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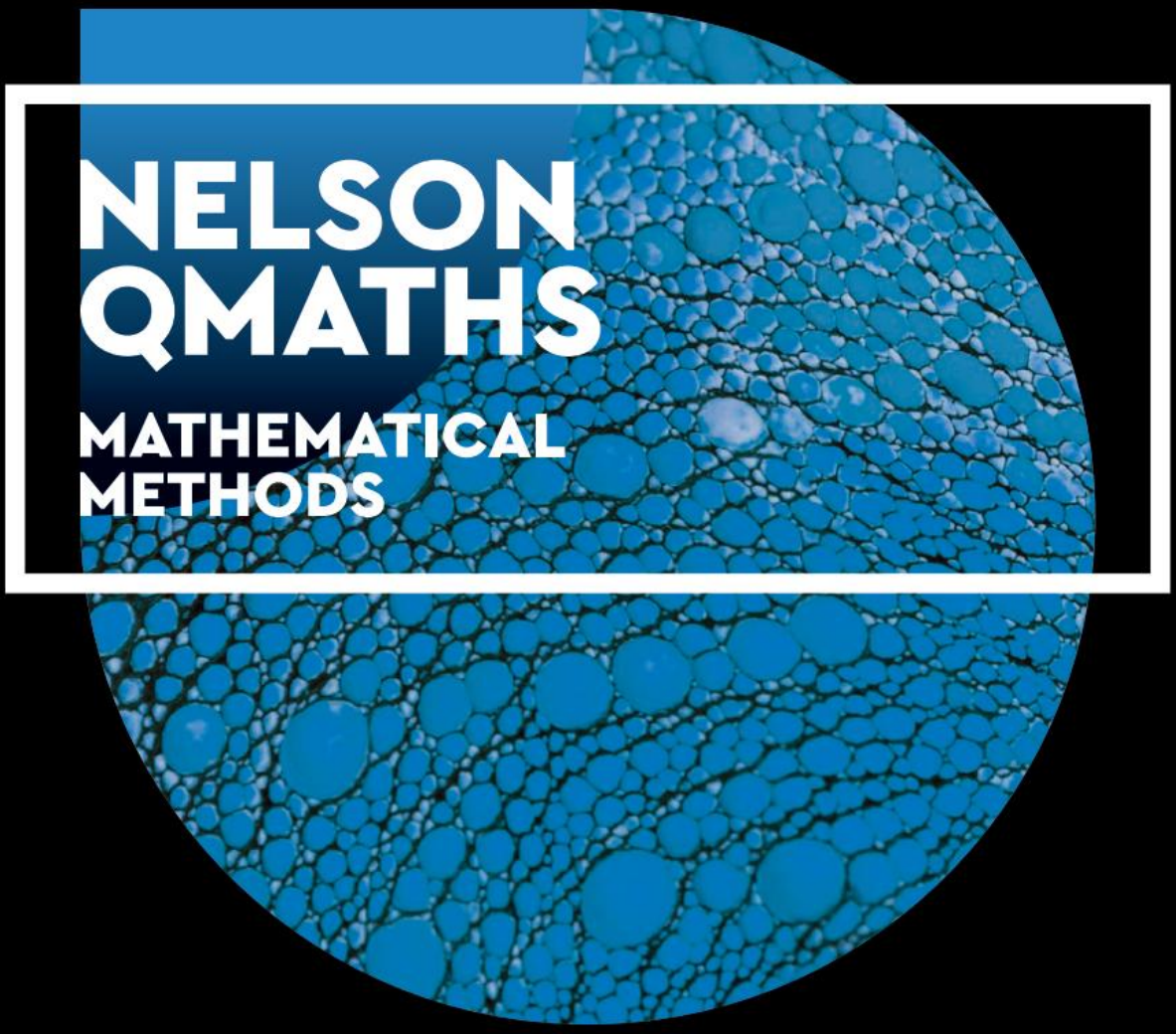


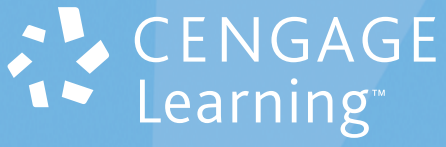
NELSON QMATHS

MATHEMATICAL METHODS

YEAR 12

Ross Brodie
Regina Edwards
Stephen Swift

4TH EDITION



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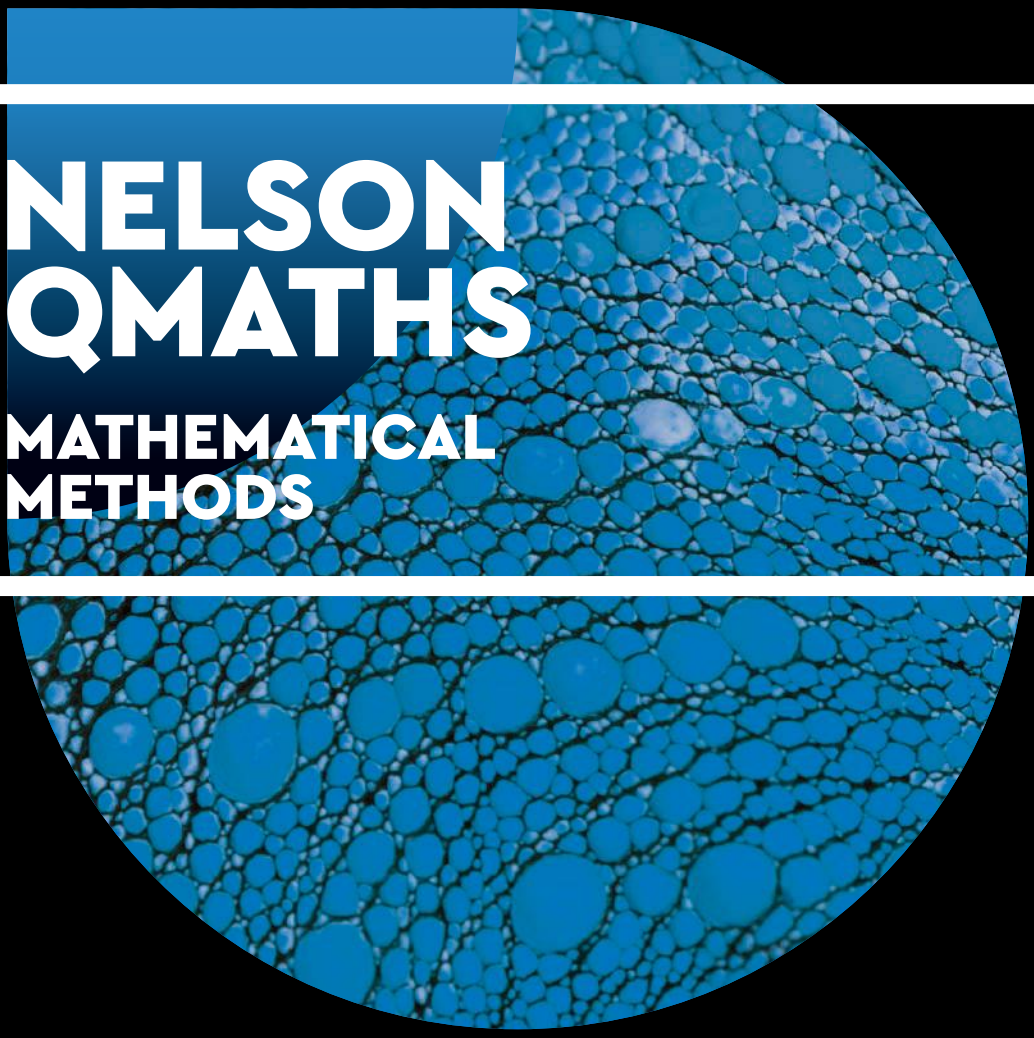

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PREFACE

Nelson QMaths, Queensland's longest-running senior mathematics series, has been rewritten for the new syllabuses and assessment procedures for implementation from 2019. Based on the Australian Curriculum, four new senior mathematics courses have been introduced into Queensland schools.

- Essential Mathematics
- General Mathematics
- Mathematical Methods
- Specialist Mathematics

With the introduction of external QCE examinations, *Nelson QMaths* will have a renewed focus on assessment, and will include features such as topic tests, practice examinations, *ExamView* question banks and chapter quizzes, video tutorials and worked solutions to all exercise questions. In this book, teachers will find familiar features such as clear worked examples, graded exercises, strong syllabus coverage, Investigations, Technology, chapter summaries, chapter reviews and a glossary/index. We wish all teachers and students using this book every success in embracing the new mathematics courses.

ABOUT THE AUTHORS

Ross Brodie has taught mathematics and been Head of Mathematics at a variety of regional and urban schools in Queensland. He is an experienced teacher and author, having co-written *Nelson QMaths 7-10*, *New QMaths 11-12* and *Nelson Senior Maths 11-12* (Australian Curriculum).

Regina Edwards teaches mathematics and science at Cleveland District State High School where she coordinates International Baccalaureate High Level Mathematics and Physics. She was a district panellist for Mathematics C and Physics, and has worked in the Information Technology industry as an analyst.

Stephen Swift has taught Mathematics, Science and Computing in urban and regional schools, and was Head of Mathematics at Wellington Point State High School. He is the lead author of *Nelson QMaths 7-10*, *New QMaths 11-12* and *Nelson Senior Maths 11-12* (Australian Curriculum).

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Amy Hawke (Brisbane SHS) and **Tara Kuhn** (Mansfield SHS) wrote many of the *NelsonNet* worksheets.

Scott Smith created the video tutorials.

Tania Eastcott and **Elizabeth Nabhan** wrote the topic tests.

Sharon Kirkby and **Dina Antoniou** wrote the worked solutions to all exercise sets.

Roger Walter wrote the ExamView questions.

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SYLLABUS REFERENCE GRID

Topics and subtopics	*Nelson QMaths 12 Mathematics Methods chapter*
UNIT 3: FURTHER CALCULUS	
The logarithmic function 2	
Logarithmic laws and logarithmic functions	**1** Logarithmic functions
Further differentiation and applications 2	
Calculus of exponential functions	**2** Derivatives of exponential functions
Calculus of logarithmic functions	**3** Derivatives of logarithmic functions
Calculus of trigonometric functions	**4** Derivatives of trigonometric functions
Differentiation rules	**5** Anti-differentiation
Integrals	
Anti-differentiation	**5** Anti-differentiation
Fundamental theorem of calculus and definite integrals	**6** Integration and applications
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General continuous random variables	**10** Continuous random variables and the normal distribution
Normal distributions	**10** Continuous random variables and the normal distribution
Interval estimates for proportions	
Random sampling	**11** Confidence intervals for proportions
Sample proportions	**11** Confidence intervals for proportions
Confidence intervals for proportions	**11** Confidence intervals for proportions

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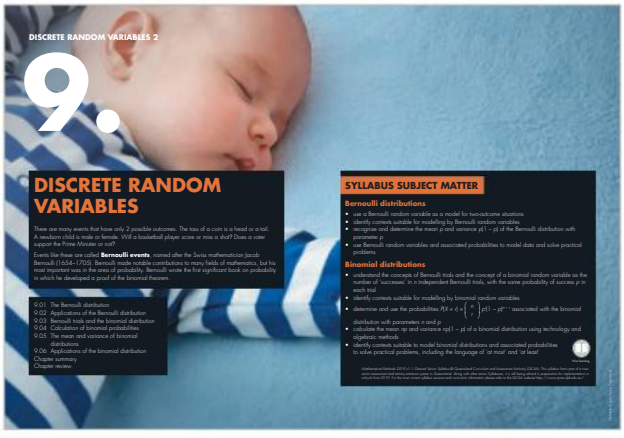
ABOUT THIS BOOK

AT THE BEGINNING OF EACH CHAPTER

- Each chapter begins on a double-page spread showing the **Chapter contents** and a list of syllabus subject matter (© Queensland Curriculum and Assessment Authority 2017)


DISCRETE RANDOM VARIABLES 2

9.

DISCRETE RANDOM VARIABLES

There are many events that have only 2 possible outcomes. The toss of a coin is a head or a tail. A newborn child is male or female. Will a basketball player score or miss a shot? Does a voter support the Prime Minister or not?

Events like these are called **Bernoulli events**, named after the Swiss mathematician Jacob Bernoulli (1654–1705). Bernoulli made notable contributions to many fields of mathematics, but his most important was in the area of probability. Bernoulli wrote the first significant book on probability in which he developed a proof of the binomial theorem.

9.01 The Bernoulli distribution
9.02 Applications of the Bernoulli distribution
9.03 Bernoulli trials and the binomial distribution
9.04 Calculation of binomial probabilities
9.05 The mean and variance of binomial distributions
9.06 Applications of the binomial distribution
Chapter summary
Chapter review

SYLLABUS SUBJECT MATTER

Bernoulli distributions

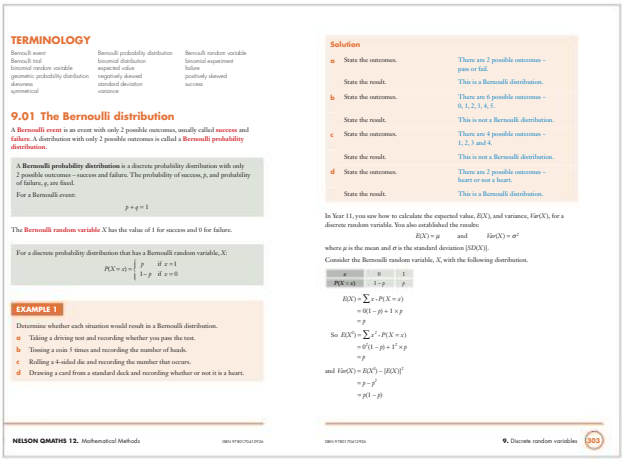
- use a Bernoulli random variable as a model for two-outcome situations
- identify contexts suitable for modelling by Bernoulli random variables
- recognise and determine the mean p and variance $p(1-p)$ of the Bernoulli distribution with parameter p
- use Bernoulli random variables and associated probabilities to model data and solve practical problems

Binomial distributions

- understand the concepts of Bernoulli trials and the concept of a binomial random variable as the number of 'successes' in n independent Bernoulli trials, with the same probability of success p in each trial
- identify contexts suitable for modelling by binomial random variables
- determine and use the probabilities $P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$ associated with the binomial distribution with parameters n and p
- calculate the mean np and variance $np(1-p)$ of a binomial distribution using technology and algebraic methods
- identify contexts suitable to model binomial distributions and associated probabilities to solve practical problems, including the language of 'at most' and 'at least'

Mathematical Methods (2019) © Queensland Curriculum and Assessment Authority (QCAA). This syllabus from part of a new senior secondary syllabus in Queensland. Along with other senior syllabuses, it is currently offered as a resource for registered schools from 2019 to 2020. For more information, syllabus updates and consultation, please refer to the QCAA website: http://www.qcaa.qld.edu.au/

- **Terminology** previews the key words and phrases from within the chapter.


TERMINOLOGY

Bernoulli event
Bernoulli trial
binomial random variable
geometric probability distribution
skewness
symmetrical
Bernoulli probability distribution
binomial distribution
expected value
negatively skewed
standard deviation
variance
Bernoulli random variable
binomial experiment
failure
positively skewed
success

9.01 The Bernoulli distribution

A **Bernoulli event** is an event with only 2 possible outcomes, usually called **success** and **failure**. A distribution with only 2 possible outcomes is called a **Bernoulli probability distribution**.

A **Bernoulli probability distribution** is a discrete probability distribution with only 2 possible outcomes – success and failure. The probability of success, p, and probability of failure, q, are fixed.

For a Bernoulli event:

$$p + q = 1$$

The **Bernoulli random variable** X has the value of 1 for success and 0 for failure.

For a discrete probability distribution that has a Bernoulli random variable, X:

$$P(X = x) = \begin{cases} p & \text{if } x = 1 \\ 1-p & \text{if } x = 0 \end{cases}$$

EXAMPLE 1

Determine whether each situation would result in a Bernoulli distribution.

a Taking a driving test and recording whether you pass the test.
b Tossing a coin 5 times and recording the number of heads.
c Rolling a 4-sided die and recording the number that occurs.
d Drawing a card from a standard deck and recording whether or not it is a heart.

Solution

a State the outcomes. — There are 2 possible outcomes – pass or fail.
State the result. — This is a Bernoulli distribution.
b State the outcomes. — There are 6 possible outcomes – 0, 1, 2, 3, 4, 5.
State the result. — This is not a Bernoulli distribution.
c State the outcomes. — There are 4 possible outcomes – 1, 2, 3 and 4.
State the result. — This is not a Bernoulli distribution.
d State the outcomes. — There are 2 possible outcomes – heart or not a heart.
State the result. — This is a Bernoulli distribution.

In Year 11, you saw how to calculate the expected value, $E(X)$, and variance, $Var(X)$, for a discrete random variable. You also established the results:

$$E(X) = \mu \quad \text{and} \quad Var(X) = \sigma^2$$

where μ is the mean and σ is the standard deviation $[SD(X)]$.

Consider the Bernoulli random variable, X, with the following distribution.

x	0	1
$P(X = x)$	$1-p$	p

$$\begin{aligned} E(X) &= \sum x \cdot P(X = x) \\ &= 0(1-p) + 1 \times p \\ &= p \end{aligned}$$

$$\begin{aligned} \text{So } E(X^2) &= \sum x^2 \cdot P(X = x) \\ &= 0^2(1-p) + 1^2 \times p \\ &= p \end{aligned}$$

$$\begin{aligned} \text{and } Var(X) &= E(X^2) - [E(X)]^2 \\ &= p - p^2 \\ &= p(1-p) \end{aligned}$$

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9. Discrete random variables 303

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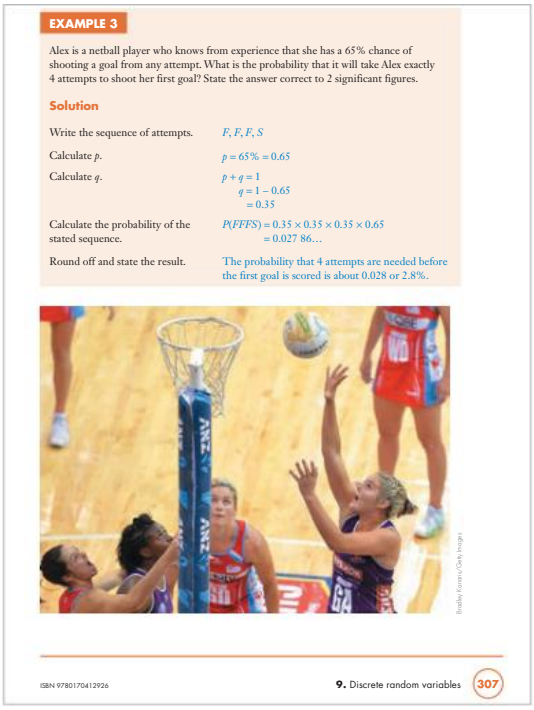
IN EACH CHAPTER

- Worked examples are explained clearly step-by-step, with the mathematical working shown on the right-hand side.
- Important facts and formulas are highlighted in a shaded box.
- Important words and phrases are printed in red and listed in the glossary at the back of the book.
- Graded exercises include **Problem solving** questions, are linked to worked examples and include exam-style problems and realistic applications.
- Worked solutions to all exercise questions are provided on the *NelsonNet* teacher website.
- **Investigations** explore the syllabus in more detail, providing ideas for modelling activities and assessment tasks.
- **Technology** promotes ICT in the classroom, using spreadsheets, graphics calculators and the Internet.
- Instructions are provided for the TI-84 Plus CE, Casio fx-CG20AU and TI-Nspire CX calculators.



EXAMPLE 3

Alex is a netball player who knows from experience that she has a 65% chance of shooting a goal from any attempt. What is the probability that it will take Alex exactly 4 attempts to shoot her first goal? State the answer correct to 2 significant figures.

Solution

Write the sequence of attempts.	F, F, F, S
Calculate p.	$p = 65\% = 0.65$
Calculate q.	$p + q = 1$ $q = 1 - 0.65$ $= 0.35$
Calculate the probability of the stated sequence.	$P(FFFS) = 0.35 \times 0.35 \times 0.35 \times 0.65$ $= 0.027\ 86\ldots$
Round off and state the result.	The probability that 4 attempts are needed before the first goal is scored is about 0.028 or 2.8%.

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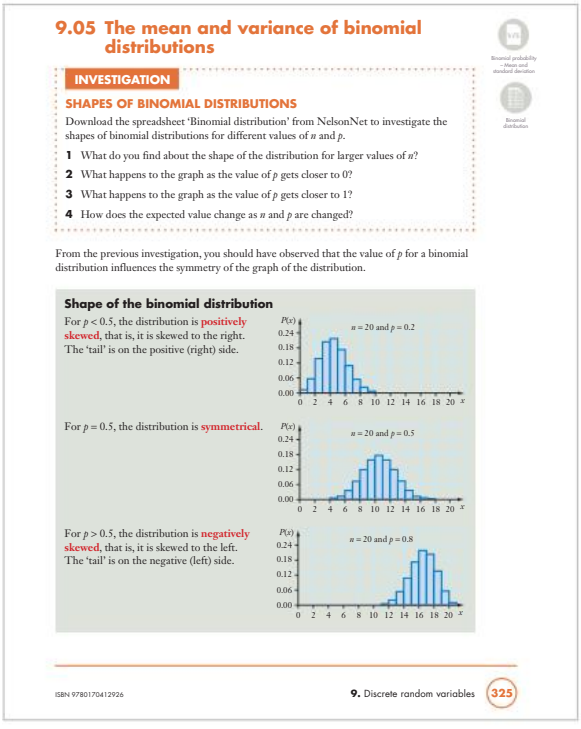

9.05 The mean and variance of binomial distributions

INVESTIGATION

SHAPES OF BINOMIAL DISTRIBUTIONS

Download the spreadsheet 'Binomial distribution' from NelsonNet to investigate the shapes of binomial distributions for different values of n and p.

1. What do you find about the shape of the distribution for larger values of n?
2. What happens to the graph as the value of p gets closer to 0?
3. What happens to the graph as the value of p gets closer to 1?
4. How does the expected value change as n and p are changed?

From the previous investigation, you should have observed that the value of p for a binomial distribution influences the symmetry of the graph of the distribution.

Shape of the binomial distribution

For $p < 0.5$, the distribution is **positively skewed**, that is, it is skewed to the right. The 'tail' is on the positive (right) side.

For $p = 0.5$, the distribution is **symmetrical**.

For $p > 0.5$, the distribution is **negatively skewed**, that is, it is skewed to the left. The 'tail' is on the negative (left) side.

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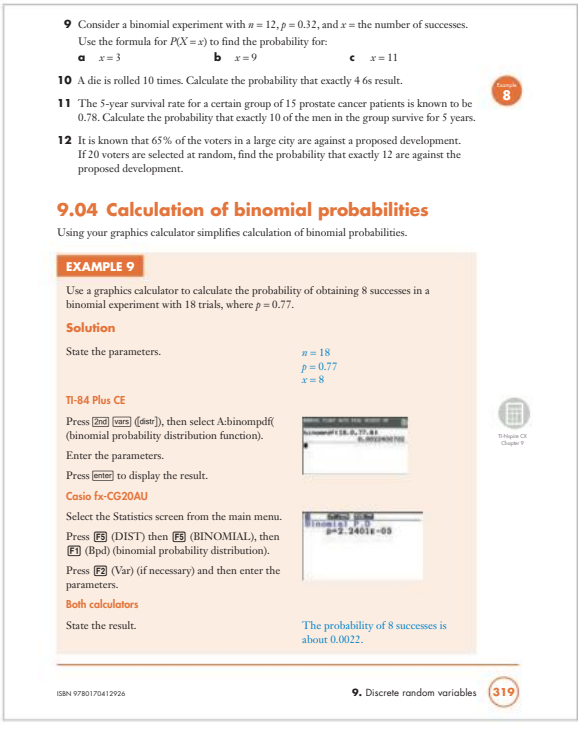

9 Consider a binomial experiment with $n = 12$, $p = 0.32$, and $x =$ the number of successes. Use the formula for $P(X = x)$ to find the probability for:

a $x = 3$ **b** $x = 9$ **c** $x = 11$

10 A die is rolled 10 times. Calculate the probability that exactly 4 6s result.

11 The 5-year survival rate for a certain group of 15 prostate cancer patients is known to be 0.78. Calculate the probability that exactly 10 of the men in the group survive for 5 years.

12 It is known that 65% of the voters in a large city are against a proposed development. If 20 voters are selected at random, find the probability that exactly 12 are against the proposed development.

9.04 Calculation of binomial probabilities

Using your graphics calculator simplifies calculation of binomial probabilities.

EXAMPLE 9

Use a graphics calculator to calculate the probability of obtaining 8 successes in a binomial experiment with 18 trials, where $p = 0.77$.

Solution

State the parameters. $n = 18$, $p = 0.77$, $x = 8$

TI-84 Plus CE

Press 2nd vars (distr]), then select A:binompdf((binomial probability distribution function).

Enter the parameters.

Press enter to display the result.

Casio fx-CG20AU

Select the Statistics screen from the main menu.

Press F5 (DIST) then F5 (BINOMIAL), then F1 (Bpd) (binomial probability distribution).

Press F2 (Var) (if necessary) and then enter the parameters.

Both calculators

State the result. The probability of 8 successes is about 0.0022.

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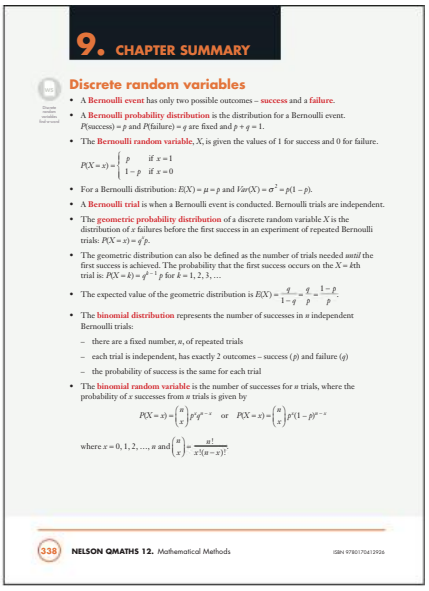
AT THE END OF EACH CHAPTER

- **Chapter summary** summarises the key concepts, skills and formulas of the chapter.
- **Chapter review** contains revision exercises and are linked to worked examples.
- **Practice examinations** after every 3 chapters revise the skills and knowledge of those chapters.



9. CHAPTER SUMMARY

Discrete random variables

- A **Bernoulli event** has only two possible outcomes – **success** and a **failure**.
- A **Bernoulli probability distribution** is the distribution for a Bernoulli event. $P(\text{success}) = p$ and $P(\text{failure}) = q$ are fixed and $p + q = 1$.
- The **Bernoulli random variable**, X, is given the values of 1 for success and 0 for failure.

$$P(X = x) = \begin{cases} p & \text{if } x = 1 \\ 1 - p & \text{if } x = 0 \end{cases}$$

- For a Bernoulli distribution: $E(X) = \mu = p$ and $Var(X) = \sigma^2 = p(1 - p)$.
- A **Bernoulli trial** is when a Bernoulli event is conducted. Bernoulli trials are independent.
- The **geometric probability distribution** of a discrete random variable X is the distribution of x failures before the first success in an experiment of repeated Bernoulli trials: $P(X = x) = q^x p$.
- The geometric distribution can also be defined as the number of trials needed *until* the first success is achieved. The probability that the first success occurs on the $X = k$th trial is: $P(X = k) = q^{k-1} p$ for $k = 1, 2, 3, \ldots$
- The expected value of the geometric distribution is $E(X) = \frac{q}{1-q} = \frac{q}{p} = \frac{1-p}{p}$.
- The **binomial distribution** represents the number of successes in n independent Bernoulli trials:
  - there are a fixed number, n, of repeated trials
  - each trial is independent, has exactly 2 outcomes – success (p) and failure (q)
  - the probability of success is the same for each trial
- The **binomial random variable** is the number of successes for n trials, where the probability of x successes from n trials is given by

$$P(X = x) = \binom{n}{x} p^x q^{n-x} \quad \text{or} \quad P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$$

where $x = 0, 1, 2, \ldots, n$ and $\binom{n}{x} = \frac{n!}{x!(n-x)!}$.

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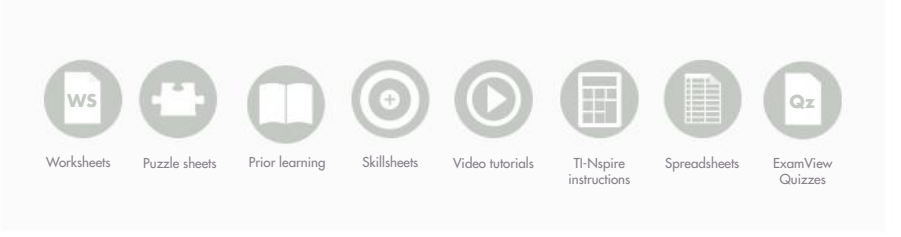
AT THE END OF THE BOOK

- **Glossary and index** includes a comprehensive dictionary of course terminology.
- **Answers** (with worked solutions on the teacher website).

NELSONNET STUDENT WEBSITE

Margin icons link to print (PDF) and multimedia resources found on the *NelsonNet* student website, **www.nelsonnet.com.au**. These include:



- **Worksheets** and **puzzle sheets** that are write-in enabled PDFs
- **Prior learning** exercises that revise prerequisite skills for the chapter
- **Skillsheets** of examples and exercises of prerequisite skills and knowledge
- **Video tutorials:** worked examples explained by 'flipped classroom' teachers
- **Graphics calculator instructions** for TI-Nspire CX calculators
- **Spreadsheets:** *Excel* files
- **ExamView quizzes:** interactive and self-marking

NELSONNET TEACHER WEBSITE

The *NelsonNet* teacher website, also at **www.nelsonnet.com.au**, contains:

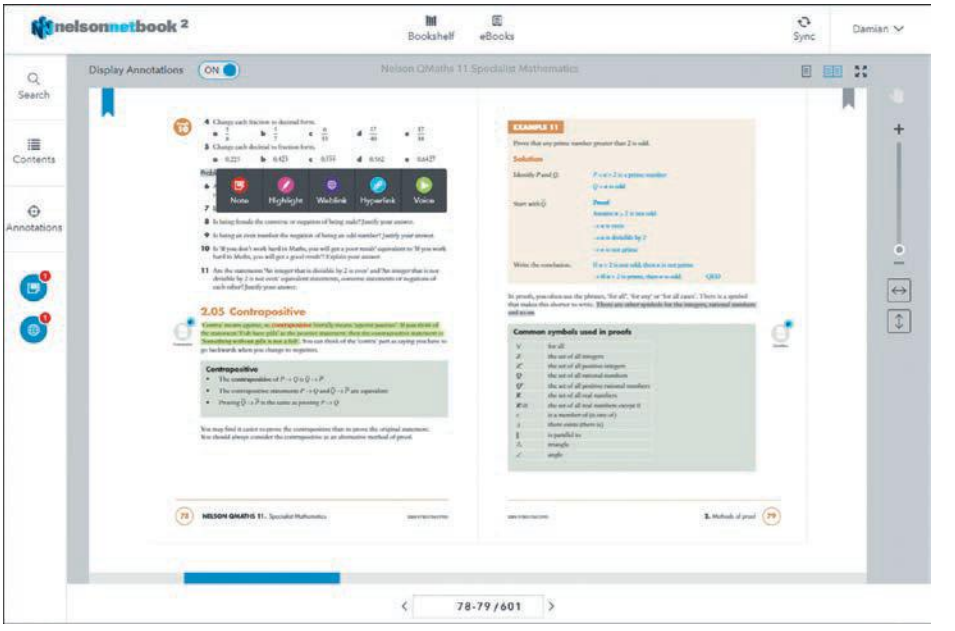
- **Teaching plan,** in Microsoft Word and PDF formats
- **Topic tests,** in Microsoft Word and PDF formats
- **Worked solutions** to each exercise set
- **Chapter PDFs** of the textbook
- **ExamView** exam-writing software and questionbanks
- **Resource Finder:** search engine for *NelsonNet* resources

Note: Complimentary access to these resources is only available to teachers who use this book as a core educational resource in their classroom. Contact your Cengage Education Consultant for information about access codes and conditions.

NELSONNETBOOK

NelsonNetBook is the web-based interactive version of this book found on *NelsonNet*.

- To each page of **NelsonNetBook** you can add notes, voice and sound bites, highlighting, weblinks and bookmarks
- **Zoom** and **Search** functions
- Chapters can be customised for different groups of students



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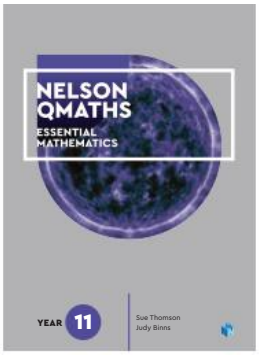

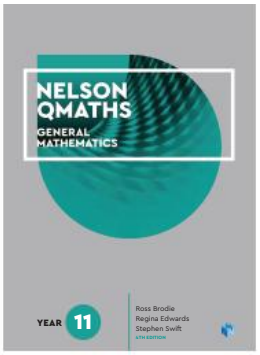

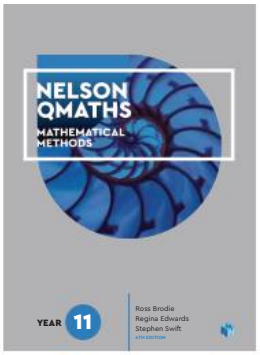

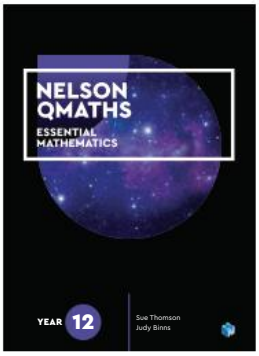

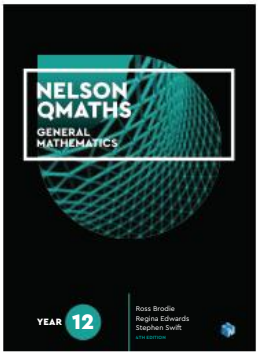

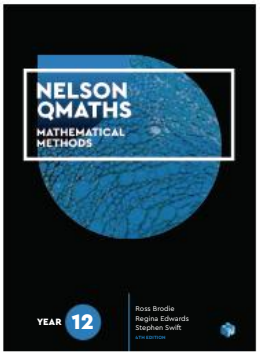

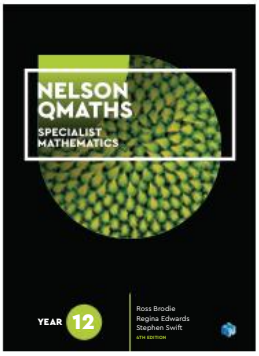

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LIST OF SYMBOLS AND ABBREVIATIONS

=	is equal to
≡	is identically equal to
≠	is not equal to
≈	is approximately equal to
<	is less than
>	is greater than
≤	is less than or equal to
≥	is greater than or equal to
AP	arithmetic progression
t_n, a_n	nth term of a sequence
S_n	the sum to n terms
GP	geometric progression
S_∞	sum to infinity
FV	future value
PV	present value
i	interest rate
$S_{\overline{n}\rceil i}$, $S_{\overline{n}\rceil}$	future value annuity factor
$a_{\overline{n}\rceil i}$, $a_{\overline{n}\rceil}$	present value annuity factor
∈	is a member of
∪	union
∩	intersection
±	plus or minus
π	pi ≈ 3.14
⊆	is a subset of
⊂	is a proper subset of
∉	is not a member of
Σ	the sum of
…	and so on
{}, ∅	the null set
U, $\mathbb{E}$	the universal set
$\boldsymbol{R}$	the real numbers
$\boldsymbol{R}^+$	the positive real numbers
$\boldsymbol{Z}$, $\boldsymbol{J}$	the integers
$\boldsymbol{Z}^+$	the positive integers
$\boldsymbol{N}$	the natural numbers
°	degree
∞	infinity
⇒	implies
⇔	if and only if
∝	is proportional to
△	the discriminant
QED	quod erat demonstrandum, statement that a proof is finished
$\sqrt{\ }$	square root, radical sign
$\overline{A}$, A'	not A, the complement of A
$\aleph_0$	aleph 0, the cardinality of N
:	such that
S	sample space
$n(A)$	the number of elements in A
$P(A)$	the probability of A
$P(A \mid B)$	the probability of A given B
$\lvert A \rvert$	the cardinality of set A
1 : 1	one-to-one
1 : n	one to n
n : 1	n to one
n : n	n to n
LHS	left-hand side
RHS	right-hand side
$\lvert a \rvert$	absolute value of a
m	gradient of a straight line
c	y-intercept of a straight line
(x, y)	Cartesian coordinates, ordered pair
${}^nC_r$, $\binom{n}{r}$	the number of combinations of r objects from n objects
$n!$	factorial n
%	percentage
$\lim_{h \to 0}$	the limit as $h \to 0$
→	gets closer to
↑	gets closer to from below
↓	gets closer to from above
$\frac{dy}{dx}$, $f'(x)$	the derivative of y, $f(x)$
$\frac{d}{dx} f(x)\Big\rvert_{x=c}$	the derivative at $x = c$
wrt	with respect to
s	displacement
v	speed, velocity
a	acceleration
$\log_{10}$	common logarithm
ln	natural logarithm
$\log_a b$	the logarithm of b to the base a
cos	trigonometric cosine ratio
sin	trigonometric sine ratio
tan	trigonometric tangent ratio
$\mu = E(X)$	expected value of the random variable X
Var(X)	variance of the random variable X
$\sigma_X = \mathrm{SD}(X)$	standard deviation of the random variable X

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LIST OF MATHEMATICAL VERBS

A glossary of 'doing words' commonly found in mathematics problems

analyse: study and state in detail the relationship of parts of a situation

apply: use knowledge or a procedure in a given situation

calculate: find a numerical value

comment: express an opinion or judgement about a statement or calculation

communicate: transmit information to others

compare: state similarities and differences and their significance

consider: take into account

construct: draw an accurate diagram or logically arrange items or ideas

convert: change from one form to another

define: give the meaning of or identify in exact terms

demonstrate: show to be correct

describe: state the features of a situation, object, pattern, event, etc

determine: find the answer or make a decision using evidence

discuss: give reasons based on evidence for and against a position or proposal

establish: introduce and develop a result

evaluate: find the value of or state the application, strengths and limitations of a solution

examine: state the details and assumptions of a situation

expand: remove brackets or change a product to an extended sum of terms

explain: state the meaning in logical detail

explore: examine or state the details and assumptions of a situation

factorise: convert to factors or change a sum of terms to a product of factors

formulate: give mathematical expression to an idea or situation

hence find/prove: find an answer or prove a result using previous answers or information supplied

identify: state the type, name or distinguishing feature of an item or situation

interpret: state a conclusion or trend from given information

investigate: establish facts, trends or conclusions from collected information

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justify: give reasons or evidence for an answer or conclusion

modify: change to accommodate different information

obtain: find an answer or conclusion

prove: use logical steps to establish the truth of

recall: remember (and state)

recognise: use knowledge to identify features of a situation

rationalise: make rational, remove surds

show that: in questions where the answer is given, to use mathematical reasoning to prove that the answer is true

simplify: reduce the size of numbers in a fraction, or reduce the size of an algebraic expression

sketch: draw a diagram that shows the general shape and includes relevant features

solve: find the answer or explanation for a problem, particularly the values of variables

substitute: replace a variable by a number to calculate an answer

translate (to mathematical form): express a situation as mathematical relationships

verify: check a solution or result, usually referring to the given situation

write/state: give the answer, formula or result without showing any working or explanation (This usually means that the answer can be found mentally, or in one step)

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1.

LOGARITHMIC FUNCTIONS

Long before calculators were invented, John Napier (1550–1617) and Henry Briggs (1561–1630) worked out a method to calculate products and quotients quickly. Their system, called **logarithms**, turned out to be a great mathematical labour-saving device. Logarithms were widely used in various calculations for engineering, science, business and economics.

Logarithms are still useful today, particularly when comparing numbers that differ by large amounts. Examples of this include the decibel scale in acoustics, the Richter scale for earthquake magnitude and the pH scale in chemistry.

1.01 Logarithm laws
1.02 Indicial equations
1.03 Logarithmic graphs
1.04 Logarithmic equations
1.05 Applications of logarithmic functions
Chapter summary
Chapter review

SYLLABUS SUBJECT MATTER

Logarithmic laws and logarithmic functions

- establish and use logarithmic laws and definitions
- interpret and use logarithmic scales such as decibels in acoustics, the Richter scale for earthquake magnitude, octaves in music, pH in chemistry
- solve equations involving indices with and without technology
- recognise the qualitative features of the graph of $y = \log_a(x)$ $(a > 1)$, including asymptotes, and of its translations $y = \log_a(x) + b$ and $y = \log_a(x + c)$
- solve equations involving logarithmic functions with and without technology
- identify contexts suitable for modelling by logarithmic functions and use them to solve practical problems; verify and evaluate the usefulness of the model using qualitative statements and quantitative analysis

Prior learning

Mathematical Methods 2019 v1.1 General Senior Syllabus © Queensland Curriculum and Assessment Authority (QCAA). This syllabus forms part of a new senior assessment and tertiary entrance system in Queensland. Along with other senior Syllabuses, it is still being refined in preparation for implementation in schools from 2019. For the most current syllabus versions and curriculum information please refer to the QCAA website https://www.qcaa.qld.edu.au/.

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TERMINOLOGY

change of base theorem
common logarithm
exponential decay
exponential equation
exponential growth
general logarithm
half-life
horizontal translation
indicial equation
logarithm
logarithmic equation
logarithmic scale
order of magnitude
Richter scale
vertical asymptote
vertical translation

WS Logarithms review

WS Logarithms

WS Logarithm laws

1.01 Logarithm laws

You examined **logarithms** in Year 11. For example, you saw that the equation $81 = 3^4$ can be written as $\log_3(81) = 4$. In this case, the **base** of both the exponential and logarithmic expressions is 3.

You also learned that when the base is 10, logarithms are called **common logarithms**. If the base of a logarithm is not stated, it is assumed to be base 10. You can use your calculator to evaluate common logarithms.

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General logarithms

The **logarithm** of x to the base a, is the power m such that $a^m = x$ (where $a > 0$ and $a \neq 1$).

This is written as $\log_a(x) = m$ (where $a > 0$, $a \neq 1$ and $x > 0$).

So

$$a^m = x \Leftrightarrow \log_a(x) = m$$

You saw that the properties of logarithms follow from the properties of indices. The index properties $a^1 = a$, $a^0 = 1$ and $a^{-1} = \frac{1}{a}$ all have equivalent logarithm properties.

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Logarithm properties

For $a > 0$ and $a \neq 1$:

$\log_a(1) = 0$ $\quad \log_a(a) = 1$ $\quad \log_a\left(\frac{1}{a}\right) = -1$

$\log_a\left(\frac{1}{x}\right) = -\log_a(x)$ $\quad a^{\log_a(x)} = x$

If $\log_a(x) = \log_a(y)$, then $x = y$. $\quad$ If $\log_a(x) = \log_b(x)$, then $a = b$.

Note: $\log_a(0)$ is undefined.

Similarly, the index laws $a^x \times a^y = a^{x+y}$, $a^x \div a^y = a^{x-y}$ and $(a^x)^n = a^{nx}$ can be used to establish the laws of logarithms.

Logarithm laws

$\log_a(xy) = \log_a(x) + \log_a(y)$ (the product law)

$\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$ (the quotient law)

$\log_a(x^n) = n \log_a(x)$ (the power law)

EXAMPLE 1

Simplify:

a $\log_5(65) + \log_5(70) - \log_5(182)$ **b** $3\log_2(9) - 3\log_2(72)$

c $\dfrac{\log_7(16)}{\log_7(256)}$ **d** $\log_3\left(\sqrt[5]{\frac{1}{243}}\right)$

Solution

a

Write the expression.	$\log_5(65) + \log_5(70) - \log_5(182)$
Use the product and quotient laws.	$= \log_5\left(\frac{65 \times 70}{182}\right)$
Simplify.	$= \log_5(25)$
Express as a power of 5.	$= \log_5(5^2)$
Use the definition of a logarithm.	$= 2$

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b Write the expression. $3\log_2(9) - 3\log_2(72)$

Use the power law. $= \log_2(9^3) - \log_2(72^3)$

Use the quotient law. $= \log_2\left(\frac{9^3}{72^3}\right)$

Simplify. $= \log_2\left(\frac{9}{72}\right)^3$

$= \log_2\left(\frac{1}{8}\right)^3$

Write as a power of 2. $= \log_2(2^{-3})^3$

Simplify. $= \log_2(2^{-9})$

Use the definition of a logarithm. $= -9$

c Write the expression. $\frac{\log_7(16)}{\log_7(256)}$

The 16 and 256 cannot be cancelled, so write them as powers. $= \frac{\log_7(2^4)}{\log_7(2^8)}$

Use the power law. $= \frac{4\log_7(2)}{8\log_7(2)}$

Cancel the logs. $= \frac{4}{8}$

Simplify. $= \frac{1}{2}$

d Write the expression. $\log_3\left(\sqrt[5]{\frac{1}{243}}\right)$

Express the surd as a fractional power. $= \log_3\left[\left(\frac{1}{243}\right)^{\frac{1}{5}}\right]$

Write as a power of 3. $= \log_3\left[\left(3^{-5}\right)^{\frac{1}{5}}\right]$

Simplify. $= \log_3(3^{-1})$

Use the definition of a logarithm. $= -1$

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EXAMPLE 2

a Expand:

i $\log_7 (3x^5y)$ **ii** $\log\left(\dfrac{\sqrt{4x+9}}{3y}\right)$

b Simplify:

i $3\log_4 (x+5) - \log_4 (x)$ **ii** $\frac{1}{3}[\log_2 (x) + \log_2 (2x+1)]$

Solution

a i Use the product law. $\log_7 (3x^5y) = \log_7 (3) + \log_7 (x^5) + \log_7 (y)$

Use the power law. $= \log_7 (3) + 5\log_7 (x) + \log_7 (y)$

ii Rewrite using a fractional power. $\log\left(\dfrac{\sqrt{4x+9}}{3y}\right) = \log\left[\dfrac{(4x+9)^{\frac{1}{2}}}{3y}\right]$

Use the quotient law. $= \log (4x+9)^{\frac{1}{2}} - \log (3y)$

Use the power law. $= \frac{1}{2}\log (4x+9) - \log (3) - \log (y)$

b i Use the power law. $3\log_4(x+5) - \log_4(x) = \log_4(x+5)^3 - \log_4(x)$

Use the quotient law. $= \log_4\left[\dfrac{(x+5)^3}{x}\right]$

ii Use the product law. $\frac{1}{3}[\log_2 (x) + \log_2 (2x+1)] = \frac{1}{3}[\log_2 [x(2x+1)]]$

Rewrite with a radical sign. $= \log_2\left(\sqrt[3]{x(2x+1)}\right)$

Change of base theorem

For $a, b, x > 0$ and $a, b \neq 0$:

$$\log_a (x) = \frac{\log_b (x)}{\log_b (a)}$$

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EXAMPLE 3

Evaluate $\log_{13}(47)$ correct to 4 decimal places.

Solution

Change to base 10. $\log_{13}(47) = \dfrac{\log_{10}(47)}{\log_{10}(13)}$

Use your calculator to evaluate. $= 1.501\,061\ldots$

≈ 1.5011

WS Logarithm laws

Exercise 1.01 Logarithm laws

1 Write in logarithm form.

a $3^4 = 81$ **b** $16^{\frac{1}{2}} = 4$ **c** $10^{-4} = 0.0001$ **d** $27^{\frac{2}{3}} = 9$

2 Write in index form.

a $\log_4(64) = 3$ **b** $\log_{27}(3) = \frac{1}{3}$ **c** $\log(0.01) = -2$ **d** $\log_8(4) = \frac{2}{3}$

3 Evaluate each logarithm.

a $\log_5(1)$ **b** $\log_2(0.5)$ **c** $2\log_3(1)$ **d** $\log_a(1), a > 0$

e $\log_5(5)$ **f** $[\log_4(4)]^2$ **g** $\log_2(8)$ **h** $\log_8(2)$

Example 1

4 Simplify each expression without using a calculator.

a $\log_4(2) + \log_4(64)$ **b** $\log_9(81) - \log_9(3)$

c $\log_2(10) + 2\log_2(4) - \log_2(80)$ **d** $\log_7[\log_5(5)]$

e $\log_7\left(\frac{5}{11}\right) + 2\log_7\left(\frac{1}{4}\right) - \log_7\left(\frac{49}{11}\right) - \log_7\left(\frac{5}{16}\right)$

f $\log_2\left(\sqrt[5]{\frac{1}{1024}}\right)$

Example 2

5 Expand each expression using the properties and laws of logarithms

a $\log_6(7y^2)$ **b** $\log_3(5xz^3)$ **c** $\log_4\left(\frac{1}{w^5}\right)$ **d** $\log_8\left(\frac{2x}{y^2z^3}\right)$

e $\log_7(x^2 - 4)$ **f** $\log_5\left[\frac{y^3(y+3)}{(y-2)^2}\right]$ **g** $\log_2\left(\sqrt{\frac{x^2+y^2}{xy}}\right)$ **h** $\log_3\left(\frac{5}{\sqrt{z^2+1}}\right)$

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6 Simplify each expression.

a $5\log_2(x) + 3\log_2(y)$ **b** $2\log_6(p) - \log_6(11) - \log_6(m)$

c $2[\log(y) + \log(y+2) - \log(y^2-1)]$ **d** $4[3\log_4(x) - \log_4(x+1) - \log_4(x^2-3)]$

e $\frac{1}{2}\log_5(17) - 3\log_5(x) - 4\log_5(y)$ **f** $\frac{1}{3}\log_7(x-3) + 3\log_7(x) + 2\log_7(3+x)$

g $\frac{1}{2}[\log_3(x) + 3\log_3(y) - 5\log_3(x-2)]$ **h** $\frac{1}{2}\log_2(x+1) + \frac{1}{2}\log_2(x-1) + \log_2(5)$

7 Simplify each expression without using a calculator.

a $a^{\log_a(x^2-3)}$ **b** $25^{\log_5(8)}$ **c** $10^{\frac{1}{2}\log(7)}$ **d** $4^{3\log_4(6)}$ **e** $3^{-2\log_3(7)}$

8 Use the change of base theorem to evaluate each expression correct to 4 decimal places.

Example 3

a $\log_2(36)$ **b** $\log_{100}(500)$ **c** $\log_{0.2}(21)$ **d** $\log_{24}(3.18)$

9 Use the change of base theorem to show that:

a $\log_3(x) = 4$ for $x = 81$ **b** $\log_x(27) = \frac{3}{2}$ for $x = 9$

10 Given that $\log_a(2) = 0.67$, $\log_a(3) = 1.07$ and $\log_a(5) = 1.56$, evaluate each expression, correct to 2 decimal places if possible.

a $\log_a(30)$ **b** $\log_a(40)$ **c** $\log_a\left(\frac{3}{10}\right)$

d $\log_a\left(\frac{1}{\sqrt{15}}\right)$ **e** $\log_a\left(-\frac{3}{5}\right)$

11 If $n = \log_3(x)$, write each expression in terms of n.

a $3x$ **b** $\log_3(3x^4)$ **c** $\log_x(243)$ **d** $\log_3\left(\frac{x^2}{27}\right)$

12 Express y in terms of x for each expression.

a $\log(y) = 2\log(4) - 5\log(x)$ **b** $\log_3(y) = 4\log_3(x) - 2$

c $\log_4(2xy) = 2.5$ **d** $\log_3\left(\frac{2x}{y}\right) + 3 = \log_3(2)$

Problem solving

13 If $\log_m(15) + \log_m(x) = 0$, find x.

14 Prove the property $\log_b(a) = \dfrac{1}{\log_a(b)}$.

15 Prove the power law for logarithms: $\log_a(x^n) = n\log_a(x)$.

16 If $\log_n(x) = 3$ and $\log_n(y) = 5$, find the exact value of:

a $\log_n(xy^2)$ **b** $\log_n(nx^3y)$ **c** $\log_n\left(\frac{x}{\sqrt{y}}\right)$ **d** $\log_n\left(\sqrt[3]{xy^3}\right)$

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WS

Solving exponential equations

1.02 Indicial equations

An **indicial equation** or **exponential equation** has the variable as part of the exponent (index) in the equation.

Some exponential equations can be solved by changing all the powers to the same base.

EXAMPLE 4

Solve each equation.

a $9^{2x-5} = 81$

b $4^x + 2^{x+3} = 48$

c $49^{m-1} = \sqrt[3]{7}$

d $2^{w+4} - 2^w = 30$

Solution

a Write the equation using base 3. $(3^2)^{2x-5} = 3^4$

Simplify. $3^{4x-10} = 3^4$

Equate the exponents. $4x - 10 = 4$

Solve for x. $4x = 14$

$x = \frac{7}{2}$

b Write the equation using base 2. $(2^2)^x + 2^{x+3} = 48$

Rearrange and simplify. $2^{2x} + 8 \times 2^x - 48 = 0$

Let $2^x = a$. $a^2 + 8a - 48 = 0$

Factorise. $(a + 12)(a - 4) = 0$

Solve for a. $a = -12$ or $a = 4$

Substitute for a. $2^x = -12$ or $2^x = 4$

$2^x > 0$, so $2^x \neq -12$ $2^x = 4$

Solve for x. $x = 2$

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c Write the equation using base 7. $7^{2(m-1)} = 7^{\frac{1}{3}}$

Equate the exponents. $2(m-1) = \frac{1}{3}$

Simplify. $6(m-1) = 1$

Expand. $6m - 6 = 1$

Solve for m. $6m = 7$

$m = \frac{7}{6}$

d Take out the common factor. $2^w(2^4 - 1) = 30$

Simplify and solve. $2^w(16 - 1) = 30$

$2^w \times 15 = 30$

$2^w = 2$

$w = 1$

It is also possible to solve exponential equations involving different bases.

