations, find the equation of the line if it passes through the points given.

(a) $(4, 1)$ and $(-1, -9)$ **(b)** $(0, 4)$ and $(1, 0)$ **(c)** $(2, -1.5)$ and $(-4, -6)$ **(d)** $(2, 4)$ and $(-6, 8)$

3.17 SOLVING SIMULTANEOUS EQUATIONS— LINEAR AND SECOND DEGREE

Equations like $y = x^2 - 5x + 6$ (a parabola when graphed), $x^2 + y^2 = 4$ (a circle when graphed) and $xy = 6$ (a rectangular hyperbola when graphed) may be intersected by a straight line. To find the intersection points, we can solve the pair of simultaneous equations.

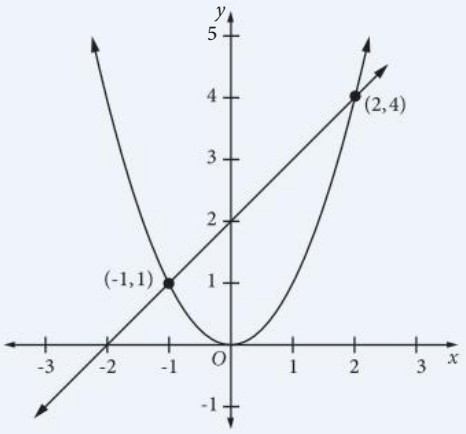
For non-linear simultaneous equations like these, the substitution method is usually the best. You might also use GeoGebra to find the solutions graphically.

Example 28

Solve the simultaneous equations $y = x^2$ and $y = x + 2$.

Solution

Graphically:



Algebraically:

$y = x^2$ (1)

$y = x + 2$ (2)

Substitute (2) into (1): $x + 2 = x^2$

$x^2 - x - 2 = 0$

$(x - 2)(x + 1) = 0$

$x = 2$ or -1

Substitute into (2): $y = 4$ or 1

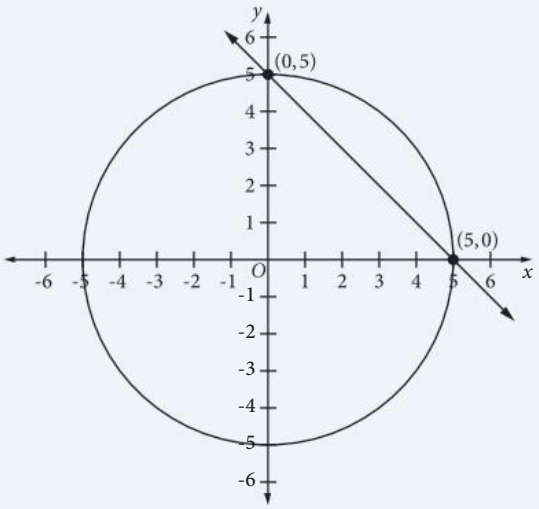
$\therefore$ Solutions are $x = 2, y = 4$ or $x = -1, y = 1$.

Example 29

Solve the simultaneous equations $x^2 + y^2 = 25$ and $x + y = 5$.

Solution

Graphically:



Algebraically:

$x^2 + y^2 = 25$ (1)

$x + y = 5$ (2)

Rewrite (2): $y = 5 - x$ (3)

Substitute (3) into (1): $x^2 + (5 - x)^2 = 25$

$x^2 + 25 - 10x + x^2 = 25$

$2x^2 - 10x = 0$

$2x(x - 5) = 0$

$x = 0, x = 5$

Substitute into (3): $y = 5, y = 0$

$\therefore$ Solutions are $x = 0, y = 5$ or $x = 5, y = 0$.

EXERCISE 3.17 SOLVING SIMULTANEOUS EQUATIONS—LINEAR AND SECOND DEGREE

Solve the following simultaneous equations. (Try to solve some using GeoGebra.)

1 $y = 5x + 6$
$y = x^2$

2 $y = 3x - 2$
$y = x^2$

3 $y = x + 5$
$y = x^2 - 3x$

4 $x + y = 15$
$y = x^2 - 6x + 1$

5 $y = x - 3$
$xy = 10$

6 $y - 2x = 1$
$x^2 + y^2 = 10$

7 $x + y = 5$
$x^2 + y^2 = 13$

8 $2x - y = 2$
$y = x^2 - x - 2$

9 $x - y = 1$
$xy = 2$

10 $y = 2x - 5$
$y = x^2 - 4x + 4$

11 $y = 2x - 6$
$x^2 - xy + 2y^2 = 16$

12 $x + y = 5$
$3x^2 + xy - y^2 = 29$

13 $y - 2x + 1 = 0$
$3y^2 - y - 2x^2 = 0$

14 $y - 4x - 8 = 0$
$y = 4 - x^2$

15 $x - y + 3 = 0$
$xy = 10$

16 $3x + y = 11$
$2x^2 - xy - y = 10$

17 $x + y = 2$
$x^2 + y^2 = 2$

18 $x = 2y - 1$
$3x^2 = x + 2y^2$

19 $x + 2y = -8$
$xy = 8$

20 $y = x + 9$
$y = x^2 - x - 6$

3.18 SOLVING SIMULTANEOUS EQUATIONS— LINEAR AND SECOND DEGREE IN THE GENERAL FORM

Example 30

Solve the simultaneous equations $3x + 2y = 4$ and $x^2 + xy - y^2 = 1$.

Solution

$3x + 2y = 4$ (1)

$x^2 + xy - y^2 = 1$ (2)

Rewrite (1): $2y = 4 - 3x$

$y = \dfrac{4 - 3x}{2}$ (3)

Substitute (3) into (2): $x^2 + \dfrac{x(4 - 3x)}{2} - \dfrac{(4 - 3x)^2}{4} = 1$

Multiply by 4: $4x^2 + 8x - 6x^2 - (16 - 24x + 9x^2) = 4$

$8x - 2x^2 - 16 + 24x - 9x^2 = 4$

$11x^2 - 32x + 20 = 0$

$(11x - 10)(x - 2) = 0$

$x = \frac{10}{11}$ or 2

Substitute into (3): $y = \dfrac{4 - \frac{30}{11}}{2} = \dfrac{44 - 30}{22} = \dfrac{7}{11}$ or $y = \dfrac{4 - 6}{2} = -1$

When rewriting equation (1), we could have instead written $x = \dfrac{4 - 2y}{3}$ and substituted for x in equation (2).

EXERCISE 3.18 SOLVING SIMULTANEOUS EQUATIONS— LINEAR AND SECOND DEGREE IN THE GENERAL FORM

Solve the following simultaneous equations. (Try to solve some using GeoGebra.)

1 $2x - 5y = 3$
$x^2 - 2y^2 - 3x = 2$

2 $3y - 4x = 0$
$x^2 + y^2 = 25$

3 $x + 2y = 3$
$xy + 2x + y = 4$

4 $3x - 2y = 2$
$x^2 - xy + y^2 = 21$

5 $3x + 2y = 1$
$xy + y^2 - y = 8$

6 $2x = 3y + 1$
$xy + x + y = 23$

CHAPTER REVIEW 3

1 Solve:

(a) $5a - 6 = 4(2a + 3)$ **(b)** $3(8a - 2) - 3(2a + 4) = 0$ **(c)** $8(x + 2) - 3(x + 5) = 2(x - 2)$

2 Solve:

(a) $\frac{x}{5} = \frac{3}{20}$ **(b)** $\frac{3x - 1}{5} = \frac{x}{20}$ **(c)** $\frac{x - 2}{x + 3} = \frac{3}{5}$

3 Solve, showing your solution on a number line:

(a) $\frac{3x - 2}{5} > 2$ **(b)** $-8 \leq 3x - 2 < 16$ **(c)** $|x - 1| > 1$

4 Solve:

(a) $x^2 = 4$ **(b)** $x^2 = 4x$ **(c)** $x^2 = 4x - 4$

(d) $(x^2 - 3x)^2 = 16$ **(e)** $(x^2 - 3x - 10)(x^2 - 3x - 4) = 0$ **(f)** $6x^2 + 7x - 3 = 0$

5 The hypotenuse of a right-angled triangle is $(x + 1)$ cm in length and the other two sides are x cm and $(x - 7)$ cm. Form an equation and solve it to find the length of each side.

6 Solve these simultaneous equations:
$$3(x - 3) = 2(2y + 1)$$
$$2(3x - 1) = 5(2y + 1) + 10$$

7 Solve the quadratic equation $2x^2 - x - 5 = 0$, giving your solutions:

(a) in simplest surd form **(b)** correct to 2 decimal places.

8 Find the solutions of the equation $\frac{x}{x + 1} - \frac{1}{x + 2} = 3$ as surds.

9 Expand and simplify $(2x - y)(x^2 - xy + y^2)$.

10 Simplify: $\frac{2}{m^2 - 4} - \frac{1}{m^2 - 3m + 2}$

11 The perimeter of a rectangle is 18 cm and its area is 20 cm^2.

(a) If the length is x cm, express the breadth in terms of x.

(b) Write the area in terms of x.

(c) Form a quadratic equation in x and solve it to find the length and breadth.

12 Solve these simultaneous equations:
$$x + y - 9 = 0$$
$$y = x^2 + 4x + 3$$

13 Solve: **(a)** $|x + 7| = 11$ **(b)** $|3x - 4| \geq 5$

CHAPTER 4
PLANE GEOMETRY

Outcomes covered:

P1 demonstrates confidence in using mathematics to obtain realistic solutions to problems

P2 provides reasoning to support conclusions which are appropriate to the context

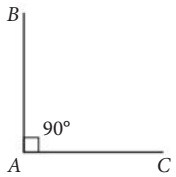
P4 chooses and applies appropriate arithmetic, algebraic, graphical, trigonometric and geometric techniques

4.1 ANGLE REVIEW

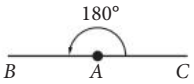
- An angle is the union of two rays that have the same endpoint. The common endpoint A is called the vertex.
- The angle can be named as angle BAC, angle CAB or angle A. This is written as $\angle BAC$, $B\hat{A}C$, $\angle CAB$, or $\angle A$.
- The unit of angle size is the degree ° and it represents $\frac{1}{360}$ of a complete turn (or 'revolution'). The magnitude (size) of an angle is the amount of turning between the arms of the angle, usually measured in degrees.

Special angles

Right angle, 90°

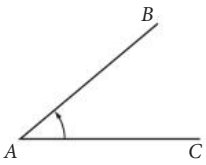


Straight angle, 180°



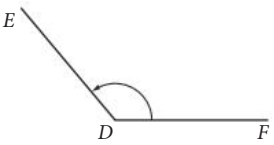
Angle of revolution, 360°

Acute angle



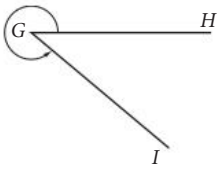
$0° < \angle BAC < 90°$

Obtuse angle



$90° < \angle EDF < 180°$

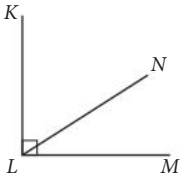
Reflex angle



$180° < \angle HGI < 360°$

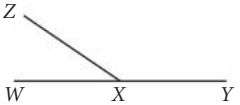
Complementary angles

- Two angles are complementary if their sum is 90°.
  $\angle KLN + \angle NLM = 90°$



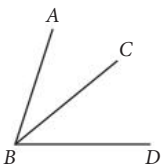
Supplementary angles

- Two angles are supplementary if their sum is 180°.
  $\angle WZX + \angle ZXY = 180°$



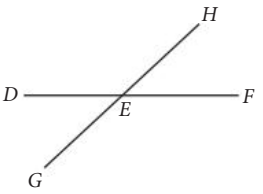
Adjacent angles

- A pair of adjacent angles have a common vertex and a common arm.
- $\angle ABC$ and $\angle CBD$ are a pair of adjacent angles. B is the vertex and BC is the common arm.
- If $\angle ABD = 90°$, then $\angle ABC$ and $\angle CBD$ are a pair of complementary adjacent angles.



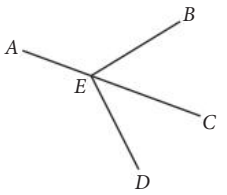
Vertically opposite angles

- Vertically opposite angles lie on opposite sides of a common vertex. Their arms form two straight lines.
- DF and GH intersect at E. $\angle DEG$ and $\angle HEF$ are a pair of vertically opposite angles.
- $\angle DEH$ and $\angle GEF$ are the other pair of vertically opposite angles.



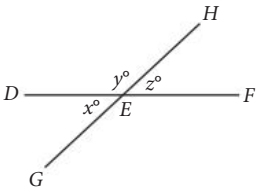
Angles at a point

- The angles around a point add up to 360°.
- $\angle AEB$, $\angle BEC$, $\angle CED$ and $\angle DEA$ are the angles at point E.
- $\angle AEB + \angle BEC + \angle CED + \angle DEA = 360°$



Theorem

When two straight lines intersect, the vertically opposite angles are equal.



Given: DF and GH intersect at E.

Aim: To prove that $\angle DEG = \angle HEF$ and that $\angle DEH = \angle GEF$.

Proof: Let $\angle DEG = x°$, $\angle DEH = y°$, $\angle HEF = z°$.

$x° + y° = 180°$ ($\angle GEH$ is a straight angle)

$y° + z° = 180°$ ($\angle DEF$ is a straight angle)

$\therefore x° + y° = y° + z°$

$\therefore x° = z°$

$\therefore \angle DEG = \angle HEF$

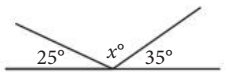
Similarly: $\angle DEH = \angle GEF$

- An **axiom** is a statement that is accepted as true without proof.
- A **theorem** is a statement that is proved to be true.

EXERCISE 4.1 ANGLE REVIEW

Find the value of each pronumeral in questions **1** to **13**.

1



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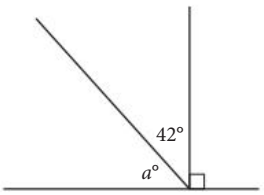

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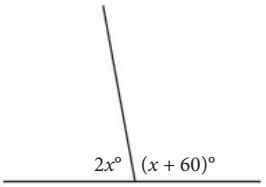

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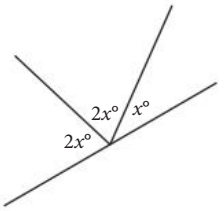

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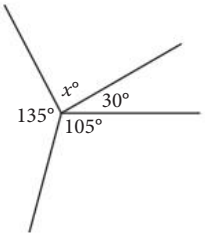

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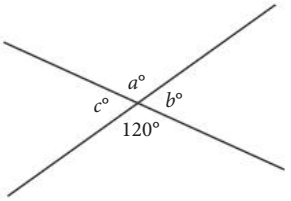

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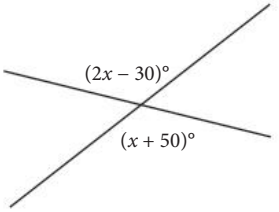

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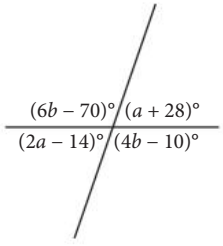

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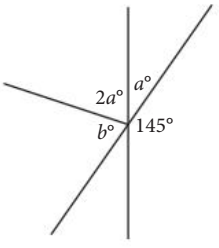

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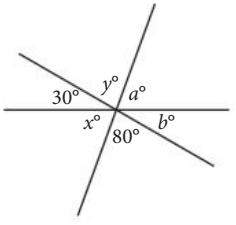

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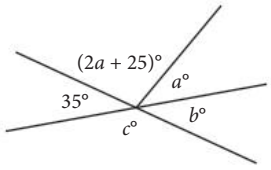

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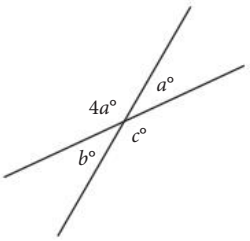

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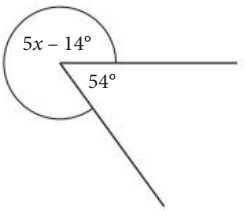


14 Write the complement of each angle:

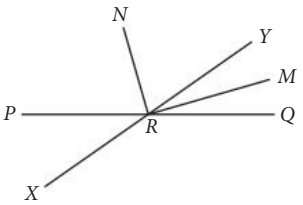
(a) 25° **(b)** 69° **(c)** 28° 14′ **(d)** 70° 53′

15 Write the supplement of each angle:

(a) 74° **(b)** 118° **(c)** 128° 32′ **(d)** 17° 17′

16 In the diagram, PQ and XY are straight lines intersecting at R. RM bisects $\angle QRY$ and RN bisects $\angle PRY$. The size of $\angle NRM$ is:

A 45° **B** 90° **C** 135° **D** 180°



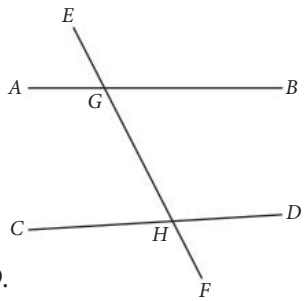
17 SOQ is a straight line. Draw OP and OR, where P and R are on opposite sides of SOQ. If $\angle POQ = \angle ROQ$, prove that $\angle POS = \angle ROS$.

18 How many pairs of adjacent angles are formed when three straight lines intersect at a point? How many pairs of vertically opposite angles are formed?

4.2 PARALLEL LINES

A **transversal** is a straight line drawn across two or more other straight lines. When a transversal cuts two other straight lines, it forms three types of pairs of angles.

Consider the transversal EF cutting the lines AB and CD at G and H respectively:



(a) Two pairs of alternate angles are formed: $\angle AGH$ and $\angle GHD$; $\angle BGH$ and $\angle GHC$.

(b) Four pairs of corresponding angles are formed: $\angle EGA$ and $\angle GHC$; $\angle EGB$ and $\angle GHD$; $\angle BGH$ and $\angle DHF$; $\angle AGH$ and $\angle CHF$.

(c) Two pairs of cointerior angles are formed: $\angle AGH$ and $\angle GHC$; $\angle BGH$ and $\angle GHD$.

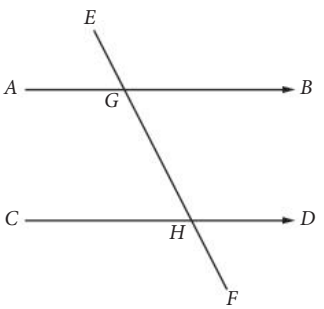
When a transversal cuts a pair of **parallel lines**, the pairs of:

(a) alternate angles are equal

(b) corresponding angles are equal

(c) cointerior angles are supplementary.

Consider $AB \parallel CD$, where EF cuts AB and CD at G and H respectively:



(a) $\angle AGH = \angle GHD$, $\angle BGH = \angle GHC$ (alternate angles)

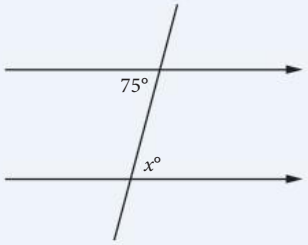
(b) $\angle EGA = \angle GHC$, $\angle EGB = \angle GHD$, $\angle BGH = \angle DHF$, $\angle AGH = \angle CHF$ (corresponding angles)

(c) $\angle AGH + \angle GHC = 180°$, $\angle BGH + \angle GHD = 180°$ (supplementary angles)

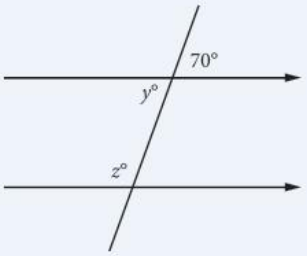
Example 1

Find the value of each pronumeral. Give reasons for your answer.

(a)



(b)



Solution

(a) $x = 75$ (alternate angles are equal, because the lines are parallel)

(b) $y = 70$ (vertically opposite angles)

$y + z = 180$ (cointerior angles supplementary, because the lines are parallel)

$70 + z = 180$

$z = 110$

Tests for parallel lines

If a transversal cuts two other straight lines and makes:

(a) a pair of alternate angles equal,

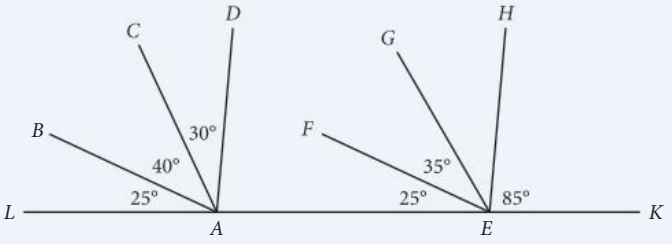
or **(b)** a pair of corresponding angles equal,

or **(c)** a pair of cointerior angles supplementary,

then the two straight lines are parallel.

Example 2

In the diagram, which pairs of lines are parallel?



Solution

1 $\angle LAB = \angle AEF = 25°$

These are a pair of equal corresponding angles.

$\therefore\ AB \parallel EF$

2 $\angle LAC = 65°$, $\angle AEG = 60°$

These corresponding angles are not equal, so AC is not parallel to EG.

3 $\angle DAE = 180° - 95°$ (LAE is a straight line)

$\angle DAE = 85°$

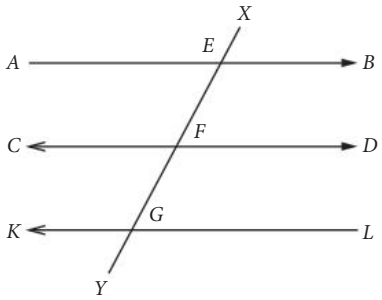
But $\angle HEK = 85°$

These are a pair of equal corresponding angles.

$\therefore\ AD \parallel EH$

Transitivity of parallel lines

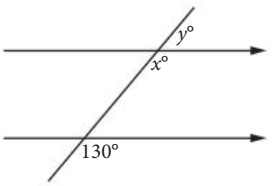
If $AB \parallel CD$ and $CD \parallel KL$ then $AB \parallel KL$.



Given:	$AB \parallel CD$, $CD \parallel KL$	
Aim:	To prove that $AB \parallel KL$.	
Construction:	Draw the transversal XY cutting AB, CD and KL at E, F and G respectively.	
Proof:	$AEX = \angle CFE$	(corresponding angles, $AB \parallel CD$)
	$\angle CFE = \angle KGF$	(corresponding angles, $CD \parallel KL$)
	$\therefore \angle AEX = \angle KGF$	(both equal $\angle CFE$)
	$\therefore AB \parallel KL$	(pair of corresponding angles equal)

EXERCISE 4.2 PARALLEL LINES

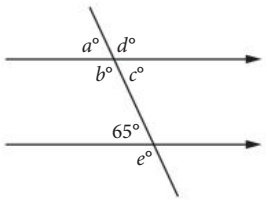
1 From the information on the diagram, select the correct set of values.



A $x = 130, y = 130$ **B** $x = 130, y = 50$
C $x = 50, y = 50$ **D** $x = 50, y = 130$

Find the value of each pronumeral in the following. Give reasons for your answers.

2



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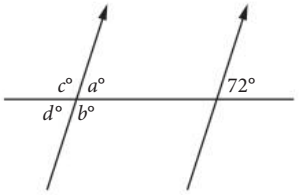

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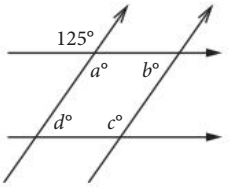

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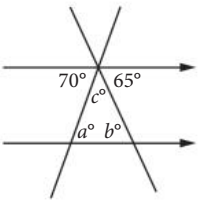

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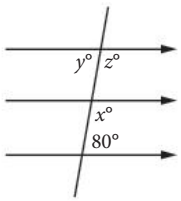

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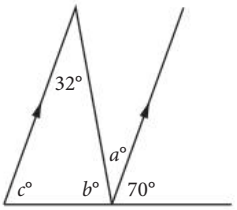

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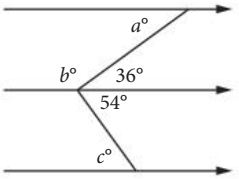

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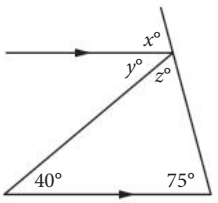

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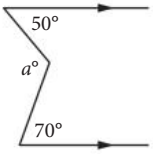

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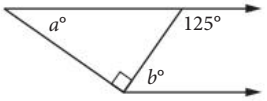

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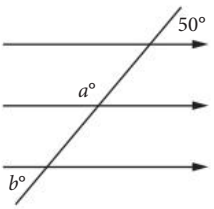
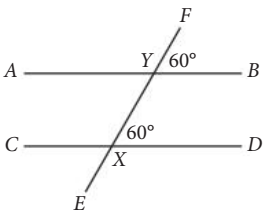
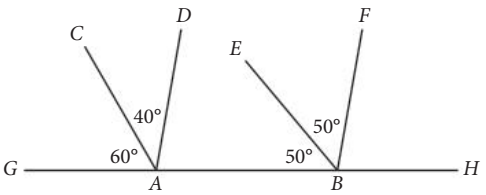


13 Which pairs of lines are parallel? Give reasons for your answers.

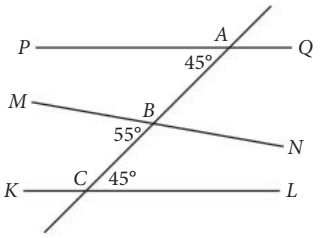
(a)



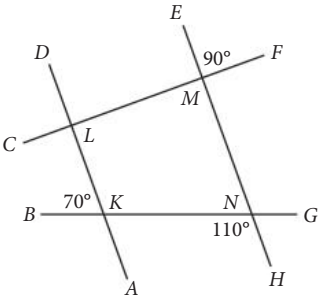
(b)



(c)



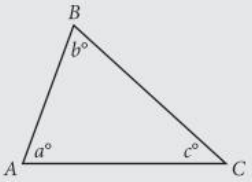
(d)



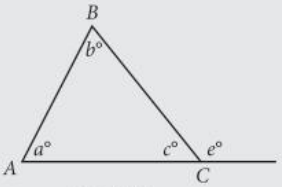
4.3 ANGLE PROPERTIES OF TRIANGLES

1 The sum of the angles of a triangle is 180°.

2 If one side of a triangle is produced, the exterior angle formed is equal to the sum of the two remote interior angles (also known as 'interior opposite angles').



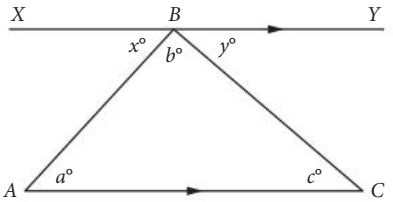
$a + b + c = 180$



$e = a + b$

Theorem

The sum of the angles of a triangle is 180°.



Given: ΔABC is any triangle.

Aim: To prove that $a + b + c = 180$.

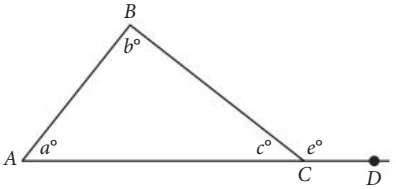
Construction: Through B, draw XY parallel to AC.

Proof:

$x = a$	(alternate angles, $XY \parallel AC$)
$y = c$	(alternate angles, $XY \parallel AC$)
$x + b + y = 180$	($\angle XBY$ is a straight angle)
$\therefore a + b + c = 180$	

Theorem

An exterior angle of a triangle is equal to the sum of the two remote interior angles.



Given: ΔABC with AC produced to D.

Aim: To prove that $e = a + b$.

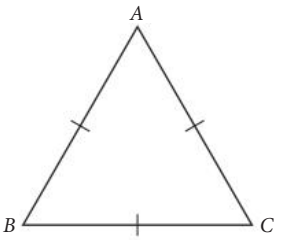
Proof:

$e + c = 180$	($\angle ACD$ is a straight angle)
$a + b + c = 180$	(angle sum of ΔABC)
$\therefore e = a + b$	

Types of triangles

Equilateral triangle

An equilateral triangle has all three sides equal in length.



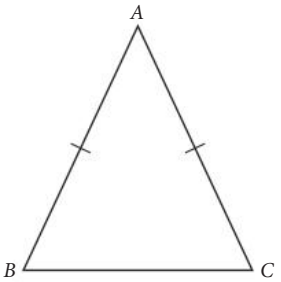
Each angle is 60°.

$AB = BC = CA$

$\angle ABC = \angle BCA = \angle CAB = 60°$

Isosceles triangle

An isosceles triangle has two sides equal in length.



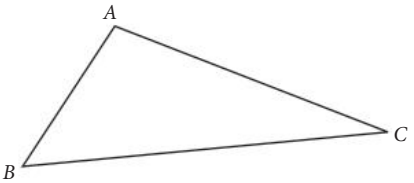
The angles opposite the equal sides are equal.

$AB = AC$

$\angle ABC = \angle ACB$

Scalene triangle

A scalene triangle has no sides equal to each other.



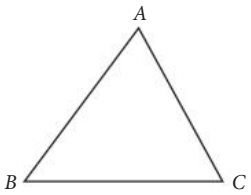
In other words, a scalene triangle has three unequal sides.
None of the angles are equal.

$AB \neq BC \neq CA$

$\angle ABC \neq \angle BCA \neq \angle CAB$

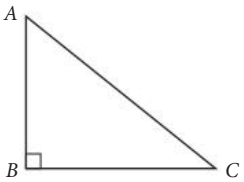
Acute-angled triangle

Three angles < 90°



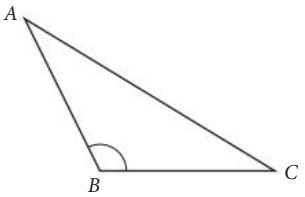
Right-angled triangle

One angle = 90°



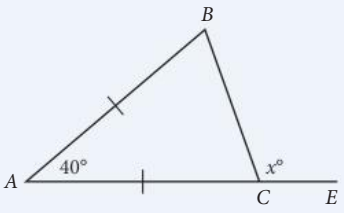
Obtuse-angled triangle

One angle > 90°



Example 3

In ΔABC, $AB = AC$ and $\angle BAC = 40°$. Find the value of x.



Solution

Because $AB = AC$, ΔABC is isosceles.

$\therefore \angle ABC = \angle ACB$

Let $\angle ABC = a°$

$\therefore 40 + 2a = 180$ (angle sum of ΔABC)

$a = 70$

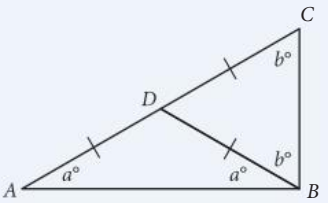
$x° = \angle BAC + \angle ABC$ (exterior angle of triangle is sum of interior opposite angles)

$x° = 40° + 70°$

$x = 110$

Example 4

In ΔABC, D is on AC such that $DA = DB = DC$. Prove that $\angle ABC = 90°$.



Solution

Aim: To prove that $\angle ABC = 90°$.

Proof:

In ΔDAB, $\angle DAB = \angle DBA$ (ΔDAB is isosceles)

In ΔDCB, $\angle DCB = \angle DBC$ (ΔDCB is isosceles)

Let $\angle DAB = a°$ and $\angle DCB = b°$

$\therefore 2a + 2b = 180$ (angle sum of ΔABC)

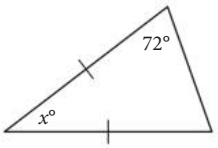
$a + b = 90$

$\therefore \angle ABC = 90°$

EXERCISE 4.3 ANGLE PROPERTIES OF TRIANGLES

Find the value of each pronumeral in questions **1** to **8**. Give reasons for your answers.

1



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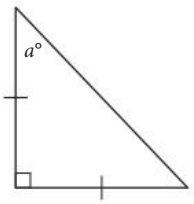

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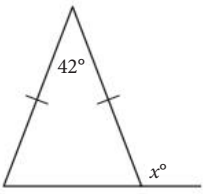

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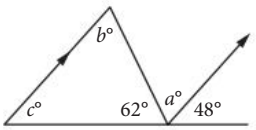

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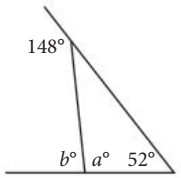

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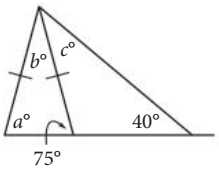

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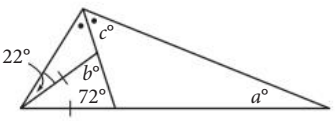

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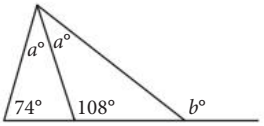
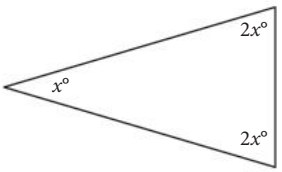


9 The value of x is:

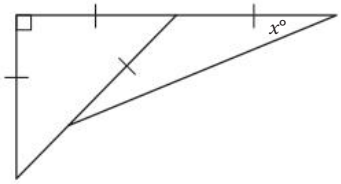
A 30 **B** 36

C 45 **D** 60



Find the value of each pronumeral in questions **10** to **12**.

10



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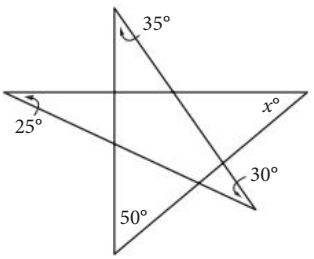

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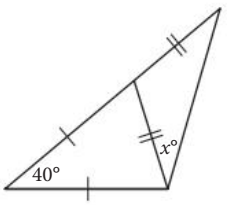
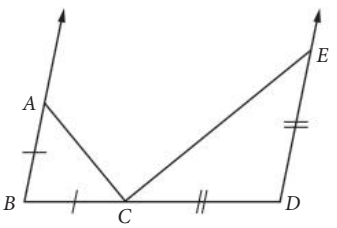
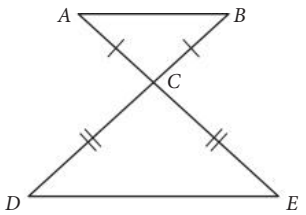


13 If $BA \parallel DE$, $AB = BC$, $CD = DE$, and also B, C and D are collinear, then prove that $\angle ACE = 90°$.
(**Collinear points** lie on the same straight line.)



14 If one angle of a triangle is equal to the sum of the other two angles, prove that the triangle is right-angled.

15 Given that $AC = CB$ and $DC = CE$, prove that $AB \parallel DE$.

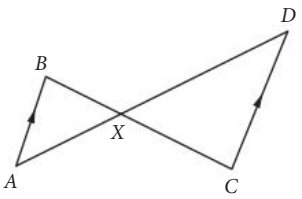


16 ABC is a right-angled isosceles triangle with the right angle at C. D and E are points on AB such that $\angle ACD = \angle BCE$. Prove that ΔCDE is isosceles.

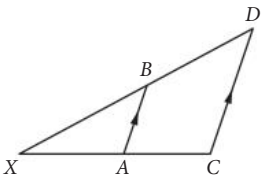
17 In ΔABC, D is a point on BC such that $BD = CD = AD$. Prove that $\angle BAC = 90°$.

18 In each of the two diagrams, prove that ΔABX and ΔCDX are equiangular, i.e. the angles are equal, given that $AB \parallel CD$.

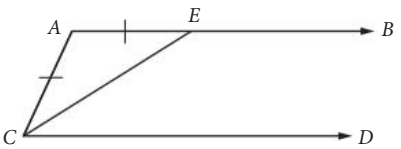
(a)



(b)



19 If $AB \parallel CD$ and $AE = AC$, prove that $\angle ACE = \angle ECD$.

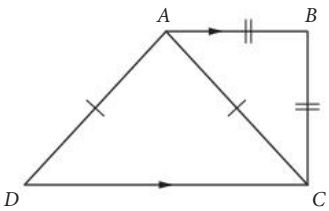


20 If $AB \parallel CD$, $AD = AC$ and $AB = BC$, prove that:

(a) $\angle ADC = \angle ACD = \angle CAB = \angle BCA$

(b) $\angle DAC = \angle ABC$.

(c) If $\angle ADC = 40°$, find the size of $\angle ABC$.



21 The three angles of a triangle are in the ratio 3 : 5 : 7. Find the magnitude of each angle.

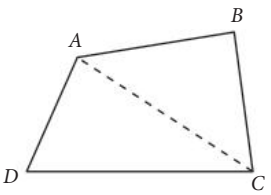
22 ABC is a triangle in which $AB = AC$. AB is produced to D so that $BD = BC$. Prove that $\angle ACB = 2\angle DCB$.

4.4 QUADRILATERALS AND POLYGONS

Quadrilaterals

A quadrilateral is a plane closed figure bounded by four straight lines.

In a quadrilateral $ABCD$, if you join the diagonal AC you create two triangles. The angles of any quadrilateral are therefore made up of the angles of two triangles, so the sum of the angles of a quadrilateral is always 360° (two times 180°).



The sum of the four angles of any quadrilateral is 360°, or four right angles.

Polygons

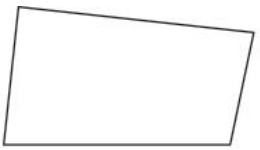
A polygon is a plane figure bounded by any number of straight lines (at least three). Most polygons have names based on the number of sides.

Triangle	3 sides	Hexagon	6 sides
Quadrilateral	4 sides	Octagon	8 sides
Pentagon	5 sides	Decagon	10 sides

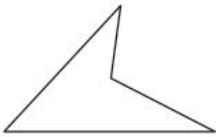
A polygon that has all sides equal and all angles equal is called a **regular polygon**. An equilateral triangle and a square are both regular polygons.

A polygon in which each angle is less than 180° is called a **convex polygon**.

A polygon in which at least one angle is more than 180° is called a **concave polygon**.



Convex polygon



Concave polygon

Theorem

The sum of the angles of a polygon with n sides is $(2n - 4)$ right angles.

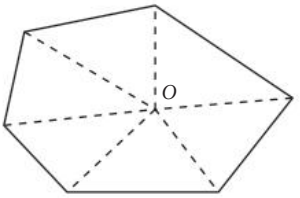
Construction: Join the vertices of the polygon to any point O inside it.

Proof: The polygon has been divided into n triangles.

The sum of the angles of each triangle is 2 right angles, so the sum of all the angles in the polygon is $2n$ right angles.

However, this includes the 4 right angles at O.

Hence, by subtraction, the sum of the angles of the polygon is $(2n - 4)$ right angles.



This can also be written as: The angle sum of any polygon is $(n - 2) \times 180°$.

The size of each angle in a regular polygon is $\frac{2n-4}{n}$ right angles, or $\frac{(n-2)\times 180°}{n}$.

Theorem

If the sides of a convex polygon are produced in order, the sum of the exterior angles formed is 4 right angles.

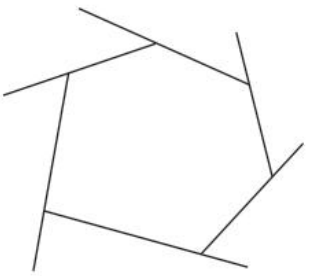
Proof: At each vertex of the polygon, the sum of the interior and exterior angles is 2 right angles.

Because there are n vertices, the sum of all the interior and exterior angles is $2n$ right angles. Therefore:

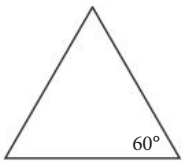
The sum of all the interior angles is $(2n - 4)$ right angles.

Hence, the sum of the exterior angles
$= 2n - (2n - 4)$ right angles $= 4$ right angles

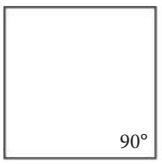
$\therefore$ the sum of the exterior angles is 360°.



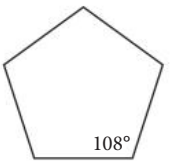
Equilateral triangle



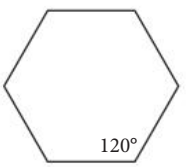
Square



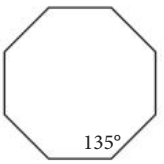
Regular pentagon



Regular hexagon



Regular octagon



Example 5

Find the size of the interior angle of a regular decagon.

Solution

Sum of interior angles of a polygon = $(2n - 4)$ right angles

Decagon, $n = 10$: angle sum $= 20 - 4$

$= 16$ right angles

$= 1440°$

Regular decagon: each angle $= \frac{1440°}{10} = 144°$

Alternatively:

Sum of exterior angles = 4 right angles = 360°

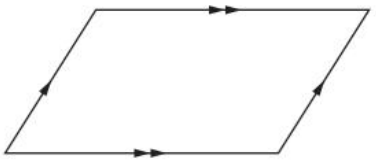
Decagon, $n = 10$: each exterior angle $= \frac{360°}{10} = 36°$

$\therefore$ each interior angle $= 180° - 36° = 144°$

Special quadrilaterals

Parallelogram

A parallelogram is a quadrilateral with both pairs of opposite sides parallel.

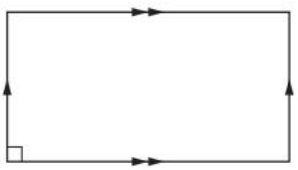


Properties:

(a) pairs of opposite sides equal
(b) pairs of opposite angles equal
(c) diagonals bisect each other

Rectangle

A rectangle is a parallelogram with one angle a right angle.

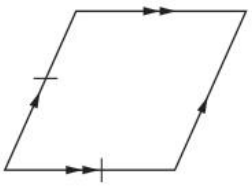


Properties:

(a) pairs of opposite sides equal
(b) each angle is 90°
(c) diagonals are equal and bisect each other

Rhombus

A rhombus is a parallelogram with a pair of adjacent sides equal.



Properties:

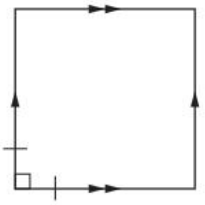
(a) all sides equal
(b) pairs of opposite angles equal
(c) diagonals bisect each other at right angles
(d) diagonals bisect the angles of the rhombus

Square

A square is a rectangle with a pair of adjacent sides equal.

OR

A square is a rhombus with one angle that is a right angle.

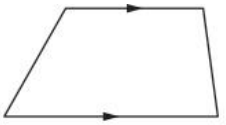


Properties:

(a) all sides equal
(b) each angle is 90°
(c) diagonals are equal and bisect each other at right angles
(d) diagonals bisect the angles of the square

Trapezium

A trapezium is a quadrilateral with one pair of opposite sides parallel.

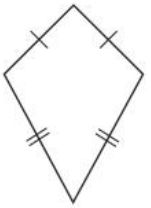


Kite

A kite is a quadrilateral with both pairs of adjacent sides equal.

Properties:

(a) one pair of opposite angles are equal (between the unequal sides)
(b) one axis of symmetry

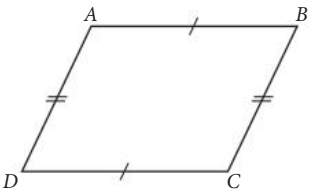


Tests for parallelograms

1 If the pairs of opposite sides of a quadrilateral are equal, then the quadrilateral is a parallelogram.

In quadrilateral $ABCD$, $AB = DC$ and $AD = BC$.

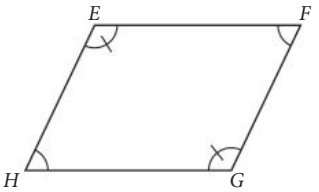
$\therefore$ $ABCD$ is a parallelogram.



2 If the pairs of opposite angles of a quadrilateral are equal, then the quadrilateral is a parallelogram.

In quadrilateral $EFGH$, $\angle EFG = \angle GHE$ and $\angle HEF = \angle FGH$.

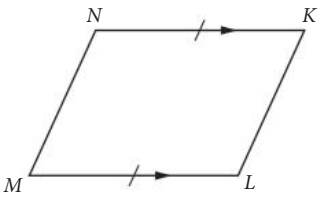
$\therefore$ $EFGH$ is a parallelogram.



3 If a quadrilateral has one pair of sides that are equal and parallel, then the quadrilateral is a parallelogram.

In quadrilateral $NKLM$, $NK \parallel ML$ and $NK = ML$.

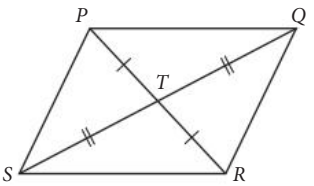
$\therefore$ $NKLM$ is a parallelogram.



4 If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.

In quadrilateral $PQRS$, $PT = TR$ and $QT = TS$.

$\therefore$ $PQRS$ is a parallelogram.

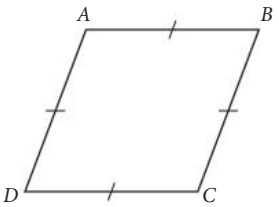


Tests for rhombuses

1 If the four sides of a quadrilateral are equal, then the quadrilateral is a rhombus.

In quadrilateral $ABCD$, $AB = BC = CD = DA$.

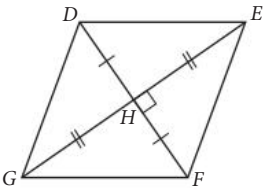
$\therefore$ $ABCD$ is a rhombus.



2 If the diagonals of a quadrilateral bisect each other at right angles, then the quadrilateral is a rhombus.

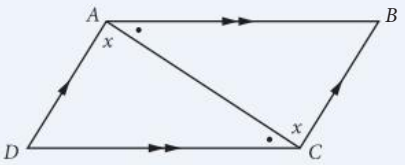
In quadrilateral $DEFG$, $DH = HF$, $EH = HG$, $\angle EHF = 90°$.

$\therefore$ $DEFG$ is a rhombus.



Example 6

Prove that the opposite angles of a parallelogram are equal.



Aim:	To prove that $\angle DAB = \angle DCB$.	
Construction:	Draw the diagonal AC.	
Proof:	$\angle BAC = \angle ACD$	(alternate angles, $AB \parallel DC$)
	$\angle DAC = \angle BCA$	(alternate angles, $AD \parallel BC$)
	$\therefore \angle BAC + \angle DAC = \angle ACD + \angle BCA$	(adding the two previous lines)
	$\therefore \angle DAB = \angle DCB$	

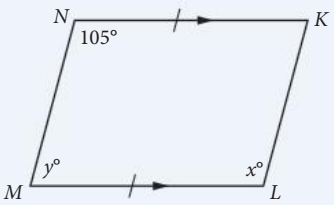
Similarly, by drawing the diagonal BD we can prove that $\angle ABC = \angle ADC$.

With this we have proved that the opposite angles of a parallelogram are equal.

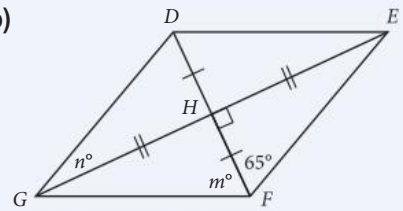
Example 7

First identify the special quadrilateral, then find the value of each pronumeral, giving reasons for your answers.

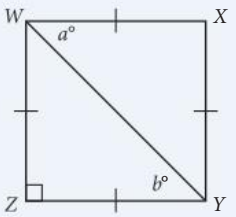
(a)



(b)



(c)



Solution

(a) $KLMN$ is a parallelogram because $NK = ML$ and $NK \parallel ML$ (a pair of opposite sides are equal and parallel).

$x = 105$	(opposite angles of a parallelogram are equal)
$y = 75$	(cointerior angles, $NK \parallel ML$)

(b) $DEFG$ is a rhombus because diagonals DF and EG bisect each other at right angles.

$m = 65$	(diagonals of a rhombus bisect its angles)
$\angle GDF = 65°$	(alternate angles are equal, $DG \parallel EF$, opposite sides of a rhombus)
$\therefore n = 25$	(angle sum of ΔGDH)

(c) $WXYZ$ is a rhombus because it is a quadrilateral with four equal sides.

$\therefore WXYZ$ is a square because it is a rhombus with a 90° angle.

$\therefore a = b = 45$	(diagonals of a square bisect its angles)

EXERCISE 4.4 QUADRILATERALS AND POLYGONS

1 Find the sum of the angles of each polygon, given the number of sides:

(a) 6 **(b)** 8 **(c)** 10 **(d)** 20

2 Calculate the number of degrees in each interior angle of a regular

(a) pentagon **(b)** hexagon **(c)** octagon **(d)** dodecagon (12 sides)

3 The interior angle of a regular polygon is 140°. How many sides does the polygon have?

A 8 **B** 9 **C** 10 **D** 12

4 Calculate the number of degrees in each exterior angle of a regular

(a) pentagon **(b)** hexagon **(c)** octagon **(d)** dodecagon (12 sides)

5 How many sides has a regular polygon, each of whose interior angles is:

(a) 90° **(b)** 108° **(c)** 165° **(d)** 160° **(e)** 168° **(f)** 170°?

6 How many sides has a regular polygon, each of whose exterior angles is:

(a) 45° **(b)** 24° **(c)** 22.5° **(d)** 10°?

7 Show that it is not possible for a regular polygon to have interior angles of 152°.

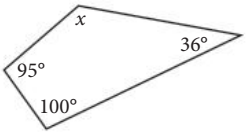
8 Each interior angle of a regular polygon is eight times an exterior angle. How many sides does the polygon have?

9 Three of the angles of a pentagon are 98°, 112° and 114°. If the other two angles are equal, their size is:

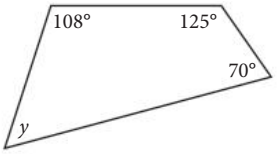
A 34° **B** 108° **C** 198° **D** 216°

10 Find the value of the pronumeral in each diagram.

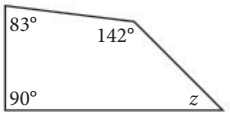
(a)



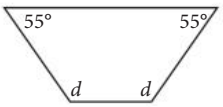
(b)



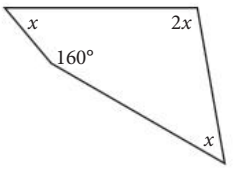
(c)



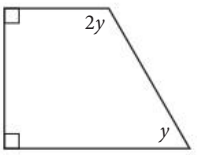
(d)



(e)



(f)



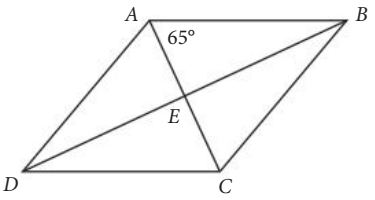
11 $ABCD$ is a trapezium in which $AB \parallel DC$. The bisectors of angles A and D intersect at E. Prove that $\angle AED = 90°$.

12 $ABCD$ is a rhombus in which $\angle BAC = 65°$.

(a) Write three other angles whose size is 65°.

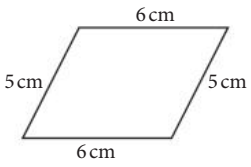
(b) Write four angles whose size is 25°.

(c) Prove that the diagonals are at right angles.

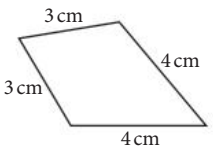


13 Write the special name for each quadrilateral, giving the test used.

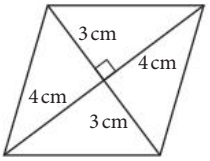
(a)



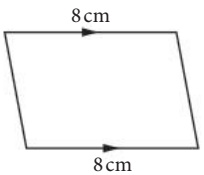
(b)



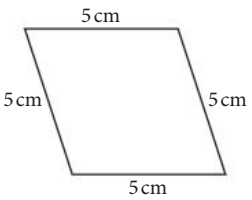
(c)



(d)



(e)



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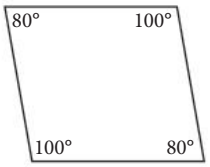
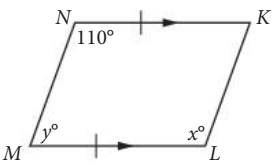
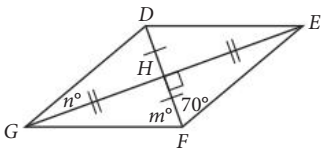
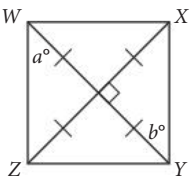
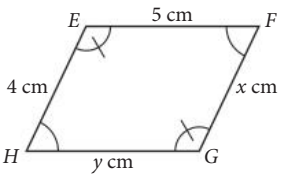


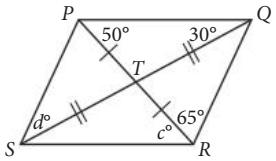
14 First identify the special quadrilateral, then find the value of each pronumeral, giving reasons for your answer.

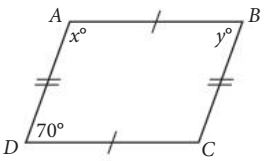
(a) 

(b) 

(c) 

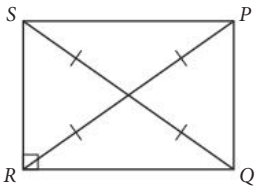
(d) 

(e) 

(f) 

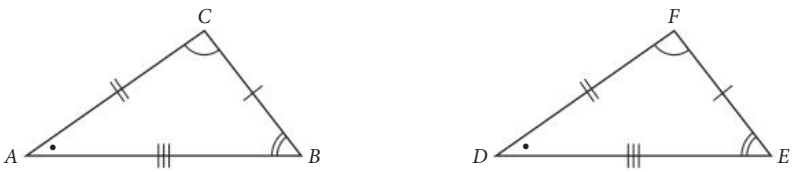
15 Based on the information shown on the diagram, the best name for *SPQR* is:

A parallelogram **B** rectangle
C rhombus **D** square



4.5 CONGRUENT TRIANGLES (PRELIMINARY)

Two plane figures are **congruent** if they are equal in size and shape. If one figure is placed on top of the other with corresponding points touching, they will fit perfectly:



The triangles *ABC* and *DEF* are congruent. The three angles of one triangle are the same as the three angles of the other, and the sides opposite these angles are equal in length:

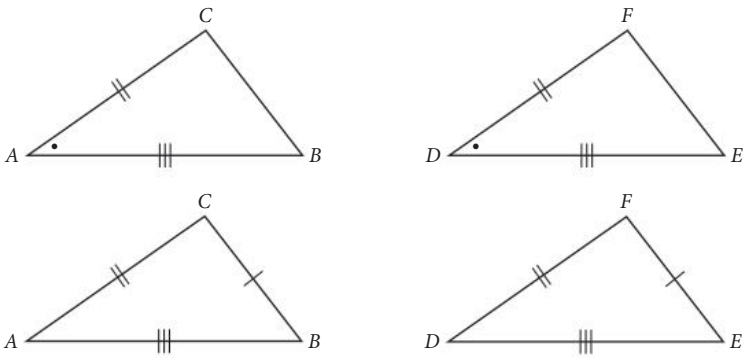
$\angle BAC = \angle EDF$ $BC = EF$
$\angle ABC = \angle DEF$ $AC = DF$
$\angle BCA = \angle EFD$ $AB = DE$

$\Delta ABC \equiv \Delta DEF$, where the symbol $\equiv$ means 'is congruent to'.

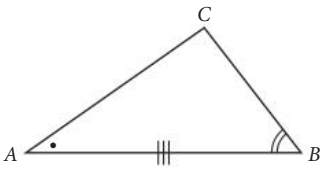
Tests for congruent triangles

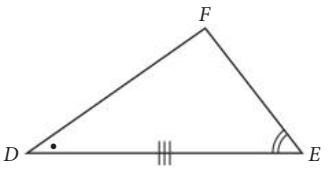
1 Two triangles are congruent if two sides and the included angle of one triangle are respectively equal to two sides and the included angle of the other triangle. (**SAS**)

2 Two triangles are congruent if the three sides of one triangle are respectively equal to the three sides of the other triangle. (**SSS**)

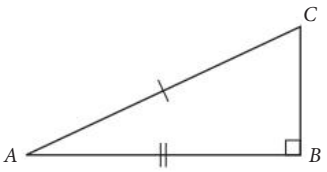


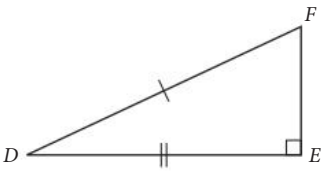
3 Two triangles are congruent if two angles and a side of one triangle are respectively equal to two angles and the corresponding side of the other triangle. (**AAS**)





4 Two triangles are congruent if the hypotenuse and one side of a right-angled triangle are equal to the hypotenuse and the corresponding side of the other right-angled triangle. (**RHS**)



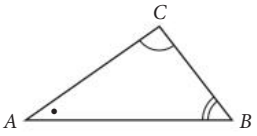


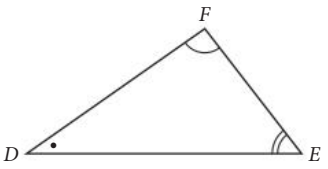
Triangle congruency tests

1	2 sides, included angle	**SAS**
2	3 sides	**SSS**
3	2 angles, corresponding side	**AAS**
4	right angle, hypotenuse, side	**RHS**

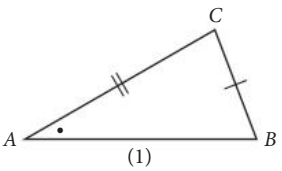
It is important to note situations where triangles are not congruent.

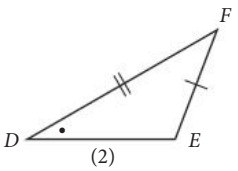
(a) Triangles with three pairs of angles equal are equiangular (or 'similar'), but are not congruent unless the sides are equal also.

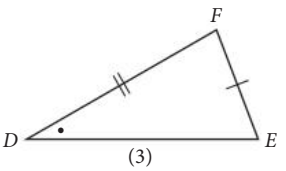




(b) Triangles with two sides and an angle equal are not necessarily congruent, if the angle is opposite one of the sides instead of included between the two sides. This is called the ambiguous case, because it is sometimes possible to draw two different triangles with these properties.





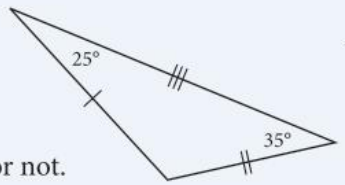


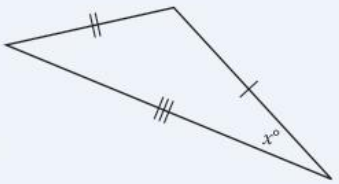
Triangles (1) and (3) are congruent, i.e. $\Delta ABC \equiv \Delta DEF$
Triangles (1) and (2) are not congruent.

The tests for congruent triangles can be used to prove geometrical facts about different figures.

Example 8

Find the value of x. Give reasons for your answer.





Solution

First, determine if the two triangles are congruent or not.

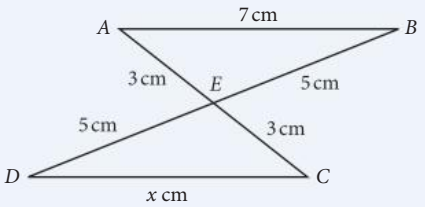
The markings on the sides of the triangles show that for each side in the first triangle there is an equal side in the second triangle.

$\therefore$ the two triangles are congruent (SSS)

$\therefore$ $x = 25$ (corresponding angles in congruent triangles) or (matching angles in congruent triangles)

Example 9

Find the value of x. Give reasons for your answer.



Solution

In ΔABE and ΔCDE:

$AE = EC$	(both 3 cm)
$\angle AEB = \angle CED$	(vertically opposite angles)
$BE = ED$	(both 5 cm)
$\therefore \Delta AEB \equiv \Delta CED$	(SAS)
$\therefore AB = DC$	(corresponding sides of congruent triangles)
$\therefore x = 7$	

Example 10

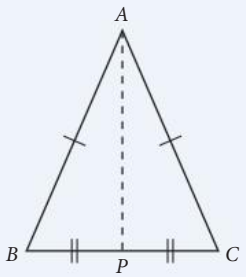
Prove that:

(a) the base angles of an isosceles triangle are equal

(b) the line drawn from the vertex of an isosceles triangle to the midpoint of the base is perpendicular to the base.

Solution

Given: An isosceles triangle ABC, with $AB = AC$. P is the midpoint of BC.



Aim: To prove that:

(a) $\angle ABC = \angle ACB$

(b) $\angle APB = 90°$.

Construction: Join AP.

Proof:

(a) In ΔABP and ΔACP, AP is a common side

$BP = CP$	(P is the midpoint of BC)
$AB = AC$	(given)
$\therefore \Delta ABP \equiv \Delta ACP$	(SSS)
$\therefore \angle ABP = \angle ACP$	(corresponding angles in congruent triangles)

(b)

$\therefore \angle APB = \angle APC$	(corresponding angles in congruent triangles)
But $\angle APB + \angle APC = 180°$	(BPC is a straight line)
$\therefore \angle APB = \angle APC = 90°$	

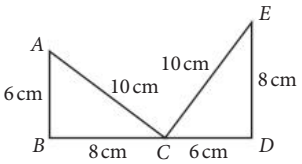
Isosceles triangle properties

1. The angles opposite the equal sides of an isosceles triangle are equal.
   (Or, the base angles of an isosceles triangle are equal.)
2. The line joining the midpoint of the base of an isosceles triangle to the opposite vertex is perpendicular to the base.
3. The line drawn from the vertex of an isosceles triangle perpendicular to the base bisects the base.
4. The line joining the midpoint of the base of an isosceles triangle to the opposite vertex bisects the angle at that vertex.

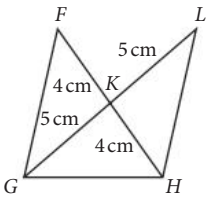
EXERCISE 4.5 CONGRUENT TRIANGLES (PRELIMINARY)

1 Name the two congruent triangles in each diagram. Give reasons for your answer.

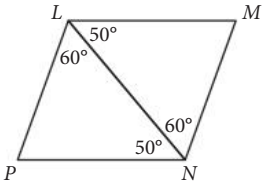
(a)



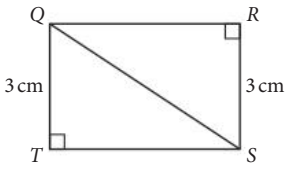
(b)



(c)



(d)



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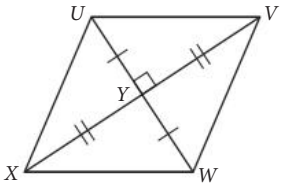
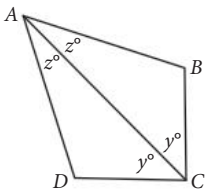
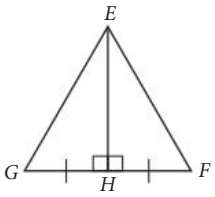


(f)



(g)



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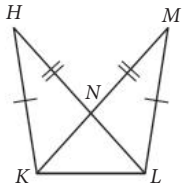
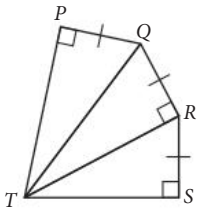


(i)

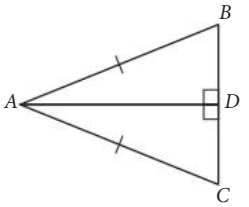


2 **(a)** Which congruence test would you use to show that $\Delta ABD \equiv \Delta ACD$?

A SSS **B** SAS

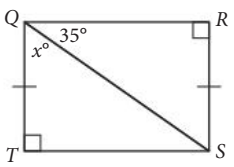
C AAS **D** RHS

(b) What can you say about BD and DC?

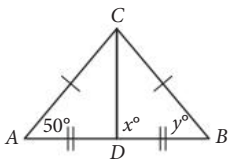


3 Find the value of each pronumeral. Justify your answers.

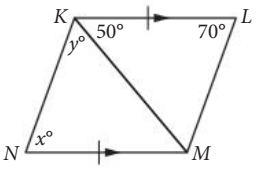
(a)



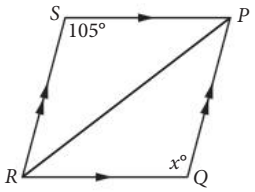
(b)



(c)



(d)



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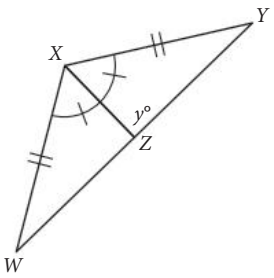
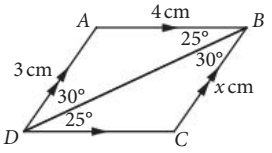
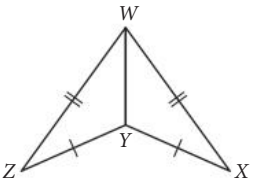


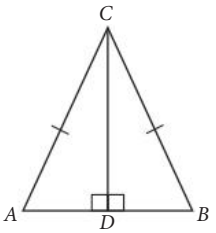
(f)



4 $WZ = WX$ and $ZY = YX$. Prove that YW bisects $\angle ZWX$.



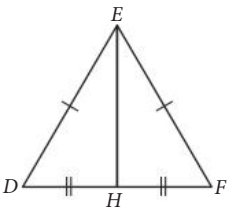
5 In ΔABC, $AC = BC$ and $\angle ADC = \angle BDC = 90°$. Prove that $\angle ACD = \angle BCD$.



6 In ΔDEF, $DE = EF = FD$ and H is the midpoint of DF. Prove that:

(a) $\angle DHE = \angle FHE = 90°$

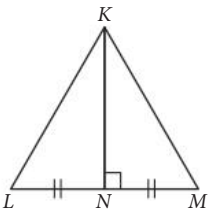
(b) $\angle DEH = 30°$



7 In ΔKLM, N is the midpoint of LM and $KN \perp LM$. Prove that:

(a) $\angle KLN = \angle KMN$

(b) NK bisects $\angle LKM$



8 $PQRS$ is a quadrilateral in which $PT = TR$ and $QT = TS$.

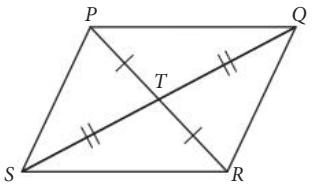
(a) Prove that $\Delta PTQ \equiv \Delta RTS$.

(b) Explain why $PQ \parallel SR$.

(c) Prove that $\Delta PTS \equiv \Delta RTQ$.

(d) Explain why $PS \parallel QR$.

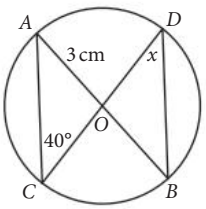
(e) What type of quadrilateral is $PQRS$?



9 AB and CD are diameters of a circle centred at O. $\angle ACO = 40°$, $AO = 3$ cm. AC and DB are chords of the circle.

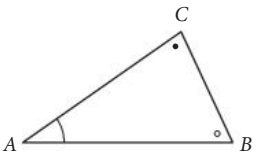
(a) Prove that $\Delta OAC \equiv \Delta OBD$.

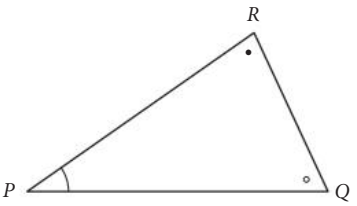
(b) Find the value of x, giving reasons for your answer.



4.6 SIMILAR TRIANGLES (PRELIMINARY)

Similar figures have exactly the same shape but are not the same size. Thus, similar figures are equiangular (have the same angles) and their pairs of matching sides are in the same ratios (proportional).





For triangles ABC and PQR:

$\angle ABC = \angle PQR$

$\angle BCA = \angle QRP$

$\angle CAB = \angle RPQ$

Hence $\Delta ABC \,|||\, \Delta PQR$. The symbol $|||$ means 'is similar to'.

Tests for similar triangles

1 Two triangles are similar if the three angles of one triangle are respectively equal to the three angles of the other triangle. (**AAA**)

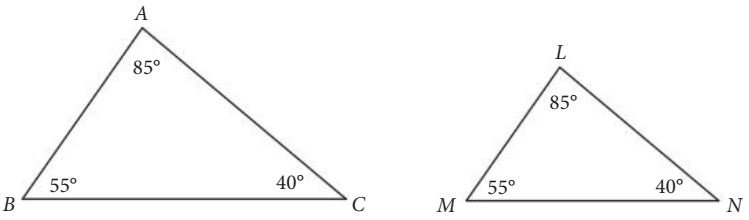
In ΔABC and ΔLMN:

$\angle ABC = \angle LMN = 55°$

$\angle BCA = \angle MNL = 40°$

$\angle CAB = \angle NLM = 85°$

$\therefore \Delta ABC \,|||\, \Delta LMN$ (AAA)



If two pairs of angles in a triangle are equal, then the third pair must also be equal (because the sum of the angles in a triangle is always 180°).

Therefore, the proof above only needed to show:

$\angle ABC = \angle LMN = 55°$

$\angle BCA = \angle MNL = 40°$

$\therefore \Delta ABC \,|||\, \Delta LMN$ (AAA)

2 Two triangles are similar if the three pairs of corresponding sides are proportional.

In ΔDEF and ΔPQR:

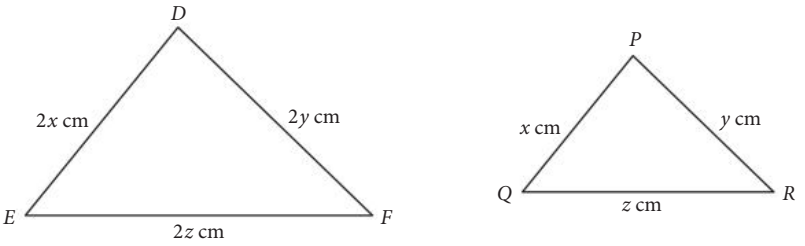
$\frac{DE}{PQ} = \frac{2x}{x} = \frac{2}{1}$

$\frac{EF}{QR} = \frac{2z}{z} = \frac{2}{1}$

$\frac{FD}{RP} = \frac{2y}{y} = \frac{2}{1}$

$\therefore \frac{DE}{PQ} = \frac{EF}{QR} = \frac{FD}{RP} = \frac{2}{1}$

$\therefore \Delta DEF \,|||\, \Delta PQR$ (3 pairs of corresponding sides proportional)



3 Two triangles are similar if two pairs of corresponding sides are proportional and the angles included by these sides are equal.

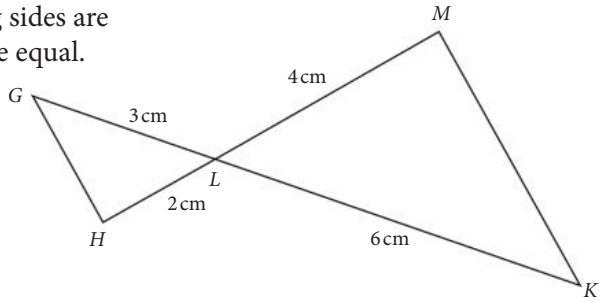
In ΔGHL and ΔMKL:

$\frac{GL}{KL} = \frac{3}{6} = \frac{1}{2}$

$\frac{HL}{ML} = \frac{2}{4} = \frac{1}{2}$

$\angle GLH = \angle KLM$ (vertically opposite angles)

$\therefore \Delta GHL \,|||\, \Delta MKL$ (2 pairs of sides proportional, included angles equal)



Comparison with congruent triangle tests

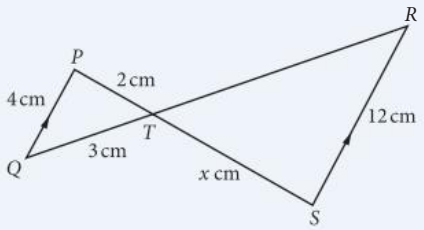
Similar triangles test **1** can be written as AAA as a reason in a proof. It is not like any test for congruent triangles.

Similar triangles test **2** is like the SSS test for congruent triangles, except that the sides are proportional instead of equal. When the sides ratio is 1:1 the triangles are congruent.

Similar triangles test **3** is like the SAS test for congruent triangles, except that the sides are proportional instead of equal, but the included angles must be equal. When the sides ratio is 1:1 the triangles are congruent.

Example 11

If $PQ \parallel RS$, find the value of x. Give reasons for your answer.



Solution

In ΔPQT and ΔSRT:

$\angle QPT = \angle RST$ (alternate angles, $PQ \parallel RS$)

$\angle PTQ = \angle STR$ (vertically opposite angles)

$\therefore \Delta PQT \,|||\, \Delta SRT$ (AAA)

$\therefore \dfrac{PQ}{SR} = \dfrac{QT}{RT} = \dfrac{TP}{TS}$ (matching sides of similar triangles are proportional)

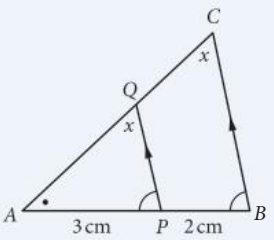
$\therefore \dfrac{4}{12} = \dfrac{3}{RT} = \dfrac{2}{x}$

$\therefore \dfrac{4}{12} = \dfrac{2}{x}$

$\therefore x = 6$

Example 12

If $PQ \parallel BC$, prove that $\Delta APQ \,|||\, \Delta ABC$.



Solution

In ΔAPQ and ΔABC:

$\angle APQ = \angle ABC$ (corresponding angles, $PQ \parallel BC$)

$\angle QAP = \angle CAB$ (common angle)

$\therefore \Delta APQ \,|||\, \Delta ABC$ (AAA)

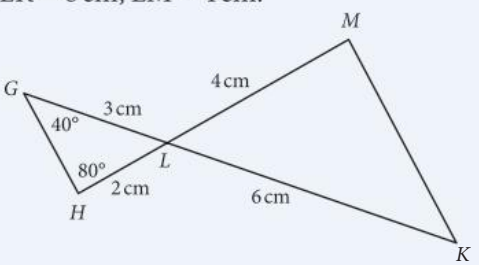
Example 13

In the diagram, $\angle HGL = 40°$, $\angle GHL = 80°$, $GL = 3$ cm, $HL = 2$ cm, $LK = 6$ cm, $LM = 4$ cm.

(a) Prove that $\Delta GLH \,|||\, \Delta KLM$.

(b) Hence find the size of $\angle LKM$.

(c) Prove that $GH \parallel MK$.



Solution

(a) In ΔGHL and ΔMKL:

$\dfrac{GL}{KL} = \dfrac{3}{6} = \dfrac{1}{2}$

$\dfrac{HL}{ML} = \dfrac{2}{4} = \dfrac{1}{2}$

$\angle GLH = \angle KLM$ (vertically opposite angles)

$\therefore \Delta GHL \,|||\, \Delta MKL$ (2 pairs of sides proportional, included angles equal)

(b) $\angle LKM = \angle HGL = 40°$ (matching angles in similar triangles)

$\therefore \angle LKM = 40°$

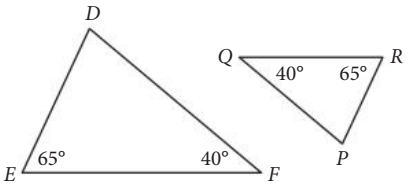
(c) $\angle LKM = \angle HGL = 40°$ are a pair of equal alternate angles between the lines GH and MK.

$\therefore GH \parallel MK$

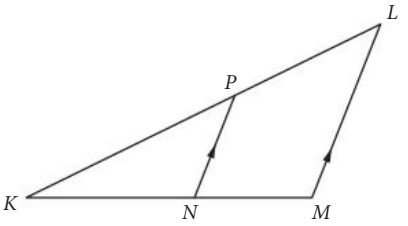
EXERCISE 4.6 SIMILAR TRIANGLES (PRELIMINARY)

1 Which pairs of triangles are similar? Why?

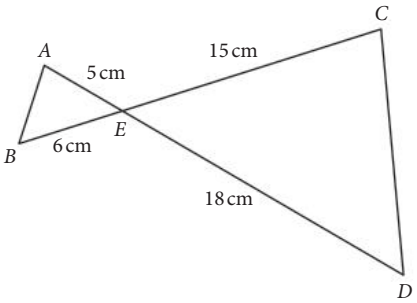
(a)



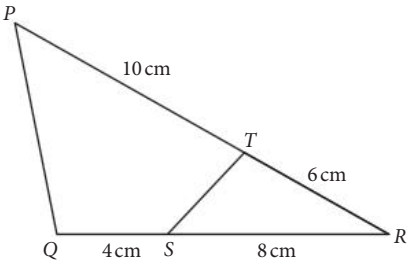
(b)



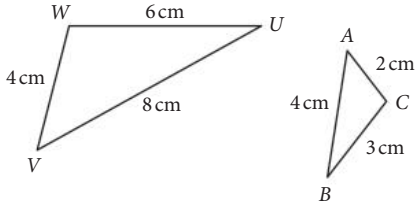
(c)



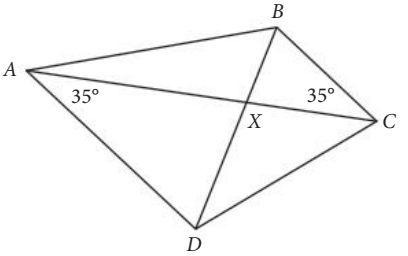
(d)



(e)

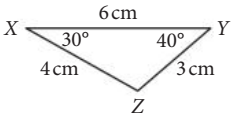


(f)

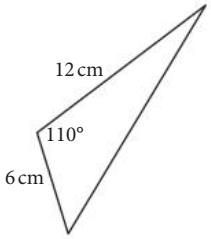


2 If the length of each side of a triangle is doubled, will the resulting triangle be similar to the original one?

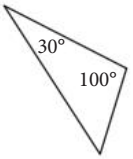
3 Which triangle is similar to ΔXYZ?



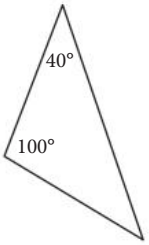
A



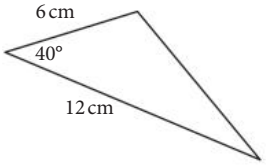
B



C

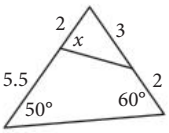


D

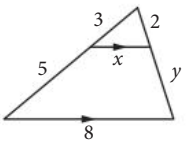


4 Find the value of each pronumeral in the following pairs of similar triangles. State the test used.

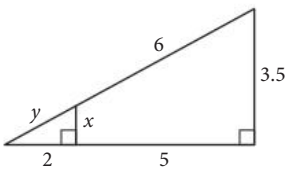
(a)



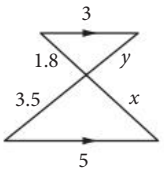
(b)



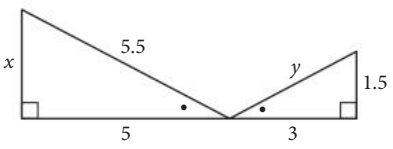
(c)



(d)

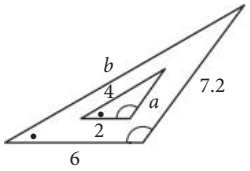


(e)

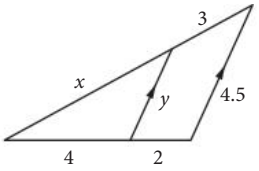


5 Find the value of each pronumeral in the following pairs of similar triangles. State the test used.

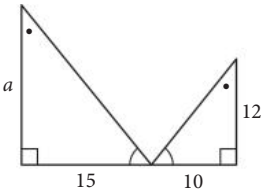
(a)



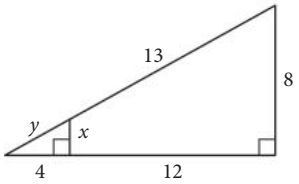
(b)



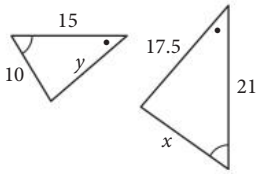
(c)



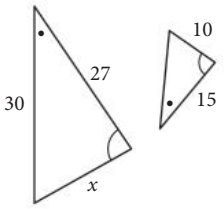
(d)



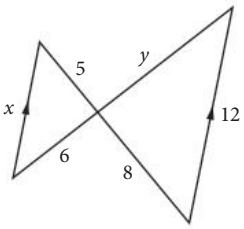
(e)



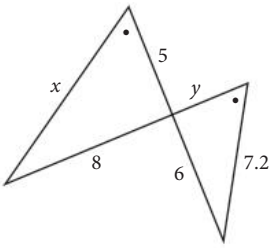
(f)



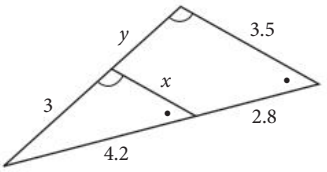
(g)



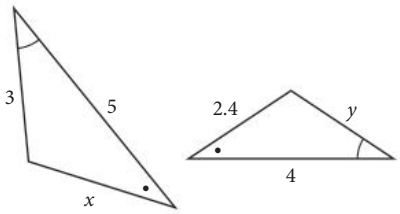
(h)



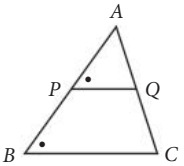
(i)



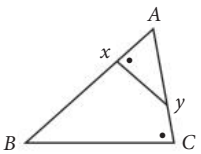
(j)



6 **(a)** If $\angle APQ = \angle ABC$, prove that $\Delta APQ \,|||\, \Delta ABC$.



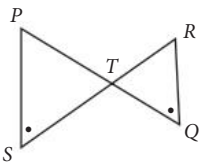
(b) If $\angle AXY = \angle ACB$, prove that $\Delta AXY \,|||\, \Delta ACB$.



7 If $\angle PST = \angle RQT$:

(a) prove that $\Delta PST \,|||\, \Delta RQT$

(b) complete the ratios $\frac{PS}{RQ} = \frac{PT}{?} = \frac{ST}{?}$.

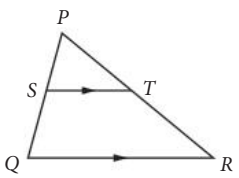


8 If $ST \parallel QR$:

(a) prove that $\Delta PST \,|||\, \Delta PQR$

(b) complete the ratios $\frac{PS}{PQ} = \frac{?}{QR} = \frac{?}{PR}$.

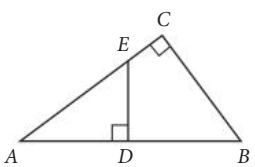
(c) If $QR = 5$ cm, $ST = 2$ cm, $SP = 1$ cm, find the length of PQ.



9 If ΔABC and ΔADE are right-angled:

(a) prove that $\Delta ABC \,|||\, \Delta AED$

(b) complete the ratios $\frac{AB}{?} = \frac{AC}{?} = \frac{BC}{?}$.



10 A line drawn parallel to the side BC of a triangle ABC cuts AB at X and AC at Y. Name the two similar triangles.

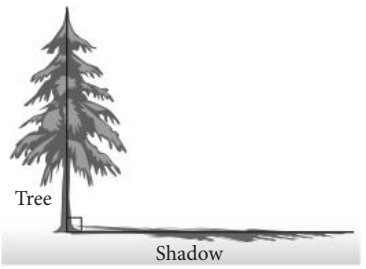
(a) If $AB = 10$ cm, $BC = 15$ cm, $AX = 6$ cm, find XY.

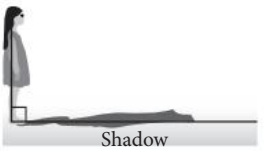
(b) If $AY = 5$ cm, $YC = 2.5$ cm, $XY = 8$ cm, find BC.

(c) If $AC = 8$ cm, $YC = 3$ cm, and BC is 5 cm longer than XY, find BC.

11 Shivani is asked to measure the height of a tree. To do this she measures the length of the tree's shadow and also the length of her own shadow.

Shivani's height is 1.6 m and the length of her shadow is 2.4 m. If the tree's shadow is 36 m long, what is the height of the tree?

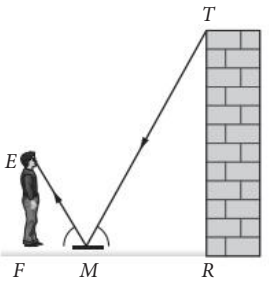




12 On an overcast day, Chris was asked to measure the height of a wall. Since there was no shadow, he used a mirror M placed on the ground.

The light from the top of the wall travels along TM and ME to his eye. Angles $\angle TMR$ and $\angle EMF$ are equal, so ΔTMR and ΔEMF are similar triangles.

If $EF = 1.6$ m, $FM = 0.8$ m and $MR = 10$ m, what is the height of the wall?



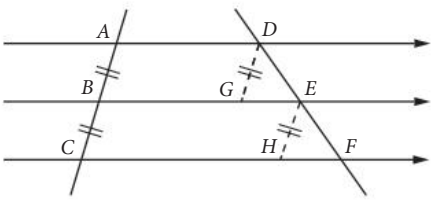
13 A stick 1 m long is placed on the ground pointing straight up and casts a shadow 60 cm long. At the same time, a building casts a shadow 3 m long. What is the height of the building?

14 Trinh is 1.2 m tall and casts a shadow 3.6 m long. Her friend Paul, who is standing next to her, is 1.6 m tall. What is the length of his shadow?

4.7 INTERCEPT PROPERTIES OF PARALLEL LINES

Theorem

If three (or more) parallel lines cut off equal intercepts on a transversal, then the same parallel lines must cut off equal intercepts on any other transversal.



Given: ABC and DEF are two transversals cutting the three parallel lines AD, BE and CF. $AB = BC$.

Aim: To prove that $DE = EF$.

Construction: Draw DG and EH parallel to AC.

Proof:

$ADGB$ is a parallelogram	(both pairs of opposite sides are parallel)
$\therefore DG = AB$	(opposite sides of a parallelogram)
$BEHC$ is a parallelogram	(both pairs of opposite sides are parallel)
$\therefore EH = BC$	(opposite sides of a parallelogram)
But $AB = BC$	(given)
$\therefore DG = EH$	

Because $AB \parallel DG$, $EH \parallel BC$ and ABC is a straight line, then $DG \parallel EH$.

In ΔDGE and ΔEHF:

$DG = EH$	(proved above)
$\angle DEG = \angle EFH$	(corresponding angles, $BE \parallel CF$)
$\angle GDE = \angle HEF$	(corresponding angles, $GD \parallel HE$)
$\therefore \Delta DGE \equiv \Delta EHF$	(AAS)
$\therefore DE = EF$	

Useful result*

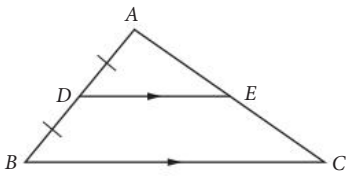
A straight line drawn through the midpoint of one side of a triangle, parallel to a second side, bisects the third side.

***Note**: this result is not an essential part of the syllabus, but it may be derived and quoted in proofs.

In ΔABC, D is the midpoint of AB. $DE \parallel BC$, where E lies on AC.

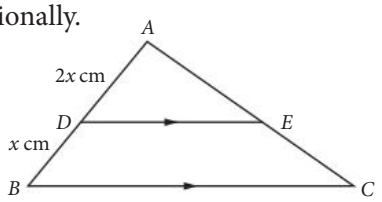
By drawing a line through A, parallel to BC, we can use the previous theorem to prove that $AE = EC$.

This also gives the result that the length of AE is half the length of AC.



Theorem*

A line through a triangle, parallel to one side, divides the other two sides proportionally.



Given: In ΔABC, $DE \parallel BC$. $AD = 2x$ cm, $DB = x$ cm.

Aim: To prove that $\frac{AD}{DB} = \frac{AE}{EC}$.

Proof: In ΔADE and ΔABC:

$\angle DAE = \angle BAC$ (common angle)

$\angle ADE = \angle ABC$ (corresponding angles, $DE \parallel BC$)

$\therefore \Delta ADE \,|||\, \Delta ABC$ (AAA)

$\therefore \frac{AD}{AB} = \frac{AE}{AC}$ (corresponding sides of similar triangles)

Now $\frac{AD}{DB} = \frac{2x}{x} = \frac{2}{1}$ (given)

And $\frac{AD}{AB} = \frac{2x}{2x + x} = \frac{2}{3}$

$\therefore \frac{AE}{AC} = \frac{2}{3}$

Hence $\frac{AE}{AC} = \frac{AE}{AE + EC} = \frac{2y}{2y + y}$

$\therefore \frac{AE}{EC} = \frac{2y}{y} = \frac{2}{1}$

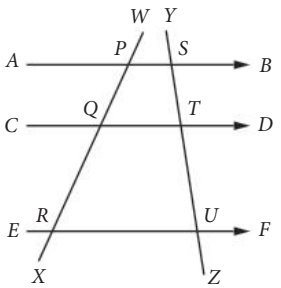
$\therefore \frac{AD}{DB} = \frac{AE}{EC}$

***Note**: this theorem is not an essential part of the syllabus, but it may be derived and quoted in proofs.

Theorem

When three (or more) parallel lines are cut by two transversals, the intercepts cut off on one transversal are proportional to the intercepts cut off on the other transversal.

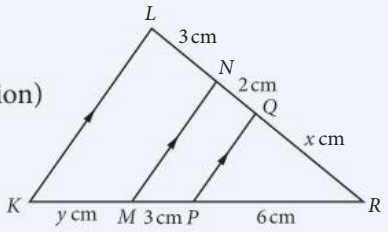
$AB \parallel CD \parallel EF$, hence $\frac{PQ}{QR} = \frac{ST}{TU}$.



This theorem may be proved as an application of the previous theorem on triangles.

Example 14

$KL \parallel MN \parallel PQ$. Find the values of x and y.



Solution

$\frac{x}{2} = \frac{6}{3}$ (line parallel to side of triangle divides other sides in proportion)

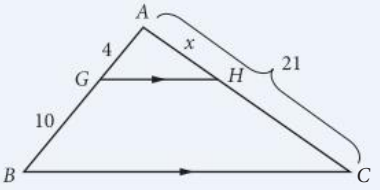
$x = 4$

$\frac{y}{3} = \frac{3}{2}$ (intercepts on parallel lines are proportional)

$y = 4.5$

Example 15

In ΔABC, $GH \parallel BC$, $AG = 4$ cm, $GB = 10$ cm, $AC = 21$ cm, $AH = x$ cm.
Find the length of AH.



Solution

$\frac{AG}{GH} = \frac{AH}{HC}$ (line parallel to side of triangle divides other sides in proportion)

$\therefore \frac{4}{10} = \frac{x}{21 - x}$

$2(21 - x) = 5x$

$42 - 2x = 5x$

$42 = 7x$

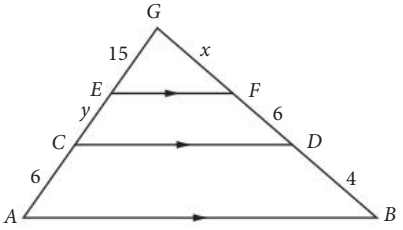
$x = 6$

$\therefore AH = 6$ cm

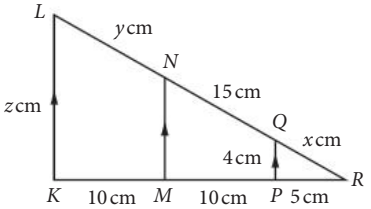
EXERCISE 4.7 INTERCEPT PROPERTIES OF PARALLEL LINES

1 Find the value of each pronumeral. Give reasons for your answers.

(a)

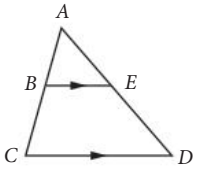


(b)

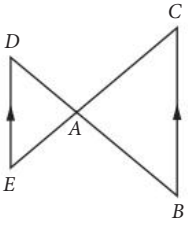


2 In which diagram is the statement $\frac{AB}{BC} = \frac{AE}{ED}$ false?

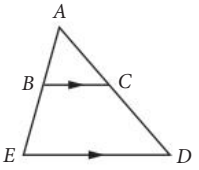
A



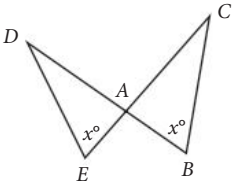
B



C

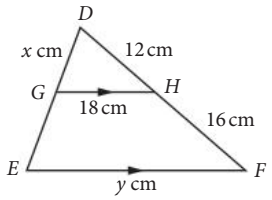


D



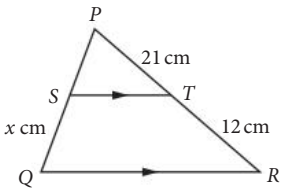
3 Find the value of each pronumeral. Give reasons for your answers.

(a)



$DE = 35$ cm

(b)

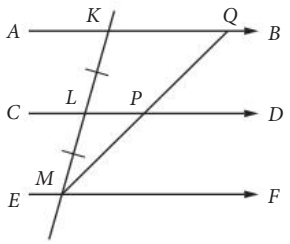


$PQ = 22$ cm

4 $AB \parallel CD \parallel EF$. $KL = LM$.

(a) Explain why $QP = PM$.

(b) Find the value of $\frac{LP}{KQ}$. Give reasons for your answer.

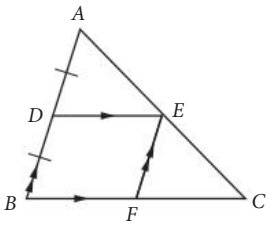


5 D is the midpoint of AB. $DE \parallel BC$ and $AB \parallel EF$.

(a) Using ΔADE and ΔABC, explain why E is the midpoint of AC.

(b) What is the ratio $BF : FC$? Give reasons.

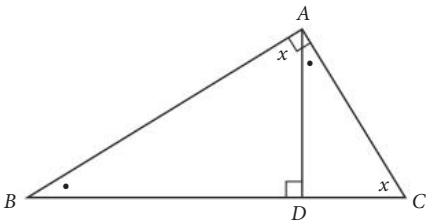
(c) Is DF parallel to AC? Why?



4.8 PYTHAGORAS' THEOREM

Theorem

The square of the hypotenuse of a right-angled triangle is equal to the sum of the squares of the other two sides.



Given: ΔABC with $\angle BAC = 90°$

Aim: To prove that $BC^2 = AB^2 + AC^2$.

Construction: Draw AD perpendicular to BC.

Proof: In ΔABD and ΔABC:

$\angle BDA = \angle BAC$ (right angles)

$\angle DBA = \angle ABC$ (common angle)

$\therefore \Delta ABD \,|||\, \Delta ABC$ (AAA)

$\therefore \frac{AB}{BC} = \frac{BD}{AB}$ (corresponding sides of similar triangles)

$\therefore AB^2 = BD \times BC$ (1)

In ΔACD and ΔABC:

$\angle ADC = \angle BAC$ (right angles)

$\angle ACD = \angle ACB$ (common angle)

$\therefore \Delta ACD \,|||\, \Delta ABC$ (AAA)

$\therefore \frac{AC}{BC} = \frac{DC}{AC}$ (corresponding sides of similar triangles)

$\therefore AC^2 = DC \times BC$ (2)

Add (1) + (2): $AB^2 + AC^2 = BD \times BC + DC \times BC$

$= BC(BD + DC)$

$= BC \times BC$

$\therefore AB^2 + AC^2 = BC^2$

Interesting results

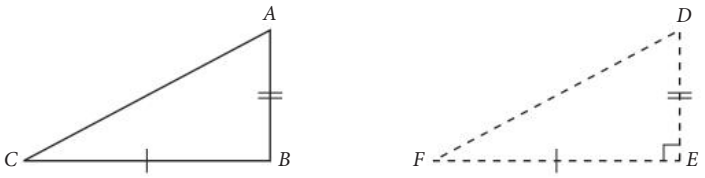
You can show that $\Delta ABD \,|||\, \Delta ADC$ because they are both similar to ΔABC.

Hence, $\frac{DA}{DC} = \frac{DB}{DA}$, so $DA^2 = DB \times DC$.

This means that the square of the perpendicular is equal to the product of the segments of the base.

Converse of Pythagoras' theorem

If the square of a side of a triangle is equal to the sum of the squares of the other two sides, the angle contained by these sides must be a right angle.



Given: ΔABC for which $AC^2 = AB^2 + BC^2$.

Aim: To prove that $\angle ABC = 90°$.

Construction: Draw ΔDEF for which $EF = BC$, $DE = AB$ and $\angle DEF = 90°$.

Proof:

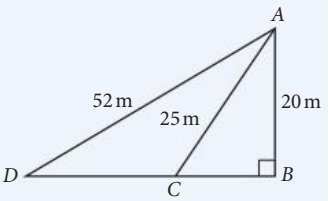
In ΔDEF, $FD^2 = DE^2 + EF^2$	(Pythagoras' theorem)
But $AC^2 = AB^2 + BC^2$	(given)
$\therefore FD^2 = AC^2$	($AB = DE$, $BC = EF$)
$\therefore FD = AC$	

In ΔABC and ΔDEF:

$AB = DE$	(by construction)
$BC = EF$	(by construction)
$AC = FD$	(proved above)
$\therefore \Delta ABC \equiv \Delta DEF$	(SSS)
$\therefore \angle ABC = \angle DEF$	(matching angles in congruent triangles)
$\therefore \angle ABC = 90°$	(because $\angle DEF = 90°$)

Example 16

If $\angle ABD = 90°$, $AB = 20$ m, $AC = 25$ m and $AD = 52$ m, calculate the length of DC.



Solution

In ΔABC:

$AC^2 = AB^2 + BC^2$

$25^2 = 20^2 + BC^2$

$BC^2 = 625 - 400$

$= 225$

$BC = 15$ m

In ΔABD:

$AD^2 = AB^2 + BD^2$

$52^2 = 20^2 + BD^2$

$BD^2 = 2704 - 400$

$= 2304$

$BD = 48$ m

Now $DC = DB - CB$

$= 48 - 15$

$DC = 33$ m

Pythagorean triads

If you recognise that $20 = 5 \times 4$ and $25 = 5 \times 5$, then using the Pythagorean triad (3, 4, 5) you can see in Example **16** above that $BC = 5 \times 3$. Also $52 = 4 \times 13$ and $20 = 4 \times 5$, so using the Pythagorean triad (5, 12, 13) you can see that $DB = 4 \times 12$.

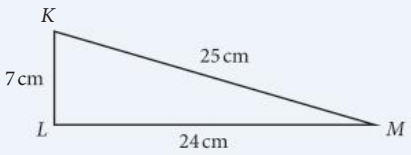
An interesting investigation is to consider the sets of triads formed from $m^2 - n^2$, $2mn$, $m^2 + n^2$, where m and n are positive integers, $m > n$. In triads with no common factors, consider questions such as: Why is only one term even? Will one term always be prime? Will one term always have a factor of 5?

Example 17

In ΔKLM, $KL = 7$ cm, $LM = 24$ cm, $MK = 25$ cm. If the triangle is right-angled, which angle is the right angle?

Solution

Sketch the information given.



KM is the longest side: $KM^2 = 25^2 = 625$

$KL^2 = 7^2 = 49$

$LM^2 = 24^2 = 576$

$KL^2 + LM^2 = 49 + 576 = 625$

$\therefore KM^2 = KL^2 + LM^2$

Hence the triangle is right-angled at L.

EXERCISE 4.8 PYTHAGORAS' THEOREM

1 Calculate the length of the diagonal of a rectangle whose sides are:

(a) 5 cm, 12 cm **(b)** 8 m, 10 m **(c)** 1.5 cm, 3.6 cm

2 Calculate the length of the diagonal of a square whose side is:

(a) 5 m **(b)** 3.6 cm

3 Given these lengths of the three sides of a triangle, which triangle is not right-angled?

A 5 cm, 12 cm, 13 cm **B** 1.3 cm, 8.4 cm, 8.5 cm

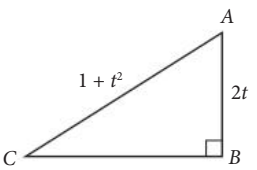
C 8 cm, 14 cm, 17 cm **D** 7 cm, 24 cm, 25 cm

4 Calculate the length of the side of a square whose diagonal is:

(a) 12 cm **(b)** $6\sqrt{2}$ cm

5 Calculate the altitude of an equilateral triangle whose side is 10 cm. (The altitude is the perpendicular distance from a vertex to a side of a triangle.)

6 In the right-angled triangle ABC, $AB = 2t$, $AC = 1 + t^2$, $\angle ABC = 90°$. Calculate the length of CB in terms of t.



7 Calculate the length of the side of a rhombus whose diagonals are:

(a) 12 cm, 16 cm **(b)** 4 cm, 9.6 cm

8 The lengths of the sides and one diagonal of a rhombus are respectively 20 cm and 24 cm. The length of the other diagonal is:

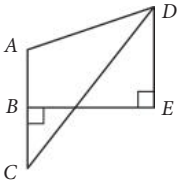
A 12 cm **B** 16 cm **C** 32 cm **D** 256 cm

9 A motorist starts from a point A and travels 32 km north, 24 km east, 25 km south and then back to A. What is the total distance travelled?

10 Two roads intersect at right angles. Two cars start from the intersection at the same time. One car travels at 60 km/h along one road and the other car travels at 80 km/h along the other road. How far apart are the cars after:

(a) 6 min **(b)** 15 min **(c)** 30 min **(d)** 36 min **(e)** 45 min **(f)** 60 min **(g)** t hours

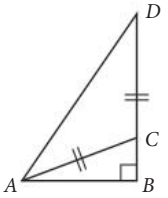
11 In the figure, $AB = 5$ cm, $BC = 6$ cm, $DE = 10$ cm, $AD = 13$ cm. Calculate the length of CD.



12 In the figure, $AB = 24$ cm, $AC = CD = 25$ cm. Calculate the length of:

(a) BC

(b) AD



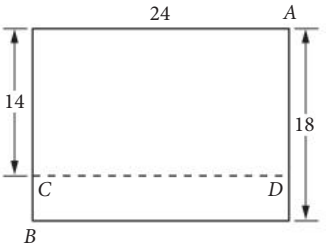
13 The two equal sides of an isosceles triangle are each 13 cm long and the base length is 10 cm. Calculate the height of the triangle.

14 Calculate the length of the altitude of an isosceles triangle whose side lengths are:

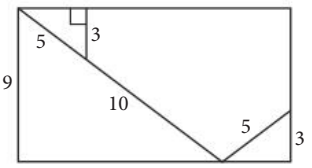
(a) 10 cm, 10 cm, 12 cm **(b)** 25 cm, 25 cm, 14 cm

(c) 5.2 cm, 5.2 cm, 4 cm **(d)** 3 cm, 3 cm, 3.6 cm

15 A rectangular sheet of paper is 24 cm by 18 cm. If it is folded flat along the line CD, then by how many cm is B made closer to A?



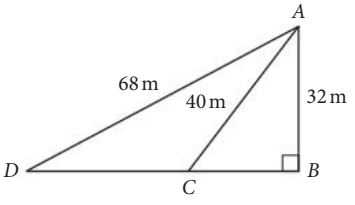
16 When a rectangular sheet of paper 16 cm by 9 cm is cut as shown, the pieces can be rearranged to form a square. What is the perimeter of this square?



17 Two vertical poles standing on horizontal ground at points 9 m apart have lengths of 6 m and 12 m. Find the length of a straight wire joining the tops of the poles.

18 The sides of a rectangular solid are 3 cm, 4 cm and 12 cm. Calculate the length of the diagonal of the rectangular solid.

19 If $\angle ABD = 90°$, $AB = 32$ m, $AC = 40$ m and $AD = 68$ m, calculate the length of DC.

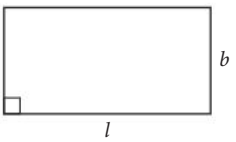


4.9 AREA FORMULAE

Area of a **rectangle** $= lb$

$=$ length $\times$ breadth

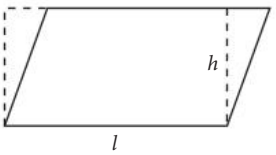
The term 'width' is often used instead of 'breadth'.



Area of **parallelogram** = area of rectangle shown

$= lh$

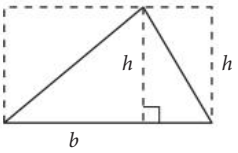
= length of base $\times$ perpendicular height



Area of **triangle** $= \frac{1}{2}$ area of rectangle shown

$= \frac{1}{2}bh$

$= \frac{1}{2}$ base $\times$ perpendicular height



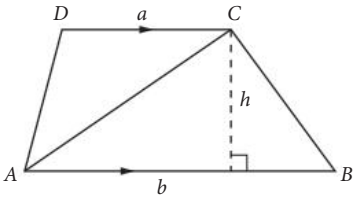
Area of **trapezium** = area of ΔADC + area of ΔABC

$= \frac{1}{2}ah + \frac{1}{2}bh$

$= \frac{1}{2}(a+b)h$

$= \frac{1}{2}$ sum of parallel sides $\times$ perpendicular height

$= \frac{1}{2}$ sum of parallel sides $\times$ distance between them



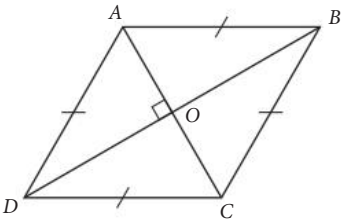
Area of **rhombus** = area of ΔADB + area of ΔBCD

$= \frac{1}{2}DB \times AO + \frac{1}{2}DB \times OC$

$= \frac{1}{2}DB(AO + OC)$

$= \frac{1}{2}DB \times AC$

$= \frac{1}{2}$ the product of the diagonals' lengths



Units for area

- $10\,000\,\text{cm}^2 = 100\,\text{cm} \times 100\,\text{cm} = 1\,\text{m}^2$
- $10\,000\,\text{m}^2 = 100\,\text{m} \times 100\,\text{m} = 1\,\text{ha}$ (hectare)

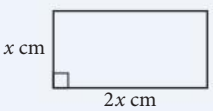
A hectare is an amount of area equivalent to a square of side length 100 m.

Example 18

A rectangle whose length is twice its width has a perimeter of 36 cm. What is its area?

Solution

Let the width be x cm, so the length is $2x$ cm.



$2 \times (x + 2x) = 36$

$3x = 18$

$x = 6$

Area $= 12 \times 6 = 72\,\text{cm}^2$

EXERCISE 4.9 AREA FORMULAE

1 Find the area and perimeter of each rectangle with these dimensions.

(a) length 20 cm, width 8 cm **(b)** length 12.2 m, breadth 150 cm

2 Find the area of each rectangle in hectares.

(a) length 150 m, width 90 m **(b)** length 3 km, width 120 m
(c) length 2 km, width 0.5 km **(d)** length 180 m, width 40 m

3 Find the perimeter of each rectangle.

(a) area 84 cm^2, length 14 cm **(b)** area 14.4 m^2, width 240 cm
(c) area 5 ha, length 1 km **(d)** area 7.2 ha, length 800 m

4 Find the area of each square.

(a) length 2.5 cm **(b)** length 1.6 m
(c) perimeter 24 cm **(d)** perimeter 140 m

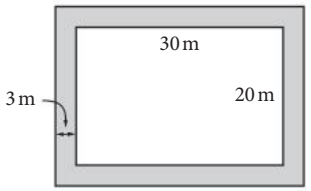
5 Find the perimeter of each square.

(a) length 3.1 cm **(b)** area 49 m^2
(c) area 16 ha **(d)** area 2.25 ha

6 A rectangular carpet is placed in a room 10 m by 8 m, leaving an uncarpeted border 0.5 m wide around it. The area of the carpet is:

A 80 m^2 **B** 71.25 m^2 **C** 63 m^2 **D** 17 m^2

7 A rectangular swimming pool 30 m by 20 m is surrounded by a rectangular concrete path 3 m wide. What is the area of the path?



8 A rectangular field has dimensions 600 m by 250 m. How many tonnes of fertiliser are needed to cover the field at a rate of 70 kg per hectare?

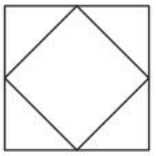
9 How many rectangular tiles 10 cm by 5 cm are needed to cover a rectangular benchtop 3 m by 2 m?

10 A square and a rectangle whose length is twice its width both have perimeters of 24 cm. Show that their areas are in the ratio 9 : 8.

11 A rectangular benchtop is covered with 400 tiles, each 10 cm by 5 cm. If square tiles of length 4 cm were used instead, how many would be needed?

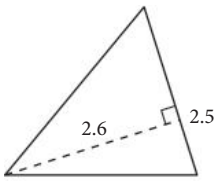
12 The dimensions of a rectangle are increased by 10%. What is the percentage increase in the area?

13 The midpoints of the sides of a 2 m square are joined to form a smaller square. What is the ratio of the areas of the two squares?

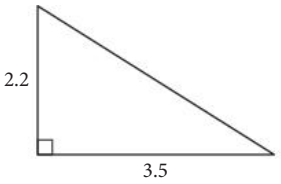


14 Calculate the area of each triangle. All measurements are in cm.

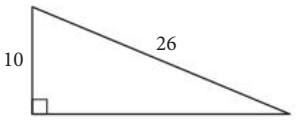
(a)



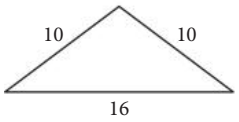
(b)



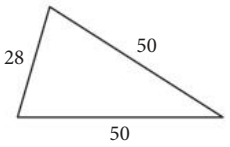
(c)



(d)



(e)



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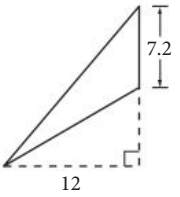
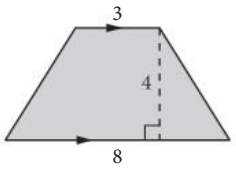
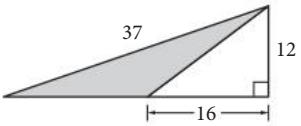


15 **(a)** Find the area of the shaded region in each diagram. Measurements are in cm.

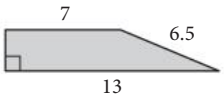
(i)



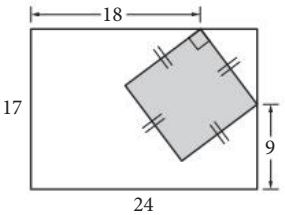
(ii)



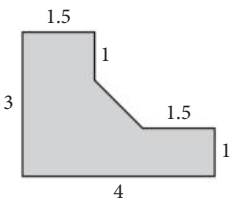
(iii)



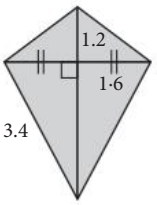
(iv)



(v)



(vi)



(b) Find the perimeter of the shaded regions in diagrams **(ii)** to **(vi)**.

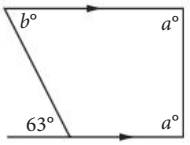
16 A lawnmower mows strips of lawn 40 cm wide. After mowing six times all the way around a rectangular lawn 32 m by 26 m, starting at the outer edge and working inwards each time around, with no gaps or overlaps, what percentage of lawn has been mowed?

17 If the length of each side of a square is increased by 50%, by what percentage is the area increased?

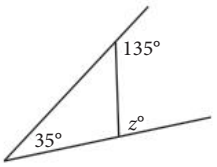
CHAPTER REVIEW 4

1 Find the value of each pronumeral.

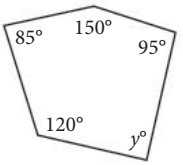
(a)



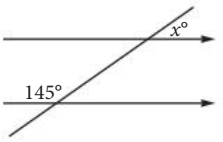
(b)



(c)

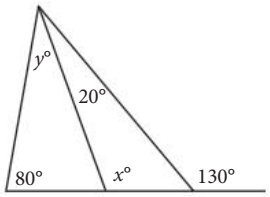


(d)

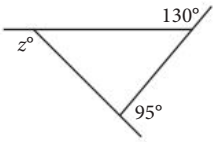


2 Find the value of each pronumeral. Give reasons for your answer.

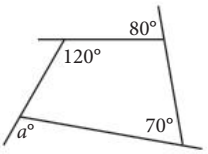
(a)



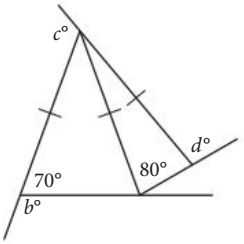
(b)



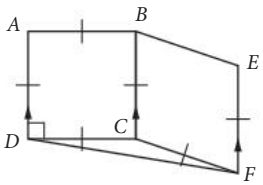
(c)



(d)



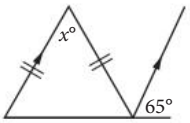
3 For the given diagram, write the best name for each of the following:



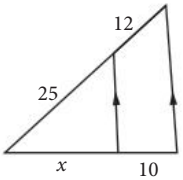
(a) *ABCD*
(b) *BEFC*
(c) *DCF*
(d) *AEFD*
(e) *BEFD*
(f) *ABEFD*
(g) *ABEC*
(h) *BDC*

4 Find the value of x in each diagram.

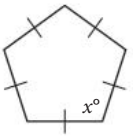
(a)



(b)



(c)

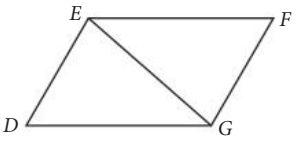


5 Calculate the length of each side of a rhombus whose diagonals are 4.8 cm and 6.4 cm long. Calculate the area of the rhombus.

6 The perimeter of a rectangle is 18 cm and its area is 20 cm^2.

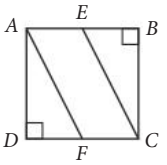
(a) If the length is x cm, express the width in terms of x.
(b) Write the area in terms of x.

7 **(a)** If $DE = GF$, $DG = EF$, prove that $\angle GDE = \angle GFE$.
(b) What sort of quadrilateral is *DEFG*?

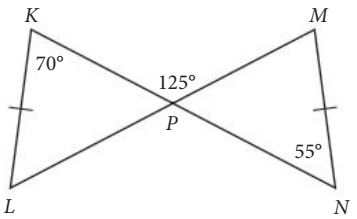


8 *ABCD* is a square. *E* is the midpoint of *AB*, and *F* is the midpoint of *DC*.

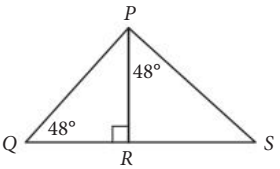
(a) Prove that $\Delta ADF \equiv \Delta CBE$.
(b) What sort of quadrilateral is *AECF*?



9 If $KL = MN$, $\angle LKP = 70°$, $\angle MNP = 55°$, prove that $LP = NP$.



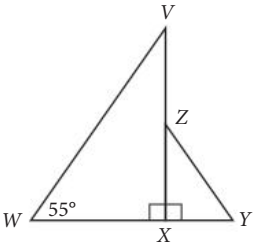
10 Is ΔPQR congruent to ΔPSR? Give reasons for your answer.



11 In the figure $\frac{VX}{ZX} = \frac{WX}{XY}$, $\angle VXW = \angle ZXY = 90°$ and $\angle VWX = 55°$.

(a) Prove that $\Delta VXW \,|||\, \Delta ZXY$.

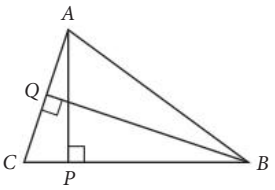
(b) What is the size of $\angle XZY$?



12 $AP \perp CB$ and $BQ \perp AC$.

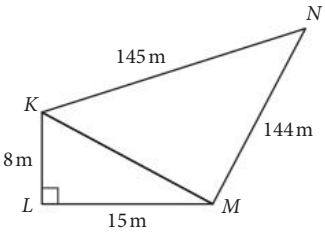
(a) Let $\angle ACP = x°$. What is the size of $\angle QBC$ in terms of x?

(b) Prove that $\frac{AC}{CB} = \frac{AP}{BQ}$.



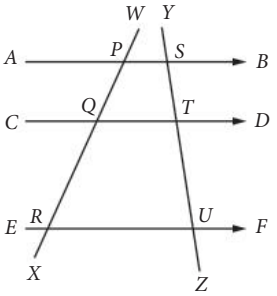
13 $ABCD$ is a square of side length 5 cm. If its diagonals intersect at E, calculate the length of AE.

14 Calculate the size of $\angle KMN$. Justify your answer.



15 **(a)** If $PQ = 10$ cm, $QR = 15$ cm and $ST = 8$ cm, calculate the length of TU.

(b) If R, C and A are collinear and $RC = 9$ cm, calculate the length of RA.



16 A prism has a base in the shape of a parallelogram. The height of the prism is 20 cm. If the distance between two parallel sides of the parallelogram is 9 cm and these sides each have a length of 17 cm, calculate the volume of the prism.

CHAPTER 5
TRIGONOMETRIC RATIOS AND APPLICATIONS

Outcomes covered:

P1 demonstrates confidence in using mathematics to obtain realistic solutions to problems

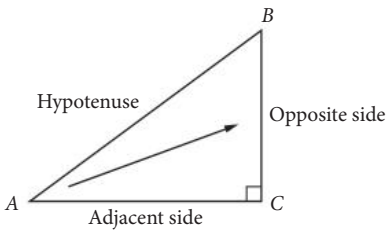
P3 performs routine arithmetic and algebraic manipulation involving surds, simple rational expressions and trigonometric identities

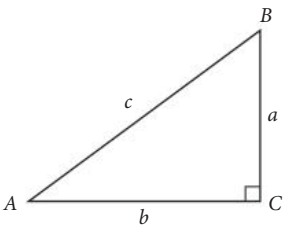
P4 chooses and applies appropriate arithmetic, algebraic, graphical, trigonometric and geometric techniques

5.1 ANGLES OF ANY MAGNITUDE

Trigonometric ratio definitions

From your study of the trigonometry of a right-angled triangle, you should already know the definitions of the trigonometric ratios for sine, cosine and tangent.





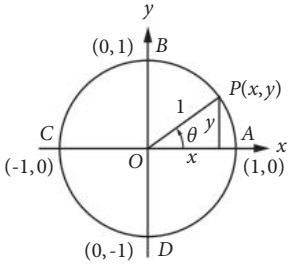
$$\sin A = \frac{\text{Opposite side}}{\text{Hypotenuse}} = \frac{a}{c} \qquad \cos A = \frac{\text{Adjacent side}}{\text{Hypotenuse}} = \frac{b}{c} \qquad \tan A = \frac{\text{Opposite side}}{\text{Adjacent side}} = \frac{a}{b}$$

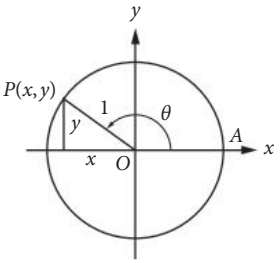
To remember these ratios you can use the mnemonic **SOHCAHTOA**, which stands for **S**in is **O**pposite over **H**ypotenuse, **C**os is **A**djacent over **H**ypotenuse, **T**an is **O**pposite over **A**djacent.

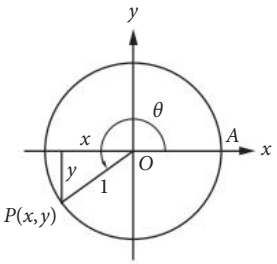
These trigonometric ratios describe the relationship between the angles and sides of a right-angled triangle, while Pythagoras' theorem $c^2 = a^2 + b^2$ describes the relationship between the side lengths without reference to the angles.

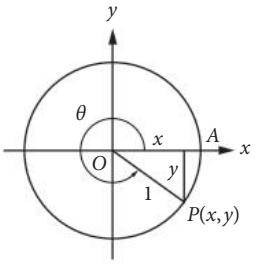
The unit circle

Consider a circle of unit radius whose centre is at the origin. The equation of the circle is $x^2 + y^2 = r^2$.









Take any point P on the circumference of this unit circle whose coordinates are (x, y). Consider the point P as starting at A and rotating in an anticlockwise direction, taking various positions around the circumference as shown in the diagrams above. In each position $\angle AOP = \theta$.

We define **cosine** and **sine** as:

$\cos\theta = x = x$-coordinate of P (the abscissa)

$\sin\theta = y = y$-coordinate of P (the ordinate)

The definitions of sine and cosine apply to angle θ of any magnitude. Also note:

- Because $-1 \leq x \leq 1$, $\cos\theta$ is between -1 and 1.
- Because $-1 \leq y \leq 1$, $\sin\theta$ is between -1 and 1.

If the point P is:

(a) at A, $\theta = 0°$, the coordinates of A are $(1, 0)$ and hence $\cos 0° = 1$, $\sin 0° = 0$

(b) at B, $\theta = 90°$, the coordinates of B are $(0, 1)$ and hence $\cos 90° = 0$, $\sin 90° = 1$

(c) at C, $\theta = 180°$, the coordinates of C are $(-1, 0)$ and hence $\cos 180° = -1$, $\sin 180° = 0$

(d) at D, $\theta = 270°$, the coordinates of D are $(0, -1)$ and hence $\cos 270° = 0$, $\sin 270° = -1$

(e) all the way around to A again, $\theta = 360°$, the coordinates of A are $(1, 0)$ and hence $\cos 360° = 1$, $\sin 360° = 0$.

Note that $\cos 360° = \cos 0° = 1$, and $\sin 360° = \sin 0° = 0$.

There are four other trigonometric ratios: tangent (tan), cotangent (cot), secant (sec) and cosecant (cosec, or sometimes csc). These can all be defined in terms of cos and sin.

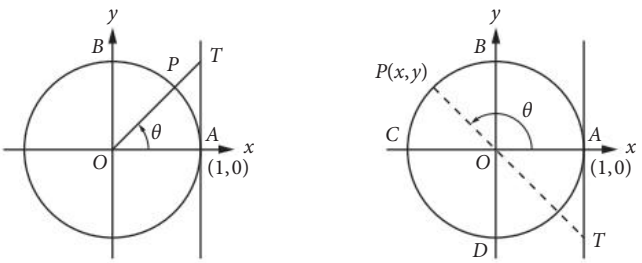
$$\tan\theta = \frac{y}{x} = \frac{\sin\theta}{\cos\theta}, \cos\theta \neq 0 \qquad \cot\theta = \frac{x}{y} = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta}, \sin\theta \neq 0$$

$$\sec\theta = \frac{1}{x} = \frac{1}{\cos\theta}, \cos\theta \neq 0 \qquad \operatorname{cosec}\theta = \frac{1}{y} = \frac{1}{\sin\theta}, \sin\theta \neq 0$$

Because sin and cos appear as denominators, there are restrictions on the values of θ in these functions. The functions tan and sec are undefined when $\cos\theta = 0$, which is when $\theta = 90°, 270°, 450°, \ldots$ (i.e. $90° + n180°$, where n is any integer); cot and cosec are undefined when $\sin\theta = 0$, which is when $\theta = 0°. 180°, 360°, \ldots$ (i.e. $n180°$, where n is any integer). The functions cot θ, sec θ and cosec θ are the reciprocals of tan θ, cos θ and sin θ respectively.

The tangent ratio can also be considered as a ratio without reference to sin or cos.

At the point $A(1, 0)$ where the unit circle cuts the x-axis, draw a tangent line AT.



If $\angle AOP = \theta$, we define tan θ as the y-coordinate of T. (The tan function gets its name from this tangent line.)

If P is at B, $\theta = 90°$, OB is parallel to AT and so tan 90° is undefined.

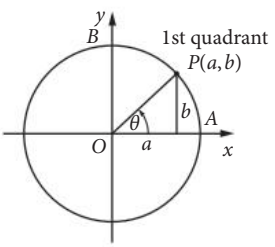
Symmetry properties of trigonometric ratios

The coordinate axes divide the circle into four quarters, called **quadrants**.

First quadrant $0° < \theta < 90°$

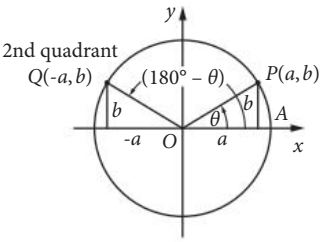
Consider the point $P(a, b)$, where P lies between A and B.

Both the x- and y-coordinates of P are positive numbers, so all the ratios are positive.



Second quadrant $90° < \theta < 180°$

Consider the point $P(a, b)$ in the first quadrant such that $\angle AOP = \theta$, and a point Q in the second quadrant such that $\angle AOQ = 180° - \theta$.



By symmetry, the coordinates of Q are $(-a, b)$. Hence:

$\cos(180° - \theta) = -a = -\cos\theta$

$\sin(180° - \theta) = b = \sin\theta$

$\tan(180° - \theta) = \dfrac{b}{-a} = -\tan\theta$

Because the triangles are congruent, we can see that for every angle in the second quadrant there is a corresponding angle in the first quadrant whose sine, cosine and tangent ratios are numerically the same. You can find this angle by subtracting the second quadrant angle from 180°. (For example, $180° - 40° = 140°$, so the angle in the first quadrant corresponding to 140° is 40°.) Because the x-coordinate is negative in the second quadrant, the values of cos and tan are now negative; sin remains positive. For example:

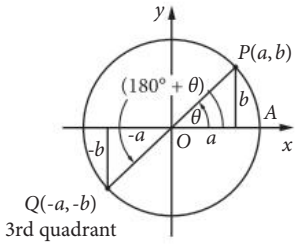
$\cos 140° = \cos(180° - 40°) = -\cos 40° \approx -0.7660$

$\sin 140° = \sin(180° - 40°) = \sin 40° \approx 0.6428$

$\tan 140° = \tan(180° - 40°) = -\tan 40° \approx -0.8391$

Third quadrant $180° < \theta < 270°$

Consider the point $P(a, b)$ in the first quadrant such that $\angle AOP = \theta$, and a point Q in the third quadrant such that $\angle AOQ = 180° + \theta$.



By symmetry, the coordinates of Q are $(-a, -b)$. Hence:

$\cos(180° + \theta) = -a = -\cos\theta$

$\sin(180° + \theta) = -b = -\sin\theta$

$\tan(180° + \theta) = \dfrac{-b}{-a} = \tan\theta$

For every angle in the third quadrant there is a corresponding angle in the first quadrant whose sine, cosine and tangent ratios are numerically the same. You can find this angle by subtracting 180° from the third quadrant angle. (For example, $220° - 180° = 40°$ and $220° = 180° + 40°$, so the angle in the first quadrant corresponding to 220° is 40°.) Because in the third quadrant $x < 0$ and $y < 0$, only tan is positive, while sin and cos are both negative. For example:

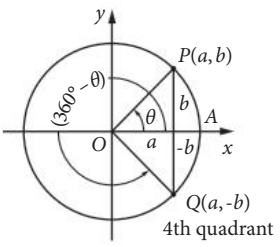
$\cos 220° = \cos(180° + 40°) = -\cos 40° \approx -0.7660$

$\sin 220° = \sin(180° + 40°) = -\sin 40° \approx -0.6428$

$\tan 220° = \tan(180° + 40°) = \tan 40° \approx 0.8391$

Fourth quadrant $270° < \theta < 360°$

Consider the point $P(a, b)$ in the first quadrant such that $\angle AOP = \theta$, and a point Q in the fourth quadrant such that $\angle AOQ = 360° - \theta$.



By symmetry, the coordinates of Q are $(a, -b)$. Hence:

$\cos(360° - \theta) = a = \cos\theta$

$\sin(360° - \theta) = -b = -\sin\theta$

$\tan(360° - \theta) = \dfrac{-b}{a} = -\tan\theta$

For every angle in the fourth quadrant there is a corresponding angle in the first quadrant whose sine, cosine and tangent ratios are numerically the same. You can find this angle by subtracting the fourth quadrant angle from 360°. (For example, $320° = 360° - 40°$, so the angle in the first quadrant corresponding to 320° is 40°.) In the fourth quadrant only cos is positive. For example:

$\cos 320° = \cos(360° - 40°) = \cos 40° \approx 0.7660$

$\sin 320° = \sin(360° - 40°) = -\sin 40° \approx -0.6428$

$\tan 320° = \tan(360° - 40°) = -\tan 40° \approx -0.8391$

Sign of the trigonometric ratios

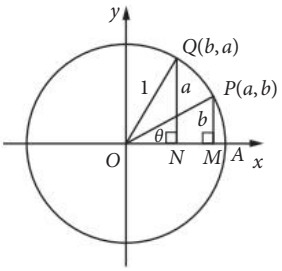
The sign of $\cos\theta$, $\sin\theta$ and $\tan\theta$ for the first four quadrants can be summarised as follows:

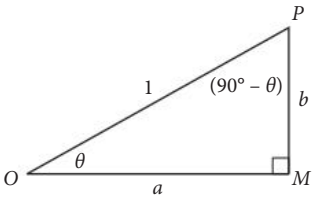
- First quadrant: All are positive (A)
- Second quadrant: $\sin\theta$ only is positive (S)
- Third quadrant: $\tan\theta$ only is positive (T)
- Fourth quadrant: $\cos\theta$ only is positive (C)

The trigonometric ratio positive signs in the different quadrants can be remembered with the mnemonic **ASTC**: **A**ll **S**tations **T**o **C**entral.

Complementary angles: θ and $(90° - \theta)$

Consider the point $P(a, b)$ on the unit circle such that $\angle AOP = \theta$, and a point Q such that $\angle AOQ = (90° - \theta)$.





From congruent triangles, the coordinates of Q are (b, a) because $ON = PM = b$ and $QN = OM = a$. Hence:

$$\sin(90° - \theta) = a = \cos\theta \qquad \cos(90° - \theta) = b = \sin\theta$$

$$\tan(90° - \theta) = \frac{a}{b} = \cot\theta \qquad \cot(90° - \theta) = \frac{b}{a} = \tan\theta$$

$$\sec(90° - \theta) = \frac{1}{b} = \operatorname{cosec}\theta \qquad \operatorname{cosec}(90° - \theta) = \frac{1}{a} = \sec\theta$$

These relationships are said to be **complementary**. This is why the prefix 'co-' is in the words cosine, cosecant and cotangent.

The ratios sine and cosine, tangent and cotangent, secant and cosecant are complementary pairs. For example:

$\sin 50° = \cos(90° - 50°) = \cos 40°$ $\qquad$ $\tan 75° = \cot(90° - 75°) = \cot 15°$

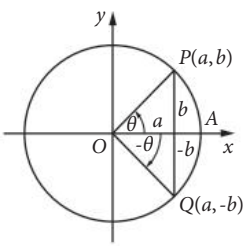
$\sec 80° = \operatorname{cosec}(90° - 10°) = \operatorname{cosec} 10°$ $\qquad$ $\cos 60° = \sin(90° - 60°) = \sin 30°$

Use your calculator to verify these results.

Negative angles: $\theta < 0°$

So far we have only considered $\theta > 0°$. If we start from the point A and rotate anticlockwise to P, then $\theta > 0°$. However, if we rotate clockwise to Q so that $\angle AOQ = \angle AOP$, then $\theta < 0°$.

Hence, by symmetry:



$$\cos(-\theta) = a = \cos\theta \qquad \sin(-\theta) = -b = -\sin\theta \qquad \tan(-\theta) = \frac{-b}{a} = -\tan\theta$$

For example:

$\cos(-40°) = \cos 40° \approx 0.7660$

$\tan(-25°) = -\tan 25° \approx -0.4663$

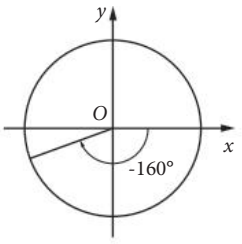
$\sin(-70°) = -\sin 70° \approx -0.9397$

$\cos(-160°) = \cos 160° = -\cos 20° \approx -0.9397$

$\tan(-245°) = -\tan 245° = -\tan 65° \approx -2.1445$

$\sin(-210°) = -\sin 210° = -(-\sin 30°) = \sin 30° = 0.5$

$\cot(-135°) = -\cot 135° = \cot 45° = \tan 45° = \frac{1}{\sqrt{2}} \approx 0.7071$



EXERCISE 5.1 ANGLES OF ANY MAGNITUDE

1 In which of the four quadrants is the following true?

(a) $\sin\theta > 0$ **(b)** $\tan\theta < 0$ **(c)** $\cos\theta < 0$ **(d)** $\sin\theta < 0$ and $\tan\theta < 0$

(e) $\sin\theta > 0$ and $\cos\theta < 0$ **(f)** $\cos\theta < 0$ and $\tan\theta > 0$ **(g)** $\cos\theta > 0$ and $\tan\theta > 0$

2 State the quadrant of each angle.

(a) 72° **(b)** 114° **(c)** 95° **(d)** 200° **(e)** 321°

(f) 183° **(g)** 83° **(h)** 216° **(i)** 300° **(j)** 155°

3 Express each of the following as a trigonometric ratio of angle A.

(a) $\sin(180° - A)$ **(b)** $\cos(90° - A)$ **(c)** $\tan(360° - A)$

(d) $\cos(180° + A)$ **(e)** $\sin(360° - A)$ **(f)** $\cot(90° - A)$

4 Use a calculator to evaluate $\sin\theta$, $\cos\theta$, $\tan\theta$, $\operatorname{cosec}\theta$, $\sec\theta$, and $\cot\theta$ for each of the following values of θ, writing each answer correct to 4 decimal places:

(a) **(i)** 125° **(ii)** 152° **(iii)** 117° **(b)** **(i)** 205° **(ii)** 217° **(iii)** 251°

(c) **(i)** 282° **(ii)** 301° **(iii)** 342° **(d)** **(i)** -25° **(ii)** -122° **(iii)** -215°

5 If $\sin\alpha = 0.2$, write the value of:

(a) $\sin(180° - \alpha)$ **(b)** $\sin(360° - \alpha)$ **(c)** $\sin(-\alpha)$

(d) $\cos(90° - \alpha)$ **(e)** $\sin(180° + \alpha)$ **(f)** $\operatorname{cosec}\alpha$

6 If $\tan\theta = t$, express in terms of t:

(a) $\cot\theta$ **(b)** $\cot(90° - \theta)$ **(c)** $\tan(180° - \theta)$

(d) $\tan(360° - \theta)$ **(e)** $\cot(180° - \theta)$ **(f)** $\tan(180° + \theta)$

7 If $\cos A = c$, express in terms of c:

(a) $\sec A$ **(b)** $\cos(-A)$ **(c)** $\cos(180° - A)$

(d) $\cos(360° - A)$ **(e)** $\sec(-A)$ **(f)** $\cos(180° + A)$

8 Use a calculator to evaluate, correct to 4 decimal places:

(a) $\tan 305°$ **(b)** $\sin 212°$ **(c)** $\cos(-140°)$

(d) $\sin(-160°)$ **(e)** $\cot 42°$ **(f)** $\cos 260°$

9 If θ is an angle in the second quadrant, state whether the following are positive or negative:

(a) $\cos(180° - \theta)$ **(b)** $\tan(180° - \theta)$ **(c)** $\sin(90° - \theta)$

(d) $\sin(360° - \theta)$ **(e)** $\cos(180° + \theta)$ **(f)** $\tan(90° - \theta)$

10 If $90° < \theta < 180°$, use a unit circle diagram to show that:

(a) $\cos(180° + \theta) = -\cos\theta$ **(b)** $\sin(360° - \theta) = -\sin\theta$

11 Which expression is equal to $\sin(360° + \theta)$?

A $\cos\theta$ **B** $-\sin\theta$ **C** $-\cos\theta$ **D** $\sin\theta$

12 Write 'correct' or 'incorrect' for each answer: $\cos\theta = \ldots$

(a) $\cos(360° - \theta)$ **(b)** $\sin(180° - \theta)$ **(c)** $-\cos(180° + \theta)$ **(d)** $\sin(90° - \theta)$

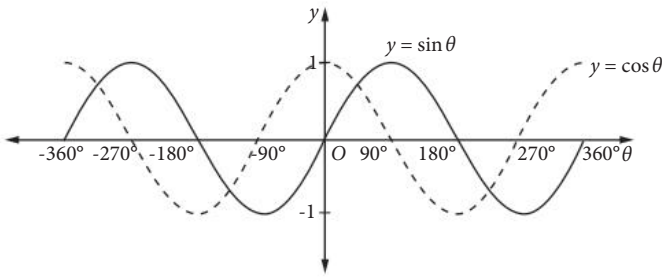
5.2 TRIGONOMETRIC GRAPHS AND EQUATIONS

Graphs of trigonometric functions

Sine (sin) and cosine (cos):

In the figure below, the graph of $y = \sin\theta$ is represented by the continuous curve, and the graph of $y = \cos\theta$ is represented by the broken curve. The graphs are drawn in the domain $-360° \le \theta \le 360°$. For a domain of real numbers, the graphs continue similarly in both directions: for example, the graph for the domain $360° \le \theta \le 720°$ is the same as the graph for the domain $0° \le \theta \le 360°$, as $720° = 360° + 360°$.

As θ increases, the values of $\sin\theta$ and $\cos\theta$ repeat after each interval or **period** of 360°. The functions sine and cosine are therefore called **periodic functions**, with the period being 360°. We saw earlier that the point $P(x, y)$ corresponds to angles θ, $360° + \theta$, $720° + \theta$ and so on, and hence $\sin(360° + \theta) = \sin\theta$ and $\cos(360° + \theta) = \cos\theta$.



The maximum and minimum values of $\sin\theta$ and $\cos\theta$ are 1 and -1 respectively. In other words their **amplitude** is 1.

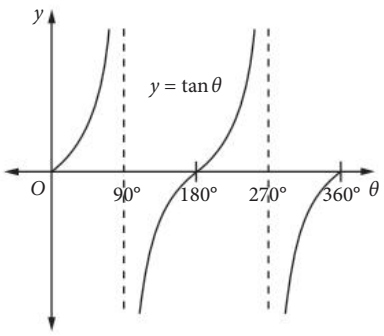
If the graph of $y = \cos\theta$ is translated 90° to the right, parallel with the θ-axis, it coincides with the graph of $y = \sin\theta$. This can be seen to follow from the fact that $\cos\theta = \sin(90° + \theta)$.

Tangent (tan):

The graph of $y = \tan\theta$ can be drawn by plotting points from a table of values or by using graphing software. But we can determine features of the graph simply from the definition of tan:

- Because $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, the graph of $y = \tan\theta$ will not exist for values of θ that make $\cos\theta = 0$. In the domain $0° \le \theta \le 360°$, $\tan\theta$ will be undefined when $\theta = 90°, 270°$.
- If $\sin\theta = 0$ then $\tan\theta = 0$, so in the domain $0° \le \theta \le 360°$ we know that $\tan\theta = 0$ when $\theta = 0°, 180°, 360°$.
- If $\sin\theta = \cos\theta$ then $\tan\theta = 1$. In the domain $0° \le \theta \le 360°$, $\tan\theta = 1$ when $\theta = 45°, 225°$; $\tan\theta = -1$ when $\theta = 135°, 315°$.

For $0° \le \theta \le 360°$, the graph of $y = \tan\theta$ is:



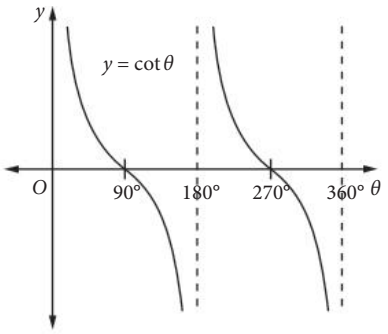
As θ increases, the values of $\tan\theta$ repeat after each interval or period of 180°. The curve has no greatest or least value, so it does not have an amplitude: $y = \tan\theta$ is **discontinuous** at $\theta = 90°, 270°$. The dotted vertical lines drawn at these values are called **asymptotes**.

Cotangent (cot):

The graph of $y = \cot\theta$ can be drawn by plotting points from a table of values or by using graphing software, but we can determine some features from its definition:

- Because $\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{\cos\theta}{\sin\theta}$, the graph of $y = \cot\theta$ will not exist for values of θ that make $\tan\theta = 0$ or $\sin\theta = 0$. In the domain $0° \le \theta \le 360°$, $\cot\theta$ will be undefined when $\theta = 0°, 180°, 360°$.
- If $\cos\theta = 0$ then $\cot\theta = 0$. In the domain $0° \le \theta \le 360°$, $\cot\theta = 0$ when $\theta = 90°, 270°$.
- If $\sin\theta = \cos\theta$ then $\cot\theta = 1$. In the domain $0° \le \theta \le 360°$, $\cot\theta = 1$ when $\theta = 45°, 225°$; $\cot\theta = -1$ when $\theta = 135°, 315°$.

For $0° \leq \theta \leq 360°$, the graph of $y = \cot\theta$ is:



As θ increases, the values of $\cot\theta$ repeat after an interval or period of 180°. The curve has no greatest or least value, so it does not have an amplitude: $y = \cot\theta$ is discontinuous at $\theta = 0°, 180°, 360°$. The dotted vertical lines drawn at these values are asymptotes.

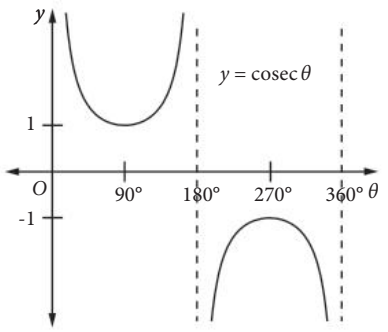
Because $\cot\theta = \tan(90° - \theta)$ and $\cot\theta = \dfrac{1}{\tan\theta}$, these two ratios are both complementary ratios and reciprocal ratios.

Cosecant (cosec):

The graph of $y = \text{cosec}\,\theta$ can be drawn by plotting points from a table of values or by using graphing software, but we can determine some features from its definition:

- Because $\text{cosec}\,\theta = \dfrac{1}{\sin\theta}$, the graph of $y = \text{cosec}\,\theta$ will not exist for values of θ that make $\sin\theta = 0$. In the domain $0° \leq \theta \leq 360°$, $\text{cosec}\,\theta$ will be undefined when $\theta = 0°, 180°, 360°$.
- If $\sin\theta = 1$ then $\text{cosec}\,\theta = 1$. In the domain $0° \leq \theta \leq 360°$, $\text{cosec}\,\theta = 1$ when $\theta = 90°$; $\text{cosec}\,\theta = -1$ when $\theta = 270°$.

For $0° \leq \theta \leq 360°$, the graph of $y = \text{cosec}\,\theta$ is:



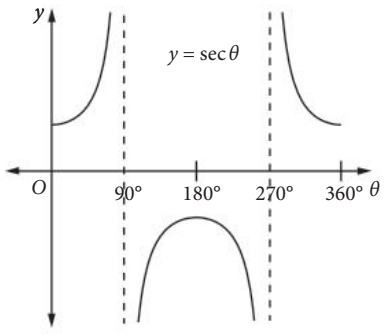
As θ increases, the values of $\text{cosec}\,\theta$ repeat after an interval or period of 360°. The curve has no greatest or least value, so it does not have an amplitude, but the range is $\text{cosec}\,\theta \leq -1$ or $\text{cosec}\,\theta \geq 1$. Also, $y = \text{cosec}\,\theta$ is discontinuous at $\theta = 0°, 180°, 360°$. The dotted vertical lines drawn at these values are asymptotes.

Secant (sec):

The graph of $y = \sec\theta$ can be drawn by plotting points from a table of values or by using graphing software, but we can determine some features from its definition:

- Because $\sec\theta = \dfrac{1}{\cos\theta}$, the graph of $y = \sec\theta$ will not exist for values of θ that make $\cos\theta = 0$. In the domain $0° \leq \theta \leq 360°$, $\sec\theta$ will be undefined when $\theta = 90°, 270°$.
- If $\cos\theta = 1$ then $\sec\theta = 1$. In the domain $0° \leq \theta \leq 360°$, $\sec\theta = 1$ when $\theta = 0°, 360°$; $\sec\theta = -1$ when $\theta = 180°$.

For $0° \leq \theta \leq 360°$, the graph of $y = \sec\theta$ is:



As θ increases, the values of $\sec\theta$ repeat after an interval or period of 360°. The curve has no greatest or least value, so it does not have an amplitude, but the range is $\sec\theta \leq -1$ or $\sec\theta \geq 1$. Also, $y = \sec\theta$ is discontinuous at $\theta = 90°, 270°$. The dotted vertical lines drawn at these values are asymptotes.

GeoGebra is an ideal way to investigate these trigonometric curves.

Trigonometric equations

We will solve some simple trigonometric equations where the values are given in degrees. Later in this course you will learn how to give answers in radians, a dimensionless unit, instead of degrees.

To solve trigonometric equations you can use the same techniques that you have used for algebraic equations.

Example 1

Solve each equation for $0° \le x \le 180°$.

(a) $3 + 2\sin x = 5\sin x$ **(b)** $\sin x = 2\cos x$ **(c)** $6\cos 2x = 3\sin 30°$

Solution

(a) $3 + 2\sin x = 5\sin x$

$3 = 3\sin x$

$\sin x = 1$

$x = 90°$

(b) $\sin x = 2\cos x$

$\frac{\sin x}{\cos x} = 2$

$\tan x = 2$

$x = 63° 26'$

(c) $6\cos 2x = 3\sin 30°$

$\cos 2x = \frac{3\sin 30°}{6}$

$\cos 2x = 0.25$

$2x = 75.52°, 360° - 75.52°$

$2x = 75.52°, 284.48°$

$x = 37.76°, 142.24°$

$x = 37° 46', 142° 14'$

Looking at the solution to **(c)** above, you might ask, 'Why two answers?' Consider that the value of x is given as between 0° and 180°, so the value of $2x$ is between 0° and $2 \times 180° = 360°$. There are thus four quadrants in which the cosine function might have solutions, but we also know that the solutions must be positive. This means that the answers can be in the first and the fourth quadrants.

Example 2

Solve each equation for $0° \le x \le 360°$.

(a) $\sec^2 x = 2$ **(b)** $2\sin x - \tan x = 0$ **(c)** $\sin x - \cos x = 0$

Solution

(a) $\sec^2 x = 2$

$\sec x = \pm\sqrt{2}$

This gives values of x in all 4 quadrants.

In the first quadrant, $\sec x = \sqrt{2}$ gives $x = 45°$.

$\therefore x = 45°, 180° - 45°, 180° + 45°, 360° - 45°$

$x = 45°, 135°, 225°, 315°$

(b) $2\sin x - \tan x = 0$

$2\sin x - \frac{\sin x}{\cos x} = 0$

$\sin x\left(\frac{2\cos x - 1}{\cos x}\right) = 0$

$\sin x(2\cos x - 1) = 0, \cos x \neq 0$

$\sin x = 0, \cos x = 0.5$

$x = 0°, 180°, 360°$ or $x = 60°, 300°$

(c) $\sin x - \cos x = 0$

$\sin x = \cos x$

$\tan x = 1$

$x = 45°, 180° + 45°$

$x = 45°, 225°$

EXERCISE 5.2 TRIGONOMETRIC GRAPHS AND EQUATIONS

If you do not have GeoGebra, you can draw graphs using other software.

1 **(a)** Complete the following table and use it to draw the graph of $y = \sin\theta$ for values of θ from 0° to 360°.

θ	0°	30°	90°	150°	180°	210°	270°	330°	360°
$\sin\theta$									

(b) Draw the graph of $y = \sin\theta$ for values of θ from 0° to 360° using GeoGebra.

2 **(a)** Complete the following table and use it to draw the graph of $y = \cos\theta$ for values of θ from 0° to 360°.

θ	0°	60°	90°	120°	180°	240°	270°	300°	360°
$\cos\theta$									

(b) Draw the graph of $y = \cos\theta$ for values of θ from 0° to 360° using GeoGebra.

3 **(a)** Use your graph in question **1** to draw the graph of $y = \text{cosec}\,\theta$ for $0° \le \theta \le 360°$.
(b) Draw the graph of $y = \text{cosec}\,\theta$ for values of θ from 0° to 360° using GeoGebra.

4 **(a)** Use your graph in question **2** to draw the graph of $y = \sec\theta$ for $0° \le \theta \le 360°$.
(b) Draw the graph of $y = \sec\theta$ for values of θ from 0° to 360° using GeoGebra.

5 **(a)** Complete the following table and use it to draw the graph of $y = \tan\theta$ for values of θ from 0° to 360°.

θ	0°	30°	90°	150°	180°	210°	270°	330°	360°
$\tan\theta$									

(b) Draw the graph of $y = \tan\theta$ for values of θ from 0° to 360° using GeoGebra.

6 **(a)** Use your graph in question **5** to draw the graph of $y = \cot\theta$ for $0° \le \theta \le 360°$.
(b) Draw the graph of $y = \cot\theta$ for values of θ from 0° to 360° using GeoGebra.

7 Repeat questions **1** to **6** using a domain of $-180° \le \theta \le 180°$.

8 Solve each equation for $0° \le x \le 180°$.

(a) $3 + 2\cos x = 5\cos x$ **(b)** $\sin x = 3\cos x$ **(c)** $6\sin 2x = 3\cos 30°$
(d) $4 - 3\tan x = \tan x$ **(e)** $3\sin x = \cos x$ **(f)** $\sin 2x = \sin 30°$

9 Repeat question **8** with $0° \le x \le 360°$.

10 The solution to $\sin x + \cos x = 0$ for $0° \le x \le 180°$ is:

A -45° **B** 45° **C** 90° **D** 135°

11 Solve each equation for $0° \le x \le 360°$.

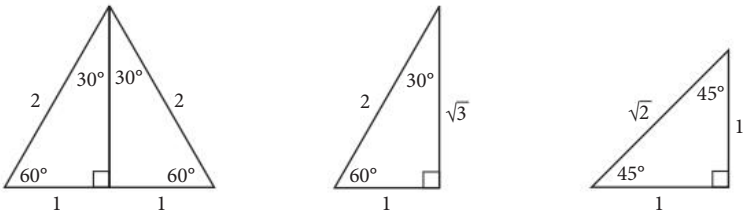
(a) $\text{cosec}^2 x = 2$ **(b)** $\sin^2 x = 1$ **(c)** $\tan^2 x = 3$
(d) $3 + 4\sin x = 9\sin x$ **(e)** $\tan^2 x - \tan x = 0$ **(f)** $\sin 2x = \cos 2x$

12 **(a)** On the same diagram draw $y = \sin x$ and $y = \frac{1}{2}$ for $0° \le x \le 360°$. Use your diagram to solve the equation $\sin x = \frac{1}{2}$ for $0° \le x \le 360°$.

(b) What line would you need to draw to solve the equation $\sin x = -\frac{1}{2}$? What are the solutions to this equation for $0° \le x \le 360°$?

5.3 EXACT VALUES OF TRIGONOMETRIC RATIOS

The exact values of sin, cos and tan for 30°, 60°, and 45° can be found from the following diagrams.



We can use Pythagoras' theorem to calculate the side lengths of the right-angled triangles and relate the lengths to the trigonometric values, as summarised in the following table.

θ	30°	45°	60°
$\sin\theta$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
$\tan\theta$	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$

You can either remember how to calculate these values from the triangles or remember the table of the values. It is also easy to obtain the exact values for the reciprocal functions cosec, sec and cot by taking the reciprocal of each value above. These are given in the following table.

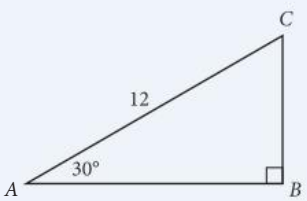
θ	30°	45°	60°
$\operatorname{cosec}\theta$	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$
$\sec\theta$	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2
$\cot\theta$	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$

Example 3

Calculate the exact length of the sides in each triangle.

Solution

(a)



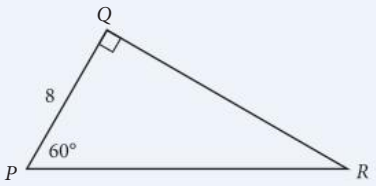
$$\sin 30° = \frac{BC}{12} \qquad \cos 30° = \frac{AB}{12}$$

$$BC = 12\sin 30° \qquad AB = 12\cos 30°$$

$$= 12 \times \frac{1}{2} \qquad = 12 \times \frac{\sqrt{3}}{2}$$

$$= 6 \qquad = 6\sqrt{3}$$

(b)



$$\tan 60° = \frac{QR}{8}$$

$$QR = 8\tan 60°$$

$$= 8\sqrt{3}$$

$$PR^2 = QR^2 + PQ^2$$

$$= \left(8\sqrt{3}\right)^2 + 8^2$$

$$= 192 + 64 = 256$$

$$PR = \sqrt{256} = 16$$

EXERCISE 5.3 EXACT VALUES OF TRIGONOMETRIC RATIOS

Give the exact answer to each of the following, expressing lengths in simplest surd form where necessary. (Do not use a calculator.)

1 In ΔABC, $B = 90°$, $A = 30°$, $AC = 20$ cm. Calculate the lengths of:

(a) BC **(b)** AB

2 In ΔABC, $C = 90°$, $A = 45°$, $BC = 10$ cm. The lengths of AC and AB are:

A $AC = 10\sqrt{2}$ cm, $AB = 10$ cm **B** $AC = 10$ cm, $AB = 10\sqrt{2}$ cm

C $AC = 10$ cm, $AB = 5\sqrt{2}$ cm **D** $AC = 10$ cm, $AB = 10$ cm

3 A vertical pole of height 15 m stands on level ground and a straight wire 30 m long joins the top of the pole to a point on the ground. Find:

(a) the distance of this point from the foot of the pole

(b) the angle the wire makes with the ground.

4 A ladder 10 m long, standing on level ground, leans against a vertical wall and makes an angle of 60° with the ground. Calculate:

(a) how high up the wall the ladder reaches

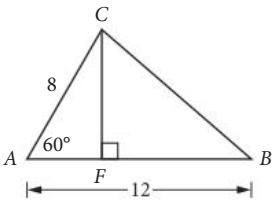
(b) the distance of the foot of the ladder from the wall.

5 In ΔABC, $AB = 12$ cm, $AC = 8$ cm, $A = 60°$. CF is drawn perpendicular to AB to meet AB at F. Calculate the length of:

(a) AF

(b) FC

(c) BC



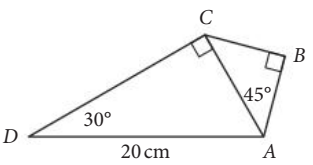
6 For the given diagram, calculate the length of:

(a) AC

(b) DC

(c) AB

(d) BC



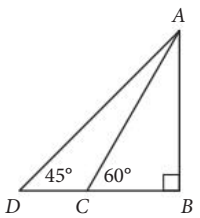
7 In the diagram, $AC = 12$ cm. Calculate the length of:

(a) AB

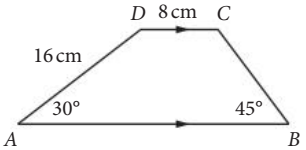
(b) BC

(c) DC

(d) AD



8 Calculate the perimeter of the trapezium $ABCD$ given that $AD = 16$ cm, $DC = 8$ cm, $A = 30°$, $B = 45°$.



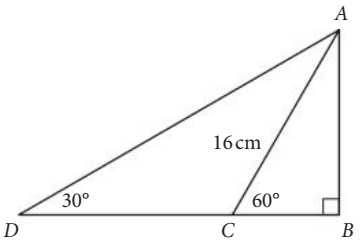
9 A stepladder stands on horizontal ground with its feet 2 m apart. If the angle formed by the legs is 60°, how high above the ground is the top of the ladder?

10 The magnitude of the angle formed by the diagonal of a rectangle and one of its longer sides is 30°. Find the dimensions of the rectangle if the length of the diagonal is 60 cm.

11 In the diagram, $AC = 16$ cm, $\angle ABD = 90°$, $\angle ACB = 60°$ and $\angle ADB = 30°$. Indicate whether each answer is correct or incorrect.

(a) $\angle DAC = 30°$ **(b)** $DC = 16$ cm

(c) $AB = 8$ cm **(d)** $CB = 8$ cm

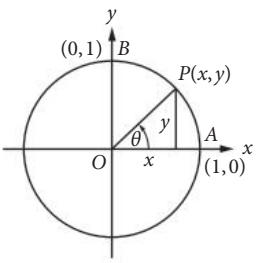


5.4 MORE TRIGONOMETRIC EXACT VALUES

We have already calculated exact values of trigonometric ratios for the first quadrant angles of 30°, 45° and 60°. We can also find exact values for 0° and 90°.

Using the unit circle:

- $\theta = 0°$ when P is at A
- $\theta = 90°$ when P is at B



Our table of exact values now becomes:

θ	0°	30°	45°	60°	90°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined

You should learn these results for sin, cos and tan. Once learnt, they can easily be used to find exact values for the reciprocal functions cosec, sec and cot.

Example 4

Find the exact value of each expression.

(a) $\sin 150°$ **(b)** $\cos 225°$ **(c)** $\tan 240°$ **(d)** $\sin 270°$ **(e)** $\cos(-300°)$ **(f)** $\cos 405°$

Solution

(a) $\sin 150° = \sin(180° - 30°) = \sin 30° = \frac{1}{2}$

(b) $\cos 225° = \cos(180° + 45°) = -\cos 45° = -\frac{1}{\sqrt{2}}$

(c) $\tan 240° = \tan(180° + 60°) = \tan 60° = \sqrt{3}$

(d) $\sin 270° = \sin(180° + 90°) = -\sin 90° = -1$

(e) $\cos(-300°) = \cos 300° = \cos 60° = \frac{1}{2}$

(f) $\cos 405° = \cos(360° + 45°) = \cos 45° = \frac{1}{\sqrt{2}}$

Example 5

Find all values of θ, for $0° \le \theta \le 360°$, for which:

(a) $\cos\theta = \frac{1}{2}$ (b) $\sin\theta = -\frac{1}{\sqrt{2}}$ (c) $\tan\theta = 1$ (d) $\cos 2\theta = -\frac{\sqrt{3}}{2}$ (e) $\sin\theta = -1$

Solution

(a) $\cos\theta > 0$, so θ is in the 1st and 4th quadrants.

$\cos\theta = \frac{1}{2}$: $\theta = 60°, 360° - 60°$

$\theta = 60°, 300°$

(b) $\sin\theta < 0$, so θ is in the 3rd and 4th quadrants.

$\sin\theta = -\frac{1}{\sqrt{2}}$: $\theta = 180° + 45°, 360° - 45°$

$\theta = 225°, 315°$

(c) $\tan\theta > 0$, so θ is in the 1st and 3rd quadrants.

$\tan\theta = 1$: $\theta = 45°, 180° + 45°$

$\theta = 45°, 225°$

(d) $0° \le \theta \le 360°$, so $0° \le 2\theta \le 720°$. Because $\cos 2\theta < 0$, 2θ is in the 2nd and 3rd quadrants.

$\cos 2\theta = -\frac{\sqrt{3}}{2}$: $2\theta = 180° - 30°, 180° + 30°, 540° - 30°, 540° + 30°$

$2\theta = 150°, 210°, 510°, 570°$

$\theta = 75°, 105°, 255°, 285°$

(e) $\sin\theta < 0$, so θ is in the 3rd and 4th quadrants.

$\sin\theta = -1$: $\theta = 180° + 90°, 360° - 90°$

$\theta = 270°$

EXERCISE 5.4 MORE TRIGONOMETRIC EXACT VALUES

For questions **1** to **4**, write the exact value without using a calculator.

1 (a) $\sin 90°$ (b) $\cos 120°$ (c) $\tan 150°$ (d) $\cos 180°$ (e) $\sin 120°$

2 (a) $\sin 180°$ (b) $\cos 210°$ (c) $\tan 225°$ (d) $\cos 240°$ (e) $\tan 180°$

3 (a) $\sin 270°$ (b) $\tan 300°$ (c) $\tan 315°$ (d) $\cos 330°$ (e) $\sin 300°$

4 (a) $\sin 360°$ (b) $\cos 390°$ (c) $\tan 405°$ (d) $\cos 450°$ (e) $\sin 420°$

5 The exact value of $\sin 210°$ is:

A $-\frac{1}{2}$ B $\frac{\sqrt{3}}{2}$ C $\frac{2}{\sqrt{3}}$ D 0

6 Write 'correct' or 'incorrect' for each answer: $-1 = \ldots$

(a) $\cos 180°$ (b) $\sin 45°$ (c) $\sin 270°$ (d) $\tan 495°$

For questions **7** to **18**, find all the correct values of θ for $0° < \theta < 360°$.

7 $\sin\theta = -\frac{\sqrt{3}}{2}$ **8** $\tan\theta = -1$ **9** $\cos\theta = -1$ **10** $\sin\theta = \cos\theta$

11 $\sin\theta = 0$ **12** $2\cos\theta + 1 = 0$ **13** $2\sin\theta = \sqrt{3}$ **14** $\sin\theta + \sqrt{3}\cos\theta = 0$

15 $\sin 2\theta = 0.5$ **16** $\cos\frac{\theta}{2} = \frac{1}{2}$ **17** $\tan 2\theta = 1$ **18** $\sin\frac{\theta}{2} = -\frac{1}{\sqrt{2}}$

5.5 TRIGONOMETRIC IDENTITIES

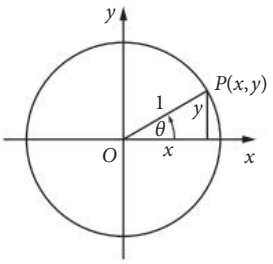
Because the equation of the unit circle is: $x^2 + y^2 = 1$

and, by definition: $x = \cos\theta, y = \sin\theta$

it follows that: $\cos^2\theta + \sin^2\theta = 1$ (1)

$\cos^2\theta = 1 - \sin^2\theta$

$\sin^2\theta = 1 - \cos^2\theta$



Dividing (1) by $\cos^2\theta$: $1 + \tan^2\theta = \sec^2\theta$ (2)

Dividing (1) by $\sin^2\theta$: $\cot^2\theta + 1 = \operatorname{cosec}^2\theta$ (3)

Statements (1), (2) and (3) are called **identities** as they are true for all values of θ.

Example 6

If $\sin\theta = \frac{3}{5}$ and $90° < \theta < 180°$, find the exact values of $\cos\theta$ and $\tan\theta$.

Solution

Using: $\cos^2\theta + \sin^2\theta = 1$

$\sin\theta = \frac{3}{5}$: $\cos^2\theta + \frac{9}{25} = 1$

$\cos^2\theta = \frac{16}{25}$

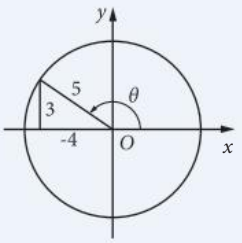
So: $\cos\theta = \pm\frac{4}{5}$; because it is given that $90° < \theta < 180°$, we have $\cos\theta = -\frac{4}{5}$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{3}{5}}{-\frac{4}{5}}$$

$\tan\theta = -\frac{3}{4}$

Alternatively: in a right-angled triangle with a side length of 3 and a hypotenuse length of 5, we can calculate that the third side length is 4. We also know from $90° < \theta < 180°$ that the angle is in the second quadrant, so the x value will be negative.

Hence from $\sin\theta = \frac{3}{5}$ we can obtain $\cos\theta = -\frac{4}{5}$ and $\tan\theta = -\frac{3}{4}$.



Example 7

Simplify:

(a) $\dfrac{1-\cos^2\theta}{1-\sin^2\theta}$ (b) $1 + \tan^2(90° - \theta)$ (c) $\sqrt{a^2 + x^2}$, for $x = a\tan\theta$ and $0° < \theta < 90°$

Solution

(a) $\dfrac{1-\cos^2\theta}{1-\sin^2\theta}$

$= \dfrac{\sin^2\theta}{\cos^2\theta}$

$= \tan^2\theta$

(b) $1 + \tan^2(90° - \theta)$

$= 1 + \cot^2\theta$

$= \operatorname{cosec}^2\theta$

(c) $\sqrt{a^2 + x^2}$

$= \sqrt{a^2 + a^2\tan^2\theta}$

$= \sqrt{a^2(1 + \tan^2\theta)}$

$= \sqrt{a^2\sec^2\theta}$

$= a\sec\theta$

Example 8

Prove the following identities.

(a) $\dfrac{1-\sin^2\theta}{\sin^2\theta+\cos^2\theta}=\cos^2\theta$ (b) $\dfrac{\cos\theta}{1-\sin\theta}-\tan\theta=\sec\theta$ (c) $\tan A\sin A+\cos A=\sec A$

Solution

(a) $\text{LHS}=\dfrac{1-\sin^2\theta}{\sin^2\theta+\cos^2\theta}$ Use $\cos^2\theta=1-\sin^2\theta$ and $\sin^2\theta+\cos^2\theta=1$

$=\dfrac{\cos^2\theta}{1}$

$=\cos^2\theta$

$=\text{RHS}$

(b) $\text{LHS}=\dfrac{\cos\theta}{1-\sin\theta}-\tan\theta$ Replace $\tan\theta$ with $\dfrac{\sin\theta}{\cos\theta}$

$=\dfrac{\cos\theta}{1-\sin\theta}-\dfrac{\sin\theta}{\cos\theta}$ Find common denominator

$=\dfrac{\cos^2\theta-\sin\theta(1-\sin\theta)}{(1-\sin\theta)\cos\theta}$ Expand

$=\dfrac{\cos^2\theta-\sin\theta+\sin^2\theta}{(1-\sin\theta)\cos\theta}$ Use $\sin^2\theta+\cos^2\theta=1$

$=\dfrac{1-\sin\theta}{(1-\sin\theta)\cos\theta}$ Cancel common factor

$=\dfrac{1}{\cos\theta}$ for $\sin\theta\neq 1$

$=\sec\theta$

$=\text{RHS}$

(c) $\text{LHS}=\tan A\sin A+\cos A$

$=\dfrac{\sin A}{\cos A}\times\sin A+\cos A$ Replace $\tan\theta$ with $\dfrac{\sin\theta}{\cos\theta}$

$=\dfrac{\sin^2 A}{\cos A}+\dfrac{\cos A}{1}$ Find common denominator

$=\dfrac{\sin^2 A+\cos^2 A}{\cos A}$

$=\dfrac{1}{\cos A}$ Use $\sin^2\theta+\cos^2\theta=1$

$=\sec A$

$=\text{RHS}$

EXERCISE 5.5 TRIGONOMETRIC IDENTITIES

1 If $\tan\theta=\frac{7}{24}$ and $180°<\theta<270°$, find the exact value of:

(a) $\sin\theta$ (b) $\cos\theta$

2 If $\sin\theta=\frac{5}{13}$ and θ is acute, indicate whether each statement is correct or incorrect.

(a) $\cos\theta=\dfrac{12}{13}$ (b) $\sec\theta=\dfrac{13}{5}$ (c) $\tan\theta=\dfrac{5}{12}$ (d) $\cot\theta=\dfrac{13}{12}$

3 If $\tan\theta = -\frac{4}{3}$ and $90° < \theta < 180°$, find the exact value of:

(a) $\cot\theta$ **(b)** $\sin\theta$ **(c)** $\cos\theta$

4 If $\cos u = \frac{2}{3}$ and u is not in the first quadrant, then $\dfrac{\cos u - 2\cot u}{\tan u - 3\sin u}$ simplifies to:

A $\dfrac{4\left(\sqrt{5}+6\right)}{15}$ **B** $\dfrac{5-2\sqrt{5}}{9}$ **C** $\dfrac{2\left(5-\sqrt{5}\right)}{21}$ **D** $\dfrac{14}{3\left(4\sqrt{5}-5\right)}$

5 Simplify:

(a) $\dfrac{\sin^2\theta + \cos^2\theta}{\tan^2\theta}$ **(b)** $\dfrac{\sin^2\theta}{1-\sin^2\theta}$ **(c)** $\dfrac{2\cot\alpha}{1+\cot^2\alpha}$ **(d)** $(\sec^2\theta - 1)\tan(90° - \theta)$

(e) $\dfrac{\sin A}{\cos A} + \dfrac{\cos A}{\sin A}$ **(f)** $\sin^3\theta + \sin\theta\cos^2\theta$ **(g)** $\operatorname{cosec}^2\theta\sin^2\theta$ **(h)** $1 - \sin^2(180° + \theta)$

6 Simplify:

(a) $\dfrac{x^2}{\sqrt{x^2 - a^2}}$ for $x = a\sec\theta$ **(b)** $\sqrt{a^2 - x^2}$ for $x = a\cos\theta$

7 If $\sec\alpha = -\frac{17}{8}$ and $90° < \alpha < 180°$, find the exact value of:

(a) $\cos\alpha$ **(b)** $\sin\alpha$ **(c)** $\tan\alpha$

8 If $180° < \theta < 270°$ and $\sin\theta = p$, express $\cos\theta$ in terms of p.

9 Find the exact value of $\sec\theta$ if $\tan\theta = 0.6$ and θ is not in the first quadrant.

10 If $\sin\theta = x$, express $\dfrac{1-\cos^2\theta}{\sec^2\theta}$ in terms of x.

11 If $a\sin^2\theta + b\cos^2\theta = c$, express $\sin\theta$ and $\cos\theta$ in terms of a, b and c.

12 If $x = a\sin\theta$ and $y = b\cos\theta$, find an expression from the two equations that does not involve θ.

13 If $\tan^2\theta + 2\sec^2\theta = 5$, find the value of $\sin^2\theta$.

14 Simplify:

(a) $(1 + \tan^2 u)(1 - \sin^2 u)$ **(b)** $\dfrac{1}{1-\sin V} + \dfrac{1}{1+\sin V}$ **(c)** $\dfrac{\sin\theta}{1+\cos\theta} + \dfrac{1+\cos\theta}{\sin\theta}$

(d) $\dfrac{1-\sin\theta}{1+\cos\theta} \times \dfrac{1+\sin\theta}{1-\cos\theta}$ **(e)** $\sin\theta\cos(90° - \theta) + \cos\theta\sin(90° - \theta)$ **(f)** $1 - \dfrac{\sin A\cos A}{\tan A}$

(g) $2\cos^2 30° - 1$ **(h)** $1 - \sin\theta\cos(90° - \theta)$

Prove the following identities:

15 $(1 - \tan x)^2 + (1 + \tan x)^2 = 2\sec^2 x$

16 $(\cot t + \operatorname{cosec} t)^2 = \dfrac{1+\cos t}{1-\cos t}$

17 $\sin^2\alpha\cos^2\beta - \cos^2\alpha\sin^2\beta = \sin^2\alpha - \sin^2\beta$

18 $\sec\theta + \tan\theta = \dfrac{1+\sin\theta}{\cos\theta}$

19 $\sin^2\theta\tan\theta + \cos^2\theta\cot\theta + 2\sin\theta\cos\theta = \tan\theta + \cot\theta$

20 $\sin\theta(1 + \tan\theta) + \cos\theta(1 + \cot\theta) = \dfrac{\sin\theta + \cos\theta}{\sin\theta\cos\theta}$

21 $\tan\theta(1 - \cot^2\theta) + \cot\theta(1 - \tan^2\theta) = 0$

22 $\dfrac{\cot\theta\cos\theta}{\cot\theta + \cos\theta} = \dfrac{\cos\theta}{1+\sin\theta}$

23 $\dfrac{1+\cot\theta}{\operatorname{cosec}\theta} - \dfrac{\sec\theta}{\tan\theta + \cot\theta} = \cos\theta$

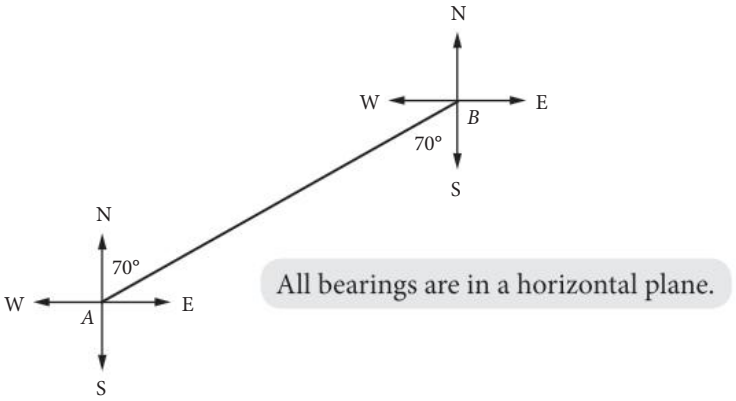
24 $\dfrac{(\cos t\cot t - \sin t\tan t)\sin t\cos t}{\cos t - \sin t} = 1 + \sin t\cos t$

5.6 DIRECTION AND BEARING

Navigators and surveyors measure direction by reference to the points of the compass: north, south, east and west. Directions are indicated in terms of the number of degrees east or west of north or south, or are measured **clockwise from north** and written in standard three-figure notation:

N 30° E or 030° means the direction is 30° east of north.

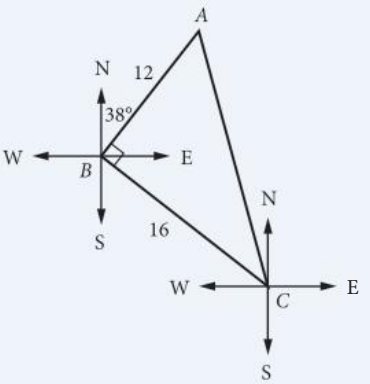
A **bearing** is a direction angle that indicates the direction of one point relative to another point. In this diagram, the bearing of B from A is N 70° E or 070°. The bearing of A from B is S 70° W or 250°.



All bearings are in a horizontal plane.

Example 9

Two yachts sail in a straight line away from a buoy B. One sails 12 km in the direction 038° and the other sails 16 km in the direction 128°.



(a) How far apart are the yachts now?

(b) What is the bearing of the first yacht, as seen from the second yacht?

Solution

(a) $\angle ABC = 128° - 38° = 90°$

Use Pythagoras' theorem for ΔABC: $AC^2 = 12^2 + 16^2 = 400$

$AC = \sqrt{400} = 20$

The two yachts are 20 km apart.

(b) To find the bearing of A from C you first must calculate the size of $\angle ACN$.

In ΔABC: $\tan(\angle ACB) = \frac{12}{16} = 0.75$

$\angle ACB = 36° 52'$

From angle sum at B: $\angle EBC = 38°$

$BE \parallel CE$, alternate angles give: $\angle BCW = 38°$

$\angle ACN = 90° - (\angle ACB + \angle BCW)$

$= 90° - (36° 52' + 38°) = 15° 8'$

The bearing of A from C is N 15° 8′ W or 344° 52′.

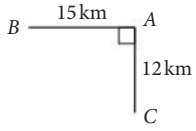
The bearing of C from A would be S 15° 8′ E or 164° 52′.

When working with bearings, draw a diagram to show all the given information. Use the diagram to calculate missing angles.

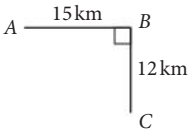
EXERCISE 5.6 DIRECTION AND BEARING

1 Two towns A and B are 15 km apart, with B due west of A. (For compass directions, 'due' means 'exactly'.) Town C is due south of B and 12 km away. Which diagram represents this information correctly?

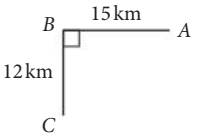
A



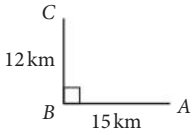
B



C

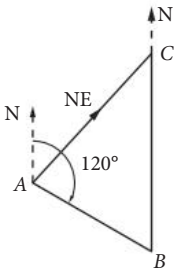


D



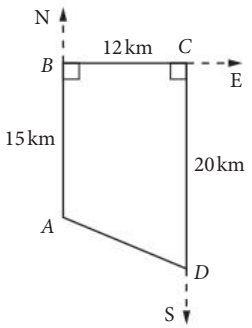
2 The bearing of B from A is 120°, the bearing of C from A is NE and the bearing of C from B is N. Find the bearing of:

(a) A from C
(b) A from B
(c) B from C

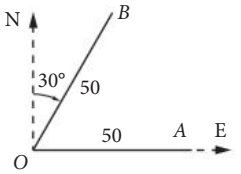


3 Two towns A and B are 15 km apart, with B due west of A. Town C is due south of B and 12 km away. Calculate the distance and bearing of A from C.

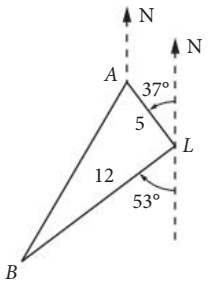
4 Ahmed cycles 15 km due north, then 12 km due east and finally 20 km due south. What are his distance and bearing from his original position?



5 On level ground, A is 50 m due east of O. The bearing of B from O is 030° and the distance of B from O is also 50 m. Find the distance and bearing of B from A.



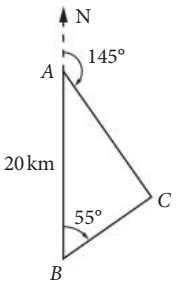
6 A is 5 km from a lighthouse on a bearing N 37° W. B is 12 km from the same lighthouse on a bearing of S 53° W. Find the distance and bearing of B from A.



7 Karen and David set out from home at the same time. Karen cycles due north at 15 km/h and David cycles due east at 20 km/h. Find:

(a) how far apart they are after 1 hour
(b) after how many minutes they are 10 km apart
(c) the bearing of Karen from David at any time.

8 A and B are two lighthouses, A being 20 km due north of B. The bearing of a ship is 145° from A and 055° from B. Calculate the distance of each lighthouse from the ship.

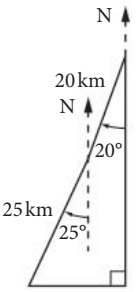


9 Two ports A and B are such that B is due west of A. A is due north of a ship C. The ship is on a course N 32° W and reaches B after travelling for 3 hours at 25 km/h. Calculate the distance between the two ports and the time it would have taken the ship to reach A from C.

10 A hiker walks 15 km from camp in the direction S 36° 52′ W and then walks 7 km due west. What are the distance and bearing of his new position from the camp?

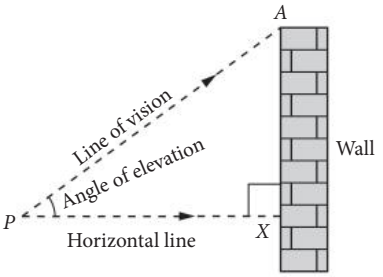
11 A ship sails for 20 km on a course S 20° W and then 25 km on a course S 25° W. Calculate:

(a) how far south the ship now is from its original position
(b) how far west the ship now is from its original position
(c) the bearing of the ship now from its original position.

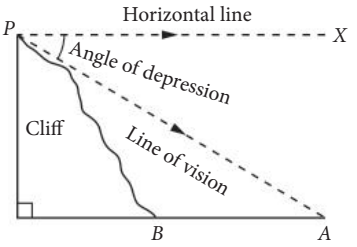


5.7 ANGLES OF ELEVATION AND DEPRESSION

(a)



(b)



In **(a)** above, if we look up at A from the point P then the **angle of elevation** of A from P is the angle between the horizontal line PX and the line of sight PA. $\angle APX$ is an angle of elevation.

The point P is the eye of the observer. A could be, for example, a point on top of a wall.

In **(b)** above, if we look down at A from the point P then the **angle of depression** of A from P is the angle between the horizontal line PX and the line of sight PA. $\angle APX$ is an angle of depression.

The point P is the eye of the observer. The observer could be, for example, at the top of a cliff looking down on a boat at A in the water below.

If we look up from A to P, then $\angle PAB$ is an angle of elevation. $\angle APX = \angle BAP$ because alternate angles between parallel lines are equal, so the angle of depression from P is the same as the angle of elevation from A.

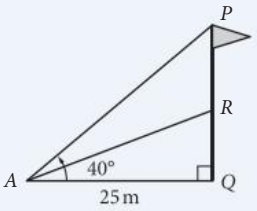
Angles of elevation and depression are in the same vertical plane.

Example 10

A point A is level with the foot of a vertical pole and 25 m away from it. The angle of elevation from point A to the top of the pole P is 40°. Calculate:

(a) the height of the pole, to the nearest centimetre

(b) the angle of elevation from A of a point R, half-way up the pole, to the nearest minute.



Solution

In the diagram, PQ is the vertical pole and $\angle PAQ$ is the angle of elevation of P from A.

(a) In ΔPAQ: $\tan 40° = \frac{PQ}{25}$

$PQ = 25 \times \tan 40° = 20.98$

The height of the pole is 20.98 m.

(b) $PQ = 2RQ$: $RQ = 10.49$

In ΔRAQ: $\tan(\angle RAQ) = \frac{10.49}{25} = 0.4196$

$\angle RAQ = 22° 46'$

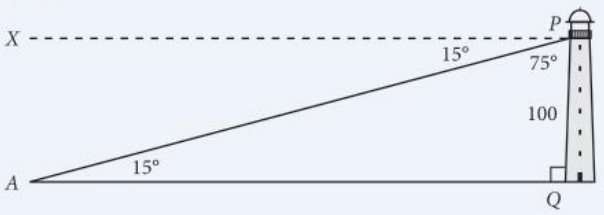
The angle of elevation of R from A is $22° 46'$.

Example 11

An observer in a lighthouse, 100 m above sea level, is watching a ship sailing towards the lighthouse. The angle of depression of the ship from the observer is 15°.

(a) How far is the ship from the lighthouse?

(b) Sometime later, the angle of depression is measured to be 25°. How far has the ship travelled in this time?



Solution

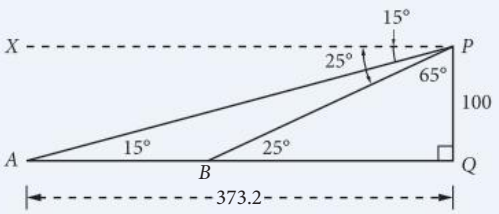
(a) In ΔPAQ: $\tan 15° = \frac{100}{AQ}$ **OR** $\tan 75° = \frac{AQ}{100}$

$AQ = \frac{100}{\tan 15°}$ $AQ = 100 \times \tan 75°$

$AQ = 373.2$ $AQ = 373.2$

The ship is 273.2 m from the lighthouse.

(b) In the diagram, the ship has moved from A to B. $\angle BPX$ is the angle of depression when the ship is at B.



In ΔBPQ: $\tan 25° = \frac{100}{BQ}$ **OR** $\tan 65° = \frac{BQ}{100}$

$BQ = \frac{100}{\tan 25°}$ $BQ = 100 \times \tan 65°$

$BQ = 214.5$ $BQ = 214.5$

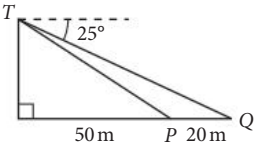
$$AB = AQ - BQ$$
$$= 373.2 - 214.5$$
$$= 158.7$$

The ship has travelled 158.7 m.

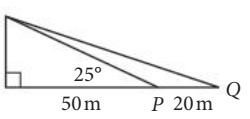
EXERCISE 5.7 ANGLES OF ELEVATION AND DEPRESSION

1 From the top of a cliff, T, a walker sees two ships P and Q at horizontal distances of 50 m and 70 m in a straight line. The angle of depression of P from T is 25°. Indicate whether each of the given diagrams is correct or incorrect.

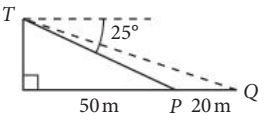
(a)



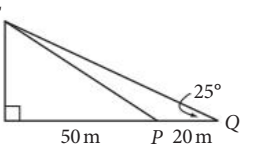
(b)



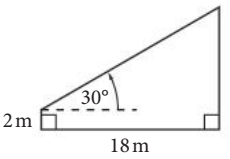
(c)



(d)

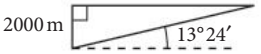


2 A person 2 m tall is standing on the ground and looking up at the top of a building. If the person is 18 m from the building and the angle of elevation of the top of the building is 30°, calculate the height of the building.



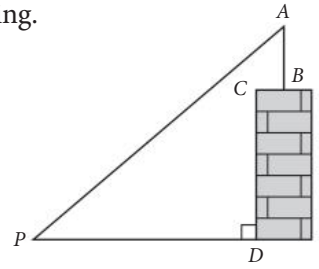
3 An aircraft flying in a horizontal straight line at an altitude of 2000 m passes directly over an observer on the ground. One minute later, the observer finds that the angle of elevation of the plane is 13° 24′. Calculate:

(a) the distance flown by the aircraft in that time
(b) the speed of the aircraft in km/h.



4 The diagram represents a vertical flagpole AB on top of a building. From a point P on the ground, the angle of elevation of A is 36° 52′. $PD = 55$ m, $CB = 5$ m, $AB = 12$ m. Calculate:

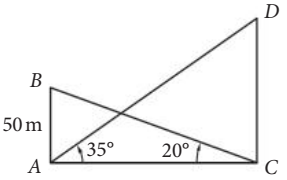
(a) the height of A above the ground
(b) the distance from A to P
(c) the angle of elevation of A from C.



5 From an aircraft 1000 m above the ground, the angles of depression of the tops of two houses (the same height) in line with the aircraft are 40° and 60° respectively. How far apart are the two houses? (Ignore the height of the houses.)

6 AB and CD are two vertical buildings with their bases A and C on level ground. The height of AB is 50 m. The angle of elevation of B as seen from C is 20° and that of D as seen from A is 35°. Calculate:

(a) the horizontal distance between the buildings
(b) the height of CD
(c) the angle of elevation of D as seen from B.



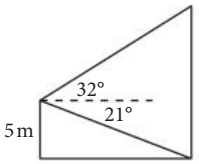
7 From the top of a cliff, T, an observer sees two ships P and Q in line with the observer and at horizontal distances of 50 m and 70 m. The angle of depression of P from T is 25°. Calculate:

(a) the vertical height of the cliff
(b) the angle of elevation of T from Q.

8 From a point 5 m above the ground, the angle of elevation of the top of a wall is 32° and the angle of depression of the bottom of the wall is 21°. Find:

(a) the horizontal distance from the point of observation to the wall

(b) the height of the wall, correct to the nearest metre.

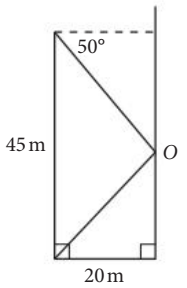


9 From a point A on the ground, the angle of elevation of the top of a tower is 38° and the angle of elevation of the top of a vertical flagpole on top of the tower is 41°. A is 80 m from the foot of the tower. The ground between A and the tower is horizontal. Calculate the length of the flagpole.

10 A city building is 45 m high. From the top of this building, the angle of depression of an object O on the wall of a building opposite is 50°. The width of the street is 20 m. Find:

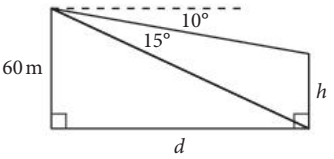
(a) the height of O above street level

(b) the angle of elevation of O from the foot of the first building.



11 Two buildings of unequal height stand at a distance apart on horizontal ground. The taller building is 60 m high and from its top an observer finds that lines of sight to the bottom and the top of the shorter building are at angles of depression of 25° and 10° respectively. Calculate:

(a) the horizontal distance apart of the buildings **(b)** the height of the shorter building.



12 From the top of a lighthouse 75 m above sea level, the angles of depression of two buoys due north of the lighthouse are 60° and 30° respectively. Find, in simplest surd form:

(a) the distance of each buoy from the lighthouse **(b)** the distance between the two buoys.

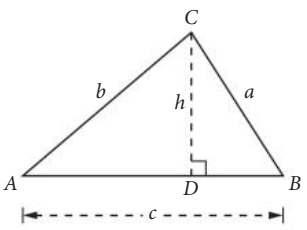
13 From a point P on horizontal ground, the angle of elevation of the top of a building 40 m high is 30°. From a point Q on the same horizontal level as P and in line with the foot of the building, the angle of elevation is 60°. Calculate the distance PQ in simplest surd form.

5.8 THE SINE RULE

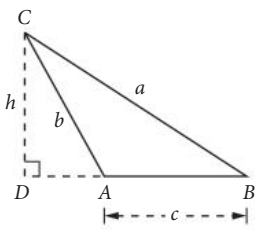
Not all triangles are right-angled. We now consider triangles that have either three acute angles or one obtuse angle and two acute angles. To do this we must establish two rules: the **sine rule** and the **cosine rule**.

The sine rule: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

In triangle ABC, a, b and c are the lengths of the sides opposite the angles of magnitudes A, B and C respectively. A, B and C are also used to label the vertices of the triangle.



Acute-angled triangle



Obtuse-angled triangle

Proof

Let h be the length of the perpendicular from C to AB. The foot of this perpendicular lies on AB if ΔABC is acute-angled (diagram above left), or on BA produced if ΔABC is obtuse-angled (diagram above right).

In the right-angled triangles ΔACD and ΔBCD:

Acute-angled triangle:

$h = b\sin A = a\sin B$

So: $b\sin A = a\sin B$

Obtuse-angled triangle:

$h = b\sin(180° - A) = a\sin B$

But: $\sin(180° - A) = \sin A$

So: $b\sin A = a\sin B$

Hence: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$

Similarly, by drawing a perpendicular from A to BC it can be seen that $\dfrac{b}{\sin B} = \dfrac{c}{\sin C}$.

Hence $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$.

The sine rule may also be written as $\dfrac{\sin A}{\sin B} = \dfrac{a}{b}$: the ratio of the sines = the ratio of the corresponding sides.

The sine rule can be used with a triangle when you are given:

(a) the size of two angles and the length of one side, **OR**

(b) the lengths of two sides and the size of an angle opposite one of these sides.

Important result

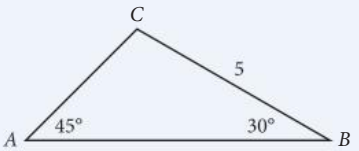
$\sin(180° - \theta) = \sin\theta$

Hence: $\sin 150° = \sin(180° - 30°) = \sin 30° = 0.5$

Solving an equation like $\sin\theta = 0.5$ gives two possible answers: $\theta = 30°$ or $(180 - 30)° = 150°$

Example 12

In ΔABC, given $A = 45°$, $B = 30°$ and $BC = 5$ cm, calculate the size of C, b and c.



Solution

Because $A + B + C = 180°$, it follows that $C = 180° - (45° + 30°) = 105°$.

Using the sine rule:

$$\frac{a}{\sin A} = \frac{b}{\sin B} \qquad \frac{c}{\sin C} = \frac{a}{\sin A}$$

$$\frac{5}{\sin 45°} = \frac{b}{\sin 30°} \qquad \frac{c}{\sin 105°} = \frac{5}{\sin 45°}$$

$$b = \frac{5\sin 30°}{\sin 45°} \qquad c = \frac{5\sin 105°}{\sin 45°}$$

$$b = 5 \times \frac{1}{2} \times \frac{\sqrt{2}}{1} \qquad c = 6.8 \text{ cm}$$

The exact value is $b = \dfrac{5\sqrt{2}}{2}$ cm, which is 3.5 cm correct to one decimal place.

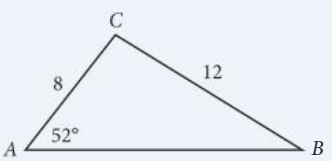
The ambiguous case

From two sides of a triangle and an angle opposite one of these sides, it is sometimes possible to construct two different solutions. This situation is investigated in the examples below.

Example 13

Solve the triangle ABC given $A = 52°$, $BC = 12$ cm and $AC = 8$ cm.

(In this context, to 'solve the triangle' is to find the sizes of all unknown angles and sides.)



Solution

Find B by writing the sine rule in the form with $\sin A$ in the numerator.

Sine rule: $\dfrac{\sin A}{a} = \dfrac{\sin B}{b}$

$$\frac{\sin 52^\circ}{12} = \frac{\sin B}{8}$$

$$\sin B = \frac{8 \sin 52^\circ}{12}$$

If needed, write as a decimal: $\sin B = 0.5253$

Because $\sin B > 0$, B is in the first or second quadrant:

$B = 31^\circ 41'$ or $180^\circ - 31^\circ 41'$

$B = 31^\circ 41'$ or $148^\circ 19'$

Check whether $B = 148^\circ 19'$ is a possible answer. The angle sum of a triangle is 180°:

$A + B = 52^\circ + 148^\circ 19' = 200^\circ 19' > 180^\circ$

Thus the only value of B is $31^\circ 41'$.

Find C using A and B:

$C = 180^\circ - (52^\circ + 31^\circ 41')$

$C = 96^\circ 19'$

Find c using the sine rule: $\dfrac{c}{\sin C} = \dfrac{a}{\sin A}$

$$\frac{c}{\sin 96^\circ 19'} = \frac{12}{\sin 52^\circ}$$

$$c = \frac{12 \sin 96^\circ 19'}{\sin 52^\circ}$$

$c = 15.14$ (correct to 2 decimal places)

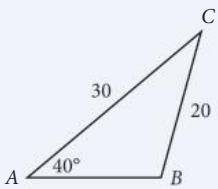
$\therefore B = 31^\circ 41'$, $C = 96^\circ 19'$, $c = 15.14$ cm

Example 14

In ΔABC, given $A = 40^\circ$, $BC = 20$ cm and $AC = 30$ cm, find the sizes of the other angles.

Solution

Draw a diagram:



By the sine rule: $\dfrac{\sin A}{a} = \dfrac{\sin B}{b}$

$$\frac{\sin 40^\circ}{20} = \frac{\sin B}{30}$$

$$\sin B = \frac{30 \sin 40^\circ}{20}$$

$B = 74^\circ 37'$ or $105^\circ 23'$

Test to see whether these values for B give a valid value for C:

When $B = 74^\circ 37'$: $C = 180^\circ - (40^\circ + 74^\circ 37') = 65^\circ 23'$

When $B = 105^\circ 23'$: $C = 180^\circ - (40^\circ + 105^\circ 23') = 34^\circ 37'$

Both these values for C are valid, so there are two possible triangles for the given information. This is an example of the ambiguous case: there is not enough information for you to find a unique solution.

TECHNOLOGY EXPLORATION

Demonstration of the sine rule

1 In GeoGebra, create a triangle using the **Polygon** tool.

2 In **Algebra** view, you should notice that the sides are labelled *a*, *b* and *c*, and the lengths of the sides are listed next to them.

3 Use the **Angle** tool to select angle *A* by selecting sides *b* and *c*. The angle will be labelled α. If the angle is a reflex angle, you can force it to take on the acute angle instead by right-clicking inside the shaded part of the angle, selecting **Object Properties** > Basic and then unselecting 'Allow Reflex Angle'. The angle and its size will also appear in **Algebra** view.

4 Repeat step **3** for angle *B* (which will be labelled β) and for angle *C* (which will be γ).

To check the sine rule for the side lengths:

5 In the input bar, enter *a_1=(b*sin(α)/sin(β))* to calculate the length of side *a* using the sine rule. This will be labelled in the **Algebra** view as a_1. (Note that you can enter Greek letters using the α symbol at the end of the input bar.)

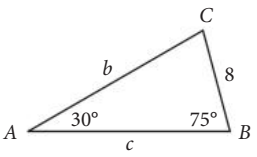
6 Compare the values of *a* and a_1.

You can use a similar method to confirm the sine rule for the other sides.

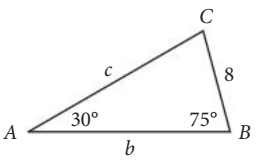
EXERCISE 5.8 THE SINE RULE

1 In the triangle *ABC*, $a = 8$, $A = 30°$ and $B = 75°$. The correct diagram is:

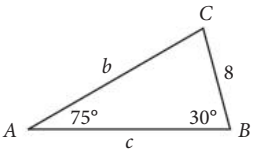
A



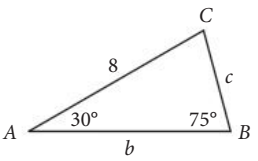
B



C

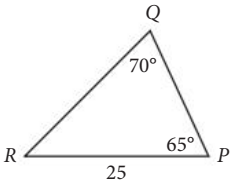


D



2 In ΔABC, $b = 10$, $B = 45°$ and $C = 120°$. Calculate the size of *A*, *a* and *c*.

3 In ΔPQR, $P = 65°$, $Q = 70°$ and $PR = 25$ cm. Calculate the length of *PQ*.

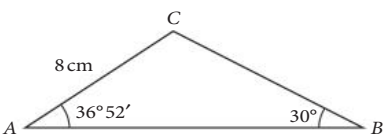


4 In ΔABC, $a = 5$, $A = 60°$ and $B = 45°$. Find the exact value of *b*.

5 In ΔABC, $\sin B = \frac{4}{5}$, $a = 6$ and $b = 9$. Find $\sin A$.

6 In ΔABC, $B = 2A$, $b = 5$ and $a = 3$. Show that $5 \sin A = 3 \sin 2A$.

7 Find the length of the longest side of a triangle *ABC* in which $A = 36° 52'$, $B = 30°$ and $AC = 8$ cm.



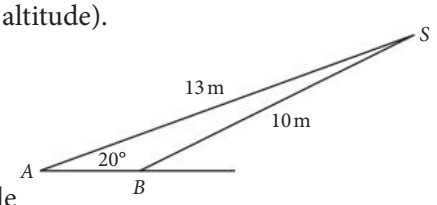
8 In a triangle *ABC*, $\angle A = 42°$, $\angle B = 28°$ and $BC = 6$ cm. Indicate whether each statement is correct or incorrect.

(a) $\frac{AB}{\sin 110°} = \frac{6}{\sin 28°}$ (b) $\frac{AB}{\sin 110°} = \frac{6}{\sin 42°}$ (c) $\frac{AB}{\sin 100°} = \frac{6}{\sin 42°}$ (d) $\frac{AB}{\sin 70°} = \frac{6}{\sin 42°}$

9 In a triangle ABC, $a = 16$, $b = 12$ and $\sin A = 0.4$. Calculate:

(a) $\sin B$ **(b)** the length of the perpendicular from C to AB (the altitude).

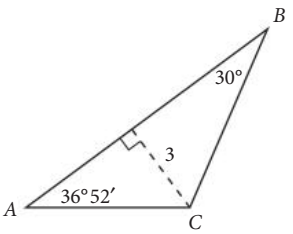
10 A wooden stake S is 13 m from a point A on a straight fence. SA makes an angle of 20° with the fence. If a goat is tethered to S by a 10 m rope, where on the fence is the closest point to A that the goat can reach?



11 In a triangle ABC, $AC = 30$ cm, $AB = 44$ cm and $B = 37°$. Find two possible values of angles A and C.

12 In the triangle PQR, $PQ = 20$ cm, $QR = 22$ cm and $R = 15°$. Find two possible values for $\angle RPQ$ and PR.

13 In the triangle ABC, $A = 36° 52'$, $B = 30°$ and the perpendicular distance from C to AB is 3 cm. Calculate the perimeter of ΔABC.



14 In ΔABC, the lengths of BC and AC are in the ratio 2 : 1 and $\sin(\angle ABC) = \frac{1}{4}$. Calculate:

(a) $\sin(\angle BAC)$ **(b)** two possible sizes for $\angle BAC$.

15 In ΔABC, $\angle B = 25°$, $\angle C = 55°$, $BC = 5$ m. Calculate:

(a) the length of AC

(b) the length of AX, where X is the foot of the perpendicular line from A to BC.

16 In ΔPQR, $PR = 3$ cm, $\angle RPQ = 40°$, $\angle PQR = 60°$. Calculate:

(a) the perimeter of ΔPQR

(b) the length of the perpendicular line from R to PQ.

17 In ΔKLM, $\angle KLM = 55°$, $\angle KML = 35°$, $KL = 10$ cm.

(a) Use the sine rule to calculate the length of KM correct to 1 decimal place.

(b) What is the size of $\angle LKM$? Comment on the significance of this result.

18 In ΔXYZ, $XY = 5$ m, $\angle ZXY = 68°$, $\angle XYZ = 82°$. Calculate the length of XZ.

19 An aircraft flies 400 km from point A to point B on a course 040°. It then flies on a course 160° from B to C, 500 km from A. Calculate:

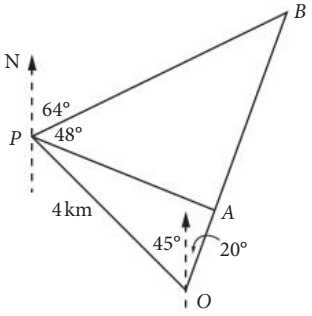
(a) the distance BC **(b)** the bearing of C from A.

20 The bearing of a ship from a lighthouse A is N 75° E. The ship's bearing from a second lighthouse B, 44 km south of A, is N 40° E. Find the distance of the ship from B.

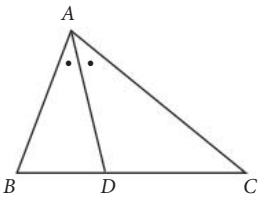
21 Three points A, B and C lie on a horizontal plane. B is 2000 m due south of A. C is on a bearing 145° from A and 052° from B. Calculate the distance of C from both A and B.

22 Two points A and B on the same bank of a river are 50 m apart. C is a point on the other bank. $\angle BAC = 80°$, $\angle ABC = 70°$. Calculate the width of the river, assuming the river is straight and its width is constant.

23 O, A and B are three points in a straight line (in that order). The bearings of A and B from O are 020°. From a point P, 4 km from O in a direction NW, the bearings of A and B are 112° and 064° respectively. Calculate the distance from A to B.



24 In ΔABC, AD bisects $\angle BAC$. Use the sine rule to prove that $\frac{AB}{AC} = \frac{BD}{DC}$.



5.9 THE COSINE RULE

For ΔABC, the cosine rule states $a^2 = b^2 + c^2 - 2bc\cos A$. In words:

- The square of one side of a triangle is equal to the sum of the squares of the other two sides minus twice the product of those two sides and the cosine of the included angle.

To find a side, use:

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

To find an angle, use:

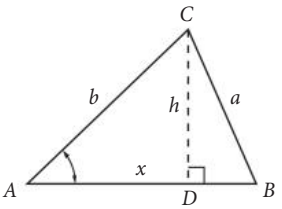
$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$
$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$
$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Proof

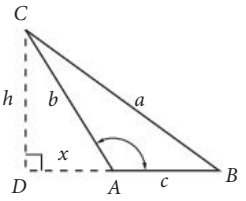
From C, draw the perpendicular to meet AB at D (for acute-angled triangles) or to meet BA produced at D (for obtuse-angled triangles).

Let $CD = h$ units and $AD = x$ units.

Apply Pythagoras' theorem to ΔBCD:



Acute-angled triangle



Obtuse-angled triangle

Acute-angled triangle:

$a^2 = h^2 + (c - x)^2$

$a^2 = h^2 + c^2 - 2cx + x^2$

But: $h^2 + x^2 = b^2$

And: $x = b\cos A$

$\therefore\ a^2 = b^2 + c^2 - 2bc\cos A$

Obtuse-angled triangle:

$a^2 = h^2 + (c + x)^2$

$a^2 = h^2 + c^2 + 2cx + x^2$

But: $h^2 + x^2 = b^2$

And: $x = b\cos(180° - A)$

$x = -b\cos A$

$\therefore\ a^2 = b^2 + c^2 - 2bc\cos A$

The cosine rule can be used with a triangle when you are given:
(a) the lengths of three sides, **OR**
(b) the lengths of two sides and the size of the included angle.

If $A = 90°$ then $\cos A = \cos 90° = 0$. The size of A determines the proportions of the sides:

- If $A < 90°$ then $a^2 < b^2 + c^2$
- If $A = 90°$ then $a^2 = b^2 + c^2$ (Pythagoras' theorem)
- If $A > 90°$ then $a^2 > b^2 + c^2$

Important result

$$\cos(180° - \theta) = -\cos\theta$$

Hence: $\cos 120° = \cos(180° - 60°) = -\cos 60° = -0.5$

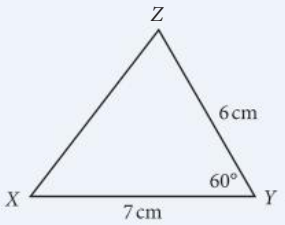
Solving an equation like $\cos\theta = -0.5$ gives $\theta = (180 - 60)° = 120°$.

Example 15

In triangle XYZ, $XY = 7$ cm, $YZ = 6$ cm and $\angle XYZ = 60°$. Calculate the exact length of XZ.

Solution

Show the information on a diagram.



Two sides and the included angle are given.

Use cosine rule: $y^2 = x^2 + z^2 - 2xz\cos Y$

$y^2 = 6^2 + 7^2 - 2 \times 6 \times 7 \times \cos 60°$

$y^2 = 36 + 49 - 84 \times 0.5$

$y^2 = 43$

$y = \sqrt{43}$

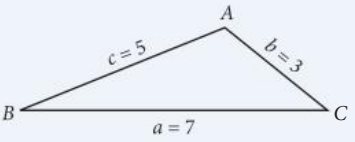
$\therefore XZ = \sqrt{43}$ cm

Example 16

Find the size of the largest angle of a triangle with side lengths 3 cm, 5 cm and 7 cm.

Solution

Show the information on a diagram.



The largest angle is opposite the longest side. Find the size of $\angle BAC$.

Use cosine rule: $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

$\cos A = \dfrac{3^2 + 5^2 - 7^2}{2 \times 3 \times 5}$

$= \dfrac{9 + 25 - 49}{30}$

$= -\dfrac{15}{30} = -\dfrac{1}{2}$

Because $\cos A < 0$, A is obtuse: $A = 180° - 60° = 120°$

The largest angle in the triangle is 120°.

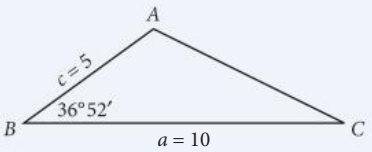
Example 17

In ΔABC, $a = 10$, $c = 5$ and $\angle B = 36° 52'$. Calculate:

(a) b **(b)** $\angle C$

Solution

Show the information on a diagram.



(a) Two sides and included angle given.

Use cosine rule: $b^2 = a^2 + c^2 - 2ac\cos B$

$b^2 = 10^2 + 5^2 - 2 \times 10 \times 5 \times \cos 36° 52'$

$b^2 = 125 - 100\cos 36° 52'$

$b^2 = 44.9966$

$b = 6.71$

(b) Because we now know the length of the three sides and the size of one angle, we can use either the sine rule or the cosine rule to find the required angle. The sine rule is usually quicker and easier to use.

$$\frac{\sin C}{c} = \frac{\sin B}{b}$$

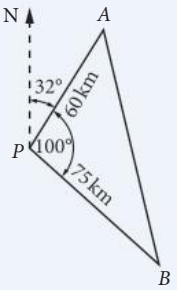
$$\frac{\sin C}{5} = \frac{\sin 36° 52'}{6.71}$$

$$\sin C = \frac{5 \sin 36° 52'}{6.71} = 0.4471$$

$$\angle C = 26° 33'$$

Example 18

Two people set out from point P at the same time. One travels at 20 km h^{-1} along a straight road in the direction 032°. The other travels at 25 km h^{-1} along another straight road in the direction 132°. Find their distance apart after 3 hours.



Solution

After 3 hours, one person is at A, 60 km from P, and the other is at B, 75 km from P, as shown above.

$\angle APB = 132° - 32° = 100°$

Two sides and the included angle are known. Use cosine rule:

$AB^2 = 60^2 + 75^2 - 2 \times 60 \times 75 \times \cos 100°$

$AB^2 = 10787.8$

$AB = 103.9$

The two people are 103.9 km apart.

TECHNOLOGY EXPLORATION

Demonstration of the cosine rule

1 In GeoGebra, create a triangle using the **Polygon** tool.

2 In **Algebra** view, you should notice that the sides are labelled *a*, *b* and *c*, and the lengths of the sides are listed next to them.

3 Use the **Angle** tool to select angle *A* by selecting sides *b* and *c*. The angle will be labelled α. If the angle is a reflex angle, you can force it to take on the acute angle instead by right-clicking inside the shaded part of the angle, selecting **Object Properties** > Basic and then unselecting 'Allow Reflex Angle'. The angle and its size will also appear in **Algebra** view.

4 Repeat step **3** for angle *B* (which will be labelled β) and for angle *C* (which will be γ).

To check the cosine rule for the side lengths:

5 In the input bar, enter *c_1=sqrt(a^2+b^2-2*a*b*cos(α))* to calculate the length of side *c* using the cosine rule. This will be labelled in the **Algebra** view as c_1.

6 Compare the values of *c* and c_1. Use the **Move** tool to select and drag a vertex. Notice how the values of *c* and c_1 change.

Do you think this result would be true for the other sides and angles, too? You can adapt the method above to test the cosine rule for the other side lengths.

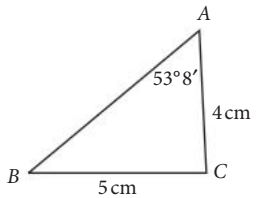
To check the cosine rule for the angle:

7 In the input bar, enter *α_1=acos(b^2+c^2-a^2)/(2*b*c)* to calculate the value of angle α using the cosine rule. (The text *acos* represents the inverse function of cos, or $\cos^{-1}$.) Unfortunately, this returns a value for α not in degrees but in radians, which is an angle measure you may not be familiar with: $180° = \pi$ radians. To convert the angle value to degrees, adjust the input bar formula to read *α_1=acos(b^2+c^2-a^2)/(2*b*c)/pi*180*.

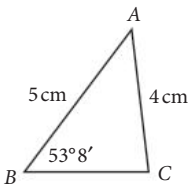
EXERCISE 5.9 THE COSINE RULE

1 In ΔABC, $b = 4$ cm, $c = 5$ cm, $\angle BAC = 53° 8'$. The correct diagram for this information is:

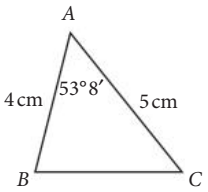
A



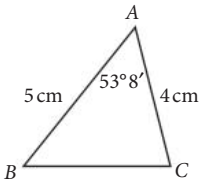
B



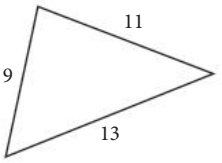
C



D

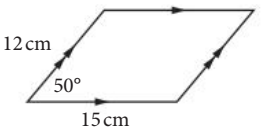


2 Calculate the cosine of the smallest angle of a triangle with side lengths 9 cm, 11 cm and 13 cm.



3 In ΔABC, $A = 36° 52'$, $b = 7$ and $c = 10$. Calculate: **(a)** a **(b)** B

4 Two adjacent sides of a parallelogram have lengths of 12 cm and 15 cm and the included angle is 50°. Calculate the lengths of the diagonals.



5 In ΔABC, $b = 4$ cm, $c = 5$ cm, $\angle BAC = 53° 8'$. Calculate the perimeter of the triangle, correct to the nearest centimetre.

6 Two sides of a triangle have lengths 3.2 cm and 4.8 cm and the included angle is 65°. Calculate the length of the third side, correct to one decimal place. Indicate whether each statement is a correct or incorrect step in the working.

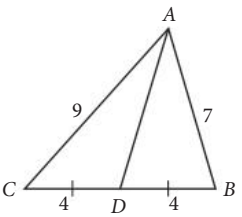
(a) $x = \dfrac{3.2 \sin 65°}{4.8}$ **(b)** $x = \dfrac{4.8 \sin 65°}{3.2}$ **(c)** $x^2 = 3.2^2 + 4.8^2 - 30.72 \cos 65°$ **(d)** $x = 4.5$ cm

7 In ΔABC, $B = 126° 52'$, $a = 12$ and $c = 15$. Find: **(a)** b **(b)** A

8 The lengths of the sides of a triangle are in the ratio 5 : 6 : 9. Find the size of the largest angle.

9 In ΔABC, $BC = 8$ cm, $AC = 9$ cm and $AB = 7$ cm. If D is the midpoint of BC, calculate:

(a) the size of $\angle ABC$
(b) the length of AD
(c) the size of $\angle DAC$.



10 The two adjacent sides of a parallelogram have lengths of 8 cm and 10 cm. If the length of the longer diagonal is 14 cm, calculate:

(a) the size of the angles of the parallelogram **(b)** the length of the other diagonal.

11 In ΔPQR, $q = 12$, $r = 5$ and $\angle QPR = 108°$. Calculate: **(a)** p **(b)** $\angle PQR$

12 In ΔABC, $BC = 11$ cm, $AC = 5$ cm and $AB = 8$ cm. P is a point on BC such that $BP = 4$ cm. Indicate whether each statement is correct or incorrect.

(a) $AP = 5$ cm **(b)** $\cos(\angle ABC) = \dfrac{10}{11}$ **(c)** $AP^2 = \dfrac{240}{11}$ **(d)** $\cos(\angle ACB) = \dfrac{41}{55}$

13 Two cars A and B depart from the same position. A travels along a straight road due east at 30 km h^{-1}. B departs 15 min after A and travels along another straight road in a north-east direction at 40 km h^{-1}. How far apart are they 15 min after B departs?

14 A, B and C are three towns such that B is 20 km from A in a direction 330° and C is 30 km from A in a direction 203° 8′. Find the distance from B to C.

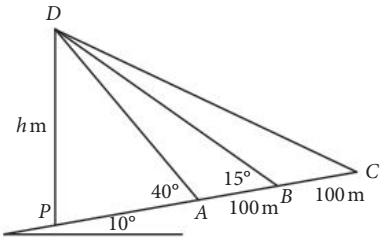
15 P and Q are two towns 50 km apart with Q due east of P. A third town R, to the north of the line joining P and Q, is 70 km from P and 30 km from Q. Find the bearing of R from: **(a)** Q **(b)** P

16 A lighthouse is 10 km north-west of a ship travelling due west at 16 km h^{-1}.

(a) How far is the ship from the lighthouse, 45 minutes later?
(b) What is the bearing of the lighthouse from the ship then?

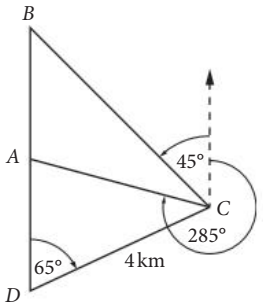
17 P, A, B and C are four points (in that order) on a straight road that runs up a hill and makes a constant angle of 10° with the horizontal. A flagpole of height h m stands at P. From A and B, the top of the flagpole has elevations of 30° and 5° respectively above the horizontal.

(a) If AB is 100 m, what is the height of the flagpole?
(b) If BC is also 100 m long, what is the elevation of the top of the flagpole from C?



18 From a point P, a person observes that the angle of elevation of the top of a cliff A is 40°. After walking 100 m towards A along a straight road inclined upwards at an angle of 15° to the horizontal, the angle of elevation of A is observed to be 50°. Find the vertical height of A above P.

19 The captain of a ship at D sailing on a bearing of 065° observes two lighthouses, A and B, in a line due north. After travelling 4 km to C, she notes that one lighthouse is on a bearing of 285° and the other 315°. Calculate the distance between the lighthouses.



20 Two cars leave a point A at the same time. One car averages 80 km h^{-1} along a straight road in the direction 025°. The other car averages 90 km h^{-1} along a straight road in the direction 135°. How far apart are they after 3 hours?

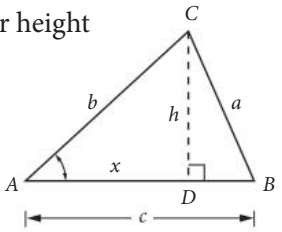
5.10 AREA OF A TRIANGLE

The formula for the area of a triangle is the same for both acute-angled and obtuse-angled triangles.

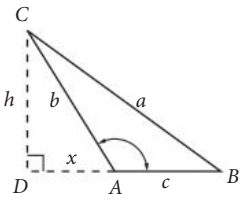
$$\text{Area of } \Delta ABC = \frac{\text{base} \times \text{altitude}}{2} = \frac{1}{2} \times \text{base} \times \text{perpendicular height}$$

In each triangle the altitude is of length h.

$$\therefore \text{ Area of } \Delta ABC = \frac{ch}{2}$$



Acute-angled triangle



Obtuse-angled triangle

Acute-angled triangle: $h = b \sin A$

Obtuse-angled triangle: $h = b \sin(180° - A) = b \sin A$

Hence: $\text{Area of } \Delta ABC = \frac{1}{2}bc \sin A = \frac{1}{2}ab \sin C = \frac{1}{2}ac \sin B$

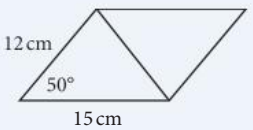
= half the product of two sides with the sine of the angle between them.

Example 19

The adjacent sides of a parallelogram are 12 cm and 15 cm and the angle between them is 50°. Calculate the area of the parallelogram correct to the nearest square centimetre.

Solution

Show the information on a diagram.



Draw the shorter diagonal.

The area of the parallelogram is twice the area of the triangle.

$$\text{Area of parallelogram} = 2 \times \frac{12 \times 15 \sin 50°}{2}$$

$$= 180 \sin 50° = 138 \text{ cm}^2$$

TECHNOLOGY EXPLORATION

Demonstration of the area of a triangle formula

1 In GeoGebra, create a triangle using the **Polygon** tool.

2 In **Algebra** view, you should notice that the sides are labelled *a*, *b* and *c*, and the lengths of the sides are listed next to them.

3 Use the **Angle** tool to select angle *A* by selecting sides *b* and *c*. The angle will be labelled α. If the angle is a reflex angle, you can force it to take on the acute angle instead by right-clicking inside the shaded part of the angle, selecting **Object Properties** > Basic and then unselecting 'Allow Reflex Angle'. The angle and its size will also appear in **Algebra** view.

4 Repeat step 3 for angle *B* (which will be labelled β) and for angle *C* (which will be γ).

To check the rule for the area of a triangle:

5 In the input bar, enter *Area=0.5*b*c*sin(α)* to calculate the area using the rule.

6 In the **Algebra** view, compare the values for *Area* and for *poly1* (the triangle).

7 Try this for all the possible combinations of sides and angles.

EXERCISE 5.10 AREA OF A TRIANGLE

1 In ΔABC, $A = 36° 52'$, $b = 7$ cm and $c = 10$ cm. Calculate:

(a) a **(b)** B **(c)** the area of ΔABC.

2 In ΔABC, $b = 4$ cm, $c = 5$ cm, $\angle BAC = 53° 8'$. Calculate the area of the triangle, correct to the nearest square centimetre.

3 In ΔABC, $BC = 11$ cm, $AC = 5$ cm and $AB = 8$ cm. Calculate:

(a) the magnitude of $\angle ABC$ **(b)** the length of the perpendicular from A to BC

(c) the area of ΔABC.

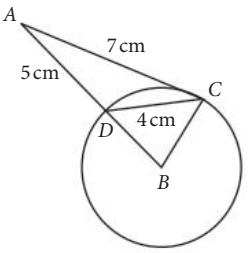
4 The area of ΔABC if $a = 6$ cm, $b = 7$ cm and $c = 11$ cm is nearest to:

A 6 cm^2 **B** 19 cm^2 **C** 20 cm^2 **D** 22 cm^2

5 ABC is a triangle in which $AC = 7$ cm. A circle, centre B and radius BC, cuts AB internally at D. $AD = 5$ cm, $DC = 4$ cm. Calculate:

(a) the length of BC

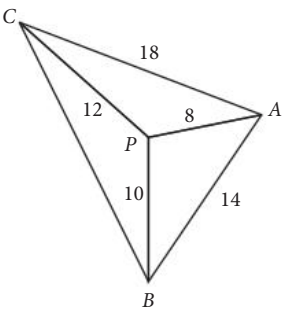
(b) the area of ΔABC.