EXAMPLE 5

Exponential equations

Solve each equation, giving the solution correct to 2 decimal places.

a $7^{3x-9} = 24$ **b** $5^{x-1} = 2^{x+3}$

Solution

a Take logarithms of both sides. $\log(7^{3x-9}) = \log(24)$

Use the laws of logarithms. $(3x - 9)\log(7) = \log(24)$

Divide both sides by log (7). $3x - 9 = \frac{\log(24)}{\log(7)}$

Use a calculator to evaluate the RHS. $3x - 9 = 1.6331\ldots$

Solve for x. $3x = 1.6331\ldots + 9$

$x = 3.5443\ldots$

≈ 3.54

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b Use the index laws.

$$5^x \times 5^{-1} = 2^x \times 2^3$$

Take logarithms of both sides.

$$\log(5^x \times 5^{-1}) = \log(2^x \times 2^3)$$

Use the logarithm laws.

$$\log(5^x) + \log(5^{-1}) = \log(2^x) + \log(2^3)$$

$$x \log(5) - \log(5) = x \log(2) + 3 \log(2)$$

Group like terms.

$$x \log(5) - x \log(2) = 3 \log(2) + \log(5)$$

Factorise.

$$x[\log(5) - \log(2)] = 3 \log(2) + \log(5)$$

Isolate x.

$$x = \frac{3\log(2) + \log(5)}{\log(5) - \log(2)}$$

Evaluate.

$$= 4.0258\ldots$$

$$\approx 4.03$$

You can also use technology to solve indicial equations.

EXAMPLE 6

TI-Nspire CX Chapter 1

Solve $9 \times 4^x - 3 = 0$ using a graphics calculator.

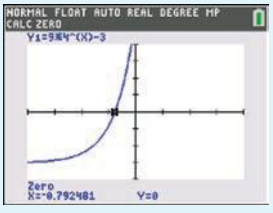
Solution

TI-84 Plus CE

Press [y=] and enter $Y_1 = 9 \times 4^x - 3$.

Set the [window] to $-4 \le x \le 4$ and $-4 \le y \le 4$ and draw the [graph].

Use [calc], **2: zero** to find the required value.



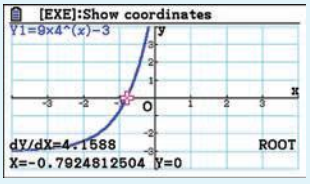
Write the answer.

Casio fx-CG20AU

Choose 5 Graph and enter $Y1 = 9 \times 4^x - 3$.

Set the **V-Window** to $-4 \le x \le 4$ and $-4 \le y \le 4$, [EXIT] and draw the graph.

Use **G-solv**, **ROOT** ([F1]) to find the required value.



$x \approx -0.792$

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Exercise 1.02 Indicial equations

1 Solve each equation.

Example 4

a $6^x = 1$ **b** $9^k = 27$ **c** $4^x = 16\sqrt{2}$ **d** $8^{-r} = \frac{1}{2}$

e $4^{2x} = 64$ **f** $4^y = \sqrt{2}$ **g** $3^{\frac{x}{2}} = 27$ **h** $\frac{1}{27^g} = \sqrt{3}$

2 Solve:

a $5^x = 5^{7x-2}$ **b** $7^{1-u} = 7^{-3}$ **c** $5^{2k-5} = 1$ **d** $81^{1-2m} = 27$

e $4^{y^2} = 4^{6-y}$ **f** $64^{x+1} = \sqrt{16}$ **g** $4^z = 16^{z+5}$ **h** $9^{5-9x} = \frac{1}{27^{x-2}}$

3 Solve each equation using factorisation.

a $4^x - 4^{x-1} = 3$ **b** $2^{m+4} - 2^m = 120$ **c** $3^n - 3^{n-1} = 54$

d $2^x - 2^{-x} = \frac{3}{2}$ **e** $2^{y+6} - 9 \times 2^y = 110$ **f** $\frac{27^k - 27^{k-1}}{3^k} = 26$

4 Solve each equation, correct to 3 decimal places where necessary.

Example 5

a $7^x = 12$ **b** $12^{3x} = 38$ **c** $3^{4x+1} = 17$

d $9^{x+2} = 4$ **e** $6^{x-1} = 4^x$ **f** $5^{3x+2} - 4^x = 0$

5 Solve each equation, correct to 3 decimal places where necessary.

a $5^{x-1} = 2^{x+1}$ **b** $9^{2x+3} = 7^{5x-1}$ **c** $12^{2x+3} = 8^{3x-1}$

d $6^{2x+5} = 11^{3x-2}$ **e** $7^{x-1} - 5^{2x+1} = 0$ **f** $21^{2x+1} - 15^{3x-1} = 0$

6 Use a graphics calculator to solve each equation correct to 3 decimal places.

Example 6

a $3^x = 17$ **b** $2^{x+1} = 11$ **c** $9^{4x+1} = 85$

d $4 \times 5^{x+3} = 14$ **e** $7 \times 4^{x-1} - 5 = 0$ **f** $7 \times 6^{2x+3} - 9 = 0$

7 Solve each equation.

a $64 \times 16^{-3x} = 16^{3x-4}$ **b** $243^{m+2} \times 9^{2m-3} = 9$ **c** $\frac{81^{3n+2}}{243^{-n}} = 3^{4-n}$

d $16^{2k-3} \times 4^{-2k+1} = 32$ **e** $6^{3b} \times 6^{-b} = 6^{-2b+1}$ **f** $3^{-2x+3} \times 3^{-2x-5} = 3^{-x-1}$

Problem solving

8 Solve $2^{2m-2} - 2^{m-2} = 3$.

9 Solve $x - 10\sqrt{x} + 21 = 0$.

10 Solve this pair of simultaneous equations for a and b.

$5^{2a-b} = \frac{1}{625}$

$10^{2b-6a} = 0.01$

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1.03 Logarithmic graphs

You have previously examined the general shape of logarithmic graphs.

INVESTIGATION

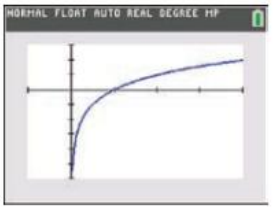
LOGARITHMIC GRAPHS

1 Use your graphics calculator to draw the graph of $y = \log_2(x)$.

TI-84 Plus CE

Select [y=] and use [catalog] to select **logBASE(**. Fill in the required information.

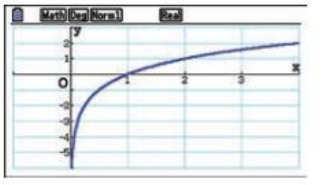
Adjust the [window] settings as required and draw the [graph].



Casio fx-CG20AU

Choose 5 Graph and use [CATALOG] to enter $Y1 = \log_2(x)$.

Set the **V-Window** as required and draw the graph.



Both calculators

2 On the same set of axes, draw the graphs of other logarithmic functions, $y = \log_a(x)$, for different values of $a > 1$.

3 Clear the graphs and then draw the graphs of $y = \log_2(x)$ and $y = \log_{0.5}(x)$ on the same set of axes.

4 On the same set of axes, draw the graphs of other logarithmic functions, $y = \log_a(x)$, for different values of a, where $0 < a < 1$.

5 Write a short statement to describe the effect that changing the value of a (for $a > 1$) has on the shape of the graph.

6 Write another statement to describe the effect that changing the value of a (for $0 < a < 1$) has on the shape of the graph.

7 Describe how the shapes of the graphs of $y = \log_a(x)$ and $\log_{\frac{1}{a}}(x)$ (where $a > 1$) are related.

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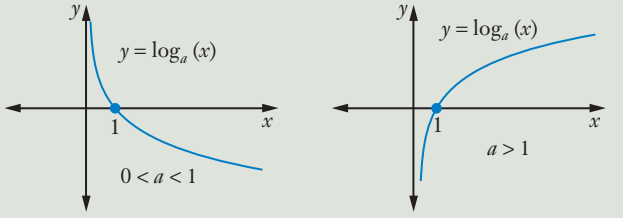
Graphs of logarithmic functions of the form $y = \log_a(x)$



- It has a zero at (1, 0).
- It has a vertical asymptote, the y-axis $x = 0$.
- For $0 < a < 1$, as $x \to 0, y \to \infty$; for $a > 1$, as $x \to 0, y \to -\infty$
- Domain: $x > 0$
- Range: $y \in \boldsymbol{R}$

The graph of $y = \log_a(x) + b$ is a **vertical translation** of the graph of $y = \log_a(x)$.

For $b > 0$, the translation is upwards and for $b < 0$, the translation is downwards.

The zero occurs when $y = 0$, so:

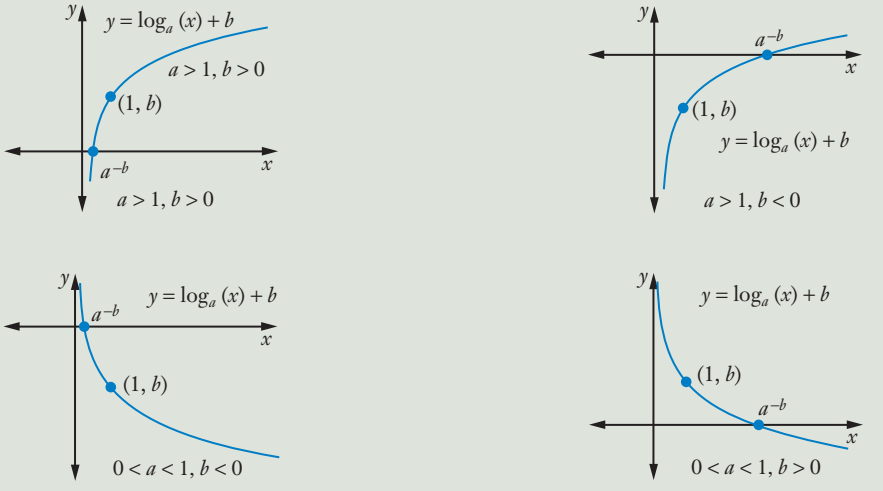
$$0 = \log_a(x) + b$$
$$\log_a(x) = -b$$
$$x = a^{-b}$$

The zero occurs at $(a^{-b}, 0)$.

When $x = 1, y = \log_a(1) + b = b$, so $(1, b)$ is always a point on the graph.

The position of the vertical asymptote (the y-axis) remains unchanged.

Vertical translation of $y = \log_a(x)$



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The graph of $y = \log_a(x + c)$ is a **horizontal translation** of the graph of $y = \log_a(x)$.

For $c > 0$, the translation is to the left and for $b < 0$, the translation is to the right.

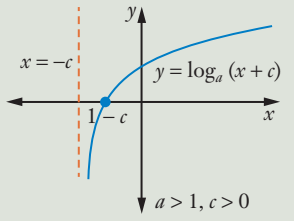
The zero occurs when $y = 0$, so:

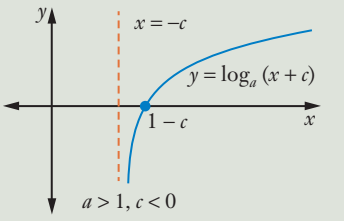
$$0 = \log_a(x + c)$$
$$x + c = 1$$
$$x = 1 - c$$

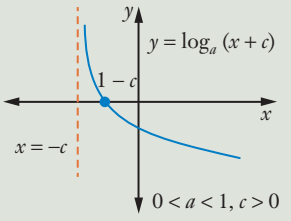
The zero occurs at $(1 - c, 0)$.

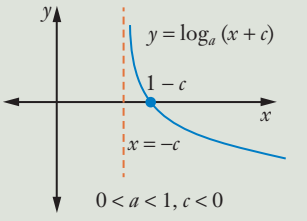
The vertical asymptote is also translated, so it becomes $x = -c$.

Horizontal translation of $y = \log_a(x)$









EXAMPLE 7

Sketch the graph of $y = \log_3(x) - 2$, labelling important features.

Solution

Compare $y = \log_3(x) - 2$ with $y = \log_a(x) + b$.	$a = 3$ (so $a > 1$) $b = -2$
Identify the translation of $y = \log_3(x)$.	$\log_3(x)$ is translated 2 units down.
Calculate the zero.	Zero $= a^{-b}$ $= 3^{-(-2)}$ $= 9$
	The graph passes through $(9, 0)$.

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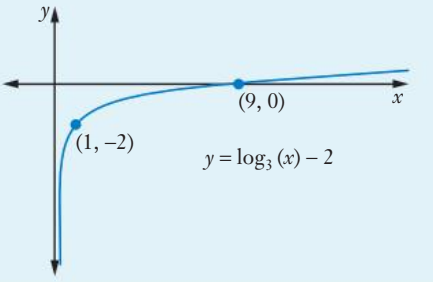
Find another point.

The graph passes through $(1, b)$.

$(1, b) = (1, -2)$

Use this information to sketch the graph.

The vertical asymptote is unchanged.



EXAMPLE 8

Sketch the graph of $y = \log_{\frac{1}{3}}(x - 2)$, labelling important features.

Solution

Compare $y = \log_{\frac{1}{3}}(x - 2)$ with $y = \log_a(x + c)$.

$a = \frac{1}{3}$ (so $0 < a < 1$)

$c = -2$

Identify the translation of $y = \log_{\frac{1}{3}}(x)$.

$\log_{\frac{1}{3}}(x)$ is translated 2 units to the right.

Calculate the zero.

$\text{Zero} = 1 - c$

$= 1 - (-2)$

$= 3$

The graph passes through $(3, 0)$.

Find another point.

$\left(\frac{1}{3}\right)^{-1} = 3$, so $\log_{\frac{1}{3}}(3) = -1$

Let $x - 2 = 3$ and solve.

$x - 2 = 3$

$x = 5$

The graph passes through $(5, -1)$.

Identify the vertical asymptote.

Vertical asymptote is at $x = -c$.

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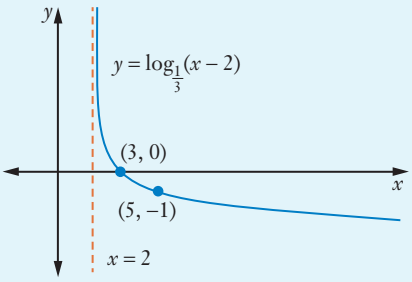
Use this information to sketch the graph.



Exercise 1.03 Logarithmic graphs

1 Find the equation of the function produced by a translation of:

a $y = \log_4(x)$, 5 units up.

b $y = \log_{\frac{1}{4}}(x)$, 3 units down.

c $y = \log_7(x)$, 2 units up.

d $y = \log_{0.6}(x)$, 7 units down.

2 Find the equation of the function produced by a translation of:

a $y = \log_5(x)$, 5 units to the left.

b $y = \log_{0.4}(x)$, 3 units to the right.

c $y = \log_{\frac{1}{8}}(x)$, 2 units to the left.

d $y = \log_3(x)$, 7 units to the right.

Example 7

3 Sketch the graph of each function, labelling important features.

a $y = \log_2(x) + 3$

b $y = \log_{0.5}(x) - 2$

c $y = \log_5(x) - 1$

d $y = \log_{0.8}(x) + 3$

e $y = \log_3(x) + 2$

f $y = \log_{0.25}(x) - 1$

Example 8

4 Sketch the graph of each function, labelling important features.

a $y = \log_3(x + 2)$

b $y = \log_{0.5}(x - 2)$

c $y = \log_{0.25}(x + 3)$

d $y = \log_5(x + 1)$

e $y = \log_4(x - 4)$

f $y = \log_{0.6}(x + 2)$

Problem solving

5 Sketch the graph of each function, labelling important features.

a $y = \log_2(x + 2) - 1$

b $y = \log_{0.5}(x - 2) + 2$

c $y = \log_3(x - 1) + 2$

d $y = \log_5(x + 2) + 1$

6 Use the laws of logarithms to describe the translation that maps $y = \log_2(x)$ onto $y = \log_2(8x)$.

7 Use the change of base law to express $y = \log_3(x)$ in terms of base 9 logarithms and hence describe the dilation that would map $y = \log_9(x)$ onto $y = \log_3(x)$.

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1.04 Logarithmic equations

WS Logarithms – Solving equations 1

WS Logarithms – Solving equations 2

When solving equations involving logarithms, you need to remember the definition of a logarithm:

$$\text{If } a^m = x, \text{ then } \log_a(x) = m \text{ (where } a > 0, a \neq 1 \text{ and } x > 0).$$

This means that a and x must both be positive.

All the equations in Example 9 have every term being a logarithm. When solving this type of equation, you use this logarithm property:

$$\text{If } \log_a(x) = \log_a(y), \text{ then } x = y.$$

EXAMPLE 9

Solve each equation.

a $\log_7(5x - 1) = 2\log_7\left(\sqrt{x}\right)$

b $\log_2(x) + \log_2(x - 2) = \log_2(6 - x)$

c $\log(4) - \log(5 - x) = \log(x)$

Solution

a Rewrite without the surd.

$$\log_7(5x - 1) = \log_7\left(\sqrt{x}\right)^2$$

$$\log_7(5x - 1) = \log_7(x)$$

Use the properties of logarithms.

$$5x - 1 = x$$

Solve for x.

$$4x = 1$$

$$x = \frac{1}{4}$$

Check the solution, starting from the most complex side.

$$\text{LHS} = \log_7\left(5 \times \frac{1}{4} - 1\right)$$

$$= \log_7\left(\frac{1}{4}\right)$$

When the solution value is substituted, LHS = RHS and there are no logarithms of negative numbers.

$$\text{RHS} = \log_7\left(\sqrt{\frac{1}{4}}\right)^2$$

$$= \log_7\left(\frac{1}{4}\right)$$

$$= \text{LHS}$$

State the result.

$$x = \frac{1}{4}$$

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b Use the properties of logarithms.

$$\log_2 [x(x-2)] = \log_2 (6-x)$$

$$x(x-2) = 6-x$$

Expand the brackets and solve for x.

$$x^2 - 2x = 6 - x$$

$$x^2 - x - 6 = 0$$

Factorise.

$$(x-3)(x+2) = 0$$

$$x = -2 \text{ or } 3$$

Check the solutions.

Let $x = -2$:

$$\text{LHS} = \log_2 (-2) + \log_2 (-2-2)$$

$\log_2(-2)$ and $\log_2(-4)$ do not exist as you cannot evaluate the logarithm of a negative number, so $x = -2$ is not a solution.

Let $x = 3$:

$$\text{LHS} = \log_2 (3) + \log_2 (3-2)$$

$$= \log_2 (3) + \log_2 (1)$$

$$= \log_2 (3) + 0$$

$$= \log_2 (3)$$

$$\text{RHS} = \log_2 (6-3)$$

$$= \log_2 (3)$$

$$= \text{LHS}$$

This solution works.

$x = 3$ is a solution.

c Use the properties of logarithms.

$$\log\left(\frac{4}{5-x}\right) = \log (x)$$

$$\frac{4}{5-x} = x$$

Simplify.

$$4 = x(5-x)$$

$$4 = 5x - x^2$$

$$x^2 - 5x + 4 = 0$$

Factorise and solve for x.

$$(x-4)(x-1) = 0$$

$$x = 1 \text{ or } 4$$

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Check the solutions.	Let $x = 1$:
	$\text{LHS} = \log(4) - \log(5 - 1)$
	$= \log(4) - \log(4)$
	$= 0$
	$\text{RHS} = \log(1)$
	$= 0$
This solution works.	$= \text{LHS}$
	Let $x = 4$:
	$\text{LHS} = \log(4) - \log(5 - 4)$
	$= \log(4) - \log(1)$
	$= \log(4)$
	$\text{RHS} = \log(4)$
This solution also works.	$= \text{LHS}$
State the result.	$x = 1$ or 4

Example 10 has equations where not all terms are logarithms. To solve these, you use the exponential form of the logarithm:

$$\log_a(x) = m \Rightarrow a^m = x$$

EXAMPLE 10

Solve each equation.

a $\log_2(5x + 12) = 6$

b $\log_3(x) + \log_3(x - 6) = 3$

c $\log_5(x^2 - 2x) = 1 + \log_5(2 - x)$

Solution

a

Rewrite the equation in index form.	$2^6 = 5x + 12$
	$64 = 5x + 12$
Solve for x.	$5x = 52$
	$x = \frac{52}{5}$

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Check the solution.

Let $x = \frac{52}{5}$:

$$\log_4\left(5 \times \frac{52}{5} + 12\right) = \log_4(64) = 3$$

$x = \frac{52}{5}$ works.

$x = \frac{52}{5}$ is a solution.

b Write with a single logarithm.

$$\log_3[x(x-6)] = \log_3(3^3)$$

Use the properties of logarithms.

$$x(x-6) = 3^3$$

Simplify and rearrange.

$$x^2 - 6x - 27 = 0$$

Factorise.

$$(x-9)(x+3) = 0$$

Solve for x.

$$x = -3 \text{ or } 9$$

Check the solutions.

Let $x = -3$:

$$\text{LHS} = \log_3(-3) + \log_3(-3-6)$$

$\log_3(-3)$ and $\log_3(-9)$ do not exist, so $x = -3$ is not a solution.

Let $x = 9$:

$$\log_3(9) + \log_3(9-6) = 2 + \log_3(3)$$
$$= 2 + 1$$
$$= 3$$

This solution works.

$x = 9$ is a solution.

c Collect the logarithmic terms.

$$\log_5(x^2 - 2x) - \log_5(2-x) = 1$$

Use the laws of logarithms.

$$\log_5\left(\frac{x^2-2x}{2-x}\right) = 1$$

$$\frac{x^2-2x}{2-x} = 5^1 = 5$$

Simplify and solve.

$$x^2 - 2x = 5(2-x)$$
$$x^2 - 2x = 10 - 5x$$
$$x^2 + 3x - 10 = 0$$
$$(x-2)(x+5) = 0$$
$$x = -5 \text{ or } 2$$

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Check the solutions.	Let $x = -5$:
	$\text{LHS} = \log_5 [(-5)^2 - 2 \times (-5)]$
	$= \log_5 (35)$
	$= \log_5 (5 \times 7)$
	$= \log_5 (5) + \log_5 (7)$
	$= 1 + \log_5 (7)$
	$\text{RHS} = 1 + \log_5 [2 - (-5)]$
	$= 1 + \log_5 (7)$
	$= \text{LHS}$
This solution works.	$x = -5$ is a solution.
	Let $x = 2$:
	$\text{LHS} = \log_5 [(2)^2 - 2 \times 2]$
This solution does not work.	$= \log_5 (0)$, which doesn't exist.
State the result.	The only solution is $x = -5$.

Exercise 1.04 Logarithmic equations

1 Solve each equation. Example 9

a $\log_5 (7x + 3) = \log_5 (5x + 9)$

b $\log_8 (x) + \log_8 (x + 6) = \log_8 (5x + 12)$

c $\log (x - 2) - \log (2x - 3) = \log (2)$

d $\log_3 [(x - 2)(x + 3)] = \log_3 (14)$

e $\log_6 (2x + 1) = \log_6 (x + 2) - \log_6 (3)$

f $\log_7 (x + 4) + \log_7 (x - 2) = \log_7 (4x)$

g $\log (3x + 5) = \log (7x - 12)$

h $\log_2 (3x + 2) = \log_2 (x - 2)$

i $\log_6 (2x) + \log_6 (4) = \log_6 (x + 12) - \log_6 (2)$

j $\log_8 (2x + 1) + \log_8 (2x - 1) = 3 \log_8 (3)$

2 Solve each equation. Example 10

a $\log_2 (5x + 7) = 5$

b $\log (4x - 1) = 3$

c $\log_3 (5x - 11) = 2$

d $\log_5 (4x + 11) = 2$

e $\log_3 (9x + 2) = 4$

f $\log_3 (4x - 9) = 3$

g $\log_3 (3x + 11) = 4$

h $\log_7 (6x - 5) = 2$

i $\log_3 (x^2 - 6x) = 3$

3 Solve each equation.

a $\log_2 (x + 5) - \log_2 (2x - 1) = 5$

b $2 \log_3 (x) - 1 = \log_3 (2x - 3)$

c $\log_4 (4x - 2) - \log_4 (3x + 1) = 5$

d $\log_2 (x + 1) - \log_2 (x - 4) = 3$

e $\log_2 (x) + \log_2 (2 - 2x) = -1$

f $\log_3 (x) + \log_3 (2x + 1) = 1$

g $\log_6 (x) + \log_6 (x + 5) = 2$

h $\log_2 (2x + 12) - \log_2 (3x) = 2$

i $\log_4 (2x - 3) - \log_4 (x + 2) = 1$

j $\log_3 (x) - \log_3 (x - 1) = 2$

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4 Solve each equation.

a $\log_6(x) + \log_6(x-9) = 2$

b $\log_4(x) + \log_4(x-12) = 3$

c $\log_6(x) + \log_6(x-1) = 1$

d $\log_6(x^2-2x) = \log_6(5x-12)$

e $\log_7(6x) - \log_7(4-x) = \log_7(3)$

f $\log(x) + \log(x+3) = \log(5x)$

g $\log_5(x-1) + \log_5(x+3) = \log_5(x^2-3x+12)$

h $[\log_3(x)]^2 = 3 - 2\log_3(x)$

i $\log_5(2x) + \log_5(x+2) = \log_5(6)$

j $\log_8(2) + \log_8(4x^2) = 2$

Problem solving

5 Solve $2^{\log_2(x-5)} + 2(x-5) - 12 = 0$.

6 Find a value of x that satisfies $\log_{10}(5^x + x - 31) = x[1 - \log_{10}(2)]$.

WS Logarithmic functions assignment

WS Logarithmic functions extension

1.05 Applications of logarithmic functions

Mel and Ethan are both art dealers and Mel has a painting that is worth $30 000 more than Ethan's painting. This may seem like a big difference, but not really if Ethan's painting is worth $2 000 000 rather than $2000.

It is often more informative to use ratios to compare amounts. If Mel's painting is worth $40 000 and Ethan's painting is worth $10 000, then Mel's is worth 4 times as much as Ethan's. This comparison informs you of the relative worth regardless of the actual amounts involved.

When ratios of quantities are so big that they differ by powers of 10, the relative size is called an **order of magnitude**.

For example, the distance from Brisbane to Perth (4000 km) is 2 orders of magnitude greater than the distance from Brisbane to Logan City (40 km). You can write this as:

$$\frac{4000}{40} = 100 = 10^2 \Rightarrow \text{order of magnitude} = 2$$

The order of magnitude is a power of 10.
So:

$$\text{Order of magnitude for 2 quantities} = \log\left(\frac{\text{quantity}_1}{\text{quantity}_2}\right) = \log\left(\frac{Q_1}{Q_2}\right)$$

Order of magnitude is especially useful when numbers differ by very large amounts.

One application of order of magnitude is in measuring earthquake intensity. The **Richter scale** is a **logarithmic scale** based on measurements by a seismograph to calculate the magnitude of an earthquake.

One of the Richter formulas used to measure the magnitude, M, of an earthquake is:

$$M(x) = \log\left(\frac{x}{x_0}\right)$$

where x is the measure of the amplitude of the earthquake wave and x_0 is the amplitude of the smallest detectable wave (or standard wave).

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Each value on the Richter scale is 10 times greater than the previous value. This means that an earthquake measuring '6' on the Richter scale actually has seismic waves 10 times greater in amplitude than of those for an earthquake measuring '5' on the Richter scale, not 1 unit higher.

EXAMPLE 11

a Find the magnitude of an earthquake (correct to 2 decimal places) if $x = 40\,000\,x_0$.

b How many times more intense than the reference intensity, x_0, is an earthquake with a magnitude of 5.7?

Solution

a

Write the formula. $M(x) = \log\left(\frac{x}{x_0}\right)$

Substitute for known values. $= \log\left(\frac{40\,000x_0}{x_0}\right)$

Simplify. $= \log(40\,000)$

Evaluate. $= 4.6020\ldots$

State the result. The magnitude of the earthquake is about 4.6.

b

Write the formula. $M(x) = \log\left(\frac{x}{x_0}\right)$

Substitute for known values. $5.7 = \log\left(\frac{x}{x_0}\right)$

Write in index form. $10^{5.7} = \frac{x}{x_0}$

Make x the subject. $10^{5.7}\,x_0 = x$

$x = 501\,187.23\ldots\,x_0$

State the result. The earthquake is over 500 000 times more intense than the standard wave.

Logarithmic scales such as the Richter scale are used in the calculation of many quantities, including the loudness of sound, the intensity of the brightness of stars, octaves in music and the hydrogen ion concentration (pH) in chemistry.

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EXAMPLE 12

Loudness, in decibels (dB), is related to the intensity, I, of sound by the equation

$$L = 10 \log\left(\frac{I}{I_0}\right)$$

where I_0 is equal to 10^{-12} watts per square metre (W/m^2), the lowest intensity of sound that can be heard by human ears.

a The bark of a large dog has a loudness of 85 dB. Calculate the intensity in W/m^2.

b If the intensity of a sound is increased by a factor of 4, what is the change in the loudness, correct to 2 decimal places?

Solution

a

Write the formula.

$$L = 10 \log\left(\frac{I}{I_0}\right)$$

Substitute for known values.

$$85 = 10 \log\left(\frac{I}{10^{-12}}\right)$$

Simplify.

$$8.5 = \log\left(10^{12} I\right)$$

Write in index form.

$$10^{8.5} = 10^{12} I$$

Solve.

$$I = 10^{8.5 - 12}$$

State the result.

$$I = 10^{-3.5} \text{ W/m}^2$$

b

Find an equation for L_1.

$$L_1 = 10 \log\left(\frac{I_1}{10^{-12}}\right)$$

$$= 10 \log\left(10^{12} I_1\right)$$

Use the laws of logarithms.

$$= 10 \log\left(10^{12}\right) + 10 \log\left(I_1\right)$$

$$= 10 \times 12 + 10 \log\left(I_1\right)$$

$$= 120 + 10 \log\left(I_1\right)$$

Find an equation for L_2.

$$L_2 = 10 \log\left(\frac{4I_1}{10^{-12}}\right)$$

$$= 10 \log\left(4 \times 10^{12} I_1\right)$$

Use the laws of logarithms.

$$= 10 \log(4) + 10 \log\left(10^{12}\right) + 10 \log\left(I_1\right)$$

$$\approx 6.02 + 10 \times 12 + 10 \log\left(I_1\right)$$

$$= 6.02 + 120 + 10 \log\left(I_1\right)$$

$$= 6.02 + L_1$$

State the result.

Increasing the intensity by a factor of 4 increases the loudness by about 6 dB.

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Exponential functions are often used to model growth and decay. For time t, the function defined by:

$$y = b \times a^{nt} \text{ (where } a > 1)$$

- represents an **exponential growth** over time if $n > 0$
- represents an **exponential decay** over time if $n < 0$.

In some models such as population growth, the initial population is represented by N_0.

iStock/Getty Images Plus/TatyanaMishchenko

For an exponential decay model, the time it takes for 50% of the initial amount of a substance to decay is called its **half-life**.

Compound interest formula

The compound interest formula is an example of an exponential growth function.

$$FV = PV\left(1+\frac{r}{n}\right)^{nt}$$

where:

FV = future value

PV = present value

r = annual interest rate (in decimal form)

n = number of times compounded per year

t = time in years

EXAMPLE 13

a If you deposit \$8000 into an account paying 6% p.a. interest compounded monthly, how many years (correct to one decimal place) will it be until there is \$10 000 in the account?

b At 5.2% annual interest compounded quarterly, how long will it take for your money to double?

Solution

a

Substitute known values into the formula.

$$10\,000 = 8000\left(1+\frac{0.06}{12}\right)^{12t}$$

Simplify.

$$1.25 = 1.005^{12t}$$

Take logarithms of both sides.

$$\log(1.25) = \log(1.005^{12t})$$

Use the laws of logarithms.

$$\log(1.25) = 12t\log(1.005)$$

Solve for t.

$$\frac{\log(1.25)}{\log(1.005)} = 12t$$

$$12t = 44.7401\ldots$$

$$t = 3.7283\ldots$$

State the result.

It will take about 3.7 years for \$8000 to grow to \$10 000 at 6% p.a. compounded monthly.

b

Write the future value in terms of the present value.

$$FV = 2PV$$

Substitute known values into the formula.

$$2PV = PV\left(1+\frac{0.052}{4}\right)^{4t}$$

Divide both sides by PV.

$$2 = \left(1+\frac{0.052}{4}\right)^{4t}$$

Take logarithms of both sides.

$$\log(2) = \log(1.013^{4t})$$

Use the laws of logarithms.

$$\log(2) = 4t\log(1.013)$$

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Solve for t.	$\frac{\log(2)}{\log(1.013)} = 4t$
	$4t = 53.6648\ldots$
	$t = 13.4162\ldots$
State the result.	It will take about 13.4 years for the money to double in value at 5.2% p.a. compounded quarterly.

Exercise 1.05 Applications of logarithmic functions

Example 11

1 An earthquake is measured with an intensity 758 times as great as a standard wave. What is the magnitude of this earthquake, correct to 1 decimal place?

Example 12

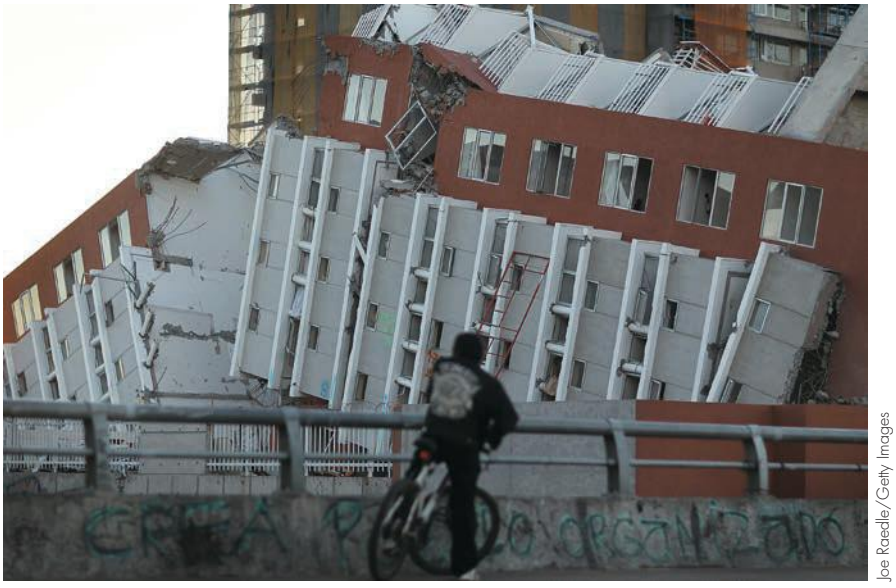
2 A drill press has a noise rating of 50 decibels and a compressor has a noise rating of 62 decibels. How many times more intense is the compressor noise than the drill press noise?

Example 13

3 How long will it take for money to double in value if it is invested at 10% p.a. compounded monthly?

Problem solving

4 An earthquake in San Francisco measured 8.3 on the Richter scale. An earthquake recorded in Chile was found to be 4 times the strength of the San Franciscan earthquake. What did the Chilean earthquake measure on the Richter scale?



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5 An earthquake in Ecuador measured 8.3 on the Richter scale and an earthquake in New Zealand measured 7.1 on the Richter scale. How much stronger was the earthquake in Ecuador?

6 A jazz band uses two different amplifiers. One is a 25 W/m^2 amplifier and the other is a 500 W/m^2 model. How many decibels louder is the 500 W/m^2 model? Give your answer correct to 2 decimal places.

7 A person's eardrum can be ruptured if it is exposed to a noise that has a loudness of 160 dB. Calculate the intensity (W/m^2) of sound that would cause your eardrum to be ruptured.

8 Octaves in music can be measured in cents, ¢. The frequencies of two notes, f_1 and f_2, are related by the equation:

$$¢ = \frac{1200}{\log(2)} \log\left(\frac{f_2}{f_1}\right)$$

Calculate the number of cents between the following pairs of notes:

a A (27.5 Hz) and B-flat (29.135 Hz) in the first octave

b middle C (261.63 Hz) and A4 (440 Hz)

c middle C (261.63 Hz) and C5 (587.33 Hz) in the next octave

9 The measure of acidity in a liquid is called the pH of the liquid. The pH is based on the amount of hydrogen ions (H+) in the liquid. The formula for pH is:

$$\text{pH} = -\log[H^+]$$

where $[H^+]$ is the concentration of hydrogen ions, given in M/litre (mole/litre).

Liquids with a low pH (down to 0) are more acidic than those with a high pH. Pure water has a pH of 7.0.

If lemon juice has a pH of 1.8, what is the concentration of hydrogen ions, correct to 3 decimal places?

10 Find the hydrogen ion concentration for each situation.

a Battery acid has a pH of zero.

b Vinegar has a pH of 3.

c Baking soda has a pH of 8.

d Household ammonia has a pH of 11.

11 The brightness of stars and other heavenly bodies has been measured by astronomers for centuries. The apparent magnitudes m_1 and m_2 of two heavenly bodies are related to their brightnesses or intensities, I_1 and I_2, by the equation

$$m_1 - m_2 = -2.5 \log_{10}\left(\frac{I_1}{I_2}\right)$$

a The planet Mercury has an apparent magnitude of –2 and a brightness of 6.31. The supernova SN1006 had an apparent magnitude of –7. Calculate the brightness of the supernova.

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b The star Proxima Centauri has an apparent magnitude of 11. Calculate the brightness of the Proxima Centauri.

c The Sun has an apparent magnitude of –27 and the full Moon has an apparent magnitude of –13. How many times brighter is the Sun than the full Moon?

12 The percentage of carbon-14 present in the remains of plants and animals is used to determine the age of the remains. The decay of radioactive carbon-14 is modelled by:

$$N(t) = N_0 \times 2.72^{-kt}$$

where N is the number of radioactive atoms present after t years and N_0 and k are positive constants.

Radioactive carbon-14 has a half-life of 5750 years.

a Calculate the value of k, correct to 4 significant figures.

b How old is a skeleton that has lost 70% of its carbon-14 atoms? Give your answer correct to the nearest year.

13 The decay of a radioactive substance is modelled by $Q(t) = Q_0 \times 2.5^{-nt}$, where Q kg is the amount of the substance present at time t years and Q_0 and n are positive constants.

a Show that the constant Q_0 represents the initial amount of the substance.

b If the half-life of the radioactive substance is 1000 years, calculate n to 2 significant figures.

c If there was 45 kg of the radioactive substance initially, how many kilograms would decay in 10 years?

14 The population of the world can be modelled using the formula:

$$P = 6.9(1.011)^t$$

where t is the number of years after 2011 and P is the world population in billions.

Estimate:

a the population in the year 2045 to the nearest hundred million

b by what year the population will be double what it was in 2011.

15 A particular bacterial colony doubles its population every 15 hours. The experiment begins with 100 bacteria cells and the number of cells is modelled by the formula $N = 100\,(2)^{\frac{t}{15}}$, where t is the number of hours since the experiment started.

After how many hours would you expect to have 300 bacteria cells?

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1. CHAPTER SUMMARY

Logarithmic functions word puzzle

Logarithmic functions

- The **logarithm** of x to the **base** a, is the **power** x such that $a^m = x$, where $a > 0$ and $a \neq 1$.

  $\log_a(x) = m$ (where $a > 0$, $a \neq 1$ and $x > 0$).

  So $a^m = x \Leftrightarrow \log_a(x) = m$

- Logarithms with base 10 are called **common logarithms**.

- The **logarithm properties** (for $a > 0$ and $a \neq 1$) are:

  $\log_a(1) = 0$ $\quad \log_a(a) = 1$ $\quad \log_a\left(\frac{1}{a}\right) = -1$

  $\log_a\left(\frac{1}{x}\right) = -\log_a(x)$ $\quad a^{\log_a(x)} = x$

  If $\log_a(x) = \log_a(y)$, then $x = y$ $\quad$ If $\log_a(x) = \log_b(x)$, then $a = b$.

- The **logarithm laws** are:

$$\log_a(xy) = \log_a(x) + \log_a(y)$$
$$\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$$
$$\log_a(x^n) = n\log_a(x)$$

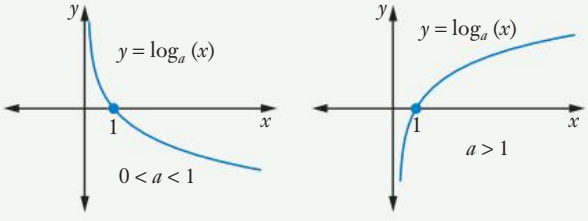
- The **change of base theorem** states:

  For $a, b, x > 0$ and $a, b \neq 0$:

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$$

- An **indicial** or **exponential equation** has the variable as part of the exponent (index) in the equation.

- Graphs of logarithmic functions of the form $y = \log_a(x)$



Zero at (1, 0) $\quad$ Vertical asymptote is the y-axis.

Domain: $x > 0$ $\quad$ Range: $y \in \boldsymbol{R}$

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- $y = \log_a(x) + b$ is a **vertical translation** of $y = \log_a(x)$.

  For $b > 0$, the translation is upwards and for $b < 0$, the translation is downwards.

  The zero occurs at $(a^{-b}, 0)$.

  $(1, b)$ is always a point on the graph.

  Vertical asymptote (the y-axis) remains unchanged.

- $y = \log_a(x + c)$ is a **horizontal translation** of $y = \log_a(x)$.

  For $c > 0$, the translation is to the left and for $b < 0$, the translation is to the right.

  The zero occurs at $(1 - c, 0)$.

  The vertical asymptote is $x = -c$.

- When solving **logarithmic equations**, you need to apply the properties and laws of logarithms and indices.

- The relative size of two quantities expressed as a power of 10, is called an **order of magnitude**.

  Order of magnitude for two quantities $= \log\left(\frac{Q_1}{Q_2}\right)$

- **Logarithmic scales** are used in the calculation of many quantities, including earthquake magnitude (**Richter scale**) the loudness of sound, the intensity of the brightness of stars, octaves in music and the hydrogen ion concentration (pH) in chemistry.

- Growth and decay models (for time t) are exponential functions of the form:

  $$y = b \times a^{nt} \text{ (where } a > 1)$$

  If $n > 0$, it is an exponential growth function.

  If $n < 0$, it is an exponential decay function.

- The time it takes for 50% of the initial amount of a substance to decay is called its **half-life**.

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1. CHAPTER REVIEW

Logarithmic functions

Example 1

1 Write each equation in logarithmic form.

a $2^5 = 32$ **b** $64^{\frac{2}{3}} = 16$ **c** $6^{-3} = \frac{1}{216}$

Example 1

2 Write each equation in index form.

a $\log_3(81) = 4$ **b** $\log_{25}(5) = \frac{1}{2}$ **c** $\log(0.001) = -3$

d $\log_8(32) = \frac{5}{3}$

Example 1

3 Evaluate each expression.

a $\log_7(1)$ **b** $\log_2(0.25)$ **c** $2\log_6(6)$

Example 1

4 Simplify each expression without using a calculator.

a $\log_3(27) + \log_3(9) - \log_3(81)$ **b** $2\log_4(7) - 2\log_4(28)$

c $\log_3\left(\sqrt[4]{\frac{1}{81}}\right)$ **d** $49^{\log_7(5)}$

Example 2

5 Expand each expression using the properties and laws of logarithms.

a $\log_5(9x^3)$ **b** $\log_7\left(\frac{3}{z^4}\right)$

c $\log_7\left[\frac{m^2(m-4)}{(m+3)^2}\right]$ **d** $\log_6\left(\sqrt{\frac{x^3-y}{2xy}}\right)$

Example 2

6 Simplify each expression.

a $2\log_2(x) + 5\log_5(y)$

b $2[\log(m) + \log(3-m) - \log(2m^3+7)]$

c $\frac{1}{2}\log_5(17) - 3\log_5(x) - 4\log_5(y)$

Example 3

7 Use the change of base theorem to evaluate $\log_7(25)$ correct to 4 decimal places.

Example 3

8 Given that $\log_m(2) = 0.73$, $\log_m(3) = 1.15$ and $\log_m(5) = 1.68$, use the properties of logarithms to evaluate each expression correct to 2 decimal places if possible.

a $\log_m(60)$ **b** $\log_m\left(\frac{8}{15}\right)$ **c** $\log_m\left(\sqrt{\frac{2}{9}}\right)$

Example 4

9 Solve each equation.

a $4^{x-1} = 64$ **b** $3^{2x-4} = 9^{2-x}$ **c** $9^x = \sqrt{3}$ **d** $9^{5-9x} = \frac{1}{27^{x-2}}$

Example 4

10 Solve each equation using factorisation.

a $5^{2x} - 24 \times 5^x = 25$ **b** $2^{2x} - 3 \times 2^x + 2 = 0$

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11 Solve each equation, correct to 3 decimal places where necessary. Example 5

a $5^{2x-3} = 84$ **b** $7^{2x-3} = 5^{3x+1}$

12 Use a graphics calculator to solve each equation, correct to 3 decimal places. Example 6

a $11^{3x+1} = 111$ **b** $5 \times 7^{2x-1} - 28 = 0$

13 Sketch the graph of each function, labelling important features. Example 7

a $y = \log_3(x) + 2$ **d** $y = \log_{0.2}(x) - 2$

14 Sketch the graph of each function, labelling important features. Example 8

a $y = \log_2(x + 3)$ **b** $y = \log_{0.6}(x - 2)$

15 Solve each equation. Example 9

a $\log_2(5x) = \log_2(2x + 9)$ **b** $\log_5(4m - 5) = \log_5(2m - 1)$

c $\log_3(16 + 2k) = \log_3(k^2 - 4k)$ **d** $\log_7(2p) + \log_7(p + 2) = \log_7(6)$

16 Solve each equation. Example 10

a $\log_2(5w + 6) = 4$ **b** $\log_3(v) + \log_3(v - 6) = 3$

c $\log_2(m + 1) - \log_2(m - 4) = 3$ **d** $\log_{12}(x) + \log_{12}(x - 1) = 1$

17 The detonation of 3 tonnes of TNT produces energy similar to an earthquake measuring 3.5 on the Richter scale ($M = 3.5$). If the energy produced by an earthquake is proportional to $10^{1.5M}$, how many megatonnes of TNT is the equivalent energy for the 1957 Alaskan earthquake which was recorded as $M = 9.1$? Answer to the nearest megatonne (1 000 000 tonnes). Example 11

18 The human threshold for loud sounds is about 130–135 dB. If you attend a rock concert where the sound intensity is measured to be 100 W/m^2, would you need ear plugs? Example 12

19 Melody deposits \$6500 into an account paying 8% annual interest compounded monthly. How much more money would be in the account after 7 years if the interest was compounded daily? Example 13

Problem solving

20 Solve this pair of simultaneous equations for a and b.

$8^b = 4^{2a+3}$

$\log_2(b) = \log_2(a) + 3$

21 Use the laws of logarithms to describe the translation that maps $y = \log_3(x)$ onto $y = \log_3(9x + 20) - 1$.

22 If $m = \frac{1}{2} + 2\log_9(x)$, express x in terms of m.

23 If $m = \log_3(3x^2)$, express $\log_x(81)$ in terms of m.

Qz Practice quiz

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2.

DERIVATIVES OF EXPONENTIAL FUNCTIONS

An old legend says that the King of India agreed to reward the inventor of chess with one grain of rice on the first square of the chessboard, 2 grains on the second square, 4 grains on the third square and so on doubling each time, up to the 64th square. What the king did not know was that the amount of rice would be astronomical, because it is an example of exponential growth.

The spread of a virus, the growth of microorganisms in a Petri dish, the decay of radioactive materials and the cooling of an object are situations involving exponential growth or decay.

2.01 The exponential function
2.02 The derivative of e^x
2.03 The derivative of $e^{f(x)}$
2.04 The derivative of a^x
2.05 Exponential growth and decay
Chapter summary
Chapter review

SYLLABUS SUBJECT MATTER

Calculus of exponential functions

- estimate the limit of $\frac{a^h - 1}{h}$ as $h \to 0$ using technology, for various values of $a > 0$
- recognise that e is the unique number a for which the above limit is 1
- define the exponential function e^x
- establish and use the formula $\frac{d}{dx}(e^x) = e^x$ and $\frac{d}{dx}(e^{f(x)}) = f'(x)\, e^{f(x)}$
- identify contexts suitable for mathematical modelling by exponential functions and their derivatives and use the model to solve practical problems; verify and evaluate the usefulness of the model using qualitative statements and quantitative analysis

Prior learning

Mathematical Methods 2019 v1.1 General Senior Syllabus © Queensland Curriculum and Assessment Authority (QCAA). This syllabus forms part of a new senior assessment and tertiary entrance system in Queensland. Along with other senior Syllabuses, it is still being refined in preparation for implementation in schools from 2019. For the most current syllabus versions and curriculum information please refer to the QCAA website https://www.qcaa.qld.edu.au/.

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TERMINOLOGY

Euler's number
exponential decay
exponential function
exponential growth
irrational
transcendental number

2.01 The exponential function

You can find the derivative of $f(x) = a^x$ using first principles.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{a^{x+h}-a^x}{h} \\ &= \lim_{h\to 0} \frac{a^x \times a^h - a^x}{h} \\ &= \lim_{h\to 0} \frac{a^x(a^h-1)}{h} \\ &= a^x \lim_{h\to 0} \frac{a^h-1}{h} \end{aligned}$$

So, the derivative exists if $\lim_{h\to 0} \frac{a^h-1}{h}$ exists. If it does, the derivative of a^x is just a^x multiplied by a constant.

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TECHNOLOGY

Derivatives of a^x

You can investigate the derivative of a^x using your graphics calculator.

1 Find the values of $\frac{a^h-1}{h}$ for $a = 0.5, 1.5, 2, 2.5, 2.7, 2.72, 3$ and 3.5 using $h = 0.001$ to estimate the values of $\lim_{h\to 0}\left(\frac{a^h-1}{h}\right)$. An example for $a = 3$ is shown below.

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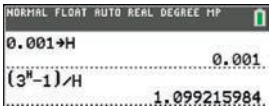

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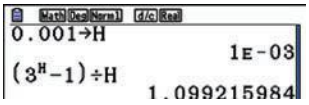

2 Now use these estimates to find an approximation to the derivative of a^x for different values of a.

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3 Using your results, graph each exponential function and its derivative on your calculator. Here is an example for $a = 3$.

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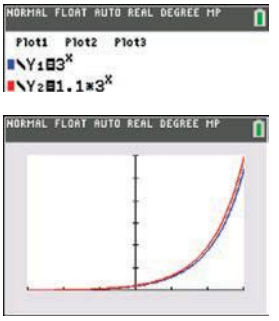

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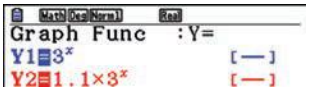

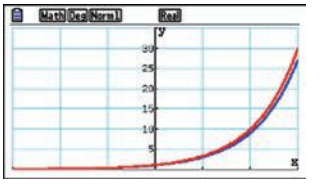

4 What happens to the value of the limit as a increases up to 3? What happens to the value of the limit as a increases beyond 3?

From your investigation, you should have noticed that near 3, $\lim_{h\to 0}\dfrac{a^h-1}{h}$ is near 1.

In fact, there exists a number for which the limit equals 1. This special constant, symbolised by the letter 'e', is often called **Euler's number** and is very important in mathematics. It first appeared obliquely in a paper on logarithms by John Napier (1550–1617), but the constant itself was discovered by Swiss mathematician Jacob Bernoulli (1654–1705) while studying compound interest. The Swiss mathematician, Leonhard Euler (1707–1783) later proved that this constant is **irrational**.

DeAgostini/Getty Images

John Napier (1550–1617)

In algebra it is common to use base 10 when dealing with exponential functions. However, in calculus, it is more convenient to use base e. Euler used a series to calculate an approximate value of e:

$$1+\frac{1}{1}+\frac{1}{1\times 2}+\frac{1}{1\times 2\times 3}+\frac{1}{1\times 2\times 3\times 4}+\ldots$$

e is a **transcendental number** – one that is not the root of any algebraic equation with rational coefficients. Euler was also responsible for naming π (another transcendental number) and for introducing the function notation $f(x)$.

The number e, like π, cannot be written down exactly. Its approximate value is given by $e \approx 2.718\ 281\ 828\ 459\ 045\ 235\ 36\ldots$ There is no pattern and it does not repeat.

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TECHNOLOGY

Euler's number

1 Enter the function $Y_1 = \left(1 + \frac{1}{X}\right)^X$ in the graphing section of your graphics calculator.

2 Use WINDOW (or V-WINDOW) to set $X_{min} = -1$, $X_{max} = 1$, $X_{scl} = 0.1$, $Y_{min} = 0$, $Y_{max} = 4$, $Y_{scl} = 1$ and graph the function.

3 Trace the graph from left to right and observe how the value of the function changes.

4 Adjust the values for X to $X_{min} = -1$, $X_{max} = 10$, $X_{scl} = 1$ and trace the value of the function for large values of X. Note how the value of the function behaves.

5 Repeat using $X_{min} = -1$, $X_{max} = 100$, $X_{scl} = 10$ and observe the value of the function.

6 Comment on your findings.

The number e may be defined as the following limit.

Definition of e

$$e = \lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^n = 2.718281...$$

You can prove that the derivative of $f(x) = e^x$ is e^x from first principles.

$$\frac{d}{dx}(e^x) = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h}$$

$$= \lim_{h \to 0} \frac{e^x \times e^h - e^x}{h}$$

$$= \lim_{h \to 0} \frac{e^x(e^h - 1)}{h}$$

$$= e^x \lim_{h \to 0} \frac{e^h - 1}{h}$$

$$= e^x \text{ because } \lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

This property makes $f(x) = e^x$ so important in mathematics, physics, engineering, economics and biology that it is called ***the*** **exponential function**.

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EXAMPLE 1

TI-Nspire CX Chapter 2

Use your calculator to evaluate $2e^2 + 3$, correct to 2 decimal places.

Solution

TI-84 Plus CE

On a calculator page, type the equation and press enter. Write one of the numbers as a decimal to force an approximate answer.

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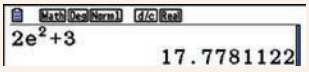
Use the Run-Matrix menu. Type the equation and press EXIT.



Write the answer.

$2e^2 + 3 \approx 17.78$

EXAMPLE 2

TI-Nspire CX Chapter 2

Use your calculator to draw the graph of $y = e^x$.

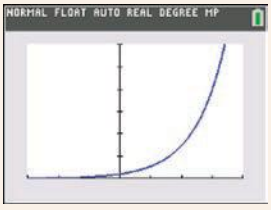
Solution

TI-84 Plus CE

Use a graph page.

Press y= and type the function into Y_1.

Set the window to $-3 \le x \le 4$ and $0 \le y \le 30$ and press graph.



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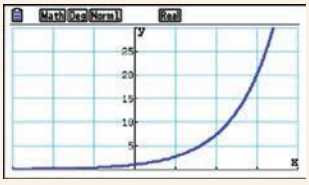
Use the Graph menu.

Set the V-Window to $-3 \le x \le 4$ and $0 \le y \le 30$ and EXIT.

Type the function into Y1 and choose DRAW.



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EXAMPLE 3

Find the exact equation of the tangent to the curve $y = e^x$ at the point $(3, e^3)$.

Solution

Find the slope at $(3, e^3)$.	At $x = 3$, $\frac{dy}{dx} = e^x = e^3$, so $m = e^3$
Write the equation of the line.	$y - y_1 = m(x - x_1)$
Substitute values.	$y - e^3 = e^3(x - 3)$
Write in standard form.	$e^3x - y - 3e^3 + e^3 = 0$
Simplify and write as a statement.	The equation of the tangent to the curve $y = e^x$ at the point $(3, e^3)$ is $e^3x - y - 2e^3 = 0$.

Exercise 2.01 The exponential function

Example 1

1 Evaluate each expression correct to 3 decimal places.

a $e^{1.5}$ **b** e^2 **c** e^{-3} **d** $\frac{1}{e}$ **e** $2e^{-2}$ **f** $-2e^2$

Example 2

2 On the same set of axes, sketch each pair of exponential functions.

a $y = e^x$ and $y = e^{-x}$ **b** $y = e^x$ and $y = -e^x$

c $y = e^x$ and $y = 2e^x$ **d** $y = e^x$ and $y = e^{2x}$

Example 3

3 Find the exact equation of the tangent to the curve $y = e^x$ at the point $(2, e^2)$.

4 For $y = e^x$, find the gradient of the tangent to the curve at the point where $x = 0.65$, correct to 2 decimal places.

5 Find the exact equation of the tangent to the curve $y = e^x$ at the point $(4, e^4)$.

2.02 The derivative of e^x

The derivative of e^x

$$\frac{d}{dx}(e^x) = e^x$$

You use the rule for the derivative of the exponential function together with other rules to calculate the derivatives of a wider range of functions.

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EXAMPLE 4

Find the derivative of each function.

a $3x^2 + 6e^x$ **b** $y = (2 + e^x)^5$

Solution

a Use the linear sum.

$$\frac{d}{dx}\left(3x^2 + 6e^x\right) = 3\frac{d}{dx}\left(x^2\right) + 6\frac{d}{dx}\left(e^x\right)$$

Simplify.

$$= 6x + 6e^x$$

b Write the function.

$$y = (2 + e^x)^5$$

Let $u = 2 + e^x$ for the chain rule.

$$\text{Let } u = 2 + e^x$$

$$y = u^5$$

Calculate $\frac{dy}{du}$.

$$\frac{dy}{du} = 5u^4$$

Calculate $\frac{du}{dx}$.

$$\frac{du}{dx} = e^x$$

Use the chain rule.

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Substitute for u and simplify.

$$= 5u^4 \times e^x$$

Write the result.

$$= 5e^x(2 + e^x)^4$$

EXAMPLE 5

Find the derivative of each function.

a $f(x) = x^2e^x$ **b** $y = \frac{e^x + 1}{2e^x - 1}$

Solution

a Write the function.

$$f(x) = x^2e^x$$

Let $u(x) = x^2$ and $v(x) = e^x$.

$$f(x) = uv$$

Calculate $\frac{du}{dx}$.

$$u' = 2x$$

Calculate $\frac{dv}{dx}$.

$$v' = e^x$$

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Use the product rule. $f'(x) = uv' + vu'$

Substitute for u, u', v' and v. $= x^2e^x + e^x \times 2x$

Write the result $= xe^x(x + 2)$

b Write the function. $y = \dfrac{e^x + 1}{2e^x - 1}$

Let $u = e^x + 1$ and $v = 2e^x - 1$. $= \dfrac{u}{v}$

Find $\dfrac{du}{dx}$ and $\dfrac{dv}{dx}$. $u' = e^x$ and $v' = 2e^x$

Use the quotient rule. $y = \dfrac{vu' - uv'}{v^2}$

Substitute for u, u', v' and v. $= \dfrac{(2e^x - 1)e^x - (e^x + 1) \times 2e^x}{(2e^x - 1)^2}$

Expand brackets in the numerator. $= \dfrac{2e^{2x} - e^x - 2e^{2x} - 2e^x}{(2e^x - 1)^2}$

Simplify and write the result. $= \dfrac{-3e^x}{(2e^x - 1)^2}$

Exercise 2.02 The derivative of e^x

Example 4

1 Find the derivative of each function.

a $3x + e^x$ **b** $\dfrac{e^x}{x}$ **c** $x^3e^x + x^2e^x + x^{-2}e^x + x^{-3}e^x$

d $3x^3 + 4e^x + 5$ **e** $9e^x$ **f** $-e^x$

g $e^x + x^2$ **h** $2x^3 - 3x^2 + 5x - e^x$ **i** $(e^x + 1)^3$

j $(5 - e^x)^9$ **k** $(2e^x - 3)^6$ **l** $(e^x + x)^4$

Example 5

2 Find the derivative of each function.

a xe^x **b** $\dfrac{e^x + 1}{e^x - 1}$ **c** $(x^2 + 3x + 1)e^x$ **d** $\dfrac{4e^x + 5}{x^2 + 1}$

e $\dfrac{e^x - x}{e^x + x}$ **f** $2x^3e^x$ **g** $(x^2 + x + 1)e^x$ **h** $x^2e^x - 2xe^x$

i x^4e^x **j** $x^2e^{x^2}$ **k** $\dfrac{e^{x^2}}{e^x}$ **l** $\dfrac{1 + e^x}{1 - e^x}$

m $e^x(1 + x)$ **n** $\dfrac{e^{6x}}{3x}$ **o** $\dfrac{2e^{5x}}{5x^3}$ **p** $\dfrac{x - 1}{e^x}$

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2.03 The derivative of $e^{f(x)}$

You can use the chain rule to differentiate functions of the form $e^{f(x)}$.

EXAMPLE 6

Find the derivative of $y = e^{x^2+2x+5}$.

Solution

Write the equation. $y = e^{x^2+2x+5}$

Let $u = x^2 + 2x + 5$. $y = e^u$

Find $\frac{dy}{du}$. $\frac{dy}{du} = e^u$

Find $\frac{du}{dx}$. $\frac{du}{dx} = 2x + 2$

Use the chain rule. $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

Substitute for known values. $= e^u \times (2x + 2)$

Substitute for u and simplify. $= (2x + 2)\, e^{x^2+2x+5}$

$= 2(x + 1)\, e^{x^2+2x+5}$

EXAMPLE 7

Find the derivative of $y = x^2 e^{2x}$.

Solution

Write the equation. $y = x^2\, e^{2x}$

Let $u = x^2$ and $v = e^{2x}$. $y = uv$

Find $\frac{du}{dx}$ and $\frac{dv}{dx}$. $u' = 2x$ and $v' = 2e^{2x}$

Use the product rule. $y' = uv' + vu'$

Substitute for u, u', v' and v. $= x^2 2e^{2x} + e^{2x} \times 2x$

Simplify to get the answer. $= 2x^2 e^{2x} + 2xe^{2x}$

$= 2xe^{2x}(x + 1)$

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Exercise 2.03 The derivative of $e^{f(x)}$

Example 6

1 Find the derivative of each function.

a	e^{x^2}	**b**	e^{-x^2+4}	**c**	$e^x(e^x+e)$	**d**	$5e^{4x-1}$
e	e^{6x-2}	**f**	e^{7-5x}	**g**	$2e^{4x+1}$	**h**	$4e^{6-2x}$
i	$-3e^{9x+4}$	**j**	$y=-2e^{6-4x}$	**k**	$10e^{6-8x}$	**l**	$-5e^{2x+4}$
m	$6e^{-7x+1}$	**n**	$2e^{\frac{x}{3}+1}$	**o**	$3e^{2-\frac{x}{5}}$	**p**	$-4e^{\frac{x}{5}+4}$
q	$5e^x-1$	**r**	$4e^{\frac{1}{x}}$	**s**	$e^{2x-1}-e^{4x^2}$	**t**	$\dfrac{4e^{-3x}}{e^{5x+1}}$
u	$(e^{2x-1})^5$	**v**	$e^{(x-1)^3}$				

Example 7

2 Find the derivative of each function.

a	$y=e^{x^2+2x}$	**b**	$y=e^{x^2-3x-2}$	**c**	$y=e^{x^2-4x}$
d	$f(x)=e^{2-6x}$	**e**	$f(x)=e^{6-3x+5x^2}$	**f**	$g(x)=e^{x^3+4x}$
g	$h(x)=3e^{4x^2-5x}$	**h**	$y=-5e^{1-2x-5x^2}$	**i**	$y=e^{(2x+1)^2}$
j	$y=e^{(4-x)^5}$	**k**	$y=e^{(x+5)^{-2}}$	**l**	$y=e^{\sqrt{x-4}}$
m	$f(x)=e^{(x+4)^{\frac{1}{3}}}$	**n**	$g(x)=e^{(x^2+4x)^2}$	**o**	$y=3xe^{x^2+4}$
p	$y=2x^2e^{x-4}$	**q**	$f(x)=(e^x-e^{x+1})^2$	**r**	$y=4x^3e^x$
s	$y=e^{x^2-1}(5+e^x)$	**t**	$f(x)=e^{2x}(e^{2x}-5e^{x^2})$	**u**	$g(x)=\sqrt{e^{4x-1}}$

3 **a** For the curve with the equation $y=(3x^2-6x)e^x$, find the x-intercepts.

b Use calculus to find the coordinates of the turning point B, correct to 3 decimal places.

c Sketch the graph of $y=(3x^2-6x)e^x$.

2.04 The derivative of a^x

You can use the chain rule to calculate the derivative of any exponential function $y=a^x$.

You can write $a=e^{\log_e(a)}$ because exponential functions and logarithm functions are inverses.

That is, if $b=\log_e(a)$, then $a=e^b=e^{\log_e(a)}$.

Write the function.	$y=a^x$
Substitute for a.	$=\left[e^{\log_e(a)}\right]^x$
Apply the index law.	$=e^{x\log_e(a)}$
Let $u=x\log_e(a)$.	$=e^u$

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Use the chain rule.

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$\frac{dy}{du} = e^u$ and $\frac{du}{dx} = \log_e(a)$

$$= e^u \times \log_e(a)$$

Substitute for u.

$$= e^{x \log_e(a)} \times \log_e(a)$$

$$= \left[e^{\log_e(a)}\right]^x \times \log_e(a)$$

Substitute for $e^{\log_e(a)}$.

$$= a^x \log_e(a)$$

Derivative of an exponential function

$$\frac{d}{dx}(a^x) = a^x \log_e(a).$$

EXAMPLE 8

Find the derivative of 5^x.

Solution

Write as an equation. $y = 5^x$

Use the rule. $\frac{dy}{dx} = 5^x \log_e(5)$

Write the answer. The derivative of 5^x is $5^x \log_e(5)$.

Exercise 2.04 The derivative of a^x

Example 8

1 Find the derivative of each function.

a $y = 10^x$ **b** $y = 4^x$ **c** $y = 7^x$

d $y = 0.4^x$ **e** $y = 1.2^x$ **f** $y = 6^{-x}$

g $y = 5^{-3x}$ **h** $y = (0.5)^{2x}$ **i** $y = 2^{0.5x}$

j $y = 3^{-4x}$

2 Find the derivative of each function.

a 2^x **b** 3^x **c** 2^{x+1}

d 5^{3x} **e** $3^x \times 5x$ **f** $3x - 3^{-x}$

g 10^{-2x+5} **h** $5^x(x^2 - x)$ **i** $(2 + x^2) \times 4^x$

j $x^2 \times 2^{3x}$ **k** $3^x \times x - 3^{x+2}$ **l** $x \times 5^x - x^2 \times 5^{-x}$

m a^{4x} **n** $x^2 a^x$ **o** $x^a a^x$

p $x^{-a} a^{-x}$ **q** $a^x - a^{2x}$

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2.05 Exponential growth and decay

Exponential functions assignment

Exponential functions extension

Many natural phenomena are modelled using simple exponential functions of the form:

$$f(x) = Ae^{kx}$$

where A and k are constants.

$f(t) = A \times e^{kt}$ is used to model **exponential growth** over time t, where A is the initial quantity of $f(t)$ and $k > 0$. It can also be used to model **exponential decay** for $k < 0$. Sometimes a base other than e can be used.

Exponential growth and decay

Exponential growth functions can be used to model bacterial growth, disease spread and financial investment amongst other things. Exponential decay functions can be used to model the radioactive decay of certain elements, the concentration and breakdown of drugs in the bloodstream, thermal cooling and other phenomena. The rate of growth or decay can be modelled by finding the derivative of the exponential function.

Exponential decay

EXAMPLE 9

The concentration of a drug measured in mg/mL in a patient's bloodstream t hours after it is administered is given by the exponential function $C(t) = 3.87e^{-0.85t}$.

a What is the initial concentration?

b What is the concentration after 1 hour?

c What is the rate of change of concentration?

d What is the rate of change of concentration after 3 hours?

Solution

a Substitute $t = 0$.

$C(t) = 3.87e^{-0.85 \times 0}$

$= 3.87$

Write the result.

The initial concentration is 3.87 mg/mL.

b Substitute $t = 1$.

$C(t) = 3.87e^{-0.85 \times 1}$

$= 1.654\ldots$

Write the result.

The concentration after 1 hour is about 1.65 mg/mL.

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c Find the derivative.

$C(t) = 3.87e^{-0.85t}$

Let $u = -0.85t$.

$C(u) = 3.87e^{u}$

Calculate $\frac{dC}{du}$.

$\frac{dC}{du} = 3.87e^{u}$

Calculate $\frac{du}{dt}$.

$\frac{du}{dt} = -0.85$

$\frac{dC}{dt} = \frac{dC}{du} \times \frac{du}{dt}$

Substitute for known values.

$= 3.87e^{u} \times (-0.85)$

Substitute for u and simplify.

$= -3.2895e^{-0.85t}$

Write the answer as a statement.

The rate of change is

$C'(t) = -3.2895e^{-0.85t}$ mg/mL per hour.

d Substitute $t = 3$.

$C'(t) = -3.2895e^{-0.85t}$

$= -3.2895e^{-0.85 \times 3}$

$= -0.2568\ldots$

Write the answer.

The rate of change of concentration after 3 hours is about –0.257 mg/mL.

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EXAMPLE 10

The increase in the number of gum trees with red flowers in a tropical region of Australia was studied. The equation $F(t) = 1000 \times 2^{0.06t}$ was used in the model, with F as the number of trees and t as the time in years.

Getty Images/Auscape/UIG

a What was the initial number of trees?

b What was the number of trees after 5 years?

c What was the rate of increase in the number of trees after 5 years?

d How long did it take for the number of trees to double?

Solution

a Substitute $t = 0$.

$$F(t) = 1000 \times 2^{0.06 \times 0}$$
$$= 1000$$

Write the answer.

There were 1000 trees in the beginning.

b Substitute $t = 5$.

$$F(t) = 1000 \times 2^{0.06 \times 5}$$
$$= 1231.1444\ldots$$

Write the answer.

There were about 1231 trees after 5 years.

c Use the derivative of an exponential function for $t = 5$.

$$F(t) = 1000 \times 2^{0.06 \times t}$$

Let $u = 0.06t$.

$$F(u) = 1000 \times 2^{u}$$

Find $\frac{dF}{du}$.

$$\frac{dF}{du} = 1000 \times \log_e(2) \times 2^{u}$$

Find $\frac{du}{dt}$.

$$\frac{du}{dt} = 0.06$$

Use the chain rule.

$$\frac{dF}{dt} = \frac{dF}{du} \times \frac{du}{dt}$$

Substitute known values.

$$= 1000 \times \log_e(2) \times 2^{u} \times 0.06$$
$$= 60 \times \log_e(2) \times 2^{0.06t}$$

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	Substitute $t = 5$.	$F'(5) = 51.201\ldots$
	Write the result.	The rate of change after 5 years is approximately 51 trees per year.
d	2000 is double the initial number of trees. Let $F(t) = 2000$.	$2000 = 1000 \times 2^{0.06t}$
	Divide both sides by 1000.	$2 = 2^{0.06t}$
	Equate indices.	$0.06t = 1$
	Solve for t.	$t = 16.667$
	Write the answer.	It took approximately 16 years and 8 months for the number of trees to double.

EXAMPLE 11

Scientists know that, given sufficient space and nutrients, the number of bacteria in a culture grows according to the function $N(t) = ae^{bt}$, where N is the number of bacteria and t is time in minutes. A particular culture has 2500 bacteria present initially. After 20 minutes, the culture has grown to 8400 bacteria.

a Determine the function for the number of bacteria present in the culture.

b How many bacteria will be present 1 hour after the initial count was made?

c Find the rate of change of the number of bacteria in the culture.

d How long will it take for the number of bacteria in the culture to double?

Solution

a Substitute (0, 2500) and solve for a.

$$2500 = ae^{b \times 0}$$

$$2500 = a$$

Substitute (20, 8400) and solve for b.

$$8400 = 2500e^{20b}$$

$$e^{20b} = \frac{8400}{2500} = 3.36$$

$$20b = \log_e(3.36)$$

$$b = \frac{\log_e(3.36)}{20} = 0.0605\ldots$$

Write the function.

$$N(t) = 2500\, e^{0.0605\ldots t}$$

b Evaluate N when $t = 60$.

$$N(60) = 2500\, e^{0.0605\ldots \times 60}$$
$$= 94\,832.64$$

Write the result.

There are about 95 000 bacteria after an hour.

c Find the derivative.

$$N'(t) = abe^{bt}$$
$$= 2500 \times 0.0605\ \ldots\ e^{0.0605\ldots t}$$
$$\approx 151\, e^{0.0605t}$$

d The doubled amount is 5000.

Solve for t.

$$5000 = 2500\, e^{0.0605\ldots t}$$
$$e^{0.0605\ldots t} = 2$$
$$0.0605\ \ldots\ t = \log_e(2)$$
$$t = \frac{\log_e(2)}{0.0605\ldots}$$
$$= 11.438\ldots$$

Write the result.

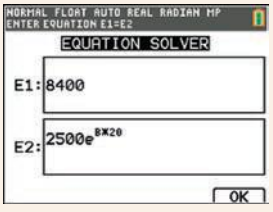
It takes about 11 minutes for the number of bacteria to double.

You can use a calculator to evaluate the unknowns as follows.

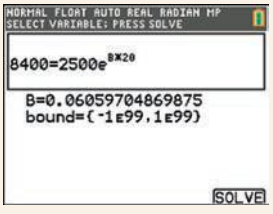
TI-Nspire CX Chapter 2

TI-84 Plus CE

Press **math** and choose **Numeric Solver**. Enter the equation and press **OK**.



Press **SOLVE**.



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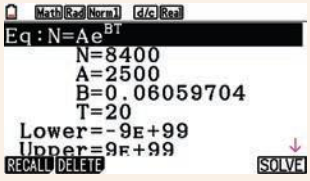
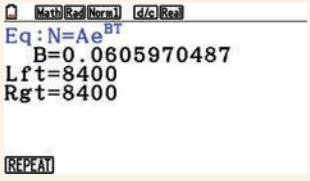
Select **EQUATION** and select **SOLVER**. Enter the equation and the values.



Press **SOLVE**.



ISBN 9780170412926

Exercise 2.05 Exponential growth and decay

Example 9

1 The decay of a uranium sample is given by $M(t) = 200e^{-0.012t}$, where M is its mass in grams and t is the time in years.

a What was the initial mass of the sample?

b What will the mass of the sample be after:

i 1 year? **ii** 10 years? **iii** 100 years?

c What is the rate of decay after:

i 1 year? **ii** 10 years? **iii** 100 years?

Example 10

2 An advertising campaign of a product helped increase sales of the product according to the formula $S(t) = 200e^{0.15t}$, where S is the number of sales and t is the time in weeks.

a How many sales were made at the beginning of the campaign?

b How many sales were made after 6 weeks of the campaign?

c What was the rate at which the sales were made after 6 weeks?

d The advertising campaign finished after 12 weeks. What was the rate of sales at this time?

Example 11

3 A breeding program is implemented to increase the number of fish in the Tamar River. The model for the population of fish over time t months is $F(t) = 500 + 2e^{0.1t}$.

a How many fish are there at the start of the program?

b What is the population after:

i 6 months? **ii** 5 years?

c Find a formula for the rate at which the population increases over time.

d At what rate is the fish population increasing after:

i 6 months? **ii** 5 years?

4 The temperature $T°$ of a chemical that cools down over t minutes is given by $T(t) = 95e^{-0.25t}$.

a Find the temperature after:

i 1 minute **ii** 5 minutes

b Find the rate at which the temperature changes after:

i 1 minute **ii** 5 minutes

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Problem solving

Example 11

5 A survey of birds in a wetlands area determined that their numbers were increasing in accordance with the function $B(m) = 1000e^{0.03m}$, where B is the number of birds and m is the number of months. After how long will there be approximately 3000 birds?

Getty Images / Martin Harvey

6 A city's population is modelled by the function $P(t) = 2000e^{0.03t}$, where t is time in years.

a After 5 years, by what percentage has the population increased?

b Find how many years it will take for the population to:

i double **ii** triple **iii** quadruple

7 A car tyre with a small puncture is inflated to a pressure of 25 units. 12 hours later, the pressure has deflated to 20 units. The pressure, P, after t hours is given by $P = P_0e^{-\lambda t}$.

a After how many hours would the pressure drop to 15 units?

b Determine the rate of pressure loss at:

i $t = 0$ **ii** $t = 12$ **iii** $t = 24$

8 A substance is heated to 80°C and then left to cool in a laboratory with a room temperature of 20°C. The temperature C at time t minutes is given by $C = 20 + 60e^{-0.25t}$.

What is the rate of cooling at:

a $t = 0$? **b** $t = 5$? **c** $t = 10$?

9 The formula for the decay of a radioactive substance over t years is given by $R(t) = R_0e^{-0.05t}$.

The half-life is the time taken for a substance to decay to half of the original quantity. What is the approximate half-life of this substance?

10 The number of apps sold by a software company after t months is given by $A(t) = 2500e^{0.36t}$. If each app costs $2.50, how long will it take for the company to make $500 000?

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2. CHAPTER SUMMARY

Derivatives of exponential functions

- $f'(x) = a^x \lim_{h \to 0} \dfrac{a^h - 1}{h}$
- $e = \lim_{n \to +\infty} \left(1 + \dfrac{1}{n}\right)^n = 2.718281...$
- $f(x) = e^x$ is known as the **exponential function**.
- The derivative of e^x is itself: $\dfrac{d}{dx}(e^x) = e^x$
- The derivative of a^x: $\dfrac{d}{dx}(a^x) = a^x \log_e(a)$
- Exponential growth and decay are modelled by the function $f(x) = Ae^{kx}$, where A and k are constants. For $k > 0$, it is exponential growth; for $k < 0$, it is exponential decay.

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2. CHAPTER REVIEW

Derivatives of exponential functions

Example 1

1 Evaluate each expression correct to 3 decimal places.

a $e^{2.5}$ **b** $4e^{-3}$

Example 2

2 Sketch $y = 2e^x$ and $y = -e^x$ on the same set of axes.

Example 3

3 Find the exact equation of the tangent to the curve $y = e^x$ at the point $(5, e^5)$.

Example 4

Example 5

4 Find the derivative of each function.

a $4x + e^x$ **b** $\dfrac{e^x}{2x}$ **c** $(e^x + 2)^2$

d $(e^x + 3x)^3$ **e** $2xe^x$ **f** $\dfrac{e^x + 2}{e^x - 2}$

g $2x^2e^x - 4xe^x$ **h** $2x^2e^{x^2}$ **i** $\dfrac{2x - 2}{2e^x}$

j $(e^x + 2)^2$

Example 6

Example 7

5 Find the derivative of each function.

a $2e^{x^2}$ **b** e^{8-3x} **c** $4e^{2x+1}$

d $y = -e^{4-2x}$ **e** $-2e^{\frac{x}{4}+5}$ **f** $\dfrac{2e^{-2x}}{e^{4x+1}}$

g $y = 2e^{x^2-2x}$ **h** $y = -3e^{1-3x-2x^2}$ **i** $y = e^{(2x+2)^{-2}}$

j $y = 2e^{\sqrt{x-4}}$ **k** $y = 2e^{\frac{1}{x}}$

Example 9

6 Find the derivative of each function, evaluating to 3 decimal places where appropriate.

a $y = 7^x$ **b** $y = 1.6^x$

c $y = (0.3)^{3x}$ **d** $y = 2^{-3x}$

Example 10

7 The area of rainforest in a tropical location is defined by the function $A(t) = 80\,000e^{-0.02t}$, where A is the area in hectares (10 000 m^2) and t is the time in years.

a What is the initial area of rainforest?

b How many hectares will there be after:

i 1 year? **ii** 10 years? **iii** 100 years?

c At what rate is the rainforest decreasing at:

i 1 year? **ii** 10 years? **iii** 100 years?

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8 An influenza virus spreads through a major city according to the function $N(t) = 85e^{0.08t}$, where N is the number of infected people and t is time in days.

Example 11

a How many people are initially affected?

b How many people are infected after:

i 5 days? **ii** 2 weeks? **iii** 12 weeks?

c What is the rate of infection after:

i 5 days? **ii** 2 weeks? **iii** 12 weeks?

d Use your calculator to determine how long it would take for there to be 100 000 cases.

Qz Practice quiz

3.

DERIVATIVES OF LOGARITHMIC FUNCTIONS

More important than common and general logarithms are **natural logarithms**, which have a base of *e*. As the name suggests, natural logarithms arise from the study of real phenomena. If the rate of change of a quantity depends on the reciprocal of the quantity itself, the quantity follows a natural logarithmic function. Such situations include learning rates, chemical reactions and periods of pendulums.

3.01 The natural logarithmic function
3.02 Exponential and natural logarithmic functions
3.03 The derivative of ln (x)
3.04 The derivative of ln $(f(x))$
3.05 The derivative of $\log_a(x)$
3.06 Applications of logarithms
Chapter summary
Chapter review

SYLLABUS SUBJECT MATTER

Calculus of logarithmic functions

- define the natural logarithm $\ln(x) = \log_e(x)$
- recognise and use the inverse relationship of the functions $y = e^x$ and $y = \ln(x)$
- establish and use the formulas $\frac{d}{dx}(\ln(x)) = \frac{1}{x}$ and $\frac{d}{dx}(\ln f(x)) = \frac{f'(x)}{f(x)}$
- use logarithmic functions and their derivatives to solve practical problems

Prior learning

Mathematical Methods 2019 v1.1 General Senior Syllabus © Queensland Curriculum and Assessment Authority (QCAA). This syllabus forms part of a new senior assessment and tertiary entrance system in Queensland. Along with other senior Syllabuses, it is still being refined in preparation for implementation in schools from 2019. For the most current syllabus versions and curriculum information please refer to the QCAA website https://www.qcaa.qld.edu.au/.

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TERMINOLOGY

exponential
ln
log-log graph
inverse function
log
natural logarithm
learning model
log graph
semi-log graph

3.01 The natural logarithmic function

Your calculator has keys named log and ln. These are both logarithm functions.
Common logarithms are base 10 and are shown as log (x).
Natural logarithms are base e and are shown as **ln (x)**.

Natural logarithms

- The **natural logarithm** of a number is the logarithm to the base $e = 2.718\ldots$
- It is shown as $\ln(x) = \log_e(x)$.
- $\ln(x) = a$ if and only if $x = e^a$.

The value of e^a is sometimes called the **antilogarithm** of a.

EXAMPLE 1

Write $\ln(6.8) \approx 1.92$ in index form.

Solution

Write the equation.	$\ln(6.8) \approx 1.92$
Write as a power of e.	$e^{1.92} \approx 6.8$

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INVESTIGATION

VALUES OF NATURAL LOGARITHMS

Use your calculator for this investigation.

Find the value of:

- $\ln(e)$
- $\ln(1)$
- $\ln\left(\frac{1}{e}\right)$
- $\ln(5)$, $\log(5)$ and $\frac{\ln(5)}{\log(5)}$
- $\ln(87)$, $\log(87)$ and $\frac{\ln(87)}{\log(87)}$
- $\ln(0.063)$, $\log(0.063)$ and $\frac{\ln(0.063)}{\log(0.063)}$

What can you say about the ratio $\frac{\ln(x)}{\log(x)}$?

- Find the value of $y = e^{\frac{\ln(5)}{\log(5)}}$
- What does this mean about the ratio $\frac{\ln(5)}{\log(5)}$?

EXAMPLE 2

Evaluate each expression correct to 4 decimal places.

a $\ln(247.8)$ **b** $\ln(0.749)$ **c** $\ln(0.000\,456)$ **d** $\ln(-5.37)$

Solution

a Use your calculator. $\ln(247.8) \approx 5.5126$

b Use your calculator. $\ln(0.749) \approx -0.2890$

c Use your calculator. $\ln(0.000\,456) \approx -7.6930$

d Only positive numbers have logarithms. $\ln(-5.37)$ is not defined.

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You can use the exponential function to reverse logarithms.

EXAMPLE 3

Solve each logarithmic equation, correct to 3 significant figures.

a $\ln(x) = 2.47$ **b** $\ln(y) = -5.2$

Solution

a Use your calculator. $x = e^{2.47} \approx 11.8$

b Use your calculator. $y = e^{-5.2} \approx 0.005\ 52$

Natural logarithms follow the same laws as general logarithms.

Natural logarithm laws

- $\ln(xy) = \ln(x) + \ln(y)$
- $\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$
- $\ln(x^a) = a\ln(x)$
- $\ln(1) = 0$
- $\ln(e) = 1$

You can prove the laws using the equivalent exponential expressions.

For the product law, choose $\ln(x) = a$ and $\ln(y) = b$ for $x, y \in \boldsymbol{R}^+$.

$\Rightarrow x = e^a, y = e^b$ (Equivalent exponential form)

$\Rightarrow xy = e^a \times e^b$

$\Rightarrow xy = e^{a+b}$ (Index laws)

$\Rightarrow \ln(xy) = a + b$ (Equivalent logarithmic form)

$\Rightarrow \ln(xy) = \ln(x) + \ln(y)$ QED

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Exercise 3.01 The natural logarithmic function

1 Write each expression in index form. Example 1

a $\ln(47.5) \approx 3.86$ **b** $\ln(0.63) \approx -0.46$

c $7.648 \approx \ln(2096)$ **d** $9.22 \approx \ln(10\,100)$

e $\ln(0.000\,489) \approx -7.62$

2 Write each expression in logarithmic form.

a $e^{-5.89} \approx 0.002\,77$ **b** $109 \approx e^{4.69}$

c $1.4 \approx e^{0.37}$ **d** $e^{9.47} \approx 13\,000$

e $e^{-2.3} \approx 0.1$

3 Evaluate each expression, correct to 3 significant figures. Example 2

a $\ln(36.8)$ **b** $\ln(0.075)$

c $\ln(498)$ **d** $\ln(2.12)$

e $\ln(0.000\,082)$

4 Find, correct to 3 significant figures, the value whose natural logarithm is: Example 3

a 3.85 **b** –6.78 **c** 1.68 **d** –0.95 **e** 0.53

5 Simplify each expression.

a $\ln(3xy) + \ln(4y)$ **b** $\ln(6pq^2) - 2\ln(q)$

c $\ln(p+q) - \ln(pq)$ **d** $2\ln(xy) - \ln(y) + 1$

e $2 - \ln(e)$ **f** $\ln(m^2 - n^2) - \ln(2mn)$

Problem solving

6 Prove that $\ln(e) = 1$.

7 Prove that $\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$ for $x, y \in \boldsymbol{R}^+$.

8 Prove that $\log_a(x) = \frac{\ln(x)}{\ln(a)}$ for $a, y \in \boldsymbol{R}^+$.

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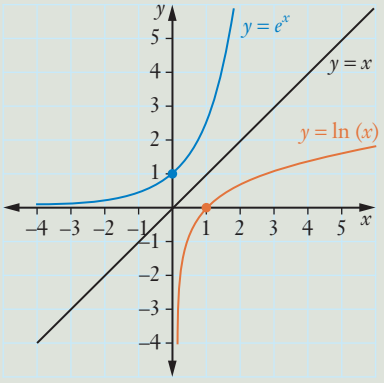
3.02 Exponential and natural logarithmic functions

$\ln(x) = y$ is equivalent to $x = e^y$. This means that the coordinates of the natural logarithm and exponential functions are exchanged.

Exponential and logarithm functions

- The natural logarithmic and the exponential are **inverse functions**.
- The point (a, b) on the graph of $y = \ln(x)$ corresponds to the point (b, a) on the graph of $y = e^x$. The coordinates of corresponding points are exchanged.
- Their graphs are the reflections of each other in the line $y = x$.



INVESTIGATION

LOGARITHM AND EXPONENTIAL GRAPHS

Use your graphics calculator for this investigation.

- Draw the graphs of $y = e^x$ and $y = \ln(x)$ on the same axes.
  - What are the x- and y- intercepts of the graphs?
  - What are the asymptotes of the graphs?
- Draw the graphs of $y = e^{x+2}$ and $y = \ln(x) - 2$ on the same axes.
  - What are the x- and y- intercepts of the graphs?
  - What are the asymptotes of the graphs?
- Draw the graphs of $y = e^x + 3$ and $y = \ln(x - 3)$ on the same axes.
  - What are the x- and y-intercepts of the graphs?
  - What are the asymptotes of the graphs?
- Draw the graphs of $y = e^{x-1} + 2$ and $y = \ln(x - 2) + 1$ on the same axes.
  - What are the x- and y-intercepts of the graphs?
  - What are the asymptotes of the graphs?

What can you conclude about the x- and y-intercepts and the asymptotes of natural logarithm and exponential graphs?

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The graph of $y = \ln(x + a) + b$

- Has a vertical asymptote at $x = -a$
- The point $(1, 0)$ on $y = \ln(x)$ is transformed to $(1 - a, b)$

When you sketch the graphs of natural logarithmic or exponential functions, show the appropriate intercepts, asymptotes, and the point corresponding to (0, 1) or (1, 0) on the graphs.

EXAMPLE 4

a What is the inverse of $y = e^{x+4}$?

b Sketch the graph of $y = e^{x+4}$ and its inverse on the same axes.

Solution

a Write in logarithmic form. $\ln(y) = x + 4$

Write as $x = \ldots$ $x = \ln(y) - 4$

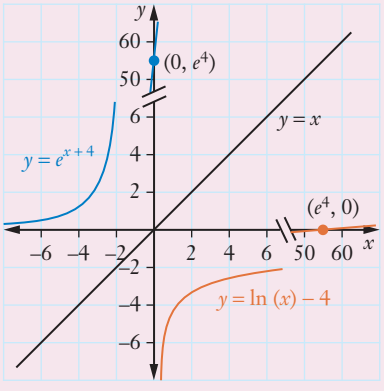
The coordinates are reversed in the inverse. The inverse is $y = \ln(x) - 4$.

b Describe the exponential. $y = e^{x+4}$ has the horizontal asymptote $y = 0$ and $(0, 1)$ is transformed to $(-4, 0)$.

Describe the logarithm. $y = \ln(y) - 4$ has the vertical asymptote $x = 0$ and $(1, 0)$ is transformed to $(1, -4)$.

Find the intercepts. $y = e^{x+4}$ has a y-intercept at $(0, e^4) \approx (0, 55)$ and $y = \ln(x) - 4$ has an x-intercept at $(e^4, 0) \approx (55, 0)$.

Sketch the functions on the same axes. Put a break in the axes to show the intercepts.



EXAMPLE 5

a What is the inverse of $y = \ln(x - 3)$?

b Sketch the graph of $y = \ln(x - 3)$ and its inverse on the same axes.

Solution

a Write in exponential form. $e^y = x - 3$

Write as $x = \ldots$ $x = e^y + 3$

The coordinates are reversed in the inverse. The inverse is $y = e^x + 3$.

b Describe the logarithmic function. $y = \ln(x - 3)$ has a vertical asymptote at $x = 3$ and $(1, 0)$ is transformed to $(4, 0)$.

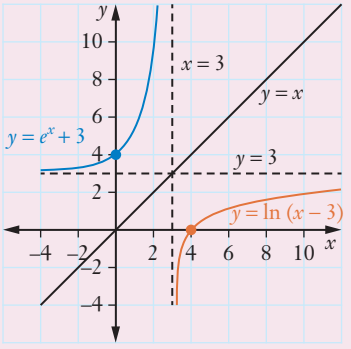
Describe the exponential function. $y = e^x + 3$ has a horizontal asymptote at $y = 3$ and $(0, 1)$ is transformed to $(0, 4)$.

Find the intercepts. $y = \ln(x - 3)$ has an intercept at $(4, 0)$ and $y = e^x + 3$ has an intercept at $(0, 4)$.

Sketch the functions on the same axes.

The exponential function crosses $x = 3$ at $(3, e^3 + 3) \approx (3, 23)$

The logarithmic function crosses $y = 3$ at about $(23, 3)$.



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INVESTIGATION

INTERSECTING LOGARITHMIC AND EXPONENTIAL FUNCTIONS

Some logarithmic functions and their inverses intersect each other. You can use your graphics calculator to investigate this situation.

TI-Nspire CX
Chapter 3

TI-84 Plus CE

Press 3 [sto→] [alpha] A to make $A = 3$.

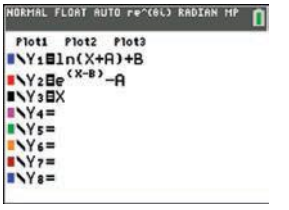
Similarly, store $B = 2$ in the memory.

Press [y=] and put $y = \ln(x + A) + B$ into Y_1,

$y = e^{(x - B)} - A$ into Y_2 and $y = x$ into Y_3.

Press [graph] to draw the graphs.

Use [trace], [zoom] or CALC to find the intersection(s).



Casio fx-CG20AU

In the Run-Matrix menu, press 3 [→] [ALPHA] A to make $A = 3$.

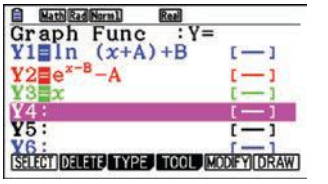
Similarly, store $B = 2$ in the memory.

Press [MENU] 5 Graph and put $y = \ln(x + A) + B$ into Y_1,

$y = e^{(x - B)} - A$ into Y_2 and $y = x$ into Y_3.

Press [F6] (DRAW) to see the graphs.

Use Trace, Zoom or G-Solv to find the intersection(s).



All calculators

Try entering in different values of A and B to investigate the intersections of other logarithmic functions and their inverses. Hint: Try setting $A = 0$ and changing B, then setting $B = 0$ and changing A before changing both.

Are the intersections always on $y = x$?

Why/Why not?

Are there values of A and B for which there is only one intersection?

What values have 2 intersections?

What values have no intersections?

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Exercise 3.02 Exponential and natural logarithmic functions

Example 4

1 **a** What is the inverse of $y = e^{x+2}$?
b Sketch the graph of $y = e^{x+2}$ and its inverse on the same axes.

2 **a** What is the inverse of $y = e^{x-3}$?
b Sketch the graph of $y = e^{x-3}$ and its inverse on the same axes.

3 **a** What is the inverse of $y = e^x + 2$?
b Sketch the graph of $y = e^x + 2$ and its inverse on the same axes.

4 **a** What is the inverse of $y = e^x - 4$?
b Sketch the graph of $y = e^x - 4$ and its inverse on the same axes

Example 5

5 **a** What is the inverse of $y = \ln(x + 1)$?
b Sketch the graph of $y = \ln(x + 1)$ and its inverse on the same axes.

6 **a** What is the inverse of $y = \ln(x) - 3$?
b Sketch the graph of $y = \ln(x) - 3$ and its inverse on the same axes.

7 **a** What is the inverse of $y = \ln(x) + 4$?
b Sketch the graph of $y = \ln(x) + 4$ and its inverse on the same axes.

Problem solving

8 Sketch the graph of $y = \ln(x + 1) - 3$ and state its domain and range.

9 What are the domain and range of $y = \ln(x - 4) + 1$?

10 Sketch the graph of $y = \ln(x + 2) + 3$ for $-1 \leq x < 2$ and state its range over this domain.

11 Sketch the graph of $y = 3 - \ln(x - 2)$.

12 Sketch the graph of $y = \ln(1 - x) + 2$.

3.03 The derivative of ln (*x*)

$\ln(2.23) = 0.8$, so $(2.23, 0.8)$ is on the graph of $y = \ln(x)$.

Also, $e^{0.8} = 2.23$, so $(0.8, 2.23)$ is the corresponding point on the graph of $y = e^x$.

Draw a tangent to e^x at $(0.8, 2.23)$.

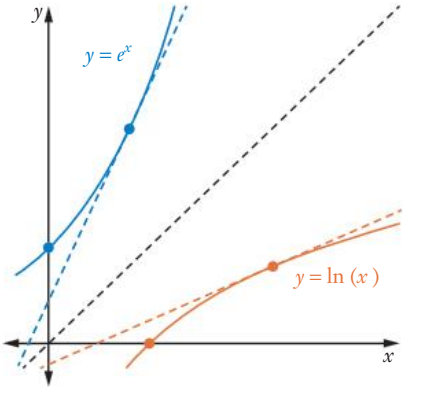
$\frac{d}{dx}(e^x) = e^x$ so the slope of the tangent is also 2.23.

The equation of the tangent is

$$y - 2.23 = 2.23(x - 0.8)$$

$$y = 2.23x - 2.23(0.8) + 2.23$$

$$y = 2.23x + 0.45$$



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The y-intercept of the tangent is (0, 0.45).

Note that $0.45 = 2.23 \times (1 - 0.8)$, which we will use to calculate the slope below.

Now draw the tangent to $y = \ln(x)$ at (2.23, 0.8).

By symmetry, the x-intercept of this tangent must be (0.45, 0).

The slope of the tangent to $y = \ln(x)$ at (2.23, 0.8) is given by

$$m = \frac{0.8 - 0}{2.23 - 0.45} = \frac{0.8}{2.23 - 2.23(1 - 0.8)} = \frac{0.8}{2.23(0.8)} = \frac{1}{2.23}.$$

Hence the derivative of $y = \ln(x)$ at (2.23, 0.8) is $\frac{1}{2.23}$.

If you choose any other point of $y = \ln(x)$, you will get the same pattern.

The derivative of $y = \ln(x)$ at any point $(a, \ln(a))$ is $\frac{1}{a}$.

The derivative of ln (x)

$$\frac{d[\ln(x)]}{dx} = \frac{1}{x}$$

Proof:

For $y = \ln(x)$, $x = e^y$, so $\frac{dx}{dy} = e^y$.

Now $\frac{dx}{dy} = \lim_{\delta y \to 0} \frac{\delta x}{\delta y}$ and since e^x and $\ln(x)$ are both continuous, you can swap $\delta x \to 0$ and $\delta y \to 0$.

At the corresponding point on the logarithmic function,

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{1}{\lim\limits_{\delta y \to 0} \frac{\delta x}{\delta y}} = \frac{1}{e^y}.$$

But $e^y = x$, so $\frac{dy}{dx} = \frac{1}{x}$ QED

EXAMPLE 6

Find the derivative of $y = \ln(x)$ at $x = 5$.

Solution

Write the general rule. $\frac{dy}{dx} = \frac{1}{x}$

Find the value for $x = 5$. $\left.\frac{dy}{dx}\right|_{x=5} = \frac{1}{5} = 0.2$

Write the answer. The derivative of $y = \ln(x)$ at $x = 5$ is 0.2.

You can use the rules for the derivatives of sums, differences and multiples of functions for derivatives involving ln (x).

EXAMPLE 7

Find the derivative of $4 \ln(x) + 9x + 2$.

Solution

Use the general derivative rules.

$$\frac{d}{dx}[4 \ln(x) + 9x + 2] = 4\frac{d}{dx}[\ln(x)] + 9 + 0$$

Use the derivative rule for ln (x).

$$= \frac{4}{x} + 9$$

$$= \frac{4 + 9x}{x}$$

You can also use the chain rule and the rules for products and quotients.

EXAMPLE 8

Find the derivative of each function.

a $\ln(5x)$ **b** $\ln(3x^4)$ **c** $x^3 \ln(x)$ **d** $\frac{\ln(x)}{x^3}$

Solution

a Use the chain rule.

Let $u = 5x$. Then $\ln(5x) = \ln(u)$.

$$\frac{d}{dx}[\ln(5x)] = \frac{d}{du}[\ln(u)] \times \frac{du}{dx}$$

$$= \frac{1}{u} \times 5$$

Substitute $u = 5x$ and simplify.

$$= \frac{1}{5x} \times 5$$

$$= \frac{1}{x}$$

b Use the chain rule.

Let $u = 3x^4$. Then $\ln(3x^4) = \ln(u)$.

$$\frac{d}{dx}[\ln(3x^4)] = \frac{d}{du}[\ln(u)] \times \frac{du}{dx}$$

$$= \frac{1}{u} \times 12x^3$$

Substitute $u = 3x^4$ and simplify.

$$= \frac{1}{3x^4} \times 12x^3$$

$$= \frac{4}{x}$$

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c Use the product rule.

Let $u(x) = x^3$ and $v(x) = \ln(x)$.

$u' = 3x^2 \qquad v' = \frac{1}{x}$

$\frac{d}{dx}[x^3 \ln(x)] = uv' + vu'$

Substitute for u, u', v' and v.

$= x^3 \times \frac{1}{x} + \ln(x)\, 3x^2$

$= x^2 + 3x^2 \ln(x)$

d Use the quotient rule.

Let $u(x) = \ln(x)$ and $v(x) = x^3$.

Then $u' = \frac{1}{x}$ and $v' = 3x^2$.

$\frac{d}{dx}\left[\frac{\ln(x)}{x^3}\right] = \frac{vu' - uv'}{v^2}$

Substitute for u, u', v' and v.

$= \frac{x^3 \times \frac{1}{x} - \ln(x) \times 3x^2}{(x^3)^2}$

Simplify.

$= \frac{x^2 - 3x^2 \ln(x)}{x^6}$

$= \frac{1 - 3\ln(x)}{x^4}$

You might find the answers for parts **a** and **b** above surprising.

You can often use the logarithm laws to make the differentiation simpler.

$\ln(5x) = \ln(5) + \ln(x)$, whose derivative is $0 + \frac{1}{x}$ because $\ln(5)$ is a constant.

$\ln(3x^4) = \ln(3) + \ln(x^4) = \ln(3) + 4\ln(x)$, so its derivative is $0 + \frac{4}{x}$. When differentiating logarithmic functions, always look for shortcuts using the logarithm laws.

Exercise 3.03 The derivative of ln (x)

1 Find the derivative of $y = \ln(x)$ at each value. Example 6

a $x = 0.5$ **b** $x = 2$ **c** $x = 4$ **d** $x = 20$ **e** $x = 0.01$

2 Find the derivative of each function. Example 7

a $7\ln(x)$ **b** $5\ln(x)$ **c** $\ln(x) + x + x^2$

d $6x^3 - 7\ln(x)$ **e** $x - x^2 + 9\ln(x)$

3 Find the derivative of each function. Example 8

a $\ln(6x)$ **b** $\ln(-4x)$ **c** $\ln\left(-\frac{3x}{4}\right)$

d $4\ln(3x)$ **e** $5\ln(3x) - 2\ln\left(\frac{x}{3}\right)$

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4 Find the derivative of each function.

a $4 \ln(x^2)$ **b** $3 \ln(x^2 + 3x + 2)$ **c** $2 \ln\left(\sqrt{3x-1}\right)$

d $3 \ln(x^2) - 2 \ln\left(\sqrt{x}\right)$ **e** $2 \ln(3x - 4)^3$

5 Differentiate each function.

a $x \ln(x)$ **b** $x^2 \ln(x^2)$ **c** $(2x^2 + 3x + 1) \ln(2x + 1)$

d $x^2 \ln(x) + x \ln(x^2)$ **e** $\sqrt{x^2+5} \ln(x^2 + 5)$

6 Differentiate each function.

a $\dfrac{\ln(x^2+3)}{x+3}$ **b** $\dfrac{\ln(x+3)}{\ln(x-3)}$ **c** $\dfrac{x^2}{\ln(x)}$

d $\dfrac{\ln(x^2+2x+5)}{(x+1)^2}$ **e** $\dfrac{\ln\left(\sqrt{3x-1}\right)}{\sqrt{3x+1}}$

7 Find the derivative of each function.

a $\ln(e^x)$ **b** $x \ln(x\, e^x)$ **c** $e^x \ln(x)$

d $e^{2x+1} \ln(2x + 1)$ **e** $3e^{4x+3} - e^{3x} \ln(4x + 3)$

Problem solving

8 Show that the derivative of $\ln(ax)$ is $\dfrac{1}{x}$.

9 Find a function whose derivative is $\ln(x)$.

10 Explain the difference between the derivatives of $\ln(x^2)$ and $[\ln(x)]^2$.

3.04 The derivative of ln (*f* (*x*))

WS

Differentiating exponential and logarithmic functions

The derivative of ln (*f* (*x*))

$$\frac{d}{dx}[\ln(f(x))] = \frac{f'(x)}{f(x)}$$

You can prove this using the chain rule.

EXAMPLE 9

Given $y = \ln(e^x + 2x + 1)$, find $\dfrac{dy}{dx}$.

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Solution

Set up the function and its derivative.

Let $f(x) = e^x + 2x + 1$.

Then $f'(x) = e^x + 2$.

State and use the rule.

$$\frac{d}{dx}[\ln(f(x))] = \frac{f'(x)}{f(x)}$$

$$\frac{dy}{dx} = \frac{e^x + 2}{e^x + 2x + 1}$$

The rule is useful with polynomial functions.

EXAMPLE 10

Differentiate $\ln(3x^4 + 2x^2 - 3)$.

Solution

Set up the function and its derivative.

Let $f(x) = 3x^4 + 2x^2 - 3$.

Then $f'(x) = 12x^3 + 4x$.

State and use the rule.

$$\frac{d}{dx}[\ln(3x^4 + 2x^2 - 3)] = \frac{f'(x)}{f(x)}$$

$$= \frac{12x^3 + 4x}{3x^4 + 2x^2 - 3}$$

Factorise where possible.

$$= \frac{4x(3x^2 + 1)}{3x^4 + 2x^2 - 3}$$

Exercise 3.04 The derivative of ln (*f* (x))

1 Find the derivative of each function. Example 9

a $\ln(3x^2 + 2x + 4)$ **b** $\ln(5e^x + 7x^2e^x)$ **c** $\ln(\ln(x))$

d $\ln(x^5)$ **e** $\ln(4x^3) + 3\ln(x^4)$

2 Find $\frac{dy}{dx}$ for each function.

a $y = \ln(x - 5)^3$ **b** $y = \ln(4x + 3)^5$ **c** $y = \ln(3x^2 + 4x + 2)^3$

d $y = \ln(x^2 + 2x + 1)^5$ **e** $y = \ln(2x^2 + 3x + 1)^4$

3 Find the derivative of each function. Example 10

a $\ln(4x^3 - 5x^2 + 2x - 3)$ **b** $\ln(x^5 + 3x^4 - 2x)$

c $\ln(7x^4 + 3x^3 - 2x^2 + 5x - 6)$

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4 Find the derivative of each function.

a $\ln(x+3)^2(x-5)^3$ **b** $[\ln(x+3)^2][\ln(x-5)^3]$

c $\dfrac{\ln(x+3)^2}{\ln(x-5)^3}$ **d** $\ln\left[\dfrac{(x+3)^2}{(x-5)^3}\right]$

5 Find $\dfrac{dy}{dx}$ for each function.

a $y=\ln\left(\sqrt{x^2-3x+4}\right)$ **b** $y=\ln(4x-5)^{\frac{3}{2}}$

c $y=[\ln(4x-5)]^{\frac{3}{2}}$ **d** $y=\sqrt[3]{\ln(x^3)}$

6 Find the derivative of each function.

a $\ln(x^{-1})$ **b** $[\ln(x)]^{-1}$

c $\ln(x^2-3x+5)^{-4}$ **d** $\ln(x^3-3x^2+5x+4)^{-0.5}$

7 What is the derivative of $(3x^2+2x+1)\ln(x^3+x^2+x-5)$?

8 Find the derivative of $(x^3+x^2+x-5)\ln(3x^2+2x+1)$.

Problem solving

9 Given $g(x)=\ln[f(x)]$, $f(5)=8$ and $f'(5)=2$, what is $g'(5)$?

10 Find a general expression for the derivative of $\ln(ax+b)^n$.

3.05 The derivative of $\log_a(x)$

You can change the base of logarithms using $\log_a(x)=\dfrac{\log_b(x)}{\log_b(a)}$.