6 P, A, B and C are four points in a plane such that angles BPA and CPA are obtuse and on opposite sides of PA. $PA = 8$ cm, $BP = 10$ cm, $PC = 12$ cm, $AB = 14$ cm and $AC = 18$ cm. Calculate:

(a) the length of BC

(b) the area of ΔABC.



7 The equal sides AB and AC of an isosceles triangle ABC are each 5 cm and $BC = 4$ cm. D is a point on AC such that $DC = 1$ cm. Calculate:

(a) the size of angle A **(b)** the length of BD **(c)** the area of ΔABC.

8 $ABCD$ is a parallelogram in which $AB = 8$ cm, $BC = 5$ cm and $\angle DAB = 60°$. Calculate:
(a) the length of the diagonals AC and BD **(b)** the area of $ABCD$.

9 Two sides of a triangular field are 60 m and 50 m and the included angle is 140°. Indicate whether each statement is correct or incorrect.
(a) Third side = 103.4 m **(b)** Area = 964.2 m^2 **(c)** Third side = 38.8 m **(d)** Area = 1149 m^2

10 The sides of a triangular field have lengths 80 m, 90 m and 100 m. Calculate the area of the field.

CHAPTER REVIEW 5

1 Simplify:
(a) $\cos(90° - \theta)$ **(b)** $\sin(270° - \theta)$ **(c)** $\cos(90° + \theta)$
(d) $\tan(\theta - 180°)$ **(e)** $\tan(180° - \theta)$ **(f)** $\sin(\theta + 180°)$

2 Write the exact value.
(a) $\tan 315°$ **(b)** $\sin 225°$ **(c)** $\cos 180°$
(d) $\tan 360°$ **(e)** $\sin 60°$ **(f)** $\cos 210°$

3 Simplify:
(a) $\dfrac{\cos\theta}{\sin(90° - \theta)}$ **(b)** $\cos(90° + \theta) + \sin\theta$

4 If $\tan\theta = \frac{3}{5}$ and $180° < \theta < 270°$, write the exact value of: **(a)** $\sin\theta$ **(b)** $\cos\theta$

5 If $\sin\alpha = 0.6$ and $0° < \alpha < 90°$, write the exact value of:
(a) $\sin(180° - \alpha)$ **(b)** $\cos(90° - \alpha)$ **(c)** $\cos(180° + \alpha)$
(d) $\tan\alpha$ **(e)** $\tan(180° - \alpha)$ **(f)** $\sin(360° - \alpha)$

6 If $\tan\theta = t$, express in terms of t:
(a) $\tan(90° - \theta)$ **(b)** $\tan(180° + \theta)$ **(c)** $\cot(180° - \theta)$
(d) $\tan(360° - \theta)$ **(e)** $\tan(-\theta)$ **(f)** $\tan(90° + \theta)$

7 Calculate the cosine of the smallest angle of the triangle with side lengths 5 cm, 6 cm and 7 cm.

8 Find the size of the largest angle of the triangle with side lengths 5 cm, 6 cm and 8 cm. Hence, show that the triangle is obtuse-angled.

9 In ΔABC, $\angle B = 53°$, $\angle C = 48°$, $AC = 8$ cm. Calculate:
(a) the length of BC **(b)** the area of ΔABC.

10 A ladder 8 m long rests against a wall and its foot makes an angle of 60° with the horizontal ground. The top of the ladder then slips down the wall until its foot makes an angle of 45° with the ground. Find, in simplest surd form, how far down the wall the ladder slips.

11 From a point A, level with the foot of a vertical pole and 30 m from it, the angle of elevation of the top of the pole is 40°. Calculate:
(a) the height of the pole **(b)** the direct distance from A to the top of the pole
(c) the angle of elevation from A of a point half-way up the pole.

12 AB and CD are two vertical buildings with their bases at A and at C on horizontal ground. The height of AB is 30 m. The angle of elevation of B as seen from C is 25° and the angle of elevation of D as seen from A is 40°. Calculate: **(a)** the horizontal distance between the buildings **(b)** the height of CD
(c) the angle of depression of B as seen from D.

13 Two yachts sail in a straight line from a buoy B. One sails 10 km in the direction 040° and the other sails 20 km in the direction 160°.

(a) How far apart are the yachts?

(b) What is the bearing of the first yacht as seen from the second yacht?

14 Prove that:

(a) $\dfrac{\sin^2\theta}{(1-\sin\theta)(1+\sin\theta)} = \tan^2\theta$ **(b)** $(\sin x+\cos x)^2+(\sin x-\cos x)^2 = 2$

15 The expression $1-\cos^2(180°-\theta)$ simplifies to:

A $\cos^2\theta$ **B** $\sin^2\theta$ **C** $1+\cos^2\theta$ **D** $\sec^2\theta$

16 Find the exact value of $\sec^2 45° + \operatorname{cosec}^2 45°$.

17 Find all values of θ between 0° and 360° for which:

(a) $\sin\theta = -0.5$ **(b)** $\cos\theta = 0$ **(c)** $\tan\theta = -1$

(d) $\sec\theta = \dfrac{2}{\sqrt{3}}$ **(e)** $\cot\theta = \sqrt{3}$ **(f)** $\operatorname{cosec}\theta = \sqrt{2}$

18 Simplify:

(a) $\dfrac{1}{\sec^2\theta}+\dfrac{1}{\operatorname{cosec}^2\theta}$ **(b)** $\dfrac{1-\sin^2\theta}{\cos^2\theta}$ **(c)** $\sec^4\theta-\sec^2\theta\tan^2\theta$

(d) $\dfrac{\tan\theta\sec\theta}{1+\tan^2\theta}$ **(e)** $\dfrac{\sec\theta}{\cos\theta}-\dfrac{\tan\theta}{\cot\theta}$ **(f)** $\dfrac{1}{\operatorname{cosec}\theta-1}-\dfrac{1}{\operatorname{cosec}\theta+1}$

19 Simplify $\dfrac{1+\sin x}{1-\sin x}$. Indicate whether each answer is correct or incorrect.

(a) $\dfrac{\cos^2 x}{1+\sin^2 x}$ **(b)** $(\sec x+\tan x)^2$ **(c)** $\dfrac{1+2\sin x+\sin^2 x}{\cos^2 x}$ **(d)** $\tan^2 x+2\tan x\sec x+\sec^2 x$

CHAPTER 6
COORDINATE GEOMETRY—STRAIGHT LINES

Outcomes covered:

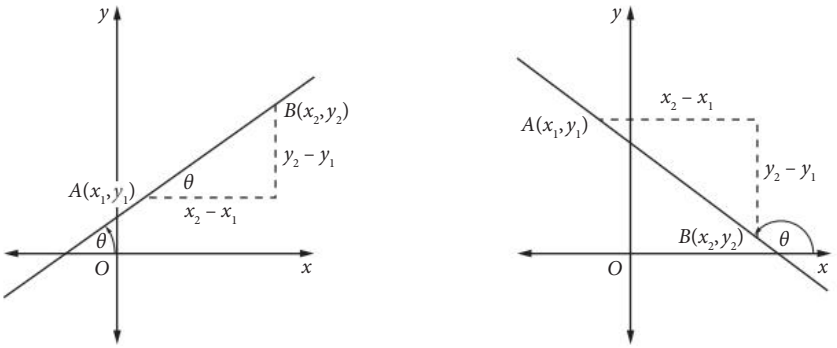
P1 demonstrates confidence in using mathematics to obtain realistic solutions to problems

P4 chooses and applies appropriate arithmetic, algebraic, graphical, trigonometric and geometric techniques

In the seventeenth century, mathematics leapt forward with two great advances: analytical geometry and the calculus. René Descartes (1596–1650) was one of the mathematicians most responsible for the creation of analytical geometry. He did this by setting up a coordinate system and applying algebra to geometry. The name of Descartes is still used today as an adjective in terms such as Cartesian plane, Cartesian coordinates and Cartesian axes.

6.1 GRADIENT OF A STRAIGHT LINE

$A(x_1, y_1)$ and $B(x_2, y_2)$ are two points on the number plane, with B to the right of A.



The **gradient** (or slope) m of AB is defined as: $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ if $x_1 \neq x_2$

If $x_1 = x_2$, the gradient is undefined and the line is vertical (that is, parallel to the y-axis).

If $y_1 = y_2$, the gradient is zero and the line is horizontal (that is, parallel to the x-axis).

Angle of inclination, θ

The angle that a line makes with the positive direction of the x-axis is called the **angle of inclination**. It is usually denoted by the symbol θ where $0° \leq \theta < 180°$. The gradient m is also equal to $\tan\theta$, hence:

$$\tan\theta = \frac{y_2 - y_1}{x_2 - x_1} \quad \text{if } x_1 \neq x_2$$

In the figure on the left above, the line rises from left to right and makes an acute angle θ with the x-axis. The values $(x_2 - x_1)$ and $(y_2 - y_1)$ are both positive numbers, so the gradient is positive.

In the figure on the right above, the line falls from left to right and makes an obtuse angle θ with the x-axis. The value $(x_2 - x_1)$ is a positive number, but $(y_2 - y_1)$ is negative, so the gradient is negative.

Example 1

Find the gradient of the line joining the given points and calculate the size of the angle of inclination.

(a) $(2,3)$ and $(4,7)$ **(b)** $(-3,5)$ and $(2,-1)$

Solution

(a) $(2,3)$ and $(4,7)$

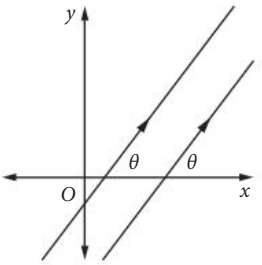
$$m = \frac{7-3}{4-2}$$
$$= 2$$
$$\tan\theta = 2$$
$$\theta = 63^\circ 26'$$

(b) $(-3,5)$ and $(2,-1)$

$$m = \frac{-1-5}{2+3}$$
$$= -1.2$$
$$\tan\theta = -1.2$$
$$\theta = 180^\circ - 50^\circ 12' = 129^\circ 48'$$

Parallel lines

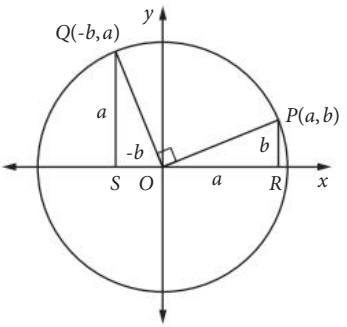
Parallel lines make angles of equal magnitude with the positive direction of the x-axis, i.e. their angle of inclination is the same.



Two lines are parallel if their gradients are equal, or $m_1 = m_2$.

Perpendicular lines

In the diagram, OP and OQ are perpendicular to each other, with gradients m_1 and m_2 respectively. $\Delta ORP \equiv \Delta QSO$ (SSS).



By definition: $\tan(\angle POR) = \frac{b}{a} = m_1$

$$\tan(\angle ROQ) = \frac{a}{-b} = m_2$$
$$m_1 m_2 = \frac{b}{a} \times \frac{a}{-b} = -1$$

i.e. $m_2 = \frac{-1}{m_1}$, provided that $m_1, m_2 \neq 0$.

Two lines are perpendicular if the product of their gradients is -1, or $m_1 m_2 = -1$.

This can also be written as $m_2 = \frac{-1}{m_1}$. Each gradient is the negative reciprocal of the other.

Example 2

The coordinates of the vertices of a triangle ABC are $(-2,1)$, $(3,2)$ and $(4,-3)$ respectively. Show that AB is perpendicular to BC.

Solution

Gradient of $AB = \frac{2-1}{3+2} = \frac{1}{5}$ Gradient of $BC = \frac{-3-2}{4-3} = -5$

Because $\frac{1}{5} \times -5 = -1$, AB is perpendicular to BC. $(AB \perp BC)$

Example 3

Given the points $P(1,3)$, $Q(5,2)$, $R(6,6)$, $S(2,7)$, show that $PQRS$ is a rectangle.

Solution

Gradient of $PQ = \frac{3-2}{1-5} = \frac{-1}{4}$ Gradient of $QR = \frac{2-6}{5-6} = 4$

Gradient of $RS = \frac{6-7}{6-2} = \frac{-1}{4}$ Gradient of $SP = \frac{7-3}{2-1} = 4$

Gradient of PQ = Gradient of RS, so $PQ \parallel RS$

Gradient of QR = Gradient of SP, so $QR \parallel SP$

$\therefore$ $PQRS$ is a parallelogram.

But: Gradient of $PQ \times$ Gradient of $QR = \frac{-1}{4} \times 4 = -1$

$\therefore$ $PQ \perp QR$ and $PQRS$ is a rectangle (parallelogram with one angle 90°).

TECHNOLOGY EXPLORATION

The slope of a line (between two points)

In GeoGebra's input bar, enter the coordinates of the points, e.g. (4, -5). Use the **Line through Two Points** tool to create a line through the points.

Measure the gradient using the **Slope** tool. The value of the gradient will be displayed in the **Algebra** view.

EXERCISE 6.1 GRADIENT OF A STRAIGHT LINE

1 Find the gradient of the line containing the given points.

(a) $(2,4), (0,6)$ **(b)** $(-3,-1), (-5,6)$ **(c)** $(-2,2), (-6,2)$ **(d)** $(2,-3), (-3,2)$

(e) $(a,b), (b,a)$ **(f)** $(3b,2c), (2b,-3c)$ **(g)** $(-1,-5), (-4,-3)$ **(h)** $(0,5), (-8,-7)$

2 Calculate the angle of inclination of the line joining the given points.

(a) $(-4,-2), (4,6)$ **(b)** $(0,5), (-2,4)$ **(c)** $(-5,6), (3,3)$

(d) $(4,5), (-2,-4)$ **(e)** $(2a,b), (2b,a)$ **(f)** $(b,c), (c,b)$

3 Calculate the gradients of the lines joining the points $(-2,3)$ and $(4,-2)$ and the points $(-1,7)$ and $(-7,12)$. These two lines are:

A parallel **B** perpendicular **C** intersecting **D** coincident

4 For each of the following, show that $ABCD$ is a parallelogram.

(a) $A(0,0), B(3,0), C(5,5), D(2,5)$ **(b)** $A(-3,-1), B(4,1), C(8,5), D(1,3)$

(c) $A(-1,4), B(4,6), C(2,7), D(-3,5)$ **(d)** $A(-2,-3), B(6,2), C(8,7), D(0,2)$

5 For each of the following, show that $PQRS$ is a trapezium.

(a) $P(-1,2), Q(3,4), R(8,-1), S(-4,-7)$ **(b)** $P(-2,3), Q(3,7), R(9,-5), S(2,-5)$

6 Show that the points $(-2,0)$, $(2,12)$ and $(-5,-9)$ are collinear.

7 For the points $A(2a,b)$, $B(a,2b)$ and $C(-a,4b)$, indicate whether each statement is correct or incorrect.

(a) $AB = BC$ **(b)** $AB + BC = AC$ **(c)** $AB \perp BC$ **(d)** A, B and C are collinear

8 For each of the following, show that ABC is a right-angled triangle.

(a) $A(2,-3), B(5,2), C(-3,0)$ **(b)** $A(-1,2), B(3,4), C(7,-4)$

9 For each of the following, show that $PQRS$ is a rectangle.

(a) $P(1,4)$, $Q(5,2)$, $R(2,-4)$, $S(-2,-2)$ **(b)** $P(4,-7)$, $Q(8,-4)$, $R(2,4)$, $S(-2,1)$

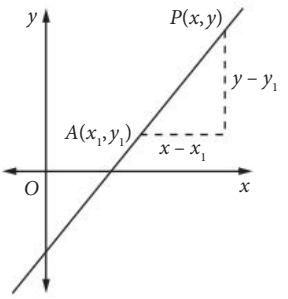
10 $A(-8,6)$, $B(2,4)$, $C(5,-7)$ and $D(-5,-3)$ are the vertices of a quadrilateral. Prove that the diagonals of the quadrilateral are perpendicular.

6.2 EQUATION OF A STRAIGHT LINE

Point-gradient form

Given the gradient of a straight line and a point on the line, you can find the equation of the line.

Let $A(x_1,y_1)$ be a given point and $P(x,y)$ be any point on a line with gradient m.



Thus: $\dfrac{y-y_1}{x-x_1}=m$

$\therefore\ y-y_1=m(x-x_1)$

The equation of the line AP must be: $y-y_1=m(x-x_1)$

This is known as the **point-gradient form**, because it is obtained using a point on the line and its gradient.

Example 4

Find the equation of the straight line through the point $(2,3)$ with a gradient of $\frac{3}{4}$.

Solution

Use the point-gradient form: $y-y_1=m(x-x_1)$

Substitute information: $y-3=\dfrac{3}{4}(x-2)$

Multiply through by 4: $4(y-3)=3(x-2)$

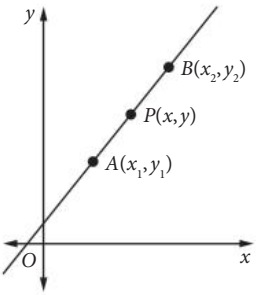
$4y-12=3x-6$

$3x-4y+6=0$ is the equation of the line.

Two-point form

Given the coordinates of two points, you can find the equation of the line that passes through those points.

Let $A(x_1,y_1)$ and $B(x_2,y_2)$ be the given points and $P(x,y)$ be any point on the line.



Gradient of $AB=\dfrac{y_2-y_1}{x_2-x_1}$ Gradient of $AP=\dfrac{y-y_1}{x-x_1}$

Because these gradients are the same, we have:

$$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$$

This must be the equation of the line AB. This equation can also be written as:

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1} \quad \text{or} \quad y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$$

You can use the form that you find the easiest to remember. This is known as the **two-point form**, because it is obtained using two points on the line.

Example 5

Find the equation of the straight line through the points (-3, 4) and (2, -6).

Solution

Use the two-point form: $\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$

Points are (-3, 4), (2, -6): $\frac{y-4}{-6-4}=\frac{x-(-3)}{2-(-3)}$

$$\frac{y-4}{-10}=\frac{x+3}{5}$$

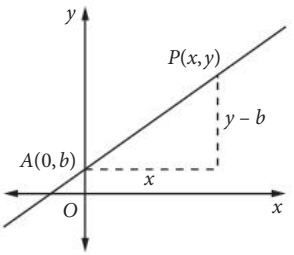
$$y-4=-2(x+3)$$

$$y-4=-2x-6$$

$2x+y+2=0$ is the equation of the line.

Gradient-intercept form of the straight line

Let the fixed point $A(0, b)$ be the point where the line AP cuts the y-axis.
Let $P(x, y)$ be any other point on the line.



Thus: $m=\frac{y-b}{x-0}$

$\therefore$ $y=mx+b$

Here b is called the y-intercept.

Example 6

For the line of the equation $3x+2y-5=0$, find:

(a) the gradient **(b)** the angle at which the line crosses the x-axis **(c)** the y-intercept.

Solution

(a) $3x+2y-5=0$

$2y=-3x+5$

$y=-\frac{3}{2}x+\frac{5}{2}$

$m=-\frac{3}{2}=-1.5$

(b) $\tan\theta=-1.5$

$\theta=180°-56°19'$

$\theta=123°41'$

(c) Using the form $y=mx+b$ gives the y-intercept as:

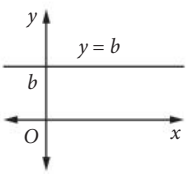
$b=\frac{5}{2}=2.5$

Line parallel to x-axis

Any line parallel to the x-axis has zero gradient, or $m=0$.

The gradient-intercept form of the straight line gives:

$y=0x+b$ or $y=b$



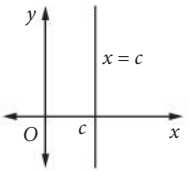
Thus $y=b$ is the equation of the straight line parallel to the x-axis passing through the point $(0, b)$.

Line parallel to *y*-axis

Any line parallel to the *y*-axis has a gradient that is undefined.

Any line parallel to the *y*-axis is a vertical line, so its equation must be of the form $x = c$.

Another way of looking at this situation is to recall that any fraction with a zero denominator is undefined. If the denominator in $m = \frac{y - y_1}{x - x_1}$ is zero, so that $x - x_1 = 0$, then the gradient is undefined.



This gives $x = x_1$ as the equation of the line through the point (x_1, y_1), which means that x is a constant. This can be written as $x = c$.

General form of the straight line equation

The equation of a straight line is a 'first degree' or linear equation. It can be written in many forms, but it is called the **general form** when written as $ax + by + c = 0$, where a, b and c are integers:

$ax + by + c = 0$
a is the coefficient of x, b is the coefficient of y and c is the constant term.

Rewriting the general form in the gradient-intercept form:

$y = \frac{-a}{b}x - \frac{c}{b}, \qquad b \neq 0$

Hence the gradient $= -\frac{a}{b}$ and the *y*-intercept $= -\frac{c}{b}$

Useful results:

1. If $a = 0$, the line is parallel to the *x*-axis.
2. If $b = 0$, the line is parallel to the *y*-axis.
3. If $c = 0$, the line passes through the origin.

Parallel and perpendicular lines

Consider the lines with equations: $ax + by + c = 0$ (1)

$a_1x + b_1y + c_1 = 0$ (2)

Writing these in gradient-intercept form: $y = \frac{-a}{b}x - \frac{c}{b}$ (1)

$y = \frac{-a_1}{b_1}x - \frac{c_1}{b_1}$ (2)

These lines have the same gradient when $\frac{a}{b} = \frac{a_1}{b_1}$, so the lines are parallel when $\frac{a}{b} = \frac{a_1}{b_1}$. The product of the gradients is -1 when $\frac{a}{b} \times \frac{a_1}{b_1} = -1$ or $aa_1 = -bb_1$, so the lines are perpendicular when $aa_1 = -bb_1$.

Useful results:

The equation of a line parallel to $ax + by + c = 0$ is $ax + by + d = 0$, where $c \neq d$.

The equation of a line perpendicular to $ax + by + c = 0$ is $bx - ay + d = 0$.

Example 7

Find the equation of the line through the point $(2, -3)$ that is:

(a) parallel to the line with equation $3x + 4y - 5 = 0$

(b) perpendicular to the line with equation $3x + 4y - 5 = 0$.

Solution

For the line equation $3x + 4y - 5 = 0$, $a = 3$ and $b = 4$.

(a) Method 1:

Parallel lines, so $m = -\frac{3}{4}$; point is $(2, -3)$.

$\therefore$ equation is: $y + 3 = -\frac{3}{4}(x - 2)$

$4y + 12 = -3x + 6$

$3x + 4y + 6 = 0$

Method 2:

The equation of the line parallel to $3x + 4y - 5 = 0$ is of the form $3x + 4y + c = 0$.

Substitute the values of the point $(2, -3)$ to find c:

$6 - 12 + c = 0$

$c = 6$

$\therefore$ equation is $3x + 4y + 6 = 0$

(b) Method 1:

Perpendicular lines, so $m = -1 \div -\frac{3}{4} = \frac{4}{3}$; point is $(2, -3)$.

$\therefore$ equation is: $y + 3 = \frac{4}{3}(x - 2)$

$3y + 9 = 4x - 8$

$4x - 3y - 17 = 0$

Method 2:

The equation of the line perpendicular to $3x + 4y - 5 = 0$ is of the form $4x + 3y + c = 0$.

Substitute the values of the point $(2, -3)$ to find c:

$8 + 9 + c = 0$

$c = -17$

$\therefore$ equation is $4x + 3y - 17 = 0$

EXERCISE 6.2 EQUATION OF A STRAIGHT LINE

1 Find the equation of the straight line with:

(a) gradient $\frac{3}{4}$, passing through $(-6, 5)$ **(b)** gradient $-\frac{1}{2}$, passing through $(4, -3)$

2 Find the equation of the straight line passing through:

(a) $(3, 3)$ and $(-4, -5)$ **(b)** $(2, -8)$ and $(7, 2)$

3 Find the equation of the straight line passing through:

(a) $(6, 6)$ with an angle of inclination of $45°$

(b) $(-2, 3)$ with an angle of inclination of $53° 8'$ $\left(\tan 53° 8' \approx \frac{4}{3}\right)$

(c) $(-5, -2)$ with an angle of inclination of $135°$

(d) $(-7, 4)$ with an angle of inclination of $143° 8'$ $\left(\tan 143° 8' \approx -\frac{3}{4}\right)$

4 Find the equation of the straight line parallel to the x-axis and passing through the point $(5, 2)$.

5 Find the equation of the straight line parallel to the y-axis and passing through the point $(-2, -4)$.

6 The equation of the straight line with x-intercept 2 and y-intercept -5 is:

A $2x - 5y - 10 = 0$ **B** $5x - 2y - 10 = 0$ **C** $2x - 5y + 10 = 0$ **D** $5x - 2y + 10 = 0$

7 Find the equation of the straight line with x-intercept -3 and y-intercept -2.

8 Write each equation in the form $y = mx + b$ and find the gradient of each line.

(a) $2x + 3y = 4$ **(b)** $3x - 2y = 7$ **(c)** $2y = 6 - 3x$ **(d)** $5y - 2x = 8$

9 Indicate whether each statement is correct or incorrect for the line $2x + 3y - 12 = 0$.

(a) $m = -\frac{2}{3}$ **(b)** x-intercept $= 6$ **(c)** y-intercept $= -4$ **(d)** passes through $(3, 2)$

10 **(a)** Show that $(2, 3)$ is on the line $2x + 3y - 13 = 0$.

(b) If $(-1, 2)$ is on the line $ax - 4y + 11 = 0$, find a.

11 Find the equation of the line containing the point $(2, -3)$ that is:

(a) parallel to the line $3x + 2y - 6 = 0$ **(b)** perpendicular to the line $3x + 2y - 6 = 0$

12 Find the equation of the line passing through the origin that is:

(a) parallel to the line $4x - 5y + 3 = 0$ **(b)** perpendicular to the line $4x - 5y + 3 = 0$

13 The coordinates of two points A and B are $(0, -2)$ and $(3, 0)$ respectively. The x-coordinate of a point C on the line AB is 6. Find:

(a) the equation of AB **(b)** the angle of inclination of AB

(c) the y-coordinate of C **(d)** the equation of the line through C that is perpendicular to AB.

14 Show that the line with equation $2x - y = 5$ is parallel to the line joining the points $(-1, 5)$ and $(1, 9)$.

15 $ABCD$ is a parallelogram. The coordinates of A, B and C are $(-1, 4)$, $(4, 6)$ and $(2, 7)$ respectively. Find:

(a) the equation of CD **(b)** the equation of AD **(c)** the coordinates of D.

16 $ABCD$ is a rectangle. The coordinates of A and B are $(1, 4)$ and $(5, 2)$ respectively. The x-coordinate of D is -2. Find:

(a) the equation of AB **(b)** the equation of AD **(c)** the y-coordinate of D

(d) the equation of BC **(e)** the equation DC **(f)** the coordinates of C.

17 $A(0, 0)$, $B(2, 1)$ and $C(1, 5)$ are the vertices of a triangle ABC. The triangle is rotated anticlockwise about the point A, through a right angle in the x–y plane. Find the equation of the image lines of:

(a) the line AC **(b)** the line AB **(c)** the line BC.

6.3 INTERSECTION OF TWO LINES

The point of intersection of two lines is found by solving the equations of the lines simultaneously.

We will look at finding the equation of a line that passes through the intersection point of two other lines and also satisfies one other condition. We will consider two different methods of doing this.

Example 8

Find the equation of the line through the point $A(4, -2)$ that also passes through the intersection point B of the lines with equations $4x + 2y + 2 = 0$ and $3x + 5y - 9 = 0$.

Solution

Method 1:

To find B, solve simultaneously: $4x + 2y + 2 = 0$ (1)

$3x + 5y - 9 = 0$ (2)

$3 \times (1)$: $12x + 6y + 6 = 0$ (3)

$4 \times (2)$: $12x + 20y - 36 = 0$ (4)

$(3) - (4)$: $-14y + 42 = 0$

$y = 3$

Substitute into (1): $4x + 6 + 2 = 0$

$x = -2$

$\therefore$ intersection point solution: $B(-2, 3)$

Gradient of AB: $m = \dfrac{3+2}{-2-4} = -\dfrac{5}{6}$

Equation of AB: $y - (-2) = -\dfrac{5}{6}(x - 4)$

$6(y + 2) = -5(x - 4)$

$6y + 12 = -5x + 20$

The equation of AB is $5x + 6y - 8 = 0$.

Method 2:

Any line passing through the intersection point of $4x + 2y + 2 = 0$ and $3x + 5y - 9 = 0$ has the equation:

$4x + 2y + 2 + k(3x + 5y - 9) = 0$

(You can check this statement by substituting some values for k and then seeing if the point B, found above, satisfies the new equation.)

The line $4x + 2y + 2 + k(3x + 5y - 9) = 0$ is to pass through the point $A(4, -2)$, so substitute the coordinate values to find k: $A(4, -2)$: $16 - 4 + 2 + k(12 - 10 - 9) = 0$

$14 - 7k = 0$

$k = 2$

Substitute $k = 2$ into the line equation: $4x + 2y + 2 + 2(3x + 5y - 9) = 0$

$10x + 12y - 16 = 0$

The equation of AB is $5x + 6y - 8 = 0$.

Two straight lines—only three possibilities

Two straight lines on the number plane must either:

(a) intersect at a point or **(b)** be parallel or **(c)** coincide.

This means that when we solve a pair of simultaneous linear equations, we will always get either:

(a) a unique solution (as the lines intersect)

or **(b)** no solution (as the lines are parallel)

or **(c)** an infinite number of solutions (as the lines coincide).

We can determine which of these three situations is the case without actually solving the equations.

For the pair of equations $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, we have:

(a) intersecting lines if $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ (the gradients are not equal)

(b) parallel lines if $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$ (the gradients are equal, but the lines are different)

(c) coincident lines if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ (the equations are equivalent).

Example 9

Consider the simultaneous equations. Discuss their solution.

(a) $3x + 2y - 5 = 0$
$6x + 4y - 16 = 0$

(b) $3x + 2y - 5 = 0$
$6x + 4y - 10 = 0$

Solution

(a) The two lines have the same gradient, $-\frac{6}{4} = -\frac{3}{2}$, but $\frac{3}{6} = \frac{2}{4} \neq \frac{5}{16}$, so the lines are parallel (not coincident) and do not intersect. Therefore there is no solution to the pair of equations.

Algebraically: $3x + 2y - 5 = 0$ (1)

$6x + 4y - 16 = 0$ (2)

$2 \times (1)$: $6x + 4y - 10 = 0$ (3)

$(2) - (3)$: $-6 = 0$

This is obviously false, so there is no solution to the pair of simultaneous equations.

(b) The two lines have the same gradient, $-\frac{6}{4} = -\frac{3}{2}$, and $\frac{3}{6} = \frac{2}{4} = \frac{5}{10}$, so the lines are coincident. Therefore the lines intersect at all points on the line $3x + 2y - 5 = 0$. Some of the points of intersection include $(1, 1)$, $(0, 2.5)$, $(-1, 4)$.

EXERCISE 6.3 INTERSECTION OF TWO LINES

1 The coordinates of the intersection point of the lines $x + 2y - 3 = 0$ and $2x - 2y - 6 = 0$ are:

A $(0, 3)$ **B** $(0, -3)$ **C** $(3, 0)$ **D** $(-3, 0)$

2 Find the equation of the line that contains the intersection point of the lines $2x + 5y - 19 = 0$ and $3x - 4y + 6 = 0$ and is parallel to the line with equation $4x - y - 8 = 0$.

3 The coordinates of the vertices A, B and C of a triangle are $(-1, 3)$, $(2, 5)$ and $(1, -1)$ respectively. Find:

(a) the equation of the perpendicular line from A to BC

(b) the equation of the line through B, parallel to this perpendicular line.

4 Consider the equations $x + y + 1 = 0$, $y = 2 - x$, $3y = 2x + 1$ and $2x - 3y + 6 = 0$.

(a) Show that these equations represent the sides of a parallelogram.

(b) Find the coordinates of the vertices of the parallelogram.

(c) Find the equations of the diagonals of the parallelogram.

5 Find the equation of the straight line that contains the intersection point of the lines $3x + 2y - 12 = 0$ and $5x - y - 7 = 0$ and that:

(a) passes through the point $(-4, -5)$ **(b)** is parallel to the line $2x - y + 4 = 0$

(c) is perpendicular to the line $y = 5$.

6 $ABCD$ is a parallelogram. The coordinates of A, B and C are $(-1, 4)$, $(4, 6)$ and $(2, 7)$ respectively. Indicate whether each statement is correct or incorrect.

(a) AD is $x + 2y - 7 = 0$ **(b)** CD is $2x - 5y + 31 = 0$

(c) gradient of $AB = 0.5$ **(d)** D is $(-3, 5)$

7 $ABCD$ is a quadrilateral. The coordinates of A, B and C are $(-8, 6)$, $(2, 4)$ and $(5, -7)$ respectively. If the diagonals are perpendicular and DC is parallel to the x-axis, find:

(a) the coordinates of D **(b)** the coordinates of the intersection point of the diagonals.

8 The equations of the sides of a triangle ABC are as follows:

AB: $5x + y - 10 = 0$; BC: $3x - 2y - 6 = 0$; CA: $x - 5y + 24 = 0$

(a) Show that angle BAC is a right angle.

(b) Find the coordinates of the foot of the perpendicular from A to BC.

9 Without actually solving the simultaneous equations, state whether the following pairs of lines intersect, are parallel or coincide.

(a) $2x - 3y - 8 = 0$, $4x - 6y - 16 = 0$

(b) $x + 3y + 7 = 0$, $2x + 7y + 16 = 0$

(c) $6x - 5y - 24 = 0$, $9x - 4y - 22 = 0$

(d) $x + y - 7 = 0$, $x + y - 8 = 0$

6.4 REGIONS AND INEQUALITIES

A straight line represents a function. If the equation of a straight line is changed to an inequality, then the function becomes a relation. It is no longer a straight line, but instead it can be represented graphically by a region in the number plane.

Regions can be graphed on the number plane by first graphing the equation of the region's boundary. Then, select a point not on the boundary. Substitute the coordinates of this point into the equation of the boundary: if the point makes the inequality true, then all points on that side of the inequality will also make it true. Shade that side of the boundary to indicate the region. (If the point does not make the inequality true, then the points on the other side of the inequality must, so shade that side instead.)

If the inequality includes '… or equal to', then the boundary is part of the region. If the inequality does not include '… or equal to', then the boundary is not part of the region, so it should be dashed to show this.

Example 10

Graph the region in the number plane represented by each inequality.

(a) $x + y \geq 1$ **(b)** $x + y < 1$ **(c)** $x + y \leq 1$ **(d)** $x + y > 1$

Solution

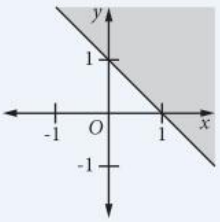
In each case, first draw the line $x + y = 1$. In parts **(b)** and **(d)**, dash the line to show that the boundary is not part of the region. Substitute the non-boundary point $(1, 1)$ into each inequality to find the region side.

(a) $x + y \geq 1$

Solid line

LHS $= 1 + 1 = 2 > 1$

Result true:
shade above the line

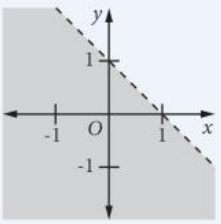


(b) $x + y < 1$

Dashed line

LHS $= 1 + 1 = 2 > 1$

Result not true:
shade below the line

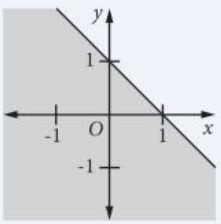


(c) $x + y \leq 1$

Solid line

LHS $= 1 + 1 = 2 > 1$

Result not true:
shade below the line

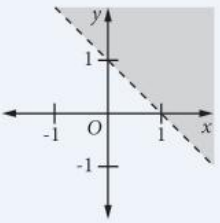


(d) $x + y > 1$

Dashed line

LHS $= 1 + 1 = 2 > 1$

Result true:
shade above the line



The regions in **(a)** and **(b)** together make the whole number plane, as do the regions in **(c)** and **(d)** together.

The boundary line divides the number plane into three sets of points: the points on the line, the points above the line and the points below the line. (If the inequality includes '… or equal to', then the boundary is part of the region; if the inequality does not include '… or equal to', then the boundary is not part of the region, so it is dashed.)

Thus the region in part **(a)** could be described as 'the set of points on or above the line with equation $x + y = 1$'. The region in part **(b)** could be described as 'the set of points below the line with equation $x + y = 1$'.

Example 11

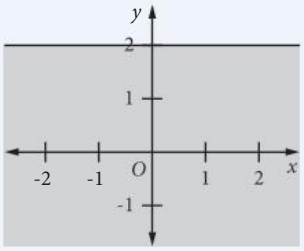
Graph the region in the number plane represented by each inequality.

(a) $y \leq 2$ **(b)** $-1 < y \leq 2$ **(c)** $x > 1$ **(d)** $x > 1$ or $x \leq -1$

Solution

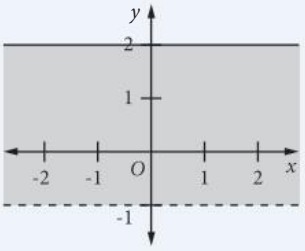
These inequalities have only one variable, so their boundaries are either horizontal lines (as in **(a)** and **(b)**) or vertical lines (as in **(c)** and **(d)**).

(a) $y \leq 2$



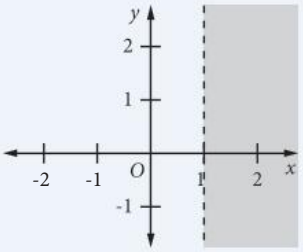
Region is below the line

(b) $-1 < y \leq 2$



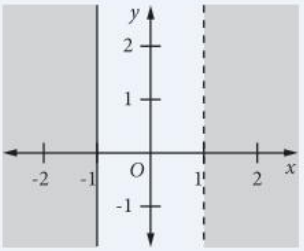
Region is between the lines

(c) $x > 1$



Region is to the right of the line

(d) $x > 1$ or $x \leq -1$



Region is outside the lines

EXERCISE 6.4 REGIONS AND INEQUALITIES

Shade the region represented by each inequality.

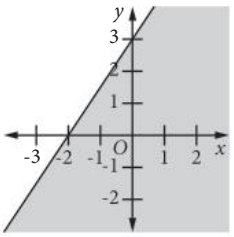
1 $y > 2x$ **2** $y < x + 1$ **3** $x \leq 3$ **4** $y \geq 1$

5 $y \leq x + 2$ **6** $2x + 3y \geq 6$ **7** $2x + y < 1$ **8** $0 \leq x \leq 2$

9 $-2 < y \leq 3$ **10** $3x - 4y \leq 6$ **11** $x - y < -2$ **12** $2y - 5x < 10$

13 $x \leq 1.5$ **14** $\frac{x}{2} + \frac{y}{3} < 1$ **15** $-3 < x + y < 2$ **16** $0 \leq x - y < 3$

17 Which inequality defines the shaded region?



A $3x - 2y + 6 \geq 0$
B $3x - 2y + 6 \leq 0$
C $3x + 2y - 6 \geq 0$
D $3x + 2y - 6 \leq 0$

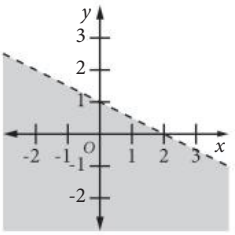
18 Shade the region in the number plane that:

(a) is above the line $y = 3x + 2$ **(b)** is on or below the line $x - 2y + 4 = 0$
(c) is above the line $y = -1$ **(d)** is on or to the left of the line $x = 3$
(e) is between the lines $2x + 3y + 6 = 0$ and $2x + 3y - 6 = 0$ **(f)** is on or above the x-axis.

19 Describe in words the regions defined by the following inequalities:

(a) $y < x + 2$ **(b)** $y \geq x$ **(c)** $x > 3$ **(d)** $y \leq 4$ **(e)** $x + 3y \leq 9$ **(f)** $-2 \leq x < 3$

20 For this graph, indicate whether each statement is correct or incorrect.



(a) The equation of the boundary is $x + 2y - 2 = 0$.
(b) The gradient of the boundary line is $\frac{1}{2}$.
(c) The inequality for the region is $x + 2y - 2 > 0$.
(d) The inequality for the region is $x + 2y - 2 < 0$.

6.5 SIMULTANEOUS LINEAR INEQUALITIES

Two linear equations may intersect at a point. The intersection of two linear inequalities is the region common to the two inequalities. This is the region where the inequalities hold simultaneously.

Example 12

(a) Sketch the region given by $y \geq x$. **(b)** Sketch the region given by $x + y < 2$.

(c) Sketch the region common to $y \geq x$ and $x + y < 2$.

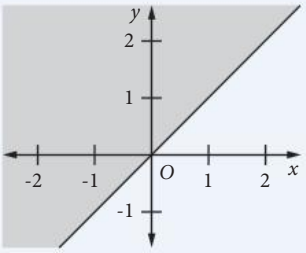
Solution

(a) $y \geq x$

Test the point $(0, 1)$

LHS $= 1 > 0 >$ RHS

Shade region above line

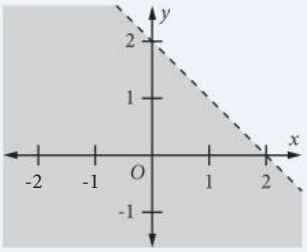


(b) $x + y < 2$

Test the point $(0, 0)$

LHS $= 0 + 0 = 0 < 2 <$ RHS

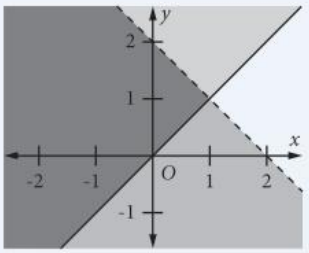
Shade region below dashed line



(c) $y \geq x$ and $x + y < 2$

Identify common region

Shade clearly, using a darker shading for the common region



Example 13

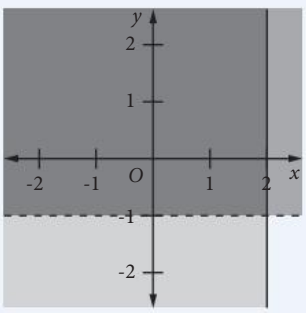
Sketch the region defined by each pair of inequalities. Describe the region in words.

(a) $x \leq 2, y > -1$ **(b)** $y \leq x + 1, x \leq 1$

Solution

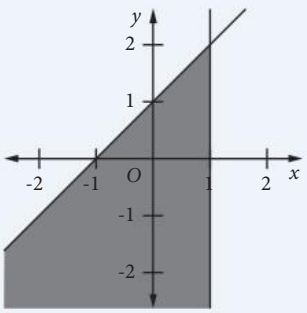
Draw each boundary and shade the two regions, then shade the common region differently.

(a) $x \leq 2, y > -1$



The region on and to the left of the line $x = 2$ that is also above the line $y = -1$.

(b) $y \leq x + 1, x \leq 1$



The region on and below the line $y = x + 1$ that is also on and to the left of the line $x = 1$.

If the shading of different regions becomes difficult to show, you should just lightly shade the original regions before darkening the final answer, as in part **(b)**.

Example 14

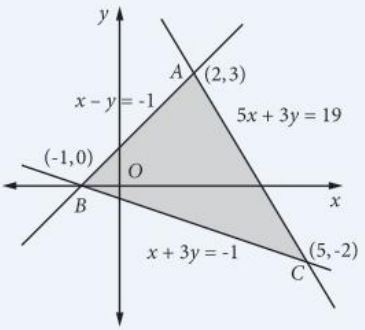
Sketch the region defined by the three inequalities $x - y \geq -1$, $x + 3y \geq -1$, $5x + 3y \leq 19$. Show the points of intersection of the lines. Describe the region in words.

Solution

To find the points of intersection of the lines, solve pairs of equations simultaneously:

- The lines $x - y = -1$ and $5x + 3y = 19$ intersect at $A(2, 3)$
- The lines $x - y = -1$ and $x + 3y = -1$ intersect at $B(-1, 0)$
- The lines $5x + 3y = 19$ and $x + 3y = -1$ intersect at $C(5, -2)$

The shaded region is the interior of the triangle bounded by the lines $x - y = -1$, $x + 3y = -1$ and $5x + 3y = 19$. The vertices of the region are the intersection points A, B and C.



TECHNOLOGY EXPLORATION

Regions in the plane

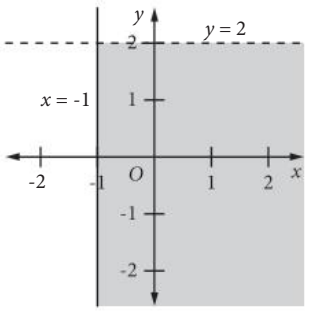
To display regions in a plane, be sure to use the latest version of GeoGebra.

1. In GeoGebra, display an inequality such as $(x - 3)^2 + y^2 > 1$ by entering *(x-3)^2+y^2<1* in the input bar. (Note that you can enter ≥ and ≤ symbols using the α symbol at the end of the input bar.)
2. To sketch a region from overlapping inequalities, enter both functions into the input bar and see where the regions overlap.

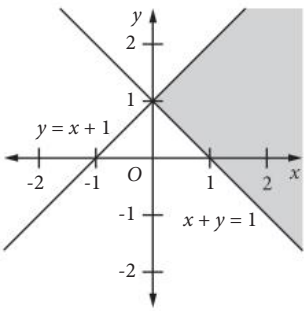
EXERCISE 6.5 SIMULTANEOUS LINEAR INEQUALITIES

1 Describe the shaded region in each diagram using both words and inequalities.

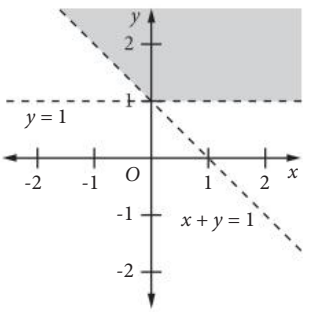
(a)



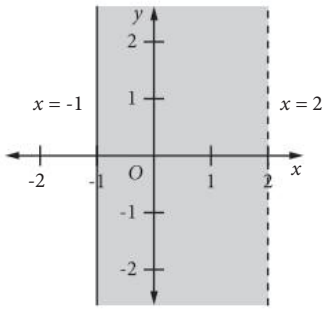
(b)



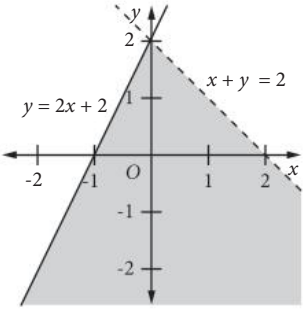
(c)



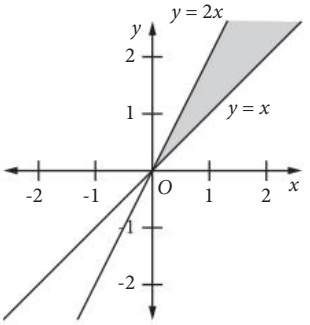
(d)



(e)

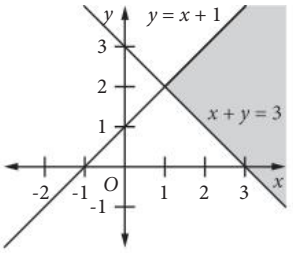


(f)



2 Which of the following points is in the shaded region?

A $(1,3)$ **B** $(1,1)$
C $(3,1)$ **D** $(-1,3)$



3 What are the inequalities that define the region graphed in question **2**?

4 Graph the regions defined by each set of inequalities. State whether each of the given points is in the region.

(a) $x + y \le 3, y \le x$
$(0,0), (2,3), (-1,-2)$

(b) $2y > x + 2, x + y > -1$
$(0,0), (0,1), (2,5)$

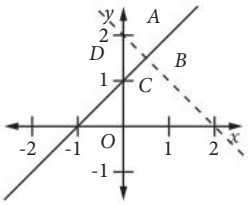
(c) $x + 2y \ge 8, y < 7$
$(0,4), (-1,1), (9,2)$

(d) $3y \le 2x + 6, x + y > 2$
$(2,0), (3,3), (4,-1)$

(e) $4x + y \le 4, x \ge -2$
$(0,0), (-3,1), (1,0)$

(f) $y > 3x + 3, x + y < 3$
$(0,3), (2,7), (-1,4)$

5 The two lines divide the number plane into four regions labelled on the diagram A, B, C and D. Indicate whether the inequalities given for each region are correct or incorrect.



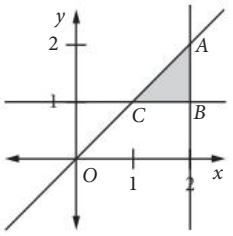
(a) $A: x + y > 2, x - y \le -1$
(b) $B: x + y > 2, x - y \le -1$
(c) $C: x + y < 2, x - y \ge -1$
(d) $D: x + y < 2, x - y \le -1$

6 Shade the region defined by the inequalities. By solving the inequalities simultaneously in pairs, find the coordinates of the vertices of the figure.

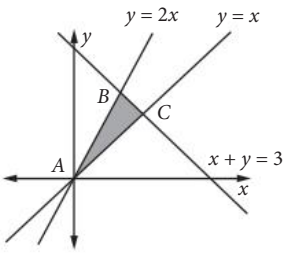
(a) $y \le x + 2, 2x + y \le 4, x + y \ge 2$
(b) $2y - x \le 4, y > 3x - 6, 3x + y > -6$
(c) $y - 2x < 4, y + 2x < 6, y \ge x + 2$
(d) $y - 3x < 3, 3x + 4y < 12, x - 2y < 4$
(e) $y \ge x - 1, x + y \le 2, x \ge 0, y \ge 0$
(f) $x + y \le 2, x + y \ge -2, x - y \le 2, x - y \ge -2$

7 Describe (in words and using inequalities) the regions that are shaded in each part. Find the coordinates of the points A, B and C in each case by solving simultaneously.

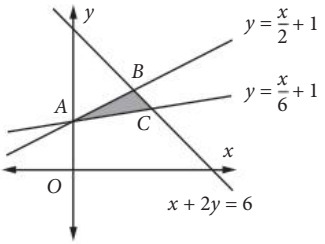
(a)



(b)



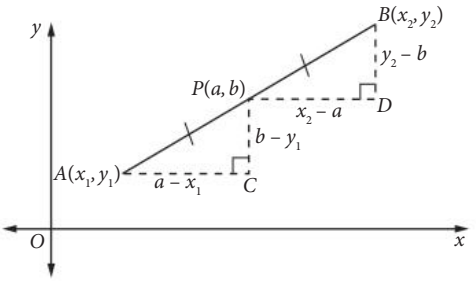
(c)



6.6 MIDPOINT OF AN INTERVAL AND DISTANCE BETWEEN TWO POINTS

Midpoint of an interval

Let $P(a,b)$ be the midpoint of the interval joining $A(x_1,y_1)$ and $B(x_2,y_2)$.



$\Delta APC \equiv \Delta PBD$

$\therefore AC = PD$ and $PC = BD$

$\therefore a - x_1 = x_2 - a$ $\quad\quad$ $\therefore b - y_1 = y_2 - b$

$2a = x_1 + x_2$ $\quad\quad$ $2b = y_1 + y_2$

$a = \dfrac{x_1 + x_2}{2}$ $\quad\quad$ $b = \dfrac{y_1 + y_2}{2}$

The coordinates of the midpoint of the interval joining (x_1, y_1) and (x_2, y_2): $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

Example 15

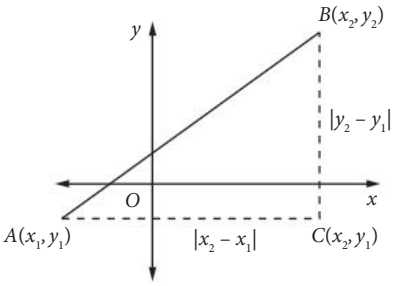
Find the coordinates of the midpoint of the interval joining the points $(-4, 2)$ and $(5, -7)$.

Solution

Coordinates are: $\left(\frac{-4+5}{2}, \frac{2-7}{2}\right)$, i.e. $\left(\frac{1}{2}, -2\frac{1}{2}\right)$

Distance between two points

Given the points $A(x_1, y_1)$ and $B(x_2, y_2)$, construct ΔABC as shown in the diagram with AC parallel to the x-axis and BC parallel to the y-axis. The coordinates of C are (x_2, y_1).



$AC = |x_2 - x_1|$, $BC = |y_2 - y_1|$, where the absolute value notation is used to make sure that the lengths are positive.

Using Pythagoras' theorem for ΔABC:

$$AB^2 = AC^2 + BC^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The distance d between two points (x_1, y_1) and (x_2, y_2) is $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

Example 16

Calculate the length of AB given $A(-2, 3)$ and $B(6, -1)$.

Solution

$A(-2, 3)$, $B(6, -1)$:

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$AB = \sqrt{(6 - -2)^2 + (-1 - 3)^2}$$

$$= \sqrt{8^2 + (-4)^2}$$

$$= \sqrt{64 + 16}$$

$$= \sqrt{80} = 4\sqrt{5}$$

EXERCISE 6.6 MIDPOINT OF AN INTERVAL AND DISTANCE BETWEEN TWO POINTS

1 Find the coordinates of the midpoint of the interval joining the points:

(a) $(5, 1)$ and $(-2, -3)$ **(b)** $(8, 8)$ and $(-4, 6)$ **(c)** $(-2, -7)$ and $(-6, -9)$

(d) $(-3, 5)$ and $(5, -3)$ **(e)** (a, b) and (c, d) **(f)** $(a + b, a)$ and $(a - b, b)$

2 The coordinates of the midpoint of the line joining $A(3, -2)$ and B are $(-4, 6)$. Find the coordinates of B.

3 P is the midpoint of the interval AB. Find the value of the unknown coordinates in each case.

(a) $A(3,6)$, $B(a,b)$, $P(9,2)$ **(b)** $A(c,d)$, $B(6,3)$, $P(1,1.5)$

(c) $A(e,-3)$, $B(-6,f)$, $P(0,3.5)$ **(d)** $A(2g,h)$, $B(4g,2h)$, $P(6,3)$

4 Find the distance between each pair of points.

(a) $(4,6)$ and $(-5,6)$ **(b)** $(5,1)$ and $(5,-4)$ **(c)** $(-5,-3)$ and $(3,-1)$

(d) $(7,-2)$, $(-6,5)$ **(e)** $(a,2a)$, $(-2a,6a)$ **(f)** (a,b), (b,a)

5 The perimeter of the triangle whose vertices are the points $(5,1)$, $(8,5)$ and $(-3,7)$ is:

A $20\sqrt{5}$ units **B** $5\sqrt{10}$ units **C** $\left(15+2\sqrt{29}\right)$ units **D** $\left(15+5\sqrt{5}\right)$ units

6 Given the points $D(-3,4)$, $E(0,2)$ and $F(-2,-1)$:

(a) use the distance formula to show that ΔDEF is right-angled

(b) use gradients to show that ΔDEF is right-angled.

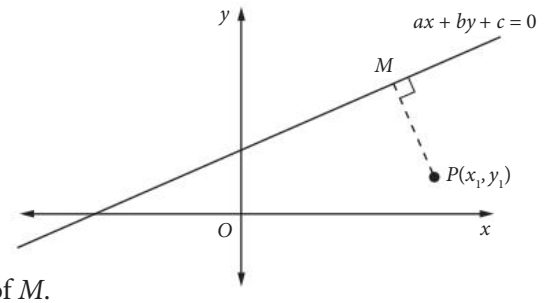
7 The vertices of a triangle are $A(2,5)$, $B(7,-3)$ and $C(-2,1)$. Find the coordinates of the midpoints of AC and AB, labelled D and E respectively. Show that DE is parallel to CB.

6.7 PERPENDICULAR DISTANCE OF A POINT FROM A LINE

The perpendicular distance of a fixed point $P(x_1, y_1)$ from a line with equation $ax + by + c = 0$ is given by the formula:

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}} \quad \text{or} \quad d = \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$$

There are a number of ways to prove this formula. One method is as follows:



1. Write the equation of PM.
2. Solve the simultaneous equations to find the coordinates of M.
3. Use the distance formula to find the length of PM.

Proof

The equation of the line through P and perpendicular to $ax + by + c = 0$ is:

$$bx - ay = bx_1 - ay_1 \qquad (1)$$

Given the line: $ax + by = -c \qquad (2)$

$b \times (1)$: $b^2x - aby = b^2x_1 - aby_1 \qquad (3)$

$a \times (2)$: $a^2x + aby = -ac \qquad (4)$

(3) + (4): $(a^2 + b^2)x = b^2x_1 - aby_1 - ac$

If $M(x_0, y_0)$ is the point of intersection: $x_0 = \dfrac{b^2x_1 - aby_1 - ac}{a^2 + b^2}$

Substitute x_0, y_0 into (2): $\dfrac{a\left(b^2x_1 - aby_1 - ac\right)}{a^2 + b^2} + by_0 = -c$

$$by_0 = \frac{-c\left(a^2 + b^2\right) - a\left(b^2x_1 - aby_1 - ac\right)}{a^2 + b^2}$$

$$by_0 = \frac{-ca^2 - cb^2 - ab^2x_1 + a^2by_1 + a^2c}{a^2 + b^2}$$

$$y_0 = \frac{-abx_1 + a^2y_1 - bc}{a^2 + b^2}$$

For the distance formula we need: $x_1 - x_0 = x_1 - \dfrac{b^2x_1 - aby_1 - ac}{a^2 + b^2}$

$$x_1 - x_0 = \frac{a^2x_1 + aby_1 + ac}{a^2 + b^2} = \frac{a(ax_1 + by_1 + c)}{a^2 + b^2}$$

and: $y_1 - y_0 = y_1 - \dfrac{-abx_1 + a^2y_1 - bc}{a^2 + b^2}$

$$y_1 - y_0 = \frac{abx_1 + b^2y_1 + bc}{a^2 + b^2} = \frac{b(ax_1 + by_1 + c)}{a^2 + b^2}$$

Hence: $(x_1 - x_0)^2 + (y_1 - y_0)^2 = \dfrac{a^2(ax_1 + by_1 + c)^2}{(a^2 + b^2)^2} + \dfrac{b^2(ax_1 + by_1 + c)^2}{(a^2 + b^2)^2}$

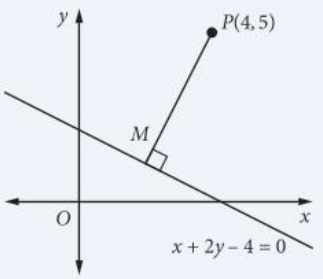
$$(x_1 - x_0)^2 + (y_1 - y_0)^2 = \frac{(ax_1 + by_1 + c)^2}{a^2 + b^2}$$

So the perpendicular distance is: $d = \dfrac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$

The sign of this distance (before the absolute value is taken) indicates which side of the line the point is on. Points with a positive distance are on one side of the line, while points with a negative distance are on the other side of the line.

Example 17

Find the distance from the point $(4, 5)$ to the line with equation $x + 2y - 4 = 0$.



Solution

Use the formula: $d = \dfrac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$

$$PM = \frac{|1 \times 4 + 2 \times 5 - 4|}{\sqrt{1^2 + 2^2}}$$

$$= \frac{|10|}{\sqrt{5}}$$

$$= 2\sqrt{5}$$

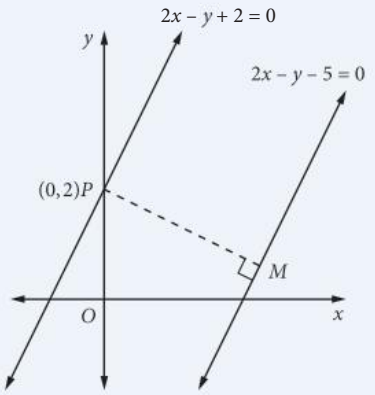
The exact distance is $2\sqrt{5}$ units.

Alternative method:

1. Because PM is perpendicular to the given line, its equation is of the form $2x - y + k = 0$.
   Substitute $M(4, 5)$ into this equation: $8 - 5 + k = 0$, $\therefore\ k = -3$
   The equation of PM is $2x - y - 3 = 0$.
2. Solve simultaneously the equations $2x - y - 3 = 0$ and $x + 2y - 4 = 0$.
   This gives $x = 2$, $y = 1$, so the coordinates of M are $(2, 1)$.
3. Use the distance formula: $PM = \sqrt{(2-4)^2 + (1-5)^2} = \sqrt{20} = 2\sqrt{5}$

Example 18

Find the (perpendicular) distance between the parallel lines $2x - y + 2 = 0$ and $2x - y - 5 = 0$.



Solution

Take any point on either line, e.g. $(0,2)$ on $2x - y + 2 = 0$.

Find the distance PM from this point $(0,2)$ to the other line $2x - y - 5 = 0$:

$$PM = \frac{|2\times 0 + (-1)\times 2 - 5|}{\sqrt{2^2 + 1^2}}$$

$$= \frac{|-7|}{\sqrt{5}} = \frac{7}{\sqrt{5}}$$

The perpendicular distance between the lines is $\frac{7\sqrt{5}}{5}$ units.

EXERCISE 6.7 PERPENDICULAR DISTANCE OF A POINT FROM A LINE

1 What is the perpendicular distance from the point $(2,1)$ to the line $3x + 4y + 5 = 0$?

A $\frac{15}{\sqrt{5}}$ units **B** 3.6 units **C** 3 units **D** $\frac{16}{\sqrt{5}}$ units

2 Find the distance between the point and the given line:

(a) $(1,4)$, $3x - 4y - 12 = 0$ **(b)** $(2,3)$, $7x - 24y + 8 = 0$
(c) $(5,6)$, $y + 2 = 0$ **(d)** $(-5,-2)$, $x - 3y - 6 = 0$
(e) $(1,-12)$, $5x - 12y + 20 = 0$ **(f)** $(0,0)$, $x + y - 4 = 0$
(g) $(-2,-3)$, $2y - x - 6 = 0$ **(h)** $(-2,-2)$, $4x - 3y = 0$

3 Find the (perpendicular) distance between each pair of parallel lines.

(a) $3x - 4y - 4 = 0$, $3x - 4y + 16 = 0$ **(b)** $7x - 24y + 8 = 0$, $7x - 24y + 58 = 0$ **(c)** $x + y + 1 = 0$, $x + y - 5 = 0$

4 $A(-2,-3)$, $B(0,3)$ and $C(2,4)$ are the vertices of ΔABC. Calculate the perpendicular distance from the vertex A to the side BC. (This is known as the altitude from A.)

5 $P(4,2)$, $Q(-8,-7)$ and $R(0,4)$ are the vertices of ΔPQR. Calculate the perpendicular distance from the point R to the side PQ.

6 $A(1,2)$ is a vertex of ΔABC. The equation of BC is $5x - 12y - 7 = 0$. If M is the foot of the perpendicular from A to BC, and $BC = 9$ units, indicate whether each of the following statements is correct or incorrect.

(a) $AM = \frac{|5 - 24 - 7|}{13}$ **(b)** $AM = \frac{|5 - 24 - 7|}{\sqrt{13}}$ **(c)** $AM = 2$ units **(d)** Area $\Delta ABC = 9$ units^2

7 Calculate the area of the triangle whose vertices are $(-1,-1)$, $(1,2)$ and $(2,1)$.

CHAPTER REVIEW 6

1 Given $A(1,5)$, $B(2,4)$, $C(-2,3)$ and $D(3,-2)$, find:

(a) the gradients of AB, AC, AD, BC, BD, CD
(b) which lines are parallel
(c) the lengths of AB, CD, AD
(d) the midpoints of AD, BC, BD.

2 $P(2,3)$, $Q(6,-1)$, $R(-4,-5)$ are the vertices of ΔPQR. M is the midpoint of PQ and N is the midpoint of PR.

(a) Find the coordinates of M and N.
(b) Show that $MN \parallel QR$.
(c) Calculate the length of QR.
(d) Show that the length of MN is half the length of QR.

3 Find the perpendicular distance from the point $(2,3)$ to the line $6x + 8y - 5 = 0$.

4 Find the equation of the line that is perpendicular to the line $3x + 4y = 5$ and also passes through the midpoint of the line segment joining the points $(3,-2)$ and $(5,8)$.

5 Find the coordinates of the intersection points of the lines $5x - 3y - 19 = 0$, $3x + 5y + 9 = 0$ and $x - 4y + 3 = 0$. Show that these intersection points are the vertices of a right-angled triangle.

6 Calculate the perpendicular distance of the point $(3,4)$ from the line $x + y - 6 = 0$.

7 Find the coordinates of the point A on the line $x = -3$ such that the line joining A to $B(3,5)$ is perpendicular to the line $2x + 5y - 12 = 0$.

8 $A(1,4)$, $B(0,3)$, $C(-4,6)$ are the vertices of ΔABC. Calculate the distance of A from the side BC and hence find the area of ΔABC.

CHAPTER 7
FUNCTIONS AND RELATIONS

Outcomes covered:

P1 demonstrates confidence in using mathematics to obtain realistic solutions to problems

P4 chooses and applies appropriate arithmetic, algebraic, graphical, trigonometric and geometric techniques

P5 understands the concept of a function and the relationship between a function and its graph

7.1 FUNCTIONS AND RELATIONS

A function is a type of mathematical object that precisely describes a relationship between variables.

Definitions

- A real function f of a real variable x assigns to each element x of a given set of real numbers **exactly one** real number y, which is called the value of the function f at x. The dependence of y on f and x is made explicit by the notation $f(x)$, which means the value of f at x.
- The set of real numbers x on which f is defined is called the **domain** of f, while the set of values $f(x)$ obtained as x varies over the domain of f is called the **range** or **image** of f.
- The variable x is called the **independent variable**, as it may be chosen freely from the domain of f, while y is called the **dependent variable**, as its value depends on the particular value chosen for x.
- A **function** may also be defined as a set of ordered pairs with the special property that no two pairs have the same first element (x value).

Finding the value of a function

Consider a function defined by the rule $f(x) = 2x - 7$. What is the value of the function when $x = 4$, $x = 0$ and $x = -1$? In the past we would have written $x = 4$, $y = 8 - 7 = 1$. With function notation, we can simply write $f(4) = 8 - 7 = 1$. Thus: $f(4) = 1, f(0) = 0 - 7 = -7, f(-1) = -2 - 7 = -9$.

The function notation $f(x)$ allows us to write, in a single statement, the value of the independent variable as well as the corresponding dependent variable.

Example 1

Plot the following points on a number plane: $(-1, 2), (0, 1), (1, 0), (2, 3), (3, -2), (4, 0)$.

Does this set of points represent a function? Write its domain and range.

Solution

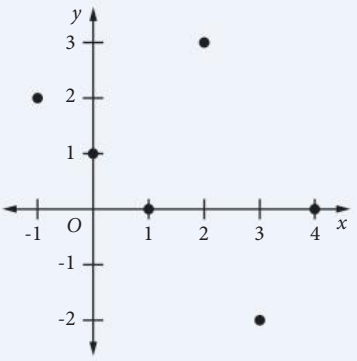
The graph shows that for each first value, x, there is only one second value, y.

This means the set of points is a function:

Domain = {-1, 0, 1, 2, 3, 4}

Range = {-2, 0, 1, 2, 3}

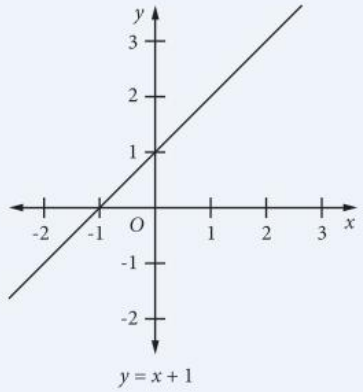
Note that 0 is the y value for two points, but it only needs to be listed once in the range.



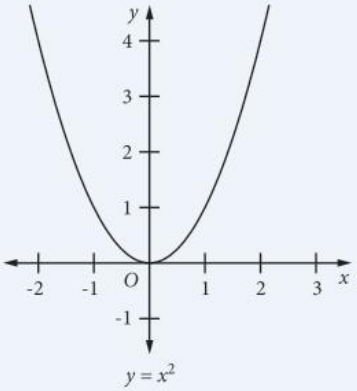
Example 2

Determine whether each graph represents a function. Write the domain and range for each.

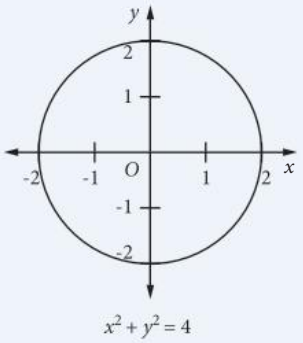
(a)



(b)



(c)



Solution

(a) Function

Domain: real numbers

Range: real numbers

Hence we can write $f(x) = x + 1$

(b) Function

Domain: real numbers

Range: real numbers, $y \geq 0$

Hence we can write $g(x) = x^2$

(c) Not a function

Domain: real numbers, $-2 \leq x \leq 2$

Range: real numbers, $-2 \leq x \leq 2$

In parts **(a)** and **(b)** the functions are different, so they have been given different labels, f and g.

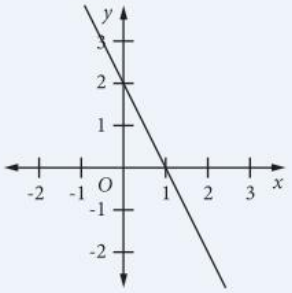
Vertical line test

A simple way to determine whether a graph represents a function is to draw vertical lines. If each vertical line cuts the graph only once, then the graph represents a function. If any of the vertical lines cut the graph more than once, then the graph does not represent a function. The x values for which vertical lines do not cut the graph are not in the domain.

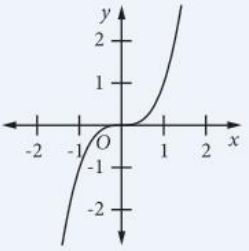
Example 3

Use the vertical line test to show that each graph represents a function. Write the domain and range for each.

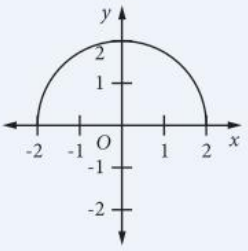
(a) $f(x) = 2 - 2x$



(b) $g(x) = x^3$



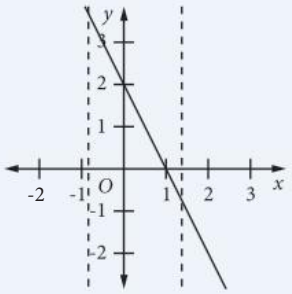
(c) $h(x) = \sqrt{4 - x^2}$



Solution

Draw several vertical lines on each graph.

(a)

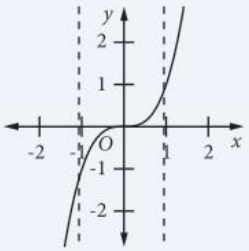


Vertical line can only cut once.

Domain: real values of x

Range: real values of x

(b)

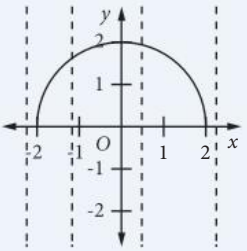


Vertical line can only cut once.

Domain: real values of x

Range: real values of x

(c)



Vertical line can only cut once, between $x = -2$ and $x = 2$.

Domain: $-2 \leq x \leq 2$

Range: $0 \leq y \leq 2$

Relations

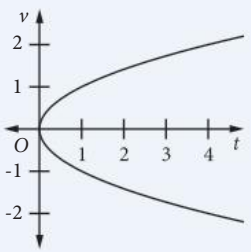
The simplest definition of a relation is that it is a multi-valued function: it is a set of ordered pairs, usually defined by some rule, but each first member of the ordered pairs can have more than one second member. If the semicircle in Example **2(c)** were a complete circle instead, then it would not be a function, but it would be a relation.

The language of functions also applies to relations, so the terms independent variable, dependent variable, domain and range have the same meaning.

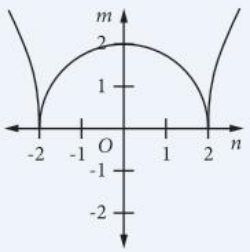
Example 4

Determine whether each graph represents a function or relation. Write the domain and range for each.

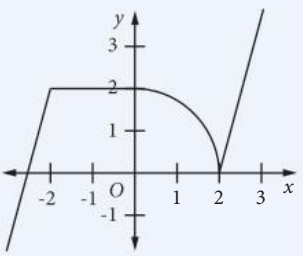
(a)



(b)



(c)



Solution

(a) Relation

Independent variable is t

Domain: real t, $t \geq 0$

Range: real numbers

(b) Function

Independent variable is n

Domain: real numbers

Range: real m, $m \geq 0$

(c) Function

Independent variable is x

Domain: real numbers

Range: real numbers

Function rules

When a function rule f is given and a domain is not specified, it is assumed that the domain of the function is the set of real numbers for which $f(x)$ defines a real number range. To find the domain, the solution of an inequality may be needed.

When the domain of f is all values of x over an interval, the graph of $y = f(x)$ is called the **curve** $y = f(x)$ and a part of the curve between two points is called an **arc**.

Example 5

State the largest possible domain for the function defined by the given rule. What is the range of each function?

(a) $f(x) = x^2$ (b) $f(x) = \frac{1}{x}$ (c) $f(x) = \sqrt{x}$ (d) $f(x) = \sqrt{4 - x^2}$ (e) $f(x) = \frac{x}{x^2 - 1}$

Solution

(a) Any real number squared is also a real number, so the domain of $f(x) = x^2$ is all real numbers.

Any real number squared is never negative, so the range of the function is all positive real numbers and zero.

(b) Fractions are not defined for a denominator of zero, so $\frac{1}{x}$ is defined for all values of x except $x = 0$.

Thus the domain of $f(x) = \frac{1}{x}$ is all real numbers except $x = 0$. We can write $f(x) = \frac{1}{x}, x \neq 0$.

Because the numerator of $f(x)$ is never zero, we have $f(x) \neq 0$. The reciprocal of every non-zero real number is another non-zero real number, so the range of the function is all real numbers except zero.

(c) Only the square roots of non-negative numbers are real, so the domain of $f(x) = \sqrt{x}$ is real x, $x \geq 0$.

The square root of zero is zero and the square root of a positive real number is another positive real number, so the range of the function is all positive real numbers and zero: $f(x) \geq 0$.

(d) For the value of $f(x) = \sqrt{4 - x^2}$ to be real, $4 - x^2 \geq 0$, so $-2 \leq x \leq 2$. Therefore the domain of $f(x) = \sqrt{4 - x^2}$ is real x, $-2 \leq x \leq 2$ (or $|x| \leq 2$).

When $x = 0$, the value of the function is $f(0) = 2$; also, $f(2) = 0$ and $f(-2) = 0$. For all other values of x in the domain, $0 < f(x) < 2$, so the range of the function is the real numbers $0 \leq f(x) \leq 2$.

(e) The function is not defined when the denominator is zero, i.e. when $x^2 - 1 = 0$. This is true for $x = \pm 1$, so the domain of $f(x) = \frac{x}{x^2 - 1}$ is real x, $x \neq \pm 1$.

$f(0) = 0$, and for all values of x in the domain the function exists. The range of the function is the set of real numbers.

TECHNOLOGY EXPLORATION

Investigation of domain and range

The domain of $f(x)$ is the set of real numbers for which the expression $f(x)$ defines a real number. You can visualise domains from your understanding that:

(a) The square root of any negative number is undefined for real numbers.

(b) $\frac{1}{0}$ is undefined.

For example:

(a) $f(x) = \sqrt{1 - x^2}$

This means that $1 - x^2 \geq 0$, thus $-1 \leq x \leq 1$.

By entering *y=sqrt(1-x^2)* into GeoGebra's input bar, you can clearly see the domain (and also the range).

(b) $f(x) = \frac{x}{1 - x^2}$

This means that $1 - x^2 \neq 0$, thus $x \neq \pm 1$.

By entering *y=x/(1-x^2)* into GeoGebra's input bar, you can clearly see the domain (and also the range). This can be highlighted further by entering *asymptote[x/(1-x^2)]* into the input bar.

A function may be defined over its domain by several different rules.

Example 6

A function is defined as $f(x)=\begin{cases} x^2, & x\le 1 \\ 2-x, & x>1 \end{cases}$. Find the domain and range of this function.

Solution

When $x\le 1, f(x)=x^2$ exists for all real values of x.

When $x>1, f(x)=2-x$ exists for all real values of x.

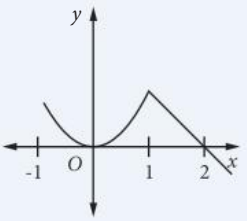
Thus the domain of the function is all real x.

When $x\le 1, f(1)=1, f(0)=0, f(-1)=1$ and $f(x)>0$ when $x\neq 0$.

When $x>1, f(1.01)=0.99. f(2)=0, f(3)=-1$ and for $x>2, f(x)<0$.

Thus the range of the function is all real numbers.

A sketch of the function shows this information more clearly.



EXERCISE 7.1 FUNCTIONS AND RELATIONS

1 Indicate whether each set of points is a function or relation. In each case write the largest possible domain and range.

(a) $\{(1,1),(2,2),(3,5),(3,7)\}$ **(b)** $\{(-3,1),(3,1),(8,7),(9,-2),(11,6)\}$

(c) $(x,5)$ for all real x **(d)** $(2,y)$ for all real y **(e)** $x^2+y^2=9$

(f) $y=\sqrt{16-x^2}$ **(g)** $y=x+2$ **(h)** $y=2-x^2$

2 Which statement is correct about the function $f(t)=t^2-9$?

A Domain is $t\ge 9$ **B** Range is $f(t)\le -9$ **C** Range is $f(t)\ge -9$ **D** Domain is $t\ge 0$

3 For each of the following functions, sketch the graph and state the largest possible domain and range.

(a) $f(x)=9-x$ **(b)** $g(x)=9-x^2$ **(c)** $h(x)=-\sqrt{4-x^2}$ **(d)** $f(t)=\sqrt{t}$

4 Consider the function defined by $f(x)=3x-6$ for all x.

(a) Find the value of $f(1), f(-2), f(a)$. **(b)** For what values of a is $f(a)=a$?

(c) For what values of x is $f(x)>x$? **(d)** Sketch the graph of f.

5 Consider the function $f(x)=x^2-1$, where x is any real number.

(a) Find $f(3), f(-3)$. **(b)** Find $f(a), f(b), f(a+b)$.

(c) Is $f(a)+f(b)=f(a+b)$ true for all a and b? **(d)** State the range of f.

6 For the function $g(x)=\sqrt{x}, x\ge 0$, state whether each statement is correct or incorrect.

(a) $g(1)=1$ **(b)** $g(9)=-3$ **(c)** $g(x^2)=x$ **(d)** $g(x+2)=\sqrt{x+2}$

7 Sketch each function and write the domain and range.

(a) $f(x)=\begin{cases} x+1 & \text{for } x\ge 1 \\ 2 & \text{for } x<1 \end{cases}$ **(b)** $f(x)=\begin{cases} \dfrac{1}{x+2} & \text{for } x\neq -2 \\ 4 & \text{for } x=-2 \end{cases}$

(c) $f(x)=\begin{cases} 2x & \text{for } x\ge 0 \\ -2x & \text{for } x<0 \end{cases}$ **(d)** $f(x)=\begin{cases} x & \text{for } x>1 \\ 1 & \text{for } -1\le x\le 1 \\ -x & \text{for } x<-1 \end{cases}$

8 State the largest possible domain and range for:

(a) $f(x)=\sqrt{x-2}$ (b) $f(x)=\sqrt{3-x}$ (c) $f(x)=\sqrt{x^2-9}$

(d) $g(x)=\frac{1}{x}$ (e) $h(t)=t^3$ (f) $g(k)=5-k^2$

9 A function is defined by the rule $f(x)=\begin{cases}1-x, & x\leq -2\\ -1, & -1<x<1\\ x+1, & x\geq 1\end{cases}$

Find the values (if they exist) for:

(a) $f(1)$ (b) $f(-1)$ (c) $f(0)$ (d) $f(2)+f(-2)$

10 A function is defined by the rule $f(x)=\begin{cases}\frac{1}{x}, & x<0\\ x, & x\geq 0\end{cases}$

Find:

(a) $f(0)$ (b) $f(2)$ (c) $f(-2)$ (d) $f(a^2)$

11 A function is defined by the rule $f(x)=\begin{cases}x-1, & x<1\\ \sqrt{x-1}, & x\geq 1\end{cases}$

Find:

(a) $f(1)$ (b) $f(-1)$ (c) $f(10)+f(-10)$ (d) $f(5)$

7.2 SKETCHING BASIC FUNCTIONS

You should be able to sketch simple linear functions from either the gradient-intercept form or the general form of the equation. You also need to be able to quickly and neatly sketch the power functions, such as $f(x)=x^2, f(x)=-x^2, f(x)=x^3, f(x)=-x^3, f(x)=x^4, f(x)=-x^4, f(x)=\frac{1}{x}$ and $f(x)=\frac{1}{x^2}$. You should already be familiar with many of these.

The following examples should also be graphed using GeoGebra or other graphing software. You may need to rewrite each equation in the form $y=f(x)$.

Example 7

Sketch each straight line. State the gradient and both axis intercepts of each.

(a) $y=2x+1$ (b) $2x+3y-6=0$ (c) $y=4-x$

Solution

(a) $y=2x+1$

Find the value of y for three different values of x: $(0,1), (2,5), (-1,-1)$

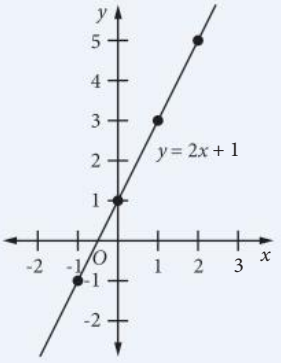
Plot these points on the number plane. Join them to obtain the line.

OR

From the form of the equation, recognise that the y-intercept is 1 and the gradient is 2.

Because the line passes through $(0,1)$ it also passes through $(1, 1+2=3)$ and $(2, 3+2=5)$. This is because the gradient is 2, which means that as x increases by 1, y increases by 2. Plot and join the points.

Gradient = 2, x-intercept = -0.5, y-intercept = 1.



(b) $2x + 3y - 6 = 0$

Find the value of y for three different values of x: $(0, 2)$, $(3, 0)$, $(-3, 4)$

Plot these points on the number plane. Join them to obtain the line.

OR

Rewrite the equation in the gradient-intercept form: $y = \frac{-2x}{3} + 2$

The gradient is a fraction, so this is not so convenient.

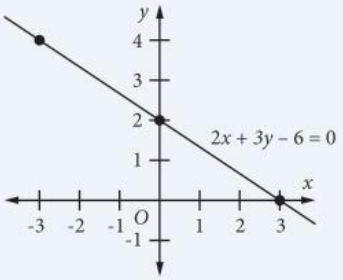
OR

Rewrite the equation by putting the constant term on the RHS of the equation and dividing by 6: the equation becomes $\frac{x}{3} + \frac{y}{2} = 1$.

This shows that the x-intercept is 3 and the y-intercept is 2. Draw a line through these intercept points to obtain the graph.

Because the line falls as x increases, the gradient is negative.

$$\text{Gradient} = -\frac{y\text{-intercept}}{x\text{-intercept}} = -\frac{2}{3},\ x\text{-intercept} = 2,\ y\text{-intercept} = 3.$$



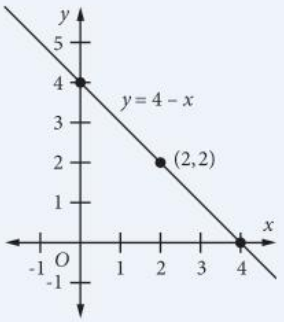
(c) $y = 4 - x$

From the equation:

gradient = -1, x-intercept = 4, y-intercept = 4.

Use this information to sketch the graph.

If you use this method, you should also find the coordinates of a third point to check that you haven't made a mistake, e.g. $(2, 2)$.



Example 8

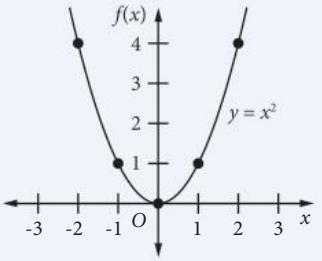
Sketch each function, showing any intercepts on the coordinate axes.

(a) $f(x) = x^2$ **(b)** $f(x) = -x^3$ **(c)** $f(x) = \frac{1}{x}$

Solution

(a) $f(x) = x^2$ is a type of curve called a parabola. Create a table of values and plot the points.

x	-2	-1	0	1	2
$f(x)$	4	1	0	1	4



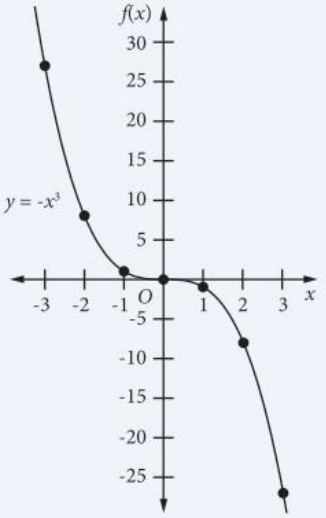
Note that $f(-x) = f(x)$. The curve is symmetrical about the y-axis. The curve passes through the point $(0, 0)$.

When $x > 0$, $f(x)$ increases as x increases, so we say that $f(x)$ is an increasing function for $x > 0$.

When $x < 0$, $f(x)$ decreases as x increases, so we say that $f(x)$ is a decreasing function for $x < 0$.

(b) $f(x) = -x^3$ is a type of curve called a cubic. Create a table of values and plot the points.

x	-3	-2	-1	0	1	2	3
$f(x)$	27	8	1	0	-1	-8	-27



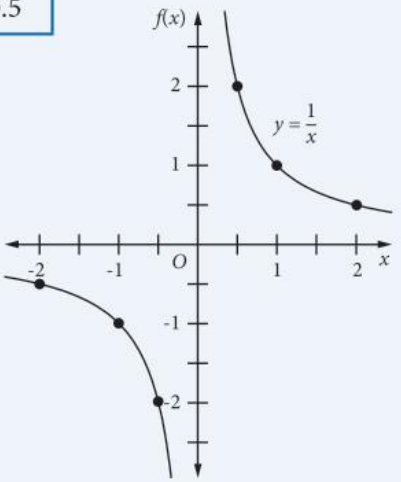
Note that $f(-x) = -f(x)$. The curve has rotational or point symmetry about the origin.

The curve passes through the point $(0, 0)$.

As x increases over the domain, the value of $f(x)$ decreases, so $f(x)$ is a decreasing function over its domain.

(c) $f(x) = \frac{1}{x}$ is a type of curve called a hyperbola. Create a table of values and plot the points.

x	-2	-1	-0.5	0	0.5	1	2
$f(x)$	-0.5	-1	-2	undefined	2	1	0.5



Note that $f(-x) = -f(x)$. The curve has rotational or point symmetry about the origin. Also note that $f(0)$ is undefined because $\frac{1}{0}$ does not exist.

The curve does not cut either axis.
As $x \to \pm\infty$, $f(x) \to 0$ and as $f(x) \to \pm\infty$, $x \to 0$.
The line $f(x) = 0$ is called a horizontal asymptote.
The line $x = 0$ is called a vertical asymptote.

When $x < 0$, $f(x)$ decreases as x increases, so we say that $f(x)$ is a decreasing function for $x < 0$.

When $x > 0$, $f(x)$ decreases as x increases, so we say that $f(x)$ is a decreasing function for $x > 0$.

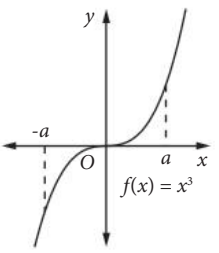
Thus $f(x)$ is a decreasing function over each part of its domain.

Odd and even functions

An **odd function** has the property that $f(-x) = -f(x)$. For example:

If $f(x) = x^3$
then $f(-x) = (-x)^3$
$= -x^3$
$= -f(x)$

Hence $f(x) = x^3$ is an odd function.

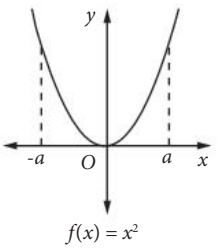


Because $f(x)$ and $f(-x)$ are opposite in sign, the graph of f for $x \leq 0$ can be obtained by rotating the graph for $x \geq 0$ through an angle of 180° about the origin.

An **even function** has the property that $f(-x) = f(x)$. For example:

If $f(x) = x^2$
then $f(-x) = (-x)^2$
$= x^2$
$= f(x)$

Hence $f(x) = x^2$ is an even function.



The graph of an even function is symmetrical about the y-axis. The graph for $x \leq 0$ can be obtained by reflecting the graph for $x \geq 0$ in the y-axis.

Note that the statement $f(-a) = f(a)$ implies that the function is defined at both $x = a$ and $x = -a$. The function $f(x) = x^2$, $x > 0$ is **not** an even function, because $f(-a)$ is not defined.

The properties of odd and even functions are useful when sketching the curves for these functions. After drawing a curve for $x \geq 0$, the other half of the curve can be drawn immediately from the odd or even symmetrical properties. Disappointingly, however, most functions are neither even nor odd.

TECHNOLOGY EXPLORATION

Odd and even functions

You can use GeoGebra to demonstrate odd and even functions. To test if a function is even, reflect it in the y-axis. For example:

1. In the input bar, enter *y=(x+2)(x-2)*.
2. Using the **Reflect Object about Line** tool  , click on the curve and then on the y-axis. The curve will be reflected.
3. If the reflected image is exactly over the original graph, then it is **even**.

You can use this procedure with any function.

To test if a function is odd, rotate it by 180°. For example:

1. In the input bar, enter *y=1/x*.
2. Select the **Intersect Two Objects** tool  and click on the origin to create a point at (0, 0).
3. Select the **Rotate Object about Point by Angle** tool  . Click on the curve and then on the origin point to rotate the curve by 180°.
4. If the rotated image is exactly over the original graph, then it is **odd**.

You can use this procedure with any function.

EXERCISE 7.2 SKETCHING BASIC FUNCTIONS

1 Sketch each function. State the gradient and the x- and y-intercepts for each.

(a) $y = 3x + 1$ **(b)** $3x + 2y - 6 = 0$ **(c)** $y = 4 - 2x$
(d) $y = x - 1$ **(e)** $4x - y - 8 = 0$ **(f)** $y = -x$
(g) $y = 3$ **(h)** $x = 4$ **(i)** $x + 2y + 5 = 0$

2 For each part of question **1**, determine whether the function is increasing, decreasing or neither. What do you notice about the gradient in each case?

3 Sketch each function, showing any intercepts on the coordinate axes. State the domain for which each function is increasing.

(a) $f(x) = x$ **(b)** $f(x) = -x^2$ **(c)** $f(x) = x^3$
(d) $f(x) = -\frac{1}{x}$ **(e)** $f(x) = x^4$ **(f)** $f(x) = \frac{1}{x^2}$

4 For each part of question **3**, determine whether the function is odd, even or neither.

5 If $f(x) = 2x^2$ and $g(x) = \frac{3}{x}$, which of the following statements is correct?

A f and g are both odd functions **B** f is an even function and g is an odd function
C f and g are both even functions **D** f and g are neither even nor odd functions

6 For $y = 2x - 3$, indicate whether each statement is correct or incorrect.

(a) gradient = 2 **(b)** x-intercept = 3 **(c)** y is an increasing function **(d)** y-intercept = 3

7.3 ABSOLUTE VALUE FUNCTIONS

In Chapter 3 we looked at $|x|$. We can now consider $f(x) = |x|$, which is called the absolute value function (or the numerical value function).

From our earlier definition of absolute value, we have: $f(x) = |x| = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$

Or, the alternative definition: $f(x) = |x| = \sqrt{x^2}$ for all real x

The domain of $f(x) = |x|$ is all real x. The range of $f(x) = |x|$ is non-negative real numbers (i.e. $f(x)$ is zero or a positive real number).

Example 9

Find the domain and range for each function. Sketch each function.

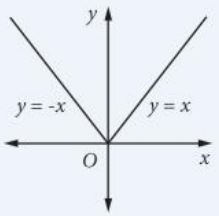
(a) $f(x) = \sqrt{x^2}$ **(b)** $f(x) = |2x - 1|$ **(c)** $f(x) = x + |x|$ **(d)** $f(x) = |x^2 - 4|$

Solution

(a) $f(x) = \sqrt{x^2} = |x| = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$

Domain: real x.

Range: $f(x)$ is zero or a positive real number.



(b) $f(x) = |2x - 1|$

$f(x) = 2x - 1$ for $2x - 1 \geq 0$

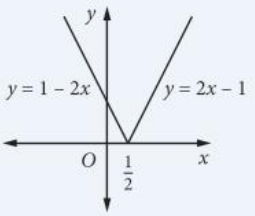
i.e. for $x \geq \frac{1}{2}$

$f(x) = -(2x - 1)$ for $2x - 1 < 0$

$f(x) = 1 - 2x$ for $x < \frac{1}{2}$

Domain: real x.

Range: $f(x)$ is zero or a positive real number.



(c) $f(x) = x + |x|$

$f(x) = x + x$ for $x \geq 0$

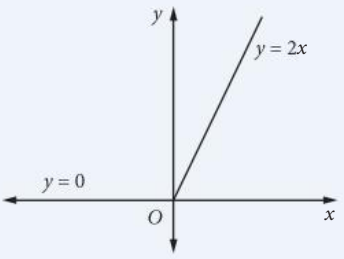
$f(x) = 2x$ for $x \geq 0$

$f(x) = x - x$ for $x < 0$

$f(x) = 0$ for $x < 0$

Domain: real x.

Range: $f(x)$ is zero or a positive real number.



(d) $f(x) = |x^2 - 4|$

$f(x) = x^2 - 4$ for $x^2 - 4 \geq 0$

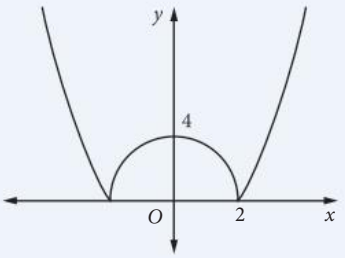
i.e. for $|x| \geq 2$

$f(x) = -(x^2 - 4)$ for $x^2 - 4 < 0$

$f(x) = 4 - x^2$ for $|x| < 2$

Domain: real x.

Range: $f(x)$ is zero or a positive real number.



This sketch can be obtained by first sketching the graph of $f(x) = x^2 - 4$. Take the part of that curve that is below the x-axis and reflect it above the x-axis.

EXERCISE 7.3 ABSOLUTE VALUE FUNCTIONS

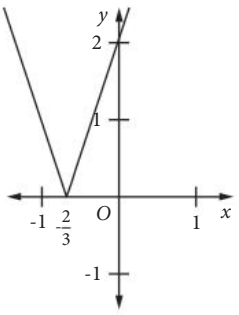
1 Sketch the graphs of the following absolute value functions defined for all x and state the range in each case.

(a) $f(x) = |x - 4|$ **(b)** $g(x) = |x| - 2$ **(c)** $h(x) = |x + 1|$ **(d)** $f(x) = \sqrt{(x+2)^2}$

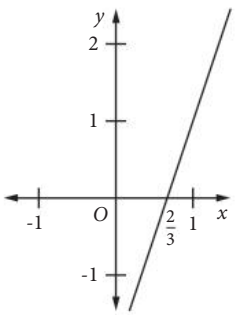
(e) $h(x) = |3x - 6|$ **(f)** $f(x) = |4 - 2x|$ **(g)** $g(x) = 4 - |2x|$ **(h)** $f(x) = 2x + |x|$

2 Which diagram is the correct sketch of $y = |3x - 2|$?

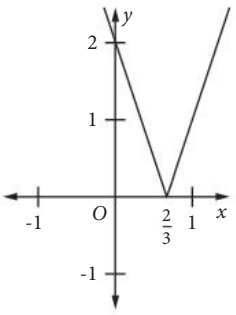
A



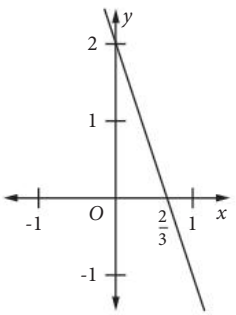
B



C



D



3 State the largest possible domain for:

(a) $f(x) = \sqrt{x-2} + \sqrt{3-x}$ **(b)** $f(x) = \dfrac{x}{|x|}$

4 State whether the following functions are odd, even or neither, defined on their largest possible domain.

(a) $f(x) = x$ **(b)** $f(x) = x + 1$ **(c)** $f(x) = |x|$

(d) $f(x) = x^3 + x$ **(e)** $f(x) = 4 - x^2$ **(f)** $f(x) = (x-2)^2$

(g) $f(x) = \sqrt{4 - x^2}$ **(h)** $f(x) = \dfrac{x}{x^2 - 1}$ **(i)** $f(x) = x^2 + x$

5 Find the largest possible range for the following functions:

(a) $f(x) = (x-3)^2$ **(b)** $f(x) = x + |x|$ **(c)** $f(x) = \sqrt{16 - x^2}$ **(d)** $f(x) = 16 - x^2$

6 Sketch the graph of $f(x) = |2x - 5|$. On the graph, indicate the values of x for which $f(x) = 3$.

7 Sketch the relation $|x| + |y| = 1$. Is this relation a function? State the greatest possible domain and the range.

8 The relation $|x| + |y| = 2$ is sketched and the equations of the four boundary lines are written. State whether each of the following statements is a correct or incorrect equation of a boundary.

(a) $x + y - 2 = 0$ **(b)** $x + y + 2 = 0$ **(c)** $x - y - 2 = 0$ **(d)** $x - y + 2 = 0$

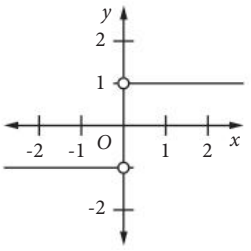
9 For the given graph, state whether each statement is correct or incorrect.

(a) The domain is real x, $x \neq 0$.

(b) The range is real y, $-1 < y < 1$.

(c) The gradient of the function is zero.

(d) The equation of the function could be $y = \dfrac{|x|}{x}$.



CHAPTER REVIEW 7

1 State the largest possible domain for the following functions:

(a) $f(x) = \sqrt{x-1}$ **(b)** $f(x) = \dfrac{1}{x^2-4}$ **(c)** $f(x) = \sqrt{25-x^2}$ **(d)** $f(x) = |x|$

2 Sketch the graph of each function given in question **1**.

3 If $g(x) = x^4 - x^2 + 1$, show that $g(x)$ is an even function.

4 Draw the graph of $y = 1 - |x|$.

5 Draw the graph of $y = x - |x|$ and state the domain of the function.

6 State the largest possible domain for each function:

(a) $f(x) = \sqrt{x+3} + \sqrt{2-x}$ **(b)** $g(x) = \dfrac{x}{|x-1|}$

7 Is the function $y = x^3 - 1$ even, odd or neither?

8 Draw the graph of $y = |x| + 1$.

CHAPTER 8
LOCUS AND REGIONS

Outcomes covered:

P1 demonstrates confidence in using mathematics to obtain realistic solutions to problems

P4 chooses and applies appropriate arithmetic, algebraic, graphical, trigonometric and geometric techniques

P5 understands the concept of a function and the relationship between a function and its graph

You have already plotted sets of points on a Cartesian number plane. Sometimes an algebraic rule has described these sets of points. The rule may have been an equation of a straight line such as $2x + 3y + 4 = 0$, a parabola such as $y = x^2$ or a circle such as $x^2 + y^2 = 9$.

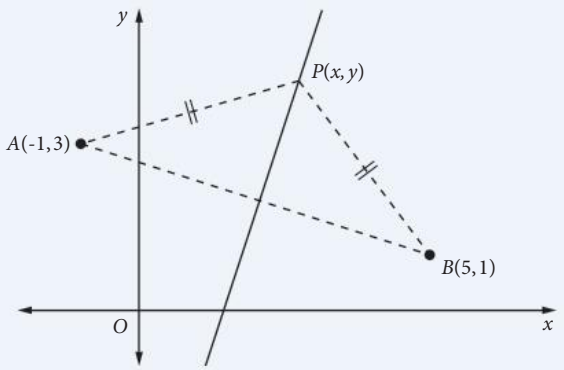
In this chapter we are going to explore function notation, functions and relations. You will graph these expressions, identify regions defined by inequalities and solve real-world problems. You should use appropriate technology to construct the graphs: this might include a ruler and pencils, a pair of compasses, a plastic template, a graphing calculator or computer graphing software such as GeoGebra.

8.1 LOCUS

A **locus** is a set of points in a plane that satisfies some geometric condition or some algebraic equation. It can be defined as the path traced out by a point moving in a plane. The Cartesian equation used to define a locus gives us the curve along which the point travels. A linear equation like $ax + by + c = 0$ means that the locus of the point $P(x, y)$ is a straight line.

Example 1

Find the locus of a point $P(x, y)$ that moves so that its distance from $A(-1, 3)$ is equal to its distance from $B(5, 1)$.



Solution

Draw a diagram and use the distance formula.

$$PA = \sqrt{(x+1)^2 + (y-3)^2}$$

$$PB = \sqrt{(x-5)^2 + (y-1)^2}$$

But: $PA = PB$

So: $\sqrt{(x+1)^2 + (y-3)^2} = \sqrt{(x-5)^2 + (y-1)^2}$

Square both sides: $(x+1)^2 + (y-3)^2 = (x-5)^2 + (y-1)^2$

$$x^2 + 2x + 1 + y^2 - 6y + 9 = x^2 - 10x + 25 + y^2 - 2y + 1$$

$$12x - 4y - 16 = 0$$

$$3x - y - 4 = 0$$

The locus of P is the straight line $3x - y - 4 = 0$.

ΔPAB will always be an isosceles triangle because $PA = PB$, as defined in the question.

Example 2

What is the locus of a point P that is always three units from the y-axis?

Solution

Draw a diagram.

$P(x, y)$ is any point on the locus.

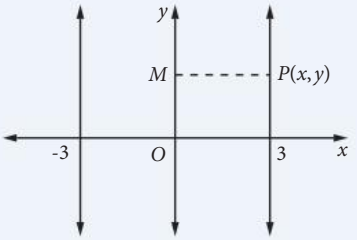
PM is the (perpendicular) distance of P from the y-axis.

$PM = 3$

$\therefore |x| = 3$

Thus the locus of P is a pair of lines with equations $x = 3$ and $x = -3$.

These can be written in a single statement as $x = \pm 3$.



EXERCISE 8.1 LOCUS

1 Find the locus of a point $P(x, y)$ that moves in a plane such that $PA = PB$, where:

(a) $A(0, 3), B(0, 7)$ **(b)** $A(2, 4), B(2, -6)$ **(c)** $A(-2, 3), B(4, 5)$ **(d)** $A(1, 7), B(4, -2)$

In each case, show that the locus is a straight line perpendicular to AB and through the midpoint of AB.

2 The locus of a point $P(x, y)$ that moves so that it is always three units above the x-axis is:

A $x = -3$ **B** $x = 3$ **C** $y = -3$ **D** $y = 3$

3 Find the equation of the locus of a point $P(x, y)$ that moves according to the following conditions:

(a) P is four units to the left of the y-axis.
(b) P is equidistant from the line $y = -5$ and from the x-axis.
(c) P is $\sqrt{2}$ units from the line with equation $y = x$.
(d) The distance of P from the y-axis is three times its distance from the x-axis.
(e) The distance of P from the line $x = 4$ is equal to its distance from the line $y = 1$.
(f) Twice the distance of P from the x-axis is three times its distance from the y-axis.
(g) The distance of P from the line $y = -5$ is three-quarters of its distance from the line $x = 2$.
(h) The distance of P from the line $x = -3$ is two-fifths of its distance from the line $y = -1$.

4 The locus of a point $P(x, y)$ that moves so that it is equidistant from $(3, 5)$ and $(-1, -1)$ is:

A $3x + 2y - 8 = 0$ **B** $2x + 3y + 8 = 0$ **C** $2x + 3y - 8 = 0$ **D** $3x + 2y + 8 = 0$

5 Find the locus of a point P that moves so that:

(a) $PA^2 - PB^2 = 5$ **(b)** $PA = PB$ for $P(x, y), A(2, 3), B(-2, -1)$

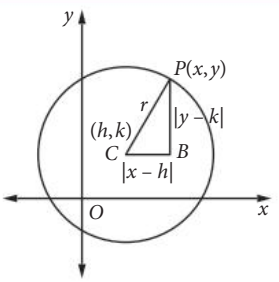
6 Given $A(2, 4), B(7, -5)$, find the equation of the locus of $P(x, y)$ if $PA = 2PB$.

7 A and B are the fixed points $(a, 0)$ and $(-a, 0)$. Find the locus of $P(x, y)$ such that the gradient of AP is twice the gradient of BP.

8.2 CIRCLES

A **circle** can be defined as the set of all points P in a plane at a given distance from a fixed point in the plane. The fixed point is the centre of the circle and the given distance is the **radius**.

Consider the circle of radius r units with its centre at $C(h, k)$. If P is a point with coordinates (x, y) on the circumference of this circle, then the distance of P from C is r units.



Applying Pythagoras' theorem to triangle CBP in the diagram gives:

$$BC^2 + BP^2 = CP^2$$

$$(x-h)^2 + (y-k)^2 = r^2$$

Thus the equation of the circle is given by $(x-h)^2 + (y-k)^2 = r^2$, with the values for x and y restricted:

- The set of values for x is given by $h - r \leq x \leq h + r$.
- The set of values for y is given by $k - r \leq y \leq k + r$.

If the centre of the circle is at the origin, then $h = 0$, $k = 0$ and the equation of the circle is $x^2 + y^2 = r^2$.

Example 3

Find the equation of the circle with centre $(-3, 4)$ and radius 6 units.

Solution

Use the result: $(x-h)^2 + (y-k)^2 = r^2$

Substitute $(-3, 4)$, $r = 6$: $(x+3)^2 + (y-4)^2 = 36$ is the equation of the circle.

Example 4

Find the coordinates of the centre and the length of the radius for the circle whose equation is $x^2 + y^2 - 4x + 10y + 14 = 0$.

Solution

Rewrite equation: $x^2 - 4x + y^2 + 10y = -14$

Complete the square for x and y: $x^2 - 4x + 4 + y^2 + 10y + 25 = -14 + 4 + 25$

Factorise: $(x-2)^2 + (y+5)^2 = 15$

The circle has its centre at $(2, -5)$ and has a radius of $\sqrt{15}$ units.

Example 5

Find the equation of the circle with centre $(3, 4)$ that passes through the point $(-1, 1)$.

Solution

Use the result: $(x-h)^2 + (y-k)^2 = r^2$

Centre is $(3, 4)$: $(x-3)^2 + (y-4)^2 = r^2$

$(-1, 1)$ satisfies equation: $(-4)^2 + (-3)^2 = r^2$

$r^2 = 25$

Equation of circle is $(x-3)^2 + (y-4)^2 = 25$.

TECHNOLOGY EXPLORATION

Locus of a set of points that are equidistant from a point

1. In GeoGebra, create an interval using the **Interval between Two Points** tool  . Let B be the point that will stay equidistant from A.
2. Add a slider using the **Slider** tool  . Select the type as 'Angle' and make sure that the 'Interval Max:' is set to 360°. This variable will be labelled α.
3. Select the **Rotate Object around Point by Angle** tool  . Click on point B and then point A. When it asks for the angle, delete the angle given and instead type α (see the drop-down menu labelled α in the text box).
4. Repeat the previous step, but this time click on point A and then point B.
5. Use the **Move** tool  to move the slider and see the motion of the points.
6. To see the locus of the moving point B', enter *locus[B',α]* into the input bar.

EXERCISE 8.2 CIRCLES

1 Find the equation of each of the following circles.

(a) centre $(3,2)$, radius 4 units
(b) centre $(-1,-4)$, radius 3 units
(c) centre $(3,-3)$, radius $\sqrt{5}$ units
(d) centre $\left(-2,\frac{5}{2}\right)$, radius $\frac{7}{2}$ units
(e) centre $\left(0,-\frac{3}{2}\right)$, radius 4 units
(f) centre $(4,0)$, radius 3 units

2 The equation of the circle with centre $(-4,4)$ and radius 6 units is:

A $(x-4)^2+(y-4)^2=36$
B $(x+4)^2+(y-4)^2=36$
C $(x-4)^2+(y+4)^2=36$
D $x^2+y^2=36$

3 Find the equation for each of the following circles.

(a) centre $(3,2)$ and passing through the point $(5,-5)$
(b) centre $(-1,4)$ and passing through the origin
(c) centre $(0,0)$ and passing through the point $(-3,4)$

4 Find the coordinates of the centre and the length of the radius for the following circles.

(a) $x^2+y^2-6x+4y-3=0$
(b) $x^2+y^2+4x+2y-4=0$
(c) $(x-3)^2+y^2=3$
(d) $(x+a)^2+(y-b)^2=8$
(e) $x^2+y^2-5x+3y-1=0$
(f) $x^2+y^2+4x+2y-5=0$
(g) $2x^2+2y^2-8x+5y+3=0$
(h) $3x^2+3y^2+9x-4y-24=0$

5 Using the fact that the centre of a circle is the midpoint of a diameter, find the equation of the circle with the diameter endpoints given.

(a) $(3,4)$ and $(9,-6)$
(b) $(0,0)$ and $(5,-3)$
(c) $(5,8)$ and $(-2,3)$

6 For the equation $x^2+y^2-6x+2y+10=0$, indicate whether each statement is correct or incorrect.

(a) centre $(3,-1)$, radius $=1$
(b) centre $(-3,1)$, radius $=0$
(c) centre $(3,-1)$, radius $=2\sqrt{5}$
(d) centre $(3,-1)$, radius $=0$

7 For the circle with equation $x^2+y^2+6x-8y=0$:

(a) find the coordinates of the centre and the length of the radius
(b) show that the origin is on the circle
(c) find the equation of the diameter that passes through the origin.

8 Find the equation of the circle that touches the x-axis at $(4,0)$ and the y-axis at $(0,4)$.

9 Show that the point $(4, -3)$ is not on the circle $x^2 + y^2 - 5x + 3y + 2 = 0$. Determine whether the point is inside or outside the circle.

10 Determine whether the origin is inside or outside the circle $x^2 + y^2 - 4x - y + 1 = 0$.

11 **(a)** Find the equation of a circle with a radius of 5 units and its centre at the point $(-1, 2)$.
(b) What is the length of the intercept cut off by this circle on the x-axis?
(c) Find the length of the tangent to this circle through the point $(4, 6)$.

(**Note**: a tangent line to a circle is perpendicular to a radius that is drawn to the point of contact.)

12 The equation of a circle is $x^2 + y^2 + 4x - 2y - 20 = 0$. Find:
(a) the length of the tangent to this circle through the point $(5, 2)$
(b) the length of the intercept on the y-axis.

13 A diameter intersects a circle at the points $(6, -4)$ and $(-2, 6)$.
(a) Find the centre and radius of the circle.
(b) What is the length of the circle's tangent from the point $(-5, 5)$?

14 The coordinates of two points A and B are $(-1, 3)$ and $(5, 7)$. Find:
(a) the coordinates of the midpoint of AB
(b) the equation of the circle of which AB is a diameter
(c) the coordinates of the intersection points of the circle with the y-axis.

15 **(a)** Find the coordinates of the centre and the length of the radius for the circle $x^2 + y^2 - 4x - 8y - 5 = 0$.
(b) The point $(3, 2)$ is the midpoint of a chord of this circle. Find the distance of the chord from the centre and the length of the chord.

8.3 FURTHER LOCUS

Example 6

Find the locus of a point P that moves so that $PA = 2PB$, given $P(x, y)$, $A(-2, 4)$, $B(4, 1)$. Show that the locus is a circle and give the radius and the coordinates of its centre.

Solution

$$PA = 2PB$$

$$\sqrt{(x+2)^2 + (y-4)^2} = 2\sqrt{(x-4)^2 + (y-1)^2}$$

Square both sides of the equation:

$$(x+2)^2 + (y-4)^2 = 4[(x-4)^2 + (y-1)^2]$$

$$x^2 + 4x + 4 + y^2 - 8y + 16 = 4(x^2 - 8x + 16 + y^2 - 2y + 1)$$

$$x^2 + 4x + 4 + y^2 - 8y + 16 = 4x^2 - 32x + 64 + 4y^2 - 8y + 4$$

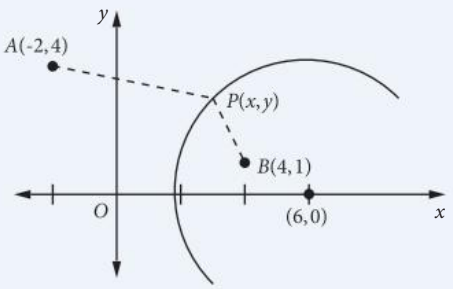
$$3x^2 - 36x + 3y^2 + 48 = 0$$

$$x^2 - 12x + y^2 + 16 = 0$$

$$(x-6)^2 + y^2 + 16 = 36$$

$$(x-6)^2 + y^2 = 20$$

Hence the locus is a circle, centre $(6, 0)$, radius $2\sqrt{5}$ units.



Example 7

$A(-1,3)$ and $B(3,1)$ are two points on the plane. Find the locus of $P(x,y)$ such that $PA \perp PB$.

Solution

Because $PA \perp PB$, $m_{PA} \times m_{PB} = -1$.

$$m_{PA} = \frac{y-3}{x+1},\ x \neq -1$$

$$m_{PB} = \frac{y-1}{x-3},\ x \neq 3$$

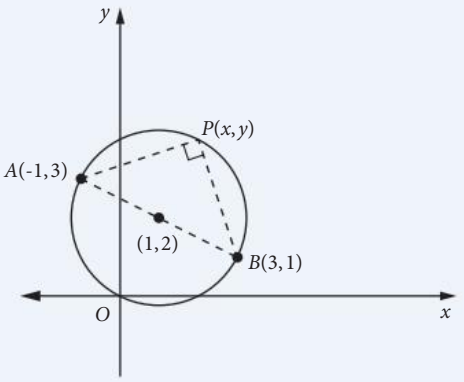
$$\therefore \frac{y-3}{x+1} \times \frac{y-1}{x-3} = -1,\ x \neq -1, 3$$

$$(y-3)(y-1) = -(x+1)(x-3)$$

$$y^2 - 4y + 3 = -x^2 + 2x + 3$$

$$x^2 - 2x + y^2 - 4y = 0$$

$$(x-1)^2 + (y-2)^2 = 5$$



The locus of P is a circle, centre $(1,2)$, radius $\sqrt{5}$ units.

This result is known as 'the angle in the semicircle' in the Extension 1 course.

Example 8

The equation of a locus is given as $x^2 - y^2 = 0$. Show that this equation represents a pair of straight lines and state their special property.

Solution

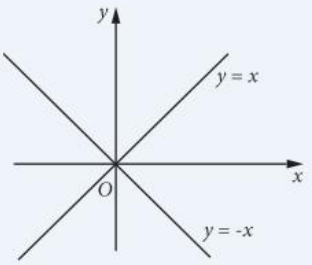
$$x^2 - y^2 = 0$$

$$(x-y)(x+y) = 0$$

$$\therefore \quad x - y = 0 \quad \text{or} \quad x + y = 0$$

$$\text{i.e.} \quad y = x \quad \text{or} \quad y = -x$$

The locus is a pair of intersecting straight lines that are perpendicular to each other.



EXERCISE 8.3 FURTHER LOCUS

1 Find the equation of the locus of a point $P(x,y)$ that is 4 units from the point $(2,-1)$.

2 What can be said about the centres of all circles that pass through the points $(2,0)$ and $(6,4)$? What is the locus of the centres?

3 Find the locus of a point $P(x,y)$ that moves in a plane so that its distance from $A(3,-1)$ is twice its distance from $B(-1,-1)$.

4 The locus of a point $P(x,y)$ that moves in a plane so that its distance from $Q(5,-2)$ is twice its distance from $R(-2,3)$ is a circle with centre and radius:

A $\left(-\frac{13}{3}, \frac{14}{3}\right), \frac{2\sqrt{74}}{3}$ **B** $\left(\frac{13}{3}, -\frac{14}{3}\right), \frac{2\sqrt{74}}{3}$ **C** $\left(-\frac{13}{3}, \frac{7}{3}\right), \frac{2\sqrt{74}}{3}$ **D** $\left(\frac{13}{3}, \frac{7}{3}\right), \frac{2\sqrt{74}}{3}$

5 Find the locus of a point $P(x,y)$ that moves in a plane such that $PA^2 + PB^2 = 44$, given $A(-3,2)$ and $B(3,-2)$. Show that the locus is a circle and find its centre and radius.

6 A ladder that is 6 m long rests with one end on the horizontal ground and its other end against a vertical wall. Considering the ground and the wall as the x- and y-axes respectively, find the locus of the possible midpoints of the ladder.

7 A point moves in a plane so that the sum of the squares of its distances from the points $(1, 2)$ and $(5, -3)$ is 45. Find the equation of the point's locus.

8 A point $P(x, y)$ moves in a plane so that the sum of the squares of its distances from the points $A(4, 0)$ and $B(-4, 0)$ is 82 units.

(a) Find the equation of the locus of P. **(b)** At what points does the locus cut the coordinate axes?

9 Find the equation of the locus of a point that moves so that its distance from the point $(-4, 2)$ is always twice its distance from the point $(5, -1)$. What point on the straight line joining the points $(-4, 2)$ and $(5, -1)$ is on the locus?

10 Given $A(4, 2)$ and $B(-2, -8)$, find the locus of a point $P(x, y)$ that moves so that the angle APB is a right angle.

11 Given $A(2, 1)$ and $B(-2, 1)$, find the equation of the locus of $P(x, y)$ if angle APB is a right angle.

12 A point $P(x, y)$ moves so that its distance from $(3, 4)$ is proportional to its distance from $(-1, 2)$. Find the equation of the locus of P if the origin is a point on the locus.

13 A is a point where the circle with equation $x^2 + y^2 = 16$ cuts the x-axis. Find the locus of the midpoints of all chords of this circle that contain A.

14 Find the equation of the locus of the midpoints of all chords of length 4 units of the circle with equation $x^2 + y^2 - 4x + 2y = 4$.

8.4 NON-LINEAR INEQUALITIES

A circle divides the number plane into two regions, a finite region called its **interior** and an infinite region called its **exterior**, as well as the set of points that make up the circle.

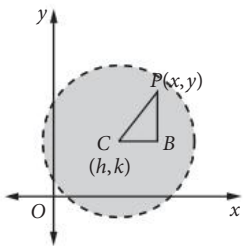
A parabola, cubic, quartic or hyperbola curve divides the number plane into two infinite regions, as well as the set of points that make up the curve.

Points inside and outside a circle

A circle divides the number plane into three sets of points: the sets of points **on** the circle, **inside** the circle and **outside** the circle. The set of points on a circle of centre $C(h, k)$ and radius r is given by the equation $(x-h)^2 + (y-k)^2 = r^2$.

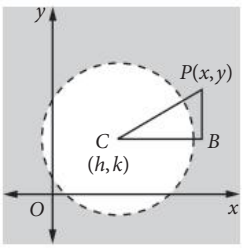
A point $P(x, y)$ lies on this circle if $CP = r$.

If $CP < r$, the point P is inside the circle.



The graph of $(x-h)^2 + (y-k)^2 < r^2$ gives the interior of the circle.

If $CP > r$, the point P is outside the circle.



The graph of $(x-h)^2 + (y-k)^2 > r^2$ gives the exterior of the circle.

Regions involving non-linear inequalities

Example 9

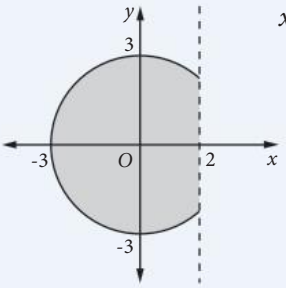
For the circle with centre $(0, 0)$ and radius 3 units, sketch the region of the Cartesian plane that includes all points on or inside the circle that are also:

(a) to the left of the line $x = 2$ **(b)** on or above the line $x + y = 3$.

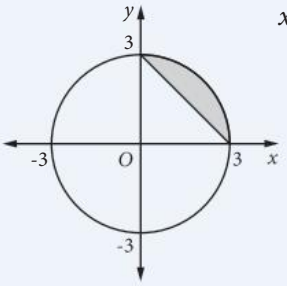
In each case give the inequalities that define the region.

Solution

(a) $x^2 + y^2 \leq 9, x < 2$



(b) $x^2 + y^2 \leq 9, x + y \geq 3$



Example 10

Sketch the region defined by $y \geq x^2$ and $y \leq 2x + 3$. Describe this region in words.

Solution

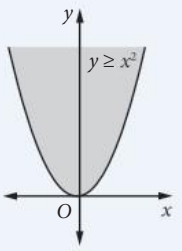
To find the points of intersection, solve simultaneously the equations $y = x^2$ and $y = 2x + 3$.

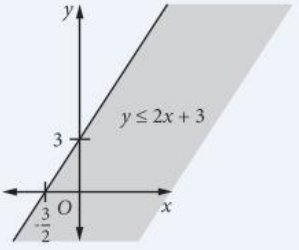
This gives: $x^2 = 2x + 3$

$x^2 - 2x - 3 = 0$

Factorise: $(x + 1)(x - 3) = 0$

Solve: $x = -1, 3$

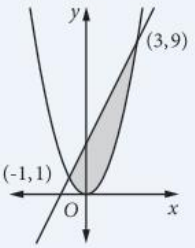




Substitute into $y = 2x + 3$: $x = -1, y = 1$; $x = 3, y = 9$

Hence the points of intersection are $(-1, 1)$ and $(3, 9)$.

The shaded region is the points on and above the parabola $y = x^2$ that are also on or below the line $y = 2x + 3$.



Example 11

Describe the region of the x–y plane whose points satisfy the inequalities $y < 2 + x - x^2$ and $y + 2x \leq 2$.

Solution

The graph of $y = 2 + x - x^2$ can be obtained by completing a table of values and then plotting points. It can also be obtained by completing the square for x and then graphing according the shape and properties of $y = x^2$: $2 + x - x^2 = 2 - \left(x^2 - x\right)$

$= 2 + \frac{1}{4} - \left(x^2 - x + \frac{1}{4}\right)$

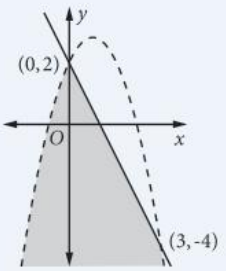
$= 2\frac{1}{4} - \left(x - \frac{1}{2}\right)^2$

Hence we can graph $y = 2\frac{1}{4} - \left(x - \frac{1}{2}\right)^2$, which is the graph of $y = x^2$ turned upside down, moved 0.5 units to the right and moved 2.25 units up:

$y < 2 + x - x^2$ is the region below this curve.

$y + 2x \leq 2$ is the region on or below the line $y + 2x = 2$.

As shown in the diagram, the required region is the region on or below the line $y + 2x = 2$ that is contained between the 'arms' of $y = 2 + x - x^2$.



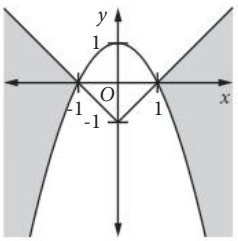
EXERCISE 8.4 NON-LINEAR INEQUALITIES

1 Sketch the region defined by each inequality.

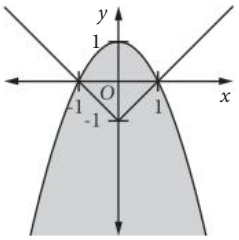
(a) $x^2 + y^2 \geq 16$ **(b)** $x^2 + y^2 < 4$ **(c)** $(x-1)^2 + y^2 > 9$

2 Which diagram shows the region satisfying $y \leq 1 - x^2$ and $y \geq |x| - 1$?

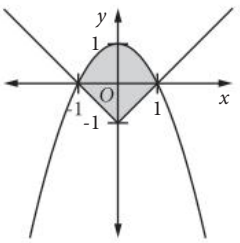
A



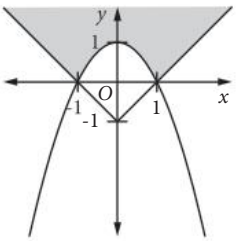
B



C



D



3 Sketch the region defined by each inequality.

(a) $(x+3)^2 + y^2 < 1$ **(b)** $y \leq x^2 + 1$ **(c)** $y \geq 9 - x^2$

(d) $y \geq |x|$ **(e)** $y < 2x + 4$ **(f)** $y < |2x + 4|$

4 Sketch the region defined by the given inequalities.

(a) $x^2 + y^2 \leq 1, x \geq 0, y \geq 0$ **(b)** $(x-1)^2 + (y-1)^2 < 1, x > 0, y > 0$

(c) $y < 4 - x^2, y \geq 0$ **(d)** $y < |x|, y > 0, 1 < x < 2$

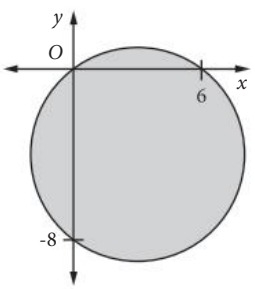
(e) $x^2 + y^2 \leq 4, y > x + 2$ **(f)** $y \geq x^2 - 4, x + y < 2$

(g) $y > |x - 2|, y > 3$ **(h)** $y \leq 1 - x^2, y \geq 0$

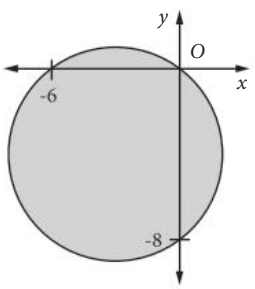
(i) $x^2 + y^2 \leq 1, y \leq 2x, x \geq 0$ **(j)** $(x-1)^2 + y^2 \geq 1, x \geq 0, y \leq 1$

5 Which graph represents the region defined by $(x-3)^2 + (y+4)^2 \leq 25$?

A



B



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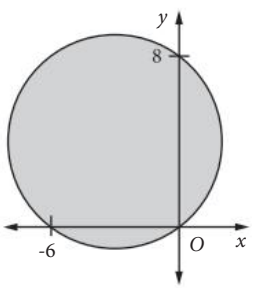

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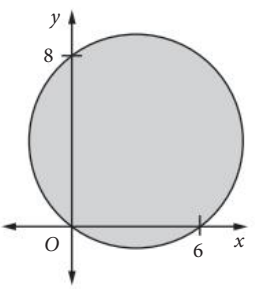


6 Sketch the region of the Cartesian plane whose boundary consists of:

(a) the curve $y = x^2$, the ordinates at $x = 1$ and $x = 2$ and the x-axis

(b) the lines $y = 1$ and $y = 3$ and the circle with centre $(0, 2)$ and radius 2 units

(c) the circle $(x-3)^2 + (y-4)^2 = 25$ and the y-axis

(d) the arc of the parabola $y = 4 - x^2$ between $x = 2$ and $x = -2$ and the x-axis

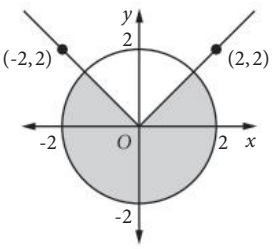
(e) the graph of $y = |x - 1|$, the x-axis and the y-axis

(f) the circle $x^2 + y^2 = 36$ and the lines $y = 6$ and $x = 6$

(g) the circle centred at $(-2, 0)$, radius 2 units, and the circle centred at $(-2, 0)$, radius 1 unit
(h) the parabola $y = x^2 - 2$ and the line $y = x$
(i) the curve $y = \sqrt{x}$, the y-axis and the line $y = 2$
(j) the semicircle $y = \sqrt{1 - x^2}$ and the x-axis.

7 For the shaded region in the diagram, state whether each statement is correct or incorrect.

(a) The shaded region is defined by $y \leq |x|$ and $x^2 + y^2 \leq 4$.
(b) The shaded region is the part of the interior of the circle of centre $(0, 0)$ and radius 2 that is below the lines given by $y = |x|$.
(c) The shaded region is defined by $y \geq |x|$ and $x^2 + y^2 \leq 4$.
(d) The shaded region is the part of the circle with centre $(0, 0)$ and radius 2, and its interior, that is on or below the lines given by $y = |x|$.



CHAPTER REVIEW 8

1 Find the equation of the locus of a point $P(x, y)$ that moves so that:

(a) its distance from $(0, 0)$ is always 5
(b) its distance from $(2, 1)$ is always 4
(c) it is equidistant from the points $A(3, 2)$ and $B(9, 5)$
(d) its distance from $(0, 3)$ is the same as its distance from $y = -3$.

2 Describe in words each locus from question **1**.

3 The equation of a circle is $x^2 + y^2 - 2x - 2y - 23 = 0$.

(a) Find the circle's centre and radius.
(b) Calculate the distance from the point $(7, -2)$ to the centre of the circle.
(c) Explain why the point $(7, -2)$ is outside the circle.
(d) Use Pythagoras' theorem to find the length of the tangent to the circle from the point $(7, -2)$. (Note: tangent $\perp$ radius drawn to point of contact.)

4 Sketch the region of the Cartesian plane that satisfies $y \geq x^2 - 1$ and $y \leq 1 - |x|$.

5 Sketch the region of the Cartesian plane bounded by curves $y = \frac{1}{x}$, $x = 1$, $x = 3$ and the x-axis.

6 Show that the straight lines $2x + y = 20$ and $x + y = 14$ intersect at $(6, 8)$. Hence sketch the region of the Cartesian plane for which $y \geq 20 - 2x$, $y \leq 14 - x$ and $y \geq 0$ are all true.

7 Sketch the region of the Cartesian plane bounded by the curves $y \geq x^2 - 4$ and $y \leq 4 - x^2$.

8 Sketch the region of the Cartesian plane bounded by the curves $y = x^2 - 4$ and $y = |x| + 1$.

9 Sketch the region in the number plane defined by $(x - 1)^2 + (y - 1)^2 < 1$ and $x > 1$.

10 Find the equation of the locus of a point $P(x, y)$ that moves so that its distance from the point $(-2, 1)$ is equal to its distance from the line $y = -1$. Express the locus in the form $(x - h)^2 = 4a(y - k)$.

11 Show algebraically that the line $y = x - 4$ is a tangent to the circle $x^2 + y^2 = 8$ and find the coordinates of the point of contact.

CHAPTER 9
QUADRATIC FUNCTIONS AND THE PARABOLA

Outcomes covered:

P1 demonstrates confidence in using mathematics to obtain realistic solutions to problems

P4 chooses and applies appropriate arithmetic, algebraic, graphical, trigonometric and geometric techniques

P5 understands the concept of a function and the relationship between a function and its graph

9.1 QUADRATIC FUNCTIONS

Definitions

$ax^2 + bx + c$, a polynomial of the second degree, is called a quadratic polynomial or a quadratic expression.

A function defined by the rule $y = ax^2 + bx + c$, where a, b and c are constants, $a \neq 0$, is called a **quadratic function**. The domain of this function is the set of real numbers (unless otherwise stated or implied).

When $y = 0$ the quadratic function becomes the quadratic equation $ax^2 + bx + c = 0$. The values of x that satisfy this equation are called the roots of the equation. They are also called the zeros of the quadratic polynomial, as they are the values that make the polynomial zero.

Example 1

Give the domain of each function.

(a) $y = 2x^2 - 3x - 5$ **(b)** $y = t^2 + 2t$ **(c)** $y = 4p^2 - 8$

(d) $A = s^2$, where s is the side length of a square.

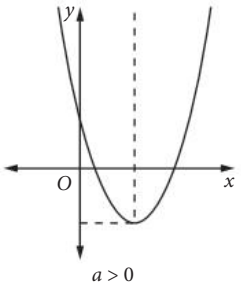
Solution

(a) Real numbers for x **(b)** Real numbers for t **(c)** Real numbers for p

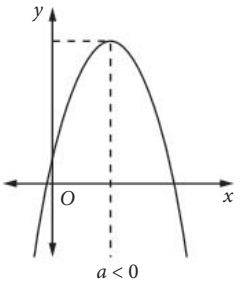
(d) Positive real numbers for s (because side length cannot be negative or zero)

Graph of a quadratic function

The graph of a quadratic function has the characteristic shape of a **parabola**. The parabola has a turning point at its vertex, where the function has a minimum value if $a > 0$ or a maximum value if $a < 0$.



$a > 0$



$a < 0$

Maximum or minimum value of a quadratic function

All quadratic polynomials can be expressed in the form $a(x + B)^2 + C$, where a, B and C are constants, $a \neq 0$.

In this form, we are effectively completing the square on x. Because $(x + B)^2$ is a perfect square, it is non-negative for all real values of x, i.e. $(x + B)^2 \geq 0$. Thus the smallest possible value of $(x + B)^2$ is zero, when $x = -B$.

(i) If $a > 0$, then the **minimum** value of $a(x + B)^2 + C$ is C.

(ii) If $a < 0$, then the **maximum** value of $a(x + B)^2 + C$ is C.

Example 2

Express $2x^2 - 4x - 5$ in the form $a(x + B)^2 + C$ and hence state its minimum value.

Solution

$$\text{Expression} = 2x^2 - 4x - 5$$

Take out the factor of 2 from terms in x: $= 2(x^2 - 2x) - 5$

Complete the square on x: $= 2(x^2 - 2x + 1) - 5 - 2 \times 1$

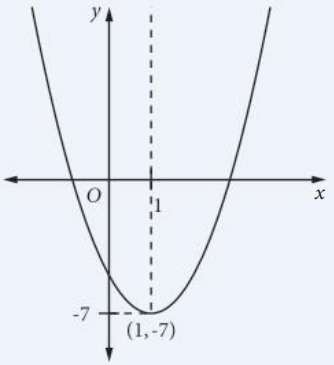
$$= 2(x - 1)^2 - 7$$

Hence the minimum value of $2x^2 - 4x - 5$ is -7, when $x = 1$.

The graph of the function $y = 2x^2 - 4x - 5$ is shown in the diagram.

The range of the function is $y \geq -7$.

The graph is symmetrical about the line $x = 1$, which is the axis of symmetry of a parabola that has its vertex at $(1, -7)$.



Example 3

Find the maximum value of $-3x^2 - 12x - 7$ by the method of completing the square.

Solution

$$\text{Expression} = -3x^2 - 12x - 7$$

Take out the factor of 3 from terms in x: $= -3(x^2 + 4x) - 7$

Complete the square on x: $= -3(x^2 + 4x + 4) - 7 + 3 \times 4$

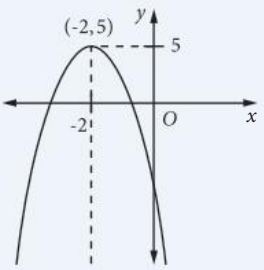
$$= -3(x + 2)^2 + 5$$

Hence the maximum value of $-3x^2 - 12x - 7$ is 5 when $x = -2$.

The graph of the function $y = -3x^2 - 12x - 7$ is shown in the diagram.

The range of the function is $y \leq -5$.

The graph is symmetrical about the line $x = -2$, which is the axis of symmetry of a parabola that has its vertex at $(-2, 5)$.



Turning point of a parabola

From the examples above, it can be seen that the turning point (minimum or maximum value) of the quadratic function $y = ax^2 + bx + c$ will occur at $x = -\dfrac{b}{2a}$.

Thus the turning point of $f(x) = ax^2 + bx + c$ has the coordinates $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.

- If $a < 0$ this will be a **maximum** turning point, so the **maximum value** of the function is $f\left(-\frac{b}{2a}\right)$.
- If $a > 0$ this will be a **minimum** turning point, so the **minimum value** of the function is $f\left(-\frac{b}{2a}\right)$.

Later in the course, you will be able to find the turning point using calculus and the fact that $\dfrac{dy}{dx} = 0$.

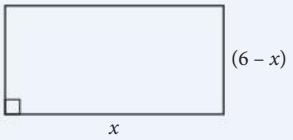
Example 4

A piece of wire of length 12 cm is bent in the shape of a rectangle. Find the maximum area of the rectangle.

Solution

Let one dimension of the rectangle be x cm.

Hence the other dimension will be $(12 \div 2 - x) = (6 - x)$ cm.

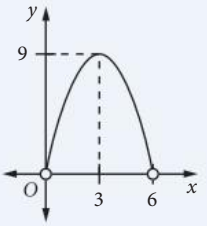


For any given x in the domain $0 < x < 6$, the area function is given by:

$$\begin{aligned} y &= x(6 - x) \qquad \text{for } 0 < x < 6 \\ &= 6x - x^2 \\ &= -(x^2 - 6x + 9) + 9 \\ &= -(x - 3)^2 + 9 \end{aligned}$$

Hence the maximum area is $9\,\text{cm}^2$, when $x = 3$ and the rectangle is a square.

The graph of the area function is shown in the diagram.



Domain: $0 < x < 6$

Range: $0 < y \leq 9$

TECHNOLOGY EXPLORATION

Axis of symmetry of the parabola

This procedure can be used to find the maximum or minimum value of a parabola.

1. In GeoGebra, use the **Slider** tool  to add three sliders a, b and c. Set each slider to have Min = -10, Max = 10 and Increment = 1.
2. In the input bar, enter *f(x)=a*(x^2)+b*(x)+c*. A parabola will appear. Look to see what happens to the parabola as you move each of the sliders (using the **Move** tool ). You may like to summarise the effect that each of the values has on the overall shape and position of the curve. What happens when $a = 0$?
3. Using the **Intersect Two Objects** tool  , find the points of intersection of the curve with the x-axis by clicking on the parabola and the x-axis. The x-intercepts will appear.
4. Using the **Perpendicular Bisector** tool  , create the perpendicular bisector of the two intersection points that you created in step **3**. This line will be the axis of symmetry. The equation of the axis of symmetry will appear in the **Algebra** view.
5. Now, create the axis of symmetry from the formula $x = -\frac{b}{2a}$ by entering *x=(-b)/(2*a)* into the input bar. What do you notice? Hopefully nothing, because the line should map perfectly onto the axis of symmetry already created in step **4**. However, the **Algebra** view should show that a second equation of $x = \ldots$ now exists and is the same as the perpendicular bisector equation.
6. Using the **Intersect Two Objects** tool  , click on the axis of symmetry and then the parabola. The resulting intersection point will be shown in the **Algebra** view and its y-coordinate will be the parabola's maximum or minimum value.

EXERCISE 9.1 QUADRATIC FUNCTIONS

1 The minimum value of $x^2 - 2x + 6$ is:

A -1 **B** 1 **C** 5 **D** 6

2 Express each of the following functions in the form $y = a(x + B)^2 + C$ and hence find the maximum or minimum value and the range.

(a) $y = 2x^2 - 4x$ **(b)** $y = -2x^2 + 8x - 3$ **(c)** $y = 7 + 16x - 4x^2$ **(d)** $y = 4x^2 + 8x - 7$
(e) $y = 8 - 2x^2$ **(f)** $y = 7 - 2x - x^2$ **(g)** $y = 2x^2 - 6x$ **(h)** $y = 6 - 10x - 5x^2$

3 The maximum value of $-x^2 + x + 11$ is:

A -0.5 **B** 0.5 **C** 10.75 **D** 11.25

4 A stone is thrown straight up from the ground. The height above the ground, $h(t)$ metres, is a function of time, t seconds ($t \geq 0$), according to the rule $h(t) = 20t - 5t^2$. Find:

(a) the domain of h **(b)** the greatest height reached.

5 The equation of the path of a cricket ball thrown at an angle of 45° to the horizontal is $y = x - \frac{x^2}{50}$, where x m and y m are the horizontal distance travelled and the vertical height respectively. Calculate the greatest vertical height reached and the horizontal distance travelled.

6 The sum of two numbers is 20. Find the numbers and their product if their product is a maximum.

7 A piece of wire 60 cm long is bent in the shape of a rectangle. Find the maximum area of the rectangle.

8 A farmer wants to make a rectangular enclosure using her existing fence as one side. If she has 20 metres of fencing material available to make the other three sides, find the area and the dimensions of the largest enclosure that can be formed.

9 A piece of wire 6 metres long is cut into two parts. One part is used to form a square and the other part is used to form a rectangle whose length is three times its breadth. Find the lengths of the two parts if the sum of the two areas is a minimum.

10 A large open area is to have a section surrounded by a rectangular fence. This rectangle is then divided into six smaller rectangles, using one dividing fence parallel to its length and two fences parallel to its breadth. If the total length of fencing available is 1200 m, find the maximum possible area.

11 A machine comes in two parts, which weigh x kg and b kg respectively. The cost c of the machine (in dollars) is given by $c = 2x + b$. The earning capacity y of the machine is given by $y = x(x + b)$. If c has the fixed value 10, express y as a function of x and hence find the value of x for which y is a maximum. Find the maximum value of y.

12 $ABCD$ is a square of unit length. Points E and F are on the sides AB and AD respectively such that $AE = AF = x$. Show that the area y of the quadrilateral $CDFE$ is given by $y = \frac{1}{2}\left(1 + x - x^2\right)$. What is this quadrilateral's greatest possible area?

9.2 PARABOLAS AND DISCRIMINANTS

Discriminants

In Chapter 3 we solved the general quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$, by 'completing the square' to obtain the quadratic formula:

$$ax^2 + bx + c = 0$$

Move constant to RHS: $ax^2 + bx = -c$

Divide by coefficient of x^2: $x^2 + \frac{b}{a}x = -\frac{c}{a}$

Add $\left(\frac{b}{2a}\right)^2$ to complete the square: $x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$

Factorise: $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$

Take square roots of both sides: $x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

We now introduce the symbol Δ (delta) for the expression $b^2 - 4ac$.

This is called the **discriminant**: $\Delta = b^2 - 4ac$.

The roots of the quadratic equation can be written as $x = \frac{-b + \sqrt{\Delta}}{2a}$ or $x = \frac{-b - \sqrt{\Delta}}{2a}$.

Case 1 If $\Delta > 0$, $\sqrt{\Delta}$ is a real number and so the roots are two real numbers.
We say that the equation has two unequal roots or two different roots.

Case 2 If $\Delta = 0$, $\sqrt{\Delta} = 0$ and so the roots consist of one real number: $-\frac{b}{2a}$.
We say that the equation has only one root or two equal roots.

Case 3 If Δ is a perfect square, $\sqrt{\Delta}$ is a rational number and so the roots are two rational numbers.
We say that the equation has two unequal, rational roots.

Case 4 If $\Delta < 0$, $\sqrt{\Delta}$ does not exist in the field of real numbers and so there are no real roots.
We say that the equation has no real roots.

TECHNOLOGY EXPLORATION

Investigating the discriminant

1. In GeoGebra, use the **Slider** tool to add three sliders *a*, *b* and *c*. Set each slider to have Min = -10, Max = 10 and Increment = 1.
2. In the input bar, enter *f(x)=a(x^2)+b(x)+c*. A parabola will appear.
3. In the input bar, enter *d=b^2-4*a*c*. This is the discriminant and will be shown as the value *d* in the **Algebra** view.
4. Use the **Move** tool to move the sliders. Notice what happens to the graph when $d < 0$. Make sure that you also consider values where $a < 0$.

Intersection of the parabola with the *x*-axis

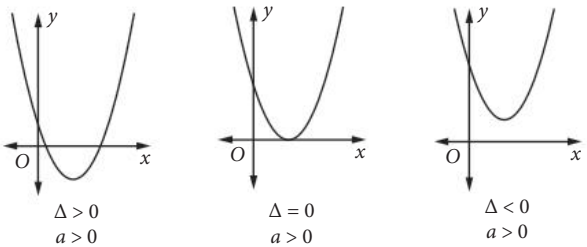
The roots of the equation $ax^2 + bx + c = 0$ are the x values of the points of intersection of the graph of the parabola $y = ax^2 + bx + c = 0$ with the x-axis. In other words, at the points where the parabola cuts the x-axis, $y = 0$.

Thus, the parabola $y = ax^2 + bx + c = 0$:

(i) cuts the x-axis at two distinct points if $\Delta > 0$

(ii) touches the x-axis at one point only (or two coincident points) if $\Delta = 0$

(iii) does not touch the x-axis if $\Delta < 0$.



$\Delta > 0$, $a > 0$ | $\Delta = 0$, $a > 0$ | $\Delta < 0$, $a > 0$

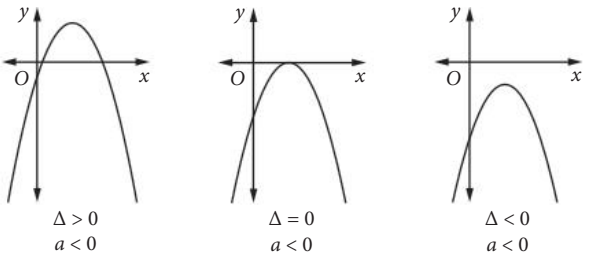
For the parabolas shown, the coefficient of x^2 is positive, so the vertex is at the bottom and the curve opens upwards.

When $a > 0$ and $\Delta < 0$, as in the third diagram shown, the graph of $f(x) = ax^2 + bx + c = 0$ is always above the x-axis, i.e. $f(x) > 0$ for all values of x. We say that f is a **positive definite** function because it is always positive.

For these parabolas the coefficient of x^2 is negative, so the vertex is at the top and the curve opens downwards.

When $a < 0$ and $\Delta < 0$, as in the third diagram at right, the graph of $f(x) = ax^2 + bx + c = 0$ is always below the x-axis, i.e. $f(x) < 0$ for all values of x. We say that f is a **negative definite** function because it is always negative.

If a quadratic function is neither positive definite nor negative definite, it is said to be **indefinite**.



Example 5

Write the discriminant of each of the following quadratic equations and hence state whether the parabola cuts the x-axis at two, one or no points.

(a) $x^2 - 5x + 6 = 0$ **(b)** $x^2 - 4x + 4 = 0$ **(c)** $2x^2 - 3x + 3 = 0$

Solution

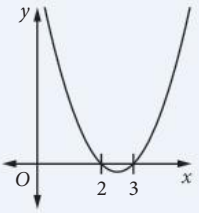
(a) $x^2 - 5x + 6 = 0$

$\Delta = b^2 - 4ac$

$\Delta = 25 - 24 = 1$

$\Delta > 0$

Two real roots, cuts x-axis at two distinct points



$y = x^2 - 5x + 6$

$y = (x - 2)(x - 3)$

Cuts at $(2, 0)$, $(3, 0)$

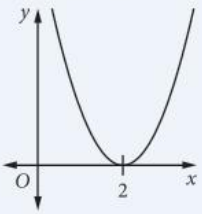
(b) $x^2 - 4x + 4 = 0$

$\Delta = b^2 - 4ac$

$\Delta = 16 - 16 = 0$

$\Delta = 0$

One real root, touches x-axis at one point



$y = x^2 - 4x + 4$

$y = (x - 2)^2$

Touches at $(2, 0)$

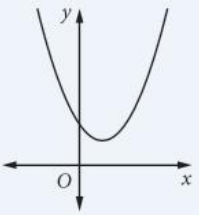
(c) $2x^2 - 3x + 3 = 0$

$\Delta = b^2 - 4ac$

$\Delta = 9 - 24 = -15$

$\Delta < 0$

No real roots, does not cut x-axis



$y = 2x^2 - 3x + 3$

Cannot factorise with real numbers, does not cut or touch x-axis.

Positive definite function, $a = 2$

The discriminant tells you whether the parabola cuts, touches, or does not cut the x-axis. To find the point at which this occurs, you need to solve the quadratic equation. Thus the discriminant tells you whether you can factorise the quadratic expression.

Example 6

Solve the following equations after first finding the discriminant.

(a) $3x^2 - 11x - 4 = 0$ **(b)** $4x^2 - 20x + 25 = 0$ **(c)** $x^2 = 2x + 5$ **(d)** $2x^2 + x + 3 = 0$

Solution

(a) $3x^2 - 11x - 4 = 0$: $\Delta = 121 + 48 = 169$

Because the discriminant is a perfect square, $3x^2 - 11x - 4$ has rational factors.

$3x^2 - 11x - 4 = 0$

$(3x + 1)(x - 4) = 0$

$x = -\frac{1}{3}$ or 4

(b) $4x^2 - 20x + 25 = 0$: $\Delta = 400 - 400 = 0$

Because the discriminant is zero, $4x^2 - 20x + 25$ has equal rational factors.

$$4x^2 - 20x + 25 = 0$$
$$(2x - 5)^2 = 0$$
$$x = 2.5$$

(c) $x^2 - 2x - 5 = 0$: $\Delta = 4 + 20 = 24$

Because $\Delta > 0$ but it is not a perfect square, the equation has two unequal, irrational roots, so it is best solved using the quadratic formula.

$$x^2 - 2x - 5 = 0$$
$$x = \frac{2 \pm \sqrt{24}}{2} = \frac{2 \pm 2\sqrt{6}}{2} = 1 \pm \sqrt{6}$$

(d) $2x^2 + x + 3 = 0$: $\Delta = 1 - 24 = -23$

Because $\Delta < 0$, there are no real roots.

EXERCISE 9.2 PARABOLAS AND DISCRIMINANTS

1 Select the equation for which $\Delta > 0$.

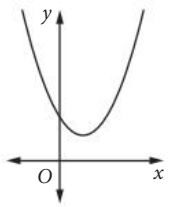
A $x^2 + 2x + 3 = 0$ **B** $x^2 + 6x + 9 = 0$ **C** $2x^2 - 4x + 5 = 0$ **D** $x^2 - 4x - 5 = 0$

2 Calculate the discriminant for each of the following equations and hence state whether the equations have two, one or no real roots.

(a) $x^2 + 6x + 2 = 0$ **(b)** $2x^2 + 3x + 4 = 0$ **(c)** $4x^2 - 12x + 9 = 0$
(d) $-3x^2 + 2x - 1 = 0$ **(e)** $2x^2 = 3x + 7$ **(f)** $4x^2 - 12x + 9 = 0$

3 For which curve can you say that $a > 0$ and $\Delta < 0$?

A



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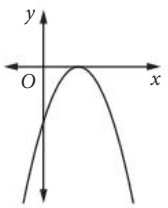

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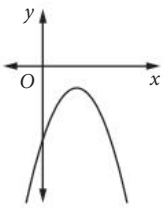

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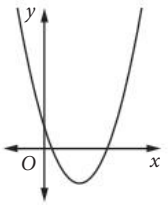


4 Without sketching the graphs of each function, determine whether or not they cross the x-axis.

(a) $y = x^2 - 5x + 2$ **(b)** $y = -4x^2 + 2x - 1$ **(c)** $y = x^2 - 6x + 9$
(d) $y = 8 - 3x - 2x^2$ **(e)** $y = 3x^2 + 2x + 5$ **(f)** $y = -x^2 - x - 1$

5 Calculate the discriminant for each of the following equations. Use this information to decide on the best solution method, then solve each equation.

(a) $x^2 + 2x - 15 = 0$ **(b)** $x^2 - 9x - 5 = 0$ **(c)** $12x^2 = 25x - 12$
(d) $4x^2 - 12x + 9 = 0$ **(e)** $7x^2 = 63$ **(f)** $x^2 - 6x = 0$
(g) $(x + 1)^2 = 4x$ **(h)** $x^2 = 4(x - 24)$ **(i)** $2x^2 - x = 25$
(j) $3x^2 = 2x + 2$ **(k)** $3x^2 - 7x - 3 = 0$ **(l)** $(x + 6)^2 = x + 6$
(m) $4x^2 = 9x - 4$ **(n)** $9x^2 + 24x + 16 = 0$ **(o)** $3x^2 + 4x = 5$
(p) $2x^2 + x - 4 = 0$ **(q)** $x^2 = 15x - 56$ **(r)** $x(2x - 3) = 0$
(s) $2x^2 + 5x + 1 = 0$ **(t)** $5x^2 - 7x + 2 = 0$ **(u)** $(x - 1)(x - 2) + (x + 1)(x + 3) = 0$

9.3 EQUATIONS REDUCIBLE TO QUADRATICS

Some non-quadratic equations with higher, even powers can be reduced to quadratic equations by making an appropriate substitution for the variable. The resulting quadratic equation can then be solved.

Example 7

Solve the equations.

(a) $(x^2-5x)^2-2(x^2-5x)-24=0$

(b) $x^4-3x^2-10=0$

(c) $4^x=12(2^x)-32$

(d) $\left(x+\frac{1}{x}\right)^2-7\left(x+\frac{1}{x}\right)+10=0$

Solution

(a) $(x^2-5x)^2-2(x^2-5x)-24=0$

Let $y=(x^2-5x)$

$\therefore y^2-2y-24=0$

$(y-6)(y+4)=0$

$y=6 \quad \text{or} \quad y=-4$

$\therefore (x^2-5x)=6 \quad \text{or} \quad (x^2-5x)=-4$

$x^2-5x-6=0 \quad \text{or} \quad x^2-5x+4=0$

$(x-6)(x+1)=0 \quad \text{or} \quad (x-4)(x-1)=0$

$\therefore x=6, -1, 4, 1$

(b) $x^4-3x^2-10=0$

Let $y=x^2$

$\therefore y^2-3y-10=0$

$(y-5)(y+2)=0$

$y=5 \quad \text{or} \quad y=-2$

$\therefore x^2=5 \quad \text{or} \quad x^2=-2$

$x=\pm\sqrt{5} \quad \text{or} \quad \text{no real roots}$

$\therefore x=\pm\sqrt{5}$

(c) $(2^x)^2-12(2^x)+32=0$

Let $y=2^x$

$\therefore y^2-12y+32=0$

$(y-8)(y-4)=0$

$y=8 \quad \text{or} \quad y=4$

$\therefore 2^x=8 \quad \text{or} \quad 2^x=4$

$2^x=2^3 \quad \text{or} \quad 2^x=2^2$

$x=3 \quad \text{or} \quad x=2$

$\therefore x=2, 3$

(d) $\left(x+\frac{1}{x}\right)^2-7\left(x+\frac{1}{x}\right)+10=0$

Let $y=x+\frac{1}{x}$

$\therefore y^2-7y+10=0$

$(y-2)(y-5)=0$

$y=2 \quad \text{or} \quad y=5$

$\therefore x+\frac{1}{x}=2 \quad \text{or} \quad x+\frac{1}{x}=5 \quad (\text{for } x\neq 0)$

$x^2+1=2x \quad \text{or} \quad x^2+1=5x$

$x^2-2x+1=0 \quad \text{or} \quad x^2-5x+1=0$

$(x-1)^2=0 \quad \text{or} \quad x=\frac{5\pm\sqrt{21}}{2}$

$\therefore x=1, \frac{5\pm\sqrt{21}}{2}$

EXERCISE 9.3 EQUATIONS REDUCIBLE TO QUADRATICS

1 Solve:

(a) $(x-2)^2-2(x-2)-15=0$

(b) $(x^2-2x)^2-11(x^2-2x)+24=0$

(c) $(x^2+2x)^2=14(x^2+2x)+15$

(d) $(x-3)^2=16$

2 The roots of $(x^2+5x)^2=6(x^2+5x)$ are:

A 0, 6 **B** 0, -5 **C** 1, -6 **D** -6, -5, 0, 1

3 Solve:

(a) $\left(x+\frac{3}{x}\right)^2=16$

(b) $3^{2x}-28(3^x)+27=0$

(c) $2^{2x}-6(2^x)+8=0$

(d) $4(2^{2x})-5(2^x)+1=0$

(e) $4^x-12(2^x)+32=0$

(f) $5(5^{2x})-26(5^x)+5=0$

(g) $2(2^{2x})=9(2^x)-4$

(h) $3(3^{2x})-28(3^x)+9=0$

4 The roots of $16(2^{2x}) - 33(2^x) + 2 = 0$ are:

A 1, 4 **B** -4, 1 **C** -1, 4 **D** -4, -1

5 Solve:

(a) $x^4 - 2x^2 - 8 = 0$ **(b)** $(2x-5)^2 = 2x - 5$ **(c)** $\left(x - \frac{4}{x}\right)^2 + 4\left(x - \frac{4}{x}\right) = 0$

(d) $x^4 + 5x^2 - 6 = 0$ **(e)** $2^{2x} = 4(2^x) + 32$ **(f)** $(x^2 - x)^2 - 8(x^2 - x) + 12 = 0$

(g) $x^6 - 28x^3 + 27 = 0$ **(h)** $x^4 - 13x^2 + 36 = 0$ **(i)** $x^4 - 5x^2 + 6 = 0$

(j) $x^4 - 2x^2 - 15 = 0$

6 Solve:

(a) $9^x - 4(3^x) + 3 = 0$ **(b)** $(x+1)^2 = 4x^2$ **(c)** $7^{2x+1} - 8(7^x) + 1 = 0$

(d) $\left(x + \frac{1}{x}\right)^2 - 5\left(x + \frac{1}{x}\right) + 6 = 0$ **(e)** $4^x - 8(2^x) + 15 = 0$ **(f)** $x^6 - 9x^3 + 8 = 0$

(g) $9^x - 4(3^{x+1}) + 27 = 0$ **(h)** $25^x - 28(5^x) + 75 = 0$

9.4 RELATIONSHIP BETWEEN ROOTS AND COEFFICIENTS

The general quadratic equation is $ax^2 + bx + c = 0$, $a \neq 0$.

Divide by a: $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$ (1)

Now let the roots of the equation be α and β, so:

$$(x - \alpha)(x - \beta) = 0$$

or: $x^2 - (\alpha + \beta)x + \alpha\beta = 0$ (2)

i.e. $x^2 - (\text{sum of roots})x + \text{product of roots} = 0$

Equating the coefficients of like powers in (1) and (2) gives:

$$\text{sum of roots} = \alpha + \beta = -\frac{b}{a} \quad (3)$$

$$\text{product of roots} = \alpha\beta = \frac{c}{a} \quad (4)$$

Equations (3) and (4) give us the relationships between the roots α, β and the coefficients a, b, c of a quadratic equation.

Example 8

Write the quadratic equation whose roots are:

(a) 3 and -2 **(b)** $3 + \sqrt{5}$ and $3 - \sqrt{5}$

Solution

(a) Sum of roots = 1

Product of roots = -6

$a = 1, b = -1, c = -6$

$\therefore$ equation is $x^2 - x - 6 = 0$

(b) Sum of roots = 6

Product of roots = 4

$a = 1, b = -6, c = 4$

$\therefore$ equation is $x^2 - 6x + 4 = 0$

Example 9

The quadratic equation $2x^2 - (m+2)x + m = 0$ has a root that is twice the other root. Find the values of m.

Solution

Let the roots be α and 2α.

Sum of roots $= 3\alpha = \frac{m+2}{2}$ Product of roots $= 2\alpha^2 = \frac{m}{2}$

Substitute $\alpha = \frac{m+2}{6}$ into $\alpha^2 = \frac{m}{4}$: $\left(\frac{m+2}{6}\right)^2 = \frac{m}{4}$

$$\frac{m^2+4m+4}{36} = \frac{m}{4}$$

$$m^2 + 4m + 4 = 9m$$

$$m^2 - 5m + 4 = 0$$

$$(m-1)(m-4) = 0$$

$$m = 1, 4$$

You can check your answer by substituting the values of m into the given equation and solving it to see that one root is twice the other.

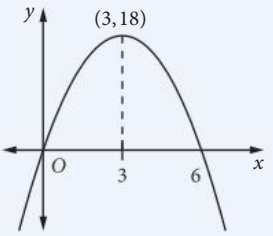
Example 10

The parabola $y = ax^2 + bx + c$ has vertex $(3, 18)$ and passes through the origin. Find the values of a, b and c.

Solution

By symmetry, the parabola cuts the x-axis again at $x = 6$.

So, $ax^2 + bx + c = 0$ has roots 0 and 6:



Sum of roots: $-\frac{b}{a} = 6, \quad \therefore b = -6a$

Product of roots: $\frac{c}{a} = 0, \quad \therefore c = 0$

At vertex, $x = 3, y = 18$: $18 = 9a + 3b$

$b = -6a$: $18 = 9a - 18a$

$\therefore a = -2, \ b = 12$

The equation of the parabola is $y = -2x^2 + 12x$.

Alternative method:

Substitute known coordinates into the equation. $y = ax^2 + bx + c$

When $x = 0, y = 0$: $c = 0$

When $x = 6, y = 0$: $36a + 6b = 0$

$b = -6a$

When $x = 3, y = 18$: $9a + 3b = 18$

$3a + b = 6$

Substitute $b = -6a$: $3a - 6a = 6$

$a = -2$

$b = 12$

EXERCISE 9.4 RELATIONSHIP BETWEEN ROOTS AND COEFFICIENTS

1 Write the quadratic equation whose roots are:

(a) 5, 2 **(b)** $\frac{1}{2}, \frac{1}{3}$ **(c)** $2 \pm \sqrt{3}$ **(d)** $p, 2p$

2 The roots of the quadratic equation $ax^2 - bx - 10 = 0$ are -1 and 5. The values of a and b are:

A $a = -2, b = -8$ **B** $a = 2, b = 8$ **C** $a = 2, b = 6$ **D** $a = -2, b = 6$

3 α and β are the roots of the equation $x^2 + 4x + 1 = 0$. Find the value of:

(a) $\alpha + \beta$ **(b)** $\alpha\beta$ **(c)** $\frac{1}{\alpha} + \frac{1}{\beta}$ after showing that $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta}$

4 The quadratic equation whose roots are p and $\frac{1}{p}$ is:

A $px^2 - px + p^2 + 1 = 0$ **B** $px^2 - x + p^2 + 1 = 0$ **C** $px^2 - (p^2 + 1)x + p = 0$ **D** $px^2 - p^2x + 1 = 0$

5 One root of the equation $x^2 + qx + r = 0$ is three times the other root. Prove that $3q^2 = 16r$.

6 The roots of the equation $x^2 - px + q = 0$ are 2 and -3. Find the value of p and q.

7 The quadratic equation $ax^2 + bx + 1 = 0$ has a root that is twice the other root. Prove that $2b^2 = 9a$.

8 Given that $x = 2$ is a root of the equation $x^2 + x + m = 0$, find the other root and the value of m.

9 If α and β are the roots of the equation $x^2 + 8x - 5 = 0$, find the value of $\frac{1}{\alpha} + \frac{1}{\beta}$.

10 Find the value of k for the equation $x^2 - (k + 2)x + 4k = 0$, if:

(a) the sum of the roots is 5 **(b)** the product of the roots is 12

(c) one root is 2 more than the other root.

11 **(a)** Show that $(\alpha + \beta)^2 - 2\alpha\beta = \alpha^2 + \beta^2$.

(b) If α and β are the roots of the equation $px^2 + qx + r = 0$, find (in terms of p, q and r):

(i) $\frac{1}{\alpha} + \frac{1}{\beta}$ **(ii)** $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$ **(iii)** $\alpha^2\beta + \alpha\beta^2$

12 The roots of the equation $2x^2 + bx + c = 0$ are -0.5 and -4. Find the values of b and c.

13 If one root of the equation $x^2 + px + q = 0$ is -2, show that $q = 2p - 4$.

14 Find the equation of the parabola that passes through the points $(-1, 0)$ and $(0, -1)$ and is symmetrical about the y-axis.

15 Find the equation of the parabola through the points $(2, 0)$, $(4, 0)$ and $(0, 8)$.

16 If one root of the equation $x^2 - mx + 2 = 0$ is double the other root, find the values of m.

17 Find p if the roots of the equation $3x^2 + px - 12 = 0$ differ by 4.

18 If α and β are the roots of the equation $x^2 + 5x - 8 = 0$, find the value of:

(a) $\alpha + \beta$ **(b)** $\alpha\beta$ **(c)** $\frac{1}{\alpha} + \frac{1}{\beta}$ **(d)** $\alpha^2\beta + \alpha\beta^2$ **(e)** $(\alpha + 2)(\beta + 2)$

19 For what values of k will the equation $x^2 - (k - 2)x - 12 = 0$ have:

(a) the sum of the roots equal to 5 **(b)** roots that are equal but opposite in sign?

9.5 SIGN OF A QUADRATIC FUNCTION

Example 11

For the graph of $y = 2x^2 - 3x - 5$, find the values of x for which:

(a) $y = 0$ **(b)** $y > 0$ **(c)** $y < 0$

Solution

(a) $2x^2 - 3x - 5 = 0$

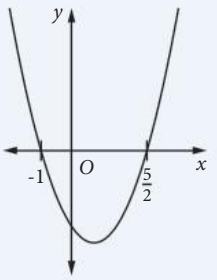
$(2x - 5)(x + 1) = 0$

$x = -1$ or 2.5

Hence the graph of $y = 2x^2 - 3x - 5$ crosses the x-axis at $x = -1$ and $x = 2.5$.

(b) From the graph it can be seen that $y > 0$ when $x < -1$ or $x > 2.5$.

(c) From the graph it can be seen that $y < 0$ when $x > -1$ or $x < 2.5$, i.e. when $-1 < x < 2.5$.



$y = 2x^2 - 3x - 5$

Example 12

Prove that $2x^2 - 4x + 5 > 0$ for all values of x.

Solution

This is the same as proving that the graph of $y = 2x^2 - 4x + 5$ does not cut the x-axis.

Consider $2x^2 - 4x + 5 = 0$: $\Delta = b^2 - 4ac$

$= 16 - 40$

< 0

Because $\Delta < 0$, $2x^2 - 4x + 5 = 0$ has no real roots.

As $a > 0$, thus $2x^2 - 4x + 5 > 0$ for all x. The function is positive definite.

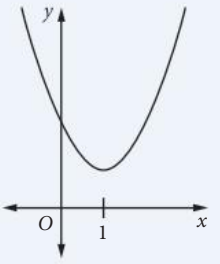
Alternatively:

Expression $= 2(x^2 - 2x) + 5$

$= 2(x^2 - 2x + 1) + 5 - 2$

$= 2(x - 1)^2 + 3$

$2(x - 1)^2 + 3$ has a minimum value of 3 when $x = 1$, so $2x^2 - 4x + 5 \geq 3$ for all x.



$y = 2x^2 - 4x + 5$

EXERCISE 9.5 SIGN OF A QUADRATIC FUNCTION

1 For what values of x are the following true?

(a) $x^2 - 3x - 10 > 0$ **(b)** $4 - x^2 \geq 0$ **(c)** $6 + 11x - 2x^2 \leq 0$ **(d)** $3x^2 + 14x < 5$

2 $4x^2 - 12x + 9 > 0$ for:

A all x **B** $x < 1.5$ **C** $x > 1.5$ **D** all x, $x \neq 1.5$

3 For the parabola $y = x^2 - (p + q)x + pq$ (where $p > q$), find the values of x for which $y \geq 0$.

4 For the parabola $y = x^2 + 4x$, find the values of x for which $y > 5$.

5 Prove that $x^2 + 2x + 7 > 0$ for all real values of x.

6 What is the relation between a and b if the graph of $y = x^2 + ax + b$:

(a) touches the x-axis **(b)** cuts the x-axis at two points?

7 For each of the following functions, find the values of x for which $y > 0$.

(a) $y = x^2 + 4x - 5$ **(b)** $y = 4x^2 - 3x - 7$ **(c)** $y = x^2 - 2x - 1$

(d) $y = -3x^2 + 2x + 1$ **(e)** $y = 1 + x - x^2$ **(f)** $y = 9 - x^2$

8 For each of the following functions, find the values of x for which $y < 0$.

(a) $y = x^2 - 4$ **(b)** $y = x^2 + 8x - 20$ **(c)** $y = 2x^2 + 4x + 1$

9 Given that $y = -2x^2 + 5x + 3$ and $-0.5 \leq x \leq 3$, which of the following is true?

A $y \geq 0$ **B** $y > 0$ **C** $y \leq 0$ **D** $y < 0$

10 For each of the following functions, show that $y > 0$ for all x.

(a) $y = x^2 + 2x + 4$ **(b)** $y = x^2 + 4$ **(c)** $y = 3x^2 + 2x + 1$

(d) $y = x^2 + x + 6$ **(e)** $y = 2x^2 - 3x + 4$ **(f)** $y = 2x^2 + 7$

9.6 FURTHER EXAMPLES INVOLVING DISCRIMINANTS

Example 13

Find the values of m for which the equation $x^2 + (m-2)x + 4 = 0$ has:

(a) one root **(b)** two roots **(c)** no roots.

Solution

(a) For one root only, $\Delta = 0$:

$$\begin{aligned}\Delta &= (m-2)^2 - 16\\ &= m^2 - 4m - 12\\ &= (m-6)(m+2)\end{aligned}$$

$\Delta = 0$ when $m = 6$ or -2

(b) For two roots, $\Delta > 0$:

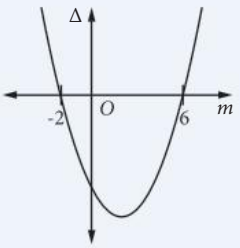
i.e. $(m-6)(m+2) > 0$

The graph of $\Delta = m^2 - 4m - 12$ cuts the m-axis at $m = 6$ and $m = -2$.

Thus $\Delta > 0$ when $m < -2$ or $m > 6$, as shown by the graph.

(c) For no roots, $\Delta < 0$:

From the graph, $\Delta < 0$ for $-2 < m < 6$.



$\Delta = m^2 - 4m - 12$

Example 14

Prove that the equation $x^2 + (k-3)x - k = 0$ has real roots for all values of k.

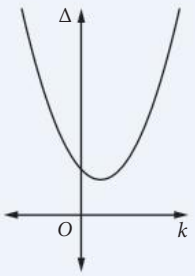
Solution

For real roots, $\Delta \geq 0$:

$$\begin{aligned}\Delta &= (k-3)^2 + 4k\\ &= k^2 - 2k + 9\end{aligned}$$

The graph of $\Delta = k^2 - 2k + 9$ does not cross the k-axis and the coefficient of k^2 is positive, so $\Delta > 0$ for all values of k.

Hence the equation $x^2 + (k-3)x - k = 0$ has real roots for all values of k.



$\Delta = k^2 - 2k + 9$

Alternatively:

$\Delta = k^2 - 2k + 9$

$= k^2 - 2k + 1 + 8$

$= (k-1)^2 + 8$

= a perfect square + a positive number

Hence $\Delta \geq 8$ for all k, so the given equation has real roots for all k.

Example 15

Show that the roots of the equation $x^2 + 2x - (m^2 - 1) = 0$ are rational for all rational values of m.

Solution

For rational roots, Δ is a perfect square: $\Delta = 4 + 4(m^2 - 1)$

$= 4m^2$

Hence Δ is a perfect square if m is a rational number.

$\therefore$ the equation $x^2 + 2x - (m^2 - 1) = 0$ has rational roots for all rational values of m.

EXERCISE 9.6 FURTHER EXAMPLES INVOLVING DISCRIMINANTS

1 Find the values of k for which the following quadratic equations have: **(i)** one root **(ii)** two roots.

(a) $x^2 - 3x + k = 0$ **(b)** $x^2 + kx + 3 = 0$

(c) $x^2 + (k-1)x - (2k+1) = 0$ **(d)** $(k-1)x^2 + (k+1)x = 1 - k$

2 The quadratic equation $(2k-3)x^2 + (k+1)x - 1 = 0$ has two roots when:

A $k > 1$ or $k < -11$ **B** $-11 < k < 1$ **C** $-11 \leq k \leq 1$ **D** $k \geq 1$ or $k \leq -11$

3 For what values of m does the quadratic equation $(5m-3)x^2 - 4mx + m + 1 = 0$ have only one root?

4 For what values of m are the roots of the following equations real?

(a) $x^2 + 2x + m^2 - 1 = 0$ **(b)** $(m-1)x^2 + (m+1)x + m - 1 = 0$ **(c)** $x^2 + 2mx + 2(m+12) = 0$

5 If the real roots of the equation $x^2 - 2mx + 3 = 0$ are a and b, find the values of m for which:

(a) $a = b$ **(b)** $a \neq b$

6 Show that the roots of the equation $4(m+1)x^2 - 4(m-1)x - 3 = 0$, for $m \neq -1$, are real for all real m.

7 Show that the roots of the equation $(3m-5)x^2 - 3m^2x + 5m^2 = 0$, for $m \neq \frac{5}{3}$, are rational if m is rational.

8 Show that the equation $x^2 - (2a+b)x + ab = 0$ has real roots for all values of a and b.

9 Find the values of p for which the equation $2x^2 - 4x + p = 0$ has:

(a) one root (i.e. two equal roots) **(b)** two distinct roots.

10 For what values of m does the equation $x^2 - 2mx + 8m - 15 = 0$ have: **(a)** one root **(b)** two roots?

11 Show that the roots of the equation $ax^2 - (a+b)x + b = 0$, for $a \neq 0$, are rational for all rational values of a and b.

12 For what values of m does the quadratic equation $x^2 + mx + (m+1)^2 = 0$ have two roots?

9.7 IDENTITY OF TWO QUADRATIC EXPRESSIONS

An equation of the nth degree has n roots and no more. A quadratic equation (second degree) has two roots: for example, the roots of $x^2 - 5x + 6 = 0$ are $x = 2$ and $x = 3$. A cubic equation (third degree) has three roots, a quartic equation (fourth degree) has four roots, and so on. The roots may be real numbers or unreal ('complex') numbers and some of them may be equal to one another.

If two polynomials of the nth degree are equal for more than n values of the variable, then they are **identically equal**, i.e. equal for all values of the variable. The sign $\equiv$ is used to indicate that the two expressions are identically equal. This also means that the coefficients of like powers of the variable in the two expressions are equal.

Theorem

If $a_1x^2 + b_1x + c_1 = a_2x^2 + b_2x + c_2$ for more than two values of x, then $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$.

(We will not prove this theorem here.)

This result has been used before when completing the square for a quadratic expression. For example:

$$2x^2 - 4x - 5 \equiv 2(x-1)^2 - 7$$

These two forms of the quadratic expression are equal for all values of x. We have merely changed the appearance of $2x^2 - 4x - 5$.

Example 16

Express $x^2 + 2x - 2$ in the form $ax(x+1) + bx^2 + c(x+1)$.

Solution

Method 1:

$$\begin{aligned} x^2 + 2x - 2 &\equiv ax(x+1) + bx^2 + c(x+1) \\ &\equiv ax^2 + ax + bx^2 + cx + c \\ &\equiv (a+b)x^2 + (a+c)x + c \end{aligned}$$

Equating coefficients of like powers of x: $a + b = 1$

$$a + c = 2$$

$$c = -2$$

Substitute for c in second coefficient equation: $a = 4$

Substitute for a in first coefficient equation: $b = -3$

Thus: $x^2 + 2x - 2 \equiv 4x(x+1) - 3x^2 - 2(x+1)$

Method 2:

The identity is true for all values of x, so select three convenient values of x to substitute into the identity to form three equations in a, b and c: $x^2 + 2x - 2 \equiv (a+b)x^2 + (a+c)x + c$

$x = 0$: $-2 = c$

$x = -1$: $-3 = b$

$x = 1$: $1 = 2a + b + 2c$

$1 = 2a + (-3) + 2 \times (-2)$

$a = 4$

Example 17

Express $x^2 + 2x - 2$ in the form $(x + a)^2 + b$.

Solution

$$x^2 + 2x - 2 \equiv (x + a)^2 + b$$
$$\equiv x^2 + 2ax + a^2 + b$$

Equating coefficients of like powers of x: $2 = 2a$ so $a = 1$

$-2 = a^2 + b$ so $b = -3$

$\therefore x^2 + 2x - 2 \equiv (x + 1)^2 - 3$

EXERCISE 9.7 IDENTITY OF TWO QUADRATIC EXPRESSIONS

1 Find the constants a, b and c such that $n^2 \equiv an(n + 1) + b(n + 1) + c$.

2 If $x^2 + 5x \equiv a(2x^2 + x) + b(3x^2 + x)$, then:

A $a = 14, b = 9$ **B** $a = -14, b = -9$ **C** $a = 14, b = -9$ **D** $a = -14, b = 9$

3 Express $x(x - 1)$ in the form $p + q(x - 1) + r(x - 1)^2$.

4 Find the constants a and b such that $x^2 + 10x + 10 \equiv a(x + 2)^2 + b(x + 1)$.

5 Express $x^4 - 6x^3 + 7x^2 + 6x - 8$ in the form $(x^2 + ax)^2 + b(x^2 + ax) + c$ and hence solve the equation $x^4 - 6x^3 + 7x^2 + 6x - 8 = 0$.

6 Solve the equation $x^4 + 4x^3 - x^2 - 10x + 6 = 0$ by expressing it in the form $(x^2 + 2x)^2 + b(x^2 + 2x) + c = 0$.

7 Express each quadratic polynomial in the form $a(x + b)^2 + c$.

(a) $x^2 - 4x + 6$ **(b)** $2x^2 - 4x - 5$ **(c)** $x^2 + x + 1$ **(d)** $2x^2 + 5x + 7$

8 Find the constants a and b such that $x^2 \equiv ax(x + 1) + bx$ for all values of x.

9 Express $x^2 + 2x - 2$ in the form $ax(x + 1) + bx^2 + c(x + 1)$.

10 Express $x^2 + 2x$ in the form $a(x - 1)(x + 2) + b(x + 2)$.

9.8 SOLUTION SET OF SIMULTANEOUS EQUATIONS

To find the points of intersection of a straight line and a parabola, you can algebraically calculate the solution set of a system consisting of the equation of the line and the equation of the parabola.

A straight line may cut a parabola at two distinct points, touch it at one point or not intersect it at all. If the line is a tangent to the parabola then the solution set has only one ordered pair (a 'double root').

Example 18

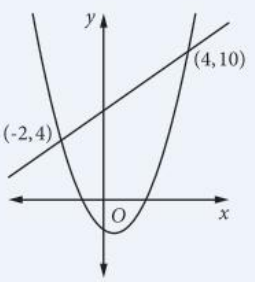
Find the points of intersection of the line $y = x + 6$ and the parabola $y = x^2 - x - 2$.

Solution

For the points of intersection: $x^2 - x - 2 = x + 6$

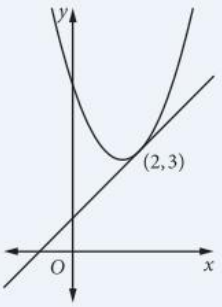
$$x^2 - 2x - 8 = 0$$
$$(x - 4)(x + 2) = 0$$
$$x = 4 \quad \text{or} \quad -2$$
$$y = 10 \quad \text{or} \quad 4$$

They intersect at the points $(4, 10)$ and $(-2, 4)$.



Example 19

Prove that the line $x - y + 1 = 0$ is a tangent to the parabola $y = x^2 - 3x + 5$.



Solution

For the points of intersection: $x^2 - 3x + 5 = x + 1$

$x^2 - 4x + 4 = 0$

$(x - 2)^2 = 0$

$x = 2, y = 3$

This is a double root, so the straight line is a tangent to the parabola at the point $(2, 3)$

Example 20

For what values of m does the line $y = mx - 6$ **(a)** touch **(b)** intersect **(c)** not intersect the parabola $y = x^2 - 2x + 3$?

Solution

For the points of intersection: $x^2 - 2x + 3 = mx - 6$

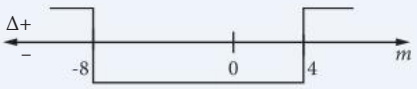
$x^2 - (m + 2)x + 9 = 0$

(a) Touch when $\Delta = 0$: $\Delta = b^2 - 4ac$

$= (m + 2)^2 - 36$

$= m^2 + 4m - 32$

$= (m + 8)(m - 4)$



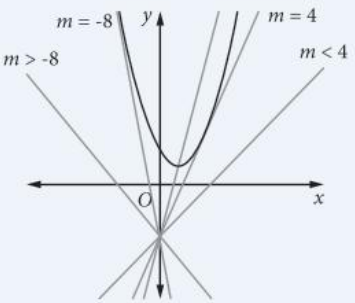
Will touch when $m = -8$ or 4

(b) Intersects when $\Delta > 0$: $(m + 8)(m - 4) > 0$

Intersects when $m < -8$ or $m > 4$

(c) Does not intersect when $\Delta < 0$: $(m + 8)(m - 4) < 0$

Does not intersect for $-8 < m < 4$.



Note: the principles involved in Examples 18, 19 and 20 apply to the intersection of a straight line and any curve whose defining equation is of the second degree (e.g. a circle, or a rational function $y = \frac{a}{x + b}$).

EXERCISE 9.8 SOLUTION SET OF SIMULTANEOUS EQUATIONS

1 Find algebraically the solution set of each system of equations.

(a) $y = 2x^2 - 3x + 4$ and $y = 12 - 3x$ **(b)** $y = 2 - x - 3x^2$ and $y = 2 - 7x$

(c) $y = x^2 + 3x - 2$ and $x + y = 3$ **(d)** $x - y = 1$ and $x^2 + xy = 6$

2 Find algebraically the coordinates of the intersection points of:

(a) the straight line $y = x - 3$ and the circle $x^2 + y^2 = 9$

(b) the straight line $y = 2x - 1$ and the parabola $y = x^2 - 3x + 5$

(c) the straight line $y = 3 - 2x$ and the parabola $y = (x - 2)^2$

3 For what value of c is the line $y = 2x + c$ a tangent to the parabola $y = x^2 - x - 2$?

A $c = -4.25$ **B** $c = 0.25$ **C** $c = -2.25$ **D** $c = 4.25$

4 For what value of c is the line $y = x + c$ a tangent to the circle $x^2 + y^2 = 4$?

5 For what value of a is the line $y = ax$ a tangent to the circle $x^2 + y^2 + 20x - 10y + 100 = 0$?

6 For what value of m is the line $y = mx$ a tangent to the parabola $y = x^2 - 8x + 25$?

7 For what value of m does the line $y = mx - 6$ **(a)** touch **(b)** intersect **(c)** not intersect the parabola $y = x^2 - 2x + 3$?

8 For what value of m does the line $y = mx + 5$ **(a)** touch **(b)** intersect **(c)** not intersect the parabola $y = 3 + 5x - 2x^2$?

9 For what value of m does the line $y = mx - 12$ **(a)** touch **(b)** intersect **(c)** not intersect the parabola $y = 2x^2 - x - 10$?

10 For what value of a does the line $y = ax$ intersect the curve with equation $y = \dfrac{2}{x-3}$?

11 For what value of a does the line $y = ax$ not meet the rectangular hyperbola $y = \dfrac{3}{x-2}$?

12 Find the equations of the two lines that contain the point $(1, 3)$ and are tangent to the parabola $y = x^2 - 2x + 5$.

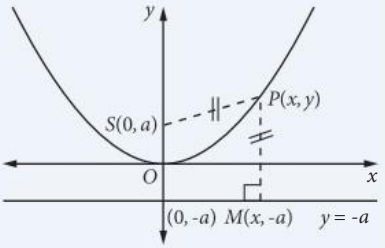
13 Prove that the parabolas $y = 2x^2 - 6x + 7$ and $y = x^2 - 2x + 3$ touch each other and find the coordinates of the point of contact.

9.9 THE PARABOLA AS A LOCUS

A parabola is the locus of a point $P(x, y)$ that moves in a plane so that its distance from a fixed point S is equal to its distance from a fixed straight line. We will consider only simple cases where the fixed straight line is parallel to the x-axis or the y-axis.

Example 21

Find the locus of a point $P(x, y)$ that moves in a plane so that its distance from the point $S(0, a)$ is equal to its distance from the line $y = -a$.



Solution

Use the distance formula: $PS = \sqrt{(x-0)^2 + (y-a)^2} = \sqrt{x^2 + (y-a)^2}$

$PM = y + a$

By definition: $PS = PM$

$\sqrt{x^2 + (y-a)^2} = y + a$

Square both sides: $x^2 + (y-a)^2 = (y+a)^2$

$x^2 + y^2 - 2ay + a^2 = y^2 + 2ay + a^2$

$x^2 = 4ay$ or $y = \dfrac{x^2}{4a}$

The equation of the locus is $x^2 = 4ay$. This equation defines a parabola with its vertex at $(0, 0)$ and an axis of symmetry that is the y-axis.

The fixed point S is called the **focus** and the fixed straight line is called the **directrix**.

The **focal length** is a, which is the distance of the focus from the vertex.

Equation: $x^2 = 4ay$

Vertex: $(0, 0)$ Focus: $(0, a)$ Focal length $= a$ Directrix: $y = -a$

For example, the equation $x^2 = 8y$ defines a parabola with focal length $a = 2$, vertex $(0, 0)$, focus $(0, 2)$, directrix $y = -2$ and axis of symmetry that is the y-axis.

Example 22

Find the locus of a point $P(x, y)$ that moves in a plane so that its distance from the point $S(a, 0)$ is equal to its distance from the line $x = -a$.

Solution

Compared to Example **21**, the point S is now on the x-axis and the line that was $y = -a$ is now parallel to the y-axis instead.

Use the distance formula: $PS = \sqrt{(x-a)^2 + y^2}$

$PM = x + a$

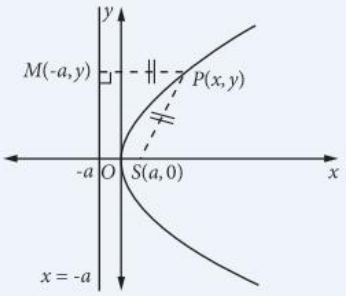
By definition: $PS = PM$

$\sqrt{(x-a)^2 + y^2} = x + a$

Square both sides: $(x-a)^2 + y^2 = (x+a)^2$

$x^2 - 2ax + a^2 + y^2 = x^2 + 2ax + a^2$

$y^2 = 4ax$



The equation of the locus is $y^2 = 4ax$.

The equation $y^2 = 4ax$ in Example **22** does not represent a function: it is a relation. It can be considered as the equation $x^2 = 4ay$ with the variables interchanged. This also represents the reflection of the parabola $x^2 = 4ay$ in the line $y = x$.

For example, the equation $y^2 = 12x$ defines a parabola with focal length $a = 3$, vertex $(0, 0)$, focus $(3, 0)$, directrix $x = -3$ and axis of symmetry that is the x-axis.

Note that in general the fixed point does not need to be on the x-axis or y-axis and the vertex does not need to be at the origin. The next example looks at this case.

Example 23

Find the locus of a point $P(x, y)$ that moves in a plane so that its distance from the point $S(2, 1)$ is equal to its distance from the line $y = -3$.

Solution

$S(2, 1)$, $y = -3$, $P(x, y)$

Use the distance formula: $PS = \sqrt{(x-2)^2 + (y-1)^2}$

$PM = y + 3$

By definition: $PS = PM$

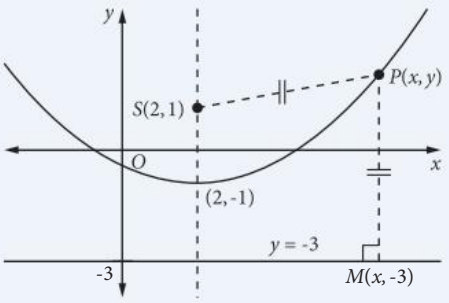
$\sqrt{(x-2)^2 + (y-1)^2} = y + 3$

Square both sides: $(x-2)^2 + (y-1)^2 = (y+3)^2$

$(x-2)^2 + y^2 - 2y + 1 = y^2 + 6y + 9$

$(x-2)^2 = 8y + 8$

$(x-2)^2 = 8(y+1)$



This equation is of the form $X^2 = 4aY$, where $X = x - 2$, $Y = y + 1$ and $a = 2$.

Thus the vertex of the parabola is $(2, -1)$.

Hence $(x-2)^2 = 8(y+1)$ is the equation of a parabola with vertex $(2, -1)$, focus $(2, 1)$, directrix $y = -3$, focal length = 2 and axis of symmetry $x = 2$.

In general, if the parabola $x^2 = 4ay$ undergoes a translation so that its vertex is at the point (h, k), then its equation becomes: $(x-h)^2 = 4a(y-k)$

This equation represents a parabola with vertex $V(h, k)$, focus $S(h, k+a)$, directrix $y = k - a$, focal length $= a$ and axis of symmetry $x = h$.

Example 24

Find the equation of the parabola with focus $(4, 1)$ and directrix the line $x = 2$.

Solution

$S(4, 1)$, $x = 2$, $P(x, y)$

Use the distance formula: $PS = \sqrt{(x-4)^2 + (y-1)^2}$

$PM = x - 2$

By definition: $PS = PM$

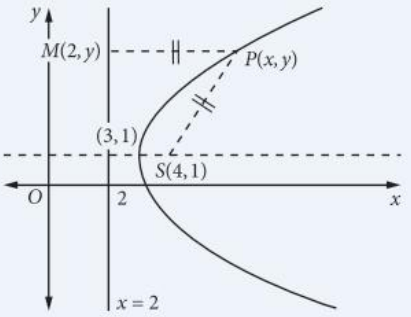
$$\sqrt{(x-4)^2 + (y-1)^2} = x - 2$$

Square both sides: $(x-4)^2 + (y-1)^2 = (x-2)^2$

$$x^2 - 8x + 16 + (y-1)^2 = x^2 - 4x + 4$$

$$(y-1)^2 = 4x - 12$$

$$(y-1)^2 = 4(x-3)$$



The locus is $(y-1)^2 = 4(x-3)$. This is in the form $Y^2 = 4aX$, where $Y = y - 1$, $X = x - 3$ and $a = 1$.

Thus the vertex is $(3, 1)$, the axis of symmetry is the line $y = 1$, and S is 1 unit to the right of the vertex.

In general, the parabola with equation $y^2 = 4ax$ has its vertex at $(0, 0)$, S is a units to the right of the vertex, and the x-axis is the axis of symmetry.

If this parabola undergoes a translation so that its vertex is at the point (h, k), then its equation becomes:

$$(y-k)^2 = 4a(x-h)$$

This means that S is a units to the right of the vertex, and the axis of symmetry is the line $y = k$.

Example 25

Show that the following equations represent parabolas and find the vertex, focus and the equation of the directrix for each.

(a) $y^2 - 2y - 2x - 4 = 0$ **(b)** $x^2 - 6x + 4y + 15 = 0$

Solution

(a) $y^2 - 2y - 2x - 4 = 0$

$y^2 - 2y + 1 = 2x + 5$

$(y-1)^2 = 2(x + 2.5)$

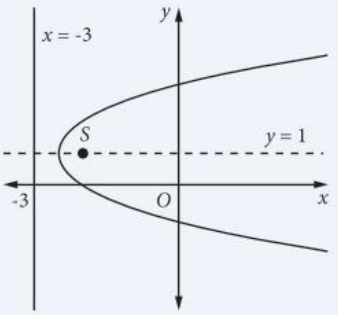
This equation is of the form $(y-k)^2 = 4a(x-h)$

The vertex is the point $(-2.5, 1)$

$4a = 2$ so $a = 0.5$, hence the focus is the point $(-2.5 + 0.5, 1) = (-2, 1)$

The equation of the directrix is $y = -2.5 - 0.5$, i.e. $y = -3$

The axis of symmetry is the line $y = 1$



(b) $x^2 - 6x + 4y + 15 = 0$

$x^2 - 6x + 9 = -4y - 6$

$(x-3)^2 = -4(y + 1.5)$

This equation is of the form $(x-h)^2 = -4a(y-k)$

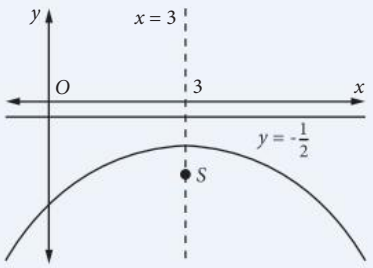
The negative sign means that the parabola opens downwards, so the focus is below the vertex.

The vertex is the point $(3, -1.5)$

$4a = 4$ so $a = 1$, hence the focus is the point $(3, -1.5 - 1) = (3, -2.5)$

The equation of the directrix is $y = -1.5 + 1$, i.e. $y = -0.5$

The axis of symmetry is the line $x = 3$



When dealing with the quadratic function we use the general form $y = ax^2 + bx + c$. When looking for the properties of a parabola, you need to 'complete the square' to change the equation into the locus form: $(x-h)^2 = 4a(y-k)$.

Example 26

Express each of the following in the form $(x-h)^2 = 4a(y-k)$ and hence state the vertex, focus and directrix.

(a) $y = x^2 - 2x + 5$ **(b)** $y = -2x^2 - 8x + 1$

Solution

(a) $y = x^2 - 2x + 5$

$= x^2 - 2x + 1 + 4$

$= (x-1)^2 + 4$

$\therefore (x-1)^2 = 1(y-4)$

The vertex is $(1, 4)$. The focal length is $a = 0.25$, so the focus is $(1, 4.25)$.

The directrix is $y = 3.75$ and the axis of symmetry is the line $x = 1$.

(b) $y = -2x^2 - 8x + 1$

$= -2(x^2 + 4x) + 1$

$= -2(x^2 + 4x + 4) + 1 + 8$

$= -2(x+2)^2 + 9$

$2(x+2)^2 = 9 - y$

$(x+2)^2 = -0.5(y-9)$

The vertex is $(-2, 9)$. $4a = 0.5$ so $a = \frac{1}{8}$ and the focus is below the vertex (due to the -0.5).

The focus is $\left(-2, 8\frac{7}{8}\right)$, the directrix is $y = 9\frac{1}{8}$ and the axis of symmetry is the line $x = -2$.

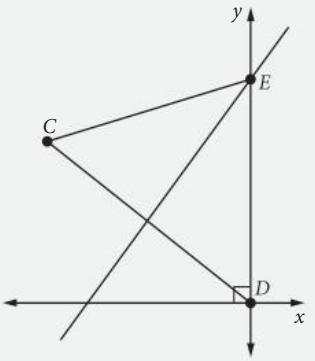
When solving problems like this, it may help to draw a quick sketch to show the relative positions of the focus and the directrix.

TECHNOLOGY EXPLORATION

Creating a parabola from the definition

1. In GeoGebra, use the **Line through Two Points** tool to create a line called *a* including points *A* and *B*. This line will represent the directrix.
2. Hide the two points *A* and *B* by clicking on the circle next to each name in the **Algebra** view or by right-clicking on each, selecting **Object Properties** and then deselecting 'Show Object'.
3. Use the **New Point** tool to create a new point *C*. This will be the focus.
4. We want the locus of the point that is equidistant from the line *a* and the point *C*. Because the distance from a line is always a perpendicular distance, we need a perpendicular line through *a*. To make this, use the **New Point** tool to create a point on the line *a*, then use the **Perpendicular Line** tool to click on this point and then on the line *a*.
5. We now need to create a point on this perpendicular line that is equidistant from both *C* and *D*. This point would form the apex of an isosceles triangle, as shown at right.

   As you can see: if we create a perpendicular bisector of *C* and *D*, then the point where this hits the perpendicular line through *D* will be equidistant from the point *C* and the line *a*. To create this new point *E*, first create the perpendicular bisector of *CD* using the **Perpendicular Bisector** tool.



6. Use the **Intersect Two Objects** tool to create the new point *E* from the intersection of the perpendicular line through *D* and the perpendicular bisector of *CD*.
7. Use the **Move** tool to drag the point *D* along the line. What shape does the point *E* trace out?
8. To see the locus of *E* as *D* varies, enter *locus[E,D]* into the input bar.

EXERCISE 9.9 THE PARABOLA AS A LOCUS

1 Sketch each parabola. Write the coordinates of the vertex and focus and the equations of the directrix and axis of symmetry.

(a) $x^2 = 8y$ **(b)** $y = \frac{x^2}{12}$ **(c)** $x^2 = 4y$ **(d)** $x^2 = -8y$

(e) $x^2 = -2y$ **(f)** $x^2 = -6y$ **(g)** $y^2 = x$ **(h)** $y^2 = -2x$

(i) $(x-1)^2 = 4(y+2)$ **(j)** $(x+3)^2 = -2y$ **(k)** $(y-2)^2 = 4(x+1)$ **(l)** $(y+4)^2 = -3(x+1)$

2 For the parabola $(x-2)^2 = 5(y+1)$, which of the following statements is not true?

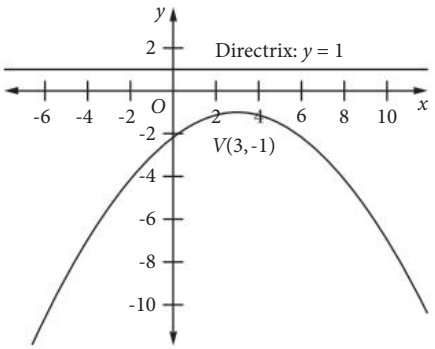
A vertex: $(2, -1)$ **B** focus: $(2, 0.25)$ **C** focal length: $a = 1.25$ **D** directrix: $y = -0.25$

3 Find the locus of a point $P(x, y)$ that moves so that:

(a) distance from the point $(0, 2)$ is equal to distance from the line $y = -2$
(b) distance from the point $(0, -4)$ is equal to distance from the line $y = 4$
(c) distance from the point $(0, 1)$ is equal to distance from the line $y = -1$
(d) distance from the point $(0, -2)$ is equal to distance from the line $y = 2$
(e) distance from the point $(-2, -4)$ is equal to distance from the line $y = 6$

4 The equation of the parabola shown in the diagram is:

A $(x+1)^2 = -8(y-3)$
B $(x-3)^2 = -8(y+1)$
C $(x+1)^2 = -8(y-3)$
D $(x-3)^2 = 8(y+1)$



5 Show that the following equations represent parabolas and find the vertex, focus and equation of the directrix for each.

(a) $y^2 - 4x + 2y - 3 = 0$ **(b)** $y^2 + 8x - 4y - 8 = 0$
(c) $x^2 - 2x + 4y + 15 = 0$ **(d)** $x^2 + 6x - 5y - 16 = 0$

6 Find the equation and sketch each of the following parabolas.

(a) vertex at $(0, 0)$, focal length 2 units, axis of symmetry that is the y-axis
(b) vertex $(0, 0)$, focus $(0, -1)$
(c) focus at $(0, 7)$, directrix the line $y = -7$
(d) vertex $(2, -1)$, focus $(2, 3)$
(e) the line $x = 2$ as axis, vertex $(2, 5)$, cuts y-axis at $y = 9$
(f) the line $x = -1$ as axis, focal length 2 units, cuts y-axis at $y = \frac{25}{8}$
(g) vertex $(2, -1)$, axis the line $x = 2$, focus $(2, 0)$
(h) the line $x = 0$ as axis, vertex $(0, 0)$, passes through the point $(-4, 2)$

7 Find the equation for the family of parabolas that have:

(a) the line $x = 2$ as axis, vertex $(2, 1)$ **(b)** the line $x = -2$ as axis, focal length 0.5 units.

8 Rewrite each of the following equations in the form $(x-h)^2 = 4a(y-k)$ and hence state the vertex, focus, directrix and axis of the parabola.

(a) $y = x^2 - 4x + 4$ **(b)** $y = x^2 + 6x + 6$ **(c)** $y = 4 - x^2$ **(d)** $4y = x^2 + 2x - 3$

CHAPTER REVIEW 9

1 By the method of 'completing the square', find:

(a) the maximum value of $5 + 4x - x^2$ **(b)** the minimum value of $2x^2 + 6x - 9$.

2 Without actually solving the following quadratic equations, determine whether each equation has two, one or no roots.

(a) $x^2 - 6x + 5 = 0$ **(b)** $2x^2 - 3x - 7 = 0$ **(c)** $x^2 - 20x + 100 = 0$ **(d)** $3x^2 + 4x - 1 = 0$

3 Solve the following equations.

(a) $(x^2 - x)^2 - 5(x^2 - x) + 6 = 0$ **(b)** $x^4 - 4x^2 - 45 = 0$ **(c)** $3^{2x} - 12(3^x) + 27 = 0$

4 For what values of k does the quadratic equation $x^2 - 5x + (k - 1) = 0$ have:

(a) two roots **(b)** one root **(c)** no roots?

5 Show that the roots of the equation $mx^2 - (m + n)x + n = 0$ are rational for all rational values of m and n.

6 Find the coordinates of the points of intersection of the line $y = 2x - 3$ and the parabola $y = x^2 - 4x + 5$.

7 If $2x^2 - 9x + 9 \equiv (ax - b)(x - b)$ for all values of x, find the values of a and b.

8 For what values of m does the line $y = mx - 5$ **(a)** touch **(b)** intersect **(c)** not intersect the parabola $y = x^2 - 5x + 4$?

9 Find the equation of the locus of the point $P(x, y)$ that moves so that its distance from the point $(-2, 1)$ is equal to its distance from the line $y = -1$. Express the locus in the form $(x - h)^2 = 4a(y - k)$ and hence show that it is a parabola. Find the vertex, focus, directrix, axis and focal length.

10 Express $4y = x^2 - 6x + 5$ in the form $(x - h)^2 = 4a(y - k)$ and hence find the vertex, focus, directrix, axis and focal length of the parabola.

11 The parabola $y = ax^2 + bx + 4$ passes through the points $(-2, 18)$ and $(1, 3)$. Find the values of a and b.

12 Show algebraically that the line $y = x - 4$ is a tangent to the circle $x^2 + y^2 = 8$ and find the coordinates of the point of contact.

CHAPTER 10
DIFFERENTIAL CALCULUS

Outcomes covered:

P1 demonstrates confidence in using mathematics to obtain realistic solutions to problems

P6 relates the derivative of a function to the slope of its graph

P7 determines the derivative of a function through routine application of the rules of differentiation

P8 understands and uses the language and notation of calculus

The study of calculus is rich in history and controversy. Isaac Newton (1642–1727) and Gottfried Wilhelm Leibniz (1646–1716) are both credited with the original development of calculus, but there was much argument at the time over who was first. Their approaches were quite different, although ultimately equivalent, and today the notation that we use comes from both sources. Important contributions to calculus were also made by mathematicians such as John Wallis, Isaac Barrow, Guillaume de l'Hôpital and later Augustin-Louis Cauchy. The work of mathematicians on differential calculus and integral calculus is the basis of much mathematics still used in a huge number of applications today.

The history of mathematics is a topic that lends itself best to individual reading and research. There are many books on the subject, as well as excellent online resources such as the History of Mathematics archive at the University of St Andrews.

10.1 CONTINUITY AND GRADIENTS OF TANGENTS

Continuity

We have drawn many different graphs in this course so far, but all of them can be divided into two special types:

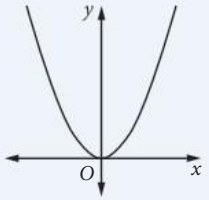
- Graphs such as $f(x) = x, f(x) = x^3, f(x) = \sin x$ and $f(x) = |x|$ are **continuous** curves: they do not have any gaps or jumps in them.
- Graphs such as $f(x) = \frac{1}{x}, f(x) = \tan x,\ f(x) = \frac{|x|}{x}$ and $f(x) = \frac{1}{x^2}$ are not continuous ('**discontinuous**') functions, because they all have gaps or jumps.

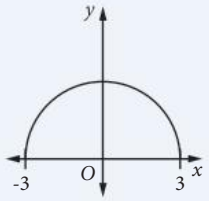
A simple way to describe a continuous function is that it can be drawn smoothly without lifting your pencil from the paper. Continuous functions like $f(x) = x, f(x) = x^3, f(x) = \sin x$ are smooth graphs, but $f(x) = |x|$ is not a smooth graph because the slope of its curve changes suddenly at $x = 0$ to create a sharp corner.

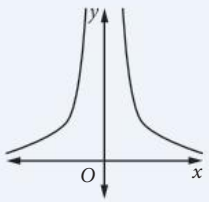
You should use appropriate software to graph all of these functions so that you can remember what they look like.

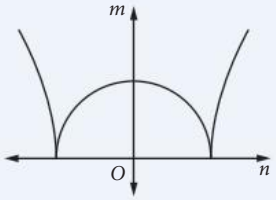
Example 1

For each graph, decide whether it is **(i)** continuous or discontinuous, and if continuous whether it is **(ii)** smooth or not smooth.

(a) 

(b) 

(c) 

(d) 

Solution

(a) **(i)** Continuous **(ii)** Smooth **(b)** **(i)** Continuous for $|x| \leq 3$ **(ii)** Smooth **(c)** **(i)** Discontinuous **(d)** **(i)** Continuous **(ii)** Not smooth

Each branch of **(c)**, if considered on its own, is a smooth continuous curve.

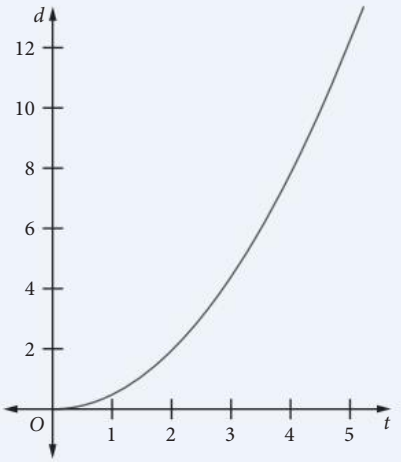
Rates of change and gradient

Smooth curves change gradually as x (the independent variable) increases. The rate of this change between two points on the curve can be calculated as the **average rate of change** over that interval. If the function increases as x increases, then the rate of change is positive. If the function decreases as x increases, then the rate of change is negative.

If we think back to the gradient of a line, a positive gradient was when y increased as x increased (moved from left to right). A negative gradient was when y decreased as x increased. Thus we can see that in general the gradient is the rate of change.

Example 2

The graph shows the distance d in kilometres that a car travels in t minutes. The function is given by the equation $d = \frac{t^2}{2}$.



The average speed of the car is the rate of change of the distance travelled over a time interval. Find the average speed over the:

(a) first minute
(b) second minute
(c) third minute
(d) first three minutes
(e) first five minutes.

Solution

(a) From the graph, find the value of d when $t = 0$, $t = 1$: $t = 0$, $d = 0$; $t = 1$, $d = 0.5$

$$\text{Average speed} = \frac{\text{distance covered}}{\text{time taken}} = \frac{0.5 - 0}{1 - 0} = 0.5 \text{ km/min}$$

(b) From the graph, find the value of d when $t = 2$: $t = 2$, $d = 2$

$t = 1$, $d = 0.5$; $t = 2$, $d = 2$: Average speed $= \frac{2 - 0.5}{2 - 1} = 1.5$ km/min

(c) From the graph, find the value of d when $t = 3$: $t = 3$, $d = 4.5$

$t = 2$, $d = 2$; $t = 3$, $d = 4.5$: Average speed $= \frac{4.5 - 2}{3 - 2} = 2.5$ km/min

(d) For $t = 0$, $d = 0$; $t = 3$, $d = 4.5$: Average speed $= \frac{4.5 - 0}{3 - 0} = 1.5$ km/min

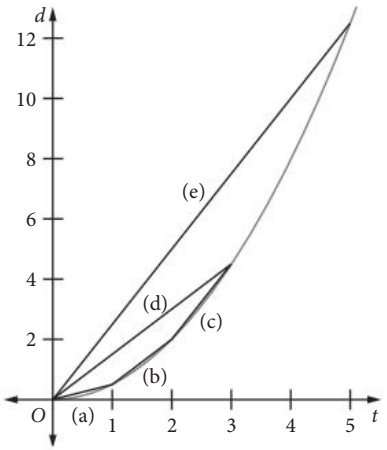
(e) From the graph, find the value of d when $t = 5$: $t = 5$, $d = 12.5$

$t = 0$, $d = 0$; $t = 5$, $d = 12.5$: Average speed $= \frac{12.5 - 0}{5 - 0} = 2.5$ km/min

The speedometer on the car would not show these speeds, because it shows the instantaneous speed. This is quite different to the average speed.

A secant is a line joining two points on a curve. A tangent to a curve is a line that just touches the curve. The tangent can be considered to be the limiting position of the secant. This means that if you fix one end of the secant and move the other end towards it, the secant becomes more and more like the tangent to the curve at the fixed end.

In Example **2** we were actually finding the gradient of each secant, as shown on the following graph:



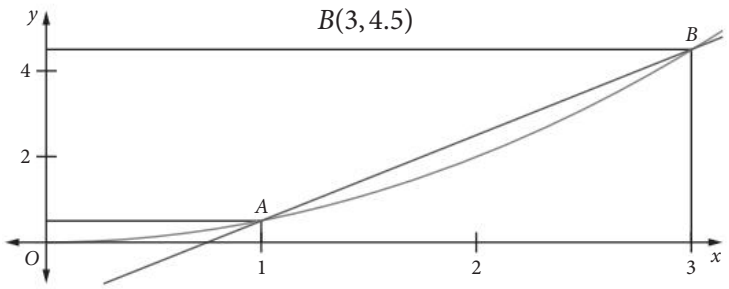
Average speed over the:

(a) first minute
(b) second minute
(c) third minute
(d) first three minutes
(e) first five minutes.

In general, this can be described as follows:

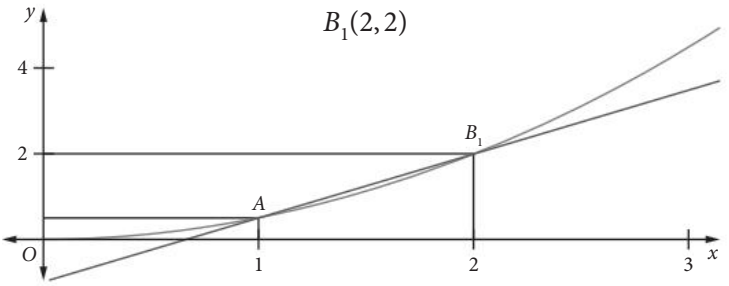
Let $A(x_1, y_1)$ be a fixed point on a curve. Take any other point $B(x, y)$ on the curve and join the points to obtain the secant AB. Let the point B get closer to the point A. This means that $|x - x_1|$ is getting smaller as the two points get closer together.

Consider the following graphs in which $A(1, 0.5)$ is a fixed point and B moves from the point $B(3, 4.5)$ to $B_1(2, 2)$ and then to $B_2(1.5, 1.125)$. By the third graph, it is hard to distinguish the secant AB from the curve.



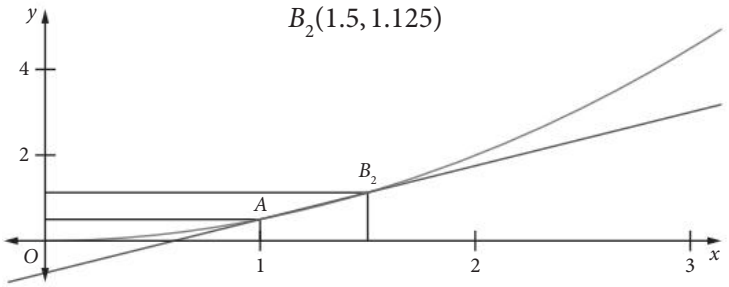
Gradient of $AB = \dfrac{4.5 - 0.5}{3 - 1} = 2$

The equation of AB is $y = 2x - 1.5$



Gradient of $AB_1 = \dfrac{2 - 0.5}{2 - 1} = 1.5$

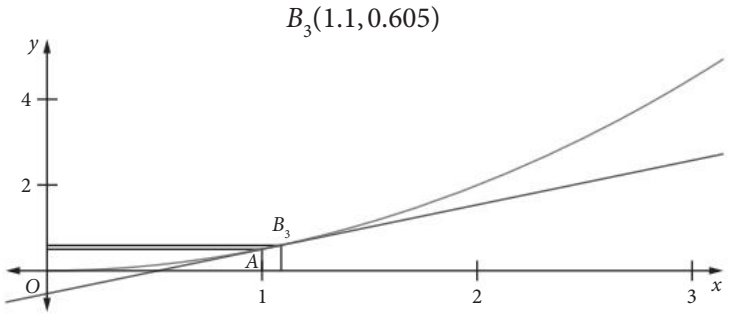
The equation of AB_1 is $y = 1.5x - 1$



Gradient of $AB_2 = \dfrac{1.125 - 0.5}{1.5 - 1} = 1.25$

The equation of AB_2 is $y = 1.25x - 0.75$

Let B move to $B_3(1.1, 0.605)$, as shown in the next graph. Here the secant and the curve appear to be the same curve. We can say that the curve here is 'locally straight'.

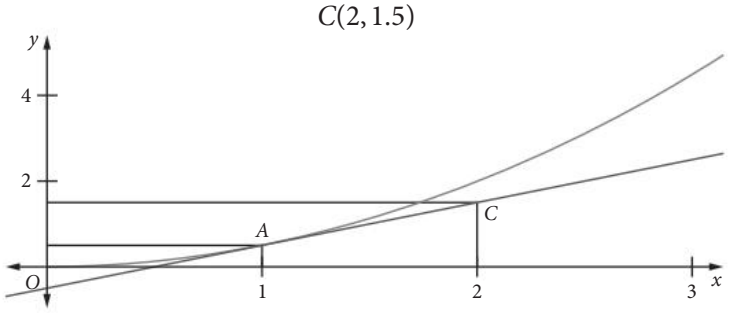


Gradient of $AB_3 = \dfrac{0.605 - 0.5}{1.1 - 1} = 1.05$

The equation of AB_3 is $y = 1.05x - 0.55$

The gradient of the secant AB has changed from 2 to 1.05 as B has moved closer to A. This suggests that as B gets closer to A, the gradient of the secant is getting closer to 1.

Let the point B move so that it coincides with the point A: the secant has now become the tangent at A. Importantly, the gradient of this tangent at A represents the **instantaneous rate of change** of the curve at the point A.



The tangent at A appears to pass through the point $C(2, 1.5)$.

Gradient of $AC = 1$

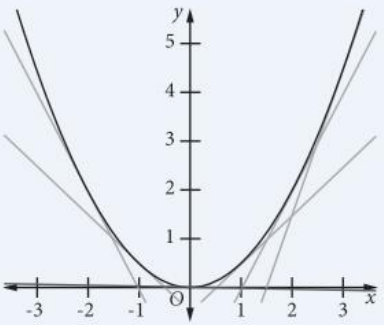
The equation of AC is $y = x - 0.5$

Given that the equation of the above curve is $y = \frac{1}{2}x^2$, use graphing software to reproduce each of the above diagrams. Investigate the point $B_4(1.05, 0.55125)$.

Repeat this investigation process for the point A approaching the point B. Does the gradient of the tangent at B appear to be 4.5?

Example 3

The graph of $y = \frac{1}{2}x^2$ is drawn for $-3 \leq x \leq 3$. Tangent lines are drawn at the points where $x = -2, -1, 0, 1, 2, 3$, as shown on the graph.



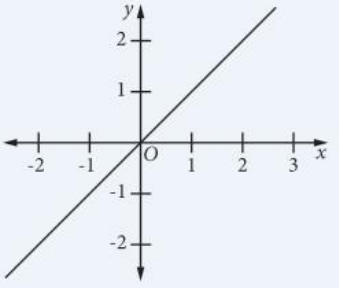
From the graph, find the slope of each of these tangents and use it to complete the following table:

x	-2	-1	0	1	2	3
m						

Plot these points on a separate number plane and write the equation of the curve that passes through them.

Solution

x	-2	-1	0	1	2	3
m	-2	-1	0	1	2	3



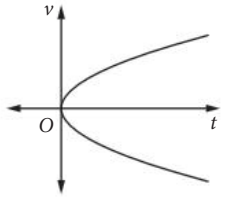
The equation of the line through these points is $y = x$.

This equation represents the gradient function for the curve, because it gives the gradient of the tangent to the curve at each point on the curve.

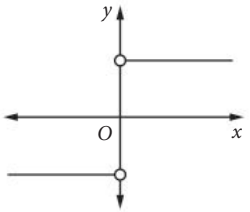
EXERCISE 10.1 CONTINUITY AND GRADIENTS OF TANGENTS

1 Indicate whether each graph is continuous or discontinuous.

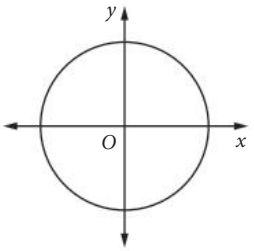
(a)



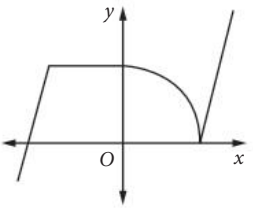
(b)



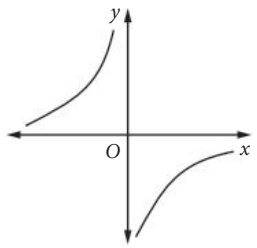
(c)



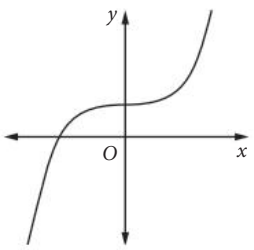
(d)



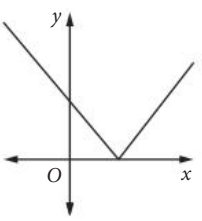
(e)



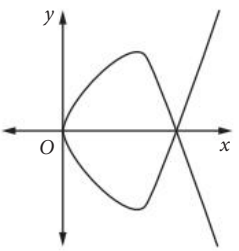
(f)



(g)



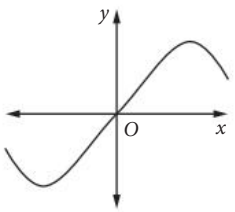
(h)



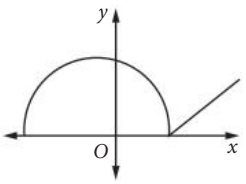
2 In question **1**, which curves could be called smooth?

3 Which graph represents a smooth continuous curve?

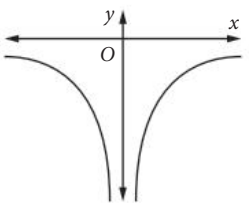
A



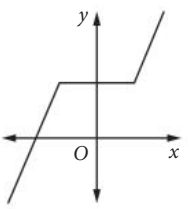
B



C



D



The following questions must be completed using GeoGebra or other graphing software.

4 Draw the graph of $y = x^2 + 1$. On it mark the point $A(1, 2)$. Consider the points $B(2.5, 7.25)$, $B_1(2, 5)$, $B_2(1.5, 3.25)$ and $B_3(1.1, 2.21)$.

(a) Find the gradient and equation of each of the secants AB, AB_1, AB_2 and AB_3.
(b) Draw a separate graph of each secant on the curve.
(c) What value does the gradient of the secant appear to approach as B gets closer to A?
(d) Check your answer to **(c)** by considering the secant AB_4, given $B_4(0.9, 1.81)$.

5 Draw the graph of $y = 4 - x^2$. On it mark the point $A(2, 0)$. Consider the points $B(0.5, 3.75)$, $B_1(1, 3)$, $B_2(1.5, 1.75)$ and $B_3(1.9, 0.39)$.

(a) Find the gradient and equation of each of the secants AB, AB_1, AB_2 and AB_3.
(b) Draw a separate graph of each secant on the curve.
(c) What value does the gradient of the secant appear to approach as B gets closer to A?
(d) Check your answer to **(c)** by considering the secant AB_4, given $B_4(2.1, -0.41)$.