It follows that

$$\frac{d}{dx}[\log_2(x)]=\frac{d}{dx}\left[\frac{\log_e(x)}{\log_e(2)}\right]$$

$$=\frac{1}{\log_e(2)}\frac{d}{dx}[\log_e(x)] \qquad \text{(as } \log_e(2)=\ln(2) \text{ is a constant)}$$

$$=\frac{1}{x\ln(2)}$$

The derivative of $\log_a(x)$

- $\log_a(x)=\dfrac{\ln(x)}{\ln(a)}$
- $\dfrac{d}{dx}[\log_a(x)]=\dfrac{1}{x\ln(a)}$

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EXAMPLE 11

Find the derivative of each function.

a $\log(x)$ **b** $\log_5(x)$ **c** $\log_2(4x+3)$

Solution

a The base for common logarithms is 10. $\log(x) = \log_{10}(x)$

Use the rule for $\frac{d}{dx}\log_a(x)$. $\frac{d}{dx}\log_{10}(x) = \frac{1}{x\ln(10)}$

Write the answer in the form of the question. $\frac{d}{dx}\log(x) = \frac{1}{x\ln(10)}$

b Use the rule for $\frac{d}{dx}\log_a(x)$. $\frac{d}{dx}[\log_5(x)] = \frac{1}{x\ln(5)}$

c Use the chain rule with $u = 4x+3$. $\frac{d}{dx}\log_2(4x+3) = \left[\frac{d}{du}\log_2(u)\right]\left[\frac{du}{dx}\right]$

Do the derivatives. $= \frac{1}{u\ln(2)} \times 4$

Substitute $u = 4x+3$. $= \frac{4}{(4x+3)\ln(2)}$

You may have to use other derivative rules as well.

EXAMPLE 12

Find $f'(x)$ for each function.

a $f(x) = \log(5x-3)^4$ **b** $f(x) = \log_4[(2x^2-3x-1)(x+6)^3]$

Solution

a Simplify $f(x)$ using a log law. $f(x) = \log(5x-3)^4 = 4\log(5x-3)$

Use the chain rule. $f'(x) = \frac{df}{du} \times \frac{du}{dx}$ where $u = 5x-3$

$= \frac{4}{u\ln(10)} \times 5$

Simplify. $= \frac{20}{(5x-3)\ln(10)}$

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b Simplify $f(x)$ using a log law.

$$f(x) = \log_4(2x^2 - 3x - 1) + 3\log_4(x + 6)$$

Use the chain rule.

$$f'(x) = \frac{1}{(2x^2 - 3x - 1)\ln(4)} \times (4x - 3) + \frac{1}{(x + 6)\ln(4)} \times 3$$

Simplify.

$$= \frac{(4x - 3)(x + 6) + 3(2x^2 - 3x - 1)}{(2x^2 - 3x - 1)(x + 6)\ln(4)}$$

$$= \frac{4x^2 + 24x - 3x - 18 + 6x^2 - 9x - 3}{(2x^2 - 3x - 1)(x + 6)\ln(4)}$$

$$= \frac{10x^2 + 12x - 21}{(2x^2 - 3x - 1)(x + 6)\ln(4)}$$

Neither of the quadratics will factorise.

Exercise 3.05 The derivative of $\log_a(x)$

Example 11

1 Find the derivative of each function.

a $3\log(x)$ **b** $\log_8(x)$ **c** $5\log_{0.5}(x)$

d $0.5\log_2(4x - 5)$ **e** $6\log(5x + 8)$

Example 12

2 Find the derivative of each function.

a $\log(x - 4)^5$ **b** $\log_4(6x^3 - 3x^2 + x + 8)$ **c** $\log_3(2x - 5)^4$

d $\log\left(\sqrt{x^2 - 1}\right)$ **e** $\log_7(3x^2 + 8)^3$

3 Find the derivative of each function.

a $\log_4[(2x - 3)(3x + 2)]$ **b** $\log[(5x - 3)^2(4x - 1)]$

c $\log_3[(3x + 2)(4x - 1)^5]$ **d** $\log[(x^2 + x + 2)(x^2 - x - 2)]$

4 Find $f'(x)$ for each function.

a $f(x) = [\log_4(2x - 3)][\log_4(3x + 2)]$ **b** $f(x) = [\log(5x - 3)^2][\log(4x - 1)]$

c $f(x) = [\log_3(3x + 2)][\log_3(4x - 1)^5]$ **d** $f(x) = [\log(x^2 + x + 2)][\log(x^2 - x - 2)]$

5 Find the derivative of each function.

a $y = \log_2\left(\frac{5x - 3}{3x - 5}\right)$ **b** $y = \frac{\log_2(5x - 3)}{\log_2(3x - 5)}$

Problem solving

6 Find the derivative of $\log_6(x^2 + 1) + \log_{36}(x^2 - 1)$.

7 Find the derivative of $f(x) = [\log_6(x^2 + 1)][\log_{36}(x^2 - 1)]$.

8 Find the derivative of $y = \frac{\log(x^2 - 3x - 4)}{\log(x + 1)^2}$.

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9 Find the derivative of $f(x) = \log_6 [(3x^2 + 5x + 1)^5(3x^2 + 5x - 3)^4]$.

10 Find the derivative of $y = [\log (3x^4 - 8x + 2)^3][\log (8x^3 - 8x^2 + 3)^4]$.

11 Find a general formula for the derivative of $f(x) = \log_a[p(x)q(x)]$.

12 Find a general formula for the derivative of $f(x) = \log_a\left[\dfrac{p(x)}{q(x)}\right]$.

13 Find the derivative of $y = \log_2\left[(3x^5 - 6x^2 + 4)^3 \cdot \sqrt[3]{5x^3 - 12x + 4}\right]$.

3.06 Applications of logarithms

Derivatives of logarithmic functions assignment

Derivatives of logarithmic functions extension

With logarithms, you might suspect a power relationship between variables. In this case, plotting logarithms of the variables will reveal the relationship's nature. This is called a **log graph**.

Suppose you find that $\ln (y) = 2 \ln (x) + 1.6$.

Then $\ln (y) = \ln (x^2) + \ln (e^{1.6}) = \ln (e^{1.6}x^2)$, so $y = e^{1.6}x^2 \approx 5x^2$.

Log graphs

If $\ln (y) = a \ln (x) + b$, then y is a power function of x given by $y = e^b x^a$.

EXAMPLE 13

Andrea left the concentrations and quantity of sulphuric acid/hydrogen peroxide/sodium thiosulphate/starch in the 'clock reaction' the same. She recorded the times taken for the solution to turn blue for different concentrations of 50 mL of potassium iodide (KI) solution. Use a log-log graph to investigate the relationship between:

a concentration and reaction time

b concentration and rate of reaction.

KI concentration (molarity), c	0.3	0.25	0.2	0.15	0.1
Reaction time (seconds), t	8	12	20	30	75

Solution

a Find the logarithms.

ln (c)	−1.20	−1.39	−1.61	−1.90	−2.30
ln (t)	2.08	2.48	3.00	3.40	4.32

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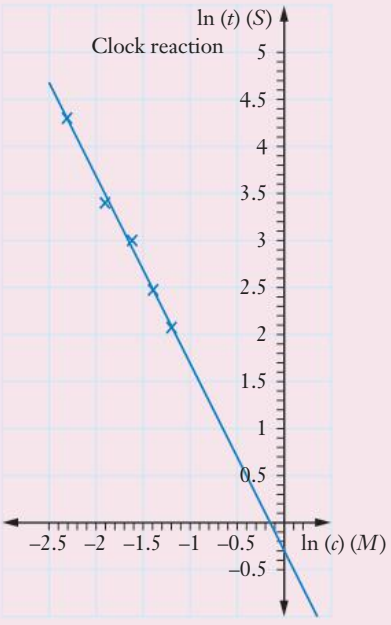
Plot the graph and draw a line of best fit.



Choose some points on the line.

The line passes through (–2, 3.7) and (0, –0.3).

Find the slope and ln (t)-intercept.

$m = \dfrac{-0.3 - 3.7}{0 - (-2)} = -2$, ln ($t$)-intercept = –0.3

Write the equation.

$\ln(t) = -2\ln(c) - 0.3$

Change the form to get t.

$t = e^{-0.3}c^{-2} \approx 0.74c^{-2} = \dfrac{0.74}{c^2}$

Write the relationship.

The reaction time is inversely proportional to the square of the concentration of potassium iodide.

b Write an expression for the rate of reaction.

$\text{Rate of reaction} = \dfrac{\text{Quantity}}{\text{Time}}$

Substitute the time.

$= \dfrac{\text{Quantity}}{0.74c^{-2}}$

$= \dfrac{\text{Quantity}}{0.74} \times c^2$

Write the relationship.

The rate of reaction is proportional to the square of the concentration of potassium iodide.

In Example 13, the temperature and concentrations of the other reactants may affect the reaction time. The first equality relationship suggests that only the concentration of KI is important. Hence, the finding should be given as a proportional statement.

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TECHNOLOGY

Regression lines

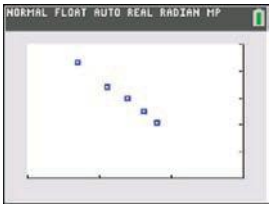
You can use your graphics calculator to find linear relationships.

TI-84 Plus CE

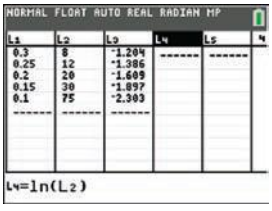
Press [stat] and choose 1: Edit from the EDIT menu.
Enter the data from Example 13 in L1 and L2.

Move the cursor across and up to highlight L3. Press [ln] [2nd] [1] [)] [enter] to put the logarithms of L1 in L3.
Put the logarithms of L2 into L4.

Press [2nd] [y=] ([stat plot]) and choose Plot 1.
Set the plot On, choose ⌐⋰, XList = [L3] and YList = [L4]. Make the [window] $-3 \le X \le 0$ and $0 \le Y \le 5$ and press [graph].



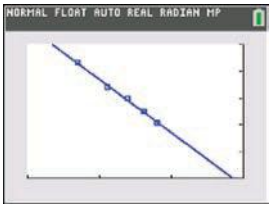
To find the line of best fit, press [stat] and choose CALC, 4: LinReg(ax+b). Make XList = [L3] and YList = [L4] and Store RegEQ: Y1 by pressing [vars], Y-VARS, 1: Function and 1: Y1.



Move the cursor down to Calculate and press [enter].

Press [graph] again to see the line of best fit.

You can also use L1, L2 with A: PwrReg instead of 4: LinReg(ax+b) to find the power function directly.

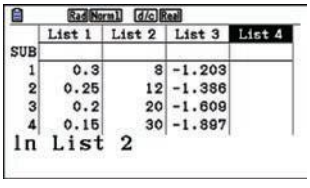


Casio fx-CG20AU

In the Statistics menu, enter the data from Example 13 in List 1 and List 2.

Move the cursor across to highlight List 3.

Press [ln] [SHIFT] [1] [1] [EXE] to put the logarithms of List 1 in List 3. Put the logarithms of List 2 in List 4.



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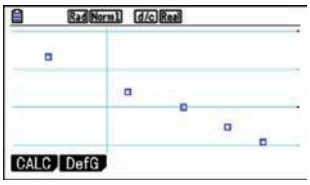
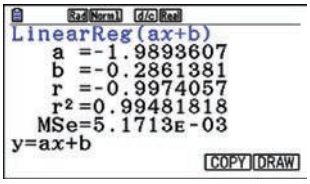
Press F1, F1 (GRAPH, SET) and change Xlist to List 3 and Ylist to List 4 using F1, 3 and F1, 4.

Choose the Mark Type you wish and press EXIT.

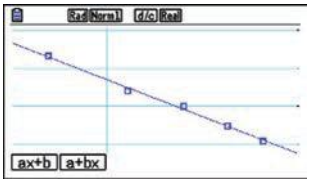
Set the V-Window to $-3 \leq X \leq 0$ and $0 \leq Y \leq 5$, EXIT and press F1 (GRAPH1) to see the graph.



To find the line of best fit, press F1, F2, F1 (CALC, X, ax+b).



To see the line of best fit, press F6 (DRAW).



You can also use List 1, List 2 with Power instead of X to find the power function directly.

If you suspect an exponential relationship between variables, plotting a graph of the logarithm of the dependent variable against the independent variable will show the relationship. This is called a **semi-log graph**.

If $y = Ae^{kx}$, then

$$\ln(y) = \ln(Ae^{kx}) = \ln(A) + \ln(e^{kx})$$

Thus $\ln(y) = kx + \ln(A)$.

Semi-log graphs

If $\ln(y) = ax + b$, then y is given by the exponential function $y = Ae^{ax}$, where $A = e^b$.

iStock/Getty Images Plus/jarun011

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EXAMPLE 14

Dom recorded the radius of a colony of bacteria on an agar plate over a period of 12 hours. Use a semi-log graph to find the relationship between the variables.

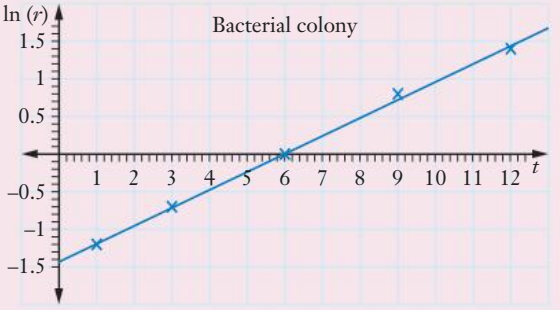
Time t (hr)	1	3	6	9	12
Radius r (cm)	0.3	0.5	1.0	2.2	3.9

Solution

Find logarithms of the radius.

t	1	3	6	9	12
$\ln(r)$	−1.2	−0.7	0	0.8	1.4

Plot the graph and draw a line of best fit.



Find the slope and y-intercept.

$$\text{Slope} = \frac{1.4-(-1.4)}{12-0} \approx 0.23$$

y-intercept $= -1.4$

Write the equation.

$$\ln(r) = 0.23t - 1.4$$

Express in usual form.

$$r = e^{2.3t-1.4} = e^{-1.4}e^{0.23t}$$

$$r \approx 0.25e^{0.23t}$$

Write the answer.

The colony grows exponentially over the period with the relationship $r \approx 0.25e^{0.23t}$.

There is obviously a limit to the growth of the colony as eventually the nutrients on the plate will be exhausted. You can only make conclusions for the period of the data.

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TECHNOLOGY

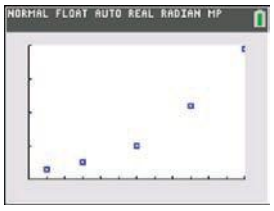
Exponential relationships

You can use your graphics calculator to find exponential relationships.

TI-Nspire CX
Chapter 3

TI-84 Plus CE

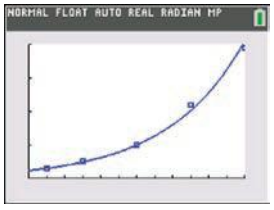
Enter the data from Example 14 in L1 and L2 as shown on page 79.



Use [stat plot] and choose Plot 1. Set to use L1 and L2 and make the [window] $0 \le X \le 12$ and $0 \le Y \le 4$ and press [graph].

To find the exponential line of best fit, press [stat] and choose CALC, 0: ExpReg. Make XList L1, YList L2 and Store RegEQ Y1 and Calculate.

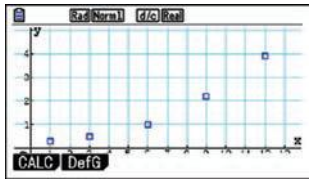
To see the exponential line of best fit, press [graph] again.



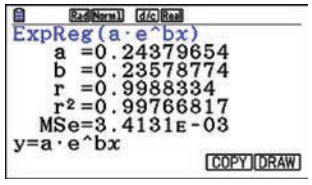
The TI finds the exponential $y = ab^x$, so to change to the form $y = ae^{kx}$, you need to write $k = \ln(b)$.

Casio fx-CG20AU

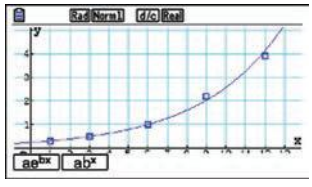
Use the statistics menu and enter the data from Example 14 in L1 and L2 as shown on pages 79–80. SET GRAPH1 to use List 1 and List 2 and make the V-WINDOW $0 \le X \le 12$ and $0 \le Y \le 4$.
[EXIT] and press [F1](GRAPH1).



To find the exponential line of best fit, press [F1] (CALC) and choose EXP and ae^{bx}.



To see the exponential line of best fit, press [F6] (DRAW).



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Consider learning to play a game. When you first start playing you improve rapidly, but as time goes on, the rate of improvement decreases. This applies to learning most skills and can often be modelled by a logarithmic function. If you start from zero at $t = 0$, the function is of the form $a \ln (t + 1)$. If you start from a level c, the function is of the type $a \ln (t + 1) + c$. More complex models may be of the form $a \ln b(t + 1)$.

EXAMPLE 15

Xi is learning to read Chinese. In the first month, she learns the meanings of 500 characters, and after 3 months has learned 1250 characters altogether. Use a logarithmic model to find approximately how long it would take her to learn a total of 2500 characters.

iStock/Getty Images Plus/brenkel

Solution

Write a possible model.

Let $n = a \ln b(t + 1)$, where n = number of characters learned and t = time in months, assuming she knows none to start with.

Write the known values.

$500 = a \ln (2b)$

$1250 = a \ln (4b)$

Solve for a and b.

$$500 = a[\ln (2) + \ln (b)] = a \ln (2) + a \ln (b)$$

$$1250 = a[\ln (4) + \ln (b)] = a \ln (4) + a \ln (b)$$

$$1250 - 500 = a \ln (4) - a \ln (2) = a \ln (2)$$

$$a = \frac{750}{\ln (2)} \approx 1080$$

$$500 = a \ln (2b)$$

$$\ln (2b) = \frac{500}{a} = \frac{500 \ln (2)}{750} = \frac{2 \ln (2)}{3}$$

$$2b = e^{\frac{2 \ln (2)}{3}}$$

$$b \approx 0.794$$

Write the model.

After t months, the number of characters that Xi will learn is given by $n \approx 1080 \ln [0.794(t + 1)]$.

Find the time to learn 2500 characters.	$2500 \approx 1080 \ln [0.794(t + 1)]$ $\ln [0.794(t + 1)] \approx 2.31$ $0.794(t + 1) \approx e^{2.31}$ $t = 11.68\ldots$ $t \approx 12$
Write the answer.	It will take Xi about a year to learn 2500 characters (unless she changes the way she is learning them).

Exercise 3.06 Applications of logarithms

Example 13

1 Daniel timed a chemical reaction for different concentrations of sodium nitrate and calculated the rates of reaction shown below.

Concentration (M)	0.01	0.02	0.05	0.1	0.2
Rate of reaction (g/s)	0.11	0.31	1.11	4	10

Use a log graph to find the power relationship.

Example 14

2 Simone suspects an exponential relationship between time and the amplitude of a pendulum. Use a semi-log graph with her observations below to find the relationship.

Time (minutes)	1	2	4	6	10
Amplitude (cm)	12	11.5	11	10	9

Example 15

3 Kerry is planning to go to France, so she is learning French online. After two weeks, she has learned 50 words and after 4 weeks she knows 95 words. Use a logarithmic model to find how long it would take her to learn a basic vocabulary of 500 words.

iStock/Getty Images Plus/Meinzahn

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Problem solving

4 The period and radius of the planets' orbits are shown below. Use a graph to find the relationship.

Planet	Period (days)	Distance (million km)
Mercury	88	58
Venus	225	108
Earth	365	150
Mars	687	228
Jupiter	4333	778
Saturn	10 759	1427
Uranus	30 685	2870
Neptune	60 191	4497
Pluto*	90 465	5900

*Considered as a planet

5 Marc and Helen Bornstein claimed that people in big cities walk faster than those in small towns. The table below shows the average speeds of people walking in some different cities and their population at the time. Use a graph to find a possible relationship.

City	Psychro, Crete	Itea, Greece	Corte, Corsica	Bastia, France	Netanya, Israel	Iraklion, Greece
Population	370	2500	5500	49 000	71 000	78 000
Speed (m/s)	0.84	0.69	1.01	1.49	1.31	1.17
City	New Haven, USA	Jerusalem, Israel	Brno, Czech Republic	Athens, Greece	Munich, Germany	Brooklyn, USA
Population	138 000	305 000	342 000	867 000	1 340 000	2 602 000
Speed (m/s)	1.34	1.35	1.47	1.59	1.71	1.54

6 Galileo noticed that the time a chandelier in Pisa Cathedral took to make one swing seemed to be constant. He timed the swings of pendulums of different lengths and divided by the number of swings to measure the period (the time for one complete swing). His observations were similar to those below. Use a graph to find the relationship.

Length (m)	0.3	0.5	1	1.5	2
Period (s)	1.1	1.4	2.0	2.5	2.8

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7 Peter measured the voltage across a capacitor as it discharged through a large resistance. His observations are shown below. Peter thought it might be an exponential decay. Use a graph to find the relationship.

Time (s)	4	8	12	20	30	40
Voltage	49	40	33	22	13	8

8 Max wants to get under the school record of 21.5 s for the 200 m track event. When he started serious training every day, he ran 24 seconds, and after 5 days got down to 22.5 seconds. Use a suitable model to find if he is likely to beat the record in another week.

9 When she started, Cordelia could sew 20 shirts in a day, but after a fortnight she increased her production to 30 shirts a day. Use a suitable model to find how much longer it will take her to reach her target of 40 shirts a day.

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3. CHAPTER SUMMARY

Derivatives of logarithmic functions

- The **natural logarithm** of a number is the logarithm to the base $e = 2.718\ldots$
- It is shown as $\ln(x) = \log_e(x)$.
- $\ln(x) = a$ if and only if $x = e^a$.

Natural logarithm laws

- $\ln(xy) = \ln(x) + \ln(y)$
- $\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$
- $\ln(x^a) = a\ln(x)$
- $\ln(1) = 0, \ln(e) = 1$

Graphing natural logarithms

- The natural logarithm and the exponential are **inverse functions**. The point (a, b) on the graph of $y = \ln(x)$ corresponds to the point (b, a) on the graph of $y = e^x$. Their graphs are reflections of each other in the line $y = x$.
- The graph of $y = \ln(x + a) + b$ has an asymptote at $x = -a$ and the point $(1, 0)$ on $y = \ln(x)$ is transformed to $(1 - a, b)$

Derivatives

- The derivative of the natural logarithm is $\frac{d}{dx}[\ln(x)] = \frac{1}{x}$
- The derivative of $\ln(f(x))$ is $\frac{d}{dx}[\ln(f(x))] = \frac{f'(x)}{f(x)}$
- The derivative of $\log_a(x) = \frac{\ln(x)}{\ln(a)}$ is $\frac{d}{dx}[\log_a(x)] = \frac{1}{x\ln(a)}$

log and semi-log graphs

- If $\ln(y) = a\ln(x) + b$, then y is a power function of x given by $y = e^b x^a$
- If $\ln(y) = ax + b$, then y is given by the exponential function $y = Ae^{ax}$, where $A = e^b$

Logarithmic learning models

- $w = a\ln(t + 1)$
- $w = a\ln(t + 1) + c$
- $w = a\ln b(t + 1)$

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3. CHAPTER REVIEW

Derivatives of logarithmic functions

Example 1

1 Write $\ln(28.4) \approx 3.35$ in index form.

Example 2

2 Evaluate each logarithm, correct to 4 decimal places.

a $\ln(13.8)$ **b** $\ln(0)$ **c** $\ln(0.0054)$

Example 3

3 Solve each logarithmic equation, correct to 3 significant figures.

a $\ln(a) = 2.98$ **b** $\ln(b) = -3.02$

Examples 4,5

4 **a** What is the inverse of $\ln(x + 2)$?

b Sketch the graph of $\ln(x + 2)$ and its inverse on the same axes.

Example 6

5 Find the derivative of $y = \ln(x)$ at $x = 15$.

6 Find the derivative of each function.

Examples 7,8

a $8\ln(x)$ **b** $\ln(25x)$ **c** $\ln(3x + 2)$

d $y = \ln(x^4)$ **e** $f(x) = \dfrac{\ln(x)}{x^3}$

Example 9

7 $f(x) = \ln(x^3 + 3x^2 - 8)$. Find $f'(x)$.

Example 10

8 Find the derivative of each function.

a $\ln[(3x - 2)^3(x + 1)^2]$ **b** $\dfrac{\ln(2x+5)^4}{\ln(x^2+3)^3}$ **c** $[\ln(x - 6)^3][\ln(x + 5)^2]$

Example 10

9 Find the derivative of each function.

a $y = \ln\left(\sqrt{6x^3 - 2x + 5}\right)$ **b** $f(x) = (5x^2 + 2x - 3)\ln(x^2 + 2x + 1)$

Example 11

10 Find the derivative of each function.

a $\log_4(x)$ **b** $\log(3x)$

Example 12

11 Find the derivative of each function.

a $\log(3x^2 - 5x + 2)^5$ **b** $\log_2\left[\dfrac{(3x-5)^2}{(2x+1)^3}\right]$ **c** $\dfrac{\log_3(6x+1)^3}{\log_3(4x^2-3)^2}$

Example 13

12 Use a log graph to find the relationship between the variables in the table below.

t	3.8	5.8	8.1	12.4	15.4	18.5	18.9
m	17.6	42.9	77.1	118	160	270	212

Example 14

13 Use a semi-log graph to find the relationship between the variables in the table below.

m	−1.3	0.3	1.2	2.6	5.5	6.9
n	2.5	5.5	20	66	210	670

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14 Raj is learning the phonetic alphabet for radio operators. He already knew Alpha (A) and Foxtrot (F) before he started, and after 2 days had learned another 10. Use a logarithmic model to find long he will take to learn all 26 codes.

Example 15

Problem solving

15 Prove that $\ln\left(\frac{1}{x}\right) = -\ln(x)$ for all $x \in R^+$.

16 Sketch the graph of $y = 4 - \ln(x + 3)$ and state its domain and range.

17 Show that the derivative of $y = (a + x)\ln(a + x)$ is $\frac{dy}{dx} = 1 + \ln(a + x)$.

18 Find the derivative of $\frac{\log_3(6x^2 + x + 1)^3}{\log_3(4x^2 - 3)^2}$.

19 To be effective, a particular antibiotic has to have a concentration of at least 5 ppm (parts per million) in the blood. The antibiotic is removed from the blood by the kidneys, and the rate of removal depends on the concentration. The concentration in a patient's blood was measured and the results are shown below. The initial amount was enough to make the concentration 20 ppm. Levels above 30 ppm are dangerous. Use a suitable graph to find the relationship and determine how often patients should be given the antibiotic.

Time (h)	0.5	1	1.5	2	2.5	3
Concentration (ppm)	18	16	14	13	11	10

20 Daniella set up two magnets to repel each other and measured the force by the compression of a spring. Her results are shown in the table below. Use a suitable graph to find the relationship between separation (x) and the force F.

Separation x (cm)	10	8	6	4	3
Force F (mm of compression)	1	1.5	3	6	11

21 A sniffer dog is being trained to identify drugs in bags at an airport. In each session, the trainer puts some drugs in 10 out of 200 bags. At the start, the dog only finds 2 bags with drugs, but after 5 days, it manages to find 6 of them. Use a suitable model to find how long it will take the dog to find all the bags with drugs.

Qz Practice quiz

WS

Year 12 formulas

Practice examination 1

Time: 90 minutes
Perusal time: 5 minutes
Marks: 50

Instructions

- Students are permitted to bring and use: pens, pencils, highlighters, erasers, sharpeners, rules and an approved graphics calculator.
- Students must show appropriate working and justification to gain full marks.
- A QCAA formula sheet is provided.
- Unless otherwise stated, numerical answers should be exact.
- Unless otherwise indicated, no diagrams in this examination are drawn to scale.
- All written responses must be in English.
- Answer all questions.
- **Students are NOT permitted to bring or use notes of any kind, correction fluid/tape, mobile phones and/or any other unauthorised electronic devices.**

Question 1 (1 mark)

Express $16^{\frac{3}{4}} = 8$ in logarithmic form.

Question 2 (2 marks)

Express each equation in index form.

a $\log_4(1024) = 5$

b $\log_{81}(3) = \frac{1}{4}$

Question 3 (1 mark)

Without using a calculator, simplify: $2\log_3(21) - 2\log_3\left(\frac{7}{3}\right)$.

Question 4 (1 mark)

Expand $\log_5\left(\frac{x^3(2x+1)}{(x-4)^2}\right)$.

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Question 5 (1 mark)

Use the change of base theorem to evaluate $\log_{11}(37)$, correct to 4 decimal places.

Question 6 (2 marks)

Solve each equation.

a $5^{2x+3} = 625$

b $81^x = \sqrt{3}$

Question 7 (1 mark) (Technology-active)

Use a graphics calculator to solve $6 \times 9^{3x-2} - 47 = 0$, correct to 3 decimal places.

Question 8 (1 mark)

Solve $[\log_3(x)]^2 - 2\log_3(x^2) = -3$.

Question 9 (2 marks) (Technology-active)

Find the gradient of the tangent to the curve $y = e^x$ at the point where $x = 0.83$, correct to 2 decimal places.

Question 10 (4 marks)

Find the derivative of:

a $y = (3 - 2e^x)^3$

b $y = \dfrac{4e^{3x}}{3x^2}$

Question 11 (4 marks)

Find the derivative of:

a $y = 2xe^{x^3+4}$

b $y = 2x^3a^{2x}$

Question 12 (2 marks)

a Express $e^{3.45} \approx 31.5$ in logarithmic form.

b Simplify $\ln\left(\dfrac{p^2}{q^3}\right) + \ln(p^2q^2)$.

Question 13 (3 marks)

Sketch the graph of $\ln(x + 2) - 1$ and state its domain and range.

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Question 14 (5 marks)

Find the derivative of:

a $y = 6 \ln(x)$

b $y = \ln(5v^2)$

c $y = \log_5(3g)$

d $y = \ln\left(\sqrt{4x-1}\right)$

Question 15 (4 marks) (Technology-active)

A formula based on Newton's Law of Cooling can be used to calculate the elapsed time since a person has died. The formula is:

$$t = -10 \ln\left(\frac{T-R}{37-R}\right)$$

where t is the time in hours since the death, T is the body's temperature measured in °C and R is the environmental temperature in °C. It is assumed that the environmental temperature is constant.

A man was found dead in a hotel room one morning.

At 11 a.m., the coroner measured the body temperature to be 28.5°C.

At 1 p.m., the temperature of the body was found to be 27.6°C. The room's temperature was 22°C. Estimate the man's time of death.

Question 16 (3 marks) (Technology-active)

A bowl of water is heated to 95°C and then left to cool, with a room temperature of 15°C. The temperature C at time t minutes is given by

$$C(t) = 15 + 80e^{-0.35t}.$$

a What is the temperature after 8 minutes?

b What is the rate of cooling at:

i $t = 5$?

ii $t = 10$?

Question 17 (3 marks)

Find the derivative of $y = \ln(2x^2 - 3x + 1) \ln(3x^3 + 5x - 8)$.

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Question 18 (3 marks) (Technology-active)

A bacterial colony on a petri dish is observed to double its population every 3 hours. Show that the number of bacteria satisfies the exponential growth law

$$P(t) = P(0)a^{kt},$$

where $P(t)$ is the bacterial colony population at time t hours, and find the values of a and k.

Question 19 (3 marks) (Technology-active)

Uranium-238 is a radioactive element and decays exponentially according to the formula

$$U(t) = U_0 e^{-0.15t},$$

where t is time in billions of years and U_0 was the amount present at the formation of the Earth. At present, there is approximately half the amount of Uranium-238 left on Earth compared to when the Earth initially formed.
Determine the approximate age of the Earth.

Question 20 (4 marks)

A Year 12 Physics student investigated the time it took a pendulum to swing back and forth. She timed 10 swings with pendulums of different lengths. Her results are shown below.

Length (m)	0.2	0.4	0.8	1.2	1.4	1.8	2.2
Time (s)	10	11	19	21	24	26	29

Use a suitable graph to find the relationship between the length of the pendulum and the period and suggest ways she could improve the accuracy.

END OF EXAMINATION

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4.

DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

The sine and cosine functions occur very frequently in nature. The electrical signals of your brain are made up of sine functions. Brain waves and other natural trigonometric functions change over time. The changes and rates of change give important information about their sources. You can't visit stars, so studying their light waves is the only way to get information about them.

4.01 The derivative of $y = \sin(x)$
4.02 The derivative of $y = \cos(x)$
4.03 The derivative of $y = \tan(x)$
4.04 Combinations of functions
4.05 Modelling with trigonometric functions
4.06 Applications of trigonometric functions
Chapter summary
Chapter review

SYLLABUS SUBJECT MATTER

Calculus of trigonometric functions

- establish the formulas $\frac{d}{dx}(\sin(x)) = \cos(x)$ and $\frac{d}{dx}(\cos(x)) = -\sin(x)$ by numerical estimations of the limits and informal proofs based on geometric constructions
- identify contexts suitable for modelling by trigonometric functions and their derivatives and use the model to solve practical problems; verify and evaluate the usefulness of the model using qualitative statements and quantitative analysis
- use trigonometric functions and their derivatives to solve practical problems

Prior learning

Mathematical Methods 2019 v1.1 General Senior Syllabus © Queensland Curriculum and Assessment Authority (QCAA). This syllabus forms part of a new senior assessment and tertiary entrance system in Queensland. Along with other senior Syllabuses, it is still being refined in preparation for implementation in schools from 2019. For the most current syllabus versions and curriculum information please refer to the QCAA website https://www.qcaa.qld.edu.au/.

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TERMINOLOGY

amplitude
cosine
exponential
periodic function
radian
tangent
angular speed
derivative
logarithm
product rule
simple harmonic motion
chain rule
differentiate
period
quotient rule
sine

4.01 The derivative of $y = \sin(x)$

Remember the **difference quotient** of a function is $\frac{f(x+h)-f(x)}{h}$. The derivative is the limit of the difference quotient as $h \to 0$.

INVESTIGATION

DIFFERENCE QUOTIENTS OF SIN (*X*)

- Make sure your calculator is set in radians for this investigation.
- Find the difference quotient of sin (x) at $x = 0.2$ for $h = 0.01$.
- Find the value of cos (0.2). What do you notice?
- Complete the table below for difference quotients with $h = 0.01$.

x	0.0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
Diff. quotient of sin (x)											
cos (x)											

- What happens if you find the difference quotients for $h = 0.001$?
- What conclusion can you make?

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Waves within waves

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TECHNOLOGY

Difference quotients of sin (*x*)

You can use your graphics calculator to find difference quotients.

TI-Nspire CX Chapter 4

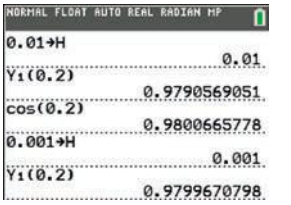
TI-84 Plus CE

Make sure your calculator is set to radians.

Press [y=] and make $Y_1 = (\sin(X+H) - \sin(X)) \div H$ and press [enter].

Press [2nd] [mode] to quit [y=].

Type 0.01[sto→] [alpha] [^] to make H = 0.01.



Now press [vars], choose Y-VARS, 1: Function and 1: Y [1]. Complete to get Y [1] (0.2) and press [enter].

Compare with cos (0.2).

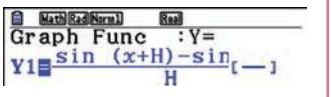
Now make $H = 0.001$ and find Y [1] (0.2) again.

Find other values of the difference quotient and compare to cos (X).

Press [y=] and make Y [2] = cos(X).
Make the [window] $-3 \le X \le 6$ and $-2 \le Y \le 2$. Press [graph] to see the graphs of the difference function in blue and the cos function in red.

Casio fx-CG20AU

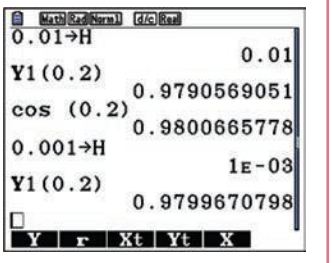
In the GRAPH menu, make $Y1 = (\sin(x+H) - \sin(x)) \div H$.



Make sure your calculator is set to radians.

Press [MENU] and choose 1: Run-Matrix.

Type 0.01 [→] [ALPHA] [F↔D] to make H = 0.01.



Now press [VARS], choose GRAPH and Y.

Complete to get Y1(0.2) and press [EXE].

Compare with cos (0.2).

Now make $H = 0.001$ and find Y [1] (0.2) again.

Find other values of the difference quotient and compare to cos (X).

Make the V-Window $-3 \le X \le 6$ and $-2 \le Y \le 2$. Press [MENU], choose 5: Graph and make Y2 = cos (*x*). Choose DRAW to see the graphs of the difference function in blue and the cos function in red.

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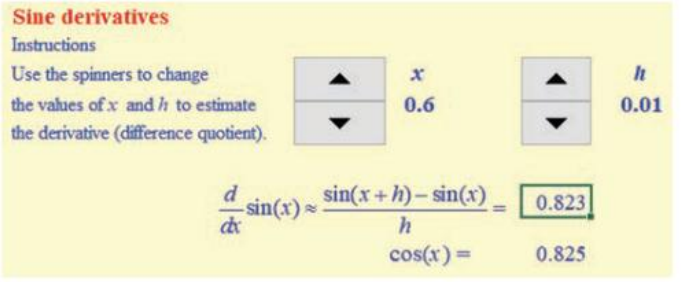
Sine derivatives

The spreadsheet 'Sine derivatives' on NelsonNet calculates difference quotients automatically.

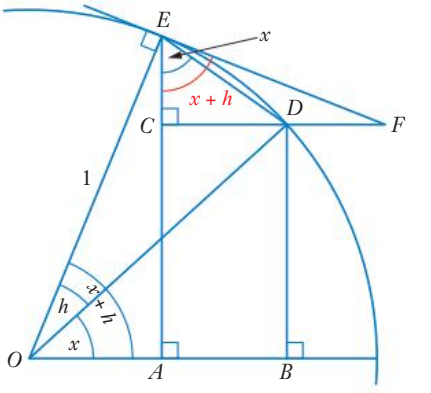


Derivative of sin (x)

$$\frac{d}{dx}[\sin(x)] = \cos(x)$$

where x is radians.

A formal proof for the derivative of sin (x) is beyond the scope of this course, but you can demonstrate it geometrically as shown below.



Draw a unit circle and angles x and $x + h$. Draw perpendiculars AE and BD from the x-axis to the radii OE and OD that form the angles x and $x + h$. Draw the line CDF from the perpendicular AE to the tangent EF. Draw the line ED.

In $\triangle OEA$, $\angle OEA = \frac{\pi}{2} - (x + h)$, so

in $\triangle EFC$, $\angle CEF = x + h$

$\triangle OED$ is isosceles, so $\angle OED = \frac{\pi - h}{2}$

$$\angle CED = \angle OED - \angle OEA$$

$$= \frac{\pi - h}{2} - \left[\frac{\pi}{2} - (x + h)\right] = x + \frac{h}{2}.$$

From the definition of sine,

$$\sin(x + h) = EA \text{ and } \sin(x) = DB = CA.$$

$$\sin(x + h) - \sin(x) = EA - CA = EC.$$

From the definition of radians, the length of the arc $\overset{\frown}{ED} = h$.

This means that $\frac{EC}{\overset{\frown}{ED}} = \frac{\sin(x+h) - \sin(x)}{h}$

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In $\triangle EFC$, $\cos(x+h) = \dfrac{EC}{EF} = \dfrac{\sin(x+h)-\sin(x)}{EF}$

In $\triangle EDC$, $\cos\left(x+\dfrac{h}{2}\right) = \dfrac{EC}{ED} = \dfrac{\sin(x+h)-\sin(x)}{ED}$

From the diagram, it is clear that $EF > \overset{\frown}{ED} > ED$, so

$$\frac{\sin(x+h)-\sin(x)}{EF} < \frac{\sin(x+h)-\sin(x)}{h} < \frac{\sin(x+h)-\sin(x)}{ED}.$$

It follows that $\cos(x+h) < \dfrac{\sin(x+h)-\sin(x)}{h} < \cos\left(x+\dfrac{h}{2}\right)$.

As $h \to 0$, $\cos(x+h) \to \cos(x)$ and $\cos\left(x+\dfrac{h}{2}\right) \to \cos(x)$.

Since $\dfrac{\sin(x+h)-\sin(x)}{h}$ is 'sandwiched' between them, $\lim\limits_{h\to 0}\dfrac{\sin(x+h)-\sin(x)}{h} = \cos(x)$.

It is obvious that you could make the same construction in other quadrants, so the demonstration works for any value of x. QED

EXAMPLE 1

Find the derivative of each function.

a $5\sin(x)$ **b** $\sin(3x)$ **c** $7\sin(2x-5)$ **d** $\sin(4\theta^\circ)$

Solution

a Use the rule $\dfrac{d}{dx}[kf(x)] = kf'(x)$. $\dfrac{d}{dx}[5\sin(x)] = 5\dfrac{d}{dx}[\sin(x)]$

Use the rule for the derivative of $\sin(x)$. $= 5\cos(x)$

b Use the chain rule with $u = 3x$. $\dfrac{d}{dx}[\sin(3x)] = \dfrac{dy}{du} \times \dfrac{du}{dx}$

Do the derivatives. $= \cos(u) \times 3$

Replace u. $= 3\cos(3x)$

c Use the rule $\dfrac{d}{dx}[kf(x)] = kf'(x)$. $\dfrac{d}{dx}[7\sin(2x-5)] = 7\dfrac{d}{dx}[\sin(2x-5)]$

Use the chain rule with $u = 2x-5$. $= 7\dfrac{df}{du} \times \dfrac{du}{dx}$

Differentiate. $= 7\cos(u) \times 2$

Replace u. $= 14\cos(2x-5)$

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d You must change θ to **radians** to use the rule for the derivative of $\sin(x)$.

$$4\theta^\circ = 4\theta \times \frac{\pi}{180} = \frac{\pi\theta}{45}$$

Substitute in the problem.

$$\frac{d}{d\theta}[\sin(4\theta^\circ)] = \frac{d}{d\theta}\left[\sin\left(\frac{\pi\theta}{45}\right)\right]$$

Write the chain rule for this case.

$$= \frac{d}{du}[\sin(u)] \times \frac{du}{d\theta}, \text{ where } u = \frac{\pi\theta}{45}$$

Do the derivatives.

$$= \cos(u) \times \frac{\pi}{45}$$

Substitute u.

$$= \frac{\pi}{45}\cos\left(\frac{\pi\theta}{45}\right)$$

Express back in degrees.

$$= \frac{\pi}{45}\cos(4\theta^\circ)$$

You will probably find it easier to communicate your method by showing substitutions for more complicated cases of the chain rule.

EXAMPLE 2

Find the derivative of each function.

a $g(x) = 7\sin(x^3 + 4x^2 - 5x + 2)$

b $f(t) = 5\sin(7t - 3)^4$

c $h(p) = 4[\sin(2p) + \sin(5p)]^3$

Solution

a Choose a variable.

Let $g(x) = 7\sin(y)$, where $y = x^3 + 4x^2 - 5x + 2$.

Write the chain rule for this case.

$$g'(x) = g'(y)\,y'(x)$$

Differentiate each part.

$$= 7\cos(y) \times (3x^2 + 8x - 5)$$

Simplify with the original variable.

$$= 7(3x^2 + 8x - 5)\cos(x^3 + 4x^2 - 5x + 2)$$

b Choose a variable.

Let $f(t) = 5\sin(p)$, where $p = (7t - 3)^4$.

Write the chain rule for this case.

$$f'(t) = f'(p)\,p'(t)$$

Differentiate each part.

$$= 5\cos(p) \times 4(7t - 3)^3 \times 7$$

Simplify with the original variable.

$$= 140(7t - 3)^3\cos(7t - 3)^4$$

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c Choose a variable. Let $h(p) = 4q^3$, where $q = \sin(2p) + \sin(5p)$.

Write the chain rule for this case. $h'(p) = h'(q)\, q'(p)$

Differentiate each part. $= 12q^2 \times [2\cos(2p) + 5\cos(5p)]$

Simplify with the original variable. $= 12[\sin(2p) + \sin(5p)]^2 \times [2\cos(2p) + 5\cos(5p)]$

You can write some general rules for derivatives of functions involving sin (x).

Derivatives of sine functions

- $\frac{d}{dx}[\sin(kx)] = k\cos(kx)$
- $\frac{d}{dx}[\sin(x°)] = \frac{\pi}{180}\cos(x°)$
- $\frac{d}{dx}(\sin[f(x)]) = f'(x)\cos[f(x)]$

Exercise 4.01 The derivative of $y = \sin(x)$

1 Find the derivative of each function. Example 1

a $6\sin(x)$ **b** $-3\sin(x)$ **c** $5 - \sin(x)$

d $\sin(4\theta)$ **e** $\sin(-7\theta)$ **f** $\frac{1}{4}\sin(8x)$

g $\frac{5}{8}\sin(10t)$ **h** $\sin(6x°)$

2 Find the derivative of each function.

a $y = 3\sin(4t)$ **b** $f(p) = -5\sin(3p)$ **c** $r(\theta) = 7\sin(2\theta°)$

d $y = \sin(4 - x)$ **e** $z = \sin(3v - 5)$ **f** $v(t) = \sin(3 - 5t)$

g $4\sin(2t + 3)$ **h** $6\sin(m - 4)$

3 Find the derivative of each function. Example 2

a $\sin(\alpha^3)$ **b** $\beta(\alpha) = \sin(4\alpha^2)$ **c** $y = \sin(2x^3 - 7)$

d $3\sin(y^2 - 5y)$ **e** $2\sin(t^3 - 3t^2 + t - 1)$ **f** $3\sin(\sqrt{x})$

g $\sin(\sqrt{x - x^3})$ **h** $f(x) = \sqrt{4\sin(3x)}$

4 Find the derivative of each function.

a $y = \sin(3x - 2)^3$ **b** $\sin(5t + 4)^6$ **c** $\sin(2m - 7)^{-1}$

d $g(v) = \sin(5 - v)^{-3}$ **e** $-5\sin(3n + 1)^4$ **f** $a(t) = 4\sin(6 - 5t)^3$

g $f(x) = 5\sin(2x + 7)^{-4}$ **h** $y = -3\sin(4x + 5)^{-2}$

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5 Find the derivative of each function.

a $[\sin(t)]^3$

b $2[\sin(4t)]^5$

c $g(x) = \sin^2(x) + 2\sin(x) - 3$

d $y = [5 - 2\sin(x)]^3$

e $y = 4\sin^2(5x - 3)$

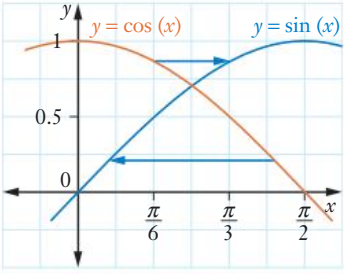
f $h(t) = [1 - 3\sin(t) + \sin(3t)]^2$

g $\sin^4(2g^3)$

h $-4\sin^5(1 - 3f)^{-2}$

4.02 The derivative of $y = \cos(x)$

The shapes of the sine and cosine graphs are the same, but they are out of phase by $\frac{\pi}{2}$.



The arrows on the graphs show that $\cos(x) = \sin\left(\frac{\pi}{2} - x\right)$.

You can also see that the slope of $y = \cos(x)$ is the same value but the opposite sign to the slope of $y = \sin(x)$ at $\left(\frac{\pi}{2} - x\right)$.

The slope of $\sin(x)$ is given by $\frac{d}{dx}[\sin(x)] = \cos(x)$, so at $\left(\frac{\pi}{2} - x\right)$, it is $\cos\left(\frac{\pi}{2} - x\right)$.

This means that $\frac{d}{dx}[\cos(x)] = \frac{d}{dx}\left[\sin\left(\frac{\pi}{2} - x\right)\right] = -\cos\left(\frac{\pi}{2} - x\right) = -\sin(x)$.

Derivative of cos (x)

$$\frac{d}{dx}[\cos(x)] = -\sin(x)$$

You can prove the rule by using the chain rule.

$$\begin{aligned}\frac{d}{dx}[\cos(x)] &= \frac{d}{dx}\left[\sin\left(\frac{\pi}{2} - x\right)\right] \\ &= \frac{d}{du}[\sin(u)] \times \frac{du}{dx}, \text{ where } u = \left(\frac{\pi}{2} - x\right) \\ &= \cos(u) \times (-1) \\ &= -\cos\left(\frac{\pi}{2} - x\right) \\ &= -\sin(x) \qquad \text{QED}\end{aligned}$$

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EXAMPLE 3

Find the derivative of each function.

a $3\cos(x)$ **b** $\cos(4x)$

c $-6\cos(3x+1)$ **d** $\cos(3\theta°)$

Solution

a Use the rule $\frac{d}{dx}[kf(x)] = kf'(x)$ $\frac{d}{dx}[3\cos(x)] = -3\sin(x)$

b Use the chain rule with $u = 4x$. $\frac{d}{dx}[\cos(4x)] = \frac{dy}{du} \times \frac{du}{dx}$

Do the derivatives. $= -\sin(u) \times 4$

Write in the usual order. $= -4\sin(4x)$

c Use the rule $\frac{d}{dx}[kf(x)] = kf'(x)$. $\frac{d}{dx}[-6\cos(3x+1)] = -6\frac{d}{dx}[\cos(3x+1)]$

Use the chain rule. $= -6 \times [-\sin(3x+1)] \times 3$

Simplify. $= 18\sin(3x+1)$

d Change $3\theta°$ to radians. $3\theta° = 3\theta \times \frac{\pi}{180} = \frac{\pi\theta}{60}$

Substitute. $\frac{d}{d\theta}[\cos(3\theta°)] = \frac{d}{d\theta}\left[\cos\left(\frac{\pi\theta}{60}\right)\right]$

Use the chain rule. $= -\frac{\pi}{60}\sin\left(\frac{\pi\theta}{60}\right)$

Express back in degrees. $= -\frac{\pi}{60}\sin(3\theta°)$

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Show the chain rule substitutions for more complex cases.

EXAMPLE 4

Find the derivative of each function.

a $h(x) = 2\cos(3x^4 + 4x^2 - 1)$

b $f(t) = -6\cos(2t + 5)^3$

c $g(p) = 4\cos^3(2p)$

Solution

a

Choose a variable.	Let $h(x) = 2\cos(u)$, where $u(x) = 3x^4 + 4x^2 - 1$.
Write the chain rule for this case.	$h'(x) = h'(u) \times u'(x)$
Differentiate each part.	$= -2\sin(u) \times (12x^3 + 8x)$
Factorise where possible.	$= -8x(3x^2 + 2)\sin(3x^4 + 4x^2 - 1)$

b

Choose a variable.	Let $f(t) = -6\cos(u)$, where $u(t) = (2t + 5)^3$.
Write the chain rule for this case.	$f'(t) = f'(u) \times u'(t)$
Differentiate each part.	$= 6\sin(u) \times 6(2t + 5)^2$
Simplify.	$= 36(2t + 5)^2 \sin(2t + 5)^3$

c

Choose a variable.	Let $g(p) = 4u^3$, where $u(p) = \cos(2p)$.
Write the chain rule for this case.	$g'(p) = g'(u) \times u'(p)$
Differentiate each part.	$= 12u^2 \times [-2\sin(2p)]$
Simplify with the original variable.	$= -24\cos^2(2p)\sin(2p)$

Derivatives of cosine functions

- $\frac{d}{dx}[\cos(kx)] = -k\sin(kx)$
- $\frac{d}{dx}[\cos(x°)] = -\frac{\pi}{180}\sin(x°)$
- $\frac{d}{dx}(\cos[f(x)]) = -f'(x)\sin[f(x)]$

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Exercise 4.02 The derivative of $y = \cos(x)$

Example 3

1 Find the derivative of each function.

a $4\cos(x)$ **b** $-5\cos(x)$

c $7 - 2\cos(x)$ **d** $\cos(3\alpha)$

e $\cos(-6\theta)$ **f** $\frac{1}{3}\cos(12x)$

g $\frac{5}{6}\cos(10t)$ **h** $\cos(5m°)$

2 Find the derivative of each function.

a $v = 8\cos(3t)$ **b** $g(x) = -2\cos(4x)$

c $r(\theta) = 8\cos(3\theta°)$ **d** $3\cos(m - 5)$

e $b = \cos(5v - 3)$ **f** $v(t) = \cos(5 - 3t)$

g $6\cos(4t + 7)$ **h** $y = \cos(7 - 2x)$

Example 4

3 Find the derivative of each function.

a $\cos(x^3)$ **b** $k(n) = \cos(5n^4)$

c $y = \cos(5x^2 - 9)$ **d** $6\cos(2y^2 - 5y - 5)$

e $4\cos(3t^3 + 4t^2 + 3t - 2)$ **f** $5\cos(\sqrt{x})$

g $\cos\left(\sqrt{x^5 - x^2}\right)$ **h** $f(x) = \sqrt{3\cos(3x)}$

4 Find the derivative of each function.

a $y = \cos(4x - 5)^2$ **b** $\cos(3t + 2)^5$

c $\cos(m - 3)^{-1}$ **d** $g(v) = \cos(4 - v)^{-4}$

e $-3\cos(2n + 5)^6$ **f** $a(t) = 3\cos(5 - 4t)^4$

g $f(x) = 8\cos(3x + 1)^{-3}$ **h** $y = -2\cos(7x + 3)^{-5}$

5 Find the derivative of each function.

a $[\cos(a)]^4$ **b** $3[\cos(2t)]^4$

c $f(x) = \cos^2(x) + 3\cos(x) - 2$ **d** $y = [6 - 5\cos(x)]^4$

e $y = 4\cos^2(3x - 5)$ **f** $h(t) = [3 - \cos(2t) + \cos(t)]^2$

g $-5\cos^6(1 - f)^{-3}$ **h** $\cos^3(4g^5)$

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WS

Differentiating trigonometric functions

4.03 The derivative of $y = \tan(x)$

You can find the derivative of tan (x) as the quotient of sin (x) and cos (x) as follows.

$$\frac{d}{dx}[\tan(x)] = \frac{d}{dx}\left[\frac{\sin(x)}{\cos(x)}\right]$$

$$= \frac{vu' - uv'}{v^2}, \text{ where } u = \sin(x) \text{ and } v = \cos(x)$$

$$= \frac{\cos(x) \times \cos(x) - \sin(x) \times [-\sin(x)]}{\cos^2(x)}, \text{ using } u' = \cos(x) \text{ and } v' = -\sin(x)$$

$$= \frac{\cos^2(x)}{\cos^2(x)} + \frac{\sin^2(x)}{\cos^2(x)}$$

$$= 1 + \tan^2(x)$$

The derivative of tan (x)

$$\frac{d}{dx}[\tan(x)] = 1 + \tan^2(x)$$

$$\text{or } \frac{d}{dx}[\tan(x)] = \frac{1}{\cos^2(x)}$$

You can prove the second form of the derivative using $\sin^2(x) + \cos^2(x) = 1$, which follows from the unit circle definitions of sine and cosine and is learned in the Specialist Mathematics course.