6 Draw the graph of $y = x^2$ for $-3 \leq x \leq 3$. Draw tangent lines at the points where $x = -2, -1, 0, 1, 2, 3$.

(a) Complete the following table, where m is the gradient of the tangent at each point x.

x	-2	-1	0	1	2	3
m						

(b) Plot these points on a number plane.
(c) Write the equation of the curve in **(b)**.

7 Draw the graph of $y = x^3$ for $-2.5 \leq x \leq 2.5$. Draw tangent lines at the points where $x = -2, -1, 0, 1, 2$.

(a) Complete the following table, where m is the gradient of the tangent at each point x.

x	-2	-1	0	1	2
m					

(b) Plot these points on a number plane.
(c) Write the equation of the curve in **(b)**.

8 Draw the graph of $y = \frac{1}{x}$ for $x > 0$. On it mark the point $A(1, 1)$. Consider the points $B\left(3, \frac{1}{3}\right)$, $B_1\left(2, \frac{1}{2}\right)$, $B_2\left(\frac{3}{2}, \frac{2}{3}\right)$ and $B_3\left(\frac{11}{10}, \frac{10}{11}\right)$.

(a) Find the gradient and equation of each of the secants AB, AB_1, AB_2 and AB_3.

(b) Draw a separate graph of each secant on the curve.

(c) What value does the gradient of the secant appear to approach as B gets closer to A?

(d) Check your answer by considering the secant AB_4, given $B_4\left(\frac{9}{10}, \frac{10}{9}\right)$.

10.2 LIMIT AND CONTINUITY

Example 4

For the function f, where $f(x) = x + 2$, find $\lim\limits_{x \to 2}(x + 2)$ (that is, the limit of the function as x approaches 2).

Solution

The domain of this function is the set of real numbers. The table shows $f(x)$ for values of x near (or 'in the **neighbourhood** of') 2.

x	1.95	1.99	1.995	$\rightarrow 2 \leftarrow$	2.005	2.01	2.05
$f(x)$	3.95	3.99	3.995	$\rightarrow 4 \leftarrow$	4.005	4.01	4.05

This table shows that as x approaches 2 from **either** below or above 2, $f(x)$ approaches 4. We can make $f(x)$ as close to 4 as we like by making x sufficiently close to 2.

Hence: $\lim\limits_{x \to 2}(x + 2) = 4$

Notice that in this example, $f(2) = 4$. Thus $\lim\limits_{x \to 2}(x + 2) = f(2)$.

In this case we say that the function is **continuous** at $x = 2$.

Example 5

For the function f where $f(x) = \frac{x^2 - 4}{x - 2}$, find $\lim\limits_{x \to 2} \frac{x^2 - 4}{x - 2}$.

Solution

If $x = 2$, this function is not defined (as the denominator is zero).

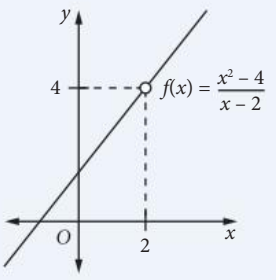
If $x \neq 2$: $\frac{x^2 - 4}{x - 2} = \frac{(x - 2)(x + 2)}{x - 2} = x + 2$

$\therefore \lim\limits_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim\limits_{x \to 2}(x + 2) = 4$

If $x \neq 2$ then $f(x) = \frac{x^2 - 4}{x - 2}$ is equivalent to $f(x) = x + 2$, so its graph is a straight line with a missing point at $x = 2$.

In this case we say that the function is **discontinuous** at $x = 2$:

$\lim\limits_{x \to 2} \frac{x^2 - 4}{x - 2} \neq f(2)$



Example 6

Investigate the following functions:

(a) $f(x)=\begin{cases} x+1 & \text{when } x \geq 1 \\ 3 & \text{when } x<1 \end{cases}$ **(b)** $f(x)=\dfrac{|x|}{x}, x \neq 0$

Solution

(a) Draw the graph of this of the function.

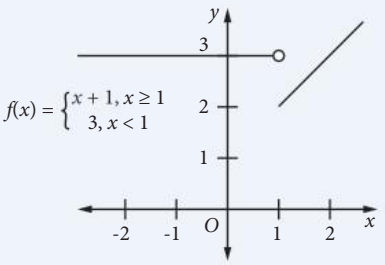
$x \geq 1$: $f(x)=x+1$

$\lim\limits_{x \to 1}(x+1)=2$

Thus as $x \to 1$ from above, $f(x) \to 2$ from above.

We can write this as $\lim\limits_{x \to 1^+} f(x)=2$

This can be confirmed by evaluating $f(1.01), f(1.001)$ etc.



$x<1$: $f(x)=3$

This means that the value of $f(x)$ is always 3, for all $x<1$.

Hence $\lim\limits_{x \to 1^-} f(x)=3$

Because $\lim\limits_{x \to 1^+} f(x) \neq \lim\limits_{x \to 1^-} f(x)$ the function is discontinuous at $x=1$.

(b)

$$f(x)=\begin{cases} \dfrac{x}{x}=1 & \text{when } x>0 \\ \dfrac{0}{0} & \text{(which is undefined) if } x=0 \\ -\dfrac{x}{x}=-1 & \text{when } x<0 \end{cases}$$

Draw the graph of the function.

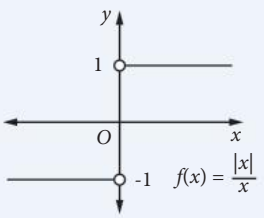
The function is defined for all values of x except $x=0$.

As $x \to 0$ from above, $f(x) \to 1$

As $x \to 0$ from below, $f(x) \to -1$

But $\lim\limits_{x \to 0} f(x)$ does not exist, because it approaches different values when approached from above or from below.

The function is discontinuous at $x=0$.



Example 7

Find $\lim\limits_{h \to 0} \dfrac{f(x+h)-f(x)}{h}$ where: **(a)** $f(x)=x^2$ **(b)** $f(x)=6+5x-2x^2$

Solution

(a)

$$\begin{aligned} f(x) &= x^2 \\ f(x+h) &= (x+h)^2 \\ &= x^2+2xh+h^2 \\ f(x+h)-f(x) &= x^2+2xh+h^2-x^2 \\ &= 2xh+h^2 \end{aligned}$$

$$\frac{f(x+h)-f(x)}{h} = \frac{2xh+h^2}{h}$$

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{2xh+h^2}{h}$$

$$= \lim_{h\to 0}\frac{h(2x+h)}{h}$$

$$= \lim_{h\to 0}(2x+h) \qquad \text{for } h \neq 0$$

$$= 2x$$

(b)

$$f(x) = 6+5x-2x^2$$

$$f(x+h) = 6+5(x+h)-2(x+h)^2$$

$$= 6+5x+5h-2x^2-4xh-2h^2$$

$$f(x+h)-f(x) = 6+5x+5h-2x^2-4xh-2h^2-(6+5x-2x^2)$$

$$= 5h-4xh-2h^2$$

$$\frac{f(x+h)-f(x)}{h} = \frac{5h-4xh-2h^2}{h}$$

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{5h-4xh-2h^2}{h}$$

$$= \lim_{h\to 0}(5-4x-2h) \qquad \text{for } h \neq 0$$

$$= 5-4x$$

Limit theorems

These theorems on limits of functions will be stated without their proofs.

Theorem 1

For the constant function f, where $f(x) = c$: $\lim\limits_{x\to a} f(x) = c$

Theorem 2

If $\lim\limits_{x\to a} f(x) = L$ and $\lim\limits_{x\to a} g(x) = M$, then: $\lim\limits_{x\to a}\big(f(x)\pm g(x)\big) = \lim\limits_{x\to a} f(x) \pm \lim\limits_{x\to a} g(x)$

$= L \pm M$

The limit of a sum = the sum of the limits.

The limit of a difference = the difference of the limits.

e.g. $\lim\limits_{x\to 2}\left(x^2-3x+5\right) = \lim\limits_{x\to 2}\left(x^2\right) - \lim\limits_{x\to 2}(3x) + \lim\limits_{x\to 2} 5$

$= 4-6+5 = 3$

Theorem 3

If $\lim\limits_{x\to a} f(x) = L$ and $\lim\limits_{x\to a} g(x) = M$, then: $\lim\limits_{x\to a}\big(f(x)\times g(x)\big) = \lim\limits_{x\to a} f(x) \times \lim\limits_{x\to a} g(x)$

$= L \times M$

The limit of a product = the product of the limits.

e.g. $\lim\limits_{x\to -1} 2x\left(x^2-4\right) = \lim\limits_{x\to -1}(2x) \times \lim\limits_{x\to -1}\left(x^2-4\right)$

$= (-2)\times(-3) = 6$

Theorem 4

If $\lim\limits_{x\to a} f(x) = L$ and $\lim\limits_{x\to a} g(x) = M$, then: $\lim\limits_{x\to a}\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$

$= \dfrac{L}{M}$ for $M \neq 0$

The limit of a quotient = the quotient of the limits.

e.g. $\lim\limits_{x\to 3}\dfrac{x^2+2}{x+1}=\dfrac{\lim\limits_{x\to 3}\left(x^2+2\right)}{\lim\limits_{x\to 3}(x+1)}$

$=\dfrac{11}{4}$

Continuity

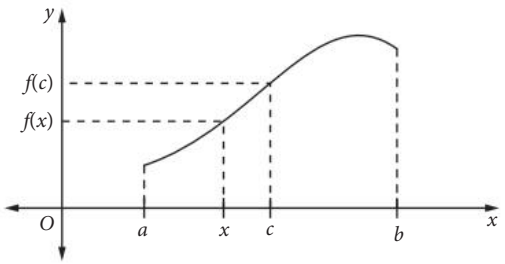
If $f(x)$ and $g(x)$ are both continuous at $x=a$, then the new functions $f(x)\pm g(x), f(x)\times g(x)$ and $\dfrac{f(x)}{g(x)}$ are also continuous at $x=a$. This is a direct consequence of the limit theorems.

Continuity at a point—formal definition

A function f that is defined in some neighbourhood of $x=c$ is said to be continuous at c if:

(a) the function has a definite value $f(c)$ at c, **AND**

(b) as $x\to c$, $f(x)\to f(c)$ as a limit, i.e. $\lim\limits_{x\to c} f(x)=f(c)$.



A function f is said to be continuous in an (open or closed) interval if it is continuous at all points of that interval. For a closed interval $[a,b]$, continuity at a and b implies that $\lim\limits_{x\to a^+} f(x)=f(a)$ and $\lim\limits_{x\to b^-} f(x)=f(b)$.

In simple language, a function is continuous in the interval $[a,b]$ if its graph can be drawn from $x=a$ to $x=b$ without lifting your pencil off the paper. (See the diagram above.) Thus all polynomial functions are continuous.

EXERCISE 10.2 LIMIT AND CONTINUITY

1 Evaluate each limit.

(a) $\lim\limits_{x\to 3}(3x)$ **(b)** $\lim\limits_{x\to -1}\left(x^2+4x\right)$ **(c)** $\lim\limits_{x\to 3}\left(9-x^2\right)$ **(d)** $\lim\limits_{x\to -2}\left(x^2-2x+1\right)$

(e) $\lim\limits_{x\to -4} x^2(x+2)$ **(f)** $\lim\limits_{h\to 2}\left(h^2-4h+4\right)$ **(g)** $\lim\limits_{a\to -1}(a+3)(a-4)$ **(h)** $\lim\limits_{x\to 3}\dfrac{x^2-5}{x+2}$

2 The value of $\lim\limits_{x\to -3}\dfrac{(x+5)(x+3)}{x+3}=\ldots$

A -2 **B** 0 **C** 2 **D** indeterminate

3 Evaluate the following limits.

(a) $\lim\limits_{x\to 3} 1$ **(b)** $\lim\limits_{x\to 0}\left(\dfrac{x^2+5x}{x}\right)$ **(c)** $\lim\limits_{x\to -2}\left(\dfrac{x^3+8}{x+2}\right)$

(d) $\lim\limits_{x\to 3}\dfrac{x^2-5x+6}{x-3}$ **(e)** $\lim\limits_{x\to 3}\dfrac{3x}{x+3}$ **(f)** $\lim\limits_{x\to 5}\dfrac{x-5}{2x^2-9x-5}$

(g) $\lim\limits_{x\to 1}\dfrac{x-1}{x^2+x-2}$ **(h)** $\lim\limits_{x\to 4}\dfrac{x-1}{x^2+x-2}$ **(i)** $\lim\limits_{x\to 1}\left(\dfrac{x^3-1}{x-1}\right)$

4 Evaluate each limit.

(a) $\lim\limits_{x\to 0} f(x)$ where $f(x)=\begin{cases} x^2+1 & \text{for } x\geq 0 \\ 1 & \text{for } x<0 \end{cases}$ **(b)** $\lim\limits_{x\to 1} f(x)$ where $f(x)=\begin{cases} 2x & \text{for } x\geq 1 \\ -2x+4 & \text{for } x<1 \end{cases}$

5 Evaluate $\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$ where:

(a) $f(x)=x^2-1$ **(b)** $f(x)=2x^2-3x+2$ **(c)** $f(x)=x^3$ **(d)** $f(x)=x(6-x)$

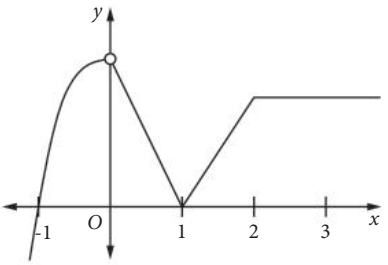
6 Show that the following limits do not exist:

(a) $\lim_{x\to 0}\dfrac{1}{x}$ **(b)** $\lim_{x\to 0} f(x)$ where $f(x)=\begin{cases}0 & \text{for } x<0\\ 1 & \text{for } x>0\end{cases}$

(c) $\lim_{x\to 0} f(x)$ where $f(x)=\begin{cases}x & \text{for } x>0\\ x+1 & \text{for } x<0\end{cases}$ **(d)** $\lim_{x\to 0} f(x)$ where $f(x)=\begin{cases}x^2+1 & \text{for } x>0\\ 2 & \text{for } x<0\end{cases}$

7 The function whose graph is shown is:

A discontinuous at $x=0$
B continuous for all x
C discontinuous at $x=2$
D continuous for all $x>0$ but discontinuous at $x=1$



10.3 GRADIENT OF A CURVE AT A POINT—FORMAL DEFINITION

The work in the previous section provides a link between the gradient of a secant at a point on a curve and the gradient of the tangent to the curve at one endpoint of the secant. As the endpoints of the secant get closer together, the gradient of the secant becomes closer to the gradient of the tangent at one endpoint.

Consider the point $P(2,4)$ on the curve $f(x)=x^2$. Let Q be a point near P on the same curve. Consider the gradient of the secant PQ for different positions of Q near P, as shown in the following tables:

x-coordinate of Q	Gradient of secant PQ
2.1	$\dfrac{2.1^2-4}{2.1-2}=4.1$
2.01	$\dfrac{2.01^2-4}{2.01-2}=4.01$
2.001	$\dfrac{2.001^2-4}{2.001-2}=4.001$
2.0001	$\dfrac{2.0001^2-4}{2.0001-2}=4.0001$

x-coordinate of Q	Gradient of secant PQ
1.9	$\dfrac{1.9^2-4}{1.9-2}=3.9$
1.99	$\dfrac{1.99^2-4}{1.99-2}=3.99$
1.999	$\dfrac{1.999^2-4}{1.999-2}=3.999$
1.9999	$\dfrac{1.9999^2-4}{1.9999-2}=3.9999$

It appears from the tables that as Q gets closer to P (that is, as x gets closer to 2), the gradient of the secant gets closer to 4. In general:

Let Q be the point $(x, f(x))$ and P be the fixed point $(2,4)$ on $f(x)=x^2$. The gradient of $PQ=\dfrac{f(x)-4}{x-2}$.

The information from our table for the function $f(x)=x^2$ indicates that as x approaches 2, the value of the gradient $\dfrac{f(x)-4}{x-2}$ approaches 4.

This sentence can be written in a new notation as $\lim_{x\to 2}\dfrac{f(x)-4}{x-2}=4$. This is read as 'the limit as x approaches 2 of function x minus 4 divided by x minus 2 is equal to 4'.

The gradient of a curve

Following the example above: if we let our points be $P(c, f(c))$ and $Q(x, f(x))$, then the gradient of the secant PQ is given by the general expression $\frac{f(x)-f(c)}{x-c}$. This expression does not exist when $x = c$ because the denominator becomes zero.

From our calculations above we have seen that the limiting value of the gradient of the secant is defined to be the gradient of the tangent at P, and is simply called the gradient of the curve at P.

The gradient of the curve $y = f(x)$ at the point $P(c, f(c))$ is defined as the limiting value $\lim\limits_{x \to c} \frac{f(x)-f(c)}{x-c} = f'(c)$, provided that this limit exists.

Thus $f'(c)$ is the slope of the tangent to the curve $y = f(x)$ at the point $x = c$.

This is also called the **derivative** of $f(x)$ at $x = c$.

Gradient and derivative notation

For the function $f(x)$, the gradient function or the derived function is given by the notation $f'(x)$. However, there are also other common notations used for this derivative function.

Let $x = c + h$, where h may be positive or negative. Thus $x - c = h$, so as $x \to c$ then $h \to 0$.

This gives $f'(c) = \lim\limits_{h \to 0} \frac{f(c+h)-f(c)}{h}$.

Also, if $\delta x = h$ and $\delta y = f(c + h) - f(c)$ when $y = f(x)$, then we can write $f'(c) = \lim\limits_{\Delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx}$.

If in $f'(c) = \lim\limits_{h \to 0} \frac{f(c+h)-f(c)}{h}$ we replace c by x, then we obtain $f'(x) = \lim\limits_{h \to 0} \frac{f(x+h)-f(x)}{h}$.

This now gives us a definition of the derivative at any point (x, y) on the curve $y = f(x)$.

The gradient at any point (x, y) on the curve $y = f(x)$ can be represented by any of the following notations: $f'(x)$, $\frac{dy}{dx}$, $\frac{d}{dx}(f(x))$.

EXERCISE 10.3 GRADIENT OF A CURVE AT A POINT—FORMAL DEFINITION

1 $P(1, 1)$ is a point on the curve $f(x) = x^2$. Complete the following table of values.

x-coordinate of Q	0.9	0.99	0.999	1.001	1.01	1.1
Gradient of secant PQ						

Use the table to find $\lim\limits_{x \to 1} \frac{f(x)-f(1)}{x-1}$.

2 $P(1, 5)$ is a point on the curve $f(x) = 2x + 3$. Complete the following table of values.

x-coordinate of Q	0.9	0.99	0.999	1.001	1.01	1.1
Gradient of secant PQ						

Use the table to find $\lim\limits_{x \to 1} \frac{f(x)-f(1)}{x-1}$.

Could you have predicted this answer from the function? Why?

3 $P(3, 9)$ is a point on the curve $f(x) = x^2$. Complete the following table of values.

x-coordinate of Q	2.9	2.99	2.999	3.001	3.01	3.1
Gradient of secant PQ						

Use the table to find $\lim\limits_{x \to 3} \frac{f(x)-f(3)}{x-3}$.

4 If $g(x) = x^2$, then $\dfrac{g(3.999) - g(4)}{3.999 - 4}$ is closest to:

A -8 **B** 0.008 **C** 0.001 **D** 8

5 $P(1, 1)$ is a point on the curve $f(x) = x^3$. Complete the following table of values.

x-coordinate of Q	0.9	0.99	0.999	1.001	1.01	1.1
Gradient of secant PQ						

Use the table to find $\lim\limits_{x \to 1} \dfrac{f(x) - f(1)}{x - 1}$.

6 $P\left(2, \frac{1}{2}\right)$ is a point on the curve $f(x) = \dfrac{1}{x}$. Complete the following table of values.

x-coordinate of Q	1.9	1.99	1.999	2.001	2.01	2.1
Gradient of secant PQ						

Use the table to find $\lim\limits_{x \to 2} \dfrac{f(x) - f(2)}{x - 2}$.

7 $P(-2, 4)$ is a point on the curve $f(x) = x^2$. Complete the following table of values.

x-coordinate of Q	-2.1	-2.01	-2.001	-1.999	-1.99	-1.9
Gradient of secant PQ						

Use the table to find $\lim\limits_{x \to -2} \dfrac{f(x) - f(-2)}{x + 2}$.

10.4 FINDING THE DERIVATIVE FROM FIRST PRINCIPLES

This process is also called 'differentiation from first principles'. We will use this to show how the general result for the derivative of x^n (powers of x) is obtained.

Example 8

Find $f'(c)$ if: **(a)** $f(x) = x$ **(b)** $f(x) = x^2$

Solution

(a) $f(x) = x$:

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

$$f'(c) = \lim_{x \to c} \frac{x - c}{x - c} = 1$$

This is a constant. It is the gradient of the straight line $y = x$.

(b) $f(x) = x^2$:

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

$$f'(c) = \lim_{x \to c} \frac{x^2 - c^2}{x - c}$$

$$= \lim_{x \to c} \frac{(x - c)(x + c)}{x - c}$$

$$= \lim_{x \to c} (x + c)$$

$$= 2c$$

Because this limit exists for all values of c in the domain of the function, we can write $f'(x) = 2x$.

Example 9

Find the gradient of the tangent at the point $(4, 16)$ on the curve $y = x^2$ using the result $f'(c) = \lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c}$.

Solution

At $(4, 16)$, $x = 4$: $f'(4) = \lim_{x \to 4} \frac{f(x) - f(4)}{x - 4}$

$$f'(4) = \lim_{x \to 4} \frac{x^2 - 16}{x - 4}$$
$$= \lim_{x \to 4} \frac{(x-4)(x+4)}{x-4}$$
$$= \lim_{x \to 4} (x + 4)$$
$$= 8$$

Example 10

Use the result $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ to find the expression for $f'(x)$, if:

(a) $f(x) = x$ **(b)** $f(x) = x^2$ **(c)** $f(x) = x^3$

Solution

(a) $f(x) = x$: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$f'(x) = \lim_{h \to 0} \frac{x + h - x}{h}$$
$$= \lim_{h \to 0} 1$$
$$= 1$$

(b) $f(x) = x^2$: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$
$$= \lim_{h \to 0} \frac{((x+h) - x)((x+h) + x)}{h}$$
$$= \lim_{h \to 0} \frac{h(2x + h)}{h}$$
$$= \lim_{h \to 0} (2x + h)$$

As $h \to 0$, $2x + h \to 2x$, so: $f'(x) = 2x$

(c) $f(x) = x^3$: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

Now $(x + h)^3 = (x + h)(x + h)^2 = x^3 + 3x^2h + 3xh^2 + h^3$

$$f'(x) = \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$$
$$= \lim_{h \to 0} \frac{h\left(3x^2 + 3xh + h^2\right)}{h}$$
$$= \lim_{h \to 0} \left(3x^2 + 3xh + h^2\right)$$
$$= 3x^2$$

OR

Use $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$: $f'(x) = \lim\limits_{h\to 0} \dfrac{((x+h)-x)\left((x+h)^2 + x(x+h) + x^2\right)}{h}$

$$= \lim_{h\to 0} \frac{h\left((x+h)^2 + x(x+h) + x^2\right)}{h}$$

$$= \lim_{h\to 0} \left((x+h)^2 + x(x+h) + x^2\right)$$

As $h \to 0$, $x + h \to x$, so: $f'(x) = x^2 + x^2 + x^2 = 3x^2$

These examples seem to suggest that there might be a rule for finding derivatives without resorting to first principles every time.

Important result

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + ab^{n-2} + b^{n-1})$$

This factorisation can be shown to be true by expanding the right-hand side and collecting like terms.

If $f(x) = x^n$, then the above factorisation can be used to find the derivative as follows:

$f(x) = x^n$: $\quad f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$

$$= \lim_{h\to 0} \frac{(x+h-x)\left((x+h)^{n-1} + (x+h)^{n-2}x + (x+h)^{n-3}x^2 + \ldots + (x+h)x^{n-2} + x^{n-1}\right)}{h}$$

$$= \lim_{h\to 0} \frac{h\left((x+h)^{n-1} + (x+h)^{n-2}x + (x+h)^{n-3}x^2 + \ldots + (x+h)x^{n-2} + x^{n-1}\right)}{h}$$

$$= \lim_{h\to 0} \left((x+h)^{n-1} + (x+h)^{n-2}x + (x+h)^{n-3}x^2 + \ldots + (x+h)x^{n-2} + x^{n-1}\right)$$

$$= \underbrace{x^{n-1} + x^{n-1} + x^{n-1} + \ldots + x^{n-1}}_{n \text{ factors}}$$

$$= nx^{n-1}$$

This process can be described in words: 'To differentiate a power of x, multiply the power by x to the power reduced by one.'

EXERCISE 10.4 FINDING THE DERIVATIVE FROM FIRST PRINCIPLES

1 Use the result $f'(c) = \lim\limits_{x\to c} \dfrac{f(x) - f(c)}{x - c}$ to find the value of:

(a) $f'(5)$ when $f(x) = x^2$ **(b)** $f'(-1)$ when $f(x) = x^2$

(c) $f'(3)$ when $f(x) = x^3$ **(d)** $f'(0)$ when $f(x) = x^3 + 2x + 1$

2 Use the result $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$ to find:

(a) $f'(-2)$ when $f(x) = x^2$ **(b)** $f'(-1)$ when $f(x) = x^3$

3 For the graph of $f(x) = 6x - 2x^2$:

(a) find the gradient of the chord joining the points whose x-coordinates are 1 and $1 + h$ respectively

(b) deduce the gradient of the curve at $x = 1$.

4 If $f(x) = 3 - 2x + 4x^2$, then $\lim\limits_{h\to 0} \dfrac{f(1+h) - f(1)}{h} = \ldots$

A -10 **B** 0 **C** 5 **D** 6

5 For the function $f(x) = 2x^2 - 4x$, find the following:

(a) $\lim\limits_{h\to 0} \dfrac{f(3+h)-f(3)}{h}$ **(b)** $\lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$

Interpret your results geometrically.

6 Find $\lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$ for the following:

(a) $f(x) = 4x^2 - 1$ **(b)** $f(x) = \dfrac{x^2}{2} - 2x - 3$ **(c)** $f(x) = x^3 - 2x^2$

7 $P(1, 1)$ and $Q(2, 8)$ are points on the curve $f(x) = x^3$. Indicate whether each statement is correct or incorrect.

(a) Gradient of $PQ = 7$ **(b)** $f'(2) = \lim\limits_{x\to 1} \dfrac{x^3 - 8}{x - 2}$ **(c)** $f'(1) = \lim\limits_{x\to 1} \dfrac{x^3 - 1}{x - 1}$ **(d)** $f'(x) = 3x^2$

10.5 CONDITIONS FOR DIFFERENTIABILITY

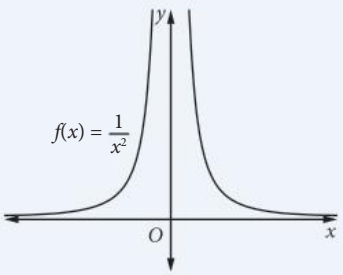
If $f(x)$ possesses a derivative $f'(x)$ for each x belonging to the domain of f, then $f(x)$ is called a **differentiable function**.

The statement '$f(x)$ is differentiable' means '$f(x)$ has a derivative at each point of its domain'.

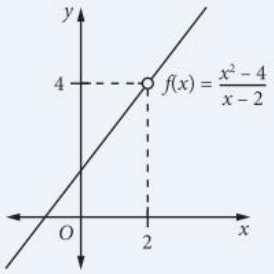
Example 11

Investigate the continuity and differentiability of the given graphs.

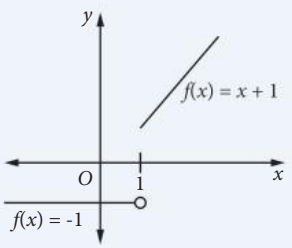
(a)



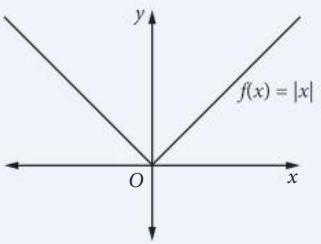
(b)



(c)



(d)



Solution

(a) $f(x) = \dfrac{1}{x^2}, x \neq 0$

$f'(0)$ cannot be found because $f(0)$ is undefined.

Not differentiable at $x = 0$. The function is discontinuous at $x = 0$.

(b) $f(x) = \dfrac{x^2 - 4}{x - 2}, x \neq 2$

$f'(2)$ cannot be found because $f(2)$ is undefined.

Not differentiable at $x = 2$. The function is discontinuous at $x = 2$.

(c) $f(x) = \begin{cases} x + 1 & \text{for } x \geq 1 \\ -1 & \text{for } x < 1 \end{cases}$

$f'(1)$ cannot be found because the left-hand derivative does not exist at $x = 1$, even though $f(1)$ is defined.

Not differentiable at $x = 1$. The function is discontinuous at $x = 1$.

(d) $f(x) = |x|$

$f'(0)$ cannot be found because the left-hand derivative is -1 and the right-hand derivative is 1.

Not differentiable at $x = 0$. The function is continuous at $x = 0$.

EXERCISE 10.5 CONDITIONS FOR DIFFERENTIABILITY

In each case write the value of $f'(a)$, if it exists, and sketch the graph of f.

1 **(a)** $f(x) = x - 2, a = 2$ **(b)** $f(x) = x^2 - 4, a = 2$ **(c)** $f(x) = \dfrac{x^2 - 4}{x + 2}, a = 2$ **(d)** $f(x) = \dfrac{x^2 - 2x}{x - 2}, a = 2$

2 $f(x) = \begin{cases} x, & x \leq 1 \\ (x-2)^2, & x > 1 \end{cases}$ for $a = 1, a = 3$

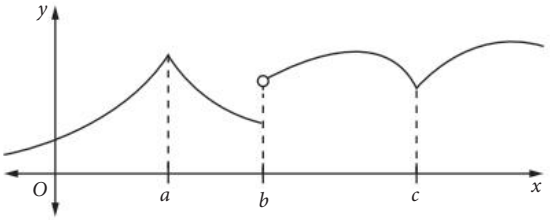
3 $f(x) = |x - 2|$ for $a = 2, a = 4$

4 $f(x) = \begin{cases} x^2, & x \leq 0 \\ x + 1, & x > 0 \end{cases}$ for $a = 0, a = -1$

5 $f(x) = \begin{cases} x^2 - 2x, & x \leq 2 \\ x - 2, & x > 2 \end{cases}$ for $a = 2, a = 3$

6 $f(x) = \begin{cases} x, & x > 3 \\ 3, & -3 \leq x \leq 3 \\ x + 1, & x < -3 \end{cases}$ for $a = 3, a = -3$

7 For the points at $x = a$, b and c, comment on the continuity and differentiability of the function given in the diagram:



10.6 MORE DERIVATIVES FROM FIRST PRINCIPLES

We have shown that if $y = x^n$, where n is a positive integer, then $\dfrac{dy}{dx} = nx^{n-1}$. In fact this result is true for all values of n.

Example 12

If $f(x) = \sqrt{x}$, find $f'(x)$ from first principles.

Solution

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$f(x) = \sqrt{x}$: $$f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

Rationalise the numerator: $$f'(x) = \lim_{h \to 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h} \times \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right)$$

$$= \lim_{h \to 0} \frac{(x+h) - x}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h \to 0} \frac{h}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

Now $\sqrt{x} = x^{\frac{1}{2}}$, so if $y = x^{\frac{1}{2}}$ then we have shown that $\frac{dy}{dx} = \frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-\frac{1}{2}}$.

Hence the rule for the derivative of x^n works for $n = \frac{1}{2}$.

Example 13

If $f(x) = \frac{1}{x}$, find $f'(x)$ from first principles.

Solution

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$f(x) = \frac{1}{x}$: $$f'(x) = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

Simplify the fraction: $$f'(x) = \lim_{h \to 0} \left(\frac{1}{h} \times \frac{x - (x+h)}{(x+h)x} \right)$$

$$= \lim_{h \to 0} \left(\frac{1}{h} \times \frac{-h}{(x+h)x} \right)$$

$$= \lim_{h \to 0} \left(\frac{-1}{(x+h)x} \right)$$

$$= \frac{-1}{x \times x} = -\frac{1}{x^2}$$

Now $\frac{1}{x} = x^{-1}$, so if $y = x^{-1}$ then we have shown that $\frac{dy}{dx} = -\frac{1}{x^2} = -1x^{-2}$.

Hence the rule for the derivative of x^n works for $n = -1$.

Example 14

If $f(x) = c$, a constant, find $f'(x)$ from first principles.

Solution

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$f(x) = c$: $$f'(x) = \lim_{h \to 0} \frac{c - c}{h} = 0$$

Now $c = cx^0$, so if $y = cx^0$ then we have shown that $\frac{dy}{dx} = 0x^{-1} = 0$.

Hence the rule for the derivative of x^n works for $n = 0$.

Derivative of x^n

If $f(x) = x^n$, then $f'(x) = nx^{n-1}$, where n is any real number.

This can also be written as $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$ or $\frac{dy}{dx} = nx^{n-1}$.

Example 15

Consider $f(x) = cx^n$, where c is a constant. Find $f'(x)$.

Solution

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$f(x) = cx^n$:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{c(x+h)^n - cx^n}{h} \\ &= c \lim_{h \to 0} \frac{(x+h)^n - x^n}{h} \\ &= c \times nx^{n-1} \end{aligned}$$

Derivative of *cg(x)*

If $y = cu$, where c is a constant and $u = g(x)$, then $\frac{dy}{dx} = c\frac{du}{dx}$.

Proof

If $u = g(x)$, then:

$$\frac{du}{dx} = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

Then:

$$\begin{aligned} \frac{dy}{dx} &= \lim_{h \to 0} \frac{cg(x+h) - cg(x)}{h} \\ &= \lim_{h \to 0} c\left(\frac{g(x+h) - g(x)}{h}\right) \\ &= c \lim_{h \to 0} \left(\frac{g(x+h) - g(x)}{h}\right) \\ &= cg'(x) = c\frac{du}{dx} \end{aligned}$$

Example 16

If $f(x) = x^2 + x$, find $f'(x)$.

Solution

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$f(x) = x^2 + x$:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{(x+h)^2 + (x+h) - (x^2 + x)}{h} \\ &= \lim_{h \to 0} \frac{x^2 + 2hx + h^2 + x + h - x^2 - x}{h} \\ &= \lim_{h \to 0} \frac{2hx + h^2 + h}{h} \\ &= \lim_{h \to 0} \frac{h(2x + 1 + h)}{h} \\ &= \lim_{h \to 0} (2x + 1 + h) \\ &= 2x + 1 \end{aligned}$$

Derivative of $f(x) \pm g(x)$

If $y = u \pm v$ where $u = f(x)$ and $v = g(x)$, then $\frac{dy}{dx} = \frac{du}{dx} \pm \frac{dv}{dx}$.

Proof

If $u = f(x)$, then: $\frac{du}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ and $\frac{dv}{dx} = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$

Then for $y = u + v$:

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{h \to 0} \frac{\big(f(x+h) + g(x+h)\big) - \big(f(x) + g(x)\big)}{h} \\
&= \lim_{h \to 0} \frac{\big(f(x+h) - f(x)\big) + \big(g(x+h) - g(x)\big)}{h} \\
&= \lim_{h \to 0} \left(\frac{\big(f(x+h) - f(x)\big)}{h} + \frac{\big(g(x+h) - g(x)\big)}{h} \right) \\
&= \lim_{h \to 0} \left(\frac{\big(f(x+h) - f(x)\big)}{h} \right) + \lim_{h \to 0} \left(\frac{\big(g(x+h) - g(x)\big)}{h} \right) \\
&= \frac{du}{dx} + \frac{dv}{dx}
\end{aligned}$$

The result follows similarly for $y = u - v$.

Summary of important results so far

1 If $y = x^n$, then $\frac{dy}{dx} = nx^{n-1}$, where n is any real number.

2 If $y = cu$, where c is a constant and $u = g(x)$, then $\frac{dy}{dx} = c\frac{du}{dx}$.

3 If $y = u \pm v$ where $u = f(x)$ and $v = g(x)$, then $\frac{dy}{dx} = \frac{du}{dx} \pm \frac{dv}{dx}$.

4 If $y = \sqrt{x}$ then $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$.

5 If $y = \frac{1}{x}$ then $\frac{dy}{dx} = -\frac{1}{x^2}$.

6 From results **1** and **3**, it follows that if $y = x^n + x^{n-1} + x^{n-2} + \ldots + x$, then $\frac{dy}{dx} = nx^{n-1} + (n-1)x^{n-2} + (n-2)x^{n-3} + \ldots + 1$.

Example 17

Find $\frac{dy}{dx}$ for each function.

(a) $y = x^4 + x^2 + x$ **(b)** $y = 5x^3 - 4x + 3$ **(c)** $y = x^5 + 6x^4 - 7x$

Solution

(a) $y = x^4 + x^2 + x$

$\frac{dy}{dx} = 4x^3 + 2x + 1$

(b) $y = 5x^3 - 4x + 3$

$\frac{dy}{dx} = 5 \times 3x^2 - 4$

$= 15x^2 - 4$

(c) $y = x^5 + 6x^4 - 7x$

$\frac{dy}{dx} = 5x^4 + 6 \times 4x^3 - 7$

$= 5x^4 + 24x^3 - 7$

Example 18

Find $\frac{dy}{dx}$ for each function.

(a) $y = \frac{1}{x^2}$ (b) $y = 3x^2 + \sqrt{x}$ (c) $y = 4 - \frac{1}{x} + \sqrt[3]{x}$

Solution

(a) $y = x^{-2}$

$\frac{dy}{dx} = -2x^{-3}$

$= -\frac{2}{x^3}$

(b) $y = 3x^2 + x^{\frac{1}{2}}$

$\frac{dy}{dx} = 3 \times 2x + \frac{1}{2}x^{-\frac{1}{2}}$

$= 6x + \frac{1}{2\sqrt{x}}$

(c) $y = 4 - x^{-1} + x^{\frac{1}{3}}$

$\frac{dy}{dx} = 0 - \left(-1x^{-2}\right) + \frac{1}{3}x^{-\frac{2}{3}}$

$= \frac{1}{x^2} + \frac{1}{3\sqrt[3]{x^2}}$

Example 19

If $y = x^3 - 6x$, find: (a) $\frac{dy}{dx}$ (b) the gradient when $x = -1$

(c) the coordinates of the points at which the gradient is 6.

Solution

(a) $y = x^3 - 6x$: $\frac{dy}{dx} = 3x^2 - 6$ (b) $x = -1$: $\frac{dy}{dx} = 3 - 6 = -3$

(c) $\frac{dy}{dx} = 6$: $3x^2 - 6 = 6$

$3x^2 = 12$

$x^2 = 4$

$x = \pm 2$

$x = 2, y = -4$; $x = -2, y = 4$. Points are $(2, -4)$ and $(-2, 4)$.

Example 20

If $f(x) = x^4 - 3x^3 + 2x$, find: (a) $f'(x)$ (b) $f'(a)$ (c) $f'(2)$ (d) $f'(-2)$

Solution

(a) $f(x) = x^4 - 3x^3 + 2x$: $f'(x) = 4x^3 - 9x^2 + 2$ (b) $f'(a) = 4a^3 - 9a^2 + 2$

(c) $f'(2) = 4 \times 8 - 9 \times 4 + 2 = 32 - 36 + 2 = -2$

(d) $f'(-2) = 4 \times (-8) - 9 \times 4 + 2 = -32 - 36 + 2 = -66$

EXERCISE 10.6 MORE DERIVATIVES FROM FIRST PRINCIPLES

1 Find the derivative of:

(a) $y = 3x^2 + 2x - 1$ (b) $y = 4x - 3x^2$ (c) $y = 7x - 4x^2$

(d) $y = x^4 + x^2 + 1$ (e) $y = x^5 - x^3 + x$ (f) $v = t^3 + 4t^2 - 2t + 5$

2 Find the derivative of:

(a) $y = x^{\frac{3}{2}}$ (b) $y = \frac{2}{x}$ (c) $y = 2\sqrt{x}$

(d) $v = \sqrt[3]{t^2}$ (e) $h(m) = \frac{1}{m^3}$ (f) $f(x) = \frac{1}{\sqrt{x}}$

3 If $g(x) = x^{\frac{4}{3}}$, then $g'(x) = \ldots$

A $g'(x) = 4x^{\frac{1}{3}}$ B $g'(x) = \dfrac{4}{3x^{\frac{1}{3}}}$ C $g'(x) = x^{\frac{1}{3}}$ D $g'(x) = \frac{4}{3}x^{\frac{1}{3}}$

4 Expand each expression and find $\dfrac{dy}{dx}$.

(a) $y = (x-1)(x+2)$ **(b)** $y = 3x(x^2-2)$ **(c)** $y = (2x-3)^2$

(d) $y = (x-4)(x+4)$ **(e)** $y = (2x-3)^3$ **(f)** $y = (x-2)(x+1)(3x+1)$

5 Write the derivative of each function.

(a) $f(x) = 3x^2 - 2x^3$ **(b)** $g(x) = 4x(1-x^3)$ **(c)** $v(t) = 3t^3 - 2t^2 + 5t + 4$

(d) $y = ax^3 + bx^2 + cx + d$ **(e)** $y = 3x + 4$ **(f)** $y = \dfrac{3x^3 - 2x^2 + x}{x}, x \neq 0$

6 Find $f'(x)$.

(a) $f(x) = x + \sqrt{x}$ **(b)** $f(x) = x^2 + \dfrac{1}{x}$ **(c)** $f(x) = x^2 + x + 1 + \dfrac{1}{x} + \dfrac{1}{x^2}$

(d) $f(x) = x^{\frac{2}{3}} + x^{\frac{1}{3}}$ **(e)** $f(x) = \left(x - \dfrac{1}{x}\right)^2$ **(f)** $f(x) = x\sqrt{x}$

7 For $f(x) = 3x^2 - 2x + 7$, indicate whether each statement is correct or incorrect.

(a) $f'(x) = 6x - 2$ **(b)** $f'(0) = 7$ **(c)** $f(1) = 8$ **(d)** $f'(2) = 10$

8 For each of the following functions, find the value of x for which $f'(x) = 0$.

(a) $f(x) = x^2 - 4$ **(b)** $f(x) = 2x^3 - 6x$ **(c)** $f(x) = x^3 - 4x^2$

9 Find the gradient of the curve $y = x^2 - x - 6$ at the points where $y = 0$.

10 If $f(x) = x^3 - x^2 - 6x + 1$, find the values of x for which $f'(x) = -5$.

11 Show that the graph of $y = x^2 + 4x - 12$ crosses the x-axis at two points. Find the gradient of the curve at these points.

12 For the graph of $f(x) = (x-1)^2$, find the values of x for which:

(a) $f(x) = 0$ **(b)** $f'(x) = 0$ **(c)** $f'(x) = -1$

13 Find the coordinates of the points on the curve $y = x^2 - 5x + 6$ at which the tangent:

(a) makes an angle of 45° with the x-axis

(b) is parallel to the line with equation $3x + y - 4 = 0$

(c) is perpendicular to the line with equation $2y - x + 3 = 0$

14 Find the x values of the points on the curve $y = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x + 1$ at which the tangent:

(a) is parallel to the x-axis

(b) makes an angle θ with the x-axis such that $\tan\theta = 2$

(c) is parallel to the line $y - 6x - 1 = 0$

15 Find the coordinates of the points on the parabola $y = x^2 - 2x - 8$ at which:

(a) the gradient is zero **(b)** the tangent is parallel to the line $2x + y = 7$

10.7 THE PRODUCT RULE

In previous exercises we have differentiated the product of two functions by first expanding the product, then differentiating term by term. Luckily there is a rule, known as the **product rule**, that allows us to differentiate products without expanding first.

The product rule:

- If u and v are functions of x, and $y = uv$, then $\dfrac{d}{dx}(uv) = v\dfrac{du}{dx} + u\dfrac{dv}{dx}$.

The product rule can also be written as $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$. You may find it easier to remember this in the form $\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$, because this is consistent with the form of the quotient rule to come later.

Example 21

If $y = x^2(3x + 4)$, find $\frac{dy}{dx}$: **(a)** by expanding the expression and differentiating term by term

(b) by using the product rule.

Solution

(a) $y = 3x^3 + 4x^2$: $\frac{dy}{dx} = 9x^2 + 8x$

(b) $y = x^2(3x + 4)$: $u = x^2$, $v = 3x + 4$

$\frac{du}{dx} = 2x$, $\frac{dv}{dx} = 3$

Product rule: $\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$

$$\frac{d}{dx}(uv) = (3x + 4) \times 2x + x^2 \times 3$$
$$= 6x^2 + 8x + 3x^2$$
$$= 9x^2 + 8x$$

The answers are the same. This example verifies the product rule.

Proof of the product rule

If $f(x) = uv$, $u = g(x)$ and $v = k(x)$ are functions of x, then:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \qquad k'(x) = \lim_{h \to 0} \frac{k(x+h) - k(x)}{h}$$

Now $f(x) = g(x)k(x)$ and $f(x + h) = g(x + h)k(x + h)$, so:

$$f'(x) = \lim_{h \to 0} \frac{g(x+h)k(x+h) - g(x)k(x)}{h}$$

$$= \lim_{h \to 0} \left\{ \frac{g(x+h)k(x+h) - g(x)k(x+h) + g(x)k(x+h) - g(x)k(x)}{h} \right\}$$

$$= \lim_{h \to 0} \left\{ \frac{g(x+h)k(x+h) - g(x)k(x+h)}{h} + \frac{g(x)k(x+h) - g(x)k(x)}{h} \right\}$$

$$= \lim_{h \to 0} \left\{ \frac{k(x+h)\big(g(x+h) - g(x)\big)}{h} \right\} + \lim_{h \to 0} \left\{ \frac{g(x)\big(k(x+h) - g(x)\big)}{h} \right\}$$

$$= k(x) \lim_{h \to 0} \left\{ \frac{g(x+h) - g(x)}{h} \right\} + g(x) \lim_{h \to 0} \left\{ \frac{k(x+h) - g(x)}{h} \right\}$$

$$= k(x)g'(x) + g(x)k'(x)$$

Hence $\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$.

Example 22

Use the product rule to differentiate $y = (3x + 2)(2x^2 - 3x + 4)$.

Solution

$y = (3x+2)(2x^2-3x+4)$: $\quad u = 3x+2 \quad v = 2x^2-3x+4$

$$\frac{du}{dx} = 3 \qquad \frac{dv}{dx} = 4x-3$$

Product rule: $\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$

$$\begin{aligned}\frac{dy}{dx} &= (2x^2-3x+4)\times 3 + (3x+2)\times(4x-3)\\ &= 6x^2-9x+12+12x^2-9x+8x-6\\ &= 18x^2-10x+6\end{aligned}$$

You can check this answer by expanding the function in the question and then differentiating term by term.

EXERCISE 10.7 THE PRODUCT RULE

1 Use the product rule to find the derivative of each function.

(a) $y = (x-2)(6x+7)$ **(b)** $f(x) = (2x+1)(x+3)$ **(c)** $y = (3x+4)(x^2-2x)$

(d) $g(x) = (x-1)(x^2-3x)$ **(e)** $y = (2x^2-5x)(x-2)$ **(f)** $f(x) = (x^2-4x)(x^2+3)$

(g) $y = (x-1)(3x+5)$ **(h)** $f(x) = (x^2-5x)(2x+3)$ **(i)** $g(x) = (4x-1)(5x^2-7)$

2 If $g(x) = (3x-1)(3x^2+1)$ then $g'(x) = \ldots$

A $g'(x) = 27x^2-6x+3$ **B** $g'(x) = 18x$

C $g'(x) = 3+6x-9x^2$ **D** $g'(x) = 9x^3-3x^2+3x-1$

3 For $f(x) = (x^3-1)(2x+3)$, indicate whether each statement is correct or incorrect.

(a) $f'(x) = 6x^2$ **(b)** $f'(x) = 8x^3+9x^2-2$ **(c)** $f'(0) = -2$ **(d)** $f(0) = -3$

4 For $g(x) = (x^2+5x)(x^3+x^2+1)$, find: **(a)** $g'(x)$ **(b)** $g'(1)$ **(c)** $g'(-2)$

5 Find $\frac{dy}{dx}$ for: **(a)** $y = \sqrt{x}(x-1)$ **(b)** $y = x\left(\sqrt{x}-1\right)$ **(c)** $y = x\left(\frac{1}{x}+1\right)$

(d) $y = \left(3\sqrt{x}+1\right)(x^2+4)$ **(e)** $y = \left(x+\frac{1}{x}\right)\left(x-\frac{1}{x}\right)$ **(f)** $y = \left(x^2+\frac{2}{x}\right)\left(1+\sqrt{x}\right)$

10.8 THE CHAIN RULE

The chain rule is also known as the 'composite function rule' or the 'function-of-a-function rule'. This rule allows us to differentiate functions such as $f(x) = (x^2-5)^7$ without having to expand.

Example 23

Differentiate each function after expanding the parentheses. Factorise your answer.

(a) $f(x) = (x^2-5)^2$ **(b)** $g(x) = (x^2-5)^3$

Solution

(a) $f(x) = (x^2-5)^2 = x^4-10x^2+25$

$$\begin{aligned}f'(x) &= 4x^3-20x\\ &= 4x(x^2-5)\\ &= 2x\times 2(x^2-5)\end{aligned}$$

(b) $g(x) = (x^4-10x^2+25)(x^2-5)$

$$\begin{aligned}&= x^6-15x^4+75x^2-125\\ g'(x) &= 6x^5-60x^3+150x\\ &= 6x(x^4-10x^2+25)\\ &= 2x\times 3(x^2-5)^2\end{aligned}$$

There is a pattern in the answers to Example **23**. In each case, $2x$ is the derivative of x^2 and the second part looks a bit like $\frac{d}{dx}\left(u^n\right) = nu^{n-1}$ where $u = h(x)$.

Let $u = (x^2 - 5)$ so that $\frac{du}{dx} = 2x$. This allows us to write:

$$f(x) = u^2 \qquad g(x) = u^3$$
$$f'(x) = 2x \times 2u \qquad g'(x) = 2x \times 3u^2$$
$$f'(x) = 2u \times \frac{du}{dx} \qquad g'(x) = 3u^2 \times \frac{du}{dx}$$

If $y = f(x) = [h(x)]^2 = k(u)$, where $u = h(x)$, then we have $y = u^2$.

Hence, the chain rule: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

OR $\frac{d}{dx}F(u) = F'(u) \times \frac{du}{dx}$ for any differentiable function F.

We can check this rule using Example **23(b)**:

$y = (x^2 - 5)^3$: $y = u^3$ where $u = x^2 - 5$

$$\frac{dy}{du} = 3u^2 \text{ and } \frac{du}{dx} = 2x$$

Chain rule: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$$\frac{dy}{dx} = 3u^2 \times 2x$$
$$= 3\left(x^2 - 5\right)^2 \times 2x = 6x\left(x^2 - 5\right)^2$$

This is the same as the derivative obtained by expanding the function and differentiating term by term.

Proof of the chain rule

The chain rule: if $y = f(u)$ and $u = g(x)$ then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.

Using first principles, we can write: $\frac{du}{dx} = \lim_{h \to 0}\left(\frac{g(x+h) - g(x)}{h}\right)$

and $\frac{dy}{du} = \lim_{k \to 0}\left(\frac{f(u+k) - f(u)}{k}\right) = \lim_{k \to 0}\left(\frac{f(g(x)+k) - f(g(x))}{k}\right)$.

If $k = g(x+h) - g(x)$ then $g(x+h) = g(x) + k$.

If $h \to 0$ then $g(x+h) \to g(x)$ and thus $k \to 0$.

Using first principles, we can write:

$$\frac{dy}{dx} = \lim_{h \to 0}\left(\frac{f(g(x+h)) - f(g(x))}{h}\right)$$
$$= \lim_{h \to 0}\left(\frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \times \frac{g(x+h) - g(x)}{h}\right)$$
$$= \lim_{h \to 0}\left(\frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)}\right) \times \lim_{h \to 0}\left(\frac{g(x+h) - g(x)}{h}\right)$$
$$= \lim_{k \to 0}\left(\frac{f(g(x)+k) - f(g(x))}{k}\right) \times \lim_{h \to 0}\left(\frac{g(x+h) - g(x)}{h}\right)$$
$$= \frac{dy}{du} \times \frac{du}{dx}$$

Example 24

Use the chain rule to find the derivative of each function.

(a) $f(x) = (2x+1)^2$ **(b)** $g(x) = (2x^2 - 3x + 1)^4$ **(c)** $y = \sqrt{4 - x^2}$

Solution

(a) Let $u = 2x + 1$ so $\frac{du}{dx} = 2$

$f(u) = u^2$ so $f'(u) = 2u$

Hence: $f'(x) = f'(u) \times \frac{du}{dx}$

$= 2u \times 2$

$= 4(2x+1)$

(b) Let $u = 2x^2 - 3x + 1$ so $\frac{du}{dx} = 4x - 3$

$g(u) = u^4$ so $g'(u) = 4u^3$

Hence: $g'(x) = g'(u) \times \frac{du}{dx}$

$= 4u^3 \times (4x - 3)$

$= 4(4x-3)\left(2x^2 - 3x + 1\right)^3$

(c) Let $u = 4 - x^2$ so $\frac{du}{dx} = -2x$

$g(u) = \sqrt{u}$ so $g'(u) = \frac{1}{2\sqrt{u}}$

Hence: $g'(x) = g'(u) \times \frac{du}{dx}$

$= \frac{1}{2\sqrt{u}} \times (-2x)$

$= \frac{-x}{\sqrt{4 - x^2}}$

Example 25

Find $f'(t)$ if $f(t) = t\sqrt{t^2 - 9}$.

Solution

This example requires both the product rule and the chain rule.

$u = t$ $\frac{du}{dt} = 1$

Chain rule: $v = \sqrt{t^2 - 9}$ $\frac{dv}{dt} = \frac{1}{2}\left(t^2 - 9\right)^{-\frac{1}{2}} \times 2t$

$= \frac{t}{\sqrt{t^2 - 9}}$

Product rule: $\frac{d}{dt}(uv) = v\frac{du}{dt} + u\frac{dv}{dt}$: $f'(t) = \sqrt{t^2 - 9} \times 1 + t \times \frac{t}{\sqrt{t^2 - 9}}$

$= \sqrt{t^2 - 9} + \frac{t^2}{\sqrt{t^2 - 9}}$

Write with common denominator: $= \frac{t^2 - 9 + t^2}{\sqrt{t^2 - 9}}$

$= \frac{2t^2 - 9}{\sqrt{t^2 - 9}}$

EXERCISE 10.8 THE CHAIN RULE

1 Use the chain rule to differentiate:

(a) $y=(x^2-4)^5$ **(b)** $f(x)=\sqrt{5x-1}$ **(c)** $y=(x^3-3x)^4$

(d) $y=(2x+5)^{-1}$ **(e)** $f(t)=\sqrt[3]{t^2+3t}$ **(f)** $g(x)=(2x^2+5x-4)^4$

(g) $f(x)=\sqrt{x^2-2x}$ **(h)** $f(t)=(t^2+4)^{-2}$ **(i)** $y=\sqrt{25-x^2}$

2 If $f(x)=(x^3-1)^5$ then $f'(x)=\ldots$

A $3x^2(x^3-1)^4$ **B** $15x^2(x^3-1)^4$ **C** $5x^2(x^3-1)^4$ **D** $15x^2$

3 Find the derivative of each function.

(a) $y=\sqrt{x^2-4}$ **(b)** $f(x)=\left(x^2+1\right)^{\frac{1}{2}}$ **(c)** $y=(1+2x)^{-1}$

(d) $v=(m^2+25)^2$ **(e)** $y=(2x-1)^5$ **(f)** $g(t)=\sqrt[3]{t+1}$

(g) $f(x)=(3x^2-2x-1)^4$ **(h)** $f(t)=(t^2+4)^{-2}$ **(i)** $y=\left(x-\frac{1}{x}\right)^4$

4 For $g(x)=x^2+5x+\sqrt[3]{x^2-4}$, indicate whether each statement is correct or incorrect.

(a) $g'(x)=\frac{2x}{3}\left(x^2-4\right)^{-\frac{2}{3}}$ **(b)** $g'(x)=4x$

(c) $g'(x)=2x+5+\frac{2x}{3}\left(x^2-4\right)^{-\frac{2}{3}}$ **(d)** $g'(x)=2x+5+\frac{2x}{3\left(x^2-4\right)^{\frac{2}{3}}}$

5 Find the derivative of each function.

(a) $y=(x-3)(3x+4)^6$ **(b)** $f(x)=x^2\sqrt{1-x^2}$ **(c)** $h(t)=t^3+(4-t)^4$

(d) $y=\sqrt{1-x}+\sqrt{1+x}$ **(e)** $y=t^2-1+(1+t)^{-1}$ **(f)** $g(x)=x\left(1+\frac{1}{x}\right)^2$

10.9 THE QUOTIENT RULE

This rule is a special application of the product rule.

Consider: If u and v are functions of x, and $y=u\times\frac{1}{v}$, $v\neq 0$, then by the product rule

$$\frac{d}{dx}\left(u\times\frac{1}{v}\right)=\frac{1}{v}\frac{du}{dx}+u\frac{d}{dx}\left(\frac{1}{v}\right)$$

Hence:
$$\begin{aligned}\frac{d}{dx}\left(\frac{u}{v}\right)&=\frac{1}{v}\frac{du}{dx}+u\frac{d}{dx}\left(v^{-1}\right)\\&=\frac{1}{v}\frac{du}{dx}+u\times\frac{-1}{v^2}\times\frac{dv}{dx}\\&=\frac{1}{v}\frac{du}{dx}-\frac{u}{v^2}\frac{dv}{dx}\\&=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}\end{aligned}$$

Hence, the quotient rule:

- If u and v are functions of x, and $y=\frac{u}{v}$, $v\neq 0$, then $\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$.

Example 26

Use the quotient rule to differentiate each function: **(a)** $y = \frac{2x+1}{4x-3}$ **(b)** $f(t) = \frac{t}{1+t^2}$

Solution

(a) $y = \frac{2x+1}{4x-3}$:

$u = 2x+1 \qquad v = 4x-3$

$\frac{du}{dx} = 2 \qquad \frac{dv}{dx} = 4$

Quotient rule: $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$$\frac{dy}{dx} = \frac{(4x-3)\times 2 - (2x+1)\times 4}{(4x-3)^2}$$

$$= \frac{8x-6-8x-4}{(4x-3)^2}$$

$$= \frac{-10}{(4x-3)^2}$$

Because the function is only defined for $x \neq \frac{3}{4}$, the same restriction applies for the derivative.

(b) $f(t) = \frac{t}{1+t^2}$:

$u = t \qquad v = 1+t^2$

$\frac{du}{dt} = 1 \qquad \frac{dv}{dt} = 2t$

Quotient rule: $f'(t) = \frac{v\frac{du}{dt} - u\frac{dv}{dt}}{v^2}$

$$f'(t) = \frac{\left(1+t^2\right)\times 1 - t\times 2t}{\left(1+t^2\right)^2}$$

$$= \frac{1+t^2-2t^2}{\left(1+t^2\right)^2}$$

$$= \frac{1-t^2}{\left(1+t^2\right)^2}$$

Example 27

Find the gradient of the tangent to the curve $y = \frac{\sqrt{4-x^2}}{x+1}$ at the point where $x = 0$.

Solution

To find $\frac{dy}{dx}$ you have to use both the chain rule and the quotient rule.

$y = \frac{\sqrt{4-x^2}}{x+1}$: $\qquad u = \sqrt{4-x^2} \qquad v = x+1$

$\frac{du}{dx} = \frac{1}{2}\left(4-x^2\right)^{-\frac{1}{2}} \times (-2x) \qquad \frac{dv}{dx} = 1$

$$= \frac{-x}{\sqrt{4-x^2}}$$

Quotient rule: $\dfrac{dy}{dx}=\dfrac{v\dfrac{du}{dx}-u\dfrac{dv}{dx}}{v^2}$

$$\frac{dy}{dx}=\frac{(x+1)\times\dfrac{-x}{\sqrt{4-x^2}}-\sqrt{4-x^2}\times 1}{(x+1)^2}$$

$$=\frac{-1}{(x+1)^2}\left(\frac{x(x+1)}{\sqrt{4-x^2}}+\sqrt{4-x^2}\right)$$

$$=\frac{-1}{(x+1)^2}\left(\frac{x^2+x+4-x^2}{\sqrt{4-x^2}}\right)$$

$$=\frac{-(x+4)}{(x+1)^2\sqrt{4-x^2}}$$

Where $x=0$: $\dfrac{dy}{dx}=\dfrac{-4}{1\times 2}=-2$

EXERCISE 10.9 THE QUOTIENT RULE

1 Use the quotient rule to differentiate each function.

(a) $y=\dfrac{x-1}{x+1}$ (b) $f(x)=\dfrac{3x-7}{4x+5}$ (c) $g(t)=\dfrac{2t+5}{t+2}$

(d) $f(x)=\dfrac{1}{x^2-5x+6}$ (e) $h(n)=\dfrac{n^2+3n+4}{2n-1}$ (f) $y=\dfrac{x}{x-3}$

(g) $y=\dfrac{4x^2}{2x+5}$ (h) $y=\dfrac{4x^2-2}{x^2+5}$ (i) $v(x)=\dfrac{x+1}{x^3-1}$

2 If $y=\dfrac{x+1}{x^2+1}$ then $\dfrac{dy}{dx}=\ldots$

A $\dfrac{1-2x-x^2}{\left(x^2+1\right)^2}$ B $\dfrac{3x^2+2x+1}{x^4+1}$ C $\dfrac{1-2x-x^2}{x^4+1}$ D $\dfrac{3x^2+2x+1}{\left(x^2+1\right)^2}$

3 Differentiate each function with respect to x.

(a) $y=\dfrac{\sqrt{x+1}}{x}$ (b) $f(x)=\dfrac{(x+1)^2}{x}$ (c) $y=\dfrac{x}{(x+1)^2}$

(d) $y=\dfrac{(2x+1)^3}{\left(3-x^2\right)^2}$ (e) $f(x)=\dfrac{x}{\sqrt{x+1}}$ (f) $y=\left(\dfrac{x+1}{x}\right)^2$

4 $f(x)=\dfrac{\sqrt{x}}{x^2+1}$. Four steps in finding the simplest form of $f'(x)$ are given. Indicate whether each step is correct or incorrect.

A $f'(x)=\dfrac{\left(x^2+1\right)\times\dfrac{1}{2\sqrt{x}}-\sqrt{x}\times 2x}{\left(x^2+1\right)^2}$ B $f'(x)=\dfrac{\dfrac{x^2+1}{2\sqrt{x}}-2x\sqrt{x}}{\left(x^2+1\right)^2}$

C $f'(x)=\dfrac{x^2+1-4x^2}{2\sqrt{x}\left(x^2+1\right)^2}$ D $f'(x)=\dfrac{5x^2+1}{2\sqrt{x}\left(x^2+1\right)^2}$

5 Find the derivative of each function.

(a) $y=(x^2-4)(x+2)$ (b) $f(x)=4x^{\frac{5}{2}}-2x^{\frac{3}{2}}+6x^{\frac{1}{2}}$ (c) $y=\sqrt{5x-1}$

(d) $g(t)=t^{\frac{2}{3}}+(2t-1)^3$ (e) $y=\dfrac{x+2}{x+5}$ (f) $f(x)=\sqrt{x^2+2}$

(g) $y=x^2-5x+\sqrt{x-2}$ (h) $f(z)=\dfrac{z+4}{2z-1}$ (i) $y=\dfrac{1}{x^2-2x+2}$

(j) $f(x)=(x^2-7x)(x+1)$ (k) $y=(x^2+1)\sqrt{x}$ (l) $g(x)=\dfrac{\sqrt{x+1}}{\sqrt{x^2+1}}$

6 Show that the gradient of the tangent to the curve $y=\dfrac{x}{x^2+1}$ is zero twice, at $x=-1$ and $x=1$.

CHAPTER REVIEW 10

1 Find the following limits. (a) $\lim\limits_{x\to\frac{1}{2}}\dfrac{1-4x^2}{1-2x}$ (b) $\lim\limits_{x\to 3}\dfrac{x^3-27}{x-3}$

2 Evaluate: (a) $\lim\limits_{h\to 0}\dfrac{2x^2h+3h}{h}$ (b) $\lim\limits_{h\to 0}\dfrac{(2+h)^2-4}{h}$ (c) $\lim\limits_{h\to 0}\dfrac{(1+h)^3-1}{h}$

3 Find $\dfrac{f(x+h)-f(x)}{h}$, $h\neq 0$, for $f(x)=2x^2-3x$.

4 For $f(x)=x^2+6x+8$, find:

(a) $f(2)$ (b) $f'(2)$ (c) $f'(c)$ (d) the value of c for which $f'(c)=-2$

5 Find $f'(x)$ for $f(x)=\sqrt{2x-1}$.

6 Given $y=(x^2-4)(3x^2-2x+1)^5$, find $\dfrac{dy}{dx}$.

7 Find the derivative of each function.

(a) $y=(x-2)^3$ (b) $f(x)=(x^2+x^3)^5$ (c) $v=\sqrt{25-t^2}$

(d) $y=(x+x^{-1})^2$ (e) $g(x)=\dfrac{1}{(x+4)^2}$ (f) $y=\dfrac{x-2}{x}$

(g) $f(x)=(x-1)^6(x+2)$ (h) $y=x^2+3x+\sqrt{x-2}$ (i) $f(x)=\dfrac{1}{x^2-2}$

(j) $u=\dfrac{2m-7}{2m+3}$ (k) $y=\dfrac{1+x^3}{x^2}$ (l) $h(t)=(t-3)\sqrt{t-3}$

8 Differentiate with respect to x: (a) x^4 (b) $x(x-1)$ (c) x^3-2x

CHAPTER 11
PLANE AND COORDINATE GEOMETRY

Outcomes covered:

H1 seeks to apply mathematical techniques to problems in a wide range of practical contexts
H2 constructs arguments to prove and justify results
H5 applies appropriate techniques from the study of calculus, geometry, probability, trigonometry and series to solve problems
H9 communicates using mathematical language, notation, diagrams and graphs

SUMMARY OF PRELIMINARY PLANE GEOMETRY

Parallel lines

When a transversal cuts a pair of parallel lines, the pairs of:

(a) alternate angles are equal
(b) corresponding angles are equal
(c) cointerior angles are supplementary.

Tests for parallel lines

If a transversal cuts two other straight lines and makes:

(a) a pair of alternate angles equal
or **(b)** a pair of corresponding angles equal
or **(c)** a pair of cointerior angles supplementary
then the two straight lines are parallel.

Transitivity of parallel lines

If $AB \parallel CD$ and $CD \parallel KL$ then $AB \parallel KL$.

Angle properties of triangles

1 The sum of the angles of a triangle is 180°.
2 If one side of a triangle is produced, the exterior angle formed is equal to the sum of the two interior opposite angles.

Isosceles triangle properties

1 Two sides equal
2 Angles opposite the equal sides (also known as 'base angles') are equal
3 Line joining the midpoint of the base to the opposite vertex is perpendicular to the base
4 Line drawn from the vertex perpendicular to the base will bisect the base
5 Line joining the midpoint of the base to the opposite vertex bisects the angle at that vertex
6 Perpendicular bisector of the base passes through the opposite vertex

Equilateral triangle properties

1 All the properties of an isosceles triangle
2 Three sides are equal
3 Each angle is 60°

Polygons

1. A polygon in which all sides are equal and all angles are equal is a **regular** polygon. An equilateral triangle and a square are both regular polygons.
2. A polygon in which each angle is less than 180° is a **convex** polygon. A polygon in which at least one angle is more than 180° is a **concave** polygon.
3. The sum of the angles of a polygon with n sides is $(2n - 4)$ right angles, or $(n - 2) \times 180°$.
4. The size of each angle in a regular polygon is $\frac{2n-4}{n}$ right angles, or $\frac{n-2}{n} \times 180°$.
5. If the sides of a convex polygon are produced in order, the sum of the exterior angles formed is 4 right angles.

Quadrilaterals

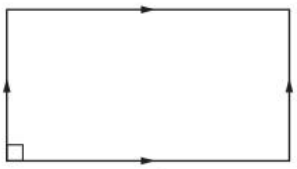
1. The sum of the four angles of any quadrilateral is 360°.
2. The sum of the exterior angles of a quadrilateral is 360°.

Special quadrilaterals and their properties

Parallelogram: a quadrilateral with both pairs of opposite sides parallel.

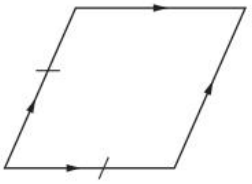
(a) pairs of opposite sides equal
(b) pairs of opposite angles equal
(c) diagonals bisect each other

Rectangle: a parallelogram for which one angle is a right angle.



(a) pairs of opposite sides equal
(b) each angle is 90°
(c) diagonals are equal and bisect each other

Rhombus: a parallelogram with a pair of adjacent sides equal.

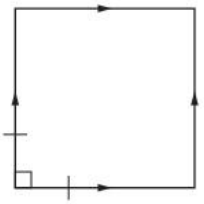


(a) all sides equal
(b) pairs of opposite angles equal
(c) diagonals bisect each other at right angles
(d) diagonals bisect the angles of the rhombus

Square: a rectangle with a pair of adjacent sides equal

OR

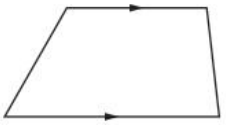
a rhombus with one angle that is a right angle.



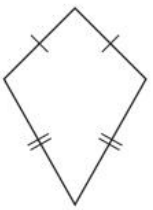
(a) all sides equal
(b) each angle is 90°
(c) diagonals are equal and bisect each other at right angles
(d) diagonals bisect the angles of the square

Trapezium: a quadrilateral with one pair of opposite sides parallel.

A trapezium with its non-parallel sides equal is called an isosceles trapezium.



Kite: a quadrilateral with both pairs of adjacent sides equal.



(a) one pair of opposite angles are equal (between the unequal sides)
(b) one axis of symmetry

Tests for a parallelogram

1. If the pairs of opposite sides of a quadrilateral are equal, then the quadrilateral is a parallelogram.
2. If the pairs of opposite angles of a quadrilateral are equal, then the quadrilateral is a parallelogram.
3. If a quadrilateral has one pair of sides that are both equal and parallel, then the quadrilateral is a parallelogram.
4. If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.

Tests for a rhombus

1. If the four sides of a quadrilateral are equal, then the quadrilateral is a rhombus.
2. If the diagonals of a quadrilateral bisect each other at right angles, then the quadrilateral is a rhombus.

Pythagoras' theorem

The square of the hypotenuse of a right-angled triangle is equal to the sum of the squares of the other two sides.

Converse of Pythagoras' theorem

If the square of a side of a triangle is equal to the sum of the squares of the other two sides, then the angle contained by these two sides is a right angle.

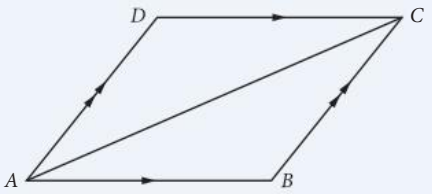
11.1 CONGRUENT TRIANGLES (HSC)

Tests for congruent triangles

1. Two triangles are congruent if two sides and the included angle of one triangle are respectively equal to two sides and the included angle of the other triangle. (**SAS**)
2. Two triangles are congruent if the three sides of one triangle are respectively equal to the three sides of the other triangle. (**SSS**)
3. Two triangles are congruent if two angles and a side of one triangle are respectively equal to two angles and the corresponding side of the other triangle. (**AAS**)
4. Two triangles are congruent if the hypotenuse and one side of a right-angled triangle are equal to the hypotenuse and corresponding side of the other right-angled triangle. (**RHS**)

Example 1

Prove that the opposite sides and angles of a parallelogram are equal.



Solution

Given: $ABCD$ is a parallelogram

Aim: To prove that:

(a) $AB = DC,\ BC = AD$

(b) $\angle ABC = \angle ADC,\ \angle DCB = \angle DAB$

Construction: Join AC.

Proof: **(a)** In ΔABC and ΔADC:

AC is a common side

$\angle BAC = \angle DCA$ (alternate angles, $AB \parallel DC$)

$\angle BCA = \angle DAC$ (alternate angles, $BC \parallel AD$)

$\therefore \Delta ABC \equiv \Delta ADC$ (AAS)

$\therefore AB = DC$ (matching sides in congruent triangles)

Similarly $BC = AD$

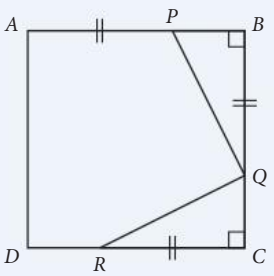
(b) $\angle ABC = \angle ADC$ (matching angles in congruent triangles)

Similarly $\angle DCB = \angle DAB$

Example 2

$ABCD$ is a square. P, Q and R are points of AB, BC and CD respectively such that $AP = BQ = CR$. Prove that:

(a) $PQ = QR$ **(b)** $\angle PQR = 90°$



Solution

(a) Because $AB = BC = CD$ and $AP = BQ = CR$, then:

$PB = AB - AP$

$QC = BC - BQ$

$PB = QC$

In ΔPBQ and ΔQCR:

$PB = QC$ (from above)

$BQ = CR$ (given)

$\angle PBQ = \angle QCR = 90°$ (given)

$\therefore \Delta PBQ \equiv \Delta QCR$ (SAS)

$\therefore PQ = QR$ (corresponding sides in congruent triangles)

(b) $\angle BPQ = \angle RQC$ (corresponding angles in congruent triangles)

But $\angle BQP = 90° - \angle BPQ$ (angle sum of ΔPBQ)

$\therefore \angle BQP = 90° - \angle RQC$

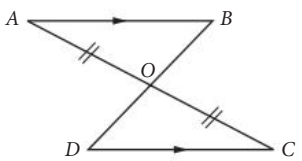
And $\angle PQR + \angle BQP + \angle RQC = 180°$ (straight angle)

$\therefore \angle PQR + 90° - \angle RQC + \angle RQC = 180°$

$\angle PQR = 90°$

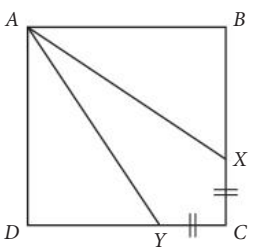
EXERCISE 11.1 CONGRUENT TRIANGLES (HSC)

1 $AB \parallel DC$ and $AO = OC$. Prove that $BO = OD$ by proving that ΔAOB and ΔCOD are congruent.



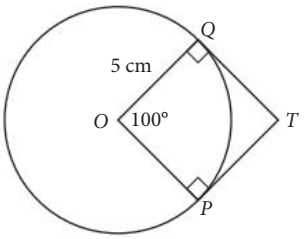
2 $ABCD$ is a square and $CX = CY$. Prove:

(a) $AX = AY$ **(b)** $\angle AXB = \angle AYD$

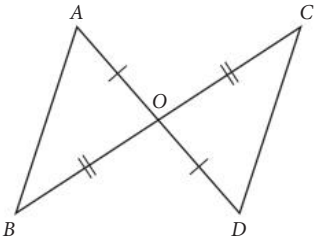


3 P and Q are points on the circumference of a circle with centre O, radius 5 cm. The lines drawn perpendicular to OP and OQ meet at T. $\angle POQ = 100°$.

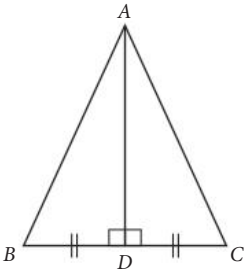
(a) Prove that $\Delta OPT \equiv \Delta OQT$. **(b)** Calculate the length of PT.



4 Two line segments AD and BC bisect each other at O. Prove that $AB = CD$ and $AB \parallel CD$.

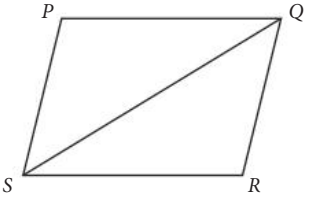


5 Prove that if the straight line drawn from the vertex of a triangle to the midpoint of the base is perpendicular to the base, then the triangle is isosceles.

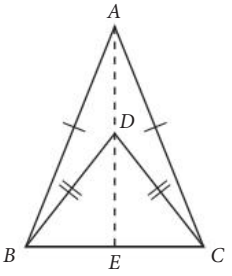


6 $PQRS$ is a quadrilateral in which $PQ = SR$ and $SP = RQ$. Prove:

(a) $\Delta PQS \equiv \Delta RSQ$
(b) $\angle PQS = \angle QSR$
(c) $PQ \parallel SR$
(d) $PQRS$ is a parallelogram.

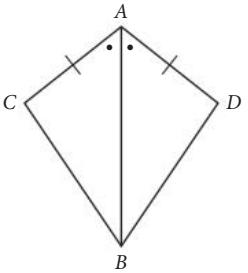


7 Given that $AB = AC$ and $DB = DC$, show that $\angle ABD = \angle ACD$.



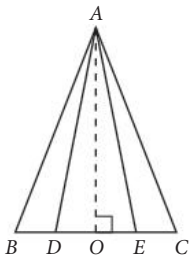
8 If $AC = AD$ and AB bisects $\angle CAD$, prove:

(a) $\Delta ABC \equiv \Delta ABD$ **(b)** $BC = BD$



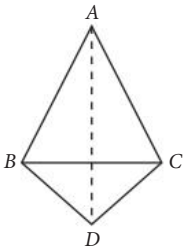
9 ABC is a triangle in which $\angle B = \angle C$. The bisector of angle A meets BC at D. Prove that ΔABD and ΔACD are congruent. Hence prove that AD is perpendicular to BC.

10 ΔABC and ΔADE are isosceles. Show that $\angle BAD = \angle EAC$.



11 ΔABC and ΔBCD are isosceles. Prove:

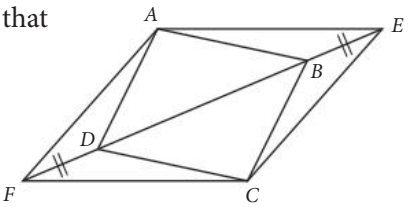
(a) $\angle ABD = \angle ACD$ **(b)** $\angle BAD = \angle CAD$



12 $ABCD$ is a square. X and Y are points on BC and CD respectively such that $BX = CY$. Prove:

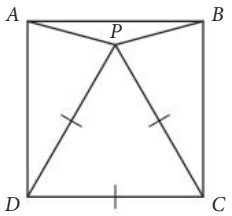
(a) $AX = BY$ **(b)** $AX \perp BY$

13 $ABCD$ is a parallelogram and $BE = DF$. Show that $AE = FC$ and that $AE \parallel FC$.

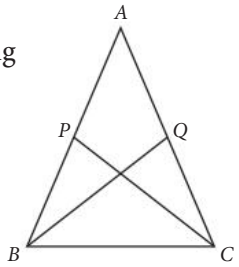


14 P is a point inside a square $ABCD$ such that ΔPDC is equilateral. Prove:

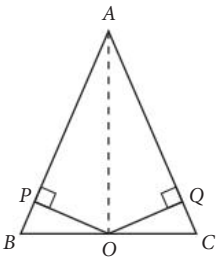
(a) $\Delta APD \equiv \Delta BPC$ **(b)** ΔAPB is isosceles.



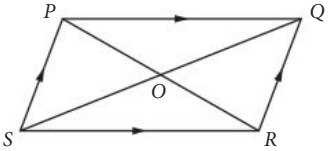
15 P and Q are the midpoints of the equal sides AB and AC of the isosceles triangle ABC. Prove that $PC = QB$ by first proving that $\Delta PBC \equiv \Delta QBC$.



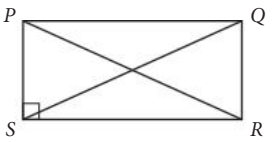
16 ABC is an isosceles triangle with $AB = AC$. O is the midpoint of BC. OP and OQ are drawn perpendicular to AB and AC respectively. Show that $OP = OQ$.



17 Prove that the diagonals of a parallelogram bisect each other. (That is, prove that in parallelogram $PQRS$, $PO = OR$ and $SO = OQ$.)



18 Prove that the diagonals of a rectangle are equal. (Hint: prove that $\Delta PQR \equiv \Delta PQS$.)

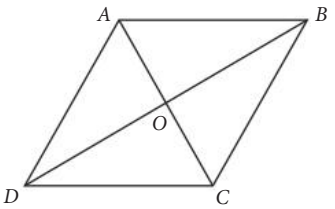


19 Prove that the diagonals of a rhombus bisect each other at right angles. That is, prove:

(a) $AO = OC$ and $DO = OB$

(b) $\angle AOB = 90°$.

(c) Also, prove that the diagonals of a rhombus bisect the angles of the rhombus, i.e. that $\angle DAO = \angle BAO$, $\angle ADO = \angle CDO$ and so on.

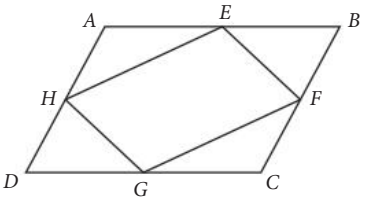


20 E, F, G and H are the midpoints of the sides AB, BC, CD and DA respectively of the parallelogram $ABCD$. Assuming that the opposite sides and angles of a parallelogram are equal, prove:

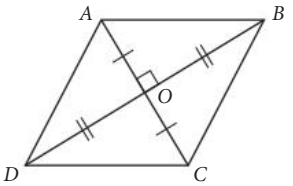
(a) $\Delta AEH \equiv \Delta CFG$

(b) $\Delta EBF \equiv \Delta GHD$

(c) $EFGH$ is a parallelogram.



21 $ABCD$ is a quadrilateral whose diagonals AC and BD bisect each other at right angles. Prove that $ABCD$ is a rhombus.



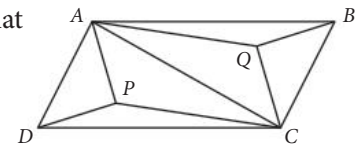
22 P and Q are two points inside a parallelogram $ABCD$ such that $AP = QC$ and $AP \parallel QC$. Prove:

(a) $\Delta APC \equiv \Delta AQC$

(b) $PC \parallel AQ$

(c) $PC = AQ$

(d) $AQCP$ is a parallelogram.

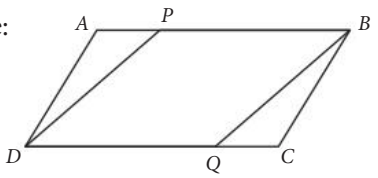


23 $ABCD$ is a parallelogram and $AP = QC$. Prove:

(a) $\Delta APD \equiv \Delta CQB$

(b) $PD = QB$

(c) $PD \parallel BQ$



24 PQR is an isosceles triangle with $PQ = PR$. ΔSQR is also isosceles, with $SQ = SR$, and S is on the side of QR opposite to P. Prove:

(a) $\Delta PQS \equiv \Delta PRS$

(b) P, X and S are collinear, where X is the midpoint of QR.

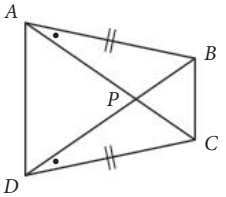
25 $ABCD$ is a quadrilateral with $AB = DC$ and $\angle BAC = \angle BDC$. Prove:

(a) $PB = PC$

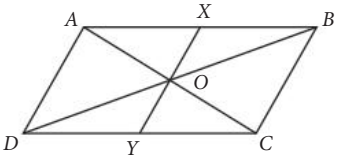
(b) $\Delta ABC \equiv \Delta DBC$

(c) $AC = BD$

(d) $PA = PD$



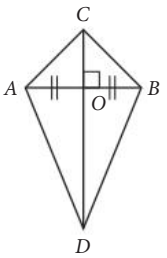
26 $ABCD$ is a parallelogram whose diagonals intersect at O. A straight line XY is drawn through O. Prove that $OX = OY$.



27 $ABCD$ is a parallelogram. P and Q are points on AB and CD respectively such that $AP = CQ$. Prove that the perpendicular distances of P and Q from the diagonal BD are equal.

28 Prove that if one pair of opposite sides of a quadrilateral are equal and parallel, then the other pair of opposite sides are also equal and parallel.

29 From the midpoint O of AB, the line CD is drawn at right angles. Prove:



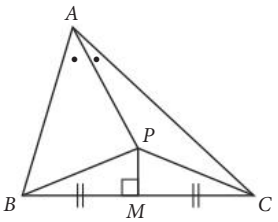
(a) $\Delta AOC \equiv \Delta BOC$
(b) $\Delta AOD \equiv \Delta BOD$
(c) $\Delta CAD \equiv \Delta CBD$

This quadrilateral $ABCD$ is called a kite.

30 From the set 'parallelogram, rectangle, rhombus, square', list all the shapes that have the given property:

(a) opposite sides equal
(b) opposite angles equal
(c) diagonals bisect each other
(d) diagonals are equal
(e) diagonals bisect the angles of the shape
(f) diagonals are at right angles
(g) diagonals bisect each other at right angles
(h) all angles are right angles.

31 ABC is an acute-angled triangle. The bisector of angle A meets the perpendicular bisector of BC at P.



(a) Prove that $\Delta PBM \equiv \Delta PCM$ and hence that $PB = PC$.
(b) Draw lines PX and PY perpendicular to AB and AC respectively. Prove that $\Delta APX \equiv \Delta APY$ and hence that $AX = AY$ and $PX = PY$.
(c) Prove that $\Delta PXB \equiv \Delta PYC$ and hence that $XB = YC$.
(d) Hence show that $AB = AC$.

You have now proved that any triangle ABC is isosceles. This is clearly a false statement, so there must be a problem with the diagram. Use geometry drawing tools in GeoGebra or other software to find the error.

11.2 SIMILAR TRIANGLES (HSC)

Tests for similar triangles

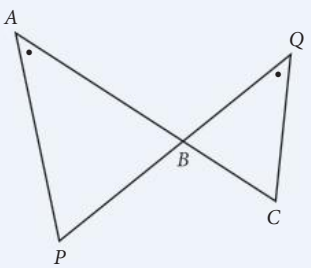
1 Two triangles are similar if the three angles of one triangle are respectively equal to the three angles of the other triangle.

2 Two triangles are similar if the three pairs of corresponding sides are respectively proportional.

3 Two triangles are similar if two pairs of corresponding sides are proportional and the angles included by these sides are equal.

Example 3

(a) $\angle PAB = \angle CQB$, $AB = 4.2$ cm, $BQ = 2.8$ cm. Prove that $\Delta APB \,|||\, \Delta QCB$.
(b) For $PB = 3$ cm and $CQ = 2.4$ cm, find the lengths of BC and AP.



Solution

(a) Given: $\angle PAB = \angle CQB$
Aim: To prove that $\Delta APB \,|||\, \Delta QCB$
Proof:

$\angle PAB = \angle CQB$		(given)
$\angle ABP = \angle QBC$		(vertically opposite angles)
$\therefore \Delta APB \,|||\, \Delta QCB$		(AAA)

(b) $\frac{PA}{CQ} = \frac{AB}{QB} = \frac{BP}{BC}$ (matching sides of similar triangles are proportional)

$\frac{PA}{CQ} = \frac{4.2}{2.8} = \frac{BP}{BC}$ or $\frac{PA}{CQ} = \frac{BP}{BC} = \frac{3}{2}$

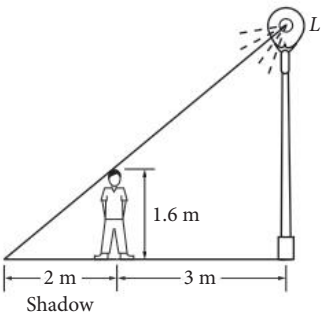
$PB = 3$ cm: $\frac{3}{BC} = \frac{3}{2}$ $\therefore BC = 2$ cm

$CQ = 2.4$ cm: $\frac{PA}{2.4} = \frac{3}{2}$ $\therefore PA = 3 \times 1.2 = 3.6$ cm

EXERCISE 11.2 SIMILAR TRIANGLES (HSC)

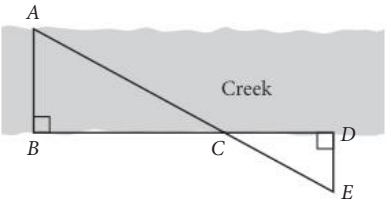
1 A boy who is 1.6 m tall stands 3 m from a street light and casts a shadow 2 m long. The height of the light above the ground is:

A 2.4 m **B** 2.7 m
C 4 m **D** 4.8 m



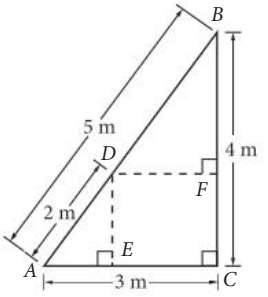
2 A girl who is 1.2 m tall stands 4.5 m from a street light and casts a shadow 1.5 m long. How high is the light above the ground?

3 To find the width of a creek, Vinh walked from B to C to D to a point E such that ECA was a straight line. If $BC = 20$ m, $CD = 10$ m, $DE = 5$ m, what is the width AB of the creek?



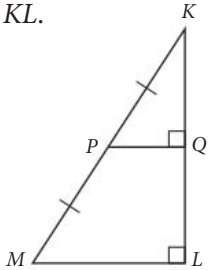
4 A ladder 5 m long leans against a wall, with the top of the ladder reaching 4 m up the wall and its foot 3 m from the wall. A person is at the point D, 2 m up the ladder.

(a) Prove that $\Delta ADE \,|||\, \Delta DBF$.
(b) Find the person's height DE above the ground.
(c) Find the person's distance DF from the wall.

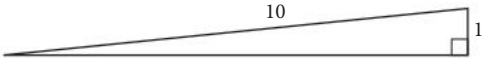


5 ΔKLM is right-angled at L. P is the midpoint of KM and $PQ \perp KL$.

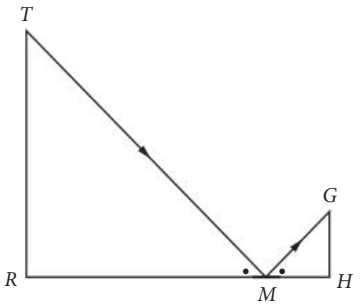
(a) Prove that $\Delta KQP \,|||\, \Delta KLM$.
(b) Explain why $KQ = QL$.
(c) Prove that ΔKPL is isosceles.
(d) Which other triangle in the figure is isosceles?



6 A road up a hillside has a constant slope of 1 in 10, i.e. for every 10 units up the hill there is a vertical rise of 1 unit. If the distance up the hillside is 80 m, what is the vertical rise?



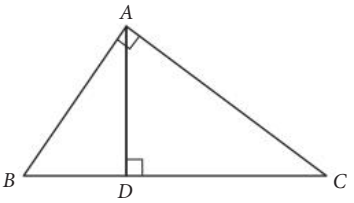
7 The diagram (not to scale) shows how GH measures the height of a tree TR, with the aid of a small mirror M placed in the ground. Gareth, whose eye level is 1.8 m above the ground, places the mirror 5 m from the tree and 30 cm from himself to see the top of the tree.



(a) Prove that $\Delta TRM \,|||\, \Delta GHM$.

(b) Calculate the height of the tree.

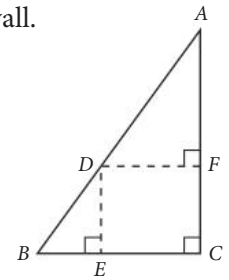
8 From the vertex A of a right-angled triangle ABC, AD is drawn perpendicular to BC.



(a) Prove that the three triangles ΔABD, ΔADC and ΔABC are similar to each another.

(b) For $AB = 15$ cm and $AD = 12$ cm, find the lengths of BD, DC and AC.

9 A ladder AB, 10 m long, rests against a wall. Point B is 6 m from the wall.

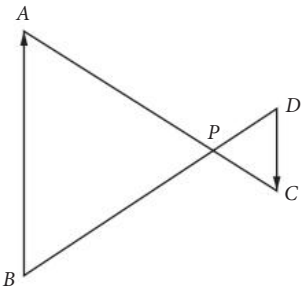


(a) How far up the wall does the ladder reach?

(b) Ruby stands on the ladder at point D, 4 m up the ladder. Prove that $\Delta ADF \,|||\, \Delta DBE$.

(c) Calculate the distances DE and DF.

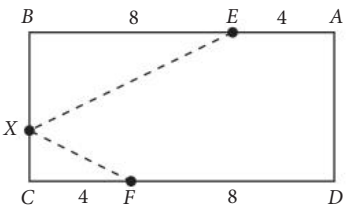
10 The diagram (not to scale) illustrates the principle of a pinhole camera. Light from the object AB passes through a small hole P to form an inverted image, DC.



(a) Prove that $\Delta APB \,|||\, \Delta DPC$.

(b) How far in front of P must an object 2.7 m high be placed so that an image 2.5 cm high is formed 3 cm behind P?

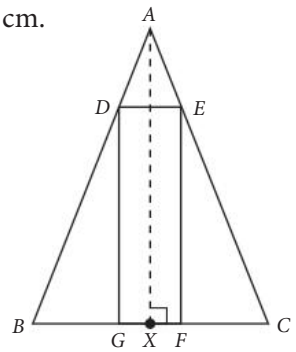
11 $ABCD$ is a billiard table whose dimensions are 12 units by 6 units. A player strikes a ball at E so that the ball hits another ball at F.



(a) Name two similar triangles in the diagram.

(b) If $AE = 4$ units and $CF = 4$ units, what is the distance BX?

12 In the diagram, ΔABC is isosceles with side lengths 13 cm, 13 cm, 10 cm. $DEFG$ is a rectangle with $EF = 9$ cm.



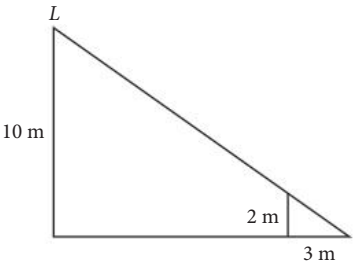
(a) Calculate the length of AX.

(b) Prove that $\Delta AXC \,|||\, \Delta EFC$.

(c) Calculate:

(i) the length of FC **(ii)** the breadth of the rectangle.

13 A person 2 m tall casts a shadow 3 m long from a light 10 m above the ground.



(a) What is the horizontal distance from the person to the light?

(b) What would be the length of the shadow of a person 1.5 m tall standing in the same position?

(c) How far horizontally from the light would a person 1.6 m tall have to stand to cast a shadow 3.2 m long?

14 A straight line through the vertex A of a parallelogram $ABCD$ meets BC at E and at DC produced at F. Prove:

(a) $\Delta ABE \,|||\, \Delta CEF$

(b) $\Delta ABE \,|||\, \Delta ADF$

(c) Complete $\frac{AB}{FD} = \frac{BE}{?}$.

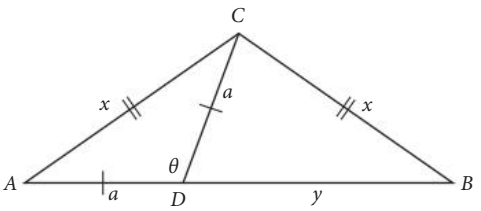
15 PQR is a triangle. A line drawn parallel to PQ meets QR at X and meets PR at Y. Through P a line is drawn parallel to QR, meeting XY produced at Z. Name three similar triangles in this description.

16 ABC is a triangle. BE and CF are drawn perpendicular to AC and AB respectively. Prove that triangles AFC and AEB are similar.

17 The sides AB and DC of a quadrilateral are produced to meet at E. If $\angle EBC = \angle ADE$, prove that $\Delta EBC \,|||\, \Delta EDA$. State two ratios that are each equal to $\frac{BC}{DA}$.

18 D, E and F are the midpoints of BC, CA and AB respectively of ΔABC. Prove that $\Delta ABC \,|||\, \Delta DEF$. What is the ratio of their corresponding sides?

19 ΔABC is isosceles with $AC = BC = x$. The point D on the side AB is chosen so that $AD = CD$. Let $AD = a$, $DB = y$ and $\angle ADC = \theta$.



(a) Show that ΔABC is similar to ΔACD.

(b) Show that $x^2 = a^2 + ay$.

(c) Use the cosine rule to find an expression for y in terms of a and θ.

11.3 HARDER INTERCEPT PROPERTIES OF PARALLEL LINES

If three (or more) parallel lines cut off equal intercepts on a transversal, then the parallel lines will cut off equal intercepts on any other transversal.

Useful results from the intercept property*

1. The straight line drawn through the midpoint of one side of a triangle, parallel to a second side, bisects the third side.
2. The straight line joining the midpoints of two sides of a triangle is parallel to the third side and half its length.
3. A line drawn parallel to one side of a triangle divides the other two sides in proportion.

***Note**: these results are not an essential part of the syllabus, but they may be derived and quoted in proofs.

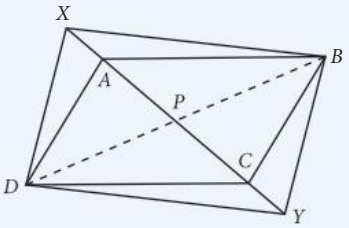
Theorem

When two transversals cut three (or more) parallel lines, the intercepts cut off on one transversal are proportional to the intercepts cut off on the other transversal.

(This theorem will not be proved here.)

Example 4

ABCD is a parallelogram in which *AC* is produced to *Y* and *CA* is produced to *X* such that $AX = CY$. Prove that *XBYD* is a parallelogram.



Solution

Join the diagonal *BD* and let it intersect *AC* at *P*.

$AP = PC$ (diagonals of parallelogram *ABCD* bisect each other)

Similarly $BP = PD$.

Because $AX = CY$, then:

$AX + AP = CY + CP$

$\therefore XP = PY$

In the quadrilateral *XBYD*, $XP = PY$ and $BP = PD$.

$\therefore$ *XBYD* is a parallelogram (diagonals bisect each other).

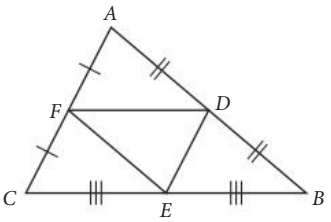
This material covers section 2.6 of the Mathematics Extension 1 course.

EXERCISE 11.3 HARDER INTERCEPT PROPERTIES OF PARALLEL LINES

1 *D*, *E* and *F* are the midpoints of the sides *AB*, *BC* and *CA* respectively.

(a) Name three parallelograms in the diagram.

(b) Name four congruent triangles in the diagram.



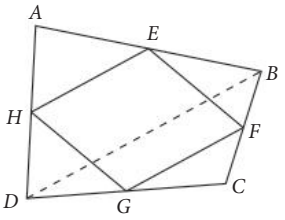
2 *E*, *F*, *G* and *H* are the midpoints of the sides *AB*, *BC*, *CD* and *DA* respectively of the quadrilateral *ABCD*. Prove:

(a) $HE \parallel DB$

(b) $GF \parallel DB$

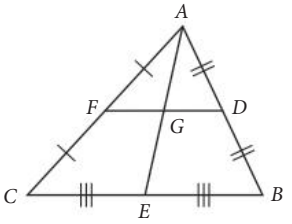
(c) $GF \parallel HE$

(d) *EFGH* is a parallelogram.



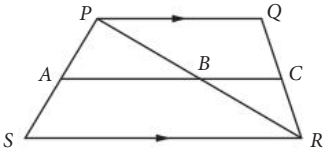
3 *D*, *E* and *F* are the midpoints of *AB*, *BC* and *CA* respectively. Prove:

(a) $FG = GD$ **(b)** $AG = GE$



4 *PQRS* is a trapezium with $PQ \parallel SR$. If *A*, *B* and *C* are the midpoints of *SP*, *PR* and *QR* respectively, prove that:

(a) $AB \parallel SR$ **(b)** $BC \parallel PQ$ **(c)** the points *A*, *B* and *C* are collinear.



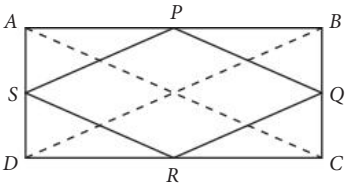
5 *PQRS* is a trapezium with $PQ \parallel SR$. If *A* and *B* are the midpoints of *SP* and *RQ* respectively, prove that:

(a) *AB* is parallel to *PQ* and *SR* **(b)** $AB = \frac{1}{2}(PQ + SR)$

6 Prove that the three straight lines joining the midpoints of the three sides of an equilateral triangle must divide the triangle into four equilateral triangles.

7 D is the midpoint of the side BC of ΔABC. Lines BX, DY and CZ are drawn from B, D and C respectively, perpendicular to any straight line through A. Prove that $XY = YZ$.

8 P, Q, R and S are the midpoints of the sides of a rectangle $ABCD$. By joining the diagonals AC and BD, prove that $PQRS$ is a rhombus.



11.4 COORDINATE METHODS IN GEOMETRY

This section uses skills developed in the Preliminary course in Chapter 6, as summarised below, to prove facts about geometrical figures using coordinate geometry.

Gradient of a straight line, *m*

- The **gradient** (or slope) m of a straight line joining the points (x_1, y_1) and (x_2, y_2) is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$, if $x_1 \neq x_2$.
- If $x_1 = x_2$, the gradient is undefined and the line is vertical (that is, parallel to the y-axis).
- If $y_1 = y_2$, the gradient is zero and the line is horizontal (that is, parallel to the x-axis).

Straight lines

- $y = mx + b$ is the equation of a straight line with gradient m and y-intercept $= b$. This is the **gradient-intercept form** of the equation of a straight line.
- $y - y_1 = m(x - x_1)$ is the equation of a line with gradient m that passes through the point (x_1, y_1). This is the **point-gradient form** of the equation of a straight line.
- $\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$ or $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$ is the equation of a line that passes through the points (x_1, y_1) and (x_2, y_2). This is the **two-point form** of the equation of a straight line.
- $ax + by + c = 0$ is the **general form** of the equation of a straight line. Here a is the coefficient of x, b is the coefficient of y and c is the constant term.
- From the general form: gradient $= -\frac{a}{b}$ and y-intercept $= -\frac{c}{b}$.
  If $a = 0$, the line is parallel to the x-axis. If $b = 0$, the line is parallel to the y-axis. If $c = 0$, the line passes through the origin.

Parallel lines

- Two lines are parallel if their gradients are equal, so $m_1 = m_2$.
- The equation of a line parallel to $ax + by + c = 0$ is $ax + by + d = 0$, where $c \neq d$.

Perpendicular lines

- Two lines are perpendicular if the product of their gradients is -1, so $m_1 m_2 = -1$.
  This can also be written as $m_2 = \frac{-1}{m_1}$. Each gradient is the negative reciprocal of the other.
- The equation of a line perpendicular to $ax + by + c = 0$ is $bx - ay + d = 0$.

Midpoint of an interval

- The coordinates of the midpoint of the interval joining (x_1, y_1) and (x_2, y_2) are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

Distance between two points

- The distance d between two points (x_1, y_1) and (x_2, y_2) is $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

Perpendicular distance of a point from a line

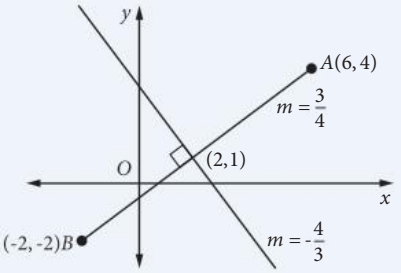
- The perpendicular distance of a fixed point $P(x_1, y_1)$ from a line with equation $ax + by + c = 0$ is given by the formula $d = \dfrac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$ or $d = \left|\dfrac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$.
- The sign of this distance (before the absolute value is taken) indicates which side of the line the point is on. Points with a positive distance are on one side of the line, while points with a negative distance are on the other side of the line.

Example 5

Find the equation of the perpendicular bisector of the line joining the points $A(6,4)$ and $B(-2,-2)$.

Solution

Draw a diagram.



Midpoint of AB is $\left(\dfrac{6+(-2)}{2}, \dfrac{4+(-2)}{2}\right)$, i.e. $(2,1)$

Gradient of AB is $\dfrac{4-(-2)}{6-(-2)} = \dfrac{6}{8} = \dfrac{3}{4}$

Gradient of perpendicular to $AB = -\dfrac{4}{3}$

Equation of perpendicular bisector: $y - y_1 = m(x - x_1)$

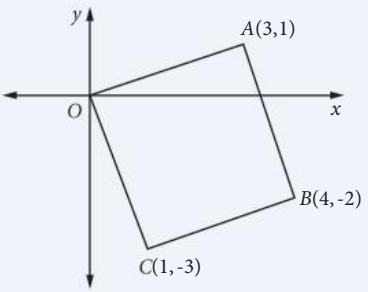
$$y - 1 = -\tfrac{4}{3}(x - 2)$$
$$3y - 3 = -4x + 8$$
$$4x + 3y - 11 = 0$$

Example 6

Show that the four points $(0,0)$, $(3,1)$, $(4,-2)$ and $(1,-3)$ are the vertices of a square.

Solution

Draw a diagram and name the vertices O, A, B and C as shown.



$OA = \sqrt{3^2 + 1^2} = \sqrt{10}$

$AB = \sqrt{(4-3)^2 + (-2-1)^2} = \sqrt{1+9} = \sqrt{10}$

$BC = \sqrt{(4-1)^2 + (-2-(-3))^2} = \sqrt{3^2 + 1^2} = \sqrt{10}$

$CO = \sqrt{1^2 + (-3)^2} = \sqrt{10}$

$\therefore OA = AB = BC = CO$, so $OABC$ is a rhombus.

Gradient of $OA = \frac{1}{3}$

Gradient of $OC = \frac{-3}{1} = -3$

$\frac{1}{3} \times (-3) = -1$, so $OA \perp OC$

$\therefore OABC$ is a rhombus with one angle a right angle, hence $OABC$ is a square.

$\therefore$ The four given points are the vertices of a square.

EXERCISE 11.4 COORDINATE METHODS IN GEOMETRY

1 Calculate the area of the triangle whose vertices are $(1,3)$, $(3,4)$ and $(0,2)$.

2 Prove that the lines $x-2y+5=0$, $x+2y-5=0$ and $2x+y-5=0$ are equidistant from the origin.

3 Prove that the points $(0,-2)$, $(6,6)$ and $(8,4)$ are the vertices of an isosceles triangle.

4 Show that the points $(2,-3)$, $(5,2)$ and $(-3,0)$ are the vertices of a right-angled triangle.

5 Show that the points $A(-1,4)$, $B(4,6)$, $C(2,7)$ and $D(-3,5)$ are the vertices of a parallelogram $ABCD$ by finding the lengths of the sides. Find the length of each diagonal.

6 Show that the points $P(-2,4)$, $Q(5,5)$ and $R(-2,-2)$ are equidistant from the point $(2,1)$.

7 Find the length of each side of a triangle whose vertices have coordinates $(-2,-3)$, $(-5,1)$ and $(6,3)$. Hence show that the triangle is right-angled.

8 The vertices of a quadrilateral are the points $A(2,-3)$, $B(5,1)$, $C(1,4)$ and $D(-2,0)$. Show that the quadrilateral is a square.

9 Show that the points $(-4,-1)$, $(2,3)$ and $(6,-3)$ are the vertices of a right-angled isosceles triangle.

10 Using only gradients, show that the points $(1,1)$, $(4,5)$, $(0,8)$ and $(-3,4)$ are the vertices of a square.

11 The coordinates of A, B and C are $(2,19)$, $(-3,7)$ and $(10,7)$ respectively.

(a) Show that ΔABC is isosceles.
(b) Find the length of the perpendicular from B to AC.

12 Show that the points $(2,-1)$, $(4,3)$ and $(5,2)$ are equidistant from the point $(3,1)$.

13 Show that the points $(-2,-2)$, $(4,6)$ and $(-4,2)$ are equidistant from the point $(1,2)$.

14 The coordinates of A, B and C are $(0,4)$, $(5,1)$ and $(1,-3)$ respectively. Find:

(a) the length of AB **(b)** the perimeter of ΔABC.

15 $A(4,3)$, $B(6,-2)$, $C(1,-5)$ and $D(-3,-2)$ are the vertices of a quadrilateral. Find the coordinates of E, F, G and H, the midpoints of AB, BC, CD and DA respectively, and prove that these points are the vertices of a parallelogram.

16 Find the equation of the perpendicular bisector of the interval joining the points $(-5,-6)$ and $(3,-2)$.

17 Find the equation of the line parallel to the line $3x+4y-6=0$ and passing through the midpoint of the interval joining the points $(-5,-6)$ and $(2,8)$.

18 The points $A(5,3)$, $B(3,-6)$, $C(-3,-2)$ and $D(-1,7)$ are the vertices of a parallelogram. Find the coordinates of the midpoint of each of its diagonals. What do you conclude?

19 Find the equation of the line perpendicular to the line $2x-5y+3=0$ and passing through the midpoint of the interval joining the points $(-8,6)$ and $(4,-2)$.

20 **(a)** Show that the triangle whose vertices are $A(3,5)$, $B(1,1)$ and $C(-1,3)$ is isosceles.
(b) Find the coordinates of the midpoint D of BC.
(c) Write the gradients of AD and BC.
(d) What property of isosceles triangles is illustrated in **(c)**?

21 $A(-3,-1)$, $B(4,1)$, $C(8,5)$ and $D(1,3)$ are the vertices of a parallelogram.

(a) Find the coordinates of the midpoints of the diagonals AC and BD.
(b) What property of parallelograms is illustrated in **(a)**?

22 **(a)** Show that the four points $(0,0)$, $(2,1)$, $(3,-1)$, $(1,-2)$ are the vertices of a square.
(b) Find the coordinates of the midpoint of each diagonal. What can you conclude?
(c) Write the gradient of each diagonal. What can you conclude?

23 $A(2,1)$, $B(1,4)$, $C(-4,-1)$ are three points.
(a) Show that $\angle BAC$ is a right angle.
(b) Find the coordinates of the midpoint D of BC.
(c) Find the distance from D to each vertex of the triangle.

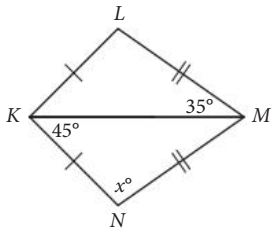
24 $A(-1,4)$, $B(4,6)$, $C(2,7)$ and $D(-3,5)$ are the vertices of a parallelogram.
(a) Find the coordinates of the midpoints P, Q, R and S of AB, BC, CD and DA respectively.
(b) Prove that $PQRS$ is also a parallelogram.

25 $A(-4,0)$, $B(4,0)$ and $C(2,4)$ are the vertices of a triangle.
(a) Find the equation of the perpendicular bisector of AB.
(b) Find the equation of the perpendicular bisector of BC.
(c) Find the coordinates of the point S where the perpendicular bisectors meet.
(d) Prove that S is equidistant from each of the vertices.

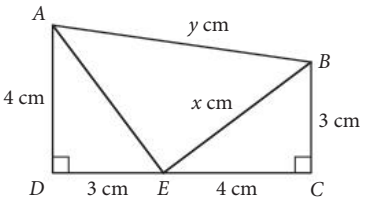
CHAPTER REVIEW 11

1 Find the value of each pronumeral. Give reasons for your answers.

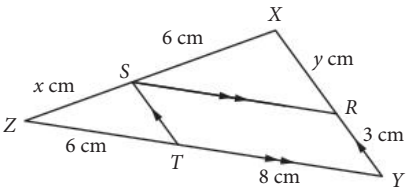
(a)



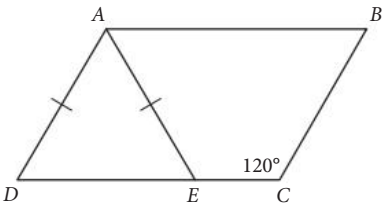
(b)



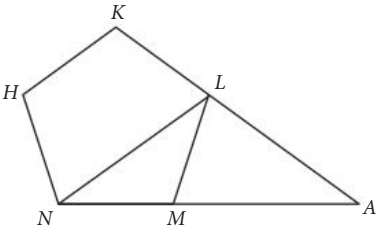
(c)



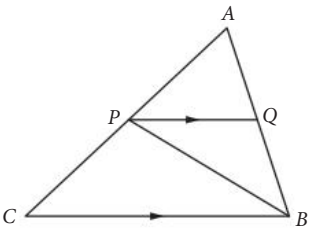
2 $ABCD$ is a parallelogram in which $\angle BCD = 120°$. E lies on DC so $AD = AE$. Prove that ΔADE is equilateral. What special property does the figure $ABCE$ have?



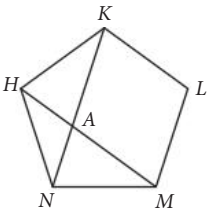
3 $HKLMN$ is a regular pentagon in which the sides NM and KL have been produced to meet at A.
(a) Calculate the size of $\angle LMN$.
(b) Hence show that ΔLNA is isosceles.



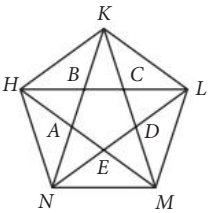
4 In ΔABC, $PQ \parallel CB$.
(a) Prove that $\Delta APQ \,|||\, \Delta ACB$.
(b) If PB bisects $\angle ABC$, prove that ΔPQB is an isosceles triangle.



5 $HKLMN$ is a regular pentagon. The diagonals HM and KN intersect at A.

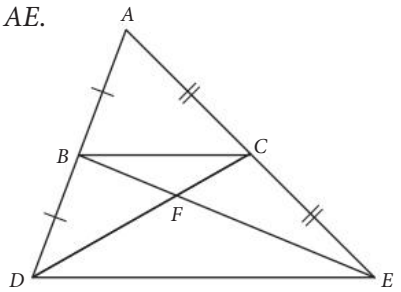


(a) Show that the size of HNM is 108°.
(b) Find the size of $\angle NMH$, giving reasons for your answer.
(c) By considering the sizes of angles, show that ΔNMA is isosceles.
(d) Hence calculate the size of $\angle KAM$.
(e) The remaining diagonals HL, KM and LN are drawn as shown. Explain why $ABCDE$ is a regular pentagon.



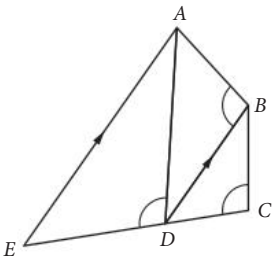
6 In ΔADE, B is the midpoint of AD and C is the midpoint of AE. The intervals BE and CD meet at F.

(a) Explain why ΔABC is similar to ΔADE.
(b) What can you say about BC and DE?
(c) Hence, or otherwise, prove that the ratio $CF : FD = 1 : 2$.



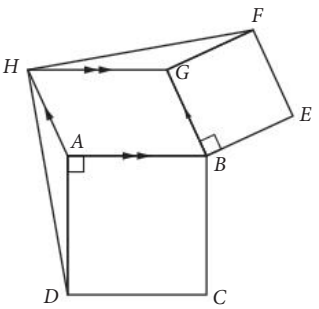
7 $AE \parallel BD$. $\angle ADE = \angle ABD = \angle BCD$.

(a) Prove that $\Delta ADE \,|||\, \Delta BCD$.
(b) Prove that $\Delta DBA \,|||\, \Delta BCD$.
(c) Hence write two ratios equal to $\dfrac{AB}{BD}$.



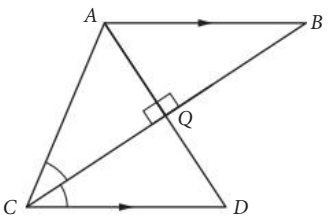
8 $ABCD$ and $GFEB$ are both squares. $HGBA$ is a parallelogram.

(a) Prove that $FG = HA$.
(b) Prove that $HF = DH$.



9 $AB \parallel CD$. BC bisects $\angle ACD$. AD and BC intersect at Q. $AD \perp BC$.

(a) Prove that $\angle ACB = \angle ABC$.
(b) Prove that $\Delta AQC \equiv \Delta AQB$.
(c) Prove that $ABDC$ is a rhombus.

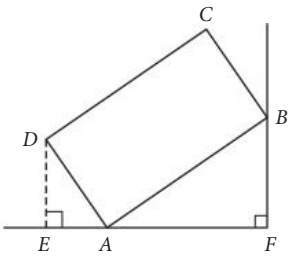


10 The hypotenuse of a right-angled triangle is $(x + 1)$ cm in length and the other two sides are x cm and $(x - 7)$ cm. Form an equation in x and solve it to find the length of each side.

11 $ABCD$ is a rectangular block 10 cm by 6 cm leaning against a vertical wall. Point D is 5 cm above the horizontal ground.

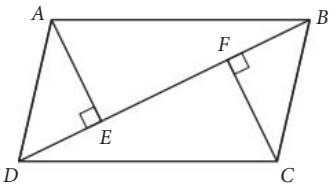
(a) Prove that ΔEAD and ΔFAB are similar.

(b) Find, in simplest surd form, the vertical height of B and C above the ground.



12 $ABCD$ is a parallelogram. AE and CF are drawn perpendicular to the diagonal DB. Prove:

(a) ΔAEB and ΔDFC are congruent **(b)** $AE = FC$



13 $A(0,0)$, $B(6,0)$, $C(8,2)$ and D are the vertices of a parallelogram $ABCD$. Find the coordinates of D.

14 Find the coordinates of the point A on the line $x = -3$ such that the line joining A to $B(3,5)$ is perpendicular to the line $2x + 5y - 12 = 0$.

15 **(a)** Calculate the distance of the point $(3,4)$ from the line $x + y - 6 = 0$.

(b) $A(1,4)$, $B(0,3)$, $C(-4,6)$ are the vertices of a triangle ABC. Calculate the distance of A from side BC and hence find the area of ΔABC.

16 **(a)** Find the equation of the perpendicular bisector of the line joining the points $A(4,2)$ and $B(-6,4)$.

(b) Prove that the point $P(-1,3)$ is on this perpendicular bisector.

(c) Find the distance of P from A and B. What do you conclude?

17 **(a)** Prove that the points $A(-2,-3)$, $B(6,2)$, $C(8,7)$ and $D(0,2)$ are the vertices of a parallelogram.

(b) Find the coordinates of the midpoints of the diagonals AC and BD.

(c) What property of parallelograms is illustrated in **(b)**?

CHAPTER 12
GEOMETRICAL APPLICATIONS OF DIFFERENTIATION

Outcomes covered:

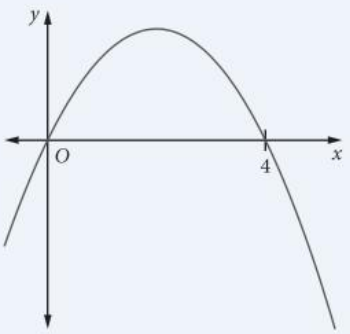
H1 seeks to apply mathematical techniques to problems in a wide range of practical contexts
H5 applies appropriate techniques from the study of calculus, geometry, probability, trigonometry and series to solve problems
H6 uses the derivative to determine the features of the graph of a function
H7 uses the features of a graph to deduce information about the derivative
H9 communicates using mathematical language, notation, diagrams and graphs

12.1 THE SIGN OF THE DERIVATIVE

Remember that the **first derivative** of a function (as studied earlier, see Chapter 10) is also known as the **gradient function**, because it allows us to calculate the gradient at any point on the curve. You have used the process of differentiation to find this gradient function or derivative of a function. This can be positive, negative, zero or undefined. Each possible value has a geometrical application.

Example 1

The sketch of the function $y = 4x - x^2$ is given in the diagram.



(a) Find $\frac{dy}{dx}$ as a function of x.

(b) Complete the following table of values for $\frac{dy}{dx}$:

x	0	1	2	3	4	5
$\frac{dy}{dx}$						

(c) Draw the graph of $\frac{dy}{dx}$ on the same diagram as a graph of y.

(d) For what values of x is **(i)** $\frac{dy}{dx} > 0$ **(ii)** $\frac{dy}{dx} = 0$ **(iii)** $\frac{dy}{dx} < 0$?

(e) Describe the function $y = 4x - x^2$ where $\frac{dy}{dx} > 0$.

(f) Describe the function $y = 4x - x^2$ where $\frac{dy}{dx} < 0$.

(g) Describe the function $y = 4x - x^2$ where $\frac{dy}{dx} = 0$.

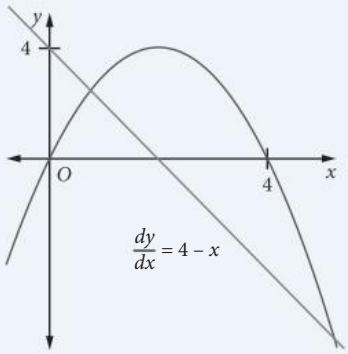
Solution

(a) $\frac{dy}{dx} = 4 - 2x$

(b)

x	0	1	2	3	4	5
$\frac{dy}{dx}$	4	2	0	-2	-4	-6

(c)



(d) **(i)** $x < 2$ **(ii)** $x = 2$ **(iii)** $x > 2$

(e) Where $\frac{dy}{dx} > 0$, y increases as x increases. The curve slopes up.

(f) Where $\frac{dy}{dx} < 0$, y decreases as x increases. The curve slopes down.

(g) Where $\frac{dy}{dx} = 0$, y neither increases nor decreases. It is at its highest point and the tangent at this point is horizontal.

The sign of the first derivative

- If $\frac{dy}{dx} > 0$ as x increases, the function is an **increasing** function.
- If $\frac{dy}{dx} < 0$ as x increases, the function is a **decreasing** function.
- If $\frac{dy}{dx} = 0$ at a given value of x, the function is **stationary** at that point; the point is called a **stationary point**. At this point, the tangent to the curve is parallel to the x-axis.

Each definition and its converse can be used to determine a function's characteristics:

- If $\frac{dy}{dx} > 0$, then the function is increasing; if the function is increasing, then $\frac{dy}{dx} > 0$.
- If $\frac{dy}{dx} < 0$, then the function is decreasing; if the function is decreasing, then $\frac{dy}{dx} < 0$.
- If $\frac{dy}{dx} = 0$ at a point, then it is a stationary point; at a stationary point, $\frac{dy}{dx} = 0$.

Example 2

For what values of x is the function $f(x) = 2x^3 - 9x^2 - 24x + 1$

(a) stationary **(b)** increasing **(c)** decreasing?

Solution

Find $f'(x)$: $f'(x) = 6x^2 - 18x - 24$

Remove common factor: $f'(x) = 6(x^2 - 3x - 4)$

Factorise: $f'(x) = 6(x + 1)(x - 4)$

(a) For stationary points, $f'(x) = 0$: $6(x+1)(x-4) = 0$

$x = -1, 4$

Stationary points occur at $x = -1$ or $x = 4$.

(b) Increasing where $f'(x) > 0$: $(x+1)(x-4) > 0$ **OR** Graphically:

Test $x = 0$: LHS $= 1 \times (-4)$

$= -4$

< 0

Graph is above axis for $x < -1$ or $x > 4$

Function is increasing for $x < -1$ or $x > 4$

Function is not increasing for $-1 < x < 4$

Hence function is increasing for $x < -1$ or $x > 4$.

(c) Decreasing where $f'(x) < 0$, so function is decreasing for $-1 < x < 4$.

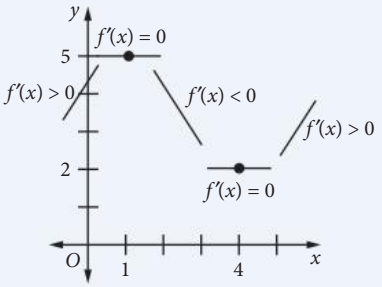
Example 3

Sketch a graph of f such that $f(4) = 2$, $f'(4) = 0$, $f(1) = 5$, $f'(1) = 0$, $f'(x) > 0$ for all $x < 1$ and for all $x > 4$; also $f'(x) < 0$ for $1 < x < 4$.

Solution

Mark this information on a number plane:

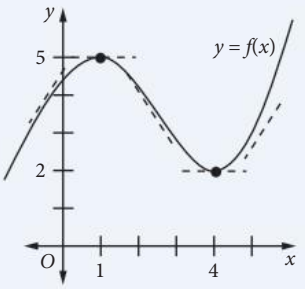
Use straight lines to indicate gradients.



Draw a curve using this information:

The only points that we know for sure on $y = f(x)$ are $(1, 5)$ and $(4, 2)$.

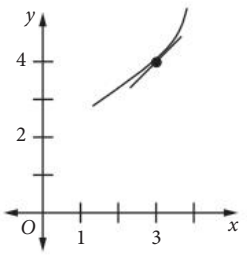
From the gradients we know that the curve changes from increasing to decreasing at $(1, 5)$ and from decreasing to increasing at $(4, 2)$. Fit the curve to this information.



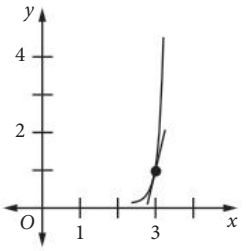
EXERCISE 12.1 THE SIGN OF THE DERIVATIVE

1 A function $f(x)$ has the following properties: $f(3) = 4$, $f'(3) = 1$. Which sketches fit the graph of $y = f(x)$ near $x = 3$?

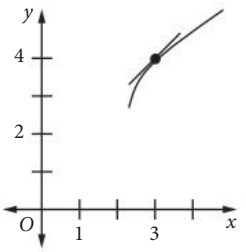
A



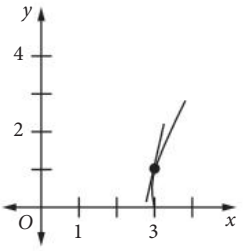
B



C



D



2 Sketch the graph of $y = f(x)$ with the following properties: $f(1) = 0$, $f'(x) = 2$ for all x. State the rule that defines the function.

3 Sketch the graph of a function given that $f(2) = 0$, $f'(2) = 0$, $f'(x) < 0$ for all $x < 2$, and $f'(x) > 0$ for all $x > 2$.

4 Sketch the graph of $y = f(x)$, such that $f(3) = 3$, $f'(3) = 0$, $f(1) = 5$, $f'(1) = 0$, $f'(x) > 0$ for all $x < 1$ and for all $x > 3$; also $f'(x) < 0$ for $1 < x < 3$.

5 For the function $f(x) = x^2 - 5x + 6$, sketch the graph of $f'(x)$ and hence find the values of x for which:

(a) $f'(x) < 0$ **(b)** $f'(x) = 0$ **(c)** $f'(x) > 0$

6 Sketch the curve $y = 4x - x^2$. For what values of x is $\frac{dy}{dx} = 0$? What is the sign of the gradient to the left and right of this point?

7 For the graph of $f(x) = 6 - 3x - x^2$, find the values of x for which the function:

(a) increases when x increases **(b)** decreases when x increases
(c) changes from increasing to decreasing.

8 For the graph of $f(x) = x^3 - 6x^2 + 9x + 2$, find:

(a) $f'(x)$
(b) the values of x for which the function increases when x increases
(c) the values of x for which the function decreases when x increases
(d) the values of x for which the function changes from increasing to decreasing.

9 For the graph of $f(x) = x^3 - 1$, find the values of x for which the function:

(a) increases when x increases **(b)** decreases when x increases
(c) changes from increasing to decreasing.

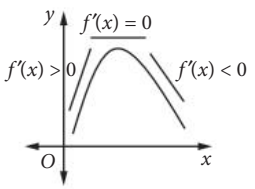
10 For the graph of $f(x) = (x - 1)^2(x + 1)$, find the values of x for which the function is:

(a) stationary **(b)** increasing **(c)** decreasing.

12.2 THE FIRST DERIVATIVE AND TURNING POINTS

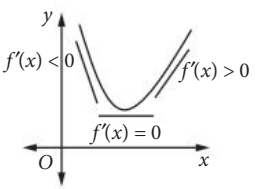
In functions we have seen so far, a stationary point is usually the point where the function changes from increasing to decreasing (or from decreasing to increasing), so that the stationary point is the highest (or lowest) point in the neighbourhood. (Here 'neighbourhood' means 'near the point on the graph'.) However, sometimes the function does not change the sign of its gradient at the stationary point. This situation will be considered later.

When a function changes from increasing to decreasing at a stationary point, the sign of $f'(x)$ changes from positive to negative.



When a function changes from decreasing to increasing at a stationary point, the sign of $f'(x)$ changes from negative to positive.

These points are called turning points. If the turning point is higher than the other points in its neighbourhood, it is called a **local maximum** turning point. If the point is lower than the other points in its neighbourhood, it is called a **local minimum** turning point.



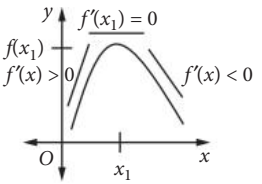
A **turning point** of $f(x)$ is a point where the curve $y = f(x)$ is locally a maximum or a minimum.

For a differentiable function $f(x)$, all turning points are stationary points.
(**However**, note that not all stationary points are turning points.)

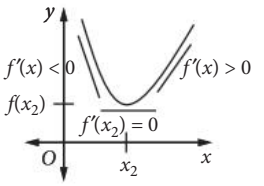
First derivative test for local maxima and minima

A turning point of the differentiable function $f(x)$ may occur when $f'(x) = 0$. The type of turning point will depend on the change in sign of $f'(x)$ as x passes through the abscissa (x-coordinate) of the stationary point, so we need to find the sign of $f'(x)$ on either side of the stationary point.

If $f'(x) = 0$ when $x = x_1$, $f'(x) > 0$ when $x < x_1$, and $f'(x) < 0$ when $x > x_1$, then the sign of $f'(x)$ changes from positive to negative as x passes through x_1. The point $(x_1, f(x_1))$ must be a **local maximum turning point**.



If $f'(x) = 0$ when $x = x_2$, $f'(x) < 0$ when $x < x_2$, and $f'(x) > 0$ when $x > x_2$, then the sign of $f'(x)$ changes from negative to positive as x passes through x_2. The point $(x_2, f(x_2))$ is a **local minimum turning point**.



Thus there are two conditions required to find turning points of the function $y = f(x)$:

1. $f'(x) = 0$ at $x = x_1$, and
2. $f'(x)$ changes sign as x passes through x_1.

The way that $f'(x)$ changes will tell you whether the turning point is a maximum or minimum turning point.

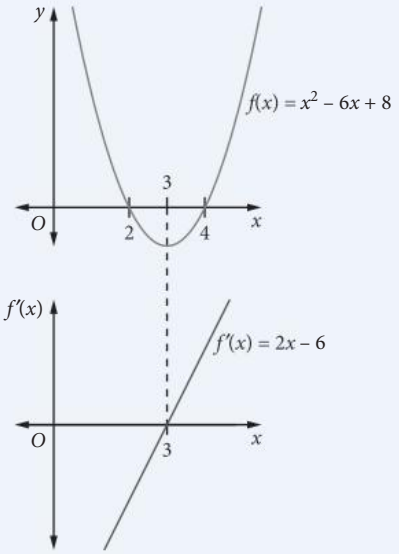
Example 4

The diagrams show the graphs of $f(x) = x^2 - 6x + 8$ and $f'(x) = 2x - 6$ drawn with a common x scale.

(a) Find the coordinates of any stationary points on $f(x)$.

(b) Use the graph of $f'(x)$ to determine the nature of the turning points (i.e. maximum or minimum).

(c) What is the least value of $f(x)$?



Solution

(a) The graph of $f'(x)$ gives that $f'(x) = 0$ when $x = 3$.

$f(3) = 9 - 18 + 8 = -1$

The coordinates of the stationary point are $(3, -1)$

(b) When $x < 3$, $f'(x) < 0$; when $x > 3$, $f'(x) > 0$.

$f'(x)$ changes from negative to positive as x passes through 3.

$\therefore$ The stationary point is a relative minimum turning point (local minimum).

(c) The least value of $f(x)$ is -1.

Example 5

A function is given by $f(x) = x^3 - 12x + 16$.

(a) Find $f'(x)$.

(b) Find the coordinates of any stationary points.

(c) Determine the nature of the stationary points.

(d) Sketch $y = f(x)$.

Solution

(a) $f'(x) = 3x^2 - 12$

(b) For stationary points, $f'(x) = 0$, so:

$$3(x^2 - 4) = 0$$
$$(x + 2)(x - 2) = 0$$
$$x = -2, 2$$

$f(-2) = -8 + 24 + 16 = 32 \qquad f(2) = 8 - 24 + 16 = 0$

Stationary points are $(-2, 32)$ and $(2, 0)$.

(c) Consider the stationary point $(-2, 32)$.

At $x = -3$, $f'(-3) = 27 - 12 = 15 > 0$

At $x = -1$, $f'(-1) = 3 - 12 = -9 < 0$

$f'(x)$ changes from +ve to −ve, so $f(x)$ has a maximum turning point at $(-2, 32)$.

Consider the stationary point $(2, 0)$.

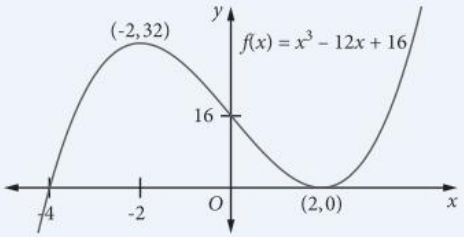
At $x = 1$, $f'(1) = 3 - 12 = -9 < 0$

At $x = 3$, $f'(3) = 27 - 12 = 15 > 0$

$f'(x)$ changes from −ve to +ve, so $f(x)$ has a minimum turning point at $(2, 0)$.

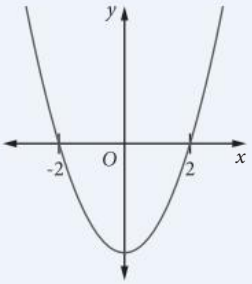
$\therefore$ $(-2, 32)$ is a maximum turning point and $(2, 0)$ is a minimum turning point.

(d)



In calculating the change in gradient, a sketch of $f'(x) = 3x^2 - 12$ could have been used to investigate the change in sign of $f'(x)$:

$x < -2$, $f'(x) > 0$
$x > -2$, $f'(x) < 0$: maximum turning point at $(-2, 32)$
$x < 2$, $f'(x) < 0$
$x > 2$, $f'(x) > 0$: minimum turning point at $(2, 0)$



When selecting values of x to substitute into $f'(x)$, you need to pick values near the abscissa (x-coordinate) of the stationary point. But how near is 'near'? In Example **5** we used values that were 1 unit either side of the stationary point, and in this example that was close enough. We could have used -2.1 and -1.9 and reached the same answer, but the calculations would have been more time-consuming.

Calculations for the stationary point at $(-2, 32)$ could have been summarised in tables as follows:

x-value	-3	-2	-1
$f'(x)$	15	0	-9
Sign of $f'(x)$	+	0	−
Direction of curve	↗	→	↘
Shape of curve			

x-value	-2.1	-2	-1.9
$f'(x)$	1.2	0	-1.2
Sign of $f'(x)$	+	0	−
Direction of curve	↗	→	↘
Shape of curve			

This would then allow us to say that the point $(-2, 32)$ is a local maximum.

Turning points—theoretical method

Another method to investigate turning points is to use the vanishingly small positive number ε (the Greek letter epsilon), defined so that ε is infinitesimally small but $\varepsilon > 0$.

To apply the ε method to Example **5**, at the stationary point $(-2, 32)$ consider $x = -2 - \varepsilon$ and $x = -2 + \varepsilon$ as the values of x either side of $x = -2$. Factorise the expression $f'(x) = 3x^2 - 12$ into $f'(x) = 3(x+2)(x-2)$ and examine the values of f':

$f'(-2-\varepsilon) = 3(-2-\varepsilon+2)(-2-\varepsilon-2) = 3(-\varepsilon)(-4-\varepsilon) > 0$, because $(-)\times(-) = (+)$

$f'(-2+\varepsilon) = 3(-2+\varepsilon+2)(-2+\varepsilon-2) = 3(\varepsilon)(-4+\varepsilon) < 0$, because $(+)\times(-) = (-)$

$f'(x)$ changes from +ve to −ve, so $f(x)$ has a maximum turning point at $(-2, 32)$.

Example 6

The function $y = 3x^2 - x^3$ is defined on the domain $-2 \le x \le 4$. Find:

(a) the stationary points and determine their nature

(b) the greatest and least values of y in the domain.

(c) Sketch the function for the given domain.

Solution

(a) $y = 3x^2 - x^3$

$\dfrac{dy}{dx} = 6x - 3x^2$

$= 3x(2-x)$

For stationary points, $\dfrac{dy}{dx} = 0$

$3x(2-x) = 0$

$\therefore x = 0$ or $x = 2$

Stationary points are $(0, 0)$ and $(2, 4)$.

For $\varepsilon > 0$:

$x = 0 - \varepsilon$, $\dfrac{dy}{dx} = 3(-\varepsilon)(2-(-\varepsilon))$

$= -3\varepsilon(2+\varepsilon)$

< 0

$x = 0 + \varepsilon$, $\dfrac{dy}{dx} = 3\varepsilon(2-\varepsilon)$

> 0

$\dfrac{dy}{dx}$ changes from −ve to +ve, so $(0, 0)$ is a minimum turning point.

$x = 2 - \varepsilon$, $\dfrac{dy}{dx} = 3(2-\varepsilon)(2-(2-\varepsilon))$

$= 3(2-\varepsilon)(\varepsilon)$

> 0

$x = 2 + \varepsilon$, $\dfrac{dy}{dx} = 3(2+\varepsilon)(2-(2+\varepsilon))$

$= 3(2+\varepsilon)(-\varepsilon)$

< 0

$\dfrac{dy}{dx}$ changes from +ve to −ve, so $(2, 4)$ is a maximum turning point.

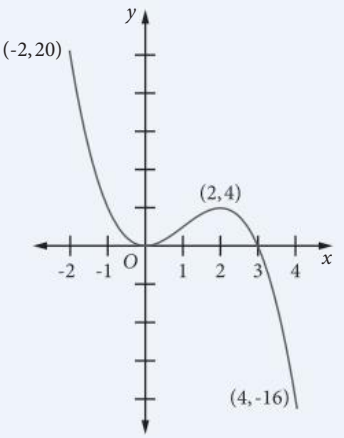
(b) The turning points give the local maximum and minimum values. The values at the endpoints of a given domain may be greater than or less than these local values. In this example we need to find the value of y at $x = -2$ and $x = 4$.

$x = -2$: $y = 12 + 8 = 20$

$x = 4$: $y = 48 - 64 = -16$

Hence the greatest value of the function in the domain is 20 and the least value is -16. These occur at the endpoints of the domain.

(c)



EXERCISE 12.2 THE FIRST DERIVATIVE AND TURNING POINTS

1 A function is given by $f(x) = x^2 - 6x + 8$.

(a) Find $f'(x)$. **(b)** Find the coordinates of any stationary points and determine their nature.
(c) Sketch $y = f(x)$.

2 If $f'(x) = x^2 - 5x - 6$ then stationary points may occur when:

A $x = 1, -6$ **B** $x = -2, -3$ **C** $x = -1, 6$ **D** $x = -3, 2$

3 A function is given by $f(x) = x^3 - 6x^2 + 16$.

(a) Find $f'(x)$. **(b)** Find the coordinates of any stationary points and determine their nature.
(c) Sketch $y = f(x)$.

4 A function is given by $f(x) = x^3 - 9x^2 + 15x + 16$. You are asked to find the stationary points and determine their nature. Four steps in answering this question are shown below. Indicate whether each step is correct or incorrect.

(a) $f'(x) = 3x^2 - 18x + 15$ **(b)** Stationary points occur when $x = -1, 5$.
(c) $(1, 23)$ is a maximum turning point. **(d)** $(5, -9)$ is a minimum turning point.

5 A function is given by $f(x) = 2x^3 - 15x^2 + 36x$.

(a) Find $f'(x)$. **(b)** Find the coordinates of any stationary points and determine their nature.
(c) Sketch $y = f(x)$.

6 A function is given by $f(x) = x^3 - x^2 - x + 1$.

(a) Find $f'(x)$. **(b)** Find the coordinates of any stationary points and determine their nature.
(c) Sketch $y = f(x)$ over the domain $-2 \leq x \leq 2$, showing the turning points.

7 Find the maximum value of $5x - 2x^2$.

8 Find the minimum value of $x(x - 2) + 3$.

9 Sketch the curve $y = x^3 - 6x^2$ over the domain $-1 \leq x \leq 6$, showing the maximum and minimum turning points.

10 Find the turning points of $y = 2x^3 + 3x^2 - 12x + 7$. Hence sketch the curve, showing the turning points and the y-intercept.

11 Sketch the curve $y = (2 - x)(1 + x^2)$, locating the turning points and the points where it crosses the coordinate axes over the domain $-1 \leq x \leq 3$.

12 Consider the function $f(x) = 9x(x - 2)^2$, $-1 \leq x \leq 3$. Find the values of x for which:

(a) $f'(x) = 0$ **(b)** $f'(x) > 0$ **(c)** $f'(x) < 0$
(d) Sketch the graph of $f(x)$ and state its range and greatest and least values.

13 Prove that the parabola $y = ax^2 + bx + c$ has a turning point at $x = \frac{-b}{2a}$.

14 Show that the hyperbola $y = \frac{1}{x}$ has no turning points. Also show that its gradient is always negative throughout its domain.

12.3 THE SECOND DERIVATIVE AND CONCAVITY

Differentiating a function once to obtain $f'(x)$ or $\frac{dy}{dx}$ gives you the first derivative of the original function.

Differentiating the first derivative gives you $f''(x)$ or $\frac{d^2y}{dx^2}$, which is called the **second derivative** of the original function. The second derivative is the rate of change of the first derivative: $\frac{d}{dx}\big(f'(x)\big)$.

Differentiating again will give the third derivative, and so on. The differentiation process may be continued for as long as a derivative exists.

Notation

Several different notations can be used for derivatives. If $y = f(x)$, then:

- the first derivative can be written $\frac{dy}{dx}$, $f'(x)$, $\frac{d}{dx}\big(f(x)\big)$ or y'
- the second derivative can be written $\frac{d}{dx}\left(\frac{dy}{dx}\right)$, $\frac{d^2y}{dx^2}$, $f''(x)$, $\frac{d}{dx}\big(f'(x)\big)$ or y''.

Example 7

Find the second derivative of each function.

(a) $y = 4x^3 - 2x^2 + 3x + 7$ **(b)** $f(x) = (2x+1)^5$ **(c)** $y = \frac{x^2}{x+1}$

Solution

(a) $y = 4x^3 - 2x^2 + 3x + 7$:

$$\frac{dy}{dx} = 12x^2 - 4x + 3$$

$$\frac{d^2y}{dx^2} = 24x - 4$$

(b) $f(x) = (2x+1)^5$:

$$f'(x) = 5(2x+1)^4 \times 2$$
$$= 10(2x+1)^4$$
$$f''(x) = 10 \times 4(2x+1)^3 \times 2$$
$$= 80(2x+1)^3$$

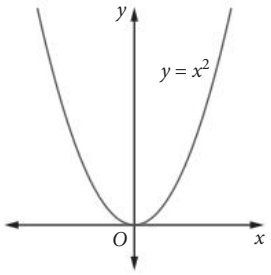
(c) $y = \frac{x^2}{x+1}$:

$$y' = \frac{2x(x+1) - x^2 \times 1}{(x+1)^2}$$
$$= \frac{2x^2 + 2x - x^2}{(x+1)^2}$$
$$= \frac{x^2 + 2x}{(x+1)^2}$$

$$y'' = \frac{(2x+2)(x+1)^2 - \left(x^2 + 2x\right) \times 2(x+1)}{(x+1)^4}$$
$$= \frac{2(x+1)^3 - 2(x+1)\left(x^2 + 2x\right)}{(x+1)^4}$$
$$= \frac{2(x+1)\left((x+1)^2 - \left(x^2 + 2x\right)\right)}{(x+1)^4}$$
$$= \frac{2\left(x^2 + 2x + 1 - x^2 - 2x\right)}{(x+1)^3}$$
$$= \frac{2}{(x+1)^3}$$

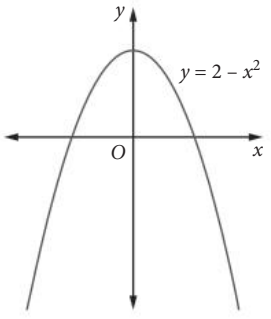
Concavity

The **concavity** of a function describes how the shape of the function's graph 'opens'. Graphs can be 'concave upwards' and 'concave downwards':



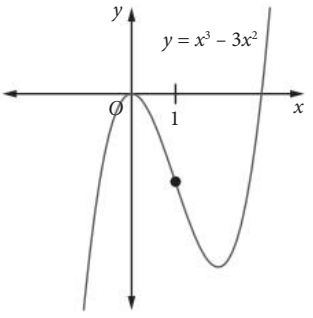
The function is concave up.

Note: has a minimum turning point.



The function is concave down.

Note: has a maximum turning point.



The left-hand part is concave down.

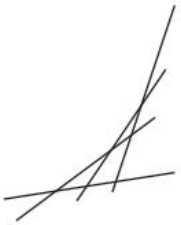
The right-hand part is concave down.

Concavity changes at $x = 1$.

The second derivative is the rate at which the first derivative is changing. This gives us information about the concavity of a function.

In each diagram below, a series of tangents have been drawn.

1 The tangents have a positive gradient and the gradient is increasing.

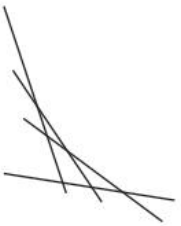


If the gradient of the tangent $= \frac{dy}{dx}$, then $\frac{dy}{dx} > 0$.

The rate at which the gradient is increasing $= \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$.

Because the gradient is increasing, $\frac{d^2y}{dx^2} > 0$: the curve is concave up.

2 The tangents have a negative gradient and the gradient is increasing (becoming less negative).



If the gradient of the tangent $= \frac{dy}{dx}$, then $\frac{dy}{dx} < 0$.

Because the gradient is increasing, $\frac{d^2y}{dx^2} > 0$: the curve is concave up.

3 The tangents have a negative gradient and the gradient is decreasing (becoming more negative).

If the gradient of the tangent $= \frac{dy}{dx}$, then $\frac{dy}{dx} < 0$.

Because the gradient is decreasing, $\frac{d^2y}{dx^2} < 0$: the curve is concave down.

4 The tangents have a positive gradient and the gradient is decreasing.

If the gradient of the tangent $= \frac{dy}{dx}$, then $\frac{dy}{dx} > 0$.

Because the gradient is decreasing, $\frac{d^2y}{dx^2} < 0$: the curve is concave down.

5 The gradient is positive.

Initially the gradient is decreasing, $\frac{d^2y}{dx^2} < 0$, but then it starts increasing, $\frac{d^2y}{dx^2} > 0$.

This means that at some point $\frac{d^2y}{dx^2} = 0$. This is also where the concavity changes from concave down to concave up, so this point is called a **point of inflexion**.

6 The gradient is negative.

Initially the gradient is increasing, $\frac{d^2y}{dx^2} > 0$, but then it starts decreasing, $\frac{d^2y}{dx^2} < 0$.

This means that at some point $\frac{d^2y}{dx^2} = 0$. This is also where the concavity changes from concave down to concave up, so this is a point of inflexion.

The sign of the second derivative

- If $\frac{d^2y}{dx^2} > 0$ on an interval then the curve is **concave upwards** on that interval.
- If $\frac{d^2y}{dx^2} < 0$ on an interval then the curve is **concave downwards** on that interval.
- If $\frac{d^2y}{dx^2} = 0$ at a point on the curve **and** the concavity changes at this point, then the point is called a **point of inflexion**.

Example 8

For what values of x is the curve given by $y = x^3 - 3x^2 + 6x + 3$:

(a) concave up **(b)** concave down?

(c) Find the coordinates of the point of inflexion. **(d)** Sketch the curve.

Solution

Find $\frac{dy}{dx}$: $\frac{dy}{dx} = 3x^2 - 6x + 6$ Find $\frac{d^2y}{dx^2}$: $\frac{d^2y}{dx^2} = 6x - 6$

(a) Concave up, $\frac{d^2y}{dx^2} > 0$: $6x - 6 > 0$, $x > 1$

(b) Concave down, $\frac{d^2y}{dx^2} < 0$: $x < 1$

The curve is concave up for $x > 1$.

(c) Inflexion point, $\frac{d^2y}{dx^2} = 0$: $x = 1$

$x < 1$, curve is concave down

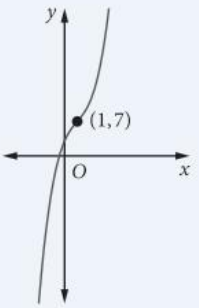
$x > 1$, curve is concave up

$\therefore$ concavity changes at $x = 1$

$x = 1$, $y = 1 - 3 + 6 + 3 = 7$

$\therefore$ point of inflexion is $(1, 7)$

(d)



EXERCISE 12.3 THE SECOND DERIVATIVE AND CONCAVITY

1 Find $f''(x)$ for each function.

(a) $f(x) = 3x^2 + 5x + 6$ **(b)** $f(x) = x^3 + 2x^2 + 4x + 2$ **(c)** $f(x) = 24 - x^2$

(d) $f(x) = x^5 + 2x^3 - 4x$ **(e)** $f(x) = 12 - x^4 + 2x^2$ **(f)** $f(x) = 5x - 4$

2 Given $f(x) = x^6 + 3x^3 - 4x + 2$, $f''(x) = \ldots$

A $6x^5 + 3x^2 - 4$ **B** $6x^5 + 9x^2 - 4$ **C** $30x^4 + 6x$ **D** $30x^4 + 18x$

3 Given $y = \frac{x^2 - 1}{x}$, find $\frac{d^2y}{dx^2}$. Indicate whether each statement below is a correct or incorrect step in finding $\frac{d^2y}{dx^2}$.

(a) $y = x - \frac{1}{x}$ **(b)** $\frac{dy}{dx} = \frac{x^2 - 1}{x^2}$ **(c)** $\frac{dy}{dx} = 1 + \frac{1}{x^2}$ **(d)** $\frac{d^2y}{dx^2} = \frac{-2}{x^2}$

4 Find $\frac{d^2y}{dx^2}$ given: **(a)** $y = \sqrt{x}$ **(b)** $y = \sqrt{x - 2}$ **(c)** $y = x\sqrt{x^2 + 1}$ **(d)** $y = \frac{1}{x}$

(e) $y = \frac{1}{x + 1}$ **(f)** $y = \frac{x}{x + 3}$ **(g)** $y = \frac{x^2 + 1}{\sqrt{x}}$ **(h)** $y = (x^2 - 1)\sqrt{x}$ **(i)** $y = \frac{\sqrt{x - 1}}{x + 1}$

5 For what values of x is $y = 5x^2 - 1$ concave up?

6 For what values of x is $y = 6 - 3x^2$ concave down?

7 For what values of x is the curve $y = x^3 + 6x^2 - x + 4$:

(a) concave up **(b)** concave down? **(c)** Find the coordinates of the point of inflexion.

8 For what values of x is the graph of $y = \sqrt{x+1}$ concave down? Explain why there is no point of inflexion.

9 For what values of x is the curve $y = \frac{1}{x}$:

(a) concave up **(b)** concave down? **(c)** Explain why there is no point of inflexion.

10 Explain why the graph of $y = \frac{1}{x^2}$ is concave up over its domain.

12.4 THE SECOND DERIVATIVE AND TURNING POINTS

Just as we compared the graphs of $f(x)$ and $f'(x)$ drawn together on the same horizontal scale, we now need to consider the graphs of $f(x)$, $f'(x)$ and $f''(x)$ together.

Consider: $y = f(x) = x^3 - x^2 - x + 1$

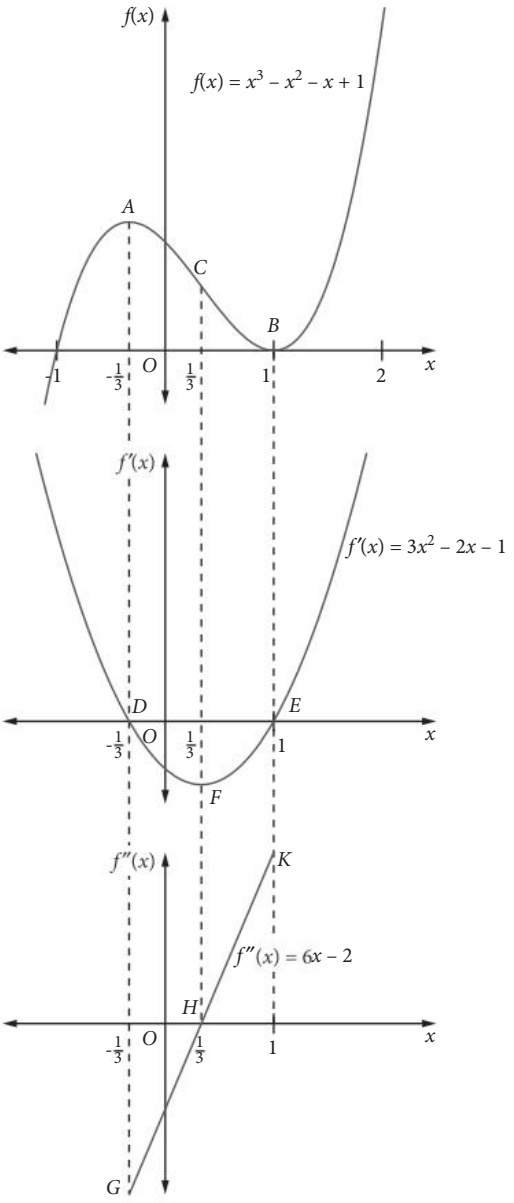
Differentiate: $\frac{dy}{dx} = y' = f'(x) = 3x^2 - 2x - 1$

Differentiate again: $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = y'' = f''(x) = 6x - 2$

For stationary points, $f'(x) = 0$: $3x^2 - 2x - 1 = 0$

Factorise: $(3x+1)(x-1) = 0$

Solve: $x = -\frac{1}{3}, 1$



On the graph, the vertical lines ADG and BEK correspond to the lines $x = -\frac{1}{3}$ and $x = 1$ respectively.

Using the first derivative test (or the graph of $f'(x)$), we can say that A is a maximum turning point and that B is a minimum turning point.

- Consider where the line ADG cuts the three curves. The point A on $f(x)$ is a maximum turning point. The point D on $f'(x)$ is where $f'(x)$ cuts the x-axis, that is, $f'(x) = 0$. At the point G on $f''(x)$, $f''(x) < 0$. Hence the curve $y = f(x)$ is concave down at a maximum turning point.
- Consider where the line BEK cuts the three curves: at point B, $f(x)$ is a minimum turning point; at point E, $f'(x)$ cuts the x-axis, i.e. $f'(x) = 0$; at point K, $f''(x) > 0$. Hence the curve $y = f(x)$ is concave up at a minimum turning point.
- Consider where the line CFH cuts the three curves: at point C, $f(x)$ seems to have its steepest tangent; at point F, $f'(x)$ has its least value, i.e. the gradient of $f(x)$ is at its most negative (steepest); at point H, $f''(x) = 0$. The concavity changes either side of C, so C is a point of inflexion on $y = f(x)$.

For $y = f(x) = x^3 - x^2 - x + 1$, we can say that the function has a maximum turning point at $\left(-\frac{1}{3}, 1\frac{5}{27}\right)$, a minimum turning point at $(1, 0)$, is concave down for $x < \frac{1}{3}$, is concave up for $x > \frac{1}{3}$ and has a point of inflexion at $\left(\frac{1}{3}, \frac{16}{27}\right)$.

Second derivative test for turning points

- If $\frac{dy}{dx} = 0$ at (x_1, y_1) and $\frac{d^2y}{dx^2} < 0$ then the point (x_1, y_1) is a maximum turning point.
- If $\frac{dy}{dx} = 0$ at (x_2, y_2) and $\frac{d^2y}{dx^2} > 0$ then the point (x_2, y_2) is a minimum turning point.
- If $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$ at (x_3, y_3) then the point may be a turning point **OR** it may be a horizontal point of inflexion. Further investigation is needed.

Point of inflexion test

- If $\frac{d^2y}{dx^2} = 0$ at (x_4, y_4) and the concavity changes either side of this point, then (x_4, y_4) is a point of inflexion.
- If $\frac{dy}{dx} = 0$ at a point of inflexion then it is called a horizontal point of inflexion.

Example 9

Investigate the stationary points of:

(a) $y = x^4$ **(b)** $y = 3x^2 - 3x - x^3$

Solution

(a) $y = x^4$: $\frac{dy}{dx} = 4x^3$; $\frac{d^2y}{dx^2} = 12x^2$

For stationary points, $\frac{dy}{dx} = 0$: $x = 0$

At $x = 0$, $\frac{d^2y}{dx^2} = 0$, so we can't identify the nature of this stationary point.

Check the sign of the first derivative: For $x < 0$, e.g. $x = -1$: $\frac{dy}{dx} = -4 < 0$

For $x > 0$, e.g. $x = 1$: $\frac{dy}{dx} = 4 > 0$

The gradient changes from negative to positive through $x = 0$, so $(0, 0)$ is a minimum turning point.

Alternative method:

Check the sign of the second derivative: For $x < 0$, e.g. $x = -1$: $\frac{d^2y}{dx^2} = 12 > 0$

For $x > 0$, e.g. $x = 1$: $\frac{d^2y}{dx^2} = 12 > 0$

The curve is concave up on both sides of the stationary point, so $(0, 0)$ is a minimum turning point.

(b) $y = 3x^2 - 3x - x^3$: $\frac{dy}{dx} = 6x - 3 - 3x^2 = -3\left(x^2 - 2x + 1\right) = -3(x-1)^2$ $\frac{d^2y}{dx^2} = 6 - 6x$

For stationary points, $\frac{dy}{dx} = 0$: $(x-1)^2 = 0$, hence $x = 1$

At $x = 1$, $\frac{d^2y}{dx^2} = 0$, so we can't identify the nature of this stationary point.

Check the sign of the second derivative: For $x < 1$, e.g. $x = 0.1$: $\frac{d^2y}{dx^2} = 6 - 0.6 = 5.4 > 0$

For $x > 1$, e.g. $x = 1.1$: $\frac{d^2y}{dx^2} = 6 - 6.6 = -0.6 < 0$

The concavity changes either side of $x = 1$, so $(1, -1)$ is a horizontal point of inflexion.

Example 10

Consider the graph of $y = 2x^4 - x + 1$.

(a) Find any turning points and points of inflexion. **(b)** For what values of x is the curve concave up?

(c) On the same set of axes sketch y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

Solution

(a) $y = 2x^4 - x + 1$: $\quad \frac{dy}{dx} = 8x^3 - 1$

For stationary points, $\frac{dy}{dx} = 0$: $\quad 8x^3 - 1 = 0$, hence $x^3 = \frac{1}{8}$ and so $x = \frac{1}{2}$.

At $x = \frac{1}{2}$, $y = 2 \times \frac{1}{2^4} - \frac{1}{2} + 1 = 1\frac{3}{8}$ $\quad \therefore$ stationary point is at $\left(\frac{1}{2}, 1\frac{3}{8}\right)$.

Find $\frac{d^2y}{dx^2}$: $\quad \frac{d^2y}{dx^2} = 24x^2$

At $x = \frac{1}{2}$, $\frac{d^2y}{dx^2} = 6 > 0$ $\quad \therefore \left(\frac{1}{2}, 1\frac{3}{8}\right)$ is a minimum turning point.

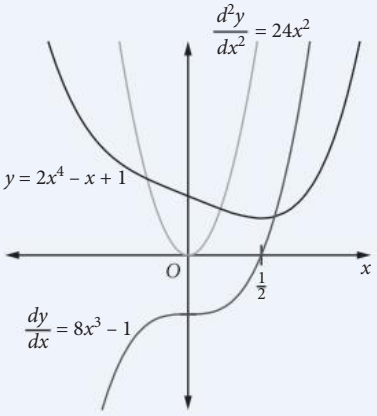
$\frac{d^2y}{dx^2} = 0$ at $x = 0$, so check concavity to see if there is an inflexion point at $(0, 1)$:

$x < 0$: $\frac{d^2y}{dx^2} > 0$ $\qquad$ $x > 0$: $\frac{d^2y}{dx^2} > 0$

The concavity does not change either side of $x = 0$, so the curve does not have a point of inflexion at $(0, 1)$.

(b) Concave up when $\frac{d^2y}{dx^2} > 0$, hence concave up for all real x.

(c)



This example shows that $\frac{d^2y}{dx^2} = 0$ is **not enough** to confirm a point of inflexion. You must also check that the concavity changes at the point.

Example 11

Sketch the graph of $y = x^3(4 - x)$, showing any turning points and points of inflexion. For what values of x is the curve concave down?

Solution

$y = 4x^3 - x^4$: $\quad \frac{dy}{dx} = 12x^2 - 4x^3$ $\qquad$ $\frac{d^2y}{dx^2} = 24x - 12x^2$

$= 4x^2(3 - x)$ $\qquad$ $= 12x(2 - x)$

For stationary points, $\frac{dy}{dx} = 0$: $\quad 4x^2(3 - x) = 0$

$x = 0$ or 3

$y = 0$ or 27 $\quad \therefore$ stationary points at $(0, 0)$ and $(3, 27)$

At $x = 0$, $\frac{d^2y}{dx^2} = 0$. Investigate further: $x = -1$: $\frac{d^2y}{dx^2} = -12(2+1) < 0$

$x = 1$: $\frac{d^2y}{dx^2} = 12(2-1) > 0$

The sign of $\frac{d^2y}{dx^2}$ changes, so the concavity changes. Hence $(0, 0)$ is a horizontal point of inflexion.

At $x = 3$: $\frac{d^2y}{dx^2} = 36(2-3) < 0$ Hence $(3, 27)$ is a relative maximum turning point.

For points of inflexion, $\frac{d^2y}{dx^2} = 0$: $12x(2-x) = 0$

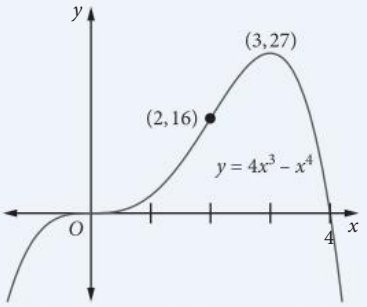
$x = 0$ or 2

$(0, 0)$ has already been identified as a horizontal point of inflexion. For $(2, 16)$:

$x = 1$: $\frac{d^2y}{dx^2} = 12(2-1) > 0$ $x = 3$: $\frac{d^2y}{dx^2} = 36(2-3) < 0$

The sign of $\frac{d^2y}{dx^2}$ changes, so the concavity changes. Hence $(2, 16)$ is a point of inflexion.

Curve is concave downwards for $x < 0$ and $x > 2$.



Summary of tests for turning points and points of inflexion

Stationary point: $\frac{dy}{dx} = 0$

Using the first derivative only:

Turning point: $\frac{dy}{dx} = 0$, $\frac{dy}{dx}$ changes sign on passing through the stationary point

Maximum turning point: $\frac{dy}{dx} = 0$, $\frac{dy}{dx}$ changes sign from positive to negative on passing through the stationary point

Minimum turning point: $\frac{dy}{dx} = 0$, $\frac{dy}{dx}$ changes sign from negative to positive on passing through the stationary point

Using the first and second derivatives:

Turning point: $\frac{dy}{dx} = 0$, $\frac{d^2y}{dx^2}$ does not change sign on passing through the stationary point

Maximum turning point: $\frac{dy}{dx} = 0$, $\frac{d^2y}{dx^2} < 0$ at the stationary point

Minimum turning point: $\frac{dy}{dx} = 0$, $\frac{d^2y}{dx^2} > 0$ at the stationary point

Point of inflexion: $\frac{d^2y}{dx^2} = 0$, $\frac{d^2y}{dx^2}$ changes sign on passing through the stationary point (i.e. concavity changes)

Special situation:

$\frac{dy}{dx} = 0$, $\frac{d^2y}{dx^2} = 0$ at a point: may be a turning point **or** a horizontal point of inflexion. You must check to see if either the gradient or the concavity changes.

EXERCISE 12.4 THE SECOND DERIVATIVE AND TURNING POINTS

1 For $y = 2x^3 + 3x^2 - 12x + 2$, find any stationary points and determine their nature. Sketch the curve, showing the turning points and any points of inflexion.

2 For what values of x is the graph of the function $f(x) = x^3 - 3x^2 + 1$ concave up?

A $x > 1$ **B** $x > -2$ **C** $x < 1$ **D** $x < -2$

3 Find the local maxima, minima and points of inflexion of $f(x) = x^2(3 - x)$. Sketch the graph of f.

4 Sketch the graph of $8y = 8 + 8x^2 - x^4$, showing any turning points.

5 A function $f(x)$ is defined by $y = x^3(x - 2)$.

(a) Find the coordinates of the turning points of $y = f(x)$.
(b) Find the coordinates of the points of inflexion.
(c) Hence sketch the graph of $y = f(x)$, showing the turning point, the points of inflexion and the points where the curve meets the x-axis.
(d) What is the minimum value of $f(x)$ for $-1 \leq x \leq 3$?

6 If $f(x) = x^3 - 6x^2 + 2$, find the values of x for which:

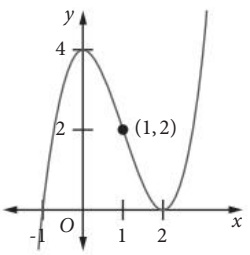
(a) $f''(x) = 0$ **(b)** $f''(x) > 0$ **(c)** $f''(x) < 0$

7 Find the coordinates of the points of inflexion of $y = x^4 - 2x^3 - 12x^2$. For what values of x is the curve concave up?

8 Let $f(x) = x^4 - x^2$.

(a) Find the coordinates of the points where the curve crosses the axes.
(b) Find the coordinates of the stationary points and determine their nature.
(c) Find the coordinates of the points of inflexion.
(d) Sketch the graph of $y = f(x)$ for $-1.5 \leq x \leq 1.5$, indicating clearly the intercepts, stationary points and points of inflexion.
(e) For what values of x is the curve concave down?

9 The graph represents a function of the form $y = ax^3 + bx^2 + cx + d$, where a, b, c and d are real numbers, $a \neq 0$. $(1, 2)$ is a point of inflexion on the curve. Indicate whether each statement below is correct or incorrect for this graph.



(a) The curve is concave up for $x > 1$.
(b) $\frac{dy}{dx} = mx(x - 2)$ where m is a positive constant.
(c) The least value of y is zero.
(d) The coordinates of the point of inflexion can be found by taking the average of the coordinates of the maximum and minimum turning points.

10 Find the turning points and points of inflexion of $y = -x^3 + 3x^2 - 3x$ and sketch its graph. Show that it crosses the x-axis at one point only. Show that $\frac{dy}{dx} < 0$ for all x except $x = 1$.

11 **(a)** Find the stationary points of $y = x^4 - 4x + 3$ and determine their nature.
(b) Show that $y = x^4 - 4x + 3$ has no points of inflexion. Comment on the concavity of the curve.

12 The revenue equation for a manufacturer is $R = \frac{80x - x^2}{4}$, where x is the number of units sold. How many units must be sold to achieve maximum revenue?

13 The revenue function for a magazine is given by $R = 4500x - 500x^2$, where x is the cost per issue of the magazine. What will be the cost per issue of the magazine to achieve maximum revenue?

14 $G = f(t)$ is the Gross Domestic Product (GDP) for Australia, where t is the number of years after 2012. The GDP growth rate $f'(t)$ for three years is given in the table.

Year	2012	2013	2014
$f'(t)$	3.9	3.2	3.1

(a) Is $f'(t)$ a decreasing or increasing function?
(b) What can you say about change in $f(t)$?
(c) Is $f(t)$ concave up or concave down?
(d) There are two different predictions for the rate of growth in 2015: that the growth rate will be 3.0, or that the growth rate will be 3.2. Sketch $G = f(t)$ for each growth rate.
(e) For which growth rate does the concavity of $f(t)$ change? What is the name of the kind of point at which this change occurs?
(f) If the growth rate for 2015 is actually 2.9, what can you say about the Gross Domestic Product for Australia?

15 A supplier has a monopoly on sales of books. The supplier's profit function is given by $P = 396x - 2.2x^2 - 400$, where x is the number of books sold.

(a) How many books must the supplier sell to maximise the profit?
(b) What is the maximum profit?
(c) If the government imposes a new 'monopoly tax' of $22 per book on the supplier, what is the new profit equation?
(d) Under the monopoly tax, how many books must the supplier now sell to maximise the profit? What is the new maximum profit?

12.5 SKETCHING RATIONAL ALGEBRAIC FUNCTIONS

Functions with the independent variable in the denominator generate curves that are not continuous and may have asymptotes. They may not have any turning points. You need to consider what happens to the function for very large positive and negative values of the variable.

Example 12

Sketch the graph of $y = \dfrac{1}{x-2}$.

Solution

Because $x - 2 \neq 0$, the function is not defined for $x = 2$, so at $x = 2$ there is a vertical asymptote.

For $x > 2$, $x - 2 > 0$, so $y > 0$. As $x \to 2$ from above, $x - 2$ is a very small positive number and so $y \to \infty$.
This can be written as: $x \to 2^+$, $y \to \infty$.

For $x < 2$, $x - 2 < 0$, so $y < 0$. As $x \to 2$ from below, $x - 2$ is a very small negative number and so $y \to -\infty$.
This can be written as: $x \to 2^-$, $y \to -\infty$.

The numerator of $y = \dfrac{1}{x-2}$ is never zero, so the curve does not cut the x-axis.

For $x = 0$, $y = -0.5$, so the y-intercept is -0.5.

As $x \to \infty$, $y \to 0$ from above; as $x \to -\infty$, $y \to 0$ from below. Thus $y = 0$ is a horizontal asymptote.

For stationary points, find $\dfrac{dy}{dx}$: $\quad \dfrac{dy}{dx} = \dfrac{-1}{(x-2)^2}$, $x \neq 2$

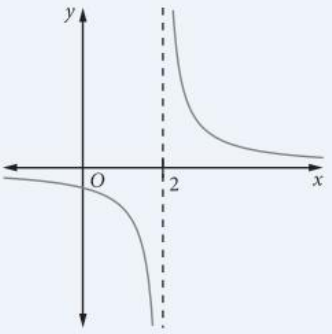
Hence $\dfrac{dy}{dx} < 0$ for all x in the domain, because $(x-2)^2 > 0$ in the domain.

Thus $y = \dfrac{1}{x-2}$ is a decreasing function in each part of its domain.

Also $\dfrac{dy}{dx} \neq 0$ in the domain, so there are no turning points.

For $x > 2$, $y > 0$; for $x < 2$, $y < 0$.

The curve is concave down for $x < 2$ and concave up for $x > 2$.



Example 13

Sketch the graph of $y = x + \frac{1}{x}$. For what values of x is the curve concave up? What is the range of the function?

Solution

$x \neq 0$: does not cut y-axis and $x = 0$ is a vertical asymptote

$y = 0$: $x + \frac{1}{x} = 0$ or $\frac{x^2 + 1}{x} = 0$

Because $x^2 + 1 \neq 0$ for real x: does not cut x-axis

As $x \to \infty$, $y \to x +$[very small amount] $\to x + 0$, so $y \to x$ from above.

As $x \to -\infty$, $y \to x -$[very small amount] $\to x - 0$, so $y \to x$ from below.

$\therefore$ $y = x$ is a sloping asymptote.

For stationary points, find $\frac{dy}{dx}$: $\frac{dy}{dx} = 1 - \frac{1}{x^2}$, $x \neq 0$

Hence for $\frac{dy}{dx} = 0$: $\frac{x^2 - 1}{x^2} = 0$, so $x = -1, 1$ (for which $y = -2, 2$)

$\therefore$ stationary points at $(-1, -2)$ and $(1, 2)$.

Second derivative: $\frac{d^2y}{dx^2} = 0 - \frac{-2}{x^3} = \frac{2}{x^3}$

At $(-1, -2)$: $\frac{d^2y}{dx^2} = -2 < 0$

$\therefore$ $(-1, -2)$ is a local maximum turning point

At $(1, 2)$: $\frac{d^2y}{dx^2} = 2 > 0$

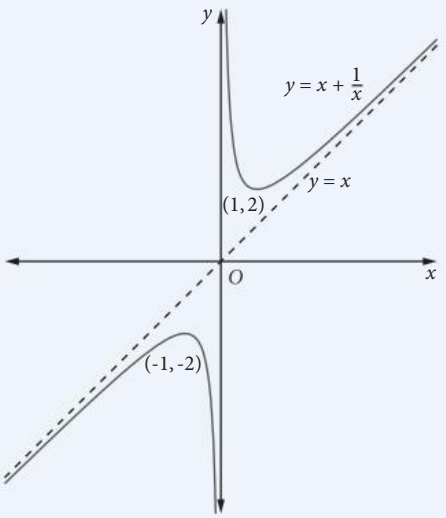
$\therefore$ $(1, 2)$ is a local minimum turning point

Because $\frac{d^2y}{dx^2} = \frac{2}{x^3}$ and $x \neq 0$, there are no points of inflexion.

The curve is concave up for $x > 0$.

The range of the function is real y, $|y| \geq 2$.

Consider: why is the curve between the lines $x = 0$ and $y = x$?



Remember that the shape of your graph sketches can be verified using GeoGebra or other graphing software.

EXERCISE 12.5 SKETCHING RATIONAL ALGEBRAIC FUNCTIONS

1 The asymptotes of $y = \frac{1}{x + 2}$ are:

A $y = 0$ and $x = -2$ **B** $y = 0$ and $x = 2$ **C** $x = 0$ and $y = -2$ **D** $x = 0$ and $y = 2$

2 Sketch the graph of each function. For what values of x is the curve concave down? State the range of each function.

(a) $y = \frac{1}{x + 2}$ **(b)** $y = \frac{1}{x - 1}$ **(c)** $y = \frac{1}{2 - x}$

3 **(a)** Show that the function $y = \frac{x - 1}{x - 2}$ can be written as $y = 1 + \frac{1}{x - 2}$.

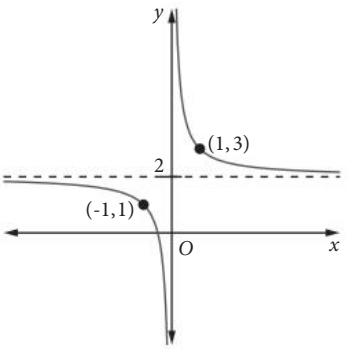
(b) Hence sketch the graph of $y = \frac{x - 1}{x - 2}$, showing all the asymptotes.

4 Sketch the graph of each function, showing all turning points and points of inflexion. For what values of x is the curve concave up? State the range of each function.

(a) $y = x + \frac{4}{x}$ **(b)** $y = x - \frac{1}{x}$ **(c)** $y = 2x + \frac{8}{x}$

5 For the function given in the sketch, state whether each statement below is correct or incorrect.

(a) The horizontal asymptote is $y = 2$.
(b) The curve is continuous.
(c) The curve is concave up for $x > 0$.
(d) The equation of the function is $y = 2 + \frac{1}{x}$.



6 Sketch the graph of each function. For what values of x is the curve concave down? State the range of each function.

(a) $y = 1 + \frac{1}{x+2}$ **(b)** $y = \frac{x-1+1}{x-1} = 1 + \frac{1}{x-1}$ **(c)** $y = \frac{x-2}{x-3}$

7 Sketch the graph of each function, showing all turning points and points of inflexion. State the range of each function.

(a) $y = x + \frac{4}{x-1}$ **(b)** $y = x + 2 + \frac{4}{x-1}$ **(c)** $y = x + 3 + \frac{1}{x-7}$

(d) $y = \frac{1}{2x-3}$ **(e)** $y = \frac{x}{4} + \frac{1}{x}$ **(f)** $y = |x| + \frac{1}{x}$

12.6 PROBLEM SOLVING WITH DERIVATIVES

A function may not always be given algebraically. Sometimes you must interpret the information given to construct the function. Remember to show clearly what each variable represents. It often helps to draw a diagram of the situation being considered.

Example 14

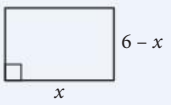
A piece of wire 12 cm long is bent in the shape of a rectangle. Find the maximum area of the rectangle.

Solution

Let $A(x)$ cm^2 be the area of the rectangle. Let the side lengths be x cm and y cm.

Because the wire is 12 cm long: $2x + 2y = 12 \quad \therefore y = (6 - x)$

Because the wire is 12 cm, the longest side of the rectangle cannot be longer than 6 cm.



Area of rectangle: $A(x) = x(6 - x)$ for $0 < x < 6$

$A(x) = 6x - x^2$

Differentiate: $A'(x) = 6 - 2x$

For stationary points, $A'(x) = 0$: $6 - 2x = 0$

$x = 3$

Differentiate again: $A''(x) = -2 < 0$ for all x

Hence $A(x)$ is concave down for all values of x in the domain. It has a maximum value when $x = 3$.

$A(3) = 3 \times 3 = 9$ cm^2

The maximum area of the rectangle is 9 cm^2.

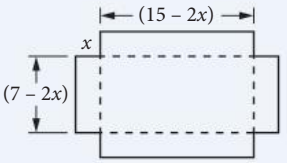
Example 15

A sheet of cardboard measures 15 cm by 7 cm. Four equal squares are cut out of the corners and the sides are turned up to form an open rectangular box. Find the edge length of the squares that were cut out to give the box a maximum volume.

Solution

Let the edge length of the squares that were cut out be x cm.

The dimensions of the base of the box will be $(15 - 2x)$ cm and $(7 - 2x)$ cm.



The height will be x cm.

Let the volume of the box be $V(x)$ cm^3. $\therefore V(x) = x(15 - 2x)(7 - 2x)$

$$= 4x^3 - 44x^2 + 105x$$

Differentiate: $V'(x) = 12x^2 - 88x + 105$

For stationary points, $V'(x) = 0$: $12x^2 - 88x + 105 = 0$

$$(2x - 3)(6x - 35) = 0$$

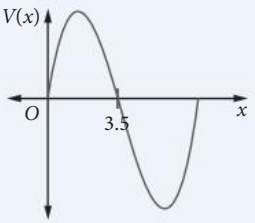
$$\therefore x = 1\tfrac{1}{2} \quad \text{or} \quad 5\tfrac{5}{6}$$

But x must be less than half the shortest side, i.e. $x < 3.5$, so we can disregard $x = 5\frac{5}{6}$. The only possible value for x is $x = 1.5$.

Use the first derivative test:

For $x < 1.5$, test $x = 1.4$: $V'(1.4) = 12 \times 1.4^2 - 88 \times 1.4 + 105 = 5.32 > 0$

For $x > 1.5$, test $x = 1.6$: $V'(1.6) = 12 \times 1.6^2 - 88 \times 1.6 + 105 = -5.08 < 0$



$V'(x)$ changes from positive to negative on passing through $x = 1.5$, so a maximum volume occurs when $x = 1.5$.

$V(1.5) = 1.5(15 - 3)(7 - 3) = 72$ cm^3

A graph of $V(x)$ for $0 \leq x \leq 7$ helps to see what is happening.

Because $V(x) > 0$, the domain of the function is $0 < x < 3.5$.

The part of the graph below the x-axis is not relevant to the problem.

EXERCISE 12.6 PROBLEM SOLVING WITH DERIVATIVES

1 A rectangular block of land is enclosed by 160 m of fencing. If the breadth of the block is x m:

(a) express the length of the block in terms of x

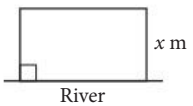
(b) find the function $A(x)$ for the area of the block

(c) find the maximum area of the block that can be fenced using this fencing.

2 The sum of two numbers is 12. If one number is x, the value of x for which the product of the two numbers is a maximum is:

A 3 **B** 6 **C** 12 **D** 36

3 A rectangular block of land has one side along a river. The other three sides are to be fenced using 160 m of fencing. If the breadth of the block is x m:

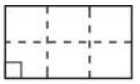


(a) express the length of the block in terms of x

(b) find the function $A(x)$ for the area of the block

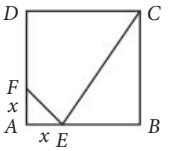
(c) find the maximum area of the block that can be fenced using this fencing.

4 A rectangular paddock is to be fenced and also divided into six smaller rectangular paddocks, with one dividing fence parallel to the length and two dividing fences parallel to the breadth (as shown in the diagram by dashed lines). The total length of fencing to be used is 120 m. If the width of the paddock is x m and the breadth is y m, find:



(a) an expression giving y as a function of x
(b) the function $A(x)$ for the area of the original paddock
(c) the maximum possible area of the paddock.
(d) To allow access to the paddocks, six gates are to be placed in the fences. Each gate is 3 m wide. What is the new maximum area of the large paddock?

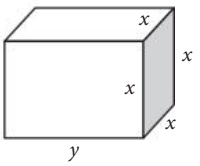
5 $ABCD$ is a square of unit length. Points E and F are on the sides AB and AD respectively so that $AE = AF = x$.



(a) Express the area of the quadrilateral $CDFE$ as a function of x.
(b) Find the greatest area that the quadrilateral can have.

6 A rectangular sheet of cardboard measures 16 cm by 6 cm. Equal squares are cut out of each corner and the sides are turned up to form an open rectangular box. What is the maximum volume of the box?

7 A block of wood in the shape of a cuboid has square ends of edge length x cm. The length of the block is y cm. The sum of length of the block and the perimeter of one end is 12 cm.



(a) Express y in terms of x.
(b) Find the volume V as a function of x.
(c) What is the largest possible volume of the block?

8 A box in the shape of a cuboid with a square base is to be made so that the sum of its dimensions $(l + b + h)$ is 20 cm. Find its maximum volume.

9 A rectangular box has a square base of edge length x cm. Its framework of 12 edges is constructed from wire of total length 36 cm. Find:

(a) the height of the box in terms of x **(b)** the volume of the box in terms of x
(c) the value of x for which the volume is a maximum.

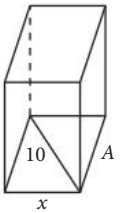
10 A closed box in the shape of a cuboid has a total surface area of 216 cm^2 and a base length that is twice the width. If the width of the base is x cm, find:

(a) the length of the base and the height in terms of x **(b)** the volume of the box in terms of x
(c) the maximum volume of the box.

11 A rectangular field is to be fenced along three sides using 300 m of fencing. The length of each equal end is x m and the length of the other side is y m. To find the dimensions of the field if its area is as large as possible, the following statements are made. Indicate whether each statement is correct or incorrect.

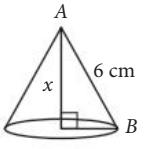
(a) $y = 300 - 2x$ **(b)** $A = 300x - 2x^2$ **(c)** $x = 75$ **(d)** Maximum area = 11 250 m^2

12 The diagonal of the base of a box in the shape of a cuboid has a length of 10 cm. One edge of the base has a length of x cm.



(a) Express, in terms of x, the length of the other edge of the base.
(b) The height of the box is equal to the length of this other edge. Find the volume of the box in terms of x.
(c) Calculate the maximum volume of the box.

13 The slant edge AB of a right circular cone is 6 cm. The vertical height of the cone is x cm.



(a) Express the radius of the base in terms of x.
(b) Express the volume of the cone in terms of x.
(c) Find the vertical height of the cone when the volume is a maximum.

14 A piece of wire of length 30 cm is cut into two sections. Each section is then bent into the shape of a square. Find the smallest possible value of the sum of the areas of the two squares.

15 A block of metal is cast into the shape of a right cylinder with a total surface area of 20π cm^2. The radius of the base is r cm and the height is h cm. The total surface area of a cylinder is given by $A = 2\pi r^2 + 2\pi rh$.

(a) Express h in terms of r. **(b)** Express the volume V in terms of r.

(c) Find the value of r for which the volume is greatest.

16 A piece of wire of length 50 cm is cut into two sections. One section is used to construct a rectangle whose dimensions are in the ratio 3 : 1; the other section is used to construct a square. Find the dimensions of the rectangle and the square so that the total enclosed area is a minimum.

17 Whale-watching boat trips go to sea with 20 or more passengers. For 20 passengers the charge is \$380 per person. For groups of more than 20, the price per person is reduced by \$12 for each additional person over 20 passengers.

(a) Show that the revenue function for the boat trip is given by $R = 620n - 12n^2$, where R is the revenue in dollars and n is the number of people in the group, $n \geq 20$.

(b) What number of passengers will produce the greatest revenue for the company?

18 Wobbly Skateboards looked at their sales in relation to their advertising budget. They found that the relationship between sales, $f(x)$, and thousands of dollars spent on advertising, x, was given by $f(x) = \frac{x^3}{3} - \frac{45x^2}{2} + 450x$, $10 \leq x \leq 40$.

(a) What number of advertising dollars can be expected to produce a maximum number of sales?

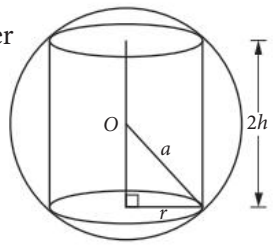
(b) What sales can be expected for that amount of advertising?

19 A company finds that the function $f(x) = x^3 - 96x^2 + 2880x$ provides a good approximation for their profit $f(x)$ in dollars, where x is the advertising expenditure in thousands of dollars.

(a) What expenditure on advertising would produce the maximum profit?

(b) What is this maximum profit?

20 A cylinder is inscribed in a sphere of radius a, centred at O. The height of the cylinder is $2h$ and the radius of the base is r, as shown in the diagram.



(a) Show that the volume V of the cylinder is given by $V = 2\pi r^2\sqrt{a^2 - r^2}$.

(b) Find the value of r for which the volume of the cylinder is a maximum. You must give reasons why your value of r gives the maximum volume.

12.7 TANGENTS AND NORMALS TO A CURVE

Differentiation gives the gradient function of a curve. This function can be evaluated at a point on the curve to obtain the gradient of the curve's **tangent** at that point. The point-gradient form of the straight line can then be used to find the equation of that tangent.

A **normal** at a point on a curve is the line that is perpendicular to the tangent at that point. Because the lines are perpendicular, the gradient of the normal can be obtained from the gradient of the tangent at that point.

Example 16

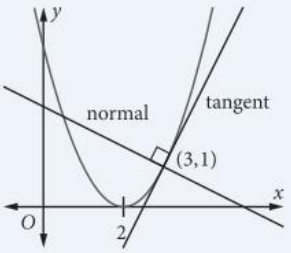
Find the equation of the tangent and the normal to the curve $y = x^2 - 4x + 4$ at the point on the curve where $x = 3$.

Solution

$x = 3$, $y = 9 - 12 + 4 = 1$ so the point is $(3, 1)$

$y = x^2 - 4x + 4$: $\frac{dy}{dx} = 2x - 4$

$x = 3$: $\frac{dy}{dx} = 6 - 4 = 2$



Equation of the tangent (from point-gradient form): $y-1=2(x-3)$

$y=2x-5$

Gradient of normal $=\dfrac{-1}{\text{gradient of tangent}}=-\dfrac{1}{2}$ (Remember $m_1\times m_2=-1$ for perpendicular lines)

Equation of the normal (from point-gradient form): $y-1=-\frac{1}{2}(x-3)$

$2y-2=-x+3$

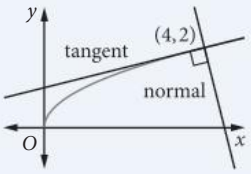
$x+2y-5=0$

Example 17

Find the equation of the tangent and normal to the curve $y=\sqrt{x}$ at the point on the curve where $x=4$.

Solution

Draw a sketch.



Point is at $x=4$, $y=2$: $y=x^{\frac{1}{2}}$: $\dfrac{dy}{dx}=\frac{1}{2}x^{-\frac{1}{2}}=\dfrac{1}{2\sqrt{x}}$

$x=4$: $\dfrac{dy}{dx}=\dfrac{1}{2\sqrt{4}}=\dfrac{1}{4}$

For $(4,2)$, $m=\frac{1}{4}$, tangent is: $y-2=\frac{1}{4}(x-4)$

$4y-8=x-4$

The equation of the tangent is $x-4y+4=0$.

For $(4,2)$, $m=-4$, normal is: $y-2=-4(x-4)$

The equation of the normal is $4x+y-18=0$.

Example 18

Find the equation of the tangent to the curve $y=\dfrac{x^3}{3}-x^2-x+1$ at the points on the curve where the tangent is parallel to the line $7x-y+5=0$.

Solution

$y=\dfrac{x^3}{3}-x^2-x+1$: $\dfrac{dy}{dx}=x^2-2x-1$

Gradient of $7x-y+5=0$: $m=7$

Solve: $x^2-2x-1=7$

$x^2-2x-8=0$

$(x-4)(x+2)=0$

$x=-2,\ 4$

At $x=2$, $y=-\frac{11}{3}$; at $x=4$, $y=\frac{7}{3}$.

At $(-2,-\frac{11}{3})$, tangent is: $y+\frac{11}{3}=7(x+2)$

$3y+11=21x+42$

$21x-3y+31=0$

At $(4,\frac{7}{3})$, tangent is: $y-\frac{7}{3}=7(x-4)$

$3y-7=21x-84$

$21x-3y-77=0$

Example 19

Find the points of intersection of the parabolas $y = x^2 - 2x$ and $y = 4x - x^2$. Sketch the parabolas and find the angle between the parabolas at their point of intersection in the first quadrant.

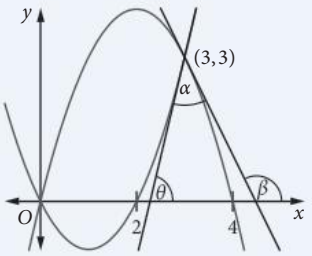
Solution

$y = x^2 - 2x$, $y = 4x - x^2$, intersection at: $x^2 - 2x = 4x - x^2$

$$2x^2 - 6x = 0$$

$$2x(x - 3) = 0$$

$$x = 0, 3 \quad y = 0, 3$$



Curves intersect at $(0,0)$ and $(3,3)$. Intersection point in the first quadrant is $(3,3)$. The angle between the curves at this point is the angle between their tangents at this point.

For $y = x^2 - 2x$: $\dfrac{dy}{dx} = 2x - 2$

At $(3,3)$: $\dfrac{dy}{dx} = 6 - 2 = 4$ Hence $\tan\theta = 4$, so $\theta = 75° 58'$

For $y = 4x - x^2$: $\dfrac{dy}{dx} = 4 - 2x$

At $(3,3)$: $\dfrac{dy}{dx} = 4 - 6 = -2$ Hence $\tan\beta = -2$, so $\beta = 180° - 63° 26' = 116° 34'$

Now $\beta = \alpha + \theta$ (exterior angle of a triangle), so $\alpha = 116° 34' - 75° 58' = 40° 36'$.

EXERCISE 12.7 TANGENTS AND NORMALS TO A CURVE

1 Find the equations of the tangent and normal to the curve $y = x^2$ at $(2,4)$.

2 Find the equations of the tangent and normal to the curve $y = 3x - x^2$ at $(0,0)$.

3 The equation of the tangent to the curve $y = x^2 - 5x + 6$ at $(3,0)$ is:

A $y = x - 3$ **B** $y = x + 3$ **C** $y = -x - 3$ **D** $y = -x + 3$

4 Find the equations of the tangent and normal to the curve $y = 2x^2 + 3x - 4$ where $x = 0$.

5 Find the equations of the tangent and normal to the curve $y = 2x^2 - 4x + 1$ where the gradient is 4.

6 Find the equations of the tangent and normal to the curve $y = \dfrac{1}{x}$ at the point where $x = -2$. Indicate whether each statement below is a correct or incorrect step in answering this question.

(a) $\dfrac{dy}{dx} = \dfrac{1}{x^2}$ **(b)** At $\left(-2, -\frac{1}{2}\right)$, $\dfrac{dy}{dx} = -\frac{1}{4}$

(c) Equation of tangent is $x + 4y + 4 = 0$ **(d)** Equation of normal is $8x - 2y + 15 = 0$

7 Find the equations of the tangent and normal to the curve $y = 3 - x - x^2$ at the point where the curve crosses the y-axis.

8 Find the equations of the tangent and normal to the curve $y = 3x^3 - 7x^2 + 2x$ at the point where $x = 2$.

9 The straight line $y = x + 2$ cuts the parabola $y = \dfrac{x^2}{2} - 2$ at two points P and Q. Find the coordinates of P and Q. Also find the equations of the tangents to the parabola at P and Q and the coordinates of the point of intersection of these tangents.

10 Find the equations of the tangents to the parabola $y = 4x - 3x^2$ at the points where the parabola cuts the x-axis.

11 Find the equations of the tangent and normal to the parabola $y = 2x^2 - 4x + 1$ at the point of zero gradient.

12 Prove that the parabolas $y = 2x^2 - 6x + 5$ and $y = x^2 - 2x + 1$ touch at a point and find the equation of their common tangent.

13 The line $y = x + 4$ cuts the parabola $y = x^2 - 2x$ at two points A and B. Find the size of the angles that the tangents to the curve at A and B make with the x-axis.

14 The line $y = x + 1$ cuts the parabola $y = x^2 - x - 2$ at two points P and Q. Find:

(a) the coordinates of P and Q **(b)** the equations of the tangents at P and Q

(c) the coordinates of the intersection point of these two tangents.

15 Find the equations of the tangents to the parabola $y = x^2 - 2x + 4$ at the points where:

(a) the gradient is zero **(b)** the tangent is parallel to the line $y = 2x + 1$.

16 Find the equations of the tangent and normal to the parabola $y = 2x^2 - 5x + 1$ at the point on the parabola where the gradient is 3.

17 The normal to the curve $y = (x + 2)^2$ at the point $A(-3, 1)$ meets the curve again at B. Find

(a) the equation of the normal **(b)** the coordinates of B

(c) the angle at B between the curve and the chord AB.

18 The line $y = x - 2$ cuts the curve $y = x^3(x - 2)$ at two points A and B. Calculate the angles that the tangents to the curve at A and B make with the x-axis and hence find the angle between the tangents.

19 Find the equations of the tangents to the curve $y = (x^2 - 1)(x - 2)$ at the points where the curve crosses the x-axis.

20 Find the coordinates of the points on the curve $y = x^2(2x - 3)$ at which the tangent is parallel to:

(a) the line $y = 12x - 1$ **(b)** the x-axis.

12.8 PRIMITIVE FUNCTIONS

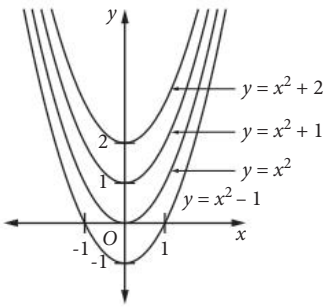
We have been finding the derivative of a function. But can we reverse the process? Given a derivative, can we find the original function that was differentiated? The short answer is: sometimes!

The process of finding the function from the derivative is called **antidifferentiation** or **finding the primitive** of the function. (In the next chapter we will also introduce the term 'indefinite integral' for this process.)

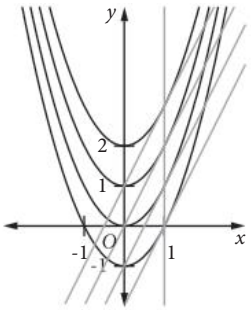
If $F(x)$ is a function such that $F'(x) = f(x)$, then $F(x)$ is called the **antiderivative** or **primitive** of $f(x)$. Similarly, $f(x)$ is the primitive of $f'(x)$. For example:

- The derivative of x^2 is $2x$, so x^2 is a primitive of $2x$.
- The derivative of $x^2 + 3$ is $2x$, so $x^2 + 3$ is a primitive of $2x$.
- For any real number C, the derivative of $x^2 + C$ is $2x$, so $x^2 + C$ is a primitive of $2x$.

These primitives all vary only by a constant. Graphically, they are the same curve translated vertically by a constant value:



If the tangent at $x = 1$ is drawn for these curves then the tangents form a system of parallel lines, because each curve has the same gradient function.



Example 20

Find the equation of the curve defined by $f'(x) = 2x$ that passes through the point $(2, 8)$.

Solution

$f'(x) = 2x$, find the primitive: $f(x) = x^2 + C$

Passes through $(2, 8)$: $8 = 2^2 + C$

$C = 4$

Hence $f(x) = x^2 + 4$ is the equation of the curve.

From our experience differentiating powers of x, we can say:

- the primitive of x^2 is $\frac{x^3}{3} + C$
- the primitive of x^3 is $\frac{x^4}{4} + C$
- the primitive of $x^{\frac{1}{2}}$ is $\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C$ or $\frac{2}{3}x^{\frac{3}{2}} + C$
- the primitive of x^n is $\frac{1}{n+1}x^{n+1} + C$

This result is easily verified by differentiating $\frac{1}{n+1}x^{n+1} + C$ to obtain x^n.

In words: 'to find the primitive of a power of x, increase the power of x by one and divide the expression by the new power'.

Example 21

Find the primitive of each expression.

(a) $5x^4 + 3x^2 + 4$ **(b)** $16 - 8x + x^3$ **(c)** $x^{10} + 4$

Solution

(a) $f'(x) = 5x^4 + 3x^2 + 4$:

$f(x) = x^5 + x^3 + 4x + C$

(b) $f'(x) = 16 - 8x + x^3$:

$f(x) = 16x - \frac{8x^2}{2} + \frac{x^4}{4} + C$

$f(x) = 16x - 4x^2 + \frac{x^4}{4} + C$

(c) $f'(x) = x^{10} + 4$:

$f(x) = \frac{x^{11}}{11} + 4x + C$

Example 22

The gradient function of a curve is $3x^2 - 2x$ and the curve passes through the point $(2, 1)$. Find the equation of the curve.

Solution

$\frac{dy}{dx} = 3x^2 - 2x$, find the primitive: $y = \frac{3x^3}{3} - \frac{2x^2}{2} + C$

Simplify: $y = x^3 - x^2 + C$

At point $x = 2, y = 1$: $1 = 8 - 4 + C$

$C = -3$

The equation of the curve is $y = x^3 - x^2 - 3$.

Example 23

Find $f(x)$ given $f'(x)$.

(a) $f'(x) = x + \frac{1}{x^2}$ **(b)** $f'(x) = x\sqrt{x} + 1$ **(c)** $f'(x) = \frac{x^3 + 2x^2 + 1}{x^2}$

Solution

(a) $f'(x) = x + \frac{1}{x^2}$, rewrite using an index: $f'(x) = x + x^{-2}$

Find primitive: $f(x) = \frac{x^2}{2} + \frac{x^{-1}}{-1} + C$

$f(x) = \frac{x^2}{2} - \frac{1}{x} + C$

(b) $f'(x) = x\sqrt{x} + 1$, rewrite using an index: $f'(x) = x^{\frac{3}{2}} + 1$

Find primitive: $f(x) = \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + x + C$

$f(x) = \frac{2}{5}x^{\frac{5}{2}} + x + C$

(c) If there is a single term in the denominator, divide the numerator by the denominator before finding the primitive.

$f'(x) = \frac{x^3 + 2x^2 + 1}{x^2}$: $f'(x) = x + 2 + \frac{1}{x^2}$

Rewrite using indices: $f'(x) = x + 2 + x^{-2}$

Find primitive: $f(x) = \frac{x^2}{2} + 2x + \frac{x^{-1}}{-1} + C$

$f(x) = \frac{x^2}{2} + 2x - \frac{1}{x} + C$

Example 24

A colony of bats has a growth rate $\frac{dN}{dt} = 80t$, where t is the number of years since 2012 and $N(t)$ is the population. If there were 3000 bats in the year 2012, how many bats are in the colony in 2042?

Solution

$t = 0, N = 3000, \frac{dN}{dt} = 80t$

Find the primitive: $N(t) = 40t^2 + C$

But $N(0) = 3000$: $3000 = C$

$N(t) = 40t^2 + 3000$

In 2042, $t = 30$: $N(30) = 40 \times 900 + 3000 = 39\,000$

There would be 39 000 bats in the colony in 2042.

EXERCISE 12.8 PRIMITIVE FUNCTIONS

1 Find the primitive of:

(a) $6x^2 - 4x + 5$ **(b)** $3 + 5x + x^2 - 3x^3$ **(c)** $x^2 - 1$

(d) $x^5 - 4x^3 - 2x^2 + 1$ **(e)** $3x - 5$ **(f)** $2x^2 - 7x + 5$

2 If $f'(x) = (x - 1)(x - 2)$, indicate whether each statement below is correct or incorrect.

(a) $f'(x) = x^2 - 3x + 2$ **(b)** $f(x) = \frac{x^3}{3} - \frac{3x^2}{2} + 2x + C$

(c) $f(x) = (x - 1)^2(x - 2)^2 + C$ **(d)** $f(x) = \frac{(x-1)^2(x-2)^2}{4} + C$

3 Find an expression for $f(x)$ given:

(a) $f'(x) = (2x + 1)^2$ **(b)** $f'(x) = 5$ **(c)** $f'(x) = x^2 + 3x$

(d) $f'(x) = (2x - 1)(x + 2)$ **(e)** $f'(x) = 4x^3 - 6x^2 + x$ **(f)** $f'(x) = (x + 3)^2$

4 Show that $\frac{(2x+1)^3}{6} = \frac{4x^3}{3} + 2x^2 + x + C$ where C is an arbitrary constant.

5 Express y in terms of x, given that:

(a) $\frac{dy}{dx} = 3 + 2x - 3x^2$ **(b)** $\frac{dy}{dx} = x^3 + 2x^2$ **(c)** $\frac{dy}{dx} = x^4 - x^3$

(d) $\frac{dy}{dx} = (x - 3)(x + 4)$ **(e)** $\frac{dy}{dx} = x^4 - \frac{1}{3}x^3 + \frac{1}{2}x^2$ **(f)** $\frac{dy}{dx} = \sqrt{x}$

6 For each of the following find $F(x)$, given that $F'(x) = f(x)$.

(a) $f(x) = x^3 - 3x^2$ **(b)** $f(x) = \frac{x^2}{5} - \frac{x^3}{4}$ **(c)** $f(x) = x^2(1 - 3x)$

(d) $f(x) = (x^2 - 1)(x^2 + 1)$ **(e)** $f(x) = x^{\frac{3}{2}} + x^{-\frac{5}{2}}$ **(f)** $f(x) = \sqrt{x} + \sqrt[3]{x}$

7 Find $f(x)$ given $f'(x) = 2x - 2$ and $f(1) = 4$.

8 If $f'(x) = 4x^2 - 3x$ and $f(-1) = 3$ then $f(x) = \ldots$

A $f(x) = \frac{4x^3}{3} - \frac{3x^2}{2} + C$ **B** $f(x) = 8x - 3$

C $f(x) = \frac{4x^3}{3} - \frac{3x^2}{2} + \frac{35}{6}$ **D** $f(x) = \frac{x^3}{12} - \frac{x^2}{6} + \frac{13}{4}$

9 At all points on a certain curve, $\frac{dy}{dx} = 4x - 6$. The point $(2, 4)$ is on the curve. Find the equation of the curve.

10 Find the equation of a curve that passes through the point $(3, 3)$ and for which the gradient function at any point $P(x, y)$ is $3x^2 - 2x + 3$.

11 For the function $f(x) = (x + 2)^2$, find the rule that defines the function F where $F'(x) = f(x)$ and $F(1) = 4$.

12 Find the equation of a curve given that $\frac{dy}{dx} = 2x + b$ at any point P and that when $x = 3$, $\frac{dy}{dx} = 2$ and $y = -3$.

13 Find the rule that defines a function f for which $f'(x) = x^2 - 2x + b$ for all x, $f'(0) = 1$ and $f(0) = 2$.

14 A curve contains the point $(0, 4)$ and its gradient is $(x - 1)(x + 2)$ at any point on the curve. Find the equation of the curve.

15 Find the primitives of:

(a) $3 + \frac{1}{x^2} - \frac{2}{x^3}$ **(b)** $\frac{x^2 + 2}{x^2}$ **(c)** $\frac{3x^3 - 2x^2 + x^{-1}}{x^2}$

(d) $\frac{1}{x^3}$ **(e)** $\frac{5}{x^2} + 2x^{\frac{1}{2}} + 3$ **(f)** $\frac{1}{\sqrt{x}} + \frac{1}{x\sqrt{x}}$

16 For $\frac{ds}{dt} = 12t^2 - 6t + 1$, find s in terms of t given that $s = 4$ when $t = 1$.

17 A new school started in 2006 with 50 students and grew at 60 students per year for the next six years. If $N(t)$ is the number of students in the school and t is the number of years after 2006, then the growth rate is given by $\frac{dN}{dt} = 60$. Use calculus to find the population of the school in 2012.

18 If velocity v is the rate of change of distance d as a function of time t, find the distance function if $v = 3t^2 + 4$ and $d = 0$ when $t = 0$.

CHAPTER REVIEW 12

1 For the graph of $y = 15x + 12x^2 - 4x^3$ for $-1 \leq x \leq 3$, find the values of x for which:

(a) y increases as x increases **(b)** y decreases as x increases

(c) y is a maximum **(d)** y is a minimum.

2 If $f(x) = 2x^3 + 3x^2 - 12x$, find the values of x for which:

(a) $f'(x) = 0$ **(b)** $f'(x) > 0$ **(c)** $f'(x) < 0$

3 Sketch the graph of $y = f(x)$ given:

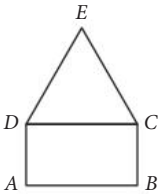
(a) $f(3) = 5$, $f'(3) = 0$, $f'(x) > 0$ for $x < 3$ and $f'(x) < 0$ for $x > 3$

(b) $f(-1) = 8$, $f'(-1) = 0$, $f(2) = 3$, $f'(2) = 0$, $f'(x) < 0$ for $-1 < x < 2$, and $f'(x) > 0$ for $x < -1$ and for $x > 2$

4 Find the coordinates of the points on the following curves where the gradient is zero. Determine whether these points are local maximum or minimum points.

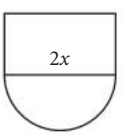
(a) $y = 3x^3 - 2x^2$ **(b)** $y = x^3 - 3x^2 - 9x$

5 A figure $ABCED$ consists of a rectangle $ABCD$ topped by an equilateral triangle CED as shown in the diagram. If the perimeter of the figure is 45 cm, find the dimensions of the rectangle when the total area is a maximum.



6 A piece of wire 8 m long is cut into two parts. One part is bent into the shape of a square and the other part is bent into a rectangle whose length is twice its breadth. Calculate the length of each part if the sum of the areas of the square and the rectangle is to be a minimum.

7 A figure consists of a semicircle with a rectangle constructed on its diameter. If the perimeter of the figure is 50 cm, find the dimensions of the rectangle such that the area of the figure is as large as possible. What is this largest area?



8 For the graph of $y = f(x)$ where $f(x) = \frac{x^3}{3} - 4x + 3$, find the following:

(a) the values of x for which $f'(x) = 0$ **(b)** the values of x for which $f'(x) < 0$
(c) any local maximum and minimum values of $f(x)$. **(d)** Sketch the curve.

9 Find the equation of the tangent to the parabola $y = 4x - x^2$ at the point where the gradient is -2.

10 Find the equation of the tangents to the curve $y = 2x^2(4 - x)$ at the point where the curve intersects the x-axis.

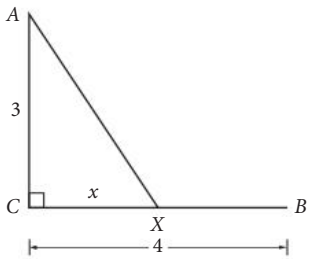
11 Sketch the graph of $y = 3x^3 - 5x^2$ for values of x in the domain $-0.5 \leq x \leq 1.5$, locating the turning points.

12 A rectangular sheet of metal measures 6 cm by 4 cm. Four equal squares are cut out of the corners and the sides are turned up to form an open rectangular box. Find the edge length of the squares cut so that the box has a maximum volume.

13 Prove that the curve $y = x^2(3 - x)$ has a horizontal tangent where $x = 2$ and crosses the y-axis at right angles at the origin.

14 Sketch the graph of $y = x^3(3 - x)$ in the domain $-1 \leq x \leq 3$, giving the coordinates of the turning points and points of inflexion.

15 Jack is in the bush at point A, 3 km from the nearest point C, which is at one end of a straight 4 km path CB. Jack wants to get to point B, the other end of the path, as quickly as possible. He can run at a speed of 20 km/h along the path CB but only at $10\sqrt{2}$ km/h in the bush off the path. He runs in a straight line through the bush from A to a point X on the path CB, then along the path from X to B.



(a) Find, in terms of x, the time taken for Jack to go from:
(i) A to X **(ii)** X to B.
(b) Find, in terms of x, the total time t hours to get from A to B.
(c) Find the position of the point X for which t is a minimum. Find this minimum time.

16 A haulage company makes frequent deliveries from Sydney to Cairns and calculates that the overhead cost \$$C$ depends on the average delivery speed v km/h according to the rule $C = v + \frac{3600}{v}$. Find the average delivery speed to minimise the overhead cost.

17 Find the primitive of:

(a) $x + 9$ **(b)** $3x^2 - 2x + 4$ **(c)** $x^4 + x^3 - 2$
(d) $(x - 2)(x + 3)$ **(e)** $(x + 2)^2$ **(f)** 7

18 Express y in terms of x given:

(a) $\frac{dy}{dx} = 5x + 4$ **(b)** $\frac{dy}{dx} = 5 - 4x + 3x^2 + x^3$ **(c)** $\frac{dy}{dx} = 2x + \sqrt{x} + 3$

19 Find $f(x)$ in terms of x given:

(a) $f'(x) = x^2 + x^3 + 1, f(0) = 2$ **(b)** $f'(x) = 3 - x + 6x^3, f(1) = 3$ **(c)** $f'(x) = 1 - \frac{1}{x^2}, f(2) = \frac{1}{2}$

20 During a storm, water flows into a 5000-litre tank at the rate of $\frac{dV}{dt}$ litres per minute, where $\frac{dV}{dt} = 140 + 13t - t^2$ and t is the time in minutes since the storm began.

(a) Find the volume of water that has flowed into the tank since the start of the storm as a function of t.
(b) How much water has flowed into the tank after 12 minutes?

CHAPTER 13
INTEGRAL CALCULUS

Outcomes covered:

H1 seeks to apply mathematical techniques to problems in a wide range of practical contexts

H5 applies appropriate techniques from the study of calculus, geometry, probability, trigonometry and series to solve problems

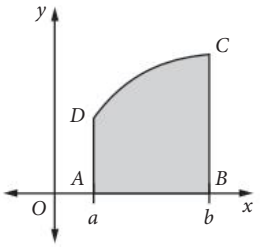
H8 uses techniques of integration to calculate areas and volumes

H9 communicates using mathematical language, notation, diagrams and graphs

You should already be familiar with the formulae used to calculate the areas of plane figures such as squares, rectangles, trapezia, triangles and circles. In this chapter we will develop a way to calculate areas bounded by curves and straight lines, using calculus.

13.1 AREA UNDER A CURVE

In the diagram below, the shaded area $ABCD$ is bounded by the continuous increasing function $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$. We will call this the area 'under the curve' between $x = a$ and $x = b$, and by convention know that this means that the other boundary is the x-axis. The lines $x = a$ and $x = b$ represent the ordinates at a and b respectively.

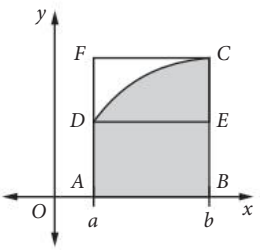


There are several ways to find an approximation to this area $ABCD$. For example:

Draw the rectangles $ABED$ and $ABCF$.

By comparing areas: Area $ABED$ < Area $ABCD$ < Area $ABCF$

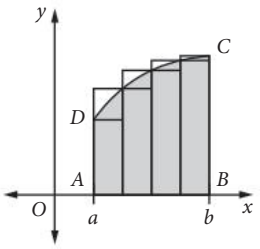
Or: $(b-a)\,f(a) <$ Area $ABCD < (b-a)\,f(b)$



Now divide the interval AB into 4 equal portions of length h, where $h = \dfrac{b-a}{4}$, as shown in the diagram. Draw each of the interior and exterior rectangles.

Total area of inside rectangles < Area $ABCD$ < Total area of outside rectangles

The sides of the rectangles are at $x = a$, $x = a + h$, $x = a + 2h$, $x = a + 3h$, $x = b$.



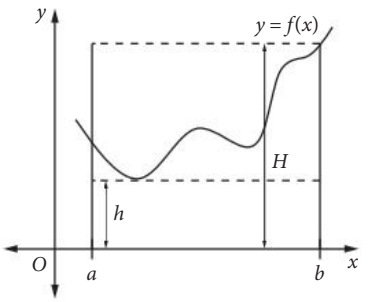
So: $h\,f(a) + h\,f(a+h) + h\,f(a+2h) + h\,f(a+3h) <$ Area $ABCD < h\,f(a+h) + h\,f(a+2h) + h\,f(a+3h) + h\,f(b)$

$h(f(a) + f(a+h) + f(a+2h) + f(a+3h)) <$ Area $ABCD < h(f(a+h) + f(a+2h) + f(a+3h) + f(b))$

This is a better estimate of the area than we obtained from the first diagram. The more parts that we divide AB into, the smaller the rectangles, and the better the approximation obtained for the area under the curve.

Now consider the continuous function $y = f(x)$, where $f(x) > 0$ for values of x in $a \le x \le b$. Over this domain, $f(x)$ will have a greatest value and a least value.

There is an area enclosed by the curve $y = f(x)$, $x = a$, $x = b$ and the x-axis. Let A be the size of this area and let h and H be the minimum and maximum values respectively of $f(x)$ in $a \le x \le b$. We then have $h(b-a) \le A \le H(b-a)$.

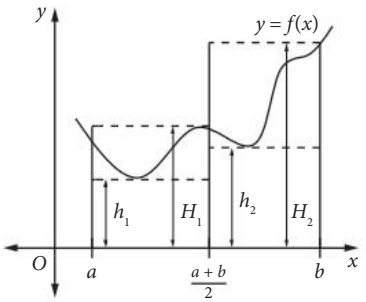


Now split $a \le x \le b$ into two subintervals, $a \le x \le \frac{a+b}{2}$ and $\frac{a+b}{2} \le x \le b$.

Take minimum and maximum values h_1, H_1 in the first subinterval and h_2, H_2 in the second subinterval as shown in the diagram.

Hence: $h(b-a) \le (h_1 + h_2) \times \frac{b-a}{2} \le A \le (H_1 + H_2) \times \frac{b-a}{2} \le H(b-a)$

The more equal subintervals we make, the closer the corresponding area sums will approach the value for A.



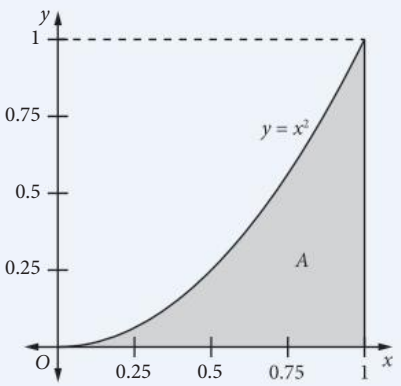
Example 1

Find an approximation of the area of the region bounded by the curve $y = x^2$, the x-axis and the ordinates at $x = 0$ and $x = 1$, using:

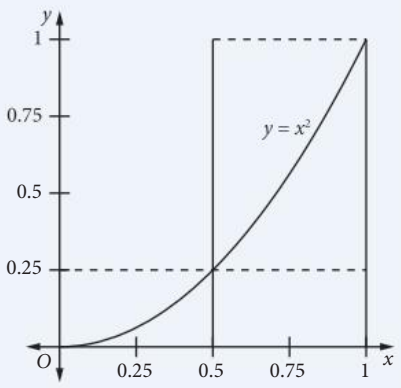
(a) one subinterval **(b)** two subintervals **(c)** four subintervals.

Solution

(a) Draw a diagram showing the region:
For $0 \le x \le 1$ and one subinterval,
$x = 0$, $y = 0$, is the smallest value;
$x = 1$, $y = 1$, is the largest value.
Hence $0 < A < 1$



(b) For $0 \le x \le 1$ and two subintervals:
$0 \le x \le 0.5$:
$x = 0$, $y = 0$ is smallest value; $x = 0.5$, $y = 0.25$ is largest value
$0.5 \le x \le 1$:
$x = 0.5$, $y = 0.25$ is smallest value; $x = 1$, $y = 1$ is largest value
Hence: $(0 + 0.25) \times 0.5 \le A \le (0.25 + 1) \times 0.5$
$0.125 \le A \le 0.625$



(c) For $0 \le x \le 1$ and four subintervals:

$0 \le x \le 0.25$: $x = 0$, $y = 0$ is smallest value

$x = 0.25$, $y = 0.0625$ is largest value

$0.25 \le x \le 0.5$: $x = 0.25$, $y = 0.0625$ is smallest value

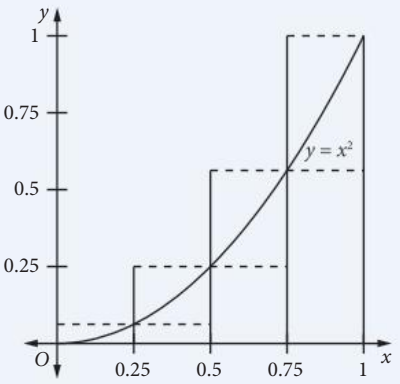
$x = 0.5$, $y = 0.25$ is largest value

$0.5 \le x \le 0.75$: $x = 0.5$, $y = 0.25$ is smallest value

$x = 0.75$, $y = 0.5625$ is largest value

$0.75 \le x \le 1$: $x = 0.75$, $y = 0.5625$ is smallest value

$x = 1$, $y = 1$ is largest value



Hence: $(0 + 0.0625 + 0.25 + 0.5625) \times 0.25 \le A \le (0.0625 + 0.25 + 0.5625 + 1) \times 0.25$

$0.21875 \le A \le 0.46875$

The area is between 0.21875 and 0.46875. Taking the average of these two values gives an approximate area of 0.34375.

EXERCISE 13.1 AREA UNDER A CURVE

1 Find an approximation for the area of the region bounded by the curve $y = x^2$, the x-axis and the ordinates at $x = 0$ and $x = 1$ using eight subintervals. Compare your answer to the answers obtained in Example **1**.

2 Find an approximation for the area of the region bounded by the curve $y = x^2$, the x-axis and the ordinates at $x = 1$ and $x = 3$ using:

(a) one subinterval **(b)** two subintervals **(c)** four subintervals.