EXAMPLE 5

Find the derivative of each function.

a 5 tan (x) **b** tan $(3x)$ **c** -2 tan $(4x - 3)$

Solution

a Use the rule $\frac{d}{dx}[k f(x)] = k f'(x)$.

$$\frac{d}{dx}[5\tan(x)] = 5[1 + \tan^2(x)]$$

b Use the chain rule with $u = 3x$.

$$\frac{d}{dx}[\tan(3x)] = \frac{dy}{du} \times \frac{du}{dx}$$

Do the derivatives.

$$= [1 + \tan^2(u)] \times 3$$

Write in the usual order.

$$= 3[1 + \tan^2(3x)]$$

c Use the rule $\frac{d}{dx}[k f(x)] = k f'(x)$.

$$\frac{d}{dx}[-2\tan(4x - 3)] = -2\frac{d}{dx}[\tan(4x - 3)]$$

Use the chain rule.

$$= -2[1 + \tan^2(4x - 3)] \times 4$$

Simplify.

$$= -8[1 + \tan^2(4x - 3)]$$

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You can use the chain rule for more complex cases.

EXAMPLE 6

Find the derivative of each function.

a $f(x) = 8 \tan (x^3 - 3x - 5)$ **b** $v = 3 \tan (1 - 4t)^2$ **c** $7 \tan^3(2p)$

Solution

a Choose a variable. Let $f(x) = f[g(x)]$, where $g(x) = x^3 - 3x - 5$.

Write the chain rule for this case. $f'(x) = f'(g) \times g'(x)$

Differentiate each part. $= 8[1 + \tan^2(x^3 - 3x - 5)] \times (3x^2 - 3)$

Factorise where possible. $= 24(x - 1)(x + 1)[1 + \tan^2(x^3 - 3x - 5)]$

b Choose a variable. Let $v = 3 \tan (u)$, where $u = (1 - 4t)^2$.

Write the chain rule for this case. $\frac{dv}{dt} = \frac{dv}{du} \times \frac{du}{dt}$

Differentiate each part. $= 3[1 + \tan^2(u)] \times [-8(1 - 4t)]$

Simplify. $= -24(1 - 4t)[1 + \tan^2(1 - 4t)^2]$

c Choose a variable. Let $7 \tan^3(2p) = 7q^3$, where $q = \tan (2p)$.

Write the chain rule for this case. $\frac{d}{dp}[7 \tan^3(2p)] = \frac{d}{dq}(7q^3) \times \frac{d}{dp}[\tan (2p)]$

Differentiate each part. $= 21q^2 \times 2[1 + \tan^2(2p)]$

Simplify. $= 42 \tan^2(2p)[1 + \tan^2(2p)]$

Derivatives of tangent functions

- $\frac{d}{dx}[\tan (kx)] = k[1 + \tan^2(kx)]$
- $\frac{d}{dx}[\tan (x^\circ)] = \frac{\pi}{180}[1 + \tan^2(x^\circ)]$
- $\frac{d}{dx}(\tan [f(x)]) = f'(x)(1 + \tan^2[f(x)])$

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Exercise 4.03 The derivative of tan (x)

Example 5

1 Find the derivative of each function.

a $9 \tan(x)$ **b** $-2 \tan(x)$

c $5 - \tan(x)$ **d** $\tan(5m)$

e $\tan(-4\theta)$ **f** $\frac{2}{5}\tan(10x)$

g $\frac{4}{9}\tan(15t)$ **h** $\tan(3m°)$

2 Find the derivative of each function.

a $v = 6 \tan(4t)$ **b** $g(x) = -3 \tan(7x)$

c $q(x) = 2 \tan(5x°)$ **d** $4 \tan(x - 5)$

e $b = \tan(4v - 1)$ **f** $a(v) = \tan(4 - 3v)$

g $6 \tan(5t + 3)$ **h** $y = \tan(8 - 5x)$

Example 6

3 Find the derivative of each function.

a $\tan(x^4)$ **b** $k(n) = \tan(3n^2)$

c $y = \tan(4x^2 - 25)$ **d** $7 \tan(3x^2 - 6x + 3)$

e $8 \tan(4t^3 - 3t^2 + 5t - 1)$ **f** $6 \tan(\sqrt{x})$

g $\tan\left(\sqrt{x^6 - x^4}\right)$ **h** $f(x) = \sqrt{8\tan(2x)}$

4 Find the derivative of each function.

a $y = \tan[(3x - 7)^3]$ **b** $\tan[(5t + 2)^4]$

c $\tan[(m - 4)^{-1}]$ **d** $g(t) = \tan[(4 - t)^{-5}]$

e $-2 \tan[(3n + 1)^3]$ **f** $a(t) = 4 \tan[(5 - 3t)^4]$

g $f(x) = 6 \tan[(4x + 1)^{-3}]$ **h** $y = -6 \tan[(3x + 4)^{-2}]$

5 Find the derivative of each function.

a $[\tan(n)]^3$ **b** $4[\tan(5t)]^2$

c $f(x) = \tan^3(x) + 3\tan^2(x) + 2$ **d** $y = [4 - \tan(x)]^3$

e $y = 3\tan^4(2x - 7)$ **f** $h(t) = [3 - \tan(4t) + \tan(t)]^3$

g $-5\tan^4[(1 - f)^{-3}]$ **h** $\tan^2(7g^3)$

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4.04 Combinations of functions

The musical note 'A' produced by a tuner for musical instruments is the sine wave $y = a \sin\left(\frac{2\pi t}{440}\right)$, where a is related to its loudness. Musical instruments do not produce pure sine waves of sound. They are combinations of different functions such as sine, cosine and exponential.

The different sounds of different instruments are due to the exact combinations.

Shutterstock.com/Trodler

You should know how to use the chain rule, product and quotient rules for combinations of different functions. Show the substitutions for more complex cases of the product and quotient rules.

EXAMPLE 7

Find the derivative of each function.

a $y = (x^2 - 5x + 1)\sin(2x)$ **b** $v(t) = \dfrac{2t^2 - 5t + 1}{3\cos(2t)}$

Solution

a Use the product rule. Let $y = uv$, where $u = x^2 - 5x + 1$ and $v = \sin(2x)$.

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Do the derivatives. $\dfrac{du}{dx} = 2x - 5, \dfrac{dv}{dx} = 2\cos(2x)$

Substitute in rule. $\dfrac{dy}{dx} = (x^2 - 5x + 1)\,2\cos(2x) + \sin(2x)\,[2x - 5]$

$= 2(x^2 - 5x + 1)\cos(2x) + (2x - 5)\sin(2x)$

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b Choose variables. Let $g = \frac{y}{z}$, where $y = 2t^2 - 5t + 1$ and $z = 3\cos(2t)$.

Write the quotient rule. $\frac{dg}{dt} = \frac{zy' - yz'}{z^2}$

Do the derivatives. $y' = 4t - 5$ and $z' = -6\sin(2t)$.

Substitute in rule. $\frac{dg}{dt} = \frac{3\cos(2t)[4t-5] - (2t^2 - 5t + 1) \times [-6\sin(2t)]}{9\cos^2(2t)}$

Factorise the top. $= \frac{3\left[(4t-5)\cos(2t) + 2(2t^2 - 5t + 1)\sin(2t)\right]}{9\cos^2(2t)}$

Simplify. $= \frac{(4t-5)\cos(2t) + 2(2t^2 - 5t + 1)\sin(2t)}{3\cos^2(2t)}$

You will also need to differentiate combinations involving trigonometric, exponential and logarithmic functions.

EXAMPLE 8

Find the derivative of each function.

a $p = 7\sin\left[\ln\left(\frac{3q-6}{2q+3}\right)\right]$

b $f(x) = 5(e^{4x-2} + 4)\cos(3x^2 + 4x - 3)$

Solution

a Choose variables. Let $p = 7\sin(t)$, where $t = \ln\left(\frac{3q-6}{2q+3}\right) = \ln\left(\frac{u}{v}\right)$.

Use the chain rule. $\frac{dp}{dq} = \frac{dp}{dt} \times \frac{dt}{dq}$

Use the quotient rule. $= \frac{d}{dt}[7\sin(t)] \times \frac{vu' - uv'}{v^2}$

Do the derivatives. $= 7\cos(t)\frac{(2q+3)3 - 2(3q-6)}{(2q+3)^2}$

Simplify. $= \frac{147\cos(t)}{(2q+3)^2}$

Substitute for t. $= \frac{147}{(2q+3)^2}\cos\left(\ln\left(\frac{3q-6}{2q+3}\right)\right)$

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b Choose variables. Let $f(x) = g(x)\cos[h(x)]$, where $g(x) = 5(e^{4x-2} + 4)$ and $h(x) = 3x^2 + 4x - 3$.

Use product rule. $f'(x) = g(x)\,h'(x)[-\sin[h(x)]] + g'(x)\cos[h(x)]$

Do derivatives. $= -5(e^{4x-2} + 4)(6x + 4)\sin[h(x)] + 5e^{4x-2}\,4\cos[h(x)]$

Factorise and substitute $h(x)$. $= 10[2e^{4x-2}\cos(3x^2 + 4x - 3) - (e^{4x-2} + 4)(3x + 2)\sin(3x^2 + 4x - 3)]$

Exercise 4.04 Combinations of functions

Example 7

1 Find the derivative of each function.

a $x^2 \sin(3x)$

b $(c^2 + c - 1)\sin(2c - 3)$

c $(5m^2 - 3m + 2)\sin(4x - 3)$

d $\sin(x - 3)\sin(x + 3)$

e $\dfrac{\sin(3x-5)}{x+4}$

f $\dfrac{2\sin(2x+3)}{(4x-1)^2}$

g $\dfrac{2x^3 - 3x^2 + 4}{\sin(4-x)}$

h $\dfrac{5\sin(1-2x)}{\sin(3x+2)}$

2 Find the derivative of each function.

a $x^3\cos(4x)$

b $(m^2 + 3m - 4)\cos(2m - 3)$

c $(5x^3 - 4x^2 + 3)\cos(3x - 5)$

d $\cos(2x + 1)\cos(3x + 1)$

e $\dfrac{\cos(5x-3)}{3x+2}$

f $\dfrac{5\cos(3-4x)}{(3x-5)^3}$

g $\dfrac{2x^2 - 3x^3 + 2}{\cos(5-3x)}$

h $\dfrac{3\cos(2-x)}{\cos(3+x)}$

3 Find the derivative of each function.

a $\cos(x) + \sin(x)$

b $\cos(x)\sin(x)$

c $\sin(x) - \cos(x)$

d $\dfrac{\sin(2x)}{\cos(3x)}$

e $\sin[\cos(x)]$

f $\cos[\sin(x)]$

g $\sqrt{\sin(x)} - \sqrt{\cos(x)}$

h $\sin^2(x)\cos^3(x)$

Example 8

4 Find the derivative of each function.

a $\sin[\ln(x)] - \ln[\sin(x)] + \sin(e^x) - e^{\sin(x)}$

b $5\sin(3x^2 + 4x - 3)(e^{4x-2} + 4)$

c $8e^{-2x}\cos(5x - 3)$

d $\sin(x)\ln(2x)\,e^{3x}\cos(4x)$

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Problem solving

5 Find the derivative of $7 \sin\left[\frac{(3q-6)^3}{(2q+3)^2}\right]$.

6 Find the derivative of $\frac{1-e^x \sin(x)}{1+e^x \cos(x)}$.

7 Find the derivative of $(3x^2 - 2x - 1)\, e^{\sin(2x+1)} + 5x^2 \ln[x \sin(x^2 + 1)]$.

8 Find the derivative of $7xe^{-0.4x^2}\{2 \sin[\ln(2x - 3)] + \cos[\ln(2x + 3)]\}$.

4.05 Modelling with trigonometric functions

In trigonometric models of periodic phenomena, you choose whether to use sine or cosine. You use the **amplitude**, **period** and any **phase shift** or **translation** to write the function.

Guidelines for trigonometric models

- Use sine if the value begins at the centre
- Use cosine if the value begins at the maximum or minimum
- Use the difference between the highest and lowest values as the amplitude
- The sine or cosine part is of the form $\sin\left[\frac{2\pi(t \pm \theta)}{T}\right]$ or $\cos\left[\frac{2\pi(t \pm \theta)}{T}\right]$, where t is the variable, T is the period and θ is the phase shift
- Check the given values when you have written the function

EXAMPLE 9

Peter watches a shadow moving backwards and forwards about a point 2 cm from a marker. It begins from that point and passes forwards through it every 10 seconds. It has a maximum displacement from the origin of 8 cm and continues with the same motion during the time it is watched.

a Write a possible function to describe its movement.

b Find its speed at 4 seconds.

Solution

a It is a period function starting from the centre. Use the sine function.

Write the period. Period = 10 s

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	Write the form of the function.	$x = A\sin\left(\frac{2\pi t}{10}\right) + a$, where x = position, t = time, a = central position and A = amplitude.
	Write the amplitude and central position.	$A = 6$ cm and $a = 2$ cm
	Write the possible function.	$x(t) = 6\sin\left(\frac{2\pi t}{10}\right) + 2$
	Simplify.	$x(t) = 6\sin\left(\frac{\pi t}{5}\right) + 2$
	Check known values.	$x(0) = 6\sin(0) + 2$ $= 2$ cm OK✓ $x(10) = 6\sin(2\pi) + 2$ $= 2$ cm OK✓
b	Differentiate to find speed.	$\frac{dx}{dt} = 6 \times \frac{\pi}{5}\cos\left(\frac{\pi t}{5}\right)$
	Substitute $t = 4$.	$\left.\frac{dx}{dt}\right\|_{t=4} = \frac{6\pi}{5} \times \cos\left(\frac{4\pi}{5}\right)$ cm/s ≈ -3.0 cm/s
	Write the answer.	The speed is $\frac{6\pi}{5} \times \cos\left(\frac{4\pi}{5}\right)$ cm/s, about 3 cm/s away from the original direction.

In real situations, periodic functions rarely have constant amplitudes.

Variable amplitude functions

- The decreasing amplitude of a periodic function often has the form Ae^{-kt}
- The increasing amplitude of a periodic function often has the form Ae^{kt}

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EXAMPLE 10

A point oscillates about a position 7 cm from the origin. It begins 19 cm from the origin and passes the 7 cm position every 3 seconds. After 18 seconds, the point only reaches 17 cm from the origin. Write a possible function to describe its movement.

Solution

It is a periodic function starting from a maximum.

Use the cosine function.

Write the period.

It is 6 seconds between 7 cm going one way to doing it again, so the period = 6 s.

Write the form of the function.

$x = A\cos\left(\frac{2\pi t}{6}\right) + a$, where x = position, t = time, a = central position = 7 and A = amplitude

A is decreasing.

$$A = A_0 e^{-kt}$$

$$\text{so } x = A_0 e^{-kt}\cos\left(\frac{2\pi t}{6}\right) + 7$$

Substitute $t = 0, x = 19$ to find A_0.

$19 = A_0 e^0 \cos(0) + 7$, so $A_0 = 12$

Substitute $t = 18, x = 17$ to find k.

$$17 = 12e^{-18k}\cos(6\pi) + 7$$

$$10 = 12e^{-18k}$$

$$e^{-18k} = \frac{10}{12}$$

$$-18k = \ln\left(\frac{10}{12}\right)$$

$$k = -\frac{\ln\left(\frac{10}{12}\right)}{18} \approx 0.0101$$

Write the possible function.

$$x(t) \approx 12e^{-0.0101t}\cos\left(\frac{2\pi t}{6}\right) + 7$$

Simplify.

$$= 12e^{-0.0101t}\cos\left(\frac{\pi t}{3}\right) + 7$$

Check known values.

$$x(0) \approx 12e^{-0.0101t}\cos(0) + 7$$

$$= 19 \text{ cm} \quad \text{OK}\checkmark$$

$$x(18) \approx 12e^{-0.0101 \times 18}\cos(6\pi) + 7$$

$$\approx 17 \text{ cm} \quad \text{OK}\checkmark$$

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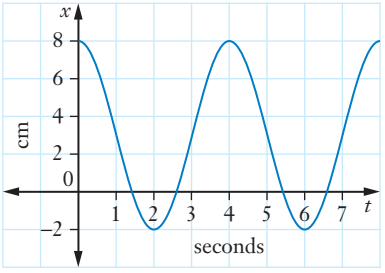
Exercise 4.05 Modelling with trigonometric functions

Example 9

1 An oscillating point begins 8 mm below the origin and rises to a maximum of 6 mm above the origin 5 seconds later. It continues to oscillate between the two extremes.
- **a** Write a possible function to describe its movement.
- **b** Find its speed 12 seconds after the motion begins.

2 This graph shows the distance of an object from the origin.

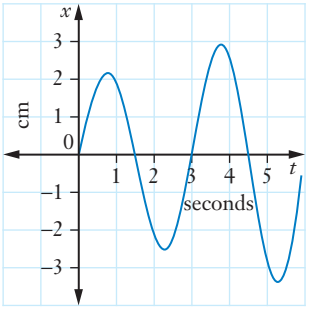


- **a** Write a possible function to describe its movement.
- **b** Find the speed 4 seconds from the start.

Example 10

3 An oscillating point begins from a high point 2 mm above the origin and doubles its amplitude every 20 seconds. It first returns to the high point after 5 seconds.
- **a** Find a possible function to describe its motion.
- **b** Find its speed 22 seconds after it begins moving.

4 This graph shows the distance of an object from the origin.



- **a** Find a function to describe its motion.
- **b** Find its speed 10 seconds after it begins moving.

Problem solving

5 An old clock pendulum bob swings 8 cm from the centre to each side, taking 1 second to go to one side and return to the centre. Write a possible function to describe the sideways motion of the bob and find its speed when the pendulum is vertical.

6 The end of a ruler is 'twanged' on the edge of a desk and starts 1.4 cm from the centre, taking 8 milliseconds to return to the same side. The period stays the same, but the amplitude of the vibration halves every 24 milliseconds. Write a possible function to describe the motion and find the speed at which the end of the ruler passes though the central position on its 40th pass through.

7 One of the speakers of an amplifier is placed too near a microphone. It begins to produce feedback of frequency 200 Hz. The loudness starts at 15 decibels and doubles every 5 seconds (so the loudness increases by 10 ln (2)). The frequency increases by 20% each 10 seconds. Write a possible function for the sound produced at time t and find the frequency and loudness after 1 minute.

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8 The centre of the string of a musical instrument is plucked so that is 4.4 mm away from the rest position and then let go. It vibrates at a frequency of 210 Hz. Write a possible function for the position of the centre of the string in the first 3 seconds of movement, before the sound begins to fade. How fast is the string moving at the extremes of its motion after 2 seconds?

4.06 Applications of trigonometric functions

WS Trigonometric functions extension

WS Trigonometric functions assignment

The derivative of a function gives the slope of the tangent. This is also the slope of the curve. You can use this to find the angle and the equation of the tangent.

EXAMPLE 11

a Find the angle between the curve $y = 4\sin\left(x + \frac{\pi}{3}\right) + 1$ and $y = 3$ at its second positive point of intersection with the line.

b Find the equation of the tangent at that point.

Solution

a Write the equation for the intersection.

$$4\sin\left(x + \frac{\pi}{3}\right) + 1 = 3$$

Solve for positive values of x.

$$\sin\left(x + \frac{\pi}{3}\right) = \frac{1}{2}$$

Think of the sine curve to obtain possible values of $x + \frac{\pi}{3}$.

$$x + \frac{\pi}{3} = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \ldots$$

Write the solutions for x.

$$x = -\frac{\pi}{6}, \frac{\pi}{2}, \frac{11\pi}{6}, \frac{5\pi}{2}, \ldots$$

Write the second *positive* point of intersection.

$$\text{The point is } \left(\frac{11\pi}{6}, 3\right)$$

Find the derivative.

$$\frac{dy}{dx} = 4\cos\left(x + \frac{\pi}{3}\right)$$

Find the value at $x = \frac{11\pi}{6}$.

$$\text{Slope at } \frac{11\pi}{6} = 4\cos\left(\frac{11\pi}{6} + \frac{\pi}{3}\right)$$
$$= 4\cos\left(\frac{13\pi}{6}\right)$$
$$= 4\cos\left(\frac{\pi}{6}\right)$$
$$= 4\frac{\sqrt{3}}{2}$$
$$= 2\sqrt{3}$$

Find the angle with the x-axis.

$$m = \tan(\theta)$$

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Write the answer. The angle is $\tan^{-1}(2\sqrt{3})$ (about 1.29 radians or 73.9°) with the positive direction of the line.

b Find the equation of the tangent.

$$y - y_1 = m(x - x_1)$$
$$y - 3 = 2\sqrt{3}\left(x - \frac{11\pi}{6}\right)$$
$$= 2\sqrt{3}x - \frac{11\pi\sqrt{3}}{3}$$

Write the answer. The equation of the tangent is $2\sqrt{3}x - y - \frac{11\pi\sqrt{3}}{3} + 3 = 0$.

Give an exact answer unless otherwise directed.

All vibrations and waves are combinations of simple movements about a central point. You can model these with sine or cosine functions. The simple types are called **simple harmonic motion**. This includes an oscillating pendulum, a vibrating spring, sound waves, light waves and the electrons in a wire carrying alternating current.

Simple harmonic motion (SHM)

- Simple harmonic motion is given by $x = a \sin(\omega t)$ or $x = a \cos(\omega t)$
- ω is called the **angular speed**, $\omega = \frac{2\pi}{T}$ for period T

ω is the lower-case Greek letter omega, not a w.

EXAMPLE 12

Applications of trigonometric functions

A particle moves in a straight line with equation $x = 2 \sin(4t)$ cm, where t is in seconds.

a Find the period of the motion.

b Find an expression for the velocity of the particle.

c When does the particle come to rest for a second time after $t = 0$?

Solution

a Use the period of the sine function. The period is given by $4t = 2\pi$

Solve. $t = \frac{\pi}{2}$

Write the answer. Period $t = \frac{\pi}{2}$ s

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b Find the derivative.

$$\frac{dx}{dt} = \frac{d}{dt}[2 \sin (4t)]$$
$$= 8 \cos (4t)$$

Write the answer.

The velocity is given by $v = 8 \cos (4t)$ cm/s.

c Find when the velocity is 0.

$$8 \cos (4t) = 0$$
$$\cos (4t) = 0$$

Find values after $t = 0$.

$$4t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots$$
$$t = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \ldots$$

Write the answer.

The second time after $t = 0$ the particle comes to rest is at $t = \frac{3\pi}{8}$ seconds.

INVESTIGATION

TIDES

The sea tides are caused by the combined effects of the gravitational attraction of the moon and Sun and the shapes of the ocean basins in the oceans. These produce up and down movements of the water in any particular place. The amplitudes and periods of the motions produced vary from place to place.

- Find the period and amplitude of the main effect of the moon
- Find the period and amplitude of the main effect of the Sun
- What are some periods and amplitudes of other effects?
- Combine the simple functions for the moon and Sun to show simple tidal variation
- What do you notice about the simple case?
- Can you explain spring tides (king tides) and neap tides?

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In some cases, you need to apply the chain rule to use related rates of change. When you are solving problems, write down as much as you can to try to see how to tackle the problem.

Problem solving

- Read the problem carefully—underline or write down important pieces of information.
- Make a sketch if you can. Write variables and information on the sketch.
- Make a list of facts and relationships
- Work out the values and variables needed to find solutions
- Isolate the variables you need
- Solve the mathematical problem, showing your steps
- Express the solution in English in terms of the problem

EXAMPLE 13

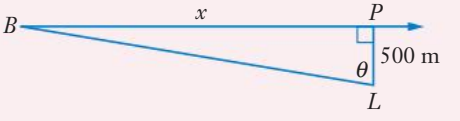
The MV *Sweetpea* is travelling in a straight line at constant speed so that it will pass a lighthouse at P, a distance of 500 m away from it.

When the angle at the lighthouse between the boat's current position and P is 83°, the angle is decreasing at 0.8°/minute.

Given that 1 nautical mile is 1852 m, find the boat's speed in knots (nautical miles/hour).

Solution

Draw a diagram and choose variables.



Let the distance from $P = x$
Let the angle at the lighthouse $= \theta$
Let the speed of the boat $= v$

Write the information given.

When $\theta = 83°$, $\dfrac{d\theta}{dt} = -0.8$ °/min

Change to required units.

When $\theta = \dfrac{83\pi}{180}$, $\dfrac{d\theta}{dt} = -\dfrac{0.8\pi \times 60}{180} = \dfrac{0.8\pi}{3}$/h

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Write any relationships.	$\tan(\theta) = \frac{x}{500}$
Isolate the variable needed for speed.	$x = 500 \tan(\theta)$ m
Change to required units.	$= \frac{500}{1852} \tan(\theta)$ nautical miles
Write the speed in terms of x.	$v = \frac{dx}{dt}$
Use the chain rule.	$= \frac{dx}{d\theta} \times \frac{d\theta}{dt}$
Substitute, ignoring sign for speed.	$= \frac{d}{d\theta}\left[\frac{500}{1852}\tan(\theta)\right] \times \frac{0.8\pi}{3}$ n.m./h
Simplify.	$= \frac{500}{1852} \times \frac{0.8\pi}{3}[1 + \tan^2(\theta)]$ knots
Evaluate with $\theta = \frac{83\pi}{180}$.	$= 15.228\ldots$ knots
Round and write the answer.	The boat is travelling at about 15 knots.

Exercise 4.06 Applications of trigonometric functions

Example 11

1 Find the equation of the tangent to the curve $y = 3\cos(x)$ at the point where $x = \frac{2\pi}{3}$.

2 Find the equation of the normal to the curve $f(x) = \sin(2x)$ at the point $\left(\frac{\pi}{2}, 0\right)$.

Example 12

3 The forward and backward movement of an air molecule in still air during the transmission of a middle C note is given by $x = a\sin(227.2t)$, where t is the time in seconds and a is the maximum displacement of the molecule. a is related to the loudness of the sound.

- **a** How long does it take for one complete cycle of the movement?
- **b** Find an expression for the speed of movement of the molecule.
- **c** How many times does the molecule have a velocity of zero in 1 second?

Example 13

4 A surveyor with a theodolite at the top of a cliff 120 m above the water observes that a boat moving directly away from him is at an angle of depression of 35° and that the angle of depression is decreasing at 20°/minute. Find the speed of the boat in km/h.

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Problem solving

5 A railway 'bumper' spring is attached to a fixed barrier at the end of a shunting track. The spring is compressed a distance of 1.5 cm when a carriage runs into the bumper and first returns to its starting position 0.2 seconds after the carriage hits the bumper.

Write an equation for the movement of the bumper and hence find the maximum speed with which the bumper moves outwards.

6 The centre of the light cast by a light suspended by a chain 3 m from a cathedral roof moves backwards and forwards across the floor with a speed of 2 sin $(5.1t)$ m/s. The light is 2 m above the floor. Find an expression for the movement of the light, which is almost horizontal. Hence find the greatest angle of the chain from the vertical.

7 From a point on the Storey Bridge 80 m above the roadway, a night-time observer can see the lights of a car moving across the bridge from Kangaroo Point. If the car is moving at 54 km/h, find the rate of increase of the angle of depression of the lights when the car is 300 m away from a point directly below the observer.

8 When a stone is dropped into a still pond, ripples move outwards from the point where the stone was dropped. At a point X, a distance of 1.8 m from the point where the stone was dropped, the water apparently moves up and down so that the difference in height between its highest and lowest points is 0.6 cm. Three ripples pass point X each second.

a Write a sinusoidal (sine or cosine) expression for the up-and-down movement of the surface at point X.

b Find an expression for the velocity with which the surface moves up and down at point X.

9 The curves $y = \sin(2x)$ and $y = \cos(2x)$ intersect at $x = \frac{\pi}{8}$.

a Find the angle between the curves at this point.

b Extend your solution to find the angle between the curves $y = \sin(5x)$ and $y = \cos(5x)$.

10 The tidal variation in the water level in a particular bay can be modelled (in metres) as $d = d_0 - 0.9\cos(0.503t)$, where d is the depth of water at a particular point, t is the time in hours after low tide and d_0 is the depth of water at the same point at half-tide. Two hours after low tide, the water reaches a mudflat sloped upwards at an angle of 2°. At what speed does it then move horizontally across the mudflat towards the shoreline?

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4. CHAPTER SUMMARY

Derivatives of trigonometric functions

The derivatives of trigonometric functions assume the variable is in radians.

- $\frac{d}{dx}[\sin(x)] = \cos(x)$
- $\frac{d}{dx}[\sin(kx)] = k\cos(kx)$
- $\frac{d}{dx}[\sin(x°)] = \frac{\pi}{180}\cos(x°)$
- $\frac{d}{dx}(\sin[f(x)]) = f'(x)\cos[f(x)]$
- $\frac{d}{dx}[\cos(x)] = -\sin(x)$
- $\frac{d}{dx}[\cos(kx)] = 1 - k\sin(kx)$
- $\frac{d}{dx}[\cos(x°)] = -\frac{\pi}{180}\sin(x°)$
- $\frac{d}{dx}(\cos[f(x)]) = -f'(x)\sin[f(x)]$
- $\frac{d}{dx}[\tan(x)] = 1 + \tan^2(x)$ or $\frac{d}{dx}[\tan(x)] = \frac{1}{\cos^2(x)}$
- $\frac{d}{dx}[\tan(kx)] = k[1 + \tan^2(kx)]$
- $\frac{d}{dx}[\tan(x°)] = \frac{\pi}{180}[1 + \tan^2(x°)]$
- $\frac{d}{dx}(\tan[f(x)]) = f'(x)(1 + \tan^2[f(x)])$
- For **trigonometric models**
  - Use sine if the value begins at the centre
  - Use cosine if the value begins at the maximum or minimum
  - Use the difference between the highest and lowest values as the amplitude
  - The sine or cosine part is of the form $\sin\left[\frac{2\pi(t \pm \theta)}{T}\right]$ or $\cos\left[\frac{2\pi(t \pm \theta)}{T}\right]$, where t is the variable, T is the period and θ is the phase shift
  - The decreasing amplitude of a periodic function often has the form $\pm Ae^{-kt}$
  - The increasing amplitude of a periodic function often has the form Ae^{kt}
  - Check the given values from the function

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- **Simple harmonic motion** is given by $x = a \sin(\omega t)$ or $x = a \cos(\omega t)$
- ω is called the **angular speed**, $\omega = \dfrac{2\pi}{T}$ for the period T
- **Problem solving guidelines**
  - Read the problem carefully—underline or write down important pieces of information.
  - Make a sketch if you can. Write variables and information on the sketch
  - Make a list of facts and relationships
  - Work out the values and variables needed to find solutions
  - Isolate the variables you need
  - Solve the mathematical problem, showing your steps
  - Express the solution in English in terms of the problem

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4. CHAPTER REVIEW

Derivatives of trigonometric functions

Example 1

1 Find the derivative of:

a $5 \sin(x)$ **b** $\sin(2x)$

Example 2

2 Find the derivative of:

a $4 \sin(x^2 - 3x + 4)$ **b** $2 \sin(3m + 7)^4$

Example 3

3 Find the derivative of:

a $4 \cos(5t)$ **b** $\cos(3\theta°)$

Example 4

4 Find the derivative of:

a $-3 \cos(3x^5 - 4x^2 + 5)$ **b** $\cos^3(3 - 2q)^5$

Example 5

5 Find the derivative of:

a $2 \tan(3t)$ **b** $-2 \tan(5a - 3)$

Example 6

6 Find the derivative of:

a $4 \tan[(5 - 2x)^3]$ **b** $3 \tan^3[(5t - 4)^2]$

Example 7

7 Find the derivative of:

a $y = (3x - 5)^2 \cos(7 - 2x)$ **b** $f(t) = \dfrac{3t^2 - 6t}{4\sin(t-1)^2}$

Example 8

8 Find the derivative of:

a $4e^{-2t} \sin(5t - 2)$ **b** $2 \ln[\cos^3(5x - 3)^4]$

Example 9

9 An object starts from a point 2 mm from the origin and moves back and forth past this point. The minimum and maximum displacements are 8 mm and –4 mm. It takes 3 seconds to travel from the minimum to the maximum and the motion continues indefinitely.

a Write a possible function to describe the motion.

b What is the displacement at $t = 7$ seconds?

Example 10

10 A particle oscillates about the origin, starting from a point 9 cm from the origin and returning to its maximum after 8 seconds. After 24 seconds the maximum has increased to 10 cm. Write a possible function to describe its movement.

Example 11

11 Find the equation for the tangent to $y = 3 \cos\left(2x - \dfrac{\pi}{6}\right)$ at $x = \dfrac{3\pi}{2}$.

Example 12

12 A particle moves in a straight line with equation $x = 3 \sin(5t)$ mm, where t is in seconds.

a What is the period of the motion?

b What is its velocity at $t = 0.4$ seconds, correct to 3 figures?

c When does the particle first come to rest?

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Problem solving

13 Find the derivative of $\dfrac{3\sin^2(3x-2)^4}{5\cos^3(x^3-x^2+5)}$.

14 What is the derivative of $m(t) = \tan^4[\ln(1-3t)^2]^3$?

15 Find the derivative of $\dfrac{\sin(1+e^{-2x})\cos(1-e^x)}{\tan(1+e^{-x})}$.

16 A feedback loop produces an increasing sinusoidal AC current. The current has a frequency of 50 Hz with an initial maximum of 0.02 milliamps. After 1 second it has increased to a maximum of 0.03 mA. Write a possible function to describe the current and find when the maximum exceeds 1 amp, correct to the nearest second.

17 A monitoring station in the Coral Sea is on a submerged mountain where the mean depth is 22 m. During a cyclone, waves over the mountain have an amplitude of 15 m and a period of 2 minutes. Find expressions for the height of the water and the speed of movement of the surface up and down.

18 A revolving searchlight is 80 m from a straight prison wall. At a point 120 m from the point on the wall directly opposite the searchlight, the light appears to move along the wall at 13.6 m/s. How long does the searchlight take for one revolution?

Practice quiz

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5.

ANTI-DIFFERENTIATION

Calculus is the branch of mathematics concerned with calculations involving varying quantities.

Calculus is made up of two related parts. The first part, called differential calculus, came about through the determination of the tangent to a curve. Differential calculus deals with problems involving the behaviour and rate of change of different quantities.

The second part, called integral calculus, involves the reverse of the process of differentiation, called anti-differentiation. Integral calculus deals with problems involving area, volume, centre of gravity, moment of inertia, work done by a force and many others.

5.01 Using the product, quotient and chain rules
5.02 The integral of x^n
5.03 The integrals of sin (x) and cos (x)
5.04 The integral of e^x
5.05 The integrals of $\frac{1}{x}$ and $\frac{1}{ax+b}$
5.06 Integration by recognition
5.07 Velocity and displacement problems
Chapter summary
Chapter review

SYLLABUS SUBJECT MATTER

Differentiation rules

- select and apply the product rule, quotient rule and chain rule to differentiate functions; express derivatives in simplest and factorised form

Anti-differentiation

- recognise anti-differentiation as the reverse of differentiation
- use the notation $\int f(x)\,dx$ for anti-derivatives or indefinite integrals
- establish and use the formula $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c$ for $n \neq -1$
- establish and use the formula $\int e^x\,dx = e^x + c$
- establish and use the formulas $\int \frac{1}{x}\,dx = \ln(x) + c$ for $x > 0$ and $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln(ax+b) + c$
- establish and use the formulas $\int \sin(x)\,dx = -\cos(x) + c$ and $\int \cos(x)\,dx = \sin(x) + c$
- understand and use the formula for indefinite integrals of the form $\int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$
- determine indefinite integrals of the form $\int f(ax+b)\,dx$
- determine $f(x)$, given $f'(x)$ and an initial condition $f(a) = b$
- determine the integral of a function using information about the derivative of the given function (integration by recognition)
- determine displacement given velocity in linear motion problems

Prior learning

Mathematical Methods 2019 v1.1 General Senior Syllabus © Queensland Curriculum and Assessment Authority (QCAA). This syllabus forms part of a new senior assessment and tertiary entrance system in Queensland. Along with other senior Syllabuses, it is still being refined in preparation for implementation in schools from 2019. For the most current syllabus versions and curriculum information please refer to the QCAA website https://www.qcaa.qld.edu.au/.

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TERMINOLOGY

anti-derivative
anti-differentiation
arbitrary constant
chain rule
composite function
constant of integration
derivative
differentiation
gradient function
indefinite integral
integration
integration by recognition
primitive function
product rule
quotient rule

5.01 Using the product, quotient and chain rules

WS Basic differentiation

WS The product rule

WS The quotient rule

WS The chain rule

You have previously seen that the **derivative** of the function $y = f(x)$ is given by:

$$f'(x) = \frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

and you used this definition to find derivatives using first principles.

You also learnt a number of rules (theorems) that can be used to **differentiate** functions.

Basic rules for finding derivatives

$$\frac{d}{dx}(k) = 0, \text{ where } k \text{ is a constant}$$

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(kx^n) = knx^{n-1}$$

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$$

$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$$

These rules can also be written using an alternative notation.

Derivative rules (alternative notation)

If $u = f(x)$ and $v = g(x)$ and k is a constant:

$$y = k \Rightarrow \frac{dy}{dx} = 0$$

$$y = kx^n \Rightarrow \frac{dy}{dx} = knx^{n-1}$$

$$y = ku \Rightarrow \frac{dy}{dx} = k\frac{du}{dx} \text{ or } y' = ku'$$

$$y = u + v \Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \text{ or } y' = u' + v'$$

$$y = u - v \Rightarrow \frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx} \text{ or } y' = u' - v'$$

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When finding the derivatives of terms involving surds or negative indices, you need to recall the index laws.

Negative and fractional powers

$$x^{-n} = \frac{1}{x^n} \qquad x^{\frac{1}{n}} = \sqrt[n]{x} \qquad x^{\frac{m}{n}} = \sqrt[n]{x^n} = \left(\sqrt[n]{x}\right)^n$$

You know how to differentiate the product of 2 functions.

Product rule

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

or $\quad (uv)' = uv' + vu'$

EXAMPLE 1

a Find the derivative of $y = 3x^3(2x - 5)$.

b If $f(x) = (2x^2 - 7x)(4x^3 + x^2 - 3x)$, calculate $f'(-2)$.

c Find $\frac{d}{dx}\left(2x^{\frac{3}{4}}\right)$.

d Calculate $f'(-27)$ if $f(x) = \sqrt[3]{x^2}$.

Solution

a

Write the function as a product.	$y = 3x^3(2x - 5) = uv$, where $u = 3x^3$ and $v = 2x - 5$
Calculate u' and v'.	$u' = 9x^2$ and $v' = 2$
Use the product rule.	$\frac{dy}{dx} = uv' + vu'$
Substitute for known values.	$= 3x^3 \times 2 + (2x - 5) \times 9x^2$
Simplify.	$= 6x^3 + 18x^3 - 45x^2$
	$= 24x^3 - 45x^2$
State the result.	$\frac{d}{dx}[3x^3(2x - 5)] = 24x^3 - 45x^2$
OR Expand the expression for y, then differentiate.	$y = 3x^3(2x - 5) = 6x^4 - 15x^3$
	$\frac{dy}{dx} = 24x^3 - 45x^2$

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b Write the function as a product. $y = (2x^2 - 7x)(4x^3 + x^2 - 3x) = uv$, where $u = 2x^2 - 7x$ and $v = 4x^3 + x^2 - 3x$

Calculate u' and v'. $u' = 4x - 7$ and $v' = 12x^2 + 2x - 3$

Use the product rule. $\frac{dy}{dx} = uv' + vu'$

Substitute for known values. $= (2x^2 - 7x)(12x^2 + 2x - 3) + (4x^3 + x^2 - 3x)(4x - 7)$

Substitute for $x = -2$. $= [2(-2)^2 - 7(-2)][12(-2)^2 + 2(-2) - 3] + [4(-2)^3 + (-2)^2 - 3(-2)][4(-2) - 7]$

Evaluate. $= (8 + 14)(48 - 4 - 3) + (-32 + 4 + 6)(-8 - 7)$

$= 1232$

You could expand and simplify the algebraic expression and then substitute for $x = -2$, but this may take longer to complete.

State the result. For $f(x) = (2x^2 - 7x)(4x^3 + x^2 - 3x)$, $f'(-2) = 1232$

c Differentiate. $\frac{d}{dx}\left(2x^{\frac{3}{4}}\right) = \frac{3}{4} \times 2x^{\frac{3}{4}-1}$

Simplify. $= \frac{3}{2}x^{-\frac{1}{4}}$

d Write in index form. $f(x) = \sqrt[3]{x^2} = x^{\frac{2}{3}}$

Find $f'(x)$. $f'(x) = \frac{2}{3}x^{\frac{2}{3}-1}$

Simplify. $= \frac{2}{3}x^{-\frac{1}{3}}$

Find $f'(-27)$. $= \frac{2}{3}(-27)^{-\frac{1}{3}}$

$= \frac{2}{3\sqrt[3]{-27}}$

$= \frac{2}{3(-3)}$

$= -\frac{2}{9}$

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You know how to differentiate the quotient of 2 functions.

Quotient rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)f'(x)-f(x)g'(x)}{\left[g(x)\right]^2}$$

or $$\left(\frac{u}{v}\right)'=\frac{vu'-uv'}{v^2}$$

EXAMPLE 2

Find the derivative of $\frac{3x-4}{2x+5}$.

Solution

Write the function as a quotient.

$y=\frac{3x-4}{2x+5}=\frac{u}{v}$, where

$u=3x-4$ and $v=2x+5$

Calculate u' and v'.

$u'=3$ and $v'=2$

Use the quotient rule.

$$\frac{dy}{dx}=\frac{vu'-uv'}{v^2}$$

Substitute for known values.

$$=\frac{(2x+5)\times 3-(3x-4)\times 2}{(2x+5)^2}$$

Simplify.

$$=\frac{6x+15-6x+8}{(2x+5)^2}$$

State the result.

$$\frac{dy}{dx}=\frac{23}{(2x+5)^2}$$

The **chain rule** is used to differentiate **composite functions** of the form $f[g(x)]$.

Chain rule

$$\frac{d}{dx}\left(f[g(x)]\right)=f'[g(x)]\,g'(x)$$

If $y=f(u)$ and $u=g(x)$, then:

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$$

WS Mixed differentiation problems

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EXAMPLE 3

Find:

a $\frac{d}{dx}[12(3x+4)^3(5x-3)^4]$

b $\frac{d}{dx}\left(\sqrt{3x^2-5x+1}\right)$

c $\frac{d}{dx}\left[2x\cos\left(\sqrt[3]{x}\right)\right]$

d $f'(0)$ if $f(x)=\sin^2(e^x)$

Solution

a Use the product rule.

Let $u=12(3x+4)^3$ and $v=(5x-3)^4$.

$y=u\times v$

Find u' using the chain rule.

$u'=3\times 12(3x+4)^2\times 3$

$=108(3x+4)^2$

Find v' using the chain rule.

$v'=4(5x-3)^3\times 5$

$=20(5x-3)^3$

Now use the product rule.

$y'=uv'+vu'$

Substitute for u', v', etc.

$=12(3x+4)^3\times 20(5x-3)^3$

$+(5x-3)^4\times 108(3x+4)^2$

Simplify.

$=240(3x+4)^3(5x-3)^3$

$+108(5x-3)^4(3x+4)^2$

Factorise.

$=12(3x+4)^2(5x-3)^3$

$\times[20(3x+4)+9(5x-3)]$

Expand brackets.

$=12(3x+4)^2(5x-3)^3$

$\times[60x+80+45x-27]$

Simplify and state the result.

$f'(x)=12(3x+4)^2(5x-3)^3(105x+53)$

b Use the chain rule.

Let $u=3x^2-5x+1$.

$y=u^{\frac{1}{2}}$

Find $y'(u)$.

$y'(u)=\frac{1}{2}u^{\frac{1}{2}-1}=\frac{1}{2}u^{-\frac{1}{2}}$

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Find $u'(x)$. $u'(x) = 6x - 5$

Use the chain rule. $y' = y'(u) \times u'(x)$

Substitute for $y'(u)$ and $u'(x)$. $= \frac{1}{2}u^{-\frac{1}{2}} \times (6x - 5)$

Substitute for u. $= \frac{1}{2}(3x^2 - 5x + 1)^{-\frac{1}{2}} \times (6x - 5)$

Simplify. $= \frac{6x - 5}{2\sqrt{3x^2 - 5x + 1}}$

c Use the product rule. Let $u = 2x$ and $v = \cos\left(\sqrt[3]{x}\right)$.

$y = u \times v$

Find u'. $u'(x) = 2$

Let $w = \sqrt[3]{x}$. $v = \cos\left(\sqrt[3]{x}\right) = \cos(w)$

Find $v'(w)$. $v'(w) = -\sin(w)$

Find $w'(x)$. $w(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$

$w'(x) = \frac{1}{3}x^{\frac{1}{3}-1}$

$= \frac{1}{3}x^{-\frac{2}{3}}$

Find v' using the chain rule. $v' = v'(w) \times w'(x)$

Substitute for known values. $= -\sin(w) \times \frac{1}{3}x^{-\frac{2}{3}}$

Substitute for w and simplify. $= -\frac{1}{3}x^{-\frac{2}{3}}\sin\left(x^{\frac{1}{3}}\right)$

Now use the product rule. $y' = uv' + vu'$

Substitute for known values. $= 2x \times -\frac{1}{3}x^{-\frac{2}{3}}\sin\left(x^{\frac{1}{3}}\right) + \cos\left(x^{\frac{1}{3}}\right) \times 2$

Simplify. $= -\frac{2}{3}\sqrt[3]{x}\sin\left(\sqrt[3]{x}\right) + 2\cos\left(\sqrt[3]{x}\right)$

d Use the chain rule twice.

$y = \sin^2(e^x) = [\sin(e^x)]^2$

Let $u = \sin(e^x)$

$y = u^2$

Let $v = e^x$

$u = \sin(v)$

Use the chain rule to find u'.

$u' = u'(v) \times v'(x)$

$= \cos(v) \times e^x$

Substitute for v.

$= e^x \cos(e^x)$

Find y'.

$y' = y'(u) \times u'(x)$

Substitute for known values.

$= 2u \times e^x \cos(e^x)$

Substitute for u.

$= 2 \sin(e^x) \times e^x \cos(e^x)$

Simplify.

$= 2e^x \sin(e^x) \cos(e^x)$

Find $f'(0)$.

$f'(0) = -2e^0 \sin(e^0) \cos(e^0)$

Evaluate.

$= -2(1) \sin(1) \cos(1)$

$= -0.9092\ 97\ldots$

State the result.

For $f(x) = \sin^2(e^x)$, $f'(0) \approx -0.909$

Exercise 5.01 Using the product, quotient and chain rules

Example 1

1 Calculate the derivative of each function.

a $(z^5 - z^3 + z - 1)(9 + 3z^2 - z^4)$

b $(3p + 4)(p^2 - 2p - 4)$

c $(5v^3 - 3v^2 + v - 9)(7v^4 - 2v^3 - v^2 + 8v + 6)$

d $(3t^2 - 2t + 5)(t^3 + 4t^2 + t - 6)(5t^4 + t^3 - 7t + 8)$

e $(4b^5 + 3b^2 - 6b + 2)(b^4 + 3b^2 - 6b^3 + 21)$

f $(3u - 8u^2 - 6u^3 + 8)(7u^3 - 4u^2 + 9u - 6)$

2 Find the derivative of each function with respect to the given variable.

a $5x^{-1}$ **b** $\frac{8}{x}$ **c** $-3x^{-4}$ **d** $\frac{3}{t^2}$

e $7m^{-10}$ **f** $\frac{-3}{u^7}$ **g** $\frac{1}{y^{20}}$ **h** $\frac{4}{v^{15}}$

i $16w^{-9}$ **j** $\frac{-8}{z^9}$

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3 Find the derivative of each function with respect to the given variable.

a $\sqrt{x}$ **b** $\sqrt[3]{m}$ **c** $7x^{\frac{1}{2}}$ **d** $5t^{1\frac{1}{3}}$

e $7p^{-\frac{2}{3}}$ **f** $\dfrac{1}{\sqrt{x}}$ **g** $-9z\sqrt{z^5}$ **h** $\dfrac{-6u}{\sqrt[4]{u^3}}$

i $4w^2\sqrt[6]{w}$ **j** $\dfrac{5m^2}{\sqrt{m}}$

Example 2

4 Find the derivative of each function with respect to the given variable.

a $\dfrac{5q+9}{5-5q}$ **b** $\dfrac{8m-7}{2m-5}$ **c** $\dfrac{3x+10}{2x-5}$

d $\dfrac{t+5}{t-5}$ **e** $\dfrac{3(2t-5)}{4(5t+8)}$ **f** $\dfrac{12-5u}{5(2u-9)}$

5 Find the derivative of each function with respect to the given variable.

a $x^3(1+\sqrt{x})$ **b** $\left(\dfrac{2}{m}+m^5\right)(m^3+1)$ **c** $\dfrac{1}{u^3+2u+1}$

d $\dfrac{x^2+1}{x^3-5}$ **e** $\dfrac{\sqrt{u}}{u^3+1}$

Example 3

6 Differentiate each function.

a $3(5a-6)^3$ **b** $-2(5x+4)^2$

c $7(y-2)^7$ **d** $-3(5q+8)^4$

e $7(7-5a)^3$ **f** $4(5g^3+3)^6$

g $6(4x^2-6x+4)^2$ **h** $-8(m^3-2m^2+4)^8$

i $-(9-3v)^6$ **j** $4(2g^2-g^3+3)^6$

7 Find the derivative of each function.

a $(5x^4-8x^3+7)^5$ **b** $(7v^3+2v^2+v-4)^7$

c $\sqrt{3t^2+t-5}$ **d** $\sqrt{6h^2-3h+1}$

e $\sqrt[3]{4v+5}$ **f** $\sqrt[4]{y^2-2y+3}$

8 Find the derivative of each function.

a $-6(v+7)^5(v-6)^4$ **b** $-3(6-5r)^3(2r+9)^{11}$

c $-5(8-6h)^7(-4-3h)^6$ **d** $(5t+8)(2t-3)$

e $-(3p+10)^7(8p+1)^8$ **f** $-(3-z)(4z+15)^6$

9 Differentiate:

a $x^3\sin(3x)$ **b** $e^x(x^2-1)$ **c** $2x^2e^{3x}$

d $2x^3\cos(2x)$ **e** $e^{2x}\sqrt{3x^2-1}$ **f** $x^2\cos^3(2x)$

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10 Differentiate:

a $\dfrac{\sin(3x)}{x^2}$ **b** $\dfrac{\cos(2x)}{3e^x - x}$ **c** $\dfrac{e^{-x}}{x-2}$

d $\dfrac{e^x}{\sin(3x+1)}$ **e** $\dfrac{\cos(2x)}{\sin(2x)}$ **f** $\dfrac{e^x+1}{\sin(2x)}$

11 Differentiate each function.

a $(3x^5 - 3x^2 + 8)(2x^2 + 5x + 2) - (2x^4 + 5x^3)(x^5 - 6)$

b $\dfrac{5c^7}{5(9-4c)^2}$

c $2(3m + 10)^7 - 5(3m + 10)^4 + 2(10 + 3m)^{-2}$

d $\dfrac{3}{2f-5} + \dfrac{5}{6f+10} - \dfrac{4}{5f-11}$

e $\dfrac{1}{(t-6)^3} - \dfrac{4}{(2t+7)^2} + \dfrac{2(3t-5)}{7(2t+9)^4}$

f $2(3w + 5)^4(3w - 1)^5 - (w - 6)^8(1 - 3w)^5$

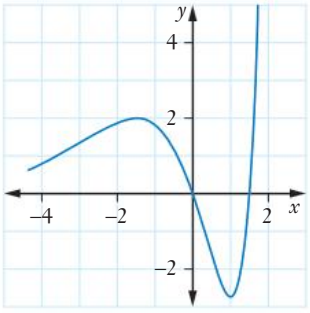
g $\dfrac{4p^5 - 5p^4 + 2p^3 - p^2 - 4p + 2}{(5p-1)^6}$

Problem solving

12 This diagram shows the graph of

$y = (2x^2 - 3x)e^x$.

Find the values of x where the gradient of the tangent to the graph is zero.

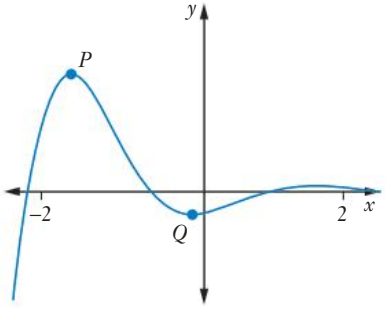


13 This diagram shows the graph of

$f(x) = \dfrac{\cos(2x+3)}{2e^x}$.

A local maximum occurs at P and a local minimum occurs at Q.

Find the coordinates of P and Q correct to 3 decimal places.



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5.02 The integral of x^n

WS Antiderivatives
WS Integrating functions
WS Finding indefinite integrals 1

You are familiar with the process of differentiation used to calculate the derivative of a function. The derivative is a new function known as the **gradient function**.

$$\text{If } y = f(x), \text{ then } f'(x) = \frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

The reverse of differentiation is known as **anti-differentiation** or **integration**.

Anti-differentiation is the process that undoes differentiation to find the original or **primitive function** or **anti-derivative**.