3 Find an approximation for the area of the region bounded by the curve $y = x^2 + 1$, the x-axis and the ordinates at $x = 0$ and $x = 1$ using:

(a) one subinterval **(b)** two subintervals **(c)** four subintervals.

4 Find an approximation for the area of the region bounded by the curve $y = x + 1$, the x-axis and the ordinates at $x = 1$ and $x = 3$ using:

(a) one subinterval **(b)** two subintervals **(c)** four subintervals.

Compare your answers to the exact area of the region.

5 Find an approximation for the area of the region bounded by the curve $y = x^3$, the x-axis and the ordinates at $x = 0$ and $x = 2$ using:

(a) one subinterval **(b)** two subintervals **(c)** four subintervals.

13.2 THE DEFINITE INTEGRAL AND THE AREA UNDER A CURVE

It can be seen that as the number of subdivisions increases, the sums of the areas of rectangles become closer to the area under the curve bounded by the two ordinates. Integral calculus is based on the infinitesimal limit of these rectangular subdivisions. The process that we are about to develop was discovered independently by Leibniz and Newton in the seventeenth century, when they were the first to make the link between the area under a curve and calculus by way of the primitive function.

Let f be a continuous and increasing function in the interval $a \le x \le b$. Divide this interval into n subintervals, where n is large. Each rectangle will have a small base of length dx (where dx is a single symbol—dx is **not** 'd multiplied by x') and a height that is close to $f(x)$ for any value of x in the base. Because f is continuous, all

values of $f(x)$ are close together if the values of x are close together. Thus the area of a typical rectangle is $f(x)\,dx$ and the sum of these areas can be written as $\sum f(x)dx$. The large Greek letter Σ (capital sigma) denotes 'the sum of'.

As n increases (and hence dx decreases), the limiting value of this sum is denoted by $\int_a^b f(x)dx$. The integral symbol $\int$ represents a large elongated S, which stands for 'sum', while the bounds a and b at either end of the $\int$ indicate the interval from $x = a$ to $x = b$ over which the sum is taken. This $\int_a^b f(x)dx$ is called the **definite integral** of the function $f(x)$ between $x = a$ and $x = b$. The value A of this definite integral is the size of the area under the curve $y = f(x)$ between $x = a$ and $x = b$.

For a definite integral, we write: $A = \int_a^b f(x)dx$ **OR** $A = \int_a^b y\,dx$

Example 2

Write the definite integral for the area of the region under the line $y = x + 1$ between the ordinates $x = 1$ and $x = 3$. By using appropriate area formulae, find the value of this area.

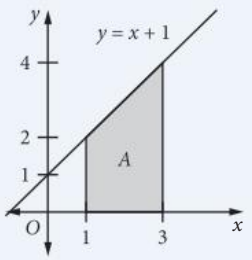
Solution

Sketch the region: bounds are $f(x) = x + 1$, $x = 1$, $x = 3$, x-axis

$$A = \int_a^b f(x)dx$$

$$= \int_1^3 (x+1)dx$$

The required area is a trapezium. $A = \frac{(2+4)}{2} \times (3-1) = 6$ units^2



Example 3

Write the definite integral for the area of the region bounded by the lines $y = 2x$, $x = t$ and the x-axis. By using appropriate area formulae, find the value of this area as a function of t.

Solution

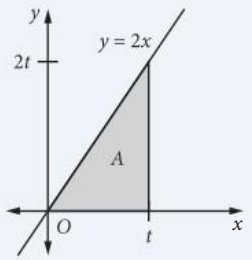
Sketch the region: bounds are $f(x) = 2x$, $x = 0$, $x = t$, x-axis

$$A = \int_a^b f(x)dx$$

$$= \int_0^t 2x\,dx$$

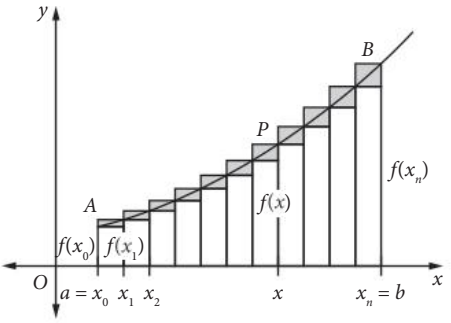
Area of triangle $= \frac{1}{2} \times t \times 2t = t^2$

It is worth noting here that x^2 is a primitive of $2x$.



Definite integral—formal development

Let f be a continuous and increasing function in the interval $a \le x \le b$.
Divide this interval into n parts, not necessarily equal, using the points $x_1, x_2, x_3, \ldots x_n$ such that $a = x_0 < x_1 < x_2 < x_3 \ldots < x_n = b$.
Using the subintervals $(x_1 - x_0), (x_2 - x_1), \ldots (x_n - x_{n-1})$ as bases, construct upper and lower rectangles as shown in the diagram.



If we denote the sum of the areas of the lower rectangles by S_L, then:

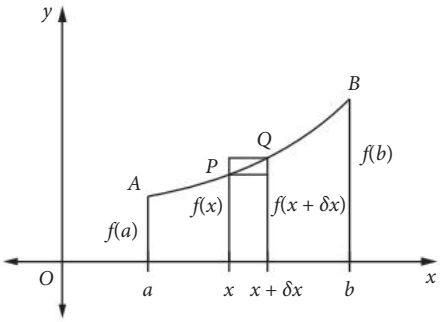
$$S_L = f(x_0)(x_1 - x_0) + f(x_1)(x_2 - x_1) + \ldots + f(x_{n-1})(x_n - x_{n-1})$$

If we denote the sum of the areas of the upper rectangles by S_U, then:

$$S_U = f(x_1)(x_1 - x_0) + f(x_2)(x_2 - x_1) + \ldots + f(x_n)(x_n - x_{n-1}).$$

It is clear from the diagram that $S_U > S_L$ and that $S_U - S_L$ equals the sum of the areas of the shaded rectangles.
If we denote the area bounded by the curve, the x-axis and the ordinates $x = a$ and $x = b$ by A, then $S_L < A < S_U$.
If we divide the interval $a \le x \le b$ into a very large number of parts, i.e. $n \to \infty$, then the sum of the areas of the shaded rectangles becomes very small: $S_U - S_L \to 0$ and S_U and S_L approach the same limit A.

Consider now a typical rectangle of width δx at any point x in the interval $a \le x \le b$. The area of a typical lower rectangle is $f(x)\,\delta x$ and the area of a typical upper rectangle is $f(x + \delta x)\,\delta x$.



Hence: $S_L = \sum_a^b f(x)\delta x$ and $S_U = \sum_a^b f(x+\delta x)\delta x$

The Σ notation here denotes 'the sum of' the areas of the n lower rectangles (S_L) or upper rectangles (S_U) in the interval $a \le x \le b$.

Thus: $\sum_a^b f(x)\delta x < A < \sum_a^b f(x+\delta x)\delta x$

As $n \to \infty$, $\delta x \to 0$ and $S_U \to S_L$, so: $A = \lim_{\delta x \to 0} \sum_a^b f(x)\delta x$

This limit is written as $A = \int_a^b f(x)\,dx$, and is read as **the integral of f from a to b**. This is known as the **definite integral**.

TECHNOLOGY EXPLORATION

Area under a curve passing through the x-axis

1. In GeoGebra, create the function $f(x) = (x - 2)(x - 4)$ by entering *f(x)=(x-2)(x-4)* into the input bar.
2. In the input bar, enter *integral[f(x),1,2]*. An area between the curve and the x-axis will be shaded between $x = 1$ and $x = 2$. An integral value a will be shown in the **Algebra** view.
3. In the input bar, enter *integral[f(x),2,4]*. The area under the curve between the two x-intercepts will be shaded. What do you notice about the integral value b in the **Algebra** view?
4. In the input bar, enter *integral[f(x),4,5]*. An area between the curve and the x-axis will be shaded between $x = 4$ and $x = 5$.
5. What is the total area shaded?
6. In the input bar, enter *integral[f(x),1,5]*. This will reshade **all** of the above areas. Note the integral value in the **Algebra** view. Make a statement about integral values and the area between the curve and the x-axis.

EXERCISE 13.2 THE DEFINITE INTEGRAL AND THE AREA UNDER THE CURVE

1 Write the definite integral that you would use to find the area of the region under the line $y = x + 2$ between the ordinates $x = 0$ and $x = 3$. By using appropriate area formulae, find the value of this area.

2 Write the definite integral for the area of the region under the line $y = 2x + 1$ between $x = 1$ and $x = 4$. By using appropriate area formulae, find the value of this area.

3 Which definite integral represents the area bounded by the curve $y = 4 - x^2$ and the x-axis?

A $\int_0^2 (4 - x^2)\,dx$ **B** $\int_{-2}^0 (4 - x^2)\,dx$ **C** $\int_{-2}^2 (4 - x^2)\,dx$ **D** $\int_{-\sqrt{2}}^{\sqrt{2}} (4 - x^2)\,dx$

4 Write the definite integral for the area of the region bounded by the lines $y = 3x$, $x = t$ and the x-axis. By using appropriate area formulae, find the value of this area.

5 Write the definite integral for the area of the region under the curve $y = x^2$ between $x = 0$ and $x = 2$. By drawing this graph on a 5 mm grid, count squares to find an approximation for this area.

6 Write the definite integral for the area of the region under the line $y = 2 - x$ between $x = 0$ and where it cuts the x-axis. By using appropriate area formulae, find the value of this area.

7 Write the definite integral for the area of the region under the curve $y = \sqrt{9 - x^2}$. By using appropriate area formulae, find the value of this area.

8 Write the definite integral for the area of the region under the curve $y = \sqrt{9 - (x - 2)^2}$. By using appropriate area formulae, find the value of this area.

13.3 THE DEFINITE INTEGRAL AND THE PRIMITIVE FUNCTION

In Example **3**, we found that $A = \int_a^b f(x)\,dx = \int_0^t 2x\,dx = t^2$. This means that the **primitive** of $2x$ is $x^2 + C$.

If $F'(x) = f(x)$, then $F(x)$ is a primitive of $f(x)$.

Here $f(x) = 2x$ so $F(x) = x^2$ and $F(t) = t^2$, $F(0) = 0$; we can see that $F(t) - F(0) = t^2 - 0 = t^2$. This leads us to a possible way to evaluate the definite integral:

$$A = \int_a^b f(x)\,dx = F(b) - F(a) \qquad \text{where } F(x) \text{ is a primitive of } f(x)$$

In Example 2 we had $A = \int_a^b f(x)\,dx = \int_1^3 (x + 1)\,dx$. By finding the area of the triangle, we found that $A = 6$ units^2.

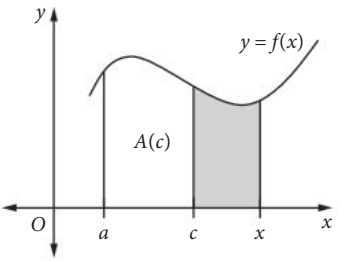
Following the definitions above, for $f(x) = x + 1$ we have $F(x) = \frac{x^2}{2} + x$. Hence $F(3) = \frac{9}{2} + 3 = 7.5$ and $F(1) = \frac{1}{2} + 1 = 1.5$:

$$F(3) - F(1) = 7.5 - 1.5 = 6$$

This verifies that $\int_1^3 (x + 1)\,dx = F(3) - F(1) = 6$ where $F(x)$ is a primitive of $x + 1$. We now need to derive this result to be sure that it holds true for all primitives.

Fundamental theorem of calculus

Let $y = f(x)$ be a continuous positive curve defined for values of x including $x \geq a$.



For $c > a$, let $A(c)$ denote the area under the curve between a and c, and $A(x)$ denote the area under the curve between a and x. The shaded area of the diagram is $A(x) - A(c)$. It can be approximated by a rectangle of height $f(x)$ and base $(x - c)$:

$$A(x) - A(c) \approx (x - c)\, f(x) \quad \text{or} \quad f(x) \approx \frac{A(x) - A(c)}{x - c}$$

If the maximum and minimum values of $f(t)$ for $c \leq t \leq x$ are M and m respectively, then:

$$m \leq \frac{A(x) - A(c)}{x - c} \leq M$$ These inequalities also hold when $x < c$.

As x approaches c, both m and M approach $f(c)$.

Hence: $\lim\limits_{x \to c} \dfrac{A(x) - A(c)}{x - c} = f(c)$

But this is the same as our earlier definition of the derivative of a function $A(x)$.

Thus we have: $A'(c) = \lim\limits_{x \to c} \dfrac{A(x) - A(c)}{x - c}$. Hence $A'(c) = f(c)$, $c > a$.

Because this statement is true for all $c > a$, the two functions $y = f(x)$ and $y = A'(x)$ are equal for all $x > a$.
Thus $A(x)$ is a primitive function of $f(x)$ for $x > a$.

If we already know a primitive function $F(x)$ of $f(x)$, then we must have $A(x) = F(x) + C$ for some constant C.
As $x \to a$, $A(x) \to 0$, $F(x) \to F(a)$, so: $0 = F(a) + C$ or $C = -F(a)$

Thus $A(x) = F(x) - F(a)$ for $x > a$.

If $b > a$, then substituting b for x we have: $A(b) = \int_a^b f(x)\,dx = F(b) - F(a)$

Hence the area and the definite integral can be found using the known primitive function $F(x)$.

Fundamental theorem of calculus:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$ where $F(x)$ is a primitive function of $f(x)$.

In deriving the above result we only considered $f(x)$ as an increasing function in the interval $a \leq x \leq b$. However, the theorem is still true for a decreasing function or for a function that is sometimes increasing and sometimes decreasing in the interval, as long as the function is continuous.

When evaluating an integral, it is convenient to set it out as follows:

$$\int_a^b f(x)\,dx = \left[F(x)\right]_a^b$$
$$= F(b) - F(a)$$

If we can find the primitive function, we can now evaluate any definite integral. Unfortunately, not all primitive functions are easy to find. At this stage we will be limited to polynomial functions and rational powers of the independent variable. Other functions will be looked at later. Later we will also use the trapezoidal rule and Simpson's rule to find numerical approximations for definite integrals.

Example 4

Evaluate:

(a) $\int_1^3 \left(x^2 - x\right) dx$ (b) $\int_{-1}^2 \left(x^3 - 2x^2 + 3x - 4\right) dx$

Solution

(a) $\int_1^3 \left(x^2 - x\right)dx = \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_1^3$

$= \left(9 - \frac{9}{2}\right) - \left(\frac{1}{3} - \frac{1}{2}\right)$

$= 4\frac{2}{3}$

(b) $\int_{-1}^2 \left(x^3 - 2x^2 + 3x - 4\right)dx = \left[\frac{x^4}{4} - \frac{2x^3}{3} + \frac{3x^2}{2} - 4x\right]_{-1}^2$

$= \left(4 - \frac{16}{3} + 6 - 8\right) - \left(\frac{1}{4} + \frac{2}{3} + \frac{3}{2} + 4\right)$

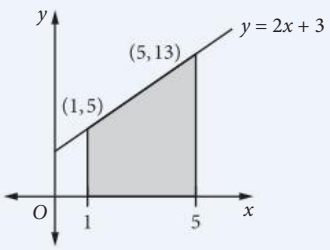
$= -9\frac{3}{4}$

Example 5

Calculate the area of the region bounded by the graph of the straight line $y = 2x + 3$, the x-axis and the ordinates $x = 1$ and $x = 5$.

Solution

Sketch the region:



$\text{Area} = \int_1^5 (2x + 3)\,dx$

$= \left[x^2 + 3x\right]_1^5$

$= (25 + 15) - (1 + 3)$

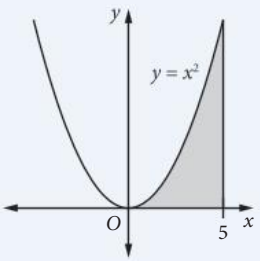
$= 36 \text{ units}^2$

Example 6

Use integration to find the area of the region enclosed by the parabola $y = x^2$, the x-axis and the line $x = 5$.

Solution

Sketch the region:



$\text{Area} = \int_0^5 x^2\,dx$

$= \left[\frac{x^3}{3}\right]_0^5$

$= \frac{5^3}{3} - 0$

$= 41\frac{2}{3} \text{ units}^2$

Example 7

Calculate the area of the region bounded by:

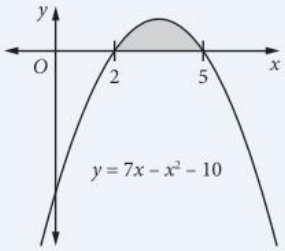
(a) the graph of the parabola $y = 7x - x^2 - 10$ and the x-axis

(b) the graph of the parabola $y = x^2 - 7x + 10$ and the x-axis.

(c) Compare the answers.

Solution

(a) Sketch the region:

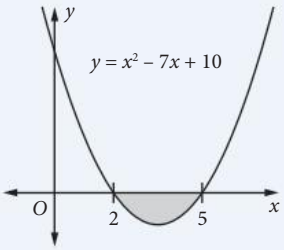


Cuts x-axis when $7x - x^2 - 10 = 0$:

$$x^2 - 7x + 10 = 0$$
$$(x-2)(x-5) = 0$$
$$x = 2, 5$$

$$\text{Area} = \int_2^5 \left(-x^2 + 7x - 10\right) dx$$
$$= \left[-\frac{x^3}{3} + \frac{7x^2}{2} - 10x\right]_2^5$$
$$= \left(-\tfrac{125}{3} + \tfrac{175}{2} - 50\right) - \left(-\tfrac{8}{3} + 14 - 20\right)$$
$$= 4.5$$

(b) Sketch the region:



Cuts x-axis when $x^2 - 7x + 10 = 0$:

$$(x-2)(x-5) = 0$$
$$x = 2, 5$$

$$\text{Area} = \int_2^5 \left(x^2 - 7x + 10\right) dx$$
$$= \left[\frac{x^3}{3} - \frac{7x^2}{2} + 10x\right]_2^5$$
$$= \left(\tfrac{125}{3} - \tfrac{175}{2} + 50\right) - \left(\tfrac{8}{3} - 14 + 20\right)$$
$$= -4.5$$

Negative area does not make sense, but we overcome this problem by taking the absolute value of the answer. The working needs to be written as:

$$\text{Area} = \left|\int_2^5 \left(x^2 - 7x + 10\right) dx\right| = |-4.5| = 4.5$$

(c) The two integrals have the same size but a different sign. The second curve and region are just the first curve and region reflected in the x-axis, so the regions must have the same area. The negative sign indicates that the region is below the x-axis.

Evaluating a definite integral and finding an area

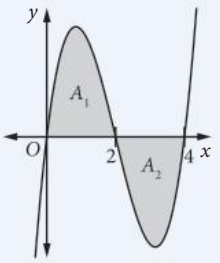
The value of an integral and the area of a region can have different signs, as Example **7** shows. When a region is entirely above the x-axis, the value of the integral is positive and is equal to the area. When the region is entirely below the x-axis, the value of the integral is negative. To find the area, we take the absolute value:

$$\text{Area} = \left|\int_a^b f(x)\,dx\right| = \left|\left[F(x)\right]_a^b\right| = |F(b) - F(a)|$$

What happens when the curve defining a region is both above and below the x-axis? In a situation like that you must find where the curve cuts the x-axis and then consider the regions above and below the x-axis separately.

Example 8

The curve $y = x^3 - 6x^2 + 8x$ cuts the x-axis at 0, 2, and 4 as shown in the diagram. By first finding the areas A_1 and A_2, find the total area enclosed by the curve and the x-axis.



Solution

$$A_1 = \left|\int_0^2 \left(x^3 - 6x^2 + 8x\right)dx\right| \qquad A_2 = \left|\int_2^4 \left(x^3 - 6x^2 + 8x\right)dx\right|$$

$$= \left|\left[\frac{x^4}{4} - 2x^3 + 4x^2\right]_0^2\right| \qquad = \left|\left[\frac{x^4}{4} - 2x^3 + 4x^2\right]_2^4\right|$$

$$= \left|(4 - 16 + 16) - (0)\right| \qquad = \left|(64 - 128 + 64) - (4 - 16 + 16)\right|$$

$$= 4 \qquad = \left|-4\right|$$

$$= 4$$

Total area $= A_1 + A_2$

$= 8$ units^2

Example 9

Find the value of the definite integral $\int_0^4 \left(x^3 - 6x^2 + 8x\right)dx$.

Solution

$$\int_0^4 \left(x^3 - 6x^2 + 8x\right)dx = \left[\frac{x^4}{4} - 2x^3 + 4x^2\right]_0^4$$

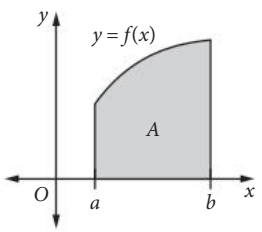
$$= (64 - 128 + 64) - (0)$$

$$= 0$$

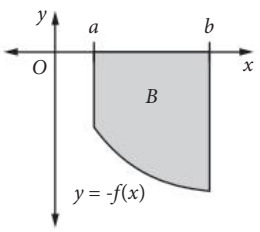
If you used this integral to find the area of the region in Example **8**, you would obtain an area of zero, which is clearly wrong. You should be aware that finding the value of a definite integral is not always the same as finding the area of a region. This is why it is important for you to sketch a region before setting up an integral to calculate its area. You must calculate the areas above and below the x-axis separately. Areas above and below the x-axis have opposite signs.

Important results

1 If $y = f(x)$ is an increasing function over the interval $a \le x \le b$, then $y = -f(x)$ is a decreasing function over the same interval. $y = -f(x)$ is the reflection of $y = f(x)$ in the x-axis. Thus the area of the region bounded by $y = f(x)$, $x = a$, $x = b$ and the x-axis is equal to the area of the region bounded by $y = -f(x)$, $x = a$, $x = b$ and the x-axis.



$$A = \left|\int_a^b f(x)\,dx\right|$$



$$B = \left|\int_a^b \left(-f(x)\right)dx\right|$$

$$B = \left|-\int_a^b f(x)\,dx\right| = \left|\int_a^b f(x)\,dx\right|$$

2 If $y = f(x)$ is a continuous function over the interval $a \le x \le b$, and $a \le c \le b$, then:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

3 If F, G are primitives of f, g respectively, then $F+G$ is a primitive of $f+g$. Thus:

$$\begin{aligned}\int_a^b \big(f(x)+g(x)\big)dx &= \big(F(b)+G(b)\big)-\big(F(a)+G(a)\big)\\ &= \big(F(b)-F(a)\big)-\big(G(b)-G(a)\big)\\ &= \int_a^b f(x)dx+\int_a^b g(x)dx\end{aligned}$$

4 If you reverse the limits of integration, you change the sign of the integral: $\int_a^b f(x)dx = -\int_b^a f(x)dx$

5 $\int_a^b f(x)dx \pm \int_a^b g(x)dx = \int_a^b \big(f(x)\pm g(x)\big)dx$

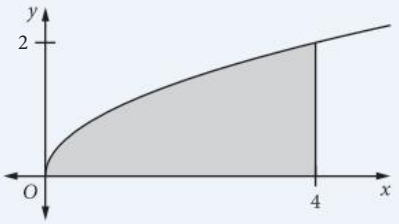
6 $\int_a^b cf(x)dx = c\int_a^b f(x)dx$ where c is a constant.

Example 10

Calculate the area between $y=\sqrt{x}$, the x-axis and $x=4$.

Solution

Sketch the region:



It is clear from the diagram that the region is above the x-axis.

Hence: $\text{Area} = \int_0^4 \sqrt{x}\,dx$

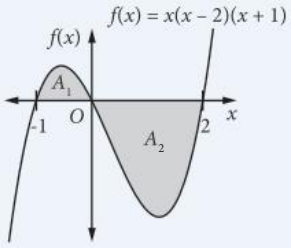
$$= \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^4 = \left[\tfrac{2}{3}x^{\frac{3}{2}}\right]_0^4 = \tfrac{2}{3}(8-0) = 5\tfrac{1}{3} \text{ units}^2$$

Example 11

Find the area of the region bounded by the curve $f(x)=x(x-2)(x+1)$ and the x-axis.

Solution

Sketch the region:



$$f(x) = x(x-2)(x+1) = x^3 - x^2 - 2x$$

$$A_1 = \int_{-1}^{0}\big(x^3-x^2-2x\big)dx = \left[\frac{x^4}{4}-\frac{x^3}{3}-x^2\right]_{-1}^{0} = 0-\left(\tfrac{1}{4}+\tfrac{1}{3}-1\right) = \tfrac{5}{12}$$

$$A_2 = \left|\int_0^2\big(x^3-x^2-2x\big)dx\right| = \left|\left[\frac{x^4}{4}-\frac{x^3}{3}-x^2\right]_0^2\right| = \left|\left(\tfrac{16}{4}-\tfrac{8}{3}-4\right)-0\right| = 2\tfrac{2}{3}$$

$$\text{Total area} = \tfrac{5}{12}+2\tfrac{2}{3} = 3\tfrac{1}{12} \text{ units}^2$$

You can show by evaluating the integral that $\int_{-1}^{2}\big(x^3-x^2-2x\big)dx \neq 3\tfrac{1}{12}$.

EXERCISE 13.3 THE DEFINITE INTEGRAL AND THE PRIMITIVE FUNCTION

1 Evaluate $\int_0^3 x^2\,dx$, $\int_3^0 x^2\,dx$ and show that $\int_0^3 x^2\,dx = -\int_3^0 x^2\,dx$.

2 Evaluate $\int_1^2 x^2\,dx$, $\int_2^3 x^2\,dx$, $\int_1^3 x^2\,dx$ and show that $\int_1^3 x^2\,dx = \int_1^2 x^2\,dx + \int_2^3 x^2\,dx$.

3 Evaluate $\int_1^3 x^2\,dx$, $\int_1^3 2x\,dx$, $\int_1^3 (x^2+2x)\,dx$ and verify that $\int_1^3 (x^2+2x)\,dx = \int_1^3 x^2\,dx + \int_1^3 2x\,dx$.

4 Evaluate $\int_0^3 (4x^2+8x)\,dx$, $4\int_0^3 (x^2+2x)\,dx$ and verify that $\int_0^3 (4x^2+8x)\,dx = 4\int_0^3 (x^2+2x)\,dx$.

5 Evaluate:

(a) $\int_0^1 (2x-5)\,dx$ (b) $\int_{-1}^4 (3x+1)\,dx$ (c) $\int_{-2}^2 (x^2-4)\,dx$

(d) $\int_1^3 (2t-4)\,dt$ (e) $\int_{-2}^3 (u^2-2u)\,du$ (f) $\int_{-2}^3 (2x+1)^2\,dx$

6 The value of $\int_0^5 (6+8x-3x^2)\,dx$ is:

A 29 B 5 C -5 D -29

7 Evaluate:

(a) $\int_{-1}^1 x(1-x^2)\,dx$ (b) $\int_{-1}^3 (2x^2-8x+8)\,dx$ (c) $\int_0^4 (x-2)^3\,dx$

(d) $\int_{-1}^1 x^2(1-x)\,dx$ (e) $\int_0^4 x^{\frac{1}{2}}\,dx$ (f) $\int_{-1}^{-\frac{1}{2}} (-x^3)\,dx$

(g) $\int_{-8}^8 x^{\frac{2}{3}}\,dx$ (h) $\int_0^1 \left(x^{\frac{1}{5}} - x^{\frac{1}{3}}\right)dx$ (i) $\int_0^1 \left(\sqrt{x} - \sqrt[3]{x}\right)dx$

8 Find an expression for:

(a) $\int_b^a 3x^2\,dx$ (b) $\int_0^2 (ax^2+bx+c)\,dx$ (c) $\int_0^a (x^2-2x)\,dx$

9 The following expressions are steps in the evaluation of $\int_2^4 \left(x+\frac{1}{x}\right)^2 dx$. Indicate whether each step is correct or incorrect.

(a) $\int_2^4 \left(x^2+2+\frac{1}{x^2}\right)dx$ (b) $\left[\frac{x^3}{3}+2x-\frac{2}{x^3}\right]_2^4$ (c) $\left[\frac{x^3}{3}+2x-\frac{1}{x}\right]_2^4$ (d) $22\frac{11}{12}$

10 Find the area of the region bounded by the x-axis and the graph of $y = 2 - x - x^2$.

11 Find the area of the region bounded by the x-axis and the graph of $y = x^2(1-x)$.

12 Find the area of the region bounded by the graph of $y = 3 + 5x - 2x^2$ and the x-axis.

13 If $\int_0^a (4-2x)\,dx = 4$, find the value of a.

14 If $\int_{-1}^a x\,dx = 0$, find the value of a.

15 If $c\int_{-2}^2 (x-5)\,dx = 1$, find the value of c.

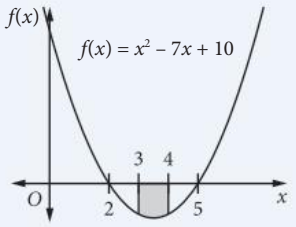
16 Calculate the area bounded by the curve $y = x^2(3-x)$.

Example 12

Calculate the area of the region bounded by the graph of the parabola $f(x) = x^2 - 7x + 10$, the x-axis, and the ordinates $x = 3$ and $x = 4$.

Solution

Sketch the region:



$$\text{Area} = \left|\int_3^4 \left(x^2 - 7x + 10\right)dx\right|$$

$$= \left|\left[\frac{x^3}{3} - \frac{7x^2}{2} + 10x\right]_3^4\right|$$

$$= \left|\left(\tfrac{64}{3} - 56 + 40\right) - \left(9 - \tfrac{63}{2} + 30\right)\right|$$

$$= 2\tfrac{1}{6}$$

Odd and even functions

For an even function, the area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$ and $x = -a$ is:

$$\text{Area} = \left|\int_{-a}^{a} f(x)\,dx\right| = 2\left|\int_0^a f(x)\,dx\right|$$

For an odd function, the area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$ and $x = -a$ is:

$$\text{Area} = \left|\int_{-a}^{0} f(x)\,dx\right| + \left|\int_0^a f(x)\,dx\right| = 2\left|\int_0^a f(x)\,dx\right|$$

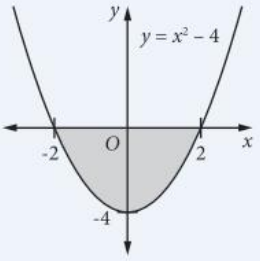
Example 13

Find the area of the region bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = -2$ and $x = 2$, for:

(a) $f(x) = x^2 - 4$ **(b)** $f(x) = x^3$

Solution

(a) Sketch the region:



$f(x) = x^2 - 4$ is an even function.

$$\int_{-2}^{0}\left(x^2 - 4\right)dx = \int_0^2\left(x^2 - 4\right)dx$$

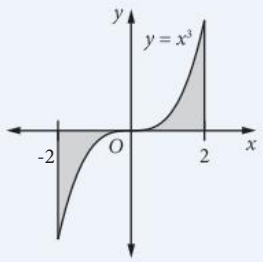
$$\text{Area} = 2\left|\int_0^2\left(x^2 - 4\right)dx\right|$$

$$= 2\left|\left[\frac{x^3}{3} - 4x\right]_0^2\right|$$

$$= 2\left|\tfrac{8}{3} - 8\right| = 2\left|-\tfrac{16}{3}\right|$$

$$= 10\tfrac{2}{3} \text{ units}^2$$

(b) Sketch the region:



$f(x) = x^3$ is an odd function.

$$\int_{-2}^{0} x^3\,dx = -\int_{0}^{2} x^3\,dx \quad \text{and} \quad \int_{-2}^{2} x^3\,dx = 0$$

$$\text{Area} = \left|\int_{-2}^{0} x^3\,dx\right| + \left|\int_{0}^{2} x^3\,dx\right| = 2\left|\int_{0}^{2} x^3\,dx\right|$$

$$= 2\left|\left[\frac{x^4}{4}\right]_0^2\right|$$

$$= 2|4-0|$$

$$= 8 \text{ units}^2$$

EXERCISE 13.4 MORE AREAS

1 Calculate the area of the region bounded by the line $y = 3x + 5$, the x-axis and the lines $x = 3$ and $x = 6$.

2 Find the area of the region bounded by the parabola $y = x^2 + 2$, the x-axis and the lines $x = -1$ and $x = 2$.

3 Calculate the area of the region bounded by the graph of $f(x) = x^2 - 4x + 4$, the x-axis and the lines $x = 1$ and $x = 4$.

4 Calculate the area of the region bounded by the curve $y = 4 - x^2$ and the x-axis.

5 Which of these integrals will give the area of the region bounded by the curve $y = 16 - x^4$ and the x-axis? Indicate whether each answer is correct or incorrect.

(a) $\int_{-2}^{2}(16-x^4)\,dx$ **(b)** $\int_{-4}^{4}(16-x^4)\,dx$ **(c)** $2\int_{0}^{2}(16-x^4)\,dx$ **(d)** $\left|\int_{-2}^{2}(16-x^4)\,dx\right|$

6 Calculate the area of the region bounded by the curve $y = -x^3$, the x-axis and the ordinates $x = -3$ and $x = 3$.

7 The value of the definite integral $\int_{-4}^{4} x^3\,dx$ is:

A -128 **B** 0 **C** 64 **D** 128

8 Calculate the area of the region bounded by the graph of $f(x) = (x-2)^3$, the x-axis, $x = 2$ and $x = 3$.

9 Find a positive number k such that the area of the region bounded by the graph of $f(x) = kx(2-x)^2$ and the x-axis is equal to 1 unit^2.

10 For the graph of $f(x) = (x+1)(x-1)^2$, calculate:

(a) the area bounded by the curve, the x-axis, $x = 0$ and $x = 0.5$
(b) the area bounded by the curve and the x-axis
(c) the area to the right of the origin bounded by the curve and the coordinate axes.

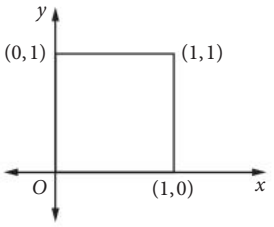
11 Find the area bounded by the parabola $y = 8x - x^2$ and the x-axis.

12 Find the area of the region bounded by the curve $y = x(x-2)^2$ and the x-axis.

13 Calculate the area of the region bounded by the curve $y = (x+1)(x-1)(x-3)$, the x-axis and the ordinates at $x = 0$ and $x = 2$.

14 Show by integration that the area of a unit square (see diagram) is:

(a) bisected by the line $y = x$
(b) trisected by the curves $y = x^2$ and $y = \sqrt{x}$
(c) divided into four equal parts by the curves $y = x^3$, $y = x$ and $y = \sqrt[3]{x}$.



13.5 APPROXIMATE METHODS OF INTEGRATION—TRAPEZOIDAL RULE

For many functions we cannot easily find the primitive. In these cases, we can use numerical methods of integration to find the approximate value of a definite integral. There are many different methods of numerical approximation; computer software that evaluates definite integrals usually uses sophisticated numerical methods.

We will look at two approximation methods: the **trapezoidal rule**, where trapezia are used to approximate the area under the curve; and **Simpson's rule**, where a parabola is fitted to sections of the curve and the area under the parabola is found.

In Example **1** we approximated the area under the curve $y = x^2$ using rectangles drawn above and below the curve. The more rectangles we used, the better the approximation obtained for the area. In Example **14** we will redo this problem using trapezia.

Recall the area of a trapezium: $\text{Area} = \dfrac{\text{sum of lengths of parallel sides}}{2} \times \text{distance between them}$

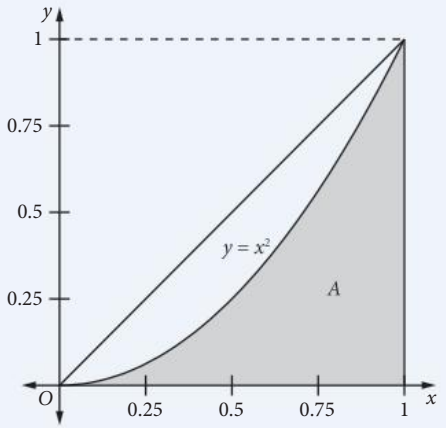
Example 14

Calculate the area of the region bounded by the curve $y = x^2$, the x-axis and the ordinates at $x = 0$ and $x = 1$, using trapezia with:

(a) one subinterval **(b)** two subintervals **(c)** four subintervals.

Solution

(a) Draw a diagram showing the region.



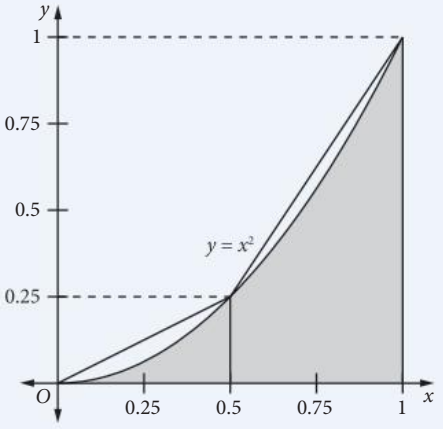
For $0 \le x \le 1$ and one subinterval, the trapezium is a triangle, but for consistency we will use the trapezium area formula.

$f(x) = x^2$: $f(0) = 0, f(1) = 1$, base $= 1 - 0 = 1$

$$\text{Area} = \frac{f(0) + f(1)}{2} \times (1 - 0)$$

$$= \frac{0 + 1}{2} \times 1 = 0.5 \text{ units}^2$$

(b) For $0 \le x \le 1$ and two subintervals:



$f(x) = x^2$: $f(0) = 0,\ f(0.5) = 0.25, f(1) = 1$,
base $= 0.5 - 0 = 1 - 0.5 = 0.5$

$$\text{Area} = \frac{f(0) + f(0.5)}{2} \times 0.5 + \frac{f(0.5) + f(1)}{2} \times 0.5$$

$$= \frac{(0 + 0.25)}{2} \times 0.5 + \frac{(0.25 + 1)}{2} \times 0.5$$

$$= 0.0625 + 0.3125 = 0.375 \text{ units}^2$$

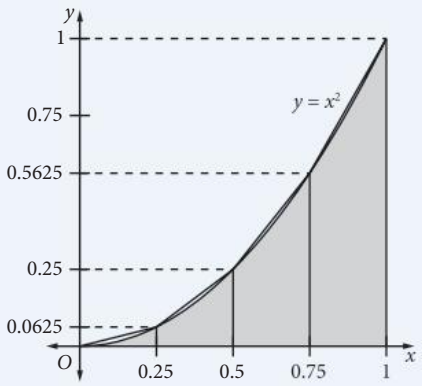
(c) For $0 \le x \le 1$ and four subintervals:

$f(x) = x^2$: $f(0) = 0, f(0.25) = 0.0625, f(0.5) = 0.25, f(0.75) = 0.5625, f(1) = 1$, base $= 0.25 - 0 = 0.25$

$$\text{Area} = \frac{f(0) + f(0.25)}{2} \times 0.25 + \frac{f(0.25) + f(0.5)}{2} \times 0.25 + \frac{f(0.5) + f(0.75)}{2} \times 0.25 + \frac{f(0.75) + f(1)}{2} \times 0.25$$

$$= \frac{0 + 0.0625}{2} \times 0.25 + \frac{0.0625 + 0.25}{2} \times 0.25 + \frac{0.25 + 0.5625}{2} \times 0.25 + \frac{0.5625 + 1}{2} \times 0.25$$

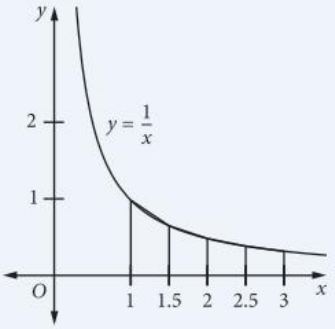
$$= 0.34375 \text{ units}^2$$



By evaluating $\int_0^1 x^2\,dx$ we know that the exact area is $\frac{1}{3}$ units^2. We can see that increasing the number of subintervals (and hence trapezia) gives a better approximation to the area.

Example 15

Use trapezia to find an approximation for the area under the curve $y = \frac{1}{x}$ between $x = 1$ and $x = 3$ using four subintervals.



Solution

Sketch the region:

$f(x) = \frac{1}{x}$: $f(1) = 1$, $f(1.5) = \frac{2}{3}$, $f(2) = 0.5$, $f(2.5) = 0.4$, $f(3) = \frac{1}{3}$,

base $= 1.5 - 1 = 0.5$

$$\text{Area} = \frac{f(1)+f(1.5)}{2}\times 0.5 + \frac{f(1.5)+f(2)}{2}\times 0.5 + \frac{f(2)+f(2.5)}{2}\times 0.5 + \frac{f(2.5)+f(3)}{2}\times 0.5$$

$$= \left(\left(1+\tfrac{2}{3}\right)+\left(\tfrac{2}{3}+\tfrac{1}{2}\right)+\left(\tfrac{1}{2}+\tfrac{2}{5}\right)+\left(\tfrac{2}{5}+\tfrac{1}{3}\right)\right)\times\tfrac{0.5}{2}$$

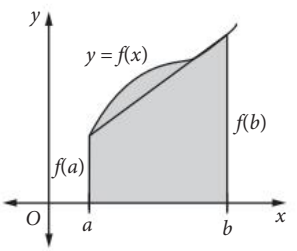
$$= \tfrac{1}{4}\left(1+2\times\tfrac{2}{3}+2\times\tfrac{1}{2}+2\times\tfrac{2}{5}+\tfrac{1}{3}\right)$$

$$= 1.11\dot{6} \text{ units}^2$$

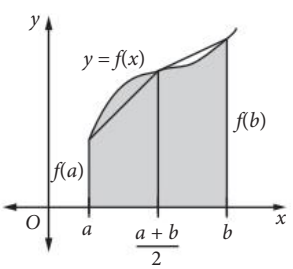
In this example we have found an approximation for the definite integral, $\int_1^3 \frac{dx}{x}$. We cannot find the primitive of $f(x) = \frac{1}{x}$ yet, but we can still find an approximate value for definite integrals involving the function.

The trapezoidal rule

If $f(x)$ is a continuous function and $f(x) \geq 0$ on the interval $a \leq x \leq b$, then we can find an approximation to the definite integral $\int_a^b f(x)\,dx$ by dividing the area into a number of trapezia of equal width, where the parallel sides of the trapezia are ordinates.



- One trapezium: $\int_a^b f(x)\,dx \approx \frac{(b-a)}{2}\big(f(a)+f(b)\big)$



- Two trapezia:

$$\int_a^b f(x)\,dx \approx \frac{(b-a)}{2}\left(\frac{f(a)+f\left(\frac{a+b}{2}\right)}{2}\right)+\frac{(b-a)}{2}\left(\frac{f\left(\frac{a+b}{2}\right)+f(b)}{2}\right)$$

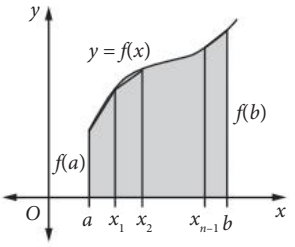
$$= \frac{(b-a)}{4}\left(f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right)$$

- n trapezia: the base of each trapezium is $\frac{b-a}{n} = h$. If $a = x_0$, $b = x_n$ and the points between are $x_1, x_2, \ldots x_{n-1}$, then the trapezoidal rule becomes:

$$\int_a^b f(x)dx \approx \frac{(b-a)}{2n}\big(f(a)+2f(x_1)+2f(x_2)+\ldots+2f(x_{n-1})+f(b)\big)$$

OR $$\int_a^b f(x)dx \approx \frac{h}{2}\big(f(a)+2f(x_1)+2f(x_2)+\ldots+2f(x_{n-1})+f(b)\big)$$

where $h = \frac{b-a}{n}$.



For $h = \frac{b-a}{n}$, we have:

$n = 1, h = b - a$: $\int_a^b f(x)dx \approx \frac{(b-a)}{2}\big(f(a)+f(b)\big) = \frac{h}{2}\big(f(a)+f(b)\big)$

$n = 2, h = \frac{b-a}{2}$: $\int_a^b f(x)dx \approx \frac{h}{2}\left(f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right)$

$n = n, h = \frac{b-a}{n}$: $\int_a^b f(x)dx \approx \frac{h}{2}\big(f(a)+2f(x_1)+2f(x_2)+\ldots+2f(x_{n-1})+f(b)\big)$ where $a = x_0$, $b = x_n$.

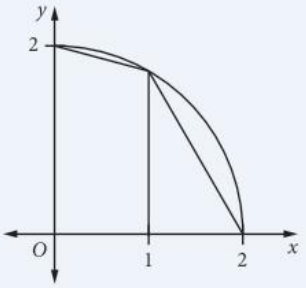
Example 16

Evaluate $\int_0^2 \sqrt{4-x^2}\,dx$ using the trapezoidal rule with:

(a) two subintervals **(b)** four subintervals. **(c)** What is the exact value of this integral?

Solution

(a) Use a table of values to sketch $f(x) = \sqrt{4-x^2}$ with two subintervals:



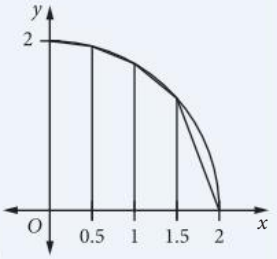
x	0	1	2
$f(x)$	2	$\sqrt{3}$	0

$h = \frac{2-0}{2} = 1$: $\int_0^2 \sqrt{4-x^2}\,dx \approx \frac{1}{2}\big(f(0)+2f(1)+f(2)\big)$

$= \frac{1}{2}\big(2+2\sqrt{3}+0\big)$

$= 2.732$

(b) Use a table of values to sketch $f(x) = \sqrt{4-x^2}$ with four subintervals:



x	0	0.5	1	1.5	2
$f(x)$	2	$\sqrt{3.75}$	$\sqrt{3}$	$\sqrt{1.75}$	0

$h = \frac{2-0}{4} = 0.5$: $\int_0^2 \sqrt{4-x^2}\,dx \approx \frac{0.5}{2}\big(f(0)+2\big(f(0.5)+f(1)+f(1.5)\big)+f(2)\big)$

$= \frac{1}{4}\left(2+2\left(\sqrt{3.75}+\sqrt{3}+\sqrt{1.75}\right)+0\right)$

$= 2.996$

(c) The region is a quarter of a circle of radius 2, so the exact value of $\int_0^2 \sqrt{4-x^2}\,dx$ is π.

Example 17

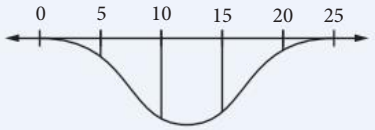
To measure the cross-sectional area of a river, a boat travels directly across the river and measures the river's depth as shown in the following table. The depth of the river at the banks is zero metres. Use the trapezoidal rule and all the values in the table to calculate the cross-sectional area of the river.

Distance from riverbank (in metres)	0	5	10	15	20	25
Depth (in metres)	0	1.2	5	4.8	1.3	0

Solution

The distance from the bank is the independent variable and the depth of the river is the dependent variable.

$$\text{Area of cross-section} = \int_0^{25} f(x)\,dx \approx \tfrac{5}{2}\big(f(0)+2\big(f(5)+f(10)+f(15)+f(20)\big)+f(25)\big)$$
$$= \tfrac{5}{2}\big(0+2(1.2+5.0+4.8+1.3)+0\big)$$
$$= 5\times 12.3$$
$$= 61.5$$



The area of the cross-section is approximately 61.5 m^2.

If an average flow rate of the river was known, then the amount of water flowing past in a given time could be calculated.

EXERCISE 13.5 APPROXIMATE METHODS OF INTEGRATION—TRAPEZOIDAL RULE

1 Evaluate $\int_0^4 \sqrt{x^2+9}\,dx$ using the trapezoidal rule and two subintervals.

2 Evaluate $\int_0^1 \frac{dx}{x^2+1}$ using the trapezoidal rule with: **(a)** two subintervals **(b)** four subintervals.

3 Indicate whether each expression is a correct or incorrect example of the use of the trapezoidal rule.

(a) $\int_1^3 \frac{dx}{\sqrt{x}} \approx \frac{3-1}{2}\left(\frac{1}{\sqrt{1}}+\frac{2}{\sqrt{2}}+\frac{1}{\sqrt{3}}\right)$ **(b)** $\int_1^3 \frac{dx}{\sqrt{x}} \approx \frac{2-1}{2}\left(\frac{1}{\sqrt{1}}+\frac{2}{\sqrt{2}}+\frac{1}{\sqrt{3}}\right)$

(c) $\int_1^3 \frac{dx}{\sqrt{x}} \approx \frac{2-1}{2}\left(\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{3}}\right)$ **(d)** $\int_1^3 \frac{dx}{\sqrt{x}} \approx \frac{0.5}{2}\left(\frac{1}{\sqrt{1}}+2\left(\frac{\sqrt{2}}{\sqrt{3}}+\frac{1}{\sqrt{2}}+\frac{\sqrt{2}}{\sqrt{5}}\right)+\frac{1}{\sqrt{3}}\right)$

4 Evaluate $\int_1^2 x^3\,dx$ by: **(a)** direct integration **(b)** using the trapezoidal rule with two subintervals.

5 Given that $f(x) = 2^x$, complete the following table:

x	0	1	2	3	4
2^x					

Using these five function values and the trapezoidal rule, estimate $\int_0^4 2^x\,dx$.

6 Evaluate $\int_{-1}^1 4^x\,dx$ using the trapezoidal rule with four subintervals.

7 Evaluate $\int_0^1 \sqrt{x}\,dx$ by: **(a)** direct integration **(b)** using the trapezoidal rule with three subintervals.

8 Evaluate $\int_0^2 \sqrt{1+x^3}\,dx$ using the trapezoidal rule with two subintervals.

9 Evaluate $\int_0^4 \frac{x}{1+x}\,dx$ using the trapezoidal rule with four subintervals.

10 Evaluate $\int_0^4 \sqrt{1+x^4}\,dx$ using the trapezoidal rule with four subintervals.

11 A mining engineer is trying to estimate the volume of earth above a coal seam prior to open-cut mining. The area to be mined is flat and rectangular. In a straight 500-metre line across the middle of the rectangle, the engineer drills down to record the depth from the ground surface to the coal seam at 50-metre intervals. The information is recorded in the following table:

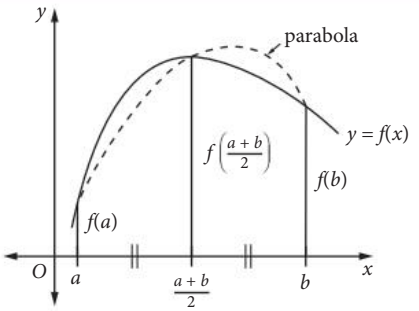
Distance from edge (m)	0	50	100	150	200	250	300	350	400	450	500
Depth (m)	200	150	100	150	100	50	50	100	150	200	250

(a) The engineer needs to calculate the vertical area beneath this line down to the coal seam. Use the trapezoidal rule to calculate this area.

(b) The rectangular area to be mined is 2.5 km long. Estimate the volume of soil to be removed (in cubic metres) to expose the coal seam. (Assume that the cross-sectional area is the same for the whole length of the rectangle.)

13.6 APPROXIMATE METHODS OF INTEGRATION—SIMPSON'S RULE

Simpson's rule is based on replacing the given function by a quadratic function: as shown in the diagram, we replace the graph of the given function from $x = a$ to $x = b$ by an arc of a parabola that intersects the given curve at $x = a$, $x = b$ and the midpoint $x = \frac{a+b}{2}$.



It can be proved that the area under the parabola through these three points is $\frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]$.

Simpson's rule then states: $\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]$

Of course, we can also break any interval into subintervals and apply Simpson's rule to each subinterval, depending on how many points on the graph (i.e. function values) we want to work with. The more subintervals, the more accurate the approximation. To apply Simpson's rule to any number of subintervals, we need an initial function value and then two more function values for each subinterval, so that each subinterval has two endpoints and a midpoint.

Example 18

Evaluate $\int_0^2 \sqrt{4-x^2}\,dx$ (as in Example **16**) using Simpson's rule, with:

(a) three function values **(b)** five function values.

Solution

Make a table of five function values for $f(x) = \sqrt{4-x^2}$:

x	0	0.5	1	1.5	2
$f(x)$	2	$\sqrt{3.75}$	$\sqrt{3}$	$\sqrt{1.75}$	0

(a) three function values: $a = 0$, $b = 2$, $\frac{a+b}{2} = 1$:

$$\int_0^2 \sqrt{4-x^2}\,dx \approx \frac{2-0}{6}\left[f(0)+4f(1)+f(2)\right]$$
$$= \tfrac{1}{3}\left[2+4\times\sqrt{3}+0\right]$$
$$= 2.976$$

(b) We break the interval $[0, 2]$ into two subintervals $[0, 1]$ and $[1, 2]$, then apply Simpson's rule to each subinterval and add the two areas. Each subinterval uses three function values: (0, 0.5, 1) and (1, 1.5, 2).

$$\int_0^1 \sqrt{4-x^2}\,dx \approx \frac{1-0}{6}\left[f(0)+4f(0.5)+f(1)\right]$$
$$= \tfrac{1}{6}\left[2+4\times\sqrt{3.75}+\sqrt{3}\right]$$
$$= 1.913$$

$$\int_1^2 \sqrt{4-x^2}\,dx \approx \frac{2-1}{6}\left[f(1)+4f(1.5)+f(2)\right]$$
$$= \tfrac{1}{6}\left[\sqrt{3}+4\times\sqrt{1.75}+0\right]$$
$$= 1.171$$

$$\int_0^2 \sqrt{4-x^2}\,dx \approx 1.913+1.171$$
$$= 3.084$$

The answer using Simpson's rule differs from the exact value (which is π) by less than 2%.

Note that Simpson's rule will give an exact value when $f(x)$ is a quadratic or cubic polynomial, just as the trapezoidal rule will give an exact value when $f(x)$ is a linear function.

Example 19

Evaluate $\int_1^2 x^2\,dx$: **(a)** by integration **(b)** using Simpson's rule with three function values.

Solution

(a) $\int_1^2 x^2\,dx = \left[\frac{1}{3}x^3\right]_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$

(b) $a = 1$, $b = 2$, $\frac{a+b}{2} = 1.5$:

$$\int_1^2 x^2\,dx \approx \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]$$
$$= \tfrac{1}{6}\left[1+4\times 2.25+4\right]$$
$$= \tfrac{7}{3}$$

These two answers are the same because Simpson's rule is exact for a quadratic function.

Example 20

Evaluate $\int_1^3 \frac{1}{x}\,dx$ using:

(a) the trapezoidal rule with four trapezia
(b) Simpson's rule with five function values.

Solution

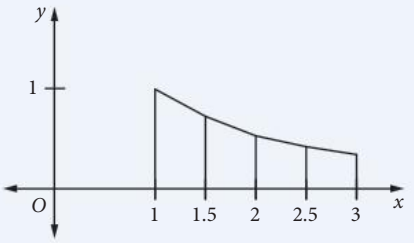
Make a table of five values for $f(x) = \frac{1}{x}$:

x	1	1.5	2	2.5	3
$f(x)$	1	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{2}{5}$	$\frac{1}{3}$

(a) $\int_1^3 \frac{1}{x}\,dx \approx \frac{0.5}{5}\left[f(1)+2f(1.5)+2f(2)+2f(2.5)+f(3)\right]$

$= \frac{1}{4}\left[1+2\times\frac{2}{3}+2\times\frac{1}{2}+2\times\frac{2}{5}+\frac{1}{3}\right]$

$= 1\frac{7}{60} \approx 1.1167$



(b) $1 \le x \le 2$: $\int_1^2 \frac{1}{x}\,dx \approx \frac{2-1}{6}\left[f(1)+4f(1.5)+f(2)\right]$

$= \frac{1}{6}\left[1+4\times\frac{2}{3}+\frac{1}{2}\right] = \frac{25}{36}$

$2 \le x \le 3$: $\int_2^3 \frac{1}{x}\,dx \approx \frac{3-2}{6}\left[f(2)+4f(2.5)+f(3)\right]$

$= \frac{1}{6}\left[\frac{1}{2}+4\times\frac{2}{5}+\frac{1}{3}\right] = \frac{73}{180}$

$\int_1^3 \frac{1}{x}\,dx \approx \frac{25}{36}+\frac{73}{180} = 1.1$

In Chapter 14 we will find that $\int_1^3 \frac{1}{x}\,dx = \log_e 3 \approx 1.099$. This exact value differs from the answer obtained using Simpson's rule by less than 0.1%, and from the trapezoidal rule answer by less than 2%.

EXERCISE 13.6 APPROXIMATE METHODS OF INTEGRATION—SIMPSON'S RULE

1 Use Simpson's rule with three function values to estimate $\int_0^4 \sqrt{x^2+9}\,dx$.

2 Evaluate $\int_0^1 \frac{dx}{x^2+1}$ using Simpson's rule with three function values.

3 Evaluate $\int_0^1 x^3\,dx$: **(a)** by integration **(b)** using Simpson's rule with three function values.
What do you observe?

4 Which expression is a correct example of the use of Simpson's rule?

A $\int_1^3 \frac{dx}{\sqrt{x}} \approx \frac{3-1}{6}\left(1+\frac{4}{\sqrt{2}}+\frac{1}{\sqrt{3}}\right)$ **B** $\int_1^3 \frac{dx}{\sqrt{x}} \approx \frac{3-1}{2}\left(1+\frac{2}{\sqrt{2}}+\frac{1}{\sqrt{3}}\right)$

C $\int_1^3 \frac{dx}{\sqrt{x}} \approx \frac{3-1}{2}\left(1+\frac{4}{\sqrt{2}}+\frac{1}{\sqrt{3}}\right)$ **D** $\int_1^3 \frac{dx}{\sqrt{x}} \approx \frac{3-1}{6}\left(1+\frac{2}{\sqrt{2}}+\frac{1}{\sqrt{3}}\right)$

5 Given that $f(x) = 2^x$, complete the following table:

x	0	1	2	3	4
2^x					

Use these five function values with Simpson's rule to estimate $\int_0^4 2^x\,dx$.

6 Evaluate $\int_{-1}^1 4^x\,dx$ using Simpson's rule with five function values.

7 Evaluate $\int_0^1 \sqrt{x}\,dx$ by: **(a)** direct integration **(b)** using Simpson's rule with three function values.

8 Evaluate $\int_0^2 \sqrt{1+x^3}\,dx$ using Simpson's rule with: **(a)** three function values **(b)** five function values.

9 Use Simpson's rule with nine function values to evaluate $\int_0^4 \frac{x}{1+x}\,dx$ correct to 5 decimal places.

10 Use Simpson's rule with five function values to evaluate $\int_0^4 \sqrt{1+x^4}\,dx$.

11 A mining engineer is trying to estimate the volume of earth above a coal seam prior to open-cut mining. The area to be mined is flat and rectangular. In a straight 500-metre line across the middle of the rectangle, the engineer drills down to record the depth from the ground surface to the coal seam at 50-metre intervals. The information is recorded in the following table:

Distance from edge (m)	0	50	100	150	200	250	300	350	400	450	500
Depth (m)	200	150	100	150	100	50	50	100	150	200	250

(a) The engineer needs to calculate the vertical area beneath this line down to the coal seam. Use Simpson's rule to calculate this area.

(b) The rectangular area to be mined is 2.5 km long. Estimate the volume of soil to be removed (in cubic metres) to expose the coal seam. (Assume that the cross-sectional area is the same for the whole length of the rectangle.)

13.7 AREA BETWEEN TWO CURVES

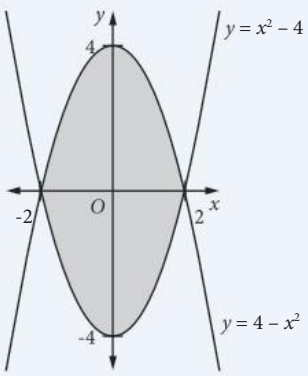
So far we have used integration to find areas bounded by a curve, an axis and (sometimes) ordinates. In this section we will use integration to calculate the area between two curves.

Example 21

Sketch the region bounded by the curves $y = 4 - x^2$ and $y = x^2 - 4$. By evaluating the appropriate definite integrals, calculate the area of this region.

Solution

Sketch the region:



Both curves cut the x-axis at $x = \pm 2$.

Call the area above the x-axis A_1 and the area below the x-axis A_2.

A_1 is bounded by $y = 4 - x^2$ and the x-axis, hence:

$$A_1 = \int_{-2}^{2} \left(4 - x^2\right) dx$$

We can see from the sketch that this area is positive, so the absolute value bars can be left out.

A_2 is bounded by $y = x^2 - 4$ and the x-axis, hence:

$$A_2 = \left| \int_{-2}^{2} \left(x^2 - 4\right) dx \right|$$

We can see from the sketch that the integral for this area will be negative, so we must include absolute value bars.

Evaluate the integrals:

$$\begin{aligned} A_1 &= \int_{-2}^{2} \left(4 - x^2\right) dx \\ &= \left[4x - \frac{x^3}{3}\right]_{-2}^{2} \\ &= \left(8 - \tfrac{8}{3}\right) - \left(-8 + \tfrac{8}{3}\right) \\ &= 16 - \tfrac{16}{3} \\ &= \tfrac{32}{3} \\ &= 10\tfrac{2}{3} \end{aligned}$$

$$\begin{aligned} A_2 &= \left| \int_{-2}^{2} \left(x^2 - 4\right) dx \right| \\ &= \left| \left[\frac{x^3}{3} - 4x\right]_{-2}^{2} \right| \\ &= \left| \left(\tfrac{8}{3} - 8\right) - \left(-\tfrac{8}{3} + 8\right) \right| \\ &= \left| \tfrac{16}{3} - 16 \right| \\ &= \left| -\tfrac{32}{3} \right| \\ &= 10\tfrac{2}{3} \end{aligned}$$

$$\begin{aligned} \text{Area of shaded region} &= A_1 + A_2 \\ &= 10\tfrac{2}{3} + 10\tfrac{2}{3} = 21\tfrac{1}{3} \text{ units}^2 \end{aligned}$$

Alternate method:

$$A_1 + A_2 = \int_{-2}^{2}\left(4-x^2\right)dx + \left|\int_{-2}^{2}\left(x^2-4\right)dx\right|$$

Now $\left|\int_{-2}^{2}\left(x^2-4\right)dx\right| = -\left(\int_{-2}^{2}\left(x^2-4\right)dx\right)$, so:

$$\begin{aligned} A_1 + A_2 &= \int_{-2}^{2}\left(4-x^2\right)dx - \int_{-2}^{2}\left(x^2-4\right)dx \\ &= \int_{-2}^{2}\left(4-x^2\right)dx + \int_{-2}^{2}\left(4-x^2\right)dx \\ &= 2\int_{-2}^{2}\left(4-x^2\right)dx \\ &= 2\left[4x - \frac{x^3}{3}\right]_{-2}^{2} = 21\tfrac{2}{3} \end{aligned}$$

The process here becomes more clear if we write $f(x) = 4 - x^2$ and $g(x) = x^2 - 4$:

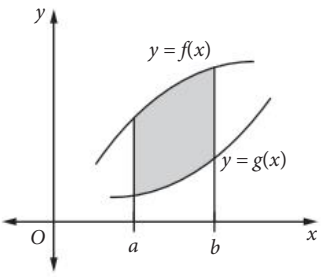
$$\begin{aligned} A_1 + A_2 &= \int_{-2}^{2} f(x)\,dx + \left|\int_{-2}^{2} g(x)\,dx\right| \\ &= \int_{-2}^{2} f(x)\,dx - \int_{-2}^{2} g(x)\,dx \\ &= \int_{-2}^{2}\left(f(x) - g(x)\right)dx \\ &= \int_{-2}^{2}\left(\left(4-x^2\right) - \left(x^2-4\right)\right)dx \\ &= \int_{-2}^{2}\left(8-2x^2\right)dx \\ &= 2\int_{-2}^{2}\left(4-x^2\right)dx = 2\left[4x - \frac{x^3}{3}\right]_{-2}^{2} = 21\tfrac{2}{3} \end{aligned}$$

Area between two curves—general result

If f and g are two continuous functions whose graphs do not intersect in the interval $a \le x \le b$, and $f(x) > g(x)$ over this interval, then the area of the region bounded by the two curves and the ordinates $x = a$ and $x = b$ is given by:

$$\begin{aligned} \text{Area} &= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx \\ &= \int_a^b \left(f(x) - g(x)\right)dx \end{aligned}$$

This result also applies if the two curves intersect only at $x = a$ and $x = b$.



Example 22

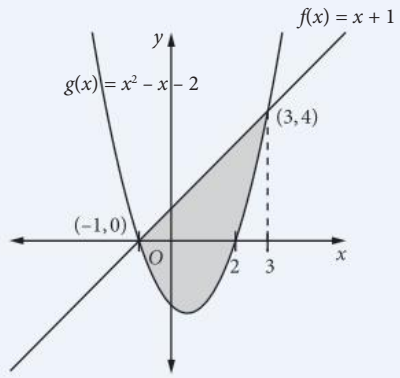
Calculate the area of the region enclosed by the graphs of $f(x) = x + 1$ and $g(x) = x^2 - x - 2$.

Solution

Find the x-values of the points of intersection of the two curves by solving $f(x)$ and $g(x)$ simultaneously:

$$\begin{aligned} x^2 - x - 2 &= x + 1 \\ x^2 - 2x - 3 &= 0 \\ (x+1)(x-3) &= 0 \\ x &= -1, 3 \\ y &= 0, 4 \end{aligned}$$

Sketch the region:



$f(x) \geq g(x)$ for $-1 \leq x \leq 3$. Area of the region A is given by:

$$A = \int_{-1}^{3} f(x)\,dx - \int_{-1}^{3} g(x)\,dx$$

OR

$$A = \int_{-1}^{3} \big(f(x) - g(x)\big)\,dx$$

Hence:

$$A = \int_{-1}^{3} \Big((x+1) - \big(x^2 - x - 2\big)\Big)\,dx$$

$$= \int_{-1}^{3} \big(-x^2 + 2x + 3\big)\,dx$$

$$= \left[-\frac{x^3}{3} + x^2 + 3x\right]_{-1}^{3}$$

$$= (-9 + 9 + 9) - \left(\tfrac{1}{3} + 1 - 3\right)$$

$$= 10\tfrac{2}{3}$$

The area between the curves is $10\frac{2}{3}$ units^2.

Example 23

Calculate the area of the region bounded by the x-axis and the curves with equations $y = \sqrt{x}$ and $y = 6 - x$.

Solution

Find the point of intersection: $\sqrt{x} = 6 - x$

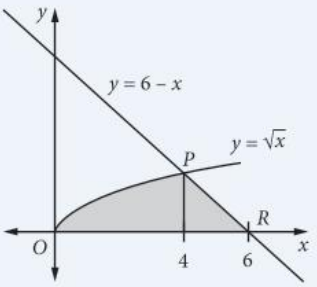
Square both sides: $x = 36 - 12x + x^2$

$$x^2 - 13x + 36 = 0$$

$$(x - 4)(x - 9) = 0$$

$$x = 4, 9$$

Only $x = 4$ is a root of $\sqrt{x} = 6 - x$. Check by substituting $x = 9$ into the equation.



Whenever you square both sides of an equation, the new equation may have roots that were not roots of the original equation. Always check your answers.

Hence the point of intersection P is $(4, 2)$.

The region consists of two areas: the area under the curve $y = \sqrt{x}$ between O and P (i.e. for $0 \leq x \leq 4$) and the area under the curve $y = 6 - x$ between P and R (i.e. for $4 \leq x \leq 6$).

$$\text{Area} = \int_0^4 \sqrt{x}\,dx + \int_4^6 (6 - x)\,dx$$

$$= \left[\tfrac{2}{3}x^{\frac{3}{2}}\right]_0^4 + \left[6x - \frac{x^2}{2}\right]_4^6$$

$$= \left(\tfrac{2}{3} \times 8 - 0\right) + \left(36 - 18 - (24 - 8)\right)$$

$$= 5\tfrac{1}{3} + 2$$

$$= 7\tfrac{1}{3}$$

The area of the region is $7\frac{1}{3}$ units^2.

TECHNOLOGY EXPLORATION

Area between two curves

Example: Find the area between *f(x)=-0.5(x-2)(x+3)* and *g(x)=0.5(x+2)(x-3)*.

1. In GeoGebra, enter both equations into the input bar.
2. Right-click on the curve of *f(x)* and select **Object Properties** to change the colour. Do the same for *g(x)* to make the colours different.
3. Using the **Intersect Two Objects** tool  , find the points where the curves intersect. The two intersection points *A* and *B* will be created.
4. In the input bar, enter *integral[f(x),x(A),x(B)]*. The area between the curve and the *x*-axis is shaded between the points *A* and *B*. Three distinct areas are shaded, two of which are outside the area that you want to find (i.e. the area between the curves).

(Note: you can change the colour and opacity of the shading by right-clicking inside a shaded area and selecting **Object Properties**.)

5. In the input bar, enter *integral[g(x),x(A),x(B)]*. Again, the area between the curve and the *x*-axis is shaded between the points *A* and *B*, and three distinct areas are shaded, two of which are outside the area that you want to find.
6. Note which shaded areas of *f(x)* and *g(x)* are positive and which are negative. Considering the signs of these areas, what would happen if you took the areas of the upper curve *f(x)* and subtracted the areas of the lower curve *g(x)*? Will you be left with the area that you want?
7. Using the integral values from the **Algebra** view, determine the area between the two curves.
8. In the input bar, now enter *integral[(f(x)-g(x)),x(A),x(B)]*. Is this numerical value (shown in the **Algebra** view) the same as your calculated value? Does the actual area shaded look the same? Discuss.

EXERCISE 13.7 AREA BETWEEN TWO CURVES

1 Calculate the area of the region bounded by the line $y = 2x$ and the parabola $y = x^2$.

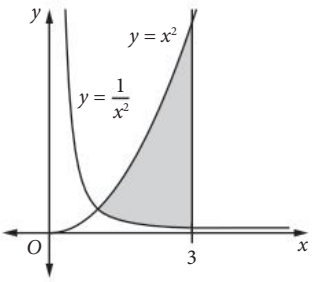
2 Calculate the area of the region bounded by the line $y = x + 1$ and the parabola $y = \frac{x^2}{4} - 2$.

3 The area of the region bounded by the line $y = x + 2$ and the parabola $y = x^2 - 4$ is given by:

A $\int_{-2}^{3}\left(6 + x - x^2\right)dx$ **B** $\int_{-3}^{2}\left(6 + x - x^2\right)dx$ **C** $\int_{-2}^{3}\left(x^2 - x - 6\right)dx$ **D** $\int_{-3}^{2}\left(x^2 - x - 6\right)dx$

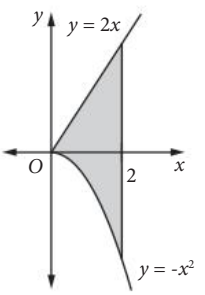
4 Calculate the area bounded by $f(x) = x^2$, $g(x) = \frac{1}{x^2}$, $x > 0$, the x-axis and the line $x = 3$.

5 Calculate the area bounded by $f(x) = x^2$, $g(x) = \frac{1}{x^2}$, $x > 0$, and the line $x = 3$.



6 Find the area enclosed by the line $y = 2x + 3$ and the parabola $y = x^2$.

7 Find the area enclosed by the line $y = 2x$, the parabola $y = -x^2$ and the line $x = 2$.



8 Calculate the area of the region enclosed by the graphs of the parabola $y = x^2 + 4$ and the line $y = 5$.

9 Calculate the area of the region enclosed by the graphs of the parabola $y = 2x^2 - 5x - 3$ and the line $y = 3x - 3$. Indicate whether each statement below is a correct or incorrect step in calculating this area.

(a) Intersection points: $(0, -3)$ and $(4, 9)$ **(b)** Area $= \int_0^4 \left(8x - 2x^2\right) dx$

(c) Area $= \int_{-3}^9 \left(8x - 2x^2\right) dx$ **(d)** Area $= 21\frac{1}{3}$ units^2

10 Calculate the area of the region enclosed by the graphs of the parabolas $f(x) = 6x^2 - 5x$ and $g(x) = 5x - 4x^2$.

11 Calculate the area of the region defined by the inequalities $y \geq x$ and $y \leq 3x - x^2$.

12 Calculate the area of the region defined by the inequalities $y \leq 6 - x$ and $y \geq x^2 - 5x + 6$.

13 A straight line through the origin cuts the parabola $y = 4x - x^2$ at the point where $x = 3$.

(a) Find the equation of this line.
(b) Calculate the area of the region bounded by the parabola and the straight line.
(c) Calculate the area of the region bounded by the parabola, the straight line and the x-axis.

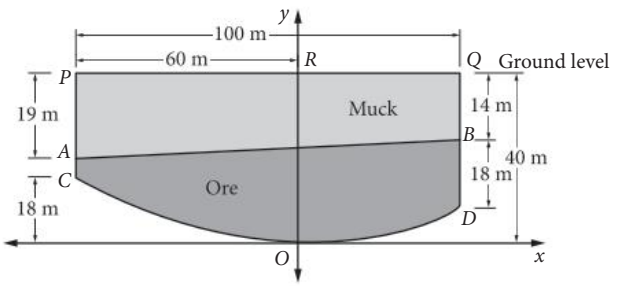
14 **(a)** Show that the area bounded by the curves $y = \sqrt{x}$, $y = \frac{1}{x}$ and $x = 3$ is given by $\int_1^3 \left(\sqrt{x} - \frac{1}{x}\right) dx$.

(b) Use the trapezoidal rule with two subintervals to find an approximation for this area correct to 2 decimal places.

15 **(a)** Show that the area between the curves $y = \frac{1}{x^2}$ and $y = \frac{-1}{x}$ for $1 \leq x \leq 4$ is given by $\int_1^4 \left(\frac{1}{x^2} + \frac{1}{x}\right) dx$.

(b) Use the trapezoidal rule with three subintervals to find an approximation for this area correct to 2 decimal places.

16 Part of a vertical section of an ore deposit takes the form shown in the diagram (not drawn to scale). The distances shown on the diagram have been indicated by vertical drilling at P and Q and by geological interpretation of the local rock formations. Taking the x-axis and y-axis as indicated, find:



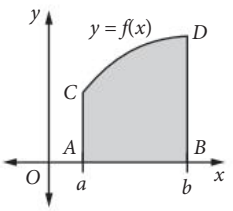
(a) the equation of the straight line AB
(b) the depth of muck at R
(c) the equation of the curve COD, assuming it is of the form $y = ax^n$
(d) the total area of the section $ABDC$ of the ore deposit.

13.8 AREA BOUNDED BY THE y-AXIS

The techniques used so far can also be used to find areas where one of the boundaries is the y-axis. We can effectively swap the roles of the x and y variables.

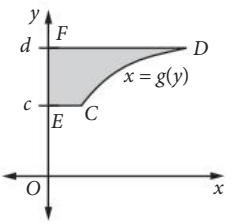
The area bounded by the curve $y = f(x)$, the x-axis and the ordinates at $x = a$ and $x = b$ is given by:

$$\text{Area} = \int_a^b f(x)\,dx \quad \textbf{OR} \quad \text{Area} = \int_a^b y\,dx$$



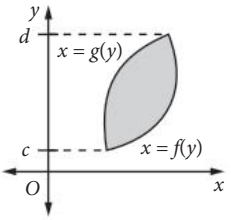
Similarly, the area bounded by the curve $x = g(y)$, the y-axis and the abscissae (i.e. horizontal lines) at $y = c$ and $y = d$ is given by:

$$\text{Area} = \int_c^d g(y)\,dy \quad \textbf{OR} \quad \text{Area} = \int_c^d x\,dy$$



For two curves $x = f(y)$ and $x = g(y)$ that intersect at $y = c$ and $y = d$, where $f(y) \geq g(y)$ over the interval $c \leq y \leq d$, the area bounded by the curves is given by:

$$\text{Area} = \int_c^d \big(f(y) - g(y)\big)\,dy$$

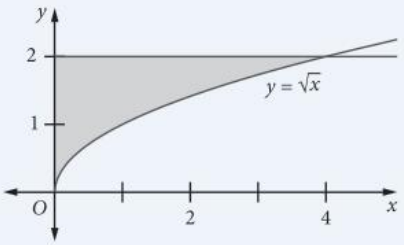


Example 24

Calculate the area of the region bounded by the curve $y = \sqrt{x}$, the y-axis and the line $y = 2$.

Solution

Sketch the region:



Write the equation as a function of y (i.e. make x the subject):

$$y = \sqrt{x} \rightarrow \sqrt{x} = y$$
$$x = y^2$$
$$\text{Area} = \int_0^2 x\,dy = \int_0^2 y^2\,dy$$
$$= \left[\frac{y^3}{3}\right]_0^2$$
$$= \tfrac{8}{3} - 0$$
$$= 2\tfrac{2}{3}$$

Area of the region is $2\frac{2}{3}$ units^2.

Example 25

Calculate the area of the region bounded by the curve $y = \sqrt{x}$, the line $y = 6 - x$ and the y-axis.

Solution

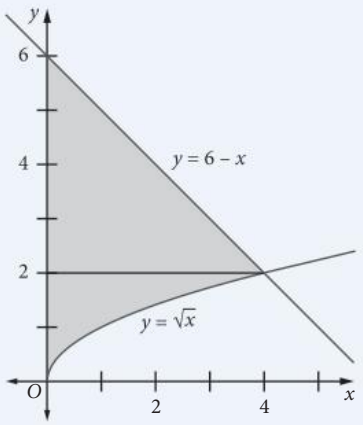
Find the point of intersection: $\sqrt{x} = 6 - x$

Square both sides: $x = 36 - 12x + x^2$

$$x^2 - 13x + 36 = 0$$
$$(x - 4)(x - 9) = 0$$
$$x = 4, 9$$

Only $x = 4$ is a root of $\sqrt{x} = 6 - x$: intersection point is $(4, 2)$.

Sketch the region:



Write the equations as functions of y:

$$y = \sqrt{x} \rightarrow y^2 = x$$
$$y = 6 - x \rightarrow x = 6 - y$$
$$\text{Area} = \int_0^2 y^2\,dy + \int_2^6 (6 - y)\,dy$$
$$= \left[\frac{y^3}{3}\right]_0^2 + \left[6y - \frac{y^2}{2}\right]_2^6$$
$$= \left(\tfrac{8}{3} - 0\right) + (36 - 18) - (12 - 2)$$
$$= 10\tfrac{2}{3}$$

Area of the region is $10\frac{2}{3}$ units^2.

EXERCISE 13.8 AREA BOUNDED BY THE y-AXIS

1 Calculate the area of the region bounded by the line $y = x + 1$, the y-axis and the line $y = 4$.

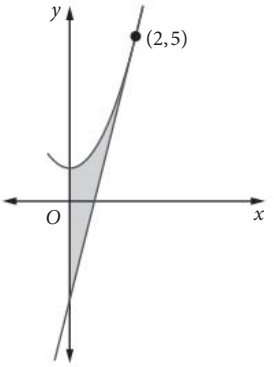
2 Calculate the area of the region bounded by the curve $y = \sqrt{x}$, the y-axis and the line $y = 3$.

3 The area of the region bounded by the curve $y = \sqrt[3]{x}$, the y-axis and the line $y = 2$ is given by:

A $\int_0^2 y\,dy$ **B** $\int_0^8 y\,dy$ **C** $\int_0^2 y^3\,dy$ **D** $\int_0^8 y^3\,dy$

4 Calculate the area of the region bounded by the curve $y = \frac{1}{x^2}$, the y-axis and the lines $y = 1$ and $y = 9$.

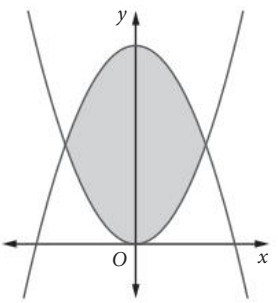
5 Calculate the area of the region bounded by the curve $y = x^2$ and the line $y = 4$.



6 **(a)** Show that the equation of the tangent to the parabola $y = x^2 + 1$ at the point where $x = 2$ is $y = 4x - 3$.
(b) Hence find the area enclosed by the parabola, the tangent and the y-axis.
(c) Find the area enclosed by the parabola, the tangent and the coordinate axes.

7 Calculate the area of the region bounded by the curve $x = \frac{1}{y}$, the y-axis and the lines $y = 1$ and $y = 4$ using the trapezoidal rule with three subintervals.

8 **(a)** Calculate the area of the region bounded by the parabolas $y = x^2$ and $y = 4 - x^2$.
(b) Calculate the area of the region bounded by the x-axis and the parabolas $y = x^2$ and $y = 4 - x^2$.



9 The area between the curves $x = \sqrt{y}$ and $y = 2x$ is calculated. Indicate whether each statement below is a correct or incorrect step in the calculation.

(a) The curves intersect at $(0, 0)$ and $(2, 4)$

(b) Area $= \int_0^4 \left(\sqrt{y} - \frac{y}{2}\right) dy$

(c) Area $= \int_0^2 \left(2x - x^2\right) dx$

(d) Area $= 1\frac{1}{3}$ units^2

10 Calculate the area of the region bounded by the lines $y = 4 - x$, $y = x$ and the y-axis.

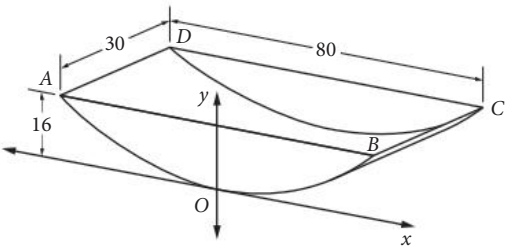
11 Calculate the area of the region bounded by the curve $x = \sqrt{y + 4}$, the y-axis and the line $y = 2$.

12 **(a)** Sketch the region defined by $y \leq 3x$, $y \geq x^2 - 4$ and $x \geq 0$.
(b) Calculate the area of this region.

13 **(a)** Sketch the region defined by $y \leq 3x$, $y \geq x^2 - 4$.
(b) Show that the area of this region is given by $\int_{-3}^{12} \left(\sqrt{y+4} - \frac{y}{3}\right) dy + 2\int_{-4}^{-3} \sqrt{y+4}\,dy$.
(c) Calculate the area of this region.
(d) What other integral could have been used to find this area? Use it to verify your answer for **(c)**.

14 Use the trapezoidal rule with five subintervals to find the area of the region bounded by $x = \sqrt{16 - y^2}$, the y-axis and the lines $y = -2$ and $y = 3$. Give your answer correct to 2 decimal places.

15 The diagram shows a container, open at the top, whose dimensions in cm are as indicated. The cross-section AOB is a parabola whose vertex is at the lowest point O. $ABCD$ is a horizontal rectangle.

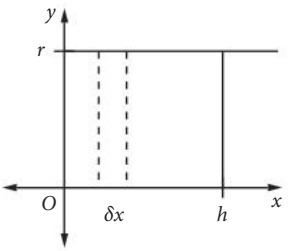


(a) Using the axes as shown, show that the equation of the curve AOB is $y = \frac{x^2}{100}$.

(b) Find, by integration, the area of the end AOB and hence the volume of the container.

13.9 VOLUME OF SOLIDS OF REVOLUTION

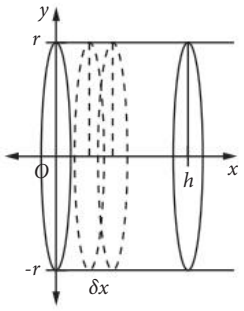
We have seen that the area of a region bounded by a line $y = r$, the x-axis and the ordinates $x = 0$ and $x = h$ can be found by adding up the areas of all the rectangles of width δx and height r between $x = 0$ and $x = h$, as δx becomes vanishingly small: $A = \lim\limits_{\delta x \to 0} \sum\limits_{0}^{h} f(x)\delta x$.



This area is given by the definite integral $A = \int_0^h r\,dx$, which is $A = \int_0^h r\,dx = [rx]_0^h = rh$.

You should recognise this as the area of a rectangle of sides r and h.

Consider what happens when the area bounded by $y = r$, the x-axis and the ordinates $x = 0$ and $x = h$ is rotated about the x-axis to form a solid of revolution, as shown in the following diagram. The solid of revolution formed is a cylinder of radius r and height h.

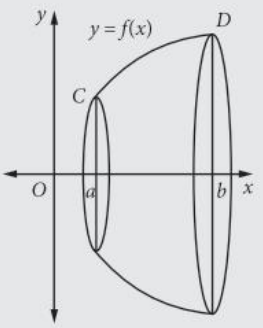


The rectangles of side r and width δx have become circular disks of radius r and thickness δx. The volume of this disk is given by $\Delta V = \pi\big(f(x)\big)^2\,\delta x$. Adding all the disks as δx gets smaller gives $V = \lim\limits_{\delta x \to 0} \sum\limits_{0}^{h} \pi\big(f(x)\big)^2\,\delta x$, which is given by the definite integral $V = \pi \int_0^h r^2\,dx$.

Thus the volume is $V = \pi \int_0^h r^2\,dx = \pi\big[r^2 x\big]_0^h = \pi r^2 h$, which you should recognise as the volume of a cylinder of radius r and height h.

When the arc CD of the curve $y = f(x)$ on the interval $a \le x \le b$ is rotated about the x-axis, the volume of the solid of revolution formed is given by:

$$V = \pi \int_a^b \big(f(x)\big)^2\,dx \qquad \textbf{OR} \qquad V = \pi \int_a^b y^2\,dx$$

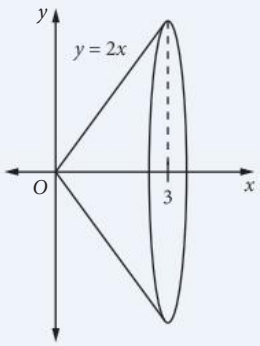


Example 26

Calculate the volume of the solid formed when the portion of the line $y = 2x$ between $x = 0$ and $x = 3$ is rotated about the x-axis. What is the name of the kind of solid formed?

Solution

Draw a diagram:



$$\text{Volume} = \pi \int_a^b y^2 \, dx$$

$$= \pi \int_0^3 (2x)^2 \, dx$$

$$= 4\pi \int_0^3 x^2 \, dx$$

$$= 4\pi \left[\frac{x^3}{3} \right]_0^3$$

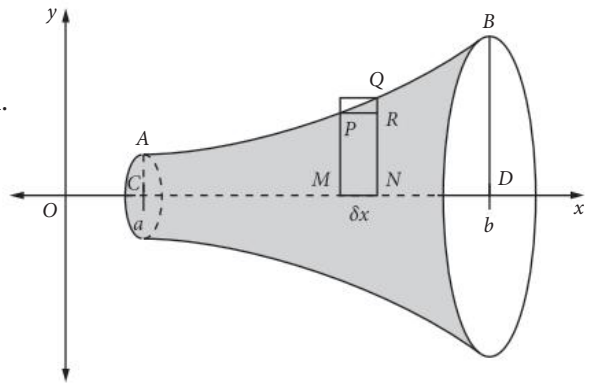
$$= 4\pi (9 - 0)$$

$$= 36\pi \text{ units}^3$$

The solid is a right circular cone of base radius 6 and height 3.

Volume of solids of revolution—formal development

Consider a continuous function f in the interval $a \leq x \leq b$. If the plane section $ABDC$ is rotated about the x-axis then a solid is generated with circular vertical cross-sections, as shown in the following diagram. This solid is called a **solid of revolution**.



$P(x, y)$ is a point on the curve $y = f(x)$ and $Q(x + \delta x, y + \delta y)$ is a point close to P. The ordinate PM describes a circle of area πy^2 and QN describes a circle of area $\pi (y + \delta y)^2$.

The typical lower rectangle $PRNM$ describes a cylinder of volume $\pi y^2 \, \delta x$ and the typical upper rectangle describes a cylinder of volume $\pi (y + \delta y)^2 \, \delta x$. If a typical layer $PQNM$ describes a solid of volume δV, then:

$$\pi y^2 \, \delta x < \delta V < \pi (y + \delta y)^2 \, \delta x$$

Thus:
$$\sum_a^b \pi y^2 \, \delta x < V < \sum_a^b \pi (y + \delta y)^2 \, \delta x$$

As $\delta x \to 0$:
$$V = \lim_{\delta x \to 0} \sum_a^b \pi y^2 \, \delta x$$

$$= \int_a^b \pi y^2 \, dx$$

$$= \pi \int_a^b y^2 \, dx$$

Hence, volume of a solid of revolution:

$$V = \pi \int_a^b y^2 \, dx \qquad \text{where } y = f(x)$$

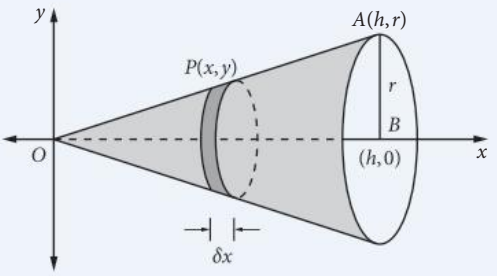
Example 27

Find the volume of a right circular cone of height h and base radius r.

Solution

The cone can be considered as a solid of revolution generated by rotating the right-angled triangle OAB about the x-axis.

The equation of OA is $y = \frac{rx}{h}$.



$$V = \pi \int_a^b y^2 \, dx$$

$$V = \pi \int_0^h \frac{r^2 x^2}{h^2} dx = \frac{\pi r^2}{h^2} \int_0^h x^2 \, dx$$

$$= \frac{\pi r^2}{h^2} \left[\frac{x^3}{3} \right]_0^h$$

$$= \frac{\pi r^2}{h^2} \times \frac{h^3}{3}$$

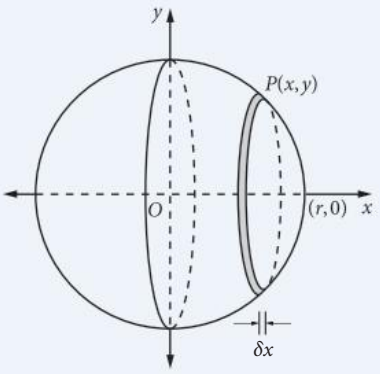
$$= \tfrac{1}{3} \pi r^2 h$$

Example 28

Find the volume of a sphere of radius r.

Solution

The volume of a sphere can be considered as the volume generated by rotating the semicircle defined by $y = \sqrt{r^2 - x^2}$, $-r \le x \le r$, about the x-axis.



Hence:

$$V = \pi \int_{-r}^{r} y^2 \, dx \qquad \text{where } y = \sqrt{r^2 - x^2}$$

$$= \pi \int_{-r}^{r} \left(r^2 - x^2 \right) dx \qquad \text{because } y^2 = r^2 - x^2$$

$$= \pi \left[r^2 x - \frac{x^3}{3} \right]_{-r}^{r}$$

$$= \pi \left(\left(r^3 - \frac{r^3}{3} \right) - \left(-r^3 + \frac{r^3}{3} \right) \right)$$

$$= \tfrac{4}{3} \pi r^3$$

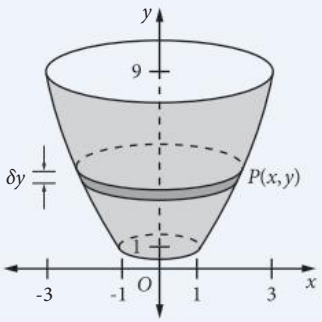
You have just proved the formula for the volume of the sphere—a formula that you have used for many years. The formula for the area of a circle $A = \pi r^2$ can also be proved using calculus, but for this you will need integration techniques that are not covered until the Extension 1 course.

Example 29

The part of the parabola $y = x^2$ between $x = 1$ and $x = 3$ is rotated about the y-axis. Calculate the volume generated.

Solution

$y = x^2$: $x = 1, y = 1; x = 3, y = 9$ $\quad V = \pi \int_1^9 x^2\,dy$ where $x^2 = y$

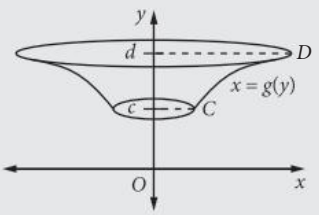


$$V = \pi \int_1^9 y\,dy$$
$$= \pi \left[\frac{y^2}{2}\right]_1^9$$
$$= \pi\left(\tfrac{81}{2} - \tfrac{1}{2}\right)$$
$$= 40\pi \text{ units}^3$$

Rotating about the *y*-axis

When the arc CD of the curve $x = g(y)$ on the interval $c \le y \le d$ is rotated about the y-axis, the volume of the solid of revolution formed is given by:

$$V = \pi \int_c^d \big(g(y)\big)^2\,dy \quad \textbf{OR} \quad V = \pi \int_c^d x^2\,dy$$

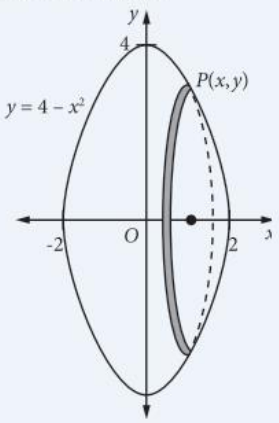


Example 30

Find the volume of the solid formed when the area bounded by the parabola $y = 4 - x^2$ and the x-axis is rotated about: **(a)** the x-axis **(b)** the y-axis.

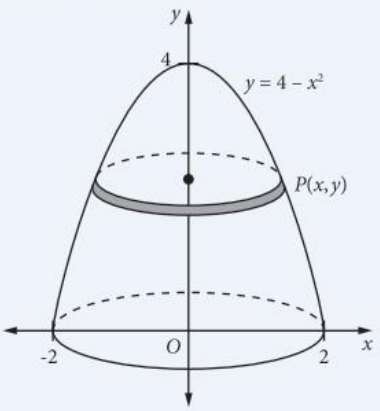
Solution

(a) Rotate about x-axis:



$$V = \pi \int_{-2}^{2} y^2\,dx \quad \text{where } y = 4 - x^2$$
$$= \pi \int_{-2}^{2} \left(16 - 8x^2 + x^4\right) dx$$
$$= \pi \left[16x - \frac{8x^3}{3} + \frac{x^5}{5}\right]_{-2}^{2}$$
$$= \pi\left(\left(32 - \tfrac{64}{3} + \tfrac{32}{5}\right) - \left(-32 + \tfrac{64}{3} - \tfrac{32}{5}\right)\right)$$
$$= \frac{512\pi}{15} \text{ units}^3$$

(b) Rotate about y-axis:



$$V = \pi \int_0^4 x^2\,dx \quad \text{where } x^2 = 4 - y$$
$$= \pi \int_0^4 (4 - y)\,dx$$
$$= \pi \left[4y - \frac{y^2}{2}\right]_0^4$$
$$= \pi\big((16 - 8) - 0\big)$$
$$= 8\pi$$

EXERCISE 13.9 VOLUME OF SOLIDS OF REVOLUTION

1 Find the volume of the solid of revolution formed by rotating about the x-axis the arc of the parabola $y = x^2$ between $x = 0$ and $x = 3$.

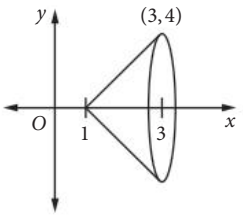
2 Find the volume of the solid of revolution formed by rotating about the x-axis the line $y = 2x$ between $x = 0$ and $x = 4$.

3 A cone is formed by rotating about the x-axis a segment of the line $y = 3x$ between $x = 0$ and $x = 4$. The definite integral used to calculate the volume of this solid is:

A $\int_0^4 9x^2\,dx$ **B** $\pi\int_0^4 3x^2\,dx$ **C** $\int_0^4 3x^2\,dx$ **D** $\pi\int_0^4 9x^2\,dx$

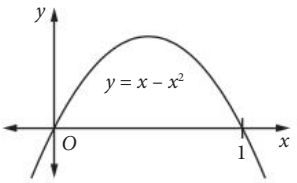
4 **(a)** Find the equation of the line passing through the points $(1, 0)$ and $(3, 4)$.

(b) A cone is formed by rotating about the x-axis the segment of the line joining the points $(1, 0)$ and $(3, 4)$. Calculate the volume of the cone.



5 The semicircle $y = \sqrt{9 - x^2}$ is rotated about the x-axis. Calculate the volume of the sphere generated.

6 The region bounded by the parabola $y = x - x^2$ and the x-axis is rotated about the x-axis. Find the volume of the solid formed.

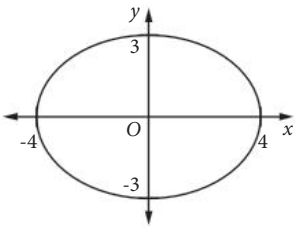


7 Find the volume of the solid formed when the region bounded by the parabola $y = 1 - x^2$ and the x-axis is rotated about: **(a)** the x-axis **(b)** the y-axis.

8 The region bounded by the parabola $y = (x - 2)^2$ and the coordinate axes is rotated about the x-axis. Find the volume of the solid generated.

9 Find the volume of the solid generated when the segment of the line joining the points $(0, 3)$ and $(6, 0)$ is rotated about: **(a)** the x-axis **(b)** the y-axis.

10 A rugby ball has a volume approximately the same as the volume generated by rotating the ellipse $9x^2 + 16y^2 = 144$ about the x-axis. Find its volume.



11 Find the volume of the solid formed when the region bounded by the parabola $y = 9 - x^2$ and the coordinate axes is rotated about: **(a)** the x-axis **(b)** the y-axis.

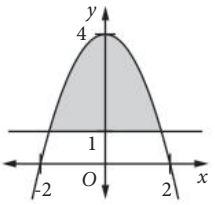
12 **(a)** Find the equation of the line through the points $(3, 0)$ and $(4, 10)$.

(b) A drinking glass has the shape of a truncated cone. The internal radii of the base and the top are 3 cm and 4 cm respectively and its depth is 10 cm. If the base of the glass sits on the x-axis, use integration to find its capacity.

(c) If the glass is filled with water to a depth of 5 cm, find the volume of water in the glass.

13 A hemispherical bowl of radius a units is filled with water to a depth of $\frac{a}{2}$ units. Use integration to find the volume of the water.

14 Find the volume of the solid formed when the region bounded by the parabola $y = 4 - x^2$ and the line $y = 1$ is rotated about the y-axis.



15 A solid is formed by rotating about the y-axis the region bounded by the parabola $y = x^2 - 2$ and the x-axis. Indicate whether each statement below is a correct or incorrect step in calculating the volume of the solid formed.

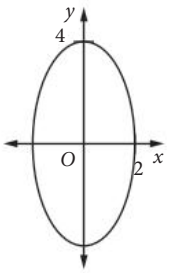
(a) $V = \pi \int_0^{\sqrt{2}} (x^2 - 2)^2 \, dx$ **(b)** $V = \pi \int_{-2}^{0} (y + 2) \, dy$ **(c)** $V = \pi \left[\frac{y^2}{2} + 2y \right]_{-2}^{0}$ **(d)** $V = 2\pi$

16 Use integration to find the volume of the sphere generated when the circle $x^2 + y^2 = 16$ is rotated about the x-axis.

17 The area under the curve $y = 2x\sqrt{1 - x^2}$ between $x = 0$ and $x = 1$ is rotated about the x-axis. Using the trapezoidal rule with four subintervals, find an approximation for the volume of the solid correct to 2 decimal places.

18 Find the volume of the solid formed when the ellipse $4x^2 + y^2 = 16$ is rotated about:

(a) the x-axis **(b)** the y-axis.

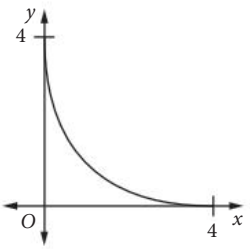


19 A region is bounded by the curve $\sqrt{x} + \sqrt{y} = 2$ and the coordinate axes.

(a) Calculate the area of the region.

(b) Calculate the volume of the solid generated when the region is rotated about the x-axis.

(c) Calculate the volume of the solid generated when the region is rotated about the y-axis.



20 The region bounded by the curve $xy = 1$, the x-axis and the lines $x = 1$ and $x = a$, for $a > 1$, is rotated about the x-axis. Find V, the volume generated. Hence find $\lim_{a \to \infty} V$.

21 The area bounded by the parabola $y = 2x - x^2$, the y-axis and the line $y = 1$ is rotated about the x-axis. Find the volume generated.

22 Find the volume of the solid generated by rotating the region bounded by the parabola $y = 1 - x^2$ and the lines $x = 1$, $y = 1$ about: **(a)** the x-axis **(b)** the y-axis.

23 Find the volume of the cone formed by rotating the segment of the line $x + 2y = 4$ that is cut off by the axes about: **(a)** the x-axis **(b)** the y-axis.

24 Use Simpson's rule with five function values to estimate the volume of the solid formed by rotating the curve $y = \frac{1}{1 + x^2}$ about the x-axis between $x = -2$ and $x = 2$.

13.10 AVERAGE VALUE OF A FUNCTION—AN APPLICATION OF INTEGRATION

(**Note**: this section is not part of the syllabus; it is given as a useful application of integration.)

We are used to finding the average of a set of numbers by adding up all the numbers and dividing by how many there are. Hence the average of the n numbers $y_1, y_2, \ldots y_n$ is given by $y_{\text{ave}} = \frac{y_1 + y_2 + \ldots + y_n}{n}$.

Similarly, to find the average value of a function $f(x)$ over the interval $a \le x \le b$, we can divide the interval $a \le x \le b$ into n equal subintervals of length $\Delta x = \frac{b-a}{n}$. Take the points $x_1, x_2, \ldots x_n$ in successive intervals and calculate the average of the numbers $f(x_1), f(x_2), \ldots, f(x_n)$: Average $= \frac{f(x_1)+f(x_2)+\ldots+f(x_n)}{n}$.

But $\Delta x = \frac{b-a}{n}$ so $n = \frac{b-a}{\Delta x}$, so we have:

$$\text{Average} = \frac{f(x_1)+f(x_2)+\ldots+f(x_n)}{n} = \frac{f(x_1)+f(x_2)+\ldots+f(x_n)}{\frac{b-a}{\Delta x}}$$

$$= \frac{f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x}{b-a}$$

$$= \frac{1}{b-a}\sum_{i=1}^{n} f(x_i)\Delta x$$

As n increases, Δx decreases and so this expression can be written as a definite integral:

$$\lim_{n\to\infty} \frac{1}{b-a}\sum_{i=1}^{n} f(x_i)\Delta x = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Thus we define the **average value** of f on the interval $a \le x \le b$ as $f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx$.

Example 31

Find the average value of each function over the interval $1 \le x \le 3$:

(a) $f(x) = x + 1$ **(b)** $g(x) = x^2 + 1$ **(c)** $h(x) = x^3 + 1$

Solution

For the integrals we will have $a = 1$, $b = 3$:

(a) $f(x) = x + 1$

$$f_{\text{ave}} = \frac{1}{3-1}\int_1^3 (x+1)\,dx$$
$$= \frac{1}{2}\left[\frac{x^2}{2} + x\right]_1^3$$
$$= \tfrac{1}{2}\left(\tfrac{9}{2} + 3 - \left(\tfrac{1}{2} + 1\right)\right)$$
$$= 3$$

(b) $g(x) = x^2 + 1$

$$g_{\text{ave}} = \frac{1}{3-1}\int_1^3 (x^2+1)\,dx$$
$$= \frac{1}{2}\left[\frac{x^3}{3} + x\right]_1^3$$
$$= \tfrac{1}{2}\left(\tfrac{27}{3} + 3 - \left(\tfrac{1}{3} + 1\right)\right)$$
$$= 5\tfrac{1}{3}$$

(c) $h(x) = x^3 + 1$

$$h_{\text{ave}} = \frac{1}{3-1}\int_1^3 (x^3+1)\,dx$$
$$= \frac{1}{2}\left[\frac{x^4}{4} + x\right]_1^3$$
$$= \tfrac{1}{2}\left(\tfrac{81}{4} + 3 - \left(\tfrac{1}{4} + 1\right)\right)$$
$$= 11$$

EXERCISE 13.10 AVERAGE VALUE OF A FUNCTION—AN APPLICATION OF INTEGRATION

1 Find the average value of f on the interval $0 \le x \le 2$ for $f(x) = x^2 - x$.

2 Find the average value of f on the interval $-2 \le x \le 2$ for $f(x) = x^4$.

3 Find the average value of f on the interval $-2 \le x \le 0$ for $f(x) = x^3$.

4 For $f(x) = x^2 - 2x + 1$, the average value of f on the interval $-1 \le x \le 3$ is:

A $-\frac{1}{3}$ **B** $\frac{1}{3}$ **C** $\frac{2}{3}$ **D** $\frac{4}{3}$

5 Find the average value of f on the interval $0 \le x \le 9$ for $f(x) = \sqrt{x}$.

6 The linear density of a rod 7 m long is given by $d(x) = \frac{12}{\sqrt{x+2}}$ kg/m, where x is measured in metres from one end of the rod. Find the average density of the rod.

7 Find the values of b for which the average value of $f(x) = 1 - 6x + 3x^2$ on the interval $1 \leq x \leq b$ is equal to 2.

13.11 INDEFINITE INTEGRALS

The notation of the definite integral and the process of finding a function's primitive are combined to give the **indefinite integral**, $\int f(x)\,dx$. This uses the integral sign $\int$ as a way of saying 'find the primitive of $f(x)$'. (Remember that we learnt how to find primitives of a function in Chapter 12.) The indefinite integral gives a function to represent all possible values of the primitive by adding C, the **constant of integration**:

If $F(x)$ is a primitive of $f(x)$, then $\int f(x)\,dx = F(x) + C$, where C is the constant of integration.

This is called the indefinite integral of $f(x)$ and represents all the primitives of $f(x)$. If further information is given, then a value for C may be calculated.

Note that the indefinite integral is a function of x, but the definite integral has a definite numerical value.

Example 32

Find: (a) $\int \left(x^3 + 3x^2 - 2x + 1\right) dx$ (b) $\int \left(\sqrt{x} - 2\right) dx$

Solution

(a) $\int \left(x^3 + 3x^2 - 2x + 1\right) dx = \frac{x^4}{4} + x^3 - x^2 + x + C$

(b) $\int \left(\sqrt{x} - 2\right) dx = \frac{2}{3}x^{\frac{3}{2}} - 2x + C$ or $\frac{2x\sqrt{x}}{3} - 2x + C$

Example 33

If $\frac{dy}{dx} = 1 + x - x^2$, find the equation of the curve y that passes through the point $(6, -40)$.

Solution

$\frac{dy}{dx} = 1 + x - x^2$: $\quad y = \int \left(1 + x - x^2\right) dx$

$$y = x + \frac{x^2}{2} - \frac{x^3}{3} + C$$

Point $(6, -40)$ is on the curve: $\quad -40 = 6 + 18 - 72 + C$

$$C = 8$$

The equation of the curve is $y = x + \frac{x^2}{2} - \frac{x^3}{3} + 8$.

EXERCISE 13.11 INDEFINITE INTEGRALS

1 Find: **(a)** $\int x\,dx$ **(b)** $\int (x^2+x+1)\,dx$ **(c)** $\int (3-x^2)\,dx$

(d) $\int (6x^5-4x^3+2x)\,dx$ **(e)** $\int dx$ **(f)** $\int x^n\,dx$

2 Find: **(a)** $\int \sqrt{x}\,dx$ **(b)** $\int \frac{1}{x^2}\,dx$ **(c)** $\int (1+\sqrt{x}+x)\,dx$

(d) $\int \left(x+\frac{1}{x^2}\right)dx$ **(e)** $\int \left(x+\frac{1}{x}\right)^2 dx$ **(f)** $\int (1-\sqrt{x})^2\,dx$

3 $\int (1+2x+3x^2)\,dx$ is equal to:

A $x+x^2+\frac{x^3}{3}+C$ **B** $x+\frac{x^2}{2}+\frac{x^3}{3}+C$ **C** $x+x^2+x^3+C$ **D** $2+6x+C$

4 If $\frac{dy}{dx}=1+x+3x^2$, find the equation of the curve that passes through the point $(2,6)$.

5 If $\frac{dy}{dx}=1+\sqrt{x}$, find the equation of the curve that passes through the point $(4,10)$.

CHAPTER REVIEW 13

1 Evaluate the following:

(a) $\int_{-1}^{3}(x^3-3x^2+2x-5)\,dx$ **(b)** $\int_{-2}^{2}(x^2-2x)^2\,dx$ **(c)** $\int_{-3}^{1}(5-2x)^3\,dx$

2 Find the area of the region bounded by the parabola $y=4x-x^2$ and the x-axis.

3 Evaluate the following:

(a) $\int_{-1}^{2}3x(2-x)\,dx$ **(b)** $4\int_{-3}^{-1}x(x+1)^2\,dx$ **(c)** $\int_{-2}^{4}(x^3-2)\,dx$

4 Find the equation of the tangent to the parabola $y=x^2$ at the point where $x=2$. By integration, find the area of the region bounded by the parabola, the tangent and the x-axis.

5 Find the area of the region defined by the inequalities $y\geq -5$ and $y\leq 4x-x^2$.

6 Calculate the area of the region bounded by the curve $y=2x^2(4-x)$ and the x-axis.

7 For the curve with equation $y=x+\frac{8}{x^2}$, find the values of x for which:

(a) $y<0$ **(b)** the gradient is negative.

(c) Calculate the area enclosed by the curve, the x-axis, and the lines $x=2$ and $x=4$.

8 Find the function f, defined for positive x only, for which $f'(x)=\left(\frac{1}{2x}+x\right)^2$ and $f(2)=6$.

9 Calculate:

(a) the area between the curve $y=x^2(3-x)$ and the x-axis

(b) the area bounded by the curve $y=x^2(3-x)$ and the lines $y=3-x$, $x=1.5$ and $x=2$.

10 Find the average value of the function $f(x)=1+\sqrt{x}$ on the interval $1\leq x\leq 9$.

11 By integration, find the volume of the solid of revolution formed from the region bounded by:

(a) the circle $x^2+y^2=1$, rotated about the x-axis

(b) the line $y=x+3$ between $x=0$ and $x=2$, rotated about the x-axis

(c) the parabola $y=x^2+3$ between $y=4$, $y=12$ and the y-axis, rotated about the y-axis.

12 **(a)** Find the values of x for which $4x - x^2 > 0$.

(b) Find the area under the graph of $y = 4x - x^2$ between $x = 1$ and $x = 2$.

(c) Find the angle between the x-axis and the tangent to the curve $y = 4x - x^2$ at $x = 1$.

13 **(a)** Sketch the graph of $y = 3x^2 - x^3$.

(b) The tangent to the curve $y = 3x^2 - x^3$ at the point A, where $x = 2$, cuts the graph at another point B. Show that the coordinates of B are $(-1, 4)$.

(c) Calculate the area enclosed by the graph and the line AB.

14 **(a)** Use the trapezoidal rule with four subintervals to find an approximation for $\int_{-2}^{2} \frac{dx}{1+x^2}$.

(b) Use Simpson's rule with five function values to find an approximation for $\int_{-2}^{2} \frac{dx}{1+x^2}$.

15 A speleologist is in a cave with a flat, circular floor. She walks across the diameter of the floor and measures the height of the cave ceiling every four metres, as recorded in the following table:

Distance from edge, x m	0	4	8	12	16	20	24
Height, y m	0	1	3	5	4	2.5	0

(a) Use the trapezoidal rule and all the values in the table to find an approximation for the area of the cave's vertical cross-section to the nearest m^2. Repeat using Simpson's rule.

(b) Considering the diameter of the cave to be the x-axis (for $-12 \leq x \leq 12$), and considering the vertical at 12 m from the edge to be the y-axis, discuss how you might use the trapezoidal rule to estimate the volume of the cave.

CHAPTER 14
EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Outcomes covered:

H1 seeks to apply mathematical techniques to problems in a wide range of practical contexts
H3 manipulates algebraic expressions involving logarithmic and exponential functions
H5 applies appropriate techniques from the study of calculus, geometry, probability, trigonometry and series to solve problems
H6 uses the derivative to determine the features of the graph of a function
H7 uses the features of a graph to deduce information about the derivative
H8 uses techniques of integration to calculate areas and volumes
H9 communicates using mathematical language, notation, diagrams and graphs

14.1 INDEX LAWS WITH INTEGERS AS INDICES

You have frequently used the following index laws, where a and b are real numbers and m and n are positive integers. These rules can now be extended to include m and n as any real numbers.

Index laws

$a^m \times a^n = a^{m+n}$ $\quad (ab)^n = a^n \times b^n$ $\quad a^{-n} = \frac{1}{a^n}$ and hence $a^{-1} = \frac{1}{a}$

$\frac{a^m}{a^n} = a^{m-n}$ $\quad a^1 = a$ $\quad a^{\frac{1}{n}} = \sqrt[n]{a}$

$\left(a^m\right)^n = a^{mn}$ $\quad a^0 = 1$ $\quad a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$

Example 1

Simplify, writing the answers with positive indices:

(a) 4^{-2} (b) $\left(\frac{1}{2}\right)^{-3}$ (c) $\frac{3^{-2} \times 6^3 \times 12^{-2}}{9^{-3} \times 2^{-1}}$

Solution

(a) 4^{-2}
$= \frac{1}{4^2}$
$= \frac{1}{\left(2^2\right)^2}$
$= \frac{1}{2^4}$

(b) $\left(\frac{1}{2}\right)^{-3}$
$= \frac{1^{-3}}{2^{-3}}$
$= 2^3$

(c) $\frac{3^{-2} \times 6^3 \times 12^{-2}}{9^{-3} \times 2^{-1}}$
$= \frac{3^{-2} \times 2^3 \times 3^3 \times 3^{-2} \times 4^{-2}}{3^{-6} \times 2^{-1}}$
$= \frac{3^{-1} \times 2^3 \times 2^{-4}}{3^{-6} \times 2^{-1}}$
$= 3^5$

Example 2

Simplify $\dfrac{x^{-1}+1}{x^{-1}-x}$, writing the answer with positive indices.

Solution

First simplify the numerator and denominator:

$$x^{-1}+1=\frac{1}{x}+1=\frac{1+x}{x}$$

$$x^{-1}-x=\frac{1}{x}-x=\frac{1-x^2}{x}$$

$$\therefore \frac{x^{-1}+1}{x^{-1}-x}=\frac{\frac{1+x}{x}}{\frac{1-x^2}{x}}=\frac{1+x}{x}\times\frac{x}{1-x^2}$$

$$=\frac{1+x}{(1+x)(1-x)}, \quad x\neq 0$$

$$=\frac{1}{1-x}, \quad x\neq -1, 0$$

EXERCISE 14.1 INDEX LAWS WITH INTEGERS AS INDICES

1 Simplify, writing your answer with positive indices.

(a) $2^3\times 4^2\times 8^2$ **(b)** $a^3b^{-2}\times(a^2b^2)^4$ **(c)** $m^2p^3\times(m^3n^2)^3\times(p^{-1})^2$ **(d)** $2^n\times 2^{2n}\times 2^{3n}$

(e) $\dfrac{\left(x^2y^3\right)^4\times(xy)^{-2}}{xy}$ **(f)** $\dfrac{\left(2m^2n\right)^3}{\left(mn^3\right)^2\times\left(4m^2\right)^2}$ **(g)** $\dfrac{5x^5y^2\times 3\left(xy^3\right)^2}{15x^{-2}y}$ **(h)** $\dfrac{\left(a^2b\right)^2\times(ab)^4}{\left(a^2b\right)^3}$

2 $\dfrac{(-2xy)^2\times 2\left(x^2y^{-1}\right)^3}{8(xy)^{-3}}$ simplifies to:

A $\dfrac{x^{11}y^2}{2}$ **B** x^8y^2 **C** $-x^{11}y^2$ **D** $x^{11}y^2$

3 Simplify:

(a) $m^2n^2p^{-2}\times(mnp^2)^{-3}$ **(b)** $\dfrac{p^2q^3r^{-3}}{p^3q^{-1}r}$ **(c)** $\dfrac{x^2yz^{-2}\times 2\left(x^2y^{-1}z\right)^2}{xyz}$ **(d)** $2^3\times(2^n)^2\times 2^{-n}$

4 Write the following as negative powers of 2.

(a) $\frac{1}{4}$ **(b)** $\frac{1}{16}$ **(c)** $\frac{1}{32}$ **(d)** 0.125 **(e)** $\frac{1}{64}$ **(f)** $\frac{1}{128}$ **(g)** 0.25 **(h)** 8^{-3}

5 Write the following as powers of 10.

(a) 100 **(b)** 10 **(c)** 1 **(d)** 0.1 **(e)** 0.01 **(f)** 0.001 **(g)** $\frac{1}{1000}$ **(h)** $\frac{1}{100\,000}$

6 Simplify, writing your answer with positive indices.

(a) $\dfrac{3^{2n}\times 25^{2n-1}}{15^{n-1}}$ **(b)** $(x^{-1}+y^{-1})(x^{-1}-y^{-1})$ **(c)** $\dfrac{2^n\times 4^{n+1}}{8^{n-2}}$ **(d)** $(x^{-2}+x^{-1})^2$

(e) $\dfrac{x-5+6x^{-1}}{1-2x^{-1}}$ **(f)** $\dfrac{x-4x^{-1}}{1+2x^{-1}}$ **(g)** $4^{-2}\times 6^3\times 8^4\times 12^{-2}$ **(h)** $\dfrac{15^{n+1}\times 25\times 5^{3n-4}}{9^{n-1}\times 25^{n-2}}$

7 For n as a positive integer, decide whether each statement is correct or incorrect.

(a) $(-1)^n=1$ when n is even **(b)** $(-1)^n=-1$ when n is odd

(c) $(-1)^n=-1$ when n is even **(d)** $(-1)^n=1$ when n is odd

8 Expand and simplify the following, expressing the results with positive indices.

(a) $(a^{-1}+b)(a^{-1}-b)$ (b) $(x^{-1}+y)(x+y^{-1})$ (c) $(x^{-2}+y^{-2})(x^{-2}-y^{-2})$

(d) $(a^2-2b^{-1})(a^{-2}-b)$ (e) $\dfrac{a^{-1}+b^{-1}}{a+b}$ (f) $\dfrac{y^{-1}+y}{1+y^2}$

14.2 INDEX LAWS WITH FRACTIONAL INDICES

Example 3

Simplify:

(a) $32^{\frac{2}{5}}$ (b) $125^{-\frac{2}{3}}$ (c) $x^{\frac{5}{2}} \times x^{-\frac{3}{4}}$

Solution

(a) $32^{\frac{2}{5}} = \left(2^5\right)^{\frac{2}{5}} = 2^2 = 4$ **OR** $32^{\frac{2}{5}} = \left(\sqrt[5]{32}\right)^2 = 2^2 = 4$

(b) $125^{-\frac{2}{3}} = \left(5^3\right)^{-\frac{2}{3}} = 5^{-2} = \dfrac{1}{5^2} = \dfrac{1}{25}$ **OR** $125^{-\frac{2}{3}} = \left(\sqrt[3]{125}\right)^{-2} = 5^{-2} = \dfrac{1}{25}$

(c) $x^{\frac{5}{2}} \times x^{-\frac{3}{4}} = x^{\frac{5}{2}-\frac{3}{4}} = x^{\frac{7}{4}}$

Example 4

Simplify:

(a) $\dfrac{5^{\frac{1}{4}} \times \sqrt{10} \times \sqrt[4]{2}}{20^{\frac{3}{4}}}$ (b) $\dfrac{3^{n-2} \times 9^{n+1}}{81^{n-1}}$ (c) $\left(x^{\frac{1}{2}} - x^{-\frac{1}{2}}\right)^2$

Solution

(a) $\dfrac{5^{\frac{1}{4}} \times \sqrt{10} \times \sqrt[4]{2}}{20^{\frac{3}{4}}} = \dfrac{5^{\frac{1}{4}} \times (2\times 5)^{\frac{1}{2}} \times 2^{\frac{1}{4}}}{\left(2^2 \times 5\right)^{\frac{3}{4}}}$

$= \dfrac{5^{\frac{1}{4}} \times 2^{\frac{1}{2}} \times 5^{\frac{1}{2}} \times 2^{\frac{1}{4}}}{2^{\frac{3}{2}} \times 5^{\frac{3}{4}}}$

$= \dfrac{1}{2^{\frac{3}{4}}}$

(b) $\dfrac{3^{n-2} \times 9^{n+1}}{81^{n-1}} = \dfrac{3^{n-2} \times \left(3^2\right)^{n+1}}{\left(3^4\right)^{n-1}}$

$= \dfrac{3^{n-2} \times 3^{2n+3}}{3^{4n-4}}$

$= \dfrac{3^{3n+1}}{3^{4n-4}}$

$= 3^{4-n}$ or $\dfrac{1}{3^{n-4}}$

(c) $\left(x^{\frac{1}{2}} - x^{-\frac{1}{2}}\right)^2 = \left(x^{\frac{1}{2}}\right)^2 - 2x^{\frac{1}{2}}x^{-\frac{1}{2}} + \left(x^{-\frac{1}{2}}\right)^2$

$= x - 2 + x^{-1}$

$= x - 2 + \dfrac{1}{x}$

OR $\left(x^{\frac{1}{2}} - x^{-\frac{1}{2}}\right)^2 = \left(\sqrt{x} - \dfrac{1}{\sqrt{x}}\right)^2$

$= \left(\dfrac{x-1}{\sqrt{x}}\right)^2$

$= \dfrac{(x-1)^2}{x} = x - 2 + \dfrac{1}{x}$

EXERCISE 14.2 INDEX LAWS WITH FRACTIONAL INDICES

1 Evaluate the following:

(a) $64^{\frac{2}{3}}$ (b) $49^{-\frac{1}{2}}$ (c) $\left(9^3\right)^{\frac{1}{2}}$ (d) $\left(\frac{1}{3}\right)^{-2}$

(e) $2^{\frac{2}{3}} \times 4^{\frac{1}{6}}$ (f) $\left(\frac{1}{16}\right)^{-\frac{3}{2}}$ (g) $\sqrt[3]{27} \times \sqrt[5]{32}$ (h) $\sqrt{6\frac{1}{4}} \times \sqrt[3]{8}$

2 The simplest correct expression for $\sqrt[5]{8} \times \sqrt[5]{4}$ is:

A $2^{\frac{3}{5}} \times 2^{\frac{2}{5}}$ B 32 C 2 D $\sqrt[5]{32}$

3 Express each of the following as simply as possible using index notation.

(a) $\sqrt[4]{36}$ (b) $\sqrt[8]{32}$ (c) $\sqrt[3]{4} \times \sqrt[6]{16}$ (d) $\sqrt{3} \times \sqrt[3]{81}$

4 Simplify the following, writing your answer with positive indices.

(a) $x^{\frac{2}{3}} \times x^{\frac{3}{2}}$ (b) $\left(a^{-1}b\right)^2 \times \left(\frac{1}{b^{-2}}\right)^{\frac{1}{2}}$ (c) $\left(x^{\frac{1}{2}}\right)^2 - \left(x^{-2}\right)^{\frac{1}{2}}$ (d) $\left(x^{\frac{1}{3}}\right)^2 \times \left(x^{-1}y^3\right)^{-1} \times x^{-\frac{5}{3}}y^2$

(e) $\left(x^{\frac{1}{2}} + y^{\frac{1}{2}}\right)\left(x^{\frac{1}{2}} - y^{\frac{1}{2}}\right)$ (f) $\sqrt[6]{x^2y^3} \times \frac{x^{\frac{1}{3}}}{y^{-\frac{1}{2}}}$ (g) $\frac{54^{\frac{1}{4}}}{6^{\frac{3}{4}} \times 12^{-\frac{1}{2}}}$ (h) $\frac{\left(x^{m+1}\right)^n \times x^{m+n}}{\left(x^m\right)^{n+1} \times x^{2n}}$

5 Simplify $\left(a^2b^{-1}\right)^{-2} \div \left(a^{-1}b^2\right)^2$. Indicate whether each statement below is correct or incorrect.

(a) $a^4b^2 \times a^2b^4$ (b) $a^{-4}b^2 \times \frac{1}{a^{-2}b^4}$ (c) $a^{-2}b^{-2}$ (d) $\frac{1}{a^2b^2}$

14.3 SOLVING EQUATIONS WITH EXPONENTS

Example 5

Solve the equations:

(a) $3^x = 27$ (b) $5^{2x} = 125^{\frac{1}{2}}$ (c) $\left(3^x - 1\right)\left(2^{2x} - \frac{1}{16}\right) = 0$

Solution

(a) $3^x = 3^3$

$x = 3$

(b) $5^{2x} = 125^{\frac{1}{2}}$

$5^{2x} = \left(5^3\right)^{\frac{1}{2}}$

$5^{2x} = 5^{\frac{3}{2}}$

$2x = \frac{3}{2}$

$x = \frac{3}{4}$

(c) $\left(3^x - 1\right)\left(2^{2x} - \frac{1}{16}\right) = 0$

$\left(3^x - 1\right)\left(2^{2x} - 2^{-4}\right) = 0$

$3^x - 1 = 0$ or $2^{2x} - 2^{-4} = 0$

$3^x = 1 = 3^0$ or $2^{2x} = 2^{-4}$

$x = 0$ or $2x = -4$

$x = 0$ or $x = -2$

EXERCISE 14.3 SOLVING EQUATIONS WITH EXPONENTS

1 Solve:

(a) $2^x = 8$ (b) $3^{x-1} = 27$ (c) $x^3 = -125$ (d) $x^{-2} = 81$

(e) $\frac{2^{x-3}}{4^{1-x}} = 1$ (f) $4^x = 32$ (g) $9^x = 27$ (h) $3^x + 5 \times 3^x = 54$

(i) $\frac{3 \times 5^x - 1}{5^x + 2} = 2$ (j) $3^x = \frac{1}{9}$ (k) $2^{-x} = \frac{1}{64}$ (l) $5^x = \frac{1}{125}$

2 The solution to the equation $9^x = \frac{1}{3}$ is:

A $x = \frac{1}{2}$ B $x = 2$ C $x = -\frac{1}{2}$ D $x = -2$

3 Solve:

(a) $2^x = \frac{1}{8}$ (b) $a^{x-3} = 1$ (c) $4^x = \frac{1}{2}$ (d) $2^x \times 4^x \times 8^x = 2^{-3}$

(e) $3^x \times 2^x = 1$ (f) $5^x = \frac{1}{125}$ (g) $16^x = 128$ (h) $5^x = 125$

(i) $8^{-x} = \frac{1}{32}$ (j) $\left(2^x - 1\right)\left(3^x - \frac{1}{9}\right) = 0$ (k) $(3^x - 9)(5^x - 1) = 0$ (l) $3^{2x+1} = \frac{1}{27}$

14.4 LOGARITHMS

Logarithms (often called 'logs') were once used mainly as a way to simplify difficult calculations, but electronic calculators and computers have made this technique obsolete. However, logarithms are still useful! You will study logarithmic functions in detail later, but at this stage we need to learn and understand the laws of logarithms. These laws are still needed to solve some equations where the variable is in the exponent (index).

Consider the statement $2^3 = 8$. This is equivalent to the statement $\log_2 8 = 3$. These two statements are different ways to write the same result. Given one statement, you can always write the other. The statement $\log_2 8 = 3$ is read as 'the logarithm of 8 to the base 2 is equal to 3'.

The following pairs of statements are all equivalent:

$$3^2 = 9 \Leftrightarrow \log_3 9 = 2 \qquad a^0 = 1 \Leftrightarrow \log_a 1 = 0$$
$$10^4 = 10\,000 \Leftrightarrow \log_{10} 10\,000 = 4 \qquad a^x = y \Leftrightarrow \log_a y = x$$

In general, if $a > 0$ then the statements $a^x = y$ and $\log_a y = x$ are equivalent. Thus we see that a logarithm is equivalent to an **index** or **exponent**.

Example 6

Without using a calculator, find the value of:

(a) $\log_2 16$ **(b)** $\log_5 125$

Solution

(a)

Let	$\log_2 16 = x$	
Then	$16 = 2^x$	
Hence	$2^4 = 2^x$	
So	$x = 4$	
Hence	$\log_2 16 = 4$	

(b)

Let	$\log_5 125 = x$	
Then	$125 = 5^x$	
Hence	$5^3 = 5^x$	
So	$x = 3$	
Hence	$\log_5 125 = 3$	

We have evaluated each logarithm by converting it to the equivalent index (exponential) function.

We can obtain the laws of logarithms from the equivalent index laws.

Index laws

Let $a^x = m$ and $a^y = n$

1 $a^x \times a^y = a^{x+y} \rightarrow mn = a^{x+y}$

2 $\dfrac{a^x}{a^y} = a^{x-y} \rightarrow \dfrac{m}{n} = a^{x-y}$

3 $\dfrac{1}{a^x} = a^{-x} \rightarrow \dfrac{1}{m} = a^{-x}$

4 $\left(a^x\right)^p = a^{xp} \rightarrow m^p = a^{xp}$

5 $a^1 = a$

6 $a^0 = 1$

Logarithm laws

Let $\log_a m = x$ and $\log_a n = y$

1 $\log_a (mn) = x + y = \log_a m + \log_a n$
$\log_a 15 = \log_a 3 + \log_a 5$

2 $\log_a \left(\dfrac{m}{n}\right) = x - y = \log_a m - \log_a n$

$\log_a \left(\frac{17}{5}\right) = \log_a 17 - \log_a 5$

3 $\log_a \left(\dfrac{1}{m}\right) = -x = -\log_a m$

$\log_a \left(\frac{1}{2}\right) = -\log_a 2$

4 $\log_a m^p = px = p \log_a m$
$\log_a 81 = \log_a (3^4) = 4 \log_a 3$

5 $\log_a a = 1$
$\log_{10} 10 = 1$

6 $\log_a 1 = 0$

Note that these laws are all for $a > 0$, and so we also have $m > 0$ and $n > 0$.

For $m > 0$ we can also see that $\log_a m > 0$; for $m = 1$, $\log_a 1 = 0$; and for $0 < m < 1$, $\log_a m < 0$. This means that m is never negative (because $a^x \geq 0$). Therefore you cannot find the logarithm of a negative number.

On most calculators you will notice two keys, **log** and **ln**. These are both logarithm keys. The **log** is the logarithm to base 10, also called the 'common logarithm'; **ln** is the logarithm to base e and is called the 'natural logarithm' (or occasionally the 'Naperian' or 'Napierian' logarithm).

Logarithms to different bases are useful in different situations, but base 10 and base e are often the most useful because of the properties of the numbers 10 and e, as we will see. It is useful to be able to convert all logarithms to a standard base. Luckily, this is possible by using the 'change of base' rule.

Change of base rule

Let $\log_a n = y$, so $n = a^y$.

Take logarithms to base b of both sides: $\log_b n = \log_b a^y$

Using the logarithm law for exponents: $\log_b n = y \log_b a$

So: $y = \dfrac{\log_b n}{\log_b a}$

Hence: $\log_a n = \dfrac{\log_b n}{\log_b a}$

This rule shows that the logarithm of a number to a given base is equal to the logarithm of the number to a new base divided by the logarithm of the old base to the new base.

Logarithm laws

$\log_a (mn) = \log_a m + \log_a n$

$\log_a \left(\dfrac{m}{n}\right) = \log_a m - \log_a n$

$\log_a m^p = p \log_a m$

$\log_a a = 1$

$\log_a 1 = 0$

$\log_a \left(\dfrac{1}{m}\right) = -\log_a m$

$\log_a n = \dfrac{\log_b n}{\log_b a}$

Example 7

Simplify:

(a) $\log_{10} 20 + \log_{10} 5$ **(b)** $\log_a 4 + \log_a 3 + \log_a 2$ **(c)** $\log_2 20 - \log_2 5$

Solution

(a) $\log_{10} 20 + \log_{10} 5$
$= \log_{10} 100$
$= \log_{10} 10^2$
$= 2 \log_{10} 10$
$= 2$

(b) $\log_a 4 + \log_a 3 + \log_a 2$
$= \log_a (4 \times 3 \times 2)$
$= \log_a 24$

(c) $\log_2 20 - \log_2 5$
$= \log_2 \left(\frac{20}{5}\right)$
$= \log_2 4$
$= \log_2 2^2$
$= 2$

Example 8

Simplify:

(a) $\log_{10} 5 + \log_{10} 4 - \log_{10} 2$ **(b)** $3 \log_{10} 2 + \log_{10} 18 - 2 \log_{10} \left(\frac{6}{5}\right)$

Solution

(a) $\log_{10} 5 + \log_{10} 4 - \log_{10} 2$
$= \log_{10}(5 \times 4) - \log_{10} 2$
$= \log_{10}\left(\frac{20}{2}\right)$
$= \log_{10} 10 = 1$

(b) $3\log_{10} 2 + \log_{10} 18 - 2\log_{10}\left(\frac{6}{5}\right)$
$= \log_{10} 8 + \log_{10} 18 - \log_{10}\left(\frac{36}{25}\right)$
$= \log_{10}\left(8 \times 18 \times \frac{25}{36}\right)$
$= \log_{10} 100 = 2$

Example 9

Evaluate $\log_2 9$.

Solution

Use the change of base rule: $\log_2 9 = \dfrac{\log_{10} 9}{\log_{10} 2}$

Use the **log** key on your calculator: $\log_2 9 = 3.1699$ (correct to 4 decimal places)

EXERCISE 14.4 LOGARITHMS

1 Simplify:

(a) $\log_3 9$ **(b)** $\log_9 3$ **(c)** $\log_2 128$ **(d)** $\log_a a$
(e) $\log_3 \frac{1}{27}$ **(f)** $\log_4 0.25$ **(g)** $\log_{\sqrt{3}} 243$ **(h)** $\log_3 \sqrt{3}$

2 $\log_8 512$ is equal to:

A 3 **B** 5 **C** 7 **D** 9

3 Simplify:

(a) $\log_5 625$ **(b)** $\log_9 243$ **(c)** $\log_a a^3$ **(d)** $\log_{10} 0.0001$

4 Simplify the following:

(a) $\log_2 16 + \log_2 8$ **(b)** $\log_{10} 2 + \log_{10} 5$ **(c)** $(\log_2 16)(\log_2 4)$ **(d)** $\log_3 54 - \log_3 18$
(e) $\dfrac{\log_a 8}{\log_a 2}$ **(f)** $\log_a 5 + \log_a \frac{1}{5}$ **(g)** $\log_2 18 - 2\log_2 3$ **(h)** $\log_3 81 \times \log_5 125$

5 Simplify $\log_{10} 125 + \log_{10} 32 - \log_{10} 4$ and state whether each of the following statements is correct or incorrect.

(a) $\log_{10} 1000$ **(b)** $\log_{10} 16\,000$ **(c)** $3\log_{10} 5 + 3\log_{10} 2$ **(d)** 3

6 Simplify the following:

(a) $\frac{1}{2}\log_{10} 16 + 2\log_{10} 5$ **(b)** $\log_2(2^x)$ **(c)** $10^{\log_{10} 3}$ **(d)** $\dfrac{\log_{10} 25}{\log_{10} 5}$

(e) $\log_{10} 125 + \log_{10} 25 + \log_{10} 5$ **(f)** $\dfrac{\log\left(x^3\right)}{\log x}$ **(g)** $\log_{10}\dfrac{1+\sqrt{5}}{2} + \log_{10}\dfrac{3+\sqrt{5}}{2}$ **(h)** $\dfrac{\log x}{\log\sqrt{x}}$

7 If $x = \log_{10} 2$ and $y = \log_{10} 3$, express the following in terms of x and y:

(a) $\log_{10}\frac{2}{3}$ **(b)** $\log_{10}\frac{4}{9}$ **(c)** $\log_{10} 15$ **(d)** $\log_{10} 54$ **(e)** $\log_{10} 5.4$ **(f)** $\log_{10} 75$
(g) $\log_{10} 150$ **(h)** $\log_{10} 0.27$ **(i)** $\log_{10} 4.5$ **(j)** $\log_{10} 0.6$ **(k)** $\log_{10} 81$ **(l)** $\log_{10} 1.8$

8 Use the change of base rule to evaluate each expression, giving your answer correct to 3 decimal places:

(a) $\log_2 5$ **(b)** $\log_3 12$ **(c)** $\log_5 20$ **(d)** $\log_4 3$ **(e)** $\log_3 16$ **(f)** $\log_6 4$ **(g)** $\log_5 3$ **(h)** $\log_2 10$

14.5 SOLVING EQUATIONS WITH LOGARITHMS

In the previous section we investigated the logarithm laws and the change of base rule. We will now see how to use these techniques in an algebraic setting to solve more difficult equations.

Note that when the notation 'log' is written without a base, then by convention we assume it represents $\log_{10}$, the common logarithm.

Example 10

Solve the equations.

(a) $\log_{10} x = \log_{10} 9 + \log_{10} 3$ **(b)** $3\log_{10} x + 4 = 7\log_{10} x$

Solution

(a)

$$\log_{10} x = \log_{10} 9 + \log_{10} 3$$

Use $\log_a m + \log_a n = \log_a (mn)$: $\log_{10} x = \log_{10} 27$

If $\log a = \log b$ then $a = b$: $x = 27$

(b)

$$3\log_{10} x + 4 = 7\log_{10} x$$

Collect like terms: $4 = 4\log_{10} x$

Simplify: $\log_{10} x = 1$

If $\log_a n = y$ then $n = a^y$: $x = 10^1 = 10$

Example 11

For what value of x is $\log_2 (x+1) - \log_2 (x-1) = 3$?

Solution

Use $\log_a m - \log_a n = \log_a \left(\frac{m}{n}\right)$: $\log_2 \left(\frac{x+1}{x-1}\right) = 3$

If $\log_a n = y$ then $n = a^y$: $\frac{x+1}{x-1} = 2^3$

$$x + 1 = 8x - 8$$

Solve equation: $x = 1\frac{2}{7}$

Remember that 'log' written without a base should be assumed to mean $\log_{10}$.

Example 12

Solve, giving answers correct to 3 decimal places: **(a)** $2^x = 7$ **(b)** $3^{x+1} = 12$

Solution

(a)

$$2^x = 7$$

Take logs to base 10: $\log_{10} 2^x = \log_{10} 7$

$$x\log_{10} 2 = \log_{10} 7$$

Divide: $x = \dfrac{\log_{10} 7}{\log_{10} 2}$

Evaluate: $x = 2.807$ (3 d.p.)

(b) $3^{x+1} = 12$

Take logs to base 10: $\log_{10} 3^{(x+1)} = \log_{10} 12$

$(x+1)\log_{10} 3 = \log_{10} 12$

Divide: $x + 1 = \dfrac{\log_{10} 12}{\log_{10} 3}$

Rearrange: $x = \dfrac{\log_{10} 12}{\log_{10} 3} - 1$

Evaluate: $x = 1.262$ (3 d.p.)

Example 13

Solve the inequalities: **(a)** $2^x > 9$ **(b)** $0.4^x < 0.3$

Solution

(a) $2^x > 9$

Take logs: $x > \log_2 9$

Change of base rule: $x > \dfrac{\log_{10} 9}{\log_{10} 2}$

$x > 3.17$ (2 d.p.)

(b) $0.4^x < 0.3$

Take logs to base 10: $x \log 0.4 < \log 0.3$

Divide: $x < \dfrac{\log_{10} 0.3}{\log_{10} 0.4}$

Although this step looks correct, it contains an error. This is because $\log_{10} 0.4 < 0$, so we have divided by a negative quantity without reversing the direction of the inequality sign.

Correct result: $x > \dfrac{\log_{10} 0.3}{\log_{10} 0.4}$

$x > 1.31$ (2 d.p.)

Be very careful when using logarithms with inequalities. Remember that if $0 < m < 1$ then $\log_a m < 0$, so when you divide by that logarithm you must reverse the direction of the inequality.

Example 14

How many years does it take for \$2000 to grow to \$3000, at 7% p.a. compound interest?

(Note that 'p.a.' = 'per annum' = per year. The compound interest formula is $A = PR^n$, where P is the initial money invested and A is the amount that P grows to after n periods of time with interest applied at r% per period. Compound interest is investigated further in Chapter 16.)

Solution

$A = 2000$, $r = 0.07$, $P = 3000$. Find n: $2000 \times 1.07^n = 3000$

$1.07^n = \frac{3}{2}$

Take logs: $n = \log_{1.07} 1.5$ **OR** $n \log_{10} 1.07 = \log_{10} 1.5$

Use change of base rule **OR** rearrange: $n = \dfrac{\log_{10} 1.5}{\log_{10} 1.07}$

$n \approx 5.993$ years

It takes just under 6 years.

EXERCISE 14.5 SOLVING EQUATIONS WITH LOGARITHMS

1 Solve for x:

(a) $\log_3 x = 3$ **(b)** $\log_x 81 = 2$ **(c)** $\log_6 x = 3$ **(d)** $\log_x 343 = 3$

(e) $\log_5 x = -3$ **(f)** $\log_3 81 = x$ **(g)** $\log_x \frac{1}{64} = -3$ **(h)** $\log_9 x = 0.25$

(i) $\log_3 27\sqrt{3} = x$ **(j)** $\log_7 x = 2.5$ **(k)** $\log_2(\log_2 x) = 2$ **(l)** $\log_2 x = \log_2 8 + \log_4 8$

2 Without using a calculator, solve each equation:

(a) $\log_{10} x = \log_{10} 4 + \log_{10} 2$ **(b)** $\log_{10} x = \log_{10} 4 - \log_{10} 2$ **(c)** $\log_{10} x = \dfrac{\log_{10} 4}{\log_{10} 2}$

(d) $\log_{10} x = \frac{1}{2}\log_{10}\left(\frac{1}{4}\right)$ **(e)** $2\log_{10} x + 3 = \log_{10}(x^5)$ **(f)** $\log_{10} x^2 = 2$

3 Find a relation between x and y that does not involve logarithms.

(a) $\log x + \log y = \log(x + y)$ **(b)** $2\log_{10} y - 3\log_{10} x = 2$ **(c)** $2\log_3 y - 3\log_3 x = 2$

(d) $2\log_{10} y + 3\log_{10} x = \log_{10}(2x)$ **(e)** $\log_5 y = 2 + \log_5 x$ **(f)** $\log y = \log 5 + 3\log x$

(g) $2\log x + 3\log y = 0$ **(h)** $\log_{10}(1 + y) - \log_{10}(1 - y) = x$

4 Solve:

(a) $\log_{10} 2 + \log_{10} 5 + \log_{10} x - \log_{10} 3 = 2$ **(b)** $2\log_{10} x + 3 = 5\log_{10} x$

(c) $\log_{10} 2 + 5\log_{10} x - \log_{10} 5 - \log_{10}(x^3) = \log_{10} 40$ **(d)** $\log_{10} x = 4\log_{10} 2 - 2\log_{10} x$

(e) $\log_{10} x - \log_{10}(x - 1) = 1$ **(f)** $\log_{10} x = 2\log_{10} 3 + \log_{10} 5 - \log_{10} 2 - 1$

5 Solve $2^{-x} = 5$. Indicate whether each statement below is correct or incorrect.

(a) $x = \dfrac{\log 5}{\log 2}$ **(b)** $x = \log_2\left(\frac{1}{5}\right)$ **(c)** $x = \dfrac{-\log 5}{\log 2}$ **(d)** -2.32 (2 d.p.)

6 Solve, correct to 3 decimal places:

(a) $2^x = 7$ **(b)** $3^x = 18$ **(c)** $5^x = 2$ **(d)** $0.4^x = 2$

(e) $6^x = 21$ **(f)** $3^{-x} = 0.1$ **(g)** $5^x = 16$ **(h)** $4^x = 5$

7 Find the values of x (to 2 decimal places) for which:

(a) $5^x > 2$ **(b)** $1.6^x \geq 0.5$ **(c)** $3^x < 0.2$ **(d)** $3^{-x} > 27$

(e) $2^x \geq 5$ **(f)** $0.25^{-x} < 1.5$ **(g)** $0.8^x < 3$ **(h)** $0.7^x \geq 0.3$

8 If $y = a10^{bx}$, then:

A $x = \log_{10} \dfrac{y}{ab}$ **B** $x = \dfrac{1}{b}\log_{10} \dfrac{y}{a}$ **C** $y = \dfrac{1}{b}\log_{10} \dfrac{x}{a}$ **D** $x = \dfrac{1}{a}\log_{10} \dfrac{y}{b}$

9 If $\log_{10} A = bt + \log_{10} P$, express A in terms of the other symbols.

10 If $\log y = \log a + n\log x$, find an expression for y.

11 If $y = \dfrac{\log x}{\log 2}$, express x in terms of y.

12 If $x = a^2\sqrt{b^3 c}$, express $\log x$ in terms of $\log a$, $\log b$ and $\log c$.

13 If $\log x = 0.6$ and $\log y = 0.2$, evaluate $\log\left(\dfrac{x^2}{\sqrt{y}}\right)$.

14 If $y = ae^{4t}$, express t in terms of a and y.

15 If $\log_b a = p$ and $c = a^2$, find the following in terms of p: **(a)** $\log_b c$ **(b)** $\log_c b$

16 If $\log_a 2 = \log_b 16$, show that $b = a^4$.

17 \$5000 is invested at 7% p.a. compound interest. How long does it take for this money to:

(a) double in value **(b)** grow to \$20 000 **(c)** grow to \$30 000?

18 \$5000 is invested at 6% p.a. compound interest. If the interest is calculated monthly, how long does it take for this money to:

(a) double in value **(b)** grow to \$20 000 **(c)** grow to \$30 000?

19 Marika and Joe deposit $4000 in an account that pays 9% p.a. compound interest, to be withdrawn when it has grown to $20 000. If the interest is calculated monthly, for how many whole months must they leave the money in the account?

20 A company is considering a merger. Currently the company earnings are $5 per share and these earnings are growing by 5% p.a. It is predicted that after the merger the earnings will drop to $4 per share, but will then grow by 7% p.a.

(a) Show that if no merger occurs, the earnings per share after n years are given by $y_n = 5 \times 1.05^n$.

(b) Show that if the merger occurs, the earnings per share after n years are given by $z_n = 4 \times 1.07^n$.

(c) If the merger goes ahead, how many years will it take for a shareholder to be better off?

14.6 EXPONENTIAL FUNCTIONS

An exponential function is a function $f(x) = a^x$, where the base a is a positive real number other than 1. Its domain is the set of real numbers and its range is the set of positive real numbers.

The exponential function can be used to model many real-life situations. The compound interest formula $A = PR^n$ is an example of this. We can also use exponential functions to model population growth and radioactive decay.

For $a > 1$: a^x increases as x increases.

As $x \to -\infty$, $a^x \to 0$ from above.

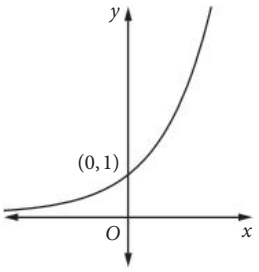
For all values of x, $a^x > 0$.

For all values of $x < 0$, $0 < a^x < 1$.

At $x = 0$, $a^0 = 1$.

$y = a^x$ cuts the y-axis at $(0, 1)$ for all values of a.

For all values of $x > 0$, $a^x > 1$.



$f(x) = a^x$, $a > 1$

For $0 < a < 1$: a^x decreases as x increases.

As $x \to \infty$, $a^x \to 0$ from above.

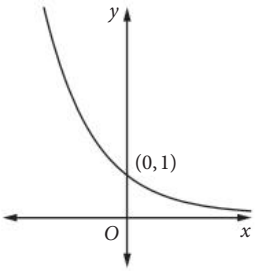
For all values of x, $a^x > 0$.

For all values of $x < 0$, $a^x > 1$.

When $x = 0$, $a^0 = 1$.

$y = a^x$ cuts the y-axis at $(0, 1)$ for all values of a.

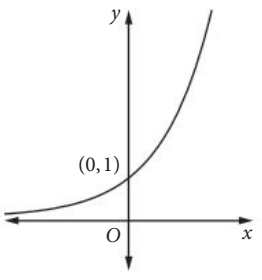
For all values of $x > 0$, $0 < a^x < 1$.



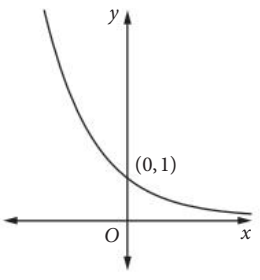
$f(x) = a^x$, $0 < a < 1$

For $a = \frac{1}{2}$, $f(x) = \left(\frac{1}{2}\right)^x = \left(2^{-1}\right)^x = 2^{-x}$, so $f(x) = a^x$ for $0 < a < 1$ is the same as $f(x) = a^{-x}$ for $a > 1$.
Hence the graphs now represent $f(x) = a^x$ and $f(x) = a^{-x}$ for $a > 1$:

From these graphs we can see that the gradient of $f(x) = a^x$ is always positive (i.e. $f'(x) > 0$) and the gradient of $f(x) = a^{-x}$ is always negative (i.e. $f'(x) < 0$). We cannot find the gradient functions yet, but the graphs tell us their signs. Also note that both graphs are concave upwards.



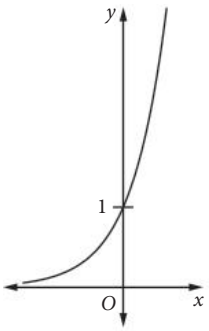
$f(x) = a^x$, $a > 1$



$f(x) = a^{-x}$, $a > 1$

Gradient of exponential functions

The diagram shows the graph of $y = 10^x$.



To find the gradient at any point P on the curve, we need to go back to our first-principles definition of the gradient function (see Chapter 10).

Let $f(x) = 10^x$ so $f(x+h) = 10^{x+h}$. From the definition of the gradient function:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{10^{x+h} - 10^x}{h}$$

$$= \lim_{h\to 0} \frac{10^x\left(10^h - 1\right)}{h}$$

$$= 10^x \lim_{h\to 0} \frac{10^h - 1}{h}$$

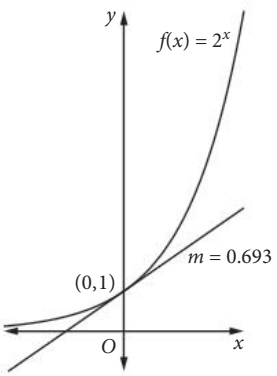
We can use a spreadsheet or a calculator to investigate the values of $\frac{10^h - 1}{h}$ as $h \to 0$:

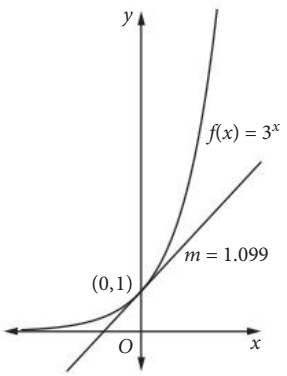
h	$\frac{10^h - 1}{h}$	h	$\frac{10^h - 1}{h}$
0.1	2.5893	-0.1	2.0567
0.05	2.4404	-0.05	2.1750
0.01	2.3293	-0.01	2.2763
0.001	2.3052	-0.001	2.2999
0.0001	2.3029	-0.0001	2.3023

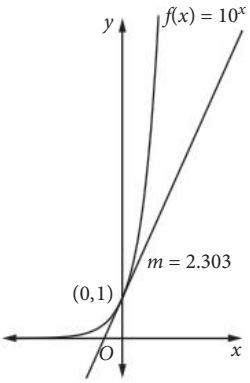
As $h \to 0$ from either positive or negative values, the value of $\frac{10^h - 1}{h}$ approaches a limit. The value of this limit is between 2.3029 and 2.3023, so a good approximation to 4 decimal places is 2.3026. This limit is the gradient at the point where the graph of $y = 10^x$ crosses the y-axis. Thus $f'(x) = 2.3026 \times 10^x$.

We can repeat this investigation of the derivative for $f(x) = 2^x$ and $f(x) = 3^x$, evaluating $\frac{2^h - 1}{h}$ and $\frac{3^h - 1}{h}$ as $h \to 0$, as shown in the following table and graphs.

$f(x)$	$f'(x)$
2^x	0.6931×2^x
3^x	1.0986×3^x
10^x	2.3026×10^x







In general, we have $\frac{d}{dx}\left(a^x\right) = ka^x$ for some positive integer k, which is the gradient of the function a^x at $(0, 1)$.

If we could find a value for a that gave $k = 1$, then that function would be its own derivative. When $a = 2$, $k = 0.693$ and when $a = 3$, $k = 1.099$, so we are looking for a value of a between 2 and 3. We will denote this special base by e, where $2 < e < 3$ such that we can write $\frac{d}{dx}\left(e^x\right) = e^x$.

Evaluation of *e*

Let $e = 10^b$, so that $e^x = 10^{bx}$. So: $\frac{d}{dx}\left(e^x\right) = \frac{d}{dx}\left(10^{bx}\right)$

Now let $y = 10^{bx} = 10^u$, where $u = bx$.

We showed earlier that $\frac{d}{dx}\left(10^x\right) = 2.3026 \times 10^x$, so $\frac{d}{du}\left(10^u\right) = 2.3026 \times 10^u$.

Using the chain rule $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$, so $\frac{dy}{dx} = 2.3026 \times 10^u \times b$ because $\frac{du}{dx} = b$.

Hence: $\frac{dy}{dx} = 2.3026b \times 10^{bx}$

Thus: $\frac{d}{dx}\left(10^{bx}\right) = 2.3026be^x$

So: $\frac{d}{dx}\left(e^x\right) = 2.3026be^x$

But we originally selected the value of e so that $\frac{d}{dx}\left(e^x\right) = e^x$, so we can now write that $e^x = 2.3026be^x$.

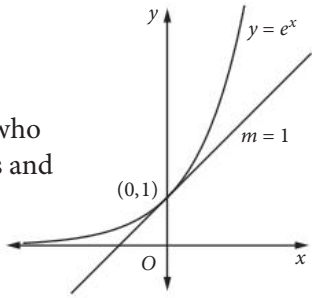
Solve this equation for b: $b = \frac{1}{2.3026} = 0.4343$ (4 decimal places)

Hence: $e = 10^{0.4343} = 2.7183$ (4 decimal places)

This number e is known as **Euler's number** after mathematician Leonhard Euler, who invented its notation. After π, e is one of the most famous transcendental numbers and scientific constants. (For more on transcendental numbers, see section 15.5.)

Here the graph of $y = e^x$ is drawn, as well as the tangent to the curve at $(0, 1)$.

The gradient of this tangent is 1, because at $(0, 1)$ we have $\frac{dy}{dx} = e^0 = 1$.



Formal derivation of *e*

Consider the function $f(x) = e^x$, where e is some positive number. Differentiating from first principles, we have:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{e^{x+h} - e^x}{h}$$

$$= \lim_{h \to 0} \frac{e^x\left(e^h - 1\right)}{h}$$

$$= e^x \lim_{h \to 0} \frac{e^h - 1}{h} \quad \text{because } e^x \text{ is independent of } h.$$

Hence: $f'(x) = f(x) \times \lim_{h \to 0} \frac{e^h - 1}{h}$

We define the value of e such that $\frac{d}{dx}\left(e^x\right) = e^x$, or $f'(x) = f(x)$.

Thus we require $\lim_{h \to 0}\left(\frac{e^h - 1}{h}\right) = 1$, which requires that $\frac{e^h - 1}{h} = 1$ for a very small value of h.

Hence: $e^h - 1 = h$ for very small h.

Recall that if n is very large $(n \to \infty)$ then the value of $\frac{1}{n}$ is very small $\left(\frac{1}{n} \to 0\right)$.

Thus for very small h, $e^h - 1 = h$ can be written as $e^{\frac{1}{n}} - 1 = \frac{1}{n}$, where n is very large.

Hence: $e^{\frac{1}{n}} = \frac{1}{n} + 1$

Raise both sides to the power n: $\left(e^{\frac{1}{n}}\right)^n = \left(\frac{1}{n} + 1\right)^n$

So: $e = \left(1 + \frac{1}{n}\right)^n$ for $n \to \infty$.

Thus we have found the value of e for which the function $f(x) = e^x$ has the property that $\frac{d}{dx}\left(e^x\right) = e^x$.

More formally, we could write: $e = \lim_{n \to \infty}\left(1 + \frac{1}{n}\right)^n$

We can use this equation to calculate values of e for increasing values of n using a calculator or a spreadsheet. Some values are given in the following table.

n	100	1000	10^4	10^5	10^6	10^9
e	2.704814	2.716923	2.718146	2.718268	2.718280	2.718282

Thus $e = 2.718\,28$ is an approximation for e correct to 5 decimal places. Most calculators will give you an approximation for e correct to 9 decimal places.

Derivative of e^{kx}, k a constant

To find the derivative of $y = e^{kx}$, we first write $y = e^{kx} = e^u$, where $u = kx$.

Use the chain rule:
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$\frac{dy}{dx} = e^u \times k = k e^x$$
$$\therefore \quad \frac{d}{dx}\left(e^{kx}\right) = ke^{kx}$$

Thus the function $f(x) = e^{kx}$ is a function whose derivative is proportional to the value of the function itself.

That is: $f'(x) = kf(x)$, or $\frac{dy}{dx} = ky$ where $y = e^{kx}$.

Consider some values for k, for example:

$k = 2$: $\frac{d}{dx}\left(e^{2x}\right) = 2e^{2x}$

$k = -3$: $\frac{d}{dx}\left(e^{-3x}\right) = -3e^{-3x}$

$k = \frac{1}{2}$: $\frac{d}{dx}\left(e^{\frac{x}{2}}\right) = \frac{1}{2}e^{\frac{x}{2}}$

Derivative of e^{ax+b}, a and b constants

Again, we first write $y = e^{ax+b} = e^u$, where $u = ax + b$.

Chain rule:
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$\frac{dy}{dx} = e^u \times a = ae^{ax+b}$$

So: $\frac{dy}{dx} = ay$, where $y = e^{ax+b}$

$$\therefore \quad \frac{d}{dx}\left(e^{ax+b}\right) = ae^{ax+b}$$

Derivative of $e^{f(x)}$

$y = e^{f(x)} = e^u$, where $u = f(x)$

Chain rule:
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$\frac{dy}{dx} = e^u \times f'(x) = f'(x)e^{f(x)}$$

So: $\frac{dy}{dx} = f'(x)e^{f(x)}$

Summary of derivatives involving e^x

$$\frac{d}{dx}\left(e^x\right)=e^x \qquad \frac{d}{dx}\left(e^{kx}\right)=ke^x \qquad \frac{d}{dx}\left(e^{ax+b}\right)=ae^{ax+b} \qquad \frac{d}{dx}\left(e^{f(x)}\right)=f'(x)e^{f(x)}$$

Example 15

Differentiate: **(a)** $y=e^{3x-4}$ **(b)** e^{x^2} **(c)** $e^{x+\sqrt{x}}$

Solution

(a) $\dfrac{dy}{dx}=3e^{3x-4}$

(b) Let $y=e^{x^2}=e^u$ where $u=x^2$.

Chain rule: $\dfrac{dy}{dx}=\dfrac{dy}{du}\times\dfrac{du}{dx}$

$\dfrac{dy}{dx}=e^u\times 2x$

$=2xe^{x^2}$

(c) Let $y=e^{x+\sqrt{x}}=e^u$ where $u=x+\sqrt{x}$.

$\dfrac{dy}{dx}=\dfrac{dy}{du}\times\dfrac{du}{dx}$

$\dfrac{dy}{dx}=e^u\times\left(1+\dfrac{1}{2\sqrt{x}}\right)$

$=\left(1+\dfrac{1}{2\sqrt{x}}\right)e^{x+\sqrt{x}}$

Example 16

Differentiate: **(a)** $(2x^2+1)e^{3x}$ **(b)** $\dfrac{e^x}{x}$

Solution

(a) Let $y=(2x^2+1)e^{3x}=uv$ where $u=2x^2+1$ and $v=e^{3x}$.

Product rule: $\dfrac{dy}{dx}=v\dfrac{du}{dx}+u\dfrac{dv}{dx}$

$\dfrac{dy}{dx}=e^{3x}\times 4x+(2x^2+1)\times 3e^{3x}$

$=(6x^2+4x+3)e^{3x}$

(b) Let $y=\dfrac{e^x}{x}=\dfrac{u}{v}$ where $u=e^x$ and $v=x$.

Quotient rule: $\dfrac{dy}{dx}=\dfrac{v\dfrac{du}{dx}-u\dfrac{dv}{dx}}{v^2}$

$\dfrac{dy}{dx}=\dfrac{x\times e^x-e^x\times 1}{x^2}$

$=\dfrac{e^x(x-1)}{x^2}$

TECHNOLOGY EXPLORATION

Exponential graphs

What is the effect of varying a, k and c in $y=ae^{kx}+c$?

1. In GeoGebra, use the **Slider** tool to create three sliders for a, k and c.
   Set a and c to have Min = -5, Max = 5 and Increment = 1.
   Set k to have Min = -5, Max = 5 and Increment = 0.5.
2. In the input bar, enter *y=a*e^(k*x)+c* (this will be labelled *f(x)*).
3. Display the asymptote by entering *asymptote[f]* into the input bar.
4. Adjust the sliders to investigate how the different values of a, k and c affect the graph.

EXERCISE 14.6 EXPONENTIAL FUNCTIONS

1 Write the derivative of:

(a) e^{4x} **(b)** $2e^{\frac{x}{2}}$ **(c)** $e^{4x}-e^{3x}$ **(d)** $2e^{3x}+e^{-x}$ **(e)** $4e^{3x}-e^{-2x}$

(f) $e^{3.2x}-e^{1.6x}$ **(g)** $3e^x-2e^{-x}$ **(h)** $4e^{2x}+\frac{1}{2}e^{-2x}$ **(i)** $e^{2x}(e^x-e^{-x})$

2 If $y = e^{x^2}$ then $\frac{dy}{dx}$ is:

A $x^2e^{x^2}$ **B** $2xe^{x^2}$ **C** x^2e^{2x} **D** $2xe^{2x}$

3 Differentiate:

(a) x^2e^{3x} **(b)** $(2x+1)e^{-x}$ **(c)** $(x^2+x+1)e^{2x}$ **(d)** xe^{-2x}

(e) $e^{-x}x^3$ **(f)** $x^3 - xe^{2x}$ **(g)** $x^2 - x^3e^{2x}$ **(h)** $\frac{e^x}{x^2}$

(i) $\frac{e^{3x}}{x}$ **(j)** $\frac{x^3}{e^x}$ **(k)** $\frac{e^{4x}}{x-1}$ **(l)** $\frac{e^x}{\sqrt{x}}$

4 Differentiate:

(a) e^{2x+3} **(b)** e^{x^2-2x} **(c)** $3e^{-x^3}$ **(d)** $2e^{3x-1}$ **(e)** $e^{3x-1} + e^{4x+2}$

(f) $\sqrt{x}e^{-x}$ **(g)** $3e^{2x^2}$ **(h)** $3e^{2x-1}$ **(i)** xe^{x^2}

5 If $y = e^{2x} + e^{8x}$, indicate whether each statement below is correct or incorrect.

(a) $\frac{dy}{dx} = 2e^{2x} + 8e^{8x}$ **(b)** $\frac{dy}{dx} = 10e^x$ **(c)** $\frac{d^2y}{dx^2} = 4e^{2x} + 64e^{8x}$ **(d)** $\frac{d^2y}{dx^2} - 10\frac{dy}{dx} + 16y = 0$

6 If $x = (1+t)e^{5t}$, prove that $\frac{d^2x}{dt^2} - 10\frac{dx}{dt} + 25x = 0$.

7 Find the equation of the tangent to the curve $y = e^x$ at the point where it crosses the y-axis.

8 Find the equation of the tangent to the curve $y = e^{-x}$ at the point where it crosses the y-axis.

9 Find the equation of the tangent to the curve $y = e^{2x}$ at the point where $x = 1$. Find also the coordinates of the points where the tangent intersects: **(a)** the x-axis **(b)** the y-axis.

10 Write the equation of the tangent and the normal to the curve $y = 2 + e^{-x}$ at the point where $x = 0$.

11 After n years, the value V of a principal of P dollars that is invested at a rate of r% per year (expressed as a decimal) and compounded continuously is given by $V = Pe^{rn}$. Show that $\frac{dV}{dn} = Vr$.

12 The expression $y = 500(1 - e^{-0.2t})$ represents the daily output of y units on day t of a production run. Find the instantaneous rate of change of the output y with respect to t.

13 In statistics, the normal probability density function is given by $f(x) = \frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}$. Find $f'(0)$.

14.7 INTEGRATING THE EXPONENTIAL FUNCTION

Indefinite integral of e^x

Because the exponential function e^x is its own derivative, it is also its own integral: $\int e^x\,dx = e^x + C$.

The integrals of related exponential functions can similarly be determined from their derivatives.

Indefinite integral of e^{kx}, k a constant

Because $\frac{d}{dx}\left(e^{kx}\right) = ke^x$, it follows that $\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C$.

Consider some values for k, for example: $k = 2$: $\int e^{2x}\,dx = \frac{1}{2}e^{2x} + C$

$k = \frac{2}{5}$: $\int e^{\frac{2x}{5}}\,dx = \frac{5}{2}e^{\frac{2x}{5}} + C$

$k = -3$: $\int e^{-3x}\,dx = -\frac{1}{3}e^{-3x} + C$

Indefinite integral of e^{ax+b}, a and b constant

Because $\frac{d}{dx}\left(e^{ax+b}\right) = ae^{ax+b}$, it follows that $\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C$.

Indefinite integral of $f'(x)\,e^{f(x)}$

Because $\frac{d}{dx}\left(e^{f(x)}\right) = f'(x)e^{f(x)}$, it follows that $\int f'(x)e^{f(x)}\,dx = e^{f(x)} + C$.

You can only find $\int e^{f(x)}\,dx$ when $f'(x)$ is a constant, i.e. if $f(x) = ax + b$.

Summary of integrals involving e^x

$$\int e^x\,dx = e^x + C \qquad \int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C$$

$$\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C \qquad \int f'(x)e^{f(x)}\,dx = e^{f(x)} + C$$

Example 17

Find: (a) $\int e^{4x-1}\,dx$ (b) $\int 3x^2e^{x^3+1}\,dx$

Solution

(a) $\frac{d}{dx}(4x-1) = 4$: $\int e^{4x-1}\,dx = \frac{1}{4}e^{4x-1} + C$

(b) $\frac{d}{dx}\left(x^3+1\right) = 3x^2$:

The integral must be of the form $\int f'(x)e^{f(x)}\,dx = e^{f(x)} + C$ where $f(x) = x^3 + 1$.

$\int 3x^2e^{x^3+1}\,dx = e^{x^3+1} + C$

Example 18

Evaluate: (a) $\int_0^2 e^{2x}\,dx$ (b) $\int_{-0.5}^{1.5}\left(e^x - e^{-x}\right)dx$

Solution

(a) $\int_0^2 e^{2x}\,dx = \left[\frac{e^{2x}}{2}\right]_0^2$

$= \frac{1}{2}\left(e^4 - e^0\right)$

$= \frac{e^4 - 1}{2}$ is the exact value

$= 26.80$ using a calculator and writing the answer correct to 2 d.p.

(b) $\int_{-0.5}^{1.5}\left(e^{x}-e^{-x}\right)dx=\left[e^{x}+e^{-x}\right]_{-0.5}^{1.5}$

$=\left(e^{1.5}+e^{-1.5}\right)-\left(e^{-0.5}+e^{0.5}\right)$

$=2.4496$ correct to 4 d.p.

It is an interesting exercise to show that the exact value of this integral is $\frac{(e+1)(e-1)^2}{e^{1.5}}$.

Important note about definite integrals

Before now, when you have evaluated a definite integral at a limit of 0 the result has usually been zero. But do not assume that an integral at 0 is always zero! As we have now seen, substituting a value of 0 into an exponential integral will often produce a value of 1 or some other non-zero constant. When evaluating definite integrals you must always take care to substitute both limit values of the integral, even when one value is zero.

EXERCISE 14.7 INTEGRATING THE EXPONENTIAL FUNCTION

1 Write a primitive function of:

(a) e^{2x} (b) e^{5x} (c) $e^{-0.4x}$ (d) $5e^{2.5x}$ (e) $e^{x}+e^{-3x}$ (f) $e^{-2x}-e^{-x}$

2 Find: (a) $\int e^{-x}\,dx$ (b) $\int e^{\frac{x}{2}}\,dx$ (c) $\int e^{-3x}\,dx$

(d) $\int\left(e^{-t}-1\right)dt$ (e) $\int\left(e^{2u}+u^{2}\right)du$ (f) $\int\left(e^{-2.5x}+e^{0.4x}\right)dx$

3 Find the value of:

(a) $\int_{-1}^{1}e^{x}\,dx$ (b) $\int_{0}^{2}e^{2x}\,dx$ (c) $\int_{-1}^{3}e^{-\frac{x}{2}}\,dx$

(d) $\int_{0}^{1}e^{1.5t}\,dt$ (e) $\int_{-0.5}^{0.5}e^{-2t}\,dt$ (f) $\int_{-1}^{0}e^{-3u}\,du$

(g) $\int_{0}^{1}\left(e^{2x}-e^{-2x}\right)dx$ (h) $\int_{-3}^{3}\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)dt$ (i) $\int_{0.5}^{1.5}\left(e^{\theta}+e^{-3\theta}\right)d\theta$

4 Indicate whether each statement below is a correct or incorrect step in the evaluation of $I=\int_{-1}^{1}\left(e^{x}-e^{-x}\right)^{2}dx$.

(a) $I=\int_{-1}^{1}\left(e^{2x}-2+e^{-2x}\right)dx$ (b) $I=\left[\frac{e^{2x}}{2}-2x-\frac{e^{-2x}}{2}\right]_{-1}^{1}$

(c) $I=\left[e^{2x}-4x-e^{-2x}\right]_{0}^{1}$ (d) $I=\frac{e^{4}-4e^{2}-1}{4e^{2}}$

14.8 APPLICATIONS OF THE EXPONENTIAL FUNCTION

Example 19

For the function $f(t)=2te^{-0.5t}$, find the value of t for which $f(t)$ has a maximum and hence calculate the maximum value. Sketch the graph of $f(t)$.

Solution

$f(t) = 2te^{-0.5t}$ Let $u = t$, $v = e^{-0.5t}$

$f'(t) = 2\left(e^{-0.5t} + t \times \left(-\frac{1}{2}\right)e^{-0.5t}\right)$

$= e^{-0.5t}(2 - t)$

For stationary points, $f'(t) = 0$: $e^{-0.5t}(2 - t) = 0$

But $e^{-0.5t} > 0$ for all t, so $t = 2$ is the only solution and $f(2) = \frac{4}{e}$

For $t < 2$: $f'(t) > 0$

For $t > 2$: $f'(t) < 0$

Gradient changes from positive to negative as x increases, so $\left(2, \frac{4}{e}\right)$ is a maximum turning point.

The maximum value of the function is $\frac{4}{e} \approx 1.472$

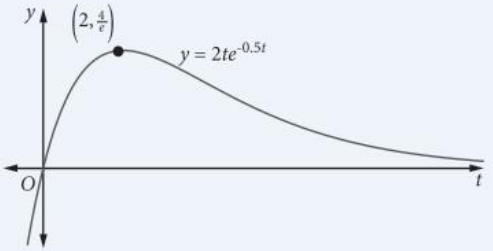
$f(t) = 0$ at $t = 0$ as $e^{-0.5t} > 0$ for all t.

$t < 0$, $f(t) < 0$ $t < 2$, $f'(t) > 0$

$t > 0$, $f(t) > 0$ $t > 2$, $f'(t) < 0$

$t \to \infty$, $f(t) \to 0$ from above

$t = 0$ is horizontal asymptote



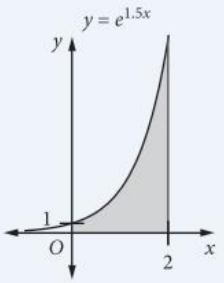
Example 20

Calculate: **(a)** the area bounded by the curve $y = e^{1.5x}$, the coordinate axes and the line $x = 2$

(b) the volume obtained by rotating this area about the x-axis.

Solution

(a) $y = e^{1.5x}$, $y = 0$, $x = 2$



$$\text{Area} = \int_0^2 e^{1.5x}\,dx$$

$$= \left[\frac{2}{3}e^{1.5x}\right]_0^2$$

$$= \frac{2}{3}\left(e^3 - e^0\right)$$

$$= \frac{2\left(e^3 - 1\right)}{3} \approx 12.72 \text{ units}^2$$

(b) Volume $= \pi \int_0^2 y^2\,dx$ where $y = e^{1.5x}$.

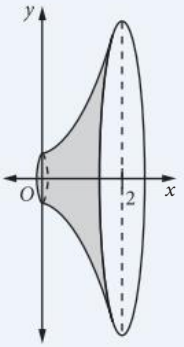
$$= \pi \int_0^2 e^{3x}\,dx$$

$$= \frac{\pi}{3}\left[e^{3x}\right]_0^2$$

$$= \frac{\pi}{3}\left(e^6 - e^0\right)$$

$$= \frac{\pi\left(e^6 - 1\right)}{3}$$

$$\approx 421.4 \text{ units}^3$$



EXERCISE 14.8 APPLICATIONS OF THE EXPONENTIAL FUNCTION

1 Find the minimum value of $(x - 2)e^x$.

2 Find the coordinates of the turning point of the curve $y = xe^{-0.5x}$ and state whether it is a maximum or minimum. Find also the values of x for which:

(a) $y > 0$ **(b)** $\frac{dy}{dx} > 0$

3 Consider the function defined by the rule $f(x) = 3 - e^{-x}$, $x \geq 0$.

(a) Find the value of $f(0)$ and $f'(0)$.

(b) Show that $f'(x) > 0$ for all values of x in the domain.

(c) What is the value of $\lim_{x \to \infty} f(x)$?

(d) Sketch the graph of $f(x)$.

4 Consider the function defined by $f(x) = e^{-x^2}$ for all values of x.

(a) Find $f'(x)$.

(b) Find the values of x for which: **(i)** $f'(x) = 0$ **(ii)** $f'(x) > 0$ **(iii)** $f'(x) < 0$

(c) Sketch the graph of the function.

5 The concentration of a certain drug in the blood at a time t hours after taking the dose is x units, where $x = 0.3te^{-1.1t}$.

(a) Determine the maximum concentration and the time at which this is reached.

(b) Plot the graph of $x = 0.3te^{-1.1t}$ for $t = 0, 0.1, 0.5, 1, 2, 3$ using graph paper or GeoGebra (or other graphing software).

(c) This drug kills germs only while its concentration is at least 0.06 units. From the graph, find the length of time during which the drug will kill germs.

6 For $y = e^t + 4e^{-t}$, find the minimum value of y. Indicate whether each of the statements below is a correct or incorrect step in solving this problem.

(a) $y' = e^t - 4e^{-t}$

(b) Stationary point when $e^t = \pm 2$

(c) $y'' = e^t + 4e^{-t}$

(d) Minimum value is 4

7 Sketch the graph of $f(t) = \dfrac{5}{2 + 3e^{-t}}$, $t \geq 0$.

(a) Show that $f'(t) > 0$ for all values of t in the domain.

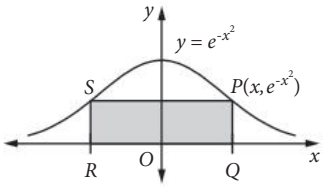
(b) Find $\lim_{t \to \infty} f(t)$.

(c) State the range of the function.

8 The rectangle $PQRS$ has two vertices on the x-axis and two on the curve $y = e^{-x^2}$. Find:

(a) the value of x for which the rectangle has a maximum area

(b) the maximum area of the rectangle.



9 Calculate the area enclosed between the curve $y = e^{2x} + e^{-2x}$, the x-axis and the lines $x = 1$ and $x = -1$.

10 **(a)** Calculate the area bounded by the curve $y = e^x$, the coordinate axes and the line $x = 2$.

(b) Write the equation of the tangent to $y = e^x$ at the point where $x = 2$.

(c) Calculate the area bounded by $y = e^x$, the coordinate axes and the tangent at $x = 2$.

(d) Calculate the area bounded by $y = e^x$, the y-axis and the line $y = e^2$.

11 Calculate the area bounded by the curves $y = e^x$, $y = e^{-x}$ and the ordinate $x = 2$.

12 Calculate the area bounded by the curve $y = e^{0.5x} - e^{-0.5x}$, the x-axis and the line $x = 1$.

13 The area under the curve $y = e^{-x}$ between $x = 0$ and $x = 1$ is rotated about the x-axis. Find the volume of the solid of revolution.

14 Find the volume generated when the curve $y = e^x$, $0.5 \leq x \leq 1.5$, is rotated about the x-axis.

15 Find the volume generated when the curve $y = e^{-0.5x}$, $-2 \leq x \leq 2$, is rotated about the x-axis.

16 Find the volume generated when the curve $y = e^x + e^{-x}$ between $x = -1$ and $x = 1$ is rotated about the x-axis.

17 Using the trapezoidal rule with two subintervals, find an approximate value for $\int_{-1}^{1} x^2 e^x \, dx$.

18 Using the trapezoidal rule with four subintervals, find an approximate value for $\int_{1}^{5} xe^{0.4x} \, dx$.

19 Use the trapezoidal rule with four subintervals to evaluate $\int_0^{0.8} xe^{-x}\,dx$.

20 Write the derivative of $(x-1)e^x$ and use your result to evaluate $\int_{-1}^{2} xe^x\,dx$.

21 **(a)** Find the area of the region bounded by the curve $y = e^{-x}$, the coordinate axes and the line $x = a$, $a > 0$.
(b) Find the limit of this area as $a \to \infty$.
(c) Find the volume of the solid generated by rotating the region in **(a)** about the x-axis and find the limit of this volume as $a \to \infty$.

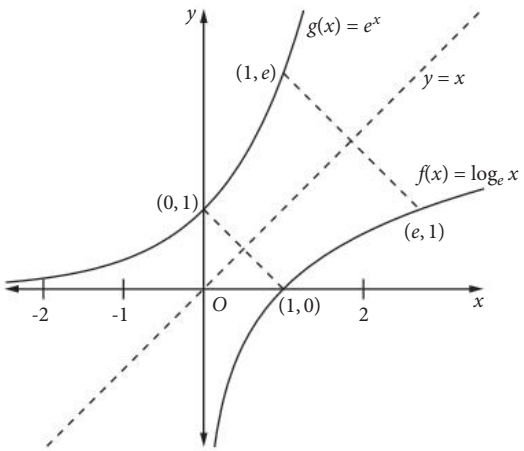
22 We define the average value of f on the interval $a \le x \le b$ as $f_{\text{ave}} = \dfrac{1}{b-a}\int_a^b f(x)\,dx$. Find the average value of each function on the interval given.

(a) $f(x) = e^x$, $0 \le x \le 2$ **(b)** $f(x) = e^{-x}$, $0 \le x \le 2$
(c) $g(x) = e^{2x}$, $1 \le x \le 5$ **(d)** $f(x) = e^x + e^{-x}$, $-2 \le x \le 2$

14.9 NATURAL LOGARITHMS

We have previously looked at the relationship between the expressions $y = a^x$ and $x = \log_a y$. We have also defined the exponential function $y = e^x$ so that $x = \log_e y$. This logarithm to the base e is called the **natural logarithm**, or sometimes the 'Naperian' (or 'Napierian') logarithm after John Napier, the Scottish mathematician who introduced logarithms in the 1600s. The function $\log_e x$ is sometimes written as $\ln x$, and calculators usually have a natural logarithm key labelled **ln**.

The diagram shows the graphs of $g(x) = e^x$ and $f(x) = \log_e x$.



These two graphs are symmetrical about the line $y = x$. We can say that $f(x)$ is the **reflection** of $g(x)$ in the line $y = x$. To 'reflect' a curve of $y = e^x$ in the line $y = x$ means that we interchange (swap) the x and y values, so the equation of the reflection becomes $x = e^y$. Because we know how the exponential and logarithmic functions are linked, we can rewrite $x = e^y$ as $y = \log_e x$.

Because of this property, $y = e^x$ and $y = \log_e x$ are known as **inverse functions**.

On the graph of $g(x) = e^x$ the point $(1, e)$ shows that $e^1 = e$.
On the graph of $f(x) = \log_e x$ the point $(e, 1)$ shows that $\log_e e = 1$.

Hence we can write that $e^{\log_e e} = e$ and in general that $e^{\log_e x} = x$.

You can verify this result by taking the point $(2, \log_e 2)$ on $y = f(x)$ and checking whether the point $(\log_e 2, 2)$ is on $y = g(x)$ using graphing software or your calculator.

Important properties of e^x and $\log_e x$

If $y = \log_e x$ then we can write:	$e^y = e^{\log_e x}$	or	$e^y = e^{\ln x}$
But $e^y = x$ so:	$x = e^{\log_e x}$	or	$x = e^{\ln x}$
Hence:	$e^{\log_e x} = x$	or	$e^{\ln x} = x$
Similarly, if $y = e^x$ then:	$\log_e y = \log_e e^x$	or	$\ln y = \ln e^x$
But $\log_e y = x$, so:	$x = \log_e e^x$	or	$x = \ln e^x$
Hence:	$\log_e e^x = x$	or	$\ln e^x = x$

Note that the operations 'square' and 'square root' are similarly inverse operations (for positive values), because when repeated after each other they return to the starting value: $\sqrt{x^2} = x$ and $\left(\sqrt{x}\right)^2 = x$. This is the same as for the exponential function and the natural logarithm function: $e^{\log_e x} = x$ and $\log_e e^x = x$.

Summary of important results—exponentials and logarithms

- If $y = e^x$ then $\log_e y = x$ or $\ln y = x$
- If $y = \log_e x$ or $y = \ln x$ then $e^y = x$
- $e^{\log_e x} = x$ or $e^{\ln x} = x$
- $\log_e e^x = x$ or $\ln e^x = x$
- $e^0 = 1$ and $\log_e 1 = 0$
- $e^x > 0$ for all x
- Domain of $y = \log_e x$ is $x > 0$

The derivative of $\log_e x$

We have seen that $g(x) = e^x$ and $f(x) = \log_e x$ define an inverse pair of functions. The graph of one of these functions is the reflection of the graph of the other function in the line $y = x$. This means that the rule for one function is obtained by writing x in place of y in the rule for the other. (This also means that their domains and ranges interchange.) This all leads to an important property relating to their gradients at any point:

Consider the chain rule: $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

Now write x in place of y: $\dfrac{d(x)}{dx} = \dfrac{dx}{du} \times \dfrac{du}{dx}$.

Hence, because $\dfrac{d}{dx}(x) = 1$, we have: $1 = \dfrac{dx}{du} \times \dfrac{du}{dx}$

Thus: $\dfrac{du}{dx} = \dfrac{1}{\frac{dx}{du}}$

This result is most frequently used in the form $\dfrac{dy}{dx} = \dfrac{1}{\frac{dx}{dy}}$. We can now apply this to the exponential and logarithmic functions.

Consider $y = \log_e x$ and rewrite it as $x = e^y$.

Differentiating with respect to y: $\dfrac{dx}{dy} = e^y = x$

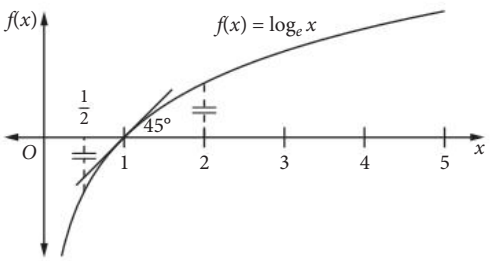
Hence: $\dfrac{dy}{dx} = \dfrac{1}{\frac{dx}{dy}} = \dfrac{1}{x}$

So we have: $\dfrac{d}{dx}(\log_e x) = \dfrac{1}{x}, \quad x > 0$

It follows that $\log_e x$ is a primitive of $\dfrac{1}{x}$, and so: $\displaystyle\int \frac{1}{x}\,dx = \log_e x + C$

This 'fills the gap' in our earlier result for integrating powers of x, $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$, $n \neq -1$, by providing the result for $n = -1$.

Graph of $f(x) = \log_e x$, $x > 0$



The slope of the curve is $\dfrac{1}{x}$, which is positive for all x in the domain ($x > 0$). The graph rises steadily with a very steep slope for small x, but this slope decreases as x increases. When x is large, the slope approaches zero.

The curve crosses the x-axis at $x = 1$ (because $\log_e 1 = 0$), at an angle of 45° because the gradient is 1.

For all $x > 1$, $f(x) > 0$

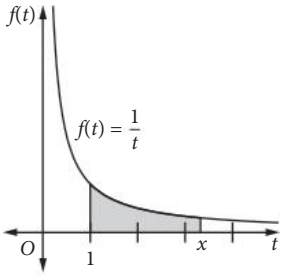
For $0 < x < 1$, $f(x) < 0$

Because $\log_e x = -\log_e\left(\frac{1}{x}\right)$, we have the interesting property that $\log_e 2 = -\log_e \frac{1}{2}$, so the lengths of the ordinates $x = 2$ and $x = \frac{1}{2}$ are equal. This is similarly true for any other value of x and its reciprocal.

Another definition for $\log_e x$

The natural logarithm of x (for $x > 0$), usually denoted by $\ln x$ or $\log_e x$, can also be defined as:

$$\log_e x = \int_1^x \frac{1}{t}\,dt, \quad t > 0$$

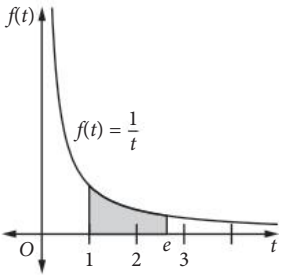


This integral shows that the natural logarithm of x is equal the area bounded by the graph of the rectangular hyperbola $f(t) = \frac{1}{t}$ (for $t > 0$), the t-axis and the ordinates $t = 1$ and $t = x$.

$$\int_1^x \frac{1}{t}\,dt = \left[\log_e t\right]_1^x = \log_e x - \log_e 1 = \log_e x$$

Similarly, the number e can be defined as:

$$\int_1^e \frac{1}{t}\,dt = \left[\log_e t\right]_1^e = \log_e e - \log_e 1 = 1, \quad t > 0$$



This integral shows that the number e is such that the area bounded by the graph of $f(t) = \frac{1}{t}$ (for $t > 0$), the t-axis and the ordinates $t = 1$ and $t = e$ is 1 square unit.

Derivative of $\log_e(ax)$, $a > 0$

Let $y = \log_e(ax)$, so $y = \log_e(a) + \log_e(x)$.

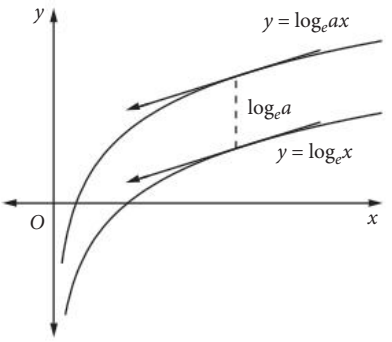
$$\frac{dy}{dx} = 0 + \frac{1}{x}$$
$$= \frac{1}{x}$$

Or, let $y = \log_e u$, where $u = ax$.

$$\frac{dy}{dx} = \frac{1}{u} \times a$$
$$= \frac{a}{ax}$$
$$= \frac{1}{x}, \quad x > 0$$

Hence:

$$\frac{d}{dx}\left(\log_e(ax)\right) = \frac{1}{x}, \quad x > 0$$



The derivatives of $\log_e x$ and $\log_e ax$ are both $\frac{1}{x}$, so the only difference between the graphs of the functions is a vertical translation of a units. If $a > 1$ then $y = \log_e(ax)$ is above $y = \log_e(x)$, but if $0 < a < 1$ then $y = \log_e(ax)$ is below $y = \log_e(x)$.

These are 'parallel curves', because they are a fixed distance apart and their gradients are equal at each corresponding point.

Example 21

Differentiate with respect to x:

(a) $x^2 \log_e(2x)$

(b) $\dfrac{\log_e 3x}{e^x}$

Solution

(a) Let $y = x^2 \log_e(2x) = uv$, where $u = x^2$ and $v = \log_e(2x)$.

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$
$$= 2x \log_e 2x + x^2 \times \frac{1}{x}, \quad x > 0$$
$$= x(2\log_e 2x + 1)$$

(b) Let $y = \dfrac{\log_e 3x}{e^x} = \dfrac{u}{v}$, where $u = \log_e(3x)$ and $v = e^x$.

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$
$$\frac{dy}{dx} = \frac{e^x \times \frac{1}{x} - \log_e 3x \times e^x}{e^{2x}}$$
$$= \frac{e^x\left(1 - x\log_e 3x\right)}{xe^{2x}}$$
$$= \frac{1 - x\log_e 3x}{xe^x}$$

Derivative of $\log_e(ax + b)$

Let $y = \log_e(ax + b) = \log_e u$, where $u = ax + b$.

Chain rule: $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

$$= \frac{1}{u} \times a$$
$$= \frac{a}{ax + b}$$

Hence:

$$\frac{d}{dx}\left(\log_e(ax+b)\right) = \frac{a}{ax+b} \text{ and } \int \frac{1}{ax+b}\,dx = \frac{1}{a}\log_e(ax+b) + C$$

Example 22

Differentiate:

(a) $\log_e(x^3 + 1)$

(b) $\log_e(x^2 + 2x - 1)$

Solution

(a) Let $y = \log_e(x^3 + 1)$
$= \log_e u$, where $u = x^3 + 1$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= \frac{1}{u} \times 3x^2$$
$$= \frac{3x^2}{x^3 + 1}$$

(b) Let $y = \log_e(x^2 + 2x - 1)$
$= \log_e u$, where $u = x^2 + 2x - 1$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= \frac{1}{u} \times (2x + 2)$$
$$= \frac{2x + 2}{x^2 + 2x - 1}$$

In Example **22**, each derivative is a quotient in which the numerator is the derivative of the denominator. It can be seen that:

$$\frac{d}{dx}\left(\log_e\left[f(x)\right]\right)=\frac{f'(x)}{f(x)}$$

The corresponding indefinite integrals are: $\int \frac{f'(x)}{f(x)}dx=\log_e\left[f(x)\right]+C$ and $\int \frac{af'(x)}{f(x)}dx=a\log_e\left[f(x)\right]+C$

Summary of logarithm derivatives

$$\frac{d}{dx}\left(\log_e x\right)=\frac{1}{x},\ x>0 \qquad \frac{d}{dx}\left(\log_e(ax+b)\right)=\frac{a}{ax+b}$$

$$\frac{d}{dx}\left(\log_e(ax)\right)=\frac{1}{x},\ x>0 \qquad \frac{d}{dx}\left(\log_e\left[f(x)\right]\right)=\frac{f'(x)}{f(x)}$$

TECHNOLOGY EXPLORATION

Logarithmic graphs

You can use GeoGebra to investigate the relationship between $\log_e x$ and $\log_n x$ using the change of base law.

1. In GeoGebra, enter *f(x)=ln(x)* into the input bar. Right-click on the curve of *f(x)* and select **Object Properties** to change the colour.
2. Use the **Slider** tool to create a slider *n* with Min = 1, Max = 20 and Increment = 0.1.
3. In the input bar, enter *g(x)=ln(x)/ln(n)*.
4. Adjust the slider to see how the two curves *f* and *g* relate to each other.

You can also use GeoGebra investigate the relationship between $\log ax$ and $\log(x-a)$. (Before you start, remember to delete all previous objects and equations so that the GeoGebra screen is empty.)

1. In GeoGebra, enter *f(x)=ln(x)* into the input bar. Right-click on the curve of *f(x)* and select **Object Properties** to change the colour.
2. Use the **Slider** tool to create a slider *a* with Min = -20, Max = 20 and Increment = 0.1.
3. In the input bar, enter *f(a*x)* (this will be labelled *g(x)*). Adjust the slider to see how this new curve reacts to changes in the value of *a*.
4. Hide the new curve *g* by clicking the circle next to its equation in the **Algebra** view.
5. In the input bar, enter *f(x-a)*. Adjust the slider to see how this curve reacts to changes in the value of *a*.

EXERCISE 14.9 NATURAL LOGARITHMS

1 Differentiate:

(a) $\log_e 2x$ **(b)** $2\log_e x$ **(c)** $\log_e x^2$

(d) $\log_e(3x-5)$ **(e)** $\log_e x+3$ **(f)** $x^2-\log_e(4x-1)$

2 State the largest possible domain of each of the following functions and find their derivatives.

(a) $f(x)=\log_e(3x+2)$ **(b)** $f(x)=\log_e(x^2+1)$ **(c)** $f(x)=\log_e(x^2-4x+4)$

(d) $f(x)=\log_e(4x+3)$ **(e)** $\ln\sqrt{x}$ **(f)** $\log_e\left(x+\sqrt{x}\right)$

3 The derivative of $\log_e(3x^2+1)$ is:

A $6x$ B $\frac{6}{x}$ C $\frac{6x}{3x^2+1}$ D $\frac{1}{x^3+x}$

4 Differentiate:

(a) $x\ln x$ (b) $x^3\ln x$ (c) $(x+2)\ln(x+2)$ (d) $(x^2+1)\ln 2x$

(e) $(2x-5)\ln x$ (f) $e^x\ln x$ (g) $e^{2x}\ln 2x$ (h) $\frac{x}{\log_e x}$

(i) $\frac{\log_e x}{x}$ (j) $\frac{\log_e x}{e^x}$ (k) $\frac{\log_e(x^2+1)}{x}$ (l) $e^x\log_e(e^x+1)$

5 Differentiate:

(a) $\log_e(x^2-2x)$ (b) $\log_e(e^x)$ (c) $\log_e(x^3)$ (d) $\log_e\left(\frac{x-3}{x+3}\right)$

(e) $\log_e(x^2-1)$ (f) $\log_e(x^4)$ (g) $e^{\log_e x}$ (h) $x^2\log_e(x^2)$

(i) $\log_e x\sqrt{x}$ (j) $\log_e(\log_e x)$ (k) $e^{x\log_e x}$ (l) $\log_e\left(\frac{e^x+1}{e^x-1}\right)$

6 Find the gradient of the curve $y=\log_e(x^2+1)$ at the point where $x=3$.

7 If $f(x)=\log_e x$, find: (a) $f'(x)$ (b) $f''(x)$ (c) $f'(2)$ (d) $f''(2)$

8 Find the equation of the tangent and normal to the curve $y=\log_e x$ at the point where it crosses the x-axis.

9 (a) On the same axes, sketch the graphs of $y=\log_e x$ and $y=\log_e\left(\frac{x}{2}\right)$ and show that their gradients are equal for all values of x.

(b) Write the equation of the tangent to the curve $y=\log_e\left(\frac{x}{2}\right)$ at the point where it crosses the x-axis.

10 Show that $y=\log_e(e^x)$ is equivalent to $y=x$ for all values of x and hence state its gradient.

11 Sketch the graph of $y=\log_e(x-1)$, stating its largest possible domain. Find the equation of the tangent at the point where $x=2$.

12 Solve: (a) $e^x=2$ (b) $e^{3x}=5$ (c) $e^{2x+3}=7$ (d) $e^{x^2-1}=10$

13 Show that $y=e^{2x}+4e^{-2x}$ has a minimum value when $x=\frac{\ln 2}{2}$. What is this minimum value?

14.10 INTEGRALS RESULTING IN LOGARITHMIC FUNCTIONS

Summary of results

$$\int \frac{1}{x}\,dx=\log_e x+C \qquad \int \frac{f'(x)}{f(x)}\,dx=\log_e\left[f(x)\right]+C$$

$$\int \frac{1}{ax+b}\,dx=\frac{1}{a}\log_e(ax+b)+C \qquad \int \frac{af'(x)}{f(x)}\,dx=a\log_e\left[f(x)\right]+C$$

Example 23

Find the indefinite integral of:

(a) $\frac{2}{2x-3}$ (b) $\frac{x}{x^2+4}$ (c) $\frac{4x-6}{x^2-3x}$ (d) $\frac{e^x}{1+e^x}$

Solution

(a) $\int \frac{2}{2x-3}dx = \int \frac{f'(x)}{f(x)}dx$ where $f(x) = 2x - 3$ and $f'(x) = 2$

$$\int \frac{2}{2x-3}dx = \log_e(2x-3) + C$$

(b) $\int \frac{x}{x^2+4}dx = \frac{1}{2}\int \frac{2x}{x^2+4}dx$

$= \frac{1}{2}\int \frac{f'(x)}{f(x)}dx$ where $f(x) = x^2 + 4$ and $f'(x) = 2x$

$= \frac{1}{2}\log_e\left(x^2+4\right) + C$

(c) $\int \frac{4x-6}{x^2-3x}dx = 2\int \frac{2x-3}{x^2-3x}dx$

$= 2\int \frac{f'(x)}{f(x)}dx$ where $f(x) = x^2 - 3x$ and $f'(x) = 2x - 3$

$= 2\log_e(x^2-3x) + C$

(d) $\int \frac{e^x}{1+e^x}dx = \int \frac{f'(x)}{f(x)}dx$ where $f(x) = 1 + e^x$ and $f'(x) = e^x$

$= \log_e(1 + e^x) + C$

Example 24

If $\frac{dy}{dx} = \frac{1}{x}$ and $y = 0$ when $x = 0.5$, express y in terms of x.

Solution

$\frac{dy}{dx} = \frac{1}{x}$

$y = \int \frac{1}{x}dx$

$= \log_e x + C$

$\therefore y = \log_e x + \log_e 2$ or $y = \log_e 2x$

When $x = 0.5$, $y = 0$, so: $0 = \log_e \frac{1}{2} + C$

$0 = -\log_e 2 + C$

$C = \log_e 2$

Example 25

The gradient of a curve at any point is $\frac{4x}{x^2+1}$ and the curve passes through the point $(0, 0)$. Find the equation of the curve.

Solution

$f'(x) = \frac{4x}{x^2+1}$

$\therefore f(x) = \int \frac{4x}{x^2+1}dx$

$= 2\int \frac{2x}{x^2+1}dx$

$= 2\log_e(x^2+1) + C$

Because $f(0) = 0$: $0 = 2\log_e 1 + C$

$C = 0$

$\therefore f(x) = 2\log_e(x^2+1)$

Example 26

Evaluate: (a) $\int_1^2 \frac{3}{x+1}dx$ (b) $\int_3^4 \frac{2x-1}{x^2-x-2}dx$ (c) $\int_2^4 \frac{x^2-1}{x}dx$

Solution

(a) $\int_1^2 \frac{3}{x+1}dx = 3\int_1^2 \frac{1}{x+1}dx$

$= 3\left[\log_e(x+1)\right]_1^2$

$= 3\left(\log_e 3 - \log_e 2\right)$

$= 3\log_e 1.5$

≈ 1.216

(b) $\int_3^4 \frac{2x-1}{x^2-x-2}dx = \left[\log_e\left(x^2-x-2\right)\right]_3^4$

$= \log_e 10 - \log_e 4$

$= \log_e 2.5$

≈ 0.916

(c) $\frac{x^2-1}{x} = \frac{x^2}{x} - \frac{1}{x} = x - \frac{1}{x}$, so: $\int_2^4 \frac{x^2-1}{x}dx = \int_2^4 \left(x - \frac{1}{x}\right)dx$

$= \left[\frac{x^2}{2} - \ln x\right]_2^4$

$= 8 - \ln 4 - (2 - \ln 2)$

$= 6 - \ln 2$

≈ 5.307

If question **(c)** had asked for an exact answer, the answer would be written as $6 - \ln 2$.

EXERCISE 14.10 INTEGRALS RESULTING IN LOGARITHMIC FUNCTIONS

1 Find the primitive of:

(a) $\frac{2}{x}$ (b) $\frac{1}{x+1}$ (c) $\frac{2}{2x+1}$ (d) $\frac{x}{x^2-4}$

(e) $\frac{1}{2x-1}$ (f) $\frac{e^x}{4+e^x}$ (g) $\frac{x^3}{x^4+1}$ (h) $\frac{e^{2x}}{4-e^{2x}}$

2 If $\frac{dy}{dx} = \frac{1}{x}$ and $y = 0$ when $x = 2$, then the correct expression for y in terms of x is:

A $y = \log_e x - 2$ **B** $y = \frac{1}{2}\log_e x$ **C** $y = \log_e\left(\frac{x}{2}\right)$ **D** $y = 2\log_e x$

3 The gradient of a curve at any point is $\frac{2}{2x+1}$ and the curve passes through the point $(1, \log_e 3)$. Find the equation of the curve.

4 Find the rule that defines $f(x)$ given that $f'(x) = \frac{x}{x^2+9}$ and $f(0) = \log_e 3$. Indicate whether each statement below is a correct on incorrect step in the solution of this problem.

(a) $f(x) = \int \frac{x}{x^2+9}dx$ (b) $f(x) = \frac{1}{2}\ln\left(x^2+9\right) + C$ (c) $C = 2\ln 3$ (d) $f(x) = \frac{1}{2}\ln\left(x^2+9\right)$

5 Evaluate:

(a) $\int_2^3 \frac{1}{x-1}dx$ (b) $\int_0^3 \frac{2}{x+3}dx$ (c) $\int_{-2}^0 \frac{dx}{5+2x}$ (d) $\int_2^4 \frac{2}{2t-3}dt$

(e) $\int_3^5 \frac{2x}{x^2-1}dx$ (f) $\int_3^6 \frac{3}{x-1}dx$ (g) $\int_5^7 \frac{x-1}{x^2-2x}dx$ (h) $\int_0^4 \frac{6}{2x+3}dx$

6 Evaluate:

(a) $\int_1^3 \left(x^2 + \frac{1}{x}\right) dx$ (b) $\int_{-3}^3 \frac{x}{x^2+1} dx$ (c) $\int_2^4 \frac{x^2-1}{x} dx$ (d) $\int_2^4 \left(\frac{1}{x} + \frac{1}{x^2}\right) dx$

(e) $\int_2^3 \left(x + \frac{1}{x-1}\right) dx$ (f) $\int_1^2 \left(x - \frac{1}{x^2}\right)^2 dx$ (g) $\int_1^3 \left(e^x + \frac{1}{x}\right) dx$ (h) $\int_1^4 \left(\sqrt{x} - \frac{1}{x}\right) dx$

14.11 APPLICATIONS OF THE LOGARITHMIC FUNCTION

Example 27

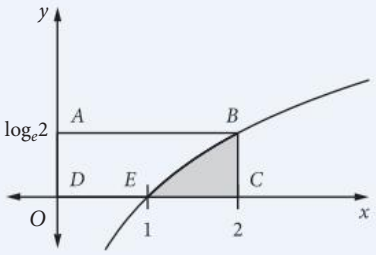
Find: **(a)** the area bounded by the curve $y = \log_e x$, the x-axis and the ordinate $x = 2$

(b) the volume of the solid of revolution formed by rotating the area bounded by the curve $y = \log_e x$, the coordinate axes and the line $y = \log_e 2$ about the y-axis.

Solution

(a) Area $= \int_1^2 \log_e x \, dx$

We don't learn how to evaluate this integral in this course. Instead, draw a diagram to see whether there may be another way to calculate the area.



We require the area of the shaded region BCE. It can be obtained by finding the area of the rectangle $ABCD$ and subtracting the area $ABED$.

Because $y = \log_e x$, we can write $x = e^y$.

At $x = 2$, $y = \log_e 2$:

$$\text{Area } ABED = \int_0^{\log_e 2} e^y \, dy$$
$$= \left[e^y\right]_0^{\log_e 2}$$
$$= e^{\log_e 2} - e^0$$
$$= 2 - 1$$
$$= 1$$

$$\text{Area } ABCD = 2\log_e 2$$

$$\therefore \text{Area } BCE = 2\log_e 2 - 1$$
$$\approx 0.386 \text{ units}^2$$

(b) Volume $= \pi \int_0^{\log_e 2} x^2 \, dy$ where $x = e^y$

$$= \pi \int_0^{\log_e 2} e^{2y} \, dy$$
$$= \frac{\pi}{2}\left[e^{2y}\right]_0^{\log_e 2}$$
$$= \frac{\pi}{2}\left(e^{2\log_e 2} - e^0\right)$$
$$= \frac{\pi}{2}(4-1) = \frac{3\pi}{2} \text{ units}^3$$

Example 28

Differentiate $x\log_e x$. Hence find the primitive of $\log_e x$ and so evaluate $\int_1^2 \log_e x\,dx$.

Solution

Let $y = x\log_e x = uv$, where $u = x$ and $v = \log_e x$.

Product rule, $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$: $\frac{d}{dx}(x\log_e x) = 1 \times \log_e x + x \times \frac{1}{x}$

$$\therefore \frac{d}{dx}(x\log_e x) = \log_e x + 1$$

Rearrange: $$\log_e x = \frac{d}{dx}(x\log_e x) - 1$$

The primitive of the derivative of a function is the function itself, so:

$$\int \frac{d}{dx}(x\log_e x)\,dx = x\log_e x + C$$

Hence: $$\int \log_e x\,dx = \int \frac{d}{dx}(x\log_e x)\,dx - \int 1\,dx$$

$$= x\log_e x - x + C$$

$$\therefore \int_1^2 \log_e x\,dx = \left[x\log_e x - x\right]_1^2$$

$$= (2\log_e 2 - 2) - (\log_e 1 - 1)$$

$$= 2\log_e 2 - 1$$

$$\approx 0.386$$

This the same answer as in Example **27(a)**, obtained by a different method.

Example 29

Find the volume generated by rotating about the x-axis the area beneath the curve $y = \frac{1}{\sqrt{x}}$ between $x = 4$ and $x = 9$.

Solution

$$\text{Volume} = \pi\int_4^9 y^2\,dx \qquad \text{where } y = \frac{1}{\sqrt{x}}$$

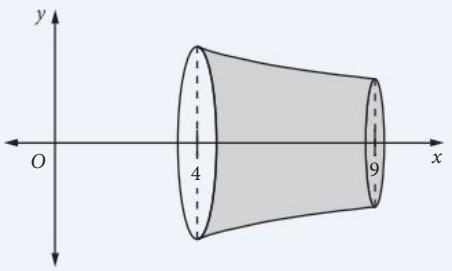
$$= \pi\int_4^9 \frac{1}{x}\,dx$$

$$= \pi\left[\log_e x\right]_4^9$$

$$= \pi(\log_e 9 - \log_e 4)$$

$$= \pi\log_e 2.25$$

$$\approx 2.548$$



EXERCISE 14.11 APPLICATIONS OF THE LOGARITHMIC FUNCTION

1 Sketch the graph of $f(x) = \frac{1}{2x-1}$ and find the area enclosed by the curve, the x-axis and the lines $x = 1$ and $x = 4$.

2 Find the area of the region enclosed by the curve $y = \frac{x}{x^2+1}$, the x-axis and the ordinates $x = 2$ and $x = 4$.

3 For the curve whose equation is $y = x + \frac{4}{x+1}$, find the area enclosed by the curve, the x-axis and the lines $x = 0$ and $x = 2$. Indicate whether each statement below is a correct or incorrect step in the solution of this problem.

(a) Area $= \int_0^2 \left(x + \frac{4}{x+1}\right) dx$ **(b)** $\left[\frac{x^2}{2} + 4\ln(x+1)\right]_0^2$ **(c)** $\left[1 - \frac{4}{(x+1)^2}\right]_0^2$ **(d)** $2 + 4\ln 3$

4 Sketch the curve $y = x + \frac{1}{x}$, $x > 0$, showing its asymptotes and the coordinates of the turning points. Also find the area enclosed by the curve, the x-axis and the lines $x = 1$ and $x = 2$.

5 The value of $\int_0^3 \frac{2x\,dx}{x^2+9} = \ldots$

A $\frac{1}{2}\ln 2$ **B** $\ln 2$ **C** $2\ln 2$ **D** $\ln 18$

6 $\int_0^1 \frac{e^x}{1+e^x} dx = \log_e c$. Find the value of c.

7 Find the area of the region enclosed by the curve $y = \frac{x}{x^2+1}$, the x-axis and the line $x = 1$.

8 Find the volume of the solid generated by rotating about the x-axis the region enclosed by the curve $y^2 = \frac{6}{x}$, the x-axis and the ordinates $x = 1$ and $x = 3$.

9 Find the volume of the solid generated by rotating about the x-axis the area beneath the curve $y = \frac{1}{\sqrt{x-2}}$ between $x = 6$ and $x = 11$.

10 **(a)** Given $a > 1$, sketch the curve $y = \log_e x$ for $1 \le x \le a$. Find the area enclosed by the curve and the lines $y = 0$ and $x = a$.

(b) The region enclosed by the curve $y = \log_e x$ and the lines $x = 0$, $y = \log_e a$ and $y = 0$ is rotated about the y-axis to form a solid of revolution. Find the volume of this solid.

11 Sketch the curve $y = \frac{1}{x^2}$ for values of x from $x = \frac{1}{2}$ to $x = 1$. This part of the curve is rotated about the y-axis to form a solid of revolution. Find its volume.

12 Sketch the curve $y = \frac{1}{\sqrt{4+x}}$ from $x = 0$ to $x = 5$. The region enclosed by the curve, the x-axis and the ordinates $x = 0$ and $x = 5$ is rotated about the x-axis. Find the volume of the solid formed.

13 The region enclosed by the curve $y = \frac{2}{\sqrt{x-7}}$ and the x-axis between $x = 8$ and $x = 10$ is rotated about the x-axis. Find the volume of the solid formed.

14 The region enclosed by the curve $y = \frac{\sqrt{x+1}}{x}$ and the x-axis between $x = 3$ and $x = 5$ is rotated about the x-axis. Find the volume of the solid formed.

15 Use the trapezoidal rule with two subintervals to find the value of $\int_1^3 x\log_e x\,dx$ correct to 3 decimal places.

16 Use Simpson's rule with five function values to find the value of $\int_1^5 \log_e x\,dx$ correct to 3 decimal places.

17 We define the average value of f on the interval $a \le x \le b$ as $f_{ave} = \frac{1}{b-a}\int_a^b f(x)dx$. Find the average value of each function on the interval given.

(a) $f(x) = \frac{1}{x}$, $1 \le x \le 3$ **(b)** $f(x) = \frac{2x}{x^2+1}$, $0 \le x \le 2$

(c) $g(x) = \frac{1}{x+1}$, $0 \le x \le 2$ **(d)** $f(x) = \frac{x}{x^2+4}$, $-2 \le x \le 2$

CHAPTER REVIEW 14

1 Find the values of x for which:

(a) $7^{x+2} = 343$ **(b)** $4^{x-2} < 128$ **(c)** $3^x \geq 12$

2 Simplify:

(a) $\log_3 18 + 2\log_3 9 - \log_3 54$ **(b)** $\log_a(xy^2) + \log_a(yz^2) - \log_a(xz^2)$

(c) $\log_{10}\dfrac{6+4\sqrt{6}}{5} + \log_{10}\dfrac{2\sqrt{6}-3}{2}$

(d) $2\log(x+1) - \log(x-1) - 2\log(y+1) + \log(y-1)$, given $x = 5$ and $y = 2$

3 Express y in terms of x: **(a)** $\log_a y = x$ **(b)** $\log_{10} y = 2 + \log_{10} x - \log_{10}(x^2)$

4 Differentiate with respect to x:

(a) $(x^2 + 2x)e^x$ **(b)** $2e^{-x}\ln x$ **(c)** $\log_e(1 + e^x)$

(d) $\log_e(x^2 + 2x)$ **(e)** $(x^2 + 3x)e^{-3x}$ **(f)** $e^{\sqrt{x}} + \log_e \sqrt{x}$

5 Evaluate: **(a)** $\int_{-2}^{2}\left(e^x - e^{-x}\right)dx$ **(b)** $\int_{-1}^{2}\left(e^x - e^{-x}\right)^2 dx$ **(c)** $\int_{1}^{3}\left(e^x + \frac{1}{x}\right)dx$

6 Sketch the graphs of $y = e^{-x}$, $y = x + 1$ and the line $x = 2$. By integration, find the area of the region bounded by them.

7 **(a)** Find the maximum value of $2xe^{-1.5x}$ and the value for which this function has a maximum value.

(b) If $f(x) = 2xe^{-1.5x}$, find $f(0)$, $f(0.5)$, $f(1)$ and hence graph the function in the domain $0 \leq x \leq 1$.

(c) Use the trapezoidal rule with two subintervals to find the value of $2\int_0^1 xe^{-1.5x}\,dx$ correct to 3 decimal places.

8 If $\theta = \theta_0 e^{-kt}$, show that $\dfrac{d\theta}{dt} = -k\theta$.

9 A car is worth \$10 000 when new. After t years, the value of the car (in dollars) is given by the formula $V = Ae^{-0.2t}$.

(a) Find the value of A. **(b)** Find the value of the car after 6 years.

(c) Find the rate in dollars per year at which the car's value is depreciating, when:

(i) $t = 6$ **(ii)** $V = 5000$

(d) How much time will it take until the car is worth only \$1000?

10 Sketch the graph of $y = e^{-\frac{x}{2}}$, $x \geq 0$, and indicate the region whose area is represented by the integral $\int_0^1 e^{-\frac{x}{2}}\,dx$. Evaluate this integral by: **(a)** direct integration

(b) Simpson's rule using three function values.

11 **(a)** Show that $\dfrac{d}{dx}\left(xe^x\right) = e^x + xe^x$. **(b)** Hence find $\int xe^x\,dx$.

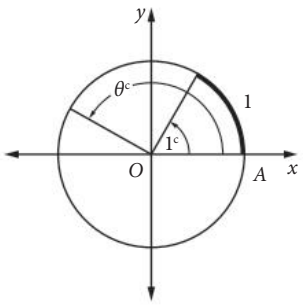
(c) We define the average value of f on the interval $a \leq x \leq b$ as $f_{\text{ave}} = \dfrac{1}{b-a}\int_a^b f(x)\,dx$. If $f(x) = xe^x$, calculate the average value of f over the interval $-1 \leq x \leq 3$.

CHAPTER 15
TRIGONOMETRIC FUNCTIONS

Outcomes covered:

H1 seeks to apply mathematical techniques to problems in a wide range of practical contexts
H4 expresses practical problems in mathematical terms based on simple given models
H5 applies appropriate techniques from the study of calculus, geometry, probability, trigonometry and series to solve problems
H6 uses the derivative to determine the features of the graph of a function
H7 uses the features of a graph to deduce information about the derivative
H8 uses techniques of integration to calculate areas and volumes
H9 communicates using mathematical language, notation, diagrams and graphs

15.1 RADIAN MEASURE OF AN ANGLE



Later in this course, when we look at the calculus of trigonometric functions, we will see that in many situations degrees are not a useful measure of angle size. Instead of using degrees, it is often better to use **radians**.

Consider a unit circle. Starting at point A and moving around the circle in an anticlockwise direction, mark an arc of length 1 unit.

The angle that this arc subtends at the circle's centre is defined to have a magnitude of 1 radian. If we mark any arc of length θ, then the angle that this arc subtends at the circle's centre has a magnitude of θ radians. In this way, the length of the arc is the same number as the size of the angle in radians.

In other words: 1 radian is the angle subtended by an arc of 1 unit in a circle of radius 1 unit.

We can write an angle of θ radians as θ^c, but the 'c' is usually left off so that the angle is written simply as θ. If an angle has no degree symbol °, you should assume that it is measured in radians.

The circumference of a circle of radius r units is given by $C = 2\pi r$. Thus the circumference of the unit circle is 2π and half the circumference is π. An arc of length π therefore subtends a straight angle of size π^c at the centre of the circle. We already know that the size of a straight angle is 180°, so $\pi^c = 180°$:

$$1^c = \left(\frac{180}{\pi}\right)^\circ \approx 57°\,18' \qquad \text{and} \qquad 1° = \left(\frac{\pi}{180}\right)^c \approx 0.017\,45^c$$

So: $\theta^c = \left(\frac{180\theta}{\pi}\right)^\circ$ and $\theta° = \left(\frac{\pi\theta}{180}\right)^c$

The relations $\pi^c = 180°$ and $1° = \left(\frac{\pi}{180}\right)^c$ enable us to convert radians to degrees and degrees to radians.

Thus, for example: $\frac{\pi}{2}^c = 90°$, $\frac{\pi}{3}^c = 60°$, $\frac{4\pi}{5}^c = \frac{4}{5}\times 180° = 144°$, $\frac{7\pi}{6}^c = \frac{7}{6}\times 180° = 210°$.

Note that angles in radians are not always given in terms of π. It is usually necessary to use a calculator and multiply by $\frac{180}{\pi}$ to convert radians to degrees.

Example 1

Convert the following to degrees: **(a)** 2.4^c **(b)** 4.6^c

Solution

(a) $2.4^c = 2.4 \times \frac{180°}{\pi}$

$= 137.5099\ldots°$

$\approx 137° 31'$

(b) $4.6^c = 4.6 \times \frac{180°}{\pi}$

$= 263.561\ldots°$

$\approx 263° 34'$

You can use a calculator to find $\sin 2.4^c$, $\cos 4.6^c$ or any other calculation using radians without converting to degrees. Most calculators will be set to calculate trigonometric functions using degrees, but they will also have a key labelled as **MODE**, **DRG** or something similar that can be used to switch between the use of degrees (often displayed as 'DEG'), radians ('RAD') and sometimes also gradians ('GRAD', which is an obscure unit equal to $\frac{9}{10}$ of a degree). If you are using calculator software on a computer, you may need to change the trigonometry calculation mode through menu settings.

Example 2

Convert to radians: **(a)** $127°$ **(b)** $214° 35'$

Solution

(a) $127° = 127 \times \frac{\pi^c}{180}$

≈ 2.217 radians

(b) $214° 35' = 214.58 \times \frac{\pi^c}{180}$

≈ 3.745 radians

Common conversions

It is important to be able to easily convert between angles in degrees and angles in radians for the common angles that give exact values in trigonometric functions.

Degrees	0°	30°	45°	60°	90°	120°	135°	150°	180°	210°	225°	240°	270°	300°	315°	330°	360°
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π

Adding 360° to an angle in degrees is the same as adding 2π to an angle in radians. Both 360° and 2π each represent a full turn.