INVESTIGATION

DOING AND UNDOING DIFFERENTIATION

If we know the derivative, $f'(x)$, we can undo the process of differentiation or work backwards from it to find a possible primitive function.

1 Copy the tables below.

$f(x)$	$f'(x)$
$5x$	
	3
x^2	
$3x^2$	
	$4x$
	$\frac{1}{2}x$
x^3	

$f(x)$	$f'(x)$
$5x^3$	
$\frac{1}{4}x^3$	
	$6x^2$
	$\frac{3}{2}x^2$
$3x^4$	
$\frac{1}{2}x^4$	
	$16x^3$

$f(x)$	$f'(x)$
	x^3
$5x^2 - 4x$	
$5x^4 + 7x^2$	
	$6x + 11$
	$9x^2 - 8x + 1$

2 Complete the tables in order.

3 As you go down the list, look for patterns that emerge.

4 Describe in words how the primitive function can be calculated, using the derivative.

How did you describe the process in the investigation?

One way would be 'Add 1 to the power of x and divide by the new power'.

Anti-derivative of ax^n

If $f'(x) = ax^n$, then $f(x) = \frac{ax^{n+1}}{n+1}$ (where $n \neq -1$) is a *possible* primitive function.

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EXAMPLE 4

Find an anti-derivative of:

a $4x^7$ **b** $3x^2 - 10x + 4$

Solution

a Write the derivative. $\frac{dy}{dx} = 4x^7$

Use the rule to find an anti-derivative. $y = \frac{4x^{7+1}}{7+1}$

Simplify. $= \frac{4x^8}{8}$

Simplify and state the result. A possible anti-derivative of $4x^7$ is $\frac{1}{2}x^8$.

b Write the derivative. $\frac{dy}{dx} = 3x^2 - 10x + 4$

Use the reverse of differentiation for each term. $y = \frac{3x^{2+1}}{2+1} - \frac{10x^{1+1}}{1+1} + \frac{4x^{0+1}}{0+1}$

Simplify. $y = \frac{3x^3}{3} - \frac{10x^2}{2} + \frac{4x^1}{1}$
$= x^3 - 5x^2 + 4x$

State the result. A possible anti-derivative of $3x^2 - 10x + 4$ is $x^3 - 5x^2 + 4x$.

In the previous investigation, you saw that the derivative of x^2 is $2x$, so *an* anti-derivative of $2x$ is x^2. But, the derivative of $x^2 + 5$ is also $2x$. In fact, there is a family of primitive functions $f(x) = x^2 + c$ (where c is a constant) whose derivatives are $2x$. The constant c is an arbitrary constant called the **constant of integration**.

Indefinite integral

The **indefinite integral** of the function $f(x)$ is $F(x) + c$.

The symbol $\int$ represents the integral.

So $F(x) = \int f(x)\,dx$, where $F'(x) = f(x)$.

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It is called the indefinite integral as the value of the constant of integration is not known.

You can combine the rule for finding an anti-derivative with the definition of the indefinite integral to state a general rule for finding the indefinite integral of a power function.

The indefinite integral (anti-derivative) of x^n

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c \text{ (where } n \neq -1)$$

Using this rule, you can see that:

$$\int 1 \, dx = \int x^0 \, dx = \frac{x^{0+1}}{0+1} + c = x + c.$$

$\int 1 \, dx$ is also written as $\int dx$.

You know a number of properties of derivatives, namely:

- $\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$
- $\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx} f(x) - \frac{d}{dx} g(x)$
- $\frac{d}{dx} kf(x) = k \frac{d}{dx} f(x)$
- $\frac{d}{dx}[af(x) + bg(x)] = a \frac{d}{dx} f(x) + b \frac{d}{dx} g(x)$

Integrals have a similar set of properties.

Properties of integrals

- $\int kf(x)\,dx = k\int f(x)\,dx$
- $\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$
- $\int [f(x) - g(x)]\,dx = \int f(x)\,dx - \int g(x)\,dx$
- $\int [af(x) + bg(x)]\,dx = a\int f(x)\,dx + b\int g(x)\,dx$

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EXAMPLE 5

a Calculate $\int (2x^3 - 5x^2 + 8)\,dx$.

b If $f(x) = \dfrac{2}{5x^3}$, find $F(x)$.

c Find y if $\dfrac{dy}{dx} = (3x - 5)^2$.

Solution

a Write as the sum or difference of integrals.

$$\int (2x^3 - 5x^2 + 8)\,dx$$
$$= \int 2x^3\,dx - \int 5x^2\,dx + \int 8\,dx$$
$$= 2\int x^3\,dx - 5\int x^2\,dx + 8\int dx$$

Use the rule for integration.

$$= 2 \times \frac{x^4}{4} - 5 \times \frac{x^3}{3} + 8 \times x + c$$

Simplify.

$$= \frac{1}{2}x^4 - \frac{5}{3}x^3 + 8x + c$$

b Write $f(x)$ using negative indices.

$$f(x) = \frac{2}{5x^3}$$
$$= \frac{2}{5}x^{-3}$$

Find the anti-derivative.

$$F(x) = \int \frac{2}{5}x^{-3}\,dx$$

Use the rule.

$$= \frac{2}{5} \times \frac{x^{-3+1}}{(-3+1)} + c$$

Simplify.

$$= \frac{2}{5} \times \frac{x^{-2}}{-2} + c$$
$$= -\frac{x^{-2}}{5} + c$$
$$F(x) = -\frac{1}{5x^2} + c$$

c Expand the RHS.

$$\frac{dy}{dx} = (3x - 5)^2$$
$$= 9x^2 - 30x + 25$$

Find the anti-derivative, y.

$$y = 9 \times \frac{x^3}{3} - 30 \times \frac{x^2}{2} + 25x + c$$

Simplify.

$$= 3x^3 - 15x^2 + 25x + c$$

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Exercise 5.02 The integral of x^n

Example 4

1 Find an anti-derivative of each function.

a $f(x) = 6x^2 + 4x + 1$ **b** $m(t) = 8t - 7$

c $g(z) = 4z^3 + 3z - 5z^4$ **d** $h(x) = 4$

e $p(y) = y^2 - y - 1$ **f** $s(v) = 3v^3 - v^4 + 4v^2 - 2$

g $v(t) = 9.8t - 0.3t^2 - 6$ **h** $j(m) = 4(m + 3)^2 + 8$

i $p(d) = 0.1d^3 + 7d - 0.38$ **j** $h(k) = 5k^2 - 2k^3 + 2k - 5$

Example 5

2 Find the indefinite integral of each function.

a $q(i) = 4i^3 - 6i^2 - 2i - 3$ **b** $f(x) = x^3 - 10x^4 - 6x^2 + 12$

c $y(u) = 2u - 5u^3 - 4u^2 + 8$ **d** $t(m) = 0.8m^2 - 6.3m^3 - 0.7m + 4$

e $r(t) = 0.07t + t^2$ **f** $v(x) = 4.9(x - 5)^2$

3 Find each integral.

a $\int (x^2 - 5)\, dx$ **b** $\int (3t + 2)\, dt$

c $\int (7 - u^3 + u^2)\, du$ **d** $\int h^4\, dh$

e $\int (p + 4)^2\, dp$ **f** $\int (z^4 - z^3 + z^2 - z + 1)\, dz$

4 Find y in each equation.

a $\frac{dy}{dx} = 4x^3 + 6x^2 - 6x + 7$ **b** $\frac{dy}{dx} = 12x^3 - 12x^2 + 12x - 12$

c $\frac{dy}{dt} = 3t + 5$ **d** $\frac{dy}{dz} = z - z^3 + 8$

e $\frac{dy}{dx} = 0.5x - 0.04x^3 + 0.7x^2 - 0.52$ **f** $\frac{dy}{dt} = 0.7t - 0.4t^2 + 0.6$

5 Find the indefinite integral of each function.

a $f(x) = (3 - x)^2$ **b** $f(x) = (2x + 5)^2$

c $f(x) = (5x - 4)(2x + 3)$ **d** $f(x) = x(x + 2)(4x - 5)$

e $f(x) = \frac{3x^2 - 7x^3}{2x}\ (x \neq 0)$ **f** $f(x) = \frac{8x^7 + 3x^4}{x^2}\ (x \neq 0)$

6 Find y in each equation.

a $\frac{dy}{dx} = x^{\frac{5}{2}}$ **b** $\frac{dy}{dx} = x^{-\frac{3}{2}}$ **c** $\frac{dy}{dx} = 2\sqrt{x^3}$

d $\frac{dy}{dx} = 5\sqrt[3]{x^4}$ **e** $\frac{dy}{dx} = 5x^3 - 5x^{-3}$ **f** $\frac{dy}{dx} = \frac{3}{7x^5}$

7 Calculate $\int \left(x - \frac{1}{x}\right)^2 dx$.

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Problem solving

8 $\frac{dy}{dx} = 4x - 5$ and (3, 2) is a point on the graph of y. Find the function y and its value at $x = 5$.

9 When $t = 1$, the value of x is -2. Find x at $t = 4$, given that $\frac{dx}{dt} = 0.6t^2 - 2t + 4$.

10 If $f(x) = \int (6x - 12x^2)\,dx$ and $f(-1) = 5$, find $f(1)$.

11 If $g'(v) = v - v^3 - 1$ and $g(v)$ has a zero at $v = 3$, find $g(4)$.

12 If $m'(r) = 4\sqrt{r}$ and (2, 6) is on the graph of $m(r)$, find $m(r)$.

13 If $\frac{dy}{dx} = 3\sqrt{x+4}$ and y is zero at $x = 5$, find an expression for y.

5.03 The integrals of sin (*x*) and cos (*x*)

In Chapter 4, you saw that:

- if $y = \sin(x)$, then $\frac{dy}{dx} = \cos(x)$
- if $y = \sin(ax + b)$, then $\frac{dy}{dx} = a\cos(ax + b)$
- if $y = \cos(x)$, then $\frac{dy}{dx} = -\sin(x)$
- if $y = \cos(ax + b)$, then $\frac{dy}{dx} = -a\sin(ax + b)$.

You get the indefinite integrals by reversing the process of differentiation.

Integrals of sin (*x*) and cos (*x*)

$$\int \cos(x)\,dx = \sin(x) + c$$

$$\int \cos(ax + b)\,dx = \frac{1}{a}\sin(ax + b) + c$$

$$\int \sin(x)\,dx = -\cos(x) + c$$

$$\int \sin(ax + b)\,dx = -\frac{1}{a}\cos(ax + b) + c$$

The integrals of cos $(ax + b)$ and sin $(ax + b)$ are both examples of the general rule:

$$\int f(ax + b)\,dx = \frac{1}{a}F(ax + b) + c, \text{ where } F(x) \text{ is the primitive of } f(ax + b).$$

To find anti-derivatives, you can either use the derivative rules in reverse, or use the indefinite integral rules above to find the results. You may find it simpler to reverse the derivative rules. When finding anti-derivatives of trigonometric functions, you should check your answers by differentiation.

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EXAMPLE 6

a Find $\int \sin(5x)\,dx$.

b Find the anti-derivative of $3\cos(12x - 7)$.

Solution

a

Use the rule for integration. $\int \sin(5x)\,dx = -\frac{1}{5}\cos(5x) + c$

Differentiate to check. $\frac{dy}{dx}\left[-\frac{1}{5}\cos(5x) + c\right] = -\frac{1}{5} \times [-\sin(5x)] \times 5 + 0$

This is the same as the original. $= \sin(5x)$ ✓

Write the answer. $\int \sin(5x)\,dx = -\frac{1}{5}\cos(5x) + c$

b

Write the problem in integral form. $\int 3\cos(12x - 7)\,dx$

Take out 3 as a factor. $= 3\int \cos(12x - 7)\,dx$

Use the rule for integration. $= 3 \times \frac{1}{12}\sin(12x - 7) + c$

Simplify. $= \frac{1}{4}\sin(12x - 7) + c$

Differentiate to check. $\frac{dy}{dx}\left[\frac{1}{4}\sin(12x - 7) + c\right]$

$= \frac{1}{4} \times \cos(12x - 7) \times 12 + 0$

This is the same as the original. $= 3\cos(12x - 7)$ ✓

Write the answer. The anti-derivative of $3\cos(12x - 7)$ is $\frac{1}{4}\sin(12x - 7) + c$.

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EXAMPLE 7

If $\frac{dy}{dx} = \frac{1}{2}\sin\left(\frac{2}{3}x+1\right)$, find y.

Solution

Use the anti-derivative. $y = \int \frac{1}{2}\sin\left(\frac{2}{3}x+1\right)dx$

Take out the factor. $= \frac{1}{2}\int \sin\left(\frac{2}{3}x+1\right)dx$

Use the rule for integration. $= \frac{1}{2}\times -\frac{1}{\frac{2}{3}}\cos\left(\frac{2}{3}x+1\right)+c$

Simplify using $1\div\frac{2}{3} = 1\times\frac{3}{2}$. $= \frac{1}{2}\times -\frac{3}{2}\cos\left(\frac{2}{3}x+1\right)+c$

Simplify. $= -\frac{3}{4}\cos\left(\frac{2}{3}x+1\right)+c$

Make sure you check the result.

Exercise 5.03 The integrals of sin (*x*) and cos (*x*)

Example 6

1 Find each integral.

a $\int \cos\left(\frac{4x}{3}\right)dx$

b $\int \frac{2}{3}\cos\left(\frac{1}{3}x\right)dx$

c $\int 6\cos(6x-5)\,dx$

d $\int 4\cos\left(\frac{1}{2}x\right)dx$

e $\int 8\sin(2x+3)\,dx$

f $\int 3\sin(9x)\,dx$

g $\int 10\sin(5x)\,dx$

h $\int \frac{1}{3}\sin\left(\frac{2}{3}x\right)dx$

Example 7

2 Find y if:

a $\frac{dy}{dx} = 3\sin\left(\frac{1}{2}x\right)$

b $\frac{dy}{dx} = \frac{1}{4}\sin\left(\frac{1}{4}x\right)$

c $\frac{dy}{dx} = \sin\left(\frac{1}{2}x\right) - 2\cos\left(\frac{1}{2}x\right)$

d $\frac{dy}{dx} = 3\cos\left(\frac{1}{2}x\right)$

e $\frac{dy}{dx} = 5\cos\left(\frac{1}{2}x\right)$

f $\frac{dy}{dx} = 1 + x + \cos(x-5)$

g $\frac{dy}{dx} = \frac{1}{6}\sin\left(\frac{1}{3}x\right)$

h $\frac{dy}{dx} = 10\cos\left(\frac{x}{5}\right)$

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3 Find each integral.

a $\int \cos(6x)\,dx$

b $\int \cos(5x-6)\,dx$

c $\int \sin(7-8x)\,dx$

d $\int [\cos(3x)+\sin(3x)]\,dx$

e $\int \left[5\sin(4x)-8\cos\left(\frac{1}{4}x\right)\right]dx$

f $\int \left[2\cos(5x)+3\sin\left(\frac{x}{3}\right)\right]dx$

Problem solving

4 The speed of the bottom of the pendulum in a grandfather clock is given by

$$v = 56.7\cos(6.3t)$$

where t is in seconds and v is in cm/s. Find an expression for the displacement from the centre if at time $t = 0$ the displacement is 0 cm. Justify your reasoning.

5 The air molecules at the open end of a flute vibrate backwards and forwards as a note is played. The average acceleration is given by

$$a = L\sin(0.016t)$$

where a is in m/s^2 and t is in seconds.

L is a quantity determined by the loudness of the note. At time $t = 0$, the molecules at the far end are moving at their fastest. Use anti-derivatives to find expressions for the velocity and displacement of these molecules at time t, and comment on the directions of the acceleration and displacement of the molecules.

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5.04 The integral of e^x

In Chapter 2, you saw that $\frac{d}{dx}(e^x) = e^x$ and $\frac{d}{dx}(e^{ax+b}) = ae^{ax+b}$.

The anti-derivative of e^x can be found by reversing the process of differentiation.

Integral of e^x

$$\int e^x \, dx = e^x + c$$

$$\int e^{ax+b} \, dx = \frac{1}{a} e^{ax+b} + c$$

The integral of e^{ax+b} is another example of the general rule:

$\int f(ax+b)\,dx = \frac{1}{a}F(ax+b) + c$, where $F(x)$ is the primitive of $f(ax+b)$.

EXAMPLE 8

Calculate:

a $\int (x^2 + e^{4x})\,dx$ **b** $\int 12e^{3x}\,dx$

Solution

a Integrate terms separately.

$$\int (x^2 + e^{4x})\,dx = \int (x^2)\,dx + \int (e^{4x})\,dx$$

Use the rules of integration.

$$= \frac{1}{3}x^3 + \frac{1}{4}e^{4x} + c$$

b Take out the factor.

$$\int 12e^{3x}\,dx = 12\int e^{3x}\,dx$$

Use the rules of integration.

$$= 12 \times \frac{1}{3}e^{3x} + c$$

Simplify.

$$= 4e^{3x} + c$$

You may need to simplify expressions before integration.

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EXAMPLE 9

Find the anti-derivative of $2(e^x - 3e^{-x})^2$.

Solution

Expand the brackets.

$$2(e^x - 3e^{-x})^2 = 2(e^{2x} - 6 + 9e^{-2x})$$
$$= 2e^{2x} - 12 + 18e^{-2x}$$

Integrate.

$$\int (2e^{2x} - 12 + 18e^{-2x})\,dx$$

Integrate separately.

$$= e^{2x} - 12x - 9e^{-2x} + c$$

In some more complex examples, you may need to differentiate a function and use the result to find the anti-derivative.

In these cases, you will need to use the fact that if $f(x) = g'(x)$, then $\int g'(x)\,dx = f(x) + c$.

EXAMPLE 10

Given that $y = e^{x^5}$, calculate $\frac{dy}{dx}$ and use this result to find $\int x^4 e^{x^5}\,dx$.

Solution

Let $u = x^5$.

$$y = e^u$$

Use the chain rule.

$$\frac{dy}{dx} = e^u \times u'$$

Calculate $\frac{du}{dx}$.

$$= e^u \times 5x^4$$

Substitute for u and rearrange.

$$= 5x^4 e^{x^5}$$

Write the result.

$$\frac{d}{dx}\left(e^{x^5}\right) = 5x^4 e^{x^5}$$

Write as an integral statement.

$$\int 5x^4 e^{x^5}\,dx = e^{x^5}$$

Take out the factor.

$$5\int x^4 e^{x^5}\,dx = e^{x^5}$$

Compare with $\int x^4 e^{x^5}\,dx$ and state the result.

$$\int x^4 e^{x^5}\,dx = \frac{1}{5}e^{x^5} + c$$

There is no need to add the constant of integration until the last step.

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Exercise 5.04 The integral of e^x

Example 8

1 Find the integral of each function.

a $e^{\frac{2x}{x}}$ **b** e^{5x} **c** $-e^{-x}$

d $e^{\frac{x}{2}}$ **e** $3e^{3x+1}$ **f** $12e^{-4x}$

g $21e^{-7x-5}$ **h** $10e^{2x+3}$

2 Find the integral of each function.

a $3x + 2e^{4x}$ **b** $2x^2 - 6e^{3x}$ **c** $e^{-4x} - e^{\frac{x}{3}}$

d $10e^{5x} + 6x$ **e** $e^{2x+1} - e^{-x}$ **f** $5e^{-2x} + 2e^{\frac{x}{4}}$

Example 9

3 Calculate:

a $\int (1+e^x)^2 dx$ **b** $\int (2+e^x)^3 dx$

c $\int (2e^x - 3)^2 dx$ **d** $\int \frac{e^x+1}{e^x} dx$

e $\int \frac{6-e^x}{e^{2x}} dx$ **f** $\int \frac{e^{2x}+2e^x+1}{e^x} dx$

4 Calculate:

a $\int \left[2e^{4x} - \sin(2x)\right] dx$ **b** $\int \frac{(2+e^x)^2}{2e^x} dx$

c $\int \left[\frac{1}{2}\cos(4x+5) - 2e^{-x}\right] dx$ **d** $\int (e^{2x} - e^x)^2 dx$

5 If $\int ae^{bx}\, dx = -3e^{4x} + 5$, find the values of a and b.

Example 10

6 Given that $y = e^{x^4}$, calculate $\frac{dy}{dx}$ and use this result to find $\int x^3 e^{x^4} dx$.

7 Find $\frac{d}{dx}\left(e^{x^2+3}\right)$ and use this result to find $\int xe^{x^2+3} dx$.

Problem solving

8 Calculate $\int 2x^4 e^{x^5-3} dx$.

9 Calculate $\int \frac{2e^{2x} + 3e^x - 2}{e^x + 2} dx$.

10 The function $y = f(x)$ has the derivative $\frac{dy}{dx} = e^{kx}$, where k is a constant.

If the tangent drawn at $(1, e^4)$ passes through the origin, find:

a the value of k

b the equation of the function.

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5.05 The integrals of $\frac{1}{x}$ and $\frac{1}{ax+b}$

Integration of reciprocals

In Chapter 3, you saw that:

- $\frac{d}{dx}[\ln(x)] = \frac{1}{x}$ (where $x > 0$)
- $\frac{d}{dx}[\ln(ax+b)] = \frac{a}{ax+b}$ (where $a, x > -\frac{b}{a}$)

The anti-derivative of $\frac{1}{x}$ can be found by reversing the process of differentiation.

Because $\ln(x)$ is only defined if $x > 0$, absolute value notation is used to define the anti-derivative of $\frac{1}{x}$.

Integral of $\frac{1}{x}$

- $\int \frac{1}{x}\,dx = \ln|x| + c$
- $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + c$

The integral of $\ln|ax+b|$ is another example of the general rule:

$\int f(ax+b)\,dx = \frac{1}{a}F(ax+b) + c$, where $F(x)$ is the primitive of $f(ax+b)$.

EXAMPLE 11

Find:

a $\int \frac{5}{2x}\,dx$ **b** $\int \frac{3}{4x+1}\,dx$

Solution

a Take out the factor.

$$\int \frac{5}{2x}\,dx = \frac{5}{2}\int \frac{1}{x}\,dx$$

Apply the integration rule.

$$= \frac{5}{2}\ln|x| + c$$

b Take out the factor.

$$\int \frac{3}{4x+1}\,dx = 3\int \frac{1}{4x+1}\,dx$$

Apply the integration rule.

$$= 3 \times \frac{1}{4}\ln|4x+1| + c$$

Simplify.

$$= \frac{3}{4}\ln|4x+1| + c$$

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EXAMPLE 12

If $\frac{dy}{dx} = \frac{4}{2x-1}$ and $y = 5$ when $x = 1$, find an expression for y in terms of x.

Solution

Take out the factor.

$$\int \frac{4}{2x-1}\,dx = 4\int \frac{1}{2x-1}\,dx$$

Apply the integration rule.

$$= 4 \times \tfrac{1}{2} \ln|2x-1| + c$$

Simplify.

$$y = 2\ln|2x-1| + c$$

Find c by substituting $x = 1$, $y = 5$.

$$5 = 2\ln|2\times 1 - 1| + c$$

Simplify.

$$5 = 2\ln|1| + c$$

$$5 = 0 + c$$

$$c = 5$$

Write the equation.

$$y = 2\ln|2x-1| + 5$$

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Exercise 5.05 The integrals of $\frac{1}{x}$ and $\frac{1}{ax+b}$

Example 11

1 Find the anti-derivative of:

a $\frac{4}{x}$ **b** $\frac{2}{3x}$ **c** $\frac{5}{7x}$

d $\frac{1}{x-4}$ **e** $\frac{5}{2x+3}$ **f** $\frac{8}{4x-5}$

2 Find:

a $\int \frac{1}{3x}\,dx$ **b** $\int \frac{3}{5x}\,dx$ **c** $\int \frac{7}{9x}\,dx$

d $\int \frac{6}{9x+2}\,dx$ **e** $\int \frac{5}{10x-3}\,dx$ **f** $\int \frac{2}{5-3x}\,dx$

3 Find:

a $\int (x+4)^{-1}\,dx$ **b** $\int (3x-5)^{-1}\,dx$ **c** $\int \frac{x}{3x+4x^2}\,dx$

d $\int 2x(5x^2+6x)^{-1}\,dx$ **e** $\int \frac{6x}{3x^2-4x}\,dx$ **f** $\int 3x(9x^2+7x)^{-1}\,dx$

4 Find:

a $\int \frac{x^2+2x-3}{x}\,dx$ **b** $\int \frac{3x^2-8x+12}{2x}\,dx$

Example 12

5 If $\frac{dy}{dx} = \frac{9}{4-3x}$ and $y = 9$ when $x = 1$, find an expression for y in terms of x.

6 Find y if $\frac{dy}{dx} = x^2 - \frac{1}{x}$ and the graph of the function passes through $\left(1, \frac{7}{3}\right)$.

7 Calculate $\int \left(x - \frac{1}{x}\right)^3 dx$.

Problem solving

8 Given that $\frac{dy}{dx} = \frac{7}{2x-4}$ and $y = 5$ when $x = -2$, find an expression for y in terms of x.

5.06 Integration by recognition

WS Finding indefinite integrals 2

Sometimes you may need to find an anti-derivative for which there is no rule for the anti-derivative. In these cases, **integration by recognition** is used. Integration by recognition involves finding the derivative of a related function and using this derivative to find the required anti-derivative.

The simplest case of the integration by recognition process is where a known anti-derivative and the required anti-derivative differ by a constant multiple, k.

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EXAMPLE 13

Find the derivative of $\sqrt{9x^2+8}$ and use the result to find $\int \frac{x}{\sqrt{9x^2+8}}\,dx$.

Solution

Write as an equation.	$y = \sqrt{9x^2+8}$
Use the chain rule.	Let $u = 9x^2 + 8$.
	$y = u^{\frac{1}{2}}$
Find $\frac{dy}{du}$.	$\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}}$
Find $\frac{du}{dx}$.	$\frac{du}{dx} = 18x$
	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
	$= \frac{1}{2}u^{-\frac{1}{2}} \times 18x$
	$= \frac{9x}{\sqrt{u}}$
Substitute for u	$= \frac{9x}{\sqrt{9x^2+8}}$
Write the result.	$\frac{d}{dx}\sqrt{9x^2+8} = \frac{9x}{\sqrt{9x^2+8}}$
Write as an integral equation.	$\int \frac{9x}{\sqrt{9x^2+8}}\,dx = \sqrt{9x^2+8}$
Use integration by recognition and add the constant of integration.	$\int \frac{x}{\sqrt{9x^2+8}}\,dx = \frac{1}{9}\sqrt{9x^2+8} + c$

In more complex cases, you will need to use composite functions. This is done by first making a substitution. This has the effect of changing the variable and the integrand (the expression being integrated). You can then use integration by recognition to finalise the process.

Recall the chain rule for the derivative of composite functions.

If $y = F(u)$ and $u = g(x)$, then:

$$\frac{d}{dx}[F(g(x))] = F'[g(x)]g'(x)$$

or $$\frac{d}{dx}[F(u)] = F'(x)\frac{du}{dx}$$

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So $\frac{du}{dx} = g'(x)$

This means $du = g'(x)dx$

This result can be used to write a rule for calculating the integral of a composite function.

The integral of a composite function

If $F(x) = f'(x)$ and $u = g(x)$, then $du = g'(x)dx$:

$$\int f[g(x)]g'(x)\,dx = \int f(u)\,du = F(u) + c = F[g(x)] + c$$

To apply this rule, you need to be able to recognise the presence of $f[g(x)]$, $g(x)$ and $g'(x)$.

For example, compare $\int \cos(x^4)4x^3 dx$ with $\int f[g(x)]g'(x)\,dx$.

In this case, $f[g(x)] = \cos(x^4)$, $g(x) = x^4$ and $g'(x) = 4x^3$, so:

$\int \cos(x^4)4x^3 dx = F[g(x)] + c = \sin(x^4) + c$.

EXAMPLE 14

Integration by substitution

Calculate:

a $\int 12x^2\sqrt{4x^3+1}\,dx$ **b** $\int \frac{8x}{x^2+3}dx$

Solution

a

Let $u = 4x^3 + 1$

$\frac{du}{dx} = 12x^2$

$du = 12x^2\,dx$

Rewrite the integral. $\int 12x^2\sqrt{4x^3+1}\,dx = \int \sqrt{4x^3+1} \times 12x^2 dx$

Use index notation. $= \int (4x^3+1)^{\frac{1}{2}} \times 12x^2\,dx$

Substitute for u. $= \int u^{\frac{1}{2}} \times 12x^2 dx$

Substitute for $12x^2\,dx$. $= \int u^{\frac{1}{2}} du$

Integrate. $= \frac{2}{3}u^{\frac{3}{2}} + c$

Substitute for u. $= \frac{2}{3}\sqrt{(4x^3+1)^3} + c$

Check the answer by differentiating. $y = \frac{2}{3}u^{\frac{3}{2}} + c$, where $u = 4x^3 + 1$

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Use the chain rule.

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= \left(\frac{2}{3} \times \frac{3}{2} \times u^{\frac{1}{2}}\right) \times 12x^2$$
$$= u^{\frac{1}{2}} \times 12x^2$$

Substitute for u.

$$= 12x^2(4x^3+1)^{\frac{1}{2}}$$
$$= 12x^2\sqrt{4x^3+1}$$

State the result.

$$\int 12x^2\sqrt{4x^3+1}\,dx = \frac{2}{3}\sqrt{(4x^3+1)^3} + c$$

b

Let $u = x^2 + 3$.

$$\frac{du}{dx} = 2x$$

$$du = 2x\,dx$$

Write the integral using u.

$$\int \frac{8x}{x^2+3}dx = \int \frac{8x\,dx}{u}$$

Use the fact that $du = 2x\,dx$.

$$= \int \frac{4du}{u}$$

Take out the factor.

$$= 4\int \frac{1}{u}du$$

Integrate.

$$= 4\ln|u| + c$$

Substitute for u.

$$= 4\ln|x^2+3| + c$$

Check the answer by differentiating.

$y = 4\ln|u| + c$, where $u = x^2 + 3$

Use the chain rule.

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= \left(\frac{4}{u}\right) \times 2x$$

Substitute for u.

$$= \frac{4}{x^2+3} \times 2x$$
$$= \frac{8x}{x^2+3}$$

State the result.

$$\int \frac{8x}{x^2+3}dx = 4\ln|x^2+3| + c$$

A similar process can be used to find integrals in the form $\int (ax+b)^n\,dx$.

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EXAMPLE 15

Calculate $\int (4x-7)^5\,dx$.

Solution

Let $u = 4x - 7$. $\int (4x-7)^5\,dx = \int u^5\,dx$

Find $\frac{du}{dx}$. $\frac{du}{dx} = 4$

Rearrange. $du = 4\,dx$

$dx = \frac{1}{4}du$

Substitute for dx. $\int u^5\,dx = \int \frac{u^5}{4}du$

Take out the factor. $= \frac{1}{4}\int u^5\,du$

Integrate. $= \frac{1}{4} \times \frac{1}{6}u^6 + c$

Substitute for u and simplify. $= \frac{1}{24}(4x-7)^6 + c$

Exercise 5.06 Integration by recognition

1 Calculate $\frac{d}{dx}\sqrt{3+x^2}$ and use the result to find $\int \frac{3x}{\sqrt{3+x^2}}dx$. Example 13

2 Calculate $\frac{d}{dx}\sqrt{x^3-6}$ and use the result to find $\int \frac{5x^2}{\sqrt{x^3-6}}dx$.

3 Calculate $\frac{d}{dx}\sqrt{3x^4+4}$ and use the result to find $\int \frac{x^3}{2\sqrt{3x^4+4}}dx$.

4 Calculate $\frac{d}{dx}(x^3+1)^5$ and use the result to find $\int 2x^2(x^3+1)^4 dx$.

5 Calculate $\frac{d}{dx}(3x^2-5x+6)^7$ and use the result to find $\int (6x-5)(3x^2-5x+6)^6 dx$.

6 Calculate: Example 14

a $\int x^3\sqrt{5x^4+3}\,dx$ **b** $\int \frac{2x}{6-5x^2}dx$ **c** $\int x^3\sin(x^4)\,dx$

d $\int \frac{\sin(2x+1)}{\cos(2x+1)}dx$ **e** $\int -\cos(x)[5-2\sin(x)]^4 dx$ **f** $\int \frac{\cos(x)}{\sin^3(x)}dx$

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Example 15

7 Calculate:

a $\int (2x+1)^4\,dx$ **b** $\int (7x-3)^6\,dx$ **c** $\int (3-5x)^3\,dx$

d $\int (9-8x)^5\,dx$ **e** $\int (3x-4)^{-3}\,dx$ **f** $\int (5x+6)^{-4}\,dx$

Problem solving

8 Show that $\int \tan(x)\,dx = \ln|\sec(x)| + c$.

9 Calculate $\int \frac{2}{x\ln(x)}\,dx$. (*Hint*: Let $u = \ln(x)$.)

10 Calculate $\int \frac{\ln(x)}{x}\,dx$.

11 Calculate $\int \frac{x}{\sqrt{x-1}}\,dx$.

12 Calculate $\int \frac{x^2}{\sqrt{x^3-4}}\,dx$.

5.07 Velocity and displacement problems

WS Displacement, velocity and acceleration

WS Anti-differentiation assignment

WS Anti-differentiation problems

You have studied the relationship between displacement (x), velocity (v) and acceleration (a) in differential calculus. You know that:

$$v = \frac{dx}{dt} \text{ and } a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

You can use anti-differentiation to reverse these equations.

Displacement and velocity as integrals

$$x = \int v\,dt \text{ and } v = \int a\,dt$$

If you know an equation for velocity, you can integrate it to find an equation for displacement, and differentiate it to find an equation for acceleration. To find the value of the constant of integration c, you need to use additional information about displacement and time.

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EXAMPLE 16

A particle moves in a straight line so that its velocity (m/s) at time t seconds is given by $v = 3t^2 - 8t - 7$, $t \geq 0$. Initially the particle is 10 metres to the right of a fixed origin.

a After how many seconds is the particle at the origin?

b Calculate the particle's acceleration when its velocity is zero.

Solution

a Calculate the displacement function using the velocity function.

$$x = \int v\,dt$$
$$= \int 3t^2 - 8t - 7\,dt$$

Integrate.

$$= \frac{3t^3}{3} - \frac{8t^2}{2} - 7t + c$$

Simplify.

$$= t^3 - 4t^2 - 7t + c$$

Use the initial condition to calculate c.

When $t = 0$, $x = 10$.

$$10 = 0^3 - 4(0^2) - 7(0) + c$$
$$= c$$

So $c = 10$

Write the displacement function.

$$x = t^3 - 4t^2 - 7t + 10$$

The particle is at the origin when $x = 0$.

$$0 = t^3 - 4t^2 - 7t + 10$$

Factorise using the factor theorem.

$$(t - 5)(t - 1)(t + 2) = 0$$

Solve.

$$t = -2, 1 \text{ or } 5$$

But $t \geq 0$, so reject $t = -2$.

$$t = 1 \text{ or } 5 \quad (\text{as } t \geq 0)$$

State the result.

The particle is at the origin at 1 s and at 5 s.

b Differentiate the velocity function to calculate the acceleration function.

$$a = \frac{dv}{dt}$$
$$a = 6t - 8$$

Let $v = 0$.

$$0 = 3t^2 - 8t - 7$$

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Use the quadratic formula.	$t = \dfrac{8 \pm \sqrt{8^2 - 4 \times 3 \times (-7)}}{2 \times 3}$
	$= \dfrac{8 \pm \sqrt{148}}{6}$
Evaluate	$= -0.6942\ldots$ or $3.3609\ldots$
But $t \geq 0$, so reject $t = -0.6942\ldots$	$t = 3.3609\ldots$
Find a.	$a = 6t - 8$
	$= 6 \times 3.3609\ldots - 8$
Evaluate and round off.	$= 12.1655\ldots$
State the result.	The acceleration is about 12.2 m/s^2 when the velocity is zero.

If you know an expression for acceleration, you will need to anti-differentiate it once to find an equation for velocity and again to find an equation for displacement.

Each time you integrate, there will be a constant of integration you have to calculate using additional information.

Displacement and velocity as integrals

EXAMPLE 17

An object moves in a straight line so that its acceleration (m/s^2) at time t seconds is given by $a = 6t + 4$ for $t \geq 0$. If $v = -4$ and $x = -7$ when $t = 0$, calculate the particle's velocity and position when $t = 2$.

Solution

Integrate the acceleration function to find the velocity function.	$v = 3t^2 + 4t + c$
Substitute $t = 0$, $v = -4$ to find c.	$-4 = 3 \times 0^2 + 4 \times 0 + c$
Solve.	$c = -4$
Write the velocity function.	$v = 3t^2 + 4t - 4$
Let $t = 2$.	$= 3 \times 2^2 + 4 \times 2 - 4$
Evaluate.	$= 16$
Integrate the velocity function to find the displacement function.	$x = t^3 + 2t^2 - 4t + d$

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Substitute $t = 0$, $x = -7$ to find d.	$-7 = 0^3 + 2(0^2) - 4(0) + d$
	$-7 = d$
	$x = t^3 + 2t^2 - 4t - 7$
Let $t = 2$.	$x = 2^3 + 2 \times 2^2 - 4 \times 2 - 7$
Evaluate.	$= 1$
State the result.	When $t = 2$, the object has a velocity of 16 m/s and a displacement of 1 m; its position is 1 m to the right of the origin.

An object that is subject to the force of gravity has an acceleration of -9.8 m/s^2, that is, 9.8 m/s^2 downwards. The positive direction is taken to be upwards and the negative direction is downwards.

EXAMPLE 18

An object, 9 m above the ground, is projected vertically upwards with a velocity of 25 m/s.

a What is the velocity of the particle at 2 s?

b When does the particle come to rest for the first time?

c What is the position of the particle at 3 s?

Solution

a

Use the acceleration of gravity.	$a = -9.8$
Find an equation for velocity.	$v = \int -9.8\,dt$
	$= -9.8t + c$
From initial conditions, $v = 25$ at $t = 0$.	$25 = -9.8 \times 0 + c$
Solve.	$c = 25$
Write the equation for velocity.	$v = -9.8t + 25$
Let $t = 2$.	$= -9.8 \times 2 + 25$
Evaluate.	$= 5.4$
State the result.	At 2 s, the object's velocity is 5.4 m/s upwards.

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b	The object comes to rest when $v = 0$.	$0 = -9.8t + 25$
	Solve for t.	$9.8t = 25$ $t = \frac{25}{9.8}$ $= 2.5510\ldots$
	Round off and state the result.	The object first comes to rest at about 2.55 s.
c	Find an equation for displacement.	$x = \int v\,dt$ $= \int (-9.8t + 25)\,dt$
	Calculate.	$= \frac{-9.8t^2}{2} + \frac{25t}{2} + c$
	Evaluate.	$= -4.9t^2 + 12.5t + c$
	From initial conditions, $x = 9$ at $t = 0$.	$9 = -4.9 \times 0^2 + 12.5 \times 0 + c$
	Solve.	$c = 9$
	Write the equation for x.	$x = -4.9t^2 + 12.5t + 9$
	Let $t = 3$.	$= -4.9 \times 3^2 + 12.5 \times 3 + 9$ $= 2.4$
	State the result.	At 3 s, the object is 2.4 m above the ground.

Exercise 5.10 Velocity and displacement problems

Example 16

1 A particle moves in a straight line so that its velocity (m/s) at time t seconds is given by

$$v(t) = 3t^2 - 4t - 4, \qquad t \geq 0.$$

Initially, the particle is 8 metres to the right of a fixed origin.

a After how many seconds is the particle at the origin?

b Calculate the particle's acceleration when its velocity is zero.

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2 The velocity, v cm/s, of a particle moving in a straight line at time t seconds is given by

$$v = 4t^2 - 10t - 6, \qquad t \geq 0.$$

Initially the particle is 27 cm to the right of a fixed origin.

a Obtain an expression for the particle's displacement at time t seconds.
b How far from its initial position is the particle after the first second?
c Determine the position of the particle when its velocity is zero.

3 The velocity of a particle moving in a horizontal line is given by

$$v = \frac{dx}{dt} = 6 - 9e^{-3t}$$

where t is the time in seconds and x is the displacement in metres.

a What is the initial ($t = 0$) velocity of the particle?
b Find the equation of the displacement (x) in terms of t if it is known that initially the particle was 4 metres to the left of the origin.

4 An object moves in a straight line and its velocity (m/s) at time t s is given by:

$$v(t) = t^2 - t - 6.$$

After 2 seconds the particle is 10 m to the left of the origin.

a What is the initial position of the particle?
b What is the position of the particle after 3 seconds?
c What is the initial acceleration of the particle?

5 An object moves in a straight line so that its acceleration (m/s^2) at time t seconds is given by $a = 5 - 8t$ for $t \geq 0$. If its initial velocity is 7 m/s and $x = 7.5$ when $t = 3$, calculate the particle's velocity and position when $t = 1$.

Example 17

6 The acceleration (a m/s^2) of a particle moving in a straight line after t seconds is given by $a = 6t - 18$, where $t \geq 0$. At $t = 1$, the velocity of the particle is 9 m/s and its displacement is 20 m to the right of the origin. Find an equation for:

a v, the particle's velocity from the origin after t seconds.
b x, the particle's displacement from the origin after t seconds.

7 A particle accelerates horizontally according to the equation $a = -e^{4t}$ (cm/s^2). Initially, the particle is 4 cm to the right of the origin, moving with a velocity of $\frac{3}{4}$ m/s.

a Find the velocity of the particle after 1 s (correct to 2 decimal places).
b Find the position of the particle after 2 s (correct to 2 decimal places).

8 An object on top of a cliff 20 m above the ground is projected vertically upwards with a velocity of 10 m/s.

Example 18

a What is the velocity of the particle after 2 s?
b When does the particle come to rest for the first time?
c The greatest height above the ground is reached when the object stops going up and begins to fall. What is the greatest height reached by the object?
d What is the height of the particle after 3 s?

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9 A pebble falls from the top of a cliff and strikes the ground 2.5 s after it begins to fall. Calculate the height of the cliff, correct to 2 decimal places.

10 An object is projected vertically upwards from the ground with an initial velocity of 210 m/s.

a What is the greatest height reached by the object?

b When does the particle return to the ground?

11 A particle is moving in a straight line. At $t = 0$, the displacement, $x = 1$ and the velocity, $v = \frac{1}{2}$. The acceleration of the particle is given by $a = \frac{1}{4}\cos(4t)$.

a Find the velocity at time t.

b Find the position at time t.

Problem solving

12 Two particles A and B are moving to the right along a straight line. Particle A is accelerating according to the equation $a_A(t) = \frac{3}{4}$ and particle B is accelerating according to the equation $a_B(t) = \frac{2}{3}$. The velocity and displacement of the particles is given by $v(t)$ and $x(t)$ respectively. If $v_A(0) = \frac{1}{2}$, $x_A(2) = 2$, $v_B(1) = \frac{5}{6}$ and $x_B(1) = \frac{1}{2}$, when will the two particles have the same position?

13 The velocity of an object at time t seconds is given by $v(t) = 2t^2 - \sqrt{t+1} + 1$ (m/s). If $x(0) = 4$, what is the position of the particle 4 seconds later?

14 The acceleration of a particle is given by $a(t) = e^{2t} - 5$ (cm/s^2). If the particle started from rest, how far is the particle from its starting point, and which way is it moving 2 seconds later?

15 A body is travelling such that its velocity is given by $v(t) = 6\sin(3t)$ (m/s) for $0 \leq t \leq \frac{\pi}{2}$. Find the *total distance* travelled from $t = 0$ to $t = \frac{\pi}{2}$ s.

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5. CHAPTER SUMMARY

Anti-differentiation

- Product rule

  $\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$

  $y = uv \Rightarrow y' = uv' + vu'$

- Quotient rule

  $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$

  $y = \frac{u}{v} \Rightarrow y' = \frac{vu' - uv'}{v^2}$

- Chain rule

  $\frac{d}{dx}(f[g(x)]) = f'[g(x)]\, g'(x)$

  $y = f(u)$ and $u = g(x) \Rightarrow \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

- Anti-differentiation (integration) is the reverse of differentiation. Anti-differentiation undoes differentiation to find the original or primitive function (anti-derivative).
- If $f'(x) = ax^n$, then $f(x) = \frac{ax^{n+1}}{n+1}$ (where $n \neq -1$) is a *possible* primitive function.
- The indefinite integral of the function $f(x)$ is $F(x) + c$, where c is an arbitrary constant or the constant of integration.
- The symbol $\int$ represents the integral, so $F(x) = \int f(x)\,dx$, where $F'(x) = f(x)$.
- The indefinite integral of x^n

  $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$ (where $n \neq -1$)
- Reversing the properties of differentiation give the following properties of integrals:
  - $\int kf(x)\,dx = k\int f(x)\,dx$
  - $\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$
  - $\int [f(x) - g(x)]\,dx = \int f(x)\,dx - \int g(x)\,dx$
  - $\int [af(x) + bg(x)]\,dx = a\int f(x)\,dx + b\int g(x)\,dx$
- $\int f(ax+b)\,dx = \frac{1}{a}F(ax+b) + c$, where $F(x)$ is the primitive of $f(ax+b)$.

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- Integrals of sin (x) and cos (x)

  $\int \cos(x)\,dx = \sin(x) + c$ and $\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + c$

  $\int \sin(x)\,dx = -\cos(x) + c$ and $\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + c$

- Integral of e^x

  $\int e^x\,dx = e^x + c$ and $\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + c$

- Integral of $\frac{1}{x}$

  – $\int \frac{1}{x}dx = \ln|x| + c$

  – $\int \frac{1}{ax+b}dx = \frac{1}{a}\ln|ax+b| + c$

- Integration by recognition involves finding the derivative of a related function and using this derivative to find the required anti-derivative.

  If $F(x) = f'(x)$ and $u = g(x)$, then $du = g'(x)dx$:

  $$\int f[g(x)]g'(x)\,dx = \int f(u)\,du = F(u) + c = F[g(x)] + c$$

- Displacement and velocity as integrals

  Displacement, $x = \int v\,dt$ and velocity, $v = \int a\,dt$

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5. CHAPTER REVIEW

Anti-differentiation

1 Calculate the derivative of: Example 1

a $(2d+1)(d^2-3d+2)$ **b** $(2m^2+3m-1)(m^3-2m^2+m-3)$

c $\frac{7}{p}$ **d** $-5u^{-3}$

e $\frac{6}{x^2}$ **f** $\sqrt{2w}$

g $\sqrt[4]{x}$ **h** $8r^{\frac{3}{2}}$

i $9x^{2\frac{1}{3}}$ **j** $\frac{-4n}{\sqrt[3]{n^2}}$

2 Find the derivative of each function with respect to the given variables. Example 2

a $\frac{3q+4}{5-3q}$ **b** $\frac{2(3t-2)}{3(4t+3)}$ **c** $x^4(1+\sqrt{x})$

d $\left(\frac{3}{x}+x^4\right)(x^2+1)$ **e** $\frac{1}{u^2+3u-1}$

3 Differentiate each function. Example 3

a $2(3a-1)^3$ **b** $-5(m^3+3m^2-2)^8$

4 Find the derivative of each function. Example 3

a $\sqrt{5x^2+x-3}$ **b** $\sqrt[5]{y^3+3y-7}$

5 Find the derivative of each function. Example 3

a $-2(3-5h)^5(1+6h)^7$ **b** $x^4 \sin(2x)$

c $e^x(x^3-1)$ **d** $e^{4x}\sqrt{5x^2+2}$

e $x^3\cos^2(3x)$

6 Differentiate each function. Example 3

a $\frac{\cos(3x)}{4e^x-x}$ **b** $\frac{e^{-2x}}{2x-1}$

7 Find an anti-derivative of each function. Example 4

a $f(x)=8x^2-3x+1$ **b** $g(m)=6m^2+2m^3-2m-5$

8 Find the indefinite integral of $q(i)=4i^3-6i^2-2i-3$. Example 5

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9 Find $\int (5-3x^2+2x^3)\,dx$.

Example 6

10 Find each integral.

a $\int \sin\left(\frac{5x}{2}\right)dx$ **b** $\int 6\cos(3x+1)\,dx$

Example 7

11 Find y if:

a $\frac{dy}{dx}=\frac{1}{6}\sin\left(\frac{1}{3}x\right)$ **b** $\frac{dy}{dx}=1+3x+\cos(2x+3)$

Example 7

12 Find $\int 2\sin(3x)+3\cos\left(\frac{x}{6}\right)dx$.

Example 8

13 Find the integral of each function.

a e^{7x} **b** $e^{\frac{x}{3}}$ **c** $20e^{-5x}$ **d** $12e^{2x-3}$

Example 8

14 Find the integral of each function.

a $6x^2+9e^{3x}$ **b** $10e^{-5x}-e^{\frac{x}{4}}$ **c** $e^{6x-5}-e^{-2x}$

Example 9

15 Calculate:

a $\int (3+e^x)^3\,dx$ **b** $\int \frac{5-e^x}{e^{3x}}\,dx$ **c** $\int\left[6e^{3x}-9\cos(3x)\right]dx$

Example 10

16 Given that $y=e^{x^5}$, calculate $\frac{dy}{dx}$ and use this result to find $\int 2x^4e^{x^5}\,dx$.

Example 11

17 Find the anti-derivative of:

a $\frac{3}{8x}$ **b** $\frac{4}{5-2x}$

Example 11

18 Find:

a $\int \frac{4}{8x+1}\,dx$ **b** $\int (5x-3)^{-1}\,dx$ **c** $\int \frac{5x}{2x^2-7x}\,dx$

d $\int 2x(6x^2+5x)^{-1}\,dx$ **e** $\int \frac{3x^2-4x+9}{x}\,dx$

Example 12

19 If $\frac{dy}{dx}=\frac{16}{9-4x}$ and $y=-5$ when $x=1$, find an equation for y in terms of x.

Example 13

20 Calculate $\frac{d}{dx}\sqrt{x^2-8}$ and use the result to find $\int \frac{5x}{3\sqrt{x^2-8}}\,dx$.

Example 13

21 Calculate $\frac{d}{dx}(3x^4+1)^3$ and use the result to find $\int x^3(3x^4+1)^2\,dx$.

Example 14

22 Calculate:

a $\int 8x^3\sqrt{5x^4+1}\,dx$ **b** $\int \frac{7x^2}{3-4x^3}\,dx$ **c** $\int 6x^4\cos(x^5)\,dx$

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23 Calculate:

Example 15

a $\int (7x-3)^6\,dx$ **b** $\int (3x-4)^{-3}\,dx$

24 A particle moves in a straight line so that its velocity (m/s) at time t seconds is given by $v(t) = 3t^2 - 4t - 13$, $t \geq 0$. Initially, the particle is 10 metres to the left of a fixed origin.

Example 16

a After how many seconds is the particle at the origin?

b Calculate correct to one decimal place the particle's acceleration when its velocity is zero.

25 An object moves in a straight line so that its acceleration (m/s^2) at time t seconds is given by $a = 7 - 4t$ for $t \geq 0$. If the initial velocity is 12 m/s and $x = 59.5$ when $t = 3$, calculate the particle's velocity and position when $t = 2$.

Example 17

26 An object on top of a building 35 m above the ground is projected vertically upwards with a velocity of 32 m/s.

Example 18

a What is the velocity of the particle at 3 s?

b When (correct to 2 decimal places) does the particle come to rest for the first time?

c What (correct to 2 decimal places) is the greatest height reached by the object?

d What is the position of the particle at 4 s?

Problem solving

27 The point (2, 19) lies on the graph of the function $y = f(x)$. If $\frac{dy}{dx} = 8x^2 - 4x + 7$, find the equation of the function and its value at $x = -2$.

28 Calculate $\int \frac{\sin\left(\sqrt{x}\right)}{\sqrt{x}}\,dx$.

29 Use the fact that $\int \tan(x)\,dx = \ln|\sec(x)| + c$ and integration by recognition to find $\int xe^{x^2}\tan(e^{x^2})\,dx$.

30 A particle moves in a straight line and its velocity in m/s is given by

$$v(t) = 6t^2 - 18t + 12 \qquad \text{for } 0 \leq t \leq 3$$

where t is in seconds. Find the total distance travelled by a particle in the 3 seconds of its motion.

Practice quiz

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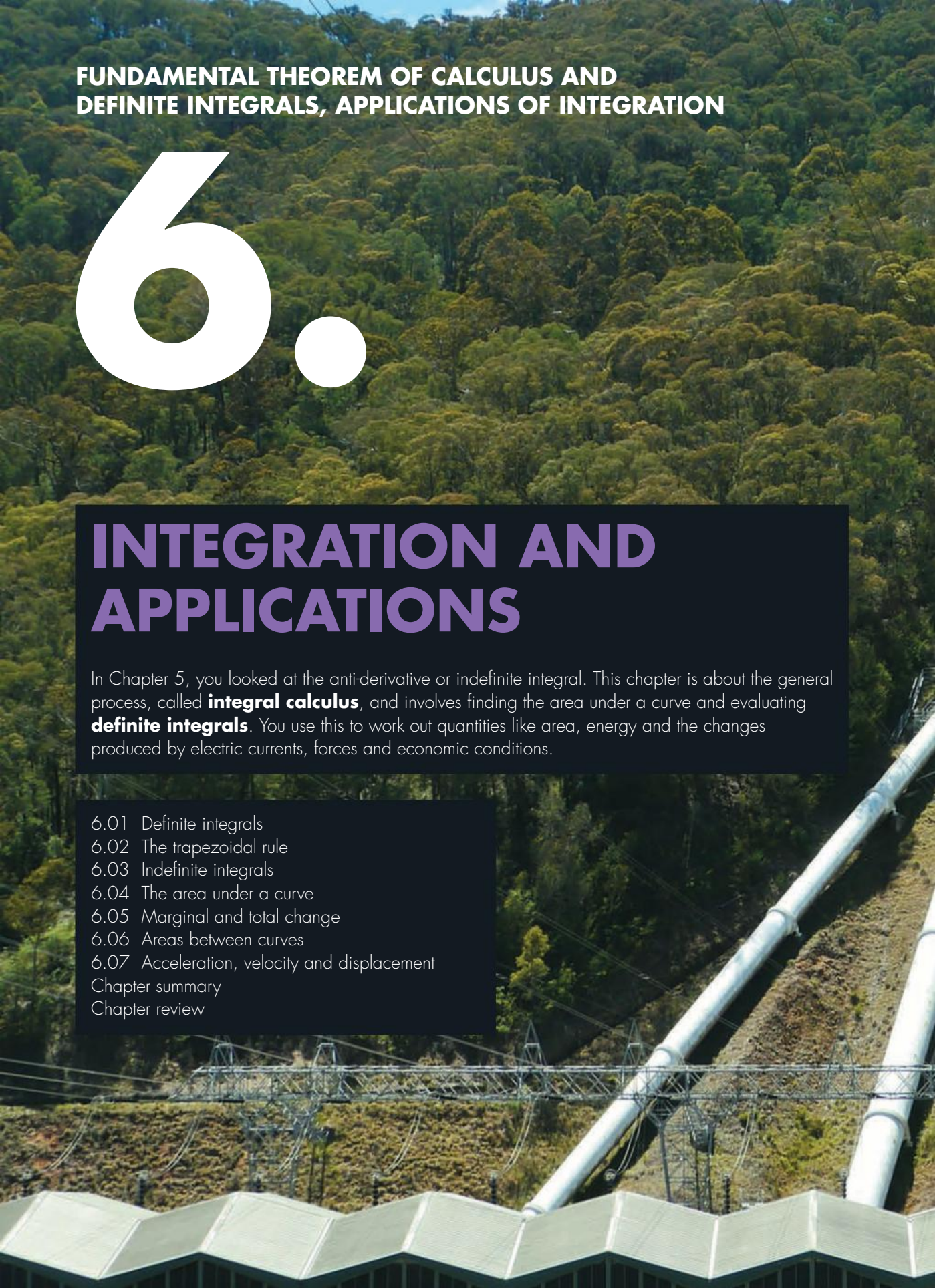


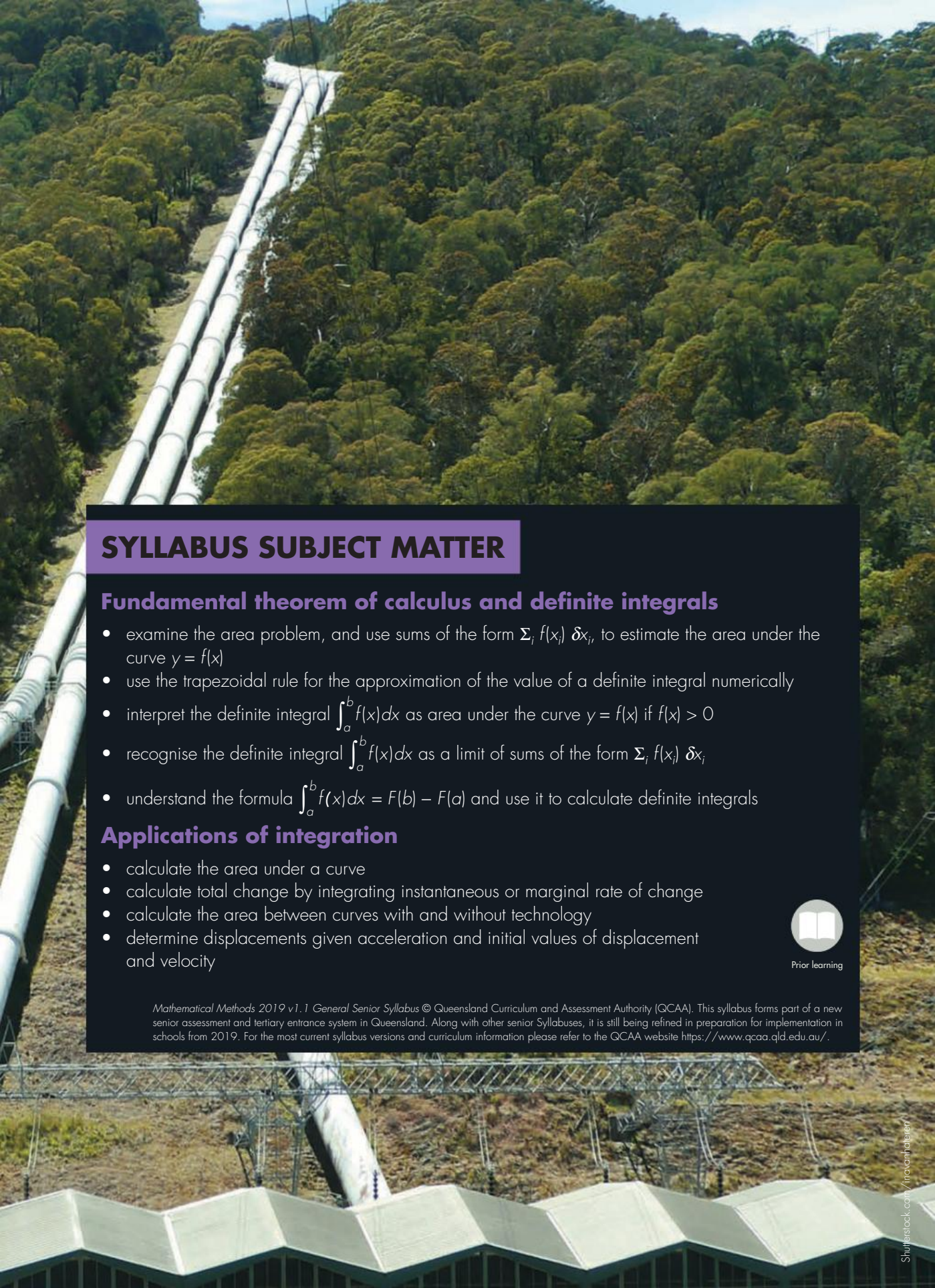
FUNDAMENTAL THEOREM OF CALCULUS AND DEFINITE INTEGRALS, APPLICATIONS OF INTEGRATION

6.

INTEGRATION AND APPLICATIONS

In Chapter 5, you looked at the anti-derivative or indefinite integral. This chapter is about the general process, called **integral calculus**, and involves finding the area under a curve and evaluating **definite integrals**. You use this to work out quantities like area, energy and the changes produced by electric currents, forces and economic conditions.

6.01 Definite integrals
6.02 The trapezoidal rule
6.03 Indefinite integrals
6.04 The area under a curve
6.05 Marginal and total change
6.06 Areas between curves
6.07 Acceleration, velocity and displacement
Chapter summary
Chapter review



SYLLABUS SUBJECT MATTER

Fundamental theorem of calculus and definite integrals

- examine the area problem, and use sums of the form $\Sigma_i f(x_i)\ \delta x_i$, to estimate the area under the curve $y = f(x)$
- use the trapezoidal rule for the approximation of the value of a definite integral numerically
- interpret the definite integral $\int_a^b f(x)dx$ as area under the curve $y = f(x)$ if $f(x) > 0$
- recognise the definite integral $\int_a^b f(x)dx$ as a limit of sums of the form $\Sigma_i f(x_i)\ \delta x_i$
- understand the formula $\int_a^b f(x)dx = F(b) - F(a)$ and use it to calculate definite integrals

Applications of integration

- calculate the area under a curve
- calculate total change by integrating instantaneous or marginal rate of change
- calculate the area between curves with and without technology
- determine displacements given acceleration and initial values of displacement and velocity

Prior learning

Mathematical Methods 2019 v1.1 General Senior Syllabus © Queensland Curriculum and Assessment Authority (QCAA). This syllabus forms part of a new senior assessment and tertiary entrance system in Queensland. Along with other senior Syllabuses, it is still being refined in preparation for implementation in schools from 2019. For the most current syllabus versions and curriculum information please refer to the QCAA website https://www.qcaa.qld.edu.au/.

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TERMINOLOGY

algebraic area
definite integral
difference function
fundamental theorem of calculus
indefinite integral
integral calculus
marginal change
marginal cost
marginal profit
marginal revenue
physical area
terminal velocity
total change
total cost
total profit
total revenue
trapezoidal rule

WS Areas using rectangles

WS Finding definite integrals

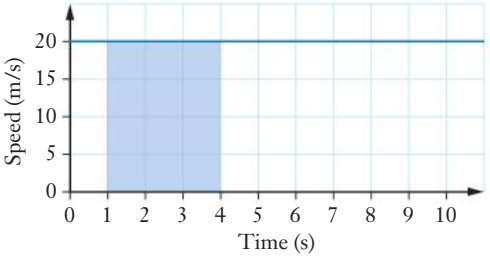
WS Definite integrals

6.01 Definite integrals

INVESTIGATION

SPEED AND DISTANCE

This speed–time graph represents a car travelling at 20 m/s (72 km/h).

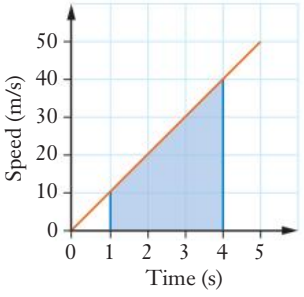


What is the shaded area under the graph between 1 and 4 seconds?

What distance does the car travel between 1 and 4 seconds?

What can you say about the distance and the area?

This speed–time graph represents a racing car whose speed increases steadily after the start.



What is its speed at $t = 1$?

What is its speed at $t = 4$?

What is the average speed?

What is the distance it travels between 1 and 4 seconds?

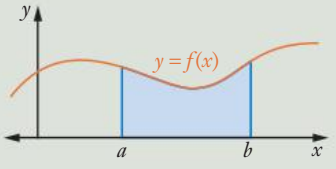
What is the area under the graph between 1 and 4 seconds?

What can you say about the relationship between the area under a speed–time graph and the distance travelled?

The area under a graph has a special name.

The definite integral

- The **definite integral** of the function $f(x)$ from $x = a$ to $x = b$ is the area enclosed by the curve $f(x)$, the lines $x = a$ and $x = b$, and the x-axis
- The symbol for the definite integral is $\int_a^b f(x)\,dx$
- The definite integral is a number



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You can find the definite integral for linear graphs using area formulas. For curved graphs, you need to use approximations. The simplest way to do this is to add the areas of thin rectangles under the curve.

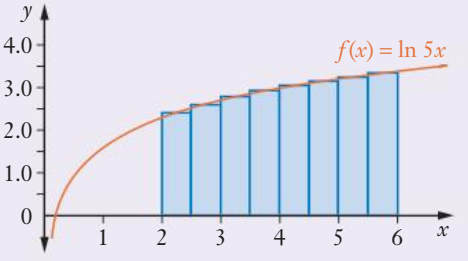
EXAMPLE 1

Find an approximate value for $\int_2^6 \ln(5x)\,dx$. Use strips 0.5 units wide and the values at the centres of the strips.

Solution

Sketch the function and strips.

Between $x = 2$ and $x = 6$, there are 8 strips.



State the centres of the strips.

The centres are at $x = 2.25, 2.75, 3.25, 3.75, 4.25, 4.75, 5.25$ and 5.75.

Calculate the values (heights) of each strip using $f(x) = \ln 5x$.

$f(2.25) \approx 2.42, f(2.75) \approx 2.62, f(3.25) \approx 2.79,$
$f(3.75) \approx 2.93, f(4.25) \approx 3.06,$
$f(4.75) \approx 3.17, f(5.25) \approx 3.27, f(5.75) \approx 3.36$

Calculate the total area of the strips using A = width × height.

$$\begin{aligned}\text{Total area} &= 0.5 \times 2.42 + 0.5 \times 2.62 + 0.5 \times 2.79 \\ &\quad + 0.5 \times 2.93 + 0.5 \times 3.06 \\ &\quad + 0.5 \times 3.17 + 0.5 \times 3.27 \\ &\quad + 0.5 \times 3.36 \\ &= 11.81\end{aligned}$$

Write the answer.

$\int_2^6 \ln(5x)\,dx \approx 11.81$

Approximations using values at the centres of strips are tedious to work out.

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Approximations of $\int_a^b f(x)\,dx$ using rectangles

- The x values at the edges of strips are usually shown as $a_0, a_1, a_2, a_3, \ldots, a_n$, for n strips of width $w = \dfrac{b-a}{n}$, $a_0 = a$ and $a_n = b$
- The x values in the centres of strips are usually shown as $a_{\frac{1}{2}}, a_{1\frac{1}{2}}, a_{2\frac{1}{2}}, \ldots, a_{n-\frac{1}{2}}$
- For **'left' rectangles** using start-values: $A \approx w[f(a_0) + f(a_1) + f(a_2) + \ldots + f(a_{n-1})]$
- For **'right' rectangles** using end-values: $A \approx w[f(a_1) + f(a_2) + f(a_3) + \ldots + f(a_n)]$
- For **'centred' rectangles** using centre-values:
  $A \approx w[f(a_{\frac{1}{2}}) + f(a_{1\frac{1}{2}}) + f(a_{2\frac{1}{2}}) + \ldots + f(a_{n-\frac{1}{2}})]$ ← Example 1 used centred rectangles.

The sums shown above are often written using Σ for 'the sum of'. The left approximation is shown as $w\sum_{i=0}^{n-1} f(a_i)$.

EXAMPLE 2

Find an approximate value for $\int_2^6 \ln(5x)\,dx$ using

a 8 left rectangles **b** 8 right rectangles

Solution

a Find the width. $w = \dfrac{b-a}{n} = \dfrac{6-2}{8} = 0.5$

Write the values of $a_0, a_1, \ldots, a_8$. $a_0 = 2, a_1 = 2.5, a_2 = 3, \ldots, a_7 = 5.5, a_8 = 6$

Write the values of the function. $f(2) \approx 2.30, f(2.5) \approx 2.53, f(3) \approx 2.71,$
$f(3.5) \approx 2.86, f(4) \approx 3.00, f(4.5) \approx 3.11,$
$f(5) \approx 3.22, f(5.5) \approx 3.31, f(6) \approx 3.40$

Find the sum. $w\sum_{i=0}^{7} f(a_i) = 0.5(2.30 + 2.53 + 2.71 + 2.86 + 3.00 + 3.11 + 3.22 + 3.31)$
$= 11.52$

Write the answer. $\int_2^6 \ln(5x)\,dx \approx 11.52$

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b Find the sum.

$$w\sum_{i=1}^{8} f(a_i) = 0.5(2.53 + 2.71 + 2.86 + 3.00 + 3.11 + 3.22 + 3.31 + 3.40)$$
$$= 12.07$$

Write the answer.

$$\int_2^6 \ln(5x)\,dx \approx 12.07$$

The approximations are different because different methods have been used. What do you think would happen to the approximations as the number of strips increases?

Your graphics calculator uses a large number of strips to approximate definite integrals. It also uses a more complicated method.

EXAMPLE 3

Use a graphics calculator to find the approximate value of $\int_{0.7}^{2.4} (x^3 + x + 3)\,dx$.

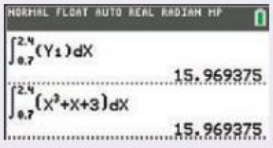
Solution

TI-84 Plus CE

Use [y=] to type the function in Y_1.

Quit y=, press [math], choose 9: fnInt(and complete the integral. Press [vars], choose Y-Vars, 1: Function and 1: Y_1 for the function.

You could also type the function instead of entering it in Y_1.



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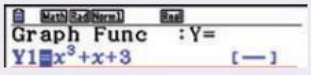
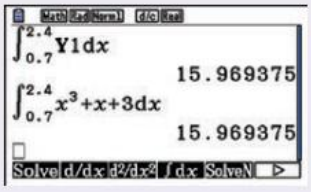
Choose the Graph menu and type the function in Y1.



Press [MENU] and choose **Run-Matrix**.

Press [OPTN], choose **CALC**, **∫dx** and complete the integral. Press [VARS], choose **GRAPH**, Y and type 1 for the function.

You could also type the function instead of entering it in Y_1.



TI-Inspire CX
Chapter 6

Exercise 6.01 Definite integrals

Example 1

1 Use centre function values with strips 0.4 units wide to find approximations for each integral.

a $\int_1^3 (3x^2 - 2x + 5)\,dx$ **b** $\int_1^5 (2x^2 + x + 3)\,dx$ **c** $\int_{1.2}^4 2^{3x}\,dx$

Example 2

2 Use left rectangles 0.4 units wide to find approximations for each integral.

a $\int_2^6 \frac{x-2}{x+1}\,dx$ **b** $\int_0^{0.8} \sin(3x)\,dx$ **c** $\int_{1.2}^4 \log(x)\,dx$

3 Use right rectangles 0.3 units wide to find approximations for each integral.

a $\int_{1.2}^{2.7} 3x^2\,dx$ **b** $\int_{1.8}^{4.2} \cos^2(x)\,dx$ **c** $\int_0^3 \sqrt{x+2}\,dx$

Example 3

4 Use your graphics calculator to find approximations for each integral.

a $\int_0^5 3e^{(x^2+1)}\,dx$ **b** $\int_0^5 (x^3 + x^2 + x + 1)\,dx$

c $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 3\tan(x)\,dx$ **d** $\int_0^{\pi} \left[\sin^2(x) + \cos^2(x)\right] dx$

e $\int_{10}^{100} \log_{10}(x)\,dx$

Problem solving

5 Use a graphics calculator to find the values of $\int_{-1}^{0} (3 + 2x - x^2)\,dx$, $\int_0^3 (3 + 2x - x^2)\,dx$ and $\int_{-1}^{3} (3 + 2x - x^2)\,dx$. What do you find?

6 Try using a graphics calculator to find the value of $\int_4^6 \frac{1}{x-2}\,dx$ and $\int_1^4 \frac{1}{x-2}\,dx$.

Explain why the first integral gives a value, but the second integral gives an error.

7 At the normal cruising height of passenger planes (8–12 km), the atmospheric pressure and density are too low for breathing. The atmospheric density at a height h m above sea level is given by

$$d \approx 1.2e^{-0.00012h} \text{ kg/m}^3$$

Nearly all air in the atmosphere is contained within the first 20 km. Use your graphics calculator to estimate the mass of air over 1 m^2 of the earth by finding the definite integral of d from 0 to 20 000 m.

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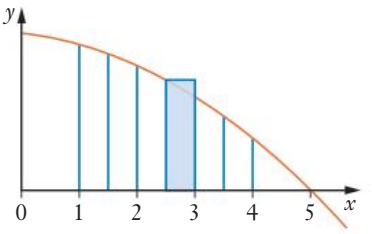
6.02 The trapezoidal rule

Divide the area for $\int_1^4 (30 - x - x^2)\,dx$ into 6 strips of width 0.5 units, as shown on the right.

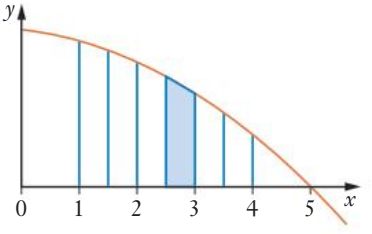


If you use left-rectangles to approximate the area, the 4th rectangle has height $f(2.5) = 21.25$, so its area is $0.5 \times 21.25 = 10.625$.

The approximation using left rectangles gives

$\int_1^4 (30 - x - x^2)\,dx \approx 65.875$.

You can improve the approximation using sloping tops to make trapeziums. The 4th trapezium is shown on the right.



The heights of each side are $f(2.5) = 21.25$ and $f(3) = 18$.

Using $A = \frac{w(a+b)}{2}$ gives an area of

$0.5 \times \frac{21.25 + 18}{2} = 9.8125$.

You can do all the strips with trapeziums to get $0.5 \times \frac{28 + 26.25}{2}$, $0.5 \times \frac{26.25 + 24}{2}$, $0.5 \times \frac{24 + 21.25}{2}$, $0.5 \times \frac{21.25 + 18}{2}$, $0.5 \times \frac{18 + 14.25}{2}$, $0.5 \times \frac{14.25 + 10}{2}$ with a total of 61.375.

You use all the heights twice (except for the ends), so you can simplify the total to

$\frac{0.5}{2} \times (28 + 2 \times 26.25 + 2 \times 24 + 2 \times 21.25 + 2 \times 18 + 2 \times 14.25 + 10)$.

This makes an easy formula for the approximation using trapeziums.

The trapezoidal rule for $\int_a^b f(x)\,dx$

$$\int_a^b f(x)\,dx \approx \tfrac{1}{2} w[f(a_0) + 2f(a_1) + 2f(a_2) + \ldots + 2f(a_{n-1}) + f(a_n)]$$

$$= \tfrac{1}{2} w[E + 2M]$$

where the interval $[a, b]$ is divided into n equal strips of width w by $a = a_0, a_1, a_2, \ldots, a_n = b$.

E is the sum of the end values of the function and M is the sum of the middle values.

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You can derive the rule using the same method as shown for $\int_1^4 (30 - x - x^2)\,dx$, but you use $f(a_0), f(a_1), f(a_2), \ldots$ instead of 28, 26.5, 24, … and w instead of 0.5.

EXAMPLE 4

Use the trapezoidal rule with 10 strips to find an approximation for $\int_4^6 (x^2 - x + 5)\,dx$.

Solution

Find the width of each strip.

$$w = \frac{6-4}{10} = 0.2$$

Write the points.

$$a_0 = 4, a_1 = 4.2, a_2 = 4.4, a_3 = 4.6, \ldots,$$
$$a_9 = 5.8, a_{10} = 6$$

Find the values.

$$f(4) = 17,\ f(4.2) = 18.44,\ f(4.4) = 19.96,$$
$$f(4.6) = 21.26,\ f(4.8) = 23.24,\ f(5) = 25,$$
$$f(5.2) = 26.84, f(5.4) = 28.76,\ f(5.6) = 30.76,$$
$$f(5.8) = 32.84, f(6) = 35$$

Find the sums.

$$E = 17 + 35 = 52$$
$$M = 18.44 + 19.96 + 21.56 + 23.24 + 25$$
$$+ 26.84 + 28.76 + 30.76 + 32.84$$
$$= 227.4$$

Write the formula.

$$\int_a^b f(x)\,dx \approx \frac{1}{2} w[E + 2M]$$

Substitute and evaluate.

$$= \tfrac{1}{2} \times 0.2(52 + 2 \times 227.4)$$
$$= 50.68$$

Write the answer.

$$\int_4^6 (x^2 - x + 5)\,dx \approx 50.68$$

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TECHNOLOGY

Graphics calculator tables

You can use your graphics calculator to find the function values. The values for Example 4 are shown below.

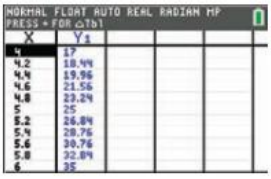
TI-84 Plus CE

Press [y=] and enter the function in Y_1.

Use tblset ([2nd][window]) to set up the x values.

Use table ([2nd][graph]) to view the table.

Use the cursor controls to move up and down the table.



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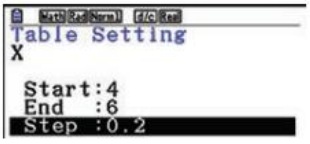
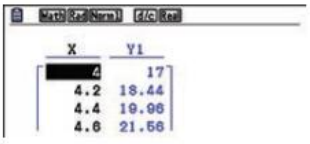
Use the table menu to enter the function in Y_1.

Choose SET to set up the x values.



Press [EXIT] and choose TABLE to see the values.

Use the cursor controls to move up and down the table.



TI-Inspire CX Chapter 6

Exercise 6.02 The trapezoidal rule

Example 4

1 Use the trapezoidal rule to find an approximation for $\int_2^7 (3x^2 - 5x + 2)\,dx$ using:

a 5 strips **b** 10 strips.

2 Use the trapezoidal rule to find an approximation for $\int_3^5 (x^3 + 2x^2 - 8x - 3)\,dx$ using:

a 5 strips **b** 10 strips.

3 Use the trapezoidal rule to find an approximation for $\int_2^4 \frac{dx}{x+1}\,dx$ using:

a 4 strips **b** 8 strips.

4 Use the trapezoidal rule to find an approximation for $\int_5^8 \frac{x-1}{x-3}\,dx$ using:

a 6 strips **b** 12 strips.

5 Use the trapezoidal rule to find an approximation for $\int_0^4 2^x\,dx$ using:

a 5 strips **b** 10 strips.

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6 Use the trapezoidal rule to find an approximation for $\int_{-1}^{1} \frac{dx}{x^2+2}$ using:

a 4 strips **b** 8 strips.

7 Use the trapezoidal rule to find an approximation for $\int_{0}^{1} \frac{dx}{\sqrt{1+x^2}}$ using:

a 5 strips **b** 10 strips.

8 Use the trapezoidal rule to find an approximation for $\int_{3}^{5} \frac{dx}{x^2-5}$ using:

a 5 strips **b** 20 strips.

Problem solving

9 Use 5, 10, and 20 strips with the trapezoidal rule to find approximations for $\int_{5}^{10} x^2\,dx$. Describe what happens to the approximation as the number of strips is increased.

10 Derive the trapezoidal rule.

6.03 Indefinite integrals

You have estimated the values of definite integrals using points at the ends or in the middle of equal-width strips. As the number of strips increases, the values of the sums get closer together. They approach a limit. The choice of the point in the interval doesn't change the limit, so you can choose any point you like.

Very small widths are shown as δx instead of w.

You can write the limit of the sums of areas of strips as $\lim_{\delta x \to 0} \sum_{i=1}^{n} f(x_i)\delta x$.

The value of n increases as the width δx decreases. The values of x_i lie between x_{i-1} and x_i.

If it exists, this limit is the definite integral. You should remember that the anti-derivative of a function is also called the **indefinite integral** (or **primitive**) of the function. They are both called integrals because they are closely related.

The fundamental theorem of calculus

- An **indefinite integral** of $f(x)$ is written as $F(x) = \int f(x)\,dx$, where $F'(x) = f(x)$
- The fundamental theorem of calculus states that the **definite integral** of $f(x)$ from a to b is given by $\int_a^b f(x)\,dx = F(b) - F(a)$
- The indefinite integral is a **function**, while the definite integral is a **number**
- Calculations are normally set out as: $\int_a^b f(x)\,dx = \left[F(x)\right]_a^b = F(b) - F(a)$

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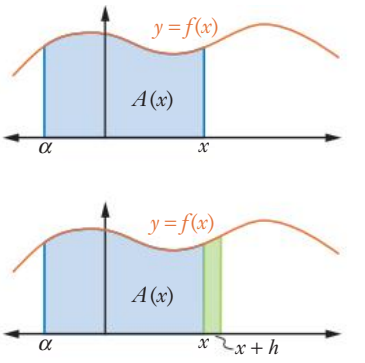
You can demonstrate the fundamental theorem using a function $A(x)$ defined as the area under the curve $f(x)$ from, say, α to x.

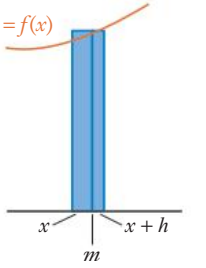
It follows that $\int_a^b f(x)\,dx = A(b) - A(a)$.

Now $A(x+h) - A(x)$ is the area of the strip between x and $x+h$, as shown on the right.



Choose a value of x, say m, between x and $x+h$ so that the area of the rectangle of height $f(m)$ and width h has the same area as the strip.

This detail is illustrated on the far right.



The area is given by both $A(x+h) - A(x)$ and $f(m) \times h$, so

$$A(x+h) - A(x) = f(m) \times h.$$

From the definition of the derivative,

$$\begin{aligned} A'(x) &= \lim_{h \to 0} \frac{A(x+h) - A(x)}{h} \\ &= \lim_{h \to 0} \frac{f(m) \times h}{h} \\ &= \lim_{h \to 0} f(m) \\ &= f(x), \text{ as } m \text{ is between } x \text{ and } x+h. \end{aligned}$$

This shows that $A(x)$ is an anti-derivative of $f(x)$.

Anti-derivatives differ from each other only by a constant, so $A(x) = F(x) + c$, where c is real.

Thus $\int_a^b f(x)\,dx = A(b) - A(a)$

$$\begin{aligned} &= [F(b) + c] - [F(a) + c] \\ &= F(b) + c - F(a) - c \\ &= F(b) - F(a) \qquad \text{QED} \end{aligned}$$

You can use any anti-derivative of $f(x)$ to calculate the definite integral from a to b. The easiest one to use is the one with a constant term of $c = 0$.

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EXAMPLE 5

Find the definite integral of $f(x) = 6x^2 - 10x + 3$ between 4 and 7.

Solution

Write in symbols.

$$\int_4^7 (6x^2 - 10x + 3)\,dx$$

Find the indefinite integral.

$$= \left[2x^3 - 5x^2 + 3x\right]_4^7$$

Substitute values and simplify.

$$= (2 \times 7^3 - 5 \times 7^2 + 3 \times 7) - (2 \times 4^3 - 5 \times 4^2 + 3 \times 4)$$

$$= 462 - 60$$

$$= 402$$

Write the answer.

The definite integral of $f(x) = 6x^2 - 10x + 3$ between 4 and 7 is 402.

You should already be familiar with the following derivative rules.

Derivatives of common functions

$$\frac{d}{dx}(x^n) = nx^{n-1} \qquad \frac{d}{dx}(kx^n) = knx^{n-1} \quad (n \neq 0)$$

$$\frac{d}{dx}[\sin(x)] = \cos(x) \qquad \frac{d}{dx}[\sin(kx)] = k\cos(kx)$$

$$\frac{d}{dx}[\cos(x)] = -\sin(x) \qquad \frac{d}{dx}[\cos(kx)] = -k\sin(kx)$$

$$\frac{d}{dx}(e^x) = e^x \qquad \frac{d}{dx}[\ln(x)] = \frac{1}{x} \quad (x > 0)$$

$$\frac{d}{dx}[\log_a(x)] = \frac{1}{x\ln(a)} \quad (a, x > 0) \qquad \frac{d}{dx}(a^x) = a^x \ln(a) \quad (a > 0)$$

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Reversing these rules gives the following rules for indefinite integrals.

Indefinite integrals of common functions

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c \quad (n \neq -1)$$

$$\int kx^n \, dx = \frac{kx^{n+1}}{n+1} + c \quad (n \neq -1)$$

$$\int \sin(x) \, dx = -\cos(x) + c$$

$$\int \sin(kx) \, dx = -\frac{1}{k}\cos(kx) + c$$

$$\int \cos(x) \, dx = \sin(x) + c$$

$$\int \cos(kx) \, dx = \frac{1}{k}\sin(kx) + c$$

$$\int e^x \, dx = e^x + c$$

$$\int \frac{1}{x} dx = \ln(x) + c \quad (x > 0)$$

$$\int a^x \, dx = \frac{a^x}{\ln(a)} + c \quad (a > 0)$$

To find integrals of more complex functions, use a guess-and-check approach. In many cases, you need only multiply or divide one of the above by a constant.

EXAMPLE 6

Find $\int 12e^{2x-5} \, dx$.

Solution

Try the derivative of e^{2x-5}.

$$\frac{d}{dx}(e^{2x-5}) = e^{2x-5} \times 2$$
$$= 2e^{2x-5}$$

Use the chain rule.

Multiply by the appropriate constant.

$$\frac{d}{dx}(6e^{2x-5}) = 6 \times 2e^{2x-5}$$
$$= 12e^{2x-5}$$

Reverse for the required integral.

$$\int 12e^{2x-5} \, dx = 6e^{2x-5} + c$$

Use the same methods for definite integrals.

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EXAMPLE 7

Find each definite integral.

a $\int_{0.1}^{3} \frac{dx}{x^2}$ **b** $\int_{0}^{\frac{\pi}{2}} \sin(2x)\,dx$ **c** $\int_{-1.5}^{1.5} 6e^{3x}\,dx$

d $\int_{0}^{4} \frac{5}{x+2}\,dx$ **e** $\int_{1}^{4} \frac{dx}{x-3}$

Solution

a Write in index form.

$$\int_{0.1}^{3} \frac{dx}{x^2} = \int_{0.1}^{3} x^{-2}\,dx$$

Write the indefinite integral.

$$= \left[-x^{-1}\right]_{0.1}^{3}$$

Evaluate at each limit and simplify.

$$= -3^{-1} - (-0.1^{-1})$$
$$= -\tfrac{1}{3} + 10$$
$$= 9\tfrac{2}{3}$$

b Write the indefinite integral.

$$\int_{0}^{\frac{\pi}{2}} \sin(2x)\,dx = \left[-\tfrac{1}{2}\cos(2x)\right]_{0}^{\frac{\pi}{2}}$$

Evaluate at each limit and simplify.

$$= (-\tfrac{1}{2} \times -1) - (-\tfrac{1}{2} \times 1)$$
$$= \tfrac{1}{2} + \tfrac{1}{2}$$
$$= 1$$

c Write the indefinite integral.

$$\int_{-1.5}^{1.5} 6e^{3x}\,dx = \left[2e^{3x}\right]_{-1.5}^{1.5}$$

Evaluate at each limit and simplify.

$$= 2e^{4.5} - 2e^{-4.5}$$
$$= 2(e^{4.5} - e^{-4.5})$$
$$\approx 180.01$$

d Write the indefinite integral.

$$\int_{0}^{4} \frac{5}{x+2}\,dx = \left[5 \ln (x+2)\right]_{0}^{4}$$

Evaluate at each limit and simplify.

$$= 5 \ln (6) - 5 \ln (2)$$
$$= 5[\ln (6) - \ln (2)]$$
$$= 5 \ln (3)$$

Remember to write an exact answer unless you are asked for an approximation.

e Write the indefinite integral.

$$\int_{1}^{4} \frac{dx}{x-3} = \left[\ln (x-3)\right]_{1}^{4}$$

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Evaluate at each limit.

$= \ln(1) - \ln(-2)$

Use the definition of logarithms.

But $\ln(-2)$ is not defined, so the integral does not exist.

The integral does not exist because the function is undefined for $x = 3$, and the integral runs from 1 to 4.

Exercise 6.03 Indefinite integrals

1 Find each definite integral.

Example 5

a $\int_1^{10} (9x+7)\,dx$ **b** $\int_0^6 8\,dx$ **c** $\int_2^{10} 5x^3\,dx$

d $\int_{-3}^{3} x^6\,dx$ **e** $\int_0^8 6x^3\,dx$ **f** $\int_5^{10} 4x^{-2}\,dx$

g $\int_{-5}^{0} (2x^2 - x)\,dx$ **h** $\int_{-12}^{12} (20 - m)\,dm$

2 Find each indefinite integral.

Example 6

a $\int (x^3 - 7x - 3)\,dx$ **b** $\int \cos(4x)\,dx$ **c** $\int 24e^{0.3x}\,dx$

d $\int \frac{1}{4x}\,dx$ **e** $\int \frac{1}{5x-2}\,dx$ **f** $\int \frac{dx}{10-7x}$

g $\int 6(x^2+1)e^{x^3+3x}\,dx$ **h** $\int \left(x + \frac{1}{2x+9}\right)dx$ **i** $\int 40t\sin\left(t^2\right)dt$

j $\int e^{\cos(x)}\sin(x)\,dx$ **k** $\int 5^{2x}\,dx$ **l** $\int e^{4-5x}\,dx$

3 Find each definite integral.

Example 7

a $\int_{2.4}^{5.8} \frac{3}{x^4}\,dx$ **b** $\int_1^4 e^{3V}\,dV$ **c** $\int_0^{\pi} \sin(x)\,dx$

d $\int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} \cos(2x)\,dx$ **e** $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} e^{3x}\,dx$ **f** $\int_1^3 10^x\,dx$

g $\int_{-1}^{2} 6e^{3t}\,dt$ **h** $\int_3^8 e^{5n}\,dn$ **i** $\int_2^5 \frac{1}{p}\,dp$

j $\int_0^5 \frac{1}{r+1}\,dr$ **k** $\int_{-1}^5 \frac{6}{3g+4}\,dg$ **l** $\int_1^{10} \left(4x + 3 - \frac{2}{x}\right)dx$

Problem solving

4 Why can't you calculate each integral?

a $\int_0^3 \frac{1}{x^2}\,dx$ **b** $\int_0^5 \frac{1}{(x-5)^2}\,dx$ **c** $\int_{-1}^3 \frac{1}{(x+1)^3}\,dx$

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5 Explain why the value you get for the integral $\int_0^4 \frac{1}{(x-2)^2}\,dx$ is not valid.

6 What is wrong with $\int_{-2}^{2} \frac{1}{x}\,dx$?

7 During a flood, the flow rate of a small creek under a road bridge increased from its normal level of 5 m^3/h to a peak after 5 hours according to the equation

$$f = 5 + 30t^2,$$

where t is the time after the increase began. Find the amount of water that flowed under the bridge in the 5 hours before it reached its peak.

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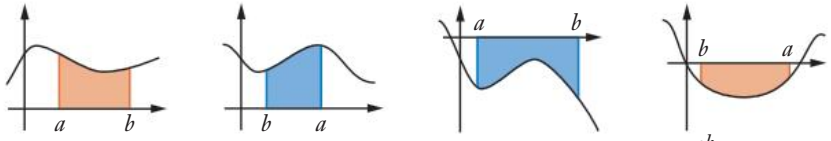
WS
Calculating physical areas

6.04 The area under a curve

For definite integrals of the form $\int_b^a f(x)\,dx$, the sums $\sum_{i=1}^{n} f(x_i)\delta x$ will be negative for intervals where $f(x)$ is negative. In cases where $b < a$, the width is backwards, so the sign is also changed. An area with a sign is called an **algebraic area** to distinguish it from a normal geometric area. The 4 cases for signs of algebraic areas are illustrated below.



$\int_a^b f(x)\,dx > 0$ $\quad$ $\int_a^b f(x)\,dx < 0$ $\quad$ $\int_a^b f(x)\,dx < 0$ $\quad$ $\int_a^b f(x)\,dx > 0$

You can show the following properties of integrals using definite integral calculations or by reversing derivatives.

Properties of integrals

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx \qquad \int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

$$\int kf(x)\,dx = k\int f(x)\,dx \qquad \int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int [f(x) - g(x)]\,dx = \int f(x)\,dx - \int g(x)\,dx$$

$$\int f(ax+b)\,dx = \frac{1}{a} \times F(ax+b) + c, \quad \text{where } F(x) = \int f(x)\,dx$$

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Area under a curve

EXAMPLE 8

Find the area under the graph of $f(x) = x^2 + 3x + 5$ between $x = 3$ and $x = 7$.

Solution

Use the integral.

$$\text{Area} = \int_3^7 (x^2 + 3x + 5)\,dx$$

Write the indefinite integral.

$$= \left[\frac{x^3}{3} + \frac{3x^2}{2} + 5x\right]_3^7$$

Substitute limits and simplify.

$$= \left(\frac{7^3}{3} + \frac{3\times 7^2}{2} + 5\times 7\right) - \left(\frac{3^3}{3} + \frac{3\times 3^2}{2} + 5\times 3\right)$$

$$= 222\frac{5}{6} - 37\frac{1}{2}$$

$$= 185\frac{1}{3} \text{ square units}$$

For $f(x) = x^3$, $f(-2) = -f(2)$, $f(-5) = -f(5)$, and so on.

This makes the **algebraic area** of $\int_{-3}^{-1} x^3\,dx < 0$, $\int_2^5 x^3\,dx > 0$ and $\int_{-4}^{4} x^3\,dx = 0$.

You need to separate a **physical area** into negative and positive parts to work it out. This means you should sketch the graph of these questions. If technology is available, you should use it to at least check your answer.

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EXAMPLE 9

Find the area enclosed by $y = x^2 - 3x - 4$, $x = 2$, $x = 5$ and $y = 0$.

Solution

$x = 2$ and $x = 5$ are vertical lines, and $y = 0$ is the x-axis.

The question is asking for physical area.

Factorise to find the zeros.

$y = x^2 - 3x - 4$

$= (x + 1)(x - 4)$

The zeros are at $x = -1$ and $x = 4$.

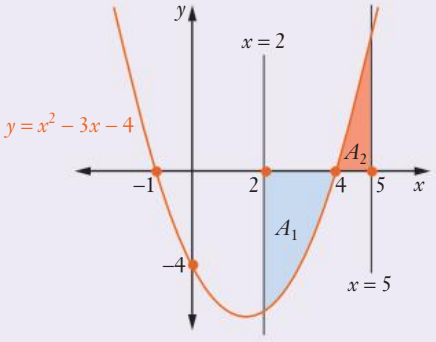
Find the y-intercept.

$y(0) = -4$

State the nature of the curve.

The function is a quadratic with a minimum.

Sketch the graph and shade the required area. Label the parts as A_1 and A_2.



Find the algebraic areas.

$$A_1 = \int_2^4 (x^2 - 3x - 4)\,dx$$

$$= \left[\frac{x^3}{3} - \frac{3x^2}{2} - 4x\right]_2^4$$

$$= -18\frac{2}{3} - \left(-11\frac{1}{3}\right)$$

$$= -7\frac{1}{3}$$

$$A_2 = \int_4^5 (x^2 - 3x - 4)\,dx$$

$$= \left[\frac{x^3}{3} - \frac{3x^2}{2} - 4x\right]_4^5$$

$$= -15\frac{5}{6} - \left(-18\frac{2}{3}\right)$$

$$= 2\frac{5}{6}$$

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Find the physical area.

The physical area $= 7\frac{1}{3} + 2\frac{5}{6} = 10\frac{1}{6}$

Write the answer.

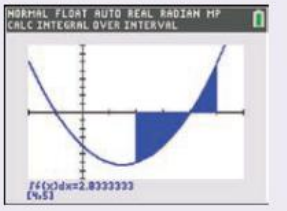
The area enclosed is $10\frac{1}{6}$ square units.

TI-84 Plus CE

Press [y=] and type the function into Y_1. Set the window to $-2 \le x \le 6$ and $-8 \le y \le 8$ and graph the function.

Use calc and 7: $\int f(x)dx$ to find the approximate areas from 2 to 4 and from 4 to 5.

Ignore signs and add the areas.



Total area $\approx 7.333 + 2.833$

$= 10.166$

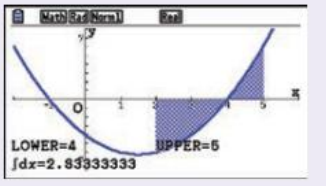
Casio fx-CG20AU

Use the Graph menu and type the function into Y1.

Set the V-Window to $-2 \le x \le 6$ and $-8 \le y \le 8$ and graph the function.

Use G-Solve and $\int dx$ to find the approximate areas from 2 to 4 and from 4 to 5.

Ignore signs and add the areas.



Total area $\approx 7.333 + 2.833$

$= 10.166$

Make sure you give square units for the area.

You can subtract the negative algebraic areas to find the physical area. If you use a graphics calculator, you need to find the areas separately.

EXAMPLE 10

Find the physical area cut off by the x-axis for the curve $f(x) = (x - 2)(x + 1)^2(x - 4)$.

Solution

The areas will be between the zeros.

The zeros are at 2, 4 and –1 (double).

State the sign changes.

$f(x)$ is positive before and after $x = -1$

$f(x)$ is positive before and negative after $x = 2$

$f(x)$ is negative before and positive after $x = 4$

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State the y-intercept.

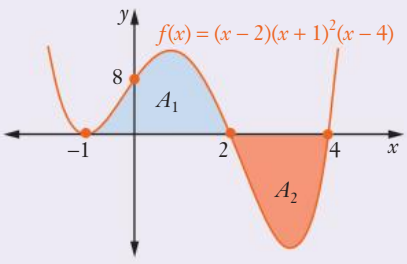
$f(0) = 8$

State the behaviour as $x \to \pm\infty$

As $x \to -\infty, f(x) \to \infty$

As $x \to \infty, f(x) \to \infty$

Sketch the graph and shade the required area. Label the parts as A_1 and A_2.



Expand the brackets to make integration possible.

$$\begin{aligned} f(x) &= (x-2)(x+1)^2(x-4) \\ &= (x^2-6x+8)(x^2+2x+1) \\ &= x^4+2x^3+x^2-6x^3-12x^2-6x+8x^2 \\ &\quad +16x+8 \\ &= x^4-4x^3-3x^2+10x+8 \end{aligned}$$

Subtract the negative algebraic area to find the physical area.

$$\begin{aligned} \text{Physical area} &= A_1 - A_2 \\ &= \int_{-1}^{2}(x^4-4x^3-3x^2+10x+8)\,dx \\ &\quad -\int_{2}^{4}(x^4-4x^3-3x^2+10x+8)\,dx \end{aligned}$$

Integrate.

$$\begin{aligned} &= \left[\frac{x^5}{5}-x^4-x^3+5x^2+8x\right]_{-1}^{2} \\ &\quad -\left[\frac{x^5}{5}-x^4-x^3+5x^2+8x\right]_{2}^{4} \\ &= \left[18\tfrac{2}{5}-\left(-3\tfrac{1}{5}\right)\right]-\left[-3\tfrac{1}{5}-18\tfrac{2}{5}\right] \\ &= 43\tfrac{1}{5} \end{aligned}$$

Write the answer.

The physical area cut off by the x-axis for the curve $f(x)$ is $43\frac{1}{5}$ square units.

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Exercise 6.04 The area under a curve

1 Find the area under each curve. Example 8

a $y = x^2 - 5x + 8$ from $x = 1$ to $x = 4$

b $f(x) = 15 + 8x - 6x^2$ between $x = -1$ and $x = 2$

c $y = 4x^3 - 3x^2 + 6x - 2$ between $x = 1$ and $x = 3$

d $f(x) = 6x^3 + 8x^2 - 2x + 8$ from $x = -1$ to $x = 2$

e $y = 7 - 6x^3$ from $x = -2$ to $x = 1$

2 Find the area enclosed by $y = 7x^2 + 19x - 6$, $x = -5$, $x = -2$ and the x-axis. Example 9

3 Use your graphics calculator to find an approximation for the area cut off:

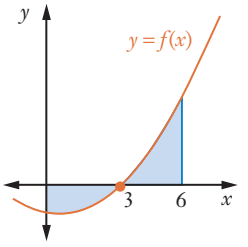
a $y = 6x - 4 - x^2$ by the x-axis

b $y = 5x^2 - x^3 - 2x - 8$ by the x-axis

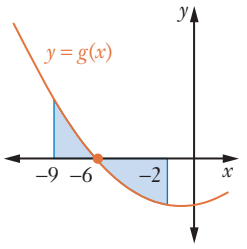
c $y = x^3 + 3x^2 - 10x - 3$ by the x-axis.

4 For each graph, state the integral used to calculate the shaded physical area.

a



b



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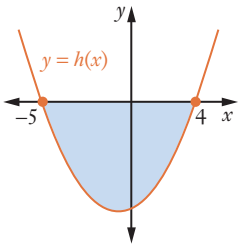

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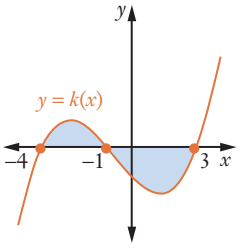
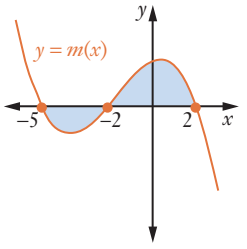
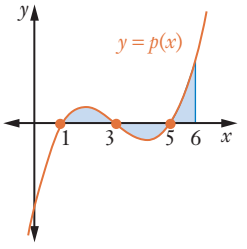


e



f



5 Find the physical area cut off by the x-axis for $y = (5 - x)(x + 1)(x - 3)$. Example 10

6 Find the area enclosed by $f(x) = 3x^2 - 12$ and the x-axis.

7 Find the physical area cut off by the x-axis for $f(x) = (x + 2)^2 (x - 1)(x + 4)$.

8 Find the physical area enclosed by $f(x) = 2x^2 + 3x - 35$, $x = 2$, $x = 5$ and $y = 0$.

9 Find the area enclosed by $y = 3x^2 + 7x - 6$ and the x-axis.

10 Find the area enclosed by $f(x) = 4x^2 - 16x + 15$ and the x-axis.

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Problem solving

11 Find the area under the curve $y = (2x+1)^{-\frac{1}{3}}$ from $x = 0$ to $x = 13$.

12 Find the area enclosed by $x = 2$, $x = -2$, the x-axis and the curve $f(x) = e^x - e^{-x}$.

13 Find the area enclosed by $y = \sin(3x)$ and the x-axis between $x = 0$ and $x = 2\pi$.

14 Find the area enclosed by $y = \dfrac{1}{x+4}$, $y = 0$, $x = -1$ and $x = 4$.

15 Find the area enclosed by $f(x) = e^{2x} - 4$, $y = 0$, $x = -1$ and $x = 2$.

16 Find the area enclosed by $y = e^{2x} + e^{-2x}$, $y = 0$, $x = -1.5$ and $x = 1.5$.

WS
Total change

6.05 Marginal and total change

The way that costs, revenue and profits change is very important in economics. There are often **economies of scale** in making large numbers of items. This means that the cost per item often decreases with the number of items produced. The cost per item is called the **marginal cost**.

Marginal and total change

- The **total change** in a quantity q for x items is usually given as a function $q(x)$
- The **marginal change** in the quantity is the cost of producing one more item
- For a quantity $q(x)$, the marginal change is $q(x+1) - q(x) \approx \dfrac{dq}{dx} = q'(x)$
- For a **total cost** function $C(x)$, the **marginal cost** is $C'(x)$
- For a **total revenue** function $R(x)$, the **marginal revenue** is $R'(x)$
- For a **total profit** function $P(x)$, the **marginal profit** is $P'(x)$

Marginal costs do not always decrease as x increases. For example, if a nursery has to use extra workers or equipment to fill a very large order for tomato seedlings, the cost per punnet could be higher than for a smaller order.

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EXAMPLE 11

A manufacturer has worked out that the total cost of producing x items per day is

$$C(x) = 0.000\,003x^3 - 0.05x^2 + 253x.$$

The items are sold for \$15.40 each.

a What is the marginal cost function?

b What is the total revenue function?

c What is the total profit function?

d What is the marginal profit function?

Solution

a Marginal cost is the derivative. $C'(x) = 0.000\,009x^2 - 0.1x + 253$

Write the answer. The marginal cost is $0.000\,009x^2 - 0.1x + 253$

b Revenue = price × number sold $R(x) = 15.4x$

Write the answer. The total revenue is $15.4x$.

c Profit = Revenue − Cost $P(x) = 15.4x - (0.000\,003x^3 - 0.05x^2 + 253x)$

$= 0.05x^2 - 0.000\,003x^3 - 237.6x$

Write the answer. The total profit is $0.05x^2 - 0.000\,003x^3 - 237.6x$

d Marginal profit is the derivative. $P'(x) = 0.1x - 0.000\,009x^2 - 237.6$

EXAMPLE 12

A kitchen appliance maker has found their marginal profit for a particular appliance is given by $\$(900 + 0.06x^2 - 58x - 0.0024x^3)$ in thousands of dollars, where x is the number made in thousands. Their break-even point is 5000 appliances. Find the maximum profit and the number they should make to achieve it.

Solution

Write the marginal profit. $P'(x) = 900 + 0.06x^2 - 58x - 0.0024x^3$

Integrate for the profit function. $P(x) = 900x + 0.02x^3 - 29x^2 - 0.0006x^4 + c$

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Use the break-even point to find c.

$P(5) = 0$

$0 = 900 \times 5 + 0.02 \times 125 - 29 \times 25 - 0.0006 \times 625 + c$

$c = -3777.125$

Write the profit function.

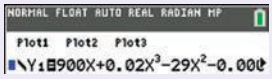
$P(x) = 900x + 0.02x^3 - 29x^2 - 0.0006x^4 - 3777.125$

Use a graphics calculator to find the maximum value.

TI-Inspire CX
Chapter 6

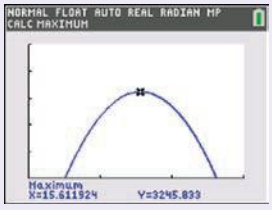
TI-84 Plus CE

Press [y=] and type the function into Y_1.



[graph] the function and use [window] or [zoom] to narrow down the maximum.

Use calc to find the maximum: $x = 15.611924 \ldots$, $P = 3245.833 \ldots$



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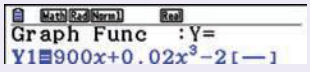
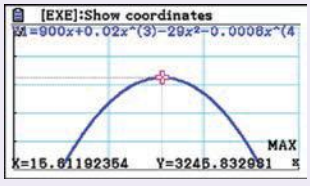
Use the Graph menu.

Type the profit function into Y1.



DRAW the function and use Zoom or V-Window to narrow down the maximum.

Use G-Solv to find the maximum: $x = 15.61192354 \ldots$, $P = 3245.832981 \ldots$



You could substitute 15.611 and 15.612 into the profit function to find the exact maximum.

State the answer.

The maximum profit is about \$3 246 000. This is achieved by making about 15 600 appliances.

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Exercise 6.05 Marginal and total change

1 The total cost of producing T-shirts for sale in world-wide chain stores is $C(x) = 0.000\,25x^4 - 17x^3 + 312\,500x^2 + 0.0015x + 36$, where x is in thousands produced and $C(x)$ is in dollars. The cost to chain stores is \$2.35 each.

Example 11

a What is the marginal cost function?

b What is the total revenue function?

c What is the total profit function?

d What is the marginal profit function?

2 The marginal cost of producing a particular item is given by $6x^2 - 114x + 780$, where x is in thousands and the cost is in thousands of dollars. The set-up cost before making any items is \$15 000. The items are sold for \$820 each.

Example 12

a What is the cost function?

b What is the profit function?

c Find the maximum profit and the number of items that must be produced to achieve it.

3 The marginal cost function for making x components per hour is given by $\$(100 - 4x)$ for up to 20 components per hour. It costs \$800 to set up the machinery.

a What is the total cost function?

b What is the cost of making 15 components per hour?

4 The marginal revenue function for the sale of items is given by $\$(84 - 27x^2 + 36x)$, where x is in thousands and the revenue is in thousands of dollars.

a What is the total revenue function?

b What is the revenue for selling 2000 items?

c What is the maximum revenue?