EXERCISE 15.1 RADIAN MEASURE OF AN ANGLE

1 Express the following angles in radians, in terms of π.

(a) $30°$ **(b)** $225°$ **(c)** $72°$ **(d)** $210°$ **(e)** $315°$ **(f)** $112° 30'$ **(g)** $330°$ **(h)** $144°$

2 Express, in degrees, the angles whose radian measures are:

(a) $\frac{3\pi}{4}$ **(b)** $\frac{7\pi}{8}$ **(c)** $\frac{6\pi}{5}$ **(d)** $\frac{3\pi}{2}$ **(e)** $\frac{5\pi}{9}$ **(f)** $\frac{11\pi}{12}$ **(g)** 1.8π **(h)** $\frac{11\pi}{8}$

3 Use a calculator to express, in degrees and minutes, the angles whose radian measures are:

(a) 0.5 **(b)** 1.82 **(c)** 3 **(d)** 4.26 **(e)** 2.72 **(f)** 3.426 **(g)** 5.24 **(h)** 4.782

4 Use a calculator to find the radian equivalent of each angle, giving your answer correct to 4 decimal places.

(a) 42° **(b)** 74° **(c)** 105° **(d)** 164° **(e)** 48° 9′ **(f)** 220° **(g)** 138° 12′ **(h)** 72° 8′

5 Indicate whether each statement is correct or is incorrect. 315° is:

(a) $\frac{5\pi}{4}$ **(b)** $\frac{7\pi}{4}$ **(c)** 5.4978 **(d)** 3.9270

6 Convert each angle into radians, in terms of π.

(a) 120° **(b)** 315° **(c)** 210° **(d)** 135° **(e)** 390° **(f)** 405° **(g)** 480° **(h)** 720°
(i) -30° **(j)** -135° **(k)** 450° **(l)** -180° **(m)** 15° **(n)** 22.5° **(o)** 345° **(p)** -67.5°

7 Convert each angle to degrees.

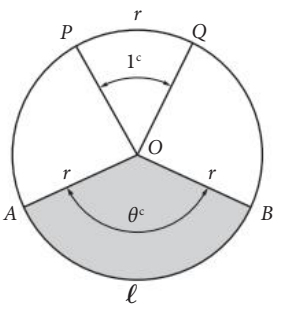
(a) $\frac{5\pi}{6}$ **(b)** $\frac{5\pi}{3}$ **(c)** $\frac{5\pi}{4}$ **(d)** 3π **(e)** $\frac{9\pi}{4}$ **(f)** $\frac{7\pi}{2}$ **(g)** $\frac{10\pi}{3}$ **(h)** $-\pi$

(i) $-\frac{3\pi}{2}$ **(j)** $-\frac{7\pi}{6}$ **(k)** $-\frac{3\pi}{4}$ **(l)** -2π **(m)** $\frac{\pi}{8}$ **(n)** $\frac{5\pi}{12}$ **(o)** $\frac{9\pi}{8}$ **(p)** $-\frac{7\pi}{12}$

15.2 ARC LENGTH AND SECTOR AREA OF A CIRCLE

Arc length

Let the arc AB subtend an angle of θ^c at the centre of a circle of radius r units.
Let the length of the arc $AB = \ell$.
Let $\angle POQ = 1^c$, hence the length of arc $PQ = r$.



Compare arcs: $\frac{\text{arc } AB}{\text{arc } PQ} = \frac{\angle AOB}{\angle POQ}$

Hence $\frac{\ell}{r} = \frac{\theta}{1}$, so:

Arc length: $\ell = r\theta$

Example 3

Find the length of an arc of a circle of radius 12 cm, if the arc subtends an angle at the centre of:

(a) 1.5 radians **(b)** 40°

Solution

(a) $r = 12, \theta = 1.5$
$\ell = r\theta$
$= 12 \times 1.5$
$= 18$
Arc length is 18 cm.

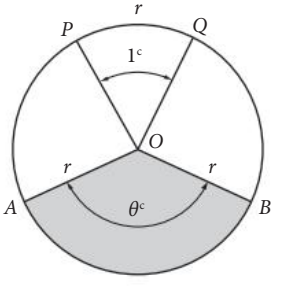
(b) $r = 12, \theta = 40 \times \frac{\pi}{180} = \frac{2\pi}{9}$
$\ell = r\theta$
$\ell = 12 \times \frac{2\pi}{9} = \frac{8\pi}{3}$
$= 8.378$
Arc length is 8.4 cm (correct to 1 decimal place).

Area of a sector

Consider a unit circle with a sector AOB.
Comparing the shaded area AOB to the area of the whole circle gives:

$\frac{\text{Area of sector } AOB}{\text{Area of circle}} = \frac{\angle AOB}{2\pi}$

$\frac{\text{Area of sector } AOB}{\pi r^2} = \frac{\theta}{2\pi}$



Area of sector $AOB = \frac{1}{2}r^2\theta$ $\quad A = \frac{1}{2}r^2\theta$

A useful variation of this formula is $A = \frac{1}{2}r^2\theta = \frac{1}{2} \times r\theta \times r = \frac{1}{2}\ell r$.

Thus if you know the arc length and the radius, you can find the area of the sector without knowing the angle.

Example 4

The length of an arc AB of a circle with centre O and radius 10 cm is 8 cm. Calculate:

(a) $\angle AOB$ in degrees and minutes **(b)** the area of the sector AOB.

Solution

(a) $r = 10, \ell = 8$

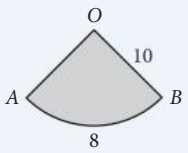
$\ell = r\theta$

$8 = 10\theta$

$\theta = 0.8^c$

$\theta = 0.8 \times \dfrac{180°}{\pi}$

$\theta = 45° 50'$



(b) $r = 10, \ell = 8, \theta = 0.8$

$A = \frac{1}{2}r^2\theta$ $\quad A = \frac{1}{2}\ell r$

$= \frac{1}{2} \times 100 \times 0.8$ **OR** $= \frac{1}{2} \times 8 \times 10$

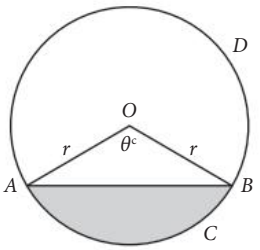
$= 40$ $\quad = 40$

Area of sector is 40 cm^2.

Area of a segment of a circle

This is an application combining the formula for the area of a sector with the formula for the area of a triangle.

The chord AB cuts the circle into two segments: minor segment ACB (the shaded portion at right) and major segment ADB.



Area of segment ACB = Area of sector AOB − Area of ΔAOB

$= \frac{1}{2}r^2\theta - \frac{1}{2}r^2 \sin\theta$

Area of segment $= \frac{1}{2}r^2(\theta - \sin\theta)$, where θ is in radians.

Example 5

An arc AB subtends an angle of 126° 52′ at the centre O of a circle with radius 6 cm. Calculate:

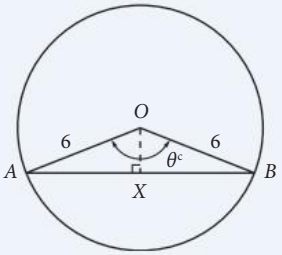
(a) the length of the arc AB, correct to 1 decimal place

(b) the length of the chord AB, correct to 1 decimal place

(c) the area of the sector AOB, to the nearest cm^2

(d) the area of the minor segment cut by the chord AB, to the nearest cm^2.

Solution



(a) $\theta = 126° 52', r = 6$

Use your calculator to convert 126° 52′ to 126.866…°, then:

$\theta = 126.867 \times \dfrac{\pi}{180} = 2.214^c$

$\ell = r\theta$

$\ell = 6 \times 2.214$

$= 13.284$

Length of arc AB = 13.3 cm.

(b) $AB = 2AX = 2 \times 6 \sin 63° 26'$
$= 10.73$
Length of chord $AB = 10.7$ cm.

(c) Area of sector $AOB = \frac{1}{2} r^2 \theta$
$= \frac{1}{2} \times 36 \times 2.214$
$= 39.85$
Area of sector $AOB = 40\,\text{cm}^2$

(d) Area of minor segment $= \frac{1}{2} r^2 (\theta - \sin\theta)$
$= \frac{1}{2} \times 36(2.214 - \sin 2.214)$ [Remember to make sure that your calculator is set to **radians**.]
$= 25.4$
Area of minor segment $= 25\,\text{cm}^2$

EXERCISE 15.2 ARC LENGTH AND SECTOR AREA OF A CIRCLE

1 Calculate, to the nearest minute of a degree, the size of the angle subtended at the centre of a circle of radius 8 cm by an arc whose length is 15 cm.

2 Find the length of an arc of a circle of radius 15 cm if the arc subtends an angle of 70° at the centre.

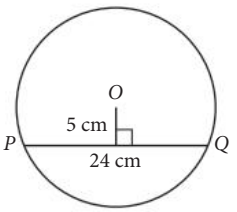
3 An arc of a circle subtends an angle of 100° at the centre. If the radius of the circle is 12 cm, then the length of the arc in terms of π is:

A 40π cm **B** $\frac{108}{5\pi}$ cm **C** $\frac{20\pi}{3}$ cm **D** $\frac{108\pi}{5}$ cm

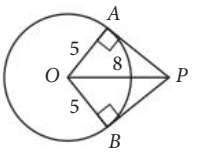
4 The area of a sector AOB of a circle with centre O, radius 20 cm is 240 cm^2. Calculate:

(a) the size of $\angle AOB$, in radians **(b)** the length of the arc AB
(c) the length of the chord AB.

5 A chord PQ, 24 cm long, is 5 cm from the centre of a circle. Calculate the length of the arc PQ correct to 1 decimal place.



6 A point P is 8 cm from the centre of a circle of radius 5 cm. Find the length of the major arc between the points where the circle touches the tangents that are drawn from P to the circle.



7 In a circle with centre O, the length of the chord AB is 16 cm. If the radius of the circle is 10 cm, calculate:

(a) the size of $\angle AOB$ **(b)** the length of the minor arc AB
(c) the area of the minor segment formed by the chord AB.

8 The length of the minute hand of a clock is 20 cm. Calculate:

(a) the length of the arc that the tip of the hand travels along in 16 minutes
(b) the shortest distance between the initial and final positions of the tip of the hand.

9 An arc AB subtends an angle of 60° at the centre O of a circle of radius 15 cm. Indicate whether each statement below is correct or incorrect.

(a) Arc $AB = 5\pi$ cm **(b)** Arc $AB = 15$ cm
(c) Chord $AB = 15$ cm **(d)** Area of sector $AOB = \frac{75\pi}{2}\,\text{cm}^2$

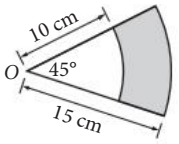
10 From a circular piece of metal 6 cm in diameter, a sector of angle 30° is removed. Find the area that remains, expressing your answer in terms of π.

11 A chord subtends an angle of 140° at the centre of a circle of radius 16 cm. Find the difference in length between the chord and the arc.

12 For the diagram, calculate (in terms of π):

(a) the area of the shaded region between the two circular arcs

(b) the perimeter of the shaded region.



13 The minute hand of a clock is 3 cm long. What area is swept out by the hand in a time interval of 40 minutes? Express your answer in terms of π.

14 A circular metal plate is cut into two segments along a chord equal in length to the radius. What is the ratio of the areas of the two segments?

15 A sheep, grazing in a paddock, is tethered to a stake by a rope 20 m long. If the stake is 10 m from a fence, find the area (to the nearest m^2) over which the sheep can graze.

16 A sector of angle 160° is cut out of a circular piece of thin cardboard of radius 10 cm. The cut edges of this sector are brought together to shape the cardboard into a cone. Find the circumference of the circular base of the cone.

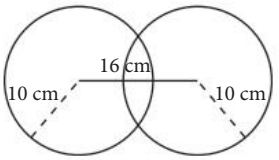
17 A piece of land is in the shape of a circular sector of radius 25 metres and angle 30°. Find the piece of land's: **(a)** perimeter **(b)** area.

18 A pendulum 40 cm long swings through an angle of 25°. Find:

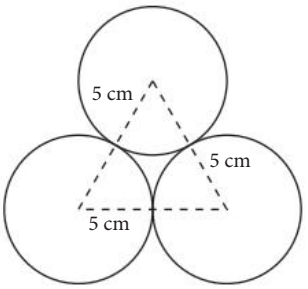
(a) the length of the arc

(b) the shortest distance between the extreme positions of the bob (bottom of the pendulum).

19 Two circles, each of radius 10 cm, have their centres 16 cm apart. Calculate the area common to the two circles.



20 Three circles are drawn, each with a radius of 5 cm. Their centres are at the vertices of an equilateral triangle with sides of length 10 cm. Calculate the area that is between the circles and inside the triangle.



21 The points K, L, N and M are on the circumference of a circle with centre O and radius r. $KL = KM = r\sqrt{3}$.

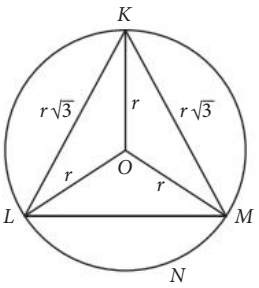
(a) Use the cosine rule, or otherwise, to find $\angle LKO$.

(b) Calculate the area of ΔKLM.

(c) Write the length of LM.

(d) By finding the area of the circle, or otherwise, calculate the area of the segment LMN.

(e) A circle with centre K is drawn through the points L and M. Calculate the area between the arc LNM and this new arc joining L and M.



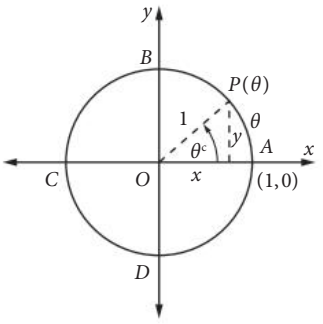
15.3 ANGLES OF ANY MAGNITUDE

This topic was introduced in Chapter 5 (section 5.1) using degrees. We can now investigate the same topic using radians.

Definitions

We can define circular or trigonometric functions as functions of real numbers. For this purpose, we consider a circle of unit radius defined by the equation $x^2 + y^2 = 1$.

Starting from the point A, we can mark an arc of length θ for any real number θ. The arc AP subtends an angle of θ^c at the centre of the circle. In this way, the same number θ measures the length of the arc as well as the number of radians in $\angle AOP$. For $\theta > 0$, we mark the arc in an anticlockwise direction; for $\theta < 0$ we mark the arc in a clockwise direction. If $P(\theta)$ is the endpoint of this arc, with coordinates (x, y), then we can define the functions cosine and sine as:



$$\cos\theta = x \qquad \sin\theta = y$$

Both these functions have the set of real numbers as their domain. Because $-1 \le x \le 1$ and $-1 \le y \le 1$, the range of each function is from -1 to 1 inclusive. Considering the unit circle diagram above and our definition of x and y:

- At A, $\theta = 0$; the coordinates of A are $(1, 0)$ and so $\cos 0 = 1$ and $\sin 0 = 0$.
- At B, $\theta = \frac{\pi}{2}$; the coordinates of B are $(0, 1)$ and so $\cos\frac{\pi}{2} = 0$ and $\sin\frac{\pi}{2} = 1$.
- At C, $\theta = \pi$; the coordinates of C are $(-1, 0)$ and so $\cos\pi = -1$ and $\sin\pi = 0$.
- At D, $\theta = \frac{3\pi}{2}$; the coordinates of D are $(0, -1)$ and so $\cos\frac{3\pi}{2} = 0$ and $\sin\frac{3\pi}{2} = -1$.
- At A again, $\theta = 2\pi$; the coordinates of A are $(1, 0)$ and so $\cos 2\pi = 1$ and $\sin 2\pi = 0$.
- The circumference of the unit circle is 2π.

Notice that $\cos 2\pi = \cos 0 = 1$ and $\sin 2\pi = \sin 0 = 0$. In fact, the points $P(\theta)$, $P(2\pi + \theta)$, $P(4\pi + \theta)$ etc. all coincide. In general, $P(\theta)$ and $P(2k\pi + \theta)$ coincide, where k is any integer. This allows us to write $\cos(2\pi + \theta) = \cos\theta$ and $\sin(2\pi + \theta) = \sin\theta$. Functions whose values recur at regular intervals like this (in this case, with a period of 2π) are called **periodic functions**.

Aside from sine and cosine, there are four other trigonometric functions: tangent, cotangent, secant and cosecant (abbreviated as tan, cot, sec and cosec, respectively) that can all be defined in terms of cos and sin:

$$\tan\theta = \frac{y}{x} = \frac{\sin\theta}{\cos\theta}, \quad \cos\theta \neq 0 \qquad \sec\theta = \frac{1}{x} = \frac{1}{\cos\theta}, \quad \cos\theta \neq 0$$

$$\cot\theta = \frac{x}{y} = \frac{\cos\theta}{\sin\theta}, \quad \sin\theta \neq 0 \qquad \operatorname{cosec}\theta = \frac{1}{y} = \frac{1}{\sin\theta}, \quad \sin\theta \neq 0$$

Thus there are restrictions on the domains of these functions. The functions tan and sec are not defined for $\cos\theta = 0$, i.e. for $\theta = 0, \frac{\pi}{2}, \frac{3\pi}{3}, \frac{5\pi}{2}, \ldots$; cot and cosec are not defined for $\sin\theta = 0$, i.e. for $\theta = 0, \pi, 2\pi, \ldots$

Also note that cot, sec and cosec are the reciprocal functions of tan, cos and sin respectively.

Symmetry properties of trigonometric functions

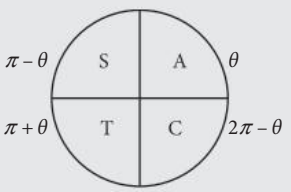
The results established in Chapter 5 for $90° - \theta$, $180° \pm \theta$, $360° \pm \theta$ and $-\theta$ are still true when we change from degrees to radians measure.

Degrees	Radians
$\sin(90° - \theta) = \cos\theta$	$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$
$\cos(90° - \theta) = \sin\theta$	$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$
$\tan(90° - \theta) = \cot\theta$	$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$
$\sin(180° - \theta) = \sin\theta$	$\sin(\pi - \theta) = \sin\theta$
$\cos(180° - \theta) = -\cos\theta$	$\cos(\pi - \theta) = -\cos\theta$
$\tan(180° - \theta) = -\tan\theta$	$\tan(\pi - \theta) = -\tan\theta$
$\sin(180° + \theta) = -\sin\theta$	$\sin(\pi + \theta) = -\sin\theta$
$\cos(180° + \theta) = -\cos\theta$	$\cos(\pi + \theta) = -\cos\theta$
$\tan(180° + \theta) = \tan\theta$	$\tan(\pi + \theta) = \tan\theta$
$\sin(360° - \theta) = -\sin\theta$	$\sin(2\pi - \theta) = -\sin\theta$
$\cos(360° - \theta) = \cos\theta$	$\cos(2\pi - \theta) = \cos\theta$
$\tan(360° - \theta) = -\tan\theta$	$\tan(2\pi - \theta) = -\tan\theta$
$\sin(360° + \theta) = \sin\theta$	$\sin(2\pi + \theta) = \sin\theta$
$\cos(360° + \theta) = \cos\theta$	$\cos(2\pi + \theta) = \cos\theta$
$\tan(360° + \theta) = \tan\theta$	$\tan(2\pi + \theta) = \tan\theta$

Sign of the trigonometric ratios

The sign of sin, cos and tan for the first four quadrants can be summarised as follows:

- First quadrant: All are positive (A)
- Second quadrant: sin only is positive (S)
- Third quadrant: tan only is positive (T)
- Fourth quadrant: cos only is positive (C)



You may find this easier to remember using the mnemonic **ASTC**: '**A**ll **S**tations **T**o **C**entral'.

Exact values in radians

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined
$\operatorname{cosec}\theta$	undefined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
$\sec\theta$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	undefined
$\cot\theta$	undefined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

Example 6

Find the exact values of: (a) $\sin\frac{5\pi}{6}$ (b) $\cos\frac{5\pi}{4}$ (c) $\tan\frac{4\pi}{3}$ (d) $\sin\frac{3\pi}{2}$ (e) $\cos\left(-\frac{5\pi}{3}\right)$ (f) $\sec\frac{2\pi}{3}$ (g) $\operatorname{cosec}\frac{11\pi}{6}$

Solution

(a) $\sin\frac{5\pi}{6} = \sin\left(\pi - \frac{\pi}{6}\right) = \sin\frac{\pi}{6} = \frac{1}{2}$

(b) $\cos\frac{5\pi}{4} = \cos\left(\pi + \frac{\pi}{4}\right) = -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$

(c) $\tan\frac{4\pi}{3} = \tan\left(\pi + \frac{\pi}{3}\right) = \tan\frac{\pi}{3} = \sqrt{3}$

(d) $\sin\frac{3\pi}{2} = \sin\left(\pi + \frac{\pi}{2}\right) = -\sin\frac{\pi}{2} = -1$

(e) $\cos\left(-\frac{5\pi}{3}\right) = \cos\frac{5\pi}{3} = \cos\left(2\pi - \frac{\pi}{3}\right) = \cos\frac{\pi}{3} = \frac{1}{2}$

(f) $\sec\frac{2\pi}{3} = \sec\left(\pi - \frac{\pi}{3}\right) = -\sec\frac{\pi}{3} = -\frac{1}{\cos\frac{\pi}{3}} = -2$

(g) $\operatorname{cosec}\frac{11\pi}{6} = \operatorname{cosec}\left(2\pi - \frac{\pi}{6}\right) = -\operatorname{cosec}\frac{\pi}{6} = -\frac{1}{\sin\frac{\pi}{6}} = -2$

Example 7

Find all values of θ, $0 < \theta < 2\pi$, for which:

(a) $\cos\theta = \frac{1}{2}$ (b) $\sin\theta = -\frac{1}{\sqrt{2}}$ (c) $\tan\theta = 1$ (d) $\sec\theta = -\frac{2}{\sqrt{3}}$ (e) $\sin\theta = -1$

Solution

(a) $\cos\theta > 0$, so θ is in the 1st and 4th quadrants: $\cos\theta = \frac{1}{2}$: $\theta = \frac{\pi}{3}, 2\pi - \frac{\pi}{3}$

$\theta = \frac{\pi}{3}, \frac{5\pi}{3}$

(b) $\sin\theta < 0$, so θ is in the 3rd and 4th quadrants: $\sin\theta = -\frac{1}{\sqrt{2}}$: $\theta = \pi + \frac{\pi}{4}, 2\pi - \frac{\pi}{4}$

$\theta = \frac{5\pi}{4}, \frac{7\pi}{4}$

(c) $\tan\theta > 0$, so θ is in the 1st and 3rd quadrants: $\tan\theta = 1$: $\theta = \frac{\pi}{4}, \pi + \frac{\pi}{4}$

$\theta = \frac{\pi}{4}, \frac{5\pi}{4}$

(d) $\sec\theta < 0$, so θ is in the 2nd and 3rd quadrants: $\sec\theta = -\frac{2}{\sqrt{3}}$ $\cos\theta = -\frac{\sqrt{3}}{2}$: $\theta = \pi - \frac{\pi}{6}, \pi + \frac{\pi}{6}$

$\theta = \frac{5\pi}{6}, \frac{7\pi}{6}$

(e) $\sin\theta < 0$, so θ is in the 3rd and 4th quadrants: $\sin\theta = -1$: $\theta = \pi + \frac{\pi}{2}, 2\pi - \frac{\pi}{2}$

$\theta = \frac{3\pi}{2}$

EXERCISE 15.3 ANGLES OF ANY MAGNITUDE

1 Express each of the following as a simpler trigonometric function.

(a) $\sin(\pi - x)$ **(b)** $\cos\left(\frac{\pi}{2} - x\right)$ **(c)** $\tan(2\pi - x)$

(d) $\cos(\pi + x)$ **(e)** $\sin(2\pi - x)$ **(f)** $\cot\left(\frac{\pi}{2} - x\right)$

2 For any angle θ, $\cos(\pi - \theta)$ is:

A $-\cos\theta$ **B** $\cos\theta$ **C** $\sin\theta$ **D** $-\sin\theta$

3 Indicate whether each statement is correct or incorrect.

(a) $\cos\left(\frac{\pi}{2} - \theta\right) = \cos\theta$ **(b)** $\cos(2\pi - \theta) = \cos\theta$ **(c)** $\sin(\pi + \theta) = \sin\theta$ **(d)** $\sin(2\pi - \theta) = -\sin\theta$

4 If $\sin\theta = 0.2$, write the value of:

(a) $\sin(\pi - x)$ **(b)** $\sin(2\pi - x)$ **(c)** $\sin(-x)$ **(d)** $\cos\left(\frac{\pi}{2} - x\right)$ **(e)** $\sin(\pi + x)$ **(f)** $\operatorname{cosec} x$

5 If $\tan\theta = t$, express the following in terms of t:

(a) $\cot\theta$ **(b)** $\cot\left(\frac{\pi}{2} - \theta\right)$ **(c)** $\tan(\pi - \theta)$ **(d)** $\tan(2\pi - \theta)$ **(e)** $\cot(\pi - \theta)$ **(f)** $\tan(\pi + \theta)$

6 If $\cos x = c$, express the following in terms of c:

(a) $\sec x$ **(b)** $\cos(-x)$ **(c)** $\cos(\pi - x)$ **(d)** $\cos(2\pi - x)$ **(e)** $\sec(-x)$ **(f)** $\cos(\pi + x)$

7 Write the exact value of:

(a) $\sin\frac{\pi}{2}$ **(b)** $\cos\frac{2\pi}{3}$ **(c)** $\tan\frac{5\pi}{6}$ **(d)** $\cos\pi$

(e) $\sec\frac{3\pi}{4}$ **(f)** $\cot\frac{5\pi}{6}$ **(g)** $\operatorname{cosec}\frac{\pi}{2}$ **(h)** $\sin\frac{2\pi}{3}$

8 Write the exact value of:

(a) $\sin\pi$ **(b)** $\cos\frac{7\pi}{6}$ **(c)** $\tan\frac{5\pi}{4}$ **(d)** $\cot\frac{4\pi}{3}$

(e) $\cos\frac{4\pi}{3}$ **(f)** $\sec\frac{5\pi}{4}$ **(g)** $\tan\pi$ **(h)** $\sin\frac{7\pi}{6}$

9 Write the exact value of:

(a) $\sin\frac{3\pi}{2}$ **(b)** $\tan\frac{5\pi}{3}$ **(c)** $\operatorname{cosec}\frac{11\pi}{6}$ **(d)** $\tan\frac{7\pi}{4}$

(e) $\cot\frac{7\pi}{4}$ **(f)** $\cos\frac{11\pi}{6}$ **(g)** $\sin\frac{5\pi}{3}$ **(h)** $\operatorname{cosec}\frac{5\pi}{3}$

10 Write the exact value of:

(a) $\sin 2\pi$ **(b)** $\sin\frac{13\pi}{6}$ **(c)** $\tan\frac{9\pi}{4}$ **(d)** $\cot\frac{7\pi}{3}$

(e) $\sec\frac{13\pi}{6}$ **(f)** $\cos\frac{5\pi}{2}$ **(g)** $\sin\frac{7\pi}{3}$ **(h)** $\tan\frac{11\pi}{4}$

11 For $\frac{\pi}{2} < x < \pi$, use a diagram of a unit circle to show that:

(a) $\cos(\pi + x) = -\cos x$ **(b)** $\sin(2\pi - x) = -\sin x$

12 If θ is an angle in the 2nd quadrant, state whether the following are positive or negative:

(a) $\cos(\pi - \theta)$ **(b)** $\tan(\pi + \theta)$ **(c)** $\sin\left(\frac{\pi}{2} - \theta\right)$

(d) $\sin(2\pi - \theta)$ **(e)** $\cos(\pi + \theta)$ **(f)** $\tan\left(\frac{\pi}{2} - \theta\right)$

13 Solve, for $0 < x < 2\pi$:

(a) $\sin x = -\frac{\sqrt{3}}{2}$ **(b)** $\tan x = -1$ **(c)** $\operatorname{cosec} x = 2$ **(d)** $\cos x = -1$

(e) $\cot x = \sqrt{3}$ **(f)** $\sec x = -\sqrt{2}$ **(g)** $\sin x = \cos x$ **(h)** $\sin x = 0$

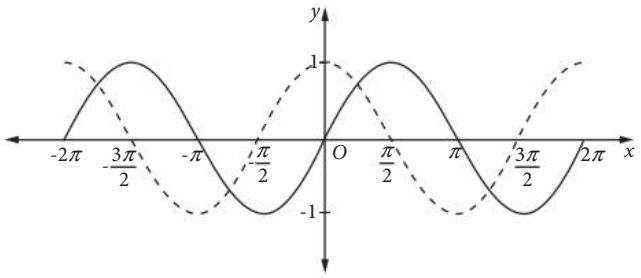
(i) $2\cos x + 1 = 0$ **(j)** $2\sin x = \sqrt{3}$ **(k)** $\sin x + \sqrt{3}\cos x = 0$ **(l)** $\operatorname{cosec} x = \sec x$

14 Using a diagram, find equivalent expressions for:

(a) $\cos\left(\frac{3\pi}{2} + x\right)$ **(b)** $\tan\left(\frac{3\pi}{2} - x\right)$ **(c)** $\sin\left(\frac{3\pi}{2} - x\right)$ **(d)** $\cot\left(\frac{3\pi}{2} + x\right)$

15.4 GRAPHS OF TRIGONOMETRIC FUNCTIONS

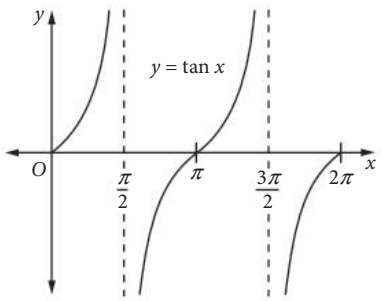
In the diagram below, the graph of $y = \sin x$ is the unbroken curve and the graph of $y = \cos x$ is the dashed curve. The graphs are drawn in the domain $-2\pi \le x \le 2\pi$. For a domain of real numbers, the graphs continue to repeat in both directions, so the graphs for the domains $-2\pi \le x \le 0$, $0 \le x \le 2\pi$ and $2\pi \le \theta \le 4\pi$ are the same.



As x increases, the values of $\sin x$ and $\cos x$ repeat themselves after an interval or **period** of 2π. Sine and cosine are therefore called **periodic functions**. This means that the points $P(\theta)$, $P(2\pi + \theta)$, $P(4\pi + \theta)$ on the unit circle all coincide and hence $\sin(2\pi + \theta) = \sin\theta$, $\cos(2\pi + \theta) = \cos\theta$ and so on.

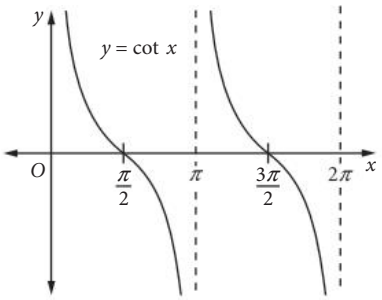
The maximum and minimum values of $\sin x$ and $\cos x$ are 1 and -1 respectively, so we say that their **amplitude** is 1.

If the graph of $y = \cos x$ is translated $\frac{\pi}{2}$ units to the right, parallel with the x-axis, it coincides with the graph of $y = \sin\theta$. You can check that this follows from $\cos x = \sin\left(\frac{\pi}{2} + x\right)$. You can also check by sketching graphs by hand or by using GeoGebra or other graphing software.



The diagram shows the graph of $y = \tan x$ for $0 \le x \le 2\pi$. As x increases, the values of $\tan x$ repeat after an interval or **period** of π. Because $\tan x = \frac{\sin x}{\cos x}$, the graph of $y = \tan x$ does not exist when $\cos x = 0$, so the tan function is undefined at $x = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

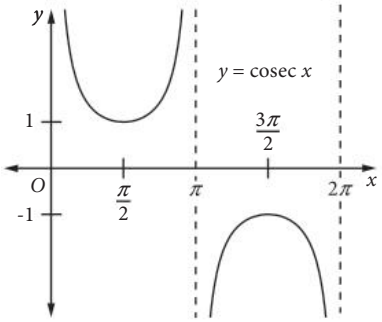
The domain of $\tan x$ is all real x except for $x = \frac{(2n-1)\pi}{2}$, where n is an integer (i.e. odd multiples of $\frac{\pi}{2}$). The range of $\tan x$ is all real y.



The diagram shows the graph of $y = \cot x$ for $0 \le x \le 2\pi$. As x increases, the values of $\cot x$ repeat after an interval or **period** of π.

The cot function is undefined at $x = 0, \pi, 2\pi$.

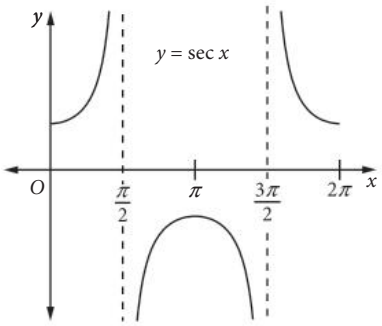
The domain of $\cot x$ is all real x except for $x = n\pi$, where n is an integer (i.e. whole multiples of π). The range of $\cot x$ is all real y.



The diagram shows the graph of $y = \operatorname{cosec} x$ for $0 \le x \le 2\pi$. As x increases, the values of $\operatorname{cosec} x$ repeat after an interval or **period** of 2π.

The cosec function is undefined at $x = 0, \pi, 2\pi$.

The domain of $\operatorname{cosec} x$ is all real x except for $x = n\pi$, where n is an integer (i.e. whole multiples of π). The range of $\operatorname{cosec} x$ is $|y| \ge 1$.



The diagram shows the graph of $y = \sec x$ for $0 \le x \le 2\pi$. As x increases, the values of $\sec x$ repeat after an interval or **period** of 2π.

The sec function is undefined at $x = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

The domain of $\sec x$ is all real x except for $x = \frac{(2n-1)\pi}{2}$, where n is an integer (i.e. odd multiples of $\frac{\pi}{2}$).

The range of $\sec x$ is $|y| \ge 1$.

The graphs of the six circular functions can be drawn as pairs of complementary functions:

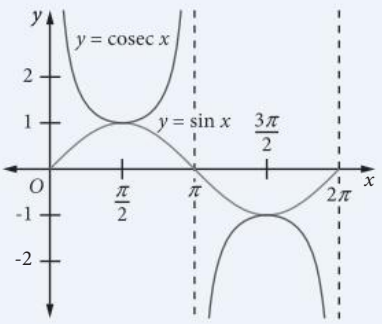
- sine and cosine—because $\cos x = \sin\left(\frac{\pi}{2} - x\right)$ and $\sin x = \cos\left(\frac{\pi}{2} - x\right)$
- tangent and cotangent—because $\cot x = \tan\left(\frac{\pi}{2} - x\right)$ and $\tan x = \cot\left(\frac{\pi}{2} - x\right)$
- cosecant and secant—because $\operatorname{cosec} x = \sec\left(\frac{\pi}{2} - x\right)$ and $\sec x = \operatorname{cosec}\left(\frac{\pi}{2} - x\right)$.

Reciprocal functions

The trigonometric functions can also be considered as pairs of reciprocal functions: sine and cosecant, cosine and secant, and tangent and cotangent. Because these can be defined according to $\operatorname{cosec} x = \frac{1}{\sin x}$, $\sec x = \frac{1}{\cos x}$ and $\cot x = \frac{1}{\tan x}$, we now have another way of sketching these trigonometric functions. For example, to draw $y = \operatorname{cosec} x$, first draw $y = \sin x$ and then take the reciprocal of each y value to obtain the graph of $y = \operatorname{cosec} x$.

Example 8

Use the graph of $y = \sin x$ to sketch the graph of $y = \operatorname{cosec} x$ for $0 \le x \le 2\pi$.



Solution

For $y = 0$, $\frac{1}{y}$ is undefined. For $y = 1$, $\frac{1}{y} = 1$, so the curves intersect at this point. For $y = -1$, $\frac{1}{y} = -1$.

For $0 < y < 1$, $\frac{1}{y} > 1$. Because $\sin\frac{\pi}{6} = \frac{1}{2}$, $\operatorname{cosec}\frac{\pi}{6} = 2$.

For $-1 < y < 0$, $\frac{1}{y} < -1$. Because $\sin\frac{7\pi}{6} = -\frac{1}{2}$, $\operatorname{cosec}\frac{7\pi}{6} = -2$.

Example 9

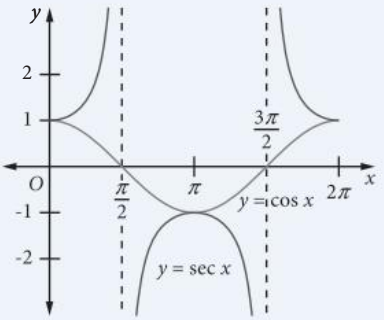
Use the graph of $y = \cos x$ to sketch the graph of $y = \sec x$ for $0 \leq x \leq 2\pi$.

Solution

For $y = 0$, $\frac{1}{y}$ is undefined. For $y = 1$, $\frac{1}{y} = 1$, so the curves intersect at this point. For $y = -1$, $\frac{1}{y} = -1$.

For $0 < y < 1$, $\frac{1}{y} > 1$. Because $\cos\frac{\pi}{3} = \frac{1}{2}$, $\sec\frac{\pi}{3} = 2$.

For $-1 < y < 0$, $\frac{1}{y} < -1$. Because $\cos\frac{2\pi}{3} = -\frac{1}{2}$, $\sec\frac{2\pi}{3} = -2$.



Example 10

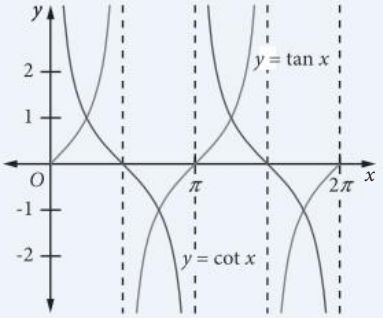
Use the graph of $y = \tan x$ to sketch the graph of $y = \cot x$ for $0 \leq x \leq 2\pi$.

Solution

For $y = 0$, $\frac{1}{y}$ is undefined. For $y = 1$, $\frac{1}{y} = 1$, so the curves intersect at this point. For $y = -1$, $\frac{1}{y} = -1$.

For $0 < y < 1$, $\frac{1}{y} > 1$. Because $\tan\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, $\cot\frac{\pi}{4} = \sqrt{2}$.

For $-1 < y < 0$, $\frac{1}{y} < -1$. Because $\tan\frac{3\pi}{4} = -\frac{1}{\sqrt{2}}$, $\cot\frac{3\pi}{4} = -\sqrt{2}$.



To investigate these graphs further, use a graphing calculator, GeoGebra or other graphing software to draw the following pairs of graphs on the same axes:

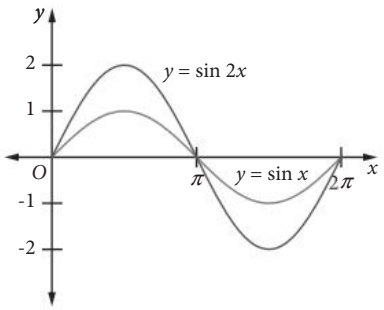
- $y = \sin x$ and $y = \operatorname{cosec} x$
- $y = \cos x$ and $y = \sec x$
- $y = \tan x$ and $y = \cot x$

Select the scale for each axis carefully. Start with $0 \leq x \leq 7$ and $-5 \leq y \leq 5$ and investigate what happens as you change the dimensions of the graph window.

Remember that as $y \to 0$, $\frac{1}{y} \to \infty$ and so the function will be undefined. The only functions that have an amplitude are $\sin x$ and $\cos x$. Also note that the period of a reciprocal function is the same as the original function.

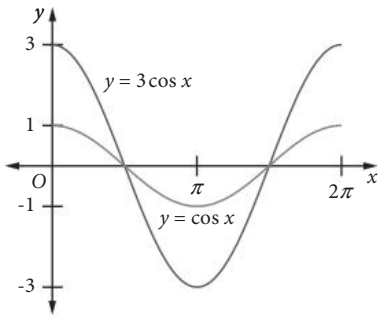
Graphs of $y = a\sin x$, $y = a\cos x$ and $y = a\tan x$

Consider the graphs of $y = \sin x$ and $y = 2\sin x$, drawn on the same axes for $0 \leq x \leq 2\pi$.



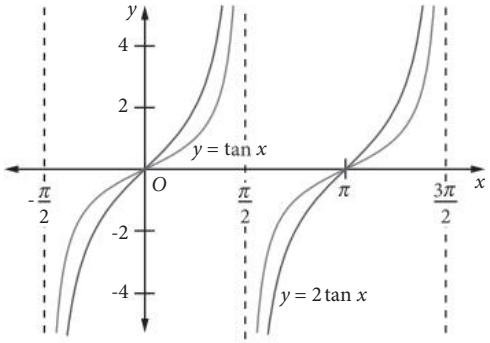
The graphs have the same shape and the same period, 2π. They both cut the x-axis at $x = 0, \pi, 2\pi$. The greatest value of $\sin x$ is 1, at $x = \frac{\pi}{2}$, while the greatest value of $2\sin x$ is 2, at $x = \frac{\pi}{2}$. The least value of $\sin x$ is -1, at $x = \frac{3\pi}{2}$, while the least value of $2\sin x$ is -2, at $x = \frac{3\pi}{2}$. The amplitude of $y = \sin x$ is 1 and the amplitude of $y = 2\sin x$ is 2.

Now consider the graphs of $y = \cos x$ and $y = 3\cos x$, drawn on the same axes for $0 \leq x \leq 2\pi$.



The graphs have the same shape and the same period, 2π. They both cut the x-axis at $x = \frac{\pi}{2}, \frac{3\pi}{2}$. The greatest value of $\cos x$ is 1, at $x = 0, 2\pi$, while the greatest value of $3\cos x$ is 3, at $x = 0, 2\pi$. The least value of $\cos x$ is -1, at $x = \pi$, while the least value of $3\cos x$ is -3, at $x = \pi$. The amplitude of $y = \cos x$ is 1 and the amplitude of $y = 3\cos x$ is 3.

Finally, consider the graphs of $y = \tan x$ and $y = 2\tan x$, drawn on the same axes for $-\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$.



The graphs have the same shape and the same period, π. They both cut the x-axis at $x = 0, \pi$. The curves have no greatest or least value, so they have no amplitude. The curves are undefined at $x = -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$. The lines $x = -\frac{\pi}{2}$, $x = \frac{\pi}{2}$, and $x = \frac{3\pi}{2}$ are asymptotes.

The effect of the '2' in $y = 2\tan x$ is to make the y value for a particular value of x twice the corresponding y value for $\tan x$.

In general, the effect of the a multiplier on a trigonometric function is to multiply the function's amplitude by a. If the function has no amplitude, as for $y = a\tan x$, then the effect is to stretch the curve vertically by a factor of a.

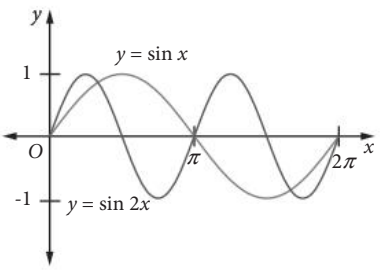
Graphs of $y = \sin bx$, $y = \cos bx$ and $y = \tan bx$

Consider the graphs of $y = \sin x$ and $y = \sin 2x$, drawn on the same axes for $0 \leq x \leq 2\pi$.

The graphs have the same shape, but in one period of $y = \sin x$ the graph of $y = \sin 2x$ occurs twice. This means that the period of $y = \sin 2x$ is half the period of $y = \sin x$. The period of $y = \sin 2x$ is π.

Both graphs cut the x-axis when $x = 0, \pi, 2\pi$. The greatest value of $\sin x$ is 1, at $x = \frac{\pi}{2}$, while the greatest value of $\sin 2x$ is also 1, at $x = \frac{\pi}{4}, \frac{5\pi}{4}$.

The least value of $\sin x$ is -1, at $x = \frac{3\pi}{2}$, while the least value of $\sin 2x$ is also -1, at $x = \frac{3\pi}{4}, \frac{7\pi}{4}$. The amplitude of both functions is 1.

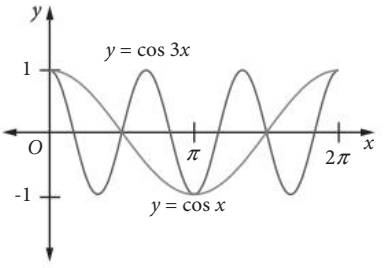


Consider the graphs of $y = \cos x$ and $y = \cos 3x$, drawn on the same axes for $0 \leq x \leq 2\pi$.

The graphs have the same shape, but in one period of $y = \cos x$ the graph of $y = \cos 3x$ occurs three times. This means that the period of $y = \cos 3x$ is one-third the period of $\cos x$. The period of $y = \cos 3x$ is $\frac{2\pi}{3}$.

Both graphs cut the x-axis at $x = \frac{\pi}{2}, \frac{3\pi}{2}$. The greatest value of $\cos x$ is 1, at $x = 0, 2\pi$, while the greatest value of $\cos 3x$ is also 1, at $x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi$.

The least value of $\cos x$ is -1, at $x = \pi$, while the least value of $\cos 3x$ is also -1, at $x = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$. The amplitude of both functions is 1.

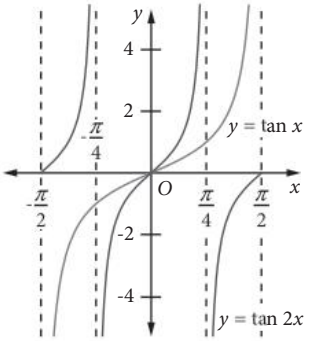


Consider the graphs of $y = \tan x$ and $y = \tan 2x$, drawn on the same axes for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.

The graphs have the same shape, but in one period of $y = \tan x$ the graph of $y = \tan 2x$ occurs twice. This means that the period of $y = \tan 2x$ is half the period of $\tan x$. The period of $y = \tan 2x$ is $\frac{\pi}{2}$.

Both graphs cut the x-axis when $x = 0$. The curves have no greatest or least value, so they have no amplitude. $y = \tan x$ is undefined at $x = -\frac{\pi}{2}, \frac{\pi}{2}$.

$y = \tan 2x$ is undefined at $x = -\frac{\pi}{4}, \frac{\pi}{4}$. The lines $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$ are asymptotes to $y = \tan x$, while the lines $x = -\frac{\pi}{4}$ and $x = \frac{\pi}{4}$ are asymptotes to $y = \tan 2x$.



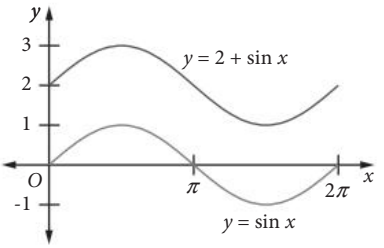
In general, the effect of the b multiplier in each equation is to divide the function's period by b. Thus the period of $y = \sin bx$ is $\frac{2\pi}{b}$, the period of $y = \cos bx$ is $\frac{2\pi}{b}$ and the period of $y = \tan bx$ is $\frac{\pi}{b}$.

Graphs of $y = \sin x + c$, $y = \cos x + c$ and $y = \tan x + c$

Consider the graphs of $y = \sin x$ and $y = 2 + \sin x$, drawn on the same axes for $0 \leq x \leq 2\pi$.

The graphs have the same shape and the same period, 2π. The greatest value of $\sin x$ is 1, at $x = \frac{\pi}{2}$, while the greatest value of $2 + \sin x$ is 3, at $x = \frac{\pi}{2}$. The least value of $\sin x$ is -1, at $x = \frac{3\pi}{2}$, while the least value of $2 + \sin x$ is 1, at $x = \frac{3\pi}{2}$. The amplitude of both curves is 1.

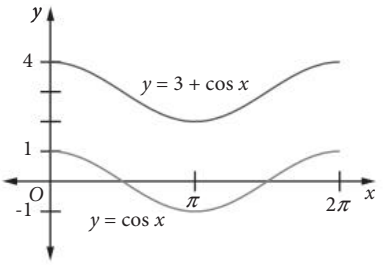
The effect of adding 2 to $\sin x$ is to translate the curve vertically upwards 2 units (i.e. 'move up 2').



Consider the graphs of $y = \cos x$ and $y = 3 + \cos x$, drawn on the same axes for $0 \leq x \leq 2\pi$.

The graphs have the same shape and the same period, 2π. The greatest value of $\cos x$ is 1, at $x = 0, 2\pi$, while the greatest value of $3 + \cos x$ is 4, at $x = 0, 2\pi$. The least value of $\cos x$ is -1, at $x = \pi$, while the least value of $3 + \cos x$ is 2, at $x = \pi$. The amplitude of both curves is 1.

The effect of adding 3 to $\cos x$ is to translate the curve vertically upwards 3 units (move up 3).



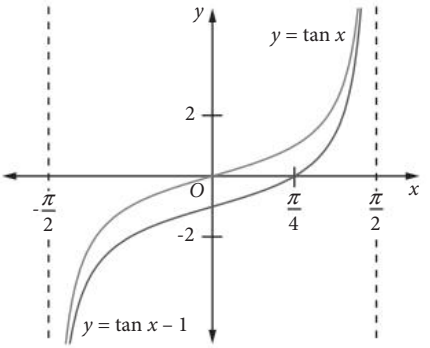
Consider the graphs of $y = \tan x$ and $y = \tan x - 1$, drawn on the same axes for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.

The graphs have the same shape and the same period, π. The curves have no greatest or least value, so they have no amplitude.

The curves are undefined at $x = -\frac{\pi}{2}, \frac{\pi}{2}$. The lines $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$ are asymptotes.

The effect of the -1 in $y = \tan x - 1$ is to translate the curve vertically downwards 1 unit (move down 1).

In general, the effect of the c in each equation is to translate the curve vertically by c units.

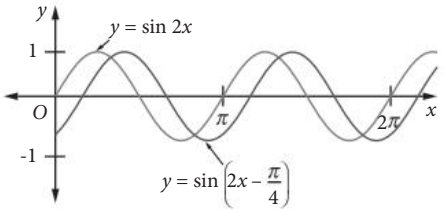


Graphs of $y = a\sin(bx + \alpha)$, $y = a\cos(bx + \alpha)$ and $y = a\tan(bx + \alpha)$

These graphs combine the effects seen so far. The value a is still the amplitude of the sine and cosine functions and affects the size of the y value in the tan function, and b still divides the period of the functions. The value α now changes the position along the x-axis where the functions occur. It is called the **phase** of the function.

Consider the graphs of $y = \sin 2x$ and $y = \sin\left(2x - \frac{\pi}{4}\right)$, drawn on the same axes for $0 \le x \le 2\pi$.

The graphs have the same shape, amplitude 1 and period π. The graph of $y = \sin 2x$ has been translated $\frac{\pi}{4}$ units to the right to obtain the graph of $y = \sin\left(2x - \frac{\pi}{4}\right)$.

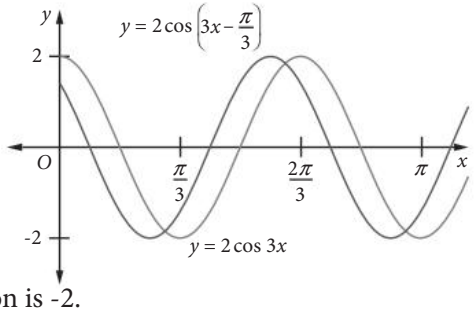


$y = \sin 2x$ cuts the x-axis when $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$.

$y = \sin\left(2x - \frac{\pi}{4}\right)$ cuts the x-axis at $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$. The greatest value of each function is 1 and the least value of each function is -1.

Consider the graphs of $y = 2\cos 3x$ and $y = 2\cos\left(3x + \frac{\pi}{3}\right)$, drawn on the same axes for $0 \le x \le 2\pi$.

The graphs have the same shape, amplitude 2 and period $\frac{2\pi}{3}$. The graph of $y = 2\cos 3x$ has been translated $\frac{\pi}{9}$ units to the left to obtain $y = 2\cos\left(3x + \frac{\pi}{3}\right)$.



$y = 2\cos 3x$ cuts the x-axis when $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$.

$y = 2\cos\left(3x + \frac{\pi}{3}\right)$ cuts the x-axis when $x = \frac{\pi}{18}, \frac{7\pi}{18}, \frac{13\pi}{18}$.

The greatest value of each function is 2 and the least value of each function is -2.

Summary of trigonometric functions

Function	Period	Amplitude	Domain	Range	Symmetry
$y = a\sin x$	2π	a	Real x	$-a \le y \le a$	Rotational symmetry
$y = a\cos x$	2π	a	Real x	$-a \le y \le a$	Symmetrical about y-axis
$y = a\tan x$	π	none	Real x, $x \ne \frac{(2n-1)\pi}{2}$ for integer n	Real y	Rotational symmetry
$y = a\sin bx$	$\frac{2\pi}{b}$	a	Real x	$-a \le y \le a$	Rotational symmetry
$y = a\cos bx$	$\frac{2\pi}{b}$	a	Real x	$-a \le y \le a$	Symmetrical about y-axis
$y = a\tan bx$	$\frac{\pi}{b}$	none	Real x, $x \ne \frac{(2n-1)\pi}{2b}$ for integer n	Real y	Rotational symmetry
$y = a\sin bx + c$	$\frac{2\pi}{b}$	a	Real x	$-a + c \le y \le a + c$	
$y = a\cos bx + c$	$\frac{2\pi}{b}$	a	Real x	$-a + c \le y \le a + c$	Symmetrical about y-axis
$y = a\tan bx + c$	$\frac{\pi}{b}$	none	Real x, $x \ne \frac{(2n-1)\pi}{2}$ for integer n	Real y	
$y = a\sin(bx + \alpha)$	$\frac{2\pi}{b}$	a	Real x	Real y	
$y = a\cos(bx + \alpha)$	$\frac{2\pi}{b}$	a	Real x	Real y	
$y = a\tan(bx + \alpha)$	$\frac{\pi}{b}$	none	Real x, $x \ne \frac{(2n-1)\pi - 2\alpha}{2b}$ for integer n	Real y	

When first sketching graphs of trigonometric functions you should draw up a table of values and plot the points, joining the points with an appropriate curve. When you are familiar with the shapes of the standard curves, you will be able to draw them confidently using only the properties of amplitude, period and asymptotes. You might also use a standard template to obtain the shape of a curve and then mark the appropriate scale on the axes.

Example 11

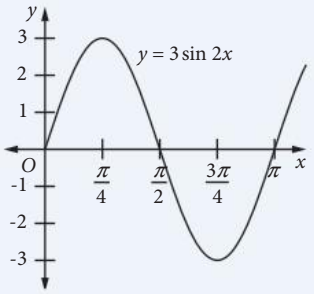
Sketch the graph of $y = 3\sin 2x$ for $0 \le x \le \pi$.

Solution

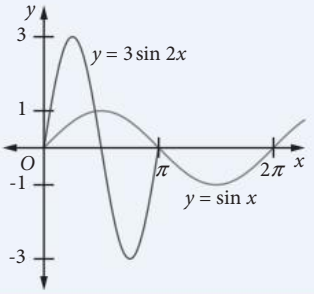
Complete a table of values.

x	0	$\frac{\pi}{12}$	$\frac{\pi}{8}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{3\pi}{8}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$	$\frac{7\pi}{12}$	$\frac{5\pi}{8}$	$\frac{2\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{8}$	$\frac{11\pi}{12}$	π
y	0	1.5	2.1	2.6	3	2.6	2.1	1.5	0	-1.5	-2.1	-2.6	-3	-2.6	-2.1	-1.5	0

Plot the points and sketch the graph, marking scales on each axis. Make sure that your curve is smooth, not pointed or jagged.



An alternative method is to first sketch $y = \sin x$ for $0 \le x \le 2\pi$. You can then draw $y = 3\sin 2x$ by noting that it has 3 times the amplitude and half the period of $y = \sin x$.



Example 12

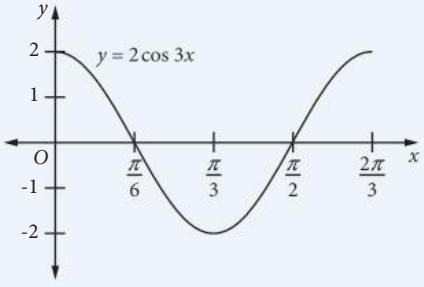
Sketch the graph of **(a)** $y = 2\cos 3x$ **(b)** $y = \tan \pi x$, showing one complete cycle.

Solution

(a) Amplitude = 2

Period = $\frac{2\pi}{3}$

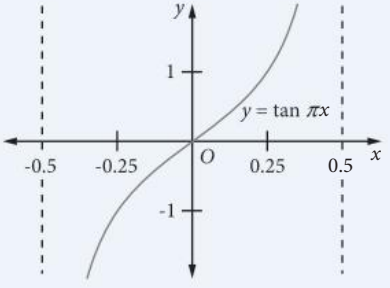
The midpoint of the cycle is $\frac{\pi}{3}$



(b) No amplitude

Period = 1 (variable is πx)

Cycle is $-0.5 < x < 0.5$



Example 13

(a) Sketch the graph of $y = \cos x$ for $0 \le x \le 2\pi$.

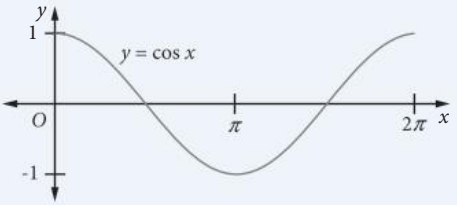
(b) On the same set of axes sketch $y = \cos\frac{x}{2}$.

(c) Hence sketch $y = -\cos\frac{x}{2}$.

(d) Use your answer to **(c)** to sketch $y = 1 - \cos\frac{x}{2}$

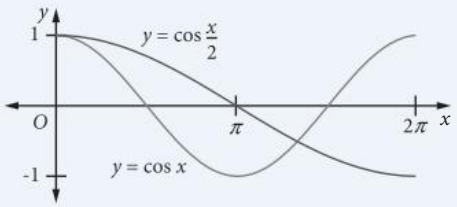
Solution

(a) $y = \cos x$ for $0 \le x \le 2\pi$.

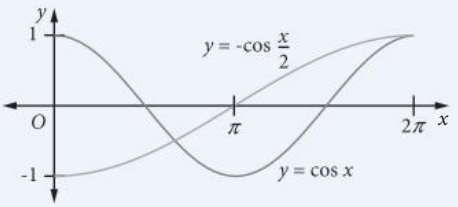


(b) The period of $y = \cos\frac{x}{2}$ is 4π, so half a cycle of the curve will fit in the domain $0 \le x \le 2\pi$.

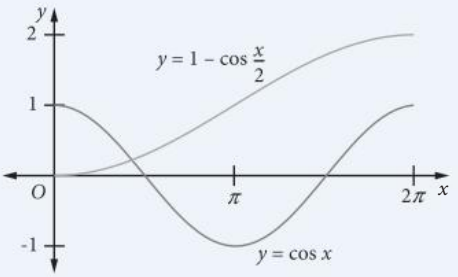
$y = \cos x$ and $y = \cos\frac{x}{2}$ both have an amplitude of 1.



(c) $y = -\cos\frac{x}{2}$ is just $y = \cos\frac{x}{2}$ flipped over (reflected in the x-axis).



(d) Obtain $y = 1 - \cos\frac{x}{2}$ by moving $y = -\cos\frac{x}{2}$ up 1 unit.



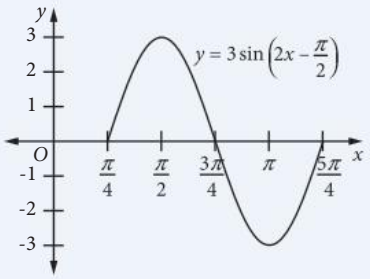
Example 14

Sketch graphs of the following, showing one complete cycle of each:

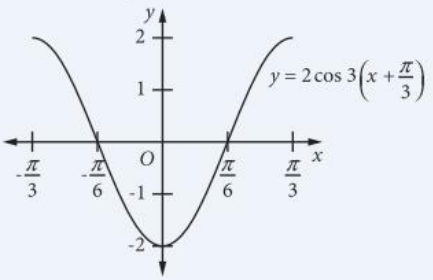
(a) $y = 3\sin\left(2x - \frac{\pi}{2}\right)$ **(b)** $y = 2\cos 3\left(x + \frac{\pi}{3}\right)$

Solution

(a)



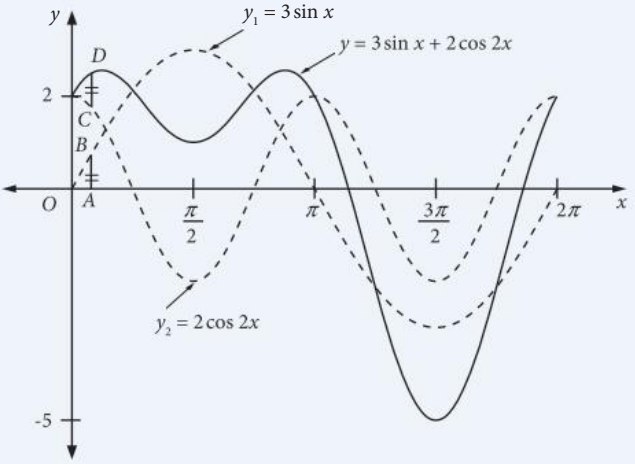
(b) Rewrite as $y = 2\cos(3x + \pi)$:



Example 15

Using the same scale and axes, sketch the graphs of $y = 3\sin x$ and $y = 2\cos 2x$. Hence sketch the graph of $y = 3\sin x + 2\cos 2x$ for $0 \le x \le 2\pi$.

Solution



The graphs of $y_1 = 3\sin x$ and $y_2 = 2\cos 2x$ are shown here as dashed lines. To obtain points on the graph of $y = 3\sin x + 2\cos 2x$ we can add the ordinates of the component curves, remembering to take the sign into account.

Let $y = y_1 + y_2$ where $y_1 = 3\sin x$ and $y_2 = 2\cos 2x$.

For example: at $x = 0$, $y = 0 + 2 = 2$; at $x = \frac{\pi}{2}$, $y = 3 - 2 = 1$; at $x = \pi$, $y = 0 + 2 = 2$;

at $x = \frac{3\pi}{2}$, $y = -3 - 2 = -5$; at $x = 2\pi$, $y = 0 + 2 = 2$; and so on.

In general, at $x = OA$, $AD = AB + AC$.

EXERCISE 15.4 GRAPHS OF TRIGONOMETRIC FUNCTIONS

1 Sketch the graph of each of the following, stating the period and amplitude of the function:

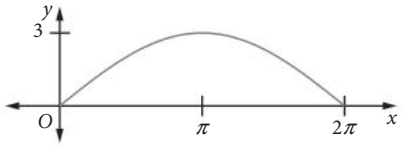
(a) $y = 4\sin x$, $0 \le x \le 2\pi$ **(b)** $y = \cos 2x$, $0 \le x \le 2\pi$ **(c)** $y = 3\tan x$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

(d) $y = 4\sin 3x$, $0 \le x \le 2\pi$ **(e)** $y = 3\cos 2x$, $0 \le x \le 2\pi$ **(f)** $y = 3\tan 2x$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

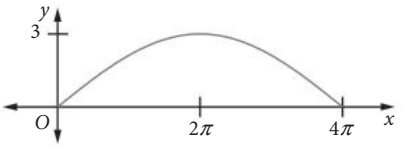
(g) $y = \sin\frac{x}{2}$, $0 \le x \le 4\pi$ **(h)** $y = \cos\frac{x}{2}$, $-2\pi \le x \le 2\pi$ **(i)** $y = \tan\frac{x}{2}$, $-\pi \le x \le \pi$

2 Which diagram shows the graph of $y = 3\sin\frac{x}{4}$ for $0 \le x \le 4\pi$?

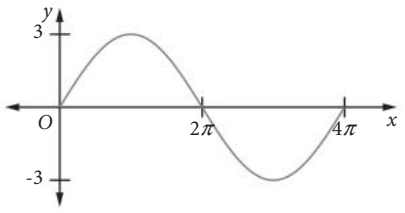
A



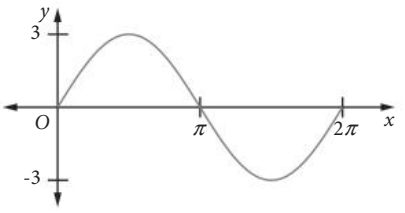
B



C



D



3 Sketch the graph of each of the following, stating the period and amplitude of each function:

(a) $y = 3\sin\frac{x}{2}, -\pi \le x \le \pi$ **(b)** $y = 2\cos\frac{x}{2}, 0 \le x \le 2\pi$ **(c)** $y = 3\tan\frac{x}{2}, 0 \le x \le 2\pi$

4 Sketch the graph of $y = \cos x$, $0 \le x \le \pi$. On the same axes, sketch the graph of:

(a) $y = -\cos x$ **(b)** $y = 1 - \cos x$

5 Sketch the graph of $y = \sin x$, $-\pi \le x \le \pi$. On the same axes, sketch the graph of:

(a) $y = 2\sin x$ **(b)** $y = -2\sin x$ **(c)** $y = 3 - 2\sin x$

6 Sketch the graph of each of the following, stating the period and amplitude of the function:

(a) $y = \sin \pi x, -1 \le x \le 1$ **(b)** $y = \cos 2\pi x, 0 \le x \le 2$ **(c)** $y = \tan\frac{\pi x}{2}, 0 \le x \le 2$

7 Sketch the graph of $y = \tan x$, $-\pi \le x \le \pi$. On the same axes, sketch the graph of:

(a) $y = 2\tan x$ **(b)** $y = -2\tan x$ **(c)** $y = 2 - 2\tan x$

8 Sketch the graph of each of the following for $0 \le x \le 2\pi$.

(a) $y = \sin x - 1$ **(b)** $y = 1 + 2\sin x$ **(c)** $y = 2 - \cos x$
(d) $y = 2\tan 2x$ **(e)** $y = 2 + \sin 2x$ **(f)** $y = \cos 2x - 1$

9 Sketch the graph of each of the following, showing one complete cycle.

(a) $y = 2\sin\left(\theta - \frac{\pi}{2}\right)$ **(b)** $y = 3\cos\left(\theta + \frac{\pi}{4}\right)$ **(c)** $y = 2\sin(\theta - \pi)$

(d) $y = 5\cos 3\left(\theta + \frac{\pi}{3}\right)$ **(e)** $y = \frac{1}{2}\tan 2(\theta + \pi)$ **(f)** $y = \sqrt{2}\sin\left(2\theta - \frac{\pi}{2}\right)$

10 Sketch the graph of each:

(a) $y = \sin 2\theta + 1$ **(b)** $y = 3\cos\theta - 2$ **(c)** $y = \frac{1}{2}\sin\left(\theta - \frac{\pi}{2}\right) + 3$

(d) $y = 2\cos 2\left(\theta - \frac{\pi}{4}\right) + 1$ **(e)** $y = 4\sin 3\left(\theta - \frac{\pi}{6}\right) - 2$ **(f)** $y = 3 - \sin\left(\theta - \frac{\pi}{2}\right)$

11 By adding ordinates, sketch the graphs of:

(a) $y = \sin\theta + \cos\theta$ **(b)** $y = 3\sin 2\theta + 4\sin\theta$ **(c)** $y = 2\cos 3\theta + 3\sin 2\theta$
(d) $y = \sin 2\theta - \cos\theta$ **(e)** $y = \frac{1}{2}\cos 2\theta - \sin\theta$ **(f)** $y = \sin\theta + \sin 2\theta$

12 Ocean tides can be approximated by the equation $y = a\sin bx + c$. Find a website that records tidal data, select a location and find the times and heights of high and low tides over a four-day period. Graph this information to see how close the graph's shape is to a sine curve.

15.5 SOLUTION OF TRIGONOMETRIC EQUATIONS

You have already solved some simple trigonometric equations in Exercise 5.2. In this section you will solve more complicated equations, which may need to be simplified using the trigonometric identities before being factorised using algebraic techniques.

You will need to remember the exact values for the common trigonometric ratios, or how to calculate them. You also need to know the signs of the values in the four quadrants (remember 'ASTC').

Example 16

Solve, for $0 \le x \le 2\pi$:

(a) $\cos x = \frac{\sqrt{3}}{2}$ **(b)** $\sin^2 x + 2\sin x + \cos^2 x = 0$ **(c)** $\sec x = -2$

Solution

(a) Solve: $\cos x = \frac{\sqrt{3}}{2}$

From the sign, angles are in 1st and 4th quadrants: $x = \frac{\pi}{6}, 2\pi - \frac{\pi}{6}$

$x = \frac{\pi}{6}, \frac{11\pi}{6}$

(b) Solve: $\sin^2 x + 2\sin x + \cos^2 x = 0$

Rearrange equation: $\sin^2 x + \cos^2 x + 2\sin x = 0$

Use the identity $\sin^2 x + \cos^2 x = 1$: $1 + 2\sin x = 0$

$2\sin x = -1$

$\sin x = -\frac{1}{2}$

From the sign, angles are in 3rd and 4th quadrants: $x = \pi + \frac{\pi}{6}, 2\pi - \frac{\pi}{6}$

$x = \frac{7\pi}{6}, \frac{11\pi}{6}$

(c) Solve: $\sec x = -2$

Change to $\cos x$: $\frac{1}{\cos x} = -2$

$\cos x = -\frac{1}{2}$

From the sign, angles are in 2nd and 3rd quadrants: $x = \pi - \frac{\pi}{3}, \pi + \frac{\pi}{3}$

$x = \frac{2\pi}{3}, \frac{4\pi}{3}$

Example 17

Solve, for $-\pi \le \theta \le \pi$: **(a)** $2\sin^2\theta = 1$ **(b)** $3\sin\theta = 2\cos\theta$ **(c)** $\sin^2\theta + 2\sin\theta + 1 = 0$

Solution

(a) Solve: $2\sin^2\theta = 1$

Rearrange: $\sin^2\theta = \frac{1}{2}$

Square root of each side: $\sin\theta = \frac{\pm 1}{\sqrt{2}}$

From the sign, angles are in all quadrants: $\theta = -\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}$ as $-\pi \le \theta \le \pi$

(b) Solve: $3\sin\theta = 2\cos\theta$

Divide both sides by $3\cos\theta$: $\frac{\sin\theta}{\cos\theta} = \frac{2}{3}$

But $\tan\theta = \frac{\sin\theta}{\cos\theta}$: $\tan\theta = \frac{2}{3}$

From the sign, angles are in 1st and 3rd quadrants: $\theta = 0.5880, -\pi + 0.5880$ as $-\pi \le \theta \le \pi$

$\theta = -2.5536, 0.5880$ (to 4 d.p.)

(c) Solve: $\sin^2\theta + 2\sin\theta + 1 = 0$

Factorise: $(\sin\theta + 1)^2 = 0$

Square root of each side: $\sin\theta + 1 = 0$

Rearrange: $\sin\theta = -1$

From the sign, angles are in 3rd and 4th quadrants: $\theta = -\frac{\pi}{2}$ as $-\pi \le \theta \le \pi$

Example 18

Solve the equation $\sin\theta = \frac{4\sin\frac{\pi}{9}}{5}$ if θ is an angle in a triangle. Give your answer correct to 2 decimal places.

Solution

Evaluate the expression to find θ:	$\theta = 0.277$
Because θ is an angle in a triangle, consider also:	$\theta = \pi - 0.277 = 2.868$
Equation implies that $\frac{\pi}{9}$ is also an angle of the triangle (via the sine rule), so the second value is too large (because $2.868 + \frac{\pi}{9} > \pi$ will not fit), hence:	$\theta = 0.28$ (to 2 d.p.)

Example 19

Solve, for $0 \le x \le \pi$: **(a)** $\cos\frac{x}{2} = \frac{1}{\sqrt{2}}$ **(b)** $\sin 2x = -\frac{1}{2}$ **(c)** $\sec^2 2x - 2\tan 2x = 0$

Solution

(a)

Solve: $\cos\frac{x}{2} = \frac{1}{\sqrt{2}}$

$0 \le x \le \pi$ so $0 \le \frac{x}{2} \le \frac{\pi}{2}$: $\frac{x}{2} = \frac{\pi}{4}$

$x = \frac{\pi}{2}$

(b)

Solve: $\sin 2x = -\frac{1}{2}$

$0 \le x \le \pi$ so $0 \le 2x \le 2\pi$: $2x = \pi + \frac{\pi}{6}, 2\pi - \frac{\pi}{6}$

$2x = \frac{7\pi}{6}, \frac{11\pi}{6}$

$x = \frac{7\pi}{12}, \frac{11\pi}{12}$

(c)

Solve: $\sec^2 2x - 2\tan 2x = 0$

Use identity $\sec^2 2x = 1 + \tan^2 2x$: $1 + \tan^2 2x - 2\tan 2x = 0$

Rearrange: $\tan^2 2x - 2\tan 2x + 1 = 0$

Factorise: $(\tan 2x - 1)^2 = 0$

$\tan 2x = 1$

$0 \le x \le \pi$ so $0 \le 2x \le 2\pi$: $2x = \frac{\pi}{4}, \pi + \frac{\pi}{4}$

$x = \frac{\pi}{8}, \frac{5\pi}{8}$

EXERCISE 15.5 SOLUTION OF TRIGONOMETRIC EQUATIONS

Give answers correct to 3 decimal places where necessary.

1 Solve, for $0 \le \theta \le 2\pi$:

(a) $\sin\theta = \frac{-1}{\sqrt{2}}$ **(b)** $\sec\theta = \frac{2}{\sqrt{3}}$ **(c)** $\cot\theta = 1$

(d) $\sin^2\theta - 2\cos\theta + \cos^2\theta = 0$ **(e)** $\sin^2\theta + \cos\theta - 1 = 0$ **(f)** $\sec^2\theta - 2\tan\theta = 0$

(g) $4\sin^2\theta = 3$ **(h)** $\sin\theta = \cos\theta$ **(i)** $\cos^2\theta - 2\cos\theta + 1 = 0$

(j) $\tan^2\theta = 3$ **(k)** $\sin\theta + \cos\theta = 0$ **(l)** $2\cos^2\theta - \cos\theta = 0$

2 What is the solution to $\operatorname{cosec} x = 3$, for $0 < x < 2\pi$?

A $x = 0.340, 2.802$ **B** $x = 19°\,28', 160°\,32'$ **C** $x = 0.340$ **D** $x = 19°\,28'$

3 Solve, for $-\pi \le \theta \le \pi$:

(a) $1 + \sin\theta = \cos^2\theta$ **(b)** $2\cos^2\theta + \sin\theta = 1$ **(c)** $4\cos\theta = \sec\theta$

(d) $2\sin^2\theta - 2 = 3\cos\theta$ **(e)** $4\sin\theta = \operatorname{cosec}\theta$ **(f)** $2\sec^2\theta = 5\tan\theta$

4 Solve, for $0 \le x \le 2\pi$:

(a) $\tan x = 2\sin x$ **(b)** $5\cos^2 x + 2\sin x - 2 = 0$ **(c)** $3\sec^2 x - 5\tan x = 5$

(d) $6\tan x = 5\operatorname{cosec} x$ **(e)** $7\sin x\cos x + \cos^2 x = 1$ **(f)** $\sin^2 x + 2\sin x\cos x = 3\cos^2 x$

5 Solve the equation $\operatorname{cosec}^2 x + \cot^2 x = 3$, for $0 \le x \le \pi$. Indicate whether each statement below is a correct or incorrect step of the solution.

(a) $2\cot^2 x = 2$ **(b)** $2\cot^2 x = 4$ **(c)** $\cot x = \pm 1$ **(d)** $x = \pm\frac{\pi}{4}$

6 Solve, for $0 \le x \le 2\pi$:

(a) $\sin\frac{x}{2} = \frac{\sqrt{3}}{2}$ **(b)** $\tan\frac{x}{2} = -\frac{1}{\sqrt{3}}$ **(c)** $\cos\frac{x}{3} = 1$

7 Solve, for $0 \le x \le \pi$:

(a) $\cos 2x = 0$ **(b)** $\tan 3x = \sqrt{3}$ **(c)** $\sin 4x = \frac{1}{\sqrt{2}}$

8 Solve for $0 \le x \le \pi$, giving your answer correct to 2 decimal places:

(a) $\sin\frac{x}{3} = 0.4$ **(b)** $\cos 3x = -0.7$ **(c)** $\tan\frac{x}{4} = 1.5$

15.6 GRAPHICAL SOLUTION OF EQUATIONS

There are usually standard techniques for solving equations associated with various functions, e.g. linear equations, quadratic equations and trigonometric equations. However, there is usually no standard technique for solving equations that combine two or more different functions, for example:

- $\sin x = 1 - 2x$
- $\cos x = \log x$

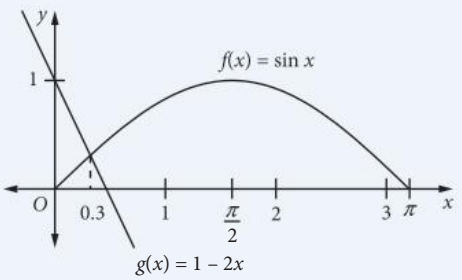
Equations like this 'transcend' (go beyond) algebraic equation-solving techniques and hence are called **transcendental equations**. We must instead use various non-algebraic methods to find approximate solutions. The most accurate method is to calculate numerical solutions using graphing software such as GeoGebra. Another method is to sketch the graphs of the two functions, use the graphs to find approximate x-values of any intersection points, then use a calculator to refine these approximate x values as closely as possible.

Example 20

Solve $\sin x = 1 - 2x$.

Solution

Sketch the functions.



This shows that the graphs of $f(x) = \sin x$ and $g(x) = 1 - 2x$ intersect at only one point, approximately $x = 0.3$, so 0.3 is an approximate solution of $\sin x = 1 - 2x$. Testing values of x to refine this approximation:

x	0.3	0.4	0.35	0.335
$\sin x$	0.2955	0.3894	0.3429	0.3288
$1 - 2x$	0.4	0.2	0.3	0.33

This table shows the values of $\sin x$ and $1 - 2x$ for values of x between $x = 0.3$ and $x = 0.4$. $x = 0.35$ is a better solution than $x = 0.3$. Further trial and error leads to a solution of $x = 0.335$.

A 'find the intersection' function in GeoGebra or another graphing software program would allow you to find this solution quickly and accurately. Unfortunately, under examination conditions you will have to use the sketching method.

EXERCISE 15.6 GRAPHICAL SOLUTION OF EQUATIONS

1 Draw the graph of $y = \sin x$ for $0 \le x \le \pi$. Use this graph to find the number of solutions in $0 \le x \le \pi$ for:

(a) $\sin x = \frac{x}{2}$ **(b)** $\sin x = \frac{x}{2} - 1$ **(c)** $\sin x = \frac{\pi}{2} - x$

2 Draw the graph of $y = \cos x$ for $-\pi \le x \le \pi$ and use it to solve the following equations.

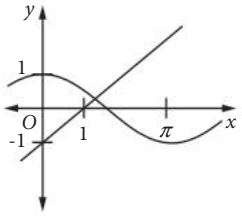
(a) $\cos x = x$ **(b)** $\cos x = \frac{x}{2}$ **(c)** $\cos x = 1 - x$

3 Use a graphical method to solve each equation. Check your answer using GeoGebra or other graphing software.

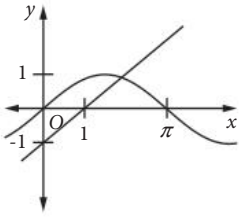
(a) $\sin 2x = \frac{x}{2}$ **(b)** $\cos x = x, x \ge 0$ **(c)** $\log_{10} x = 2 - x^2$ **(d)** $\tan x = 1 - x, 0 \le x \le 2\pi$

4 Which graph could be used to solve the equation $\sin x = x - 1$?

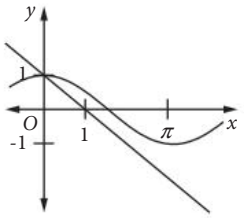
A



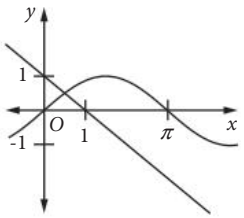
B



C



D



5 By drawing appropriate graphs, determine the number of solutions to the equation $\sin 2x = 1 - \frac{x}{4}$ in the domain $0 \le x \le 2\pi$.

6 A rectangular strip of metal 9 cm wide is bent to form a water channel. The section perpendicular to the length is a circular arc whose chord is 6 cm long.

(a) Show that if the circular arc subtends an angle of $2x$ radians at the centre of the circle then $3 \sin x = 2x$.

(b) Solve the equation in part **(a)** graphically, giving your answer correct to 1 decimal place.

(c) Find the area of the cross-section of the channel in cm^2.

7 A semicircle of radius r is divided into two parts of equal area by a chord parallel to the base (diameter).

(a) If the chord subtends an angle of x radians at the centre, prove that $x - \frac{\pi}{2} = \sin x$.

(b) Solve the equation in part **(a)** graphically, giving your answer correct to 1 decimal place.

8 The chord of a segment of a circle subtends an angle of $\frac{\pi}{2} + x$ radians at the centre. The area of the segment is one-quarter the area of the circle.

(a) Prove that $x = \cos x$.

(b) Solve the equation in part **(a)** graphically, giving your answer correct to 2 decimal places.

9 Show from a rough sketch that the equation $2 - \frac{x}{2} = \tan x$ has a solution that is approximately 1. Using a calculator, find this solution correct to 2 decimal places.

10 Show graphically that the equation $8 \log_{10}(0.1x + 0.5) = 2 - x$ has a solution between $x = 2$ and $x = 4$. Find this solution correct to 2 decimal places.

15.7 APPROXIMATIONS WHEN *x* IS SMALL

It may seem strange that the expression $\lim_{x \to 0} \frac{\sin x}{x}$ has a particular value. As you look at this fraction, which appears to be approaching $\frac{0}{0}$, the natural reaction is to say that it is undefined, does not exist, or could be anything. However, consider the following table of values (where x is in radians):

x	0.1	0.01	0.001	0.0001	-0.0001	-0.001	-0.01	-0.1
$\frac{\sin x}{x}$	0.998334	0.999983	0.9999998	0.999999998	0.999999998	0.9999998	0.999983	0.998334

As $x \to 0$ from above (through positive values) it seems safe to say that $\frac{\sin x}{x} \to 1$. As $x \to 0$ from below (through negative values) it also seems safe to say that $\frac{\sin x}{x} \to 1$. We can use a spreadsheet to continue this investigation with even smaller values for x, and ultimately we have the result: $\lim_{x \to 0} \frac{\sin x}{x} = 1$

Consider the following table of values for $\frac{\tan x}{x}$.

x	0.1	0.01	0.001	0.0001	-0.0001	-0.001	-0.01	-0.1
$\frac{\tan x}{x}$	1.003347	1.000033	1.0000003	1.000000003	1.000000003	1.0000003	1.000033	1.003347

As $x \to 0$ from above (through positive values) it seems safe to say that $\frac{\tan x}{x} \to 1$. As $x \to 0$ from below (through negative values) it seems safe to say that $\frac{\tan x}{x} \to 1$. Again, we can use a spreadsheet to continue this investigation with even smaller values for x, and we have the result: $\lim_{x \to 0} \frac{\tan x}{x} = 1$

From these tables of values it would be reasonable to say that when x is small, $\frac{\tan x}{x} > \frac{\sin x}{x}$. This leads to another useful result: when x is small, we have $\tan x > \sin x$ for small x.

Consider the following table of values for $\frac{1 - \cos x}{x^2}$.

x	0.1	0.01	0.001	-0.001	-0.01	-0.1
$\frac{1 - \cos x}{x^2}$	0.49958	0.4999958	0.49999999	0.49999999	0.4999958	0.49958

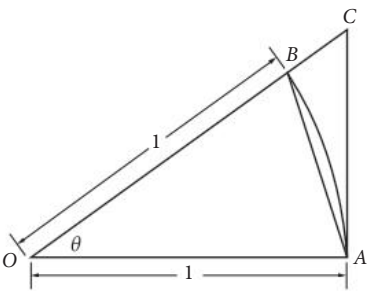
As $x \to 0$ from above (through positive values) we have $\frac{1 - \cos x}{x^2} \to 0.5$. As $x \to 0$ from below (through negative values) we have $\frac{1 - \cos x}{x^2} \to 0.5$. A spreadsheet investigation can confirm this for even smaller values of x, hence we have the result: $\lim_{x \to 0} \frac{1 - \cos x}{x^2} = 0.5$

Derivation of trigonometry limit results

Consider a sector OAB of a circle of unit radius.

The tangent at A meets OB produced at C.

Let the size of $\angle AOB$ be θ radians.



$\therefore$ Area $\Delta AOB <$ area of sector $AOB <$ area ΔAOC

$\frac{1}{2} \sin\theta < \frac{\theta}{2} < \frac{1}{2} \tan\theta \qquad \left(0 < \theta < \frac{\pi}{2}\right)$

Hence: $\sin\theta < \theta < \tan\theta$

Take reciprocals: $\frac{1}{\sin\theta} > \frac{1}{\theta} > \frac{1}{\tan\theta}$

or $\frac{1}{\sin\theta} > \frac{1}{\theta} > \frac{\cos\theta}{\sin\theta}$

Hence: $1 > \frac{\sin\theta}{\theta} > \cos\theta$

As $\theta \to 0$, $\cos\theta \to 1$, and as $\frac{\sin\theta}{\theta}$ is between 1 and $\cos\theta$, then: $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

Another way of thinking about this limit is to see that when θ is small, $\sin\theta \approx \theta$ and $\cos\theta \approx 1$.

Looking again at: $\frac{1}{\sin\theta} > \frac{1}{\theta} > \frac{1}{\tan\theta}$

$$\frac{\tan\theta}{\sin\theta} > \frac{\tan\theta}{\theta} > 1$$

Hence: $\frac{1}{\cos\theta} > \frac{\tan\theta}{\theta} > 1$

As $\theta \to 0$, $\cos\theta \to 1$, and because $\frac{\tan\theta}{\theta}$ is between 1 and $\cos\theta$, then: $\lim_{\theta\to 0}\frac{\tan\theta}{\theta} = 1$

Now consider $\frac{\sin^2\theta}{\theta^2} = \frac{1-\cos^2\theta}{\theta^2} = \frac{(1-\cos\theta)(1+\cos\theta)}{\theta^2}$.

This can be written: $\frac{(1-\cos\theta)(1+\cos\theta)}{\theta^2} = \frac{\sin^2\theta}{\theta^2}$

Hence: $\frac{1-\cos\theta}{\theta^2} = \frac{1}{1+\cos\theta}\left(\frac{\sin\theta}{\theta}\right)^2$

Thus as $x \to 0$: $\frac{1-\cos\theta}{\theta^2} \to \frac{1}{1+1}(1)^2 = \frac{1}{2}$

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta^2} = \frac{1}{2}$$

Some of these results we will use later to find the derivative of $\sin x$, $\cos x$ and $\tan x$ from first principles.

Example 21

Find: (a) $\lim_{x\to 0}\frac{\sin 3x}{3x}$ (b) $\lim_{x\to 0}\frac{\sin 4x}{2x}$ (c) $\lim_{x\to 0}\frac{\tan 3x}{6x}$

Solution

(a) $\lim_{x\to 0}\frac{\sin 3x}{3x}$
$= \lim_{x\to 0}\frac{\sin\theta}{\theta}$ where $\theta = 3x$
$= 1$

(b) $\lim_{x\to 0}\frac{\sin 4x}{2x}$
$= 2 \times \lim_{x\to 0}\frac{\sin 4x}{4x}$
$= 2 \times 1 = 2$

(c) $\lim_{x\to 0}\frac{\tan 3x}{6x}$
$= \frac{1}{2} \times \lim_{x\to 0}\frac{\tan 3x}{3x}$
$= \frac{1}{2} \times 1 = \frac{1}{2}$

EXERCISE 15.7 APPROXIMATIONS WHEN *x* IS SMALL

1 Evaluate the following limits.

(a) $\lim_{x\to 0}\frac{\sin 2x}{x}$ (b) $\lim_{x\to 0}\frac{\sin x}{3x}$ (c) $\lim_{\theta\to 0}\frac{\sin 5\theta}{2\theta}$ (d) $\lim_{x\to 0}\frac{\tan 2x}{2x}$

(e) $\lim_{x\to 0}\frac{\sin\frac{x}{2}}{x}$ (f) $\lim_{x\to 0}\frac{\sin\frac{x}{3}}{3x}$ (g) $\lim_{x\to 0}\frac{\tan 3x}{x}$ (h) $\lim_{x\to 0}\frac{3\sin 2x}{4x}$

2 Indicate whether each statement is correct or incorrect.

(a) $\sin(\pi - x) = -\sin x$ (b) $\sin(\pi - x) = \sin x$ (c) $\lim_{x\to 0}\frac{\sin(\pi - x)}{x} = -1$ (d) $\lim_{x\to 0}\frac{\sin(\pi - x)}{x} = 1$

3 Evaluate the following limits.

(a) $\lim_{x\to 0}\frac{1-\cos 2x}{x^2}$ (b) $\lim_{h\to 0}\frac{\tan 2h}{3h}$ (c) $\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta^2}$ (d) $\lim_{x\to 0}\frac{\sin^2 x}{x}$

(e) $\lim_{x\to 0}\frac{\sin(\pi + x)}{x}$ (f) $\lim_{x\to 0}\frac{\cos\left(\frac{\pi}{2} - x\right)}{x}$ (g) $\lim_{x\to 0}\frac{1-\sin\left(\frac{\pi}{2} - x\right)}{x^2}$ (h) $\lim_{x\to 0}\frac{2\sin\frac{x}{2}}{x}$

15.8 DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

The result in Example **22**, verified and then derived below, is needed to differentiate $\sin x$ from first principles. You do not need to memorise this result for this course, but you may be asked to use it in a question where it is given.

Example 22

Show that $\sin(x+y) = \sin x \cos y + \cos x \sin y$, for:

(a) $x = \frac{\pi}{3}, y = \frac{\pi}{6}$ (b) $x = \frac{\pi}{2}, y = \frac{\pi}{4}$ (c) $x = \frac{5\pi}{6}, y = \frac{\pi}{3}$

Solution

(a) $x = \frac{\pi}{3}, y = \frac{\pi}{6}$:

$$\text{LHS: } \sin\left(\frac{\pi}{3} + \frac{\pi}{6}\right) = \sin\frac{\pi}{2} = 1$$

$$\text{RHS: } \sin\frac{\pi}{3}\cos\frac{\pi}{6} + \cos\frac{\pi}{3}\sin\frac{\pi}{6}$$

$$= \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} + \frac{1}{2} \times \frac{1}{2}$$

$$= \frac{3}{4} + \frac{1}{4} = 1$$

$$= \text{LHS}$$

(b) $x = \frac{\pi}{2}, y = \frac{\pi}{4}$:

$$\text{LHS: } \sin\left(\frac{\pi}{2} + \frac{\pi}{4}\right) = \sin\frac{3\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\text{RHS: } \sin\frac{\pi}{2}\cos\frac{\pi}{4} + \cos\frac{\pi}{2}\sin\frac{\pi}{4}$$

$$= 1 \times \frac{1}{\sqrt{2}} + 0 \times \frac{1}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}$$

$$= \text{LHS}$$

(c) $x = \frac{5\pi}{6}, y = \frac{\pi}{3}$:

$$\text{LHS: } \sin\left(\frac{5\pi}{6} + \frac{\pi}{3}\right) = \sin\frac{7\pi}{6} = -\frac{1}{2}$$

$$\text{RHS: } \sin\frac{5\pi}{6}\cos\frac{\pi}{3} + \cos\frac{5\pi}{6}\sin\frac{\pi}{3}$$

$$= \frac{1}{2} \times \frac{1}{2} + \left(\frac{-\sqrt{3}}{2}\right) \times \frac{\sqrt{3}}{2}$$

$$= \frac{1}{4} - \frac{3}{4} = -\frac{1}{2}$$

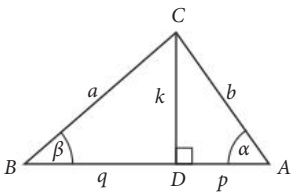
$$= \text{LHS}$$

Important result

This verifies that $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$.

Proof of result: $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

Consider ΔABC:



$$\text{Area } \Delta ABC = \tfrac{1}{2}ab\sin C$$

$$= \tfrac{1}{2}ab\sin\big(\pi-(\alpha+\beta)\big)$$

$\sin(\pi-\theta)=\sin\theta$:

$$= \tfrac{1}{2}ab\sin(\alpha+\beta)$$

But:

$$\text{Area } \Delta ABC = \tfrac{1}{2}kp + \tfrac{1}{2}kq$$

Hence:

$$\tfrac{1}{2}ab\sin(\alpha+\beta) = \tfrac{1}{2}kp + \tfrac{1}{2}kq$$

$$\therefore \sin(\alpha+\beta) = \frac{kp}{ab} + \frac{kq}{ab}$$

$$\sin(\alpha+\beta) = \frac{k}{a}\times\frac{p}{b} + \frac{k}{b}\times\frac{q}{a}$$

$$\sin(\alpha+\beta) = \sin\beta\cos\alpha + \sin\alpha\cos\beta$$

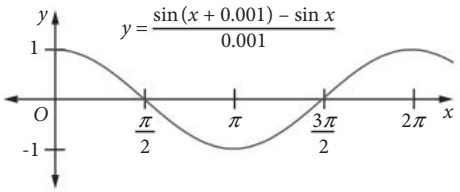
Hence we have the general result:

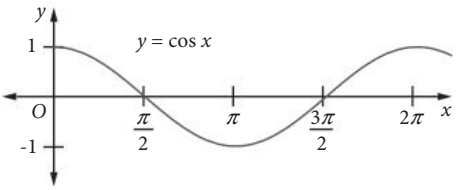
$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

Note: this result is a part of the Mathematics Extension 1 course. It does not need to be memorised for this course.

Derivative of $\sin x$

First, draw the graphs of $y = \dfrac{\sin(x+0.001)-\sin x}{0.001}$ and $y=\cos x$ for $0 \le x \le 2\pi$.





These two graphs appear the same. If you use graphing software to draw the two graphs on the same set of axes, you will be unable to distinguish between them. It seems reasonable to assume the result:

$$\frac{d}{dx}(\sin x) = \cos x$$

Proof

If $y=\sin x$, then from first principles we have $\dfrac{dy}{dx} = \lim\limits_{h\to 0}\dfrac{\sin(x+h)-\sin x}{h}$.

From Example **22** we have the result $\sin(x+h) = \sin x\cos h + \cos x\sin h$, so:

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{\sin x\cos h + \cos x\sin h - \sin x}{h}$$

$$= \lim_{h\to 0}\frac{\sin x(\cos h-1) + \cos x\sin h}{h}$$

$$= \lim_{h\to 0}\frac{\sin x(\cos h-1)}{h} + \lim_{h\to 0}\frac{\cos x\sin h}{h}$$

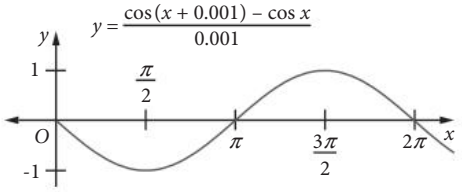
$$= \sin x\lim_{h\to 0}\frac{h(\cos h-1)}{h^2} + \cos x\lim_{h\to 0}\frac{\sin h}{h}$$

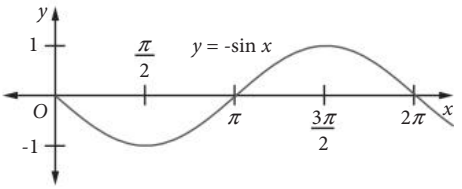
But $\lim\limits_{x\to 0}\dfrac{\sin x}{x} = 1$ and $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2} = \dfrac{1}{2}$, so: $\dfrac{dy}{dx} = \sin x\times 0\times -\tfrac{1}{2} + \cos x\times 1$

$$\therefore \frac{d}{dx}(\sin x) = \cos x$$

Derivative of cos *x*

First, draw the graphs of $y = \dfrac{\cos(x + 0.001) - \cos x}{0.001}$ and $y = -\sin x$ for $0 \le x \le 2\pi$.





These two graphs appear the same. If you use graphing software to draw the two graphs on the same set of axes, you will be unable to distinguish between them. It seems reasonable to assume the result:

$$\frac{d}{dx}(\cos x) = -\sin x$$

Proof

If $y = \cos x$, then from first principles we have $\dfrac{dy}{dx} = \lim\limits_{h \to 0} \dfrac{\cos(x + h) - \cos x}{h}$. But this result can be derived more easily using the fact that $\cos x = \sin\left(\frac{\pi}{2} - x\right)$ with the chain rule.

If $y = \cos x$, then: $y = \sin\left(\frac{\pi}{2} - x\right)$

Differentiate using chain rule: $\dfrac{dy}{dx} = -\cos\left(\frac{\pi}{2} - x\right)$

$\therefore \dfrac{dy}{dx} = -\sin x$

Derivative of tan *x*

This result can be derived from $\tan x = \dfrac{\sin x}{\cos x}$ using the quotient rule.

If $y = \tan x$, then: $y = \dfrac{\sin x}{\cos x}$

Differentiate using quotient rule:

$$\frac{dy}{dx} = \frac{\cos x \times \cos x - \sin x \times (-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$\therefore \frac{d}{dx}(\tan x) = \sec^2 x$$

Summary of trigonometric derivatives

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\sin(ax + b)) = a\cos(ax + b)$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\cos(ax + b)) = -a\sin(ax + b)$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\tan(ax + b)) = a\sec^2(ax + b)$$

Example 23

Differentiate: **(a)** $f(x) = \sin(2x+3)$ **(b)** $f(x) = 5\cos x^2$ **(c)** $f(x) = 3\tan 4x$

Solution

(a) $f(x) = \sin(2x+3)$

$f'(x) = 2\cos(2x+3)$

(b) $f(x) = 5\cos x^2$

$f'(x) = -5\sin x^2 \times 2x$

$= -10x\sin x^2$

(c) $f(x) = 3\tan 4x$

$f'(x) = 12\sec^2 4x$

Example 24

Differentiate: **(a)** $f(x) = \sin^2 x$ **(b)** $f(x) = \sin x \cos x$ **(c)** $f(x) = \cos \pi x - \tan \pi x$

Solution

(a) $f(x) = \sin^2 x$

$f(x) = (\sin x)^2$

$f'(x) = 2\sin x \cos x$

(b) $f(x) = \sin x \cos x$

$f'(x) = \cos x \cos x + \sin x(-\sin x)$

$= \cos^2 x - \sin^2 x$

(c) $f(x) = \cos \pi x - \tan \pi x$

$f'(x) = -\pi \sin \pi x - \pi \sec^2 \pi x$

$= -\pi(\sin \pi x + \sec^2 \pi x)$

Example 25

Differentiate: **(a)** $f(x) = \dfrac{\sin(3x-1)}{\cos x}$ **(b)** $f(x) = \dfrac{e^x}{\sin x}$ **(c)** $f(x) = \dfrac{\log_e x}{\tan x}$

Solution

(a) $f(x) = \dfrac{\sin(3x-1)}{\cos x}$

$f'(x) = \dfrac{3\cos(3x-1)\cos x - \sin(3x-1)(-\sin x)}{\cos^2 x}$

$= \dfrac{3\cos(3x-1)\cos x + \sin(3x-1)\sin x}{\cos^2 x}$

(b) $f(x) = \dfrac{e^x}{\sin x}$

$f'(x) = \dfrac{e^x \sin x - e^x \cos x}{\sin^2 x}$

$= \dfrac{e^x(\sin x - \cos x)}{\sin^2 x}$

(c) $f(x) = \dfrac{\log_e x}{\tan x}$

$f'(x) = \dfrac{\frac{1}{x} \times \tan x - \sec^2 x \log_e x}{\tan^2 x}$

$= \dfrac{\tan x - x\sec^2 x \log_e x}{x\tan^2 x}$

TECHNOLOGY EXPLORATION

Derivative of $\sin x$ and $\cos x$

1. In GeoGebra, enter *f(x)=sin(x)* into the input bar.
2. Next, enter *f′(x)* into the input bar. Right-click on the curve of *f′(x)* and select **Object Properties** to change the colour.
3. In the input bar, enter *g(x)=cos(x)*. What do you notice about this curve?
4. Enter *g′(x)* into the input bar. Hide the curves of *f′(x)* and *g(x)* by clicking the circles next to their equations in the **Algebra** view.
4. What relationship do you notice between *f(x)* and *g′(x)*?
5. Write statements to clearly articulate the results that you have discovered.

EXERCISE 15.8 DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

1 Differentiate with respect to x:

(a) $\sin 3x$ **(b)** $3\sin x$ **(c)** $\cos 2x$ **(d)** $2\cos x$

(e) $\sin x + 4\cos x$ **(f)** $\tan 2x$ **(g)** $\sin 2x - \cos 2x$ **(h)** $\sin\left(x + \frac{\pi}{4}\right)$

2 The derivative of $\cos^2 5t$ is:

A $-10\sin 5t\cos 5t$ **B** $-10\cos 5t$ **C** $-5\sin 5t\cos 5t$ **D** $-2\sin 5t\cos 5t$

3 Differentiate with respect to x:

(a) $\sin x\cos x$ **(b)** $x\sin x$ **(c)** $2x\tan x$ **(d)** $x^2\cos x$

(e) $\dfrac{x}{\sin x}$ **(f)** $\dfrac{\sin x}{x}$ **(g)** $x^3\sin 2x$ **(h)** $\sin(x^3)$

4 Differentiate with respect to t:

(a) $\sin\frac{t}{2} + \frac{1}{2}\cos t$ **(b)** $\cos^3 t$ **(c)** $\cos(t^2+1)$ **(d)** $\sin\left(2t + \frac{\pi}{2}\right)$

(e) $t^2 + \tan\frac{t}{2}$ **(f)** $(t^2-1)\cos 3t$ **(g)** $\cos\left(2t + \frac{\pi}{3}\right)$ **(h)** $\cos(3t-2)$

5 Differentiate with respect to x:

(a) $\cos^2 2x$ **(b)** $\sin^2 3x$ **(c)** $\cos^3 x$ **(d)** $\cos(x^3)$

(e) $\sin 2x\cos 2x$ **(f)** $3x + 2\cos x - \sin\frac{x}{2}$ **(g)** $2\cos^2 x$ **(h)** $\dfrac{\sin x}{x^2}$

(i) $\tan(x^2-1)$ **(j)** $x\cos\frac{x}{2}$ **(k)** $\sin x(1+\cos x)$ **(l)** $\frac{1}{2}\cos 2x + x\sin x$

6 Find $f'(x)$ for $f(x) = 3\sin\frac{x}{2} - 4\cos\frac{3x}{2} - x^3$. Indicate whether each statement below is a correct or incorrect step in finding $f'(x)$.

(a) $\dfrac{d}{dx}\left(3\sin\frac{x}{2}\right) = \frac{3}{2}\cos\frac{x}{2}$ **(b)** $\dfrac{d}{dx}\left(4\cos\frac{3x}{2}\right) = 6\sin\frac{3x}{2}$

(c) $f'(x) = \frac{3}{2}\cos\frac{3x}{2} - 6\sin\frac{3x}{2} - 3x^2$ **(d)** $f'(x) = \frac{3}{2}\cos\frac{x}{2} + 6\sin\frac{3x}{2} - 3x^2$

7 Differentiate with respect to x:

(a) $\sqrt{\sin 2x}$ **(b)** $(\sin x - \cos x)^2$ **(c)** $\sin^2 x + \cos^2 x$

(d) $\sin^2 x - \cos^2 x$ **(e)** $\sin(\cos x)$ **(f)** $\cos(\sin x)$

(g) $\tan(\pi - x)$ **(h)** $\sqrt{1-\cos x}$ **(i)** $\tan\sqrt{x}$

8 Differentiate with respect to x:

(a) $e^x\sin x$ **(b)** $e^{2x}\cos\frac{x}{2}$ **(c)** $e^{-x}\sin 3x$ **(d)** $e^x\cos 4x$ **(e)** $(\cos x + \sin x)e^{-x}$

(f) $e^{\sin 2x}$ **(g)** $e^{\cos x}$ **(h)** $e^{\sin x + \cos x}$ **(i)** $\sin x\ln x$ **(j)** $\log_e(\sin x)$

(k) $\tan x\ln x$ **(l)** $\cos x\ln 2x$ **(m)** $\log_e(\cos x)$ **(n)** $\log_e(\tan x)$ **(o)** $\cos(\log_e x)$

(p) $\sin(\log_e x)$ **(q)** $\tan(\log_e x)$ **(r)** $\dfrac{\log_e x}{\cos x}$

9 $f(x)$ is defined by the rule $f(x) = e^{-x}\cos x$ over the domain $-\frac{\pi}{2} \le x \le \frac{3\pi}{2}$.

(a) Find the values $f(0)$, $f\left(\frac{\pi}{2}\right)$, $f(\pi)$. **(b)** Find $f'(x)$.

(c) Show that $f'(0) = -1$, $f'\left(\frac{3\pi}{4}\right) = 0$ and $f'\left(-\frac{\pi}{4}\right) = 0$. **(d)** Sketch the graph of $f(x)$.

(e) Find the maximum of $f(x)$ over the domain and the value of x where it occurs.

10 Find all the values of x between 0 and 2π for which $\log_e(\sin x)$ is defined and state the maximum value of $\log_e(\sin x)$.

11 Given $y = \log_e(1 + \sin x)$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

12 If $y = 4\cos 3t$, prove that $\frac{d^2y}{dt^2} + 9y = 0$.

13 The concentration C of insects per square metre in a forest is given by $C(t) = 1000\left(\cos\left(\frac{\pi(t-8)}{2}\right) + 2\right)^2 - 1000$, for $8 \leq t \leq 16$, where t is the number of hours after midnight. What is the minimum concentration of insects and when does it occur?

15.9 PRIMITIVES OF TRIGONOMETRIC FUNCTIONS

Because $\frac{d}{dx}(\sin x) = \cos x$, $\int \cos x\, dx = \sin x + C$.

Because $\frac{d}{dx}(\cos x) = -\sin x$, $\int \sin x\, dx = -\cos x + C$.

Because $\frac{d}{dx}(\tan x) = \sec^2 x$, $\int \sec^2 x\, dx = \tan x + C$

Thus, provided $a \neq 0$:

$\frac{d}{dx}(\sin(ax+b)) = a\cos(ax+b)$, so $\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C$

$\frac{d}{dx}(\cos(ax+b)) = -a\sin(ax+b)$, so $\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + C$

$\frac{d}{dx}(\tan(ax+b)) = a\sec^2(ax+b)$, so $\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + C$

Summary of trigonometric integrals

$\int \sin x\, dx = -\cos x + C$	$\int \sin ax\, dx = -\frac{1}{a}\cos ax + C$	$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + C$
$\int \cos x\, dx = \sin x + C$	$\int \cos ax\, dx = \frac{1}{a}\sin ax + C$	$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C$
$\int \sec^2 x\, dx = \tan x + C$	$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax + C$	$\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + C$

Example 26

Find: (a) $\int (\sin x + 2\cos x)\,dx$ (b) $\int (\cos 2x + \sin 3x)\,dx$ (c) $\int \sin\frac{x}{2}\,dx$ (d) $\int \left(2\sin x + 3\cos\frac{x}{2}\right)dx$

Solution

(a) $\int (\sin x + 2\cos x)\,dx$

$= -\cos x + 2\sin x + C$

(b) $\int (\cos 2x + \sin 3x)\,dx$

$= \frac{1}{2}\sin 2x - \frac{1}{3}\cos 3x + C$

(c) $\int \sin\frac{x}{2}\,dx$

$= -2\cos\frac{x}{2} + C$

(d) $\int \left(2\sin x + 3\cos\frac{x}{2}\right)dx$

$= -2\cos x + 6\sin\frac{x}{2} + C$

Example 27

Evaluate: (a) $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos x\,dx$ (b) $\int_0^{\pi} \sin 2x\,dx$ (c) $\int_0^{\frac{\pi}{4}} \sec^2 x\,dx$

(d) $\int_0^{\frac{\pi}{2}} \cos\left(2x-\frac{\pi}{2}\right)dx$ (e) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (\cos 2x - 3\sin x)\,dx$

Solution

(a) $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos x\,dx = \left[\sin x\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$

$= \sin\frac{\pi}{2} - \sin\frac{\pi}{3}$

$= 1 - \frac{\sqrt{3}}{2}$

(b) $\int_0^{\pi} \sin 2x\,dx = \left[-\frac{1}{2}\cos 2x\right]_0^{\pi}$

$= -\frac{1}{2}\cos 2\pi + \frac{1}{2}\cos 0$

$= -\frac{1}{2} + \frac{1}{2} = 0$

(c) $\int_0^{\frac{\pi}{4}} \sec^2 x\,dx = \left[\tan x\right]_0^{\frac{\pi}{4}}$

$= \tan\frac{\pi}{4} - \tan 0$

$= 1 - 0 = 1$

(d) $\int_0^{\frac{\pi}{2}} \cos\left(2x-\frac{\pi}{2}\right)dx = \left[\frac{1}{2}\sin\left(2x-\frac{\pi}{2}\right)\right]_0^{\frac{\pi}{2}}$

$= \frac{1}{2}\sin\frac{\pi}{2} - \frac{1}{2}\sin\left(-\frac{\pi}{2}\right)$

$= \frac{1}{2} + \frac{1}{2} = 1$

(e) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (\cos 2x - 3\sin x)\,dx = \left[\frac{1}{2}\sin 2x + 3\cos x\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$

$= \left(\frac{1}{2}\sin\frac{2\pi}{3} + 3\cos\frac{\pi}{3}\right) - \left(\frac{1}{2}\sin\frac{\pi}{3} + 3\cos\frac{\pi}{6}\right)$

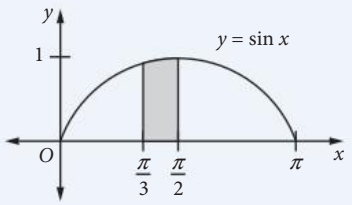
$= \frac{\sqrt{3}}{4} + \frac{3}{2} - \left(\frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{2}\right)$

$= \frac{3-3\sqrt{3}}{2}$

Example 28

Calculate the area bounded by the curve $y = \sin x$, the x-axis and the ordinates $x = \frac{\pi}{3}$ and $x = \frac{\pi}{2}$.

Solution



$\text{Area} = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin x\,dx$

$= \left[-\cos x\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$

$= \left(-\cos\frac{\pi}{2}\right) - \left(-\cos\frac{\pi}{3}\right)$

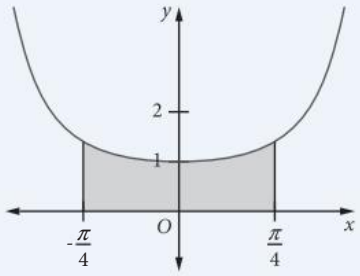
$= 0 + \frac{1}{2} = \frac{1}{2}$

$\therefore$ Area $= \frac{1}{2}$ unit^2

Example 29

Calculate the exact volume of the solid of revolution formed when the region bounded by $y = \sec x$, $x = -\frac{\pi}{4}$, $x = \frac{\pi}{4}$ and the x-axis is rotated about the x-axis.

Solution



$$V = \pi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} y^2 \, dx$$

$$= \pi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^2 x \, dx$$

$$= \pi \left[\tan x \right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= \pi \left(1 - (-1)\right) = 2\pi$$

Volume of solid of revolution is 2π units^3.

EXERCISE 15.9 PRIMITIVES OF TRIGONOMETRIC FUNCTIONS

1 Write the primitive function of:

(a) $\sin 2x$ **(b)** $\cos 3x$ **(c)** $\sec^2 x$ **(d)** $\sin x + \cos x$

(e) $2\sin x - 3\cos x$ **(f)** $\sin\left(x + \frac{\pi}{4}\right)$ **(g)** $\cos \frac{x}{2}$ **(h)** $2\sin 2x$

2 The primitive of $3\cos \frac{x}{3}$ is:

A $-\sin \frac{x}{3}$ **B** $-9\sin \frac{x}{3}$ **C** $\sin \frac{x}{3}$ **D** $9\sin \frac{x}{3}$

3 Find:

(a) $\int \left(\sin \frac{x}{4} + \cos \frac{x}{4}\right) dx$ **(b)** $\int (\sin x - \cos 2x) \, dx$ **(c)** $\int \sin\left(2x + \frac{\pi}{2}\right) dx$

(d) $\int \cos\left(2x - \frac{\pi}{4}\right) dx$ **(e)** $\int \sec^2 3x \, dx$ **(f)** $\int \left(\frac{1}{2}\sin 2x - \cos x\right) dx$

(g) $\int \left(\cos \frac{x}{2} - \frac{1}{2}\sin 2x\right) dx$ **(h)** $\int \left(x^2 + \sin 2x\right) dx$ **(i)** $\int \left(\sqrt{x} - \frac{1}{2}\cos x\right) dx$

4 Evaluate:

(a) $\int_0^{\pi} \sin x \, dx$ **(b)** $\int_0^{\frac{\pi}{3}} \sec^2 x \, dx$ **(c)** $\int_{\frac{\pi}{3}}^{\pi} \cos \frac{x}{2} \, dx$

(d) $\int_0^{\frac{\pi}{2}} \cos x \, dx$ **(e)** $\int_0^{\pi} (\sin x + \cos x) \, dx$ **(f)** $\int_{-\pi}^{\pi} \sin x \, dx$

(g) $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (\cos x - \sin 2x) \, dx$ **(h)** $\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} (\sin 2x - \cos 2x) \, dx$ **(i)** $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (2\sin x + \sin 2x) \, dx$

(j) $\int_0^{\pi} \left(\sin \frac{x}{4} + \cos \frac{x}{4}\right) dx$ **(k)** $\int_0^{\frac{\pi}{3}} \left(3\cos 3x - \frac{\sin 2x}{2}\right) dx$ **(l)** $\int_{-\pi}^{\pi} \left(\frac{\sin x}{2} - \cos x\right) dx$

5 Evaluate $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(3\sin 2x + 4\cos 2x\right) dx$. Indicate whether each statement below is a correct or incorrect step in evaluating this integral.

(a) $I = 2\int_0^{\frac{\pi}{2}} \left(3\sin 2x + 4\cos 2x\right) dx$ **(b)** $\left[\frac{-3\cos 2x}{2} + 2\sin 2x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$

(c) $\frac{-3\cos \pi}{2} + 2\sin \pi + \frac{3\cos(-\pi)}{2} - 2\sin(-\pi)$ **(d)** 0

6 The average value of a function $f(x)$ between $x = a$ and $x = b$ is defined as $\frac{1}{b-a}\int_a^b f(x) \, dx$. Find the average value of $f(x) = 2\cos x + 3\sin x$ between $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$.

7 By integration, find the area bounded by the curve $y = \cos x$ and the x-axis between $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$.

8 The region under the curve $y = \sec x$, $0 \leq x \leq \frac{\pi}{4}$ is rotated about the x-axis. What is the volume of the solid of revolution?

9 Sketch the graph of $f(x) = 1 - \cos x$, $0 \leq x \leq \pi$. Evaluate $\int_0^{\pi} f(x)\,dx$ and indicate on the sketch the area represented by this integral. What is the exact value of this area?

10 Sketch the graph of $f(x) = \sqrt{1 - \cos^2 x}$, $0 \leq x \leq 2\pi$, and evaluate $\int_{\pi}^{\frac{3\pi}{2}} f(x)\,dx$. Illustrate your result on your sketch.

11 Evaluate $\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} \left(\frac{\cos 2x}{4} + \cos x\right) dx$.

12 The area bounded by the curve $y = \sqrt{\sin x}$ between $x = 0$, $x = \pi$ and the x-axis is rotated about the x-axis. Find the volume of the solid of revolution.

13 Find the volume of the solid generated by rotating the curve $y = \sqrt{\cos x}$ between $x = 0$, $x = \frac{\pi}{2}$ and the x-axis about the x-axis.

14 Calculate the area of the region enclosed by the graph of $y = \cos 2x$, the x-axis and the ordinates $x = 0$ and $x = \frac{\pi}{4}$.

15 Calculate the area bounded by the given curves, the x-axis and the given ordinates:

(a) $y = 1 + \cos x$, $x = 0$, $x = \pi$ **(b)** $y = 1 + \dfrac{\sin 2x}{2}$, $x = 0$, $x = \frac{\pi}{2}$

(c) $y = \dfrac{1 + \cos x}{2}$, $x = -\frac{\pi}{4}$, $x = \frac{\pi}{4}$ **(d)** $y = a \sin \dfrac{\pi x}{b}$, $x = 0$, $x = \frac{b}{2}$

16 Find the points of intersection of the curves $y = \sin 2\theta$ and $y = \cos 2\theta$ for values of θ in the interval $0 \leq \theta \leq \pi$. Calculate the area between the two curves for $0 \leq x \leq \pi$.

17 The average value of a function $f(x)$ in the interval $[a, b]$ is defined as $\dfrac{1}{b-a}\int_a^b f(x)\,dx$. Find the average value of $f(x) = 2\cos 2x + \sin 2x$ in the interval $\frac{\pi}{6} \leq x \leq \frac{\pi}{3}$.

18 Find the points of intersection of the curves $y = \sin\theta$ and $y = \cos\theta$ for $0 \leq \theta \leq 2\pi$ and calculate the area between the two curves.

19 Use the trapezoidal rule with two subintervals to estimate $\int_0^{\pi} \sqrt{\sin x}\,dx$ correct to 2 decimal places.

20 Calculate the value of $\dfrac{1}{2 + \cos x}$ for $x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$, and use Simpson's rule with 5 function values to estimate $\int_0^{\pi} \dfrac{dx}{2 + \cos x}$.

15.10 APPLICATIONS OF TRIGONOMETRIC FUNCTIONS

Example 30

Find $\int \dfrac{\sin x}{\cos x}\,dx$.

Solution

$\int \dfrac{\sin x}{\cos x}\,dx$ is of the form $\int \dfrac{f'(x)}{f(x)}\,dx = \log_e\big(f(x)\big) + C$, where $f(x) = \cos x$ and $f'(x) = -\sin x$.

Write: $$\int \frac{\sin x}{\cos x}\,dx = -\int \frac{-\sin x}{\cos x}\,dx$$
$$= -\log_e(\cos x) + C$$

Example 31

Given the function $y = 3\sin t$, for $t > 0$:

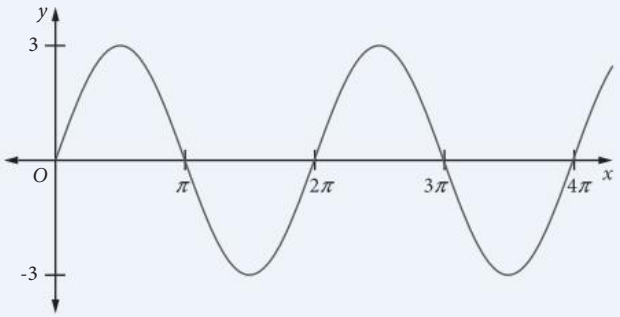
(a) Sketch the graph of this function for $0 \le t \le 4\pi$.

(b) Find the greatest and least values of the function and where they occur for $0 \le t \le 4\pi$.

(c) Describe the behaviour of the curve if y is the distance in metres to the right of a fixed point after a time t hours.

Solution

(a) $y = 3\sin t$, for $0 \le t \le 4\pi$:



(b) $\frac{dy}{dt} = 3\cos t$. For stationary points $\frac{dy}{dt} = 0$, so $\cos t = 0$.

For $0 \le t \le 4\pi$: $t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$

$\frac{d^2y}{dt^2} = -3\sin t$: at $t = \frac{\pi}{2}, \frac{5\pi}{2}$ we have $\frac{d^2y}{dt^2} < 0$ $\therefore$ maximum turning point

at $t = \frac{3\pi}{2}, \frac{7\pi}{2}$ we have $\frac{d^2y}{dt^2} > 0$ $\therefore$ minimum turning point

At $t = \frac{\pi}{2}, \frac{5\pi}{2}$, $y = 3$. The greatest value of the function is 3 and occurs at $t = \frac{\pi}{2}, \frac{5\pi}{2}$.

At $t = \frac{3\pi}{2}, \frac{7\pi}{2}$, $y = -3$. The least value of the function is -3 and occurs at $t = \frac{3\pi}{2}, \frac{7\pi}{2}$.

(c) As t increases, y increases and then decreases in a repeating pattern. The function is periodic with a period of 2π hours (i.e. about 6 h 17 min). Its greatest distance from the fixed point is 3 metres in either direction. After moving 3 metres to the right, the object moves back through its starting point until it reaches a point 3 metres to the left; it then starts moving back to the right again until the pattern repeats.

The kind of wave-like repeating movement described in Example **31** is called **simple harmonic motion**. It is a type of motion that can be described using sine and cosine functions. It occurs naturally in the motion of pendulums and masses on springs, and it can also be used to model ocean tides and other wave-like movements.

Example 32

The tide at a point on the NSW coast can be modelled using the equation $y = a\cos nt$. At Nobby's Beach in NSW, over two consecutive days, the average difference between high and low tides is 1.6 metres and the average time between high tide and low tide is 6.25 hours.

(a) What is the amplitude of the tide function at Nobby's Beach?

(b) How much time passes between successive high tides (i.e. the period) and what is the value of n?

(c) Use this information to obtain the Nobby's Beach tide function and draw its graph.

(d) If the depth of water at low tide is 0.2 metres, what is the depth of the water 2 hours after low tide?

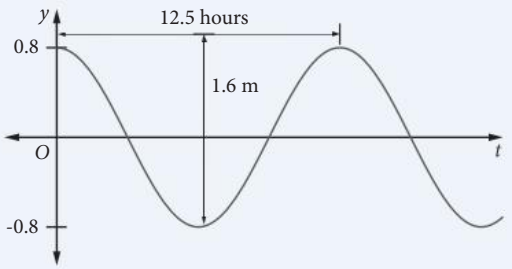
Solution

(a) $2a = 1.6$ so $a = 0.8$

(b) Time between successive high tides $= 2 \times 6.25 = 12.5$ hours

Period $T = \frac{2\pi}{n}$ so $12.5 = \frac{2\pi}{n}$ hence $n = \frac{4\pi}{25}$.

(c) $y = \frac{4}{5}\cos\frac{4\pi t}{25}$



(d) From the graph, low tide occurs at $t = 6.25$ and $y = -0.8$.
2 hours after low tide means $t = 6.25 + 2 = 8.25$

$t = 8.25$: $\quad y = \frac{4}{5}\cos\frac{33\pi}{25} = -0.43$ (to 2 d.p.)

Hence the depth is $0.8 - 0.43 = 0.37$ m above low tide.
$\therefore$ depth of the water 2 hours after low tide $= 0.2 + 0.37 = 0.57$ m (to 2 d.p.)

EXERCISE 15.10 APPLICATIONS OF TRIGONOMETRIC FUNCTIONS

1 Find: **(a)** $\int \frac{\sin x}{1-\cos x}\,dx$ **(b)** $\int \frac{\cos x}{\sin x}\,dx$

2 Evaluate: **(a)** $\int_0^{\frac{\pi}{6}} \frac{\cos x}{1+\sin x}\,dx$ **(b)** $\int_0^{\frac{\pi}{2}} \frac{\sin x}{2-\cos x}\,dx$

3 Write the derivative of $\log_e(\cos x)$ and hence find the primitive function of $\tan x$. Use this result to find the area enclosed by the curve $y = \tan x$, the x-axis and the lines $x = 0$ and $x = \frac{\pi}{3}$.

4 Use the trapezoidal rule with two subintervals to estimate the value of:

(a) $\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \sin^2 x\,dx$ **(b)** $\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \cos^2 x\,dx$

5 Use the trapezoidal rule with four subintervals to find the volume of the solid generated when the area bounded by the curve $y = \sin x$ between $x = 0$, $x = \pi$ and the x-axis is rotated about the x-axis.

6 Find the equation of the tangent to the curve $y = \tan x$ at $x = \frac{\pi}{4}$.

7 For $f(x) = \sin x + \cos x$ over the domain $0 \le x \le 2\pi$, find:

(a) $f'(x)$ **(b)** $f''(x)$
(c) the coordinates of any turning points
(d) the coordinates of any points of inflexion
(e) the maximum value of $f(x)$

8 For $y = e^{\sin x}$, find the equation of the normal to the curve at the point where $x = 0$.

9 The gradient of a curve is given by $\frac{dy}{dx} = 2\sin 3x$. If the curve passes through the point $(\frac{\pi}{3}, 3)$, find the equation of the curve.

10 Given $\int_0^1 \frac{4}{1+x^2}\,dx = \pi$, find an approximate value of π using the trapezoidal rule with two subintervals.

11 Complete the table of values for $f(x) = x \sin x$.

Use Simpson's rule with these five values to find an approximate value of $\int_0^{\pi} x \sin x \, dx$.

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
$f(x)$					

12 The length of one arch of the curve $y = \sin x$ is given by $\int_0^{\pi} \sqrt{1 + \cos^2 x} \, dx$. Find this length using the trapezoidal rule with four subintervals.

13 Find all the points on the graph of $y = 2 \sin x + \sin^2 x$, $0 \leq x \leq 4\pi$, at which the tangent is horizontal.

14 The tide at a point on the WA coast can be modelled using the equation $y = a \cos nt$. At Cable Beach in WA, over two consecutive days, the average difference between high and low tides is 9.0 metres and the average time between high tide and low tide is 6.1 hours.

(a) What is the amplitude of the tide function at Cable Beach?
(b) How much time passes between successive high tides (i.e. the period) and what is the value of n?
(c) Use this information to obtain the tide function and draw its graph.
(d) If the depth of water at low tide is 0.5 metres, what is the depth of the water 1 hour after low tide?

CHAPTER REVIEW 15

1 The minute hand of a clock is 1.2 metres long. How far does the tip move in 40 minutes?

2 An arc AB of length 6 cm subtends an angle of 56° at the centre of a circle. Calculate:

(a) the length of the radius **(b)** the length of the chord AB.

3 Find the exact value of $\sec^2 \frac{\pi}{4} + \operatorname{cosec}^2 \frac{\pi}{4}$.

4 Solve, for $-\pi \leq x \leq \pi$, $6 \cos^2 x - 5 \cos x + 1 = 0$.

5 An arc AB of a sector of a circle is $\frac{\pi}{4}$ metres long and subtends an angle of 60° at the centre, O, of the circle. Calculate:

(a) the length of the radius
(b) the area of the sector AOB (correct to 1 decimal place)
(c) the length of the chord AB (correct to 1 decimal place).

6 Find all values of θ between 0 and 2π for which:

(a) $\sin \theta = -0.5$ **(b)** $\cos \theta = 0$ **(c)** $\tan \theta = -1$
(d) $\sec \theta = \frac{2}{\sqrt{3}}$ **(e)** $\cot \theta = \sqrt{3}$ **(f)** $\operatorname{cosec} \theta = \sqrt{2}$

7 Draw the graph of $y = \cos 2x$ for $-\pi \leq x \leq \pi$. On the same set of axes, draw the graph of $y = -\frac{x}{2}$.
Use your graphs to solve the equation $\cos 2x = -\frac{x}{2}$.

8 An arc AB of a circle, centre C, is 20 cm long.

(a) If the chord AB is 15 cm long and subtends an angle of x radians at C, show that $8 \sin \frac{x}{2} = 3x$.
(b) Solve the equation in part **(a)** graphically, giving your answer correct to 1 decimal place.

9 Differentiate:

(a) $\sin x + \tan 2x$ **(b)** $3 \cos 4x - 5 \sin 2x$ **(c)** $x \cos x$
(d) $\frac{\cos x}{\sin x}$ **(e)** $2e^{-x} \cos 3x$ **(f)** $\log_e (\sin 2x)$

10 Find: **(a)** $\int 3 \sin \frac{x}{2} \, dx$ **(b)** $\int \left(x + \sec^2 2x\right) dx$ **(c)** $\int \frac{\cos t}{\sin t} \, dt$

11 Evaluate: **(a)** $\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} (\sin 2x - \cos 2x) \, dx$ **(b)** $\int_0^{\frac{\pi}{2}} (\cos 2x - x) \, dx$ **(c)** $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sec^2 x}{\tan x} \, dx$

12 The average value of a function $f(x)$ in the interval $[a, b]$ is defined as $\frac{1}{b-a}\int_a^b f(x)\,dx$. Find the average value of $f(x) = 2\cos 2x + \sin 2x$ between $x = \frac{\pi}{4}$ and $x = \frac{\pi}{2}$.

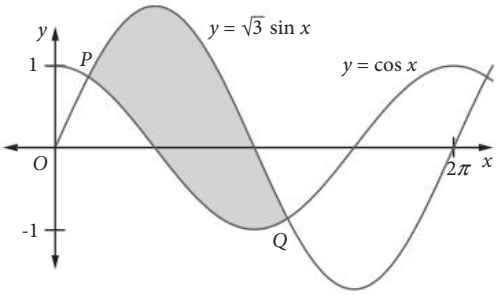
13 Evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cos(x + \pi)\,dx$.

14 The region under the curve $y = \tan x$ between $x = \frac{\pi}{6}$ and $x = \frac{\pi}{3}$ is rotated about the x-axis. What is the volume of the solid of revolution? (**Hint:** remember that $\sec^2 x = 1 + \tan^2 x$.)

15 The diagram shows the graphs of $y = \sqrt{3}\sin x$ and $y = \cos x$. The first two points of intersection to the right of the y-axis are labelled P and Q.

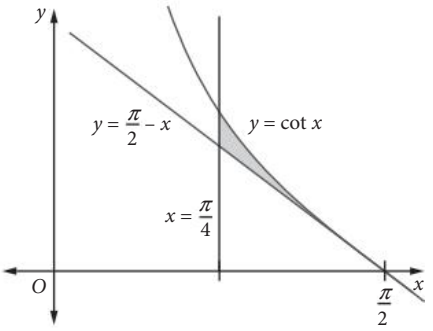
(a) Solve the equation $\sqrt{3}\sin x = \cos x$ to find the abscissae of P and Q.

(b) Find the area of the shaded region in the diagram.



16 **(a)** Show that $\frac{d}{dx}\left(\log_e(\sin x)\right) = \cot x$.

(b) The shaded region in the diagram is bounded by the curve $y = \cot x$ and the lines $y = \frac{\pi}{2} - x$ and $x = \frac{\pi}{4}$. Using the result of part **(a)**, or otherwise, find the exact area of the shaded region.



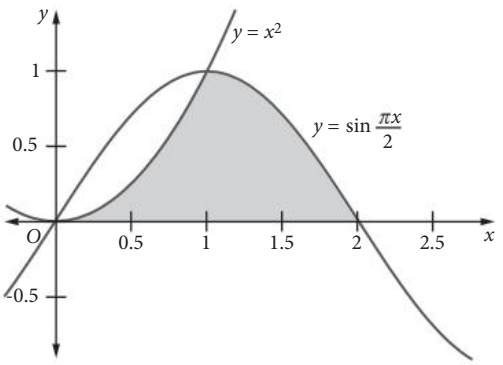
17 The shaded region in the diagram is bounded by the curves $y = \sin\frac{\pi x}{2}$, $y = x^2$ and the x-axis.

(a) Show that the two curves meet at $x = 1$.

(b) Calculate the exact area of the shaded region.

(c) Write the definite integral(s) that will give the volume of the solid of revolution formed when the shaded region is rotated about the x-axis. **Do not evaluate the integral(s).**

(d) Use the trapezoidal rule with four subintervals over $0 \le x \le 2$ to find an approximate value for this volume in terms of π.



18 Use Simpson's rule with 3 function values to estimate the volume of the solid generated when the region bounded by $y = \sqrt{\sin x}$, $x = 0$, $x = \pi$ and the x-axis is rotated about the x-axis. Find the exact volume by integration and hence calculate the percentage error involved in using Simpson's approximation here.

CHAPTER 16
SERIES AND APPLICATIONS

Outcomes covered:

H1 seeks to apply mathematical techniques to problems in a wide range of practical contexts

H4 expresses practical problems in mathematical terms based on simple given models

H5 applies appropriate techniques from the study of calculus, geometry, probability, trigonometry and series to solve problems

H9 communicates using mathematical language, notation, diagrams and graphs

Sequences

The word **sequence** occurs often in everyday language. We say 'a sequence of events'; we complete tasks in a certain sequence; the order of chapters in this book is the sequence in which the Mathematics syllabus is treated.

Whenever the word 'sequence' is used, we are considering a set of objects, ideas, steps or events in some definite order. This order can be associated with the set of positive integers $\{1, 2, 3, \ldots\}$, as is indicated when we use the terms first, second, third etc. to describe a position in the sequence of events.

The term 'sequence' is not used in this syllabus, but mathematicians frequently refer to a **series** as being the collection of the terms of a sequence. When referring to a pattern, the word 'sequence' is often the best term to use.

16.1 SERIES AND SIGMA NOTATION (Σ)

Series

When we write a set of numbers in order according to some rule, the collection of numbers is called a **series** and each number in the collection is called a **term** of the series. In other words, a series is the sum of the terms of its corresponding sequence.

For example, $1 + 4 + 9 + 16 + \ldots + n^2$ is a series in which the first term is 1, the second term is 4, the third term is 9 and the nth term is n^2.

We use the symbols T_n or u_n to represent the nth term of a series. Thus in the series $1 + 4 + 9 + 16 + \ldots + n^2$, $T_1 = 1$, $T_2 = 4$, $T_3 = 9$ and $T_n = n^2$. T_n is also called the **general term** of the series, because it allows us to find any other term of the series according to its rule.

$S_n = T_1 + T_2 + T_3 + \ldots + T_n$ is the sum of the series.

Example 1

What are the first three terms of the series whose general term is given by $T_n = n^2 + 2$?

Solution

$T_1 = 1^2 + 2 = 3$ $\quad T_2 = 2^2 + 2 = 6$ $\quad T_3 = 3^2 + 2 = 11$

This series can be written as $S_n = 3 + 6 + 11 + \ldots + (n^2 + 2)$.

Example 2

Find the nth term of the series $4 + 7 + 10 + 13 + \ldots$

Solution

Because the terms increase by 3 each time, we can rewrite each term as a number plus an increasing multiple of 3:

$T_1 = 4$ $\quad$ $T_2 = 7$ $\quad$ $T_3 = 10$

$= 1 + 3$ $\quad$ $= 1 + 6$ $\quad$ $= 1 + 9$

$= 1 + 3 \times 1$ $\quad$ $= 1 + 3 \times 2$ $\quad$ $= 1 + 3 \times 3$

$\therefore T_n = 1 + 3n = 3n + 1$

Sigma notation (Σ)

The symbol Σ (the Greek capital letter sigma) is used in mathematics to mean 'the sum of'. When we sum the terms of a sequence we have a series, so sigma notation provides us with a shorter way to write the series.

We can rewrite the series discussed above using sigma notation:

$$\sum_{k=1}^{n} k^2 = 1 + 4 + 9 + 16 + \ldots + n^2$$

$$\sum_{k=1}^{n} \left(k^2 + 2\right) = 3 + 6 + 11 + \ldots + (n^2 + 2)$$

$$\sum_{k=1}^{n} (3k + 1) = 4 + 7 + 10 + \ldots + (3n + 1)$$

Each sigma means 'the sum of these terms, starting with the value underneath the Σ and continuing to the value above the Σ'. For example:

- $\sum_{n=1}^{5} x^n$ denotes the sum of terms of the form x^n, where n has the values 1, 2, 3, 4, 5.
  Thus: $\sum_{n=1}^{5} x^n = x^1 + x^2 + x^3 + x^4 + x^5$
- $\sum_{k=0}^{n} (2k + 1)$ denotes the sum of terms of the form $2k + 1$, where
  $k = 0, 1, 2, 3, \ldots n$ (so k is an integer from 0 to n inclusive): $\sum_{k=0}^{n} (2k + 1) = 1 + 3 + 5 + \ldots + (2n + 1)$
  The right-hand side is the expansion of the series defined by $\sum_{k=0}^{n} (2k + 1)$.
- $\sum_{r=1}^{10} rx^{r-1} = 1 + 2x + 3x^2 + \ldots + 10x^9$

It is also useful to use sigma notation to obtain a shorter expression for the sum of a number of terms.

- $1^2 + 2^2 + 3^2 + \ldots + n^2 = \sum_{k=1}^{n} k^2$ $\quad$ The left-hand expression has n terms.
- $1 \times 2 + 2 \times 3 + 3 \times 4 + \ldots + 10 \times 11 = \sum_{n=1}^{10} n(n + 1)$ $\quad$ The left-hand expression has 10 terms.
- $2 + 2x + 2x^2 + \ldots + 2x^8 = \sum_{k=0}^{8} 2x^k$ $\quad$ The left-hand expression has 9 terms.

Example 3

Evaluate: (a) $\sum_{n=2}^{5} n^2$ $\quad$ (b) $\sum_{n=1}^{5} (2n - 1)$

Solution

Write the series in full, then add the terms.

(a) $\sum_{n=2}^{5} n^2 = 2^2 + 3^2 + 4^2 + 5^2$
$= 4 + 9 + 16 + 25$
$= 54$

(b) $\sum_{n=1}^{5} (2n-1) = 1 + 3 + 5 + 7 + 9$
$= 25$

EXERCISE 16.1 SERIES AND SIGMA NOTATION

1 Write expansions of the series defined by the following:

(a) $\sum_{k=1}^{4} k^2$ (b) $\sum_{r=0}^{5} 3^r$ (c) $\sum_{k=1}^{p} (2k-1)$ (d) $\sum_{k=1}^{5} k(k+1)$

(e) $\sum_{r=1}^{8} rx^r$ (f) $\sum_{r=1}^{k} \frac{1}{x^r}$ (g) $\sum_{k=1}^{p} (2k+1)^2$ (h) $\sum_{k=1}^{n+1} (3k-2)$

2 Indicate whether each statement below is a correct or an incorrect expression for $1 + x + x^2 + \ldots + x^{10}$.

(a) $\sum_{n=0}^{10} x^n$ (b) $\sum_{n=1}^{10} x^n$ (c) $\sum_{n=1}^{11} x^{n-1}$ (d) $\sum_{n=1}^{10} x^{n-1}$

3 Use sigma notation to represent each of the following:

(a) $1^2 + 2^2 + 3^2 + \ldots + 9^2$ (b) $1 \times 3 + 2 \times 4 + 3 \times 5 + \ldots + 10 \times 12$ (c) $1 + 6 + 11 + \ldots + (5p - 4)$

(d) $\frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \frac{1}{4 \times 5} + \ldots + \frac{1}{p(p+1)}$ (e) $2x^2 + 3x^3 + 4x^4 + \ldots + 12x^{12}$ (f) $a + ar + ar^2 + \ldots + ar^{n-1}$

4 Evaluate: (a) $\sum_{n=1}^{4} n^2$ (b) $\sum_{n=1}^{6} (2n+1)$ (c) $\sum_{k=1}^{4} (3k-2)$ (d) $\sum_{r=1}^{4} 2^r$

(e) $\sum_{n=0}^{5} n^2$ (f) $\sum_{n=0}^{5} (2n-1)$ (g) $\sum_{n=1}^{4} \left(n^2 + n\right)$ (h) $\sum_{n=1}^{6} (12-3n)$ (i) $\sum_{r=1}^{4} r^r$

16.2 ARITHMETIC SERIES

An **arithmetic series** is a set of terms in which each term is formed by adding a constant number to the preceding term. The series starts with the **first term**, which is usually denoted by a. The constant amount added each time is called the **common difference** and is usually denoted by d.

Thus $S_n = a + (a + d) + (a + 2d) + \ldots + (a + (n-1)d)$ is a series with a first term a and a common difference d.

- The nth term of this series is $T_n = a + (n-1)d$.
- The common difference is given by $d = T_2 - T_1 = T_3 - T_2 = \ldots = T_n - T_{n-1}$.
- Also note that $S_n = T_1 + T_2 + T_3 + \ldots + T_n$.

We call S_n the 'sum to n terms' of the series. This is a finite series.

Using sigma notation, we have: $S_n = \sum_{k=1}^{n} [a + (k-1)d]$

Sum of an arithmetic series

We can obtain a formula for S_n so that we don't always have to add the terms to find the sum. First, note that the nth term of a series is sometimes called the 'last' term and denoted by l, so $T_1 = a$ and $T_n = l = a + (n-1)d$.

Write: $S_n = a + (a + d) + (a + 2d) + \ldots + (a + (n-3)d) + (a + (n-2)d) + (a + (n-1)d)$

Let $l = a + (n-1)d$, so: $S_n = a + (a + d) + (a + 2d) + \ldots + (l - 2d) + (l - d) + l$

Write this series in reverse: $S_n = l + (l-d) + (l-2d) + \ldots + (a+2d) + (a+d) + a$

Add the two expressions: $2S_n = (a+l) + (a+l) + (a+l) + \ldots + (a+l) + (a+l) + (a+l)$

There are n factors of $(a+l)$: $\therefore 2S_n = n(a+l)$

$\therefore S_n = \frac{n}{2}(a+l)$

But $l = a + (n-1)d$, so: $S_n = \frac{n}{2}(a + a + (n-1)d)$

$$\therefore S_n = \frac{n}{2}[2a + (n-1)d]$$

It is also important to realise that $S_n = S_{n-1} + T_n$, so $T_n = S_n - S_{n-1}$, $n > 1$.

Summary of formulae

- $a + (a+d) + (a+2d) + \ldots + (a+(n-1)d)$ is an arithmetic series
- d is the common difference
- $T_1 = a$
- $T_n = a + (n-1)d$
- $S_n = \frac{n}{2}[2a + (n-1)d]$
- $S_n = \frac{n}{2}(a+l)$ where $l = a + (n-1)d$
- $T_n = S_n - S_{n-1}$, $n > 1$

Example 4

For each arithmetic series, calculate the common difference.

(a) $1 + 7 + 13 + 19 + \ldots$ **(b)** $17 + 14 + 11 + 8 + \ldots$ **(c)** $-5 - 3\frac{1}{2} - 2 - \frac{1}{2} + 1 + \ldots$

Solution

Because it is given that the series is arithmetic, find the difference between any pair of consecutive terms. Check your answer using another pair of terms.

(a) $d = 7 - 1 = 13 - 7 = 19 - 13 = 6$

(b) $d = 14 - 17 = 11 - 14 = 8 - 11 = -3$

(c) $d = -3\frac{1}{2} - (-5) = -2 - (-3\frac{1}{2}) = -\frac{1}{2} - (-2) = 1 - (-\frac{1}{2}) = 1\frac{1}{2}$

Example 5

Write the next three terms in each arithmetic series:

(a) $37 + 44 + 51 + 58 + \ldots$ **(b)** $26 + 21 + 16 + \ldots$ **(c)** $3x + 7x + 11x + 15x + \ldots$

Solution

(a) $d = 44 - 37 = 7$

$T_5 = 58 + 7 = 65$ $T_6 = 65 + 7 = 72$ $T_7 = 72 + 7 = 79$

(b) $d = 21 - 26 = -5$

$T_4 = 16 - 5 = 11$ $T_5 = 11 - 5 = 6$ $T_6 = 6 - 5 = 1$

(c) $d = 7x - 3x = 4x$

$T_5 = 15x + 4x = 19x$ $T_6 = 19x + 4x = 23x$ $T_7 = 23x + 4x = 27x$

Example 6

Which series are arithmetic?

(a) $15 + 23 + 31 + 39 + \ldots$ **(b)** $121 + 110 + 100 + 91 + \ldots$ **(c)** $3.6 + 2.5 + 1.4 + 0.3 + \ldots$

Solution

(a) $T_2 - T_1 = 23 - 15 = 8$ $T_3 - T_2 = 31 - 23 = 8$ $T_4 - T_3 = 39 - 31 = 8$

The difference between consecutive terms is a constant.

$15 + 23 + 31 + 39 + \ldots$ is an arithmetic series.

(b) $T_2 - T_1 = 121 - 110 = 11$ $T_3 - T_2 = 110 - 100 = 10$ $T_4 - T_3 = 100 - 91 = 9$

Although the difference between consecutive terms follows a pattern that allows you to obtain further terms, the series is not arithmetic.

(c) $T_2 - T_1 = 2.5 - 3.6 = -1.1$ $T_3 - T_2 = 1.4 - 2.5 = -1.1$ $T_4 - T_3 = 0.3 - 1.4 = -1.1$

The difference between consecutive terms is a constant.

$3.6 + 2.5 + 1.4 + 0.3 + \ldots$ is an arithmetic series.

Example 7

Find the sum of the first twenty terms of the series $3 + 5 + 7 + \ldots$

Solution

$5 - 3 = 7 - 5 = 2$, so the series is arithmetic with $a = 3$, $d = 2$

$n = 20$: $S_n = \frac{n}{2}(2a + (n-1)d)$

$$S_n = \frac{20}{2}(6 + 19 \times 2)$$

$$= 440$$

Example 8

How many terms of the series $4 + 7 + 10 + \ldots$ must be taken to give a sum of 531?

Solution

$7 - 4 = 10 - 7 = 3$, so the series is arithmetic with $a = 4$, $d = 3$

$S_n = 531$: $S_n = \frac{n}{2}(2a + (n-1)d)$

$$531 = \frac{n}{2}(8 + (n-1)3)$$

$$1062 = n(3n + 5)$$

$$3n^2 + 5n - 1062 = 0$$

$$(3n + 59)(n - 18) = 0$$

$$n = -19\tfrac{2}{3}, 18$$

n must be a positive integer, so $n = 18$ and 18 terms must be taken.

Example 9

Find the sum of the first twenty terms of an arithmetic series, given that the tenth term is 39 and the sum of the first ten terms is 165.

Solution

$T_{10} = 39$: $a + 9d = 39$ (1)

$S_{10} = 165$: $\frac{10}{2}(2a + 9d) = 165$

So: $2a + 9d = 33$ (2)

(2) − (1): $a = -6$

Substitute into (1): $-6 + 9d = 39$

$d = 5$

$S_n = \frac{n}{2}(2a + (n-1)d)$: $S_{20} = \frac{20}{2}(-12 + 19 \times 5)$

$= 10(-12 + 95)$

$= 830$

Example 10

Find the first three terms of the arithmetic series defined by $S_n = 2n^2 + n$.

Solution

$T_n = S_n - S_{n-1}$

$S_n = 2n^2 + n$

and $S_{n-1} = 2(n-1)^2 + (n-1)$

$S_{n-1} = 2n^2 - 3n + 1$

$\therefore T_n = 2n^2 + n - (2n^2 - 3n + 1)$

$T_n = 4n - 1$

$\therefore T_1 = 3,\ T_2 = 7,\ T_3 = 11$

Hence the series is $3 + 7 + 11 + \ldots$

Alternatively:

$S_n = 2n^2 + n$

$S_1 = 3 = T_1$

$S_2 = 10$

$T_2 = S_2 - S_1 = 10 - 3 = 7$

$S_3 = 21$

$T_3 = S_3 - S_2 = 21 - 10 = 11$

Hence the series is $3 + 7 + 11 + \ldots$

EXERCISE 16.2 ARITHMETIC SERIES

1 Which of the following are arithmetic series?

(a) $5 + 2 - 1 - 4 - \ldots$ **(b)** $7 + 17 + 27 + 37 + \ldots$ **(c)** $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots$ **(d)** $\frac{5}{8} + 1 + 1\frac{3}{8} + 1\frac{3}{4} + \ldots$

2 The sixth term of the arithmetic series $2 + 9 + 16 + 23 + \ldots$ is:

A 30 **B** 37 **C** 44 **D** 51

3 Find the eighth term and the fourteenth term of the arithmetic series $8 + 14 + 20 + \ldots$

4 For the arithmetic series $17.2 + 16.6 + 16 + \ldots$, find T_5 and T_{11}.

5 Find the arithmetic series for which $T_5 = 17$ and $T_{12} = 52$.

6 Find T_6 of an arithmetic series given that $T_3 = 5.6$ and $T_{12} = -7$.

7 The first and second terms of an arithmetic series are p and q respectively. Find the tenth term.

8 Find the $(n + 2)$-th term of the series $14 + 11 + 8 + \ldots$

9 Find the value of p so that $(p + 5) + (4p + 3) + (8p - 2)$ is an arithmetic series. Indicate whether each statement below is a correct or incorrect step in finding the value of p.

(a) $(4p + 3) - (p + 5) = (8p - 2) - (4p + 3)$ **(b)** $3p + 8 = 4p + 1$

(c) $3p - 2 = 4p - 5$ **(d)** $p = 3$

10 Find the seventh term of an arithmetic series whose fifth term is m and whose eleventh term is n.

11 Find the sum of the first 16 terms of the arithmetic series $3 + 4\frac{1}{4} + 5\frac{1}{2} + \ldots$.

12 Find the sum of the first 12 terms of the arithmetic series in which the first term is 8 and the twelfth term is 41.

13 Find the sum of all integers between 20 and 50 that are divisible by 3.

14 The first three terms of an arithmetic series are $-2 + 3 + 8 + \ldots$

(a) Find the 60th term. **(b)** Hence, or otherwise, find the sum of the first 60 terms of the series.

15 The sum of the first 40 terms of the arithmetic series $40 + 36 + 32 + \ldots$ is:

A 3920 **B** -1520 **C** -2320 **D** 4720

16 A body falling freely from a height travels 4.9 metres in the first second, 14.7 metres in the second second and 24.5 metres in the third second. How far has it fallen:

(a) after six seconds **(b)** between the fifth and the sixth second?

17 Find the sum of the first 50 terms of an arithmetic series, given that the fifteenth term is 34 and the sum of the first eight terms is 20.

18 Find the sum of the first 20 terms of an arithmetic series whose eighth term is 6 and whose twelfth term is 9.

19 Show that the sum of the first n odd positive integers is a perfect square.

20 Evaluate: **(a)** $\sum_{k=1}^{10}(3k-7)$ **(b)** $\sum_{k=1}^{8}(4k+1)$ **(c)** $\sum_{k=1}^{n}(4k-1)$

(d) $\sum_{k=1}^{n}(6k+2)$ **(e)** $\sum_{k=1}^{n}(2k-1)$ **(f)** $\sum_{k=1}^{10}(3k+2)$

21 The first term of an arithmetic series is 7, the common difference is 2 and the sum of the first n terms is 247. Find the value of n.

22 In an arithmetic series, $T_3 = -2$ and $T_9 = 28$. How many terms of this series are required to give a sum of 1092?

23 The sum of the first three terms of the arithmetic series $a + (a + d) + (a + 2d) + \ldots$ is 15 and the product of the three terms is 105. Indicate whether each statement below is a correct or incorrect step in finding the terms of the series.

(a) $a + d = 5$ **(b)** $a(5 + d) = 21$ **(c)** $a^2 - 10a + 21 = 0$ **(d)** The series is $3 + 5 + 7 + \ldots$

24 Find the sum of the integers between 0 and 101 that are:

(a) divisible by 2 **(b)** divisible by 5 **(c)** divisible by 2 and 5 **(d)** divisible by 2 or 5 but not both.

25 The sum of the first six terms of an arithmetic series is -12 and the sum of the first fourteen terms is 196. Find:

(a) the sum of n terms **(b)** the smallest value of n for the sum of the series to exceed 250.

26 The sum of the magnitudes of the angles of a pentagon (five-sided polygon) is 540°. The magnitudes of the angles form the terms of an arithmetic series. If the largest angle has a magnitude of 136°, find the magnitude of each of the other four angles.

27 The sum of the first four terms of an arithmetic series is 34 and the sum of the next four terms is 146. Find the sum of the ninth and tenth terms.

28 The rungs of a ladder decrease in length uniformly from 40 cm at the bottom rung to 30 cm at the top rung. How many rungs are there if the total length of all the rungs is 5.25 m?

29 How many terms of the series $7 + 9 + 11 + \ldots$ must be added to give 352?

30 The first three terms of an arithmetic series are 45, 41, 37. If the nth term is 1, find the value of n and the sum of these n terms.

31 The track of a vinyl record is in the shape of a spiral curve, but it may be considered as a number of concentric circles of minimum and maximum radius 5.25 cm and 10.5 cm respectively. The record rotates at $33\frac{1}{3}$ revolutions per minute and takes 18 minutes to play from start to finish. Find an approximation to the length of the track.

32 Given $S_n = 3n^2 - 11n$, find T_n and hence show that the series is arithmetic.

33 A series is such that the sum of its first n terms is $n(3n + 2)$, where n is a positive integer. Prove that the series is arithmetic and find the eighth term.

34 The sum of the first nine terms of an arithmetic series is 81. If the sum of the first and third terms is zero, find the first term and the common difference.

35 Find the sum of all the integers between 200 and 400 that are divisible by 6.

36 The first and second terms of an arithmetic series are a and b respectively. If the nth term is c, express n in terms of a, b and c, and hence find the sum of these n terms.

37 Find the first three terms of an arithmetic series in which the fifth term is three times the second term, and the sum of the first six terms is 36.

38 Logs of wood are stacked in a pile so that there are 15 logs on the top row, 16 on the next row, 17 on the next, and so on. If there are 246 logs altogether:

(a) how many rows are there **(b)** how many logs are on the bottom row?

39 The lengths of the sides of a right-angled triangle form the terms of an arithmetic series. If the hypotenuse is 15 cm, what is the length of each of the other two sides?

40 How many terms of the series $6 + 10 + 14 + \ldots$ must be taken to give a sum of 880?

41 The sum of the first n terms of the series $30 + 26 + 22 + \ldots$ is 120. Find two possible values of n.

42 Find the sum of: **(a)** the first n odd positive integers **(b)** the first n even positive integers

(c) the first n positive integers, and find this value of n if the sum is 210.

43 For a potato race, a straight line is marked on the ground from a point A and points B, C, D, … are marked on the line so that $AB = BC = CD = \ldots = 2$ metres. A potato is placed at each of the points B, C, D, … A runner has to start from A and bring each potato back to a basket at A, carrying only one potato at a time. Find the number of potatoes so that the total distance run during the race will be 480 metres.

44 Cans of fruit in a supermarket display are stacked so that there are 3 cans in the top row, 5 in the next row, 7 in the next row and so on. If there are 10 rows in the display, find:

(a) the number of cans in the bottom row **(b)** the total number of cans in the display.

45 The first term of an arithmetic series is 5. The ratio of the sum of the first four terms to the sum of the first ten terms is 8 : 35. Find the common difference.

46 The angles of a hexagon form the terms of an arithmetic series. If the smallest angle is 95°, find the size of each of the other angles. (Hint: you need to know the angle sum of a hexagon.)

16.3 FINITE GEOMETRIC SERIES

A **geometric series** is a set of terms in which each term is formed by multiplying the preceding term by a constant number. The series starts with the **first term**, which is usually denoted by a. The constant multiplier is called the **common ratio** and is usually denoted by r.

Thus $S_n = a + ar + ar^2 + \ldots + ar^{n-1}$ is a series with first term a and common ratio r.

- The nth term of this series is $T_n = ar^{n-1}$
- The common ratio is given by $r = \frac{T_2}{T_1} = \frac{T_3}{T_2} = \frac{T_4}{T_3} = \ldots - \frac{T_n}{T_{n-1}}$
- Also note that $S_n = T_1 + T_2 + T_3 + \ldots + T_n$

We call S_n the 'sum to n terms' of the series. This is a finite series.

$S_\infty = T_1 + T_2 + T_3 + \ldots + T_n + \ldots$ is an infinite geometric series.

Using sigma notation, we have: $S_n = \sum_{k=1}^{n} ar^{k-1}$

Thus $T_1 = a$ and $T_n = ar^{n-1}$.

Sum of a geometric series

We can obtain a formula for S_n so that we don't always have to add the terms to find the sum.

Write: $S_n = a + ar + ar^2 + \ldots + ar^{n-1}$ (1)

Multiply both sides of (1) by r: $rS_n = ar + ar^2 + \ldots + ar^{n-1} + ar^n$ (2)

(1) − (2): $S_n - rS_n = a - ar^n$

$S_n(1 - r) = a(1 - r^n)$

Hence: $S_n = \frac{a(1-r^n)}{1-r}$ for $r < 1$

Alternatively, finding (2) − (1) gives: $S_n = \frac{a(r^n - 1)}{r-1}$ for $r > 1$

Note that when $r = 1$, the series becomes $a + a + a + \ldots$ to n terms and $S_n = na$.

It is also important to realise that $S_n = S_{n-1} + T_n$, so $T_n = S_n - S_{n-1}$, $n > 1$.

Summary of formulae

- $a + ar + ar^2 + \ldots + ar^{n-1}$ is a geometric series
- r is the common ratio
- $T_1 = a$
- $T_n = ar^{n-1}$
- $S_n = \frac{a(1-r^n)}{1-r}$ for $r < 1$
- $S_n = \frac{a(r^n - 1)}{r-1}$ for $r > 1$
- $T_n = S_n - S_{n-1}$, $n > 1$

Example 11

For each geometric series, write the common ratio.

(a) $1+2+4+8+\ldots$ **(b)** $x^2+x^4+x^6+x^8+\ldots$ **(c)** $0.1-0.01+0.001-0.0001+\ldots$

Solution

It is given that the series is geometric, so find the ratio between any pair of consecutive terms. Check your answer using another pair.

(a) $\frac{T_2}{T_1}=\frac{2}{1}=2;\quad \frac{T_3}{T_2}=\frac{4}{2}=2;\quad \frac{T_4}{T_3}=\frac{8}{4}=2;\quad r=2$

(b) $\frac{T_2}{T_1}=\frac{x^4}{x^2}=x^2;\quad \frac{T_3}{T_2}=\frac{x^6}{x^4}=x^2;\quad \frac{T_4}{T_3}=\frac{x^8}{x^6}=x^2;\quad r=x^2$

(c) $\frac{T_2}{T_1}=\frac{-0.01}{0.1}=-0.1;\quad \frac{T_3}{T_2}=\frac{0.001}{-0.01}=-0.1;\quad \frac{T_4}{T_3}=\frac{-0.0001}{0.001}=-0.1;\quad r=-0.1$

Example 12

Write the next three terms in each geometric series.

(a) $3+9+27+\ldots$ **(b)** $16+8+4+\ldots$ **(c)** $\frac{8}{25}-\frac{4}{5}+2-\frac{1}{5}+\ldots$

Solution

(a) $r=\frac{9}{3}=3:\quad T_4=27\times 3=81\quad T_5=81\times 3=243\quad T_6=243\times 3=729$

(b) $r=\frac{8}{16}=\frac{1}{2}:\quad T_4=4\times\frac{1}{2}=2\quad T_5=2\times\frac{1}{2}=1\quad T_6=1\times\frac{1}{2}=\frac{1}{2}$

(c) $r=\frac{-\frac{4}{5}}{\frac{8}{25}}=-\frac{5}{2}:\quad T_5=-\frac{1}{5}\times\left(-\frac{5}{2}\right)=\frac{1}{2}\quad T_6=\frac{1}{2}\times\left(-\frac{5}{2}\right)=-\frac{5}{4}\quad T_7=-\frac{5}{4}\times\left(-\frac{5}{2}\right)=\frac{25}{8}$

Example 13

By finding the ratio of successive terms, determine whether or not each series is a geometric series.

(a) $1+2+6+24+\ldots$ **(b)** $2-1+\frac{1}{2}-\frac{1}{4}+\ldots$ **(c)** $x+(x^2+1)+(x^3+2)+(x^4+3)+\ldots$

Solution

(a) $1+2+6+24+\ldots\quad \frac{T_2}{T_1}=\frac{2}{1}=2;\quad \frac{T_3}{T_2}=\frac{6}{2}=3;\quad \frac{T_4}{T_3}=\frac{24}{6}=4;$

There is no common ratio between successive terms, so the sequence is not geometric.

(b) $2-1+\frac{1}{2}-\frac{1}{4}+\ldots\quad \frac{T_2}{T_1}=\frac{-1}{2};\quad \frac{T_3}{T_2}=\frac{\frac{1}{2}}{-1}=\frac{-1}{2};\quad \frac{T_4}{T_3}=\frac{\frac{-1}{4}}{\frac{1}{2}}=\frac{-1}{2};$

There is a common ratio between successive terms of $-\frac{1}{2}$, so the sequence is geometric with first term 2 and common ratio $-\frac{1}{2}$.

(c) $x+(x^2+1)+(x^3+2)+(x^4+3)+\ldots\quad \frac{T_2}{T_1}=\frac{x^2+1}{x};\quad \frac{T_3}{T_2}=\frac{x^3+2}{x^2+1};\quad \frac{T_4}{T_3}=\frac{x^4+3}{x^3+2};$

There is no common ratio between successive terms, so the sequence is not geometric.

However, this series appears to have a pattern. We can discover the pattern by looking at each term as two separate parts: a part involving x and a constant. We can consider these as two separate series $x+x^2+x^3+x^4+\ldots$ and $0+1+2+3+\ldots$

The first is a geometric series with $a=1$ and $r=x$; the second is an arithmetic series with $a=0$ and $d=1$. Hence for the original series we can write $T_n=x^n+(n-1)$.

Example 14

Find the sum of the first eight terms of the geometric series $3 + 6 + 12 + \ldots$

Solution

$a = 3, r = 2, n = 8$: $\quad S_n = \dfrac{a\left(r^n - 1\right)}{r - 1}$

$$S_8 = \frac{3\left(2^8 - 1\right)}{2 - 1}$$

$$S_8 = 3(2^8 - 1) = 765$$

Example 15

For the series $4 + 2 + 1 + \ldots$, find S_6 and S_{10}.

Solution

$\dfrac{T_2}{T_1} = \dfrac{2}{4} = \dfrac{1}{2}$ and $\dfrac{T_3}{T_2} = \dfrac{1}{2}$, so the series is geometric with $a = 4$ and $r = \frac{1}{2}$.

Now: $\quad S_n = \dfrac{a\left(1 - r^n\right)}{1 - r}$

For $n = 6$: $\quad S_6 = \dfrac{4\left(1 - \frac{1}{2^6}\right)}{1 - \frac{1}{2}}$

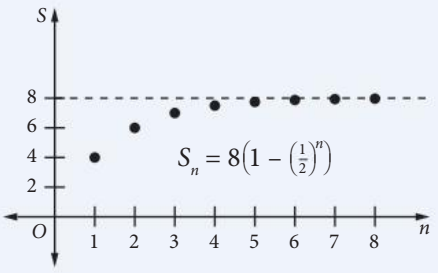
$$S_6 = 8\left(1 - \tfrac{1}{64}\right) = \tfrac{63}{8} = 7\tfrac{7}{8}$$

For $n = 10$: $\quad S_{10} = \dfrac{4\left(1 - \frac{1}{2^{10}}\right)}{1 - \frac{1}{2}}$

$$S_{10} = 8\left(1 - \frac{1}{1024}\right) = \frac{1023}{128} = 7\tfrac{127}{128}$$

These values of S_n for $n = 6$ and $n = 10$ appear to be getting closer to 8.

The graph of $S_n = 8\left(1 - \dfrac{1}{2^n}\right)$ for $n = 1, 2, 3, \ldots 10$ is shown here.



EXERCISE 16.3 FINITE GEOMETRIC SERIES

1 Find the sum of the first six terms of the series $4 + 6 + 9 + \ldots$

2 Find the sum of the first ten terms of the series $8 - 4 + 2 - \ldots$

3 The sum of the first ten terms of the series $2 + 2\sqrt{3} + 6 + \ldots$ is:

A $242\left(\sqrt{3} + 1\right)$ **B** $162\sqrt{3}$ **C** $244\left(\sqrt{3} + 1\right)$ **D** $121\left(\sqrt{3} + 1\right)$

4 Evaluate the series $16 - 8 + 4 - 2 + \ldots + \frac{1}{16}$.

5 Find the sum of nine terms of the series $3 + 3^{\frac{4}{3}} + 3^{\frac{2}{3}} + \ldots$

6 The first four terms of a series are $x + \dfrac{2x^2}{3} + \dfrac{4x^3}{9} + \dfrac{8x^4}{27}$. Find the sum of n terms and evaluate the sum when $n = 6$ and $x = 3$.

7 A pump removes one-half of the water remaining in a tank in 10 minutes. What fraction of the original amount of water will be left in the tank after one hour?

8 A pump removes one-quarter of the water from a tank every 15 minutes. If the tank initially holds 256 000 litres, how much water will remain in the tank after one hour?

9 How many terms of the series $6 + 3 + \frac{3}{2} + \ldots$ must be taken to give a sum of $11\frac{13}{16}$? Indicate whether each statement below is a correct or incorrect step in solving this problem.

(a) $a = 6, r = \frac{1}{2}$ **(b)** $\frac{189}{16} = 12\left(1 - \frac{1}{2^n}\right)$ **(c)** $\frac{1}{2^n} = 1 + \frac{63}{64}$ **(d)** $n = 6$

10 Three numbers a, b and c, whose sum is 15, are successive terms of a geometric series, while b, a and c are successive terms of an arithmetic series. Find the values of a, b and c.

11 The sum of the first eight terms of a geometric series is seventeen times the sum of its first four terms. Find the common ratio.

12 Find the sum of the first ten terms of the series $\log_{10} 3 + \log_{10} 6 + \log_{10} 12 + \ldots$

13 Find the sum of the first seven terms of the series $\log_{10} 27 + \log_{10} 9 + \log_{10} 3 + \ldots$

14 An A0-sized sheet of paper is folded along its long side and then cut to create two sheets of A1-sized paper. Each sheet of A1-sized paper is folded along its long side and then cut to create two sheets of A2-sized paper. This process is repeated many times.

(a) How many sheets of A3-sized paper are created?

(b) I have eleven sheets of A0-sized paper. I cut each sheet into one of the eleven different sizes A0, A1, A2, … A10, creating as many sheets of each size as possible from each sheet of A0.

(i) How many sheets of A10 are created?

(ii) I stack all the sheets of paper on top of each other. with the A0 sheet on the bottom and the A10 sheets on top. How many sheets of paper are in the pile?

(iii) If a pack of 500 sheets of A4 paper is 55 mm thick, approximately how high is the stack of sheets in part **(ii)**?

16.4 INFINITE GEOMETRIC SERIES

In Example **15** we observed that for $a = 4$, $r = \frac{1}{2}$:

$S_1 = 4, S_2 = 7, S_3 = 7, S_4 = 7\frac{1}{2}, S_5 = 7\frac{3}{4}, S_6 = 7\frac{7}{8}, S_7 = 7\frac{15}{16}, S_{10} = 7\frac{127}{128}$

It appears that as n increases, $S_n = 7 +$ a fraction less than 1, so that as $n \to \infty$, $S_n \to 8$. Because the series approaches a limiting value, we say that it **converges**.

We define S_∞, the limiting sum of S_n, as $S_\infty = \lim_{n \to \infty} S_n$. Hence we have $S_\infty = 8$.

Consider a piece of string 8 cm long. It is trimmed so that the first piece cut off is 4 cm long. The remainder is then cut in half so that the next piece is 2 cm long. The remainder is then cut in half so that the next piece is 1 cm long, and so on. You can imagine that there will always be a piece left over to be cut in half again, however small; but as the number of pieces cut off becomes increasingly large, their total length will get closer and closer to 8. That is:

$S_\infty = 4 + 2 + 1 + \frac{1}{2} + \frac{1}{4} + \ldots = 8$

We have seen that $S_n = \dfrac{a\left(1 - r^n\right)}{1 - r}$, which can be written as $S_n = \dfrac{a}{1 - r} - \dfrac{ar^n}{1 - r}$.

If $|r| < 1$, i.e. r is between -1 and 1, then $r^n \to 0$ as $n \to \infty$. In this example we are looking at $r = \frac{1}{2}$, so $r^n = \left(\frac{1}{2}\right)^n = \frac{1}{2^n}$:

n	1	4	10	$\to \infty$
$\frac{1}{2^n}$	$\frac{1}{2}$	$\frac{1}{16}$	$\frac{1}{1024}$	$\to 0$

As $n \to \infty$, $\frac{1}{2^n} \to 0$. Thus $\lim_{n\to\infty} r^n = 0$ and so $\lim_{n\to\infty} \frac{ar^n}{1-r} = 0$, which means that the geometric series converges for $|r| < 1$:

$$S_\infty = \frac{a}{1-r} \qquad \text{for } |r| < 1$$

If $|r| > 1$ then $r^n \to \infty$ and there is no limiting sum, so the geometric series does **not** converge for $|r| > 1$.

Example 16

A rubber ball is dropped from a height of 20 m. Each time it strikes the ground it rebounds $\frac{3}{4}$ of the distance of the previous fall. Find the total distance it travels.

Solution

For the downward motion: $a = 20$, $r = \frac{3}{4}$, $n \to \infty$, $S_\infty = \frac{a}{1-r}$

$$S_\infty = \frac{20}{1-\frac{3}{4}} = 80$$

For the upward motion: $a = 15$, $r = \frac{3}{4}$, $n \to \infty$, $S_\infty = \frac{a}{1-r}$

$$S_\infty = \frac{15}{1-\frac{3}{4}} = 60$$

Total distance = 80 m + 60 m = 140 m.

Example 17

Find the values of x for which the series $1 + (x-3)^2 + (x-3)^4 + \ldots$ converges.

Solution

For the series to be geometric, $a = 1$ and $r = (x-3)^2$.

Series converges for $|r| < 1$:

$$(x-3)^2 < 1$$
$$(x-3)^2 - 1 < 0$$
$$(x-3-1)(x-3+1) < 0$$
$$(x-4)(x-2) < 0$$
$$\therefore 2 < x < 4$$

Example 18

Express as a rational number: **(a)** $0.\dot{2}\dot{3}$ **(b)** $0.5\dot{7}$

Solution

(a) $0.\dot{2}\dot{3} = 0.23 + 0.0023 + 0.000\,023 + \ldots$

Geometric series with $a = 0.23$, $r = \frac{1}{100}$, $S_\infty = \frac{a}{1-r}$

$$0.\dot{2}\dot{3} = \frac{0.23}{1-\frac{1}{100}}$$
$$= \frac{23}{99}$$

(b) $0.5\dot{7} = 0.5 + 0.07 + 0.007 + 0.0007 + \ldots$

Geometric series with $a = 0.07$, $r = 0.1$, $S_\infty = \dfrac{a}{1-r}$

$$0.5\dot{7} = 0.5 + \frac{0.07}{1-0.1}$$

$$= \frac{1}{2} + \frac{7}{90} = \frac{52}{90} = \frac{26}{45}$$

EXERCISE 16.4 INFINITE GEOMETRIC SERIES

1 Evaluate the following: **(a)** $8 - 4 + 2 - \ldots$ **(b)** $4 + 3 + 2\frac{1}{4} + \ldots$

(c) $25 - 10 + 4 + \ldots$ **(d)** $\left(\sqrt{3}+1\right) + 1 + \dfrac{\left(\sqrt{3}-1\right)}{2} + \ldots$

2 If $1 + 2x + 4x^2 + \ldots = \frac{3}{4}$, find the value of x.

3 Find the first three terms of a geometric series given that the sum of the first four terms is $21\frac{2}{3}$ and the sum to infinity is 27.

4 The limiting sum of the series $6 + 3 + \frac{3}{2} + \ldots$ is:

A 10.5 **B** 11.25 **C** 11.625 **D** 12

5 The sum of the first four terms of a geometric series is 30 and the sum of the infinite series is 32. Find the first three terms.

6 Find the sum of the series $12 + 8 + 5\frac{1}{3} + \ldots$

7 Find the sum of the series $1 + \dfrac{1}{a+1} + \dfrac{1}{(a+1)^2} + \ldots$ For what values of a does this infinite series have a sum?

8 For the geometric series $\left(\sqrt{5}+\sqrt{3}\right) + \left(\sqrt{5}-\sqrt{3}\right) + \dfrac{\left(\sqrt{5}-\sqrt{3}\right)^2}{\sqrt{5}+\sqrt{3}} + \ldots$, find the common ratio and the sum of the infinite series.

9 Find the fractional equivalent of: **(a)** $2.3\dot{8}$ **(b)** $4.\dot{6}\dot{2}$ **(c)** $0.417\,17\ldots$

10 $0.323\,232\ldots = \dfrac{p}{q}$, where p and q are integers with no common factor. Find the value of p and q.

11 Show that $1.2888\ldots$ is a rational number by expressing it in the form $\dfrac{m}{n}$, where m and n are integers with no common factor.

12 Evaluate $\dfrac{1 + 2 + 3 + \ldots + 10}{1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{512}}$.

16.5 COMPOUND INTEREST APPLICATIONS

A practical example of a geometric sequence is the growth of money invested at compound interest. If \$100 is invested at the start of a year at compound interest of 8% p.a. (where 'p.a.' = 'per annum' = per year), its value is:

\$100 × 1.08 at the end of the first year,

\$100 × $(1.08)^2$ at the end of the second year,

\$100 × $(1.08)^3$ at the end of the third year,

\$100 × $(1.08)^n$ at the end of n years.

In general, if $\$P$ is invested at compound interest of $r\%$ p.a., it grows to:

PR at the end of the first year, where $R = 1 + \frac{r}{100}$,

PR^2 at the end of the second year,

PR^3 at the end of the third year,

PR^n at the end of n years.

These amounts form a geometric sequence whose common ratio is R. Thus we have the compound interest formula:

$$A_n = PR^n \text{ or } A_n = P\left(1 + \frac{r}{100}\right)^n$$

Here P is the initial amount and A_n is the amount that P grows to after n periods of time, where interest is applied at $r\%$ per period. The period is not necessarily one year:

18% p.a. = 9% per 6 months = 4.5% per 3 months = 1.5% per month, etc.

(For compound interest, this division of interest into shorter time periods is not precisely correct, but for simplicity we will calculate it as shown above.)

A_n is the nth term of the sequence. The corresponding series is investigated in the following examples.

Example 19

At the beginning of each year, Clementine invests \$500. If compound interest is paid at 8% p.a., calculate:

(a) the value of the first investment at the end of 10 years

(b) the accumulated value of her total investment at the end of 10 years.

Solution

(a) Value of first investment at end of first year $= 500 \times 1.08$

Value of first investment at end of second year $= 500 \times 1.08^2$

Value of first investment at end of tenth year $= 500 \times 1.08^{10}$

$= 1079.46$

The value of Clementine's first investment at the end of 10 years is \$1079.46.

(b) Value of last investment at end of tenth year $= 500 \times 1.08$ (1 year's interest)

Value of second-last investment at end of tenth year $= 500 \times 1.08^2$ (2 years' interest)

Value of first investment at end of tenth year $= 500 \times 1.08^{10}$ (10 years' interest)

The sum of all ten investments $= 500 \times 1.08 + 500 \times 1.08^2 + 500 \times 1.08^3 + \ldots + 500 \times 1.08^{10}$

This is a finite geometric series of ten terms, with first term 500×1.08 and common ratio 1.08.

$$S_{10} = \frac{a\left(R^{10} - 1\right)}{R - 1}$$

$$= \frac{500 \times 1.08\left(1.08^{10} - 1\right)}{1.08 - 1}$$

$$= \frac{500 \times 1.08\left(1.08^{10} - 1\right)}{0.08}$$

$$= 7822.74$$

The total investment is worth \$7822.74 after 10 years.

In general, if someone invests $\$P$ at the beginning of each period of time at $r\%$ per period compound interest, the sum V of all investments at the end of n periods of time is given by:

$$\begin{aligned} V &= PR + PR^2 + PR^3 + \ldots + PR^n \\ &= PR(1 + R + R^2 + \ldots + R^{n-1}) \\ &= \frac{PR\left(R^n - 1\right)}{R - 1} \qquad \text{where } R = 1 + \frac{r}{100} \end{aligned}$$

Note that the process of obtaining this result is the important thing to remember, not the result.

Example 20

Your employer offers you two different salary packages: Package 1 offers a raise of \$1300 each year for 5 years, while Package 2 offers a raise of \$100 per month for 5 years.

(a) Which package will give you the largest salary after 5 years?

(b) Package 3 offers an annual increase of 4%. What further information will you need to determine the best of all three packages?

Solution

(a) Package 1: starting salary = $\$P$, increase \$1300 p.a. for 5 years

Salary after 5 years = $P + 5 \times 1300 = \$(P + 6500)$

Package 2: starting salary = $\$P$, increase \$100 p.m. for 60 months

Salary after 5 years = $P + 60 \times 100 = \$(P + 6000)$

Package 1 offers the largest salary after 5 years.

(b) To compare the first two salary packages to a percentage increase you will need to know the starting salary.

Example 21

Georgia borrows \$5000 and agrees to make repayments of \$100 at the end of each month, calculated from the date of the loan. Interest is charged on the unpaid debt at 1.5% per month.

(a) How much does Georgia still owe after the eighth repayment?

(b) How much time will it take to pay off the loan?

(c) If Georgia doubles the repayments, how much time will it take to pay off the loan?

Solution

(a) $P = 5000$, $r = 0.015$, $R = 1.015$

$$\begin{aligned} \text{Amount owing after first repayment} &= 5000 \times 1.015 - 100 \\ \text{Amount owing after second repayment} &= (5000 \times 1.015 - 100) \times 1.015 - 100 \\ &= 5000 \times 1.015^2 - 100(1 + 1.015) \\ \text{Amount owing after third repayment} &= (5000 \times 1.015^2 - 100(1 + 1.015)) \times 1.015 - 100 \\ &= 5000 \times 1.015^3 - 100(1 + 1.015 + 1.015^2) \\ \text{Amount owing after eighth repayment} &= 5000 \times 1.015^8 - 100(1 + 1.015 + 1.015^2 + \ldots + 1.015^7) \end{aligned}$$

Now $1 + 1.015 + 1.015^2 + \ldots + 1.015^7$ is a geometric series with $a = 1$, $r = 1.015$, $n = 8$.

$$\begin{aligned} \therefore \text{ Amount owing after eighth repayment} &= 5000 \times 1.015^8 - \frac{100\left(1.015^8 - 1\right)}{1.015 - 1} \\ &= 5000 \times 1.015^8 - \frac{100\left(1.015^8 - 1\right)}{0.015} \\ &= 4789.18 \end{aligned}$$

Thus Georgia still owes \$4789.18 after the eighth repayment.

(b) From part **(a)** we can say that after n repayments, the amount owing $= 5000 \times 1.015^n - \dfrac{100\left(1.015^n - 1\right)}{0.015}$

When the loan is paid off, the amount owing is 0, so we must solve $5000 \times 1.015^n - \dfrac{100\left(1.015^n - 1\right)}{0.015} = 0$

Solve: $0.015 \times 5000 \times 1.015^n - 100 \times 1.015^n + 100 = 0$

$$1.015^n(100 - 75) = 100$$

$$1.015^n = 4$$

$$n \log_{e10} 1.015 = \log_{10} 4$$

$$n = \frac{\log_{10} 4}{\log_{10} 1.015} = 93.1$$

Georgia would take 94 months (7 years 10 months) to pay off the loan. The final payment is less than \$100.

(c) As for part **(b)**, but replacing the repayment value 100 with 200:

Solve: $0.015 \times 5000 \times 1.015^n - 200 \times 1.015^n + 200 = 0$

$$1.015^n(200 - 75) = 200$$

$$1.015^n = \frac{200}{125}$$

$$1.015^n = 1.6$$

$$n = \frac{\log_{10} 1.6}{\log_{10} 1.015} = 31.6$$

Georgia would take 32 months (2 years 8 months) to pay off the loan. Doubling the repayments reduces the time to pay off the loan by almost two-thirds.

EXERCISE 16.5 COMPOUND INTEREST APPLICATIONS

1 At the beginning of 2013, Amélie invests \$1000. If compound interest is paid at 6% p.a., the value of the investment after 8 years is:

A \$60 **B** \$480 **C** \$1593.85 **D** \$10491.32

2 In 2006, 5000 students entered for a particular examination. The number increased each year by 20% of the number who entered the previous year. Calculate:

(a) the number who entered in 2011

(b) the total number who entered between 2006 and 2011 inclusive.

3 If the amount of wheat harvested in a certain area in 2012 is 4.2 thousand tonnes and it is anticipated that this will increase annually by 2.5%, estimate:

(a) the amount harvested in 2020

(b) the total amount harvested over the 9 years from 2012 to 2020 inclusive.

4 At the beginning of 2011, a mining town had a population of 15 000. It was estimated that this would increase each year by 8% of its population at the beginning of the year. What should the population be at the beginning of 2019?

5 The value of a new car is \$35 000. Its value depreciates (falls) each year by 15% of its value at the beginning of that year. After how many years will its value be \$15 000?

6 Megan is on a fixed salary of $83 000 per year and is offered a performance bonus of 5% of this salary each year. Paul is on a salary of $83 000 per year with an annual increase of 3% p.a.

(a) How much has each person earned after working 4 years, if Megan receives her bonus in the last pay for each year?

(b) When will Paul's annual earnings first exceed Megan's annual earnings, if Megan receives her bonus every year?

7 Kris contributes towards a pension for his retirement by depositing into a superannuation fund an amount of $5000 on each of his 44 birthdays from his 21st to his 64th inclusive.

(a) If the money is invested at 7% p.a. compound interest, how much money is in Kris's superannuation fund on his 65th birthday?

(b) As Kris is about to make the deposit on his 43rd birthday, he is told that he can only expect to earn 3% p.a. compound interest for the remainder of the time. How much less will be in the superannuation fund on his 65th birthday?

8 Two employees start work on a salary of $40 000 per year. Eleanor decides to take an increase of 4% p.a. while Henry accepts an annual increase of $2000 per year.

(a) What are their salaries at the start of the fourth year of employment?

(b) What is the total earned by each person after they have been working for 10 years?

(c) When does Eleanor's salary first exceed Henry's?

9 Tranh borrowed $20 000 on 1 January 2008. He agreed that on 1 January in each succeeding year he would pay back $3000 and add to the debt 6% p.a. interest on the amount owing during the year just completed. Find:

(a) the amount still owing immediately after 1 January 2013

(b) the number of years necessary to pay off the debt.

10 Kathryn borrows $200 000, to be repaid in equal monthly instalments. The interest rate is 8.4% p.a., calculated monthly.

(a) Show that the interest for the first month is $1400.

(b) Why should Kathryn's payments be more than $1400 per month?

(c) Kathryn decides to pay $3000 per month off the loan. Show that the amount owing after two repayments, $\$A_2$, is given by $A_2 = 200\,000 \times 1.007^2 - 3000(1 + 1.007)$.

(d) Hence find an expression for $\$A_n$, the amount owing after the nth repayment.

(e) How long will it take Kathryn to pay off the loan?

11 Phuong decides to set up a trust fund to provide for the education of her grandchildren. She deposits $60 at the beginning of each month into an account that earns 7.2% p.a. compounded monthly. The trust fund will mature at the end of the month of her final investment, 20 years after her first investment, so Phuong will need to make 240 monthly deposits.

(a) After 20 years, what is the value of the first $60 deposited?

(b) Write a geometric series for the value of all the deposits and calculate the final value of the trust fund.

16.6 FURTHER APPLICATIONS OF SERIES

Annuities

An **annuity** (from the same Latin word that gives us 'per annum' and 'annual') is a series of payments made at equal intervals of time: payments are traditionally once per year (hence 'annuity'), but may also be half-yearly, quarterly or at more frequent intervals.

In this course, we define an annuity as a form of investment in which periodical equal contributions are made to or taken from the investment, with interest compounding at the conclusion of each period.

For example, a person may deposit the same amount of money each time period (e.g. year) into a superannuation fund or an investment account, with the interest compounding at the end of the time period. At some point in the future, the funds may be taken out of the account as a lump sum or as a regular payment or pension.

Example 22

Sam invests $5000 on the condition that she is repaid the money in 10 equal quarterly instalments, with the first amount to be repaid 3 months from the investment date. If the investment earns interest at the rate of 4% per quarter, what is the amount of each instalment?

Solution

$P = 5000$, $r = 0.04$, $R = 1.04$, repayment amount (instalment) is Q per quarter

Amount owing after first repayment: $= 5000 \times 1.04 - Q$

Amount owing after second repayment: $= (5000 \times 1.04 - Q) \times 1.04 - Q$

$= 5000 \times 1.04^2 - Q(1 + 1.04)$

Amount owing after third repayment: $= (5000 \times 1.04^2 - Q(1 + 1.04)) \times 1.04 - Q$

$= 5000 \times 1.04^3 - Q(1 + 1.04 + 1.04^2)$

Amount owing after tenth repayment: $= 5000 \times 1.04^{10} - Q(1 + 1.04 + 1.04^2 + \ldots + 1.04^9)$

Now $1 + 1.04 + 1.04^2 + \ldots + 1.04^9$ is a geometric series with $a = 1$, $r = 1.04$, $n = 10$.

After the tenth payment, the amount owing is zero, so $A_{10} = 0$.

$\therefore$ Amount owing after tenth repayment: $= 5000 \times 1.04^{10} - \dfrac{Q\left(1.04^{10} - 1\right)}{1.04 - 1} = 0$

Hence: $\dfrac{Q\left(1.04^{10} - 1\right)}{1.04 - 1} = 5000 \times 1.04^{10}$

$$Q = \frac{5000 \times 1.04^{10} \times 0.04}{1.04^{10} - 1} = 616.45$$

Thus Sam receives $616.45 per quarter.

This result can be generalised into a formula if we let $P = 5000$, $R = 1.04$. with n repayments:

Amount owing after n repayments $= P \times R^n - \dfrac{Q\left(R^n - 1\right)}{R - 1} = 0$

So repayments are: $Q = \dfrac{PR^n(R - 1)}{R^n - 1}$

Although this formula could have been used from the start, it is important to know how it is obtained.

Example 23

When Peggy started work she began paying $120 at the beginning of each month into a superannuation fund. These contributions are compounded monthly at an interest rate of 6% p.a. Peggy intends to retire after having worked for 40 years.

(a) Let $\$P$ be the final value of Peggy's superannuation when she retires after 40 years (480 months). Show that $\$P = \$240\,174$ to the next dollar.

(b) After working for 20 years, Peggy decides that she needs to have $750 000 in her fund before she can retire. At this stage the fund has only $55 722. Peggy decides to increase the amount that she pays into the fund for the next 20 years to $\$M$ at the beginning of each month. The contributions will continue to attract the same interest rate of 6% p.a. compounded monthly. At the end of n months after starting the new contributions, the amount in the fund is $\$A_n$.

(i) Show that $A_2 = 55\,722 \times 1.005^2 + M(1.005 + 1.005^2)$

(ii) Find the value of M so that Peggy will have $750 000 in her fund after the remaining 20 years (240 months).

Solution

(a) Monthly investment = \$120, $r = 0.005\%$ p.m., $n = 480$

Value of last investment at end of 40th year: $= 120 \times 1.005$ (1 month's interest)

Value of second-last investment at end of 40th year: $= 120 \times 1.005^2$ (2 months' interest)

Value of first investment at end of 40th year: $= 120 \times 1.005^{480}$ (480 months' interest)

Sum of the investments $= 120 \times 1.005 + 1200 \times 1.005^2 + 120 \times 1.005^3 + \ldots + 120 \times 1.005^{480}$

This is a finite geometric series with first term 120×1.005, common ratio 1.005, $n = 480$.

$$P = S_{480} = \frac{a\left(R^{480} - 1\right)}{R - 1}$$

$$= \frac{120 \times 1.005\left(1.05^{480} - 1\right)}{1.005 - 1}$$

$$= \frac{120 \times 1.005\left(1.005^{480} - 1\right)}{0.005}$$

$$= 240173.78$$

$\$P = \$240\,174$ (to the next dollar)

(b) (i) Monthly investment = \$$M$, $r = 0.005\%$ p.m., $n = 2$, initial balance = \$55 722

Value of balance at end of 2 months: $= 55\,722 \times 1.005^2$

Value of last investment at end of 1 month: $= M \times 1.005$

Value of second-last investment at end of 2 months: $= M \times 1.005^2$

$\therefore A_2 = 55\,722 \times 1.005^2 + M(1.005 + 1.005^2)$

(ii) $A_{240} = 55\,722 \times 1.005^{240} + M(1.005 + 1.005^2 + 1.005^3 + \ldots + 1.005^{240})$

$$A_{240} = 55\,722 \times 1.005^{240} + \frac{M \times 1.005\left(1.005^{240} - 1\right)}{1.005 - 1}$$

We require $A_{240} = 750\,000$:

$$750\,000 = 55\,722 \times 1.005^{240} + \frac{M \times 1.005\left(1.005^{240} - 1\right)}{0.005}$$

$$750\,000 - 55\,722 \times 1.005^{240} = \frac{M \times 1.005\left(1.005^{240} - 1\right)}{0.005}$$

$$M = \frac{0.005\left(750\,000 - 55\,722 \times 1.005^{240}\right)}{1.005\left(1.005^{240} - 1\right)}$$

$$= 1217.93$$

The monthly payment would need to be \$1218 per month (to the next dollar).

EXERCISE 16.6 FURTHER APPLICATIONS OF SERIES

1 Noor invests \$10 000 on the condition that the money is repaid in 12 equal quarterly instalments. If the investment earns interest at the rate of 1% per quarter, what is the amount of each instalment?

2 Ava invests \$20 000 on the condition that she is repaid the money in 16 equal quarterly instalments. If the investment earns interest at the rate of 0.75% per quarter, what is the amount of each instalment?

3 **(a)** When Nazeera started a new job, \$400 was deposited into her superannuation fund at the beginning of each month. The money was invested at 0.4% per month, compounded monthly.
Let \$$Y$ be the value of the investment after 360 months, when Nazeera retires. Show that $Y = 322\,142.43$.

(b) After retirement, Nazeera withdraws \$3000 from her superannuation fund at the end of each month without making any further deposits. The account continues to earn interest at 0.4% per month.
Let \$$A_n$ be the amount of money left in the account n months after Nazeera's retirement.

(i) Show that $A_n = (Y - 750\,000) \times 1.004^n + 750\,000$.

(ii) For how many months after retirement will there be money left in the account?

4 One year ago William and Kate borrowed \$400 000 to buy an apartment. The interest rate was 6% p.a., compounded monthly. They agreed to repay the loan over 25 years with equal monthly repayments of \$2578.

(a) Calculate how much money William and Kate owed after their first monthly repayment.

(b) After making their twelfth monthly repayment, William and Kate owe \$392 870. The interest rate now increases to 9% p.a., compounded monthly. The amount \$$A_n$ owing on the loan after the nth monthly repayment is now calculated using the formula $A_n = 392\,870 \times 1.0075^n - 1.0075^{n-1}M - \ldots - 1.005M - M$, where \$$M$ is the monthly repayment and $n = 1, 2, \ldots 288$. (Do **not** prove this formula.)
Calculate the monthly repayment if the loan is to be repaid over the remaining 24 years (288 months).

(c) William and Kate now decide to increase their monthly repayments to \$3500. How long will it take them to repay \$392 870?

(d) How much money will William and Kate save over the term of the loan by making these higher monthly repayments, compared to keeping the repayments at the amount calculated in **(b)**?

5 Which will give the better financial result after 20 years: a lump sum of \$100 000 invested at 5% p.a. compounded annually, or a monthly payment of \$600 with interest at 5% p.a. compounded monthly?

6 A lottery offers a prize of \$100 000 immediately, or \$10 000 now plus \$10 000 per year for the next 11 years. You take the prize of \$100 000, keep \$10 000 to spend over the next year and invest the remaining \$90 000 as an annuity at 5% p.a. You plan to withdraw \$10 000 at the end of each year for the next 11 years.

(a) Is this a realistic plan?

(b) How much money is left in your annuity after the eleventh payment?

(c) How much money is left in your annuity after the twelfth payment?

Another practical use for series—the rule of 72

(**Note**: the following information is not part of the syllabus.)

The '**rule of 72**' is a rule that can be used to find roughly how much time it will take for an investment to double in value if invested at a particular interest rate. It should technically be the rule of $100 \ln 2$ (or 69.3), but 72 is used instead because it has more factors, which makes it easier to divide without a calculator than 69.3. (A good rule for approximations is that they must be easy to use.)

The rule of 72 says that if you divide 72 by the interest rate, the answer will be a good approximation of how much time it will take to double your investment. For example:

- At 12% p.a. it takes $\approx 72 \div 12 = 6$ years to double your money.
- At 6% p.a. it takes ≈ 12 years to double your money.

The rule also works for depreciation by half. For example, if the depreciation rate of a car is 15% then it will take $\approx 72 \div 15 = 4.8$ years for the car to halve in value.

Why does this rule work? Consider:

$$A = P(1 + r)^n \qquad \text{where } r \text{ is a decimal}$$

To double in value to $2P$: $2P = P(1 + r)^n$

$$2 = (1 + r)^n$$

Take $\log_e$ of both sides: $\log_e 2 = n \log_e (1 + r)$

$$n = \frac{\log_e 2}{\log_e (1 + r)}$$

When r is small, we have $\log_e(1+r) \approx r$, so that $n \approx \frac{\log_e 2}{r} = \frac{0.693}{r}$. If we consider $r = \frac{\alpha}{100}$, meaning that the interest rate is $\alpha\%$, then we have:

$$n \approx \frac{0.693}{\alpha} \times 100 = \frac{69.3}{\alpha} \approx \frac{72}{\text{interest rate}}$$

Therefore 72 divided by the interest rate is approximately equal to the length of time n for the investment to double.

As this table of values shows, for various values of r and n we have:

r	0.01	0.03	0.06	0.10	0.20
n	69.7	23.4	11.9	7.3	3.8
$r \times n$	0.697	0.702	0.714	0.730	0.760
$r\% \times n$	69.7	70.2	71.4	73.0	76.0

In practice, 72 is a good approximation for rates up to about 12%.

Knowing the rule of 72 can help people to avoid bad investment decisions. It is a good backup for that other important rule: 'if it seems too good to be true, then it probably is'.

Consider the case of the 'Wattle' scheme in the late 1990s, which was one of Australia's worst investment disasters. More than 3000 Australians invested almost $200 million in an illegal investment scheme that promised a return of 50% per year, paid each month.

Using the rule of 72, $72 \div 50 \approx 1.5$. This means that an investor in the scheme would double their money in less than 1.5 years. It sounds too good to be true, and it was. The scheme was supposed to make its profits by lending funds to new businesses, but this never happened; instead, the initial investors were paid using the funds collected from later investors. (This type of scam is so common, it has a name: it is called a 'Ponzi' scheme.) The initial investors, impressed with their profits, put even more money into the scheme and encouraged their friends to do so too. Of course, the scheme soon turned into a disaster for investors and its promoter was sentenced to jail for 10 years.

CHAPTER REVIEW 16

1 Find the sum of all the numbers between 20 and 200 that are divisible by 9.

2 Evaluate $6 + 3 + 1.5 + \ldots$

3 A ball is dropped from a height of 20 m and rebounds to a height of 18 m. If every time it rebounds it rises to nine-tenths of its previous height, calculate the total distance it could travel.

4 Three numbers whose sum is 15 are successive terms of an arithmetic series. If 1, 1 and 4 are added to these three numbers respectively, the resulting numbers are successive terms of a geometric series. Find the numbers.

5 If $\sqrt{3} - 1$ and $2 - \sqrt{3}$ are the first two terms of a geometric series, write the next two terms and the sum to infinity in simplest surd form.

6 Express 0.2333… in the form $\frac{m}{n}$, where m and n are integers.

7 The first, third and ninth terms of an arithmetic series are also the first three terms of a geometric series. Find the common ratio of the geometric series.

8 For the function defined by $S_n = n^2 - 3n$ for $n = 1, 2, 3, \ldots$ find T_n and hence show that the series is arithmetic.

9 For a geometric series, the second term is 6 and the fifth term is 48. Find the sum of the first five terms.

10 The population of a certain town in the year 2010 was 24 000. Every year its population increases by 25% of its population during the previous year. What will the town's population be in 2030?

11 Find the sum of the series $1 - \frac{1}{4} + \frac{1}{16} - \frac{1}{64} + \ldots$

12 Find the sum of the first n terms of a series given $T_r = 2^r + 2r - 1$.

13 Sanjay borrows $5000 at an interest rate of 1.5% per month and pays it off in equal monthly instalments. What should the instalment be in order to pay off the loan at the end of 3 years?

14 **(a)** At the beginning of each year, $100 is placed in an investment fund. Calculate the accumulated value at the end of 12 years if the interest is 6% p.a. compounded yearly.

(b) If the $100 due at the beginning of the fifth year was not placed in the fund, what would be the accumulated value at the end of 12 years?

15 A family borrows $300 000 from a bank. Interest is charged at 6% p.a.

(a) How much must be repaid each year, rounded to the next dollar, if the loan is to be repaid over 30 years?

(b) How much of the loan will remain after the 18th payment?

(c) If repayments are made at $40 000 per year, how long will it take to repay the loan? What will be the last repayment?

16 **(a)** Find the limiting sum of the geometric series $3 + \frac{3}{\sqrt{3}+1} + \frac{3}{(\sqrt{3}+1)^2} + \ldots$

(b) Explain why the geometric series $3 + \frac{3}{\sqrt{3}-1} + \frac{3}{(\sqrt{3}-1)^2} + \ldots$ does not have a limiting sum.

17 Merv retires with a lump sum of $200 000. The money is invested in a fund that pays interest each month at a rate of 6% p.a., and Merv receives a fixed monthly payment of $\$M$ from the fund. Thus the amount left in the fund after the first monthly payment is $\$(201\,000 - M)$.

(a) Find a formula for the amount $\$A_n$ left in the fund after n monthly payments.

(b) Merv chooses the value of M so that there will be nothing left in the fund at the end of the 15th year (after 180 payments). Find the value of M.

CHAPTER 17
APPLICATIONS OF CALCULUS TO THE PHYSICAL WORLD

Outcomes covered:

H1 seeks to apply mathematical techniques to problems in a wide range of practical contexts
H3 manipulates algebraic expressions involving logarithmic and exponential functions
H4 expresses practical problems in mathematical terms based on simple given models
H5 applies appropriate techniques from the study of calculus, geometry, probability, trigonometry and series to solve problems
H6 uses the derivative to determine the features of the graph of a function
H7 uses the features of a graph to deduce information about the derivative
H8 uses techniques of integration to calculate areas and volumes
H9 communicates using mathematical language, notation, diagrams and graphs

17.1 GRADIENT AS A RATE OF CHANGE

Consider the linear function $y = f(x) = 3x + 2$. An increase of h in the value of x leads to an increase of $3h$ in the value of y. For any value of h, $h \neq 0$, y increases by an amount equal to three times the increase in x.

We say that 'the rate of change of y with respect to x is 3'. The 3 is the gradient of the line. Because the gradient in this case is constant, y increases at a constant rate.

(This should remind you of differentiation from first principles, where the gradient of the function $= \frac{f(x+h) - f(x)}{h}$.)

Consider the function $y = f(x) = 2x^2$ at the point $(1, 2)$:

$f(x) = 2x^2 \qquad f(1) = 2 \qquad f(1+h) = 2(1+h)^2 = 2 + 4h + 2h^2$

$k = f(1+h) - f(1) = 2(1+h)^2 - 2 = 4h + 2h^2$

$\frac{k}{h} = 4 + 2h,\ h \neq 0$

Thus when x changes by an amount h, from 1 to $1 + h$, $f(x)$ changes by an amount $k = 4h + 2h^2$; that is, the value of the function changes by an amount that is $(4 + 2h)$ times the amount of the change in x.

The 'average rate of change of y with respect to x' $= 4 + 2h$. This rate varies with h.

As $h \to 0$, $4 + 2h \to 4$, i.e. $f'(1) = 4$. The number 4 in this case gives us:

(a) the gradient of the tangent at $x = 1$
(b) the rate of change of y with respect to x at $x = 1$.

This concept of the derivative as a 'rate of change' is very important in differential calculus. For example, there are many practical situations in which the change in a physical quantity depends on time:

- If a vessel is being filled with water, the volume V of the water in the vessel is a function of time. $\frac{dV}{dt}$ is the rate at which the volume changes, which may or may not be a constant rate.
- The population P of a town may increase or decrease with time. $\frac{dP}{dt}$ is the rate of change of population with respect to time.

- As a spherical balloon is being inflated, $\frac{dV}{dt}$ is the instantaneous rate of change of its volume with respect to time. Its spherical radius is also increasing, so $\frac{dr}{dt}$ is the rate of change of the radius with respect to time. Because a sphere's volume is given by $V = \frac{4}{3}\pi r^3$, we have $\frac{dV}{dr} = 4\pi r^2$, where $\frac{dV}{dr}$ measures the rate of change of volume with respect to the radius.

Note: when dealing with time t, we will only consider $t \geq 0$, so we will use the term **initially** to mean 'at time $t = 0$'.

Rates of change have many applications in physics and chemistry. Boyle's law for gases states that 'the pressure P of a given mass of gas varies inversely with the volume V', so that P is a function of V where:

$$P = \frac{k}{V} = kV^{-1}$$

$$\frac{dP}{dV} = -\frac{k}{V^2} = \text{rate of change of pressure with respect to volume.}$$

The negative sign indicates that an increase in V leads to a decrease in P.

Proportion

If two variables are in **direct proportion**, this means that their absolute values vary in the same direction: as the magnitude of the independent variable increases, the magnitude of the dependent variable will increase proportionally (and vice versa). Mathematically, the two variables are linear functions of each other (e.g. $y = kx$).

Inverse proportion means that the absolute values of the variables vary in opposite directions: as the magnitude of the independent variable increases, the magnitude of the dependent variable will decrease proportionally (and vice versa). Mathematically, the two variables are the inverse (reciprocal) of each other $\left(\text{e.g. } y = \frac{k}{x}\right)$.

Example 1

The volume V litres of water in a tank is given by $V = 4t + 30$, where t is in seconds.

(a) How much water is in the tank initially? **(b)** At what rate is water flowing into the tank?

Solution

(a) 'Initially' means $t = 0$, as we are dealing with time $t \geq 0$.

When $t = 0$: $V = 30$

Initially there is 30 L of water in the tank.

(b) If $V = 4t + 30$: $\frac{dV}{dt} = 4$

Water is flowing into the tank at a constant rate of 4 litres per second (L s^{-1}).

Example 2

A large cube of ice has an edge length of 10 cm. It melts so that its volume decreases at a constant rate of 25 cm^3 per hour. Find:

(a) the volume V at time t **(b)** the time required to completely melt the ice.

Solution

(a) Because the volume decreases at a constant rate of 25 cm^3 per hour, $\frac{dV}{dt} = -25$.

$$\therefore V = -\int 25\,dt$$

$$= -25t + C$$

When $t = 0$, $V = 10^3 = 1000$: $1000 = C$

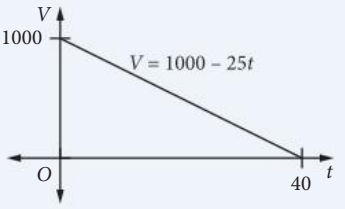
$$\therefore V = 1000 - 25t$$

(b) When $V = 0$: $0 = 1000 - 25t$

$t = 40$

The volume at any time t is given by $V = 1000 - 25t$, $0 \le t \le 40$.

Note that the domain of the volume function is restricted to values of t from 0 to 40. The volume function is a linear function and its graph has a constant gradient of -25.



Example 3

A chemical solution is being filtered into an empty beaker. The rate at which the level of the solution is rising in the beaker is given by $\frac{dh}{dt} = \frac{1}{10}\left(1 - \frac{t}{100}\right)$, where h is the depth of the solution in cm and t is the time in seconds after the solution starts flowing through the filter.

(a) What is the depth of the solution in the beaker after 40 seconds?

(b) What will be the depth of the solution in the beaker when the solution has stopped flowing through the filter?

Solution 1

(a) When $t = 0$, $h = 0$: $\frac{dh}{dt} = \frac{1}{10}\left(1 - \frac{t}{100}\right)$

$$h = \int \frac{1}{10}\left(1 - \frac{t}{100}\right) dt$$

$$= \frac{1}{10}\left(t - \frac{t^2}{200}\right) + C$$

When $t = 0$, $h = 0$: $\therefore C = 0$

Hence: $h = \frac{1}{10}\left(t - \frac{t^2}{200}\right)$

When $t = 40$: $h = \frac{1}{10}\left(40 - \frac{1600}{200}\right)$

$h = 3.2$

The depth of the solution after 40 seconds is 3.2 cm.

(b) The solution will have stopped flowing when $\frac{dh}{dt} = 0$, as at this time the depth of the solution has stopped changing.

When $\frac{dh}{dt} = 0$: $\frac{1}{10}\left(1 - \frac{t}{100}\right) = 0$

$t = 100$

Stops flowing after 100 seconds.

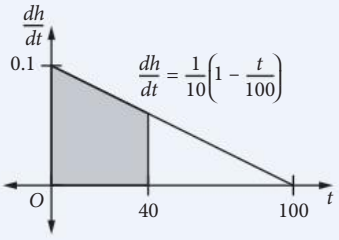
When $t = 100$: $h = \frac{1}{10}\left(100 - \frac{100^2}{200}\right) = 5$

The depth when it stops flowing is 5 cm.

Solution 2

(a) $h = \int dh = \int \frac{dh}{dt}\, dt$, so the area under $\frac{dh}{dt}$ can be used to find the value of h between appropriate limits.

Sketch $\frac{dh}{dt} = \frac{1}{10}\left(1 - \frac{t}{100}\right)$:



The shaded region is bounded by $\frac{dh}{dt} = \frac{1}{10}\left(1 - \frac{t}{100}\right)$, $\frac{dh}{dt} = 0$, $t = 0$, $t = 40$, because we want the depth after 40 seconds.

$$\text{Shaded area} = \int_0^{40} \frac{1}{10}\left(t - \frac{t}{100}\right) dt$$

$$= \frac{1}{10}\left[t - \frac{t^2}{200}\right]_0^{40}$$

$$= \frac{1}{10}\left(40 - \frac{1600}{200} - 0\right)$$

$$= 3.2$$

The shaded area represents the depth of the solution after 40 seconds, so:

$$h = \int_0^{40} \frac{1}{10}\left(t - \frac{t}{100}\right) dt = 3.2 \text{ cm}$$

(b) From the graph we can see that $\frac{dh}{dt} = 0$ when $t = 100$, so this is when the water has stopped flowing.

$$h = \int_0^{100} \frac{1}{10}\left(t - \frac{t}{100}\right)dt$$
$$= \frac{1}{10}\left[t - \frac{t^2}{200}\right]_0^{100}$$
$$= \tfrac{1}{10}\left(100 - \tfrac{10000}{200} - 0\right)$$
$$= 5\,\text{cm}$$

EXERCISE 17.1 GRADIENT AS A RATE OF CHANGE

1 A trench is being dug by a team of labourers who remove V cubic metres of soil in t minutes, where $V = 10t - \frac{t^2}{20}$.

(a) State the domain of the function, i.e. the values of t during which soil is being removed.
(b) At what rate is the soil being removed at the end of 40 minutes?
(c) Are the labourers working at a constant rate?
(d) What is their initial rate of work, i.e. when $t = 0$?
(e) At what time are they removing soil at the rate of $5\,\text{m}^3$ per minute?

2 When some concentrated chemical solutions are allowed to evaporate slowly, crystals are formed. The surface area of a particular crystal is given by $A = 0.6t^2$, where A is in mm^2 and t is in days. The rate at which the surface is increasing after 4 days is:

A $9.6\,\text{mm}^2/\text{day}$ **B** $12.8\,\text{mm}^2/\text{day}$ **C** $0.6\,\text{mm}^2/\text{day}$ **D** $4.8\,\text{mm}^2/\text{day}$

3 A cube of ice has an edge length of 10 cm. It melts so that its volume decreases at a constant rate and the block remains a cube. If the edge length measures 5 cm after 70 minutes, find:

(a) the rate at which the volume decreases **(b)** the volume at any time t.

4 A water tank is being emptied. The quantity Q litres of water remaining in the tank at any time t minutes after it starts to empty is given by $Q(t) = 1000(20 - t)^2$, $t \geq 0$.

(a) At what rate is the tank being emptied at any time t?
(b) How much time does it take to empty the tank? (When is $V = 0$?)
(c) At what time is the water flowing out at a rate of 20 000 litres per minute?
(d) What is the average rate at which the water flows out in the first 5 minutes?

5 The following table shows the temperature T degrees Celsius (°C) of water in a vessel, initially at 100°C, after t minutes.

t (min)	0	5	10	15	20	25	30
T (°C)	100	85	74	64	56	48	44

Plot the graph of this data to show the relation between temperature and time. From the graph, estimate the rate at which the temperature is falling:

(a) after 15 minutes **(b)** when the temperature is 80°C.

6 A machine manufactures items at a variable rate given by $\frac{dQ}{dt} = 2t + 1$, $t \geq 0$, where Q is the number of items manufactured in a time t minutes.

(a) At what rate is the machine working: **(i)** initially **(ii)** after 10 minutes?
(b) What is the total number of items manufactured in the first 10 minutes?

7 If the area of a circle is given by $A = \pi r^2$, show that the rate of change of the area with respect to the radius, $\frac{dA}{dr}$, is proportional to the radius. Find this rate when the radius is 2 cm.

8 A right circular cylinder of volume V has height h and radius of its base r. Find:

(a) the rate of change of volume with respect to height, if the radius of the base is constant

(b) the rate of change of volume with respect to the radius of the base, if the height is constant.

9 The pressure P and volume V of a given mass of gas at constant temperature are connected by the formula $PV = 500$. Find $\frac{dP}{dV}$ when $V = 10$.

10 Using the straight-line method of depreciation, it is found that after t years have elapsed the value V of a certain machine is given by $V = 40\,000 - 5000t$, where $0 \leq t \leq 8$.

(a) Find $\frac{dV}{dt}$ and interpret your result. **(b)** What is the value of the machine after 3 years?

11 The revenue function for a particular manufacturer is $R = x\left(15 - \frac{x}{30}\right)$, where x is the number of units of the product sold. If the marginal revenue is given by $\frac{dR}{dx}$, find the marginal revenue when:

(a) $x = 6$ **(b)** $x = 15$ **(c)** $x = 225$

12 The rate at which carbon dioxide is produced by the action of yeast in a dough is given by $\frac{dV}{dt} = \frac{1}{10\,000}\left(200t - t^2\right)$, where V cm^3 is the volume of carbon dioxide produced after t seconds. What is the volume of carbon dioxide produced in the first 3 minutes after the yeast starts to work? Indicate whether each statement below is a correct or incorrect step in the solution of this problem.

(a) $V = \frac{1}{10\,000}\int_0^4 \left(200t - t^2\right)dt$ **(b)** $V = \frac{1}{10\,000}\int_0^{180} \left(200t - t^2\right)dt$

(c) $V = \left[100t^2 - \frac{t^3}{3}\right]_0^{180}$ **(d)** $V = 129.6\text{ cm}^3$

13 The sluice gates of a dam are operated by an automatic program that controls the flow of water out of the dam. The program is set at 7 a.m. one morning so that t hours after 7 a.m. the flow of water will be given by $\frac{dV}{dt} = 500 - 15t^2 + t^3$ megalitres (ML) per hour.

(a) If no water flows from the dam before 7 a.m., calculate:

(i) the flow of the water at 9 a.m.

(ii) the total volume of water released between 7 a.m. and 9 a.m.

(b) **(i)** Sketch $\frac{dV}{dt} = 500 - 15t^2 + t^3$ for $0 \leq t \leq 10$.

(ii) When does the flow of water stop?

(iii) If the sluice gates close at the moment when $\frac{dV}{dt} = 0$, how much water has been released altogether?

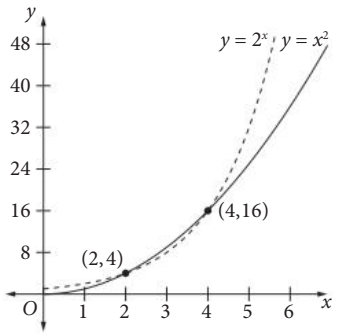
14 The current of I amperes (or 'amps', symbol A) flowing in an electrical circuit is given by $I = \frac{dQ}{dt}$, where Q is the charge on a capacitor and t is the time in seconds. For a particular electrical circuit, $Q = 4\sin \pi t$. Calculate the current flowing in the circuit after 2 seconds.

17.2 EXPONENTIAL GROWTH AND DECAY

A quantity exhibits **exponential growth** when it increases by a constant percentage of its whole over a given time period. This means that the larger the quantity at the start, the bigger the increase will be. For example: if a country of 100 million people and a country of 10 million people are each growing their populations at an annual rate of 10%, then the first country's population increases by 10 million next year while the second country's population increases by only 1 million.

The following table shows values of x^2 (quadratic growth) and 2^x (exponential growth) for selected values of x. Corresponding graphs for $x \geq 0$ are shown below. Note how exponential growth far exceeds quadratic growth as x increases beyond the value $x = 4$.

x	0	1	2	3	4	5	6	10	100
x^2	0	1	4	9	16	25	36	100	10^4
2^x	1	2	4	8	16	32	64	1024	$\approx 10^{30}$



In 1972, a political think-tank called the Club of Rome published *The Limits to Growth*, a controversial but influential study of the possible interactions between various kinds of linear and exponential growth and decline in global systems of population, pollution, natural resources, food and other factors.

The general principle behind their models of exponential growth can be appreciated if we imagine the vertical axis of the exponential graph above to represent the growth of a quantity (e.g. population or pollution) and the horizontal axis to represent time. An increase of 1 unit of x ('time') from $x = 1$ to $x = 2$ causes only a small absolute increase in y, from $y = 2$ to $y = 4$. You could imagine this might represent world population growth in past centuries. However, an increase of one unit of 'time' from $x = 10$ to $x = 11$ causes y to increase by more than 1000. This illustrates the enormous increases that can occur when an exponentially growing variable becomes large enough to reach the steeply rising part of the curve. (Of course, global population growth is more complicated than a simple exponential curve, but the general principle of exponential change is the important point to remember.) Exponential decline can be equally dramatic, which accounts for concern about the use of non-renewable resources.

There are many examples of exponential growth and decay in the physical world, including population growth, the growth of bacteria, radioactive decay, rates of heating and cooling, the decrease in atmospheric pressure at higher altitudes, the decrease in light intensity through water or glass, and the dying away of sound vibrations. In each case, the quantities follow the exponential law:

- $y = Ae^{kx}$ for exponential growth
- $y = Ae^{-kx}$ for exponential decay (or 'exponential decline').

For $y = Ae^{kx}$, we have: $\dfrac{dy}{dx} = kAe^{kx}$

$= ky$

$\therefore \dfrac{dy}{dx} = ky$ is the exponential rate of change.

This means that exponential growth occurs when the rate of change of a quantity y with respect to another quantity x is proportional to y. For example:

- The rate of growth of a colony of bacteria is proportional to the number of bacteria N present at any time, i.e. $\dfrac{dN}{dt} = kN$.
- The rate of decay of a radioactive isotope is proportional to the mass of that isotope present at any time, i.e. $\dfrac{dM}{dt} = -kM$. (The negative sign indicates decay, meaning the amount is reducing over time.)
- The rate of cooling of a body is proportional to the difference between the temperature of the body and the temperature of the surrounding medium, i.e. $\dfrac{d\theta}{dt} = -k\theta$, where θ is the temperature difference at any time. (This is Newton's 'law of cooling'.)
- The rate of decrease of atmospheric pressure with respect to height above sea level is proportional to the pressure at that height, i.e. $\dfrac{dP}{dh} = -kP$.

- Light passing through a transparent medium loses its intensity as it goes on. The rate of loss of light intensity with respect to the distance is proportional to the light's intensity at the distance, i.e. $\frac{dI}{dx} = -kI$.

In each case we have a **differential equation** of the form $\frac{dy}{dx} = ky$. A differential equation is an equation that involves derivatives.

It is important to understand the role played by the constants A and k in the equation $y = Ae^{kx}$.

- When $x = 0$, $y = Ae^0 = A$, so if $x \geq 0$ then A is the initial value of y.
- Because $\frac{dy}{dx} = ky$, k is the **growth rate** and influences the slope of the curve.

For $y = Ae^{kx}$: • A is the value of y when $x = 0$ • k is the growth rate.

Example 4

The annual growth rate of the population of two towns P and Q are 10% and 5% respectively of their populations at any time. If the initial population of P is 20 000 and of Q is 10 000, find their populations 3 years later.

Solution

Let N be the population at any time t years.

For P: $\frac{dN}{dt} = 0.1N$

$\therefore N = Ae^{0.1t}$

At $t = 0$: $N = 20\,000$, so $A = 20\,000$

So: $N = 20\,000e^{0.1t}$

At $t = 3$: $N = 20\,000e^{0.3}$

$N \approx 27\,000$

For Q: $\frac{dN}{dt} = 0.05N$

$\therefore N = Ae^{0.05t}$

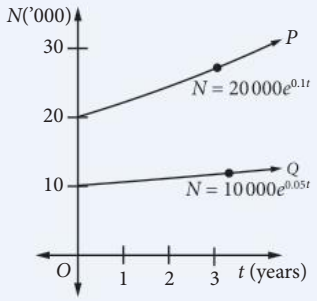
At $t = 0$: $N = 10\,000$, so $A = 10\,000$

So: $N = 10\,000e^{0.05t}$

At $t = 3$: $N = 10\,000e^{0.15}$

$N \approx 12\,000$

The answers are given to the nearest thousand.



The graphs indicate the roles played by A and k. Because A is the value of N when $t = 0$, it is the point where the curve crosses the N-axis.

The graph for P is steeper than the graph for Q. This indicates the role of k, the growth rate. The population of P grows more rapidly than the population of Q.

Note that exponential growth or decay is continuous, so k indicates the **instantaneous** rate of change. Compare this to the growth of money invested at compound interest: interest is calculated periodically, so change only happens at fixed intervals (e.g. daily, monthly, quarterly, yearly).

Example 5

A vessel containing water is being emptied. The volume $V(t)$ cubic metres of water remaining in the vessel after t minutes is given by $V(t) = Ae^{-kt}$.

(a) If $V(0) = 100$, find the value of A. **(b)** If $V(5) = 90$, find the value of k. **(c)** Find $V(20)$.

Solution

(a) When $t = 0$, $V = 100$, so: $A = 100$

(b) When $t = 5$, $V = 90$, so: $90 = 100e^{-5k}$

$0.9 = e^{-5k}$

$\therefore\ -5k = \log_e 0.9$

$k = -0.2 \log_e 0.9 \approx 0.02$

(c) $V(t) \approx 100e^{-0.02t}$

$V(20) \approx 100e^{-0.4} \approx 67$ (Note that without rounding, this answer is closer to 66.)

Example 6

The pressure of the atmosphere, measured as P kilopascals (kPa), decreases with the altitude h km above sea level approximately according to $P = 101e^{-0.2h}$. Find the rate at which the air pressure falls with respect to height above sea level when: **(a)** $h = 5$ **(b)** $P = 20$

Solution

$P = 101e^{-0.2h}$

$\frac{dP}{dh} = -0.2 \times 101e^{-0.2h}$

$\frac{dP}{dh} = -20.2e^{-0.2h}$ (1) Here (1) gives $\frac{dP}{dh}$ as a function of h, while

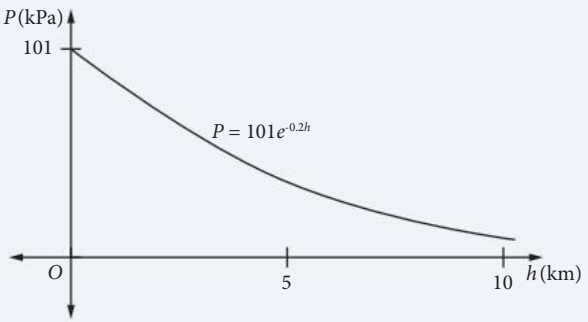
Or: $\frac{dP}{dh} = -0.2P$ (2) (2) gives $\frac{dP}{dh}$ as a function of P.

(a) $h = 5$: $\frac{dP}{dh} = -20.2e^{-1} = -7.43$

The pressure falls at a rate of 7.43 kPa/km when $h = 5$.

(b) $P = 20$: $\frac{dP}{dh} = -0.2 \times 20 = -4$

The pressure falls at a rate of 4 kPa/km when $P = 20$.



Example 7

The mass M of a radioactive substance is initially 10 grams. Twenty years later the mass of remaining radioactive substance is 9.6 grams.

(a) Find the annual decay rate, given that the rate of decay of a radioactive substance is proportional to the mass of the substance present at any time.

(b) In how many years will the mass of radioactive substance be halved?

Solution

We are told $\frac{dM}{dt} = -kM$. We know that $M = Ae^{-kt}$ is a solution to this differential equation.

(a) $t = 0$, $M = 10$, so: $A = 10$

$t = 20$, $M = 9.6$: $9.6 = 10e^{-20k}$

$\therefore\ -20k = \log_e 0.96$

$k = -\frac{1}{20} \log_e 0.96 \approx 0.002$

(b) If $k = 0.002$: $M = 10e^{-0.002t}$

$M = 5$: $5 = 10e^{-0.002t}$

$e^{-0.002t} = 0.5$ or $e^{0.002t} = 2$

$\therefore 0.002t = \log_e 2$

$t = 500 \log_e 2 \approx 347$

The mass of radioactive substance will be halved after about 347 years.

The time taken for half of an amount of radioactive substance to decay is called the **half-life** of the substance.

EXERCISE 17.2 EXPONENTIAL GROWTH AND DECAY

1 If $\frac{dy}{dx} = 2y$ and $y = 5$ when $x = 0$, express y as a function of x.

2 If $\frac{dN}{dt} = -0.5N$ and $N = 100$ when $t = 0$, then N expressed as a function of t is:

A $N = 100e^{0.5t}$ **B** $N = 100e^{-0.5t}$ **C** $N = 0.5e^{100t}$ **D** $N = 0.5e^{-100t}$

3 If $\frac{dQ}{dt} = 0.4Q$ and $Q = 50$ when $t = 0$, express Q as a function of t.

4 If $\frac{dy}{dt} = -3y$ and $y = 20$ when $t = 0$, express y as a function of t.

5 If $\frac{dM}{dt} = -0.01M$ and $M = 10$ when $t = 0$, express M as a function of t.

6 If $y = Ae^{-kt}$, $y = 1000$ when $t = 0$, and $y = 368$ when $t = 2$, find the values of A and k.

7 If $N = Ae^{kt}$, $N = 200$ when $t = 0$, and $N = 1478$ when $t = 5$, find the values of A and k.

8 If $P = Ae^{-kh}$, $P = 76$ when $h = 0$, and $P = 28$ when $h = 5$, indicate whether each of the following statements is correct or incorrect.

(a) $A = 76$ **(b)** $e^{5k} = \frac{7}{19}$ **(c)** $k = 0.2 \log_e\left(\frac{19}{7}\right)$ **(d)** $k \approx 0.20$

9 The population of a city increases at a rate that is proportional to the current population. If the population of the city was 100 000 in the year 2000 and 120 000 in 2010, express the population P in terms of t years after 2000.

10 In a certain bacterial culture, the rate of increase of bacteria is proportional to the number of bacteria present.

(a) If the number of bacteria doubles every 3 hours, find the hourly growth rate.
(b) If the original bacteria population is 10^4, how many are there after 9 hours?
(c) After how many hours are there 4×10^4 bacteria?

11 The rate of decay of a radioactive isotope is proportional to the amount of the isotope present at any time. If one half of a given quantity of the isotope decays in 1600 years, what percentage will decay in 100 years?

12 The number of bacteria N in a colony after t minutes is given by $N = 10\,000e^{0.05t}$. Find:

(a) the number of bacteria after 10 minutes
(b) the time required for the original number to double.
(c) Find the rate at which the colony increases when: **(i)** $t = 10$ **(ii)** $N = 20\,000$

13 A vessel filled with liquid is being emptied. The volume V cubic metres of liquid remaining after t minutes is given by $V = V_0 e^{-kt}$.

(a) Show that $\frac{dV}{dt} = -kV$.

(b) If one-quarter of the vessel is emptied in the first 5 minutes, what fraction remains after 10 minutes?

(c) At what rate is the liquid flowing out:

(i) after 10 minutes **(ii)** when one-quarter of the vessel is empty.

14 For a period of its life, the increase in the diameter of a tree approximately follows the rule $D(t) = Ae^{kt}$, where $D(t)$ is the diameter of the tree t years after the beginning of this period.

(a) If the diameter is initially 50 cm, find the value of A.

(b) If $D'(t) = 0.1D(t)$, find the value of k.

(c) After how many years is the diameter 61 cm?

15 The charge Q (measured in coulombs) on the plate of a condenser t seconds after it starts to discharge is given by the formula $Q = Ae^{-kt}$.

(a) If the original charge is 5000 coulombs, find the value of A.

(b) If $\frac{dQ}{dt} = -2000$ when $Q = 1000$, find the value of k.

(c) Find the rate of discharge when $Q = 5000$.

16 The rate of increase in the number N of bacteria in a certain culture is given by $\frac{dN}{dt} = 0.15N$, where t is time in hours.

(a) If the original number of bacteria is 1000, express N as a function of t.

(b) After how many hours has the original number of bacteria doubled? What is the rate of increase at this time?

17 Sunlight transmitted into water loses intensity as it penetrates to greater depths according to the law $I(d) = I(0)e^{-kd}$, where $I(d)$ is the intensity at depth d metres below the surface. If $I(300) = 0.3I(0)$, find:

(a) the value of k **(b)** the depth at which the intensity would be decreased by one-half.

18 The rate of increase of the population $P(t)$ of a particular island is given by the equation $\frac{d}{dt}P(t) = kP(t)$, where t is time in years. In 2000 the population was 1000 and in 2010 it had decreased to 800.

(a) Find k, the annual growth rate. **(b)** In how many years will the population be half that in 2000?

19 A substance decomposes at a rate equal to k times the mass of the substance present. If initially the mass is M, find the mass m at time t. If $k = 0.1$, find the value of t for which $m = \frac{M}{2}$.

20 A heated body is cooling. The excess of its temperature above that of its surroundings is $\theta = Ae^{-kt}$, where t is in minutes.

(a) At time $t = 0$, $\theta = 80$. Find A.

(b) If the temperature of the surroundings is 20°C and the body cools to 70°C in 10 minutes, find:

(i) the body's temperature after 20 minutes **(ii)** the time taken to reach 60°C.

21 The number N of bacteria in a colony grows according to the rule $\frac{dN}{dt} = kN$. If the original number increases from 4000 to 8000 in 4 days, find the number after another 4 days.

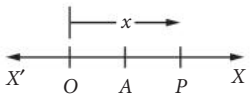
22 A population of size N is decreasing according to the rule $\frac{dN}{dt} = -\frac{N}{100}$, where t is the time in days. If the population is initially of size N_0, find how much time it takes for the size to be halved, to the next day.

23 A radioactive substance decays at a rate that is proportional to the mass of radioactive substance present at any time. If 10% decays in 200 years, what percentage of the original radioactive mass will remain after 1000 years?

17.3 MOTION OF A PARTICLE IN A STRAIGHT LINE

Particle is the term used for a body that behaves such that all forces acting on the body can be regarded as acting through a single point. This means that we can represent the body as a single point, regardless of its actual size and shape. This definition of a particle means that quite large bodies, e.g. trains, can still be classified as 'particles' provided this condition applies.

Displacement



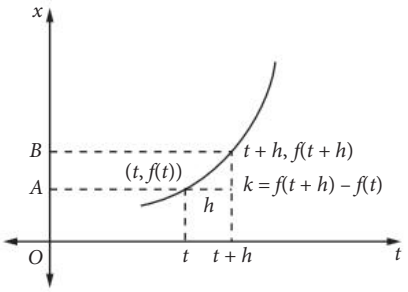
Consider a particle, which can be represented by a point P, moving in a straight line $X'OX$.

The **displacement** x is the particle's position relative to the fixed point O. It may be a positive or negative number, according to whether P is to the right or left of O. The origin of the motion is not necessarily at O, so when $t = 0$, P may be (for example) at the point A.

Displacement is defined as the position relative to a starting point. It can be positive or negative.
Displacement does not necessarily represent the total distance travelled.
Unlike displacement, distance is always a positive quantity.

Velocity

Consider the equation $x = f(t)$ that gives the position coordinate x of a particle moving in a straight line at time t.



At time t, the particle is at A, and at time $(t + h)$ the particle is at B, as shown in the diagram. Thus in the small time interval h the particle has changed its position by an amount $k = f(t + h) - f(t)$.

The average **velocity** in this time interval $= \frac{k}{h} = \frac{f(t+h)-f(t)}{h}$, $h \neq 0$.

The instantaneous velocity of the particle at time t is defined by $\lim_{h\to 0} \frac{f(t+h)-f(t)}{h}$. It may be denoted by $v(t)$, $f'(t)$, $\frac{dx}{dt}$ or $\dot{x}$.

Velocity is defined as the rate of change of position (i.e. of displacement) with respect to time, or as the time rate of change of position in a given direction.

$$v(t) = f'(t) = \frac{dx}{dt} = \dot{x} = \lim_{h\to 0} \frac{f(t+h)-f(t)}{h}$$

Velocity can be positive or negative, depending on the direction of travel.
Speed is the magnitude of the velocity and is always positive.

Example 8

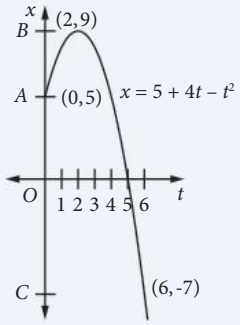
Consider the equation $x = 5 + 4t - t^2$, which defines the displacement x metres from O at time t seconds (for $t \geq 0$) of a particle moving in a straight line.

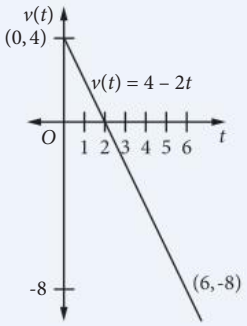
(a) Find the velocity function. **(b)** Discuss the sign of the velocity over $0 \leq t \leq 6$.

Solution

(a) For $x = 5 + 4t - t^2$: $\quad v = \dfrac{dx}{dt} = 4 - 2t$

(b)





The diagram at left is the graph of the displacement function. The diagram at right is the graph of the velocity function.

- When $t = 0$, $v = 4$ and the particle is at point A.
- When $t = 2$, $v = 0$ and the particle is at rest at B. For the first 2 seconds of motion the particle is moving in a positive direction with a positive velocity.
- When $t = 3$, $v = -2$. This negative velocity means that the particle is moving in a negative direction (i.e. 'backwards') with a speed of $2\,\text{m s}^{-1}$.
- When $t = 6$, $v = -8$. The particle is moving with a speed of $8\,\text{m s}^{-1}$ in the negative direction and is at the point C.

Over $0 \leq t \leq 6$, the particle moves from A to B to C. Its final displacement from O is -7 m. The total distance travelled is 20 m, which includes the distance from A to B (4 m) plus the distance from B to C (16 m).

After the particle was at rest (at $t = 2$ seconds), it reversed its direction and was then moving in the opposite direction.

Also, when $t = 3$, $v = -2$ and $x = 8$. The particle has moved from a displacement of 9 m to a displacement of 8 m. The displacement has decreased, so its rate of change (i.e. velocity) is negative.

Acceleration

Acceleration is defined as the rate of change of velocity with respect to time. Acceleration, like velocity, can be positive or negative. Positive acceleration indicates that the velocity is increasing, while negative acceleration indicates that the velocity is decreasing, which is often called deceleration or retardation.

(Note that 'increasing velocity' is not necessarily 'faster speed', it only means acceleration in the direction of positive displacement.)

If we denote the velocity by $v(t)$, then the average acceleration over the time interval from t to $(t + h)$ is $\dfrac{v(t+h) - v(t)}{h}$.

The instantaneous acceleration at time t is defined by $\lim\limits_{h \to 0} \dfrac{v(t+h) - v(t)}{h}$. It may be denoted by $v'(t)$, $a(t)$, $f''(t)$, $\dfrac{dv}{dt}$, $\dfrac{d^2x}{dt^2}$ or $\ddot{x}$:

$$a(t) = v'(t) = \frac{d^2x}{dt^2} = \ddot{x} = \lim_{h \to 0} \frac{v(t+h) - v(t)}{h}$$

Example 9

A particle is moving in a straight line such that its displacement x m from a fixed point O on the line at time t seconds (for $t \geq 0$) is given by $x = t^3 - 12t + 16$. Find:

(a) the particle's initial displacement, velocity and acceleration

(b) the time when its velocity is zero, and its displacement and acceleration at this time.

Solution

(a) $x = t^3 - 12t + 16$

$v = \dfrac{dx}{dt} = 3t^2 - 12$

$a = \dfrac{dv}{dt} = 6t$

When $t = 0$: $x = 16$, $v = -12$, $a = 0$

Initially the particle is 16 m from O, moving with a velocity of $-12\ \text{m s}^{-1}$ and with zero acceleration.

(b) $v = 0$: $3t^2 - 12 = 0$

Factorise: $3(t-2)(t+2) = 0$

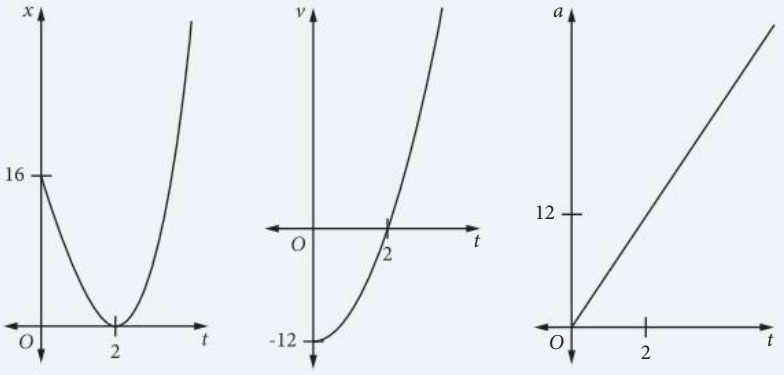
$\therefore\ t = 2, -2$

But $t \geq 0$, so the particle is at rest at 2 seconds.

When $t = 2$: $x = 8 - 24 + 16 = 0$, so the particle is at O.

$a = 12$, so acceleration is $12\ \text{ms}^{-2}$.

It is worth looking at the graphs of the functions for x, v and a:



From the graphs we can see that the acceleration, after initially being zero, is positive and increasing at a constant rate. As $a > 0$, the velocity is increasing at all times. During the first two seconds the particle moves from 16 metres on the positive side of O back to O, with an increasing negative velocity. After 2 seconds the particle is at rest at O ($v = 0$). It then moves back in a positive direction with an increasing velocity.

In Example 9 above, the displacement function was given and the velocity and acceleration functions were obtained by differentiation. If you are given the velocity function, you can obtain the acceleration function by differentiation and the displacement function by integration. If you are given the acceleration function, you can obtain the velocity and displacement functions by integration. When you find the indefinite integral, you will need to be given additional information about the problem so that the value of the arbitrary constant of integration, C, can be found.

Summary of important motion terms

'initially':	$t = 0$	'at the origin':	$x = 0$
'at rest':	$v = 0$	'velocity is constant':	$a = 0$

Example 10

A particle starts from rest 5 m from a fixed point O and moves in a straight line with an acceleration $a\ \text{m s}^{-1}$, where $a = 3t - 4$. Find the velocity and position of the particle at any time t.

Solution

When $t = 0, v = 0, x = 5$: $a = 3t - 4$

Integrate for v: $v = \int (3t - 4)\,dt$

$$v = \frac{3t^2}{2} - 4t + C_1$$

When $t = 0, v = 0$: $0 = 0 - 0 + C_1$

$C_1 = 0$

Hence: $v = \frac{3t^2}{2} - 4t$

Integrate for x: $x = \int \left(\frac{3t^2}{2} - 4t \right) dt$

$$x = \frac{t^3}{2} - 2t^2 + C_2$$

When $t = 0, x = 5$: $5 = 0 - 0 + C_2$

$C_2 = 5$

Hence $x = \frac{t^3}{2} - 2t^2 + 5$

Units and symbols

Physical quantity	Unit	Symbol
Time	s	t
Displacement	cm, m	x (or s in Physics)
Velocity	cm s^{-1}, m s^{-1}	v, $\frac{dx}{dt}$, $\dot{x}$
Acceleration	cm s^{-2}, m s^{-2}	a, $\frac{dv}{dt}$, $\frac{d^2x}{dt^2}$, $\ddot{x}$

Note that 's' is the abbreviation for second, 'cm' for centimetre and 'm' for metre.

Constant acceleration due to gravity = $9.8\,\text{m s}^{-2}$ ($\approx 10\,\text{m s}^{-2}$)

Example 11

A ball is projected vertically upwards, with a velocity of $25\,\text{m s}^{-1}$, from the top of a building 30 m high. If the acceleration due to gravity is taken to be $10\,\text{m s}^{-2}$, find:

(a) the time taken for the ball to reach its highest point, and the height of this highest point

(b) how much time the ball will take to reach the ground

(c) the speed with which the ball hits the ground

(d) when the ball is 60 m above the ground.

Solution

Take the upward direction as positive. Thus when $t = 0$, $a = -10$, $v = 25$, $x = 30$.

$$a = \ddot{x} = -10$$

Integrate for v: $v = \dot{x} = -\int 10\,dt$

$$v = \dot{x} = -10t + C_1$$

When $t = 0$, $v = 25$: $C_1 = 25$

$$\dot{x} = 25 - 10t$$

Integrate for x: $x = \int (25 - 10t)\,dt$

$$x = 25t - 5t^2 + C_2$$

When $t = 0$, $x = 30$: $C_2 = 30$

$$x = 30 + 25t - 5t^2$$

or $x = 5(6 + 5t - t^2)$

(a) At the highest points, $\dot{x} = 0$: $0 = 25 - 10t$

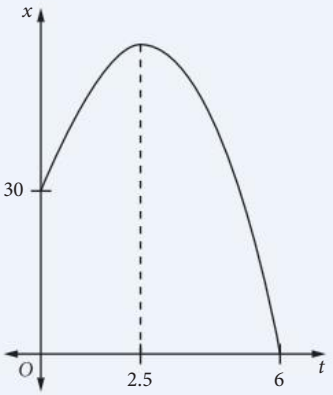
$t = 2.5$

It takes 2.5 seconds to reach the highest point.

When $t = 2.5$: $x = 5(6 + 12.5 - 6.25)$

$x = 61.25$

Highest point is 61.25 m above the ground.



(b) Reaches the ground when $x = 0$: $5(6 + 5t - t^2) = 0$

$t^2 - 5t - 6 = 0$

$(t - 6)(t + 1) = 0$

Because $t \geq 0$, the ball reaches the ground after 6 seconds.

(c) When $t = 6$, find $\dot{x}$: $\dot{x} = 25 - 60 = -35$

The ball is falling when it hits the ground, hence the negative velocity, and it strikes the ground with a speed of 35 m s^{-1}.

(d) When $x = 60$: $60 = 5(6 + 5t - t^2)$

$12 = 6 + 5t - t^2$

$t^2 - 5t + 6 = 0$

$(t - 2)(t + 3) = 0$

$t = 2$ or 3

The ball is 60 m above the ground at 2 seconds (on the way up) and again at 3 seconds (on the way down).

EXERCISE 17.3 MOTION OF A PARTICLE IN A STRAIGHT LINE

1 A particle moves in a straight line so that its displacement x m from a fixed point O on the line at any time t seconds ($t \geq 0$) is given by $x = t^2 - 5t + 6$. Find:

(a) its initial displacement **(b)** its initial velocity

(c) when it first passes through O and with what velocity

(d) when it passes through O the second time and with what velocity

(e) when and where its velocity is zero.

2 The displacement x m at time t seconds ($t \geq 0$) of a particle moving in a straight line is given by $x = 2t^3 - t^2 + 4t + 1$. Its acceleration is given by:

A $a = 2t^3 - t^2 + 4t + 1$ **B** $a = 6t^2 - 2t + 4$ **C** $a = 12t - 2$ **D** $a = 12$

3 The displacement x m at time t seconds ($t \geq 0$) of a particle moving in a straight line is given by $x = t^2 - 5t + 4$.

(a) At what time is its velocity zero? **(b)** What is the acceleration at this time?
(c) What is the distance travelled in the first 4 seconds? **(d)** At what time is the velocity 8 m s^{-1}?

4 A point moving in a straight line is distant x m from the origin O at time t, where $x = 2t^3 - 15t^2 + 36t$.

(a) Find the velocity and acceleration at any time t.
(b) Find the initial velocity and acceleration. **(c)** At what times is the velocity zero?
(d) At what time is the acceleration zero? Find the velocity and position at this time.
(e) During what interval of time is the velocity negative?

5 The displacement x m at time t seconds of a particle moving in a straight line is given by $x = 2t^3 - 9t^2 + 12t + 6$. Find:

(a) when its acceleration is zero, and its velocity at this time
(b) when its velocity is zero, and its acceleration at this time.

6 Two bodies move along a straight path, starting at the same time, so that their displacement x m from a fixed point O at any time t is given by $x = t + 6$ and $x = t^2 + 4$ respectively. At what times are they:

(a) together **(b)** travelling with the same velocity?

7 A body starts from O and moves in a straight line. At any time t its velocity is given by $\dot{x} = 6t - 4$. Indicate whether each statement below is correct or incorrect.

(a) $x = 3t^2 - 4t + C$ **(b)** $x = 3t^2 - 4t$ **(c)** $\ddot{x} = 3t^2 - 4t$ **(d)** $\ddot{x} = 6$

8 A body starts from O and moves in a straight line. At any time t, its velocity is $t^2 - 4t^3$. Find, in terms of t:

(a) the displacement x **(b)** the acceleration.

9 The velocity v m s^{-1} at time t seconds ($t \geq 0$) of a body moving in a straight line is given by $v = 6t^2 + 6t - 12$. Its initial displacement is 7 m from O. Find:

(a) the displacement and acceleration at any time t
(b) the acceleration when the velocity is zero **(c)** the initial velocity and acceleration.

10 The acceleration of a body moving in a straight line is $10 - 2t$ m s^{-2} at any time t seconds. The body is initially at zero displacement with a velocity of 11 m s^{-1}. Find:

(a) the velocity and displacement at any time t
(b) when the body has zero velocity, and its displacement at this time.

11 A body is projected vertically upwards with an initial velocity of 30 m s^{-1}. It rises with a deceleration of 10 m s^{-2}. Find:

(a) its velocity at any time t **(b)** its height h m above the point of projection at any time t
(c) the greatest height reached **(d)** the time taken to return to the point of projection.

12 A particle is projected vertically upwards from a point O with an velocity of 25 m s^{-1} and a downward acceleration of 10 m s^{-2}.

(a) Find its velocity and height above O at any time t.
(b) What maximum height does the particle reach?
(c) At what time has its velocity been reduced to half the velocity of projection?

13 The velocity v m s^{-1} of a body moving in a straight line is given by $v = 3t^2 - 2t - 1$. The body initially has a displacement 1 m from O. Find:

(a) the displacement and acceleration at any time t
(b) when the body has zero displacement, and its velocity and acceleration at this time
(c) the distance travelled in the first two seconds.

14 The velocity $v(t)$ of a particle moving in a straight line at any time $t \geq 0$ is $v(t) = 12t^2 - 6t + 1$. Find its position $s(t)$ given that $s(1) = 4$.

15 The acceleration of a particle moving in a straight line is $10 - 2t$ ms^{-2} at any time $t \geq 0$. The particle starts from O with a velocity of 24 m s^{-1}. At what time is its velocity zero, and what is its displacement at this time?

16 Two cars A and B travel along a straight road in the same direction. Their respective distances x km from a fixed point O at any time t hours are given by the following rules:

A: $x = 50t - 20t^2$ B: $x = 80t^2 + 20t$

(a) Calculate each car's speed at the point O.

(b) At what time are the cars travelling at the same speed?

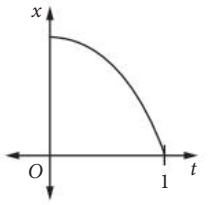
(c) Both cars reach a point Q at the same time. Calculate the distance from O to Q.

(d) A third car, travelling at uniform speed, is 2 km ahead of A and B when they pass the point O. If this car arrives at Q at the same time as A and B, find a rule connecting x and t for it.

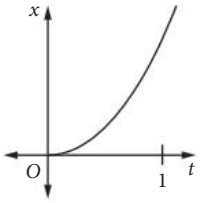
17 A particle moves in a straight line so that its displacement $x(t)$ from a fixed point in the line at time $t \geq 0$ is given by $x(t) = 3 + 4t - 5\sqrt{t^2 + 4}$. Find the particle's displacement when it comes to rest.

18 A particle is moving so that, for $0 < t < 1$, its velocity is positive and its acceleration is negative. Which graph could represent the displacement function of this particle?

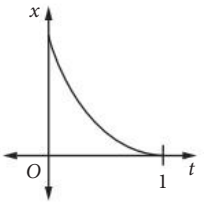
A



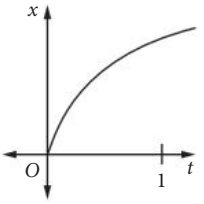
B



C



D



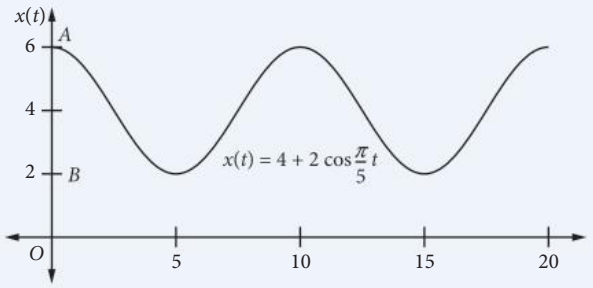
17.4 OTHER EXAMPLES OF MOTION

Example 12

A particle moves in a straight line so that at any time t seconds ($t \geq 0$), its displacement x metres from a fixed point O in the line is given by $x(t) = 4 + 2\cos\frac{\pi t}{5}$, $0 \leq t \leq 20$. Sketch the graph of the displacement and find when the velocity is zero. Discuss the motion.

Solution

$n = \frac{\pi}{5}$; Period $= \frac{2\pi}{n} = 2\pi \div \left(\frac{\pi}{5}\right) = 10$



The graph is a cosine curve with amplitude 2 and period 10 that has been raised 4 units.

Velocity: $v = \dot{x}(t) = -\frac{2\pi}{5}\sin\frac{\pi t}{5}$

When $v = 0$: $\sin\frac{\pi t}{5} = 0$

$\frac{\pi t}{5} = 0, \pi, 2\pi, 3\pi, 4\pi, \ldots$

$t = 0, 5, 10, 15, 20$

The particle is initially at rest at A, 6 metres from O. It then moves from A to B in 5 seconds and stops at B. It then returns to A in the next 5 seconds. It repeats this motion over the next 10 seconds. In general, the particle oscillates repeatedly between $x = 2$ and $x = 6$ every 10 seconds.

The period of the function tells you how long it takes for the motion to repeat itself. Repetitive back-and-forth motion of this type is called simple harmonic motion.

Example 13

A particle moves in a straight line so that at any time t seconds, its velocity v m s^{-1} is given by $v = e^{-t}$. If initially the particle is 2 metres from a fixed point O in the line, find its position x at any time t. Sketch the graph of x as a function of t.

Solution

$$v = \frac{dx}{dt} = e^{-t}$$

$$\therefore\ x = \int e^{-t}\, dt$$

$$x = -e^{-t} + C$$

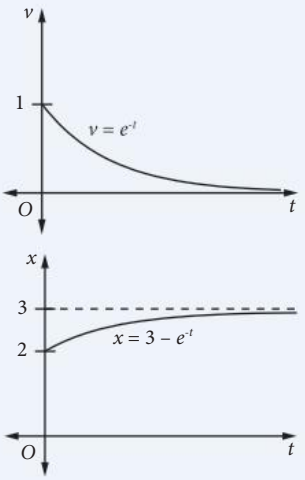
When $t = 0$, $x = 2$: $\quad 2 = -1 + C$

$$C = 3$$

$$x = 3 - e^{-t}$$

The velocity–time graph for $v = e^{-t}$ shows that initially the particle moves with a velocity of 1 m s^{-1}, but as time goes by this velocity gets smaller and smaller so that as $t \to \infty$, $v \to 0$ from above.

The displacement–time graph shows that the particle starts at $x = 2$ and as time goes by it moves closer and closer to $x = 3$, but it never reaches it. As $t \to \infty$, $x \to 3$ from below.



Displacement, velocity, acceleration—important links

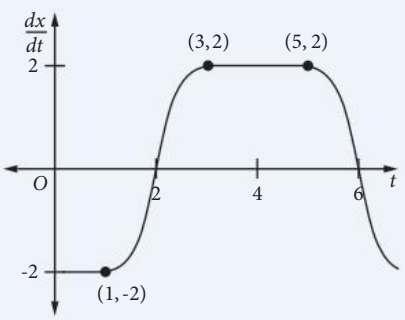
Given the displacement function, you can find the velocity and acceleration functions by differentiating with respect to time.

Given the velocity function, you can find the displacement function by integrating with respect to time, and you can find the acceleration function by differentiating with respect to time.

If the information is given as a graph, then you must remember that the definite integral is linked to the area under the graph, while the derivative is linked to the gradient of the curve. If you are given the graph of the velocity $\frac{dx}{dt}$ against time, then the value of the definite integral gives the displacement and the slope of the curve gives the acceleration.

Example 14

The graph shows the velocity $\frac{dx}{dt}$ of a particle as a function of time. Initially the particle is at the origin.



(a) At what times is the velocity zero?

(b) At what time is the displacement x from the origin a maximum?

(c) What is the displacement when $t = 4$? What does this tell you?

(d) When does the particle have zero acceleration?

(e) At what time is the acceleration the greatest?

(f) Use the trapezoidal rule to estimate the displacement when $t = 6$.

Solution

(a) $\frac{dx}{dt} = 0$: graph shows this at $t = 2, 6$

Hence the velocity is zero (particle at rest) at 2 seconds and again at 6 seconds.

(b) Maximum and minimum displacement occur when $\frac{dx}{dt} = 0$.

From the graph, the value of $x = \int_0^2 \frac{dx}{dt}\,dt < 0$ so the displacement at $t = 2$ is a minimum.

The value of $x = \int_0^6 \frac{dx}{dt}\,dt > 0$ so the displacement at $t = 6$ is a maximum.

(c) From the symmetry of the graph, $x = \int_0^4 \frac{dx}{dt}\,dt = 0$, so the displacement is zero and the particle is again at the origin.

(d) The particle has zero acceleration when the velocity is constant, i.e. when the graph is horizontal. Acceleration is zero for $0 \leq t \leq 1$, $3 \leq t \leq 5$.

(e) The velocity graph is steepest at $t = 2$ and $t = 6$. At $t = 2$ the slope > 0; at $t = 6$ the slope < 0. Greatest acceleration is at $t = 2$.

(f) As the particle is back at the origin when $t = 4$, the displacement when $t = 6$ is given by $x = \int_4^6 \frac{dx}{dt}\,dt$.

From the graph we have the points $(4, 2)$, $(5, 2)$, $(6, 0)$: $x = \int_4^6 \frac{dx}{dt}\,dt \approx \frac{1}{2}(2 + 2 \times 2 + 0) = 3$

The particle is about 3 units on the positive side of the origin.

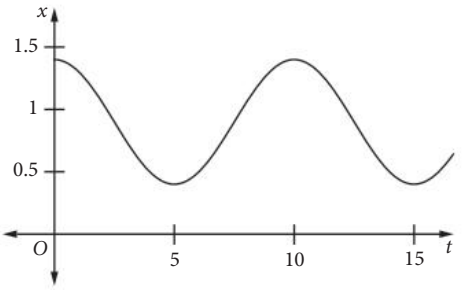
EXERCISE 17.4 OTHER EXAMPLES OF MOTION

1 The displacement x of a particle at time t is given by $x = 6\cos 4t + 3$. Find:

(a) the velocity and acceleration at any time t **(b)** the position of the particle when $t = 0$

(c) the values that x can take **(d)** the time when the particle first reaches the position $x = 0$.

2 Consider the graph.



Which of the following functions does this graph represent?

A $x = 0.9 + \cos\frac{\pi t}{10}$ **B** $x = 0.9 + \cos\frac{\pi t}{5}$

C $x = 0.9 + 0.5\cos\frac{\pi t}{10}$ **D** $x = 0.9 + 0.5\cos\frac{\pi t}{5}$

3 The position x metres of a particle relative to a fixed point O at any time t is $x = 4 - \cos 2t$.

(a) Sketch the graph of x as a function of t in the domain $0 \leq t \leq 2\pi$.

(b) Find the times when the particle is at rest.

(c) Express the acceleration in terms of: **(i)** t **(ii)** x

4 A driver takes 3 hours to travel the distance between two points A and B on a country road. At time t hours after passing A, her speed v km h^{-1} is given by $v = 60 + 40e^{-t}$.

(a) Calculate the speeds when she passes points A and B.

(b) Write the acceleration in terms of: **(i)** t **(ii)** v

(c) Sketch the velocity–time curve and comment on the motion for large t.

(d) Calculate the distance from A to B.

5 A particle moves in a straight line. At time t seconds, its displacement x cm from a fixed point O in the line is given by $x = 5\sin\frac{\pi t}{2}$, $0 \le t \le 4$.

(a) Sketch the graph of its displacement at any time t. **(b)** At what times is the particle at rest?
(c) Find the speed when $t = 2.5$. **(d)** Express the acceleration in terms of: **(i)** t **(ii)** x

6 A particle moves in a straight line so that at time t its displacement from a fixed origin is x and its velocity is v. If its acceleration is $2\sin t$, and $v = 1$ and $x = 1$ when $t = 0$, find x as a function of t.

7 A particle moves in a straight line so that at time $t \ge 0$ its velocity v is given by $v = \frac{1}{1+t}$.

(a) If its distance from a fixed point O in the line is x, show that $x = \log_e(1 + t)$, given that $x = 0$ when $t = 0$.
(b) Show that as t increases, the particle is moving away from O.
(c) Find the acceleration at $t = 0$.

8 A particle moves in a straight line so that at time t its displacement from a fixed origin O is x, where $x = 2 + t - 2\cos t$.

(a) Write the velocity and acceleration at any time t.
(b) Find its initial displacement, velocity and acceleration.

9 A particle moves in a straight line so that its displacement x from a fixed origin at any time t is given by $x(t) = 2(1 - e^{-t})$.

(a) Find $x(0)$, $\dot{x}(0)$ and $\ddot{x}(0)$. **(b)** Sketch the graph of $x(t)$. **(c)** Find t when $x(t) = 1$.

10 A particle moves in a straight line. At time t its displacement from a fixed origin on the line is x, where $x = 2 - 2\sin 2t$, $0 \le t \le 2\pi$.

(a) Draw the graph of x as a function of t.
(b) Show that the particle oscillates between $x = 0$ and $x = 4$.
(c) For what values of t is the velocity zero?
(d) Express the acceleration in terms of: **(i)** t **(ii)** x

11 A particle moves in a straight line. At time t its displacement from a fixed origin in the line is x and its velocity is v, where $v = e^{-t}$.

(a) Express x in terms of t, given that $x = 1$ when $t = 0$. Draw the graph of the function.
(b) Show that $\frac{d^2x}{dt^2} = x - 2$.

12 A particle moves in a straight line. At time t its distance x from a fixed point O in the line is given by $x = 4\log_e(1 + t) - 2t$, $t \ge 0$.

(a) Find x when $t = 0$. **(b)** Find the initial velocity, i.e. for $t = 0$.
(c) Find the acceleration at any time t and show that it is always negative.
(d) At what time is the velocity zero?

13 A particle moves in a straight line so that at time t its displacement from a fixed origin is x and its velocity is v.

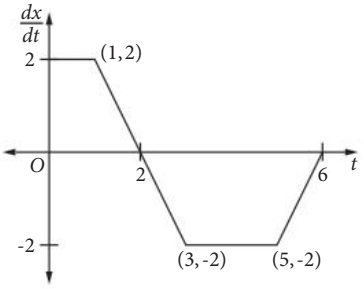
(a) If its acceleration is $2\cos t$, and $v = 1$ and $x = 0$ when $t = 0$, find x in terms of t.
(b) If its acceleration is $-3e^{-t}$ and $v = 0$ when $t = 0$, find the time at which $v = -2$.

14 A body falls from rest. Its velocity v at any time t is given by $v = 49(1 - e^{-0.5t})$. Find:

(a) the acceleration at any time t **(b)** the initial velocity
(c) the terminal velocity, i.e. its velocity as $t \to \infty$ **(d)** the distance fallen in the first 5 seconds.

15 The acceleration of a particle moving in a straight line is given by $\frac{d^2x}{dt^2} = 12\cos 2t$. Initially $v = 0$ and $x = 6$. Find its velocity v and displacement x at any time t seconds and sketch the graph of the displacement. How many times does the particle change direction in the first 10 seconds?

16 The graph shows the velocity $\frac{dx}{dt}$ of a particle as a function of time. Initially the particle is at the origin.

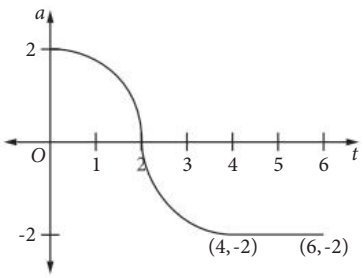


(a) At what time is the displacement x from the origin a maximum?

(b) At what time does the particle return to the origin? Justify your answer.

(c) Draw a sketch of the acceleration $\frac{d^2x}{dt^2}$ as a function of time for $0 \leq t \leq 6$.

17 A particle moves along the x-axis. Initially it is at rest at the origin. The graph shows the acceleration a of the particle as a function of time t for $0 \leq t \leq 6$.

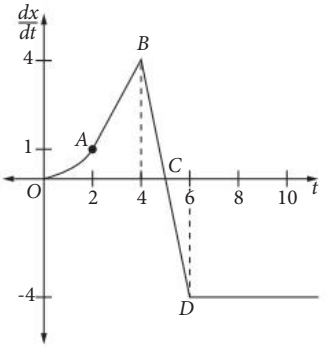


(a) Write the time at which the velocity of the particle is a maximum.

(b) At what time during the interval $0 < t \leq 6$ is the particle at rest?

(c) At what time during the interval $0 \leq t \leq 6$ is the particle farthest from the origin? Give brief reasons for your answer.

18 An object is moving on the x-axis. The graph shows the velocity $\frac{dx}{dt}$ of the object as a function of time t. The coordinates of the points shown on the graph are $A(2,1)$, $B(4,4)$, $C(5,0)$ and $D(6,-4)$. The velocity is constant for $t \geq 6$.



(a) Using the trapezoidal rule, estimate the distance travelled between $t = 0$ and $t = 4$.

(b) How far is it from B to C?

(c) The object is initially at the origin. During which time(s) is the displacement of the object decreasing?

(d) Estimate the time at which the object returns to the origin. Justify your answer.

(e) Sketch the displacement x as a function of time.

19 The table shows the velocity (in metres per second) of a moving object, evaluated at one-second intervals.

t	0	1	2	3	4	5	6
v	0	4.6	5.7	8.0	9.9	12.7	18.2

(a) Use the trapezoidal rule to estimate the distance travelled over the time interval $0 \leq t \leq 6$.

(b) Use Simpson's rule to estimate the distance travelled over the time interval $0 \leq t \leq 6$.

CHAPTER REVIEW 17

1 A body starts from rest and moves in a straight line so that its velocity v m s^{-1} after t seconds is given by $v = 2t + 6t^2$. Calculate:

(a) its acceleration at the end of the first second

(b) its displacement after 5 seconds, given that the body is initially at zero displacement.

2 The acceleration of a particle moving in a straight line is $6 - 8t$ m s^{-2} at any time $t \geq 0$. The particle starts from O with a velocity of 10 m s^{-1}. When is its velocity zero, and what is its displacement at that time?

3 The velocity v m s^{-1} of a particle moving in a straight line is given by $v = 6t^2 - 4t + 1$ $(t \geq 0)$. The particle initially has a displacement -10 m from O. Find:

(a) the displacement and acceleration at any time t
(b) the acceleration when the velocity is 3 m s^{-1}
(c) the velocity when the acceleration is 20 m s^{-1}.

4 The position x of a particle moving along a straight line at any time t is given by $x = 3 + 6\cos\frac{\pi t}{6}$.

(a) Find the position of the particle for values of $t = 0, 2, 4, 6, 8, 10, 12$.
(b) Find the velocity and acceleration of the particle when it first reaches the position $x = 0$.

5 A particle is moving along the x-axis and is initially at the origin. Its velocity v metres per second at time t seconds is given by $v = \frac{2t}{9 + t^2}$.

(a) What is the initial velocity of the particle?
(b) Find an expression for the acceleration of the particle. **(c)** When is the acceleration zero?
(d) What is the maximum velocity attained by the particle and when does it occur?
(e) Find the position of the particle when $t = 3$.

6 The growth of the number of internet users in the USA was modelled as an exponential function $N = Ae^{kt}$, where N is the estimate for the number of internet users (in millions) and t is the time in years after 1 January 2001.

(a) At the start of 2001 $(t = 0)$ there were 124 million internet users in the USA. At the start of 2009 there were 220 million. Find A and k.
(b) How many internet users would you expect there to be at the start of 2012?
(c) In what year would you expect the number of internet users in the USA to first exceed 300 million?

7 A full water tank holds 4000 litres. When the tap is turned on, water flows out from the tank at a rate of $\frac{dV}{dt} = 110 + 17t - t^2$ litres per minute, where t is the time in minutes since the tap was turned on.

(a) At what time is the tank emptying at a rate of 50 litres per minute?
(b) Find the volume of water that has flowed out of the since the tap was turned on as a function of t.
(c) How much water has flowed out of the tank 12 minutes after the tap was turned on?
(d) When does the water stop flowing out of the tank?
(e) How much water is left in the tank (to the nearest litre) when the water stops flowing out of the tank?

8 A particle is initially at rest at the origin. Its acceleration as a function of time t is given by $\ddot{x} = 9\sin 3t$.

(a) Show that the velocity of the particle is given by $\dot{x} = 3 - 3\cos 3t$.
(b) Sketch the graph of the velocity for $0 \leq t \leq \frac{4\pi}{3}$ and determine the time at which the particle first comes to rest after $t = 0$.
(c) Find the distance travelled by the particle between $t = 0$ and the time found in **(b)**.
(d) Find an expression for x as a function of t.

9 The number N of bacteria in a colony grows according to the rule $\frac{dN}{dt} = kN$. If the original number increases from 5000 to 10 000 in 6 days, find the number after another 6 days.

10 A population of size N is decreasing according to the rule $\frac{dN}{dt} = -\frac{N}{40}$, where t is the time in days. If the population is initially of size N_0, find how long it takes for the size to be halved (rounded to the next day).

11 A radioactive substance decays at a rate that is proportional to the mass of radioactive substance present at any time. If 10% decays in 400 years, what percentage of the original radioactive mass will remain after 1000 years?

CHAPTER 18
PROBABILITY

Outcomes covered:

H1 seeks to apply mathematical techniques to problems in a wide range of practical contexts

H4 expresses practical problems in mathematical terms based on simple given models

H5 applies appropriate techniques from the study of calculus, geometry, probability, trigonometry and series to solve problems

H9 communicates using mathematical language, notation, diagrams and graphs

18.1 INTRODUCTION TO PROBABILITY

We often make statements like the following:

- It will **probably** rain today.
- I have a **good chance** of passing my exams.
- There is a **50-50 chance** that heads will come up when I toss a coin.
- I have a **slight chance** of being dux of the class.

In each case there is some doubt about the outcome, but the degree of doubt is different. **Probability** is about this doubt: it is the study of events that may or may not happen, rather than of events that will happen or that have already happened.

If we toss a coin, the result must be heads (H) or tails (T). There are only two possible outcomes, H or T, and each outcome is **equally likely**. We say that the probability of heads is $\frac{1}{2}$, and we can write this as:

$$P(\text{H}) = \frac{1}{2}$$

This probability be expressed in words in a variety of ways:

- There is a 50-50 chance that heads will turn up.
- There is a 1-in-2 chance that heads will turn up.
- The odds that heads will turn up are 1 to 1.
- It is an even money bet that heads will turn up.

Before beginning a game of football, cricket, basketball etc., the team captains usually toss a coin. Why?

Equally likely events are events where each outcome has the same chance of occurring.

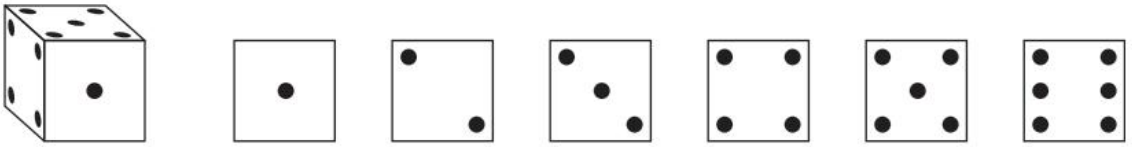
Coin experiment

Toss a coin 10 times, 50 times, 100 times, 200 times and 500 times, counting the number of heads tossed each time. (This can be simulated using a calculator or spreadsheet.) Complete the following table:

Number of coin tosses	10	50	100	200	500
Number of heads					

If you toss the coin 10 times, how many heads would you expect? If you toss the coin 50 times, how many heads would you expect? Would you be surprised if you didn't get exactly the number that you expected?

An American soldier, while a prisoner of war in World War II, passed the time by tossing a coin 1000 times. He performed this experiment ten times and obtained the following numbers of heads: 502, 511, 497, 529, 504, 476, 507, 528, 504, 529. Are these numbers surprising?



Die Faces

When we say 'a die' (plural: dice), we mean a cube with six faces numbered 1 to 6, as shown above. If a die is rolled on a flat surface, one of the six numbers must appear on the top. (For example, the number 5 appears on top of the die in the picture above.)

- Each number from 1 to 6 is **equally likely** to turn up. Why?
- What is the probability that a 5 will appear on top?

There are six possible outcomes. Of these six outcomes, the outcome that we are interested in is a 5. We call this the **favourable** outcome. The probability of a 5 appearing is $\frac{1}{6}$, so we write that the probability of a favourable outcome is $P(5) = \frac{1}{6}$.

This can be expressed in a variety of ways:

- There is a 1-in-6 chance that a 5 will turn up.
- The odds are 5 to 1 against a 5 turning up. This means that for every 1 favourable outcome there are 5 unfavourable outcomes.

We can define an outcome's probability as being the ratio of the number of favourable outcomes to the number of possible outcomes, assuming that the outcomes are equally likely. Thus, the probability $P(A)$ of a particular result A is:

$$P(A) = \frac{\text{number of favourable outcomes}}{\text{number of possible outcomes}} \quad \textbf{OR} \quad P(\text{Event}) = \frac{\text{number of ways Event can occur}}{\text{number of possible outcomes}}$$

Dice experiment

Roll a die 6 times, 60 times, 120 times, 300 times and 600 times, counting the number of times a 5 (or another particular number) is rolled. (This can be simulated using a calculator or spreadsheet.) Complete the following table:

Number of rolls of the die	6	60	120	240	300	600
Number of times 5 appears						

If you roll a die 6 times, how many 5s would you expect? If you roll a die 60 times, how many 5s would you expect? Would you be surprised if you didn't get exactly the number you expected?

From the definition of probability as the number of favourable outcomes divided by the number of possible outcomes, we can see that if the number of favourable outcomes is equal to the number of possible outcomes then the probability is 1. If the probability is 1, then the event is **certain** to happen. For example:

- It is certain that the Sun will rise in the east tomorrow.
- The probability of obtaining a number less than 7 when a die is rolled $= \frac{6}{6} = 1$ (a certainty).

List some other situations where the probability is 1.

If there are no favourable outcomes, then the probability is 0. For example, many athletes can run very fast, but no athlete can run at the speed of light. The probability that an athlete will run faster than light is 0. Similarly, the probability that a 7 will appear when a normal die is rolled $= \frac{0}{6} = 0$.

Probabilities must always be from 0 to 1 inclusive:

- $0 \leq P(\text{Event}) \leq 1$
- If $P(\text{Event}) = 0$, the event is impossible.
- If $P(\text{Event}) = 1$, the event is certain.

To win Lotto®, you must pick six winning numbers out of the numbers 1 to 45. The number of different ways that you can choose 6 numbers from 45 numbers is 8 145 060, but only one of these ways is favourable (i.e. will win):

$P(\text{6 winning numbers}) = \frac{1}{8145060}$, $\approx 0.000\,000\,1$, which is rather small.

The following table shows the number of live births and the number of males and females born in Australia over the three-year period 2000 to 2002.

Year	Live births	Males	Females	Proportion of males
2000	249 636	128 190	121 446	0.514
2001	246 394	126 298	120 096	0.513
2002	250 988	128 623	122 365	0.512

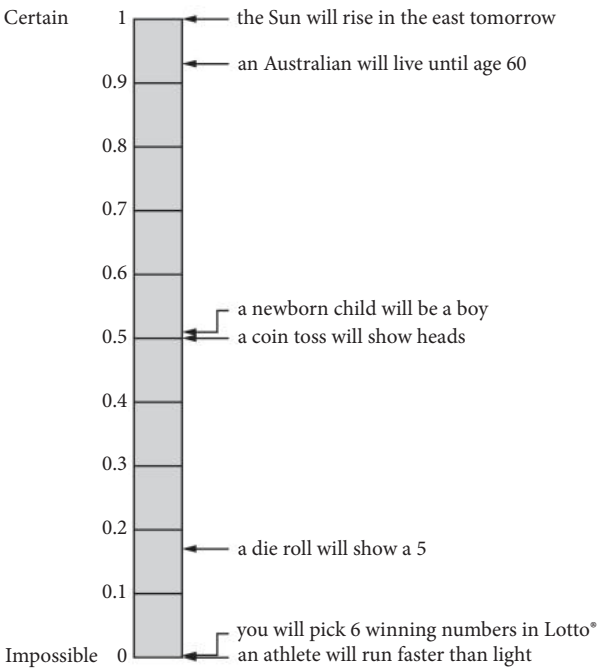
With rare exceptions, statistics over the years indicate a slight excess of male over female births.

$P(\text{newborn is male}) = \frac{\text{number of males born}}{\text{total number of births}} \approx 0.51$

Insurance companies prepare 'life tables', which show the number of Australian people per 100 000 who are statistically expected to be alive at any given age. For example, a table like this could show that 92 515 people out of every 100 000 people born are expected to be still alive after 60 years:

$P(\text{an Australian will live until age 60}) = \frac{92515}{100000} = 0.93$

The following scale shows the probabilities of the events discussed.



Example 1

A die is rolled on a flat surface. What is the probability that the number appearing is:

(a) odd **(b)** divisible by 3

(c) odd or divisible by 3 **(d)** odd and divisible by 3?

Solution

The six possible outcomes from rolling a die are equally likely.

(a) There are three odd numbers (1, 3, 5), so there are 3 favourable outcomes.

$P(\text{odd}) = \frac{3}{6} = \frac{1}{2}$

(b) There are two numbers divisible by 3 (3, 6), so there are 2 favourable outcomes.

$P(\text{divisible by 3}) = \frac{2}{6} = \frac{1}{3}$

(c) 1, 3, 5, 6 are favourable, so there are 4 favourable outcomes.

$P(\text{odd or divisible by 3}) = \frac{4}{6} = \frac{2}{3}$

(d) 3 is the only number that is odd and also divisible by 3, so there is only 1 favourable outcome.

$P(\text{odd and divisible by 3}) = \frac{1}{6}$

Complementary events

The **complement of event** A is the event 'A **does not occur**', and is denoted by $\bar{A}$.

If an experiment has n possible outcomes, m of which are associated with event A, then $(n - m)$ outcomes are associated with event $\bar{A}$, and so:

$$P(A) = \frac{m}{n} \qquad P(\bar{A}) = \frac{n-m}{n} = \frac{n}{n} - \frac{m}{n} = 1 - \frac{m}{n} = 1 - P(A)$$

Thus:

$$P(A) + P(\bar{A}) = 1$$

In general, $P(\text{Event does not occur}) = 1 - P(\text{Event occurs})$, or:

$$P(\bar{A}) = 1 - P(A)$$

Example 2

A die is rolled on a flat surface. What is the probability that the number rolled is **not** 4?

Solution

$P(4) = \frac{1}{6}$ $\qquad$ $P(A) = \frac{1}{6}$

$\therefore P(\text{not } 4) = 1 - \frac{1}{6} = \frac{5}{6}$ $\qquad$ $P(\bar{A}) = 1 - \frac{1}{6} = \frac{5}{6}$

Example 3

From a standard pack of 52 playing cards, one card is drawn at random. What is the probability that it is:

(a) a diamond **(b)** not a spade **(c)** an ace

(d) a diamond **or** an ace **(e)** a diamond **and** an ace **(f)** a heart **and** the ace of clubs?

Solution

There are 52 possible outcomes.

(a) There are 13 diamonds: $P(\text{diamond}) = \frac{13}{52} = \frac{1}{4}$

(b) There are 13 spades, so there are 39 cards that are not spades: $P(\text{not spade}) = \frac{39}{52} = \frac{3}{4}$
Or, using complementary events: $P(\text{not spade}) = 1 - P(\text{spade}) = 1 - \frac{1}{4} = \frac{3}{4}$

(c) There are 4 aces: $P(\text{ace}) = \frac{4}{52} = \frac{1}{13}$

(d) There are 13 diamonds and 4 aces, but one ace is also a diamond, so there are $13 + 3 = 16$ favourable outcomes: $P(\text{diamond } \textbf{or} \text{ ace}) = \frac{16}{52} = \frac{4}{13}$

(e) The ace of diamonds is the only card that is a diamond and also an ace: $P(\text{diamond } \textbf{and} \text{ ace}) = \frac{1}{52}$

(f) There is no card that is a heart and also the ace of clubs: $P(\text{heart } \textbf{and} \text{ ace of clubs}) = 0$

EXERCISE 18.1 INTRODUCTION TO PROBABILITY

1 Select 20 lines of text from a page in a book. Count the number of times each letter of the alphabet appears. Would you expect each of the 26 letters of the alphabet to appear approximately the same number of times? If a letter in your 20 lines is selected at random, would you say that the probability that the letter is 'e' is the same as the probability that it is 'z'?

2 Select a page in a telephone directory (or search at random on a telephone directory website) and count the number of times that each digit 0, 1, 2, 3, … 9 appears as the last digit of a phone number. Prepare a table showing your results.

(a) Did you expect results like these?
(b) Would similar results have occurred if the first digit of each number was recorded? Why?
(c) What would have happened if the first letter of each person's name was recorded?

3 What is wrong with the following arguments?

(a) The probability that a child born in Australia is born in New South Wales is $\frac{1}{7}$.

(b) Because there are fewer road accidents in Hobart than in Sydney, it is safer to drive in Hobart.

4 Which statement has a probability closest to 1?

A The cost of buying a house will increase this year.
B You will win the lottery.
C A person will be born in Sydney in the next hour.
D It will rain in Sydney tomorrow.

5 Perform the following experiment:
From a bag containing 7 black marbles and 3 white marbles, withdraw a marble, note its colour and then replace it in the bag. Shake the marbles and then withdraw a marble again. Repeat this 10, 50, 100, 150 and 200 times to complete the following table.

Number of withdrawals	10	50	100	150	200
Number of black marbles withdrawn					

How many black marbles do you expect to get in each case?

6 Perform the following experiment:
From a standard pack of 52 playing cards, withdraw a card and note whether it is a diamond, heart, spade or club. Replace the card in the pack, shuffle the cards and then withdraw another card. Repeat this 20, 40, 60, 80 and 100 times to complete the following table.

Number of withdrawals	20	40	60	80	100
Number of hearts					

How many hearts do you expect to get in each case?

7 A box contains 50 matches, of which 40 are 'live' (ready to be struck) and the remainder 'dead' (used). A match is selected at random. Indicate whether each statement below is correct or incorrect.

(a) $P(\text{live}) = \frac{4}{5}$ **(b)** $P(\text{dead}) = \frac{4}{5}$ **(c)** $P(\text{live}) = \frac{1}{5}$ **(d)** $P(\text{dead}) = \frac{1}{5}$

8 A carton contains a dozen eggs, of which three are brown and the rest are white. An egg is chosen at random from the carton. What is the probability that it is:

(a) brown **(b)** white?

9 A bag contains 2 white marbles, 2 black marbles and 1 red marble. A marble is selected from the bag. What is the probability that it is:

(a) red **(b)** black **(c)** not white
(d) red or black **(e)** not black, nor white **(f)** red or white?

10 A set of 20 cards is numbered 1, 2, 3, … 20. A card is drawn at random from the set. What is the probability that the number on the card is divisible by:

(a) 3 **(b)** 5 **(c)** 3 or 5 or both **(d)** 3 and 5?

11 A normal die is rolled on a flat surface. What is the probability that the number appearing is:

(a) greater than 2 **(b)** greater than 3 but less than 5
(c) odd **(d)** odd and divisible by 3?

12 A set of 10 cards is numbered 3, 4, 5, … 12. A card is drawn at random. What is the probability that the number on the card is:

(a) even **(b)** odd **(c)** odd or even
(d) greater than 7 **(e)** divisible by 3 **(f)** even and divisible by 3
(g) divisible by 5 **(h)** odd or divisible by 5, but not both?

13 A letter is chosen at random from the letters of the word SUNDAY. What is the probability that it is:

(a) a vowel **(b)** a consonant **(c)** D or A?

14 A card is drawn at random from a standard pack of 52 playing cards. What is the probability that it is:

(a) a heart **(b)** a king **(c)** a heart or a king
(d) a heart **and** a king **(e)** the queen of diamonds?

15 A number is chosen at random from the numbers 7, 8, 9, … 15. What is the probability that the number is:

(a) odd **(b)** odd and divisible by 5 **(c)** odd but not divisible by 5
(d) a multiple of 2 **(e)** a multiple of 3 **(f)** a multiple of both 2 and 3
(g) a multiple of 2 or 3 **(h)** a multiple of 2 or 3, but not both?

16 A bag contains 5 blue marbles, 3 red marbles and 2 green marbles. One marble is drawn at random from the bag. What is the probability that it is:

(a) blue **(b)** red **(c)** not green **(d)** blue or green **(e)** not red, nor blue?

17 A box contains marbles of the same size but different colours: red, white and blue. If a marble is drawn at random, the probability that it is red is the same as the probability that it is white and twice the probability that it is blue.

(a) What is the smallest number of marbles that the box could have?
(b) If a marble is chosen at random, what is the probability that it is:
(i) red **(ii)** white **(iii)** red or white **(iv)** not blue?

18 A bag contains 10 balls, of which 6 are red, 3 are yellow and 1 is white. A ball is drawn at random. What is the probability that it is:

(a) red **(b)** not yellow **(c)** white or yellow
(d) neither red nor white **(e)** not red?

19 A bag contains 12 discs, of which 7 are black, 3 are white and 2 are red. A black disc is withdrawn. What is the probability that the next disc withdrawn is:

(a) black **(b)** not white **(c)** neither black nor white?

20 A survey of a certain district showed that 6% of families there have 1 child, 38% have 2 children, 42% have 3 children, and 10% have more than 3 children. A family from the district is selected at random. What is the probability that it will have:

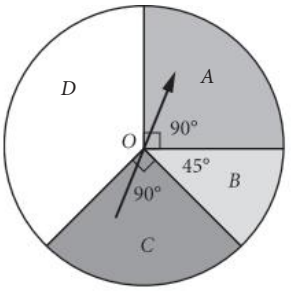
(a) some children **(b)** no children

(c) at least 2 children **(d)** not more than 2 children?

21 The diagram shows a circular disc divided into four sections A, B, C and D. A pointer is free to spin, pivoted at the centre O. If the pointer is spun, find the probability that the end of the pointer stops inside:

(a) A **(b)** B **(c)** A or B

(d) not C **(e)** B, C or D

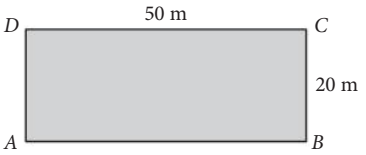


22 A horse is grazing inside an enclosed rectangular paddock 50 m by 20 m and is free to move anywhere inside the paddock. Assuming that the horse's position is random, what is the probability that at any given time the horse is:

(a) more than 5 m from the fence

(b) less than 5 m from the fence

(c) not more than 5 m from a corner?

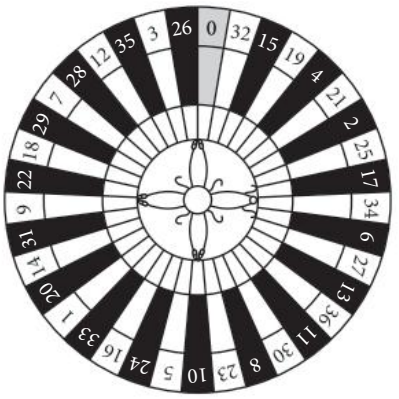


23 A French roulette wheel has 37 compartments around its rim. One of these is numbered 0 and is coloured green. The others are numbered 1 to 36, with half of these 36 compartments coloured red and the other half coloured black. The wheel is spun in one direction and a small ball is rolled in the opposite direction. The chances of the ball falling into any of the 37 compartments are equally likely. Find the probability that the ball will land on:

(a) black **(b)** an odd number

(c) a multiple of 3 **(d)** any number from 1 to 12

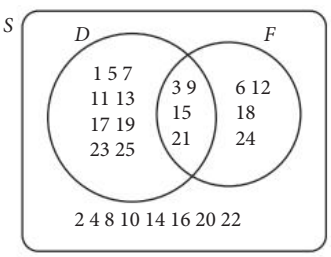
(e) 15



18.2 VENN DIAGRAMS

Venn diagrams are a visual way to represent sets of data. We can name the sets with letters and talk about various properties of the sets. Venn diagrams are introduced here as an efficient tool to be used when analysing data to find the probability of events.

If $S = \{1, 2, 3, 4, \ldots 25\}$ (i.e. whole numbers 1 to 25), $D = \{1, 3, 5, \ldots 25\}$ (i.e. odd numbers from 1 to 25), $F = \{3, 6, 9, \ldots 24\}$ (i.e. multiples of 3 between 1 and 25), then $n(S) = 25$, $n(D) = 13$ and $n(F) = 8$, where the notation $n(S)$ means the number of members of set S. This information can be shown on a Venn diagram as follows:



The outer group contains all the members of set S; the inner group on the left contains all the members of set D; the inner group on the right contains all the members of set F. The area where the inner groups overlap contains the members that are in both sets D and F. The region outside the groups of D and F contain all the members of set S that are in neither set D nor set F.

To enter information onto a Venn diagram like this it is best to first find the members common to both sets D and F, then put the rest of the data into D and F. The remaining members of S that were in neither D nor F can then be put into the region outside D and F.

The set made by combining sets D and F is called the **union** of the two sets, written $D \cup F$. It consists of all the elements that are in D or in F or in both D and F:

$D \cup F = \{1, 3, 5, 6, 7, 9, 11, 12, 13, 15, 17, 18, 19, 21, 23, 24, 25\}$

The set made up of the elements common to sets D and F is called the **intersection** of the two sets, written $D \cap F$:

$D \cap F = \{3, 9, 15, 21\}$

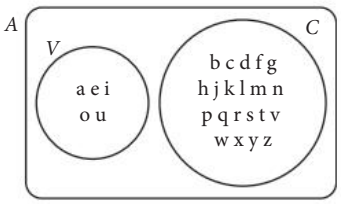
The intersection contains numbers that belong to both sets, i.e. the odd multiples of 3 between 1 and 25.

Now $n(D) = 13$, $n(F) = 8$, $n(D \cup F) = 17$, $n(D \cap F) = 4$. Hence:

$$n(D \cup F) = n(D) + n(F) - n(D \cap F)$$

Because sets D and F contain some common elements, the two sets are not mutually exclusive. Also, sets D and F are each **subsets** of set S, because all their elements are in set S.

Note that sets do not always overlap. If A = {letters of the alphabet), V = {vowels} and C = {consonants}, this information can be shown on a Venn diagram:



Sets V and C have no members in common, so they do not overlap.
Hence $V \cap C = \{\}$, where $\{\}$ is called the empty set.

All members of set A are in either set V or set C. Because of this, sets V and C are called **mutually exclusive** or **disjoint**.

Now $n(V) = 5$, $n(C) = 21$, $n(V \cup C) = 26$, $n(V \cap C) = 0$.

Hence, for disjoint or mutually exclusive sets:

$$n(V \cup C) = n(V) + n(C)$$

Sets V and C are each subsets of set A, because all their elements are in set A. They are also called **complementary** sets, because together they make up set A with no overlapping.

Example 4

A is the set of possible sums from the numbers showing when two dice are rolled. B is the set of odd sums and C is the set of sums that are more than 7. Show this information on a Venn diagram and find:

(a) $B \cup C$ **(b)** $B \cap C$

Solution

List the elements in each set:

$A = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$,

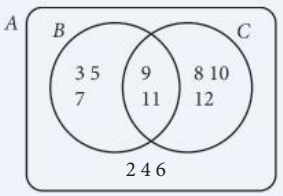
$B = \{3, 5, 7, 9, 11\}$, $C = \{8, 9, 10, 11, 12\}$

Elements common to B and C: $\{9, 11\}$

(a) $B \cup C = \{3, 5, 7, 8, 9, 10, 11, 12\}$

(b) $B \cap C = \{9, 11\}$

Venn diagram:



Example 5

A die is rolled on a flat surface. A is the set of even results, B is the set of odd results and C is the set of results less than 4. Show this information on a Venn diagram and find:

(a) $A \cup C$ **(b)** $C \cup B$ **(c)** $A \cup B \cup C$ **(d)** $A \cap B$

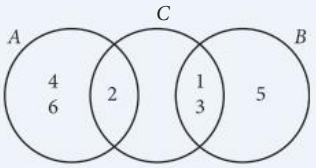
Solution

List the sets: $A = \{2, 4, 6\}$, $B = \{1, 3, 5\}$, $C = \{1, 2, 3\}$

There are 3 intersecting sets, so 3 groups needed.

Common elements between sets: 2 is common to A and C

1, 3 are common to B and C

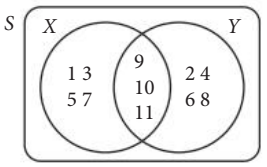


(a) $A \cup C = \{1, 2, 3, 4, 6\}$ **(b)** $C \cup B = \{1, 2, 3, 5\}$

(c) $A \cup B \cup C = \{1, 2, 3, 4, 5, 6\}$ **(d)** $A \cap B = \{\}$

EXERCISE 18.2 VENN DIAGRAMS

1 For the Venn diagram shown, the set S is given by:



A $S = \{1, 3, 5, 7, 9, 10, 11\}$ **B** $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$

C $S = \{2, 4, 6, 8, 9, 10, 11\}$ **D** $S = \{9, 10, 11\}$

To show the information given in questions **2** to **5**, draw a Venn diagram for each.

2 S is the set of possible sums from the numbers showing when two dice are rolled, D is the set of even sums, F is the set of sums less than 6. Find: **(a)** $D \cup F$ **(b)** $D \cap F$

3 A coin is tossed and a die is rolled. S is the set of all possible results obtained, H is the set of all results containing heads, G is the set of all results containing an odd number. Find:

(a) $H \cup G$ **(b)** $n(H \cap G)$

4 S is the set of all possible outcomes on a French roulette wheel, B is the set of black numbers, R is the set of red numbers. (For a description of a French roulette wheel, see Exercise 18.1 question **23**.) Find:

(a) $B \cup R$ **(b)** $B \cap R$ **(c)** $n(B \cup R)$ **(d)** $n(B \cap R)$ **(e)** $n(S)$

5 $S = \{1, 2, 3, \ldots 25\}$, $E = \{\text{even numbers in set } S\}$, $P = \{\text{perfect squares in set } S\}$. Find:

(a) $n(S)$ **(b)** $n(E)$ **(c)** $n(P)$ **(d)** $n(E \cap P)$ **(e)** $n(E \cup P)$

6 In a group of 25 students, 18 study French, 12 study German and 5 study neither French nor German. Find how many students:

(a) study only French **(b)** study only German **(c)** study only one language.

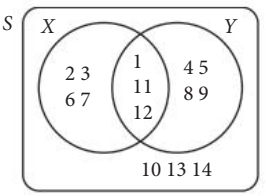
7 $S = \{1, 2, 3, \ldots 20\}$, $K = \{\text{multiples of 3 in set } S\}$, $L = \{\text{multiples of 6 in set } S\}$. Find:

(a) $K \cup L$ **(b)** $K \cap L$ **(c)** $n(K \cap L)$

8 A die is rolled on a flat surface. A is the set of even results, B is the set of odd results, C is the set of results more than 4. Find: **(a)** $A \cap B$ **(b)** $A \cup B$ **(c)** $B \cap C$ **(d)** $n(A \cap C)$

9 Two dice are rolled. Of the results, $E = \{\text{even sums}\}$, $F = \{\text{odd sums}\}$, $G = \{\text{sums where one die shows a 6}\}$. Find: **(a)** $n(E \cap F)$ **(b)** $n(F \cap G)$ **(c)** $n(E \cup F)$ **(d)** $n(E \cup G)$

10 For the Venn diagram shown, indicate whether each statement is correct or incorrect.



(a) $S = \{1, 2, 3, \ldots 12\}$ **(b)** $X \cup Y = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12\}$

(c) $X \cap Y = \{1, 11, 12\}$ **(d)** $S = \{1, 2, 3, \ldots 14\}$

18.3 FINITE SAMPLE SPACES

A set of favourable outcomes can be considered an **event**. The rolling of even numbers with a die (3 favourable outcomes), the drawing of a heart from a standard pack of cards (13 favourable outcomes) or the tossing of heads with a coin (1 favourable outcome) are all examples of an event.

The rolling of a die, the drawing of cards from a pack, the tossing of a coin—that is, activities that will produce various events—are called **experiments** or **trials**.

In modern probability theory, all possible outcomes of an experiment are considered as points in a space, called a **sample space** S. If S contains a finite number of points n and if the outcomes of an experiment are equally likely, then we can assign each point (called a **sample point**) a probability of $\frac{1}{n}$. The sum of the probabilities of all the sample points is therefore 1.

If S is the sample space and A is a set of points drawn from the sample space (i.e. A is an event), then:

$$P(A) = \frac{\text{number of sample points in } A}{\text{number of sample points in } S} = \frac{n(A)}{n(S)}$$
$$= \frac{\text{number of outcomes corresponding to } A}{\text{total number of possible outcomes}}$$

If $S = \{a, b, c, \ldots x, y, z\}$ then $n(S) = 26$.

Let $A = \{a, e, i, o, u\}$ so $n(A) = 5$: $P(A) = \frac{n(A)}{n(S)} = \frac{5}{26}$

Thus the probability of selecting a vowel from S is $\frac{5}{26}$.

Mutually exclusive events

If two or more events cannot occur simultaneously then the events are said to be **mutually exclusive** or **disjoint**. In the language of the sample space, the events have no points in common.

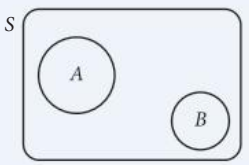
If a coin is tossed, either heads or tails may turn up, but both events cannot occur at the same time. If a card is drawn at random from a pack, it may be a heart or the ace of spades or some other card, but if the card is a heart then this excludes the possibility of it being the ace of spades. The two events 'heart' and 'ace of spades' are mutually exclusive. However, the events 'heart' and 'ace' are not mutually exclusive. Why?

Example 6

If one card is drawn at random from a standard pack of 52 playing cards, what is the probability that it is a heart or the ace of spades?

Solution

These two events are mutually exclusive, because they can't happen together.



$S = \{\text{playing cards}\}$ $\quad n(S) = 52$

$A = \{\text{heart}\}$ $\quad n(A) = 13$

$B = \{\text{ace of spades}\}$ $\quad n(B) = 1$

$A \cup B = A$ or $B = \{\text{heart or ace of spades}\}$

$n(A \text{ or } B) = 14$

$$P(A \text{ or } B) = \frac{n(A \text{ or } B)}{n(S)} = \frac{14}{52} = \frac{7}{26}$$

A and B are disjoint sets, so they don't intersect.

Now $P(A) = \frac{13}{52}$, $P(B) = \frac{1}{52}$ and so $P(A) + P(B) = \frac{13}{52} + \frac{1}{52} = \frac{14}{52} = \frac{7}{26}$

Because the sets A and B have no points in common and thus are mutually exclusive, then:

$P(A \text{ or } B) = P(A) + P(B)$

Example 7

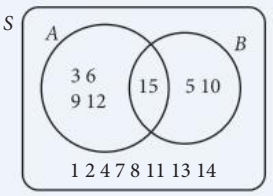
From a set of 15 cards numbered 1 to 15, one card is drawn at random. What is the probability that it is a multiple of 3 or 5 or both?

Solution

$S = \{1, 2, 3, \ldots 15\}$ $\quad n(S) = 15$

$A = \{3, 6, 9, 12, 15\}$ $\quad n(A) = 5$

$B = \{5, 10, 15\}$ $\quad n(B) = 3$



One card, 15, is common to both sets A and B:

$A \cap B = A$ and $B = \{15\}$ $\quad n(A \text{ and } B) = 1$

$A \cup B = A$ or $B = \{3, 5, 6, 9, 10, 12, 15\}$ $\quad n(A \text{ or } B) = 7$

$\therefore P(A \text{ or } B) = \dfrac{n(A \text{ or } B)}{n(S)} = \dfrac{7}{15}$

The notation 'A and B' means the elements common to both sets (in Example **7** above, the multiples of 3 and 5). This is also known as the intersection of the two sets.

The notation 'A or B' means the elements in either set (in the example above, the multiples of 3 or 5 or both). This is also known as the union of the two sets.

We have $n(A \text{ or } B) = n(A) + n(B) - n(A \text{ and } B)$, because the sets are not mutually exclusive. Thus:

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

This formula also works for mutually exclusive sets, where $P(A \text{ and } B) = 0$ and so the result becomes:

$$P(A \text{ or } B) = P(A) + P(B)$$

Example 8

A card is drawn at random from a standard pack of 52 playing cards. R is the event 'drawing a red card,' H is the event 'drawing the ace of hearts,' A is the event 'drawing an ace.' Find:

(a) $\overline{R}$ **(b)** $\overline{A}$ **(c)** $P(R)$ **(d)** $P(\overline{R})$

(e) $P(R \text{ or } A)$ **(f)** $P(R \text{ or } H)$ **(g)** $P(R \text{ and } A)$ **(h)** $P(\overline{R} \text{ or } H)$

Solution

(a) $\overline{R} = \{\text{not red}\} = \{\text{black}\}$ **(b)** $\overline{A} = \{\text{not an ace}\}$ **(c)** $n(R) = 26, \quad P(R) = \frac{26}{52} = \frac{1}{2}$

(d) $n(\overline{R}) = 26, \quad P(\overline{R}) = \frac{26}{52} = \frac{1}{2}$ **OR** $P(\overline{R}) = 1 - P(R) = 1 - \frac{1}{2} = \frac{1}{2}$

(e) $n(R \cup A) = 26 + 4 - 2 = 28, \quad P(R \text{ or } A) = \dfrac{n(R \cup A)}{52} = \frac{28}{52} = \frac{7}{13}$

(f) $n(R \cup H) = n(R) = 26$ because the ace of hearts is a red card.

$$P(R \text{ or } H) = \frac{n(R \cup H)}{52} = \frac{26}{52} = \frac{1}{2}$$

(g) $n(R \cap A) = 2$ because there are 2 red aces.

$$P(R \text{ and } A) = \frac{n(R \cap A)}{52} = \frac{2}{52} = \frac{1}{26}$$

(h) $\overline{R} \cup H = \{\text{black or ace of hearts}\}, \quad P(\overline{R} \text{ or } H) = \dfrac{n(\overline{R} \cup H)}{52} = \frac{27}{52}$

EXERCISE 18.3 FINITE SAMPLE SPACES

1 A card is drawn at random from a standard pack of 52 playing cards. A is the event 'drawing a heart', B the event 'drawing the ace of clubs', C is the event 'drawing an ace'. Find the following:

(a) $P(A)$ **(b)** $P(B)$ **(c)** $P(C)$ **(d)** $P(A \text{ or } B)$ **(e)** $P(A \text{ or } C)$

(f) $P(B \text{ or } C)$ **(g)** Which pairs of events A and B, A and C, or B and C are mutually exclusive?

2 From a set of 17 cards numbered 1, 2, 3, … 17, one card is selected at random. A is the event 'a multiple of 3', B is the event 'a multiple of 8', C is the event 'a multiple of 5'. Find the following:

(a) $P(A)$ **(b)** $P(B)$ **(c)** $P(C)$ **(d)** $P(\overline{A})$ **(e)** $P(\overline{B})$

(f) $P(A \text{ or } B)$ **(g)** $P(A \text{ or } C)$ **(h)** $P(\overline{A} \text{ or } B)$ **(i)** $P(A \text{ or } \overline{C})$ **(j)** $P(\overline{B} \text{ or } \overline{C})$

(k) Which pairs of the events A, B and C are mutually exclusive?

3 A die is rolled on a flat surface. A is the event 'an even number', B is the event 'an odd number', C is the event 'a multiple of 5'. Which of the following statements is true?

A $P(A \text{ or } B) = 0.5$ **B** $P(\overline{A} \text{ or } B) = 1$ **C** $P(A \text{ or } C) = 0.5$ **D** $P(\overline{B} \text{ or } \overline{C}) = \frac{5}{6}$

4 A container holds 8 red marbles, 7 white marbles and 5 black marbles. One marble is drawn at random from the container. Indicate whether each statement below is correct or incorrect.

(a) $P(\text{red or black}) = \frac{3}{4}$ **(b)** $P(\text{not white}) = \frac{1}{4}$

(c) $P(\text{neither black nor white}) = \frac{2}{5}$ **(d)** $P(\text{not red}) = \frac{3}{5}$

5 A coin is tossed three times. A is the event 'at least two heads', B is the event 'head, tail, head', C is the event 'not more than one head'. Find:

(a) $P(A)$ **(b)** $P(B)$ **(c)** $P(C)$ **(d)** $P(A \text{ or } B)$ **(e)** $P(A \text{ or } C)$ **(f)** $P(B \text{ or } C)$

(g) $P(A \text{ or } B \text{ or } C)$ **(h)** Which pairs of the events A, B and C are mutually exclusive?

6 An integer is chosen at random from the first 50 positive integers. A is the event 'divisible by 2', B is the event 'divisible by 3', C is the event 'divisible by 5'. Find:

(a) $P(A \text{ and } B)$ **(b)** $P(A \text{ and } C)$ **(c)** $P(B \text{ and } C)$ **(d)** $P(A \text{ and } B \text{ and } C)$

(e) $P(A \text{ or } B)$ **(f)** $P(A \text{ or } C)$ **(g)** $P(B \text{ or } C)$ **(h)** $P(A \text{ or } B \text{ or } C)$

7 A die is rolled twice and the numbers rolled are added together. A is the event 'the sum is 4 or more', B is the event 'the sum is less than 6', C is the event 'the two numbers rolled are the same'. Find:

(a) $P(A \text{ and } B)$ **(b)** $P(A \text{ and } C)$ **(c)** $P(B \text{ and } C)$ **(d)** $P(A \text{ and } B \text{ and } C)$

(e) $P(A \text{ or } B)$ **(f)** $P(A \text{ or } C)$ **(g)** $P(B \text{ or } C)$ **(h)** $P(A \text{ or } B \text{ or } C)$

8 A die is rolled twice and the numbers rolled are added together. A is the event 'the sum is greater than 5', B is the event 'the sum is less than 8'. Find:

(a) $P(A)$ **(b)** $P(B)$ **(c)** $P(A \text{ and } B)$ **(d)** $P(A \text{ or } B)$

(e) Hence show that $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$. Are A and B mutually exclusive?

9 If $P(A) + P(B) = 1$, does it follow that either event A or event B must occur?

10 $P(A \text{ or } B) = \frac{15}{16}$, $P(A \text{ and } B) = \frac{1}{8}$, $P(A) = \frac{3}{8}$. Find $P(B)$.

11 In a group of 50 students, 30 study Mathematics, 25 study Physics and 20 study both Mathematics and Physics. One student is selected at random from the group. What is the probability that this student studies:

(a) Mathematics but not Physics **(b)** Physics but not Mathematics

(c) neither Physics nor Mathematics?

12 From a group of 100 students, 50 study History, 30 study English and 20 study both. If a student is selected at random from the group, what is the probability that the student studies:

(a) at least one of these subjects **(b)** History but not English

(c) History, given that the student also studies English?

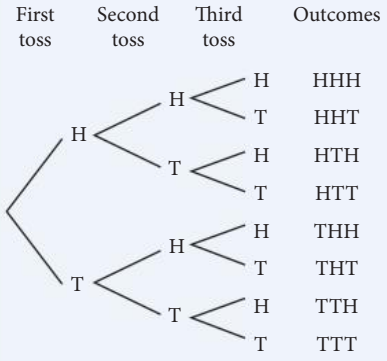
18.4 SUCCESSIVE OUTCOMES

It is important to be able to organise information so that probabilities can be calculated easily. A **tree diagram** is one of the most common ways to do this.

Example 9

A coin is tossed three times. Use a tree diagram to show the possible outcomes.

Solution



Each time a coin is tossed, it can land as either a head (H) or a tail (T).

There are 8 possible outcomes ($2 \times 2 \times 2 = 2^3$) for this experiment. How you describe the outcomes depends on whether or not the order of the events is important.

If the order is important, then the outcomes HHT, HTH and THH are three different outcomes. If order is not important, then these three results represent the same single outcome: 2 heads and 1 tail.

When order is not important, there are only 4 possible outcomes for this experiment: 3 heads, 2 heads and 1 tail, 1 head and 2 tails, or 3 tails (no heads).

As well as listing outcomes, a tree diagram can be used to show the probability of each outcome in a repeated experiment. The fundamental counting principle (or 'multiplication principle'—see below) means that you can multiply the successive probabilities along the branches to obtain the probability of a final outcome.

Example 10

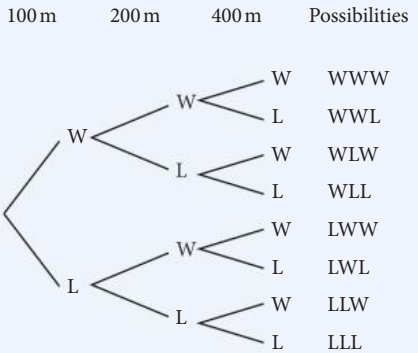
At a school sports day Joely runs in the 100 m, 200 m and 400 m races. In terms of win (W) and loss (L), there are two possible outcomes for each race.

(a) How many possibilities are there for the three races?

(b) What are the probabilities of Joely winning 3, 2, 1 or no races?

Solution

(a) There are 8 possible outcomes ($2 \times 2 \times 2$). WWW represents a win in each of the three races; WWL, for example, represents a win in 100 m and 200 m and a loss in the 400 m.



(b) The question does not say that Joely has an equally likely chance of winning each race, so the question cannot be answered due to lack of information.

However, we can find the probabilities if we assume that she has a 50-50 chance of winning each race, in which case:

$P(\text{wins 3 races}) = \frac{1}{8}$ $P(\text{wins 2 races}) = \frac{3}{8}$

$P(\text{wins 1 race}) = \frac{3}{8}$ $P(\text{wins 0 races}) = \frac{1}{8}$

Multiplication principle

Tree diagrams can be used to calculate probabilities. To obtain the probability of a final outcome, you can multiply the probabilities of the successive intermediate outcomes along the branches. For example:

Number of ways a two-part outcome can occur = $m \times n$

(where m = number of ways the first part can occur,
n = number of ways the second part can occur)

Example 11

A coin is tossed twice. Draw a tree diagram to show the possible outcomes, marking the probability of each outcome on each branch. Use your tree diagram to find:

(a) $P(2 \text{ heads})$ **(b)** $P(1 \text{ head})$ **(c)** $P(0 \text{ heads})$

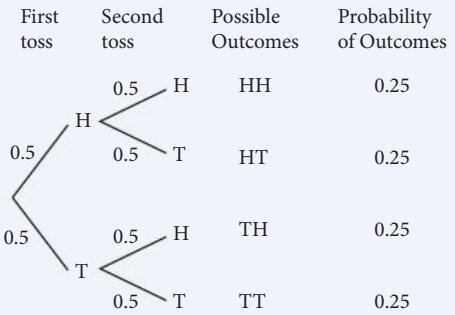
Solution

$P(\text{H}) = P(\text{T}) = 0.5$

(a) $P(2 \text{ heads}) = P(\text{H}) \times P(\text{H}) = 0.5 \times 0.5 = 0.25$

(b) $P(1 \text{ head}) = P(\text{HT}) + P(\text{TH}) = 0.5 \times 0.5 + 0.5 \times 0.5$
$= 0.5$

(c) $P(0 \text{ heads}) = P(\text{T}) \times P(\text{T}) = 0.5 \times 0.5 = 0.25$



Example 12

A card is drawn from a standard pack of 52 playing cards and a coin is tossed. If you are only concerned with the suit of the card (i.e. whether hearts, diamonds, clubs or spades), draw a tree diagram to show the possible outcomes, marking the probability of each outcome on each branch. Use your tree diagram to find:

(a) $P(\text{spade and heads})$ **(b)** $P(\text{red card and tails})$ **(c)** $P(\text{not a club and heads})$

Solution

Multiply successive probabilities along the branches.

(a) $P(\text{spade and heads}) = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$

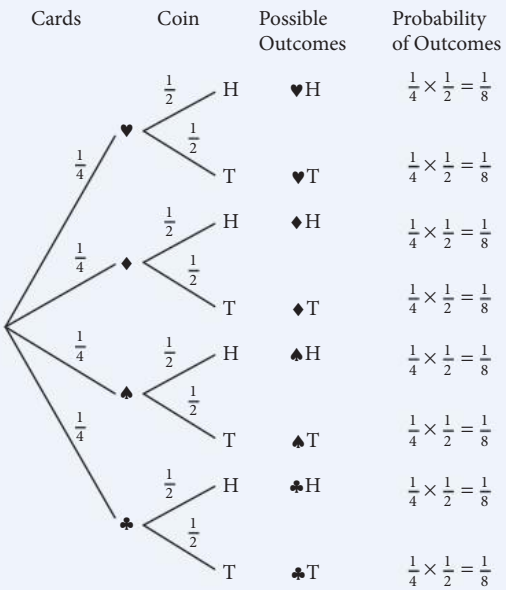
(b) $P(\text{red card and tails}) = \frac{1}{4} \times \frac{1}{2} + \frac{1}{4} \times \frac{1}{2} = \frac{1}{4}$

OR, using successive probabilities:

$P(\text{red card}) = \frac{1}{2}$, $P(\text{tails}) = \frac{1}{2}$

$P(\text{red card and tails}) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

(c) $P(\text{not a club and heads}) = 3 \times \frac{1}{4} \times \frac{1}{2} = \frac{3}{8}$



EXERCISE 18.4 SUCCESSIVE OUTCOMES

1 A coin is tossed and a die is rolled at the same time. Draw a tree diagram to show the outcomes, marking the probability of the outcome on each branch. If each outcome is equally likely, find the probability of:

(a) a head and an even number **(b)** a tail and a number greater than 4.

2 Two cubes each have two faces painted red, another two faces painted white and the remaining two faces painted blue. Both cubes are rolled on a table like dice. Draw a tree diagram to show the possible outcomes of the colours that are rolled. If each outcome is equally likely, what is the probability that:

(a) both colours rolled are the same **(b)** one colour rolled is red and the other is white?

3 A committee consisting of one boy and one girl is to be selected from three girls (Anna, Kate and Sue-Lin) and four boys (Brent, Jimmy, Mustafa and Rajiv). What is the probability that the committee consists of Kate and Rajiv?

4 A restaurant has a set menu that offers a choice of three different entrees, three different main courses and three different desserts. The entrées are vegetable soup, spring rolls, and calamari; the main courses are roast beef, roast eggplant, and spaghetti bolognese; the desserts are cheesecake, ice-cream, and fruit salad. Draw a tree diagram to show all the menu choices that include one item from each course.

(a) If you must choose one item from each course, how many different meal choices are there?

If each choice is equally likely, find the probability of selecting:

(b) vegetable soup, roast beef, and either cheesecake or ice-cream

(c) spring rolls, not roast beef, and fruit salad.

5 A die is rolled twice and the numbers rolled are added together. Which of the following events has a probability of $\frac{1}{18}$?

A both numbers rolled are the same **B** the sum is 5 **C** the sum is 3 **D** the sum is 12

6 A coin is tossed three times. Illustrate on a tree diagram the eight possible outcomes. Use the tree diagram to find the probability of tossing:

(a) 3 tails **(b)** 1 head **(c)** 2 heads **(d)** 3 heads

7 Three students Kristian, Soung-ho and Amélie work independently on a mathematics problem. Each student has a 50-50 chance of success. Draw a tree diagram and use it to indicate whether each of the following statements is correct or incorrect.

(a) $P(\text{all successful}) = \frac{1}{2}$ **(b)** $P(\text{only Kristian and Amélie successful}) = \frac{1}{8}$

(c) $P(\text{none successful}) = \frac{3}{8}$ **(d)** $P(\text{at least one is successful}) = \frac{7}{8}$

8 One cube has 4 red faces and 2 white faces, another has 3 red and 3 white faces, and a third cube has 2 red and 4 white faces. The three cubes are rolled like dice.

(a) Draw a tree diagram to show all possible outcomes and the associated probabilities.

What is the probability that:

(b) 3 red faces are rolled **(c)** only one cube rolls a white face

(d) more red faces than white faces are rolled?

9 A 50-cent coin, a 20-cent coin and a 10-cent coin are tossed. What is the probability of:

(a) 3 heads **(b)** a head only with the 50-cent coin **(c)** at least 2 tails?

10 A card is drawn at random from a standard pack of 52 playing cards. It is replaced, the pack is shuffled and a card is drawn again. What is the probability that:

(a) both cards are spades **(b)** neither card is a heart **(c)** both cards are the same suit?

18.5 INDEPENDENT EVENTS

Two events are **independent** if the outcome of each event is not affected by the outcome of the other event. In other words:

Events A and B are independent if:	$P(A \text{ and } B) = P(A) \times P(B)$
Or, writing $P(A \text{ and } B)$ as the 'product' of A and B:	$P(AB) = P(A) \times P(B)$

Many of the questions in Exercise 8.4 involved independent events.

Example 13

If two dice are rolled, what is the probability of rolling an even number with the first die and a 3 or a 5 with the second die?

Solution

The sample space consists of $6 \times 6 = 36$ points, as listed below. Each sample point is represented for convenience below as (**a**, b), where **a** is the number on the first die and b is the number on the second die:

(**1**, 1)	(**2**, 1)	(**3**, 1)	(**4**, 1)	(**5**, 1)	(**6**, 1)
(**1**, 2)	(**2**, 2)	(**3**, 2)	(**4**, 2)	(**5**, 2)	(**6**, 2)
(**1**, 3)	(**2**, 3)	(**3**, 3)	(**4**, 3)	(**5**, 3)	(**6**, 3)
(**1**, 4)	(**2**, 4)	(**3**, 4)	(**4**, 4)	(**5**, 4)	(**6**, 4)
(**1**, 5)	(**2**, 5)	(**3**, 5)	(**4**, 5)	(**5**, 5)	(**6**, 5)
(**1**, 6)	(**2**, 6)	(**3**, 6)	(**4**, 6)	(**5**, 6)	(**6**, 6)

There are 18 points that correspond to event A, an even number with the first die and any number with the second die. There are 12 points that correspond to event B, any number with the first die and a 3 or a 5 with the second die.

Thus: $P(A) = \frac{18}{36} = \frac{1}{2}$ and $P(B) = \frac{12}{36} = \frac{1}{3}$.

The intersection of A and B is the set of 6 points above that are shaded.

$A \cap B = \{(2, 3), (2, 5), (4, 3), (4, 5), (6, 3), (6, 5)\}$

Hence: $P(A \text{ and } B) = \frac{6}{36} = \frac{1}{6}$

And: $P(A) \times P(B) = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6} = P(A \text{ and } B)$

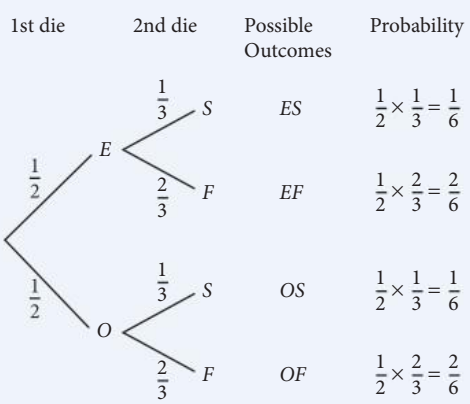
This result conforms to our definition of independent events, as well as to the everyday meaning of the word 'independent'. Clearly, whichever number the first die happens to roll will have absolutely no influence on the second die's roll.

Alternatively:

This example could also be solved by a careful observation of the 36 sample points. Observe that $\frac{1}{2}$ of the outcomes contain an even number with the first die; then note that $\frac{1}{3}$ of this $\frac{1}{2}$ (that is, $\frac{1}{6}$) contain an even number with the first die **and** a 3 or a 5 with the second die.

We can think of rolling two dice as a two-stage process, which can be represented by a tree diagram with two sets of branches. For convenience, we label an even number on the first die E, an odd number on the first die O, a 3 or a 5 with the second die a success S, and 'not a 3 or a 5' a failure F.

The tree diagram shows the four possible outcomes, but they are not all equally likely. An even number E with the first die and a 3 or a 5 with the second die is given by ES with probability $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$.



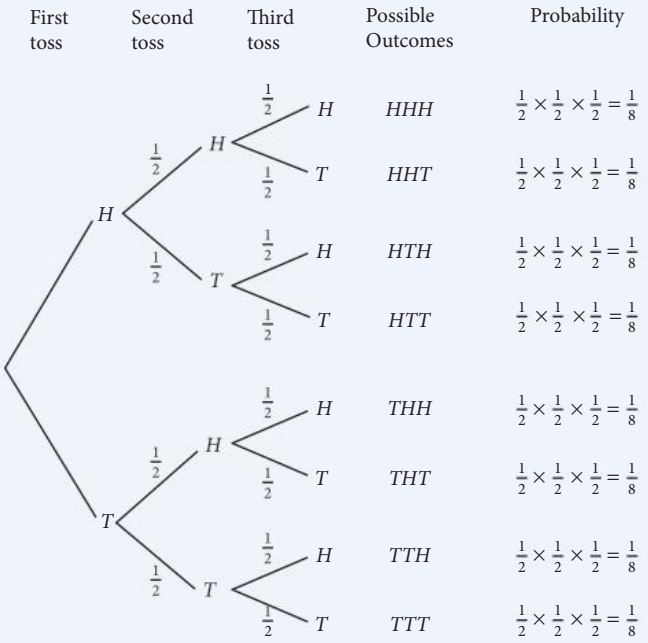
Example 14

A fair coin is tossed 3 times. What is the probability of:

(a) 2 heads and 1 tail **(b)** at least 1 head?

Solution

The experiment of tossing a coin 3 times can be represented by a tree diagram with 3 sets of branches. As usual, H = head and T = tail:



Hence $S = \{(HHH), (HHT), (HTH), (HTT), (THH), (THT), (TTH), (TTT)\}$

$\therefore n(S) = 8$ and each of the 8 outcomes is equally likely.

(a) From observation of the 8 sample points, we can pick out the points corresponding to the event A, '2 heads and 1 tail': $A = \{(HHT), (HTH), (THH)\}$

$$\therefore P(\text{2 heads and 1 tail}) = \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} = \tfrac{3}{8}$$

(b) Each of the 8 outcomes except (TTT) has at least 1 head: $\therefore P(\text{at least 1 head}) = 1 - P(\text{0 heads})$

$$= 1 - \tfrac{1}{8} = \tfrac{7}{8}$$

Example 15

Two friends, Suzy and Dimitri, often play golf and tennis with each other. Overall, Suzy tends to win 3 rounds of golf out of every 5 and 1 game of tennis out of every 4. If Suzy and Dimitri play one round of golf and one game of tennis, find the probability that Suzy:

(a) wins both **(b)** loses both

(c) wins only the round of golf **(d)** wins either the golf or the tennis but not both.

Solution

Assume that no game ends in a draw.

Let A be the event 'Suzy wins golf', so $\overline{A}$ is the event 'Suzy loses golf'. From the overall probabilities:

$P(A) = \frac{3}{5}$ and $P(\overline{A}) = \frac{2}{5}$

Let B be the event 'Suzy wins tennis', so $\overline{B}$ is the event 'Suzy loses tennis'.

$P(B) = \frac{1}{4}$ and $P(\overline{B}) = \frac{3}{4}$

Assuming that A and B are independent events (which seems reasonable), then:

(a) $P(A \text{ and } B) = P(A) \times P(B)$
$= \frac{3}{5} \times \frac{1}{4}$
$= \frac{3}{20}$

(b) $P(\bar{A} \text{ and } \bar{B}) = P(\bar{A}) \times P(\bar{B})$
$= \frac{2}{5} \times \frac{3}{4}$
$= \frac{3}{10}$

(c) $P(A \text{ and } \bar{B}) = P(A) \times P(\bar{B})$
$= \frac{3}{5} \times \frac{3}{4}$
$= \frac{9}{20}$

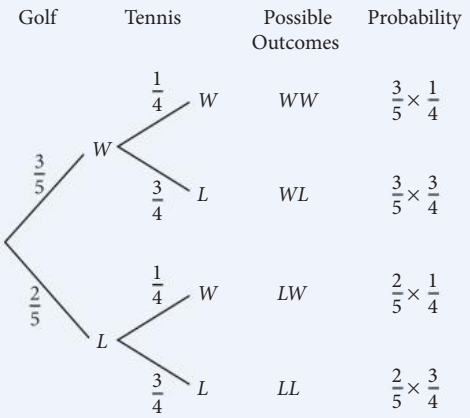
(d) For this outcome, Suzy either **(i)** wins golf and loses tennis, or **(ii)** loses golf and wins tennis.

(i) $P(A \text{ and } \bar{B}) = P(A) \times P(\bar{B})$
$= \frac{3}{5} \times \frac{3}{4}$
$= \frac{9}{20}$

(ii) $P(\bar{A} \text{ and } B) = P(\bar{A}) \times P(B)$
$= \frac{2}{5} \times \frac{1}{4}$
$= \frac{2}{20}$

(i) and **(ii)** are mutually exclusive, so $P((A \text{ and } \bar{B}) \text{ or } (\bar{A} \text{ and } B)) = \frac{9}{20} + \frac{2}{20} = \frac{11}{20}$.

This problem can also be solved using the tree diagram method. There are four possible outcomes (not all equally likely); if we denote a win by Suzy as W and a loss as L, then the possible outcomes are WW, WL, LW and LL. Their respective probabilities can be illustrated by the following tree diagram.



Example 16

A die is rolled, a coin is tossed and a letter is taken from the set {a, b, c}. Find the probability of each event described below.

(a) Event A: a 3 with the die
(b) Event B: a head with the coin
(c) Event C: 'b' from the set {a, b, c}
(d) A and B and C.

Solution

Events A, B and C are independent.

(a) $P(A) = \frac{1}{6}$ **(b)** $P(B) = \frac{1}{2}$ **(c)** $P(C) = \frac{1}{3}$

(d) $P(A \text{ and } B \text{ and } C) = P(A) \times P(B) \times P(C)$
$= \frac{1}{6} \times \frac{1}{2} \times \frac{1}{3}$
$= \frac{1}{36}$

EXERCISE 18.5 INDEPENDENT EVENTS

1 I toss a coin and roll a die. What is the probability of:

(a) a head with the coin
(b) a number greater than 4 with the die
(c) a head and a number greater than 4?

2 In a class of 25 boys and 15 girls, 8 boys and 7 girls wear glasses. One student from the class is selected. Indicate whether each statement below is correct or incorrect.

(a) $P(\text{boy}) = \frac{5}{8}$ **(b)** $P(\text{wears glasses}) = \frac{3}{8}$
(c) $P(\text{boy and wears glasses}) = \frac{1}{5}$ **(d)** $P(\text{girl and does not wear glasses}) = \frac{7}{40}$

3 A coin is tossed three times. Find the probability of each event:

(a) A: at least two heads **(b)** B: head, tail, head **(c)** C: not more than one head

4 A red die and a blue die are rolled on a table. Find the probability of each event:

(a) the same number with each die **(b)** the sum of the numbers exceeds 9
(c) a 3 with the red die and a 4 with the blue die
(d) an odd number with the red die and an even number with the blue die
(e) the sum of the numbers is less than 2.

5 A coin is tossed and a die is rolled. The probability of 'a head and a number greater than 4' or 'a tail and a number not exceeding 3' is:

A $\frac{1}{24}$ **B** $\frac{1}{6}$ **C** $\frac{1}{4}$ **D** $\frac{5}{12}$

6 A die is rolled and a number is selected at random from the set $\{1, 2, 3, 4, 5\}$. The die number and the selected number are added together to find the score. Find the probability of each event:

(a) A: an even number is selected from the set **(b)** B: the die rolls an odd number
(c) C: the score exceeds 9 **(d)** D: the score is 10

7 A coin is tossed three times. A is the event 'at least two tails'; B is the event 'three heads or three tails'; C is the event 'at least one tail'. Which of the following are independent?

(a) A and B **(b)** A and C **(c)** B and C

8 Consider two spinners. One spinner shows the letters A, B, C, D and E, and the other spinner shows the digits 1, 2, 3, 4 and 5. When the spinners are spun, it is equally likely that they will stop on any letter or number. Find the probability of each event:

(a) B and an even number **(b)** C or D, and an odd number
(c) E and an even number, or C and a number greater than 3
(d) a consonant and an odd number, or a consonant and a number greater than 2.

9 A coin is tossed and a die is rolled. What is the probability of 'a head and a number greater than 3' or 'a tail and a number not exceeding 4'?

10 In a large school, 25% of the students ride bicycles to school and 40% of the students have fair hair. One student is selected at random. What is the probability that the student:

(a) has fair hair and rides a bicycle to school
(b) does not have fair hair and does not ride a bicycle to school
(c) has fair hair but does not ride a bicycle to school
(d) rides a bicycle to school but does not have fair hair?

11 A die is loaded so that the probability of rolling a 6 is $\frac{3}{10}$, the probability of a 5 is $\frac{3}{10}$, and each of the other numbers is equally likely. If the die is rolled twice, find the probability of:

(a) two 6s **(b)** no 6s **(c)** at least one 6 **(d)** the sum of the two numbers being six.

12 One card is drawn at random from a standard pack of 52 playing cards. The card is replaced in the pack and the pack is shuffled. A second card is then drawn. What is the probability that:

(a) both cards are diamonds **(b)** neither card is a diamond
(c) only one of the cards is a diamond **(d)** only the first card is a diamond
(e) only the second card is a diamond **(f)** at least one of the cards is a diamond?

13 For a certain species of bird, there is a chance of 4 in 5 that a fledgling will survive the first month after birth. From a brood of 3 fledglings, what is the probability that:

(a) all will survive **(b)** none will survive **(c)** at least one will survive?

14 An athlete competes in races over 100 m, 200 m and 400 m, and she estimates her chances of winning as $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$ respectively. Using these probabilities, calculate the probability that:

(a) she wins all three races **(b)** she loses all three races
(c) she wins the 100 m race and loses the others **(d)** she wins the 400 m race and loses the others
(e) she wins the 100 m race and the 200 m race but loses the 400 m race.

15 Cube A has 4 red faces and 2 white faces; cube B has 3 red faces and 3 white faces; cube C has 2 red faces and 4 white faces. The three cubes are rolled like dice. What is the probability of rolling:

(a) 3 red faces **(b)** 3 white faces **(c)** red with A and B, white with C
(d) red with A, white with B and C **(e)** at least one red face?

16 A coin is tossed four times. What is the probability of:

(a) 4 heads **(b)** 4 tails **(c)** head, tail, head, tail (in that order)
(d) heads in the first 3 tosses but not in the fourth **(e)** heads in any one of the 4 tosses?

17 A die is tossed three times. What is the probability of:

(a) 3 sixes **(b)** 0 sixes **(c)** 3 odd numbers **(d)** 3 even numbers
(e) a six in the first two tosses only **(f)** a six, not a six, a six (in that order)?

18 Trinh and Oscar play three tennis matches. Trinh's chance of winning any one match is $\frac{2}{3}$. What is the probability that Trinh:

(a) wins all three matches **(b)** loses all three matches
(c) wins the first and third but loses the second **(d)** loses the first and wins the other two?

19 A container holds 2 blue, 3 black and 5 white balls. A ball is withdrawn and then replaced in the container. This is repeated three times. Find the probability of each event:

(a) 3 white balls **(b)** a black ball in the first two drawings, but not in the third
(c) white, black, blue (in that order) **(d)** white, black, white (in that order)
(e) not more than two white balls **(f)** a white or a black ball each time.

20 A survey finds that, on average, 2 out of every 3 people interviewed are in favour of a certain proposal. If a random group of 3 people are interviewed, what is the probability that:

(a) all will be in favour of the proposal **(b)** none will be in favour
(c) the first and third people interviewed will be in favour, but not the second?

21 A student estimates that the chances of passing English, Mathematics and Physics are respectively $\frac{9}{10}$, $\frac{4}{5}$ and $\frac{3}{4}$. Estimate the probability of the events:

(a) passes English only **(b)** passes English and Mathematics only
(c) passes all three subjects **(d)** fails every subject.

22 The probability of a disease being cured with drug A is 0.8 and with drug B is 0.6. A randomly selected patient with the disease is treated with drug A and another is treated with drug B. What is the probability that:

(a) neither patient is cured **(b)** both patients are cured?

23 One container holds 2 red cubes and 4 blue cubes and a second container holds 4 red cubes and 3 blue cubes. One cube is selected at random from each of the two containers. What is the probability that one of the cubes is red?

24 The probability that a certain woman will be alive in 20 years is $\frac{2}{3}$ and the probability that a certain man will be alive is $\frac{3}{5}$. What is the probability that in 20 years' time:

(a) both will be alive **(b)** only one will be alive **(c)** at least one will be alive?

25 A man finds that he is late for work on 10% of occasions if he was on time the previous day, but late on 20% of occasions if he was late the previous day. Given that he was on time on Monday and worked on Tuesday, what is the probability that he is on time on Wednesday? Illustrate using a tree diagram.

26 A certain factory has three machines A, B and C, which manufacture 25%, 35% and 40% respectively of the factory's products. Of the machines' products, 5%, 4% and 2% respectively are defective. A product from the machines is selected at random. What is the probability that:

(a) it was manufactured by A and is defective **(b)** it was manufactured by B and is not defective?

27 There are three containers A, B and C. A contains 3 black and 2 white cubes; B contains 3 black and 1 white cube; C contains 3 black and 3 white cubes.

(a) A container is chosen at random and from it a cube is chosen at random. Illustrate this two-stage process with a probability tree diagram.

(b) What is the probability that the cube is black?

28 On average, a student misses her bus to school once every eight weeks (where there are five days per school week). Find the probability that she:

(a) does not miss the bus on any one morning **(b)** catches the bus on two successive mornings

(c) catches the bus each morning for a week (5 days) **(d)** misses the bus on at least one morning in a week.

29 To open a locked safe requires a correct 3-digit combination. If a combination is chosen at random, calculate the probability of:

(a) succeeding at the first attempt **(b)** failing at the first attempt.

30 The first race at Randwick has 13 horses running and the second race has 16 horses. Assuming that all horses have an equal chance of winning, calculate the probability of predicting:

(a) a double (i.e. a winner in each race)

(b) a quinella (i.e. the two horses that come first and second, in either order) in the first race

(c) a quinella in the second race **(d)** a quinella in both races.

31 To gain a driver licence in NSW, you must pass both a touch-screen computer-based test and a practical driving test. Statistics show that 70% of learners pass the computer-based test on the first attempt; of those who fail, 90% pass on the second attempt. Also, 60% of learners pass their first practical test and 80% pass their second practical test. The computer-based and practical tests are independent. Calculate the probability of:

(a) passing the computer-based test on the second attempt

(b) passing the computer-based test after no more than two attempts

(c) needing to take a third computer-based test

(d) passing the practical test on the second attempt

(e) receiving a licence after taking one practical test and two computer-based tests.

18.6 DEPENDENT EVENTS

Example 17

Two dice are rolled. A is the event '5 with the first die' and B is the event 'sum of the numbers on the two dice exceeds 10'.

(a) Find $P(A \text{ and } B)$ (also written $P(AB)$).

(b) Are A and B independent?

(c) Find the probability that the sum of the numbers exceeds 10, given that the first die rolls a 5.

Solution

(a) Rolling two dice has 36 possible outcomes.

$A = \{(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6)\}$, so $P(A) = \frac{6}{36} = \frac{1}{6}$

$B = \{(5, 6), (6, 5), (6, 6)\}$, so $P(B) = \frac{3}{36} = \frac{1}{12}$

A and $B = A \cap B = \{5, 6)\}$, so $P(A \text{ and } B) = \frac{1}{36}$

Note that in this case $P(A \text{ and } B) \neq P(A) \times P(B)$.

(b) Because $P(A \text{ and } B) \neq P(A) \times P(B)$, events A and B are not independent. We say they are **dependent**. When two events have common members, they are not independent.

(c) We are asked to find the probability of event B given that event A has happened. This can be written as $P(B \mid A)$.

Because A has occurred, the only possible events in this case are (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6). Of these 6 outcomes, only one is favourable: (5, 6).

Hence $P(B \mid A) = \frac{1}{36}$

For dependent events:

$$P(AB) = P(A) \times P(B \mid A)$$

where $P(B \mid A)$ is the probability of B occurring given that A has already occurred.

Sampling without replacement from a small population

Example 18

From a set of 5 cards numbered 1, 2, 3, 4, 5, two cards are selected at random **without replacement** (i.e. without putting the first card back before taking the second). What is the probability that both cards are odd-numbered cards?

Solution

Let A be the event 'odd number first card' and B the event 'odd number second card'. We must find $P(A \text{ and } B)$. First: $P(A) = \frac{3}{5}$

If the first card is odd, then this leaves 2 odd cards in the remaining 4 cards, hence: $P(B) = \frac{2}{4}$

Thus: $P(A \text{ and } B) = \frac{3}{5} \times \frac{2}{4} = \frac{3}{10}$

It may have been easier to say $P(\text{odd, odd}) = \frac{3}{5} \times \frac{2}{4} = \frac{3}{10}$.

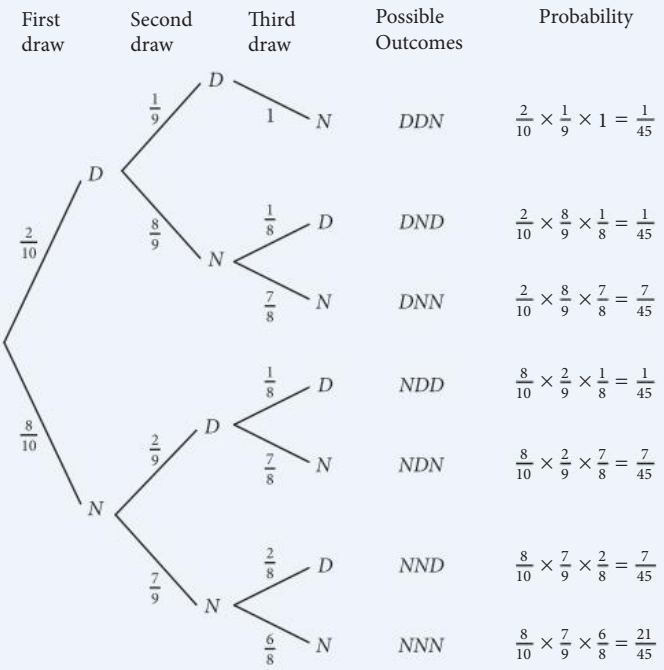
Example 19

A carton contains 10 batteries, 2 of which are defective. A sample of 3 batteries is drawn at random from the carton. Find the probability that not more than one battery is defective, if the sampling is done:

(a) without replacement **(b)** with replacement.

Solution

(a) Let D denote 'defective' and N 'non-defective'. We can represent the outcomes using a tree diagram:

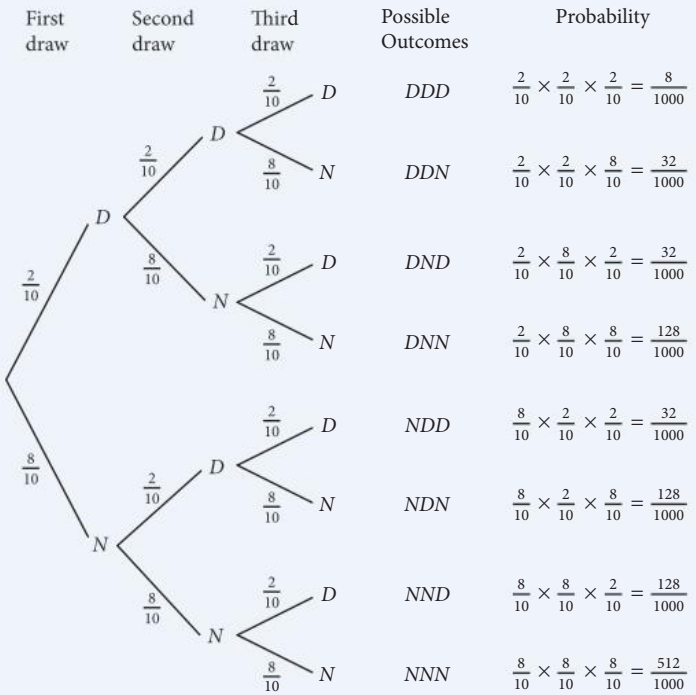


There is no branch for DDD because there any only 2 defective batteries in the carton.

The statement 'not more than 1 defective' is the same as '0 defective or 1 defective'.

$$\therefore \text{Required probability} = P(DNN) + P(NDN) + P(NND) + P(NNN)$$
$$= \frac{7}{45} + \frac{7}{45} + \frac{7}{45} + \frac{21}{45}$$
$$= \frac{42}{45} = \frac{14}{15}$$

(b) Because each battery is replaced in the carton before the next withdrawal is made, the probability of a defective battery being withdrawn remains the same each time: $\frac{2}{10}$ for each withdrawal. We can again represent the outcomes using a tree diagram (note the difference compared to part **(a)**):



$$\therefore \text{Required probability} = P(DNN) + P(NDN) + P(NND) + P(NNN)$$
$$= \frac{128}{1000} + \frac{128}{1000} + \frac{128}{1000} + \frac{512}{1000}$$
$$= 0.896$$

Alternatively:

$$\begin{aligned}\text{Required probability} &= 1 - P(2 \text{ defectives} + 3 \text{ defectives}) \\ &= 1 - [P(DDN) + P(DND) + P(NDD) + P(DDD)] \\ &= 1 - \left[\tfrac{32}{1000} + \tfrac{32}{1000} + \tfrac{32}{1000} + \tfrac{8}{1000}\right] \\ &= 0.896\end{aligned}$$

Sampling without replacement from a large population

Example 20

A shoe manufacturer makes two types of shoes: sneakers and boots. Of the shoes at the factory, 60% are sneakers. If a random sample of 3 pairs of shoes is taken from the factory, find the probability that the sample is such that:

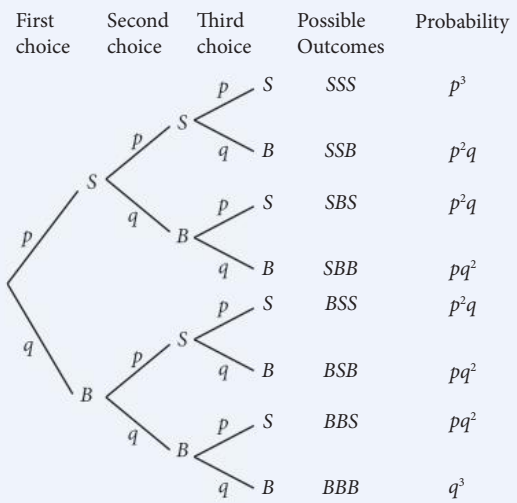
(a) there are exactly 2 pairs of sneakers

(b) there is at least 1 pair of boots

(c) the first two pairs chosen are sneakers and the third pair are boots.

Solution

Let $p = 0.6$ and $q = 0.4$ be the respective probabilities of choosing a pair of sneakers and a pair of boots.



(a) $$\begin{aligned}\text{Required probability} &= P(SSB) + P(SBS) + P(BSS) \\ &= 3p^2q \\ &= 3 \times 0.6^2 \times 0.4 \\ &= 0.432\end{aligned}$$

(b) $$\begin{aligned}\text{Required probability} &= 1 - P(0 \text{ boots}) \\ &= 1 - P(SSS) \\ &= 1 - p^3 \\ &= 1 - 0.6^3 \\ &= 0.784\end{aligned}$$

(c) $$\begin{aligned}\text{Required probability} &= P(SSB) \\ &= p^2q \\ &= 0.6^2 \times 0.4 \\ &= 0.144\end{aligned}$$

EXERCISE 18.6 DEPENDENT EVENTS

1 A box contains 5 black cubes and 3 red cubes. Two cubes are drawn at random from the box. Find the probability that:

(a) both cubes are black **(b)** both cubes are the same colour

(c) both cubes are different colours.

2 A box contains 6 green balls and 4 white balls. A batch of two balls is drawn at random from the box. The probability that the two balls are the same colour is:

A $\frac{1}{3}$ **B** $\frac{7}{15}$ **C** $\frac{2}{15}$ **D** $\frac{2}{45}$

3 An angler has caught 15 fish, of which 3 are undersized. A random sample of 3 fish is taken without replacement by an inspector. The angler is fined if one or more of the fish in the sample is undersized. What is the probability that the angler is fined?

4 A carton contains a dozen eggs, 3 of which have a double yolk. If 3 eggs are taken to make a cake, find the probability that all 3 eggs will have double yolks.

5 In a group of 9 people, 3 have brown hair and 6 have black hair. A random sample of 2 people is selected from the group. What is the probability that:

(a) the first person selected has black hair **(b)** both people selected have black hair
(c) one person has brown hair and the other has black hair?

6 From a standard pack of 52 playing cards, two cards are selected at random without replacement. Indicate whether each statement below is correct or incorrect.

(a) $P(\text{both cards are diamonds}) = \frac{1}{17}$ **(b)** $P(\text{both cards are the same suit}) = \frac{1}{17}$
(c) $P(\text{one card is a spade and the other is a club}) = \frac{13}{102}$ **(d)** $P(\text{both cards are different suits}) = \frac{13}{17}$

7 A container holds 3 white balls, 4 red balls and 5 black balls. Two balls are drawn at random without replacement. What is the probability that the balls are:

(a) both white **(b)** both red **(c)** both black?

8 From 7 teachers and 5 pupils, a random selection of 2 people is made. What is the probability that:

(a) they are both teachers **(b)** they are both pupils **(c)** one is a teacher and the other is a pupil?
(d) How can the answer to **(c)** be deduced from the answers to **(a)** and **(b)**?

9 From the letters of the word 'PROMISE', three letters are chosen at random, one at a time. What is the probability that the three letters are:

(a) vowels **(b)** consonants **(c)** vowel, consonant, vowel (in that order)?

10 A punter correctly picked first and second in a race of 10 horses. What is the probability of this, if all the horses were equally likely to win?

11 From a group of 5 boys and 6 girls, two are selected at random for a class committee. What is the probability that a boy and a girl are selected?

12 A carton contains 10 electric lights, 3 of which are defective. Two are drawn at random. What is the probability that:

(a) the first drawn is defective **(b)** both are defective
(c) neither is defective **(d)** exactly one is defective?

13 In a raffle, 20 tickets are sold and there are 2 prizes. If I buy 5 tickets, what is the probability that I win at least one of the prizes?

14 Group A contains 10 gorillas and 5 chimpanzees. Group B contains 4 gorillas and 6 chimpanzees. Two apes are selected at random from the groups. What is the probability that:

(a) they are both gorillas, if they are selected from Group A
(b) they are both chimpanzees, if they are selected from Group B
(c) one is a gorilla, the other is a chimpanzee, if one is selected from each group?

15 A box of 10 chocolates contains 4 hard-centred and 6 soft-centred chocolates. If two chocolates are selected at random, what is the probability that:

(a) they both have hard centres **(b)** they both have soft centres
(c) one has a soft centre and the other has a hard centre?

16 In a lottery game, three numbers are selected at random from 1, 2, 3, 4, … 40. Find the probability that the three selected numbers are all even.

17 A sample of three items is selected at random without replacement from a batch of ten items, four of which are defective. Find the probability that there is at most one defective item in the sample.

18 In a race with nine runners, if every possible order of finishing is equally likely, find the probability of picking the runners who come first, second and third in the correct order.

19 Three cards are drawn at random from a standard pack of 52 playing cards. What is the probability that they are all from the same suit?

20 A bag contains 6 red balls and 4 white balls. A random sample of 3 is withdrawn. What is the probability that the balls are the same colour, if the sampling is done:

(a) without replacement **(b)** with replacement?

21 Three cards are drawn at random from a standard pack of 52 playing cards. What is the probability that 3 aces are drawn, if the drawing is done:

(a) without replacement **(b)** with replacement?

22 A bag contains a large number of five-cent coins and ten-cent coins in a ratio of 2 to 3. If three coins are randomly selected from the bag, find the probability that:

(a) two of them are five-cent coins **(b)** at least two of them are five-cent coins
(c) not more than two of them are five-cent coins.

23 A manufacturer finds that 10% of the products made in a factory are defective. If three products are taken at random, what is the probability that:

(a) all are defective **(b)** none are defective **(c)** more are defective than are non-defective?

24 A hand of three cards dealt from a standard pack of 52 playing cards contains the ace of clubs. What is the probability of this happening?

25 In a raffle, 30 tickets are sold and there are two prizes. What is the probability that a person who buys five tickets wins:

(a) neither prize **(b)** both prizes **(c)** at least one prize?

26 From a set of 10 cards numbered 1 to 10, two cards are drawn at random without replacement. What is the probability that:

(a) both numbers are even **(b)** one is even and the other is odd
(c) the sum of the two numbers is 12?

27 Three cards are taken without replacement from a standard pack of 52 playing cards. What is the probability of taking:

(a) exactly 3 hearts **(b)** exactly 3 aces **(c)** at least 1 heart?

28 A manufacturer of metal pistons finds that on average 20% of pistons are rejected because they are either oversize or undersize. What is the probability that a batch of three pistons will contain:

(a) no more than two rejects **(b)** at least two rejects?

29 It is known that 7 out of 10 students from a certain school will go on to university. If a group of 3 students is chosen at random from this school, find the probability that:

(a) all will go on to university **(b)** some will go on to university.

30 An archer finds that her ratio of success to failure in hitting a bullseye is 9 to 1. If 3 arrows are shot, what is the probability of:

(a) 3 successes **(b)** at least 2 successes **(c)** not more than 1 success?

31 In an assortment of bananas, the ratio of ripe bananas to unripe bananas is 3 to 5. If 3 bananas are chosen at random, find the probability that:

(a) exactly 2 will be ripe **(b)** at least 1 will be unripe.

32 In an opinion poll, the ratio of those in favour to those against a particular proposal was 7 to 3. If three randomly chosen people are interviewed about the proposal, what is the probability that:

(a) all will be in favour **(b)** the majority will be in favour

(c) not more than two will be against the proposal?

33 On average, at a particular beach it rains on 2 days out of every 7. Find the probability that on a given weekend it will rain:

(a) on both days **(b)** on at least one day.

34 A container holds a number of cubes: 60% are white and the remainder are black. Two cubes are randomly selected without replacement. What is the probability that:

(a) they are the same colour **(b)** they are different colours?

35 A container holds 10 cubes: 6 are white and the remainder are black. Two cubes are randomly selected without replacement. What is the probability that:

(a) they are the same colour **(b)** they are different colours?

36 In a group of 10 birds, 5 are red and 5 are green. If two birds are selected at random, what is the probability that:

(a) they are both red **(b)** at least one is green?

37 In a large flock of birds, 50% are red and 50% are green. If two birds are selected at random, what is the probability that:

(a) they are both red **(b)** at least one is green?

38 Container X holds 1 white cube and 2 black cubes. Container Y holds 2 white cubes and 1 black cube. A container is selected at random and from it two cubes are selected without replacement. Draw a tree diagram to represent this three-stage process and find the probability that both cubes drawn are:

(a) the same colour **(b)** different colours.

39 A and B play a set of tennis. When a player wins two games, the set is won. If A has probability 0.6 of winning any one game, what is the probability that A wins the set? Construct a tree diagram.

40 Of two coins A and B, A is a fair coin while B is loaded with a probability of 0.6 for heads. A coin is chosen at random and tossed twice. What is the probability of tossing two heads?

CHAPTER REVIEW 18

1 The combination lock on a safe has three concentric circular discs, each showing the digits 0 to 9. Only one combination of digits will open the safe. What is the probability of opening the safe at my first attempt if I do not know the combination?

2 One student has a pencil whose cross-section is square and whose faces are coloured black, white, green and red. Another student has a pencil whose cross-section is hexagonal (six-sided) and whose faces are coloured black, white, green, red, yellow and orange. Both pencils are rolled on a flat surface and the colours appearing uppermost are noted. Find the probability of each event:

(a) both colours uppermost are the same
(b) both colours uppermost are different
(c) black is uppermost on at least one of the pencils
(d) neither black nor white appears uppermost.

3 Four names beginning with B and five names beginning with G are in a hat. If two names are randomly drawn without replacement, find the probability that:

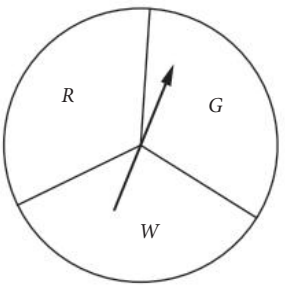
(a) two G names are drawn out **(b)** at least one B name is drawn out.

4 A jar contains red buttons and white buttons in a ratio of 3 : 2. If three buttons are chosen at random from the jar, find the probability that:

(a) exactly two are red **(b)** not more than one is white.

5 A box contains four balls marked 1, 2, 3, 4. A spinner has an equal chance of spinning *R*, *G* or *W*. A ball is selected at random from the box and the spinner is spun. Illustrate on a tree diagram the 12 possible outcomes.
Find the probability of each outcome:

(a) *R* and an odd number
(b) *W* and a number greater than 2, or *R* and an even number.



6 In a group of 25 students, 18 study French, 12 study German and 5 study neither French nor German. If a student is chosen at random, what is the probability that the student studies:

(a) French only **(b)** German only
(c) French or German or both **(d)** both French and German?

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} \quad n \neq -1;\ x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

SUMMARY—PRELIMINARY

1 ARITHMETIC AND SURDS

Rational and irrational numbers

- Rational numbers can be expressed in the form $\frac{a}{b}$ where a and b are integers and $b \neq 0$. They include positive and negative integers, fractions, terminating decimals and recurring decimals.
- Irrational numbers are numbers that are not rational, like $\sqrt{2}, \sqrt{15}, 2\sqrt{7}$ and π. They cannot be represented as a fraction. Irrational numbers such as $\sqrt{2}, \sqrt{15}, 2\sqrt{7}$ (i.e. any square root or higher root) are called surds; π is called a transcendental number (i.e. it cannot be solved algebraically).

Basic rules for surds

If a and b are positive numbers then:

(a) $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$

(b) $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

(c) $\sqrt{a^2} = a$

(d) $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$

(e) $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$

(f) $\sqrt{a} \times \sqrt{a} = a$

(g) $\sqrt{a} \div \sqrt{b} = \frac{\sqrt{a}}{\sqrt{b}} = \frac{\sqrt{a} \times \sqrt{b}}{\sqrt{b} \times \sqrt{b}} = \frac{\sqrt{ab}}{b}$ to write the expression with a rational denominator.

Adding and subtracting surds

- Surds of the same kind can be added and subtracted just like pronumerals, using the distributive law to collect like terms:

  $ab + ac = a(b + c)$

- You can only add and subtract like algebraic terms, such as $2a + 3a - a = 4a$. Similarly, you can only add and subtract like surds, such as $2\sqrt{2} + 3\sqrt{2} = 5\sqrt{2}$. It may be necessary to first simplify the surd terms by removing perfect square factors.

Rationalising surds

- The expressions $\sqrt{a} \times \sqrt{a}$ and $\left(\sqrt{a} - \sqrt{b}\right)\left(\sqrt{a} + \sqrt{b}\right)$ both have rational answers (that is, answers that do not involve surds): $\sqrt{a} \times \sqrt{a} = a$ and $\left(\sqrt{a} - \sqrt{b}\right)\left(\sqrt{a} + \sqrt{b}\right) = \left(\sqrt{a}\right)^2 - \left(\sqrt{b}\right)^2 = a - b$.
- The expansion $\left(\sqrt{a} - \sqrt{b}\right)\left(\sqrt{a} + \sqrt{b}\right) = a - b$ is known as the 'difference of two squares'.
- When you have a surd expression in the denominator of a fraction, it is normal to make the denominator into a rational number. This process is called **rationalising the denominator**.
- The expressions $\sqrt{a} - \sqrt{b}$ and $\sqrt{a} + \sqrt{b}$ are called conjugate surds, with each expression being the conjugate of the other. To convert any surd into a rational number, multiply the surd by its conjugate.

Scientific notation (standard form)

- A number in scientific notation is expressed as a product of a number (between 1 and 10) and a power of 10, e.g. 2.4×10^{-7}.

  The power of 10 is determined by the change in the place value of the digits. This is the number of digits that must be moved across the decimal point to obtain the new number (between 1 and 10) from the original number.

Significant figures (s.f.)

- Significant figures are the non-zero digits used in a number. Zeros are only considered significant if they occur between non-zero digits or if they represent greater precision in an answer. For example: 2874 is correct to 4 s.f.; 301 is 3 s.f.; 30 100 is 3 s.f.; 30.10 is 4 s.f.
- To write an answer correct to (e.g.) 3 s.f., look at the next (fourth) digit and round it off: if is less than 5, replace it and all following digits with zeros; if it is 5 or more, increase the third digit by one and replace all the following digits with zeros.

2 ALGEBRA

Common terms

- A **monomial** is an expression that contains only **one** term, e.g. $5x$, x^2, $2ab$, $5a^2b^3$.
- A **binomial** is an expression that contains **two** terms added or subtracted, e.g. $x + y$, $3a - 2b$, $x^2 + 1$, $3y - 4$.
- A **trinomial** is an expression that contains **three** terms added or subtracted, e.g. $x^2 - 5x + 6$, $x + y - 4$, $4x^2 - 2xy + y^2$, $m + n - p$.
- A **quadratic trinomial** is a trinomial of the form $ax^2 + bx + c$ ($a \neq 0, b \neq 0, c \neq 0$). a is the coefficient of x^2, b is the coefficient of x, and c is the constant term.

Standard results for basic polynomials

$(x+m)(x+n) = x^2 + (m+n)x + mn$

$(a+b)^2 = a^2 + 2ab + b^2$

$(a-b)^2 = a^2 - 2ab + b^2$

$(a-b)(a+b) = a^2 - b^2$

Factorising using the difference of two squares

$a^2 - b^2 = (a-b)(a+b)$

Factorising quadratic trinomials

To factorise quadratic trinomials, you must remember how to expand binomial products and then work backwards. We know the following:

- $(x+m)(x+n) = x^2 + (m+n)x + mn =$ $x^2 +$ (sum of m and n)$x +$ (the product of m and n)
- $(x-m)(x-n) = x^2 - (m+n)x + mn =$ $x^2 +$ (sum of $-m$ and $-n$)$x +$ (the product of $-m$ and $-n$)
- $(x+m)(x-n) = x^2 + (m-n)x - mn =$ $x^2 +$ (sum of m and $-n$)$x +$ (the product of m and $-n$)

To factorise $x^2 + 5x + 6$ you must write it in the form $(x+m)(x+n)$, where $m+n = 5$ and $mn = 6$. This means you must find two numbers whose sum is 5 and whose product is 6.

Mixed factorisations

To factorise quadratic expressions, you may need to use **one or more** of the following techniques:

- remove a common factor
- group in pairs and remove common factors
- difference of two squares
- quadratic trinomials.

Sum and difference of two cubes

$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

Algebraic fractions

To **simplify** algebraic fractions:

1 Factorise the numerator and denominator.

2 Cancel any common factors.

Adding and subtracting algebraic fractions

- When adding and subtracting fractions, first rewrite each fraction with the same (common) denominator, then add or subtract the numerators.
- More complex algebraic fractions require you to factorise the denominator before you find the common denominator. Write each fraction with the common denominator before you add or subtract the numerators.

3 EQUATIONS AND INEQUALITIES

Linear equations in one variable

- To **solve** a linear equation in one variable is to find the value of the variable that will make the equation true.
- For equations involving **fractions**, write each term with the common denominator and then solve the equation given by the numerators.

  e.g. $\frac{x}{3} = \frac{4}{5}$ becomes $\frac{5x}{15} = \frac{12}{15}$ and you can solve $5x = 12$.

  If the equation has only one term on each side of the equals sign, you can multiply each numerator by the other term's denominator to achieve the same result.

Quadratic equations

- An equation of the form $ax^2 + bx + c = 0$, $a \neq 0$ is called a quadratic equation in x. The values of x that make this equation true are called the solutions or the roots of the equation.
- A quadratic equation can be solved by factorising the quadratic expression and making the factored expression equal to zero: if $AB = 0$, then $A = 0$ or $B = 0$ or $A = B = 0$.
- The equation $ax^2 + bx + c = 0$, $a \neq 0$ is the general quadratic equation. Its roots are: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Rules for inequalities

If both sides of an inequality are multiplied or divided by a negative number, then the direction of the inequality is reversed.

If $a > b$, then:		If $a < b$, then:	
$a + c > b + c$		$a + c < b + c$	
$a - c > b - c$		$a - c < b - c$	
$ac > bc$	if $c > 0$	$ac < bc$	if $c > 0$
$ac < bc$	if $c < 0$	$ac > bc$	if $c < 0$
$\frac{a}{c} > \frac{b}{c}$	if $c > 0$	$\frac{a}{c} < \frac{b}{c}$	if $c > 0$
$\frac{a}{c} < \frac{b}{c}$	if $c < 0$	$\frac{a}{c} > \frac{b}{c}$	if $c < 0$

On number lines:

- $a > b$ means that a is to the right of b.
- $a < b$ means that a is to the left of b.
- $x \geq 2$ is shown by a solid circle over 2 and an arrow to the right
- $x > 2$ is shown by an empty circle over 2 and the arrow to the right.

Square roots

- If $a \geq 0$, then $\sqrt{a}$ is a non-negative number such that $\left(\sqrt{a}\right)^2 = a$. $\sqrt{a}$ is called the positive square root of a. The value of $\sqrt{a}$ will always be either positive or zero (non-negative).
- An equation like $x^2 = 9$ always has two solutions: the positive and negative square roots of 9.
  $x = \sqrt{9} = 3$ and $x = -\sqrt{9} = -3$.
- Thus we have:

$$\begin{aligned}\sqrt{x^2} &= x \quad \text{if } x > 0\\ &= -x \quad \text{if } x < 0\\ &= 0 \quad \text{if } x = 0\end{aligned}$$

  Here $-x$ means the opposite sign of x, hence $-x > 0$.

Absolute value

- The absolute value (also called 'modulus') of a real number x is written $|x|$. It is the non-negative number that defines the magnitude of the given number.
- Thus we have:

$$\begin{aligned}|x| &= x \quad \text{if } x > 0\\ &= -x \quad \text{if } x < 0\\ &= 0 \quad \text{if } x = 0\end{aligned}$$

  This is identical to $\sqrt{x^2}$, so it leads to another definition: $|x| = \sqrt{x^2}$
- Because x is a real number, it can be represented by a point on the number line, and $|x|$ is the distance of the point x from the origin. Distance is always positive, so $|x| > 0$ for all $x \neq 0$.
- For example: the expression $|x| = 2$ describes a distance of 2 from the origin. The two points that are 2 units distant from 0 are $x = 0 + 2 = 2$ and $x = 0 - 2 = -2$.
- In general:

$$\begin{aligned}|x - y| &= x - y \quad \text{if } x > y\\ &= y - x \quad \text{if } x < y\\ &= 0 \quad \text{if } x = y\end{aligned}$$

  Thus e.g. $|5 - 3| = |3 - 5| = 2$.

Absolute value—important results

1 $|xy| = |x| \times |y|$

2 $|x + y| \leq |x| + |y|$ (the **triangle inequality**) and $|x + y| = |x| + |y|$ if and only if x and y are either zero or have the same sign.

Algebraic denominators

- A fraction cannot have zero as a denominator. If we write a fraction with an algebraic expression as the denominator, then that fraction is undefined when that algebraic expression is zero.
  e.g. $\frac{1}{x}$ is undefined when $x = 0$; the expression is defined for all x, $x \neq 0$.
  e.g. $\frac{x-1}{x(x-1)}$ is undefined for $x = 0$ or 1; the expression is defined for all x, $x \neq 0, 1$.
- When an expression is undefined for some values, you must write the restrictions in the answer.

Simultaneous equations

The most common algebraic methods of solving a simultaneous pair of linear equations in two variables are (a) elimination and (b) substitution.

(a) Solving simultaneous equations by elimination:

- If the coefficients have the same size but opposite sign, eliminate by addition.
- If the coefficients are equal, eliminate by subtraction.

(b) Solving simultaneous equations by substitution:

- Rewrite one of the equations to make it equal to one of the variables, then substitute this into the other equation.

Solving simultaneous equations where one is not linear

- Equations like $y = x^2 - 5x + 6$ (a parabola when graphed), $x^2 + y^2 = 4$ (a circle when graphed) and $xy = 6$ (a rectangular hyperbola when graphed) may be intersected by a straight line. To find the intersection points, solve the pair of simultaneous equations.
- To solve non-linear simultaneous equations, the substitution method is usually best. You might also find the solutions graphically (by hand or by using GeoGebra or other graphing software).

4 PLANE GEOMETRY

Angles

- Two angles are **complementary** if their sum is 90°.
- Two angles are **supplementary** if their sum is 180°.
- A pair of **adjacent** angles have a common vertex and a common arm.
- The angles around a point add up to 360°.
- When two straight lines intersect, the vertically opposite angles are equal.

Transversals and parallel lines

A **transversal** is a straight line drawn across two or more other straight lines.

When a transversal cuts a pair of **parallel lines**, the pairs of:

(a) alternate angles are equal

(b) corresponding angles are equal

(c) cointerior angles are supplementary.

Tests for parallel lines

If a transversal cuts two other straight lines and makes:

(a) a pair of alternate angles equal, **or**

(b) a pair of corresponding angles equal, **or**

(c) a pair of cointerior angles supplementary, then the two straight lines are **parallel**.

Transitivity of parallel lines

- If $AB \parallel CD$ and $CD \parallel KL$ then $AB \parallel KL$.

Angle properties of triangles

- The sum of the angles of a triangle is 180°.
- If one side of a triangle is produced, the exterior angle formed is equal to the sum of the two remote interior angles (also known as 'interior opposite angles').
- An exterior angle of a triangle is equal to the sum of the two remote interior angles.

Types of triangles

- An **equilateral** triangle has all three sides equal in length.

  Each angle is 60°.
- An **isosceles** triangle has two sides equal in length. The angles opposite the equal sides are equal. (See more *Isosceles triangle properties* below, page 461.)
- An **acute-angled** triangle has each of its angles less than 90°.
- A **right-angled** triangle has one angle 90°.
- An **obtuse-angled** triangle has one angle greater than 90°.

Polygons

- A polygon is a plane figure bounded by any number of straight lines (at least three).
- A polygon that has all sides equal and all angles equal is called a **regular polygon**. An equilateral triangle and a square are both regular polygons.
- A polygon in which each angle is less than 180° is called a **convex polygon**.
- A polygon in which at least one angle is more than 180° is called a **concave polygon**.
- The sum of the angles of a polygon with n sides is $(2n - 4)$ right angles.

  The size of each angle in a regular polygon is $\frac{2n-4}{n}$ right angles.
- If the sides of a convex polygon are produced in order, the sum of the exterior angles formed is 4 right angles.

Quadrilaterals

- A quadrilateral is a plane closed figure bounded by four straight lines.
- The sum of the four angles of any quadrilateral is 360°, or four right angles.

A **parallelogram** is a quadrilateral with both pairs of opposite sides parallel. Properties:

(a) pairs of opposite sides equal

(b) pairs of opposite angles equal

(c) diagonals bisect each other

A **rectangle** is a parallelogram with one angle a right angle. Properties:

(a) pairs of opposite sides equal

(b) every angle is 90°

(c) diagonals bisect each other and are equal

A **rhombus** is a parallelogram with a pair of adjacent sides equal. Properties:

(a) all sides equal

(b) pairs of opposite angles equal

(c) diagonals bisect each other at right angles

(d) diagonals bisect the angles of the rhombus

A **square** is a rectangle with a pair of adjacent sides equal; also, a rhombus with one angle a right angle. Properties:

(a) all sides equal

(b) each angle is 90°

(c) diagonals bisect each other and are equal

(d) diagonals bisect the angles of the square

A **trapezium** is a quadrilateral with one pair of opposite sides parallel.

A **kite** is a quadrilateral with both pairs of adjacent sides equal. Properties:

(a) one pair of opposite angles are equal (between the unequal sides)

(b) an axis of symmetry

Tests for parallelograms

- If the pairs of opposite sides of a quadrilateral are equal, then the quadrilateral is a parallelogram.
- If the pairs of opposite angles of a quadrilateral are equal, then the quadrilateral is a parallelogram.
- If a quadrilateral has one pair of sides that are equal and parallel, then the quadrilateral is a parallelogram.
- If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.

Tests for rhombuses

- If the four sides of a quadrilateral are equal, then the quadrilateral is a rhombus.
- If the diagonals of a quadrilateral bisect each other at right angles, then the quadrilateral is a rhombus.

Congruent shapes

Two plane figures are **congruent** if they are equal in all respects. If one figure is placed on top of the other, with corresponding points touching, they fit perfectly.

Tests for congruent triangles

- Two triangles are congruent if two sides and the included angle of one triangle are respectively equal to two sides and the included angle of the other triangle. (**SAS**)
- Two triangles are congruent if the three sides of one triangle are respectively equal to the three sides of the other triangle. (**SSS**)
- Two triangles are congruent if two angles and a side of one triangle are respectively equal to two angles and the corresponding side of the other triangle. (**AAS**)
- Two triangles are congruent if the hypotenuse and one side of a right-angled triangle are equal to the hypotenuse and the corresponding side of the other right-angled triangle. (**RHS**)

Non-congruent triangles

- Triangles with three pairs of angles equal are equiangular (or 'similar'), but are not congruent unless the sides are equal too.
- Triangles with two sides and an angle equal are not necessarily congruent, if the angle is opposite one of the sides instead of included between the two sides. This is called the **ambiguous** case, because it is sometimes possible to draw two different triangles with these properties.

Isosceles triangle properties

- The angles opposite the equal sides of an isosceles triangle are equal. (Or, the base angles of an isosceles triangle are equal.)
- The line joining the midpoint of the base of an isosceles triangle to the opposite vertex is perpendicular to the base.
- The line drawn from the vertex of an isosceles triangle perpendicular to the base bisects the base.
- The line joining the midpoint of the base of an isosceles triangle to the opposite vertex bisects the angle at that vertex.

Similar shapes

Two plane figures are **similar** if they have exactly the same shape but are not the same size. Thus, similar figures are equiangular (have the same angles) and their pairs of matching sides are in the same ratios (proportional).

Tests for similar triangles

- Two triangles are similar if the three angles of one triangle are respectively equal to the three angles of the other triangle. (**AAA**)
- Two triangles are similar if the three pairs of corresponding sides are proportional.
- Two triangles are similar if two pairs of corresponding sides are proportional and the angles included by these sides are equal.

Intercept properties of parallel lines

- If three (or more) parallel lines cut equal intercepts on a transversal, then the same parallel lines must cut equal intercepts on any other transversal.
- A straight line drawn through the midpoint of one side of a triangle, parallel to a second side, bisects the third side.
- A line through a triangle, parallel to one side, divides the other two sides proportionally.
- When three (or more) parallel lines cut two transversals, the intercepts cut on one transversal are proportional to the intercepts cut on the other transversal.

Pythagoras' theorem

The square of the hypotenuse of a right-angled triangle is equal to the sum of the squares of the other two sides.

Converse of Pythagoras' theorem

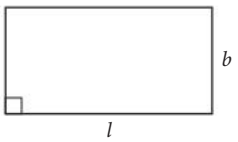
If the square of a side of a triangle is equal to the sum of the squares of the other two sides, the angle contained by these sides must be a right angle.

Area formulae

Area of a **rectangle** $= lb$

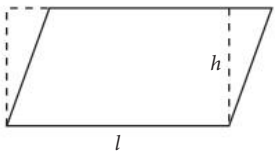
$=$ length $\times$ breadth

The term 'width' is often used instead of 'breadth'.



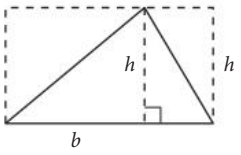
Area of **parallelogram** $= lh$

$=$ length of base $\times$ perpendicular height



Area of **triangle** $= \frac{1}{2}bh$

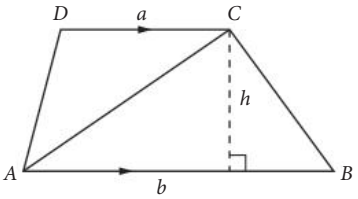
$= \frac{1}{2}$ base $\times$ perpendicular height



Area of **trapezium** $= \frac{1}{2}(a+b)h$

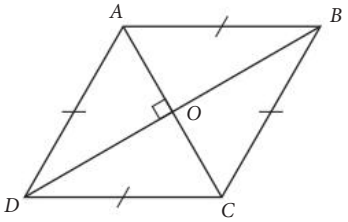
$= \frac{1}{2}$ sum of parallel sides $\times$ perpendicular height

$= \frac{1}{2}$ sum of parallel sides $\times$ distance between them



Area of **rhombus** $= \frac{1}{2}DB \times AC$

$= \frac{1}{2}$ the product of the diagonals' lengths



Units for area

- $10\,000\,\text{cm}^2 = 100\,\text{cm} \times 100\,\text{cm} = 1\,\text{m}^2$
- $10\,000\,\text{m}^2 = 100\,\text{m} \times 100\,\text{m} = 1\,\text{ha}$ (hectare)

  A hectare is an amount of area equivalent to a square of side length 100 m.

5 TRIGONOMETRIC RATIOS AND APPLICATIONS

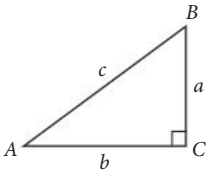
Trigonometric ratio definitions

These trigonometric ratios describe the relationship between the angles and sides of a right-angled triangle:

$$\sin A = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{a}{c}$$

$$\cos A = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{b}{c}$$

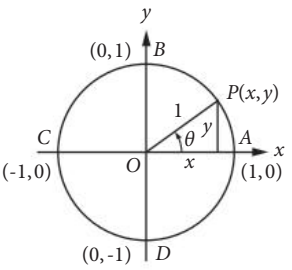
$$\tan A = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{a}{b}$$



To remember these ratios you can use the mnemonic **SOHCAHTOA**, which stands for **S**in is **O**pposite over **H**ypotenuse, **C**os is **A**djacent over **H**ypotenuse, **T**an is **O**pposite over **A**djacent.

The unit circle

A unit circle is a circle of unit radius whose centre is at the origin. The equation of the circle is $x^2 + y^2 = r^2$.



Take any point P on the circumference of this unit circle whose coordinates are (x, y). Consider the point P as starting at A and rotating in an anticlockwise direction, taking various positions around the circumference as shown in the diagrams above. In each position $\angle AOP = \theta$. Then we can define:

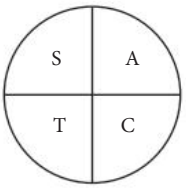
- $\cos\theta = x = x$-coordinate of P (the abscissa)
- $\sin\theta = y = y$-coordinate of P (the ordinate)
- $\tan\theta = \frac{y}{x} = \frac{\sin\theta}{\cos\theta}$, $\cos\theta \neq 0$
- $\cot\theta = \frac{x}{y} = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta}$, $\sin\theta \neq 0$
- $\sec\theta = \frac{1}{x} = \frac{1}{\cos\theta}$, $\cos\theta \neq 0$
- $\text{cosec}\,\theta = \frac{1}{y} = \frac{1}{\sin\theta}$, $\sin\theta \neq 0$

Sign of the trigonometric ratios

The sign of cos, sin and tan for the first four quadrants can be summarised as follows:

- First quadrant: All are positive (A)
- Second quadrant: sin only is positive (S)
- Third quadrant: tan only is positive (T)
- Fourth quadrant: cos only is positive (C)

The positive signs can be remembered with the mnemonic **ASTC**: **A**ll **S**tations **T**o **C**entral.



Complementary angles: θ and $(90° - \theta)$

$\sin(90° - \theta) = \cos\theta$ $\quad$ $\cos(90° - \theta) = \sin\theta$

$\tan(90° - \theta) = \cot\theta$ $\quad$ $\cot(90° - \theta) = \tan\theta$

$\sec(90° - \theta) = \operatorname{cosec}\theta$ $\quad$ $\operatorname{cosec}(90° - \theta) = \sec\theta$

Negative angles: $\theta < 0°$

- $\cos(-\theta) = \cos\theta$
- $\sin(-\theta) = -\sin\theta$
- $\tan(-\theta) = -\tan\theta$

Exact values of trigonometric ratios

θ	0°	30°	45°	60°	90°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined

Trigonometric identities

- $\cos^2\theta + \sin^2\theta = 1$
- $\cos^2\theta = 1 - \sin^2\theta$
- $\sin^2\theta = 1 - \cos^2\theta$
- $1 + \tan^2\theta = \sec^2\theta$
- $\cot^2\theta + 1 = \operatorname{cosec}^2\theta$

The sine rule

The sine rule: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

The sine rule can be used with a triangle when you are given:

(a) the size of two angles and the length of one side, **OR**

(b) the lengths of two sides and the size of an angle opposite one of these sides.

The cosine rule

For ΔABC, the cosine rule states $a^2 = b^2 + c^2 - 2bc\cos A$.
In words:

The square of one side of a triangle is equal to the sum of the squares of the other two sides minus twice the product of those two sides and the cosine of the included angle.

- To find a side, use: $a^2 = b^2 + c^2 - 2bc\cos A$
- To find an angle, use: $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

The cosine rule can be used with a triangle when you are given:

(a) the lengths of three sides, **OR**

(b) the lengths of two sides and the size of the included angle.

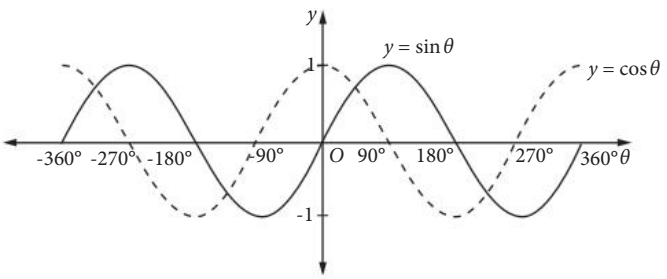
Area of a triangle

Area of $\Delta ABC = \frac{bc\sin A}{2}$

= half the product of two sides and the sine of the angle between them.

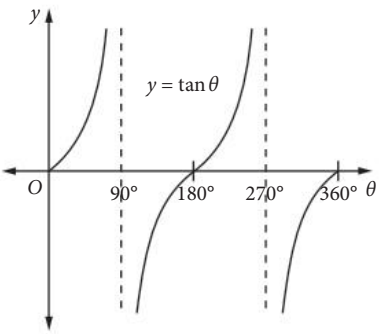
Graphs of trigonometric functions

- $y = \sin\theta$ and $y = \cos\theta$



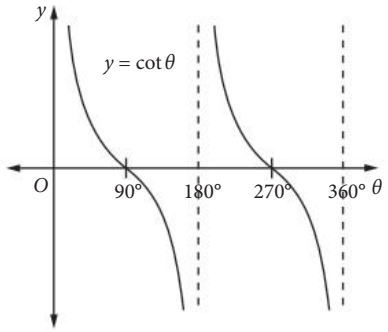
Continuous curves
Period = 360° $\quad$ Amplitude = 1

- $y = \tan\theta$



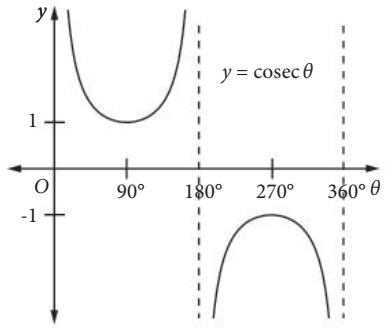
Discontinuous, asymptotes at discontinuities
Period = 180° $\quad$ No amplitude

- $y = \cot\theta$



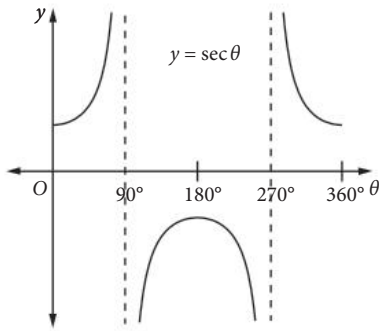
Discontinuous, asymptotes at discontinuities
Period = 180° No amplitude

- $y = \operatorname{cosec}\theta$



Discontinuous, asymptotes at discontinuities
Period = 360° No amplitude

- $y = \sec\theta$



Discontinuous, asymptotes at discontinuities
Period = 360° No amplitude

6 COORDINATE GEOMETRY—STRAIGHT LINES

Gradient of a straight line, *m*

- The **gradient** (or slope) m of a straight line joining the points (x_1, y_1) and (x_2, y_2) is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$, if $x_1 \neq x_2$.
- If $x_1 = x_2$, the gradient is undefined and the line is vertical (that is, parallel to the y-axis).
- If $y_1 = y_2$, the gradient is zero and the line is horizontal (that is, parallel to the x-axis).

Angle of inclination, θ

- The angle that a line makes with the positive direction of the x-axis is called the **angle of inclination** and usually denoted by θ, where $0° \leq \theta < 180°$.
- The gradient m is also equal to $\tan\theta$, hence $\tan\theta = \frac{y_2 - y_1}{x_2 - x_1}$, if $x_1 \neq x_2$.

Straight line—gradient-intercept form

- $y = mx + b$ is the equation of a straight line with gradient m that intersects the y-axis at $y = b$.

Straight line—point-gradient form

- $y - y_1 = m(x - x_1)$ is the equation of a line with gradient m that passes through the point (x_1, y_1).

Straight line—two-point form

- $\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$ or $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$ is the equation of a line that passes through the points (x_1, y_1) and (x_2, y_2).

Straight line—general form

- $ax + by + c = 0$ is called the **general form** of the equation of a straight line. Here a is the coefficient of x, b is the coefficient of y and c is the constant term
- From the general form: gradient $= -\frac{a}{b}$ and y-intercept $= -\frac{c}{b}$
- If $a = 0$, the line is parallel to the x-axis.
- If $b = 0$, the line is parallel to the y-axis.
- If $c = 0$, the line passes through the origin.

Parallel lines

- Two lines are parallel if their gradients are equal, so $m_1 = m_2$.
- The equation of a line parallel to $ax + by + c = 0$ is $ax + by + d = 0$, where $c \neq d$.

Perpendicular lines

- Two lines are perpendicular if the product of their gradients is -1, so $m_1 m_2 = -1$.

  This can also be written as $m_2 = \frac{-1}{m_1}$. Each gradient is the negative reciprocal of the other.
- The equation of a line perpendicular to $ax + by + c = 0$ is $bx - ay + d = 0$.

Intersection of two lines

Two straight lines must either:

(a) intersect at a point, **OR**

(b) be parallel, **OR**

(c) coincide.

When you solve a pair of simultaneous linear equations, you will always get either:

(a) a unique solution (as the lines intersect), **OR**

(b) no solution (as the lines are parallel), **OR**

(c) an infinite number of solutions (as the lines coincide).

You can determine which situation is the case without actually solving the equations.

For $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, the lines are:

(a) intersecting lines if $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ (the gradients are not equal)

(b) parallel lines if $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$ (the gradients are equal, but the lines are different)

(c) coincident lines if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ (the equations are equivalent).

Regions and inequalities

- A straight line represents a function. If the equation is changed to an inequality, then the function becomes a relation and is represented by a region in the number plane.
- A boundary line divides the number plane into three sets of points: points on the line, points above the line and points below the line.
- An inequality with $\geq$ or $\leq$ includes the boundary line as part of the region; an inequality with $>$ or $<$ does not include the boundary line as part of the region, so the line is dashed.

Simultaneous linear inequalities

- Two linear equations may intersect at a point. The intersection of two linear inequalities is the region common to the two inequalities. This is the region where the inequalities hold simultaneously.

Midpoint of an interval

- The coordinates of the midpoint of the interval joining (x_1, y_1) and (x_1, y_1) are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

Distance between two points

- The distance d between two points (x_1, y_1) and (x_2, y_2) is $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

Perpendicular distance of a point from a line

- The perpendicular distance of a fixed point $P(x_1, y_1)$ from a line with equation $ax + by + c = 0$ is given by the formula $d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$ or $d = \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$.
- The sign of this distance (before the absolute value is taken) indicates which side of the line the point is on. Points with a positive distance are on one side of the line, while points with a negative distance are on the other side of the line.

7 FUNCTIONS AND RELATIONS

Functions

- A function is a type of mathematical object that precisely describes a relationship between variables.
- A real function f of a real variable x assigns to each element x of a given set of real numbers **exactly one** real number y, which is the value of the function f at x. The notation $f(x)$ means the value of f at x: $y = f(x)$.
- The set of real numbers x on which f is defined is called the **domain** of f, while the set of values $f(x)$ obtained from x over the domain of f is called the **range** or **image** of f.
- The variable x is the **independent variable**, because it can be chosen freely from the domain of f, while y is the **dependent variable**, because its value depends on x.
- A **function** can also be defined as a set of ordered pairs with the special property that no two pairs have the same first element (x value).

Vertical line test

- If you can draw a vertical line to cut the graph more than once, then the graph does not represent a function.
- The x values for which a vertical line cannot cut the graph are not in the domain.

Relations

- A relation is like a multi-valued function: it is a set of ordered pairs, usually defined by some rule, but each first member of the ordered pairs can have more than one second member.
- The language of functions also applies to relations, so the terms independent variable, dependent variable, domain and range have the same meanings.

Function rules

- When a function rule f is given and a domain is not specified, it is assumed that the domain of the function is the set of real numbers for which $f(x)$ defines a real number range. To find the domain, the solution of an inequality may be needed.
- When the domain of f is all values of x over an interval, the graph of $y = f(x)$ is called the **curve** $y = f(x)$ and a part of the curve between two points is called an **arc**.

Odd and even functions

- An **odd function** has the property that $f(-x) = -f(x)$. The graph of f for $x \le 0$ can be obtained by rotating the graph for $x \ge 0$ through an angle of 180° about the origin.
- An **even function** has the property that $f(-x) = f(x)$. The graph of an even function is symmetrical about the y-axis. The graph for $x \le 0$ can be obtained by reflecting the portion for $x \ge 0$ in the y-axis.
- Note that the statement $f(-a) = f(a)$ implies that the function is defined at both $x = a$ and $x = -a$.

Absolute value functions

- $f(x) = |x|$ is the absolute value function (or 'numerical value function').
- $f(x) = |x| = \begin{cases} x & \text{for } x \ge 0 \\ -x & \text{for } x < 0 \end{cases}$ or equivalently

  $f(x) = |x| = \sqrt{x^2}$ for all real x.
- The domain of $f(x) = |x|$ is all real x. The range of $f(x) = |x|$ is non-negative real numbers (i.e. $f(x)$ is zero or a positive real number).

8 LOCUS AND REGIONS

Locus

- A **locus** is a set of points in a plane that satisfies some geometric condition or some algebraic equation. It can be defined as the path traced out by a point moving in a plane.
- The Cartesian equation used to define a locus gives us the curve along which the point travels. A linear equation like $ax + by + c = 0$ means that the locus of the point is a straight line.

The circle

- A circle can be defined as the set of all points P in a plane at a given distance from a fixed point in the plane. The fixed point is the centre of the circle and the given distance is the radius.
- The equation of a circle is given by $(x - h)^2 + (y - k)^2 = r^2$, with the values for x and y restricted so that:

  $h - r \le x \le h + r$

  $k - r \le y \le k + r$
- If the centre of the circle is at the origin, then $h = 0$, $k = 0$ and the circle equation is $x^2 + y^2 = r^2$.

Non-linear inequalities and regions

- A circle divides the number plane into two regions, a finite region called its **interior** and an infinite region called its **exterior**, as well as the set of points that make up the circle.
- A parabola, cubic, quartic or hyperbola curve divides the number plane into two infinite regions, as well as the set of points that make up the curve.

9 QUADRATIC FUNCTIONS AND THE PARABOLA

Quadratic functions

- $ax^2 + bx + c$, a polynomial of the second degree, is called a quadratic polynomial or a quadratic expression.
- A function defined by the rule $y = ax^2 + bx + c$, where a, b and c are constants, $a \ne 0$, is a **quadratic function**. The domain of this function is the set of real numbers (unless otherwise stated or implied).
- When $y = 0$ the quadratic function becomes the quadratic equation $ax^2 + bx + c = 0$. The values of x that satisfy this equation are called the roots, solutions or zeros of the equation.
- The graph of a quadratic function has the characteristic shape of a **parabola**. The parabola has a turning point at its vertex, where the function has a minimum value if $a > 0$ or a maximum value if $a < 0$.

Maximum or minimum value of a quadratic function

All quadratic polynomials can be expressed in the form $a(x + B)^2 + C$, where a, B and C are constants, $a \ne 0$. Because $(x + B)^2$ is a perfect square, it is non-negative for all real x, i.e. $(x + B)^2 \ge 0$. Thus the smallest possible value of $(x + B)^2$ is zero, when $x = -B$, and so:

(a) If $a > 0$, the **minimum** value of $a(x + B)^2 + C$ is C.

(b) If $a < 0$, the **maximum** value of $a(x + B)^2 + C$ is C.

Turning point of a parabola

The turning point (minimum or maximum value) of the quadratic function $y = ax^2 + bx + c$ occurs at $x = -\frac{b}{2a}$, so it has the coordinates $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.

(a) If $a < 0$ this will be a **maximum** turning point, so the **maximum value** of the function is $f\left(-\frac{b}{2a}\right)$.

(b) If $a > 0$ this will be a **minimum** turning point, so the **minimum value** of the function is $f\left(-\frac{b}{2a}\right)$.

Quadratic equations

- The roots of the general quadratic equation $ax^2 + bx + c = 0$, $a \ne 0$ are $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

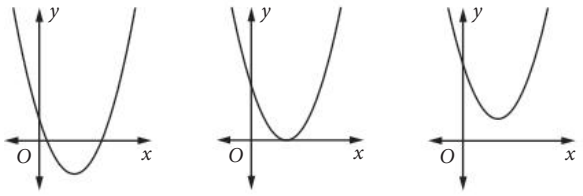
Discriminants

- The **discriminant** of a quadratic equation is $\Delta = b^2 - 4ac$, so the roots of the quadratic equation can be written $x = \dfrac{-b + \sqrt{\Delta}}{2a}$ and $x = \dfrac{-b - \sqrt{\Delta}}{2a}$.

- If $\Delta > 0$, $\sqrt{\Delta}$ is a real number and so the roots are two real numbers. We say that the equation has two unequal roots, or two different roots.
- If $\Delta = 0$, $\sqrt{\Delta} = 0$ and so the roots consist of one real number: $-\frac{b}{2a}$. We say that the equation has only one root, or two equal roots.
- If Δ is a perfect square, $\sqrt{\Delta}$ is a rational number and so the roots are two rational numbers. We say that the equation has two unequal, rational roots.
- If $\Delta < 0$, $\sqrt{\Delta}$ does not exist in the field of real numbers and so there are no real roots. We say that the equation has no real roots.

Intersection of the parabola with the *x*-axis

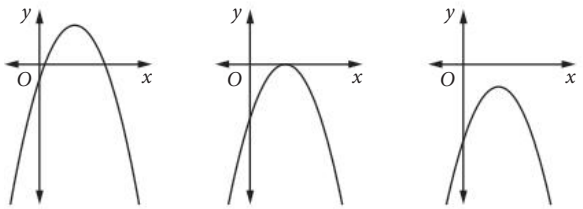
- The roots of $ax^2 + bx + c = 0$ are the x values of the intersection points of the graph of $y = ax^2 + bx + c = 0$ with the x-axis. Hence at the points where the parabola cuts the x-axis, $y = 0$. Thus, the parabola:
  (a) cuts the x-axis at two distinct points if $\Delta > 0$
  (b) touches the x-axis at one point only (two coincident points) if $\Delta = 0$
  (c) does not touch the x-axis if $\Delta < 0$.



$\Delta > 0$ $\Delta = 0$ $\Delta < 0$

$a > 0$ $a > 0$ $a > 0$

When the coefficient of x^2 is positive, the vertex is at the bottom and the curve opens upwards.



$\Delta > 0$ $\Delta = 0$ $\Delta < 0$

$a < 0$ $a < 0$ $a < 0$

When the coefficient of x^2 is negative, the vertex is at the top and the curve opens downwards.

- When $a > 0$ and $\Delta < 0$, the graph of $f(x) = ax^2 + bx + c = 0$ is always above the x-axis, i.e. $f(x) > 0$ for all values of x. We say f is a **positive definite** function.
- When $a < 0$ and $\Delta < 0$, the graph of $f(x) = ax^2 + bx + c = 0$ is always below the x-axis, i.e. $f(x) < 0$ for all values of x. We say f is a **negative definite** function.
- If a quadratic function is neither positive definite nor negative definite, it is **indefinite**.

Relationship between roots and coefficients

- If the roots of $ax^2 + bx + c = 0$, $a \neq 0$, are α and β, then:

$$\text{sum of roots} = \alpha + \beta = -\frac{b}{a}$$

$$\text{product of roots} = \alpha\beta = \frac{c}{a}$$

Identity of two quadratic expressions

- An equation of the nth degree has n roots and no more: a quadratic equation (second degree) has two roots, a cubic equation has three roots, and so on. The roots may be real numbers or unreal ('complex') numbers and some of them may be equal to one another.
- If two polynomials of degree n are equal for more than n values of the variable, then they are **identically equal**, i.e. equal for all values of the variable. The sign $\equiv$ indicates 'identically equal'. This also means that the coefficients of like powers of the variable in the expressions are equal, so:
- If $a_1x^2 + b_1x + c_1 = a_2x^2 + b_2x + c_2$ for more than two values of x, then $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$.

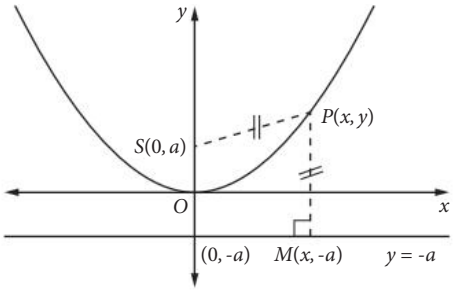
Solution set of simultaneous equations

- To find the intersection points of a straight line and a parabola, you can algebraically calculate the solution set of a system consisting of the line equation and the parabola equation.
- A straight line may cut a parabola at two distinct points, touch it at one point or not intersect it at all. If the line is a tangent to the parabola then the solution set has only one ordered pair (a 'double root').

The parabola as a locus

- A parabola is the locus of a point $P(x, y)$ that moves in a plane so that its distance from a fixed point S is equal to its distance from a fixed straight line.
- The fixed point S is called the **focus** and the fixed straight line is called the **directrix**. The distance a of the focus from the vertex is the **focal length**.

- The locus of a point $P(x, y)$ that moves in a plane so that its distance from the point $S(0, a)$ is equal to its distance from the line $y = -a$ is the parabola $x^2 = 4ay$.



  $x^2 = 4ay$ defines a parabola with its vertex at $(0, 0)$ and an axis of symmetry that is the y-axis.
- If the parabola $x^2 = 4ay$ undergoes a translation so that its vertex is at the point (h, k), then its equation becomes $(x - h)^2 = 4a(y - k)$.
  This represents a parabola with vertex $V(h, k)$, focus $S(h, k + a)$, directrix $y = k - a$, focal length $= a$ and axis of symmetry $x = h$.
- $y^2 = 4ax$ defines a parabola with its vertex at $(0, 0)$, focus $S(a, 0)$, directrix $x = -a$ and an axis of symmetry that is the x-axis.
- If the parabola $y^2 = 4ax$ undergoes a translation so that its vertex is at the point (h, k), then its equation becomes $(y - k)^2 = 4a(x - h)$.
  This represents a parabola with vertex $V(h, k)$, focus $S(h + a, k)$, directrix $x = h - a$, focal length $= a$ and axis of symmetry $y = k$.

10 DIFFERENTIAL CALCULUS

Continuity

- Functions such as $f(x) = x, f(x) = x^3, f(x) = \sin x$ and $f(x) = |x|$ are **continuous**: their graphs do not have any gaps or jumps in them.
- Functions such as $f(x) = \frac{1}{x}, f(x) = \tan x$, $f(x) = \frac{1}{x^2}$ and $f(x) = \frac{|x|}{x}$ are **not continuous**, because their graphs all have gaps or jumps.
- A simple way to describe a continuous function is that it can be drawn smoothly without lifting your pencil from the paper. Continuous functions like $f(x) = x, f(x) = x^3, f(x) = \sin x$ are smooth graphs, but $f(x) = |x|$ is not a smooth graph because the slope of its curve changes suddenly at $x = 0$ to create a sharp corner.

Limit theorems

1 For the constant function f, where $x = c$:
$$\lim_{x \to a} f(x) = c$$

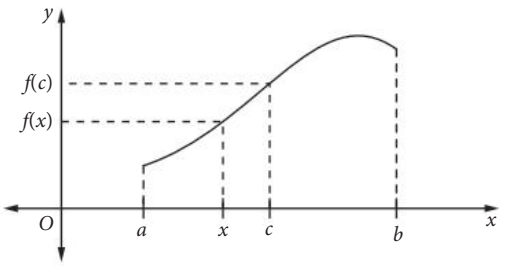
2 If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, then:
$$\lim_{x \to a} \big(f(x) \pm g(x) \big) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$$
$$= L \pm M$$
The limit of a sum = the sum of the limits.
The limit of a difference = the difference of the limits.

3 If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, then:
$$\lim_{x \to a} \big(f(x) \times g(x) \big) = \lim_{x \to a} f(x) \times \lim_{x \to a} g(x)$$
$$= L \times M$$
The limit of a product = the product of the limits.

4 If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, then:
$$\lim_{x \to a} \left(\frac{f(x)}{g(x)} \right) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$$
$$= \frac{L}{M} \qquad \text{for } M \neq 0$$
The limit of a quotient = the quotient of the limits.

Continuity at a point—formal definition

- A function f that is defined in some neighbourhood of $x = c$ is continuous at c if:
  (a) the function has a definite value $f(c)$ at c, **AND**
  (b) as $x \to c$, $f(x) \to f(c)$ as a limit, i.e. $\lim_{x \to c} f(x) = f(c)$.



- A function f is continuous in an (open or closed) interval if it is continuous at all points of that interval. For a closed interval $[a, b]$, continuity at a and b implies that $\lim_{x \to a^+} f(x) = f(a)$ and $\lim_{x \to b^-} f(x) = f(b)$.
- In simple language, a function is continuous in the interval $[a, b]$ if its graph can be drawn from $x = a$ to $x = b$ without lifting your pencil off the paper.
- All polynomial functions are continuous.

Gradient and derivative notation

The gradient at any point (x, y) on the curve $y = f(x)$ can be represented by any of the following notations:
$f'(x)$, $\frac{dy}{dx}$, y', $\frac{d}{dx}\big(f(x)\big)$

Finding the derivative from first principles

- The derivative at any point (x, y) on the curve $y = f(x)$ is given by $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.
- If $f(x)$ possesses a derivative $f'(x)$ for each x belonging to the domain of f, then $f(x)$ is called a **differentiable function**.
- The statement '$f(x)$ is differentiable' means '$f(x)$ has a derivative at each point of its domain'.

Differentiation results

- **Binomial factorisation:** $a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + ab^{n-2} + b^{n-1})$

1. If $y = x^n$, then $\frac{dy}{dx} = nx^{n-1}$ where n is any real number
2. If $y = cu$, where c is a constant and $u = g(x)$, then $\frac{dy}{dx} = c\frac{du}{dx}$
3. If $y = u \pm v$ where $u = f(x)$ and $v = g(x)$, then $\frac{dy}{dx} = \frac{du}{dx} \pm \frac{dv}{dx}$
4. If $y = \sqrt{x}$ then $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$
5. If $y = \frac{1}{x}$ then $\frac{dy}{dx} = -\frac{1}{x^2}$
6. From results **1** and **3**, it follows that if $y = x^n + x^{n-1} + x^{n-2} + \ldots + x$, then $\frac{dy}{dx} = nx^{n-1} + (n-1)x^{n-2} + (n-2)x^{n-3} + \ldots + 1$
7. **Product rule:** If u and v are functions of x, and $y = uv$, then $\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$
8. **Chain rule:** $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ **OR** $\frac{d}{dx}F(u) = F'(u) \times \frac{du}{dx}$ for any differentiable function F
9. **Quotient rule:** If u and v are functions of x, and $y = \frac{u}{v}$, $v \neq 0$, then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

SUMMARY—HSC

Material covered in Chapter 11 Plane geometry is summarised in the Preliminary course under Chapter 4 Plane geometry.

12 GEOMETRICAL APPLICATIONS OF DIFFERENTIATION

The sign of the derivative

- If $\frac{dy}{dx} > 0$ as x increases, the function is an **increasing** function.
- If $\frac{dy}{dx} < 0$ as x increases, the function is a **decreasing** function.
- If $\frac{dy}{dx} = 0$ at a given value of x, the function is **stationary** at that point. At this **stationary point**, the tangent to the curve is parallel to the x-axis.

These results and their converses can determine a function's characteristics:

- If $\frac{dy}{dx} > 0$, then the function is increasing; if the function is increasing, then $\frac{dy}{dx} > 0$.
- If $\frac{dy}{dx} < 0$, then the function is decreasing; if the function is decreasing, then $\frac{dy}{dx} < 0$.
- If $\frac{dy}{dx} = 0$, then the point is a stationary point; at a stationary point, $\frac{dy}{dx} = 0$.

Turning points

- A turning point of $f(x)$ is a point where the curve $y = f(x)$ is locally a maximum or a minimum.
- For a differentiable function $f(x)$, all turning points are stationary points.
  (**However**, not all stationary points are turning points.)

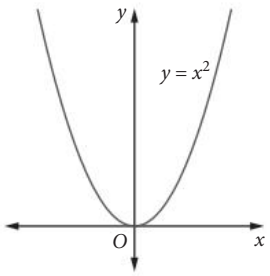
The second derivative

- Differentiating a function once to obtain $f'(x)$ or $\frac{dy}{dx}$ gives you the first derivative of the original function.
- Differentiating the first derivative gives you $f''(x)$ or $\frac{d^2y}{dx^2}$, which is the **second derivative** of the original function. This is the rate of change of the first derivative: $\frac{d}{dx}\big(f'(x)\big)$.

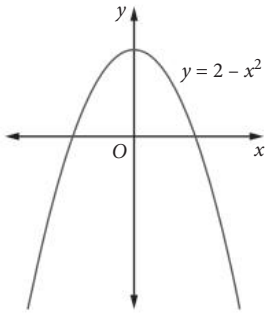
- If $y = f(x)$, then the first derivative can be written $\frac{dy}{dx}$, $f'(x)$, $\frac{d}{dx}\big(f(x)\big)$ or y'.
- If $y = f(x)$, then the second derivative can be written $\frac{d}{dx}\left(\frac{dy}{dx}\right)$, $\frac{d^2y}{dx^2}$, $f''(x)$, $\frac{d}{dx}\big(f'(x)\big)$ or y''.

Concavity

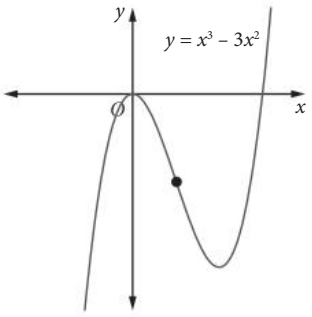
The **concavity** of a function describes how the shape of the function's graph 'opens'. Graphs can be 'concave up' and 'concave down'.



The function is concave up.
Note: has a minimum turning point.



The function is concave down.
Note: has a maximum turning point.



The left-hand part is concave down.
The right-hand part is concave up.
Concavity changes at $x = 1$.

The sign of the second derivative

- If $\frac{d^2y}{dx^2} > 0$ on an interval then the curve is concave up on that interval.
- If $\frac{d^2y}{dx^2} < 0$ on an interval then the curve is concave down on that interval.
- If $\frac{d^2y}{dx^2} = 0$ at a point on the curve **and** the concavity changes at this point, then the point is called a **point of inflexion**.

Tests for stationary points—summary

- For stationary points: $\frac{dy}{dx} = 0$

Using only the first derivative:

- Turning points: $\frac{dy}{dx} = 0$ and $\frac{dy}{dx}$ changes sign on passing through the point.
- Maximum turning points: $\frac{dy}{dx} = 0$ and $\frac{dy}{dx}$ changes from positive to negative through the point.
- Minimum turning points: $\frac{dy}{dx} = 0$ and $\frac{dy}{dx}$ changes from negative to positive through the point.

Using the first and second derivatives:

- Turning points: $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2}$ does not changes sign on passing through the point.
- Maximum turning points: $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$ at the point.
- Minimum turning points: $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$ at the point.
- Point of inflexion: if $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$ at the point, it may be a turning point **OR** a horizontal point of inflexion. If $\frac{d^2y}{dx^2}$ also changes sign on passing through the point, then it is a point of inflexion.

Sketching rational algebraic functions

Functions with the independent variable in the denominator generate curves that are not continuous and may have asymptotes. They may not have any turning points. You need to consider what happens to the function for very large positive and negative values of the variable.

Tangents and normals to a curve

- Differentiation gives the gradient function of a curve. This function can be evaluated at a point on the curve to obtain the gradient of the curve's **tangent** at that point. The point-gradient form of the straight line can then be used to find the equation of that tangent.

- A **normal** to a point on a curve is the line that is perpendicular to the tangent at that point. Because the lines are perpendicular, the gradient of the normal can be obtained from the gradient of the tangent.

Primitive functions

- The process of finding the function from the derivative is called **antidifferentiation**, **finding the primitive** of the function or **finding the indefinite integral**.
- If $F(x)$ is a function such that $F'(x) = f(x)$, then $F(x)$ is called the **antiderivative** or **primitive** of $f(x)$:

$$F(x) = \int f(x)\,dx$$

Similarly, $f(x)$ is the primitive of $f'(x)$.

- The primitive of x^n is $\frac{1}{n+1}x^{n+1} + C$, $n \neq 1$.

13 INTEGRAL CALCULUS

The definite integral and the area under a curve

- $\int_a^b f(x)\,dx$ is called the **definite integral** of the function $f(x)$ between $x = a$ and $x = b$.
- The value A of this definite integral is the size of the area under the curve $y = f(x)$ between $x = a$ and $x = b$.
- For a definite integral, we write:

$$A = \int_a^b f(x)\,dx \quad \textbf{OR} \quad A = \int_a^b y\,dx$$

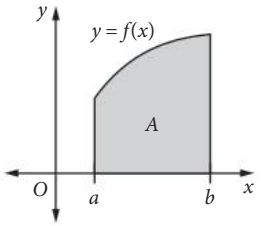
Fundamental theorem of calculus

- $\int_a^b f(x)\,dx = F(b) - F(a)$ where $F(x)$ is a primitive function of $f(x)$.
- When evaluating an integral, it is convenient to set it out as:

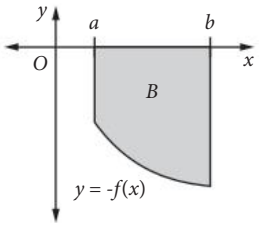
$$\int_a^b f(x)\,dx = \left[F(x)\right]_a^b$$
$$= F(b) - F(a)$$

Integrals—important results

- If $y = f(x)$ is an increasing function over the interval $a \leq x \leq b$, then $y = -f(x)$ is a decreasing function over the same interval.
- $y = -f(x)$ is the reflection of $y = f(x)$ in the x-axis. Thus the area of the region bounded by $y = f(x)$, $x = a$, $x = b$ and the x-axis is equal to the area of the region bounded by $y = -f(x)$, $x = a$, $x = b$ and the x-axis:



$$A = \left|\int_a^b f(x)\,dx\right|$$



$$B = \left|\int_a^b \left(-f(x)\right)dx\right|$$

$$B = \left|-\int_a^b f(x)\,dx\right| = \left|\int_a^b f(x)\,dx\right|$$

- If $y = f(x)$ is a continuous function over the interval $a \leq x \leq b$, and $a \leq c \leq b$, then:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

- If F, G are primitives of f, g respectively, then $F + G$ is a primitive of $f + g$. Thus:

$$\begin{aligned}\int_a^b \left(f(x) + g(x)\right)dx &= \left(F(b) + G(b)\right) - \left(F(a) + G(a)\right)\\ &= \left(F(b) - F(a)\right) - \left(G(b) - G(a)\right)\\ &= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx\end{aligned}$$

- If you reverse the limits of integration, you change the sign of the integral:

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

- $\int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx = \int_a^b \left(f(x) \pm g(x)\right)dx$
- $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$ where c is a constant.

Odd and even functions

- For an even function, the area bounded by the curve $y = f(x)$, the x-axis, $x = a$ and $x = -a$ is:

$$\text{Area} = \left|\int_{-a}^{a} f(x)\,dx\right| = 2\left|\int_{0}^{a} f(x)\,dx\right|$$

- For an odd function, the area bounded by the curve $y = f(x)$, the x-axis, $x = a$ and $x = -a$ is:

$$\text{Area} = \left|\int_{-a}^{0} f(x)\,dx\right| + \left|\int_{0}^{a} f(x)\,dx\right| = 2\left|\int_{0}^{a} f(x)\,dx\right|$$

Trapezoidal rule

If $f(x)$ is a continuous function and $f(x) \geq 0$ on the interval $a \leq x \leq b$, then you can find an approximation to the definite integral $\int_a^b f(x)\,dx$ by dividing the area into a number of trapezia of equal width, where the parallel sides of the trapezia are ordinates.

- One trapezium:

$$\int_a^b f(x)\,dx \approx \frac{(b-a)}{2}\big(f(a)+f(b)\big)$$

- Two trapezia:

$$\int_a^b f(x)\,dx \approx \frac{(b-a)}{2}\left(\frac{f(a)+f\left(\frac{a+b}{2}\right)}{2}\right)+\frac{(b-a)}{2}\left(\frac{f\left(\frac{a+b}{2}\right)+f(b)}{2}\right)$$

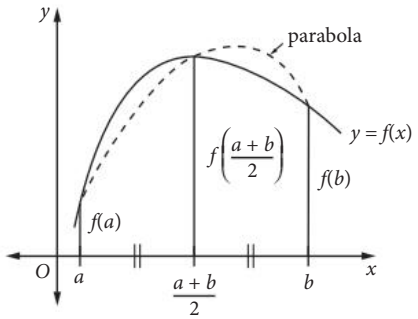
$$= \frac{(b-a)}{4}\left(f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right)$$

- n trapezia: the base of each trapezium is $\frac{b-a}{n} = h$. For $a = x_0$, $b = x_n$ and other points are $x_1, x_2, \ldots x_{n-1}$, so the trapezoidal rule is:

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\big(f(a)+2f(x_1)+2f(x_2)+\ldots+2f(x_{n-1})+f(b)\big) \qquad \text{where } h = \frac{b-a}{n}$$

Simpson's rule

Simpson's rule approximates the given function by a quadratic function: as shown below, we replace $f(x)$ from $x = a$ to $x = b$ by an arc of a parabola that intersects f at $x = a$, $x = b$ and the midpoint $x = \frac{a+b}{2}$:



Simpson's rule then states:

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]$$

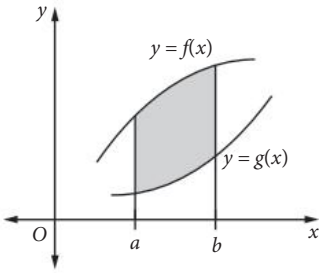
You can also break any interval into subintervals and apply Simpson's rule to each subinterval.

Area between two curves

If f and g are two continuous functions whose graphs do not intersect in the interval $a \leq x \leq b$, and $f(x) > g(x)$ over this interval, then the area of the region bounded by the two curves and the ordinates $x = a$ and $x = b$ is given by:

$$\text{Area} = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

$$= \int_a^b \big(f(x)-g(x)\big)\,dx$$



This result also applies if the two curves intersect only at $x = a$ and $x = b$.

Area bounded by the *y*-axis

- To find areas where one of the boundaries is the *y*-axis, you can effectively swap the roles of the *x* and *y* variables.
- The area bounded by the curve $y = f(x)$, the *x*-axis and the ordinates at $x = a$ and $x = b$ is given by:

$$\text{Area} = \int_a^b f(x)\,dx = \int_a^b y\,dx$$

- Similarly, the area bounded by the curve $x = g(y)$, the *y*-axis and the abscissae at $y = c$ and $y = d$ is given by:

$$\text{Area} = \int_c^d g(y)\,dy = \int_c^d x\,dy$$

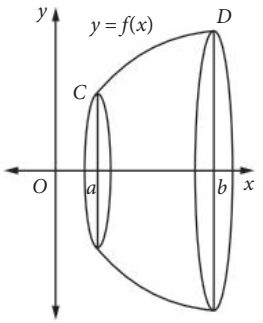
- For two curves $x = f(y)$ and $x = g(y)$ that intersect at $y = c$ and $y = d$, where $f(y) \geq g(y)$ over the interval $c \leq y \leq d$, the area bounded by the curves is given by:

$$\text{Area} = \int_c^d \big(f(y) - g(y)\big)\,dy$$

Volume of solids of revolution

- When the arc *CD* of the curve $y = f(x)$ on the interval $a \leq x \leq b$ is rotated about the *x*-axis, the volume of the solid of revolution formed is given by:

$$V = \pi \int_a^b \big(f(x)\big)^2\,dx = \pi \int_a^b y^2\,dx$$



- When the arc *CD* of the curve $x = g(y)$ on the interval $c \leq y \leq d$ is rotated about the *y*-axis, the volume of the solid of revolution formed is given by:

$$V = \pi \int_c^d \big(g(y)\big)^2\,dy = \pi \int_c^d x^2\,dy$$

The indefinite integral

- The notation $\int f(x)\,dx$ represents the **indefinite integral**. This uses the integral sign $\int$ to say 'find the primitive of $f(x)$'. It gives a function to represent all possible values of the primitive by adding *C*, the **constant of integration**.
- If $F(x)$ is a primitive of $f(x)$, then:

$$\int f(x)\,dx = F(x) + C$$

- The indefinite integral is a function of *x*, whereas the definite integral has a (definite) value.

14 EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Index laws

Let $a^x = m$ and $a^y = n$

1 $a^x \times a^y = a^{x+y} \rightarrow mn = a^{x+y}$

2 $\dfrac{a^x}{a^y} = a^{x-y} \rightarrow \dfrac{m}{n} = a^{x-y}$

3 $\dfrac{1}{a^x} = a^{-x} \rightarrow \dfrac{1}{m} = a^{-x}$

4 $\left(a^x\right)^p = a^{xp} \rightarrow m^p = a^{xp}$

5 $a^1 = a$

6 $a^0 = 1$

7 $a^{\frac{1}{n}} = \sqrt[n]{a}$

8 $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$

9 $(ab)^x = a^x \times b^x$

Logarithm laws

Let $\log_a m = x$ and $\log_a n = y$

1 $\log_a(mn) = x + y = \log_a m + \log_a n$
$\log_a 15 = \log_a 3 + \log_a 5$

2 $\log_a\left(\dfrac{m}{n}\right) = x - y = \log_a m - \log_a n$
$\log_a\left(\frac{17}{5}\right) = \log_a 17 - \log_a 5$

3 $\log_a\left(\dfrac{1}{m}\right) = -x = -\log_a m$
$\log_a\left(\frac{1}{2}\right) = -\log_a 2$

4 $\log_a m^p = px = p\log_a m$
$\log_a 81 = \log_a(3^4) = 4\log_a 3$

5 $\log_a a = 1$
$\log_{10} 10 = 1$

6 $\log_a 1 = 0$

Note that these laws are all for $a > 0$, and so we also have $m > 0$ and $n > 0$.

- For $m > 0$, $\log_a m > 0$; for $m = 1$, $\log_a 1 = 0$; for $0 < m < 1$, $\log_a m < 0$.
  Because $a^x \geq 0$, m is never negative, so you cannot find the logarithm of a negative number.
- Most calculators have two logarithm keys: **log** (which is logarithm to base 10, also called 'common logarithm') and **ln** (which is logarithm to base e, usually called 'natural logarithm').

Change of base rule

- $\log_a n = \dfrac{\log_b n}{\log_b a}$

Exponential function (Euler's number *e*)

- $e = \lim\limits_{n \to \infty} \left(1 + \dfrac{1}{n}\right)^n$
- $e = 2.718\,28$ is an approximation correct to 5 decimal places.

Derivatives involving e^x

- $\dfrac{d}{dx}\left(e^x\right) = e^x$
- $\dfrac{d}{dx}\left(e^{kx}\right) = ke^x$
- $\dfrac{d}{dx}\left(e^{ax+b}\right) = ae^{ax+b}$
- $\dfrac{d}{dx}\left(e^{f(x)}\right) = f'(x)e^{f(x)}$

Integrals involving e^x

- $\int e^x \, dx = e^x + C$
- $\int e^{kx} \, dx = \dfrac{1}{k}e^{kx} + C$
- $\int e^{ax+b} \, dx = \dfrac{1}{a}e^{ax+b} + C$
- $\int f'(x)e^{f(x)} \, dx = e^{f(x)} + C$

Caution: do not assume an integral at 0 is always zero; exponential integrals at 0 often produce a non-zero constant. Always evaluate both limit values of a definite integral.

Important properties of e^x and $\log_e x$

- If $y = e^x$ then $\log_e y = \ln y = x$
- If $y = \log_e x = \ln x$ then $e^y = x$
- $e^{\log_e x} = e^{\ln x} = x$
- $\log_e e^x = \ln e^x = x$
- $e^0 = 1$ and $\log_e 1 = 0$
- $e^x > 0$ for all x
- Domain of $y = \log_e x$ is $x > 0$

Reciprocal property of $\dfrac{dy}{dx}$

- $\dfrac{dy}{dx} = \dfrac{1}{\dfrac{dx}{dy}}$

Definitions for $\log_e x$

- The natural logarithm of x (for $x > 0$), usually denoted by $\ln x$ or $\log_e x$, can also be defined as:

  $$\log_e x = \int_1^x \frac{1}{t} dt, \; t > 0$$

- Similarly, the number e can be defined as:

  $$\int_1^e \frac{1}{t} dt = \left[\log_e t\right]_1^e = \log_e e - \log_e 1 = 1, \; t > 0$$

- The number e is such that the area bounded by the graph of $f(t) = \dfrac{1}{t}$ (for $t > 0$), the t-axis and the ordinates $t = 1$ and $t = e$ is 1 square unit.

Derivatives involving $\log_e x$

- $\dfrac{d}{dx}\left(\log_e x\right) = \dfrac{1}{x}, \; x > 0$
- $\dfrac{d}{dx}\left(\log_e (ax)\right) = \dfrac{1}{x}, \; x > 0$
- $\dfrac{d}{dx}\left(\log_e (ax+b)\right) = \dfrac{a}{ax+b}$
- $\dfrac{d}{dx}\left(\log_e \left[f(x)\right]\right) = \dfrac{f'(x)}{f(x)}$

Integrals involving $\log_e x$

- $\int \dfrac{1}{x} dx = \log_e x + C$
- $\int \dfrac{1}{ax+b} dx = \dfrac{1}{a}\log_e (ax+b) + C$
- $\int \dfrac{f'(x)}{f(x)} dx = \log_e \left[f(x)\right] + C$
- $\int \dfrac{af'(x)}{f(x)} dx = a\log_e \left[f(x)\right] + C$

Radian measure of an angle

1 radian is the angle subtended by an arc of 1 unit in a circle of radius 1 unit, so $\pi^c = 180°$:

$1^c = \left(\frac{180}{\pi}\right)^\circ \approx 57°\,18'$ and $1° = \left(\frac{\pi}{180}\right)^c \approx 0.017\,45$

So: $\theta^c = \left(\frac{180\theta}{\pi}\right)^\circ$ and $\theta° = \left(\frac{\pi\theta}{180}\right)^c$

Common conversions

Degrees	0°	30°	45°	60°	90°	120°	135°	150°	180°	210°	225°	240°	270°	300°	315°	330°	360°
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π

Adding 360° to an angle in degrees is the same as adding 2π to an angle in radians. Both 360° and 2π each represent a full turn.

Arc length

- Arc length $\ell = r\theta$
  The angle θ (in radians) is subtended at the centre of a circle of radius r units.

Area of a sector

- Area of a sector $A = \frac{1}{2}r^2\theta$
  The angle θ (in radians) is subtended at the centre of a circle of radius r units.
- Because $\ell = r\theta$, $A = \frac{1}{2}r^2\theta = \frac{r\ell}{2}$.

Area of a segment of a circle

- Area of segment $= \frac{1}{2}r^2(\theta - \sin\theta)$
  The angle θ (in radians) is subtended at the centre of a circle of radius r units.

Symmetry properties of trigonometric functions

Degrees	Radians
$\sin(90° - \theta) = \cos\theta$	$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$
$\cos(90° - \theta) = \sin\theta$	$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$
$\tan(90° - \theta) = \cot\theta$	$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$
$\sin(180° - \theta) = \sin\theta$	$\sin(\pi - \theta) = \sin\theta$
$\cos(180° - \theta) = -\cos\theta$	$\cos(\pi - \theta) = -\cos\theta$
$\tan(180° - \theta) = -\tan\theta$	$\tan(\pi - \theta) = -\tan\theta$
$\sin(180° + \theta) = -\sin\theta$	$\sin(\pi + \theta) = -\sin\theta$
$\cos(180° + \theta) = -\cos\theta$	$\cos(\pi + \theta) = -\cos\theta$

Degrees	Radians
$\tan(180° + \theta) = \tan\theta$	$\tan(\pi + \theta) = \tan\theta$
$\sin(360° - \theta) = -\sin\theta$	$\sin(2\pi - \theta) = -\sin\theta$
$\cos(360° - \theta) = \cos\theta$	$\cos(2\pi - \theta) = \cos\theta$
$\tan(360° - \theta) = -\tan\theta$	$\tan(2\pi - \theta) = -\tan\theta$
$\sin(360° + \theta) = \sin\theta$	$\sin(2\pi + \theta) = \sin\theta$
$\cos(360° + \theta) = \cos\theta$	$\cos(2\pi + \theta) = \cos\theta$
$\tan(360° + \theta) = \tan\theta$	$\tan(2\pi + \theta) = \tan\theta$

Sign of the trigonometric ratios

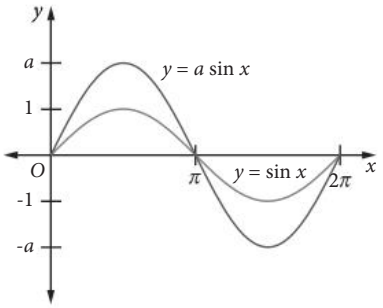
For radians as well as for degrees, the sign of cos, sin and tan for the first four quadrants can be remembered with the mnemonic **ASTC**: **A**ll **S**tations **T**o **C**entral.

Exact values in radians

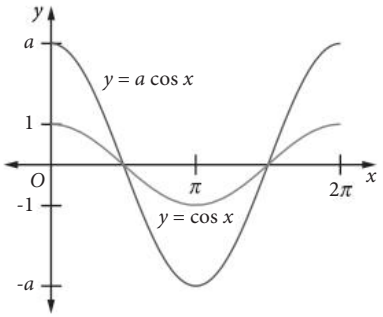
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined
$\operatorname{cosec}\theta$	undefined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
$\sec\theta$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	undefined
$\cot\theta$	undefined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

Graphs of $y = a\sin x$, $y = a\cos x$ and $y = a\tan x$

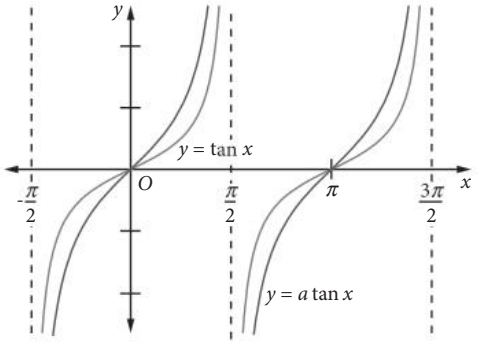
The graphs of $y = \sin x$ and $y = a\sin x$, drawn on the same axes for $0 \le x \le 2\pi$:



The graphs of $y = \cos x$ and $y = a\cos x$, drawn on the same axes for $0 \le x \le 2\pi$:



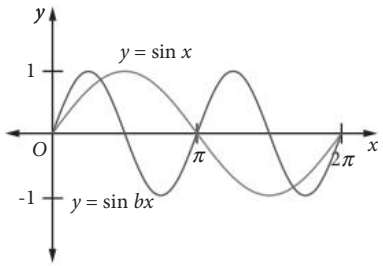
The graphs of $y = \tan x$ and $y = a\tan x$, drawn on the same axes for $-\frac{\pi}{2} \le x \le \frac{3\pi}{2}$:



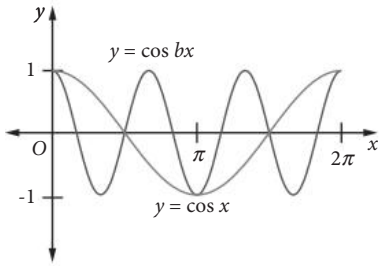
- The effect of the a is to multiply the amplitude by a. If the function has no amplitude, as for $y = a\tan x$, then the curve is stretched vertically by a factor of a.

Graphs of $y = \sin bx$, $y = \cos bx$ and $y = \tan bx$

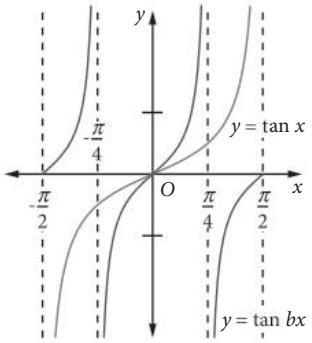
The graphs of $y = \sin x$ and $y = \sin bx$, drawn on the same axes for $0 \le x \le 2\pi$:



The graphs of $y = \cos x$ and $y = \cos bx$, drawn on the same axes for $0 \le x \le 2\pi$:



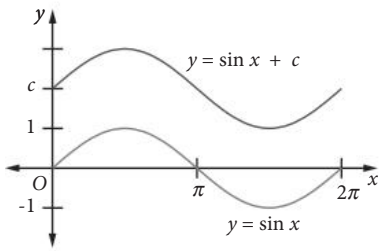
The graphs of $y = \tan x$ and $y = \tan bx$, drawn on the same axes for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$:



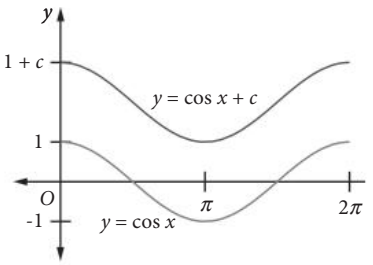
- The effect of the b is to divide the function's period by b. Thus the period of $y = \sin bx$ is $\frac{2\pi}{b}$, the period of $y = \cos bx$ is $\frac{2\pi}{b}$ and the period of $y = \tan bx$ is $\frac{\pi}{b}$.

Graphs of $y = \sin x + c$, $y = \cos x + c$ and $y = \tan x + c$

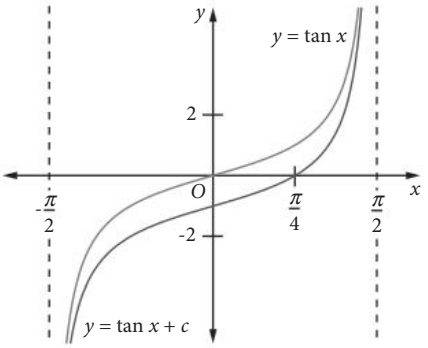
The graphs of $y = \sin x$ and $y = \sin x + c$, drawn on the same axes for $0 \le x \le 2\pi$:



The graphs of $y = \cos x$ and $y = \cos x + c$, drawn on the same axes for $0 \le x \le 2\pi$:



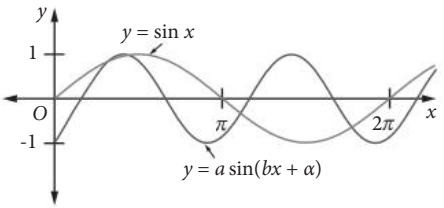
The graphs of $y = \tan x$ and $y = \tan x + c$, drawn on the same axes for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$:



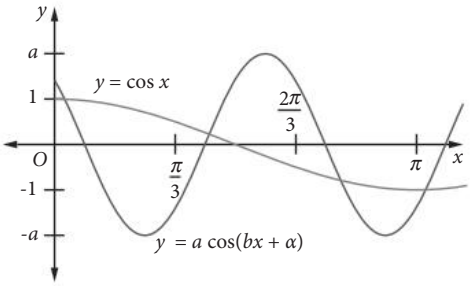
- The effect of the c in each equation is to translate the curve vertically by c units.

Graphs of $y = a\sin(bx + \alpha)$, $y = a\cos(bx + \alpha)$ and $y = a\tan(bx + \alpha)$

The graphs of $y = \sin x$ and $y = a\sin(bx + \alpha)$, drawn on the same axes for $0 \le x \le 2\pi$:



The graphs of $y = \cos x$ and $y = a\cos(bx + \alpha)$, drawn on the same axes for $0 \le x \le 2\pi$:



- The value α changes the position of the function along the x-axis. It is called the **phase** of the function.

Summary of trigonometric functions

Function	Period	Amplitude	Domain	Range	Symmetry
$y = a\sin x$	2π	a	Real x	$-a \le y \le a$	Rotational symmetry
$y = a\cos x$	2π	a	Real x	$-a \le y \le a$	Symmetrical about y-axis
$y = a\tan x$	π	none	Real x, $x \neq \frac{(2n-1)\pi}{2}$ for integer n	Real y	Rotational symmetry
$y = a\sin bx$	$\frac{2\pi}{b}$	a	Real x	$-a \le y \le a$	Rotational symmetry
$y = a\cos bx$	$\frac{2\pi}{b}$	a	Real x	$-a \le y \le a$	Symmetrical about y-axis
$y = a\tan bx$	$\frac{\pi}{b}$	none	Real x, $x \neq \frac{(2n-1)\pi}{2b}$ for integer n	Real y	Rotational symmetry
$y = a\sin bx + c$	$\frac{2\pi}{b}$	a	Real x	$-a + c \le y \le a + c$	
$y = a\cos bx + c$	$\frac{2\pi}{b}$	a	Real x	$-a + c \le y \le a + c$	Symmetrical about y-axis
$y = a\tan bx + c$	$\frac{\pi}{b}$	none	Real x, $x \neq \frac{(2n-1)\pi}{2}$ for integer n	Real y	
$y = a\sin(bx + \alpha)$	$\frac{2\pi}{b}$	a	Real x	Real y	
$y = a\cos(bx + \alpha)$	$\frac{2\pi}{b}$	a	Real x	Real y	
$y = a\tan(bx + \alpha)$	$\frac{\pi}{b}$	none	Real x, $x \neq \frac{(2n-1)\pi - 2\alpha}{2b}$ for integer n	Real y	

Approximations when x is small

- $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$
- $\lim\limits_{x\to 0}\frac{\tan x}{x} = 1$
- Thus: $\tan x > \sin x$ for small x
- $\lim\limits_{x\to 0}\frac{1-\cos x}{x^2} = 0.5$

Derivatives of trigonometric functions

- $\frac{d}{dx}(\sin x) = \cos x$
- $\frac{d}{dx}(\cos x) = -\sin x$
- $\frac{d}{dx}(\tan x) = \sec^2 x$
- $\frac{d}{dx}(\sin(ax+b)) = a\cos(ax+b)$
- $\frac{d}{dx}(\cos(ax+b)) = -a\sin(ax+b)$
- $\frac{d}{dx}(\tan(ax+b)) = a\sec^2(ax+b)$

Integrals of trigonometric functions

- $\int \sin x\,dx = -\cos x + C$
- $\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax) + C$
- $\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + C$
- $\int \cos x\,dx = \sin x + C$
- $\int \cos ax\,dx = \frac{1}{a}\sin ax + C$
- $\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C$
- $\int \sec^2 x\,dx = \tan x + C$
- $\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C$
- $\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + C$

16 SERIES AND APPLICATIONS

Arithmetic series—summary of formulae

- $a + (a + d) + (a + 2d) + \ldots + (a + (n - 1)d)$ is an arithmetic series
- d is the common difference
- $T_1 = a$ is the first term
- $T_n = a + (n - 1)d$
- $S_n = \frac{n}{2}(a + l)$ where $l = a + (n - 1)d$
- $S_n = \frac{n}{2}[2a + (n-1)d]$ or $n\left(a + \frac{n-1}{2}d\right)$
- $S_n = \sum_{k=1}^{n}[a + (k-1)d]$
- $T_n = S_n - S_{n-1}$, $n > 1$

Geometric series—summary of formulae

- $a + ar + ar^2 + \ldots + ar^{n-1}$ is a geometric series
- r is the common ratio
- $T_1 = a$ is the first term
- $T_n = ar^{n-1}$
- $S_n = \frac{a(1 - r^n)}{1 - r}$ for $r < 1$
- $S_n = \frac{a(r^n - 1)}{r - 1}$ for $r > 1$
- $S_n = \sum_{k=1}^{n} ar^{k-1}$
- $T_n = S_n - S_{n-1}$, $n > 1$

Infinite geometric series

- If $\lim_{n\to\infty} r^n = 0$ then $\lim_{n\to\infty} \frac{ar^n}{1 - r} = 0$, so the geometric series converges for $|r| < 1$:

$$S_\infty = \frac{a}{1 - r} \qquad \text{for } |r| < 1$$

- If $|r| > 1$ then $r^n \to \infty$ and there is no limiting sum, so the geometric series does **not** converge for $|r| > 1$.

Compound interest

- The compound interest formula: $A_n = PR^n$

If someone invests \$$P$ at the beginning of each period of time at r% per period compound interest, the sum V of all investments at the end of n periods of time is:

$$V = PR + PR^2 + PR^3 + \ldots + PR^n = \frac{PR(R^n - 1)}{R - 1}$$

where $R = 1 + \frac{r}{100}$

Note that the **process** of obtaining this result is the important thing to remember, not the result.

17 APPLICATIONS OF CALCULUS TO THE PHYSICAL WORLD

Variables in proportion

- **Direct proportion** means that the absolute values of the variables change in the same direction: as the magnitude of the independent variable increases, the magnitude of the dependent variable increases proportionally (and vice versa). Mathematically, the two variables are linear functions of each other (e.g. $y = kx$).
- **Inverse proportion** means that the absolute values of the variables change oppositely (inversely) to each other: as the magnitude of the independent variable increases, the magnitude of the dependent variable decreases proportionally (and vice versa). Mathematically, the two variables are reciprocal functions of each other (e.g. $y = \frac{k}{x}$).

Exponential growth and decay

$\frac{dy}{dx} = ky$ means that the rate of change of a quantity y with respect to another quantity x is proportional to y. The solution of this differential equation is the exponential rate of change:

- $y = Ae^{kx}$ for exponential growth
- $y = Ae^{-kx}$ for exponential decline (or 'exponential decay').

For $y = Ae^{kx}$, we have $\frac{dy}{dx} = kAe^{kx} = ky$:

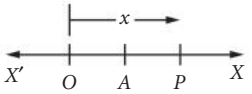
- A is the value of y when $x = 0$ (i.e. the initial value)
- k is the **growth rate**.

Motion of a particle in a straight line

A **particle** is a body that behaves such that all forces acting on it can be regarded as acting through a single point. This means we can represent the body as a single point, regardless of its actual size and shape. Quite large bodies, e.g. trains, can still be classified as 'particles' provided this condition applies.

Displacement

- For a particle represented by a point P, moving in a straight line $X'OX$:



The **displacement** x is the particle's position relative to the fixed point O. It may be a positive or negative number, according to whether P is to the right or left of O, so it does not necessarily represent the total **distance** travelled. Unlike displacement, distance is always a positive quantity.

- Note that the origin of the motion (when $t = 0$) is not necessarily at O.

Velocity

- **Velocity** is defined as the rate of change of position (i.e. of displacement) with respect to time, or as the time rate of change of position in a given direction:

$$v(t) = f'(t) = \frac{dx}{dt} = \dot{x} = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$$

- Velocity can be positive or negative, depending on the direction of travel. Speed is the magnitude of the velocity and is always positive.

Acceleration

- **Acceleration** is defined as the rate of change of velocity with respect to time:

$$a(t) = v'(t) = \frac{d^2x}{dt^2} = \frac{dv}{dt} = \ddot{x} = \lim_{h \to 0} \frac{v(t+h) - v(t)}{h}$$

- Acceleration can be positive or negative. Positive acceleration means the velocity is increasing (i.e. in the direction of positive displacement), while negative acceleration means the velocity is decreasing, which is often called deceleration or retardation.

Important motion terms

- 'initially': $t = 0$
- 'at the origin': $x = 0$
- 'at rest': $v = 0$
- 'velocity is constant': $a = 0$

Units and symbols

Physical quantity	Unit	Symbol
Time	s	t
Displacement	cm, m	x (or s in Physics)
Velocity	cm s^{-1}, m s^{-1}	v, $\frac{dx}{dt}$, $\dot{x}$
Acceleration	cm s^{-2}, m s^{-2}	a, $\frac{dv}{dt}$, $\frac{d^2x}{dt^2}$, $\ddot{x}$

- Note that 's' is the abbreviation for second, 'cm' for centimetre and 'm' for metre.
- Constant acceleration due to gravity = $9.8\ \text{m s}^{-2}$ ($\approx 10\ \text{m s}^{-2}$)

Displacement, velocity, acceleration—important links

- Given the displacement function, you can find velocity and acceleration functions by differentiating with respect to time.
- Given the velocity function, you can find the displacement function by integrating with respect to time; you can find the acceleration function by differentiating with respect to time.
- If motion information is given as a graph, you must remember that the definite integral is linked to the area under the graph, while the derivative is linked to the gradient of the curve.
- For example: given the graph of velocity $\frac{dx}{dt}$ against time, then the value of the definite integral gives the displacement and the slope of the curve gives the acceleration.

18 PROBABILITY

- The probability $P(A)$ of a particular result A is:

$$P(A) = \frac{\text{number of favourable outcomes}}{\text{number of possible outcomes}},\ 0 \leq P(A) \leq 1$$

- If $P(A) = 0$, then A is impossible.
- If $P(A) = 1$, then A is certain.

Complementary events

- The **complement** of event A is the event 'A **does not occur**', and is denoted by $\bar{A}$. A and $\bar{A}$ are complementary events.
- $P(A) + P(\bar{A}) = 1$
- $P(\bar{A}) = 1 - P(A)$

Mutually exclusive events

- If two or more events cannot occur simultaneously then the events are called **mutually exclusive** or **disjoint**.
- If the events A and B have no points in common and thus are mutually exclusive, then $P(A \text{ or } B) = P(A) + P(B)$.
- If the events A and B have points in common, then $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$.

Multiplication principle

- Tree diagrams can be used to calculate probabilities. To obtain the probability of a final outcome, you can multiply the probabilities of the successive intermediate outcomes along the branches.
- Number of ways a two-part outcome can occur = $m \times n$
  (where m = number of ways the first part can occur, n = number of ways the second part can occur)

Independent events

- Two events are **independent** if the outcome of each event is not affected by the outcome of the other event.
- If events A and B are independent, then:
  $P(A \text{ and } B) = P(A) \times P(B) = P(AB)$

Dependent events

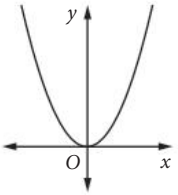
- Two events are **dependent** if the outcome of one event is affected by the outcome of the other event.
- $P(B \mid A)$ is the probability of B occurring given that A has already occurred.
- $P(AB) = P(A) \times P(B \mid A)$

Venn diagrams

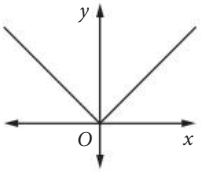
- The set made by combining sets D and F is the **union**, written $D \cup F$. It consists of all the elements that are in D or in F or in both D and F.
- The set of the elements common to sets D and F is the **intersection**, written $D \cap F$.
- $n(D \cup F) = n(D) + n(F) - n(D \cap F)$
  where n is the number of elements in a set D or F.
- If sets are **mutually exclusive** or **disjoint**, then $n(D \cap F) = 0$ and so $n(D \cup F) = n(D) + n(F)$.

SOME BASIC CURVES

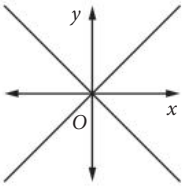
Parabola, $y = x^2$



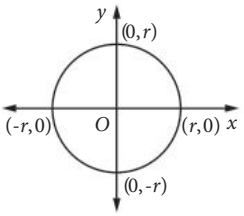
Absolute value, $y = |x|$



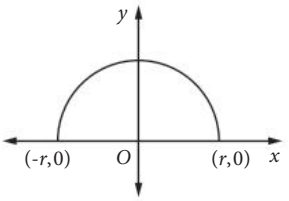
Pair of lines, $x^2 - y^2 = 0$



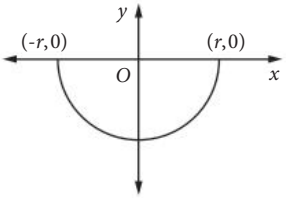
Circle, $x^2 + y^2 = r^2$



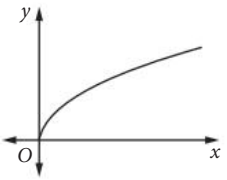
Semicircle, $y = \sqrt{r^2 - x^2}$



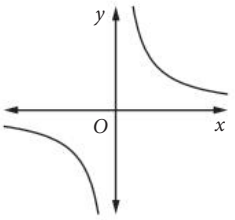
Semicircle, $y = -\sqrt{r^2 - x^2}$



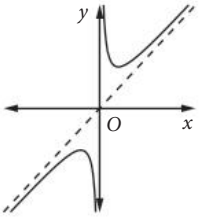
Positive square root, $y = \sqrt{x}$, $x \geq 0$



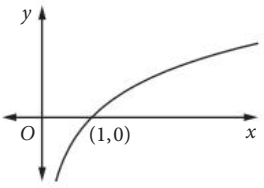
Rectangular hyperbola, $y = \frac{1}{x}$, $x \neq 0$



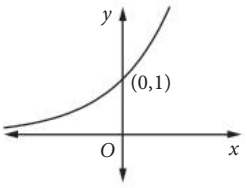
$y = x + \frac{1}{x}$, $x \neq 0$



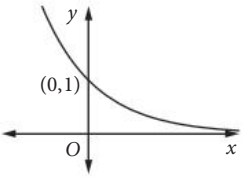
Logarithm, $y = \log_e x$, $x > 0$



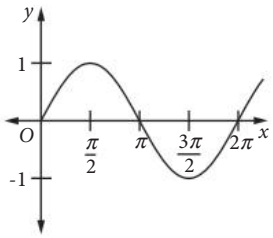
Exponential, $y = e^x$



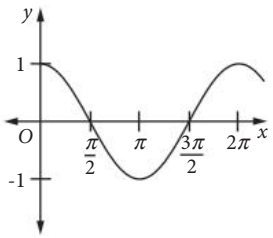
Exponential, $y = e^{-x}$



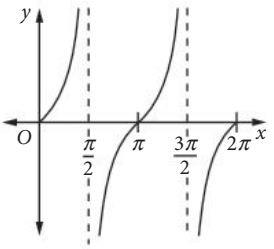
Sine, $y = \sin x$, $0 \leq x \leq 2\pi$



Cosine, $y = \cos x$, $0 \leq x \leq 2\pi$



Tangent, $y = \tan x$, $0 \leq x \leq 2\pi$, $x \neq \frac{\pi}{2}, \frac{3\pi}{2}$



MATHEMATICS COURSE OUTCOMES

Objectives	Preliminary outcomes	HSC outcomes
Students will develop:	A student:	A student:
• appreciation of the scope, usefulness, beauty and elegance of mathematics	P1 • demonstrates confidence in using mathematics to obtain realistic solutions to problems	H1 • seeks to apply mathematical techniques to problems in a wide range of practical contexts
• the ability to reason in a broad range of mathematical contexts	P2 • provides reasoning to support conclusions which are appropriate to the context	H2 • constructs arguments to prove and justify results
• skill in applying mathematical techniques to the solution of practical problems	P3 • performs routine arithmetic and algebraic manipulation involving surds, simple rational expressions and trigonometric identities P4 • chooses and applies appropriate arithmetic, algebraic, graphical, trigonometric and geometric techniques	H3 • manipulates algebraic expressions involving logarithmic and exponential functions H4 • expresses practical problems in mathematical terms based on simple given models H5 • applies appropriate techniques from the study of calculus, geometry, probability, trigonometry and series to solve problems
• understanding of the key concepts of calculus and the ability to differentiate and integrate a range of functions	P5 • understands the concept of a function and the relationship between a function and its graph P6 • relates the derivative of a function to the slope of its graph P7 • determines the derivative of a function through routine application of the rules of differentiation	H6 • uses the derivative to determine the features of the graph of a function H7 • uses the features of a graph to deduce information about the derivative H8 • uses techniques of integration to calculate areas and volumes
• the ability to interpret and communicate mathematics in a variety of forms	P8 • understands and uses the language and notation of calculus	H9 • communicates using mathematical language, notation, diagrams and graphs

MATHEMATICS PERFORMANCE BANDS

The typical performance in this band:

BAND 6

- Exhibits extensive knowledge and skills appropriate to the Mathematics course
- Uses sophisticated multi-step reasoning
- Integrates ideas of calculus with strong algebraic, deductive and modelling skills to successfully solve difficult problems
- Exhibits excellent problem solving skills
- Communicates effectively using appropriate mathematical language, notation, diagrams and graphs

BAND 5

- Exhibits sound knowledge and skills appropriate to the Mathematics course
- Uses multi-step logical reasoning in both numerical and theoretical contexts such as problems in calculus, geometry and probability
- Combines ideas of calculus with algebraic, deductive and modelling skills to successfully solve many difficult problems
- Exhibits a wide range of problem solving skills such as applications of series
- Communicates effectively using mathematical language, notation, diagrams, and graphs

BAND 4

- Exhibits the manipulative skills and knowledge base appropriate to the Mathematics course
- Uses logical reasoning in both numerical and theoretical contexts such as problems in calculus and geometry
- Identifies appropriate approaches to the solution of difficult problems
- Uses calculus and other methods to determine the features of, and to graph, a wide range of functions
- Successfully applies calculus and other appropriate ideas to model practical problems
- Communicates using mathematical language, notation, diagrams and graphs

BAND 3

- Consistently applies arithmetic and algebraic procedures correctly
- Applies geometrical reasoning in a numerical context
- Graphs functions such as $3\sin 2x$, $\log x$ and e^x
- Consistently applies rules of differentiation and basic integration correctly
- Uses calculus to determine the features of, and to graph, functions such as cubic polynomials
- Solves simple problems involving series

BAND 2

- Correctly applies arithmetic and basic algebraic procedures
- Recalls many of the formulae and algorithms appropriate to the Mathematics course, such as Simpson's rule, the sine rule, and the cosine rule
- Graphs simple functions such as linear functions, quadratics, $\sin x$ and $\cos x$
- Finds derivatives of basic functions such as polynomials, $\sin x$ and e^x
- Uses the rules of differentiation such as the product rule
- Solves numerical problems involving the geometry of triangles

MATHEMATICS PERFORMANCE BANDS

This table shows approximate relative skill levels in the performance bands.

Skill level	Band 2	Band 3	Band 4	Band 5	Band 6
1	Correctly applies arithmetic and basic algebraic procedures	Consistently applies arithmetic and algebraic procedures correctly	Exhibits the manipulative skills and knowledge base appropriate to the Mathematics course	Exhibits sound knowledge and skills appropriate to the Mathematics course	Exhibits extensive knowledge and skills appropriate to the Mathematics course
2	Recalls many of the formulae and algorithms appropriate to the Mathematics course, such as Simpson's rule, the sine rule, and the cosine rule	Applies geometrical reasoning in a numerical context	Uses logical reasoning in both numerical and theoretical contexts such as problems in calculus and geometry	Uses multi-step logical reasoning in both numerical and theoretical contexts such as problems in calculus, geometry and probability	Uses sophisticated multi-step reasoning
3	Graphs simple functions such as linear functions, quadratics, $\sin x$ and $\cos x$	Graphs functions such as $3\sin 2x$, $\log x$ and e^x			
4	Finds derivatives of basic functions such as polynomials, $\sin x$ and e^x	Consistently applies rules of differentiation and basic integration correctly	Uses calculus and other methods to determine the features of, and to graph, a wide range of functions		
5	Uses the rules of differentiation such as the product rule	Uses calculus to determine the features of, and to graph, functions such as cubic polynomials	Successfully applies calculus and other appropriate ideas to model practical problems	Combines ideas of calculus with algebraic, deductive and modelling skills to successfully solve many difficult problems	Integrates ideas of calculus with strong algebraic, deductive and modelling skills to successfully solve difficult problems
6	Solves numerical problems involving the geometry of triangles	Solves simple problems involving series	Identifies appropriate approaches to the solution of difficult problems	Exhibits a wide range of problem solving skills such as applications of series	Exhibits excellent problem solving skills
7			Communicates using mathematical language, notation, diagrams and graphs	Communicates effectively using mathematical language, notation, diagrams, and graphs	Communicates effectively using appropriate mathematical language, notation, diagrams and graphs

ANSWERS

CHAPTER 1

EXERCISE 1.1

1 (a) (i) 0.6 (ii) 60% (b) (i) 0.875 (ii) 87.5% (c) (i) 0.72 (ii) 72% (d) (i) 0.025 (ii) 2.5% (e) (i) 0.3125 (ii) 31.25% (f) (i) 0.65 (ii) 65% (g) (i) 3.6 (ii) 360% (h) (i) 4.25 (ii) 425%

2 (a) (i) $\frac{1}{4}$ (ii) 25% (b) (i) $\frac{37}{20}$ (ii) 185% (c) (i) $\frac{3}{8}$ (ii) 37.5% (d) (i) $\frac{23}{100}$ (ii) 23% (e) (i) $\frac{7}{4}$ (ii) 175% (f) (i) $\frac{1}{40}$ (ii) 2.5% (g) (i) $\frac{49}{80}$ (ii) 61.25% (h) (i) $\frac{91}{50}$ (ii) 182%

3 (a) (i) 0.4 (ii) $\frac{2}{5}$ (b) (i) 0.01 (ii) $\frac{1}{100}$ (c) (i) 0.1 (ii) $\frac{1}{10}$ (d) (i) 0.325 (ii) $\frac{13}{40}$ (e) (i) 0.65 (ii) $\frac{13}{20}$ (f) (i) 0.075 (ii) $\frac{3}{40}$ (g) (i) 1.16 (ii) $\frac{29}{25}$ (h) (i) 1.125 (ii) $\frac{9}{8}$

4 B

EXERCISE 1.2

1 $\frac{5}{9}, \frac{2}{3}, \frac{7}{12}, \frac{4}{11}, \frac{2}{13}, \frac{4}{15}$

2 (a) $0.\dot{6}$ (b) $0.\dot{7}$ (c) $0.\dot{3}\dot{6}$ (d) $0.1\dot{3}$ (e) $0.8\dot{3}$ (f) $0.\dot{6}92307$ (g) $0.41\dot{6}$ (h) $0.6\dot{1}$

3 (a) $\frac{5}{9}$ (b) $\frac{8}{45}$ (c) $\frac{68}{99}$ (d) $\frac{19}{90}$ (e) $\frac{17}{33}$ (f) $\frac{7}{12}$ (g) $\frac{136}{99}$ (h) $\frac{323}{90}$

4 $\frac{2}{7} = 0.\dot{2}85714$, $\frac{3}{7} = 0.\dot{4}28571\dot{4}$, $\frac{4}{7} = 0.\dot{5}71428$, $\frac{5}{7} = 0.\dot{7}14285$, $\frac{6}{7} = 0.8571\dot{4}2$

5 $\frac{1}{13} = 0.0\dot{7}6923$, $\frac{2}{13} = 0.1\dot{5}3846$, $\frac{3}{13} = 0.2\dot{3}0769$, $\frac{4}{13} = 0.3076\dot{9}2$, $\frac{5}{13} = 0.3846\dot{1}5$, $\frac{6}{13} = 0.4615\dot{3}8$, $\frac{7}{13} = 0.5384\dot{6}1$, $\frac{8}{13} = 0.6153\dot{8}4$, $\frac{9}{13} = 0.692307$, $\frac{10}{13} = 0.7692\dot{3}0$, $\frac{11}{13} = 0.8461\dot{5}3$, $\frac{12}{13} = 0.923076$

There are two sets of repeating digits: one for the numerators 1, 3, 4, 9, 10, 12, and one for the numerators 2, 5, 6, 7, 8, 11. The sum of pairs of fractions for each set is $0.\dot{9} = 1$.

6 $\frac{1}{17} = 0.0\dot{5}8823529411764\dot{7}$, $\frac{2}{17} = 0.1\dot{1}7647058823529\dot{4}$, $\frac{3}{17} = 0.1\dot{7}6470588235294\dot{1}$, $\frac{4}{17} = 0.2\dot{3}5294117647058\dot{8}$, $\frac{5}{17} = 0.2941176470588235$, $\frac{6}{17} = 0.3\dot{5}2941176470588\dot{2}$, $\frac{7}{17} = 0.4\dot{1}1764705882352\dot{9}$, $\frac{8}{17} = 0.4705882352941176$, $\frac{9}{17} = 0.5\dot{2}9411764705882\dot{3}$, $\frac{10}{17} = 0.5882352941176470$, $\frac{11}{17} = 0.6\dot{4}7058823529411\dot{7}$, $\frac{12}{17} = 0.7058823529411764$, $\frac{13}{17} = 0.\dot{7}64705882352941\dot{1}$, $\frac{14}{17} = 0.8235294117647058$, $\frac{15}{17} = 0.\dot{8}82352941176470\dot{5}$, $\frac{16}{17} = 0.941176470588235\dot{2}$

16 digits repeat. The pattern is difficult to find using only a calculator. Using a spreadsheet set to 17 decimal places, you can obtain just enough information to extrapolate the result, but it needs to be checked by long division to verify that it repeats.

7 C

EXERCISE 1.3

1 (a) 3×10^3 (b) 4.682×10^2 (c) 1.2×10^7 (d) 4.2×10^6 (e) 1.0×10^{-4} (f) 4.1×10^{-3} (g) 2.2×10^{-7} (h) 8.65×10^{-5} (i) 2.155×10^9 (j) 9.3×10^{-4} (k) 2.25×10^9 (l) 5×10^{-8}

2 B

3 (a) 2.41×10^9 (b) 1.8×10^5 (c) 5×10^{-9} (d) 3.52×10^{14} (e) 1.031×10^{-2} (f) 1.12×10^{-7}

4 (a) 4.0×10^6 (b) 3×10^6 (c) 1.44×10^8 (d) 5×10^8

5 B

6 4.023594×10^{13} km

7 1.488×10^{11} m

8 25.6 h

EXERCISE 1.4

1 (a) 9730 (b) 86400 (c) 18.1 (d) 0.373

2 (a) 0.973 (b) 0.864 (c) 18.071 (d) 0.370

3 B

4 (a) 1 (b) 3 (c) 4 (d) 3

5 (a) $61.441 \approx 61$ (b) $1.240 \approx 1.2$ (c) $28.902 \approx 29$

6 (a) $1.255 \approx 1.26$ (b) $0.085 \approx 0.09$ (c) $3.587 \approx 3.59$

7 (a) $1.199 \times 10^9 \approx 1.20 \times 10^9$ (b) $1.655 \times 10^9 \approx 1.66 \times 10^9$. 3 significant figures, because this was the least accurate measurement given in the question.

8 (a) 1.1 (b) $5\frac{1}{2}$ (c) $1\frac{3}{5}$ (d) 0.2 (e) $\frac{9}{16}$ (f) $7\frac{9}{16}$ (g) 9.61 (h) 0.0049 (i) 1.331 (j) $42\frac{7}{8}$ (k) 5 (l) $2\frac{1}{2}$

9 (a) 7.81 (b) 2.42 (c) 140.61 (d) 12.70 (e) 4.63 (f) 0.81 (g) 4.64 (h) 0.32

EXERCISE 1.5

1 (a) 400 (b) 0.7 (c) 3000 (d) 9 (e) 50 000 (f) 0.006 (g) 0.02 (h) 0.2

2 C

3 (a) 480 (b) 2100 (c) 80 (d) 500 (e) 14 (f) 0.4 (g) 10 000 (h) 400 (i) 4 (j) 4500 (k) 1 (l) 1

EXERCISE 1.6

1 $2\sqrt{2}$ **2** $2\sqrt{5}$ **3** $3\sqrt{3}$ **4** $4\sqrt{2}$

5 C **6** $3\sqrt{5}$ **7** $6\sqrt{2}$ **8** $2\sqrt{21}$

9 $7\sqrt{2}$ **10** $6\sqrt{3}$ **11** $5\sqrt{5}$ **12** $9\sqrt{2}$

13 $10\sqrt{2}$ **14** $5\sqrt{64 \times 2} = 40\sqrt{2}$ **15** $80\sqrt{2}$

16 $10\sqrt{6}$ **17** $6\sqrt{13}$ **18** D **19** $\frac{\sqrt{64 \times 5}}{2} = 4\sqrt{5}$

20 $\sqrt{7}$ **21** $12\sqrt{3}$ **22** $\frac{\sqrt{10}}{3}$ **23** B

24 (a) correct (b) incorrect (c) correct (d) incorrect

25 $\sqrt{15}$ **26** 4 **27** $2\sqrt{3}$ **28** $2\sqrt{15}$

29 $8\sqrt{6}$ **30** 8 **31** $10\sqrt{10}$ **32** $24\sqrt{2}$

33 $6\sqrt{35}$ **34** $12\sqrt{6}$ **35** $8\sqrt{6}$ **36** 40

37 $\frac{\sqrt{6}}{2}$ **38** $\frac{2\sqrt{3}}{3}$ **39** 2 **40** $\sqrt{7}$

41 4 **42** $\sqrt{3}$ **43** 1 **44** 3

EXERCISE 1.7

1 $7\sqrt{3}$ **2** $7\sqrt{7}$ **3** $6\sqrt{5}$ **4** $3\sqrt{2} + 3\sqrt{3}$

5 $4\sqrt{5} - 5\sqrt{2}$ **6** $\sqrt{2}$ **7** A **8** $3\sqrt{5}$

9 $6\sqrt{2}$ **10** $7\sqrt{3}$ **11** $7\sqrt{3}$ **12** $\sqrt{2}$ **13** 0

14 (a) incorrect (b) incorrect (c) correct (d) correct

15 $4\sqrt{5} + 4\sqrt{7}$ **16** $8\sqrt{3} - 3\sqrt{5}$ **17** $3\sqrt{5} + 3\sqrt{2}$

18 $5\sqrt{15} - 2\sqrt{10}$ **19** $5\sqrt{7}$ **20** $\sqrt{2} + \sqrt{3}$

21 $12\sqrt{3}$ **22** $5\sqrt{6} - 10\sqrt{2}$ **23** $6\sqrt{6}$

24 $\sqrt{7} + 3\sqrt{5}$ **25** $15\sqrt{5} - 8\sqrt{2}$ **26** $7\sqrt{2} - 4\sqrt{5} - 2\sqrt{3}$

27 $5\sqrt{5}$ **28** $9\sqrt{3}$ **29** $-\sqrt{6}$ **30** 0

EXERCISE 1.8

1 $\sqrt{10} + \sqrt{15}$ **2** $5 + \sqrt{10}$ **3** 6 **4** $\sqrt{6} - 3\sqrt{2}$

5 $3\sqrt{2} - 2\sqrt{6}$ **6** $14\sqrt{5} - 7$ **7** B **8** $12\sqrt{3} - 3\sqrt{10}$

9 $a + \sqrt{ab}$ **10** $x - \sqrt{xy}$ **11** $\sqrt{35} - \sqrt{10} + \sqrt{21} - \sqrt{6}$

12 $\sqrt{6} + 4 + \sqrt{21} + 2\sqrt{14}$ **13** $\sqrt{6} + 3\sqrt{3} - \sqrt{2} - 3$

14 $16 + 7\sqrt{5}$ **15** $-3 - 4\sqrt{3}$ **16** $\sqrt{30} + \sqrt{5} + 3\sqrt{2} + \sqrt{3}$

17 $9 - 3\sqrt{7}$ **18** $8 - 3\sqrt{6}$ **19** $20 + 6\sqrt{5} - 2\sqrt{10} - 3\sqrt{2}$

20 $5\sqrt{6} - 8\sqrt{2}$ **21** $4 + 2\sqrt{3}$ **22** $7 - 2\sqrt{10}$ **23** $27 + 12\sqrt{2}$

24 $39 + 12\sqrt{3}$ **25** 1 **26** 1 **27** 2 **28** 7

29 21 **30** 4 **31** 3 **32** D **33** $8 - 2\sqrt{15}$

34 $18 + 2\sqrt{77}$ **35** $4 + 3\sqrt{3}$ **36** 1 **37** 1

38 $11 + 4\sqrt{6}$ **39** 37 **40** $\sqrt{30} - \sqrt{5} + 4\sqrt{3} - 2\sqrt{2}$

41 $3 + 15\sqrt{2}$

42 (a) incorrect (b) correct (c) correct (d) incorrect

43 34 **44** $82 + 12\sqrt{42}$ **45** $15 - 25\sqrt{3}$

46 $35 + 12\sqrt{6}$ **47** 11 **48** $13 - 4\sqrt{3}$

EXERCISE 1.9

1 $\frac{2\sqrt{3}}{3}$ 2 $\frac{\sqrt{15}}{3}$ 3 $\sqrt{3}$ 4 $\sqrt{3}+\sqrt{2}$

5 $\frac{2\sqrt{7}-\sqrt{6}}{28-6}=\frac{2\sqrt{7}-\sqrt{6}}{22}$ 6 $\sqrt{5}-2$ 7 $\frac{2\sqrt{5}+3\sqrt{2}}{2}$

8 $\frac{3(\sqrt{10}+\sqrt{6})}{2}$ 9 $\sqrt{30}+3\sqrt{3}$ 10 $\frac{6\sqrt{55}+8\sqrt{10}}{67}$

11 $\frac{(\sqrt{3}+2)^2}{3-4}=-(7+4\sqrt{3})$ 12 $\frac{(4\sqrt{2}+3\sqrt{5})(2\sqrt{5}+\sqrt{2})}{20-2}=\frac{38+11\sqrt{10}}{18}$

13 $\frac{3\sqrt{35}+2\sqrt{14}-30-4\sqrt{10}}{37}$ 14 $5+2\sqrt{6}$

15 $\frac{25\sqrt{15}+45+75+9\sqrt{15}}{125-27}=\frac{17\sqrt{15}+60}{49}$ 16 $\frac{18+4\sqrt{3}}{23}$ 17 $\frac{12\sqrt{3}-9\sqrt{2}}{5}$

18 $\frac{9+\sqrt{11}}{5}$ 19 $-\frac{2+\sqrt{6}}{2}$ 20 $\frac{18-3\sqrt{6}}{10}$

21 $\frac{\sqrt{3}}{2\sqrt{6}-4\sqrt{3}}=\frac{1}{2(\sqrt{2}-2)}=-\frac{\sqrt{2}+2}{4}$ 22 $3-2\sqrt{2}$ 23 $\frac{11-2\sqrt{10}}{9}$

24 $\frac{6+5\sqrt{2}}{7}$ 25 $\frac{11-2\sqrt{14}}{13}$ 26 $\frac{10\sqrt{2}+\sqrt{30}+2\sqrt{30}+3\sqrt{2}}{40-6}=\frac{13\sqrt{2}+3\sqrt{30}}{34}$

27 $\frac{12\sqrt{3}+2\sqrt{30}-45\sqrt{2}-15\sqrt{5}}{39}$ 28 A

29 $\left(\sqrt{3}+1\right)^2-\frac{1}{\left(\sqrt{3}+1\right)^2}=4+2\sqrt{3}-\frac{1}{4+2\sqrt{3}}=\frac{6+5\sqrt{3}}{2}$ 30 $\frac{17\sqrt{2}}{6}$

31 $\frac{7\sqrt{5}-15}{4}$ 32 LHS $=\left(2\sqrt{2}-3\right)^2+6\left(2\sqrt{2}-3\right)+1$

$=8-12\sqrt{2}+9+12\sqrt{2}-18+1=0=$ RHS

33 (a) incorrect (b) correct (c) incorrect (d) correct

34 LHS $=\left(\sqrt{5}-1\right)^3+3\left(\sqrt{5}-1\right)^2-2\left(\sqrt{5}-1\right)-4$

$=\left(8\sqrt{5}-16\right)+3\left(6-2\sqrt{5}\right)-2\sqrt{5}+2-4$

$=8\sqrt{5}-16+18-6\sqrt{5}-2\sqrt{5}-2$

$=0=$ RHS

35 LHS $=2\left(\frac{1}{\sqrt{3}-1}\right)^2-2\left(\frac{1}{\sqrt{3}-1}\right)-1=\frac{2\left(\sqrt{3}+1\right)^2}{(3-1)^2}-\frac{2\left(\sqrt{3}+1\right)}{3-1}-1$

$=\frac{4+2\sqrt{3}}{2}-\sqrt{3}-1-1$

$=2+\sqrt{3}-\sqrt{3}-2=0=$ RHS

36 $\frac{2\sqrt{3}+1}{11}+\frac{3\left(\sqrt{3}-1\right)}{2}=\frac{4\sqrt{3}+2+33\sqrt{3}-33}{22}=\frac{37\sqrt{3}-31}{22}$ 37 $\frac{4\sqrt{10}}{3}$

38 $\frac{3\sqrt{6}-7}{3\sqrt{6}+7}=\frac{103-42\sqrt{6}}{5}$ 39 $\frac{5\sqrt{10}-14}{9}$ 40 $\frac{21+4\sqrt{5}}{19}-\frac{7+\sqrt{5}}{11}=\frac{98+25\sqrt{5}}{209}$

41 $4\sqrt{2}-1$ 42 $\frac{51\sqrt{3}+3\sqrt{15}-98}{17}$

43 $\frac{2}{\sqrt{2}-\sqrt{3}}-\frac{3}{\sqrt{5}-\sqrt{2}}=-\left(3\sqrt{2}+2\sqrt{3}+\sqrt{5}\right)$ 44 $\frac{\sqrt{3}-1}{2}$

CHAPTER REVIEW 1

1 6×10^{-7} 2 (a) 1 (b) $\frac{8}{15}$ (c) $\frac{2}{3}$

3 (a) $\frac{2}{3}$ (b) $\frac{1202}{999}$ (c) $\frac{3578}{49995}$

4 (a) $2.484\approx2.48$ (b) $0.0375\approx0.04$ (c) $1.17558\approx1.18$

5 (a) $9\sqrt{3}-9\sqrt{6}$ (b) $\frac{14\sqrt{5}}{5}$ (c) $\frac{7-2\sqrt{10}}{3}$ (d) 15

6 $57-12\sqrt{15}$ 7 D 8 $\frac{3-\sqrt{2}}{7}$ 9 289

10 LHS $=\left(\frac{\sqrt{5}-1}{2}\right)^3+3\left(\frac{\sqrt{5}-1}{2}\right)^2+\left(\frac{\sqrt{5}-1}{2}\right)-2$

$=\frac{8\sqrt{5}-16}{8}+\frac{18-6\sqrt{5}}{4}+\frac{\sqrt{5}-1}{2}-2$

$=\sqrt{5}-2+\frac{9-3\sqrt{5}}{2}+\frac{\sqrt{5}-1}{2}-2$

$=\sqrt{5}-4+4-\sqrt{5}=0=$ RHS

CHAPTER 2

EXERCISE 2.1

1 $10x+15$ 2 $10x-5$ 3 $3a-3b$

4 $9ab+9a$ 5 $4xy$ 6 $6mn$

7 $5a^2b-3ab^2$ 8 $8xy^2-2xy$ 9 $x^2y+6x^2y^2$

10 $6abc$ 11 p^2q-5pq 12 $6mn+2m$

13 $5x^2-x$ 14 $6a^2-8a$ 15 $5y^2-2x^2$

16 $3x^2+2xy$ 17 $2a-3b$ 18 $2x-4y$

19 $15n-2m$ 20 $10x+11y$ 21 $5x-20$

22 $15a+15b$ 23 $11k+1$ 24 $2a^2+7a$

25 $11x^2-13xy$ 26 $7a^2+2ab$ 27 $a+4b$

28 $3y-2x$ 29 $5x^2-5xy$ 30 $9x^2+4x$

31 $9m^2-8mn$ 32 $27x-42$ 33 $-x^2-8x-12$

34 $3x^2-10x+4$ 35 $x^2+14x-1$ 36 $4x+9y-12$

37 C 38 B

EXERCISE 2.2

1 64 2 96 3 10 4 (a) 140 (b) 5

5 38.5 6 554.2 7 (a) 25.6 (b) 3 8 38

9 (a) 29.3 (b) 8 10 15 11 7.3 12 55.0

13 (a) 7.0 (b) 9.6 14 24 15 (a) 7.6 (b) 5

16 $6\frac{2}{3}\approx6.7$ 17 2 18 65 19 92.4

20 3.6 21 33.0 22 (a) 4 (b) 39.8

23 1.0 24 (a) $11\frac{1}{9}\approx11.1$ (b) 100 25 1210

EXERCISE 2.3

1 x^2+6x+5 2 x^2-5x+6 3 a^2+a-12

4 a^2+a-20 5 x^2-4x+4 6 $y^2+14y+49$

7 $2x^2+13x+15$ 8 $3x^2-10x+8$ 9 $6m^2+11m-7$

10 $35x^2-11x-6$ 11 $9x^2+12x+4$ 12 $4p^2-81$

13 $6x^2+13x+6$ 14 $2x^2+x-1$ 15 $16p^2-40p+25$

16 $9x^2+24x+16$ 17 $2x^3-3x^2-8x-3$

18 $6x^3-22x^2+24x-8$ 19 x^3-4x 20 x^3-4x^2+5x-2

21 $2x^3-12x^2+22x-12$ 22 $x^4-2x^3+2x^2-10x-15$

23 x^3+2x^2-4x-8 24 $x^6-3x^4y^2+3x^2y^4-y^6$ 25 C

26 (a) incorrect (b) correct (c) incorrect (d) correct

EXERCISE 2.4

1 $(a+b)(x+2)$ 2 $(3a-m)(2b-3c)$ 3 $(p+q-r)(a+b)$

4 $(x^2+4)(2x-1)$ 5 $(a+b)(x+4)$ 6 $(x+z)(x-y)$

7 $(2x+1)(y+z)$ 8 $(a-c)(a-b)$ 9 $(2x+5y)(2-5y)$

10 $(a^2+b^2)(a+3b)$ 11 $(c-2d)(a-2b)$ 12 $(3y+7)(x-2)$

13 $(x-z)(x-2y)$ 14 $(a^2-b)(a-b)$ 15 $(2m+pn)(n+p)$

16 $(x^2+4)(x+3)$ 17 $(pq+5)(p-q)$ 18 $(m^2+n)(p+1)$

19 $(x^2+1)(y+1)$ 20 $(a-4)(b-3)$ 21 $(2-y)(x-3y)$

22 A 23 (a) incorrect (b) correct (c) correct (d) incorrect

EXERCISE 2.5

1 $(m-1)(m+1)$ 2 $(x-4)(x+4)$ 3 $(8-m)(8+m)$

4 $(3a-5)(3a+5)$ 5 $(x-0.6)(x+0.6)$ 6 $(ab-c)(ab+c)$

7 $(3x-2y)(3x+2y)$ 8 $(x-2)(x+4)$ 9 $(x-yz)(x+yz)$

10 $\left(\frac{a}{5}-1\right)\left(\frac{a}{5}+1\right)$ 11 $\left(p-\frac{1}{2}\right)\left(p+\frac{1}{2}\right)$ 12 $\left(\frac{x}{2}-\frac{1}{3}\right)\left(\frac{x}{2}+\frac{1}{3}\right)$

13 $a(a+4)$ 14 $(x-y-z)(x+y+z)$

15 $98\times100=9800$ 16 $46\times1000=46\,000$

17 $ab(a-b)(a+b)$ 18 $3a(2a-b)(2a+b)$

19 $3y(x-3)(x+3)$ 20 $(x+y-2)(x+y+2)$

21 $b(2a-b)$ 22 $(x-3)(x+3)(x-y)$

23 $(x-2)(x+2)(x+3)$ 24 $(p-4)(p+4)(q-1)$

25 $x(a-1)(a+1)$ 26 $3(4a-5b)(4a+5b)$

27 $h(2+h)$ 28 $\left(\frac{x}{5}-y\right)\left(\frac{x}{5}+y\right)$ 29 D

30 (a) correct (b) correct (c) incorrect (d) correct

EXERCISE 2.6

1 $(y-5)(y^2+5y+25)$ 2 $(z+1)(z^2-z+1)$

3 $(2p+3)(4p^2-6p+9)$ 4 $(6-a)(36+6a+a^2)$

5 $(2x+3)(x^2+3x+39)$ 6 $(x+7)(7x^2-x+13)$

7 $(b-a)(b+a)(b^4+b^2a^2+a^4)$

$=(b-a)(b+a)(b^2+ba+a^2)(b^2-ba+a^2)$

8 $8(2a+b)(4a^2-2ab+b^2)$

9 $\frac{4\pi}{3}\left(R-r\right)\left(R^2+Rr+r^2\right)$ 10 $p^4x^4(p-x)(p^2+px+x^2)$

11 $(x^2+y^2)(x^4-x^2y^2+y^4)$ 12 $\left(\frac{2}{a}-\frac{3}{b}\right)\left(\frac{4}{a^2}+\frac{6}{ab}+\frac{9}{b^2}\right)$

13 $(a-b)(m+n)(a^2+ab+b^2)(m^2-mn+n^2)$

14 $(2x-3)(2x+3)(x-1)(x^2+x+1)$

15 $h(3x^2+3xh+h^2)$ 16 $(2a-b)(a^2-ab+b^2)$

17 $2b(3a^2+b^2)$ 18 $2(12x^2+1)$
19 $x(12-6x+x^2)$ 20 $a^2b(ab-1)(a^2b^2+ab+1)$
21 $2(x-y+3)(x^2-2xy+y^2-3x+3y+9)$ 22 B 23 D

EXERCISE 2.7

1 $(x+1)(x+3)$ 2 $(x+3)(x+7)$ 3 $(x+3)(x+8)$
4 $(x+5)(x+1)$ 5 $(a+6)(a+1)$ 6 $(a+4)(a+8)$
7 $(m+5)(m+4)$ 8 $(p+6)(p+3)$ 9 $(p+2)(p+9)$
10 $(p+1)(p+18)$ 11 $(x+4)(x+3)$ 12 $(x+1)(x+12)$
13 $(x+6)(x+2)$ 14 $(x-3)(x-4)$ 15 $(x-12)(x-1)$
16 $(x-2)(x-6)$ 17 $(m-5)(m-4)$ 18 $(m-10)(m-2)$
19 $(m-20)(m-1)$ 20 $(x-13)(x-1)$ 21 $(p+5)(p-3)$
22 $(p+15)(p-1)$ 23 $(p-5)(p+3)$ 24 $(p-15)(p+1)$
25 $(x-7)(x+5)$ 26 $(x-5)(x+2)$ 27 $(x+9)(x+8)$
28 $(a-6)(a+2)$ 29 $(x-6)(x-1)$ 30 $(x-9)(x+8)$
31 $(x+12)(x-6)$ 32 $(x-24)(x+3)$ 33 $(a+10)(a+3)$
34 $(x-7)(x+6)$ 35 $(x-21)(x+2)$ 36 $(x+21)(x-2)$
37 D 38 (a) incorrect (b) correct (c) incorrect (d) correct

EXERCISE 2.8

1 $(2x+1)(x+1)$ 2 $(3x-1)(x+4)$ 3 $(2x+3)(x+2)$
4 $(4a+1)(a+3)$ 5 $(4x+1)(x+1)$ 6 $(3a-2)(a-1)$
7 $(4x-1)(2x-3)$ 8 $(2x+1)(x-5)$ 9 $(13c+6)(c-1)$
10 $(4x+5)(2x+1)$ 11 $(3x+4)(2x+3)$ 12 $(3x-1)(x-4)$
13 $(3x-2)(x-5)$ 14 $(3a+7)(2a-9)$ 15 $(3x+1)(x-4)$
16 $(5x-8)(2x+1)$ 17 $(2x-1)(x+2)$ 18 $(2x-3)^2$
19 $(3x-2)^2$ 20 $(x-2)(2x-5)$ 21 $(6x-1)(x-14)$
22 $(5m+9)(3m-2)$ 23 $(5a+2)(2a-9)$ 24 $(y-3)(y+1)$
25 $2(3y-1)(2y+3)$ 26 $(3x-2)(2x-7)$ 27 $(3x-4)(2x-7)$
28 $(6x-7)(x-2)$ 29 $2(3x-7)(x-1)$ 30 $(4x+3)(2x-1)$
31 $(p+3)(6p+7)$ 32 $(5a+2)(2a-3)$ 33 $(6y-1)(2y+5)$
34 $(8x-9)(3x-4)$ 35 $(5x-3)(3x-2)$ 36 $(3x+1)(x-1)$
37 $(3x+5)(3x-2)$ 38 $(2x-1)(x-4)$ 39 $(5p+3)(2p+1)$
40 $(5x+3)(x-1)$ 41 C
42 (a) correct (b) correct (c) correct (d) incorrect

EXERCISE 2.9

1 $x(x-3)$ 2 $2a(a-2)(a+2)$ 3 $3x(x+3)$
4 $x(x-8)$ 5 $(x-3)(x+3)$ 6 $(x-9)(x+1)$
7 $3y(x-2y)(x+2y)$ 8 $5xy(x-2y)(x+2y)$
9 $(1-b-c)(1+b+c)$ 10 $(10x-1)(x+1)$
11 $a(a+2b)$ 12 $6(x-2)(x+2)$ 13 $(a-7)(a+6)$
14 $(a-b)(m+n)$ 15 $2x(x+8)(x-1)$ 16 $3a(a+7)(a+1)$
17 $(x+2y-2)(x+2y+2)$ 18 $(x-5)(x+2)$
19 $(a-1)(a+1)$ 20 $ab(b+c+d)$ 21 $(x-6y)(x+6y)$
22 $x(x+1)$ 23 $(x+4)(x-3)$ 24 $(x+y)(y-z)$
25 $4(x-15)(x+8)$ 26 $b(x-7y)^2$ 27 $3y(2y-1)(y+1)$
28 $2y(3y+1)(y+4)$ 29 $15(a-2)(a+2)$
30 $mn(3-5mn)(3+5mn)$ 31 $5x(a-5)(a+5)$
32 $4xy$ 33 $5t(t+9)(t-8)$ 34 $(m+6)(m-n)$
35 $(x-2)(x+2)(x+3)$ 36 $(x-l)(mx-y)$ 37 D
38 $4r^2-\pi r^2=(4-\pi)r^2$ 39 $(\pi-2)r^2$ 40 $\pi(R-r)(R+r)$
41 $(8-\pi)r^2$ 42 $\pi a(a-b)$ 43 $\pi R^2-4r^2=\pi(R-2r)(R+2r)$
44 $16r-4\pi r^2=4(4-\pi)r^2$ 45 $a(a+b+c)$
46 (a) correct (b) correct (c) incorrect (d) correct

EXERCISE 2.10

1 $2a-b$ 2 $\frac{3x+2y}{3}$ 3 7 4 $\frac{2x-y}{2y}$ 5 $\frac{2(2a-b)}{3a}$
6 2 7 $\frac{1}{a}$ 8 $1+m$ 9 $m-n$ 10 $p-q$
11 $\frac{x+y}{2}$ 12 $\frac{2(s-6)}{r+s}$ 13 $\frac{x-y}{x+y}$ 14 k 15 $\frac{x-3}{x}$
16 $\frac{5a}{b}$ 17 $\frac{4x}{x+y}$ 18 $\frac{(x-6)(x-1)}{(x-6)(x+6)}=\frac{x-1}{x+6}$ 19 $\frac{a}{b}$ 20 $\frac{a-b}{a}$
21 $\frac{x+1}{x-4}$ 22 $\frac{(x-2)(x-4)}{(x-2)(x+1)}=\frac{x-4}{x+1}$ 23 $\frac{x+1}{x-2}$ 24 $\frac{x-2}{x+4}$
25 $\frac{x+2}{x-5}$ 26 $\frac{4xy(x-2)(x+2)}{(x+4)(x-2)}=\frac{4xy(x+2)}{x+4}$ 27 1 28 $\frac{x^2}{y}$
29 C 30 $\frac{a(2a-3b)}{b(a-b)}\times\frac{2a(a-b)}{2(2a-3b)}=\frac{a^2}{b}$ 31 $\frac{5x(3x-y)}{10xy}\times\frac{2y}{3x-y}=1$
32 $\frac{4x(3x-1)}{x(3x-1)}\times\frac{5x^2y^2}{10x^2y}=2y$ 33 1 34 $\frac{a-b}{b}$
35 (a) incorrect (b) incorrect (c) incorrect (d) correct
36 $\frac{x^2-xy+y^2}{x-y}$ 37 $\frac{2}{2x-1}$ 38 $\frac{2}{x^2+xy+y^2}$
39 $3x^2+3xh+h^2$ 40 $\frac{3x^2-3xy+y^2}{2x-y}$

EXERCISE 2.11

1 $\frac{x}{30}$ 2 $\frac{7x}{8}$ 3 $\frac{10a+24a-5a}{30}=\frac{29a}{30}$ 4 $\frac{11y}{12}$
5 $\frac{3(a+2)-5(a-1)}{15}=\frac{11-2a}{15}$ 6 $\frac{3x+y}{6}$ 7 $\frac{3x+1}{12}$
8 $\frac{7m-3n}{10}$ 9 $\frac{6x+3y-4(x+y)}{12}=\frac{2x-y}{12}$ 10 $\frac{-(a+8b)}{18}$
11 $\frac{7a+11b}{12}$ 12 $\frac{3}{3x}-\frac{2}{3x}=\frac{1}{3x}$ 13 $\frac{3a+1}{a^2}$ 14 $\frac{1-2a}{ab}$
15 $\frac{m^2-n^2}{mn}$ 16 $\frac{4z+3x}{xyz}$ 17 $\frac{5b-2a}{a^2b^2}$ 18 $\frac{4a-11}{6a}$
19 $\frac{3+2x+2}{3(x+1)}=\frac{2x+5}{3(x+1)}$ 20 $\frac{3}{x}-\frac{1}{x^2}=\frac{3x-1}{x^2}$ 21 B

EXERCISE 2.12

1 $(x-3)(x+3)$ 2 $x^2-25=(x-5)(x+5)$ 3 $x(x-2)$
4 $6(x-2)$ 5 $x^2-4x=x(x-4)$ 6 $x(x-4)(x+4)$
7 $x^2+4x+4=(x+2)^2$ 8 $(x-y)(x+y)$ 9 $xy(x-y)(x+y)$
10 A 11 $\frac{2a}{(a-b)(a+b)}$ 12 $\frac{3(x+y)-2(x-y)}{(x-y)(x+y)}=\frac{x+5y}{(x-y)(x+y)}$
13 $\frac{x+y}{x-y}$ 14 $\frac{x^2+2xy-y^2}{(x-y)(x+y)}$ 15 $\frac{3a-b+a+b}{(a-b)(a+b)}=\frac{4a}{(a-b)(a+b)}$
16 $\frac{3-x}{(x-2)(x+2)}$ 17 $\frac{1}{x+y}$ 18 $\frac{3+2(x-2)}{(x-2)^2}=\frac{2x-1}{(x-2)^2}$
19 $\frac{4}{(x-1)(x-3)(x+1)}$ 20 $\frac{4x+7}{(x-2)(x+3)}$ 21 $\frac{-2y}{(x+y)(x-y)}$
22 $\frac{7a-15}{2(a-3)(a+3)}$ 23 $\frac{x+20}{(x-5)(x+5)}$ 24 $\frac{22}{(3x-2)(4x+1)}$
25 $\frac{-a(a+1)}{(3a-4)(2a-3)}$ 26 $\frac{y+x-y}{x(x-y)}=\frac{1}{x-y}$ 27 $\frac{2(x-2)}{(x-4)(x+4)}$
28 $\frac{2}{x-2}$ 29 $\frac{a+2+a+4}{(a+3)(a+2)}=\frac{2}{a+2}$ 30 $\frac{x-7}{(x-2)(x+2)(x-1)}$
31 $\frac{(x+1)^2-(x-1)^2}{(x-1)(x+1)}=\frac{4x}{(x-1)(x+1)}$ 32 $\frac{5x+13}{(x-2)(x+2)}$
33 $\frac{3x-2(x-4)}{(x-4)(x+4)}=\frac{x+8}{(x-4)(x+4)}$ 34 $\frac{5(x-5)-6}{3x(x-5)}=\frac{5x-31}{3x(x-5)}$
35 (a) incorrect (b) correct (c) incorrect (d) incorrect

CHAPTER REVIEW 2

1 (a) $4x^2-20x+25$ (b) $x^2-4x-21$ (c) $6y^2+11y+4$
(d) $25x^2-16$ (e) $2x^3-3x^2y+3xy^2-y^3$
2 (a) $(9-2a)(9+2a)$ (b) $5y(2x-1)$ (c) $5xy(x-2y-1)$
(d) $(a-4)(a-14)$ (e) $(4x-1)(4x+1)$ (f) $(3x+7)(x-1)$
(g) $(a-b)(a+b)+2(a-b)=(a-b)(a+b+2)$
(h) $x(3+7x-6x^2)=x(3-2x)(1+3x)$
(i) $(2a-3)(4a^2+6a+9)$
(j) $(2-x-h)(4+2(x+h)+(x+h)^2)$
(k) $(4y+3)(2y-3)$ (l) $x(x-2)(x^2+2x+4)$
3 $\sqrt{25}=5$
4 (a) $\frac{3x^3}{4a^2}\times\frac{a(y-1)}{xy^2}\times\frac{4ay^2}{3(y-1)}=x^2$ (b) $\frac{20x-6x-9+2x}{12}=\frac{16x-9}{12}$ (c) $2n$
(d) $\frac{4-2x}{x^2}$ (e) $\frac{3}{x+1}+\frac{1}{(x+1)^2}=\frac{3x+4}{(x+1)^2}$ (f) $\frac{x(2x-y)}{y(x-y)}\times\frac{2x(x-y)}{2(2x-3y)}=\frac{x^2}{y}$
(g) $\frac{a^2-ab+b^2}{a-b}$ (h) $\frac{m-4}{(m-2)(m+2)(m-1)}$ (i) $\frac{2}{(x-4)(x-6)}$
5 (a) $a-b$ (b) $a-\frac{2\sqrt{a}}{\sqrt{b}}+\frac{1}{b}$ 6 $\frac{3}{8}$ 7 (a) 416 (b) 25

CHAPTER 3

EXERCISE 3.1

1 4.5 2 4.6 3 $3x+9=27-9x$, $12x=18$, $x=1.5$
4 5 5 -8.75 6 1 7 -1.25 8 -1

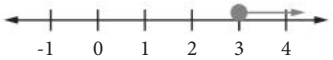
9 -1 **10** $12a + 8 - 18 + 6a = 8$, $18a = 18$, $a = 1$
11 2.5 **12** 1.5 **13** 2.25 **14** 6.5
15 $4x + 20 - x + 1 = 9$, $3x = -12$, $x = -4$ **16** 4
17 -1 **18** -14.75
19 $12x - 18 = 8 - 6x - 2$, $18x = 24$, $x = 1\frac{1}{3}$ **20** -1.5

EXERCISE 3.2

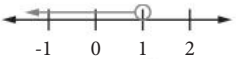
1 24 **2** 40 **3** 30 **4** $14x + 35 = 3x + 6 + 84$, $x = 5$
5 $2\frac{1}{3}$ **6** $\frac{1}{6}$ **7** -0.75 **8** $10\frac{2}{3}$ **9** $4\frac{6}{7}$
10 3 **11** 4.6 **12** $4(x + 4) - 2(3 - 4x) = 5 - x$, $x = -\frac{5}{13}$
13 -2 **14** -3.5 **15** 3.25 **16** -2
17 -0.25 **18** 13 **19** -0.5 **20** -2.5
21 $-\frac{1}{7}$ **22** C **23** 15 **24** -4.2
25 $11\frac{2}{3}$ **26** $3(x + 2) - 2(x - 2) = 1$, $x \neq \pm 2$, $x = -9$
27 -7 **28** $\frac{2}{3}$ **29** 1
30 $x + 2 + x + 1 = 1$, $x \neq -1$, $x \neq -2$, no solution **31** 0.5
32 (a) incorrect (b) incorrect (c) incorrect (d) incorrect

EXERCISE 3.3

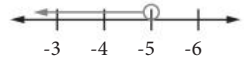
1 $x \geq 3$



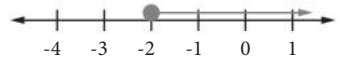
2 $x < 1$



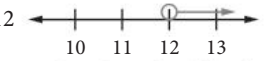
3 $x < -5$



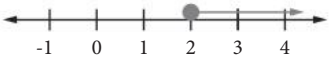
4 $x \geq -2$



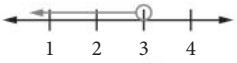
5 $x > 12$



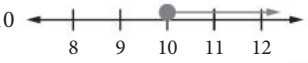
6 $x \geq 2$



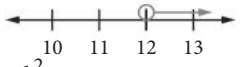
7 $x < 3$



8 $x \geq 10$

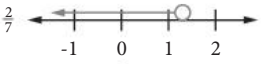


9 $4x - 3x > 12$, $x > 12$



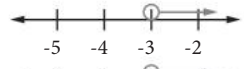
$1\frac{2}{7}$

10 $x < 1\frac{2}{7}$

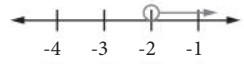


11 $x \geq 1.5$

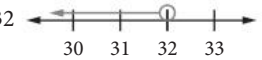
12 $x > -3$



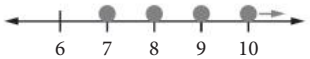
13 $x > -2$



14 $x < 32$

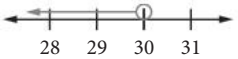


15 $x > 6$ so $x = 7, 8, 9 \ldots$

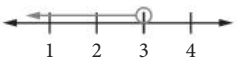


16 D

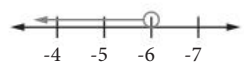
17 $-x > -30$, $x < 30$



18 $x < 3$

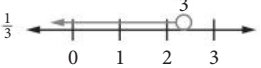


19 $x < -6$

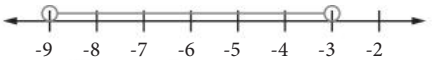


$2\frac{1}{3}$

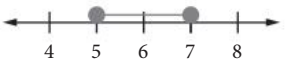
20 $x < 2\frac{1}{3}$



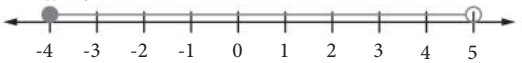
21 $-9 < x < -3$



22 $5 \leq x \leq 7$



23 $-4 \leq x < 5$



24 (a) incorrect (b) correct (c) incorrect (d) correct
25 $5 < x - 5 < 12$, $x = 11, 12, 13, 14, 15, 16$
26 $8 < x < 16$
27 $x + (x + 1) \leq 35$, $x = 1, 2, 3, \ldots 17$
28 5, 6, 7, 8, 9
29 $\frac{x+5}{5} \leq \frac{x+13}{9}$, $x \leq 5$, $x = 5$
30 base length = x cm, equal sides = y cm
$2y = x + 4$, $2y + x < 80$
$0 < x < 38$ where x is an integer
31 $7 < x + (x + 1) + (x + 2) < 25$, $x = 2, 3, 4, 5, 6, 7$
32 $2 < x < 18$

EXERCISE 3.4

1 0, 5 **2** 2, 3 **3** 0, -0.5 **4** -2.5, 7
5 0, 4.5 **6** 0, -1 **7** a, b **8** $3a$, $-2b$
9 2, -2 **10** 5.5, -5.5 **11** 1 **12** -1.5

EXERCISE 3.5

1 1, -1 **2** 5, -5 **3** 7, -7
4 4, -4 **5** C **6** 1.5, -1.5
7 2.5, -2.5 **8** 2.5, -2.5 **9** 1.5, -1.5
10 1, -1 **11** $\frac{1}{4}$, $-\frac{1}{4}$ **12** 4, -4
13 1.4, -1.4
14 (a) correct (b) correct (c) incorrect (d) correct
15 6, -2 **16** 7, -7 **17** 3, -3
18 1, -0.6 **19** -1, -5 **20** $\sqrt{5}$, $-\sqrt{5}$
21 $\sqrt{2}$, $-\sqrt{2}$ **22** $-1 + 2\sqrt{2}$, $-1 - 2\sqrt{2}$

EXERCISE 3.6

1 0, 6 **2** 0, 5 **3** 0, -5 **4** 0, -10
5 D **6** 0, 2.5 **7** 0, 7 **8** 0, 7
9 0, -10 **10** 0, 4 **11** 0, 0.5 **12** 0, -0.2
13 0, -3 **14** 0, $\frac{5}{12}$ **15** 0, 15

EXERCISE 3.7

1 1, 2 **2** 1, 5 **3** 4, -2 **4** 1, 3
5 3 **6** 1, 4 **7** -1, -8 **8** -1, $\frac{5}{9}$
9 B **10** 2, -6 **11** 2, 0.2 **12** 3.5, -0.5
13 2.5, -2 **14** -5 **15** -1, -4 **16** 3.5, -1.5
17 1, $8\frac{1}{3}$ **18** 4 **19** -4, -1.2 **20** 12, $1\frac{2}{3}$
21 3, -10 **22** $5x^2 - 8x + 3 = 0$, $(5x - 3)(x - 1) = 0$, $x = 1, 0.6$
23 $2x^2 - 11x - 6 = 0$, $(2x + 1)(x - 6) = 0$, $x = 6, -0.5$
24 1, -6 **25** -9, $2\frac{2}{3}$ **26** 3, 5 **27** 3, -2.5
28 2, -6 **29** 0.5, -1.5
30 $x^2 + 2x + 1 = 4x$, $x^2 - 2x + 1 = 0$, $(x - 1)^2 = 0$, $x = 1$
31 $(x + 6)(x + 6 - 1) = 0$, $x = -5, -6$ **32** $\frac{2}{3}$, $-2\frac{1}{2}$ **33** 4, $\frac{6}{7}$

EXERCISE 3.8

1 4 **2** 9 **3** 49 **4** 1 **5** A
6 $\frac{1}{4}$ **7** $6\frac{1}{4}$ **8** $2\frac{1}{4}$ **9** $12\frac{1}{4}$ **10** $\frac{1}{4}$
11 a^2 **12** b^2 **13** $\frac{c^2}{4}$
14 (a) incorrect (b) correct (c) correct (d) incorrect

EXERCISE 3.9

1 1, 5 **2** 4, -2 **3** 1, -5 **4** 2, -6 **5** D
6 -3, 7 **7** 1, 25 **8** 1, 2 **9** 3, -4
10 1, 4 **11** 3, -10 **12** -1, 12 **13** -2, 5
14 2, 5 **15** 8, -9 **16** -1, 11 **17** 0, 10
18 (a) correct (b) incorrect (c) correct (d) correct

EXERCISE 3.10

1 3.24, -1.24 **2** $-2 \pm 2\sqrt{2}$ **3** 2.79, -1.79 **4** $3 \pm \sqrt{7}$
5 4.79, 0.21 **6** $-1 \pm \sqrt{3}$ **7** 5.24, 0.76 **8** $\frac{-1 \pm \sqrt{5}}{2}$
9 6.74, -0.74 **10** $-2 \pm \sqrt{5}$ **11** 3.45, -1.45 **12** $\frac{-3 \pm \sqrt{33}}{2}$

EXERCISE 3.11

1 1.85, -1.35 **2** $\frac{-3 \pm \sqrt{19}}{2}$ **3** 0.78, -1.28 **4** $\frac{-3 \pm \sqrt{17}}{4}$
5 1.85, -0.18 **6** $\frac{-2 \pm \sqrt{19}}{3}$ **7** 1.54, -0.87 **8** $\frac{3 \pm \sqrt{7}}{2}$
9 2.70, -0.37 **10** $\frac{-1 \pm \sqrt{6}}{2}$ **11** 3.71, -1.21 **12** $\frac{1 \pm \sqrt{7}}{3}$
13 1.19, -1.69 **14** $\frac{4 \pm \sqrt{7}}{3}$ **15** 1.27, 0.39

EXERCISE 3.12

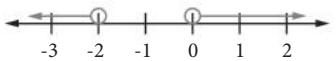
1 -1, -5 **2** 2, -4 **3** 7, -1 **4** 2, 5
5 0.41, -2.41 **6** $3 \pm \sqrt{5}$ **7** 3.45, -1.45 **8** $\frac{-5 \pm \sqrt{29}}{2}$
9 4.16, -2.16 **10** $-2 \pm \sqrt{2}$ **11** 7, 8 **12** 3, -5
13 -0.22, -2.28 **14** $\frac{4 \pm \sqrt{10}}{2}$ **15** $-\frac{1}{2}$, -1 **16** 0, 1.5
17 -1.22, 0.55 **18** 1, -2.5 **19** -0.17, -5.83 **20** 4
21 1.5, -1 **22** $\frac{7 \pm \sqrt{105}}{10}$ **23** 1.64, 0.61 **24** $-\frac{1}{3}$, 4
25 -1.5, 1 **26** $\frac{-3 \pm \sqrt{17}}{2}$ **27** -2.82, -0.18 **28** $\frac{3 \pm \sqrt{15}}{2}$
29 2.73, -0.73 **30** $3 \pm \sqrt{11}$ **31** 2.35, -0.85 **32** $\frac{-5 \pm \sqrt{15}}{2}$
33 3, -20 **34** $\frac{-9 \pm \sqrt{21}}{6}$ **35** ±2.24 **36** $\frac{-1 \pm \sqrt{5}}{2}$

EXERCISE 3.13

1 (a) 6 (b) 4 (c) 3 (d) 8 (e) 2 **2** D
3 (a) 5 (b) 9 (c) 7 **4** 3 **5** 8, 11 or -8, -11
6 8, 9 or -8, -9 **7** (a) 2 s, 6 s (b) 4 s **8** 6 **9** 4 m
10 1 m **11** 4 cm **12** 4 cm, 12 cm **13** 12 cm, 16 cm, 20 cm
14 (a) $(20 - x)$ cm (b) $20x - x^2$ (c) $x^2 - 20x + 84 = 0$, 14 cm, 6 cm

EXERCISE 3.14

1 9 **2** 2.5 **3** $6x - 4$ if $x > \frac{2}{3}$; $4 - 6x$ if $x < \frac{2}{3}$; 0 if $x = \frac{2}{3}$
4 $2 - \sqrt{3}$ **5** $(x + y)$ if $x + y > 0$; $-(x + y)$ if $x + y < 0$; 0 if $x + y = 0$
6 $(x + y)$ if $x, y > 0$; $(x - y)$ if $x > 0, y < 0$; $(y - x)$ if $x < 0, y > 0$; $-(x + y)$ if $x, y < 0$; 0 if $x = y = 0$
7 $2x$ if $x \geq 0$; 0 if $x < 0$ **8** $2x$ if $x \geq 5$; 10 if $-5 < x < 5$; $-2x$ if $x \leq -5$
9 8 **10** $2x + 3$ if $x \geq -1.5$; $-(2x + 3)$ if $x < -1.5$ **11** $x - 3$
12 (a) $x - 2 = 3$ or $-(x - 2) = 3$, $x = 5, -1$ (b) $x = -10, 4$
(c) $x = -1, 9$ (d) $x = -9, -5$ (e) $x = 6$ (f) $x = 4, 6$
(g) $x = -1$ (h) $x = -13, -7$ (i) 0.5, -1.5 (j) 4, 1 (k) 0.6, -1
(l) $x = \frac{-1}{3}, 3$ (m) $x = \frac{-1}{3}$ (n) $x = \frac{-4}{3}, 1$ (o) $x = \frac{1}{4}$ (p) $x = -2, 11$
13 C
14 (a) $x - 1 < 3$ or $x - 1 > -3$; $-2 < x < 4$ (b) $y < -6$ or $y > 2$
(c) $4 \leq t \leq 8$ (d) $x \leq -6$ or $x \geq -2$ (e) real m
(f) $-5 \leq 3 - x \leq 5$; $-8 \leq -x \leq 2$; $-2 \leq x \leq 8$ (g) $y = -1$
(h) $-10 < x < -4$ (i) $3 < 2x + 1 < -3$; $x < -2$ or $x > 1$
(j) $\frac{4}{3} < z < 2$ (k) $x \leq -2$ or $x \geq \frac{1}{2}$ (l) $-1 < t < \frac{7}{3}$ (m) no solution
(n) $x < -2.6$ or $x > 1$ (o) real x, $x \neq \frac{1}{2}$ (p) $x \leq -2$ or $x \geq 9$
15



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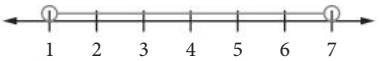

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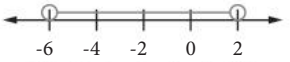

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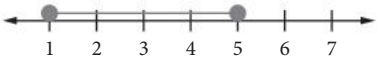

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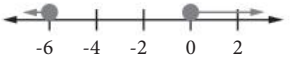

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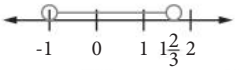

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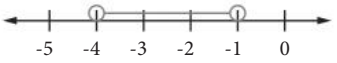

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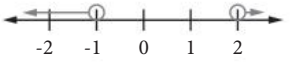

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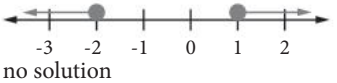
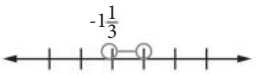
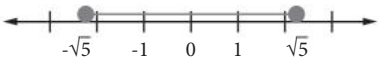


27 no solution
28

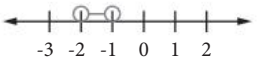
29



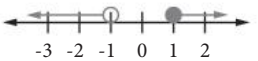
30 $-\sqrt{5} \leq x \leq \sqrt{5}$



31 $x + 3 > 1$, $x + 3 < -1$ and $-3 < 2x + 5 < 3$
$x > -2$, $x < -4$ and $-8 < 2x < -2$
$x > -2$ or $x < -4$ and $-4 < x < -1$
Hence $-2 < x < -1$:



32 $-3 < 2x + 5 < 3$ or $2 + 4x \geq 6$, $2 + 4x \leq -6$
$-8 < 2x < -2$ or $4x \geq 4$, $4x \leq -8$
$-4 < x < -1$ or $x \geq 1$, $x \leq -2$
Hence $x < -1$ or $x \geq 1$



33 (a) correct (b) incorrect (c) correct (d) correct
34 1 for all $x \neq 0$ **35** 1 if $x > 0$; -1 if $x < 0$
36 $\frac{x-4}{x}$ if $x < 0$; $\frac{4-x}{x}$ if $0 < x < 4$; $\frac{x-4}{x}$ if $x \geq 4$
37 1 if $x < 1$; -1 if $x > 1$ **38** $x - 5$ if $x \geq 5$; $5 - x$ if $x < 5$
39 $x + 1$ if $x < -1$ or $x > 1$; $-(x + 1)$ if $-1 \leq x < 1$

EXERCISE 3.15

1 $x = -2, y = 1$ **2** $x = 14, y = 4$ **3** $x = 5, y = -2$
4 $x = 0.5, y = -3.5$ **5** $m = -2, n = 0$ **6** $y = 2, x = 5$
7 $x = 2, y = -3$ **8** $x = 1, y = 2$ **9** C
10 $x = 0.5, y = 3$ **11** $x = 5.5, y = 1$ **12** $m = 4, n = 1\frac{1}{3}$
13 $x = 1, y = 2$ **14** $x = 8, y = -1$ **15** $x = 9, y = 3$
16 $x = 2, y = -4$ **17** $a = 8, b = 3$ **18** $x = 7, y = 2.5$
19 $x = -8, y = 3$ **20** $a = 5, b = 3$ **21** $x = 7, y = 4$

EXERCISE 3.16

1 250, 200 **2** 6, 2 **3** C **4** 6, 30
5 600, 400 **6** 2, 4 **7** $\frac{7}{12}$ **8** 6, 12 **9** $880, $360
10 (a) $y = 2x - 7$ (b) $y = -4x + 4$ (c) $y = \frac{3}{4}x - 3$ (d) $y = -\frac{x}{2} + 5$

EXERCISE 3.17

1 $x = -1, y = 1; x = 6, y = 36$ **2** $x = 1, y = 1; x = 2, y = 4$
3 $x = -1, y = 4; x = 5, y = 10$ **4** $x = -2, y = 17; x = 7, y = 8$
5 $x = -2, y = -5; x = 5, y = 2$ **6** $x = 1, y = 3; x = -1.8, y = -2.6$
7 $x = 2, y = 3; x = 3, y = 2$ **8** $x = 0, y = -2; x = 3, y = 4$
9 $x = 2, y = 1; x = -1, y = -2$ **10** $x = 3, y = 1$
11 $x = 2, y = -2; x = 4, y = 2$ **12** $x = 3, y = 2; x = -18, y = 23$
13 $x = 1, y = 1; x = 0.4, y = -0.2$ **14** $x = -2, y = 0$
15 $x = 2, y = 5; x = -5, y = -2$ **16** $x = 3, y = 2; x = -1.4, y = 15.2$
17 $x = 1, y = 1$ **18** $x = 1, y = 1; x = -0.2, y = 0.4$
19 $x = -4, y = -2$ **20** $x = 5, y = 14; x = -3, y = 6$

EXERCISE 3.18

1 $x = 4, y = 1; x = -1, y = -1$ **2** $x = 3, y = 4; x = -3, y = -4$
3 $x = 1, y = 1; x = 5, y = -1$ **4** $x = 4, y = 5; x = -2\frac{6}{7}, y = -5\frac{2}{7}$
5 $x = 3, y = -4; x = -3\frac{2}{3}, y = 6$ **6** $x = 5, y = 3; x = -7, y = -5$

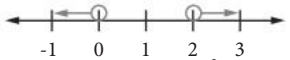
CHAPTER REVIEW 3

1 (a) -6 (b) 1 (c) $-\frac{5}{3}$ **2** (a) $\frac{3}{4}$ (b) $\frac{4}{11}$ (c) 9.5

3 (a) $x > 4$

(b) $-2 \le x < 6$

(c) $x < 0, x > 2$



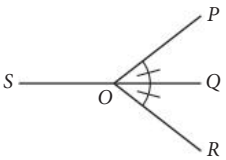
4 (a) $x = \pm 2$ (b) $x = 0, 4$ (c) $x = 2$ (d) $x^2 - 3x = \pm 4, x = -1, 4$
(e) $(x-5)(x+2)(x-4)(x+1) = 0, x = -2, -1, 4, 5$
(f) $x = -1.5, \frac{1}{3}$

5 $(x+1)^2 = x^2 + (x-7)^2, x = 12$. Sides 5 cm, 12 cm, 13 cm
6 $x = 7, y = 2.5$ **7** (a) $x = \frac{1 \pm \sqrt{41}}{4}$ (b) $x = -1.35, 1.85$
8 $x = \frac{-4 \pm \sqrt{2}}{2}$ **9** $2x^3 - 3x^2y + 3xy^2 - y^3$ **10** $\frac{m-4}{(m-2)(m+2)(m-1)}$
11 (a) $9 - x$ (b) $9x - x^2$ (c) 5 cm, 4 cm
12 $x = -6, y = 15; x = 1, y = 8$ **13** (a) $x = -18, 4$ (b) $x \le -\frac{1}{3}, x \ge 3$

CHAPTER 4

EXERCISE 4.1

1 $x = 120$ **2** $a = 48$ **3** $x = 40$ **4** $x = 36$ **5** $x = 90$
6 $a = 120, b = 60, c = 60$ **7** $x = 80$ **8** $a = 42, b = 30$
9 $a = 35, b = 75$ **10** $a = 70, b = 30, x = 70, y = 80$
11 $a = 40, b = 35, c = 145$ **12** $a = 36, b = 36, c = 144$
13 $x = 64°$ **14** (a) 65° (b) 21° (c) 61°46′ (d) 19°7′
15 (a) 106° (b) 62° (c) 51°28′ (d) 162°43′ **16** B
17 $\angle POS + \angle POQ = 180°$ (straight angle)
$\angle ROS + \angle ROQ = 180°$ (straight angle)
But $\angle POQ = \angle ROQ$ (by construction)
$\therefore \angle POS = \angle ROS$



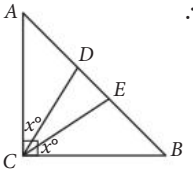
18 6, 6

EXERCISE 4.2

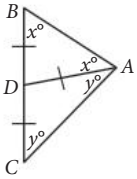
1 B **2** $a = 65, b = 115, c = 65, d = 115, e = 115$
3 $a = 72, b = 108, c = 108, d = 72$
4 $a = 125, b = 55, c = 125, d = 55$
5 $a = 70, b = 65, c = 45$ **6** $x = 100, y = 80, z = 100$
7 $a = 32, b = 78, c = 70$ **8** $a = 36, b = 144, c = 54$
9 $x = 75, y = 40, z = 65$ **10** $a = 120$
11 $a = 35, b = 55$ **12** $a = 130, b = 50$
13 (a) $AB \parallel CD$ (corresponding angles equal to 60°)
(b) $\angle GAD = 100° = \angle ABF$.
$AD \parallel BF$ (corresponding angles equal)
(c) $PQ \parallel KL$ ($\angle PAC = \angle ACL = 45°$ are equal alternate angles)
(d) $\angle KNE = 70°$ (HNE is a straight line),
$AD \parallel HE$ (corresponding angles equal)

EXERCISE 4.3

1 $x = 36$ (base angles of isosceles triangle equal, angle sum of triangle is 180°)
2 $a = 45$ (base angles of isosceles triangle equal, angle sum of triangle is 180°)
3 $x = 111$ (base angles of isosceles triangle equal, exterior angle of triangle equals sum of interior opposite angles)
4 $a = 70$ (straight angle), $b = 70$ (alternate angles),
$c = 48$ (corresponding angles)
5 $a = 96$ (exterior angle of triangle), $b = 84$ (straight angle)
6 $a = 75$ (base angles of isosceles triangle),
$b = 30$ (angle sum of triangle),
$c = 35$ (exterior angle of triangle)
7 $a = 22$ (exterior angle of triangle),
$b = 72$ (base angles of isosceles triangle),
$c = 50$ (angle sum of triangle, equal angles marked on figure)
8 $a = 34$ (exterior angle of triangle),
$b = 142$ (exterior angle of triangle)
9 B **10** $x = 22.5$ **11** $x = 40$ **12** $x = 35$
13 $\angle BAC = \angle BCA$ (base angles of isosceles triangle)
$\therefore \angle ABC + 2\angle BCA = 180°$ (angle sum of triangle)
$\therefore \angle ABC = 180° - 2\angle BCA$
$\angle DEC = \angle DCE$ (base angles of isosceles triangle)
$\therefore \angle CDE + 2\angle DCE = 180°$ (angle sum of triangle)
$\therefore \angle CDE = 180° - 2\angle DCE$
$\angle ABC + \angle CDE = 180°$ (cointerior angles, $BA \parallel DE$)
$\therefore 180° - 2\angle BCA + 180° - 2\angle DCE = 180°$
$180° = 2\angle BCA + 2\angle DCE$
$\therefore \angle BCA + \angle DCE = 90°$
$\therefore \angle ACE = 90°$ ($\angle BCD$ is a straight angle)
14 In ΔABC, let $\angle ABC = x°$, $\angle BCA = y°$. Hence $\angle CAB = x° + y°$
But $x° + y° + (x° + y°) = 180°$ (angle sum of ΔABC)
So $x° + y° = 90°$
Hence the triangle is right-angled.
15 $\angle ACB = \angle DCE$ (vertically opposite angles)
$\angle CAB = \angle CBA$ (base angles of isosceles triangle)
$\angle CDE = \angle CED$ (base angles of isosceles triangle)
But $\angle CAB + \angle CBA = \angle CDE + \angle CED$ (angle sum of triangle)
$\therefore 2\angle CAB = 2\angle CED$
So $\angle CAB = \angle CED$
$\therefore AB \parallel DE$ (pair of alternate angles equal)
16 Let $\angle ACD = \angle BCE = x°$
Because ΔACB is a right-angled isosceles triangle,
$\angle CAB = \angle CBA = 45°$
$\therefore \angle CDE = (45 + x)°$ (exterior angle of triangle equals sum of interior opposite angles)
Similarly $\angle CED = (45 + x)°$
$\therefore \angle CDE = \angle CED$, so ΔCDE is isosceles.



17 Let $\angle DBA = x°$, so $\angle DAB = x°$ (base angles of isosceles ΔDAB)
Let $\angle DCA = y°$, so $\angle DAC = y°$ (base angles of isosceles ΔDAC)
$x° + (x° + y°) + x° = 180°$ (angle sum of ΔABC)
$\therefore 2(x° + y°) = 180°$
$x° + y° = 90°$



$\therefore \angle BAC = 90°$

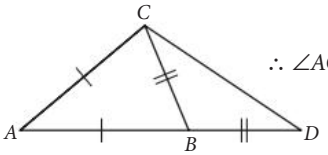
18 (a) In ΔABX and ΔDCX:
$\angle ABX = \angle DCX$ (alternate angles, $AB \parallel CD$)
$\angle AXB = \angle DXC$ (vertically opposite angles)
$\angle BAX = \angle CDX$ (alternate angles, $AB \parallel CD$)
$\therefore \Delta ABX$ and ΔDCX are equiangular.
(b) In ΔABX and ΔDCX:
$\angle ABX = \angle DCX$ (corresponding angles, $AB \parallel DC$)
$\angle AXB = \angle DXC$ (common angle)
$\angle BAX = \angle CDX$ (corresponding angles, $AB \parallel DC$)
$\therefore \Delta ABX$ and ΔDCX are equiangular.

19 $\angle ACE = \angle AEC$ (base angles of isosceles triangle)
$\angle AEC = \angle ECD$ (alternate angles, $AB \parallel CD$)
$\therefore \angle ACE = \angle ECD$

20 (a) In ΔADC, $\angle ADC = \angle ACD$ (base angles of isosceles triangle)
$\angle ACD = \angle CAB$ (alternate angles, $DC \parallel AB$)
In ΔABC, $\angle CAB = \angle BCA$ (base angles of isosceles triangle)
$\therefore \angle ADC = \angle ACD = \angle CAB = \angle BCA$
(b) $\angle DAC = 180° - 2\angle ACD$ (angle sum of ΔADC)
$\therefore \angle DAC = 180° - 2\angle CAB$
But $\angle ABC = 180° - 2\angle CAB$ (angle sum of ΔABC)
$\therefore \angle DAC = \angle ABC$
(c) $\angle ABC = 180° - 2 \times 40° = 100°$

21 $3x + 5x + 7x = 180$, $15x = 180$, $x = 12$
Angles are 36°, 60°, 84°

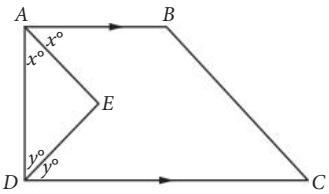
22 Let $\angle DCB = x°$, hence $\angle CDB = x°$ (base angles of isosceles triangle)
$\therefore \angle ABC = 2x°$ (exterior angle of triangle equals sum of interior opposite angles)
$\angle ACB = \angle ABC$ (base angles of isosceles triangle)



$\therefore \angle ACB = 2x° = 2\angle DCB$

EXERCISE 4.4

1 (a) 720° (b) 1080° (c) 1440° (d) 3240°
2 (a) 108° (b) 120° (c) 135° (d) 150°
3 B **4** (a) 72° (b) 60° (c) 45° (d) 30°
5 (a) 4 (b) 5 (c) 24 (d) 18 (e) 30 (f) 36
6 (a) 8 (b) 15 (c) 16 (d) 36
7 $\frac{2n-4}{n} \times 90 = 152$, $180n - 360 = 152n$, $28n = 360$, $n = 12.8$
Because n is not an integer, no regular polygon exists whose angles are 152°.
8 18 **9** B
10 (a) $x = 129°$ (b) $y = 57°$ (c) $z = 45°$ (d) $d = 125°$
(e) $x = 50°$ (f) $y = 60°$
11 Let $\angle BAE = x°$, $\therefore \angle DAE = x°$ and $\angle BAD = 2x°$
Let $\angle CDE = y°$, $\therefore \angle ADE = y°$ and $\angle ADC = 2y°$
$\angle BAD + \angle ADC = 180°$ (cointerior angles, $AB \parallel DC$)
$\therefore 2x + 2y = 180$, $x + y = 90$



$\angle AED = 180° - (x° + y°) = 90°$ (angle sum of triangle)

12 (a) $\angle ACD$, $\angle ACB$, $\angle DAC$ (b) $\angle ABD$, $\angle BDC$, $\angle ADB$, $\angle CBD$
(c) $\angle BCA = \angle BAC = 65°$ (base angles of isosceles triangle, $BA = BC$, sides of a rhombus)
$\therefore \angle ABC = 50°$ (angle sum of triangle)
$\therefore \angle ABE = \angle CBE = 25°$ (diagonals of rhombus bisect its angles)
$\therefore \angle AEB = 90°$ (angle sum of ΔABE)
$\therefore$ Diagonals intersect at right angles.

13 (a) Parallelogram, both pairs of opposite sides equal
(b) Kite, two pairs of adjacent sides equal
(c) Rhombus, diagonals bisect each other at right angles
(d) Parallelogram, one pair of opposite sides equal and parallel
(e) Rhombus, four sides equal
(f) Parallelogram, both pairs of opposite angles equal

14 (a) $NKLM$ is a parallelogram (one pair of opposite sides equal and parallel)
$x = 110$ (opposite angles of a parallelogram)
$y = 70$ (cointerior angles $NK \parallel ML$)
(b) $DEFG$ is a rhombus (diagonals bisect each other at right angles)
$m = 70$ (diagonals bisect angles of rhombus)
$n = 20$ (alternate angles $DG \parallel EF$, angle sum of triangle)
(c) $WXYZ$ is a square (diagonals equal and bisect each other at right angles)
$a = b = 45$ (diagonals bisect angles of square)
(d) $EFGH$ is a parallelogram (both pairs of opposite angles equal)
$x = 4$, $y = 5$ (opposite sides of a parallelogram)
(e) $PQRS$ is a parallelogram (diagonals bisect each other)
$c = 50$ (alternate angles, $PQ \parallel SR$)
$d + 65 + 80 = 180$, $\therefore d = 35$
(alternate angles, exterior angle of a triangle, angle sum of ΔPST)
(f) $ABCD$ is a parallelogram (both pairs of opposite sides equal)
$x = 110$ (cointerior angles, $AB \parallel DC$)
$y = 70$ (opposite angles of parallelogram)

15 B

EXERCISE 4.5

1 (a) $\Delta ABC \equiv \Delta CDE$ (SSS) (b) $\Delta FKG \equiv \Delta HKL$ (SAS)
(c) $\Delta LPN \equiv \Delta NML$ (AAS) (d) $\Delta QTS \equiv \Delta SRQ$ (RHS)
(e) $\Delta UYX \equiv \Delta UYV \equiv \Delta WYV \equiv \Delta WYX$ (SAS),
$\Delta XUV \equiv \Delta XWV$, $\Delta UXW \equiv \Delta UVW$
(f) $\Delta ADC \equiv \Delta ABC$ (AAS) (g) $\Delta EHG \equiv \Delta EHF$ (SAS)
(h) $\Delta HKL \equiv \Delta MLK$ (SSS) (i) $\Delta PQT \equiv \Delta RQT$ (RHS)
2 (a) D (b) BD and DC are the same length.
3 (a) $x = 55$ (angle sum of congruent triangles, RHS)
(b) $y = 50$ (base angles of isosceles triangle)
$x = 90$ (congruent triangles, SSS, straight angle)
(c) $x = 70$ (opposite angles of parallelogram)
$y = 60$ (alternate angles $KL \parallel NM$, angle sum of triangle)
(d) $x = 105$ (opposite angles of parallelogram)
(e) $y = 90$ (bisector of vertical angle of isosceles triangle $\perp$ the base)
(f) $x = 3$ (opposite sides of parallelogram)

4 In ΔWYZ and ΔWYX:
$WZ = WX$ (given)
WY is a common side
$YZ = YX$ (given)
$\therefore \Delta WYZ \equiv \Delta WYX$ (SSS)
$\therefore \angle ZWY = \angle XWY$ (corresponding angles in congruent triangles)
$\therefore$ YW bisects $\angle ZWX$

5 In ΔADC and ΔBDC:
$\angle ADC = \angle BDC = 90°$
$AC = BC$ hypotenuse (given)
CD is a common side
$\therefore \Delta ADC \equiv \Delta BDC$ (RHS)
$\therefore \angle ACD = \angle BCD$ (matching angles in congruent triangles)

6 (a) In ΔDHE and ΔFHE:
$DE = FE$ (given)
EH is a common side
$HD = HF$ (given)
$\therefore \Delta DHE \equiv \Delta FHE$ (SSS)
$\therefore \angle DHE = \angle FHE$ (corresponding angles in congruent triangles)
But $\angle DHE + \angle FHE = 180°$ (straight angle)
$\therefore \angle DHE = \angle FHE = 90°$

(b) ΔDEF is equilateral as the three sides are equal
$\therefore \angle DEF = 60°$
But $\angle DEH = \angle FEH$ (corresponding angles in congruent triangles)
$\therefore \angle DEH = \angle FEH = 30°$ ($\angle DEF = 60°$)

7 (a) In ΔKLN and ΔKMN:
$LN = NM$ (N is midpoint of LM)
$\angle KNL = \angle KNM = 90°$ ($KN \perp LM$)
KN is a common side
$\therefore \Delta KLN \equiv \Delta KMN$ (SAS)
$\therefore \angle KLN = \angle KMN$ (matching angles in congruent triangles)

(b) $\angle LKN = \angle MKN$ (matching angles in congruent triangles)
$\therefore$ NK bisects $\angle LKM$

8 (a) In ΔPTQ and ΔRTS:
$PT = RT$ (given)
$\angle PTQ = \angle RTS$ (vertically opposite angles)
$TQ = TS$ (given)
$\therefore \Delta PTQ \equiv \Delta RTS$ (SAS)

(b) $\angle QPT = \angle TRS$ (matching angles in congruent triangles)
These are equal alternate angles, so $PQ \parallel SR$

(c) In ΔPTS and ΔRTQ:
$PT = RT$ (given)
$\angle PTS = \angle RTQ$ (vertically opposite angles)
$TS = TQ$ (given)
$\therefore \Delta PTS \equiv \Delta RTQ$ (SAS)

(d) $\angle SPT = \angle TRQ$ (matching angles in congruent triangles)
These are equal alternate angles, so $PS \parallel QR$

(e) $PQRS$ is a parallelogram (both pairs of opposite sides parallel)

9 (a) In ΔOAC and ΔOBD:
$OA = OB$ (radii)
$\angle AOC = \angle BOD$ (vertically opposite angles)
$OC = OD$ (radii)
$\therefore \Delta OAC \equiv \Delta OBD$ (SAS)

(b) $x = 40°$ (matching angles in congruent triangles)

EXERCISE 4.6

1 (a) $\angle DEF = \angle PRQ = 65°$, $\angle EFD = \angle RQP = 40°$, $\Delta DEF \,|||\, \Delta PRQ$ (AAA)

(b) $\angle PKN = \angle LKM$ (same angle), $\angle KPN = \angle KLM$ (corresponding angles, $NP \parallel ML$), $\Delta KNP \,|||\, \Delta KML$ (AAA)

(c) $\frac{AE}{EC} = \frac{1}{3}$, $\frac{BE}{ED} = \frac{1}{3}$, $\angle AEB = \angle CED$ (vertically opposite angles), $\Delta AEB \,|||\, \Delta CED$ (sides proportional, included angles equal)

(d) $\frac{RS}{RP} = \frac{8}{16} = \frac{1}{2}$, $\frac{RT}{RQ} = \frac{6}{12} = \frac{1}{2}$, $\angle SRT = \angle PRQ$ (same angle), $\Delta SRT \,|||\, \Delta PRQ$ (sides proportional, included angles equal)

(e) $\frac{UW}{CB} = \frac{2}{1}$, $\frac{UV}{BA} = \frac{2}{1}$, $\frac{VW}{AC} = \frac{2}{1}$, $\Delta WUV \,|||\, \Delta CBA$ (3 pairs of sides proportional)

(f) $\angle DAX = \angle BCX = 35°$, $\angle AXD = \angle CXB$ (vertically opposite angles), $\Delta AXD \,|||\, \Delta CXB$ (AAA)

2 yes 3 D

4 (a) $x = 60°$, two sides proportional, included angles equal
(b) $x = 3$, $y = 3\frac{1}{3}$, AAA (c) $x = 1$, $y = 2.4$, AAA
(d) $x = 3$, $y = 2.1$, AAA (e) $x = 2.5$, $y = 3.3$, AAA

5 (a) $a = 2.4$, $b = 12$ (b) $x = 6$, $y = 3$
(c) $a = 18$ (d) $x = 2$, $y = 4\frac{1}{3}$
(e) $x = 14$, $y = 12.5$ (f) $x = 18$
(g) $x = 7.5$, $y = 9.6$ (h) $x = 9.6$, $y = 3.75$
(i) $x = 2.1$, $y = 2$ (j) $x = 3$, $y = 2.4$

6 (a) $\angle APQ = \angle ABC$ (given), $\angle PAQ = \angle BAC$ (same angle), $\Delta APQ \,|||\, \Delta ABC$ (AAA)

(b) $\angle AXY = \angle ACB$ (given), $\angle XAY = \angle CAB$ (same angle), $\Delta AXY \,|||\, \Delta ACB$ (AAA)

7 (a) $\angle PST = \angle RQT$ (given),
$\angle PTS = \angle QTR$ (vertically opposite angles),
$\Delta PST \,|||\, \Delta QRT$

(b) $\frac{PS}{RQ} = \frac{PT}{RT} = \frac{ST}{QT}$

8 (a) $\angle PST = \angle PQR$ (corresponding angles, $ST \parallel QR$),
$\angle SPT = \angle QPR$ (common angle),
$\Delta PST \,|||\, \Delta PQR$ (AAA)

(b) $\frac{PS}{PQ} = \frac{ST}{QR} = \frac{PT}{PR}$ (c) $\frac{1}{PQ} = \frac{2}{5}$, $PQ = 2.5$ cm

9 (a) $\angle BAC = \angle EAD$ (common angle), $\angle ACB = \angle ADE = 90°$, $\Delta ABC \,|||\, \Delta AED$ (AAA)

(b) $\frac{AB}{AE} = \frac{AC}{AD} = \frac{BC}{ED}$

10 $\Delta ABC \,|||\, \Delta AXY$
(a) $XY = 9$ cm (b) $BC = 12$ cm (c) $BC = 13\frac{1}{3}$ cm

11 24 m 12 20 m 13 5 m 14 4.8 m

EXERCISE 4.7

1 (a) $\frac{y}{6} = \frac{6}{4}$, $y = 9$; $\frac{x}{6} = \frac{15}{9}$, $x = 10$ (intercepts on parallel lines are proportional)

(b) $\frac{x}{15} = \frac{5}{10}$, $x = 7.5$; $\frac{y}{15} = \frac{10}{10}$, $y = 15$; $\frac{z}{4} = \frac{25}{5}$, $z = 20$ (intercepts on parallel lines are proportional)

2 B

3 (a) $\frac{x}{35} = \frac{12}{28}$, $x = 15$ (corresponding sides of similar triangles are proportional)
$\frac{y}{18} = \frac{28}{12}$, $y = 42$ (corresponding sides of similar triangles are proportional)

(b) $\frac{22 - x}{22} = \frac{21}{33}$, $x = 8$ (intercepts on parallel lines are proportional)

4 (a) $QP = PM$. If a set of parallel lines cut equal intercepts on one transversal, then they cut equal intercepts on any other transversal.

(b) $\frac{LP}{KQ} = \frac{1}{2}$. Because $ML = LK$, then $MK = 2ML$, so $\frac{ML}{MK} = \frac{1}{2}$. Because $\Delta LMP \,|||\, \Delta KMQ$ (AAA), then $\frac{LP}{KQ} = \frac{ML}{MK}$.

5 (a) If a line is drawn through A parallel to DE, then the set of parallel lines cut equal intercepts on AB. Thus they cut equal intercepts on AC so that $AE = EC$. Hence E is the midpoint of AC.

(b) Similarly F is the midpoint of BC. Hence $BF : FC = 1 : 1$.

(c) D and F are the midpoints of sides of the triangle, so $DF \parallel AC$.

EXERCISE 4.8

1 (a) 13 cm (b) $2\sqrt{41}$ m (c) 3.9 cm
2 (a) $5\sqrt{2}$ m (b) $\frac{18\sqrt{2}}{5}$ cm $\approx$ 5.1 cm
3 C **4** (a) $6\sqrt{2}$ cm (b) 6 cm
5 $h^2 = 10^2 - 5^2$, $h = 5\sqrt{3}$ cm **6** $CB = 1 - t^2$
7 (a) 10 cm (b) 5.2 cm **8** C **9** 106 km
10 (a) 10 km (b) 25 km (c) 50 km (d) 60 km
(e) 75 km (f) 100 km (g) $100t$ km
11 20 cm **12** (a) 7 cm (b) 40 cm **13** 12 cm
14 (a) 8 cm (b) 24 cm (c) 4.8 cm (d) 2.4 cm
15 4 cm **16** 48 cm **17** $3\sqrt{13}$ m **18** 13 cm **19** 36 m

EXERCISE 4.9

1 (a) A = 160 cm^2, P = 56 cm (b) A = 18.3 m^2, P = 27.4 m
2 (a) 1.35 (b) 36 (c) 100 (d) 0.72
3 (a) 40 cm (b) 16.8 m (c) 2.1 km (d) 1780 m
4 (a) 6.25 cm^2 (b) 2.56 m^2 (c) 36 cm^2 (d) 1225 m^2
5 (a) 12.4 cm (b) 28 m (c) 1.6 km (d) 600 m
6 C **7** 336 m^2 **8** 1.05 **9** 1200
10 square area = 36 cm^2; rectangle area = 32 cm^2; 36 : 32 = 9 : 8
11 1250 **12** 21% **13** 2 : 1
14 (a) 3.25 cm^2 (b) 3.85 cm^2 (c) 120 cm^2 (d) 48 cm^2
(e) 672 cm^2 (f) 43.2 cm^2
15 (a) (i) 22 cm^2 (ii) 114 cm^2 (iii) 25 cm^2 (iv) 100 cm^2
(v) 7.5 cm^2 (vi) 6.72 cm^2
(b) (ii) 76 cm (iii) 29 cm (iv) 40 cm (v) 13.4 cm (vi) 10.8 cm
16 30.7% **17** 125%

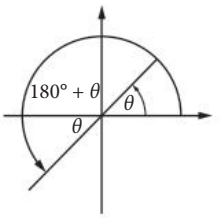
CHAPTER REVIEW 4

1 (a) $a = 90$, $b = 63$ (b) $z = 80$ (c) $y = 90$ (d) $x = 35$
2 (a) $x = 110$ (exterior angle of triangle is sum of interior opposite angles), $y = 30$ (same reason)
(b) $z = 135$ (sum of exterior angles is 360°)
(c) $a = 110$ (angle sum of quadrilateral, straight angles)
(d) $b = 110$ (straight angle), $d = 100$ (isosceles triangle, straight angle), $c = 120$ (sum of exterior angles is 360°)
3 (a) square (b) rhombus (c) isosceles triangle
(d) parallelogram (e) quadrilateral (f) pentagon
(g) quadrilateral (h) right-angled isosceles triangle
4 (a) $x = 50$ (b) $x = 20\frac{5}{6}$ (c) $x = 108$
5 side length = 4 cm, area = 15.36 cm^2
6 (a) width = $(9 - x)$ cm
(b) area = $x(9 - x) = (9x - x^2)$ cm^2
7 (a) In ΔDEG and ΔFGE:
$DE = FG$ (given)
EG is a common side
$GD = EF$ (given)
$\therefore \Delta DEG \equiv \Delta FGE$ (SSS)
$\therefore \angle GDE = \angle GFE$ (matching angles in congruent triangles)
(b) parallelogram
8 (a) In ΔADF and ΔCBE:
$AD = CB$ (sides of a square)
$\angle ADF = \angle CBE = 90°$ (given)
$DF = BE$ (F and E are midpoints of sides of the square)
$\therefore \Delta ADF \equiv \Delta CBE$ (SAS)
(b) parallelogram (both pairs of opposite sides equal)
9 $\angle KLP = 55°$ (exterior angle of triangle is sum of interior opposite angles)
In ΔKPL and ΔMPN:
$\angle KLP = \angle MNP = 55°$
$\angle KPL = \angle MPN$ (vertically opposite angles)
$KL = MN$ (given)
$\therefore \Delta KPL \equiv \Delta MPN$ (AAS)
$\therefore LP = NP$ (matching sides in congruent triangles)
10 The common side PR is not opposite the 48° angle in both triangles, so the triangles are not congruent. They can be shown to be similar.
11 (a) In ΔVXW and ΔZXY:
$\frac{VX}{ZX} = \frac{WX}{XY}$ (given)
$\angle VXW = \angle ZXY$ (both 90°)
$\therefore \Delta VXW \,|||\, \Delta ZXY$ (two pairs of sides proportional, included angles equal)
(b) $\angle ZYX = 55°$ (matching angles in similar triangles)
$\therefore \angle XZY = 35°$ (angle sum of triangle)
12 (a) $\angle QBC = (90 - x)°$
(b) In ΔACP and ΔBCQ:
$\angle ACP = \angle BCQ = x°$ (common angle)
$\angle APC = \angle BQC = 90°$ (given)
$\therefore \Delta ACP \,|||\, \Delta BCQ$ (AAA)
$\therefore \frac{AC}{CB} = \frac{AP}{BQ}$ (corresponding sides in similar triangles are proportional)
13 3.5 cm **14** $KM = 17$ m, $KM^2 + MN^2 = KN^2$, $\therefore \angle KMN = 90°$
15 (a) 12 cm (b) 15 cm **16** 3060 cm^3

CHAPTER 5

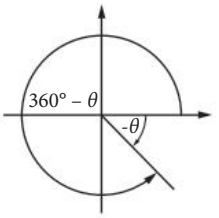
EXERCISE 5.1

1 (a) 1st, 2nd (b) 2nd, 4th (c) 2nd, 3rd (d) 4th
(e) 2nd (f) 3rd (g) 1st
2 (a) 1st (b) 2nd (c) 2nd (d) 3rd (e) 4th (f) 3rd
(g) 1st (h) 3rd (i) 4th (j) 2nd
3 (a) $\sin A$ (b) $\sin A$ (c) $-\tan A$ (d) $-\cos A$
(e) $-\sin A$ (f) $\tan A$
4 (a) (i) 0.8912, -0.5736, -1.4281, 1.2208, -1.7434, -0.7002
(ii) 0.4695, -0.8829, -0.5317, 2.1301, -1.1326, -1.8807
(iii) 0.8910, -0.4540, -1.9626, 1.1223, -2.2027, -0.5095
(b) (i) -0.4226, -0.9063, 0.4663, -2.3662, -1.1034, 2.1445
(ii) -0.6018, -0.7986, 0.7536, -1.6616, -1.2521, 1.3270
(iii) -0.9455, -0.3256, 2.9042, -1.0576, -3.0716, 0.3443
(c) (i) -0.9781, 0.2079, -4.7046, -1.0223, 4.8097, -0.2126
(ii) -0.8572, 0.5150, -1.6643, -1.1666, 1.9416, -0.6009
(iii) -0.3090, 0.9511, -0.3249, -3.2361, 1.0515, -3.0777
(d) (i) -0.4226, 0.9063, -0.4663, -2.3662, 1.1034, -2.1445
(ii) -0.8480, -0.5299, 1.6003, -1.1792, -1.8871, 0.6249
(iii) 0.5736, -0.8192, -0.7002, 1.7434, -1.2208, -1.4281
5 (a) 0.2 (b) -0.2 (c) -0.2 (d) 0.2 (e) -0.2 (f) 5
6 (a) $\frac{1}{t}$ (b) t (c) $-t$ (d) $-t$ (e) $-\frac{1}{t}$ (f) t
7 (a) $\frac{1}{c}$ (b) c (c) $-c$ (d) c (e) $\frac{1}{c}$ (f) $-c$
8 (a) -1.4281 (b) -0.5299 (c) -0.7660 (d) -0.3420
(e) 1.1106 (f) -0.1736
9 (a) positive (b) positive (c) negative (d) negative
(e) positive (f) negative
10 (a)



$\cos(180° + \theta) = -\cos\theta$

(b)



$\sin(360° - \theta) = -\sin\theta$

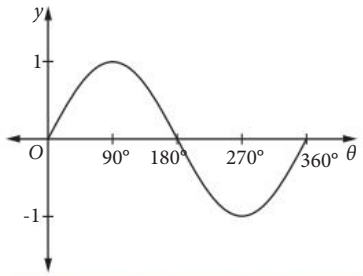
11 D
12 (a) correct (b) incorrect (c) correct (d) correct

EXERCISE 5.2

1 (a)

θ	0°	30°	90°	150°	180°	210°	270°	330°	360°
$\sin\theta$	0	0.5	1	0.5	0	-0.5	-1	-0.5	0

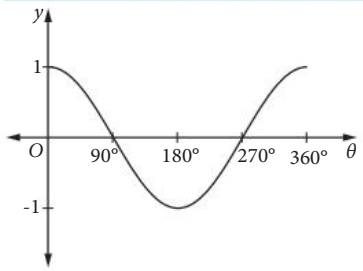
(b)



2 (a)

θ	0°	60°	90°	120°	180°	240°	270°	300°	360°
$\cos\theta$	1	0.5	0	-0.5	-1	-0.5	0	0.5	1

(b)



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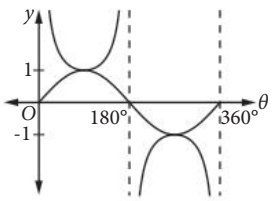

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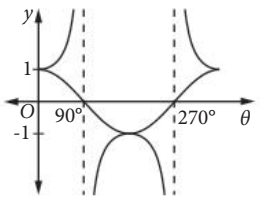
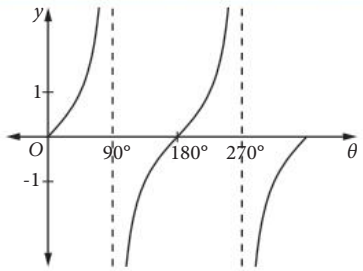


5 (a)

θ	0°	30°	90°	150°	180°	210°	270°	330°	360°
$\tan\theta$	0	0.577	undefined	-0.577	0	0.577	undefined	-0.577	0

(b)



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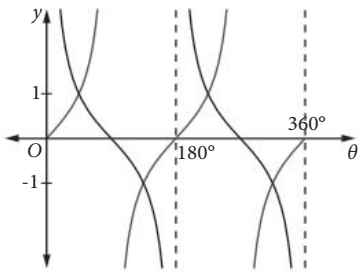
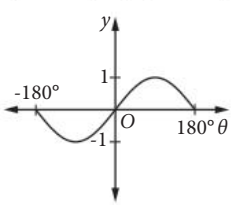
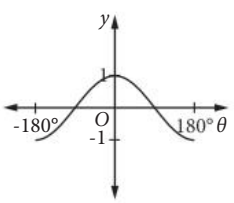


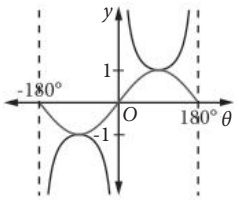
7 (a)



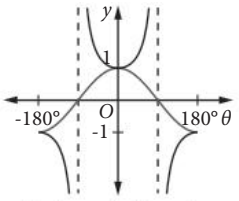
(b)



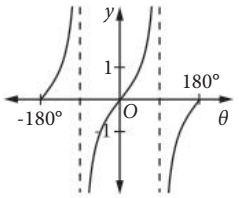
(c)



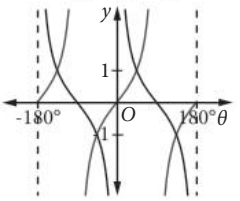
(d)



(e)



(f)



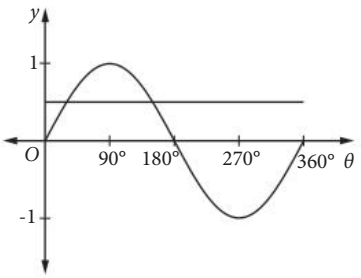
8 (a) $x = 0°$ (b) $x = 71°\,34'$ (c) $x = 12°\,50', 77°\,10'$
(d) $x = 45°$ (e) $x = 18°\,26'$ (f) $x = 15°, 75°$

9 (a) $x = 0°, 360°$ (b) $x = 71°\,34', 251°\,34'$
(c) $x = 12°\,50', 77°\,10', 192°\,50', 257°\,10'$
(d) $x = 45°, 225°$ (e) $x = 18°\,26', 198°\,26'$
(f) $x = 15°, 75°, 195°, 255°$

10 D

11 (a) $x = 45°, 135°, 225°, 315°$ (b) $x = 90°, 270°$
(c) $x = 60°, 120°, 240°, 300°$ (d) $x = 36°\,52', 143°\,8'$
(e) $x = 0°, 45°, 180°, 225°, 360°$
(f) $x = 22.5°, 112.5°, 202.5°, 292.5°$

12 (a)



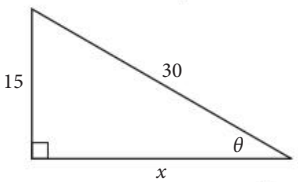
$x = 30°, 150°$

(b) Draw $y = -0.5$. $x = 210°, 330°$

EXERCISE 5.3

1 (a) 10 cm (b) $10\sqrt{3}$ cm **2** B

3



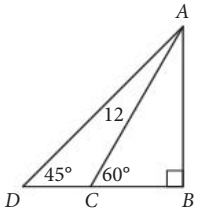
(a) $x^2 = 900 - 225$, $x = 15\sqrt{3}$ m (b) $\sin\theta = \frac{15}{30}$, $\theta = 30°$

4 (a) $5\sqrt{3}$ m (b) 5 m

5 (a) $\frac{AF}{8} = \cos 60°$, $AF = 4$ cm (b) $\frac{FC}{8} = \sin 60°$, $FC = 4\sqrt{3}$ cm
(c) $FB = 12 - 4 = 8$ cm; $BC^2 = 8^2 + \left(4\sqrt{3}\right)^2 = 112$; $BC = 4\sqrt{7}$ cm

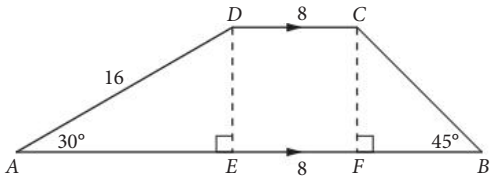
6 (a) $\frac{AC}{20} = \sin 30°$, $AC = 10$ cm (b) $\frac{DC}{20} = \cos 30°$, $DC = 10\sqrt{3}$ cm
(c) $\frac{AB}{10} = \cos 45°$, $AB = 5\sqrt{2}$ cm (d) $BC = AB = 5\sqrt{2}$ cm

7



(a) $\frac{AB}{12} = \sin 60°$, $AB = 6\sqrt{3}$ cm
(b) $\frac{BC}{12} = \cos 60°$, $BC = 6$ cm
(c) $DB = AB = 6\sqrt{3}$, $DC = \left(6\sqrt{3} - 6\right)$cm
(d) $\frac{AB}{AD} = \sin 45°$, $\frac{AD}{6\sqrt{3}} = \frac{1}{\sin 45°}$, $AD = 6\sqrt{3} \times \sqrt{2} = 6\sqrt{6}$ cm

8



$\frac{DE}{16} = \sin 30°$, $DE = 8$, $\therefore$ $CF = 8$ and $FB = 8$

$\frac{8}{CB} = \sin 45°$, $CB = 8\sqrt{2}$

$\frac{AE}{16} = \cos 30°$, $AE = 8\sqrt{3}$, $\therefore$ $AB = 8\sqrt{3} + 8 + 8 = 16 + 8\sqrt{3}$

Perimeter $= 16 + 8 + 8\sqrt{2} + 16 + 8\sqrt{3} = \left(40 + 8\sqrt{2} + 8\sqrt{3}\right)$ cm

9 $\sqrt{3}$ m **10** $30\sqrt{3}$ cm, 30 cm

11 (a) correct (b) correct (c) incorrect (d) correct

EXERCISE 5.4

1 (a) 1 (b) $-\frac{1}{2}$ (c) $-\frac{1}{\sqrt{3}}$ (d) -1 (e) $\frac{\sqrt{3}}{2}$

2 (a) 0 (b) $-\frac{\sqrt{3}}{2}$ (c) 1 (d) $-\frac{1}{2}$ (e) 0

3 (a) -1 (b) $-\sqrt{3}$ (c) -1 (d) $\frac{\sqrt{3}}{2}$ (e) $-\frac{\sqrt{3}}{2}$

4 (a) 0 (b) $\frac{\sqrt{3}}{2}$ (c) 1 (d) 0 (e) $\frac{\sqrt{3}}{2}$

5 A **6** (a) correct (b) incorrect (c) correct (d) correct

7 240°, 300° **8** 135°, 315° **9** 180° **10** 45°, 225°

11 180° **12** 120°, 240° **13** 60°, 120° **14** 120°, 300°

15 15°, 75°, 195°, 255° **16** 120°

17 22.5°, 112.5°, 202.5°, 292.5° **18** no solution

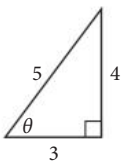
EXERCISE 5.5

1 (a) $\sec^2\theta = 1 + \left(\frac{7}{24}\right)^2$, $\sec^2\theta = \frac{625}{576}$, 3rd quadrant so $\cos\theta = -\frac{24}{25}$, $\sin\theta = -\frac{7}{25}$

(b) $\cos\theta = -\frac{24}{25}$

2 (a) correct (b) incorrect (c) correct (d) incorrect

3



(a) $\cot\theta = -\frac{3}{4}$

(b) 2nd quadrant, $\sin\theta > 0$, $\sin\theta = \frac{4}{5}$

(c) 2nd quadrant, $\cos\theta < 0$, $\cos\theta = -\frac{3}{5}$

4 A

5 (a) $\cot^2\theta$ (b) $\tan^2\theta$

(c) $\frac{2\cot\alpha}{\operatorname{cosec}^2\alpha} = \frac{2\cos\alpha}{\sin\alpha} \times \sin^2\alpha = 2\sin\alpha\cos\alpha$ (d) $\tan\theta$

(e) $\frac{1}{\sin A\cos A}$ (f) $\sin\theta\left(\sin^2\theta + \cos^2\theta\right) = \sin\theta$ (g) 1

(h) $1 - (-\sin\theta)^2 = 1 - \sin^2\theta = \cos^2\theta$

6 (a) $\frac{a^2\sec^2\theta}{\sqrt{a^2\sec^2\theta - a^2}} = \frac{a^2\sec^2\theta}{a\sqrt{\tan^2\theta}} = \frac{a\sec^2\theta}{\tan\theta} = a\sec\theta\operatorname{cosec}\theta$

(b) $a\sin\theta$

7 (a) $-\frac{8}{17}$ (b) $\frac{15}{17}$ (c) $-\frac{15}{8}$

8 3rd quadrant, $\cos^2\theta = 1 - p^2$, $\cos\theta = -\sqrt{1-p^2}$

9 $-\frac{\sqrt{34}}{5}$ **10** $\sin\theta = x$, $\cos^2\theta = 1 - x^2$; $\sin^2\theta\cos^2\theta = x^2(1-x^2)$

11 $a\sin^2\theta + b(1-\sin^2\theta) = c$, $(a-b)\sin^2\theta = c - b$,

$\sin\theta = \pm\sqrt{\frac{c-b}{a-b}}$, $\cos\theta = \pm\sqrt{\frac{c-a}{b-a}}$

12 Use $\sin^2\theta + \cos^2\theta = 1$, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

13 $\sec^2\theta - 1 + 2\sec^2\theta = 5$, $3\sec^2\theta = 6$, $\cos^2\theta = 0.5$, $\sin^2\theta = \frac{1}{2}$

14 (a) 1 (b) $2\sec^2 V$ (c) $2\operatorname{cosec}\theta$ (d) $\cot^2\theta$ (e) 1

(f) $\sin^2 A$ (g) $\frac{1}{2}$ (h) $\cos^2\theta$

15 LHS $= 1 - 2\tan x + \tan^2 x + 1 + 2\tan x + \tan^2 x = 2 + 2\tan^2 x$

$= 2\sec^2 x =$ RHS

16 LHS $= \left(\frac{\cos t}{\sin t} + \frac{1}{\sin t}\right)^2 = \frac{(1+\cos t)^2}{\sin^2 t} = \frac{(1+\cos t)^2}{1-\cos^2 t}$

$= \frac{(1+\cos t)^2}{(1-\cos t)(1+\cos t)} = \frac{1+\cos t}{1-\cos t} =$ RHS

17 LHS $= \sin^2\alpha(1-\sin^2\beta) - (1-\sin^2\alpha)\sin^2\beta$

$= \sin^2\alpha - \sin^2\alpha\sin^2\beta - \sin^2\beta + \sin^2\alpha\sin^2\beta =$ RHS

18 LHS $= \frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta} = \frac{1+\sin\theta}{\cos\theta} =$ RHS

19 LHS $= \frac{\sin^3\theta}{\cos\theta} + \frac{\cos^3\theta}{\sin\theta} + 2\sin\theta\cos\theta = \frac{\left(\sin^2\theta + \cos^2\theta\right)^2}{\sin\theta\cos\theta}$

$= \frac{1}{\sin\theta\cos\theta} = \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta} =$ RHS

20 LHS $= \frac{\sin\theta(\cos\theta + \sin\theta)}{\cos\theta} + \frac{\cos\theta(\sin\theta + \cos\theta)}{\sin\theta} = \frac{\sin^2\theta(\cos\theta + \sin\theta) + \cos^2\theta(\sin\theta + \cos\theta)}{\sin\theta\cos\theta}$

$= \frac{\sin\theta + \cos\theta}{\sin\theta\cos\theta} =$ RHS

21 LHS $= \frac{\tan\theta(\tan^2\theta - 1)}{\tan^2\theta} + \frac{1}{\tan\theta}\left(1 - \tan^2\theta\right) = \frac{\tan^2\theta - 1 + 1 - \tan^2\theta}{\tan\theta} = 0 =$ RHS

22 LHS $= \frac{\cos\theta}{\sin\theta} \times \cos\theta \times \frac{\sin\theta}{\cos\theta + \sin\theta\cos\theta} = \frac{\cos\theta}{1+\sin\theta} =$ RHS

23 LHS $= \sin\theta + \cos\theta - \frac{\sin\theta}{\sin^2\theta + \cos^2\theta} = \sin\theta + \cos\theta - \sin\theta =$ RHS

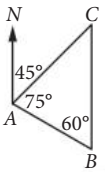
24 LHS $= \left(\frac{\cos^2 t}{\sin t} - \frac{\sin^2 t}{\cos t}\right) \times \frac{\sin t\cos t}{\cos t - \sin t} = \frac{\cos^3 t - \sin^3 t}{\sin t\cos t} \times \frac{\sin t\cos t}{\cos t - \sin t}$

$= \sin^2 t + \sin t\cos t + \cos^2 t = 1 + \sin t\cos t =$ RHS

EXERCISE 5.6

1 C

2



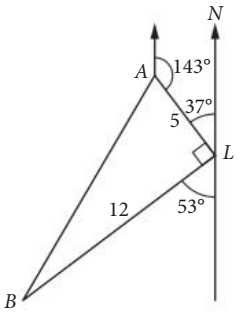
(a) SW (b) N 60° W or 300° (c) S

3 $AC = 3\sqrt{41} \approx 19.2$ km, $\tan\angle BCA = 1.25$; bearing is 051° 20′

4 distance = 13 km, $\tan\theta = \frac{5}{12}$, $\theta = 22°\,37'$; bearing is 112° 37′

5 50 m, N 30° W

6



$\angle ALB = 90°$, $AB = 13$ km,
$\tan\angle LAB = 2.4$,
$\angle LAB = 67°\,23'$

Bearing of B from $A = 143° + 67°\,23' = 210°\,23'$

7 (a) 25 km (b) 24 min (c) 306° 52′

8 16.4 km, 11.5 km

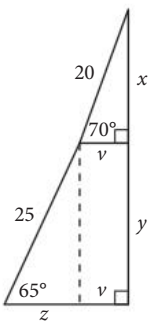
9 $BC = 75$ km, $AB = 75\sin 32° = 39.74$ km

$AC = 75\cos 32° = 63.60$ km

Time $= 63.60 \div 25 = 2.54$ h

10 20 km, 233° 8′

11



(a) $x = 20\sin 70 = 18.79$,
$y = 25\sin 65° = 22.66$,
distance south = 41.45 km

(b) $v = 20\cos 70° = 6.84$,
$z = 25\cos 65° = 10.57$,
distance west = 17.41 km

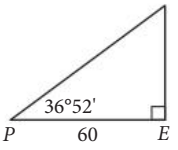
(c) $\tan\theta = \frac{17.41}{41.45}$, $\theta = 22°\,47'$,
bearing is 202° 47′

EXERCISE 5.7

1 (a) incorrect (b) correct (c) correct (d) incorrect

2 12.4 m **3** (a) 8395 m (b) 504 km/h

4



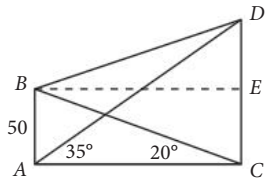
(a) $AE = 60\tan 36°\,52' = 45$ m

(b) $AP^2 = 60^2 + 45^2$, $AP = 75$ m

(c) $\tan\theta = 2.4$, angle of elevation = 67° 23′

5 614.4 m

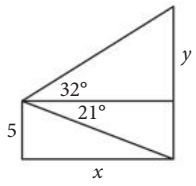
6



(a) $AC = 50\tan 70° = 137.4$ km
(b) $CD = 137.4\tan 35° = 96.2$ m
(c) $DE = 46.2$ m, $\tan\angle DBE = \frac{46.2}{137.4}$, $\angle DBE = 18°\,35'$

7 (a) 23.32 m (b) 18° 26′

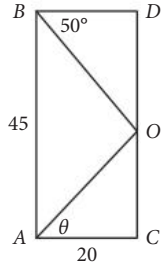
8



(a) $x = 5\tan 69° = 13$ m
(b) $y = 13\tan 32° = 8$ m;
height $= 5 + 8 = 13$ m

9 7 m

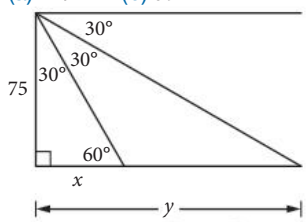
10



(a) $OD = 20\tan 50° = 23.84$ m, $CO = 21.16$ m
(b) $\tan\theta = \frac{21.16}{20}$, $\angle CAO = 46°\,37'$

11 (a) 129 m (b) 37 m

12



(a) $x = 75\tan 30° = 25\sqrt{3}$ m,
$y = 75\tan 60° = 75\sqrt{3}$ m
(b) $y - x = 50\sqrt{3}$ m

13 $PQ = 40\sqrt{3} - \frac{40\sqrt{3}}{3} = \frac{80\sqrt{3}}{3}$ m

EXERCISE 5.8

1 A **2** $A = 15°$, $a = 3.66$, $c = 12.2$

3 $\frac{PQ}{\sin 45} = \frac{25}{\sin 70}$; $PQ = 18.8$ cm

4 $\frac{b}{\sin 45} = \frac{5}{\sin 60}$; $b = \frac{5\sqrt{6}}{3}$

5 $\frac{\sin A}{6} = \frac{\frac{4}{5}}{9}$; $\sin A = \frac{8}{15}$

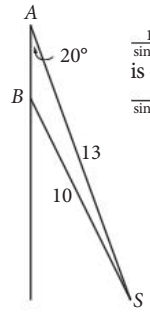
6 $\frac{3}{\sin A} = \frac{5}{\sin 2A}$

7 $\frac{c}{\sin 113°8'} = \frac{8}{\sin 30°}$; $c = 14.7$ cm

8 (a) incorrect (b) correct (c) incorrect (d) correct

9 (a) $\frac{\sin B}{12} = \frac{0.4}{16}$; $\sin B = 0.3$ (b) $CD = 16\sin B = 4.8$

10



$\frac{10}{\sin 20°} = \frac{13}{\sin B}$; $B = 26°\,24'$, $153°\,36'$; nearest A when B is obtuse;
$\frac{AB}{\sin 6°24'} = \frac{10}{\sin 20°}$, $AB = 3.26$ m

11 $\frac{\sin C}{44} = \frac{\sin 37}{30}$, $C = 61°\,58'$ or $118°\,2'$, $A = 81°\,2'$ or $24°\,58'$

12 $\sin\angle RPQ = \frac{11\sin 15°}{10}$: $\angle RPQ = 16°\,32'$, $PR = \frac{20\sin 148°28'}{\sin 15°} = 40.41$ cm;
$\angle RPQ = 163°\,28'$, $PR = \frac{20\sin 1°32'}{\sin 15°} = 2.07$ cm

13 $BC = \frac{3}{\sin 30°} = 6$, $AC = \frac{3}{\sin 36°52'} = 5$, $AB = \frac{5\sin 113°8'}{\sin 30°} = 9.2$;
perimeter $= 20.2$ cm

14 (a) 0.5 (b) 30°, 150°

15 (a) $AC = \frac{5\sin 25°}{\sin 100°} = 2.15$ m (b) $AX = 2.15\sin 55° = 1.76$ m

16 (a) $PQ = \frac{3\sin 80°}{\sin 60°} = 3.4$, $RQ = \frac{3\sin 40°}{\sin 60°} = 2.2$, perimeter $= 8.6$ cm
(b) $RS = 3\sin 40° = 1.9$ cm

17 (a) 14.3 cm (b) 90°; triangle is right-angled, sine rule still works

18 $XZ = \frac{5\sin 82°}{\sin 30°} = 9.90$ m

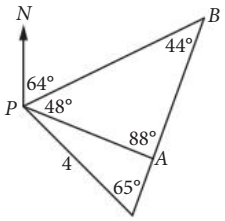
19 (a) $\sin\angle ACB = \frac{4\sin 60°}{5}$, $\angle ACB = 43°\,51'$, $\angle BAC = 76°\,9'$,
$BC = \frac{500\sin 76°9'}{\sin 60°} = 561$ km
(b) $40° + 76°\,9' = 116°\,9'$

20 $SB = \frac{44\sin 105°}{\sin 35°} = 74$ km

21 $CA = \frac{2000\sin 52°}{\sin 93°} = 1578$ m, $CB = \frac{2000\sin 35°}{\sin 93°} = 1149$ m

22 $AC = \frac{50\sin 70°}{\sin 30°} = 94$ m, width $= 94\sin 80° = 93$ m

23



$\frac{AP}{\sin 65°} = \frac{4}{\sin 92°}$, $AP = \frac{4\sin 65°}{\sin 92°}$
$\frac{AB}{\sin 48°} = \frac{AP}{\sin 44°}$, $AB = \frac{4\sin 65°\sin 48°}{\sin 44°\sin 92°} = 3.88$ km

24 $\sin\angle BAD = \frac{BD\sin\angle B}{AD}$, $\sin\angle DAC = \frac{DC\sin\angle ADC}{AC}$,
$\therefore \frac{BD\sin\angle B}{AD} = \frac{DC\sin\angle ADC}{AC}$
$\sin\angle ADC = \sin\angle ADB$, so $\frac{\sin\angle B}{\sin\angle ADB} = \frac{DC\times AD}{BD\times AC}$,
$\therefore \frac{AD}{AB} = \frac{DC\times AD}{BD\times AC}$, $\therefore \frac{AB}{AC} = \frac{BD}{DC}$

EXERCISE 5.9

1 D **2** $\cos\theta = \frac{11^2+13^2-9^2}{2\times 11\times 13} = 0.7308$

3 (a) $a^2 = 7^2 + 10^2 - 2\times 7\times 10\cos 36°\,52'$, $a = 6.1$
(b) $\sin B = \frac{7\sin 36°52'}{6.1}$, $\angle B = 43°\,31'$ **4** 11.7 cm, 24.5 cm

5 $a^2 = 4^2 + 5^2 - 2\times 4\times 5\cos 53°\,8'$, $a = 4.1$ cm, perimeter $= 13$ cm

6 (a) incorrect (b) incorrect (c) correct (d) correct

7 (a) 24.2 (b) $\sin A = \frac{12\sin 126°52'}{24.2}$, $\angle A = 23°\,22'$

8 $\cos\theta = \frac{5^2+6^2-9^2}{2\times 5\times 6}$, $\theta = 109°\,28'$

9 (a) $\cos B = \frac{7^2+8^2-9^2}{2\times 7\times 8}$, $\angle ABC = 73°\,24'$
(b) $AD^2 = 7^2 + 4^2 - 2\times 7\times 4\cos 73°\,24'$, $AD = 7$ cm
(c) $\cos\angle DAC = \frac{9^2+7^2-4^2}{2\times 9\times 7} = \frac{19}{21}$, $\angle DAC = 25°\,13'$

10 (a) $\cos\theta = \frac{8^2+10^2-14^2}{160} = \frac{-1}{5}$, $\theta = 101°\,32'$, $78°\,28'$
(b) $d^2 = 8^2 + 10^2 - 2\times 8\times 10\cos 78°\,28'$, $d = 11.5$ cm

11 (a) $p^2 = 169 - 120\cos 108°$, $p = 14.4$
(b) $\sin\angle PQR = \frac{12\sin 108°}{14.4}$, $\angle PQR = 52°\,25'$

12 (a) incorrect (b) correct (c) correct (d) correct

13 $AB^2 = 325 - 300\cos 45°$, $AB = 10.6$ km

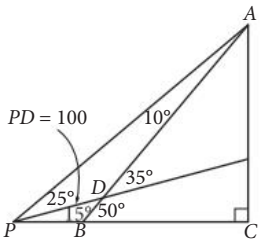
14 $BC^2 = 1300 - 1200\cos 126°\,52'$, $BC = 44.9$ km

15 (a) $\cos\angle PQR = \frac{30^2+50^2-70^2}{2\times 30\times 50} = \frac{-1}{2}$, $\angle PQR = 120°$, 030°
(b) $\cos\angle RPQ = \frac{70^2+50^2-30^2}{2\times 70\times 50}$, $\angle RPQ = 21°\,47'$, $068°\,13'$

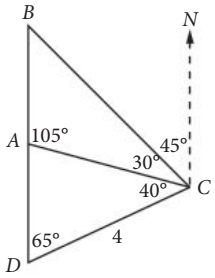
16 (a) 8.62 km (b) 034° 53′

17 (a) $AD = \frac{100\sin 15°}{\sin 25°}$, $h = \frac{AD\sin 40°}{\sin 80°} = \frac{100\sin 15°\sin 40°}{\sin 25°\sin 80°} = 40$ m
(b) $DB = \frac{100\sin 140°}{\sin 25°} = 152.1$ m
$DC^2 = DB^2 + 100^2 - 200\times DB\cos 165°$, $DC = 250$ m
$\sin\angle BDC = \frac{100\sin 165°}{DC}$, $\angle BDC = 5°\,57'$
$\angle BCD = 15° - 5°\,57' = 9°\,3'$
angle of elevation $= 9°\,3' - 10° = -0°\,57'$
angle of depression $= 0°\,57'$

18



$AP = \frac{100\sin 145°}{\sin 10°}$;
$AC = AP\sin 40° = \frac{100\sin 145°\sin 40°}{\sin 10°}$
$= 212.3$ m

19 

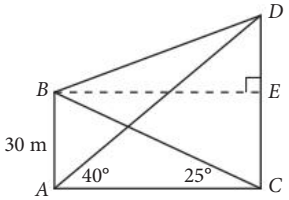
$AC = \frac{4\sin 65°}{\sin 75°}$,
$AB = \frac{AC\sin 30°}{\sin 45°} = \frac{4\sin 65°\sin 30°}{\sin 75°\sin 45°} = 2.654\,\text{km}$

20 $\angle BAC = 110°$, $BC = 418\,\text{km}$

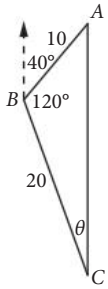
EXERCISE 5.10

1 (a) $a = 6.1$ (b) $\angle B = 43°\,31'$
(c) Area $= 35\sin 36°\,52' = 21\,\text{cm}^2$
2 $8\,\text{cm}^2$ **3** (a) $24°\,37'$ (b) 3.33 cm (c) $18.3\,\text{cm}^2$
4 B
5 (a) $\cos\angle ADC = \frac{5^2+4^2-7^2}{2\times5\times4} = -0.2$, $\angle ADC = 101°\,32'$,
$\angle BDC = \angle DCB = 78°\,28'$,
$\angle DBC = 23°\,4'$; $BC = \frac{4\sin 78°28'}{\sin 23°4'} = 10\,\text{cm}$
(b) $29.4\,\text{cm}^2$
6 (a) $\angle APB = 101°\,32'$, $\angle APC = 127°\,10'$, $\angle BPC = 131°\,18'$,
$BC^2 = 244 - 240\cos 131°\,18'$, $BC = 20\,\text{cm}$
(b) $122.5\,\text{cm}^2$
7 (a) $47°\,09'$ (b) 3.7 cm (c) $9.2\,\text{cm}^2$
8 (a) $AC = 11.4\,\text{cm}$, $BD = 7\,\text{cm}$ (b) $20\sqrt{3}\,\text{cm}^2$
9 (a) correct (b) correct (c) incorrect (d) incorrect
10 smallest angle $= 49°\,28'$, area $= 3420\,\text{m}^2$

CHAPTER REVIEW 5

1 (a) $\sin\theta$ (b) $-\cos\theta$ (c) $-\sin\theta$ (d) $\tan\theta$ (e) $-\tan\theta$ (f) $-\sin\theta$
2 (a) -1 (b) $-\frac{1}{\sqrt{2}}$ (c) -1 (d) 0 (e) $\frac{\sqrt{3}}{2}$ (f) $-\frac{\sqrt{3}}{2}$
3 (a) 1 (b) 0 **4** (a) $-\frac{3}{\sqrt{34}}$ (b) $-\frac{5}{\sqrt{34}}$
5 (a) 0.6 (b) 0.6 (c) -0.8 (d) $\frac{3}{4}$ (e) -0.75 (f) -0.6
6 (a) $\frac{1}{t}$ (b) t (c) $-\frac{1}{t}$ (d) $-t$ (e) $-t$ (f) $-\frac{1}{t}$
7 $\cos\theta = \frac{49+36-25}{2\times7\times6} = \frac{5}{7}$
8 $\cos\theta = \frac{25+36-64}{2\times5\times6} = \frac{-1}{20} < 0$; cosine of angle opposite largest side < 0, so angle is obtuse; $\theta = 92°\,52'$
9 (a) $BC = \frac{8\sin 79°}{\sin 53°} = 9.8\,\text{cm}$ (b) area $= 39.2\sin 48° = 29.1\,\text{cm}^2$
10 $\frac{8\sqrt{3}}{2} - \frac{8}{\sqrt{2}} = 4\left(\sqrt{3}-\sqrt{2}\right)$
11 (a) 25.17 m (b) $\frac{30}{\cos 40°} = 39.16\,\text{m}$ (c) $\tan\theta = \frac{12.585}{30}$, $\theta = 22°\,45'$
12 

(a) $AC = 30\tan 65° = 64.3\,\text{m}$
(b) $CD = 64.3\tan 40° = 54\,\text{m}$
(c) $\tan\angle DBE = \frac{24}{64.3}$, angle of depression $= 20°\,28'$

13 

(a) $AC = 10\sqrt{7} = 26.46\,\text{km}$
(b) $\sin\theta = \frac{10\sin 120°}{10\sqrt{7}}$, $\theta = 19°\,6'$,
bearing of A from $C = 359°\,6'$

14 (a) LHS $= \frac{\sin^2\theta}{1-\sin^2\theta} = \frac{\sin^2\theta}{\cos^2\theta} = \tan^2\theta =$ RHS
(b) LHS $= \sin^2 x + 2\sin x\cos x + \cos^2 x + \sin^2 x - 2\sin x\cos x + \cos^2 x$
$= 2(\sin^2 x + \cos^2 x) = 2 =$ RHS

15 B **16** 4
17 (a) 210°, 330° (b) 90°, 270° (c) 135°, 315°
(d) 30°, 330° (e) 30°, 210° (f) 45°, 135°
18 (a) 1 (b) 1 (c) $\sec^2\theta$ (d) $\sin\theta$ (e) 1 (f) $2\tan^2\theta$
19 (a) incorrect (b) correct (c) correct (d) correct

CHAPTER 6

EXERCISE 6.1

1 (a) -1 (b) -3.5 (c) 0 (d) -1 (e) -1 (f) $\frac{5c}{b}$ (g) $\frac{-2}{3}$ (h) 1.5
2 (a) $\tan\theta = 1$, $\theta = 45°$ (b) $26°\,34'$ (c) $159°\,27'$ (d) $56°\,19'$
(e) $153°\,26'$ (f) 135°
3 A **4–6** Answers will vary.
7 (a) incorrect (b) correct (c) incorrect (d) correct
8–10 Answers will vary.

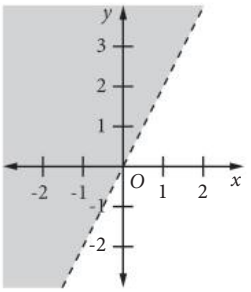
EXERCISE 6.2

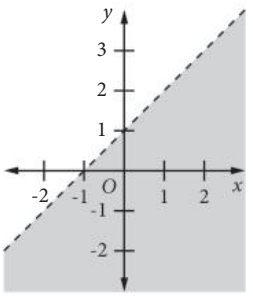
1 (a) $3x - 4y + 38 = 0$ (b) $x + 2y + 2 = 0$
2 (a) $8x - 7y - 3 = 0$ (b) $2x - y - 12 = 0$
3 (a) $x - y = 0$ (b) $m = \frac{4}{3}$, $4x - 3y + 17 = 0$ (c) $x + y + 7 = 0$
(d) $3x + 4y + 5 = 0$
4 $y = 2$ **5** $x = -2$ **6** B **7** $2x + 3y + 6 = 0$
8 (a) $y = \frac{-2}{3}x + \frac{4}{3}$, $\frac{-2}{3}$ (b) $y = \frac{3}{2}x - \frac{7}{2}$, 1.5
(c) $y = \frac{-3}{2}x + 3$, -1.5 (d) $y = \frac{2}{5}x + \frac{8}{5}$, 0.4
9 (a) correct (b) correct (c) incorrect (d) correct
10 (a) Answers will vary. (b) 3
11 (a) $3x + 2y = 0$ (b) $2x - 3y - 13 = 0$
12 (a) $4x - 5y = 0$ (b) $5x + 4y = 0$
13 (a) $2x - 3y - 6 = 0$ (b) $33°\,41'$ (c) 2 (d) $3x + 2y - 22 = 0$
14 Answers will vary.
15 (a) $2x - 5y + 31 = 0$ (b) $x + 2y - 7 = 0$ (c) (-3, 5)
16 (a) $x + 2y - 9 = 0$ (b) $2x - y + 2 = 0$ (c) -2
(d) $2x - y - 8 = 0$ (e) $x + 2y + 6 = 0$ (f) (2, -4)
17 (a) $x + 5y = 0$ (b) $2x + y = 0$ (c) $x - 4y + 9 = 0$

EXERCISE 6.3

1 C **2** $4x - y - 5 = 0$
3 (a) $m_{BC} = 6$; $y - 3 = \frac{-1}{6}(x+1)$, $x + 6y - 17 = 0$
(b) $x + 6y - 32 = 0$
4 (a) $m_1 = -1 = m_2$; $m_3 = \frac{2}{3} = m_4$; first two lines are parallel, last two lines are parallel, no lines are coincident, so lines are sides of a parallelogram. (b) $\left(\frac{-4}{5}, \frac{-1}{5}\right)$, $\left(-1\frac{4}{5}, \frac{4}{5}\right)$, (1, 1), (0, 2)
(c) $x - 14y + 13 = 0$, $11x - 4y + 8 = 0$
5 (a) $4x - 3y + 1 = 0$ (b) $2x - y - 1 = 0$ (c) $x = 2$
6 (a) correct (b) correct (c) incorrect (d) correct
7 (a) (-9, -7) (b) (-2, 0)
8 (a) Answers will vary. (b) (4, 3)
9 (a) coincide (b) intersect (c) intersect (d) parallel

EXERCISE 6.4

1 

2 

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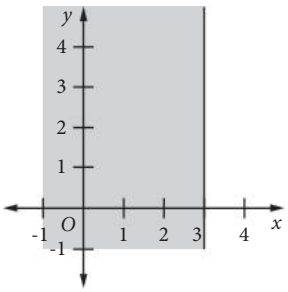

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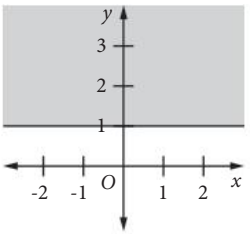

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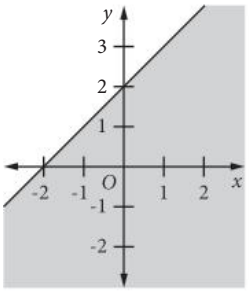

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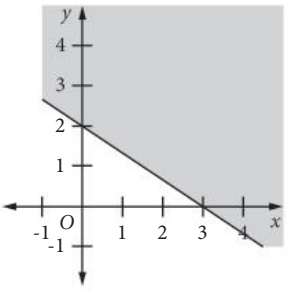

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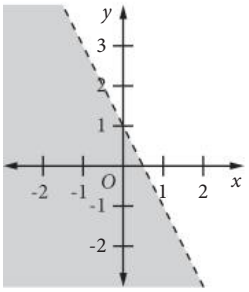

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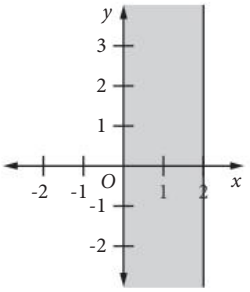

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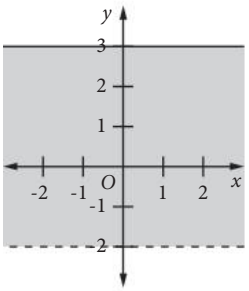

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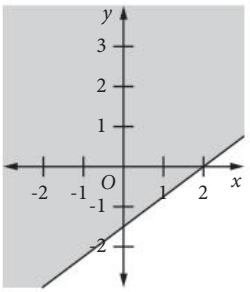

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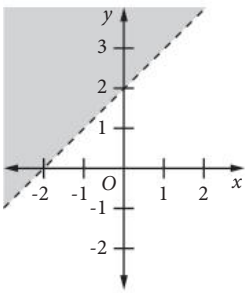

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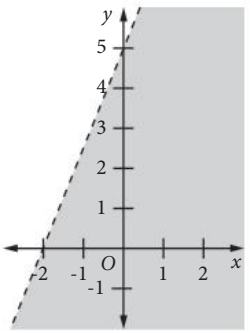

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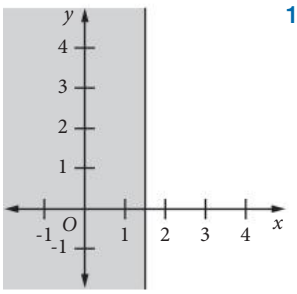

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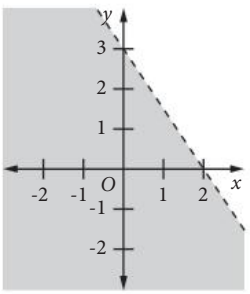

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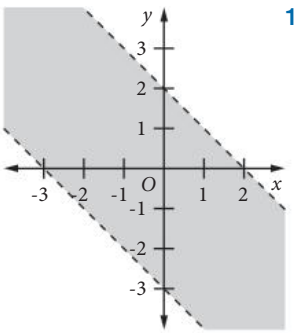

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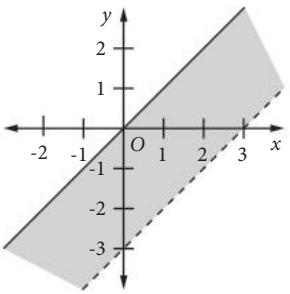
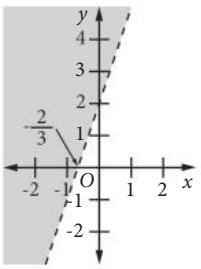


17 A

18 (a)



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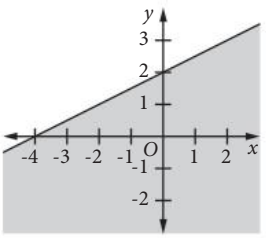

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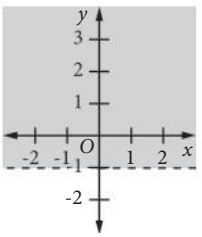

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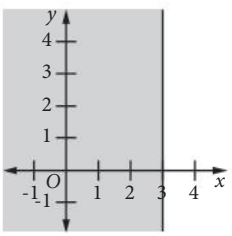
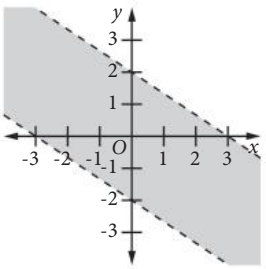
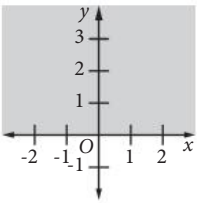


(e)



(f)



19 (a) The region below the line $y = x + 2$.
(b) The region on and above the line $y = x$.
(c) The region to the right of the line $x = 3$.
(d) The region on and below the line $y = 4$.
(e) The region on and below the line $x + 3y = 9$.
(f) The region on and to the right of the line $x = -2$ that is also to the left of the line $x = 3$.

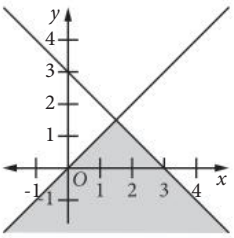
20 (a) correct (b) incorrect (c) incorrect (d) correct

EXERCISE 6.5

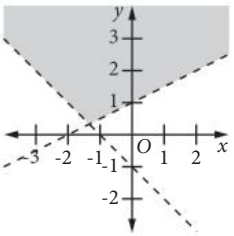
1 (a) The region on and to the right of the line $x = -1$ that is also below the line $y = 2$.
(b) The region on and above the line $x + y = 1$ that is also on and below the line $y = x + 1$.
(c) The region above the line $y = 1$ that is also above the line $x + y = 1$.
(d) The region on and to the right of the line $x = -1$ that is also to the left of the line $x = 2$.
(e) The region on and below the line $y = 2x + 2$ that is also below the line $x + y = 2$.
(f) The region on and above the line $y = x$ that is also on and below the line $y = 2x$.

2 C **3** $x + y \geq 3, y \leq x + 1$

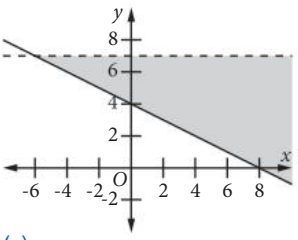
4 (a) yes, no, yes



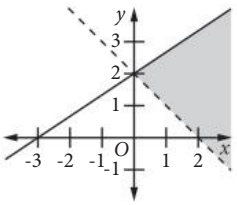
(b) no, no, yes



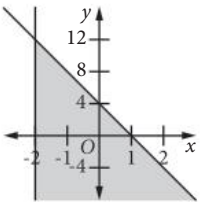
(c) yes, no, yes



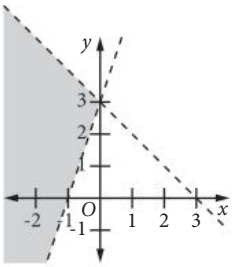
(d) no, yes, yes



(e) yes, no, yes

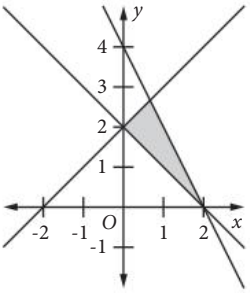


(f) no, no, no



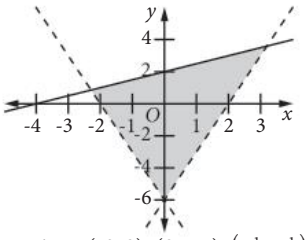
5 (a) correct (b) incorrect (c) correct (d) correct

6 (a)



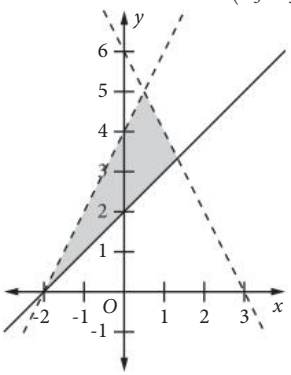
$y = x + 2$, $2x + y = 4$;
$2x + x + 2 = 4$, $3x = 2$:
$x = \frac{2}{3}$, $y = 2\frac{2}{3}$
vertices: $\left(\frac{2}{3}, 2\frac{2}{3}\right)$, $(2,0)$, $(0,2)$

(b)

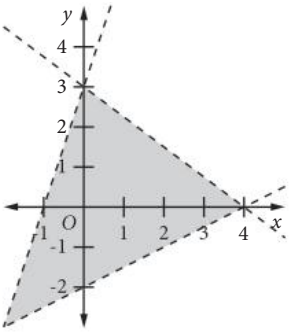


$2y - x = 4$, $y = 3x - 6$;
$6x - 12 - x = 4$, $5x = 16$:
$x = 3\frac{1}{5}$, $y = 3\frac{3}{5}$
$2y - x = 4$, $3x + y = -6$;
$6x - 12 - x = 4$, $7x = -16$:
$x = -2\frac{2}{7}$, $y = \frac{6}{7}$
vertices: $\left(3\frac{1}{5}, 3\frac{3}{5}\right)$, $\left(-2\frac{2}{7}, \frac{6}{7}\right)$, $(0,-6)$

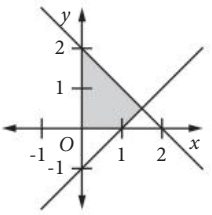
(c) vertices: $(-2,0)$, $(0.5,5)$, $\left(1\frac{1}{3}, 3\frac{1}{3}\right)$



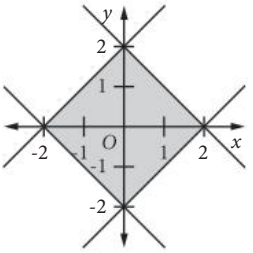
(d) vertices: $(-2,-3)$, $(4,0)$, $(0,3)$



(e) vertices: $(0,0)$, $(1,0)$, $(1.5,0.5)$, $(0,2)$



(f) vertices: $(-2,0)$, $(0,2)$, $(2,0)$, $(0,-2)$



7 (a) the region bounded by the lines $y = x$, $x = 2$ and $y = 1$;
$y \le x$, $x \le 2$, $y \ge 1$; $A(2,2)$, $B(2,1)$, $C(1,1)$
(b) the region bounded by the lines $y = 2x$, $y = x$ and $x + y = 3$;
$y \le 2x$, $y \ge x$, $x + y \le 3$; $A(0,0)$, $B(1,2)$, $C(1.5,1.5)$
(c) the region bounded by the lines $y = \frac{x}{2} + 1$, $y = \frac{x}{6} + 1$ and $x + 2y = 6$; $y \le \frac{x}{2} + 1$, $y \ge \frac{x}{6} + 1$, $x + 2y \le 6$; $A(0,1)$, $B(2,2)$, $C(3,1.5)$

EXERCISE 6.6

1 (a) $(1.5,-1)$ (b) $(2,7)$ (c) $(-4,-8)$ (d) $(1,1)$ (e) $\left(\frac{a+c}{2}, \frac{b+d}{2}\right)$ (f) $\left(a, \frac{a+b}{2}\right)$

2 $(-11,14)$

3 (a) $(15,-2)$ (b) $(-4,0)$ (c) $e = 6$, $f = 10$; $A(6,-3)$, $B(-6,10)$
(d) $g = 2$, $h = 2$; $A(4,2)$, $B(8,4)$

4 (a) 9 (b) 5 (c) $\sqrt{8^2 + 2^2} = 2\sqrt{17}$ (d) $\sqrt{218}$ (e) $5a$ (f) $\sqrt{2(a-b)^2}$

5 D

6 (a) $DE = \sqrt{13}$, $EF = \sqrt{13}$, $FD = \sqrt{26}$; hence $FD^2 = DE^2 + EF^2$, so ΔDEF is right-angled at E
(b) $m_{DE} = -\frac{2}{3}$, $m_{EF} = \frac{3}{2}$, $m_{FD} = -5$. $m_{DE} \times m_{EF} = -\frac{2}{3} \times \frac{3}{2} = -1$;
$\therefore DE \perp EF$

7 $D(0,3)$, $E(4.5,1)$; $m_{DE} = -\frac{4}{9}$, $m_{CB} = -\frac{4}{9}$, so $DE \parallel CB$

EXERCISE 6.7

1 C

2 (a) 5 units (b) 2 units (c) 8 units (d) $\frac{\sqrt{10}}{2}$ units
(e) 13 units (f) $2\sqrt{2}$ units (g) $2\sqrt{5}$ units (h) 0.4 units

3 (a) 4 units (b) 2 units (c) $3\sqrt{2}$ units

4 equation of BC is $x - 2y + 6 = 0$; $2\sqrt{5}$ units

5 4 units

6 (a) correct (b) incorrect (c) correct (d) correct

7 2.5 units^2

CHAPTER REVIEW 6

1 (a) -1, $\frac{2}{3}$, -3.5, 0.25, -6, -1 (b) $AB \parallel CD$ (c) $\sqrt{2}$, $5\sqrt{2}$, $\sqrt{53}$
(d) $(2,1.5)$, $(0,3.5)$, $(2.5,1)$

2 (a) $M(4,1)$, $N(-1,-1)$
(b) gradient of $MN = 0.4$, gradient of $QR = 0.4$, $\therefore MN \parallel QR$
(c) $QR = 2\sqrt{29}$ (d) $MN = \sqrt{29}$, $\therefore MN = \frac{QR}{2}$

3 3.1 units **4** midpoint $(4, 3)$, line $4x - 3y - 7 = 0$

5 $A(2, -3)$, $B(5, 2)$, $C(-3, 0)$; gradient of $CA = -0.6$, gradient of $BC = 0.25$, gradient of $AB = \frac{5}{3}$;
Grad $CA \times$ Grad $AB = \frac{-3}{5} \times \frac{5}{3} = -1$
$\therefore AB \perp CA$, so triangle is right-angled at A

6 $\frac{\sqrt{2}}{2}$ units **7** $A(-3, y_1)$, $\frac{5-y_1}{6} \times \frac{-2}{5} = -1$, $y_1 = -10$: $A(-3, -10)$

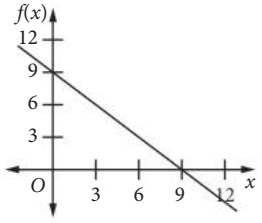
8 BC: $3x + 4y - 12 = 0$; distance = 1.4 units, $BC = 5$ units, area = 3.5 units^2

CHAPTER 7

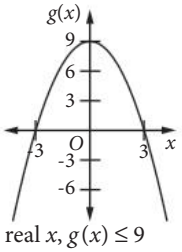
EXERCISE 7.1

1 (a) relation, domain $\{1, 2, 3\}$, range $\{1, 2, 5, 7\}$
(b) function, domain $\{-3, 3, 8, 9, 11\}$, range $\{-2, 1, 6, 7\}$
(c) function, domain: real x, range $\{5\}$
(d) relation, domain $\{2\}$, range: real y
(e) relation, domain $-3 \le x \le 3$, range $-3 \le y \le 3$
(f) function, domain $-4 \le x \le 4$, range $0 \le y \le 4$
(g) function, domain: real x, range: real y
(h) function, domain: real x, range $y \le 2$

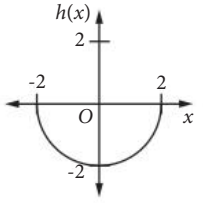
2 C

3 (a) 

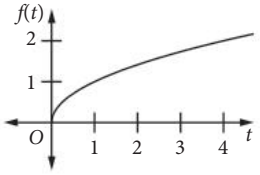
real x, real $f(x)$

(b) 

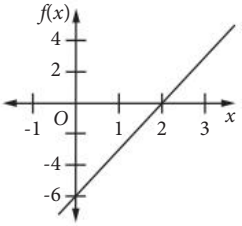
real x, $g(x) \le 9$

(c) 

$-2 \le x \le 2$, $-2 \le h(x) \le 0$

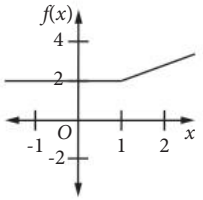
(d) 

$t \ge 0$, $f(t) \ge 0$

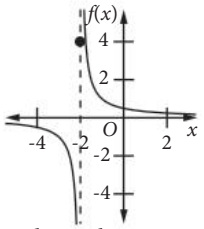
4 (a) -3, -12, $3a - 6$
(b) 3
(c) $x > 3$
(d) 

5 (a) 8, 8 (b) $a^2 - 1$, $b^2 - 1$, $(a + b)^2 - 1$ (c) no (d) $f(x) \ge -1$

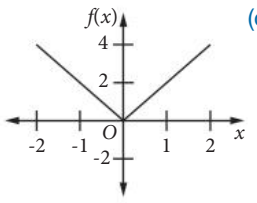
6 (a) correct (b) incorrect (c) correct (d) correct

7 (a) 

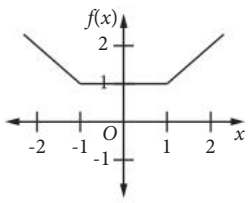
real x, $y \ge 2$

(b) 

real x, real y, $y \ne 0$

(c) 

real x, $y \ge 0$

(d) 

real x, $y \ge 1$

8 (a) $x \ge 2$, $f(x) \ge 0$ (b) $x \le 3$, $f(x) \ge 0$ (c) $|x| \ge 3$, $f(x) \ge 0$
(d) real x, $x \ne 0$; real y, $g(x) \ne 0$ (e) real t, real $h(t)$
(f) real k, $g(k) \le 5$

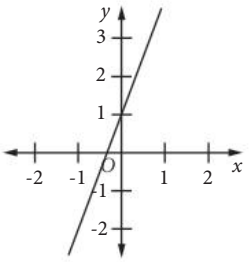
9 (a) 2 (b) does not exist (c) -1 (d) 6

10 (a) 0 (b) 2 (c) -0.5 (d) a^2

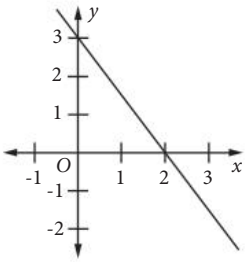
11 (a) 0 (b) -2 (c) -8 (d) 2

EXERCISE 7.2

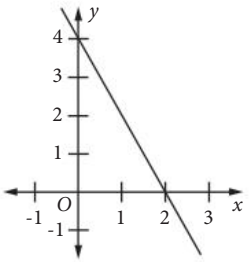
1 (a) $m = 3$; $-\frac{1}{3}$; 1



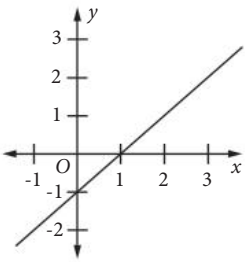
(b) $m = -1.5$; 2; 3



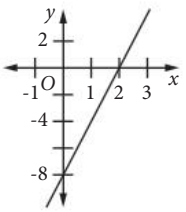
(c) $m = -2$; 2; 4



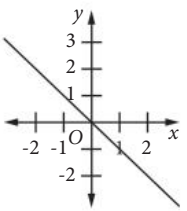
(d) $m = 1$; 1; -1



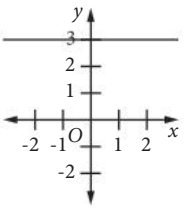
(e) $m = 4$; 2; -8



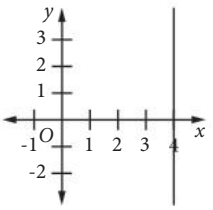
(f) $m = -1$; 0; 0



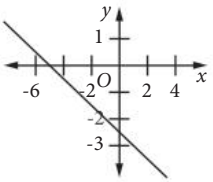
(g) $m = 0$; none; 3



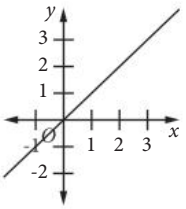
(h) $m =$ undefined; 4; none



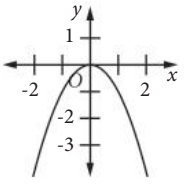
(i) $m = -0.5$; -5; -2.5



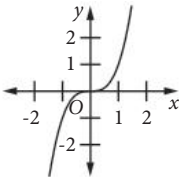
2 (a) increasing (b) decreasing (c) decreasing
(d) increasing (e) increasing (f) decreasing
(g) neither (h) neither (i) decreasing

3 (a) 

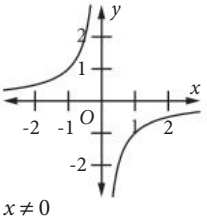
all x

(b) 

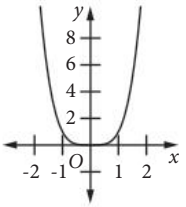
$x < 0$

(c) 

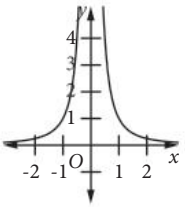
all x

(d) 

$x \neq 0$

(e) 

$x > 0$

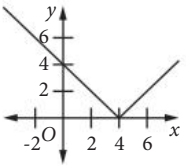
(f) 

$x < 0$

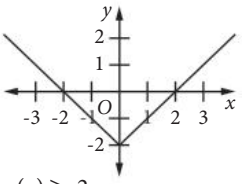
4 (a) odd (b) even (c) odd (d) odd (e) even (f) even

5 B **6** (a) correct (b) incorrect (c) correct (d) incorrect

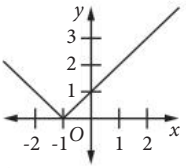
EXERCISE 7.3

1 (a) 

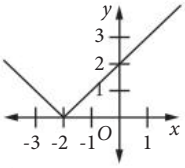
$f(x) \geq 0$

(b) 

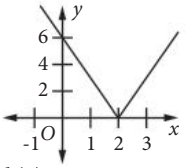
$g(x) \geq -2$

(c) 

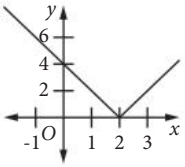
$h(x) \geq 0$

(d) 

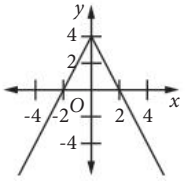
$f(x) \geq 0$

(e) 

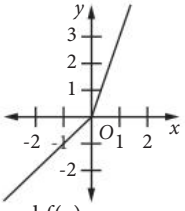
$h(x) \geq 0$

(f) 

$f(x) \geq 0$

(g) 

$g(x) \leq 4$

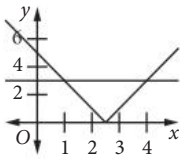
(h) 

real $f(x)$

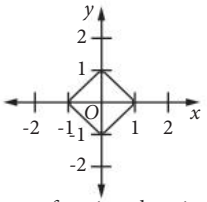
2 C **3** (a) $2 \leq x \leq 3$ (b) real x, $x \neq 0$

4 (a) odd (b) neither (c) odd (d) neither (e) even (f) neither (g) even (h) neither (i) neither

5 (a) $f(x) \geq 0$ (b) $f(x) = 2x$ for $x \geq 0$, $f(x) = 0$ for $x < 0$; range $f(x) \geq 0$ (c) $0 \leq f(x) \leq 4$ (d) $f(x) \leq 16$

6 

$f(x) = 3$ for $x = 1$ or 4

7 

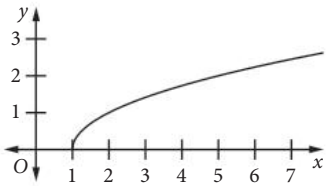
not a function; domain $-1 \leq x \leq 1$, range $-1 \leq y \leq 1$

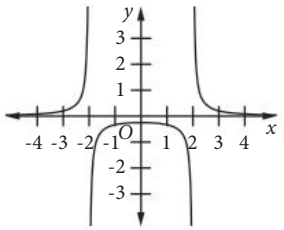
8 (a) correct (b) correct (c) correct (d) correct

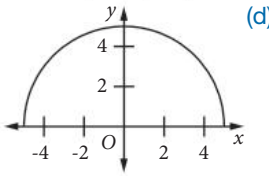
9 (a) correct (b) incorrect (c) correct (d) correct

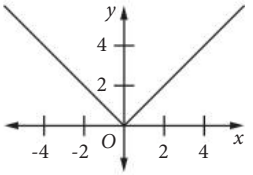
CHAPTER REVIEW 7

1 (a) $x \geq 1$ (b) real x, $x \neq \pm 2$ (c) $-5 \leq x \leq 5$ (d) real x

2 (a) 

(b) 

(c) 

(d) 

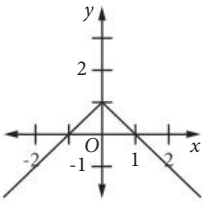
3 $g(x) = x^4 - x^2 + 1$

$g(-x) = (-x)^4 - (-x)^2 + 1$

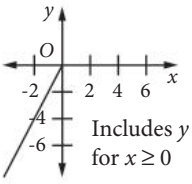
$= x^4 - x^2 + 1$

$= g(x)$

Hence $g(x)$ is an even function.

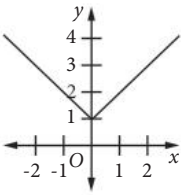
4 

5 all x



Includes $y = 0$ for $x \geq 0$

6 (a) $-3 \leq x \leq 2$ (b) real x, $x \neq 1$ **7** neither

8 

CHAPTER 8

EXERCISE 8.1

1 (a) $y = 5$ (b) $y = -1$ (c) $3x + y - 7 = 0$ (d) $x - 3y + 5 = 0$

2 D

3 (a) $x = -4$ (b) $y = -2.5$ (c) $y = x \pm 2$ (d) $x \pm 3y = 0$
(e) $x - y - 3 = 0$, $x + y - 5 = 0$ (f) $3x \pm 2y = 0$
(g) $3x - 4y - 26 = 0$, $3x + 4y + 14 = 0$
(h) $5x - 2y + 13 = 0$, $5x + 2y + 17 = 0$

4 C **5** (a) $8x + 8y - 3 = 0$ (b) $x + y - 1 = 0$

6 $3x^2 - 52x + 3y^2 + 48y + 276 = 0$ **7** $x = 3a$

EXERCISE 8.2

1 (a) $(x-3)^2 + (y-2)^2 = 16$ (b) $(x+1)^2 + (y+4)^2 = 9$
(c) $(x-3)^2 + (y+3)^2 = 5$ (d) $(x+2)^2 + \left(y - \frac{5}{2}\right)^2 = \frac{49}{4}$
(e) $x^2 + \left(y + \frac{3}{2}\right)^2 = 16$ (f) $(x-4)^2 + y^2 = 9$

2 B

3 (a) $(x-3)^2 + (y-2)^2 = 53$ (b) $(x+1)^2 + (y-4)^2 = 17$
(c) $x^2 + y^2 = 25$

4 (a) $(3,-2)$, $r=4$ (b) $(-2,-1)$, $r=3$ (c) $(3,0)$, $r=\sqrt{3}$
(d) $(-a,b)$, $r=2\sqrt{2}$ (e) $(2.5,-1.5)$, $r=\frac{\sqrt{38}}{2}$
(f) $(-2,-1)$, $r=\sqrt{10}$ (g) $\left(2,-1\frac{1}{4}\right)$, $r=\frac{\sqrt{65}}{4}$
(h) $\left(-1\frac{1}{2},\frac{2}{3}\right)$, $r=\frac{\sqrt{385}}{6}$

5 (a) $(x-6)^2+(y+1)^2=34$ (b) $\left(x-\frac{5}{2}\right)^2+\left(y+\frac{3}{2}\right)^2=\frac{17}{2}$
(c) $\left(x-\frac{3}{2}\right)^2+\left(y-\frac{11}{2}\right)^2=\frac{37}{2}$

6 (a) incorrect (b) incorrect (c) incorrect (d) correct

7 (a) $(-3,4)$, $r=5$ (b) LHS $=0=$ RHS (c) $4x+3y=0$

8 $(x-4)^2+(y-4)^2=16$ 9 inside 10 outside

11 (a) $x^2+y^2+2x-4y-20=0$ (b) $2\sqrt{21}$ units (c) 4 units

12 (a) 5 units (b) $2\sqrt{21}$ units

13 (a) $(2,1)$, $\sqrt{41}$ units (b) $2\sqrt{6}$ units

14 (a) $(2,5)$ (b) $x^2+y^2-4x-10y+16=0$ (c) $(0,2)$, $(0,8)$

15 (a) $(2,4)$, 5 units (b) $\sqrt{5}$ units, $4\sqrt{5}$ units

EXERCISE 8.3

1 $x^2+y^2-4x+2y-11=0$

2 centres are on a line, $x+y-6=0$

3 $3x^2+3y^2+14x+6y-2=0$

4 A

5 $x^2+y^2=9$; $(0,0)$, 3 units

6 arc of circle $x^2+y^2=9$, $0\le x\le 3$, $0\le y\le 3$

7 $x^2+y^2-6x+y-3=0$

8 (a) $x^2+y^2=25$ (b) $(\pm5,0)$, $(0,\pm5)$

9 $x^2+y^2-16x+4y+28=0$, point is $(2,0)$

10 $x^2+y^2-2x+6y-24=0$

11 $x^2+y^2-2y-3=0$

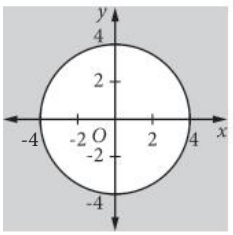
12 $x^2+y^2+4x-3y=0$

13 $x^2+y^2-4x=0$ for $A(4,0)$, $x^2+y^2+4x=0$ for $A(-4,0)$

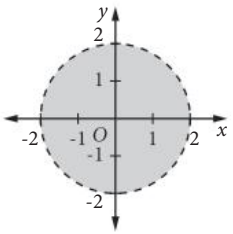
14 $x^2+y^2-4x+2y=0$

EXERCISE 8.4

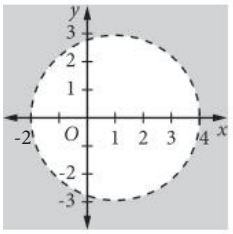
1 (a)



(b)

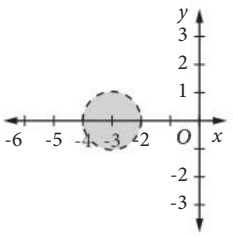


(c)

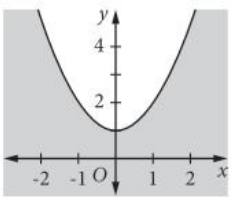


2 C

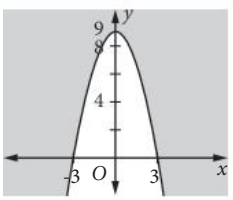
3 (a)



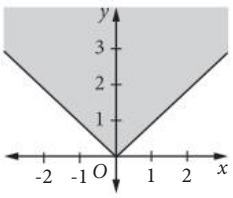
(b)



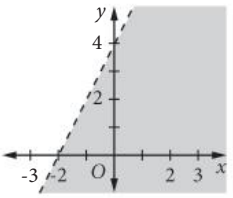
(c)



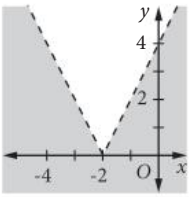
(d)



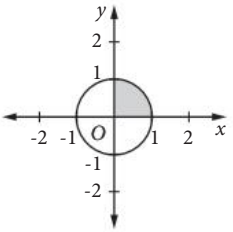
(e)



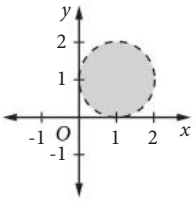
(f)



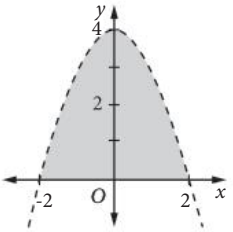
4 (a)



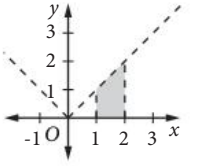
(b)



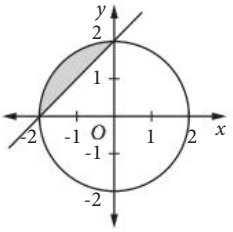
(c)



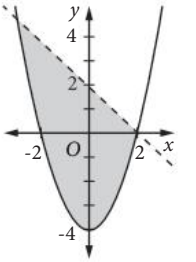
(d)



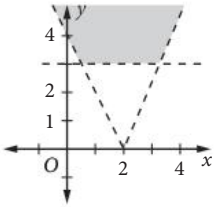
(e)



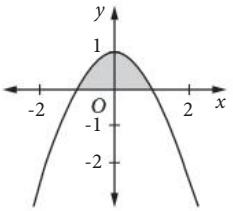
(f)



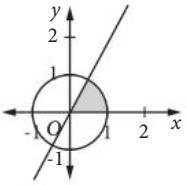
(g)



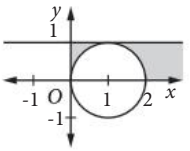
(h)



(i)

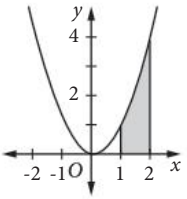


(j)

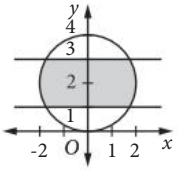


5 A

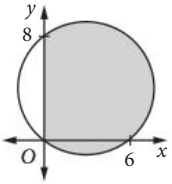
6 (a)



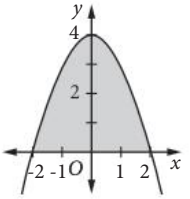
(b)

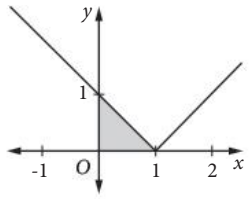


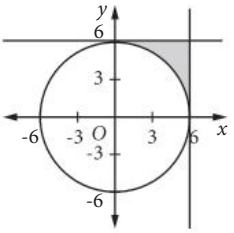
(c)

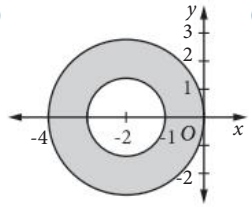


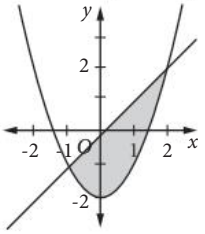
(d)

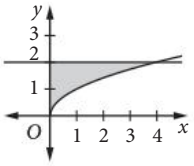


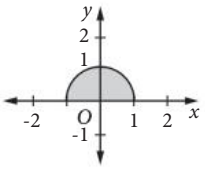
(e) 

(f) 

(g) 

(h) 

(i) 

(j) 

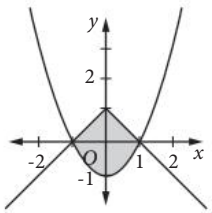
7 (a) correct (b) incorrect (c) incorrect (d) correct

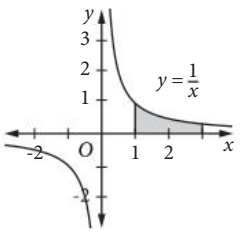
CHAPTER REVIEW 8

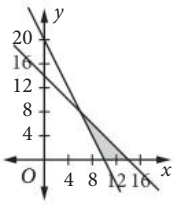
1 (a) $x^2 + y^2 = 25$ (b) $(x-2)^2 + (y-1)^2 = 16$
(c) $4x + 2y - 31 = 0$ (d) $x^2 = 12y$

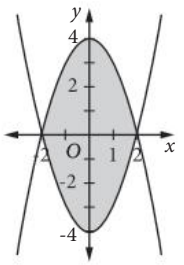
2 (a) circle, centre at origin, radius 5 units
(b) circle, centre (2, 1), radius 4 units
(c) straight line, gradient = -2, y-intercept = 15.5
(d) parabola, vertex (0, 0), focus (0, 3)

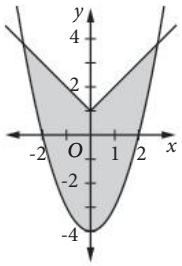
3 (a) $(x-1)^2 + (y-1)^2 = 25$; centre (1, 1), radius = 5 units
(b) $\sqrt{45} = 3\sqrt{5}$ units
(c) $\sqrt{45} \approx 6.7 > 5$; distance from point to centre > radius, so point is outside circle
(d) $2\sqrt{5}$ units

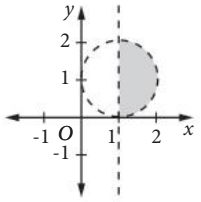
4 

5 

6 (6, 8) satisfies both equations:


7 

8 

9 

10 $(x+2)^2 = 4y$

11 $x^2 + (x-4)^2 = 8$, $2x^2 - 8x + 8 = 0$, $(x-2)^2 = 0$, $x = 2$; line touches circle at (2, -2)

CHAPTER 9

EXERCISE 9.1

1 C

2 (a) $2(x-1)^2 - 2$; min = -2; range $y \geq -2$
(b) $-2(x-2)^2 + 5$; max = 5; range $y \leq 5$
(c) $-4(x-2)^2 + 23$; max = 23; range $y \leq 23$
(d) $4(x+1)^2 - 11$; min = -11; range $y \geq -11$
(e) $-2x^2 + 8$; max = 8; range $y \leq 8$
(f) $-(x+1)^2 + 8$; max = 8; range $y \leq 8$
(g) $2(x-1.5)^2 - 4.5$; min = -4.5; range $y \geq -4.5$
(h) $-5(x+1)^2 + 11$; max = 11; range $y \leq 11$

3 D **4** (a) $0 \leq t \leq 4$ (b) 20 m **5** 12.5 m, 50 m

6 10, 10; 100 **7** 225 cm^2 **8** 50 m^2; 5 m, 10 m

9 $2\frac{4}{7}$ m, $3\frac{3}{7}$ m **10** 30 000 m^2 **11** $y = 10x - x^2$; $x = 5$, $y = 25$

12 area $= 1 - \frac{x^2}{2} - \frac{1-x}{2} = \frac{1+x-x^2}{2}$; $y = -\frac{1}{2}\left(x - \frac{1}{2}\right)^2 + \frac{5}{8}$; 0.625 units^2

EXERCISE 9.2

1 D

2 (a) 28, two (b) -23, none (c) 0, one (d) -8, none
(e) 65, two (f) 0, one

3 A

4 (a) yes (b) no (c) touches (d) yes (e) no (f) no

5 (a) 64; 3, -5 (b) 101; $\frac{9 \pm \sqrt{101}}{2}$ (c) 49; $\frac{3}{4}$, $1\frac{1}{3}$ (d) 0; 1.5
(e) 1764; ±3 (f) 36; 0, 6 (g) 0; 1 (h) -368; none
(i) 201; $\frac{1 \pm \sqrt{201}}{4}$ (j) 28; $\frac{1 \pm \sqrt{7}}{3}$ (k) 85; $\frac{7 \pm \sqrt{85}}{2}$ (l) 1; -5, -6
(m) 17; $\frac{9 \pm \sqrt{17}}{8}$ (n) 0; $-1\frac{1}{3}$ (o) 76; $\frac{-2 \pm \sqrt{19}}{3}$
(p) 33; $\frac{-1 \pm \sqrt{33}}{4}$ (q) 1; 7, 8 (r) 9; 0, 1.5 (s) 17; $\frac{-5 \pm \sqrt{17}}{4}$
(t) 9; 1, 0.4 (u) -39; none

EXERCISE 9.3

1 (a) -1, 7 (b) -2, -1, 3, 4 (c) -5, -1, 3 (d) -1, 7

2 D

3 (a) ±1, ±3 (b) 0, 3 (c) 1, 2 (d) -2, 0 (e) 2, 3 (f) -1, 1
(g) -1, 2 (h) -1, 2

4 B

5 (a) ±2 (b) 2.5, 3 (c) $\pm 2, -2 \pm 2\sqrt{2}$ (d) ±1 (e) 3
(f) ±2, -1, 3 (g) 1, 3 (h) ±2, ±3 (i) $\pm\sqrt{2}, \pm\sqrt{3}$ (j) $\pm\sqrt{5}$

6 (a) 0, 1 (b) 1, $-\frac{1}{3}$ (c) -1, 0 (d) 1, $\frac{3 \pm \sqrt{5}}{2}$
(e) $\log_2 3, \log_2 5$ (f) 1, 2 (g) 1, 2 (h) 2, $\log_5 3$

EXERCISE 9.4

1 (a) $x^2 - 7x + 10 = 0$ (b) $6x^2 - 5x + 1 = 0$
(c) $x^2 - 4x + 1 = 0$ (d) $x^2 - 3px + 2p^2 = 0$

2 B **3** (a) -4 (b) 1 (c) -4 **4** C

5 Roots α, 3α; $4\alpha = -q$, $3\alpha^2 = r$;
$3 \times \left(\frac{-q}{4}\right)^2 = r$, $3q^2 = 16r$

6 $p = -1$, $q = -6$

7 Roots α, 2α; $3\alpha = -\frac{b}{a}$, $2\alpha^2 = \frac{1}{a}$;
$2 \times \left(-\frac{b}{3a}\right)^2 = \frac{1}{a}$, $2b^2 = 9a$

8 -3; -6 **9** 1.6 **10** (a) 3 (b) 3 (c) 0 or 12

11 (a) $(\alpha+\beta)^2 - 2\alpha\beta = \alpha^2 + 2\alpha\beta + \beta^2 - 2\alpha\beta = \alpha^2 + \beta^2$
(b) (i) $-\frac{q}{r}$ (ii) $\frac{q^2 - 2rp}{r^2}$ (iii) $-\frac{qr}{p^2}$

12 $b = 9$, $c = 4$

13 Roots α, -2; $\alpha - 2 = -p$, $-2\alpha = q$;
$q = -2(2 - p) = 2p - 4$

14 $y = x^2 - 1$ **15** $y = x^2 - 6x + 8$ **16** ±3 **17** 0

18 (a) -5 (b) -8 (c) $\frac{5}{8}$ (d) 40 (e) -14

19 (a) 7 (b) 2

EXERCISE 9.5

1 (a) $x > 5$ or $x < -2$ (b) $-2 \le x \le 2$ (c) $x \le -0.5$ or $x \ge 6$
(d) $-5 < x < \frac{1}{3}$
2 D **3** $x \ge p$ or $x \le q$ **4** $x > 1$ or $x < -5$
5 $\Delta = 4 - 4 \times 1 \times 7 = -24 < 0$
$a > 0$, so it is positive definite
6 (a) $a^2 = 4b$ (b) $a^2 > 4b$
7 (a) $x < -5$ or $x > 1$ (b) $x < -1$ or $x > 1.75$
(c) $x < 1 - \sqrt{2}$ or $x > 1 + \sqrt{2}$ (d) $-\frac{1}{3} < x < 1$
(e) $\frac{1-\sqrt{5}}{2} < x < \frac{1+\sqrt{5}}{2}$ (f) $-3 < x < 3$
8 (a) $-2 < x < 2$ (b) $-10 < x < 2$ (c) $\frac{-2-\sqrt{2}}{2} < x < \frac{-2+\sqrt{2}}{2}$
9 A
10 (a) $\Delta = 4 - 16 < 0$ (b) $\Delta = 0 - 16 < 0$ (c) $\Delta = 4 - 12 < 0$
(d) $\Delta = 1 - 24 < 0$ (e) $\Delta = 9 - 32 < 0$ (f) $\Delta = 0 - 56 < 0$

EXERCISE 9.6

1 (a) (i) $k = 2.25$ (ii) $k < 2.25$ (b) (i) $k = \pm 2\sqrt{3}$ (ii) $k > 2\sqrt{3}$ or $k < -2\sqrt{3}$ (c) (i) $k = -1, -5$ (ii) $k < -5$ or $k > -1$
(d) (i) $k = \frac{1}{3}, 3$ (ii) $\frac{1}{3} < k < 3$
2 A **3** $m = 1$ or -3
4 (a) $-\sqrt{2} \le m \le \sqrt{2}$ (b) $\frac{1}{3} \le m \le 3$ (c) $m \le -4$ or $m > 6$
5 (a) $m = \pm\sqrt{3}$ (b) $m < -\sqrt{3}$, $m > \sqrt{3}$
6 $\Delta_1 = 16(m-1)^2 - 4 \times 4(m+1) \times (-3) = 16(m^2 + m + 4)$
For $m^2 + m + 4$: $\Delta_2 = 1 - 16 < 0$
$\therefore m^2 + m + 4 > 0$ for all real m.
Hence $\Delta_1 > 0$ for all real m and the original equation has real roots for all real m.
7 $\Delta = 9m^4 - 4(3m-5)(5m^2) = m^2(3m-10)^2 > 0$ for real m, $m \ne \frac{5}{3}$
Because Δ is a perfect square, the roots of the equation are rational if m is rational, $m \ne \frac{5}{3}$.
8 $\Delta = (2a+b)^2 - 4ab = 4a^2 + b^2 \ge 0$ for all real values of a and b,
$\therefore$ equation has real roots.
9 (a) $p = 2$ (b) $p < 2$
10 (a) $m = 3$ or 5 (b) $m > 5$ or $m < 3$
11 $\Delta = (a+b)^2 - 4ab = a^2 - 2ab + b^2 = (a-b)^2 \ge 0$; roots real.
Because Δ is a perfect square, the roots of the equation are rational for all rational values of a and b.
12 $-2 < m < -\frac{2}{3}$

EXERCISE 9.7

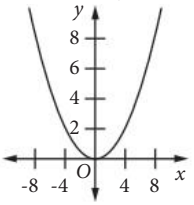
1 $a = 1, b = -1, c = 1$ **2** C
3 $p = 0, q = 1, r = 1$: $(x-1) + (x-1)^2$ **4** $a = 1, b = 6$
5 $a = -3, b = -2, c = -8$: $(x^2 - 3x)^2 - 2(x^2 - 3x) - 8$; $x = -1, 1, 2, 4$
6 $b = -5, c = 6$: $(x^2 + 2x)^2 - 5(x^2 + 2x) + 6 = 0$; $x = 1, -3, -1 \pm \sqrt{3}$
7 (a) $(x-2)^2 + 2$ (b) $2(x-1)^2 - 7$ (c) $(x+0.5)^2 + 0.75$
(d) $2(x+1.25)^2 + 3.875$
8 $a = 1, b = -1$ **9** $a = 4, b = -3, c = -2$: $4x(x+1) - 3x^2 - 2(x+1)$
10 $a = 1, b = 1$: $(x-1)(x+2) + (x+2)$

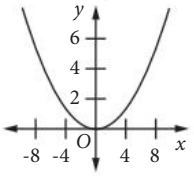
EXERCISE 9.8

1 (a) $(2, 6), (-2, 18)$ (b) $(0, 2), (2, -12)$ (c) $(1, 2), (-5, 8)$
(d) $(2, 1), (-1.5, -2.5)$
2 (a) $(0, -3), (3, 0)$ (b) $(2, 3), (3, 5)$ (c) $(1, 1)$
3 A **4** $c = \pm 2\sqrt{2}$ at $(\pm\sqrt{2}, \mp\sqrt{2})$ **5** $0, -1\frac{1}{3}$ **6** $2, -18$
7 (a) 4 or -8 (b) $m > 4$ or $m < -8$ (c) $-8 < m < 4$
8 (a) 1 or 9 (b) $m > 9$ or $m < 1$ (c) $1 < m < 9$
9 (a) -5 or 3 (b) $m > 3$ or $m < -5$ (c) $-5 < m < 3$
10 $a > 0$ or $a < -\frac{8}{9}$ **11** $-3 < a < 0$
12 $y = 2x + 1$; $y = -2x + 5$
13 $2x^2 - 6x + 7 = x^2 - 2x + 3$; $x^2 - 4x + 4 = 0$, $(x-2)^2 = 0$;
$\Delta = 0$, touches $x = 2$; point is $(2, 3)$

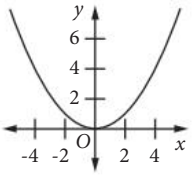
EXERCISE 9.9

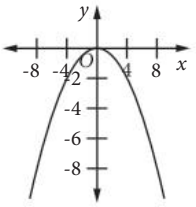
1 (a) $(0,0), (0,2), y = -2, x = 0$ (b) $(0,0), (0,3), y = -3, x = 0$



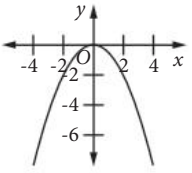


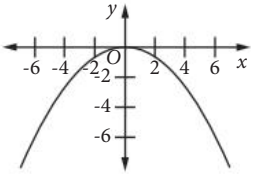
(c) $(0,0), (0,1), y = -1, x = 0$ (d) $(0,0), (0,-2), y = 2, x = 0$



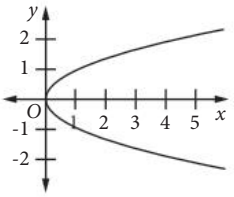


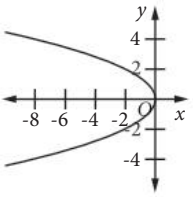
(e) $(0,0), (0,-0.5), y = 0.5, x = 0$ (f) $(0,0), (0,-1.5), y = 1.5, x = 0$



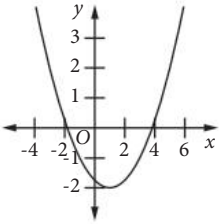


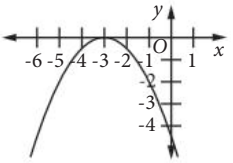
(g) $(0,0), (0.25,0), x = -0.25, y = 0$ (h) $(0,0), (-0.5,0), x = 0.5, y = 0$



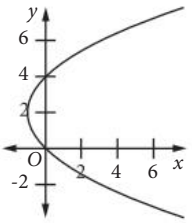


(i) $(1,-2), (1,-1), y = -3, x = 1$ (j) $(-3,0), (-3,-0.5), y = 0.5, x = -3$

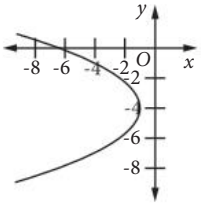




(k) $(-1,2), (0,2), x = -2, y = 2$



(l) $(-1,-4), (-1.75,-4), x = -0.25, y = -4$



2 D
3 (a) $x^2 = 8y$ (b) $x^2 = -16y$ (c) $x^2 = 4y$ (d) $x^2 = -8y$
(e) $(x+2)^2 = -20(y-1)$
4 B
5 (a) $(y+1)^2 = 4(x+1)$; $(-1,-1), (0,-1), x = -2$
(b) $(y-2)^2 = -8(x-1.5)$; $(1.5,2), (-0.5,2), x = 3.5$
(c) $(x-1)^2 = -4(y+3.5)$; $(1,-3.5), (1,-4.5), y = -2.5$
(d) $(x+3)^2 = 5(y+5)$; $(-3,-5), (-3,-3.75), y = -6.25$

6 (a) $x^2 = \pm 8y$ (b) $x^2 = -4y$ (c) $x^2 = 28y$
(d) $(x-2)^2 = 16(y+1)$ (e) $(x-2)^2 = y-5$
(f) $(x+1)^2 = 8(y-3)$ (g) $(x-2)^2 = 4(y+1)$ (h) $x^2 = 8y$

7 (a) $(x-2)^2 = \pm 4a(y-1)$ (b) $(x+2)^2 = \pm 2(y-k)$

8 (a) $(x-2)^2 = y$; $(2,0)$, $\left(2, \frac{1}{4}\right)$, $y = -\frac{1}{4}$, $x = 2$
(b) $(x+3)^2 = y+3$; $(-3,-3)$, $\left(-3, -2\frac{3}{4}\right)$, $y = -3\frac{1}{4}$, $x = -3$
(c) $x^2 = -(y-4)$; $(0,4)$, $\left(0, 3\frac{3}{4}\right)$, $y = 4\frac{1}{4}$, $x = 0$
(d) $(x+1)^2 = 4(y+1)$; $(-1,-1)$, $(-1,0)$, $y = -2$, $x = -1$

CHAPTER REVIEW 9

1 (a) 9 (b) -13.5

2 (a) two (b) two (c) one (d) two

3 (a) $-1, 2, \frac{1 \pm \sqrt{13}}{2}$ (b) ± 3 (c) 1, 2

4 (a) $k < 7\frac{1}{4}$ (b) $k = 7\frac{1}{4}$ (c) $k > 7\frac{1}{4}$

5 $\Delta = (m+n)^2 - 4mn = m^2 + 2mn + n^2 - 4mn = m^2 - 2mn + n^2$
$= (m-n)^2$; a perfect square, so roots are rational

6 $(2,1)$, $(4,5)$ 7 $a = 2$, $b = 3$

8 (a) $m = 1$ or -11 (b) $m < -11$ or $m > 1$ (c) $-11 < m < 1$

9 $(x+2)^2 = 4y$; vertex $(-2,0)$, focus $(-2,1)$, directrix $y = -1$, axis $x = -2$, focal length 1 unit

10 $(x-3)^2 = 4(y+1)$; vertex $(3,-1)$, focus $(3,0)$, directrix $y = -2$, axis $x = 3$, focal length 1 unit

11 $a = 2$, $b = -3$

12 $x^2 + (x-4)^2 = 8$, $x^2 + x^2 - 8x + 16 = 8$, $2x^2 - 8x + 8 = 0$, $x^2 - 4x + 4 = 0$, $(x-2)^2 = 0$; double root, so line touches circle, hence a tangent; point is $(2,-2)$

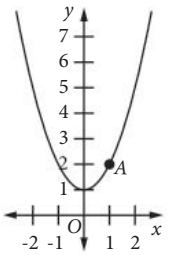
CHAPTER 10

EXERCISE 10.1

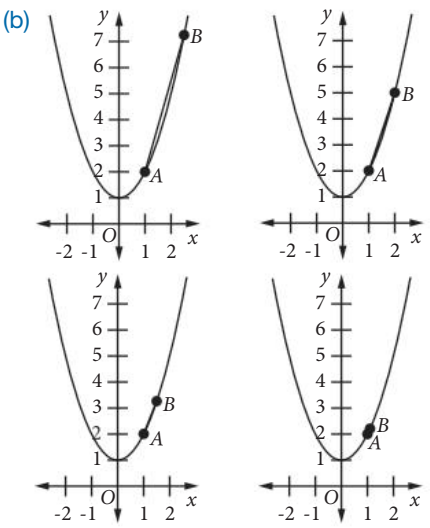
1 (a) continuous (b) discontinuous (c) continuous
(d) continuous (e) discontinuous (f) continuous
(g) continuous (h) continuous

2 (a), (c), (f), (h)

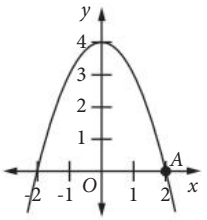
3 A

4 

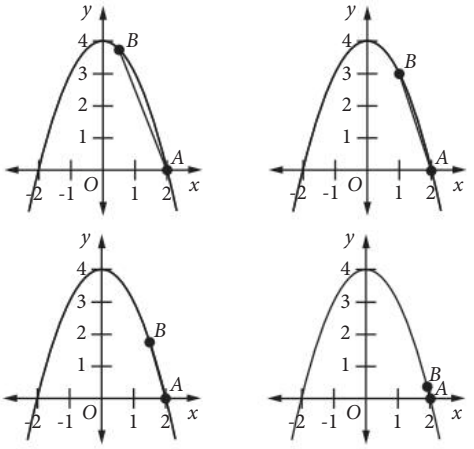
(a) $m_{AB} = 3.5$, $y = 3.5x - 1.5$
$m_{AB_1} = 3$, $y = 3x - 1$
$m_{AB_2} = 2.5$, $y = 2.5x - 0.5$
$m_{AB_3} = 2.1$, $y = 2.1x - 0.1$

(b) 

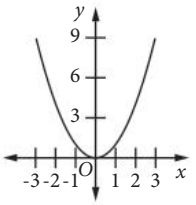
(c) 2 (d) $m_{AB_4} = 1.9$

5 

(a) $m_{AB} = -2.5$, $y = -2.5x + 5$
$m_{AB_1} = -3$, $y = -3x + 6$
$m_{AB_2} = -3.5$, $y = -3.5x + 7$
$m_{AB_3} = -3.9$, $y = -3.9x + 7.8$

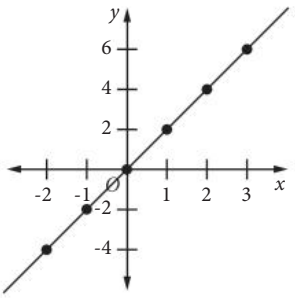
(b) 

(c) -4 (d) $m_{AB_4} = -4.1$

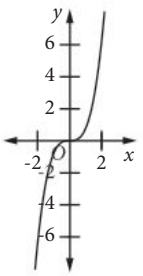
6 

(a)

x	-2	-1	0	1	2	3
m	-4	-2	0	2	4	6

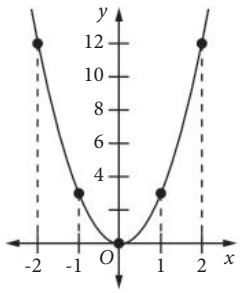
(b) 

(c) $y = 2x$

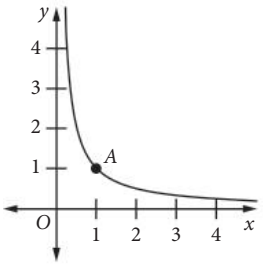
7 

(a)

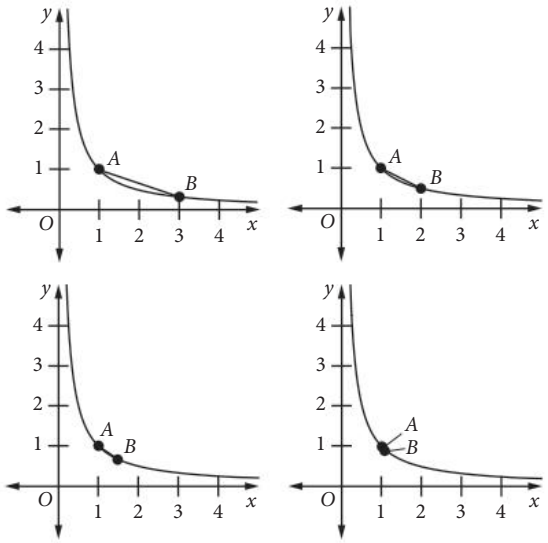
x	-2	-1	0	1	2
m	12	3	0	3	12

(b) 

(c) $y = 3x^2$

8 

(a) $m_{AB} = -\frac{1}{3}$, $3y = 4 - x$
$m_{AB_1} = -\frac{1}{2}$, $2y = 3 - x$
$m_{AB_2} = -\frac{2}{3}$, $3y = 5 - 2x$
$m_{AB_3} = -\frac{10}{11}$, $11y = 21 - 10x$

(b) 

(c) -1 (d) $m_{AB_4} = -\frac{10}{9}$

EXERCISE 10.2

1 (a) 9 (b) -3 (c) 0 (d) 9 (e) -32 (f) 0 (g) -10 (h) 0.8
2 C
3 (a) 1 (b) 5 (c) 12 (d) 1 (e) 1.5 (f) $\frac{1}{11}$ (g) $\frac{1}{3}$ (h) $\frac{1}{6}$ (i) 3
4 (a) 1 (b) 2
5 (a) $2x$ (b) $4x - 3$ (c) $3x^2$ (d) $6 - 2x$
7 A

EXERCISE 10.3

Values shown in these tables may be approximate.

1 2

x-coordinate of Q	0.9	0.99	0.999	1.001	1.01	1.1
Gradient of secant PQ	1.9	1.99	1.999	2.001	2.01	2.1

2 2

x-coordinate of Q	0.9	0.99	0.999	1.001	1.01	1.1
Gradient of secant PQ	2	2	2	2	2	2

3 6

x-coordinate of Q	2.9	2.99	2.999	3.001	3.01	3.1
Gradient of secant PQ	5.9	5.99	5.999	6.001	6.01	6.1

4 D

5 3

x-coordinate of Q	0.9	0.99	0.999	1.001	1.01	1.1
Gradient of secant PQ	2.71	2.97	2.997	3.003	3.03	3.31

6 -0.25

x-coordinate of Q	1.9	1.99	1.999	2.001	2.01	2.1
Gradient of secant PQ	-0.263	-0.251	-0.250	-0.250	-0.249	-0.238

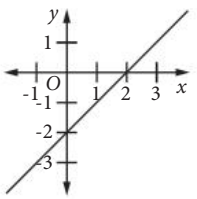
7 -4

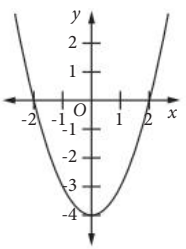
x-coordinate of Q	-2.1	-2.01	-2.001	-1.999	-1.99	-1.9
Gradient of secant PQ	-4.1	-4.01	-4.001	-3.999	-3.99	-3.9

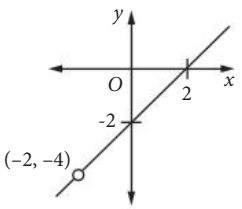
EXERCISE 10.4

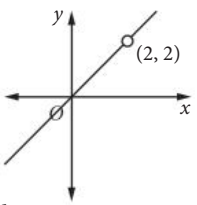
1 (a) 10 (b) -2 (c) 27 (d) 2
2 (a) -4 (b) 3
3 (a) $2 - 2h$ (b) 2 **4** D
5 (a) 8, gradient of tangent at (3, 6)
(b) $4x - 4$, gradient of tangent at $(x, f(x))$
6 (a) $8x$ (b) $x - 2$ (c) $3x^2 - 4x$
7 (a) correct (b) incorrect (c) correct (d) correct

EXERCISE 10.5

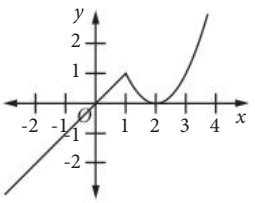
1 (a) 1 

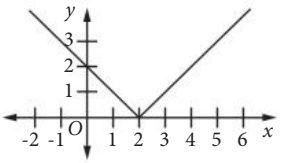
(b) 4 

(c) 1 

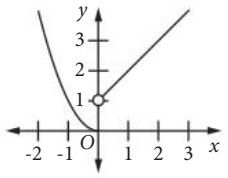
(d) 

does not exist

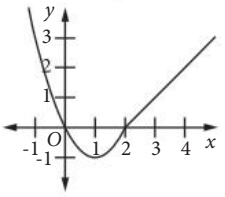
2 $f'(3) = 2$ 

3 $f'(4) = 1$ 

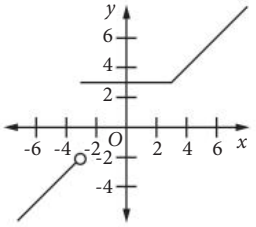
4 $f'(-1) = -2$



5 $f'(3) = 1$



6 not differentiable at $x = 3$, $x = -3$



7 continuous at a and c, discontinuous at b; no derivative at a, b, c

EXERCISE 10.6

1 (a) $6x + 2$ (b) $4 - 6x$ (c) $7 - 8x$ (d) $4x^3 + 2x$
(e) $5x^4 - 3x^2 + 1$ (f) $3t^2 + 8t - 2$

2 (a) $\frac{3\sqrt{x}}{2}$ (b) $-\frac{2}{x^2}$ (c) $\frac{1}{\sqrt{x}}$ (d) $\frac{2}{3\sqrt[3]{t}}$ (e) $-\frac{3}{m^4}$ (f) $\frac{-1}{2x\sqrt{x}}$

3 D

4 (a) $2x + 1$ (b) $9x^2 - 6$ (c) $4(2x - 3)$ (d) $2x$
(e) $24x^2 - 72x + 54$ (f) $9x^2 - 4x - 7$

5 (a) $6x - 6x^2$ (b) $4 - 16x^3$ (c) $9t^2 - 4t + 5$ (d) $3ax^2 + 2bx + c$
(e) 3 (f) $6x - 2$

6 (a) $1 + \frac{1}{2\sqrt{x}}$ (b) $2x - \frac{1}{x^2}$ (c) $2x + 1 - \frac{1}{x^2} - \frac{2}{x^3}$
(d) $\frac{2}{3\sqrt[3]{x}} - \frac{1}{3x\sqrt[3]{x}}$ (e) $2x - \frac{2}{x^3}$ (f) $\frac{3\sqrt{x}}{2}$

7 (a) correct (b) incorrect (c) correct (d) correct

8 (a) $f'(x) = 2x$, $x = 0$ (b) $f'(x) = 6x^2 - 6$, $x = \pm 1$
(c) $f'(x) = 3x^2 - 8x$, $x = 0, 2\frac{2}{3}$

9 $\frac{dy}{dx} = 2x - 1$: at $(3, 0)$, $\frac{dy}{dx} = 5$; at $(-2, 0)$, $\frac{dy}{dx} = -5$

10 $f'(x) = 3x^2 - 2x - 6$, $x = -\frac{1}{3}, 1$

11 $x^2 + 4x - 12 = 0$ for $x = -6, 2$; $\frac{dy}{dx} = 2x + 4$
$x = -6$, $\frac{dy}{dx} = -8$; $x = 2$, $\frac{dy}{dx} = 8$

12 (a) $x = 1$ (b) $x = 1$ (c) $x = 0.5$

13 $\frac{dy}{dx} = 2x - 5$ (a) $(3, 0)$ (b) $(1, 2)$ (c) $\frac{3}{2}, \frac{3}{4}$

14 $\frac{dy}{dx} = x^2 - 3x + 2$
(a) $x = 1, 2$ (b) $x^2 - 3x + 2 = 2$; $x = 0, 3$ (c) $x = -1, 4$

15 (a) $(1, -9)$ (b) $(0, -8)$

EXERCISE 10.7

1 (a) $12x - 5$ (b) $4x + 7$ (c) $9x^2 - 4x - 8$ (d) $3x^2 - 8x + 3$
(e) $6x^2 - 18x + 10$ (f) $4x^3 - 12x^2 + 6x - 12$ (g) $6x + 2$
(h) $6x^2 - 14x - 15$ (i) $60x^2 - 10x - 28$

2 A 3 (a) incorrect (b) correct (c) correct (d) correct

4 (a) $5x^4 + 24x^3 + 15x^2 + 2x + 5$ (b) 51 (c) -51

5 (a) $\sqrt{x} + \frac{x-1}{2\sqrt{x}} = \frac{3\sqrt{x}}{2} - \frac{1}{2\sqrt{x}}$ (b) $\frac{3\sqrt{x}}{2} - 1$ (c) 1
(d) $\frac{15x\sqrt{x}}{2} + 2x + \frac{6}{\sqrt{x}}$ (e) $2x + \frac{2}{x^3}$ (f) $\frac{5x\sqrt{x}}{2} + 2x - \frac{1}{x\sqrt{x}} - \frac{2}{x^2}$

EXERCISE 10.8

1 (a) $10x(x^2 - 4)^4$ (b) $\frac{5}{2\sqrt{5x-1}}$ (c) $12(x^2 - 1)(x^3 - 3x)^3$
(d) $\frac{-2}{(2x+5)^2}$ (e) $\frac{2t+3}{3\sqrt[3]{(t^2+3t)^2}}$ (f) $4(4x + 5)(2x^2 + 5x - 4)^3$
(g) $f'(x) = \frac{1}{2} \times (2x - 2)(x^2 - 2x)^{-\frac{1}{2}} = \frac{x-1}{\sqrt{x^2-2x}}$
(h) $\frac{-4t}{(t^2+4)^3}$ (i) $\frac{-x}{\sqrt{25-x^2}}$

2 B

3 (a) $\frac{x}{\sqrt{x^2-4}}$ (b) $\frac{x}{\sqrt{x^2+1}}$ (c) $\frac{-2}{(1+2x)^2}$ (d) $4m(m^2 + 25)$
(e) $10(2x - 1)^4$ (f) $\frac{1}{3\sqrt[3]{(t+1)^2}}$ (g) $8(3x - 1)(3x^2 - 2x - 1)^3$
(h) $\frac{-4t}{(t^2+4)^3}$ (i) $4\left(1 + \frac{1}{x^2}\right)\left(x - \frac{1}{x}\right)^3$

4 (a) incorrect (b) incorrect (c) correct (d) correct

5 (a) $\frac{dy}{dx} = (3x + 4)^6 + (x - 3) \times 6(3x + 4)^5 \times 3$
$= (21x - 50)(3x + 4)^5$
(b) $f'(x) = 2x\sqrt{1-x^2} + \frac{x^2 \times (-2x)}{2\sqrt{1-x^2}} = \frac{x(2-3x^2)}{\sqrt{1-x^2}}$
(c) $3t^2 - 4(4 - t)^3$ (d) $\frac{-1}{2\sqrt{1-x}} + \frac{1}{2\sqrt{1+x}} = \frac{\sqrt{1-x}-\sqrt{1+x}}{2\sqrt{1-x^2}}$
(e) $2t - \frac{1}{(1+t)^2}$ (f) $g'(x) = \left(1 + \frac{1}{x}\right)^2 + 2x\left(1 + \frac{1}{x}\right)\left(\frac{-1}{x^2}\right) = 1 - \frac{1}{x^2}$

EXERCISE 10.9

1 (a) $\frac{dy}{dx} = \frac{1(x+1)-1(x-1)}{(x+1)^2} = \frac{2}{(x+1)^2}$ (b) $\frac{43}{(4x+5)^2}$ (c) $\frac{-1}{(t+2)^2}$
(d) $\frac{5-2x}{(x^2-5x+6)^2}$ (e) $h'(n) = \frac{(2n+3)(2n-1)-2(n^2+3n+4)}{(2n-1)^2} = \frac{2n^2-2n-11}{(2n-1)^2}$
(f) $\frac{-3}{(x-3)^2}$ (g) $\frac{8x(x+5)}{(2x+5)^2}$ (h) $\frac{44x}{(x^2+5)^2}$ (i) $\frac{-(2x^3+3x^2+1)}{(x^3-1)^2}$

2 A

3 (a) $\frac{dy}{dx} = \frac{\frac{x}{2\sqrt{x+1}} - \sqrt{x+1}}{x^2} = \frac{-(x+2)}{2x^2\sqrt{x+1}}$
(b) $f(x) = x + 2 + \frac{1}{x}$, $f'(x) = 1 - \frac{1}{x^2}$ (c) $\frac{1-x}{(x+1)^3}$
(d) $\frac{2(2x+1)^2(x^2+2x+9)}{(3-x^2)^3}$ (e) $\frac{x+2}{2(x+1)^{\frac{3}{2}}}$
(f) $y = \left(1 + \frac{1}{x}\right)^2$, $\frac{dy}{dx} = 2\left(1 + \frac{1}{x}\right) \times -\frac{1}{x^2} = \frac{-2(x+1)}{x^3}$

4 (a) correct (b) correct (c) correct (d) incorrect

5 (a) $3x^2 + 4x - 4$ (b) $10x^{\frac{3}{2}} - 3\sqrt{x} + \frac{3}{\sqrt{x}}$ (c) $\frac{5}{2\sqrt{5x-1}}$
(d) $\frac{2}{3\sqrt[3]{t}} + 6(2t - 1)^2$ (e) $\frac{3}{(x+5)^2}$ (f) $\frac{x}{\sqrt{x^2+2}}$
(g) $2x - 5 + \frac{1}{2\sqrt{x-2}}$ (h) $\frac{-9}{(2z-1)^2}$ (i) $\frac{2-2x}{(x^2-2x+2)^2}$
(j) $3x^2 - 12x - 7$ (k) $\frac{5x\sqrt{x}}{2} + \frac{1}{2\sqrt{x}} = \frac{5x^2+1}{2\sqrt{x}}$
(l) $g'(x) = \frac{\frac{1}{2\sqrt{x+1}} \times \sqrt{x^2+1} - \sqrt{x+1} \times \frac{x}{\sqrt{x^2+1}}}{x^2+1} = \frac{1-2x-x^2}{2\sqrt{x+1}(x^2+1)^{\frac{3}{2}}}$

6 $\frac{dy}{dx} = \frac{1-x^2}{(x^2+1)^2}$; $\frac{dy}{dx} = 0$ at $x = -1, 1$

CHAPTER REVIEW 10

1 (a) 2 (b) 27 2 (a) $2x^2 + 3$ (b) 4 (c) 3

3 $4x - 3 + 2h$ 4 (a) 24 (b) 10 (c) $2c + 6$ (d) $c = -4$

5 $\frac{1}{\sqrt{2x-1}}$ 6 $(36x^3 - 14x^2 - 118x + 40)(3x^2 - 2x + 1)^4$

7 (a) $3(x - 2)^2$ (b) $5x^9(2 + 3x)(1 + x)^4$ (c) $\frac{-t}{\sqrt{25-t^2}}$
(d) $2\left(x + x^{-1}\right)\left(1 - \frac{1}{x^2}\right) = \frac{2(x^4-1)}{x^3}$ (e) $\frac{-2}{(x+4)^3}$ (f) $\frac{2}{x^2}$
(g) $(7x + 11)(x - 1)^5$ (h) $2x + 3 + \frac{1}{2\sqrt{x-2}}$ (i) $\frac{-2x}{(x^2-2)^2}$
(j) $\frac{20}{(2m+3)^2}$ (k) $1 - \frac{2}{x^3}$ (l) $\frac{3\sqrt{t}-3}{2}$

8 (a) $4x^3$ (b) $2x - 1$ (c) $3x^2 - 2$

CHAPTER 11

EXERCISE 11.1

1 In ΔAOB and ΔDOC:
$\angle ABO = \angle ODC$ (alternate angles, $AB \parallel DC$)
$\angle AOB = \angle COD$ (vertically opposite angles)
$AO = OC$ (given)
$\therefore \Delta AOB \equiv \Delta DOC$ (AAS)
$\therefore BO = OD$ (matching sides of congruent triangles)

2 (a) $BX = BC - XC$, $DY = DC - YC$
Because $BC = DC$ (sides of a square) and
$XC = YC$ (given), then $BX = DY$.
In ΔABX and ΔADY:
$BX = DY$ (proved above)
$\angle ABX = \angle ADY = 90°$ (angles of a square)
$AB = AD$ (sides of a square)
$\therefore \Delta ABX \equiv \Delta ADY$ (SAS)
$\therefore AX = AY$ (matching sides of congruent triangles)

(b) $\angle AXB = \angle AYD$ (matching angles of congruent triangles)

3 (a) Join OT. In ΔOPT and ΔOQT:
$\angle OPT = \angle OQT = 90°$ (given)
OT is a common side
$OP = OQ = 5$ cm (radii of circle)
$\therefore \Delta OPT \equiv \Delta OQT$ (RHS)

(b) $\angle POT = \angle QOT$ (corresponding angles in congruent triangles)
$\therefore \angle POT = 50°$ ($\angle POQ = 100°$)
$\frac{PT}{5} = \tan 50°$
$PT = 5.96\text{ cm} \approx 6\text{ cm}$

4 In ΔAOB and ΔDOC:
$AO = OD$ (given)
$\angle AOB = \angle DOC$ (vertically opposite angles)
$BO = OC$ (given)
$\therefore \Delta AOB \equiv \Delta DOC$ (SAS)
$\therefore AB = CD$ (corresponding sides in congruent triangles)
$\angle ABC = \angle BCD$ (matching angles in congruent triangles)
But these are equal alternate angles between AB and CD, so $AB \parallel CD$.

5 To prove that $AB = AC$:
In ΔADB and ΔADC, AD is a common side
$\angle ADB = \angle ADC = 90°$ (given)
$DB = DC$ (D is midpoint of BC)
$\therefore \Delta ADB \equiv \Delta ADC$ (SAS)
$\therefore AB = AC$ (matching sides in congruent triangles)
$\therefore \Delta ABC$ is isosceles.

6 (a) In ΔPQS and ΔRSQ:
$PQ = RS$ (given)
QS is a common side
$SP = QR$ (given)
$\therefore \Delta PQS \equiv \Delta RSQ$ (SSS)

(b) $\angle PQS = \angle QSR$ (matching angles in congruent triangles)

(c) $\angle PQS$ and $\angle QSR$ are equal alternate angles between PQ and SR
$\therefore PQ \parallel SR$

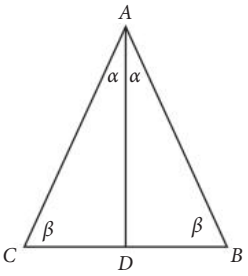
(d) PQ and SR are both equal and parallel, thus $PQRS$ is a parallelogram.

7 $\angle ABC = \angle ACB$ (base angles of isosceles triangle ABC)
$\angle DBC = \angle DCB$ (base angles of isosceles triangle DBC)
$\therefore \angle ABC - \angle DBC = \angle ACB - \angle DCB$
$\therefore \angle ABD = \angle ACD$

8 (a) In ΔABC and ΔABD:
AB is a common side
$\angle BAC = \angle BAD$ (given)
$AC = AD$ (given)
$\therefore \Delta ABC \equiv \Delta ABD$ (SAS)

(b) $BC = BD$ (matching sides in congruent triangles)

9



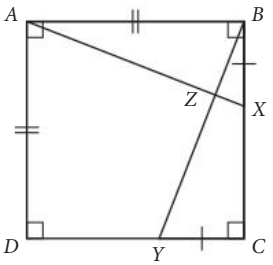
In ΔADC and ΔADB:
$\angle ACD = \angle ABD$ (given)
$\angle CAD = \angle BAD$ (construction)
AD is a common side
$\therefore \Delta ADC \equiv \Delta ADB$ (AAS)
$\therefore \angle ADC = \angle ADB$ (matching angles in congruent triangles)
But $\angle ADC + \angle ADB = 180°$ (CDB is a straight line)
$\therefore \angle ADC = \angle ADB = 90°$
$\therefore AD \perp BC$

10 Because ΔADE is isosceles,
then $\angle ADE = \angle AED$ (base angles of an isosceles triangle)
Because ΔABC is isosceles,
then $\angle ABC = \angle ACB$ (base angles of an isosceles triangle)
Now $\angle ADE = \angle ABD + \angle BAD$
(exterior angle of triangle is sum of interior opposite angles)
Similarly, $\angle AED = \angle ACE + \angle EAC$
Because $\angle AED = \angle ADE$ and $\angle ABD = \angle ACE$ then
$\angle BAD = \angle EAC$

11 (a) $\angle ABC = \angle ACB$ (base angles of an isosceles triangle)
Similarly, $\angle CBD = \angle BCD$
$\angle ABD = \angle ABC + \angle CBD$
$\angle ACD = \angle ACB + \angle BCD$
$\therefore \angle ABD = \angle ACD$

(b) In ΔABD and ΔACD
$AB = AC$ (equal sides of isosceles triangle)
$BD = DC$ (equal sides of isosceles triangle)
AD is a common side
$\therefore \Delta ABD \equiv \Delta ACD$ (SSS)
$\therefore \angle BAD = \angle CAD$ (matching angles in congruent triangles)

12



(a) In ΔABX and ΔBCY:
$AB = BC$ (sides of a square)
$\angle ABC = \angle BCY = 90°$ (angles of a square)
$BX = CY$ (construction)
$\therefore \Delta ABX \equiv \Delta BCY$ (SAS)
$\therefore AX = BY$ (corresponding sides in congruent triangles)

(b) Let AX and BY intersect at Z.
$\angle ZBX = \angle XAB$ (corresponding angles in congruent triangles)
But $\angle XAB + \angle AXB = 90°$ (ΔABX is right-angled at B)
$\therefore \angle ZBX + \angle AXB = 90°$
$\therefore \angle BZX = 90°$ (angle sum of ΔZBX)
$\therefore AX \perp BY$

13 Join AC and let it intersect DB at G.
$AG = GC$ and $BG = GD$ (diagonals of parallelogram $ABCD$ bisect each other)
Because $BE = DF$ then $EG = FG$ ($EB + BG = GD + DF$)
$\therefore AC$ and EF bisect each other so that $AECF$ is a parallelogram
$\therefore AE = FC$ (opposite sides of parallelogram $AECF$)
And $AE \parallel FC$ (opposite sides of parallelogram $AECF$)
There is a proof for this question that does not involve a construction.

14 (a) $\angle ADC = \angle BCD = 90°$ (angles of a square)
$\angle PDC = \angle PCD = 60°$ (angles in an equilateral triangle)
$\therefore \angle ADP = \angle BCP = 30°$
In ΔAPD and ΔBPC:
$AD = BC$ (sides of a square)
$\angle ADP = \angle BCP = 30°$ (from above)
$DP = CP$ (sides of an equilateral triangle)
$\therefore \Delta APD \equiv \Delta BPC$ (SAS)

(b) $PA = PB$ (matching sides in congruent triangles)
$\therefore \Delta APB$ is isosceles

15 In ΔPBC and ΔQBC:
$PB = QC$ (P and Q are the midpoints of equal sides)
$\angle PBC = \angle QCB$ (base angles of isosceles triangle)
BC is a common side
$\therefore \Delta PBC \equiv \Delta QBC$ (SAS)
$\therefore PC = QB$ (matching sides in congruent triangles)

16 In ΔOPB and ΔOQC:
$\angle PBO = \angle QCO$ (base angles of isosceles ΔABC)
$\angle OPB = \angle OQC = 90°$ (given)
$\angle BO = OC$ (O is midpoint of BC)
$\therefore \Delta OPB \equiv \Delta OQC$ (AAS)
$\therefore OP = OQ$ (matching sides in congruent triangles)

17 In ΔPQO and ΔRSO
$\angle PQO = \angle OSR$ (alternate angles, $PQ \parallel SR$)
$\angle POQ = \angle ROS$ (vertically opposite angles)
$PQ = SR$ (opposite sides of a parallelogram)
$\therefore \Delta PQO \equiv \Delta RSO$ (AAS)
$\therefore PO = OR$ and $QO = OS$ (matching sides in congruent triangles)
$\therefore$ The diagonals of a parallelogram bisect each other.

18 In ΔPQR and ΔPQS:
PQ is a common side
$\angle PQR = \angle QPS = 90°$ (angles of a rectangle)
$QR = PS$ (opposite sides of a rectangle)
$\therefore \Delta PQR \equiv \Delta PQS$ (SAS)
$\therefore PR = QS$ (matching sides in congruent triangles)
$\therefore$ The diagonals of a rectangle are equal.

19 (a) In ΔAOB and ΔCOD:
$\angle BAO = \angle OCD$
(alternate angles, $AB \parallel DC$, opposite sides of a rhombus)
$\angle AOB = \angle COD$ (vertically opposite angles)
$AB = DC$ (opposite sides of a rhombus)
$\therefore \Delta AOB \equiv \Delta COD$ (AAS)
$\therefore AO = OC$ and $BO = OD$ (matching sides in congruent triangles)

(b) $DA = DC$ (sides of a rhombus)
$\therefore \angle DAO = \angle DCO$ (base angles of isosceles ΔADC)
In ΔAOB and ΔAOD
$AD = AB$ (sides of a rhombus)
$\angle BAO = \angle DAO$ (both equal $\angle OCD$)
AO is a common side
$\therefore \Delta AOB \equiv \Delta AOD$ (SAS)
$\therefore \angle AOB = \angle AOD$ (matching angles in congruent triangles)
But $\angle AOB + \angle AOD = 180°$ (BOD a straight line)
$\therefore \angle AOB = \angle AOD = 90°$
$\therefore$ The diagonals of a rhombus bisect each other at right angles.

(c) $\angle BAO = \angle DAO$ (both equal $\angle OCD$)
Similarly, $\angle BCO = \angle DCO$
$\angle ABD = \angle BDC$ (alternate angles, $AB \parallel DC$)
$AD = AB$ (sides of a rhombus)
$\therefore \angle ADO = \angle ABO$ (base angles of isosceles ΔADB)
$\therefore \angle ADO = \angle CDO$
Similarly $\angle ABO = \angle CDO$
$\therefore$ The diagonals of a rhombus bisect the angles of the rhombus.

20 (a) In ΔAEH and ΔCFG:
$AH = FC$ (H and F are midpoints of equal sides of the parallelogram)
Similarly, $AE = CG$
$\angle HAE = \angle FCG$ (opposite angles of a parallelogram)
$\therefore \Delta AEH \equiv \Delta CFG$ (SAS)

(b) In ΔEBF and ΔGHD:
$BE = DG$ (E and G are midpoints of equal sides of the parallelogram)
Similarly, $BF = DH$
$\angle EBF = \angle GDH$ (opposite angles of a parallelogram)
$\therefore \Delta EBF \equiv \Delta GHD$ (SAS)

(c) From (a), $HE = GF$ (matching sides in congruent triangles)
From (b), $EF = HG$ (matching sides in congruent triangles)
$\therefore EFGH$ is a parallelogram (both pairs of opposite sides equal)

21 In ΔABO and ΔCBO
$AO = CO$ (given)
$\angle AOB = \angle COB = 90°$ (given)
OB is a common side
$\therefore \Delta ABO \equiv \Delta CBO$ (SAS)
$\therefore AB = BC$ (matching sides in congruent triangles)
Similarly $\therefore \Delta ADO \equiv \Delta CDO$ (SAS)
$\therefore AD = DC$ (matching sides in congruent triangles)
Similarly $\therefore \Delta ADO \equiv \Delta ABO$ (SAS)
$\therefore AD = AB$ (matching sides in congruent triangles)
$\therefore AB = BC = CD = DA$
$\therefore ABCD$ is a rhombus (all sides equal)

22 (a) In ΔAPC and ΔAQC:
$AP = QC$ (given)
$\angle PAC = \angle QCA$ (alternate angles, $AP \parallel QC$)
AC is a common side
$\therefore \Delta APC \equiv \Delta AQC$ (SAS)

(b) $\angle PCA = \angle CAQ$ (matching angles in congruent triangles)
These are equal alternate angles, $\therefore PC \parallel AQ$.

(c) $PC = AQ$ (matching sides in congruent triangles)

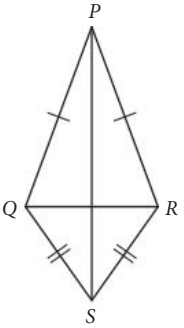
(d) $\therefore ACQP$ is a parallelogram (both pairs of opposite sides parallel)

23 (a) In ΔAPD and ΔCQB:
$DA = CB$ (opposite sides of a parallelogram)
$\angle DAP = \angle BCQ$ (opposite angles of a parallelogram)
$AP = QC$ (given)
$\therefore \Delta APD \equiv \Delta CQB$ (SAS)

(b) $\therefore PD = BQ$ (matching sides in congruent triangles)

(c) $\angle APD = \angle PDQ$ (alternate angles, $AP \parallel DC$)
But $\angle APD = \angle CQB$ (matching angles in congruent triangles)
$\therefore \angle PDQ = \angle BQC$
These are equal corresponding angles between PD and BQ, thus $PD \parallel BQ$.

24



(a) In ΔPQS and ΔPRS:
$PQ = PR$ (given)
$QS = RS$ (given)
PS is a common side
$\therefore \Delta PQS \equiv \Delta PRS$ (SSS)

(b) $PQSR$ is a kite
$\therefore PS$ bisects QR
$\therefore X$ the midpoint of QR is on PS
$\therefore P$, X and S are collinear

25 (a) In ΔABP and ΔDCP:
$\angle BAP = \angle CDP$ (given)
$\angle APB = \angle DPC$ (vertically opposite angles)
$AB = DC$ (given)
$\therefore \Delta ABP \equiv \Delta DCP$ (AAS)
$\therefore PB = PC$ (matching sides in congruent triangles)

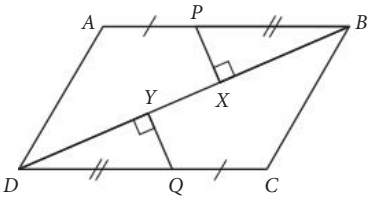
(b) ΔPBC is isosceles ($PB = PC$)
$\therefore \angle PBC = \angle PCB$ (angles opposite equal sides in isosceles triangle)
In ΔABC and ΔDCB:
$\angle BAC = \angle CDB$ (given)
$\angle ACB = \angle DBC$ (from above)
$AB = DC$ (given)
$\therefore \Delta ABC \equiv \Delta DCB$ (AAS)

(c) $\therefore AC = DB$ (matching sides in congruent triangles)

(d) $AP = AC - PC$ and $DP = DB - PB$
$\therefore AP = DP$

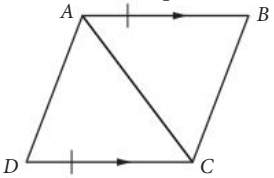
26 In ΔAOX and ΔCOY:
$\angle XAO = \angle OCY$ (alternate angles, $AB \parallel DC$)
$\angle AOX = \angle COY$ (vertically opposite angles)
$AO = OC$ (diagonals of a parallelogram bisect each other)
$\therefore \Delta AOX \equiv \Delta COY$ (AAS)
$\therefore OX = OY$ (matching sides in congruent triangles)

27



Let X and Y be the feet of the perpendiculars from P and Q to BD respectively.
Because $AB = DC$ and $AP = CQ$, thus $PB = DQ$.
In ΔBPX and ΔDQY:
$\angle PBX = \angle YDQ$ (alternate angles, $AB \parallel DC$)
$\angle BXP = \angle DYQ = 90°$ (given)
$PB = QD$ (from above)
$\therefore \Delta BPX \equiv \Delta DQY$ (AAS)
$\therefore PX = QY$ (matching sides in congruent triangles)
$\therefore P$ and Q are equidistant from BD.

28



Given: $AB = DC$, $AB \parallel DC$
Aim: To prove that $AD = BC$ and $AD \parallel BC$.
In ΔABC and ΔCDA:
$AB = DC$ (given)
$\angle BAC = \angle DCA$ (alternate angles, $AB \parallel DC$)
AC is a common side
$\therefore \Delta ABC \equiv \Delta CDA$ (SAS)
$\therefore AD = BC$ (matching sides in congruent triangles)
$\angle BCA = \angle DAC$ (matching angles in congruent triangles)
But these are equal alternate angles between AD and BC, so $AD \parallel BC$.

29 (a) In ΔAOC and ΔBOC:
$AO = OB$ (given)
$\angle AOC = \angle BOC = 90°$ (given)
OC is a common side
$\therefore \Delta AOC \equiv \Delta BOC$ (SAS)

(b) In ΔAOD and ΔBOD:
$AO = OB$ (given)
$\angle AOD = \angle BOD = 90°$ (given)
OD is a common side
$\therefore \Delta AOD \equiv \Delta BOD$ (SAS)

(c) In ΔCAD and ΔCBD:
$CA = CB$ (matching sides in congruent triangles)
$AD = BD$ (matching sides in congruent triangles)
DC is a common side
$\therefore \Delta CAD \equiv \Delta CBD$ (SSS)

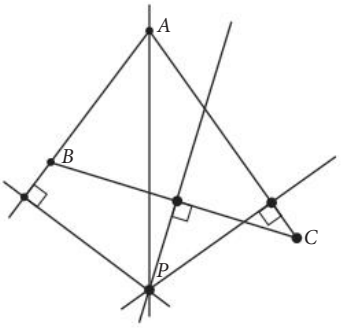
30 (a) parallelogram, rectangle, rhombus, square
(b) parallelogram, rectangle, rhombus, square
(c) parallelogram, rectangle, rhombus, square
(d) rectangle, square (e) rhombus, square
(f) rhombus, square (g) rhombus, square
(h) rectangle, square

31 (a) In ΔPBM and ΔPCM:
$BM = CM$ (given)
$\angle BMP = \angle CMP = 90°$ (given)
MP is a common side
$\therefore \Delta PBM \equiv \Delta PCM$ (SAS)
$\therefore PB = PC$ (matching sides in congruent triangles)

(b) In ΔAPX and ΔAPY:
$\angle AXP = \angle AYP = 90°$ (construction)
$\angle XAP = \angle YAP$ (given)
AP is a common side
$\therefore \Delta APX \equiv \Delta APY$ (AAS)
$\therefore AX = AY$ (matching sides in congruent triangles)
and $PX = PY$ (matching sides in congruent triangles)

(c) In ΔPXB and ΔPYC:
$\angle BXP = \angle CYP = 90°$ (construction)
$PB = PC$ (from (a))
$PX = PY$ (from (b))
$\therefore \Delta PXB \equiv \Delta PYC$ (RHS)
$\therefore XB = YC$ (matching sides in congruent triangles)

(d) $AB = AX + YB = AY + YC = AC$



The image shows that the assumption that P was inside ΔABC was incorrect, so the diagram was incorrect.

EXERCISE 11.2

1 C **2** 4.8 m **3** $\frac{AB}{5} = \frac{20}{10}$, $AB = 10$ m

4 (a) In ΔADE and ΔBDF:
$\angle AED = \angle DFB = 90°$ (given)
$\angle ADE = \angle DBF$ (corresponding angles, $ED \parallel CB$)
$\therefore \Delta ADE \,|||\, \Delta BDF$ (AAA)

(b) $\frac{DE}{FB} = \frac{AD}{DB}$ (matching sides in similar triangles proportional)
$\frac{DE}{4 - DE} = \frac{2}{3}$, $3DE = 8 - 2DE$, $5DE = 8$, $DE = 1.6$ m

(c) $\frac{DF}{3 - DF} = \frac{3}{2}$, $2DF = 9 - 3DF$, $5DF = 9$, $DF = 1.8$ m

5 (a) In ΔKQP and ΔKLM:
$\angle KQP = \angle KLM = 90°$ (given)
$\angle PKQ = \angle MKL$ (same angle)
$\therefore \Delta KQP \,|||\, \Delta KLM$ (AAA)

(b) $\frac{KQ}{KL} = \frac{1}{2}$ (matching sides in similar triangles proportional)
$\therefore KQ = QL$

(c) In ΔKPQ and ΔLPQ:
$KQ = QL$ (from (b))
$\angle KQP = \angle LQP = 90°$ (given)
PQ is a common side
$\therefore \Delta KPQ \equiv \Delta LPQ$ (SAS)
$\therefore PK = PL$ (matching sides in congruent triangles)
$\therefore \Delta KPL$ is isosceles
(d) ΔMPL as $PM = PL$

6 8 m

7 (a) Assume that the tree and the person are vertical.
In ΔTRM and ΔGHM:
$\angle TRM = \angle GHM = 90°$ (given)
$\angle TMR = \angle GMH$ (given)
$\therefore \Delta TRM \,|||\, \Delta GHM$ (AAA)
(b) $\frac{TR}{1.8} = \frac{5}{0.3}$ (matching sides in similar triangles proportional), $TR = 30$ m

8 (a) In ΔABD and ΔABC:
$\angle ADB = \angle CAB = 90°$ (given)
$\angle ABD = \angle CBA$ (same angle)
$\therefore \Delta ABD \,|||\, \Delta ABC$ (AAA)
In ΔDAC and ΔABC:
$\angle CDA = \angle BAC = 90°$ (given)
$\angle ACD = \angle BCA$ (same angle)
$\therefore \Delta DAC \,|||\, \Delta ABC$ (AAA)
$\therefore \Delta ABD$, ΔABC and ΔDAC are similar
(b) $BD = 9$ cm (Pythagoras' theorem)
$\frac{DC}{12} = \frac{12}{9}$, $DC = 16$ cm
$\frac{AC}{15} = \frac{12}{9}$, $AC = 20$ cm

9 (a) $AC = 8$ m (Pythagoras' theorem)
(b) In ΔADF and ΔDBE:
$\angle AFD = \angle DEB = 90°$ (given)
$\angle ADF = \angle DBE$ (corresponding angles, $DF \parallel BC$)
$\therefore \Delta ADF \,|||\, \Delta DBE$ (AAA)
(c) $\frac{DE}{8 - DE} = \frac{4}{6}$, $3DE = 16 - 2DE$, $5DE = 16$, $DE = 3.2$ m
$AF = 8 - 3.2 = 4.8$
$DF^2 + 4.8^2 = 6^2$, $DF = 3.6$ m

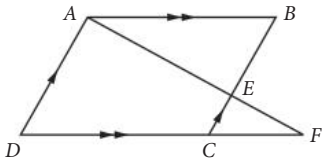
10 (a) In ΔAPB and ΔDPC:
$\angle APB = \angle CPD$ (vertically opposite angles)
$\angle BAP = \angle PCD$ (alternate angles, $AB \parallel DC$)
$\therefore \Delta APB \,|||\, \Delta DPC$ (AAA)
(b) $\frac{d}{3} = \frac{270}{2.5}$, $d = 3.24$ m

11 (a) ΔXCF and ΔXBE
(b) $\frac{BX}{6 - BX} = \frac{8}{4}$, $BX = 12 - 2BX$, $BX = 4$ units

12 (a) $AX^2 = 13^2 - 5^2$, $AX = 12$ cm
(b) In ΔAXC and ΔEFC:
$\angle AXC = \angle EFC = 90°$ (given)
$\angle ACX = \angle ECF$ (same angle) $\therefore \Delta AXC \,|||\, \Delta EFC$ (AAA)
(c) (i) $\frac{FC}{5} = \frac{9}{12}$ (matching sides in similar triangles proportional), $FC = 3.75$ cm
(ii) $GF = 2 \times (5 - 3.75) = 2.5$ cm

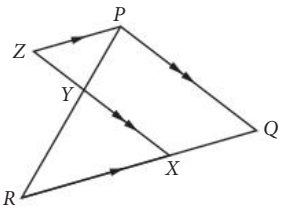
13 (a) $d = (15 - 3)$ m $= 12$ m (b) $\frac{10}{1.5} = \frac{12 + x}{x}$, $x = 2\frac{2}{17}$ m
(c) $\frac{y + 3.2}{3.2} = \frac{10}{1.6}$, $y = 16.8$ m

14



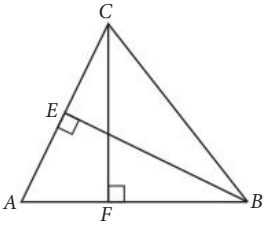
(a) In ΔABE and ΔCEF:
$\angle ABE = \angle ECF$ (alternate angles, $AB \parallel DC$)
$\angle AEB = \angle FEC$ (vertically opposite angles)
$\therefore \Delta ABE \,|||\, \Delta CEF$ (AAA)
(b) In ΔABE and ΔADF:
$\angle AEB = \angle EAD$ (alternate angles, $AD \parallel BC$)
$\angle BAE = \angle AFD$ (alternate angles, $AB \parallel DF$)
$\therefore \Delta ABE \,|||\, \Delta ADF$ (AAA)
(c) $\frac{AB}{FD} = \frac{BE}{DA}$

15



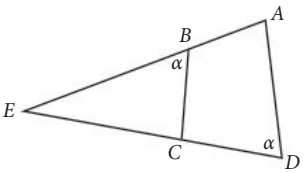
ΔZPY, ΔXRY, ΔQRP

16



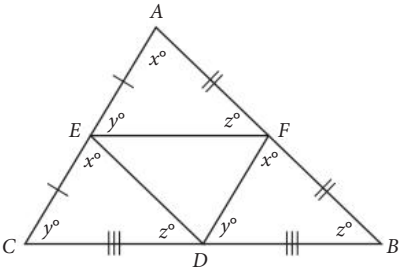
In ΔAFC and ΔAEB:
$\angle CAF = \angle BAE$ (same angle)
$\angle CFA = \angle BEA = 90°$ (construction)
$\therefore \Delta AFC \,|||\, \Delta AEB$ (AAA)

17



In ΔEBC and ΔEDA:
$\angle EBC = \angle EDA$ (given)
$\angle BEC = \angle DEA$ (same angle)
$\therefore \Delta EBC \,|||\, \Delta EDA$ (AAA)
$\frac{BC}{DA} = \frac{CE}{AE} = \frac{EB}{ED}$

18



In ΔABC and ΔAFE:
$\angle CAB = \angle EAF$ (same angle)
$\frac{AC}{AE} = \frac{2}{1}$ (E is the midpoint of AC)
$\frac{AB}{AF} = \frac{2}{1}$ (F is the midpoint of AB)
$\therefore \Delta ABC \,|||\, \Delta AFE$ (two pairs of sides proportional, included angles equal)
$\therefore \angle CAB = \angle EAF = x°$, $\angle ACB = \angle AEF = y°$,
$\angle ABC = \angle AFE = z°$ (matching angles in similar triangles)
Similarly, $\Delta ABC \,|||\, \Delta EDC$
$\therefore \angle CED = x°$, $\angle ECD = y°$, $\angle EDC = z°$ (matching angles in similar triangles)
Similarly, $\Delta ABC \,|||\, \Delta FBD$
$\therefore \angle DFB = x°$, $\angle FDB = y°$, $\angle FBD = z°$ (matching angles in similar triangles)
Now $\angle AEF + \angle FED + \angle DEC = 180°$ (AEC is a straight line)
$\therefore y° + \angle FED + x° = 180°$ so $\angle FED = z°$ because $x° + y° + z° = 180°$ (angle sum of triangle)
Similarly $\angle EFD = y°$ and $\angle EDF = x°$

$\therefore \Delta ABC \,|||\, \Delta DEF$ as they are equiangular

$\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD} = \frac{2}{1}$

19 (a) $\angle CAB = \angle CBA$ (base angle of isosceles triangle ABC)

$\angle CAD = \angle DCA$ (base angle of isosceles triangle ACD)

$\angle CAB = \angle CBA = \angle DCA$

In ΔABC and ΔACD:

$\angle BAC = \angle CAD$ (same angle)

$\angle ABC = \angle ACD$ (from above)

$\therefore \Delta ABC \,|||\, \Delta ACD$ (AAA)

(b) $\frac{BA}{CA} = \frac{BC}{DA}$ (matching sides in similar triangles proportional)

$\frac{y+a}{x} = \frac{x}{a}$ or $ay + a^2 = x^2$

(c) In ΔACD, using the cosine rule: $x^2 = a^2 + a^2 - 2aa\cos\theta$

So $ay + a^2 = 2a^2 - 2a^2\cos\theta$

$\therefore y = a - 2a\cos\theta$

EXERCISE 11.3

1 (a) $ADEF$, $FDEC$, $FDBE$

(b) ΔAFD, ΔFCE, ΔDEB, ΔEDF

2 (a) In ΔAHE and ΔADB:

$\angle HAE = \angle DAB$ (same angle)

$\frac{AH}{AD} = \frac{AE}{AB} = \frac{1}{2}$ (H and E are the midpoints of the sides)

$\therefore \Delta AHE \,|||\, \Delta ADB$ (two pairs of sides proportional, included angles equal)

$\therefore \angle AHE = \angle ADB$ (matching angles in similar triangles)

But these are a pair of equal corresponding angles on HE and DB, hence $HE \parallel DB$.

(b) Similarly in ΔCFG and ΔCBD, we can show that $\angle CFG = \angle CBD$, and hence $GF \parallel DB$.

(c) Because $HE \parallel DB$ and $GF \parallel DB$, then by the transitivity of parallel lines $GF \parallel HE$.

(d) By drawing the diagonal AC and repeating parts (a), (b) and (c) you can prove that $EF \parallel HG$.

$\therefore EFGH$ is a parallelogram because it is a quadrilateral with both pairs of opposite sides parallel.

3 (a) In ΔAFD and ΔACB:

$\angle FAD = \angle CAB$ (same angle)

$\frac{AF}{AC} = \frac{AD}{AB} = \frac{1}{2}$ (F and D are the midpoints of the sides)

$\therefore \Delta AFD \,|||\, \Delta ACB$ (two pairs of sides proportional, included angles equal)

$\therefore \angle AFD = \angle ACB$ (matching angles in similar triangles)

Similarly $\angle ADF = \angle ABC$

In ΔAFG and ΔACE:

$\angle FAG = \angle CAE$ (same angle)

$\angle AFG = \angle ACE$ (from above)

$\therefore \Delta AFG \,|||\, \Delta ACE$ (AAA)

Because $\frac{AF}{AC} = \frac{1}{2}$ then $\frac{FG}{CE} = \frac{1}{2}$ so $FG = \frac{CE}{2}$

Similarly, $\Delta ADG \,|||\, \Delta ABE$ (AAA) and $GD = \frac{BE}{2}$

But $CE = EB$ (given) so $FG = GD$.

(b) From the similar triangles above, $\frac{AG}{AE} = \frac{AF}{AC} = \frac{1}{2}$

$\therefore AG = GE$

4 (a) In ΔPAB and ΔPSR:

$\angle APB = \angle SPR$ (same angle)

$PA : PS = PB : PR = 1 : 2$ (A and B are midpoints of the sides)

$\therefore \Delta PAB \,|||\, \Delta PSR$ (two pairs of sides proportional, included angles equal)

$\therefore \angle PAB = \angle PSR$ (matching angles in similar triangles)

But these are equal corresponding angles, so $AB \parallel SR$.

(b) In ΔRCB and ΔRQP:

$\angle CRB = \angle QRP$ (same angle)

$RC : RQ = RB : RP = 1 : 2$ (C and B are midpoints of the sides)

$\therefore \Delta RCB \,|||\, \Delta RQP$ (two pairs of sides proportional, included angles equal)

$\therefore \angle RCB = \angle RQP$ (matching angles in similar triangles)

But these are equal corresponding angles, so $BC \parallel PQ$.

(c) $PQ \parallel SR$ (given), $AB \parallel SR$, $BC \parallel PQ$, $\therefore AB \parallel BC$

Because AB and BC have a point in common and are parallel to each other, ABC is a straight line and A, B and C are collinear.

5 (a) Join PR. The structure of the proof is now the same as 4 (a) together with 4(b).

(b) Let the perpendicular distance from P to AB be h. The perpendicular distance from A to SR is also h (because PQ, AB and SR cut equal intercepts on any transversal).

$\therefore$ perpendicular distance from P to SR is $2h$

Area of $PQBA = \frac{PQ + AB}{2} \times h$

Area of $ABSR = \frac{AB + SR}{2} \times h$

Area of $PQRS = \frac{PQ + SR}{2} \times 2h$

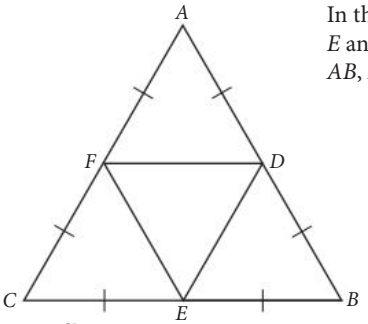
$\therefore (PQ + SR) \times h = \frac{PQ + AB}{2} \times h + \frac{AB + SR}{2} \times h$

$2PQ + 2SR = PQ + AB + AB + SR$

$2AB = PQ + SR$

$\therefore AB = \frac{1}{2}(PQ + SR)$

6



In the equilateral ΔABC, D, E and F are midpoints of AB, BC and CA respectively.

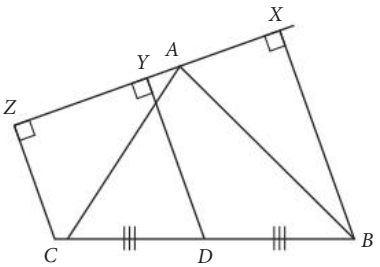
$FD = \frac{CB}{2}$ (line joining midpoints of sides of a triangle is parallel to third side and half its length)

$FD = CE = EB$

Similarly $DE = \frac{AC}{2}$ and $EF = \frac{AB}{2}$

So $DE = AF$ and $EF = AD$.

7



$\angle CZX = \angle DYZ = 90°$ (construction)

$\therefore CZ \parallel DY$ (pair of supplementary cointerior angles between CZ and DY)

$\angle DYZ = \angle BXY = 90°$ (construction)

$\therefore DY \parallel BX$ (pair of equal corresponding angles on XB and DY)

$\therefore CZ \parallel DY \parallel BX$

$CD = DB$ (D is the midpoint of CB)

$\therefore ZY = YX$

(if parallel lines cut equal intercepts on one transversal, then they cut equal intercepts on any other transversal)

8 In ΔABD, $PS \parallel BD$ (line joining midpoints of sides of a triangle is parallel to the third side)

Similarly in ΔCBD, $QR \parallel BD$

$\therefore PS \parallel QR$ (both parallel to BD)

In ΔBCA, $QP \parallel CA$ (line joining midpoints of sides of a triangle is parallel to the third side)

Similarly in ΔDCA, $SR \parallel CA$
$\therefore$ $QP \parallel RS$ (both parallel to CA)
$\therefore$ $PQRS$ is a parallelogram (both pairs of opposite sides parallel)

In ΔAPS and ΔBPQ:
$AS = BQ$ (halves of equal sides)
$\angle SAP = \angle QBP = 90°$ (angles of a rectangle)
$\therefore$ $AP = BP$ (halves of equal sides)
$\therefore$ $\Delta APS \equiv \Delta BPQ$ (SAS)
$\therefore$ $SP = PQ$ (matching sides of congruent triangles)
$\therefore$ $PQRS$ is a rhombus (parallelogram with pair of adjacent sides equal)

EXERCISE 11.4

1 side: equation $x - 2y + 5 = 0$, length $= \sqrt{5}$; height $= \frac{1}{\sqrt{5}}$
area $= \frac{1}{2} \times \sqrt{5} \times \frac{1}{\sqrt{5}} = 0.5$ units^2

2 $d = \frac{5}{\sqrt{5}} = \sqrt{5}$ in each case

3 sides are 10, $2\sqrt{2}$, 10: isosceles triangle

4 gradients are $\frac{5}{3}$, $\frac{1}{4}$, $-\frac{3}{5}$; $\frac{5}{3} \times -\frac{3}{5} = -1$, sides perpendicular; triangle is right-angled

5 sides $\sqrt{29}$, $\sqrt{5}$, $\sqrt{29}$, $\sqrt{5}$; diagonals $3\sqrt{2}$, $5\sqrt{2}$

6 distance is 5

7 sides 5, 10, $5\sqrt{5}$; $\left(5\sqrt{5}\right)^2 = 10^2 + 5^2$ $\therefore$ triangle is right-angled

8 $AB = BC = CD = DA = 5$; $m_{AB} = \frac{4}{3}$, $m_{BC} = -\frac{3}{4}$,
$m_{AB} \times m_{BC} = \frac{4}{3} \times \left(-\frac{3}{4}\right) = -1$; four sides equal, angle 90°
$\therefore$ square

9 sides $\sqrt{52} = 2\sqrt{13}$, $2\sqrt{13}$, $\sqrt{104} = 2\sqrt{26}$: isosceles
$\left(\sqrt{104}\right)^2 = \left(\sqrt{52}\right)^2 + \left(\sqrt{52}\right)^2$, so right-angled

10 gradients of sides $\frac{4}{3}$, $-\frac{3}{4}$, $\frac{4}{3}$, $-\frac{3}{4}$; two pairs of sides parallel;
$\frac{4}{3} \times \left(-\frac{3}{4}\right) = -1$ so sides perpendicular; $\therefore$ square

11 (a) $AB = 13$, $BC = 13$, $CA = 4\sqrt{13}$: isosceles (b) $3\sqrt{13}$

12 distance is $\sqrt{5}$ **13** distance is 5

14 (a) $\sqrt{34}$ (b) $9\sqrt{2} + \sqrt{34}$

15 $E(5, 0.5)$, $F(3.5, -3.5)$, $G(-1, -3.5)$, $H(0.5, 0.5)$;
$m_{EF} = \frac{8}{3}$, $m_{FG} = 0$, $m_{GH} = \frac{8}{3}$, $m_{EH} = 0$; opposite sides parallel;
$\therefore$ parallelogram

16 $2x + y + 6 = 0$ **17** $6x + 8y + 1 = 0$

18 $(1, 0.5)$; diagonals bisect each other

19 $5x + 2y + 6 = 0$

20 (a) $AB = 2\sqrt{5}$, $BC = 2\sqrt{2}$, $CA = 2\sqrt{5}$; $AB = AC$;
ΔABC is isosceles
(b) $(0, 2)$ (c) 1, -1
(d) base $\perp$ line drawn from vertex to midpoint of base

21 (a) $(2.5, 2)$ (b) diagonals bisect each other

22 (a) sides each $\sqrt{5}$, gradients 0.5 and -2; sides perpendicular;
$\therefore$ square (b) $(1.5, -0.5)$ (c) diagonals bisect each other

23 (a) $m_{AB} = -3$, $m_{AC} = \frac{1}{3}$, $m_{AB} \times m_{AC} = -1$ so $BA \perp AC$; $\angle BAC = 90°$
(b) $(-1.5, 1.5)$ (c) $\frac{5\sqrt{2}}{2}$

24 (a) $(1.5, 5)$, $(3, 6.5)$, $(-0.5, 6)$, $(-2.\ 4.5)$
(b) $m_{PQ} = 1$, $m_{QR} = \frac{1}{7}$, $m_{SR} = 1$, $m_{PS} = \frac{1}{7}$; pairs of opposite sides parallel, so $PQRS$ is a parallelogram

25 (a) $x = 0$ (b) $x - 2y + 1 = 0$ (c) $(0, 0.5)$ (d) distance $= \frac{\sqrt{65}}{2}$

CHAPTER REVIEW 11

1 (a) $\Delta KLM \equiv \Delta KNM$ (SSS)
$x = 100$ (angle sum of triangle)
(b) $x = 5$ (Pythagoras' theorem)
$AE = 5$ cm (Pythagoras' theorem)
$\angle AEB = 90°$ (congruent triangles and straight angle)
$y = 5\sqrt{2}$ (Pythagoras' theorem)
(c) $\frac{x}{6} = \frac{6}{8}$, (proportional intercepts), $x = 4.5$
$\frac{y}{3} = \frac{6}{4.5}$, (proportional intercepts), $y = 4$

2 $\angle ADC = 60°$ (cointerior angles supplementary, $AB \parallel DC$)
$\therefore$ $\angle AED = 60°$ (base angles of an isosceles triangle)
$\therefore$ $\angle DAE = 60°$ (angle sum of ΔADE)
$AE = BC$ (both equal AD)
$ABCE$ is a trapezium with its non-parallel sides equal; it is an isosceles trapezium.

3 (a) angle sum of $HKLMN = 6$ right angles $= 540°$
$\angle LMN = 540° \div 5 = 108°$
(b) $NM = ML$ (sides of a regular pentagon)
$\therefore$ $\angle MNL = \angle MLN$ (angles opposite equal sides in isosceles triangle ΔNML)
$\therefore$ $\angle MNL = \angle MLN = 36°$ (angle sum of ΔNML)
$\angle ALM = 72°$ (exterior angle of regular pentagon)
$\therefore$ $\angle LAN = 36°$ (angle sum of ΔLNA)
$\therefore$ $\angle LNA = \angle LAN = 36°$, so ΔLNA is isosceles

4 (a) In ΔAPQ and ΔACB:
$\angle APQ = \angle ACB$ (corresponding angles, $PQ \parallel CB$)
$\angle PAQ = \angle CAB$ (same angle)
$\therefore$ $\Delta APQ \,|||\, \Delta ACB$ (AAA)
(b) $\angle QPB = \angle PBC$ ($PQ \parallel CB$)
But $\angle QPB = \angle PBC$ (PB bisects $\angle ABC$)
$\therefore$ $\angle QPB = \angle QPB$
$\therefore$ ΔPQB is isosceles

5 (a) angle sum of $HKLMN = 6$ right angles $= 540°$
$\angle HNM = 540° \div 5 = 108°$
(b) ΔNMH is isosceles ($NM = NH$, sides of a regular pentagon)
$\therefore$ $\angle NHM = \angle NMH = 36°$ (angle sum of triangle)
(c) Using the method of (b), $\angle HNK = 36°$
$\therefore$ $\angle NAM = 72°$ (exterior angle of ΔAHN)
$\angle ANM = 108° - 36° = 72°$
$\therefore$ ΔNMA is isosceles
(d) $\angle KAM = 108°$ (KAN a straight line)
(e) From (d) we can obtain that each of the angles of $ABCDE$ is 108°, $\therefore$ $ABCDE$ is a regular pentagon.

6 (a) In ΔABC and ΔADE:
$\angle BAC = \angle DAE$ (same angle)
$\frac{AB}{AD} = \frac{AC}{AE} = \frac{1}{2}$
$\therefore$ $\Delta ABC \,|||\, \Delta ADE$ (two pairs of sides proportional, included angles equal)
(b) $BC \parallel DE$
(c) $\frac{BC}{DE} = \frac{1}{2}$ (matching sides of similar triangles)
In ΔBCF and ΔEDF:
$\angle BFC = \angle EFD$ (vertically opposite angles)
$\angle CBF = \angle FED$ (alternate angles, $BC \parallel DE$)
$\therefore$ $\Delta BCF \,|||\, \Delta EDF$ (AAA)
$\therefore$ $\frac{CF}{FD} = \frac{BC}{DE} = \frac{1}{2}$

7 (a) In ΔADE and ΔBCD:
$\angle ADE = \angle BCD$ (given)
$\angle AED = \angle BDC$ (corresponding angles, $AE \parallel BD$)
$\therefore$ $\Delta ADE \,|||\, \Delta BCD$ (AAA)
(b) $\angle EAD = \angle ADB$ (alternate angles, $AE \parallel BD$)
$\angle ADE = \angle ABD$ (given)
$\therefore$ $\angle AED = \angle BAD$ (angle sum of triangle
In ΔDBA and ΔBCD:
$\angle ABD = \angle DCB$ (given)
$\angle BAD = \angle BDC$ (both equal $\angle AED$)
$\therefore$ $\Delta DBA \,|||\, \Delta BCD$ (AAA)
(c) $\frac{AB}{BD} = \frac{ED}{AD} = \frac{DC}{BC}$

8 (a) $FG = GB$ (sides of a square)
$GB = HA$ (opposite sides of a parallelogram)
$\therefore$ $FG = HA$

(b) $HG = AB$ (opposite sides of a parallelogram)
$AD = AB$ (sides of a square)
$\therefore GH = AD$
Let $\angle HGF = x°$
$\angle HGB = 360° - (90° + x°) = 270° - x°$ (angles at a point)
$\therefore \angle HAB = 270° - x°$ (opposite angles of a parallelogram)
$\therefore \angle HAD = 360° - (270° - x° + 90°)$ (angles at a point)
$\therefore \angle HAD = x°$
In ΔFGH and ΔHAD:
$FG = HA$ (from part (a))
$\angle HGF = \angle HAD = x°$ (from above)
$GH = AD$
$\therefore \Delta FGH \equiv \Delta HAD$ (SAS)
$\therefore HF = DH$ (matching sides of congruent triangles)

9 (a) $\angle ABC = \angle BCD$ (alternate angles, $AB \parallel CD$)
But $\angle BCD = \angle BCA$ (BC bisects $\angle ACD$)
$\therefore \angle ACB = \angle ABC$
(b) In ΔAQC and ΔAQB:
$\angle ACQ = \angle ABQ$ (from above)
$\angle AQC = \angle AQB = 90°$ (given)
AQ is a common side
$\therefore \Delta AQC \equiv \Delta AQB$ (AAS)
$\therefore CQ = QB$ (matching sides of congruent triangles)
(c) In ΔAQC and ΔDQC:
$\angle ACQ = \angle DCQ$ (BC bisects $\angle ACD$)
$\angle AQC = \angle DQC = 90°$ (given)
CQ is a common side
$\therefore \Delta AQC \equiv \Delta DQC$ (AAS)
$\therefore AQ = QD$ (matching sides of congruent triangles)
$\therefore AD$ and BC bisect each other at right angles, hence $ABDC$ is a rhombus.

10 $(x+1)^2 = x^2 + (x-7)^2$, $x = 4, 12$
$x = 4$ is not possible because $x > 7$; sides are 13 cm, 12 cm, 5 cm

11 (a) Let $\angle EAD = x°$, so $\angle EDA = (90 - x)°$
$\angle DAB = 90°$ (angle of a rectangle)
$\angle BAF + 90° + x° = 180°$ (EAF is a straight line)
$\therefore \angle BAF = (90 - x)°$
In ΔEAD and ΔFAB:
$\angle DEA = \angle BFA = 90°$ (given)
$\angle EDA = \angle FAB$ (both $x°$)
$\therefore \Delta EAD \,|||\, \Delta FAB$ (AAA)
(b) $EA^2 = 36 - 25$, $EA = \sqrt{11}$ cm
$\frac{BF}{\sqrt{11}} = \frac{10}{6}$ (matching sides of similar triangles are proportional)
$BF = \frac{5\sqrt{11}}{3}$ cm
vertical height of $B = \frac{5\sqrt{11}}{3}$ cm
vertical height of $C = 5 + \frac{5\sqrt{11}}{3}$ cm

12 (a) In ΔAEB and ΔDFG:
$\angle AED = \angle CFB = 90°$ (given)
$\angle ADE = \angle CBF$ (alternate angles, $AD \parallel BC$, opposite sides of a parallelogram)
$AD = BC$ (opposite sides of a parallelogram)
$\therefore \Delta AEB \equiv \Delta DFG$ (AAS)
(b) $AE = CF$ (matching sides of congruent triangles)

13 The abscissa of A is 6 less than that of B; the ordinates are the same. $\therefore D(8-6, 2)$, $D(2, 2)$

14 (-3, -10)

15 (a) $\frac{\sqrt{2}}{2}$ (b) distance = 1.4 units; area = 3.5 units^2

16 (a) $5x - y + 8 = 0$
(b) $x = -1$, $y = 3$: LHS = -5 − 3 + 8 = 0 = RHS, P is on perpendicular bisector
(c) $PA = PB = \sqrt{26}$; points on the perpendicular bisector are equidistant from A and B

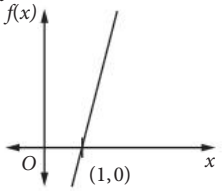
17 (a) $m_{AB} = \frac{5}{8}$, $m_{BC} = \frac{5}{2}$, $m_{DC} = \frac{5}{8}$, $m_{AD} = \frac{5}{2}$; pairs of opposite sides are parallel, so $ABCD$ is a parallelogram
(b) (3, 2) (c) diagonals bisect each other

CHAPTER 12

EXERCISE 12.1

1 A, C

2 $f(x) = 2x - 2$



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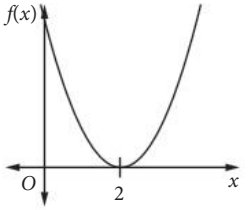

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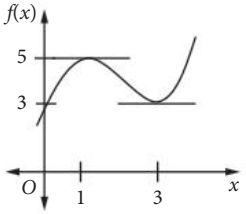

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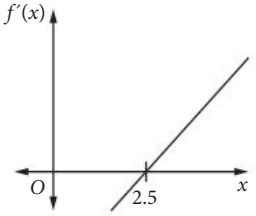
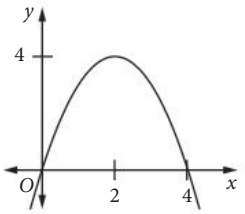


$f'(x) = 2x - 5$
(a) $x < 2.5$ (b) $x = 2.5$
(c) $x > 2.5$

6



$x = 2$; positive, negative

7 (a) $x < -1.5$ (b) $x > -1.5$ (c) $x = -1.5$

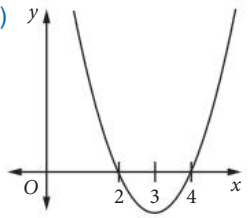
8 (a) $f'(x) = 3x^2 - 12x + 9$ (b) $x < 1$, $x > 3$
(c) $1 < x < 3$ (d) $x = 1$

9 (a) real x (b) none (c) never

10 (a) $x = -\frac{1}{3}, 1$ (b) $x < -\frac{1}{3}, x > 1$ (c) $-\frac{1}{3} < x < 1$

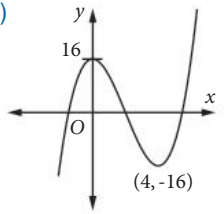
EXERCISE 12.2

1 (a) $f'(x) = 2x - 6$
(b) (3, -1) minimum turning point
(c)



2 C

3 (a) $f'(x) = 3x^2 - 12x$
(b) (0, 16) local maximum; (4, -16) local minimum
(c)

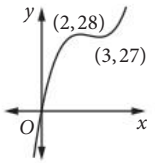


4 (a) correct (b) incorrect (c) correct (d) correct

5 (a) $f'(x) = 6x^2 - 30x + 36$

(b) (2, 28) local maximum; (3, 27) local minimum

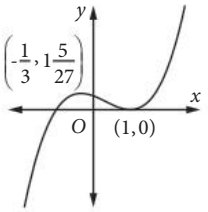
(c)



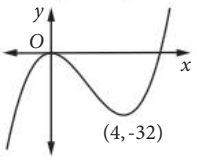
6 (a) $f'(x) = 3x^2 - 2x - 1$

(b) $-\frac{1}{3}, 1\frac{5}{27}$ local maximum; (1, 0) local minimum

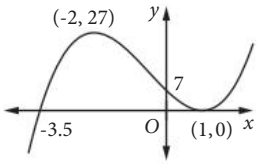
(c)



7 $3\frac{1}{8}$ 8 2 9



10 (-2, 27) maximum; (1, 0) minimum



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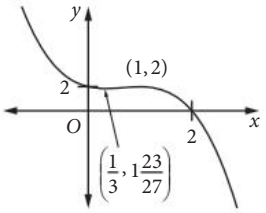
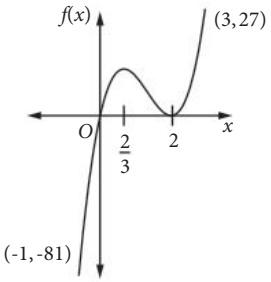


$y' = -(3x^2 - 4x + 1)$: $\frac{1}{3}, 1\frac{23}{27}$ minimum, (1, 2) maximum

12 (a) $x = \frac{2}{3}, 2$ (b) $x < \frac{2}{3}, x > 2$ (c) $\frac{2}{3} < x < 2$

(d)



range = 108, greatest value = 27, least value = -81

13 $y' = 2ax + b$: $y' = 0$ where $2ax + b = 0$, $x = \frac{-b}{2a}$;

$y'' = 2a$ so the turning point is minimum if $a > 0$, maximum if $a < 0$

14 $\frac{dy}{dx} = -\frac{1}{x^2}$: $\frac{dy}{dx}$ is never zero, hence no stationary points, hence no turning points; $\frac{dy}{dx} < 0$ for all x in the domain

EXERCISE 12.3

1 (a) $f'(x) = 6x + 5$, $f''(x) = 6$

(b) $f'(x) = 3x^2 + 4x + 4$, $f''(x) = 6x + 4$

(c) $f'(x) = -2x$, $f''(x) = -2$

(d) $f'(x) = 5x^4 + 6x^2 - 4$, $f''(x) = 20x^3 + 12x$

(e) $f'(x) = -4x^3 + 4x$, $f''(x) = -12x^2 + 4$

(f) $f'(x) = 5$, $f''(x) = 0$

2 D

3 (a) correct (b) incorrect (c) correct (d) incorrect

4 (a) $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$, $\frac{d^2y}{dx^2} = \frac{-1}{4x\sqrt{x}}$ (b) $\frac{dy}{dx} = \frac{1}{2\sqrt{x-2}}$, $\frac{d^2y}{dx^2} = \frac{-1}{4(x-2)\sqrt{x-2}}$

(c) $\frac{dy}{dx} = \frac{2x^2+1}{\sqrt{x^2+1}}$, $\frac{d^2y}{dx^2} = \frac{4x\sqrt{x^2+1} - \frac{2x(2x^2+1)}{2\sqrt{x^2+1}}}{x^2+1} = \frac{x(2x^2-3)}{(x^2+1)^{\frac{3}{2}}}$

(d) $\frac{dy}{dx} = -\frac{1}{x^2}$, $\frac{d^2y}{dx^2} = \frac{2}{x^3}$ (e) $\frac{dy}{dx} = \frac{-1}{(x+1)^2}$, $\frac{d^2y}{dx^2} = \frac{2}{(x+1)^3}$

(f) $\frac{dy}{dx} = \frac{3}{(x+3)^2}$, $\frac{d^2y}{dx^2} = \frac{-6}{(x+3)^3}$ (g) $\frac{dy}{dx} = \frac{3\sqrt{x}}{2} - \frac{1}{2\sqrt{x}}$, $\frac{d^2y}{dx^2} = \frac{3x+1}{4x\sqrt{x}}$

(h) $\frac{dy}{dx} = \frac{5x^{\frac{3}{2}}}{2} - \frac{x^{-\frac{1}{2}}}{2}$, $\frac{d^2y}{dx^2} = \frac{15x+1}{4\sqrt{x}}$ (i) $\frac{dy}{dx} = \frac{1}{\sqrt{x-1}(x+1)^2}$,

$\frac{d^2y}{dx^2} = -\frac{1}{2}(x-1)^{-\frac{3}{2}}(x+1)^{-2} + (x-1)^{-\frac{1}{2}} \times (-2)(x+1)^{-3} = \frac{3-5x}{2(x-1)^{\frac{3}{2}}(x+1)^3}$

5 real x 6 real x 7 (a) $x > -2$ (b) $x < -2$ (c) (-2, 22)

8 $\frac{d^2y}{dx^2} = \frac{-1}{4(x+1)^{\frac{3}{2}}}$; concave down for $x > -1$

$\frac{d^2y}{dx^2} \neq 0$ for x in the domain $\therefore$ no point of inflexion

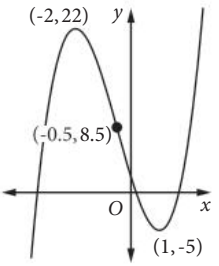
9 (a) $\frac{d^2y}{dx^2} = \frac{2}{x^3}$; $x > 0$ (b) $x < 0$

(c) $\frac{d^2y}{dx^2} \neq 0$ for x in the domain $\therefore$ no point of inflexion

10 $\frac{d^2y}{dx^2} = \frac{2}{x^4}$; $\frac{d^2y}{dx^2} > 0$ for x in the domain $\therefore$ concave up

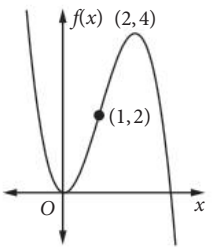
EXERCISE 12.4

1 (-2, 22) maximum; (1, -5) minimum; (-0.5, 8.5) inflexion

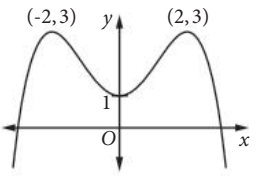


2 A

3 (0, 0) minimum; (2, 4) maximum; (1, 2) inflexion



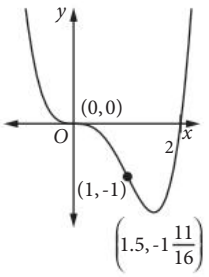
4 (0, 1) min; (2, 3) max; (-2, 3) max;



5 (a) $\left(1\frac{1}{2}, -1\frac{11}{16}\right)$ minimum

(b) (0, 0) horizontal inflexion, (1, -1) inflexion

(c)



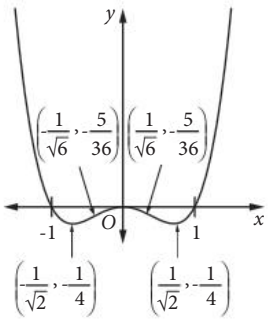
(d) $-1\frac{11}{16}$

6 (a) $x = 2$ (b) $x > 2$ (c) $x < 2$

7 (-1, -9), (2, -48); $x < -1$, $x > 2$

8 (a) (-1, 0), (0, 0), (1, 0)

(b) (0, 0) max; $\left(\pm\frac{1}{\sqrt{2}}, -\frac{1}{4}\right)$ min (c) $\left(\pm\frac{1}{\sqrt{6}}, -\frac{5}{36}\right)$

(d) 

(e) $-\frac{\sqrt{6}}{6} < x < \frac{\sqrt{6}}{6}$

9 (a) correct (b) correct (c) incorrect (d) correct

10 $\frac{dy}{dx} = -3(x-1)^2 < 0$ for all $x \neq 1$

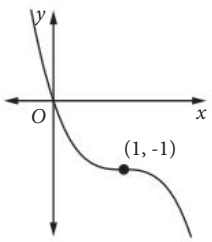
$\frac{d^2y}{dx^2} = -6(x-1) = 0$ where $x = 1$;

concavity changes, so $(1, -1)$ is point of inflexion.

$y = -x(x^2 - 3x + 3)$, $\Delta = 9 - 12 < 0$,

$x^2 - 3x + 3 = 0$ has no real roots;

curve only cuts x-axis at $(1, -1)$



11 (a) $(1, 0)$ min

(b) $y'' = 12x^2$: $y'' = 0$ at $x = 0$ but concavity does not change

$\therefore$ no point of inflexion, curve is always concave up

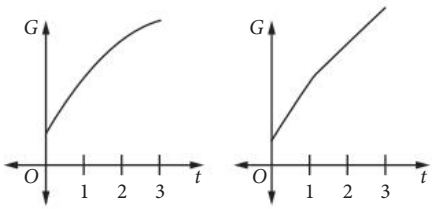
12 $\frac{dR}{dx} = \frac{40 - x}{2}$: $(40, 400)$ stationary point;

$\frac{d^2R}{dx^2} = -\frac{1}{2} < 0$: maximum revenue at $x = 40$

13 \$4.50

14 (a) decreasing (b) it is increasing at a decreasing rate

(c) concave down

(d) 

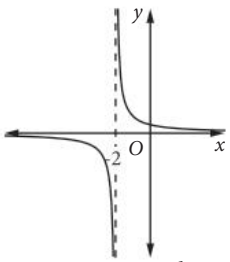
(e) $f'(t) = 3.2$, point of inflexion

(f) it is increasing at a decreasing rate

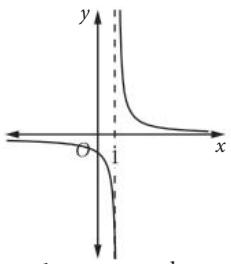
15 (a) 90 (b) \$17 420 (c) $P = 374x - 2.2x^2 - 400$ (d) 85, \$15 495

EXERCISE 12.5

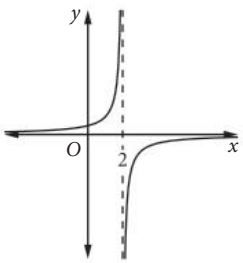
1 A

2 (a) 

$x < -2$ range: real y, $y \neq 0$

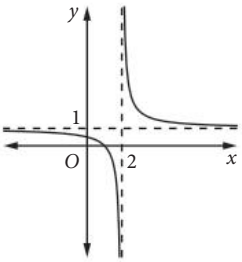
(b) 

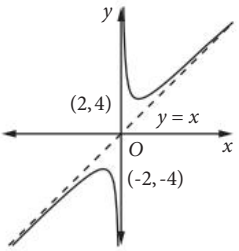
$x < 1$ range: real y, $y \neq 0$

(c) 

$x > 2$ range: real y, $y \neq 0$

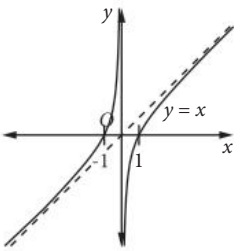
3 (a) $1 + \frac{1}{x-2} = \frac{x-2+1}{x-2} = \frac{x-1}{x-2}$

(b) 

4 (a) 

no inflexions, $x > 0$

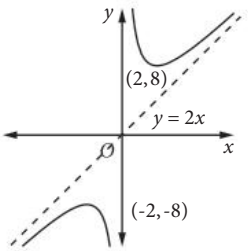
range: real y, $|y| \geq 2$

(b) 

no turning points,

no inflexions, $x < 0$

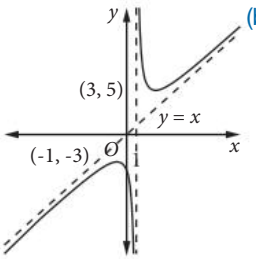
range: real y

(c) 

no inflexions, $x > 0$

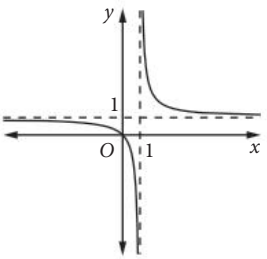
range: real y, $|y| \geq 8$

5 (a) correct (b) incorrect (c) correct (d) correct

6 (a) 

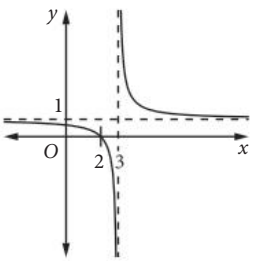
$x < 1$ range:

$y \leq -3$, $y \geq 5$

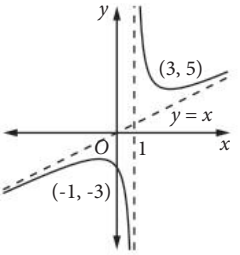
(b) 

$x < 1$ range: real y,

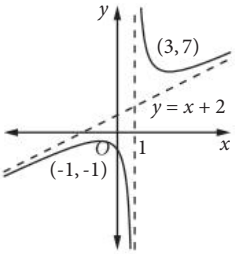
$y \neq 1$

(c) 

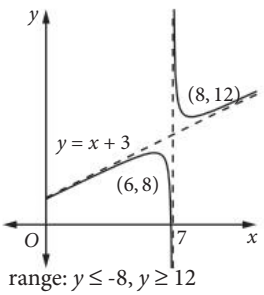
$x < 3$ range: real y, $y \neq 1$

7 (a) 

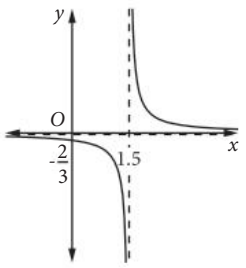
range: $y \leq -3$, $y \geq 5$

(b) 

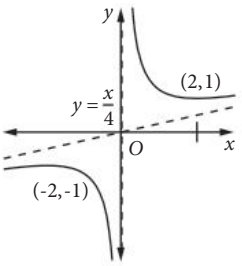
range: $y \leq -1$, $y \geq 7$

(c) 

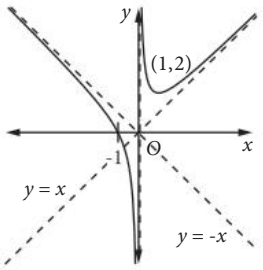
range: $y \le -8, y \ge 12$

(d) 

range: real $y, y \ne 0$

(e) 

range: $y \le -1, y \ge 1$

(f) 

range: real y

EXERCISE 12.6

1 (a) $l = 80 - x$ (b) $A(x) = x(80 - x) = 80x - x^2$
(c) $A'(x) = 80 - 2x$, $x = 40$, max area $= 1600\,\text{m}^2$

2 B

3 (a) $l = 160 - 2x$ (b) $A(x) = x(160 - 2x)$ (c) $3200\,\text{m}^2$

4 (a) $y = \frac{120 - 4x}{3}$ (b) $A(x) = \frac{x(120 - 4x)}{3}$
(c) $x = 15$, $y = 20$, $A = 300\,\text{m}^2$
(d) $A(x) = \frac{x(138 - 4x)}{3}$, $x = 17.25$, $y = 23$, $A = 396.75\,\text{m}^2$

5 (a) $A = 1 - \frac{x^2}{2} - \frac{1 - x}{2} = \frac{1 + x - x^2}{2}$ (b) $x = \frac{1}{2}$, max area $= \frac{5}{8}$

6 squares of side x cm, $V = x(6 - 2x)(16 - 2x)$, $0 < x < 3$;
$x = \frac{4}{3}$, max volume $= \frac{1600}{27}\,\text{cm}^3$

7 (a) $y = 12 - 4x$, $0 < x < 3$ (b) $V = 12x^2 - 4x^3$
(c) $x = 2$, max volume $= 16\,\text{cm}^2$

8 box $x \times x \times y$: $y = 20 - 2x$, $0 < x < 10$
$V = x^2(20 - 2x)$: $x = \frac{20}{3}$, max volume $\frac{8000}{27}\,\text{cm}^3$

9 (a) $8x + 4y = 36$, $y = 9 - 2x$, $0 < x < 4.5$
(b) $V = x^2(9 - 2x)$ (c) $x = 3$

10 (a) length $= 2x$, height $= y$, $216 = 4x^2 + 6xy$, $y = \frac{108 - 2x^2}{3x}$
(b) $V = \frac{2x(108 - 2x^2)}{3}$ (c) $144\sqrt{2}\,\text{cm}^3$

11 (a) correct (b) correct (c) correct (d) correct

12 (a) $y = \sqrt{100 - x^2}$ (b) $V = 100x - x^3$ (c) $\frac{2000\sqrt{3}}{9}\,\text{cm}^3$

13 (a) $r = \sqrt{36 - x^2}$ (b) $V = \frac{\pi x(36 - x^2)}{3}$ (c) $x = 2\sqrt{3}$ cm

14 pieces x cm, $(30 - x)$ cm; $A = \frac{x^2}{16} + \frac{(30 - x)^2}{16}$: $x = 15$, minimum area $28\frac{1}{8}\,\text{cm}^2$

15 (a) $2\pi r^2 + 2\pi rh = 20\pi$, $h = \frac{10 - r^2}{r}$
(b) $V = \pi(10r - r^3)$ (c) $r = \frac{\sqrt{30}}{3}$

16 rectangle $x \times 3x$; square sides $\frac{25 - 4x}{2}$;
$A = 3x^2 + \frac{(25 - 4x)^2}{4}$; $A' = 14x - 50$; $x = \frac{25}{7}$
rectangle $3\frac{4}{7}\,\text{cm} \times 10\frac{5}{7}\,\text{cm}$; square sides $5\frac{5}{14}\,\text{cm}$

17 (a) $R = (380 - 12(n - 20)) \times n = 620n - 12n^2$
(b) R maximum at $n = 25.8$: $n = 25$, $R = \$8000$; $n = 26$, $R = \$8008$; group size is 26

18 (a) \$15 000 (b) $f(15) = 2812.5$, function is approximate so answer is 2812 or 2813

19 (a) $f'(x) = 3x^2 - 192x + 2880 = 3(x - 24)(x - 40)$, \$24 000
(b) \$27 648

20 (a) $h^2 + r^2 = a^2$, $h = \sqrt{a^2 - r^2}$, $V = \pi r^2 \times 2h = 2\pi r^2\sqrt{a^2 - r^2}$
(b) $V'(r) = 2\pi\left(2r\sqrt{a^2 - r^2} + \frac{r^2 \times (-2r)}{2\sqrt{a^2 - r^2}}\right) = 2\pi r\left(2a^2 - 3r^2\right)$;
$3r^2 = 2a^2$, $r = \frac{a\sqrt{6}}{3}$
$V''(r) = 2\pi(2a^2 - 9r^2)$
$V''\left(\frac{a\sqrt{6}}{3}\right) = 2\pi\left(2a^2 - \frac{3 \times 6a^2}{3}\right) < 0$;
$r = \frac{a\sqrt{6}}{3}$ gives max volume

EXERCISE 12.7

1 $4x - y - 4 = 0$, $x + 4y - 18 = 0$

2 $y = 3x$, $x + 3y = 0$ 3 A

4 $(0, -4)$: $3x - y - 4 = 0$, $x + 3y + 12 = 0$

5 $(2, 1)$: $y = 4x - 7$, $x + 4y - 6 = 0$

6 (a) incorrect (b) correct (c) correct (d) correct

7 $(0, 3)$: $x + y - 3 = 0$, $y = x - 3$

8 $(2, 0)$: $y = 10x - 20$, $x + 10y - 2 = 0$

9 $P(-2, 0)$, $Q(4, 6)$;
tangent at P: $2x + y + 4 = 0$
tangent at Q: $y = 4x - 10$
intersection at $(1, -6)$

10 $(0, 0)$, $\frac{4}{3}$; $y' = 4 - 6x$; $y = 4x$, $12x + 3y - 16 = 0$

11 $(1, -1)$: $y = -1$, $x = 1$

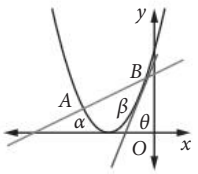
12 $2x^2 - 6x + 5 = x^2 - 2x + 1$, $(x - 2)^2 = 0$, $x = 2$, $y = 1$: intersect at $(2, 1)$
$y_1' = 4x - 6$: at $(2, 1)$ $y_1' = 2$
$y_2' = 2x - 2$: at $(2, 1)$ $y_2' = 2$
curves have the same gradient at their point of intersection, so they touch at that point; equation of tangent: $y = 2x - 3$

13 $A(-1, 3)$, $B(4, 8)$; $y' = 2x - 2$; at A: $y' = -4$, angle $= 104°\,2'$
at B: $y' = 6$, angle $= 80°\,32'$

14 (a) $P(-1, 0)$, $Q(0, 4)$ (b) $3x + y + 3 = 0$, $y = 5x - 11$ (c) $(1, -6)$

15 (a) $y = 3$ (b) $y = 2x$

16 $4x - 5 = 3$: $(2, -1)$
tangent: $y = 3x - 7$
normal: $x + 3y + 1 = 0$

17 (a) $x - 2y + 5 = 0$ (b) $\left(-\frac{1}{2}, 2\frac{1}{4}\right)$
(c) 

at B: $m = 3$, $\tan\theta = 3$, $\theta = 71°\,34'$, $m_{AB} = 0.5$
$\tan\alpha = 0.5$, $\alpha = 26°\,34'$, $\beta = 71°\,34' - 26°\,34' = 45°$

18 $A(1, -1)$, $B(2, 0)$
tangent at A: $m = -2$, $\alpha = 116°\,34'$
tangent at B: $m = 8$, $\beta = 82°\,52'$
angle between tangents $= 116°\,34' - 82°\,52' = 33°\,42'$

19 $(-1, 0)$, tangent $y = 6x + 6$; $(1, 0)$, tangent $y = 2 - 2x$;
$(2, 0)$, tangent $y = 3x - 6$

20 $y' = 6x^2 - 6x$ (a) $(-1, -5)$, $(2, 4)$ (b) $(0, 0)$, $(1, -1)$

EXERCISE 12.8

1 (a) $2x^3 - 2x^2 + 5x + C$ (b) $3x + \frac{5x^2}{2} + \frac{x^3}{3} - \frac{3x^4}{4} + C$
(c) $\frac{x^3}{3} - x + C$ (d) $\frac{x^6}{6} - x^4 - \frac{2x^3}{3} + x + C$ (e) $\frac{3x^2}{2} - 5x + C$
(f) $\frac{2x^3}{3} - \frac{7x^2}{2} + 5x + C$

2 (a) correct (b) correct (c) incorrect (d) incorrect

3 (a) $\frac{4x^3}{3} + 2x^2 + x + C$ (b) $5x + C$ (c) $\frac{x^3}{3} + \frac{3x^2}{2} + C$
(d) $\frac{2x^3}{3} + \frac{3x^2}{2} - 2x + C$ (e) $x^4 - 2x^3 + \frac{x^2}{2} + C$
(f) $\frac{x^3}{3} + 3x^2 + 9x + C$

4 LHS $= \frac{8x^3 + 12x^2 + 6x + 1}{6} = \frac{4x^3}{2} + 2x^2 + x + 1$,
which differs from RHS only by a constant

5 (a) $3x + x^2 - x^3 + C$ (b) $\frac{x^4}{4} + \frac{2x^3}{3} + C$ (c) $\frac{x^5}{5} - \frac{x^4}{4} + C$
(d) $\frac{x^3}{3} + \frac{x^2}{2} - 12x + C$ (e) $\frac{x^5}{5} - \frac{x^4}{12} + \frac{x^3}{6} + C$ (f) $\frac{2x^{\frac{3}{2}}}{3} + C$

6 (a) $\frac{x^4}{4} - x^3 + C$ (b) $\frac{x^3}{15} - \frac{x^4}{16} + C$ (c) $\frac{x^3}{3} - \frac{3x^4}{4} + C$
(d) $\frac{x^5}{5} - x + C$ (e) $-2x^{-\frac{1}{2}} - \frac{2x^{-\frac{3}{2}}}{3} + C$ (f) $\frac{2x^{\frac{3}{2}}}{3} + \frac{3x^{\frac{4}{3}}}{4} + C$

7 $x^2 - 2x + 5$ **8** C **9** $y = 2x^2 - 6x + 8$

10 $y = x^3 - x^2 + 3x - 24$ **11** $F(x) = \frac{(x+2)^3}{3} - 5$ **12** $y = x^2 - 4$

13 $f(x) = \frac{x^3}{3} - x^2 + x + 2$ **14** $y = \frac{x^3}{3} + \frac{x^2}{2} - 2x + 4$

15 (a) $3x - \frac{1}{x} + \frac{1}{x^2} + C$ (b) $\frac{x^2-2}{x} + C = x - \frac{2}{x} + C$
(c) $\frac{3x^2}{2} - 2x - \frac{1}{2x^2} + C$ (d) $C - \frac{1}{2x^2}$ (e) $\frac{4x^{\frac{3}{2}}}{3} + 3x - \frac{5}{x} + C$
(f) $2\sqrt{x} - \frac{2}{\sqrt{x}} + C$

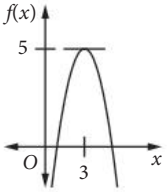
16 $s = 4t^3 - 3t^2 + t + 2$ **17** $N = 60t + 50,410$ **18** $d = t^3 + 4t$

CHAPTER REVIEW 12

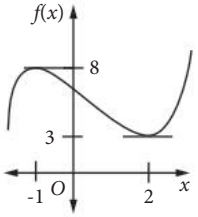
1 $y' = 15 + 24x - 12x^2$
(a) $-0.5 < x < 2.5$ (b) $x < -0.5, x > 2.5$ (c) $x = 2.5$ (d) $x = -0.5$

2 $f'(x) = 6x^2 + 6x - 12$
(a) $x = -2, 1$ (b) $x < -2, x > 1$ (c) $-2 < x < 1$

3 (a) $f(x)$



(b) $f(x)$



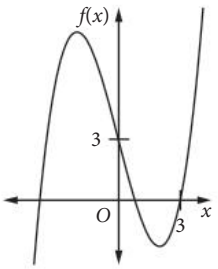
4 (a) (0, 0) max; $\left(\frac{4}{9}, \frac{-32}{243}\right)$ min (b) (-1, 5) max; (3, -27) min

5 $DC = x$ cm, $AD = y$ cm; $3x + 2y = 45$, $y = \frac{45-3x}{2}$
$A = xy + \frac{x^2 \sin 60°}{2} = \frac{90x - (6-\sqrt{3})x^2}{4}$
$A' = \frac{45-(6-\sqrt{3})x}{2}$: $x = \frac{15(6+\sqrt{3})}{11}$, $y = \frac{45(5-\sqrt{3})}{22}$

6 square side x m, rectangle y m × $2y$ m:
$4x + 6y = 8$, $y = \frac{4-2x}{3}$
$A = x^2 + \frac{2(4-2x)^2}{9}$, $A' = \frac{34x-32}{9}$, $x = \frac{16}{17}$
part lengths: $3\frac{13}{17}$ m, $4\frac{4}{17}$ m

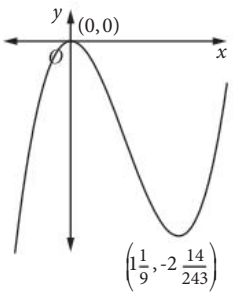
7 height = y cm; $2x + 2y + \pi x = 50$, $y = \frac{50-(\pi+2)x}{2}$;
$A = \frac{100x-(\pi+4)x^2}{2}$, $A' = 50 - (\pi+4)x$, $x = \frac{50}{\pi+4}$
rectangle: $\frac{100}{\pi+4}$ by $\frac{50}{\pi+4}$, greatest area $\frac{1250}{\pi+4}$ cm^2

8 $f'(x) = x^2 - 4$ (a) $x = \pm 2$ (b) $-2 < x < 2$
(c) $8\frac{1}{3}$, $f(x) = 8\frac{1}{3}$, local max; $(2, -2\frac{1}{3})$, $f(x) = -2\frac{1}{3}$, local min
(d)



9 $2x + y - 9 = 0$ **10** $y = 132 - 32x$; $y = 0$ touches axis

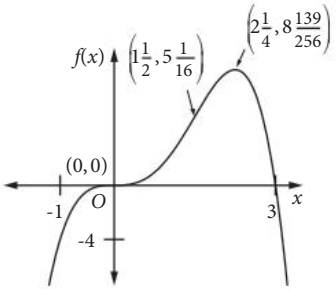
11



12 side of square = x cm: $V = x(4-2x)(6-2x)$, $x = \frac{5-\sqrt{7}}{3}$ cm

13 $y' = 6x - 3x^2 = 3x(2-x)$: $y' = 0$ at $x = 2$
At $x = 0$, $y = 0$, $y' = 0$, so the curve is horizontal where it cuts the y-axis at the origin.

14



$y' = 9x^2 - 4x^3 = x^2(9-4x)$, $y'' = 18x - 12x^2 = 6x(3-2x)$:
$\left(2\frac{1}{4}, 8\frac{139}{256}\right)$ maximum turning point;
(0, 0) horizontal inflexion, $\left(1\frac{1}{2}, 5\frac{1}{16}\right)$ inflexion

15 $AX = \sqrt{9+x^2}$, $XB = 4 - x$
(a) (i) Time A to $X = \frac{\sqrt{9+x^2}}{10\sqrt{2}}$ hours
(ii) Time X to $B = \frac{4-x}{20}$ hours
(b) Total time A to B: $t = \frac{\sqrt{9+x^2}}{10\sqrt{2}} + \frac{4-x}{20}$
(c) $\frac{dt}{dx} = \frac{x}{10\sqrt{2}\sqrt{9+x^2}} - \frac{1}{20}$, $x = 3$ km, $t = 0.35$ h $= 21$ min

16 $v = 60$ km/h

17 (a) $\frac{x^2}{2} + 9x + C$ (b) $x^3 - x^2 + 4x + C$ (c) $\frac{x^5}{5} + \frac{x^4}{4} - 2x + C$
(d) $\frac{x^3}{3} + \frac{x^2}{2} - 6x + C$ (e) $\frac{(x+2)^3}{3} + C = \frac{x^3}{3} + 2x^2 + 4x + C$ (f) $7x + C$

18 (a) $\frac{5x^2}{2} + 4x + C$ (b) $5x - 2x^2 + x^3 + \frac{x^4}{4} + C$
(c) $x^2 + \frac{2x\sqrt{x}}{3} + 3x + C$

19 (a) $f(x) = \frac{x^3}{3} + \frac{x^4}{4} + x + 2$ (b) $f(x) = 3x - \frac{x^2}{2} + \frac{3x^4}{2} - 1$
(c) $f(x) = x + \frac{1}{x} - 2$

20 (a) $V = 140t + \frac{13t^2}{2} - \frac{t^3}{3}$ (b) 2040 litres

CHAPTER 13

EXERCISE 13.1

1 $0.2734 <$ area < 0.3984

x	x^2
0	0
0.125	0.015 625
0.25	0.0625
0.375	0.140 625
0.5	0.25
0.625	0.390 625
0.75	0.5625
0.875	0.765 625
1	1

2 (a) 10 (b) 9 (c) 8.75
3 (a) 1.5 (b) 1.375 (c) 1.343 75
4 (a) 6 (b) 6 (c) 6 **5** (a) 8 (b) 5 (c) 4.25

EXERCISE 13.2

1 $\int_0^3 (x+2)\,dx = 10.5$ **2** $\int_1^4 (2x+1)\,dx = 18$ **3** C

4 $\int_0^t 3x\,dx = \frac{3t^2}{2}$ **5** $\int_0^2 x^2\,dx$, 2.6 **6** $\int_0^2 (2-x)\,dx = 2$

7 $\int_{-3}^{3} \sqrt{9-x^2}\,dx = \frac{9\pi}{2}$ **8** $\int_{-1}^{5} \sqrt{9-(x-2)^2}\,dx = \frac{9\pi}{2}$

EXERCISE 13.3

1 9, -9; $9 = -(-9)$ **2** $2\frac{1}{3}, 6\frac{1}{3}, 8\frac{2}{3}$; $2\frac{1}{3} + 6\frac{1}{3} = 8\frac{2}{3}$

3 $8\frac{2}{3}, 8, 16\frac{2}{3}$; $8\frac{2}{3} + 8 = 16\frac{2}{3}$ **4** 72, 72; $72 = 72$

5 (a) -4 (b) 27.5 (c) $-10\frac{2}{3}$ (d) 0 (e) $6\frac{2}{3}$ (f) $61\frac{2}{3}$ **6** B

7 (a) $\int_{-1}^{1}\left(x - x^3\right)dx = 0$ (b) $18\frac{2}{3}$

(c) $\int_{0}^{4}\left(x^3 - 6x^2 + 12x - 8\right)dx = \left[\frac{x^4}{4} - 2x^3 + 6x^2 - 8x\right]_{0}^{4} = 0$

(d) $\int_{-1}^{1}\left(x^2 - x^3\right)dx = \frac{2}{3}$ (e) $\frac{16}{3}$ (f) $\frac{15}{64}$ (g) 38.4 (h) $\frac{1}{12}$ (i) $-\frac{1}{12}$

8 (a) $a^3 - b^3$ (b) $\frac{8a}{3} + 2b + 2c$ (c) $\frac{a^3}{3} - a^2$

9 (a) correct (b) incorrect (c) correct (d) correct

10 $\int_{-2}^{1}\left(2 - x - x^2\right)dx = \left[2x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-2}^{1} = 4.5$

11 $\int_{0}^{1} x^2(1-x)\,dx = \frac{1}{12}$ **12** $\int_{-0.5}^{3}\left(3 + 5x - 2x^2\right)dx = 14\frac{7}{24}$

13 $a = 2$ **14** $a = 1$ **15** $c = -\frac{1}{20}$ **16** 6.75

EXERCISE 13.4

1 55.5 **2** $\int_{-1}^{2}\left(x^2 + 2\right)dx = 9$

3 $\int_{1}^{4}\left(x^2 - 4x + 4\right)dx = 3$ **4** $\int_{-2}^{2}\left(4 - x^2\right)dx = 10\frac{2}{3}$

5 (a) correct (b) incorrect (c) correct (d) correct

6 $2\int_{-3}^{0}\left(-x^3\right)dx = 40.5$ **7** B **8** 0.25 **9** $k = 0.75$

10 (a) $\int_{0}^{0.5}\left(x^3 - x^2 - x + 1\right)dx = \frac{67}{192}$

(b) $\int_{-1}^{1}\left(x^3 - x^2 - x + 1\right)dx = 1\frac{1}{3}$

(c) $\int_{0}^{1}\left(x^3 - x^2 - x + 1\right)dx = \frac{5}{12}$

11 $\int_{0}^{8}\left(8x - x^2\right)dx = 85\frac{1}{3}$

12 $\int_{0}^{2} x(x-2)^2\,dx = \left[\frac{x^4}{4} - \frac{4x^3}{3} + 2x^2\right]_{0}^{2} = 1\frac{1}{3}$

13 $\int_{0}^{1}(x+1)(x-1)(x-3)\,dx + \left|\int_{1}^{2}(x+1)(x-1)(x-3)\,dx\right|$

$= 1\frac{3}{4} + \left|0 - 1\frac{3}{4}\right| = 3\frac{1}{2}$

14 Area of square = 1 units^2

(a) $\int_{0}^{1} x\,dx = \left[\frac{x^2}{2}\right]_{0}^{1} = \frac{1}{2}$

(b) $\int_{0}^{1} x^2\,dx = \left[\frac{x^3}{3}\right]_{0}^{1} = \frac{1}{3}$, $\int_{0}^{1}\sqrt{x}\,dx = \left[\frac{2x^{\frac{3}{2}}}{3}\right]_{0}^{1} = \frac{2}{3}$;

square is trisected

(c) $\int_{0}^{1} x^3\,dx = \left[\frac{x^4}{4}\right]_{0}^{1} = \frac{1}{4}$, $\int_{0}^{1} x\,dx = \left[\frac{x^2}{2}\right]_{0}^{1} = \frac{1}{2}$,

$\int_{0}^{1}\sqrt[3]{x}\,dx = \left[\frac{3x^{\frac{4}{3}}}{4}\right]_{0}^{1} = \frac{3}{4}$; four equal parts

EXERCISE 13.5

1 $8 + 2\sqrt{13}$ **2** (a) $\frac{31}{40}$ (b) 0.783

3 (a) incorrect (b) correct (c) incorrect (d) correct

4 (a) $3\frac{3}{4}$ (b) $3\frac{15}{16}$

5

x	0	1	2	3	4
2^x	1	2	4	8	16

$\frac{1}{2}(1 + 2(2 + 4 + 8) + 16) = 22.5$

6 2.8125 **7** (a) $\frac{2}{3}$ (b) 0.631 **8** 3.414

9 $2\frac{19}{60}$ **10** 23.11

11 (a) $\frac{50}{2}(200 + 2(150 + 100 + 150 + 100 + 50 + 50 + 100$

$+ 150 + 200) + 250) = 63750\,\text{m}^2$

(b) $159\,375\,000\,\text{m}^3$

EXERCISE 13.6

1 14.95 **2** 0.783

3 (a) 0.25 (b) 0.25 **4** A

5

x	0	1	2	3	4
2^x	1	2	4	8	16

21.67

6 2.708 **7** (a) $\frac{2}{3}$ (b) 0.638

8 (a) 3.219 (b) 3.240 **9** 2.389 15 **10** 22.37

11 (a) $\frac{50}{3}(200 + 4\times150 + 2\times100 + 4\times150 + 2\times100 + 4\times50$

$+ 2\times50 + 4\times100 + 2\times150 + 4\times200 + 250) = 64167\,\text{m}^2$

(b) $160\,400\,000\,\text{m}^3$

EXERCISE 13.7

1 $\int_{0}^{2}\left(2x - x^2\right)dx = \frac{4}{3}$

2 $\int_{-2}^{6}\left(3 + x - \frac{x^2}{4}\right)dx = \left[3x + \frac{x^2}{2} - \frac{x^3}{12}\right]_{-2}^{6} = 21\frac{1}{3}$

3 A **4** $\int_{0}^{1} x^2\,dx + \int_{1}^{3}\frac{dx}{x^2} = 1$

5 $\int_{1}^{3}\left(x^2 - \frac{1}{x^2}\right)dx = 8$ **6** $\int_{-1}^{3}\left(3 + 2x - x^2\right)dx = 10\frac{2}{3}$

7 $\int_{0}^{2}\left(2x + x^2\right)dx = 6\frac{2}{3}$ **8** $2\int_{0}^{1}\left(1 - x^2\right)dx = \frac{4}{3}$

9 (a) correct (b) correct (c) incorrect (d) correct

10 $\int_{0}^{1}\left(10x - 10x^2\right)dx = \frac{5}{3}$ **11** $\int_{0}^{2}\left(2x - x^2\right)dx = \frac{4}{3}$

12 $\int_{0}^{4}\left(4x - x^2\right)dx = 10\frac{2}{3}$

13 (a) $y = x$ (b) $\int_{0}^{3}\left(3x - x^2\right)dx = 4.5$

(c) $\int_{0}^{3} x\,dx + \int_{3}^{4}\left(4x - x^2\right)dx = 6\frac{1}{6}$

14 (a) intersection at $\sqrt{x} = \frac{1}{x}$: $x^3 = 1$, $x = 1$; for $x > 1$, $\sqrt{x} > \frac{1}{x}$

(b) 1.61

15 (a) curves do not intersect; $\frac{1}{x^2} > \frac{-1}{x}$ for $1 \le x \le 4$

(b) $\frac{1}{2}\left(2 + 2\left(\frac{3}{4} + \frac{4}{9}\right) + \frac{5}{16}\right) \approx 2.35$

16 (a) $x - 20y + 480 = 0$ (b) 16 m (c) $y = \frac{x^2}{200}$ (d) $1883\frac{1}{3}\,\text{m}^2$

EXERCISE 13.8

1 $\int_{1}^{4}(y-1)\,dy = 4.5$ **2** $\int_{0}^{3} y^2\,dy = 9$ **3** C

4 $\int_{1}^{9}\frac{dy}{\sqrt{y}} = 4$ **5** $2\int_{0}^{4}\sqrt{y}\,dy = 10\frac{2}{3}$

6 (a) $(2,5)$: $\frac{dy}{dx} = 2x$, $m = 4$; $y - 5 = 4(x-2)$, $y = 4x - 3$

(b) $\int_{-3}^{5}\left(\frac{y+3}{4}\right)dy + \int_{1}^{5}\sqrt{y-1}\,dy$

$= \frac{1}{4}\left[\frac{y^2}{2} + 3y\right]_{-3}^{5} + \left[\frac{2}{3}(y-1)^{\frac{3}{2}}\right]_{1}^{5} = 2\frac{2}{3}$

(c) $1\frac{13}{24}$

7 $\int_{1}^{4}\frac{dy}{y} \approx \frac{1}{2}\left(1 + 2\left(\frac{1}{2} + \frac{1}{3}\right) + \frac{1}{4}\right) = 1\frac{11}{24}$

8 (a) $x^2 = 4 - x^2$, $x = \pm\sqrt{2}$;

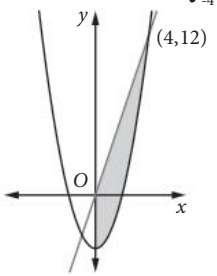
$\int_{-\sqrt{2}}^{\sqrt{2}} \left(4 - 2x^2\right) dx = 2\left[4x - \frac{2x^3}{3}\right]_0^{\sqrt{2}} = \frac{16\sqrt{2}}{3}$

(b) $2\left\{\int_0^{\sqrt{2}} x^2\, dx + \int_{\sqrt{2}}^{2} \left(4 - x^2\right) dx\right\} = \frac{16\left(2-\sqrt{2}\right)}{3}$

9 (a) correct (b) correct (c) correct (d) correct

10 4 11 $\int_{-4}^{2} \sqrt{y+4}\, dy = 4\sqrt{6}$

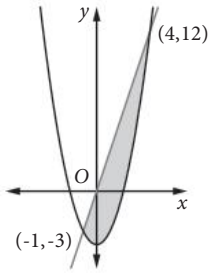
12 (a)



(b) $x^2 - 4 = 3x$, $x = -1, 4$:

$\int_0^4 \left(4 + 3x - x^2\right) dx = \left[4x + \frac{3x^2}{2} - \frac{x^3}{3}\right]_0^4 = 18\frac{2}{3}$

13 (a)



(b) intersection at (-1, -3), (4, 12):

$A = \int_{-4}^{-3} \sqrt{y+4}\, dy - \int_{-4}^{-3} -\sqrt{y+4}\, dy + \int_{-3}^{12} \left(\sqrt{y+4} - \frac{y}{3}\right) dy$

$= 2\int_{-4}^{-3} \sqrt{y+4}\, dy + \int_{-3}^{12} \left(\sqrt{y+4} - \frac{y}{3}\right) dy$

(c) $2\left[\frac{2}{3}(y+4)^{\frac{3}{2}}\right]_{-4}^{-3} + \left[\frac{2}{3}(y+4)^{\frac{3}{2}} - \frac{y^2}{6}\right]_{-3}^{12} = 20\frac{5}{6}$

(d) $\int_{-1}^{4} \left(3x + 4 - x^2\right) dx = 20\frac{5}{6}$

14 $\frac{1}{2}\left[2\sqrt{3} + 2\left(\sqrt{15} + 4 + \sqrt{15} + 2\sqrt{3}\right) + \sqrt{7}\right] \approx 18.25$ units^2

15 (a) $B(40, 16)$, LHS = 16, RHS = $\frac{40^2}{100} = 16$ = LHS

(b) $A = 2\int_0^{16} 10\sqrt{y}\, dy = \frac{2560}{3}$ cm^2; $V = 25\,600$ cm^3

EXERCISE 13.9

1 $\pi \int_0^3 x^4\, dx = \pi\left[\frac{x^5}{5}\right]_0^3 = \frac{243\pi}{5}$ 2 $\frac{256\pi}{3}$ 3 D

4 (a) $y = 2x - 2$ (b) $4\pi \int_1^3 (x-1)^2\, dx = \frac{32\pi}{3}$

5 $\pi \int_{-3}^{3} \left(9 - x^2\right) dx = 36\pi$ 6 $\frac{\pi}{30}$

7 (a) $\pi \int_{-1}^{1} \left(1 - x^2\right)^2 dx = 2\pi\left[x - \frac{2x^3}{3} + \frac{x^5}{5}\right]_0^1 = \frac{16\pi}{15}$

(b) $\pi \int_0^1 (1-y)\, dy = \frac{\pi}{2}$

8 $\pi \int_0^2 (x-2)^4\, dx = \frac{32\pi}{5}$

9 (a) $\pi \int_0^6 \left(3 - \frac{x}{2}\right)^2 dx = \pi\left[9x - \frac{3x^2}{2} + \frac{x^3}{12}\right]_0^6 = 18\pi$

(b) $4\pi \int_0^3 (3-y)^2\, dy = 36\pi$

10 $\pi \int_{-4}^{4} \left(\frac{144 - 9x^2}{16}\right) dx = \frac{\pi}{8}\left[144x - 3x^3\right]_0^4 = 48\pi$

11 (a) $2\pi \int_0^3 \left(81 - 18x^2 + x^4\right) dx = \frac{1296\pi}{5}$

(b) $\pi \int_0^9 (9-y)\, dy = \frac{81\pi}{2}$

12 (a) $y = 10x - 30$ (b) $\pi \int_0^{10} \left(\frac{y+30}{10}\right)^2 dy = \frac{370\pi}{3}$ (c) $\frac{635\pi}{12}$

13 $\pi \int_{-a}^{\frac{-a}{2}} \left(a^2 - y^2\right) dy = \frac{5a^3\pi}{24}$

14 $\pi \int_1^4 (4-y)\, dy = \frac{9\pi}{2}$

15 (a) incorrect (b) correct (c) correct (d) correct

16 $\frac{256\pi}{3}$

17 $\pi \int_0^1 \left(4x^2 - 4x^4\right) dx \approx 1.37$

18 (a) $\pi \int_{-2}^{2} \left(16 - 4x^2\right) dx = \frac{128\pi}{3}$ (b) $\frac{\pi}{2} \int_0^4 \left(16 - y^2\right) dy = \frac{64\pi}{3}$

19 (a) $\int_0^4 \left(2 - \sqrt{x}\right)^2 dx = \int_0^4 \left(4 - 4\sqrt{x} + x\right) dx = 2\frac{2}{3}$

(b) $V = \pi \int_0^4 \left(4 - 4\sqrt{x} + x\right)^2 dx$

$= \pi \int_0^4 \left(16 - 32\sqrt{x} + 24x - 8x^{\frac{3}{2}} + x^2\right) dx = \frac{64\pi}{15}$

(c) $\frac{64\pi}{15}$

20 $V = \pi \int_1^a \frac{dx}{x^2} = \pi\left[\frac{-1}{x}\right]_1^a = \pi\left(1 - \frac{1}{a}\right)$, $\lim_{a\to\infty} V = \pi$

21 $\pi - \pi \int_0^1 \left(2x - x^2\right)^2 dx = \frac{7\pi}{15}$ 22 (a) $\frac{7\pi}{15}$ (b) $\frac{\pi}{2}$

23 (a) $\pi \int_0^4 \left(\frac{4-x}{2}\right)^2 dx = \frac{16\pi}{3}$ (b) $\pi \int_0^2 (4 - 2y)^2\, dy = \frac{32\pi}{3}$

24 $\frac{34\pi}{25}$

EXERCISE 13.10

1 $\frac{1}{2}\int_0^2 \left(x^2 - x\right) dx = \frac{1}{3}$ 2 $\frac{1}{4}\int_{-2}^{2} x^4\, dx = \frac{1}{2}\left[\frac{x^5}{5}\right]_0^2 = 3\frac{1}{5}$

3 $\frac{1}{2}\int_{-2}^{0} x^3\, dx = -2$ 4 D 5 $\frac{1}{9}\int_0^9 \sqrt{x}\, dx = 2$

6 $\frac{1}{7}\int_0^7 \frac{12\, dx}{\sqrt{x+2}} = \frac{24}{7}\left(3 - \sqrt{2}\right)$ 7 $\frac{1}{b-1}\int_1^b \left(1 - 6x + 3x^2\right) dx = 2,\ b = 3$

EXERCISE 13.11

1 (a) $\frac{x^2}{2} + C$ (b) $\frac{x^3}{3} + \frac{x^2}{2} + x + C$ (c) $3x - \frac{x^3}{3} + C$

(d) $x^6 - x^4 + x^2 + C$ (e) $x + C$ (f) $\frac{x^{n+1}}{n+1} + C$

2 (a) $\frac{2}{3}x^{\frac{3}{2}} + C$ (b) $C - \frac{1}{x}$ (c) $x + \frac{2x\sqrt{x}}{3} + \frac{x^2}{2} + C$

(d) $\frac{x^2}{2} - \frac{1}{x} + C$ (e) $\int \left(x^2 + 2 + \frac{1}{x^2}\right) dx = \frac{x^3}{3} + 2x - \frac{1}{x} + C$

(f) $\int \left(1 - 2\sqrt{x} + x\right) dx = x - \frac{4}{3}x^{\frac{3}{2}} + \frac{x^2}{2} + C$

3 C

4 $y = x + \frac{x^2}{2} + x^3 - 6$

5 $y = x + \frac{2x\sqrt{x}}{3} + \frac{2}{3}$ or $y = x + \frac{2}{3}x^{\frac{3}{2}} + \frac{2}{3}$

CHAPTER REVIEW 13

1 (a) -20 (b) $\int_{-2}^{2} \left(x^4 - 4x^3 + 4x^2\right) dx = \frac{512}{15}$

(c) $\left[\frac{(5-2x)^4}{-2\times 4}\right]_{-3}^{1} = 1820$ 2 $10\frac{2}{3}$

3 (a) 0 (b) $4\int_{-3}^{-1} \left(x^3 + 2x^2 + x\right) dx = -26\frac{2}{3}$ (c) 48

4 $y = 4x - 4$, $A = \int_0^2 x^2\, dx - \int_1^2 (4x - 4)\, dx = \frac{2}{3}$

5 $\int_{-1}^{5} \left(4x - x^2 + 5\right) dx = 36$

6 $\int_0^4 \left(8x^2 - 2x^3\right) dx = 42\frac{2}{3}$

7 (a) $x < -2$ (b) $\frac{dy}{dx} = 1 - \frac{16}{x^3}$; $0 < x < 2\sqrt[3]{2}$ (c) $\int_2^4 \left(x + \frac{8}{x^2}\right) dx = 8$

8 $f(x) = \frac{x^3}{3} + x + \frac{35}{24} - \frac{1}{4x}$

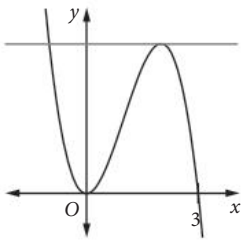
9 (a) 6.75 (b) $\int_{1.5}^{2} \left(3x^2 - x^3 - 3 + x\right) dx = 1\frac{17}{64}$

10 $\frac{19}{6}$

11 (a) $\frac{4\pi}{3}$ (b) $\frac{98\pi}{3}$ (c) 40π

12 (a) $0 < x < 4$ (b) $3\frac{2}{3}$ (c) $\tan\theta = 2$, $\theta = 63°26'$

13 (a)



(b) $\frac{dy}{dx} = 6x - 3x^2$; at $x = 2$, $\frac{dy}{dx} = 0$; $x = 2$, $y = 4$, $A(2, 4)$
For B, $y = 4$; from the sketch $x = -1$.
Check by substitution: LHS = 3 + 1 = 4 = RHS, so $B(-1, 4)$.

(c) $\int_{-1}^{2} \left(4 - 3x^2 + x^3\right) dx = 6.75$

14 (a) 2.2 (b) $\frac{32}{15}$

15 (a) $62\,\text{m}^2$, $64\,\text{m}^2$

(b) Rotating about the y-axis, the subintervals for the trapezia would not be equal and the curve is not the same above and below the axis. Calculating the volume for each region and taking the average would give an approximate volume:

$$V = \pi \int_0^5 x^2\,dy \approx \pi\left[\frac{2.5}{2}\left(12^2 + 8^2\right) + \frac{4-2.5}{2}\left(8^2 + 4^2\right) + \frac{5-4}{2}\left(4^2 + 0^2\right)\right] \approx 288\pi \text{ for one side.}$$

CHAPTER 14

EXERCISE 14.1

1 (a) $2^3 \times 2^4 \times 2^6 = 2^{13}$ (b) $a^{11}b^6$ (c) $m^{11}n^6p$ (d) 2^{6n}
(e) x^5y^9 (f) $\frac{1}{2n^3}$ (g) x^9y^7 (h) a^2b^3

2 D

3 (a) $\frac{m^2n^2p^{-2}}{m^3n^3p^6} = \frac{1}{mnp^8}$ (b) $\frac{q^4}{pr^4}$ (c) $\frac{2x^5}{y^2z}$ (d) 2^{n+3}

4 (a) 2^{-2} (b) 2^{-4} (c) 2^{-5} (d) 2^{-3} (e) 2^{-6} (f) 2^{-7}
(g) 2^{-2} (h) 2^{-9}

5 (a) 10^2 (b) 10^1 (c) 10^0 (d) 10^{-1} (e) 10^{-2} (f) 10^{-3}
(g) 10^{-3} (h) 10^{-5}

6 (a) $\frac{3^{2n} \times 5^{4n-2}}{3^{n-1} \times 5^{n-1}} = 3^{n+1} \times 5^{3n-1}$ (b) $\frac{1}{x^2} - \frac{1}{y^2}$ (c) 2^8
(d) $\frac{1}{x^4} + \frac{2}{x^3} + \frac{1}{x^2}$ (e) $x - 3$ (f) $x - 2$ (g) 3×2^7 (h) $\frac{5^{2n+3}}{3^{n-3}}$

7 (a) correct (b) correct (c) incorrect (d) incorrect

8 (a) $\frac{1}{a^2} - b^2$ (b) $2 + xy + \frac{1}{xy}$ (c) $\frac{1}{x^4} - \frac{1}{y^4}$
(d) $3 - a^2b - \frac{2}{a^2b}$ (e) $\frac{1}{ab}$ (f) $\frac{1}{y}$

EXERCISE 14.2

1 (a) 16 (b) $\frac{1}{7}$ (c) 27 (d) 9 (e) 2 (f) 64 (g) 6 (h) 5

2 C

3 (a) $6^{\frac{1}{2}}$ (b) $2^{\frac{5}{8}}$ (c) $2^{\frac{4}{3}}$ (d) $3^{\frac{11}{8}}$

4 (a) $x^{\frac{13}{6}}$ (b) $\frac{b^3}{a^2}$ (c) $x - \frac{1}{x}$ (d) $\frac{1}{y}$ (e) $x - y$ (f) $x^{\frac{2}{3}}y$
(g) $6^{\frac{1}{2}}$ (h) 1

5 (a) incorrect (b) correct (c) correct (d) correct

EXERCISE 14.3

1 (a) 3 (b) 4 (c) -5 (d) $\pm\frac{1}{9}$ (e) $1\frac{2}{3}$ (f) 2.5 (g) 1.5
(h) 2 (i) 1 (j) -2 (k) 6 (l) -3

2 C

3 (a) -3 (b) 3 (c) $\frac{-1}{2}$ (d) $\frac{-1}{2}$ (e) 0 (f) -3 (g) $1\frac{3}{4}$
(h) 3 (i) $1\frac{2}{3}$ (j) -2, 0 (k) 0, 2 (l) -2

EXERCISE 14.4

1 (a) 2 (b) $\frac{1}{2}$ (c) 7 (d) 1 (e) -3 (f) -1 (g) 10 (h) $\frac{1}{2}$

2 A

3 (a) 4 (b) $2\frac{1}{2}$ (c) 3 (d) -4

4 (a) 7 (b) 1 (c) 8 (d) 1 (e) 3 (f) 0 (g) 1 (h) 12

5 (a) correct (b) incorrect (c) correct (d) correct

6 (a) 2 (b) x (c) 3 (d) 2 (e) $6\log_{10}5$ (f) 3
(g) $\log_{10}\left(2 + \sqrt{5}\right)$ (h) 2

7 (a) $x - y$ (b) $2x - 2y$ (c) $y - x + 1$ (d) $x + 3y$
(e) $x + 3y - 1$ (f) $y - 2x + 2$ (g) $y - x + 2$ (h) $3y - 2$
(i) $2y - x$ (j) $x + y - 1$ (k) $4y$ (l) $x + 2y - 1$

8 (a) 2.322 (b) 2.262 (c) 1.861 (d) 0.793 (e) 2.524
(f) 0.774 (g) 0.683 (h) 3.322

EXERCISE 14.5

1 (a) 27 (b) 9 (c) 216 (d) 7 (e) $\frac{1}{125}$ (f) 4 (g) 4
(h) $\sqrt{3}$ (i) 3.5 (j) $49\sqrt{7}$ (k) 16 (l) $16\sqrt{2}$

2 (a) 8 (b) 2 (c) 100 (d) 0.5 (e) 10 (f) ± 10

3 (a) $y = \frac{x}{x-1}$ (b) $y = 10x^{\frac{3}{2}}$ (c) $y = 3x^{\frac{3}{2}}$ (d) $y = \frac{\sqrt{2}}{x}$
(e) $y = 25x$ (f) $y = 5x^3$ (g) $x^2y^3 = 1$ (h) $y = \frac{10^x - 1}{10^x + 1}$

4 (a) 30 (b) 10 (c) 10 (d) $\sqrt[3]{16}$ (e) $1\frac{1}{9}$ (f) $2\frac{1}{4}$

5 (a) incorrect (b) correct (c) correct (d) correct

6 (a) 2.807 (b) 2.631 (c) 0.431 (d) -0.756 (e) 1.699
(f) 2.096 (g) 1.723 (h) 1.161

7 (a) $x > 0.43$ (b) $x \geq -1.47$ (c) $x < -1.46$ (d) $x < -3$
(e) $x \geq 2.32$ (f) $x < 0.29$ (g) $x > -4.92$ (h) $x \leq 3.38$

8 B **9** $A = P \times 10^{bt}$ **10** $y = ax^n$

11 $x = 2^y$ **12** $\log x = 2\log a + \frac{3\log b}{2} + \frac{\log c}{2}$ **13** 1.1

14 $t = \frac{1}{4}\log_e\frac{y}{a}$ **15** (a) $2p$ (b) $\frac{1}{2p}$

16 $\log_a 2 = \log_b 16$, $\log_a 2 = \frac{\log_u 16}{\log_u b}$, $\log_a 2 = \frac{4\log_u 2}{\log_u b}$, $\log_a b = 4$, $b = a^4$

17 (a) $2 = 1.07^n$, $n = \frac{\log_{10} 2}{\log_{10} 1.07} = 10.24$ years
(b) $n = \frac{\log_{10} 4}{\log_{10} 1.07} = 20.49$ years (c) 26.48 years

18 (a) $2 = 1.005^n$, $n = \frac{\log_{10} 2}{\log_{10} 1.005} = 139$ months
(b) 278 months (c) 359.2 months

19 $5 = 1.0075^n$, 216 months

20 (a) $y_n = 5 \times (1 + 0.05)^n$ (b) $z_n = 4 \times (1 + 0.07)^n$
(c) $4 \times 1.07^n = 5 \times 1.05^n$, $\left(\frac{1.07}{1.05}\right)^n = 1.25$,
$n = \frac{\log_{10} 1.25}{\log_{10} 1.07 - \log_{10} 1.05}$, $n = 12$ years

EXERCISE 14.6

1 (a) $4e^{4x}$ (b) $e^{\frac{x}{2}}$ (c) $4e^{4x} - 3e^{3x}$ (d) $6e^{3x} - e^{-x}$
(e) $12e^{3x} + 2e^{-2x}$ (f) $3.2e^{3.2x} - 1.6e^{1.6x}$ (g) $3e^x + 2e^{-x}$
(h) $8e^{2x} - e^{-2x}$ (i) $3e^{3x} - e^x$

2 B

3 (a) $2xe^{3x} + 3x^2e^{3x} = xe^{3x}(2 + 3x)$ (b) $(1 - 2x)e^{-x}$
(c) $(2x^2 + 4x + 3)e^{2x}$ (d) $(1 - 2x)e^{-2x}$ (e) $x^2e^{-x}(3 - x)$
(f) $3x^2 - e^{2x} - 2xe^{2x}$ (g) $2x - 3x^2e^{2x} - 2x^3e^{2x}$ (h) $\frac{(x-2)e^x}{x^3}$
(i) $\frac{(3x-1)e^{3x}}{x^2}$ (j) $\frac{x^2(3-x)}{e^x}$ (k) $\frac{(4x-5)e^{4x}}{(x-1)^2}$ (l) $\frac{\sqrt{x}e^x - \frac{e^x}{2\sqrt{x}}}{x} = \frac{(2x-1)e^x}{2x\sqrt{x}}$

4 (a) $2e^{2x+3}$ (b) $2(x-1)e^{x^2-2x}$ (c) $-9x^2e^{-x^3}$ (d) $6e^{3x-1}$
(e) $3e^{3x-1} + 4e^{4x+2}$ (f) $\frac{1}{2\sqrt{x}}e^{-x} - \sqrt{x}e^{-x} = \frac{(1-2x)e^{-x}}{2\sqrt{x}}$
(g) $12xe^{2x^2}$ (h) $6e^{2x-1}$ (i) $\left(1 + 2x^2\right)e^{x^2}$

5 (a) correct (b) incorrect (c) correct (d) correct

6 $\frac{dx}{dt} = (6 + 5t)e^{5t}$, $\frac{d^2x}{dt^2} = (35 + 25t)e^{5t}$;
$(35 + 25t)e^{5t} - 10\left((6 + 5t)e^{5t}\right) + 25(1 + t)e^{5t} = 0$

7 $y=x+1$ 8 $y=1-x$

9 $y=2e^2x-e^2$ (a) $(0.5,0)$ (b) $(0,-e^2)$

10 $y=3-x,\ y=x+3$ 11 $\frac{dV}{dn}=p\times re^{rn}=r\times Pe^{rn}=rV$

12 $\frac{dy}{dt}=500\left(-(-0.2)e^{-0.2t}\right)=100e^{-0.2t}$

13 $f'(x)=\frac{1}{\sqrt{2\pi}}\times\frac{-2x}{2}e^{\frac{-x^2}{2}}=\frac{-x}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}$, $f'(0)=\frac{0}{\sqrt{2\pi}}e^0=0$

EXERCISE 14.7

1 (a) $\frac{e^{2x}}{2}$ (b) $\frac{e^{5x}}{5}$ (c) $\frac{-5e^{-0.4x}}{2}$ (d) $2e^{2.5x}$ (e) $e^x-\frac{e^{-3x}}{3}$ (f) $e^{-x}-\frac{e^{-2x}}{2}$

2 (a) $-e^{-x}+C$ (b) $2e^{\frac{x}{2}}+C$ (c) $C-\frac{e^{-3x}}{3}$

(d) $-e^{-t}-t+C$ (e) $\frac{e^{2u}}{2}+\frac{u^3}{3}+C$ (f) $\frac{-2e^{-2.5}}{5}+\frac{5e^{0.4x}}{2}+C$

3 (a) $\frac{e^2-1}{e}$ (b) $\frac{e^4-1}{2}$ (c) $\frac{2\left(e^2-1\right)}{e^{\frac{3}{2}}}$ (d) $\frac{2\left(e^{1.5}-1\right)}{3}$

(e) $\frac{e^2-1}{2e}$ (f) $\frac{e^3-1}{3}$ (g) $\frac{\left(e^2-1\right)^2}{2e^2}$ (h) 0 (i) 2.904

4 (a) correct (b) correct (c) correct (d) incorrect

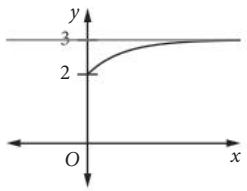
EXERCISE 14.8

1 $-e$

2 $\frac{dy}{dx}=\frac{(2-x)e^{-0.5x}}{2}$: $(2,\frac{2}{e})$, maximum (a) $x>0$ (b) $x<2$

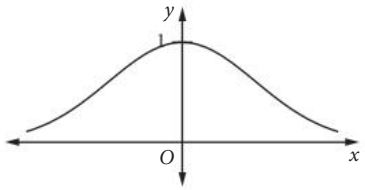
3 (a) $f(0)=2,\ f'(0)=1$ (b) $f'(x)=e^{-x}>0$ for all x (c) 3

(d)



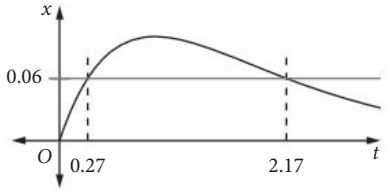
4 (a) $f'(x)=-2xe^{-x^2}$ (b) (i) $x=0$ (ii) $x<0$ (iii) $x>0$

(c)



5 (a) $\frac{dx}{dt}=(0.3-0.33t)e^{-1.1t}$, $\frac{d^2x}{dt^2}=(0.363t-0.66)e^{-1.1t}$:
maximum concentration 0.1 units after 0.91 hours

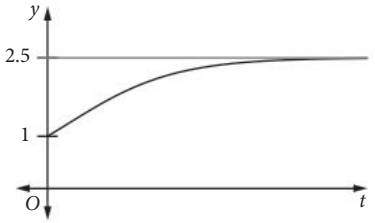
(b)



(c) $0.27<t<2.17$

6 (a) correct (b) incorrect (c) correct (d) correct

7



(a) $f'(t)=\frac{-5\times\left(-3e^{-t}\right)}{\left(2+3e^{-t}\right)^2}=\frac{15e^{-t}}{\left(2+3e^{-t}\right)^2}>0$ because $e^{-t}>0$ and

$\left(2+3e^{-t}\right)^2>0$ for all t

(b) 2.5 (c) $1\leq f(t)<2.5$

8 (a) $A=2xe^{-x^2}$, $\frac{dA}{dx}=2e^{-x^2}\left(1-2x^2\right)$, $\frac{d^2A}{dx^2}=4xe^{-x^2}\left(2x^2-3\right)$:
$x=\pm\frac{1}{\sqrt{2}}$ from symmetry take $x=\frac{1}{\sqrt{2}}$: maximum area when
$x=\frac{1}{\sqrt{2}}$ (b) maximum area $=\sqrt{2}e^{-0.5}\approx0.86$ units^2

9 e^2-e^{-2}

10 (a) e^2-1 (b) $y=e^2x-e^2$

(c) $A=\int_0^2 e^x\,dx-\int_1^2\left(e^2x-e^2\right)dx=\frac{e^2}{2}-1$

(d) $2e^2-(e^2-1)=e^2+1$

11 $e^2+e^{-2}-2$ 12 $\int_0^1\left(e^{0.5x}-e^{-0.5x}\right)dx=2\left(e^{0.5}+e^{-0.5}-2\right)\approx0.511$

13 $V=\pi\int_0^1 e^{-2x}\,dx=\frac{\pi\left(e^2-1\right)}{2e^2}$ 14 $\frac{\pi}{2}\left(e^3-e\right)$

15 $\frac{\pi\left(e^4-1\right)}{e^2}$ 16 $V=\pi\int_{-1}^1\left(e^{2x}+2+e^{-2x}\right)dx=\pi\left(e^2+4\right)$

17 1.54 18 $\frac{1}{2}\left(e^{0.4}+2\left(2e^{0.8}+3e^{1.2}+4e^{1.6}\right)+5e^2\right)\approx53.44$

19 $\frac{0.2}{2}\left(0+2\left(0.2e^{-0.2}+0.4e^{-0.4}+0.6e^{-0.6}\right)+0.8e^{-0.8}\right)\approx0.188$

20 $\frac{d}{dx}\left((x-1)e^x\right)=xe^x$, $\int_{-1}^2 xe^x\,dx=\left[(x-1)e^x\right]_{-1}^2=e^2+2e^{-1}$

21 (a) $1-e^{-a}$ (b) 1 (c) $V=\frac{\pi}{2}\left(1-e^{-2a}\right)$, $\frac{\pi}{2}$

22 (a) $\frac{e^2-1}{2}$ (b) $\frac{e^2-1}{2e^2}$ (c) $\frac{e^{10}-e}{8}$ (d) 0

EXERCISE 14.9

1 (a) $\frac{1}{x}$ (b) $\frac{2}{x}$ (c) $\frac{2}{x}$ (d) $\frac{3}{3x-5}$ (e) $\frac{1}{x}$ (f) $2x-\frac{4}{4x-1}$

2 (a) $x>-\frac{2}{3}$, $\frac{3}{3x+2}$ (b) all real x, $\frac{2x}{x^2+1}$ (c) $x\neq2$, $\frac{2}{x-2}$

(d) $x>-\frac{3}{4}$, $\frac{4}{4x+3}$ (e) $x>0$, $\frac{1}{2x}$ (f) $x>0$, $\frac{2\sqrt{x+1}}{2x\left(\sqrt{x}+1\right)}$

3 C

4 (a) $\ln x+1$ (b) $x^2(3\ln x+1)$ (c) $\ln(x+2)+1$

(d) $2x\ln2x+x+\frac{1}{x}$ (e) $2\ln x+2-\frac{5}{x}$ (f) $\frac{e^x}{x}\left(x\ln x+1\right)$

(g) $\frac{e^{2x}}{x}\left(2x\ln2x+1\right)$ (h) $\frac{\log_e x-1}{(\log_e x)^2}$ (i) $\frac{1-\log_e x}{x^2}$ (j) $\frac{1-x\log_e x}{xe^x}$

(k) $\frac{x\times\frac{2x}{x^2+1}-\log_e\left(x^2+1\right)}{x^2}=\frac{2x^2-\left(x^2+1\right)\log_e\left(x^2+1\right)}{x^2\left(x^2+1\right)}$

(l) $e^x\log_e\left(e^x+1\right)+\frac{e^{2x}}{e^x+1}$

5 (a) $\frac{2x-2}{x^2-2x}$ (b) 1 (c) $\frac{3}{x}$ (d) $\frac{6}{x^2-9}$ (e) $\frac{2x}{x^2-1}$

(f) $\frac{4}{x}$ (g) 1 (h) $2x(2\log_e x+1)$ (i) $\frac{3}{2x}$ (j) $\frac{1}{x\log_e x}$

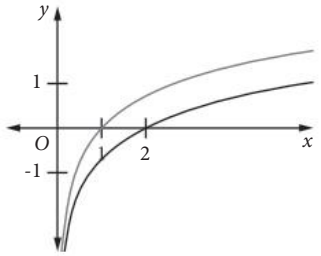
(k) $(\log_e x+1)e^{x\log_e x}$ (l) $\frac{e^x}{e^x+1}-\frac{e^x}{e^x-1}=\frac{2e^x}{1-e^{2x}}$

6 $\frac{dy}{dx}=\frac{2x}{x^2+1}$, $m=0.6$

7 (a) $\frac{1}{x}$ (b) $\frac{-1}{x^2}$ (c) 0.5 (d) -0.25

8 $y=x-1,\ y=1-x$

9 (a)



$\frac{d}{dx}\left(\log_e x\right)=\frac{1}{x}$

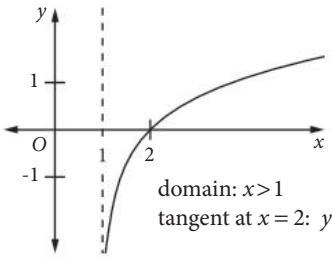
$\frac{d}{dx}\left(\log_e\frac{x}{2}\right)=\frac{1}{\frac{x}{2}}\times\frac{1}{2}=\frac{1}{x}$

The gradients are equal.

(b) $x-2y-2=0$

10 $\log_e(e^x)=x\log_e e=x$, so $y=\log_e(e^x)$ is the same as $y=x$;
gradient = 1

11



domain: $x>1$

tangent at $x=2$: $y=x-2$

12 (a) $x=\ln2\approx0.693$ (b) $x=\frac{\log_e5}{3}\approx0.536$

(c) $x=\frac{\log_e7-3}{2}\approx-0.527$ (d) $x=\pm\sqrt{1+\ln10}\approx\pm1.817$

13 $y' = 2e^{2x} - 8e^{-2x}$, $2(e^{2x} - 4e^{-2x}) = 0$: $(e^x - 2e^{-x})(e^x + 2e^{2x}) = 0$:
$e^x - 2e^{-x} = 0$, $e^{2x} = 2$, $x = \frac{\ln 2}{2}$; minimum value = 4

EXERCISE 14.10

1 (a) $2\ln x + C$ (b) $\ln(x+1) + C$ (c) $\ln(2x+1) + C$
(d) $\frac{1}{2}\ln\left(x^2 - 4\right) + C$ (e) $\frac{1}{2}\ln(2x-1) + C$ (f) $\ln\left(4 + e^x\right) + C$
(g) $\frac{1}{4}\ln\left(x^4 + 1\right) + C$ (h) $-\frac{1}{2}\ln\left(4 - e^{2x}\right) + C$

2 C

3 $y = \log_e(2x+1)$

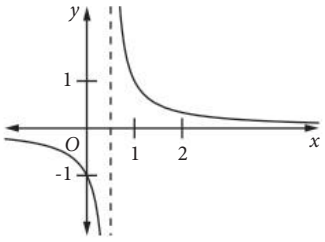
4 (a) correct (b) correct (c) incorrect (d) correct

5 (a) $\ln 2$ (b) $2\ln 2$ (c) $\frac{1}{2}\log_e 5$ (d) $\ln 5$ (e) $\ln 3$
(f) $3\ln 2.5$ (g) $\frac{1}{2}\log_e \frac{7}{3}$ (h) $3\log_e \frac{11}{3}$

6 (a) $\frac{26 + 3\ln 3}{3}$ (b) 0 (c) $6 - \ln 2$ (d) $\ln 2 + 0.25$ (e) $\ln 2 + 2.5$
(f) $\int_1^2 \left(x^2 - \frac{2}{x} + \frac{1}{x^4}\right) dx = \left[\frac{x^3}{3} - 2\log_e x - \frac{1}{3x^3}\right]_1^2 = \frac{21}{8} - 2\log_e 2$
(g) $e^3 - e + \log_e 3$ (h) $\frac{14}{3} - 2\ln 2$

EXERCISE 14.11

1

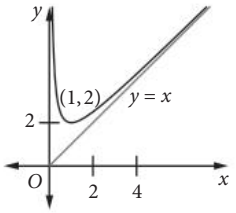


area $= \int_1^4 \frac{dx}{2x-1} = \frac{\ln 7}{2}$

2 $\int_2^4 \frac{x\,dx}{x^2+1} = \frac{1}{2}\left[\ln\left(x^2+1\right)\right]_2^4 = \frac{1}{2}\ln\frac{17}{5}$

3 (a) correct (b) correct (c) incorrect (d) correct

4

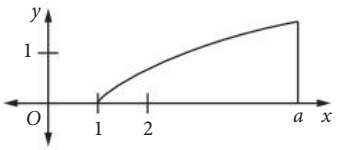


$y' = \frac{x^2-1}{x^2}$, $y'' = \frac{2}{x^3}$.
turning point $(1, 2)$
area $= \int_1^2 \left(x + \frac{1}{x}\right) dx = 1.5 + \ln 2$

5 B **6** $\frac{1+e}{2}$ **7** $\int_0^1 \frac{x\,dx}{x^2+1} = \frac{1}{2}\left[\log_e\left(x^2+1\right)\right]_0^1 = \frac{\log_e 2}{2}$

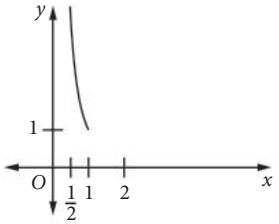
8 $\pi\int_1^3 \frac{6}{x}\,dx = 6\pi\log_e 3$ **9** $\pi\int_6^{11} \frac{dx}{x-2} = \pi\log_e 2.25$ units^3

10 (a)



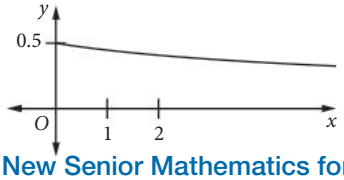
area $= a\ln a - a + 1$
(b) $V = \pi\int_0^{\log_e a} e^{2y}\,dy = \pi\left(a^2 - 1\right)$

11



$V = \pi\ln 4$

12



$V = \pi\ln 2.25$

13 $4\pi\ln 3$ **14** $\pi\left(\frac{2}{15} + \log_e \frac{5}{3}\right)$

15 $2\ln 2 + 1.5\ln 3 \approx 3.034$ **16** 4.041

17 (a) $\frac{\log_e 3}{2}$ (b) $\frac{\log_e 5}{2}$ (c) $\frac{\log_e 3}{2}$ (d) 0

CHAPTER REVIEW 14

1 (a) 1 (b) $x < 5.5$ (c) $x \geq 2.26$

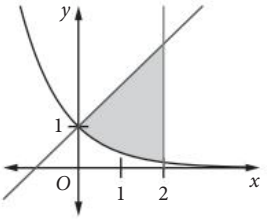
2 (a) 3 (b) $3\log_a y$ (c) $\log_{10} 3$ (d) 0

3 (a) $y = a^x$ (b) $y = \frac{100}{x}$

4 (a) $(x^2 + 4x + 2)e^x$ (b) $\frac{2e^{-x}(1 - x\ln x)}{x}$ (c) $\frac{e^x}{1+e^x}$
(d) $\frac{2x+2}{x^2+2x}$ (e) $e^{-3x}(3 - 7x - 3x^2)$ (f) $\frac{1}{2\sqrt{x}}e^{\sqrt{x}} + \frac{1}{2x}$

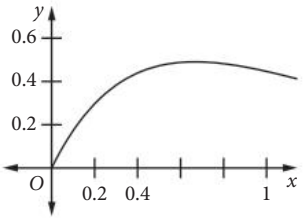
5 (a) 0 (b) 24.92 (c) $e^3 + \ln 3 - e$

6



area $= \int_0^2 \left(x + 1 - e^{-x}\right) dx = \frac{3e^2+1}{e^2}$

7 (a) 0.4905 at $x = \frac{2}{3}$
(b) $f(0) = 0$, $f(0.5) = 0.472$, $f(1) = 0.446$

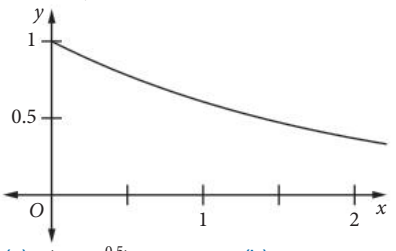


(c) 0.348

8 $\frac{d}{dt}(\theta_0 e^{-kt}) = -k\theta_0 e^{-kt} = -k\theta$

9 (a) \$10 000 (b) \$3012
(c) $\frac{dV}{dt} = -0.2V$ (i) \$602.39 per year (ii) \$1000 per year
(d) 11.5 years

10



(a) $2(1 - e^{-0.5}) \approx 0.7868$ (b) 0.7870

11 (a) $\frac{d}{dx}\left(xe^x\right) = e^x + xe^x$
(b) $\int xe^x\,dx = \int\left(\frac{d}{dx}\left(xe^x\right) - e^x\right)dx = xe^x - e^x + C$ (c) $\frac{e^4+1}{2e}$

CHAPTER 15

EXERCISE 15.1

1 (a) $\frac{\pi}{6}$ (b) $\frac{5\pi}{4}$ (c) $\frac{2\pi}{5}$ (d) $\frac{7\pi}{6}$ (e) $\frac{7\pi}{4}$ (f) $\frac{5\pi}{8}$
(g) $\frac{11\pi}{6}$ (h) $\frac{4\pi}{5}$

2 (a) 135° (b) 157.5° (c) 216° (d) 270° (e) 100° (f) 165°
(g) 324° (h) 247.5°

3 (a) 28° 39′ (b) 104° 17′ (c) 171° 53′ (d) 244° 5′
(e) 155° 51′ (f) 196° 18′ (g) 300° 14′ (h) 273° 59′

4 (a) 0.7330 (b) 1.2915 (c) 1.8326 (d) 2.8623 (e) 0.8404
(f) 3.8397 (g) 2.4120 (h) 1.2590

5 (a) incorrect (b) correct (c) correct (d) incorrect

6 (a) $\frac{2\pi}{3}$ (b) $\frac{7\pi}{4}$ (c) $\frac{7\pi}{6}$ (d) $\frac{3\pi}{4}$ (e) $\frac{13\pi}{6}$ (f) $\frac{9\pi}{4}$
(g) $\frac{8\pi}{3}$ (h) 4π (i) $-\frac{\pi}{6}$ (j) $-\frac{3\pi}{4}$ (k) $\frac{5\pi}{2}$ (l) $-\pi$ (m) $\frac{\pi}{12}$
(n) $\frac{\pi}{8}$ (o) $\frac{23\pi}{12}$ (p) $-\frac{3\pi}{8}$

7 (a) 150° (b) 300° (c) 225° (d) 540° (e) 405° (f) 630°
(g) 600° (h) -180° (i) -270° (j) -210° (k) -135° (l) -360°
(m) 22.5° (n) 75° (o) 202.5° (p) -105°

EXERCISE 15.2

1 107° 26′ 2 $15 \times 70\pi \div 180 = 18.3$ cm 3 C

4 (a) $\angle AOB = \frac{2 \times 240}{400}$ rad $= 1.2$ rad (b) 24 cm
(c) $AB^2 = 800 - 800\cos 1.2$, $AB = 22.6$ cm

5 30.6 cm

6 $\cos\angle POA = \frac{5}{8}$, $\angle AOB = 1.79$ rad
major arc $= 10\pi - 5 \times 1.79 = 22.5$ cm

7 (a) $\cos\angle AOB = -0.28$, $\angle AOB = 106° 16'$ (b) 18.5 cm (c) 44.7 cm^2

8 16 min $= 96°$ (a) 33.5 cm (b) $\frac{20\sin 96°}{\sin 42°} = 29.7$ cm

9 (a) correct (b) incorrect (c) correct (d) correct

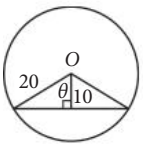
10 area $= \frac{1}{2} \times 9 \times \frac{11\pi}{6} = \frac{33\pi}{4}$ cm^2

11 arc $= 16 \times \frac{140\pi}{180} = 39.1$, chord $= \frac{16\sin 140°}{\sin 20°} = 30.1$, difference $= 9$ cm

12 (a) $\frac{125\pi}{8}$ cm^2 (b) $\left(10 + \frac{25\pi}{4}\right)$cm

13 6πcm^2

14 small sector $= \frac{\pi r^2}{6}$, large sector $= \frac{5\pi r^2}{6}$, triangle $= \frac{\sqrt{3}r^2}{4}$;
ratio is $\left(2\pi - 3\sqrt{3}\right):\left(10\pi + 3\sqrt{3}\right)$

15 

$\cos\theta = 0.5$, $\theta = \frac{\pi}{3}$
major sector $= \frac{1}{2} \times 20^2 \times \frac{4\pi}{3} = \frac{800\pi}{3}$ m^2,
triangle $= \frac{1}{2} \times 20^2 \times \sin\frac{2\pi}{3} = 100\sqrt{3}$ m^2
area grazed $= \frac{800\pi}{3} + 100\sqrt{3} = 1011$ m^2

16 circumference $= \frac{80\pi}{9} = 27.9$ cm

17 (a) 63.1 m (b) 163.6 m^2

18 (a) $\frac{25}{180} \times \pi \times 40 = 17.5$ cm (b) $\frac{40\sin 25°}{\sin 77.5°} = 17.3$ cm

19 area $= 2 \times \frac{1}{2} \times 100(1.287 - \sin 1.287) = 32.7$ cm^2

20 triangle area $= \frac{1}{2} \times 100 \times \frac{\sqrt{3}}{2} = 25\sqrt{3}$,
sector area $= \frac{1}{2} \times 25 \times \frac{\pi}{3} = \frac{25\pi}{6}$
area $= 25\left(\sqrt{3} - \frac{\pi}{2}\right) = 4.0$ cm^2

21 (a) $\cos\angle LKO = \frac{3r^2 + r^2 - r^2}{2\sqrt{3}r^2} = \frac{\sqrt{3}}{2}$: $\angle LKO = 30°$ (b) $\frac{3\sqrt{3}r^2}{4}$
(c) $LM = r\sqrt{3}$ (d) $\frac{4\pi r^2 - 3\sqrt{3}r^2}{12}$
(e) area $= \frac{4\pi r^2 - 3\sqrt{3}r^2}{12} - \left(\frac{\pi r^2}{2} - \frac{3\sqrt{3}r^2}{4}\right) = \frac{3\sqrt{3}r^2 - \pi r^2}{6}$

EXERCISE 15.3

1 (a) $\sin x$ (b) $\sin x$ (c) $-\tan x$ (d) $-\cos x$ (e) $-\sin x$ (f) $\tan x$

2 A

3 (a) incorrect (b) correct (c) incorrect (d) correct

4 (a) 0.2 (b) -0.2 (c) -0.2 (d) 0.2 (e) -0.2 (f) 5

5 (a) $\frac{1}{t}$ (b) t (c) $-t$ (d) $-t$ (e) $-\frac{1}{t}$ (f) t

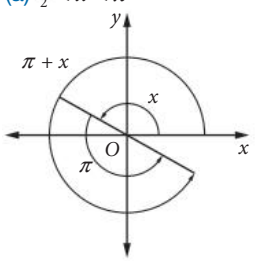
6 (a) $\frac{1}{c}$ (b) c (c) $-c$ (d) c (e) $\frac{1}{c}$ (f) $-c$

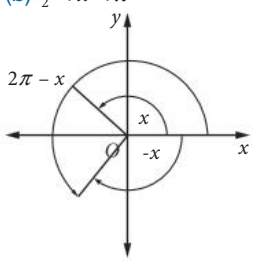
7 (a) 1 (b) $-\frac{1}{2}$ (c) $-\frac{1}{\sqrt{3}}$ (d) -1 (e) $-\sqrt{2}$ (f) $-\sqrt{3}$
(g) 1 (h) $\frac{\sqrt{3}}{2}$

8 (a) 0 (b) $-\frac{\sqrt{3}}{2}$ (c) 1 (d) $\frac{1}{\sqrt{3}}$ (e) $-\frac{1}{2}$ (f) $-\sqrt{2}$ (g) 0 (h) $-\frac{1}{2}$

9 (a) -1 (b) $-\sqrt{3}$ (c) -2 (d) -1 (e) -1 (f) $\frac{\sqrt{3}}{2}$
(g) $-\frac{\sqrt{3}}{2}$ (h) $-\frac{2}{\sqrt{3}}$

10 (a) 0 (b) $\frac{1}{2}$ (c) 1 (d) $\frac{1}{\sqrt{3}}$ (e) $\frac{2}{\sqrt{3}}$ (f) 0 (g) $\frac{\sqrt{3}}{2}$ (h) -1

11 (a) $\frac{\pi}{2} < x < \pi$ 

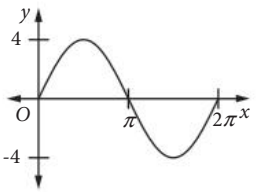
(b) $\frac{\pi}{2} < x < \pi$ 

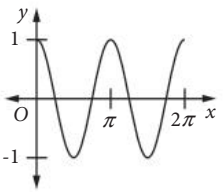
12 (a) positive (b) negative (c) negative (d) negative
(e) positive (f) negative

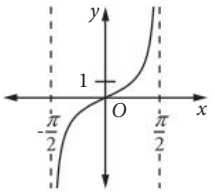
13 (a) $\frac{4\pi}{3}, \frac{5\pi}{3}$ (b) $\frac{3\pi}{4}, \frac{7\pi}{4}$ (c) $\frac{\pi}{6}, \frac{5\pi}{6}$ (d) π (e) $\frac{\pi}{6}, \frac{7\pi}{6}$
(f) $\frac{3\pi}{4}, \frac{5\pi}{4}$ (g) $\frac{\pi}{4}, \frac{5\pi}{4}$ (h) $0, \pi, 2\pi$ (i) $\frac{2\pi}{3}, \frac{4\pi}{3}$ (j) $\frac{\pi}{3}, \frac{2\pi}{3}$
(k) $\frac{2\pi}{3}, \frac{5\pi}{3}$ (l) $\frac{\pi}{4}, \frac{5\pi}{4}$

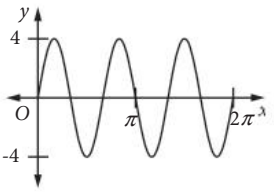
14 (a) $\sin x$ (b) $\cot x$ (c) $-\cos x$ (d) $-\tan x$

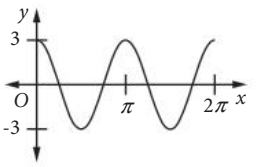
EXERCISE 15.4

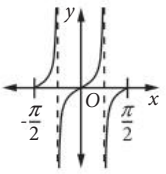
1 (a) period 2π, amplitude 4


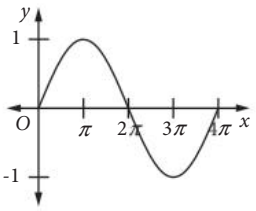
(b) period π, amplitude 1


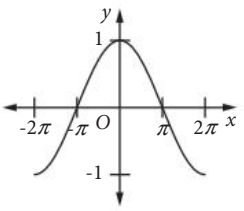
(c) period π, no amplitude


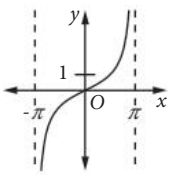
(d) period $\frac{2\pi}{3}$, amplitude 4


(e) period π, amplitude 3


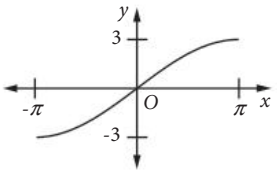
(f) period $\frac{\pi}{2}$, no amplitude


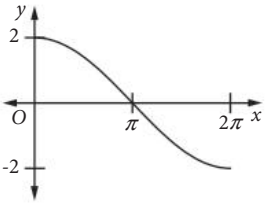
(g) period 4π, amplitude 1


(h) period 4π, amplitude 1


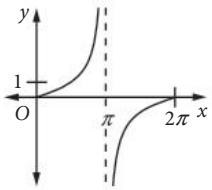
(i) period 2π, no amplitude


2 B

3 (a) period 4π, amplitude 3


(b) period 4π, amplitude 2


(c) period 2π, no amplitude



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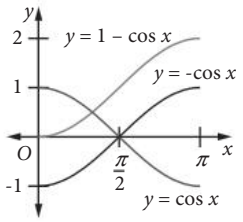

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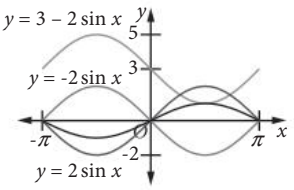
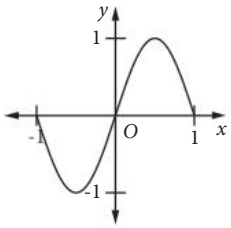
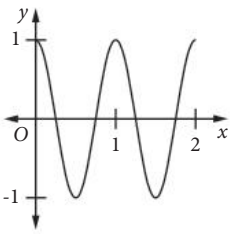


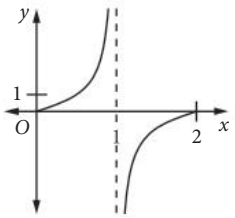
6 (a) period 2, amplitude 1



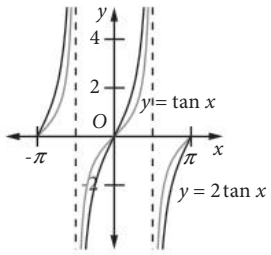
(b) period 1, amplitude 1



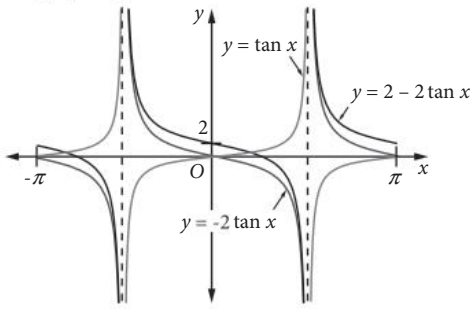
(c) period 2, no amplitude



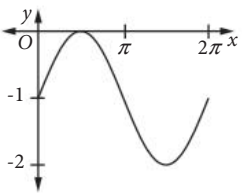
7 (a)



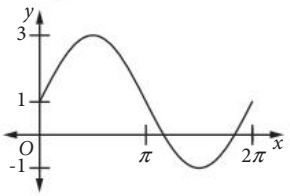
(b)–(c)



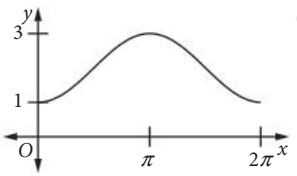
8 (a)



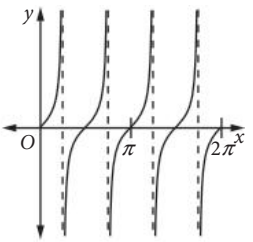
(b)



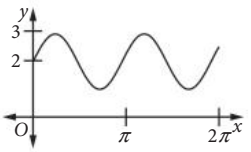
(c)



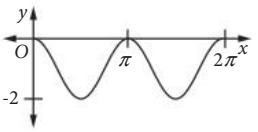
(d)



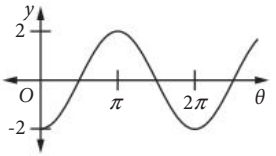
(e)



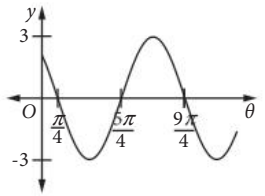
(f)



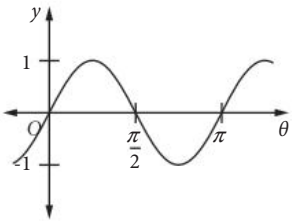
9 (a)



(b)



(c)



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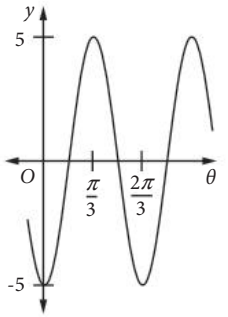
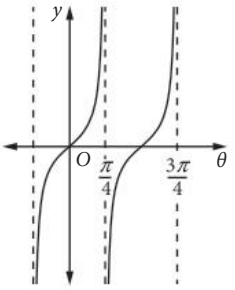
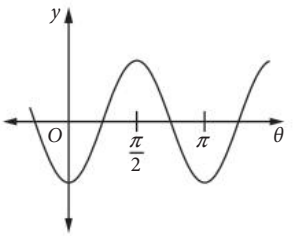


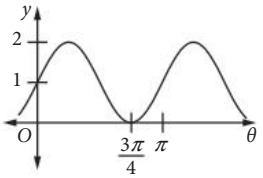
(e)



(f)



10 (a)



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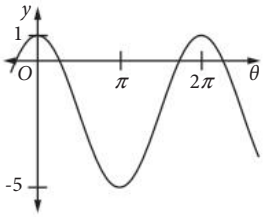
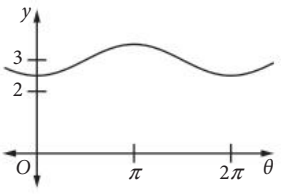
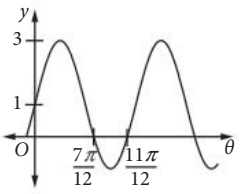


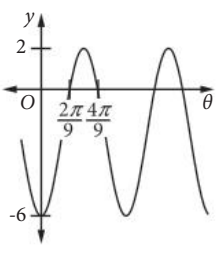
(c)



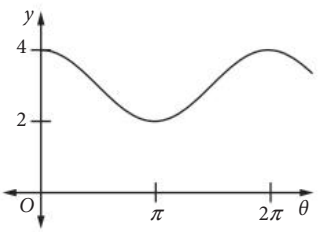
(d)

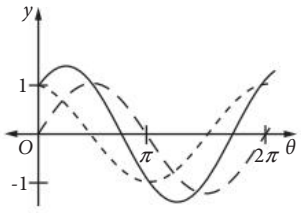


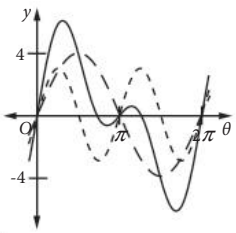
(e)

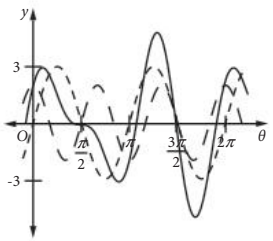


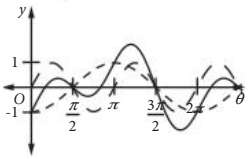
(f)

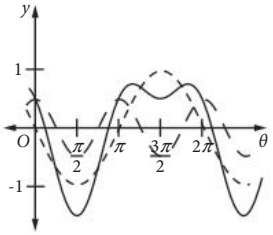


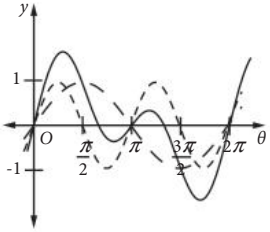
11 (a) 

(b) 

(c) 

(d) 

(e) 

(f) 

12 Answers will vary.

EXERCISE 15.5

1 (a) $\frac{5\pi}{4}, \frac{7\pi}{4}$ (b) $\frac{\pi}{6}, \frac{11\pi}{6}$ (c) $\frac{\pi}{4}, \frac{5\pi}{4}$
(d) $1 - 2\cos\theta = 0$; $\cos\theta = 0.5$; $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$
(e) $\cos\theta(1 - \cos\theta) = 0$; $\theta = 0, \frac{\pi}{2}, \frac{3\pi}{2}, 2\pi$
(f) $\frac{\pi}{4}, \frac{5\pi}{4}$ (g) $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$ (h) $\frac{\pi}{4}, \frac{5\pi}{4}$ (i) $0, 2\pi$
(j) $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$ (k) $\frac{3\pi}{4}, \frac{7\pi}{4}$
(l) $\cos\theta(2\cos\theta - 1) = 0$; $\theta = 0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{3}$

2 A

3 (a) $\sin\theta(1 + \sin\theta) = 0$; $\theta = -\pi, -\frac{\pi}{2}, 0, \pi$
(b) $2\sin^2\theta - \sin\theta - 1 = 0$; $\theta = -\frac{5\pi}{6}, -\frac{\pi}{6}, \frac{\pi}{2}$
(c) $4\cos^2\theta = 1$; $\theta = -\frac{2\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}$
(d) $\cos\theta(3 + 2\cos\theta) = 0$; $\theta = -\frac{\pi}{2}, \frac{\pi}{2}$
(e) $\theta = -\frac{5\pi}{6}, -\frac{\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}$
(f) $(2\tan\theta - 1)(\tan\theta - 2) = 0$; $\theta = -2.678, -2.034, 0.464, 1.107$

4 (a) $\sin 2x = 2\sin x\cos x$; $\sin x(2\cos x - 1) = 0$, $\sin x = 0$, $\cos x = 0.5$; $x = 0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}, 2\pi$
(b) $(5\sin x + 3)(\sin x - 1) = 0$; $x = 3.786, 5.640, \frac{\pi}{2}$
(c) $\tan x = \frac{-1}{3}, 2$; $x = 1.107, 2.820, 4.249, 5.961$
(d) $6\sin^2 x = 5\cos x$; $x = 0.841, 5.442$
(e) $\sin x = 0$, $\tan x = 7$; $x = 0, 1.429, \pi, 4.570, 2\pi$
(f) $\tan x = -3, 1$; $x = \frac{\pi}{4}, 1.893, \frac{5\pi}{4}, 5.034$

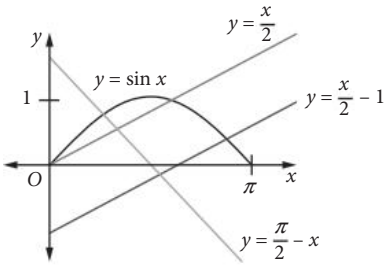
5 (a) correct (b) incorrect (c) correct (d) incorrect

6 (a) $\frac{x}{2} = \frac{\pi}{3}, \frac{2\pi}{3}$; $x = \frac{2\pi}{3}, \frac{4\pi}{3}$ (b) $x = \frac{5\pi}{3}$ (c) $x = 0$

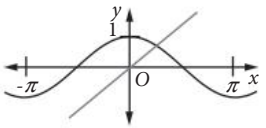
7 (a) $2x = \frac{\pi}{2}, \frac{3\pi}{2}$; $x = \frac{\pi}{4}, \frac{3\pi}{4}$ (b) $x = \frac{\pi}{9}, \frac{4\pi}{9}, \frac{7\pi}{9}$ (c) $x = \frac{\pi}{16}, \frac{3\pi}{16}, \frac{9\pi}{16}, \frac{11\pi}{16}$

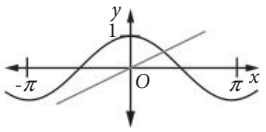
8 (a) $x = 1.23$ (b) $x = 0.78, 1.31, 2.88$ (c) no solution

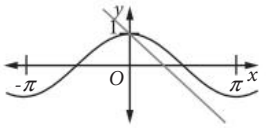
EXERCISE 15.6

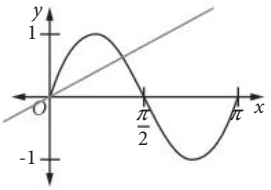
1 

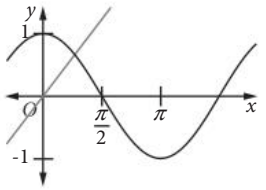
(a) 2 (b) 1 (c) 1

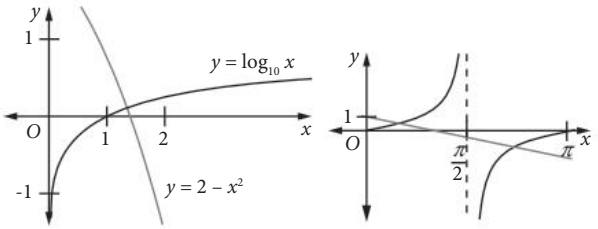
2 (a) $x = 0.74$ 

(b) $x = 0.74$ 

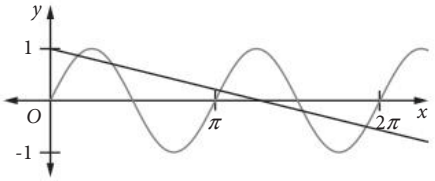
(c) $x = 0$ 

3 (a) $x = 1.24$ 

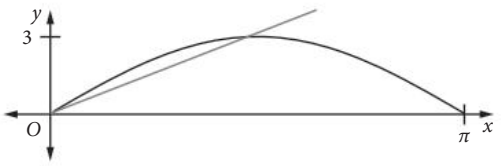
(b) $x = 0.74$ 

(c) $x = 1.366$ (d) $x = 0.48, 2.25$ 

4 B

5 draw $y = \sin 2x$ and $y = 1 - \frac{x}{4}$: 5 solutions 

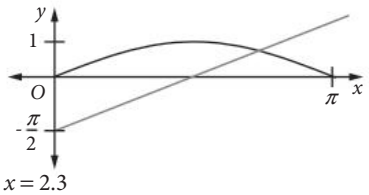
6 (a) $2rx = 9$, $3 = r\sin x$; $\frac{9}{2x} = \frac{3}{\sin x}$ so $3\sin x = 2x$

(b) 

$x = 1.5$ rad

(c) area $= \frac{1}{2} \times \left(\frac{9}{2x}\right)^2 (2x - \sin 2x) = 4.5(3 - \sin 3) = 12.9\,\text{cm}^2$

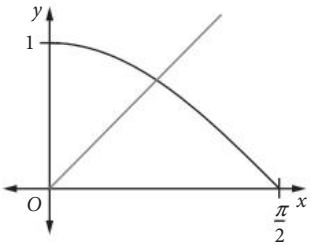
7 (a) $\frac{r^2}{2}(x - \sin x) = \frac{\pi r^2}{4}$, $x - \sin x = \frac{\pi}{2}$ so $x - \frac{\pi}{2} = \sin x$

(b) 

$x = 2.3$

8 (a) $\frac{r^2}{2}\left(\frac{\pi}{2}+x-\sin\left(\frac{\pi}{2}+x\right)\right)=\frac{\pi r^2}{4}$, $\frac{\pi}{2}+x-\cos x=\frac{\pi}{2}$, $x=\cos x$

(b)



$x=0.74$

9 0.99 10 2.84

EXERCISE 15.7

1 (a) 2 (b) $\frac{1}{3}$ (c) 2.5 (d) 1 (e) 0.5 (f) $\frac{1}{9}$ (g) 3 (h) 1.5

2 (a) incorrect (b) correct (c) incorrect (d) correct

3 (a) $4\lim\limits_{x\to 0}\frac{1-\cos 2x}{(2x)^2}=\frac{4}{2}=2$ (b) $\frac{2}{3}$ (c) $\frac{1}{2}$

(d) $\lim\limits_{x\to 0}\left(\frac{\sin x}{x}\times\sin x\right)=1\times 0=0$ (e) $\lim\limits_{x\to 0}\frac{-\sin x}{x}=-1$

(f) $\lim\limits_{x\to 0}\frac{\sin x}{x}=1$ (g) $\lim\limits_{x\to 0}\frac{1-\cos x}{x^2}=\frac{1}{2}$ (h) 1

EXERCISE 15.8

1 (a) $3\cos 3x$ (b) $3\cos x$ (c) $-2\sin 2x$ (d) $-2\sin x$

(e) $\cos x-4\sin x$ (f) $2\sec^2 2x$

(g) $2\cos 2x+2\sin 2x$ (h) $\cos\left(x+\frac{\pi}{4}\right)$

2 A

3 (a) $\cos^2 x-\sin^2 x$ (b) $\sin x+x\cos x$

(c) $2\tan x+2x\sec^2 x$ (d) $2x\cos x-x^2\sin x$

(e) $\frac{\sin x-x\cos x}{\sin^2 x}$ (f) $\frac{x\cos x-\sin x}{x^2}$

(g) $3x^2\sin 2x+2x^3\cos 2x$ (h) $3x^2\cos(x^3)$

4 (a) $\frac{1}{2}\cos\frac{t}{2}-\frac{1}{2}\sin t$ (b) $-3\cos^2 t\sin t$ (c) $-2t\sin(t^2+1)$

(d) $2\cos\left(2t+\frac{\pi}{2}\right)=-2\sin 2t$ (e) $2t+\frac{1}{2}\sec^2\frac{t}{2}$

(f) $2t\cos 3t-3(t^2-1)\sin 3t$ (g) $-2\sin\left(2t+\frac{\pi}{3}\right)$ (h) $-3\sin(3t-2)$

5 (a) $-4\sin 2x\cos 2x$ (b) $6\sin 3x\cos 3x$ (c) $-3\sin x\cos^2 x$

(d) $-3x^2\sin(x^3)$ (e) $2\cos^2 2x-2\sin^2 2x$ (f) $3-2\sin x-\frac{1}{2}\cos\frac{x}{2}$

(g) $-4\sin x\cos x$ (h) $\frac{x^2\cos x-2x\sin x}{x^4}=\frac{x\cos x-2\sin x}{x^3}$

(i) $2x\sec^2(x^2-1)$ (j) $\cos\frac{x}{2}-\frac{x}{2}\sin\frac{x}{2}$

(k) $\cos x(1+\cos x)+\sin x(-\sin x)=\cos x+\cos^2 x-\sin^2 x$

(l) $-\sin 2x+\sin x+x\cos x$

6 (a) correct (b) incorrect (c) incorrect (d) correct

7 (a) $\frac{\cos 2x}{\sqrt{\sin 2x}}$ (b) $2(\sin^2 x-\cos^2 x)$ (c) 0 (d) $4\sin x\cos x$

(e) $-\sin x\cos(\cos x)$ (f) $-\cos x\sin(\sin x)$ (g) $-\sec^2 x$

(h) $\frac{\sin x}{2\sqrt{1-\cos x}}$ (i) $\frac{1}{2\sqrt{x}}\sec^2\sqrt{x}$

8 (a) $e^x(\sin x+\cos x)$ (b) $e^{2x}\left(2\cos\frac{x}{2}-\frac{1}{2}\sin\frac{x}{2}\right)$

(c) $e^{-x}(3\cos 3x-\sin 3x)$ (d) $e^x(\cos 4x-4\sin 4x)$

(e) $-2e^{-x}(\sin x)$ (f) $2e^{\sin 2x}\cos 2x$ (g) $-e^{\cos x}\sin x$

(h) $(\cos x-\sin x)e^{\sin x+\cos x}$ (i) $\cos x\ln x+\frac{\sin x}{x}$

(j) $\frac{\cos x}{\sin x}=\cot x$ (k) $\sec^2 x\ln x+\frac{\tan x}{x}$

(l) $-\sin x\ln 2x+\frac{\cos x}{x}$ (m) $\frac{-\sin x}{\cos x}=-\tan x$

(n) $\frac{\sec^2 x}{\tan x}=\frac{1}{\cos^2 x}\times\frac{\cos x}{\sin x}=\frac{1}{\sin x\cos x}$ (o) $\frac{-\sin(\log_e x)}{x}$ (p) $\frac{\cos(\log_e x)}{x}$

(q) $\frac{\sec^2(\log_e x)}{x}$ (r) $\frac{\frac{\cos x}{x}+\sin x\log_e x}{\cos^2 x}=\frac{\cos x+x\sin x\log_e x}{x\cos^2 x}$

9 (a) $f(0)=1$, $f\left(\frac{\pi}{2}\right)=0$, $f(\pi)=-e^{-p}$

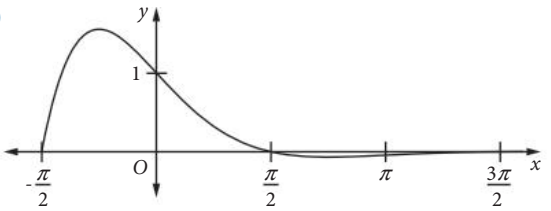
(b) $f'(x)=-e^{-x}(\cos x+\sin x)$

(c) $f'(0)=-e^0(\cos 0+\sin 0)=-1$;

$f'\left(\frac{3\pi}{4}\right)=-e^{-\frac{3\pi}{4}}\left(\cos\frac{3\pi}{4}+\sin\frac{3\pi}{4}\right)=-e^{-\frac{3\pi}{4}}\left(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)=0$;

$f'\left(\frac{-\pi}{4}\right)=-e^{\frac{\pi}{4}}\left(\cos\frac{-\pi}{4}+\sin\frac{-\pi}{4}\right)=-e^{\frac{\pi}{4}}\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\right)=0$

(d)



(e) 1.55 at $x=-\frac{\pi}{4}$

10 $\sin x>0$, $0<x<\pi$; $\frac{d}{dx}\left(\log_e(\sin x)\right)=\frac{\cos x}{\sin x}=\cot x$; stationary points at $x=\frac{\pi}{2}$

$\frac{d^2}{dx^2}\left(\log_e(\sin x)\right)=\frac{-\sin^2 x-\cos^2 x}{\sin^2 x}=\frac{-1}{\sin^2 x}=-\operatorname{cosec}^2 x$

at $x=\frac{\pi}{2}$, $\frac{d^2}{dx^2}\left(\log_e(\sin x)\right)<0$, so this is the maximum value $=0$

11 $\frac{dy}{dx}=\frac{\cos x}{1+\sin x}$, $\frac{d^2y}{dx^2}=\frac{-\sin x(1+\sin x)-\cos x\times\cos x}{(1+\sin x)^2}=\frac{-\sin x-\sin^2 x-\cos^2 x}{(1+\sin x)^2}=\frac{-1}{1+\sin x}$

12 $\frac{dy}{dt}=-12\sin 3t$, $\frac{d^2y}{dt^2}=-36\cos 3t$;

$\frac{d^2y}{dt^2}+9y=-36\cos 3t+9\times 4\cos 3t=0$

13 $\frac{dC}{dt}=-1000\pi\left(\cos\left(\frac{\pi(t-8)}{2}\right)+2\right)\times\sin\left(\frac{\pi(t-8)}{2}\right)$;

$t=10$ or 14; $C=0$

EXERCISE 15.9

1 (a) $C-\frac{\cos 2x}{2}$ (b) $\frac{\sin 3x}{3}+C$ (c) $\tan x+C$

(d) $\sin x-\cos x+C$ (e) $-2\cos x-3\sin x+C$

(f) $C-\cos\left(x+\frac{\pi}{4}\right)$ (g) $2\sin\frac{x}{2}+C$ (h) $C-\cos 2x$

2 D

3 (a) $4\sin\frac{x}{4}-4\cos\frac{x}{4}+C$ (b) $C-\cos x-\frac{\sin 2x}{2}$

(c) $-\frac{1}{2}\cos\left(2x+\frac{\pi}{2}\right)+C=\frac{\sin 2x}{2}+C$ (d) $\frac{1}{2}\sin\left(2x-\frac{\pi}{4}\right)+C$

(e) $\frac{\tan 3x}{3}+C$ (f) $C-\frac{1}{4}\cos 2x-\sin x$ (g) $2\sin\frac{x}{2}+\frac{1}{4}\cos 2x+C$

(h) $\frac{x^3}{3}-\frac{\cos 2x}{2}+C$ (i) $\frac{2}{3}x^{\frac{3}{2}}-\frac{\sin x}{2}+C$

4 (a) 2 (b) $\sqrt{3}$ (c) 1 (d) 1 (e) 2 (f) 0 (g) -0.25

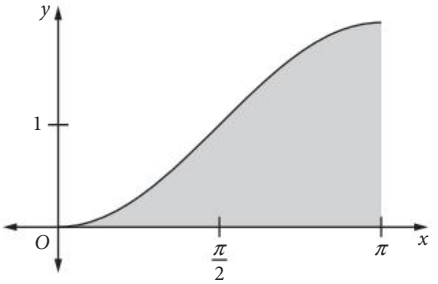
(h) $\frac{\sqrt{2}-1}{2}$ (i) 0 (j) 4 (k) $\frac{-3}{8}$ (l) 0

5 (a) incorrect (b) correct (c) correct (d) correct

6 $\frac{1}{\pi}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}(2\cos x+3\sin x)\,dx=\frac{4}{\pi}$ 7 $2\int_0^{\frac{\pi}{2}}\cos x\,dx=2$

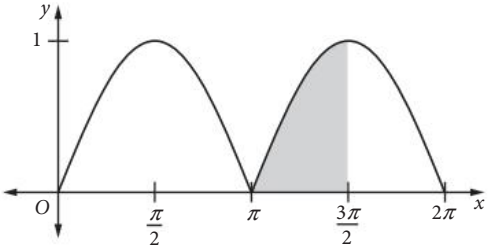
8 π units^3

9



$\int_0^{\pi}(1-\cos x)\,dx=\pi$

10



$\int_{\pi}^{\frac{3\pi}{2}}\sqrt{1-\cos^2 x}\,dx=\int_{\pi}^{\frac{3\pi}{2}}|\sin x|\,dx$

$=\int_{\pi}^{\frac{3\pi}{2}}(-\sin x)\,dx=\left[\cos x\right]_{\pi}^{\frac{3\pi}{2}}=1$

11 $\frac{-9\sqrt{3}}{8}$ **12** 2π **13** π **14** 0.5

15 (a) π (b) $\frac{\pi+1}{2}$ (c) $\frac{\pi+2\sqrt{2}}{4}$ (d) $\frac{ab}{\pi}$

16 $\tan 2\theta = 1$, $\theta = \frac{\pi}{8}, \frac{5\pi}{8}, \left(\frac{\pi}{8}, \frac{\sqrt{2}}{2}\right), \left(\frac{5\pi}{8}, \frac{-\sqrt{2}}{2}\right)$

$\text{area} = \int_{\frac{\pi}{8}}^{\frac{5\pi}{8}} (\sin 2\theta - \cos 2\theta)\,d\theta = \sqrt{2}$

17 $\frac{6}{\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (2\cos 2x + \sin 2x)\,dx = \frac{3}{\pi}$

18 $\left(\frac{\pi}{4}, \frac{\sqrt{2}}{2}\right), \left(\frac{5\pi}{4}, \frac{-\sqrt{2}}{2}\right)$

$\text{area} = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin\theta - \cos\theta)\,d\theta = 2\sqrt{2}$

19 $\frac{\pi}{2} \approx 1.57$

20 $f(0) = \frac{1}{3}$, $f\left(\frac{\pi}{4}\right) = \frac{2}{4+\sqrt{2}}$, $f\left(\frac{\pi}{2}\right) = \frac{1}{2}$, $f\left(\frac{3\pi}{4}\right) = \frac{2}{4-\sqrt{2}}$, $f(\pi) = 1$

$\int_0^{\pi} \frac{dx}{2+\cos x} \approx \frac{\pi}{12}\left(\frac{1}{3} + 4\left(\frac{2}{4+\sqrt{2}} + \frac{2}{4-\sqrt{2}}\right) + 2\times\frac{1}{2} + 1\right) = \frac{\pi}{12}\left(\frac{7}{3} + \frac{8}{7}\right) \approx 0.91$

EXERCISE 15.10

1 (a) $\ln(1-\cos x) + C$ (b) $\ln(\sin x) + C$

2 (a) $\left[\log_e(1+\sin x)\right]_0^{\frac{\pi}{6}} = \log_e 1.5$ (b) $\ln 2$

3 $\frac{d}{dx}\left(\log_e(\cos x)\right) = \frac{-\sin x}{\cos x} = -\tan x$, $\int \tan x\,dx = C - \log_e(\cos x)$

$\text{area} = \int_0^{\frac{\pi}{3}} \tan x\,dx = \left[-\log_e(\cos x)\right]_0^{\frac{\pi}{3}} = -\log_e\frac{1}{2} + \log_e 1 = \log_e 2$

4 (a) $f\left(\frac{\pi}{6}\right) = \frac{1}{4}$, $f\left(\frac{\pi}{2}\right) = 1$, $f\left(\frac{5\pi}{6}\right) = \frac{1}{4}$

$\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \sin^2 x\,dx \approx \frac{\pi}{6}\left(\frac{1}{4} + 2 + \frac{1}{4}\right) = \frac{5\pi}{12} \approx 1.31$

(b) $f\left(\frac{-\pi}{3}\right) = \frac{1}{4}$, $f(0) = 1$, $f\left(\frac{\pi}{3}\right) = \frac{1}{4}$

$\int_{\frac{-\pi}{3}}^{\frac{\pi}{3}} \cos^2 x\,dx \approx \frac{\pi}{6}\left(\frac{1}{4} + 2 + \frac{1}{4}\right) = \frac{5\pi}{12} \approx 1.31$

5 $V = \pi\int_0^{\pi} \sin^2 x\,dx \approx \pi\times\frac{\pi}{8}\left(0 + 2\left(\frac{1}{2}+1+\frac{1}{2}\right) + 0\right) = \frac{\pi^2}{2} \approx 4.93$

6 $4x - 2y + 2 - \pi = 0$

7 (a) $\cos x - \sin x$ (b) $-\sin x - \cos x$ (c) $\left(\frac{\pi}{4}, \sqrt{2}\right), \left(\frac{5\pi}{4}, -\sqrt{2}\right)$

(d) $\left(\frac{3\pi}{4}, 0\right), \left(\frac{7\pi}{4}, 0\right)$ (e) $\sqrt{2}$

8 $y = 1 - x$ **9** $y = \frac{7 - 2\cos 3x}{3}$

10 $\pi \approx \frac{1}{4}\left(4 + 2\times\frac{16}{5} + 2\right) = 3.1$

11

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
$f(x)$	0	$\frac{\pi\sqrt{2}}{8}$	$\frac{\pi}{2}$	$\frac{3\pi\sqrt{2}}{8}$	0

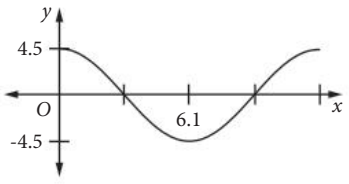
$\frac{\pi^2(1+2\sqrt{2})}{12} \approx 3.15$

12 $\frac{\pi}{8}\left(\sqrt{2} + 2\left(\frac{\sqrt{5}}{2} + 1 + \frac{\sqrt{5}}{2}\right) + \sqrt{2}\right) = \frac{\pi}{4}\left(\sqrt{2} + \sqrt{5} + 1\right) \approx 3.65$

13 $\frac{dy}{dx} = 2\cos x + 2\sin x\cos x = 2\cos x(1+\sin x)$; $\left(\frac{\pi}{2}, 3\right), \left(\frac{3\pi}{2}, -1\right), \left(\frac{5\pi}{2}, 3\right), \left(\frac{7\pi}{2}, -1\right)$

14 (a) 4.5 m (b) 12.2 hours; $n = \frac{10\pi}{61}$

(c) $y = 4.5\cos\frac{10\pi t}{61}$



(d) low tide at $t = 6.1$; at $t = 7.1$, $y = -3.92$:
depth $= 0.5 + (4.5 - 3.92) = 1.08$ m

CHAPTER REVIEW 15

1 $1.6\pi = 5.03$ m

2 (a) 6.1 cm (b) $122\sin 28° = 5.73$ cm

3 4

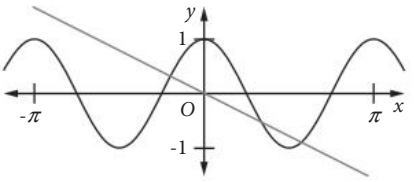
4 $x = -1.231, -\frac{\pi}{3}, \frac{\pi}{3}, 1.231$

5 (a) 0.75 m (b) $\frac{3\pi}{32} = 0.3\,\text{m}^2$ (c) 0.75 m

6 (a) $\frac{7\pi}{6}, \frac{11\pi}{6}$ (b) $\frac{\pi}{2}, \frac{3\pi}{2}$ (c) $\frac{3\pi}{4}, \frac{7\pi}{4}$ (d) $\frac{\pi}{6}, \frac{11\pi}{6}$

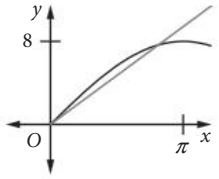
(e) $\frac{\pi}{6}, \frac{7\pi}{6}$ (f) $\frac{\pi}{4}, \frac{3\pi}{4}$

7



$x = -0.63, 1.07, 1.8$

8 (a) $\frac{20}{x} = \frac{7.5}{\sin\frac{x}{2}}$, $8\sin\frac{x}{2} = 3x$ (b)



$x = 2.6$

9 (a) $\cos x + 2\sec^2 2x$ (b) $-12\sin 4x - 10\cos 2x$

(c) $\cos x - x\sin x$ (d) $\frac{-1}{\sin^2 x} = -\text{cosec}^2 x$

(e) $-2e^{-x}(\cos 3x + 3\sin 3x)$ (f) $\frac{2\cos 2x}{\sin 2x} = 2\cot 2x$

10 (a) $C - 6\cos\frac{x}{2}$ (b) $\frac{x^2}{2} + \frac{1}{2}\tan 2x + C$ (c) $\ln(\sin t) + C$

11 (a) $\frac{\sqrt{2}-1}{2}$ (b) $\frac{-\pi^2}{8}$ (c) $\log_e 3$

12 $\frac{4}{\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (2\cos 2x + \sin 2x)\,dx = -\frac{2}{\pi}$ **13** $\frac{\sqrt{2}-\sqrt{3}}{2}$

14 $V = \pi\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \tan^2 x\,dx = \pi\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(\sec^2 x - 1\right)dx = \pi\left[\tan x - x\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$

$= \pi\left(\sqrt{3} - \frac{\pi}{3} - \left(\frac{\sqrt{3}}{3} - \frac{\pi}{6}\right)\right) = \frac{\pi(4\sqrt{3}-\pi)}{6}$

15 (a) $x = \frac{\pi}{6}, \frac{7\pi}{6}$

(b) $A = \int_{\frac{\pi}{6}}^{\frac{7\pi}{6}} \left(\sqrt{3}\sin x - \cos x\right)dx = \left[-\sqrt{3}\cos x - \sin x\right]_{\frac{\pi}{6}}^{\frac{7\pi}{6}}$

$= 2\left(\sqrt{3}\times\frac{\sqrt{3}}{2} + \frac{1}{2}\right) = 4$

16 (a) $\frac{d}{dx}\left(\log_e(\sin x)\right) = \frac{1}{\sin x}\times\cos x = \cot x$

(b) $A = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\cot x - \frac{\pi}{2} + x\right)dx = \left[\log_e(\sin x) - \frac{\pi x}{2} + \frac{x^2}{2}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$

$= \frac{1}{2}\log_e 2 - \frac{\pi^2}{32}$

17 (a) $x = 1$: $y = \sin\frac{\pi x}{2} = \sin\frac{\pi}{2} = 1$; $y = x^2 = 1^2 = 1$;
curves meet at $x = 1$, $y = 1$

(b) $A = \int_0^1 x^2\,dx + \int_1^2 \sin\frac{\pi x}{2}\,dx = \frac{\pi+6}{3\pi}$

(c) $V = \pi\int_0^1 x^4\,dx + \pi\int_1^2 \sin^2\frac{\pi x}{2}\,dx$

(d) $V \approx \frac{\pi}{4}\left(0 + 2\times\frac{1}{16} + 1\right) + \frac{\pi}{4}\left(1 + 2\times\frac{1}{2} + 0\right) = \frac{25\pi}{32}$

18 6.58, 2π, 4.7%

CHAPTER 16

EXERCISE 16.1

1 (a) $1 + 4 + 9 + 16$ (b) $1 + 3 + 3^2 + 3^3 + 3^4 + 3^5$

(c) $1 + 3 + 5 + \ldots + (2p-1)$ (d) $2 + 6 + 12 + 20 + 30$

(e) $x + 2x^2 + 3x^3 + \ldots + 8x^8$ (f) $\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \ldots + \frac{1}{x^k}$

(g) $9 + 25 + 49 + \ldots + (2p+1)^2$ (h) $1 + 4 + 7 + \ldots + (3n+1)$

2 (a) correct (b) incorrect (c) correct (d) incorrect

3 (a) $\sum_{k=1}^{9} k^2$ (b) $\sum_{k=1}^{10} k(k+2)$ (c) $\sum_{k=1}^{p} (5k-4)$

(d) $\sum_{k=2}^{p} \frac{1}{k(k+1)}$ (e) $\sum_{r=2}^{12} rx^r$ (f) $\sum_{k=1}^{n} ar^{k-1}$

4 (a) $1 + 4 + 9 + 16 = 30$ (b) $3 + 5 + 7 + 9 + 11 + 13 = 48$
(c) 22 (d) 30 (e) 55 (f) 24 (g) 40 (h) 9 (i) 288

EXERCISE 16.2

1 (a) arithmetic (b) arithmetic (c) not arithmetic (d) arithmetic

2 B **3** 50, 86 **4** 14.8, 11.2 **5** -3, 2, 7, …

6 1.4 **7** $9q - 8p$ **8** $11 - 3n$

9 (a) correct (b) incorrect (c) correct (d) correct

10 $a=\frac{5m-2n}{3}$, $d=\frac{n-m}{6}$, $T_7=\frac{2m+n}{3}$ **11** $8\left(6+15\times\frac{5}{4}\right)=198$

12 294 **13** 345 **14** (a) 293 (b) 8730

15 B **16** (a) 176.4 m (b) 9.8 m **17** 3275

18 157.5 **19** $1+3+5+(2n-1)=n^2$

20 (a) 95 (b) 152 (c) $n(2n+1)$ (d) $n(3n+5)$ (e) n^2 (f) 185

21 13 **22** 24

23 (a) correct (b) correct (c) correct (d) correct

24 (a) 2550 (b) 1050 (c) 550 (d) 3050

25 (a) $a=-12$, $d=4$, $S_n=2n(n-7)$ (b) $n=16$

26 $\frac{5}{2}(a+136)=540$; 80°, 94°, 108°, 122°

27 $S_8=180$; $a=-2$, $d=7$, $S_{10}=295$
$T_9+T_{10}=115$

28 15 **29** 16 **30** $n=12$, $S_{12}=276$

31 600 revolutions (playing from the outside in): $a=21\pi$, $l=10.5\pi$, $n=600$, $S_{600}\approx 297$ m

32 $T_n=6n-14$: -8, -2, 4, …
$a=-8$, $d=6$

33 $T_n=6n-1$, $a=5$, $d=6$, $T_8=47$

34 $a=-3$, $d=3$ **35** $\frac{33}{2}(204+396)=9900$

36 $c=a+(n-1)(b-a)$; $n=\frac{b+c-2a}{b-a}$, $S_n=\frac{(b+c-2a)(a+c)}{2(b-a)}$

37 1, 3, 5 **38** (a) $n^2+29n-492=0$, $n=12$ (b) 26

39 9 cm, 12 cm **40** 20 **41** $n=6, 10$

42 (a) n^2 (b) n^2+n (c) $\frac{n(n+1)}{2}$, 20 **43** 15

44 (a) 21 (b) 120 **45** $\frac{2(10+3d)}{5(10+9d)}=\frac{8}{35}$. $d=2$

46 105°, 115°, 125°, 135°, 145°

EXERCISE 16.3

1 geometric, $a=4$, $r=1.5$: $S_6=\frac{4\left(1.5^6-1\right)}{0.5}=83\frac{1}{8}$

2 geometric, $a=8$, $r=-0.5$: $5\frac{21}{64}$ **3** A **4** $n=9$; $10\frac{11}{16}$

5 176.37 **6** $\frac{x\left[\left(\frac{2x}{3}\right)^n-1\right]}{\frac{2x}{3}-1}$, 189 **7** $\frac{1}{64}$

8 removed $\frac{a}{4}, \frac{3a}{16}, \frac{9a}{64}, \frac{27a}{256}$; remaining $\frac{3a}{4}, \frac{9a}{16}, \frac{27a}{64}, \frac{81a}{256}$; 81 000 L

9 (a) correct (b) correct (c) incorrect (d) correct

10 $b^2=ac$, $a+b+c=15$; $2a=b+c$; $a=5$, $b=-10$, $c=20$

11 $r=\pm 2$ **12** $10\log_{10}3+45\log_{10}2$ **13** 0

14 (a) 8 (b) (i) 1024 (ii) $1+2+2^2+\ldots+2^{10}=2047$ (iii) 22.5 cm

EXERCISE 16.4

1 (a) $5\frac{1}{3}$ (b) 16 (c) $17\frac{6}{7}$ (d) $\frac{4\sqrt{3}+6}{3}$ **2** $-\frac{1}{6}$

3 $\frac{a}{1-r}=27$, $\frac{a}{1-r}\left(1-r^4\right)=\frac{65}{3}$; 9, 6, 4; 45, -30, 20 **4** D

5 16, 8, 4; 48, -24, 12 **6** 36 **7** $\frac{a+1}{a}$; $a<-2$ or $a>0$

8 $r=4-\sqrt{15}$; $\frac{4\sqrt{3}+3\sqrt{5}}{3}$ **9** (a) $2\frac{7}{18}$ (b) $4\frac{62}{99}$ (c) $\frac{413}{990}$

10 $p=32$, $q=99$ **11** $\frac{58}{45}$ **12** $27\frac{49}{93}$

EXERCISE 16.5

1 C **2** (a) 12 442 (b) 49 650 **3** (a) 5100 (b) 41 800

4 $27\,764\approx 28\,000$ **5** $0.85^n=\frac{3}{7}$; 5 to 6 years

6 (a) Megan earns $\$83\,000\times 4+\$4150\times 4=\$348\,600$
Paul earns $\$83\,000(1+1.03+1.03^2+1.03^3)=\$347\,241.04$
Paul earns the most
(b) 3rd year

7 (a) \$1 423 747 (b) $\frac{5000\times 1.07\left(1.07^{22}-1\right)\times 1.03^{22}}{0.07}+\frac{5000\times 1.03\left(1.03^{22}-1\right)}{0.03}$
$=\$659\,630$; \$764 117 less

8 (a) Henry earns \$46 000, Eleanor earns \$44 994.56
(b) Henry earns \$490 000, Eleanor earns \$480 244.28
(c) start of 13th year
(graph $y=40\,000+2000x$ and $y=40\,000\times 1.04^x$, find the intersection point)

9 (a) \$9853.23 (b) 9 years

10 (a) $200\,000\times 0.007=\$1400$
(b) to pay off more than the interest so that the balance owed is reduced
(c) $A_1=200\,000\times 1.007-3000$;
$A_2=(200\,000\times 1.007-3000)\times 1.007-3000$
$=200\,000\times 1.007^2-3000(1+1.007)$
(d) $A_n=200\,000\times 1.007^n-\frac{3000\left(1.007^n-1\right)}{0.007}$
(e) $n=\frac{\log_{10}15-\log_{10}8}{\log_{10}1.007}=90.1$; 91 months

11 (a) $60\times 1.006^{240}=\$252.15$ (b) $\frac{60\times 1.006\left(1.006^{240}-1\right)}{0.006}=\$32\,217.89$

EXERCISE 16.6

1 $Q=\frac{10000\times 1.01^{12}\times 0.01}{1.01^{12}-1}=\$888.49$

2 $Q=\frac{20000\times 1.0075^{16}\times 0.0075}{1.0075^{16}-1}=\$1331.18$

3 (a) $Y=\frac{400\times 1.004\left(1.004^{360}-1\right)}{0.004}=322142.43$
(b) (i) $A_n=Y\times 1.004^n-3000(1+1.004+\ldots+1.004^{n-1})$
$A_n=Y\times 1.004^n-\frac{3000\left(1.004^n-1\right)}{0.004}$
$=(Y-750000)\times 1.004^n+750000$
(ii) $1.004^n=\frac{750000}{427857.57}$, $n\log_{10}1.004=\log_{10}\left(\frac{750000}{427857.57}\right)$, $n=140.6$
The money would be finished at the end of the 141st month.

4 (a) \$399 422 (b) $M=\frac{392870\times 0.0075\times 1.0075^{288}}{1.0075^{288}-1}=3334.15$
(c) $1.0075^n=\frac{3500}{3500-392870\times 0.0075}$,
$n=\log_{10}\left(\frac{3500}{3500-392870\times 0.0075}\right)\div\log_{10}1.0075=246.8$
It will take 247 months at the higher rate.
(d) amount saved = $288\times\$3334.15-247\times\$3500=$ \$95 735.20, about \$95 700

5 lump sum becomes \$265 329.77, monthly payment becomes \$247 647.79; lump sum investment is best

6 (a) value after 11 years
$=90000\times 1.05^{11}-\frac{10000\left(1.05^{11}-1\right)}{0.05}=\$11862.67$; plan is realistic
(b) \$11 862.67 (c) $90000\times 1.05^{12}-\frac{10000\left(1.05^{12}-1\right)}{0.05}=\$2455.80$

CHAPTER REVIEW 16

1 2250 **2** 12 **3** 380 m **4** 2, 5, 8; 11, 5, -1

5 $\frac{3\sqrt{3}-5}{2}, \frac{7-4\sqrt{3}}{2}, \frac{2\sqrt{3}}{3}$ **6** $\frac{7}{30}$ **7** $r=3$

8 $T_n=2n-4$, $a=-2$, $d=2$ **9** $a=3$, $r=2$, $S_5=93$

10 2 082 000 **11** 0.8

12 $S_n=2\left(2^n-1\right)+\frac{n}{2}(1+2n-1)=2^{n+1}+n^2-2$ **13** \$180.76

14 (a) \$1788.21
(b) value at end of 5th year $=\frac{100\times 1.06\left(1.06^4-1\right)}{0.06}\times 1.06=\$491.53$
This grows to \$739.08 after 12 years. \$100 now invested for remaining 7 years:
value $=\frac{100\times 1.06\left(1.06^7-1\right)}{0.06}=\$889.75$
total value after 12 years = \$739.08 + \$889.75 = \$1628.83
OR, \$100 not invested would have become
$\$100\times 1.06^8=\$159.38$
accumulated value = \$1788.21 − \$159.38 = \$1628.83

15 (a) \$21 795 (b) \$182 713.05
(c) $n=\log_e\left(\frac{20}{11}\right)\div\log_e 1.06=10.26$; it will be repaid in 11 years.
amount owing after 10 years
$=300000\times 1.06^{10}-\frac{40000\left(1.06^{10}-1\right)}{0.06}=\$10022.51$
This would accrue interest for 1 year, so final payment = \$10 623.86

16 (a) $3+\sqrt{3}$
(b) common ratio $r=\frac{1}{\sqrt{3}-1}=\frac{\sqrt{3}+1}{2}>1$, so there is no limiting sum

17 (a) $A_n = 200\,000 \times 1.005^n - \frac{M\left(1.005^n - 1\right)}{0.005}$

(b) $0 = 200\,000 \times 1.005^{180} - \frac{M\left(1.005^{180} - 1\right)}{0.005}$,

$M = \frac{0.005 \times 200\,000 \times 1.005^{180}}{1.005^{180} - 1} = 1687.71$

Merv receives \$1687.71 per month.

CHAPTER 17

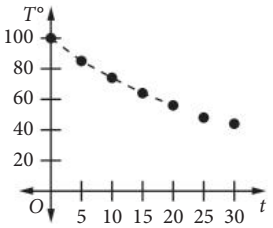
EXERCISE 17.1

1 (a) $0 \le t \le 200$ (b) $6\,\text{m}^3\,\text{min}^{-1}$ (c) no (d) $10\,\text{m}^3\,\text{min}^{-1}$ (e) 50 min

2 D

3 (a) $12.5\,\text{cm}^3\,\text{min}^{-1}$ (b) $V = 1000 - 12.5t$, $0 \le t \le 80$

4 (a) $2000(20 - t)\,\text{L min}^{-1}$ (b) 20 min (c) 10 min (d) $35\,000\,\text{L min}^{-1}$

5 

(a) $1.8°\,\text{min}^{-1}$ (b) $2.2°\,\text{min}^{-1}$

6 (a) (i) 1 item/min (ii) 21 items/min (b) $\int_0^{10}(2t+1)\,dt = 110$

7 $A = \pi r^2$, $\frac{dA}{dr} = 2\pi r \propto r$; $4\pi\,\text{cm}^2/\text{cm}$

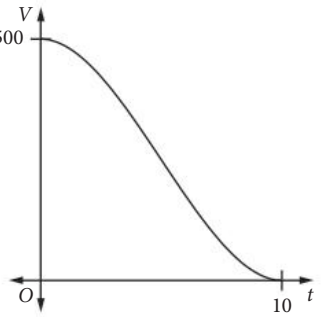
8 (a) $\frac{dV}{dh} = \pi r^2$ (b) $\frac{dV}{dr} = 2\pi rh$ **9** $\frac{dP}{dv} = -\frac{500}{v^2} = -5$

10 (a) -5000; depreciating by \$5000 per year (b) \$25 000

11 $\frac{dR}{dx} = 15 - \frac{x}{15}$ (a) 14.6 (b) 14 (c) 0

12 (a) incorrect (b) correct (c) incorrect (d) correct

13 (a) (i) $448\,\text{ML h}^{-1}$ (ii) $\int_0^2 \left(500 - 15t^2 + t^3\right)dt = 964\,\text{ML}$

(b) (i) 

(ii) $t = 10$, so 5 p.m.

(iii) $\int_0^{10} \left(500 - 15t^2 + t^3\right)dt = 2500\,\text{ML}$

14 $I = 4\pi\cos\pi t$; $t = 2$, $I = 4\pi$ A

EXERCISE 17.2

Logarithm values are approximate.

1 $y = 5e^{2x}$ **2** B **3** $Q = 50e^{0.4t}$ **4** $y = 20e^{-3t}$

5 $M = 10e^{-0.01t}$ **6** $A = 1000$, $k = 0.5$ **7** $A = 200$, $k = 0.4$

8 (a) correct (b) incorrect (c) correct (d) correct

9 $P = 100\,000e^{0.018t}$ **10** (a) $\frac{dN}{dt} = 0.23N$ (b) 8×10^4 (c) 6

11 $k = \frac{\ln 2}{1600}$, 4.24%

12 (a) 16 500 (b) 13.86 min (c) (i) 824/min (ii) 1000/min

13 (a) $\frac{d}{dt}\left(V_0e^{-kt}\right) = -kV_0e^{-kt} = -kV$ (b) $k = \frac{1}{5}\log_e\left(\frac{4}{3}\right)$, $\frac{9}{16}$

(c) (i) $0.032V_0\,\text{m}^3\,\text{min}^{-1}$ (ii) $0.043V_0\,\text{m}^3\,\text{min}^{-1}$

14 (a) 50 (b) 0.1 (c) 2 years

15 (a) 5000 (b) 2 (c) 10 000 coulombs per second

16 (a) $N = 1000e^{0.15t}$ (b) 4.62 h, 300/h

17 (a) $k = \frac{1}{300}\ln\left(\frac{10}{3}\right) \approx 0.004$ (b) 173 m

18 (a) $k = \frac{-1}{10}\ln\left(\frac{5}{4}\right) \approx -0.02$ (b) 35

19 $m = Me^{-kt}$, 6.9 **20** (a) 80 (b) (i) 51.25°C (ii) 14.8 min

21 16 000 **22** 70 days **23** 59%

EXERCISE 17.3

1 (a) $x = 6$ m (b) $v = -5\,\text{m s}^{-1}$ (c) $t = 2$ s, $v = -1\,\text{m s}^{-1}$ (d) $t = 3$ s, $v = 1\,\text{m s}^{-1}$ (e) $t = 2.5$ s, $x = -0.25$ m

2 C

3 (a) $t = 2.5$ s (b) $a = 2\,\text{m s}^{-2}$ (c) $4 + 2 \times 2.25 = 8.5$ m (d) $t = 6.5$ s

4 (a) $\dot{x} = 6t^2 - 30t + 36$, $\ddot{x} = 12t - 30$ (b) $36\,\text{m s}^{-1}$, $-30\,\text{m s}^{-2}$ (c) $t = 2$, 3 s (d) $t = 2.5$ s, $v = -1.5\,\text{m s}^{-1}$, $x = 27.5$ m (e) $2 < t < 3$

5 (a) $t = 1.5$ s, $v = -1.5\,\text{m s}^{-1}$ (b) $t = 1$ s, $a = -6\,\text{m s}^{-2}$; $t = 2$ s, $a = 6\,\text{m s}^{-2}$

6 (a) $t = 2$ s (b) $t = 0.5$ s

7 (a) correct (b) incorrect (c) incorrect (d) correct

8 (a) $x = \frac{t^3}{3} - t^4$ (b) $\ddot{x} = 2t - 12t^2$

9 (a) $x = 2t^3 + 3t^2 - 12t + 7$, $a = 12t + 6$

(b) $t = 1$, $a = 18\,\text{m s}^{-2}$ (c) $v = -12\,\text{m s}^{-1}$, $a = 6\,\text{m s}^{-2}$

10 (a) $\dot{x} = 11 + 10t - t^2$, $x = 11t + 5t^2 - \frac{t^3}{3}$ (b) $t = 11$, $x = 282\frac{1}{3}$

11 (a) $v = 30 - 10t$ (b) $h = 30t - 5t^2$ (c) $t = 3$, $h = 45$ m (d) 6 s

12 (a) $v = 25 - 10t$, $x = 25t - 5t^2$ (b) 31.25 m (c) $t = 1.25$ s

13 (a) $x = t^3 - t^2 - t + 1$, $a = 6t - 2$

(b) $t = 1$ s, $v = 0\,\text{m s}^{-1}$, $a = 4\,\text{m s}^{-2}$ (c) $1 + 3 = 4$ m

14 $s(t) = 4t^3 - 3t^2 + t + 2$

15 $v = 24 + 10t - t^2$, $x = 24t + 5t^2 - \frac{t^3}{3}$, $t = 12$ s, $x = 432$ m

16 (a) $50\,\text{km h}^{-1}$, $20\,\text{km h}^{-1}$ (b) 9 min (c) 13.2 km

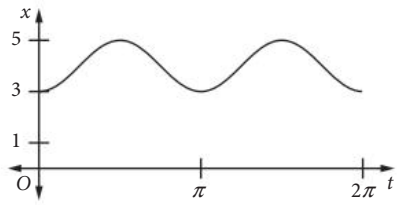
(d) $x = \frac{112t}{3} + 2$

17 $\dot{x} = 4 - \frac{5t}{\sqrt{t^2+4}}$, $t = 2\frac{2}{3}$ s, $x = -3$ **18** D

EXERCISE 17.4

1 (a) $\dot{x} = -24\sin 4t$, $\ddot{x} = -96\cos 4t$ (b) $x = 9$ (c) $-3 \le x \le 9$ (d) $t = \frac{\pi}{6}$

2 D

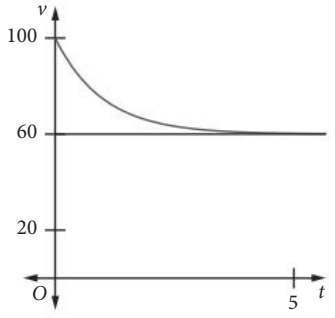
3 (a) 

(b) $\dot{x} = 2\sin 2t$: $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$

(c) (i) $\ddot{x} = 4\cos 2t$ (ii) $\ddot{x} = 4(4 - x)$

4 (a) A: $v = 100\,\text{km h}^{-1}$; B: $v = 62\,\text{km h}^{-1}$

(b) (i) $\ddot{x} = -40e^{-t}$ (ii) $\ddot{x} = 60 - v$

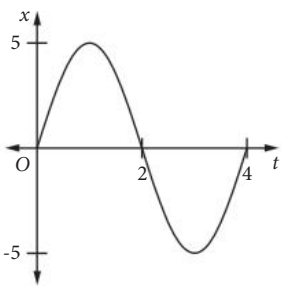
(c) 

As t increases, the velocity decreases towards $60\,\text{km h}^{-1}$.

(d) $x = \int_0^3 \left(60 + 40e^{-t}\right)dt = \left[60t - 40e^{-t}\right]_0^3 = 220 - 40e^{-3}$

≈ 218 km

5 (a)



(b) $\dot{x} = \frac{5\pi}{2}\cos\frac{\pi t}{2}$: $t = 1, 3$ s

(c) $\frac{5\pi\sqrt{2}}{4}$ cm s^{-1}

(d) (i) $\ddot{x} = -\frac{5\pi^2}{4}\sin\frac{\pi t}{2}$

(ii) $\ddot{x} = -\frac{\pi^2 x}{4}$

6 $a = 2\sin t$, $v = 3 - 2\cos t$, $x = 1 + 3t - 2\sin t$

7 (a) $\frac{dx}{dt} = \frac{1}{1+t}$, $x = \int \frac{dt}{1+t} = \log_e(1+t) + C$, $C = 0$, $x = \log_e(1+t)$

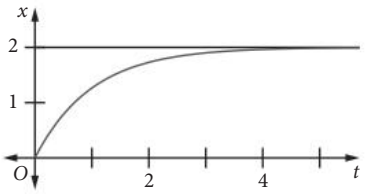
(b) $\lim_{t\to\infty} \log_e(1+t) \to \infty$, moving away in positive direction

(c) $\ddot{x} = \frac{-1}{(1+t)^2}$: $t = 0$, $\ddot{x} = -1$

8 (a) $\dot{x} = 1 + 2\sin t$, $\ddot{x} = 2\cos t$ (b) $t = 0$, $x = 0$: $\dot{x} = 1$, $\ddot{x} = 2$

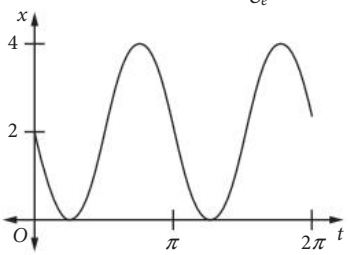
9 (a) $x(0) = 0$, $\dot{x}(0) = 2$, $\ddot{x}(0) = -2$

(b)



(c) $1 = 2 - 2e^{-t}$, $2e^{-t} = 1$, $t = \log_e 2$

10 (a)



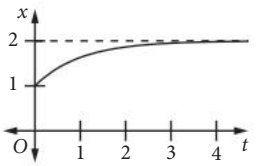
(b) $\dot{x} = -4\cos 2t$: $\dot{x} = 0$ at $t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

$t = \frac{\pi}{4}$, $x = 0$; $t = \frac{3\pi}{4}$, $x = 4$

$\therefore$ particle oscillates between $x = 0$ and $x = 4$

(c) $t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ (d) (i) $\ddot{x} = 8\sin 2t$ (ii) $\ddot{x} = 4(2 - x)$

11 (a) $x = 2 - e^{-t}$



(b) $\frac{d^2x}{dt^2} = -e^{-t} = x - 2$

12 (a) $x = 0$ (b) $\dot{x} = \frac{4}{1+t} - 2$, $\dot{x} = 2$

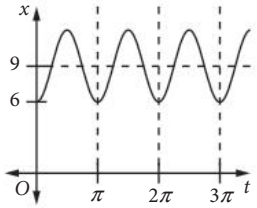
(c) $\ddot{x} = \frac{-4}{(1+t)^2}$; as $(1+t)^2 > 0$ for $t \geq 0$, $\ddot{x} < 0$ for all $t \geq 0$

(d) $t = 1$

13 (a) $v = 1 + 2\sin t$, $x = 2 + t - 2\cos t$ (b) $v = 3e^{-t} - 3$, $t = \log_e 3$

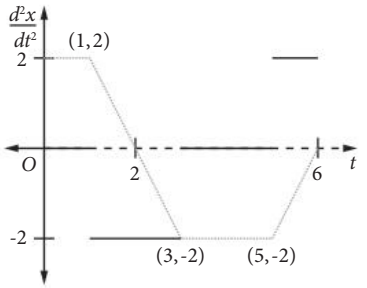
14 (a) $a = \frac{49}{2}e^{-0.5t}$ (b) 0 (c) 49 (d) $x = 49\int_0^5 \left(1 - e^{-0.5t}\right)dt \approx 155$

15 $v = 6\sin 2t$, $x = 9 - 3\cos 2t$: 6 times



16 (a) $t = 2$ (b) $t = 4$; when the area below the t-axis equals the area above the axis, the displacement is zero: $x = \int_0^4 \frac{dx}{dt}\,dt = 0$

(c)



17 (a) $t = 2$; $v = \int_0^2 a\,dt$ is largest value of v

(b) $t = 4$; $v = \int_0^4 a\,dt = 0$

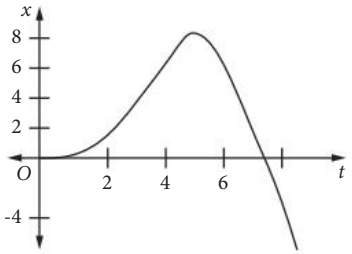
(c) $t = 4$: x is maximum when $v = 0$ and $a < 0$

18 (a) distance $\approx \frac{2}{2}(0 + 2\times 1 + 4) = 6$ (b) 2 (c) $t > 5$

(d) when the area under the curve to the right of D has a magnitude of 6, at time t_1:

$\int_0^{t_1} \frac{dx}{dt}\,dt = 0$; $4(t_1 - 6) = 6$, $t_1 = 7.5$, so when $t \approx 7.5$

(e)



19 (a) distance $\approx \frac{1}{2}(0 + 2(4.6 + 5.7 + 8 + 9.9 + 12.7) + 18.2) = 50$ m

(b) distance $\approx \frac{1}{3}(0 + 4(4.6 + 8 + 12.7) + 2(5.7 + 9.9) + 18.2) = 50.2$ m

CHAPTER REVIEW 17

1 (a) 14 m s^{-2} (b) 275 m **2** $t = 2.5$ s, $x = 22\frac{11}{12}$ m

3 (a) $x = 2t^3 - 2t^2 + t - 10$, $a = 12t - 4$ (b) 8 m s^{-2} (c) 17 m s^{-1}

4 (a) 9, 6, 0, -3, 0, 6, 9 (b) $t = 4$, $v = \frac{-\pi\sqrt{3}}{2}$, $a = \frac{\pi^2}{12}$

5 (a) 0 (b) $a = \frac{18 - 2t^2}{(9 + t^2)^2}$ (c) $t = 3$ (d) $t = 3$, $v = \frac{1}{3}$ m s^{-1}

(e) $x = \int_0^3 \frac{2t\,dt}{9 + t^2} = \log_e 18 - \log_e 9 = \log_e 2$

6 (a) $A = 124$ million, $k = \frac{1}{8}\log_e\left(\frac{55}{31}\right) \approx 0.072$

(b) 273 million (c) $t = \frac{8\left(\log_e 75 - \log_e 31\right)}{\log_e 55 - \log_e 31} \approx 12.3$, 2013

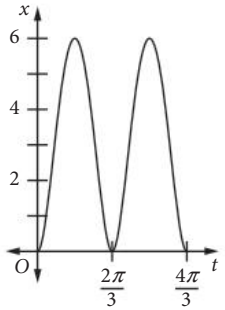
7 (a) $t = 20$ min (b) $V = 110t + \frac{17t^2}{2} - \frac{t^3}{3}$ (c) 1968 L

(d) $t = 22$ min (e) 1015 L

8 (a) $\dot{x} = \int 9\sin 3t\,dt$, $\dot{x} = -3\cos 3t + C$: $t = 0$, $v = 0$, $C = 3$:

$\dot{x} = 3 - 3\cos 3t$

(b)



$t = \frac{2\pi}{3}$

(c) distance $= \int_0^{\frac{2\pi}{3}} (3-3\cos 3t)\,dt = \left[3t - \sin 3t\right]_0^{\frac{2\pi}{3}} = 2\pi$

(d) $x = 3t - \sin 3t$

9 $A = 5000$, $k = \frac{1}{6}\ln 2$, $N = 5000e^{2\ln 2} = 5000e^{\ln 4} = 5000 \times 4 = 20000$

10 $k = \frac{1}{40}$, $\frac{N_0}{2} = N_0 e^{-\frac{t}{40}}$, $2 = e^{\frac{t}{40}}$, $t = 40\ln 2 \approx 28$ days

11 $t = 0$, $M = A$; $k = \frac{1}{400}\ln\frac{10}{9}$;

$M = A \times e^{-\frac{1000}{400}\ln\frac{10}{9}} = Ae^{2.5\ln\frac{9}{10}} = A\left(\frac{9}{10}\right)^{2.5} \approx 0.768A$; 77%

CHAPTER 18

EXERCISE 18.1

1 no, no **2** Answers will vary.

3 (a) states have different populations

(b) there are fewer cars in Hobart

4 C **5** 7, 35, 70, 105, 140 **6** 5, 10, 15, 20, 25

7 (a) correct (b) incorrect (c) incorrect (d) correct

8 (a) $\frac{1}{4}$ (b) $\frac{3}{4}$

9 (a) $\frac{1}{5}$ (b) $\frac{2}{5}$ (c) $\frac{3}{5}$ (d) $\frac{3}{5}$ (e) $\frac{1}{5}$ (f) $\frac{3}{5}$

10 (a) $\frac{3}{10}$ (b) $\frac{1}{5}$ (c) $\frac{9}{20}$ (d) $\frac{1}{20}$

11 (a) $\frac{2}{3}$ (b) $\frac{1}{6}$ (c) $\frac{1}{2}$ (d) $\frac{1}{6}$

12 (a) $\frac{1}{2}$ (b) $\frac{1}{2}$ (c) 1 (d) $\frac{1}{2}$ (e) $\frac{2}{5}$ (f) $\frac{1}{5}$ (g) $\frac{1}{5}$ (h) $\frac{1}{2}$

13 (a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) $\frac{1}{3}$

14 (a) $\frac{1}{4}$ (b) $\frac{1}{13}$ (c) $\frac{4}{13}$ (d) $\frac{1}{52}$ (e) $\frac{1}{52}$

15 (a) $\frac{5}{9}$ (b) $\frac{1}{9}$ (c) $\frac{4}{9}$ (d) $\frac{4}{9}$ (e) $\frac{1}{3}$ (f) $\frac{1}{9}$ (g) $\frac{2}{3}$ (h) $\frac{5}{9}$

16 (a) $\frac{1}{2}$ (b) $\frac{3}{10}$ (c) $\frac{4}{5}$ (d) $\frac{7}{10}$ (e) $\frac{1}{5}$

17 (a) 5 (b) (i) $\frac{2}{5}$ (ii) $\frac{2}{5}$ (iii) $\frac{4}{5}$ (iv) $\frac{4}{5}$

18 (a) $\frac{3}{5}$ (b) $\frac{7}{10}$ (c) $\frac{2}{5}$ (d) $\frac{3}{10}$ (e) $\frac{2}{5}$

19 (a) $\frac{6}{11}$ (b) $\frac{8}{11}$ (c) $\frac{2}{11}$

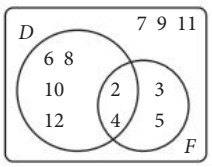
20 (a) 0.96 (b) 0.04 (c) 0.9 (d) 0.48

21 (a) $\frac{1}{4}$ (b) $\frac{1}{8}$ (c) $\frac{3}{8}$ (d) $\frac{3}{4}$ (e) $\frac{3}{4}$

22 (a) 0.4 (b) 0.6 (c) 0.079

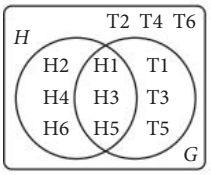
23 (a) $\frac{18}{37}$ (b) $\frac{18}{37}$ (c) $\frac{12}{37}$ (d) $\frac{12}{37}$ (e) $\frac{1}{37}$

EXERCISE 18.2

1 B **2** 

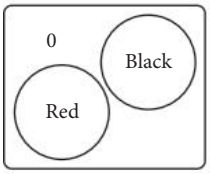
(a) {2, 3, 4, 5, 6, 8, 10, 12}

(b) {2, 4}

3 

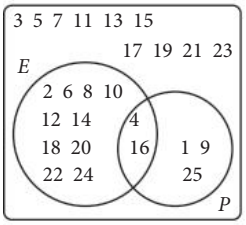
(a) {H1, H2, H3, H4, H5, H6, T1, T3, T5}

(b) 3

4 

(a) {1, 2, 3, …36} (b) {} (c) 36

(d) 0 (e) 37

5 

(a) 25 (b) 12 (c) 5

(d) 2 (e) 15

6 (a) $n(s) = 25$; $n(F \cup G) = 25 - 5 = 20$; $20 = 18 + 12 - n(F \cap G)$; $n(F \cap G) = 10$; French only $= 18 - 10 = 8$

(b) German only $= 12 - 10 = 2$ (c) $8 + 2 = 10$

7 (a) $\{3, 6, 9, 12, 15, 18\} = K$ (b) $\{6, 12, 18\} = L$ (c) 3

8 (a) $\{\}$ (b) $\{1, 2, 3, 4, 5, 6\}$ (c) $\{5\}$ (d) 1

9 $E = \{2, 4, 6, 8, 10, 12\}$, $F = \{3, 5, 7, 9, 11\}$, $G = \{7, 8, 9, 10, 11, 12\}$

(a) 0 (b) 3 (c) 11 (d) 9

10 (a) incorrect (b) correct (c) correct (d) correct

EXERCISE 18.3

1 (a) $\frac{1}{4}$ (b) $\frac{1}{52}$ (c) $\frac{1}{13}$ (d) $\frac{7}{26}$ (e) $\frac{4}{13}$ (f) $\frac{1}{13}$ (g) A and B

2 (a) $\frac{5}{17}$ (b) $\frac{2}{17}$ (c) $\frac{3}{17}$ (d) $\frac{12}{17}$ (e) $\frac{15}{17}$ (f) $\frac{7}{17}$ (g) $\frac{7}{17}$ (h) $\frac{12}{17}$

(i) $\frac{15}{17}$ (j) 1 (k) A and B, B and C

3 D

4 (a) incorrect (b) incorrect (c) correct (d) correct

5 (a) $\frac{1}{2}$ (b) $\frac{1}{8}$ (c) $\frac{1}{2}$ (d) $\frac{1}{2}$ (e) 1 (f) $\frac{5}{8}$ (g) 1

(h) A and C, B and C

6 (a) $\frac{4}{25}$ (b) $\frac{1}{10}$ (c) $\frac{3}{50}$ (d) $\frac{1}{50}$ (e) $\frac{33}{50}$ (f) $\frac{3}{5}$ (g) $\frac{23}{50}$ (h) $\frac{18}{25}$

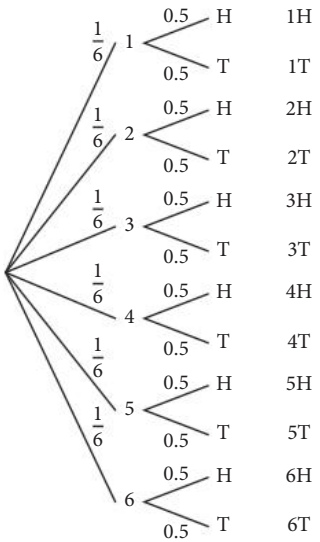
7 (a) $\frac{7}{36}$ (b) $\frac{5}{36}$ (c) $\frac{1}{18}$ (d) $\frac{1}{36}$ (e) 1 (f) $\frac{17}{18}$ (g) $\frac{7}{18}$ (h) 1

8 (a) $\frac{13}{18}$ (b) $\frac{7}{12}$ (c) $\frac{11}{36}$ (d) 1 (e) no **9** no

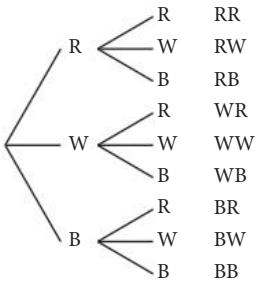
10 $\frac{11}{16}$ **11** (a) $\frac{1}{5}$ (b) $\frac{1}{10}$ (c) $\frac{3}{10}$ **12** (a) $\frac{3}{5}$ (b) $\frac{3}{10}$ (c) $\frac{2}{3}$

EXERCISE 18.4

1 Die Coin Outcomes (a) $\frac{1}{4}$ (b) $\frac{1}{6}$



2 Cube 1 Cube 2 Outcomes (a) $\frac{1}{3}$ (b) $\frac{2}{9}$

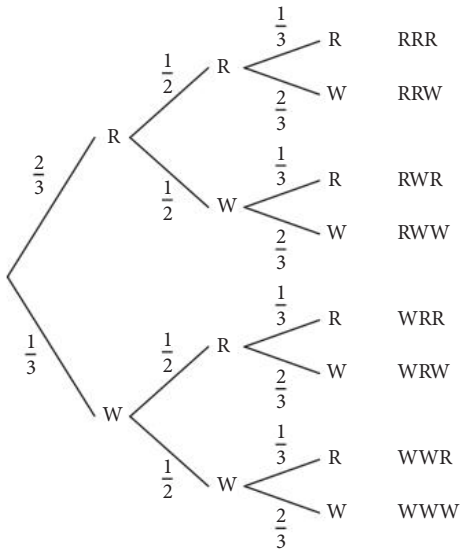


3 $\frac{1}{12}$ **4** (a) 27 (b) $\frac{1}{3} \times \frac{1}{3} \times \frac{2}{3} = \frac{2}{27}$ (c) $\frac{1}{3} \times \frac{2}{3} \times \frac{1}{3} = \frac{2}{27}$

5 C **6** (a) $\frac{1}{8}$ (b) $\frac{3}{8}$ (c) $\frac{3}{8}$ (d) $\frac{1}{8}$

7 (a) incorrect (b) correct (c) incorrect (d) correct

8 (a) Cube 1 Cube 2 Cube 3 Outcomes

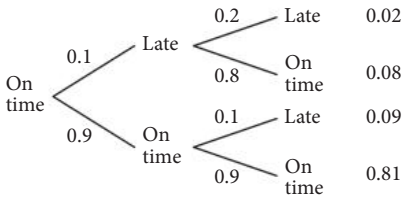


(b) $\frac{2}{3}\times\frac{1}{2}\times\frac{1}{3}=\frac{1}{9}$ (c) $\frac{2}{3}\times\frac{1}{2}\times\frac{2}{3}+\frac{2}{3}\times\frac{1}{2}\times\frac{1}{3}+\frac{1}{3}\times\frac{1}{2}\times\frac{1}{3}=\frac{7}{18}$
(d) $\frac{1}{9}+\frac{7}{18}=\frac{1}{2}$

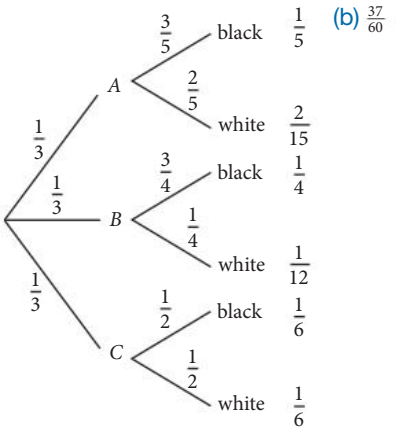
9 (a) $\frac{1}{8}$ (b) $\frac{1}{8}$ (c) $\frac{1}{2}$ **10** (a) $\frac{1}{16}$ (b) $\frac{9}{16}$ (c) $\frac{1}{4}$

EXERCISE 18.5

1 (a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{1}{6}$
2 (a) correct (b) correct (c) correct (d) incorrect
3 (a) $\frac{1}{2}$ (b) $\frac{1}{8}$ (c) $\frac{1}{2}$
4 (a) $\frac{1}{6}$ (b) $\frac{1}{6}$ (c) $\frac{1}{36}$ (d) $\frac{1}{4}$ (e) 0
5 D
6 (a) $\frac{2}{5}$ (b) $\frac{1}{2}$ (c) $\frac{1}{10}$ (d) $\frac{1}{15}$
7 (a) yes (b) no (c) no
8 (a) $\frac{2}{25}$ (b) $\frac{6}{25}$ (c) $\frac{4}{25}$ (d) $\frac{12}{25}$ **9** $\frac{7}{12}$
10 (a) $\frac{1}{10}$ (b) $\frac{9}{20}$ (c) $\frac{3}{10}$ (d) $\frac{3}{20}$
11 (a) $\frac{9}{100}$ (b) $\frac{49}{100}$ (c) $\frac{51}{100}$ (d) $\frac{9}{100}$
12 (a) $\frac{1}{16}$ (b) $\frac{9}{16}$ (c) $\frac{3}{8}$ (d) $\frac{3}{16}$ (e) $\frac{3}{16}$ (f) $\frac{7}{16}$
13 (a) $\frac{64}{125}$ (b) $\frac{1}{125}$ (c) $\frac{124}{125}$
14 (a) $\frac{1}{24}$ (b) $\frac{1}{4}$ (c) $\frac{1}{4}$ (d) $\frac{1}{12}$ (e) $\frac{1}{8}$
15 (a) $\frac{1}{9}$ (b) $\frac{1}{9}$ (c) $\frac{2}{9}$ (d) $\frac{2}{9}$ (e) $\frac{8}{9}$
16 (a) $\frac{1}{16}$ (b) $\frac{1}{16}$ (c) $\frac{1}{16}$ (d) $\frac{1}{16}$ (e) $\frac{15}{16}$
17 (a) $\frac{1}{216}$ (b) $\frac{125}{216}$ (c) $\frac{1}{8}$ (d) $\frac{1}{8}$ (e) $\frac{5}{216}$ (f) $\frac{5}{216}$
18 (a) $\frac{8}{27}$ (b) $\frac{1}{27}$ (c) $\frac{4}{27}$ (d) $\frac{4}{27}$
19 (a) $\frac{1}{8}$ (b) $\frac{63}{1000}$ (c) $\frac{3}{100}$ (d) $\frac{3}{40}$ (e) $\frac{7}{8}$ (f) $\frac{64}{125}$
20 (a) $\frac{8}{27}$ (b) $\frac{1}{27}$ (c) $\frac{4}{27}$
21 (a) $\frac{9}{200}$ (b) $\frac{9}{50}$ (c) $\frac{27}{50}$ (d) $\frac{1}{200}$
22 (a) 0.08 (b) 0.48 **23** $\frac{11}{21}$
24 (a) $\frac{2}{5}$ (b) $\frac{7}{15}$ (c) $\frac{13}{15}$
25 0.89 Monday Tuesday Wednesday



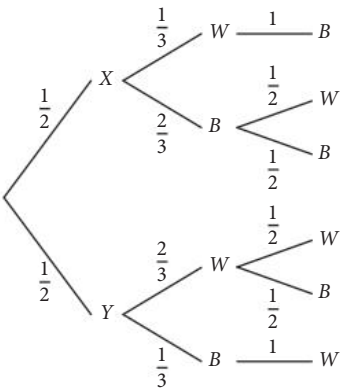
26 (a) 0.0125 (b) 0.336
27 (a)



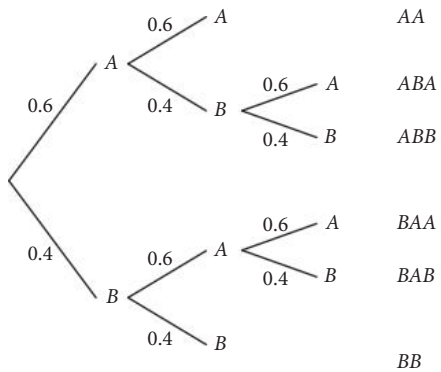
(b) $\frac{37}{60}$
28 (a) $\frac{39}{40}$ (b) $\left(\frac{39}{40}\right)^2$ (c) $\left(\frac{39}{40}\right)^5$ (d) $1-\left(\frac{39}{40}\right)^5$
29 (a) $\frac{1}{1000}$ (b) $\frac{999}{1000}$
30 (a) $\frac{1}{208}$ (b) $\frac{1}{78}$ (c) $\frac{1}{120}$ (d) $\frac{1}{9360}$
31 (a) 0.27 (b) 0.97 (c) 0.03 (d) 0.32 (e) 0.162

EXERCISE 18.6

1 (a) $\frac{5}{14}$ (b) $\frac{13}{28}$ (c) $\frac{15}{28}$ **2** B **3** $\frac{47}{91}$ **4** $\frac{1}{220}$
5 (a) $\frac{2}{3}$ (b) $\frac{5}{12}$ (c) $\frac{1}{2}$
6 (a) correct (b) incorrect (c) correct (d) correct
7 (a) $\frac{1}{22}$ (b) $\frac{1}{11}$ (c) $\frac{5}{33}$ **8** (a) $\frac{7}{22}$ (b) $\frac{5}{33}$ (c) $\frac{35}{66}$
(d) The possibilities 'one teacher and one pupil', 'both teachers' and 'both pupils' are the only possibilities, so
$P(\text{one teacher one pupil}) = 1 - P(\text{both teachers}) - P(\text{both pupils})$.
9 (a) $\frac{1}{35}$ (b) $\frac{4}{35}$ (c) $\frac{4}{35}$ **10** $\frac{1}{90}$ **11** $\frac{6}{11}$
12 (a) $\frac{3}{10}$ (b) $\frac{1}{15}$ (c) $\frac{7}{15}$ (d) $\frac{7}{15}$ **13** $\frac{17}{38}$
14 (a) $\frac{3}{7}$ (b) $\frac{1}{3}$ (c) $\frac{8}{15}$ **15** (a) $\frac{2}{15}$ (b) $\frac{1}{3}$ (c) $\frac{8}{15}$
16 $\frac{3}{26}$ **17** $\frac{2}{3}$ **18** $\frac{1}{504}$ **19** $\frac{22}{425}$
20 (a) $\frac{1}{5}$ (b) $\frac{7}{25}$ **21** (a) $\frac{1}{5525}$ (b) $\frac{1}{2197}$
22 (a) $\frac{36}{125}$ (b) $\frac{44}{125}$ (c) $\frac{117}{125}$ **23** (a) 0.001 (b) 0.729 (c) 0.028
24 $\frac{3}{52}$ **25** (a) $\frac{20}{29}$ (b) $\frac{2}{87}$ (c) $\frac{9}{29}$ **26** (a) $\frac{2}{9}$ (b) $\frac{5}{9}$ (c) $\frac{4}{45}$
27 (a) $\frac{11}{850}$ (b) $\frac{1}{5525}$ (c) $\frac{997}{1700}$ **28** (a) $\frac{124}{125}$ (b) $\frac{13}{125}$
29 (a) 0.343 (b) 0.973 **30** (a) 0.729 (b) 0.972 (c) 0.028
31 (a) $\frac{135}{512}$ (b) $\frac{485}{512}$ **32** (a) 0.343 (b) 0.784 (c) 0.973
33 (a) $\frac{4}{49}$ (b) $\frac{24}{49}$ **34** (a) $\frac{13}{25}$ (b) $\frac{12}{25}$ **35** (a) $\frac{7}{15}$ (b) $\frac{8}{15}$
36 (a) $\frac{2}{9}$ (b) $\frac{7}{9}$ **37** (a) $\frac{1}{4}$ (b) $\frac{3}{4}$
38 (a) $\frac{1}{3}$ (b) $\frac{2}{3}$



39 0.648



40 0.305

CHAPTER REVIEW 18

1 0.001

2 (a) $\frac{1}{6}$ (b) $\frac{5}{6}$ (c) $\frac{3}{8}$ (d) $\frac{1}{3}$

3 (a) $\frac{5}{18}$ (b) $\frac{13}{18}$

4 (a) $\frac{54}{125}$ (b) $\frac{81}{125}$

5 (a) $\frac{1}{6}$ (b) $\frac{1}{3}$

6 (a) $\frac{8}{25}$ (b) $\frac{2}{25}$ (c) $\frac{4}{5}$ (d) $\frac{2}{5}$

GLOSSARY

A

abscissa
The first member of an ordered pair on the number plane; for coordinates (x, y) the abscissa is x. Plural *abscissae*.

absolute value
1 Geometrically, the distance from 0 on the number line.

2 Algebraically, the size of the expression inside the absolute value symbols (always a positive number).

acceleration
The rate of change of velocity with respect to time, $\frac{dv}{dt}$, standard units m s^{-2}.

amplitude
The amplitude of a sine or cosine function is half the difference between the greatest and least values of the function. The amplitude of $y = a \sin nx$ is a.

annuity
A form of investment in which periodical equal contributions are made to or taken from an account, with interest compounding at the conclusion of each period.

antidifferentiation
The reverse process to differentiation, also known as finding the primitive function.

arc of curve
A portion of a curve joining two points.

area under a curve
The region in the number plane bounded by a curve, the x-axis and two x values (abscissae).

arithmetic series
A series formed by adding a set of terms, where each term is obtained by adding a particular number to the previous term.

asymptote
A line that a function approaches but never reaches.

C

Cartesian coordinates
An ordered pair of numbers (x, y) used to locate a point in the number plane.

chain rule
A rule used to find the derivative of a function of a function. If $y = f(u)$ and $u = g(x)$ then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.

change of base
The function $\log_a n = \frac{\log_b n}{\log_b a}$ can be used to change the base of a logarithm from base b to base a.

chord
A straight line joining two points on a circle.

circular measure
The size of an angle, measured in radians.

common difference
The difference between pairs of successive terms of an arithmetic series.

common ratio
The ratio of successive terms of a geometric series.

complementary angles
Two angles whose sum is 90° or $\frac{\pi}{2}$ radians (i.e. a right angle).

complementary events
Two events such that one or the other event must occur, but not both. The sum of their probabilities is one.

composite events
Events that consist of two or more simple events.

composite function rule
See *chain rule*.

concave downwards
If $f''(x) < 0$ then the curve of f is concave downwards.

concave upwards
If $f''(x) > 0$ then the curve of f is concave upwards.

concavity
The way a curve 'turns' or 'opens', either *concave upwards* or *concave downwards*.

continuity
A function is said to be continuous (or that it 'has continuity') if for every x value in the domain there exists a corresponding y value, i.e. there are no gaps or holes.

continuous function
A function with a curve that has no gaps. A continuous curve can be drawn without lifting your pen from the paper.

cosecant (cosec)
The reciprocal of the sine function.

cotangent (cot)
The reciprocal of the tangent function.

curve
A joined set of points representing a function or a relation.

D

decreasing function
A function that always has a negative gradient.

definite integral
An integral written $\int_a^b f(x)\,dx$, where a is the lower limit and b is the upper limit of the integration.

dependent variable
The variable that depends on the independent variable. In $y = f(x)$, x is the independent variable and y is the dependent variable.

derivative at a point
The value of the derivative at a particular point; the gradient of the tangent to the curve at that point.

derived function
The result of differentiation; gives the gradient of the tangent at any point on the curve. Also known as 'gradient function'.

differentiable function
A function that has a derivative at all points in the given domain.

differentiation
The process of finding the gradient function from a function.

differentiation from first principles
Finding the derivative using the formula $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

discontinuous function
A curve that is not continuous; a curve that has gaps, holes or jumps.

discrete random variable
A variable with a fixed number of possible outcomes, e.g. selecting from the positive integers less than 10.

discriminant
For a quadratic equation $y = ax^2 + bx + c$, the discriminant is $\Delta = b^2 - 4ac$.

displacement
The signed distance (positive or negative) of a particle from the origin.

domain
The set of values of the independent variable (x) for which a function exists.

E

equally likely outcomes
Outcomes that have the same chance of occurring.

Euler's number
The irrational number $e \approx 2.718\,28\ldots$ defined by $e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$ such that the function $y = e^x$ is its own derivative, or $\frac{d}{dx}\left(e^x\right) = e^x$.

even function
A function is even if $f(-x) = f(x)$ for all values of x in the domain. The function is also symmetrical about the y-axis.

exponential decay
Exponential decay (or 'exponential decline') occurs when a quantity decreases by a constant percentage over time.

exponential function
A function with its variable as an index, e.g. $y = a^x$.

exponential growth
Exponential growth occurs when a quantity increases by a constant percentage over time.

extrapolate
To predict the value of a function outside the given domain.

F

first derivative test
If $f'(x) = 0$, then the point at x is a stationary point. If the sign of $f'(x)$ changes from positive to negative through x, then it is a local maximum turning point; if the sign of $f'(x)$ changes from negative to positive through x, then it is a local minimum turning point.

function
A relationship between two variables, e.g. x and y, such that there is one y value for each x value.

fundamental counting principle
If one event can occur in m ways and another event can occur in n ways, then the two-stage event (both events together) can occur in $m \times n$ ways. Also called 'multiplication principle'.

fundamental theorem of calculus
$\int_a^b f(x)\,dx = \left[F(x)\right]_a^b = F(b) - F(a)$, where $F(x)$ is the primitive function of $f(x)$.

G

general term
An expression that allows you to calculate any of the terms of a series.

geometric series
A series formed by adding a set of terms, where each term is obtained by multiplying the previous term by the same number each time.

gradient of a secant
The slope of the line joining two points on a curve.

gradient of a tangent
The slope of the line that touches a point on a curve. This is the limiting position of the secant, where two points become the same point.

greatest value of a function
For a given domain, the value of a function at its highest maximum turning point or highest endpoint, whichever is the greatest.

H

half-life
The time taken for half of the atoms in a radioactive substance to decay.

horizontal point of inflexion
The point on a curve where $\frac{dy}{dx} = 0$ and at which the gradient has the same sign on either side of the point.

I

increasing function
A function that always has a positive gradient.

indefinite integral
The primitive of a function, written $\int f(x)\,dx$.

independent events
Events are independent if the occurrence or non-occurrence of one event cannot change the probability of the occurrence of another event, i.e. the events have no effect on each other.

independent variable
The variable that we choose to substitute into a function to generate the dependent variable.

integration
See *antidifferentiation*. The reverse process to differentiation, used to find the area under a curve.

interpolate
To predict the value of a function in the given domain using the data available when a given rule or function is not available.

L

limiting sum
For a geometric series with a common ratio $|r| < 1$, the value that the sum of the series approaches as the number of terms increases to infinity.

local maximum
The highest value of a function in a certain neighbourhood. Occurs at a maximum turning point and is the value of the function at this point.

local minimum
The lowest value of a function in a certain neighbourhood. Occurs at a minimum turning point and is the value of the function at this point.

logarithmic function
The function $f(x) = \log_a x$, where $a^{\log_a x} = \log_a\left(a^x\right) = x$.

M

maximum turning point
A point on a curve where $\frac{dy}{dx} = 0$ and the gradient changes from positive to negative as x increases through the point. If $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$ then it is a maximum turning point; if $\frac{d^2y}{dx^2} = 0$ then it may or may not be a maximum turning point.

minimum turning point
A point on a curve where $\frac{dy}{dx} = 0$ and the gradient changes from negative to positive as x increases through the point. If $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$ then it is a minimum turning point; if $\frac{d^2y}{dx^2} = 0$ then it may or may not be a minimum turning point.

mutually exclusive events
Two events are mutually exclusive (or 'disjoint') if membership of one event excludes membership of the other, so that they cannot occur simultaneously.

N

natural logarithm
A logarithm to base e, written $\log_e x$ or $\ln x$. Also called 'Naperian' or 'Napierian' logarithm.

normal
The line perpendicular to the tangent at a point of contact on a curve.

O

odd function
A function is odd if $f(-x) = -f(x)$ for all values of x in the domain.

ordinate
The second member of an ordered pair on the number plane; for coordinates (x, y) the ordinate is y.

oscillation
A movement from one extreme position to another and then back again, made by an object with a wave-like or vibrating motion.

P

particle
A body that behaves such that all forces acting on the body can be regarded as acting through a single point. This means that the body can be represented as a single point, regardless of its actual size and shape.

period
The interval over which a trigonometric graph repeats itself. The period of $y = a \sin nx$ is $\frac{2\pi}{n}$.

periodic phenomenon
Something that repeats itself in a regular (periodic) way.

point of inflexion
A point on a curve where $\frac{d^2y}{dx^2} = 0$ and the concavity changes.

primitive function
If $F'(x) = f(x)$, then $F(x)$ is the primitive function of $f(x)$.

product rule
A rule used to find the derivative of the product of two functions. If $y = uv$, where u and v are functions of x, then $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$.

Q

quadratic formula
A formula that gives the roots of the general quadratic equation $ax^2 + bx + c = 0$: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

quotient rule
A rule used to find the derivative of the quotient of two functions. If $y = \frac{u}{v}$, where u and v are functions of x, then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$.

R

radian
The angle at the centre of a unit circle subtended by an arc of unit length. 2π radians = 360°.

range
The set of values of a function for a given domain.

rate of change
The rate that compares the change in one variable (the dependent variable) with the change in another other variable (the independent variable). The rate of change of a function is the gradient of its curve.

relation
A set of points for which a given value of x may have more than one y value. A circle is a relation. Also called 'multi-valued function'.

S

sample space
All possible outcomes of an experiment.

secant (sec)
1 A straight line joining two points on a curve.

2 The reciprocal of the cosine function.

second derivative
A derivative of a derivative, written $\frac{d^2y}{dx^2}$ or $f''(x)$.

sector
A portion of a circle bounded by two radii.

segment
A portion of a circle cut off by a chord.

series
The sum of the terms of a sequence.

sigma notation (Σ)
The symbol Σ (the Greek capital letter sigma) means 'the sum of'. It is specifically used in series notation to mean 'all these terms added together, starting with the value underneath the Σ and continuing to the value above the Σ'.

simple harmonic motion
A kind of motion where a particle oscillates about an equilibrium position.

Simpson's rule
An approximation to the definite integral made by finding the area under a parabola formed from three points on the curve of f:
$\int_a^b f(x)dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$

solid of revolution
A three-dimensional solid formed by rotating an area under a curve about one of the coordinate axes.

stationary point
A point on a curve at which $\frac{dy}{dx} = 0$.

subinterval
A section of an interval of a definite integral that has been divided into smaller parts, as when using Simpson's rule or the trapezoidal rule.

sum to *n* terms
To add together the terms of a series, from the first term to the *n*th term.

T

tangent
A line that touches a curve only at the point of contact and does not pass through (cut) the curve.

term
A single part of a series.

time rate of change
The rate at which something is changing over time.

trapezoidal rule
An approximation to the definite integral made by finding the area of a trapezoidal shape between two points on the curve:

$$\int_a^b f(x)\,dx \approx \frac{(b-a)}{2}\big(f(a)+f(b)\big)$$

tree diagram
A diagram that shows outcomes and probabilities of multiple and/or successive events by connecting possible outcomes along 'branches'.

trigonometric function
A function involving sin, cos, tan, cosec, sec or cot.

trigonometric identity
A general relation involving trigonometric functions that is true for all values, e.g. $\sin^2 x + \cos^2 x = 1$

turning point
The point on a curve where $\frac{dy}{dx} = 0$ and at which the gradient changes sign on either side of the point.

V

velocity
The rate of change of displacement with respect to time, standard units $\mathrm{m\,s^{-1}}$.

Venn diagram
A diagram that shows closed curves (e.g. circles or other shapes) around members of sets, using the shapes' intersections to show the common elements of different sets.

vertical line test
If a vertical line cannot pass through a curve more than once, then the curve represents a function.