Problem solving

5 A shade-cloth maker has a marginal cost of $20xe^{0.01x^2}$ per roll of 90% UV cloth. It costs \$1500 to set up the 90% rolls. What is the cost of manufacturing 20 rolls?

6 The marginal profit of a manufacturer of fencing systems is given by $400(1 - e^{-0.015x})$.

What is the profit from making 500 systems?

7 The marginal cost of producing USB drives is given by $0.000\,015n^2 - 0.011n + 0.5$ per day for between 100 and 1500 per day. It costs \$2000 per day even if no drives are produced.

Compare the average and marginal costs of producing 800 drives per day and comment on your results.

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8 A fashion designer with outlets in all major cities in Australia has worked out that if she makes more than 50 of any particular item, the price she can charge decreases sharply.

The marginal revenue function for a particular dress is given by

$$R'(x) = 2750 - 75x + 1.5x^2 - 0.01x^3.$$

The total cost function is

$$C(x) = 4200 + 505x.$$

Find the number she should make to maximise profit and the maximum profit.

WS

Calculating areas between curves

6.06 Areas between curves

You might need to find the intersections to find an enclosed area or one between two curves.

Area between curves $f(x)$ and $g(x)$

You can calculate the area between 2 curves as:

- the difference of the areas under the curves

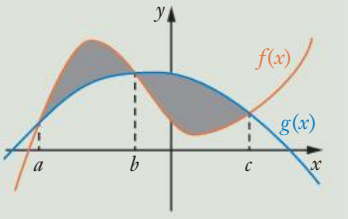
$$\text{Area} = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx + \int_b^c g(x)\,dx - \int_b^c f(x)\,dx$$

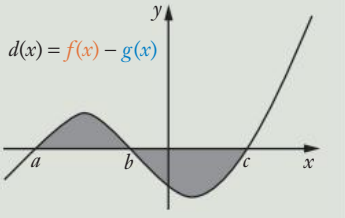
or

- the area under the **difference function** $d(x) = f(x) - g(x)$

$$\text{Area} = \int_a^b d(x)\,dx - \int_b^c d(x)\,dx$$

The zeros of the difference function are at the x-values of the intersections of the functions.





You can use the difference function because:

$$\int f(x)\,dx - \int g(x)\,dx = \int [f(x) - g(x)]\,dx = \int d(x)\,dx$$

and

$$\int g(x)\,dx - \int f(x)\,dx = \int [g(x) - f(x)]\,dx = -\int d(x)\,dx$$

You can get the right signs from a sketch of the functions.

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EXAMPLE 13

Find the area cut off by $x - y + 3 = 0$ for $f(x) = x^2 - 6x + 13$.

Solution

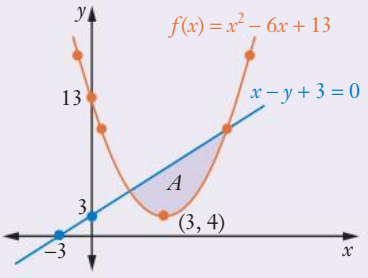
Check for zeros of $f(x)$.

$\Delta = -16$, so there are no intersections

You can use completing the square to find the minimum, but it may be quicker to use a table of values.

x	-1	1	3	5	7
$f(x)$	20	8	4	8	20
y	2	4	6	8	10

Use the points to sketch the graphs on the same axes.



Change the form of the straight line.

$$x - y + 3 = 0$$
$$y = x + 3$$

Equate the functions to find the intersections.

$$x + 3 = x^2 - 6x + 13$$
$$x^2 - 7x + 10 = 0$$
$$(x - 2)(x - 5) = 0$$
$$x = 2 \text{ or } x = 5$$

Method 1

Find the area under each function between the intersections. Subtract the areas.

$$\text{Area} = \int_2^5 (x+3)\,dx - \int_2^5 (x^2 - 6x + 13)\,dx$$
$$= \left[\frac{x^2}{2} + 3x\right]_2^5 - \left[\frac{x^3}{3} - 3x^2 + 13x\right]_2^5$$
$$= (27\tfrac{1}{2} - 8) - (31\tfrac{2}{3} - 16\tfrac{2}{3})$$
$$= 19\tfrac{1}{2} - 15$$
$$= 4\tfrac{1}{2}$$

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Method 2

Find the difference function.

$$d(x) = (x + 3) - (x^2 - 6x + 13)$$
$$= -x^2 + 7x - 10$$

Find the area under the difference function between the intersections.

$$\text{Area} = \int_2^5 (-x^2 + 7x - 10)\,dx$$
$$= \left[-\frac{x^3}{3} + \frac{7x^2}{2} - 10x\right]_2^5$$
$$= (-4\tfrac{1}{6} - (-8\tfrac{2}{3}))$$
$$= 4\tfrac{1}{2}$$

Write the answer.

The area cut off by $x - y + 3 = 0$ for $f(x) = x^2 - 6x + 13$ is $4\frac{1}{2}$ square units.

You can use your graphics calculator to check your calculations. It uses many strips to get an approximation with a method that is much more accurate than the trapezoidal rule.

EXAMPLE 14

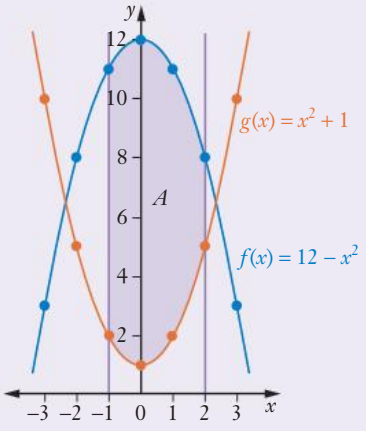
Find the area between $f(x) = 12 - x^2$ and $g(x) = x^2 + 1$ and the lines $x = -1$ and $x = 2$.

Solution

Use a table of values to sketch the graphs.

x	−3	−2	−1	0	1	2	3
$f(x)$	3	8	11	12	11	8	3
$g(x)$	10	5	2	1	2	5	10

Sketch the graphs.



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Find the area.

$$\text{Area} = \int_{-1}^{2} (12 - x^2)\,dx - \int_{-1}^{2} (x^2 + 1)\,dx$$
$$= \left[12x - \frac{x^3}{3}\right]_{-1}^{2} - \left[\frac{x^3}{3} + x\right]_{-1}^{2}$$
$$= [21\tfrac{1}{3} - (-11\tfrac{2}{3})] - [4\tfrac{2}{3} - (-1\tfrac{1}{3})]$$
$$= 33 - 6$$
$$= 27$$

Check using your graphics calculator. If you needed only an approximate answer, you could use your calculator for the whole question.

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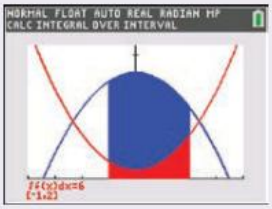

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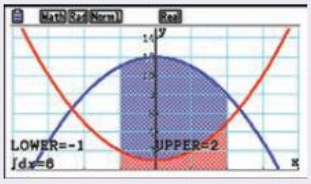


TI-Inspire CX Chapter 6

Find the difference of the values.

Area = 33 − 6 = 27

Write the answer.

The area between $f(x) = 12 - x^2$, $g(x) = x^2 + 1$, $x = -1$ and $x = 2$ is 27 square units.

You can use a graphics calculator to find the intersections of more complex functions.

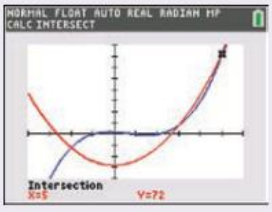
EXAMPLE 15

Find the area between $f(x) = (x + 1)(x - 1)(x - 2)$ and $g(x) = 4x^2 - 28$.

Solution

Enter the functions in your graphics calculator to find the intersections.

TI-84 Plus CE



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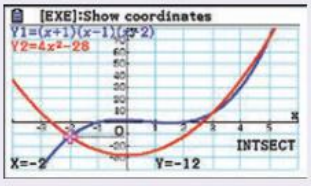

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State the intersections.	The functions intersect at $(-2, -12)$, $(3, 8)$ and $(5, 72)$.
Expand $f(x)$ before integrating.	$f(x) = (x + 1)(x - 1)(x - 2)$ $= (x^2 - 1)(x - 2)$ $= x^3 - 2x^2 - x + 2$
Find the difference function.	$d(x) = f(x) - g(x)$ $= x^3 - 2x^2 - x + 2 - (4x^2 - 28)$ $= x^3 - 6x^2 - x + 30$
State the sign between the limits.	$d(x) > 0$ for $-2 < x < 3$, $d(x) < 0$ for $3 < x < 5$.
Find the area.	$\text{Area} = \int_{-2}^{3} (x^3 - 6x^2 - x + 30)\,dx$ $- \int_{3}^{5} (x^3 - 6x^2 - x + 30)\,dx$ $= \left[\frac{x^4}{4} - 2x^3 - \frac{x^2}{2} + 30x\right]_{-2}^{3}$ $- \left[\frac{x^4}{4} - 2x^3 - \frac{x^2}{2} + 30x\right]_{3}^{5}$ $= \left[51\frac{3}{4} - (-42)\right] - \left[43\frac{3}{4} - 51\frac{3}{4}\right]$ $= 93\frac{3}{4} - (-8)$ $= 101\frac{3}{4}$
State the answer.	The area between $f(x) = (x + 1)(x - 1)(x - 2)$ and $g(x) = 4x^2 - 28$ is $101\frac{3}{4}$ square units.

Exercise 6.06 Areas between curves

Example 13

1 Find the area cut off by $2x + y = 12$ for $y = 2x^2 - 12x + 20$.

2 Find the area cut off by:

a $y = 2x - 1$ for $f(x) = 2x + 8 - x^2$

b $y = 4x + 26$ for $f(x) = 4x^2 + 24x + 26$

c $y = 4x + 13$ for $y = 4x^2 + 12x + 8$.

Example 14

3 Find the area between $f(x) = 3 - x^3$, $g(x) = x^2 - 9$, $x = -3$ and $x = 1$.

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4 Find the area between $f(x) = (x + 2)(x - 2)^2$ and $y = x + 2$.

Example 15

5 Find the area between $f(x) = 2x^2 - 8$, $g(x) = x^2 - 2x + 6$, $x = -3$ and $x = 2$.

6 Find the area between $y = (x + 1)(x - 1)(x - 2)$ and $y = 2 + 5x - x^2$.

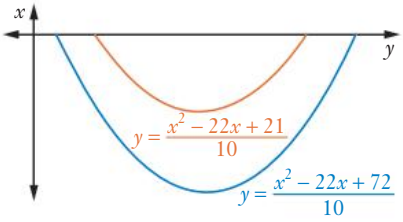
7 Find the area between $f(x) = 4x^2 + 4x + 5$ and $y = 5 - 2x$.

8 Find the area between $y = 3x^2 + 2x - 21$ and $y = (x + 1)(x + 3)(x - 3)$.

9 Find the area between $f(x) = x^2 - 2x + 2$, $g(x) = 4 + 4x - x^2$, the y-axis and $x = 3$.

Problem solving

10 A sweeping 'circular' driveway actually has two parabolas as its edges to allow room for parking near the house. The x-axis is the edge of the roadway, as shown in the diagram.



Estimate the cost of concreting to a depth of 15 cm with concrete costing \$426/m^3.

11 Another driveway sweeps from a narrow entrance to a double garage with its doors perpendicular to the road (the x-axis). The driveway can be modelled as the area between the positive axes and the curves $y = \frac{12 - x^2}{4}$ and $y = \frac{36 - x^2}{4}$ (in metres)

Estimate the cost of concreting to a depth of 10 cm with concrete costing \$394/m^3.

12 A cam shaft has a cross-section that is a good approximation to the area between the curves $y = 0.75x^2 - 3.75x + 6$ and $y = 5x - 1 - x^2$, where x is in centimetres.

The shaft is 32 cm long. It is made of high-tensile steel with a density of 7920 kg/m^3.

Find the mass of the shaft.

6.07 Acceleration, velocity and displacement

You already know the differential relationships between displacement, velocity and acceleration: $v = \frac{ds}{dt}$ and $a = \frac{dv}{dt}$. You can reverse them to get the integral relationships.

For acceleration a, velocity v and displacement s:

- Displacement is the integral of velocity: $s = \int v\,dt$
- Velocity is the integral of displacement: $v = \int a\,dt$

Near the Earth, the acceleration of gravity is taken as the constant $g = 9.8$ m/s^2 downwards.

WS Displacement, velocity and acceleration

WS Integration assignment

WS Integration problems

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EXAMPLE 16

An object is given an initial speed of 18 m/s upwards from a height of 16 m and is acted on by gravity alone.

a Find expressions for the velocity and displacement at time t.

Find, correct to 2 decimal places:

b when the velocity is 0.

c the object's height when the velocity is 0.

d when the object hits the ground.

e the object's velocity when it hits the ground.

Solution

a Find velocity from acceleration, taking the upwards direction as positive.

$$v = \int -g\,dt$$
$$= -gt + c$$

Use $t = 0$, $v = 18$ to find c.

$$18 = -9.8 \times 0 + c$$
$$c = 18$$

Write the equation for velocity.

$$v = -9.8t + 18 = 18 - 9.8t$$

Find displacement from velocity.

$$s = \int (18 - 9.8t)\,dt$$
$$= 18t - 4.9t^2 + d$$

Use $t = 0$, $s = 16$ to find d.

$$16 = 18 \times 0 - 4.9 \times 0^2 + d$$
$$d = 16$$

Write the equation for displacement.

$$s = 18t - 4.9t^2 + 16$$

b Substitute $v = 0$.

$$0 = 18 - 9.8t$$
$$t = \frac{18}{9.8} = 1.836\ldots$$

Write the answer.

The velocity is 0 at about 1.84 seconds.

c Substitute $t = 1.836\ldots$ into s.

$$s = 18 \times 1.836\ldots - 4.9 \times (1.836\ldots)^2 + 16$$
$$= 32.530\ldots$$

Write the answer.

The height when $v = 0$ is about 32.53 m.

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d Substitute $h = 0$.

$0 = 18t - 4.9t^2 + 16$

$4.9t^2 - 18t - 16 = 0$

Use the quadratic formula.

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Substitute values and simplify.

$$= \frac{18 \pm \sqrt{18^2 - 4 \times 4.9 \times (-16)}}{2 \times 4.9}$$

$$= \frac{18 \pm \sqrt{637.6}}{9.8}$$

$= 4.413\ldots$ or $-0.739\ldots$

Discard negative answer.

But $t > 0$, so $t \approx 4.41$.

Write the answer.

The object hits the ground at about 4.41 seconds.

e Write the equation for velocity.

$v = 18 - 9.8t$

Substitute $t = 4.413\ldots$

$v = 18 - 9.8 \times 4.413\ldots$

$= -25.247\ldots$ m/s

Write the answer.

The object's velocity when it hit the ground was -25.25 m/s.

Air friction is proportional to mass and inversely proportional to cross-sectional area. For very small objects, the ratio of $\frac{m}{A}$ is much less than for larger objects of the same density. This means they have a low **terminal velocity** under gravity. The terminal velocity is the limit of the speed an object can reach under gravity.

EXAMPLE 17

The acceleration of a dust particle falling from rest is given by $a = 9.8e^{-kt}$. The terminal velocity is about 2 cm/s.

a Find an expression for the velocity and distance fallen (from rest).

b Find the value of k.

c Find an expression for the distance fallen.

d How far does the particle fall in the 1st second?

e How far does it fall in the 2nd second?

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Solution

a Use the integral to find velocity.

$$v = \int a\,dt$$
$$= \int 9.8e^{-kt}\,dt$$
$$= -\frac{9.8}{k}e^{-kt} + c$$

Use initial conditions to find c.

$v = 0$ at $t = 0$

$$-\frac{9.8}{k}e^{0} + c = 0$$
$$c = \frac{9.8}{k}$$

Write the expression for velocity.

$$v = -\frac{9.8}{k}e^{-kt} + \frac{9.8}{k}$$
$$= \frac{9.8}{k}(1 - e^{-kt})$$

b The terminal velocity is the value of v as $t \to \infty$.

As $t \to \infty$, $e^{-kt} \to 0$

$$\text{Terminal velocity} = \frac{9.8}{k}$$

Substitute $v = 0.02$ to find k.

$$\frac{9.8}{k} = 0.02$$
$$k = \frac{9.8}{0.02} = 490$$

c Substitute $k = 490$ in the equation for velocity and simplify.

$$v = \frac{9.8}{490}(1 - e^{-490t})$$
$$= 0.02(1 - e^{-490t})$$

Use the integral to find displacement.

$$s = \int v\,dt$$
$$= \int \left(0.02 - 0.02e^{-490t}\right) dt$$
$$= 0.02t + \frac{1}{24\,500}e^{-490t} + d$$

Substitute $t = 0$, $s = 0$ to find d.

$s = 0$ at $t = 0$

$$0.02 \times 0 + \frac{1}{24\,500}e^{0} + d = 0$$
$$d = -\frac{1}{24\,500}$$

Write the answer.

$$s = 0.02t + \frac{1}{24\,500}(e^{-490t} - 1)$$

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d Substitute $t = 1$.

$$s = 0.02 + \frac{1}{24\,500}(e^{-490} - 1)$$
$$= 0.019\,959\ldots$$
$$\approx 0.02 \text{ m}$$

Write the answer.

It falls almost 2 cm in the first second.

e Substitute $t = 2$.

$$s = 0.04 + \frac{1}{24\,500}(e^{-980} - 1)$$
$$= 0.039\,995\ldots$$

Subtract the distance in the 1st second.

Distance in 2nd second
$= 0.039\,995\ldots - 0.019\,9591\ldots$
$= 0.0200\ldots$

Write the answer.

It falls about 2 cm in the 2nd second.

Very small falling objects attain terminal velocity in a very short time.

INVESTIGATION

TERMINAL VELOCITIES

Investigate the terminal velocities of:

- pollen particles
- sand particles
- pebbles and stones
- falling aircraft
- parachutes
- objects falling through water

Your teacher may want you to write a report.

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Exercise 6.07 Acceleration, velocity and displacement

Example 16

1 An object is thrown upwards at 15 m/s from a height of 2 m. After it reaches maximum height, it lands in the gutter of the old Queenslander house at a height of 5 m. Find, correct to 2 decimal places:

- **a** an expression for its velocity.
- **b** the time to reach maximum height.
- **c** an expression for its height.
- **d** the maximum height.
- **e** the time taken before it lands in the gutter.
- **f** the velocity when it lands.

Example 17

2 The acceleration of an object is given by $a = 3.5t + 0.6$ m/s^2. It begins from rest and accelerates for 30 seconds.

- **a** Find an expression for the velocity of the object.
- **b** Find the velocity reached after 10, 20 and 30 seconds.
- **c** Find an expression for the displacement.
- **d** Find the displacement after 10, 20 and 30 seconds.

3 The acceleration of an object falling through the water in a river is 3 m/s^2. It initially entered the water at a velocity of 6 m/s downwards.

- **a** Find an expression for the velocity of the object.
- **b** Assuming that the water is deep enough, find the time taken for the velocity to double from the initial velocity.
- **c** Find an expression for the depth of the object.
- **d** The depth of the water is actually 11 m. How long does it take (correct to 2 decimal places) to reach half-way to the bottom?
- **e** What is its speed at that point?
- **f** How long (correct to 2 decimal places) does it take to reach the bottom?
- **g** What is the speed with which it hits the bottom?

There is deep thick mud at the bottom of the water, so the object slows at 22.5 m/s^2 during its passage through the mud.

- **h** Find expressions for its velocity and displacement in the mud.
- **i** How far does it penetrate the mud, correct to 2 decimal places?

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4 A dragcar has a velocity of 8 m/s. Its acceleration is given by $a = 1.6t + 3$ m/s^2 for 9 seconds. It then travels at a constant speed for 2 seconds before slowing with an acceleration of –4.1 m/s^2 until it comes to rest.

- **a** Find expressions for its velocity and displacement in the first part of its movement.
- **b** Find the velocity and displacement at 9 seconds.
- **c** Find the displacement at 11 seconds.
- **d** Find expressions for its velocity and displacement in the last part of its movement.
- **e** Find how long it takes to come to rest after it starts to slow down.
- **f** Find its total displacement when it comes to rest.

Problem solving

5 The electric tilt-train from Brisbane to Rockhampton has an acceleration of 0.69 m/s^2 and a braking deceleration of 1.05 m/s^2. The train operates at a top speed of 160 km/h. There are no regular stops on the 109 km from Gladstone to Rockhampton. How long should it take to travel between Gladstone and Rockhampton?

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6 A high-speed personnel lift in a mine can accelerate downwards at 6 m/s^2 and is not allowed to exceed a speed of 20 m/s. It can brake to decelerate at 4.5 m/s^2. The current seam being mined is 1.8 km deep. How long do the miners take to reach the bottom?

7 The acceleration of a fireworks skyrocket increases as it ascends because the mass decreases as the fuel is burned. It is given by the expression $a = \frac{400}{4.5 - t} - 9.8$ m/s^2 for the first 4 seconds. The fuel is all spent after 2 seconds, when the rocket explodes in a blaze of colour. Approximately how high does it get before the explosion? (Hint: Use a numerical method to integrate the velocity)

8 The terminal velocity of someone free-falling is about 50 m/s. The acceleration of someone falling is $a = 9.8e^{-kt}$ m/s downwards. It is normal to open a parachute at 500 m above the ground to give time to open the reserve chute if the main one fails. Show that after jumping from a plane 2 km above the ground, you should open your parachute at 35 seconds.

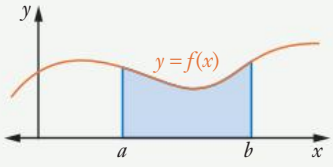
6. CHAPTER SUMMARY

Integration and applications

- **Integral calculus** is the complementary process to differentiation, the calculation of areas under curves

Definite integrals

- The **definite integral** of the function $f(x)$ from $x = a$ to $x = b$ is the area enclosed by the curve $f(x)$, the lines $x = a$ and $x = b$, and the x-axis
- The symbol for the definite integral is $\int_a^b f(x)\,dx$
- The definite integral is a number
- An indefinite integral of $f(x)$ is written as $F(x) = \int f(x)\,dx$, where $F'(x) = f(x)$
- Indefinite integrals are functions



- $\int_a^b f(x)\,dx \approx w[f(a_0) + f(a_1) + f(a_2) + \ldots + f(a_{n-1})] = w\sum_{i=0}^{n-1} f(a_i)$ for left rectangles
- $\int_a^b f(x)\,dx \approx w[f(a_1) + f(a_2) + f(a_3) + \ldots + f(a_n)]$ for right rectangles
- $\int_a^b f(x)\,dx \approx w[f(a_{\frac{1}{2}}) + f(a_{1\frac{1}{2}}) + f(a_{2\frac{1}{2}}) + \ldots + f(a_{n-\frac{1}{2}})]$ for centred rectangles

  where the interval $[a, b]$ is divided into n equal strips of width w by $a = a_0, a_1, a_2, \ldots, a_n = b$ and the half values are at the centres of the strips
- The trapezoidal rule is

  $\int_a^b f(x)\,dx \approx \frac{1}{2}w[f(a_0) + 2f(a_1) + 2f(a_2) + \ldots + 2f(a_n - 1) + f(a_n)] = \frac{1}{2}w[E + 2M]$

  where the interval $[a, b]$ is divided into n equal strips of width w by $a = a_0, a_1, a_2, \ldots, a_n = b$. E is the sum of the end values of the function and M is the sum of the middle values.

The fundamental theorem of calculus

- An indefinite integral of $f(x)$ is written as $F(x) = \int f(x)\,dx$, where $F'(x) = f(x)$
- The definite integral of $f(x)$ from a to b is given by $\int_a^b f(x)\,dx = F(b) - F(a)$
- The indefinite integral is a function, while the definite integral is a number
- Calculations are normally set out as: $\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$

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- **Indefinite integrals of common functions**

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c \quad (n \neq -1)$$

$$\int kx^n\,dx = \frac{kx^{n+1}}{n+1} + c \quad (n \neq -1)$$

$$\int \sin(x)\,dx = -\cos(x) + c$$

$$\int \sin(kx)\,dx = -\frac{1}{k}\cos(kx) + c$$

$$\int \cos(x)\,dx = \sin(x) + c$$

$$\int \cos(kx)\,dx = \frac{1}{k}\sin(kx) + c$$

$$\int e^x\,dx = e^x + c$$

$$\int \frac{1}{x}\,dx = \ln(x) + c \quad (x > 0)$$

$$\int a^x\,dx = \frac{a^x}{\ln(a)} + c \quad (a > 0)$$

- **Properties of integrals**

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

$$\int kf(x)\,dx = k\int f(x)\,dx$$

$$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int [f(x) - g(x)]\,dx = \int f(x)\,dx - \int g(x)\,dx$$

$$\int f(ax+b)\,dx = \frac{1}{a} \times F(ax+b) + c, \quad \text{where } F(x) = \int f(x)\,dx$$

- **Physical area** is unsigned, but **algebraic area** has a sign
- The **total change** in a quantity q for x items is usually given as a function $q(x)$
- The **marginal change** in the quantity is the cost of producing one more item
- For a quantity $q(x)$, the marginal change is $q(x+1) - q(x) \approx \frac{dq}{dx} = q'(x)$
- For a **total cost** function $C(x)$, the **marginal cost** is $C'(x)$
- For a **total revenue** function $R(x)$, the **marginal revenue** is $R'(x)$
- For a **total profit** function $P(x)$, the **marginal profit** is $P'(x)$
- The area under the **difference function** $d(x) = f(x) - g(x)$ is the same as the area between $f(x)$ and $g(x)$
- Displacement is the integral of velocity: $s = \int v\,dt$
- Velocity is the integral of displacement: $v = \int a\,dt$
- The **terminal velocity** is the limit of the speed an object can reach under gravity

6. CHAPTER REVIEW

Integration and applications

Example 1

1 Find an approximation for the definite integral of $y = x^2 + 3$ from $x = 1$ to $x = 3$ using strips of width 0.4 units and centre function values.

Example 2

2 Approximate the area under $y = x^2 + 5$ between 1 and 4 using 6 left rectangles.

Example 3

3 Use your graphics calculator to find an approximation for $\int_2^5 3e^{0.5x}\,dx$.

Example 4

4 Use the trapezoidal rule to find approximations for $\int_0^2 (4x^2 + 2x + 1)\,dx$ using 10 strips.

Examples 5-7

5 Evaluate each integral.

a $\int_{-1}^{2} (12x^2 - 6x + 1)\,dx$ **b** $\int_1^9 x^{-\frac{1}{2}}\,dx$ **c** $\int_5^{10} \frac{dx}{(x-4)^2}$

d $\int_0^{\frac{\pi}{3}} \sin(3x)\,dx$ **e** $\int_0^{\frac{\pi}{4}} \sin\left(2x + \frac{\pi}{3}\right)dx$ **f** $\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 12\cos(3x)\,dx$

g $\int_3^8 \frac{dx}{x}$ **h** $\int_4^6 \frac{dx}{2x-7}$ **i** $\int_0^5 3 \times 5^{0.2x+0.1}\,dx$

j $\int_{-1}^{4} \frac{2x+3}{x^2+3x+5}$

Example 8

6 Find the area under the curve $y = 8x^3 - 3x^2 + 6x$ between $x = 3$ and $x = 7$.

Example 9

7 Find the area enclosed by $f(x) = 3(x-2)^2$, $x = -1$, $x = 1$ and the x-axis.

Example 10

8 Find the physical area cut off $f(x) = (x+3)(x-1)(x-2)$ by the x-axis.

Examples 11,12

9 The marginal cost of producing particular components is given by $y = 0.000\,375x^2 - 0.015x + 0.15$, where x is in thousands and the costs are in thousands of dollars. The set-up costs before making any items is \$20 000. The items are sold for \$47 each.

a What is the total cost of producing the first 5000 items?

b What is the profit function?

c What is the marginal profit?

Example 13

10 Find the area cut off by $y = x + 2$ for $f(x) = x^2 - 2x - 2$.

Examples 14,15

11 Find the area between the curves $f(x) = 3x^2 - 8x - 3$ and $g(x) = 2x^2 - 5x + 7$.

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12 An object is thrown upwards from a height of 5 m. It has an initial velocity of 17 m/s.

Example 16

a Find expressions for the velocity and displacement at time t.

Find correct to 2 decimal places:

b the time when the velocity is 0.

c the height when the velocity is 0.

d the time when the height is 0.

e the velocity when the height is 0.

13 The terminal velocity of an object falling from a height of 3 m is about 50 cm/s. Its acceleration is given by $a = -9.8e^{-kt}$.

Example 17

a Find an expression for the velocity at time t.

b Find the value of k.

c Find an expression for the height of the object.

d Find the heights after 1, 2, and 3 seconds.

e Estimate the time taken to fall to a height of 0.

Problem solving

14 The shape of a *shade sail* is formed by three intersecting parabolas. The parabolas have equations $y = x^2 - 8x + 16$, $y = x^2 + 8x + 16$ and $y = 2 - 0.125x^2$, where all dimensions are in metres. The cost of the shade cloth is \$30/m^2. Find the cost of the sail and comment on your result.

15 The *learning curve* for the production of wheelbarrows in a factory is given by $f(x) = 2.8x^{-0.3}$, where x is the number of wheelbarrows produced and $f(x)$ is the number of work-hours needed to produce the next wheelbarrow. Find the time needed to produce the second 50 wheelbarrows.

16 The approximate acceleration of a pebble falling through a viscous liquid is given by $a = 4(t + 1)^{-3}$ cm/s^2, where $t \geq 0$. Find the time taken to fall the first 10 cm through the liquid and the speed of the pebble at that point, correct to 2 decimal places.

17 The deceleration of the payload due to air friction when an experimental parachute attached to it opens is given by $a = \left(1 - \frac{v_0}{v_t}\right)e^{-kt}$, where k is a constant, v_0 is the speed when the parachute opens and v_t is the terminal velocity. The assembly has a terminal velocity of 20 m/s and the initial speed is 55 m/s.

Find the time required (correct to 2 decimal places) for the speed to reduce to 10% more than the terminal velocity and the distance (correct to the nearest metre) fallen in this time.

Qz Practice quiz

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WS
Year 12 formulas

Practice examination 2 ●●○○

Time: 90 minutes
Perusal time: 5 minutes
Marks: 50

Instructions

- Students are permitted to bring and use: pens, pencils, highlighters, erasers, sharpeners, rules and an approved graphics calculator.
- Students must show appropriate working and justification to gain full marks.
- A QCAA formula sheet is provided.
- Unless otherwise stated, numerical answers should be exact.
- Unless otherwise indicated, no diagrams in this examination are drawn to scale.
- All written responses must be in English.
- Answer all questions.
- **Students are NOT permitted to bring or use notes of any kind, correction fluid/tape, mobile phones and/or any other unauthorised electronic devices.**

Question 1 (3 marks)

Find the derivative of:

a $3 \sin(x)$ **b** $\cos(5m)$ **c** $\tan(x)$

Question 2 (3 marks)

Find the derivative of:

a $\cos^2(3x + 2)$ **b** $\sin(g^2 + 3g + 1)$ **c** $2 \tan(\theta°)$

Question 3 (2 marks)

Find the derivative of:

a $4 \sin(3x - 5)^3$ **b** $2\sqrt{\cos(2p)}$

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Question 4 (3 marks)

Differentiate each expression.

a $-8m^{-4}$ **b** $\dfrac{3(2x-5)}{2(7x+1)}$ **c** $-3(z^2+3z-2)^7$

Question 5 (3 marks)

Find y if:

a $\dfrac{dy}{dx}=\cos\left(\dfrac{3x}{2}\right)$ **b** $\dfrac{dy}{dx}=4x+\sin(4x-5)$ **c** $\dfrac{dy}{dx}=4e^{2x-5}$

Question 6 (2 marks)

Find the anti-derivative of:

a $\dfrac{7}{8-3x}$ **b** $\dfrac{4-e^x}{e^{2x}}$

Question 7 (2 marks)

a Find the remainder when x^2-2x+5 is divided by $x-3$.

b Use the result from **a** to calculate $\int\dfrac{x^2-2x+5}{x-3}\,dx$.

Question 8 (2 marks)

Find $\dfrac{d}{dx}\sqrt{x^3-4}$ and use the result to find $\int\dfrac{x^2}{\sqrt{x^3-4}}\,dx$.

Question 9 (1 marks)

Calculate $\int\dfrac{4x}{x^2+2}\,dx$.

Question 10 (2 marks) (Technology-active)

Find an approximation for $\int_1^5(x^2+3)\,dx$ using 8 left rectangles.

Question 11 (2 marks) (Technology-active)

Use the trapezoidal rule with 10 strips to find an approximation for $\int_0^4 3^x\,dx$.

Question 12 (2 marks)

Find $\int_1^5\dfrac{3}{x}\,dx$.

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Question 13 (1 marks)

Find the area under $y = x^3 + 3x^2 + 1$ from $x = 1$ to $x = 4$.

Question 14 (2 marks)

Find the area cut off $y = x^2 + x - 5$ by $y = 2x + 1$.

Question 15 (3 marks)

Find the derivatives of

a $5 \ln [2 \cos (6x + 5)^2]$ **b** $\dfrac{5\sin(3x-2)^3}{3\cos^3(2x)}$

Question 16 (2 marks)

The point $(-1, -2)$ lies on the graph of the function $y = f(x)$.

If $\dfrac{dy}{dx} = 12x^2 - 4x$, find the equation of the function.

Question 17 (2 marks)

A particle moves in a straight line and its velocity (m/s) is given by

$$v(t) = t^2 - 4t + 5 \qquad \text{for } 0 \le t \le 4, \text{ where } t \text{ is in seconds.}$$

Find the total distance travelled by the particle in the first 3 seconds of its motion.

Question 18 (3 marks) (Technology-active)

The marginal cost of an item in dollars is given by

$$\frac{x^3 - 450x^2 + 67\,500x - 3\,375\,000}{3\,375\,000} + 2.$$

The set-up cost for the item is $2000.

What is the cost of making the first 100 items?

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Question 19 (3 marks)

The water in a pond produces waves after an object is thrown in. When the waves first reach one edge of the pond, they displace the water upwards from the original position by 5 cm. After 20 seconds, the displacement has reduced to 4 cm. The waves are 2 seconds apart and the edge of the pond has a slope of 15°. Find the horizontal speed at which the edge of the water is moving across the slope 35 seconds after they first reach the edge of the pond.

Question 20 (2 marks)

Find $\int xe^{x^2}\,dx$.

Question 21 (2 marks)

Find $\int \frac{4x^2}{\sqrt{x^3+2}}\,dx$.

Question 22 (3 marks)

The acceleration of a rocket t seconds after the first stage has separated is given by

$$a = 30e^{0.2t} \text{ for } t = 0 \text{ to } 45 \text{ seconds.}$$

The height and velocity at $t = 0$ are 4000 m and 200 m/s respectively.

What are the height and velocity of the rocket 25 seconds after the first stage separation?

END OF EXAMINATION

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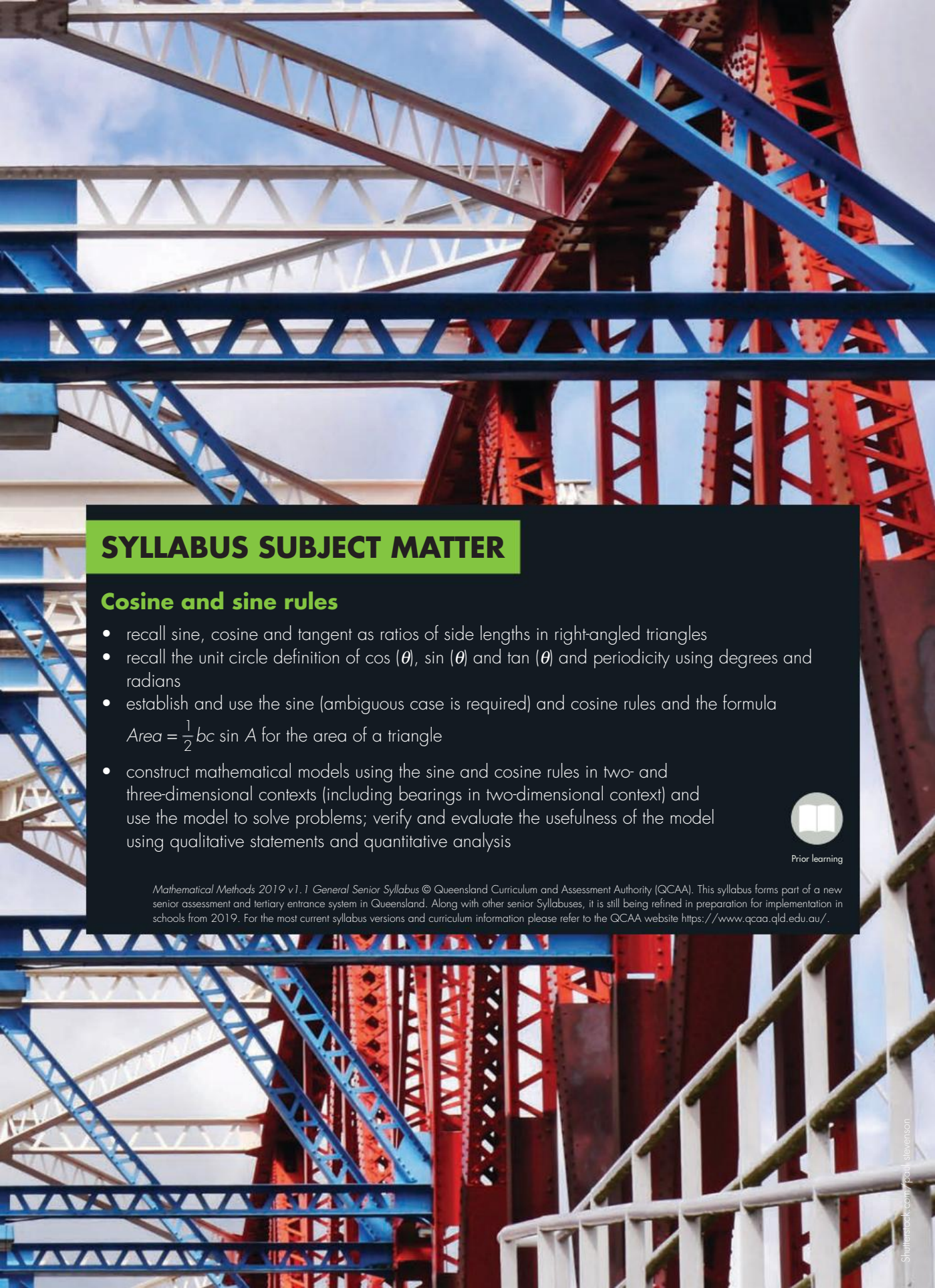
7.

THE SINE AND COSINE RULES

In this chapter you will study the trigonometry of general triangles, including triangles that are not right-angled. Applications of trigonometry occur very frequently in our built environment, in land surveys and navigation.

7.01 Trigonometry of right-angled triangles
7.02 Trigonometry and the unit circle
7.03 The area of a triangle
7.04 The sine rule
7.05 The cosine rule
7.06 2D applications of trigonometry
7.07 3D applications of trigonometry
Chapter summary
Chapter review



SYLLABUS SUBJECT MATTER

Cosine and sine rules

- recall sine, cosine and tangent as ratios of side lengths in right-angled triangles
- recall the unit circle definition of cos (θ), sin (θ) and tan (θ) and periodicity using degrees and radians
- establish and use the sine (ambiguous case is required) and cosine rules and the formula $Area = \frac{1}{2}bc \sin A$ for the area of a triangle
- construct mathematical models using the sine and cosine rules in two- and three-dimensional contexts (including bearings in two-dimensional context) and use the model to solve problems; verify and evaluate the usefulness of the model using qualitative statements and quantitative analysis

Prior learning

Mathematical Methods 2019 v1.1 General Senior Syllabus © Queensland Curriculum and Assessment Authority (QCAA). This syllabus forms part of a new senior assessment and tertiary entrance system in Queensland. Along with other senior Syllabuses, it is still being refined in preparation for implementation in schools from 2019. For the most current syllabus versions and curriculum information please refer to the QCAA website https://www.qcaa.qld.edu.au/.

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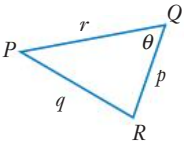
TERMINOLOGY

adjacent
angle of elevation
hypotenuse opposite
radian
triangle definitions
well-conditioned triangle
ambiguous case
bearing
period
reference angle
trigonometric ratios
angle of depression
cosine rule
problem solving
sine rule
unit circle

7.01 Trigonometry of right-angled triangles

This section is a review of work from Year 10.



This triangle has vertices P, Q and R and sides p, q and r. You name the vertices of a triangle with capital letters. The names of the sides are the lower-case letter of the opposite vertex.

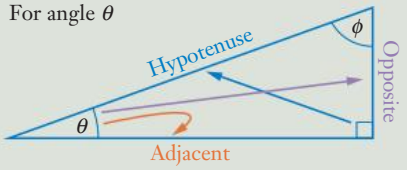
For example, you could write some of the measurements as $PR = 2$ cm or $q = 2$ cm, $\angle Q \approx 79°$, $\angle PQR \approx 79°$ or $\theta = 79°$. The triangle is written with the vertices in order as $\triangle PQR$ or $\triangle QRP$ or $\triangle RPQ$.

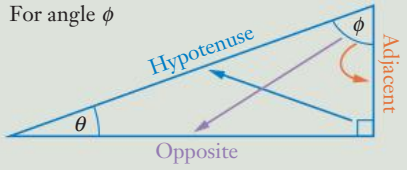
You should also remember the relationships between the sides and angles of right-angled triangles.

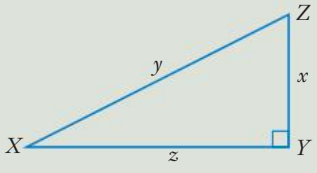
Right-angled triangles

- The **hypotenuse** is the side opposite the right angle
- The **opposite** is the side opposite the angle under consideration
- The **adjacent** is the remaining side, next to the angle
- $\sin(X) = \dfrac{\text{opposite side}}{\text{hypotenuse}} = \dfrac{x}{y}$
- $\cos(X) = \dfrac{\text{adjacent side}}{\text{hypotenuse}} = \dfrac{z}{y}$
- $\tan(X) = \dfrac{\text{opposite side}}{\text{adjacent side}} = \dfrac{x}{z}$

You can remember the ratios with a mnemonic like SOH CAH TOA (sock-ah-towa)







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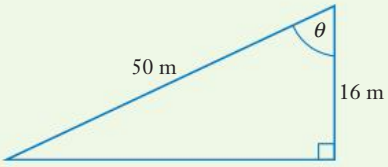
EXAMPLE 1

Find θ in this triangle, correct to 2 decimal places.



Solution

The 50 m side is the hypotenuse and the 16 m side is the adjacent, so use cosine.	$\cos(\theta) = \frac{16}{50}$
Make sure your calculator is set in degrees and remember to use brackets.	$\theta = \cos^{-1}(16 \div 50)$
	$= 71.3370\ldots$
Round and write the answer.	θ is about 71.34°.

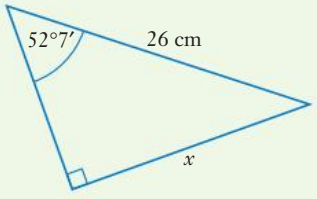
The **exact** answer would be written as $\cos^{-1}\left(\frac{8}{25}\right)$.

When you find a side, the known and required sides tell you which trig ratio to use.

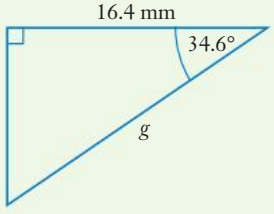
EXAMPLE 2

Find the value of the pronumeral in each triangle.

a



b



Solution

a

The known side is the hypotenuse and the side you want is the opposite, so use sine.	$\sin(52° 7') = \frac{x}{26}$
Multiply by 26 to find x.	$x = 26 \sin(52° 7')$
Use your calculator.	$= 20.5208\ldots$
Round appropriately.	x is about 20.5 cm.

Strictly speaking, the answer should be given as 21 cm because the length was only given correct to 2 significant figures.

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b The known side is the adjacent and the side you want is the hypotenuse, so use cosine.

$$\cos(34.6°) = \frac{16.4}{g}$$

Solve for g.

$$g\cos(34.6°) = 16.4$$

$$g = \frac{16.4}{\cos(34.6°)}$$

Use your calculator.

$$= 19.9237\ldots$$

Round appropriately and write the answer.

g is about 19.9 mm.

You should remember the conventions for **bearings**, **angles of elevation** and **angles of depression**.

WS A page of bearings

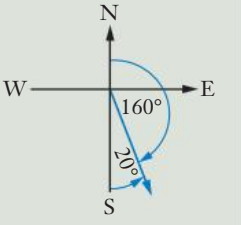
WS Bearings match-up

True bearings

WS Elevations and bearings

Bearings, angles of depression and elevation

- The **bearing** of an object is the 3-digit clockwise angle you turn from due north to face the object. The letter T shows that the bearing is from true north.
- You can also use the angle from north or south towards east or west, so 160° T = S 20° E as shown on the diagram.
- The **angle of elevation** of an object is the angle you must look up from the horizontal to see the object.
- The **angle of depression** of an object is measured downwards from the horizontal.



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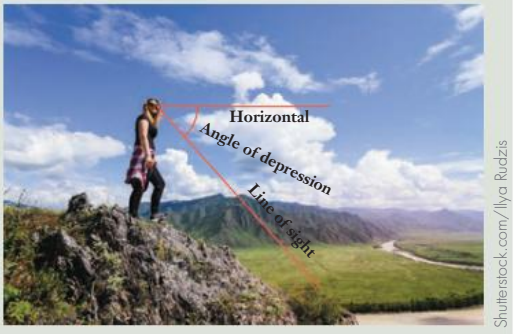

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Always draw a diagram for trigonometry problems.

EXAMPLE 3

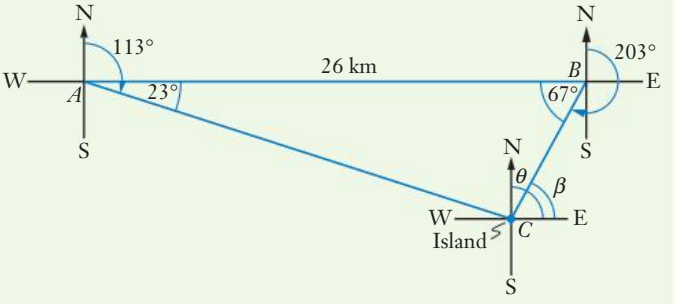
A boat B is 26 km due east of boat A. Boat A sets off for an island in the direction 113°T, and B takes the course 203° T for the same island.

a After they meet, how far has A travelled?

b What is the bearing from the island to B's starting point?

Solution

a Draw a diagram showing the information. Label everything.



Find the third angle in $\triangle ACB$.

$\angle BCA = 180° - (23° + 67°)$

$= 90°$

A travels along AC.

AC is the adjacent for 23° and $AB = 26$ km is the hypotenuse.

$$\cos(23°) = \frac{AC}{26}$$

$AC = 26\cos(23°)$

$= 23.9331\ldots$

Round the answer.

Boat A travelled about 24 km.

b Find θ.

$\theta = 90° - 67°$

$= 23°$

Write the answer.

The bearing from the island to B's starting point is 023° T.

You should round off answers to the accuracy given in questions like the above. Show the 'exact' answer in your working.

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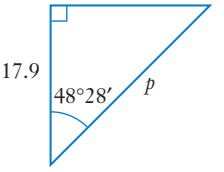
Exercise 7.01 Trigonometry of right-angled triangles

Examples 1,2

1 Find the value of each pronumeral.

a



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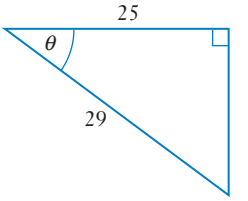

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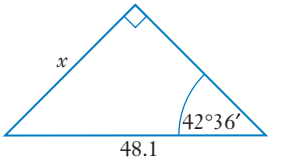

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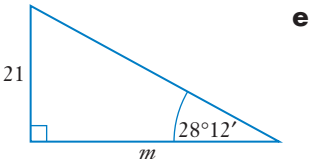

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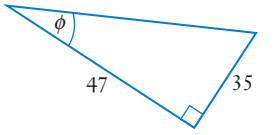

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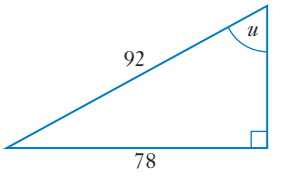
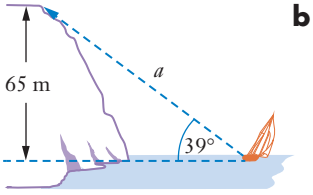


2 Find the value of each pronumeral.

a



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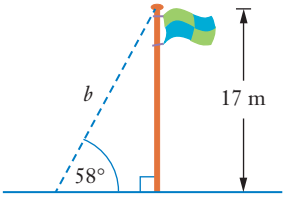

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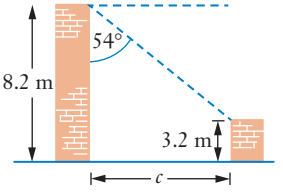

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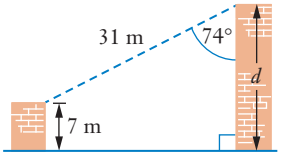

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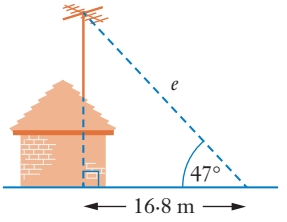

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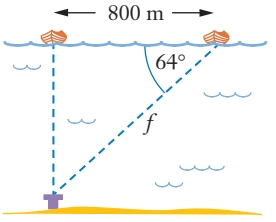
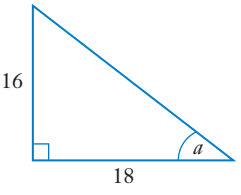


3 Find the value of each pronumeral.

a



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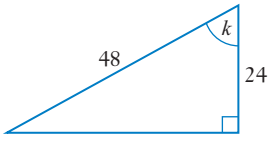

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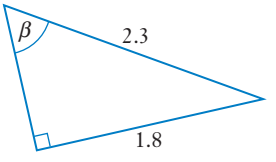

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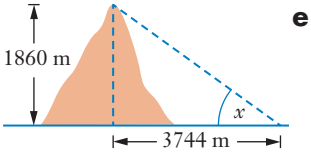

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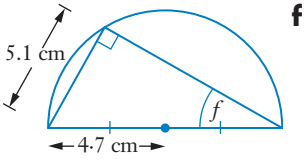

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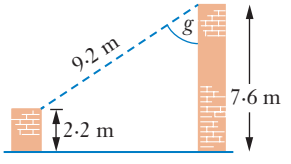

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4 In $\triangle PMQ$, $M = 90°$, $Q = 60°$ and $PQ = 40$ cm. Find the length of:

a MQ **b** PM

5 In $\triangle MNO$, $MN = 24$ mm, $MO = 16$ mm and $M = 60°$. OQ is drawn perpendicular to MN so that it meets MN at Q. Find the length of:

a MQ **b** QO **c** NO

Example 3

6 A fishing boat leaves port and travels on a bearing of N 60° E for 1.6 km until it is due north of a buoy. The buoy is due east of the port.

a What is the bearing of the port from the boat?

b How far does the boat have to travel to reach the buoy?

c How far is it from the buoy to the port?

7 A motorist travels 8 km due north from a petrol station and then 6 km due west. What is the bearing of the motorist from the petrol station?

8 Two towns, X and Y, are 25 km apart, with Y being due north of X. Town Z is due east of X and 60 km away. Find the distance and bearing of Y from Z.

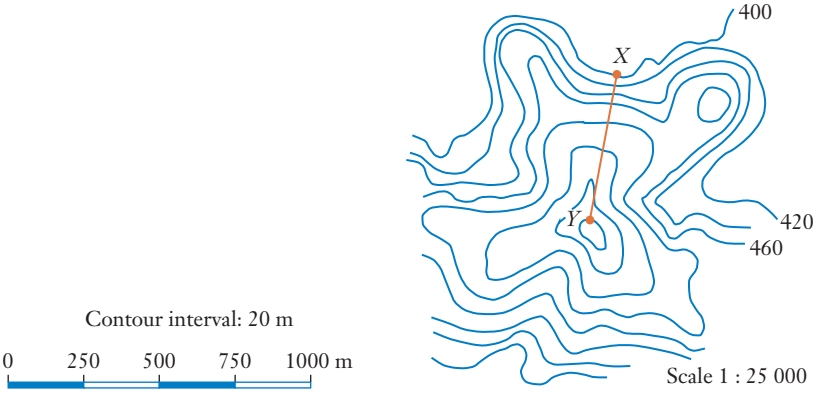
9 A flagpole has a guy wire to stabilise it. The wire is connected 11.2 m above the ground. The other end is attached to a spike in the level ground. The angle of elevation from the spike to the wire is 65°. How far is the spike from the pole?

10 A worker in a cherry picker spots his mate's bald spot at an angle of depression of 40°. His mate is 1.8 m tall and 13.2 m away horizontally. How high is the worker in the cherry picker?

Problem solving

11 A civil engineering company is using a tunnel-boring machine to bore a tunnel into a hill that is inclined at an angle of 15° to the horizontal tunnel. After the tunnel has been drilled 90 m into the hill, a ventilation shaft needs to be driven vertically up to the surface. What will be the depth of the shaft?

12 This diagram shows part of a contour map. An all-terrain vehicle can manage slopes inclined at less than 18° to the horizontal. Would it be possible for the vehicle to proceed directly from X to Y on the route shown? Explain your reasoning.



13 A pendulum 1.2 m long is designed for a grandfather clock. It swings through an angle of 12°. How wide should the case of the clock be?

14 Julio and Maria left school at the same time. Julio cycled due west at 18 km/h and Maria cycled due south at 24 km/h. How long did it take for them to be 15 km apart?

15 An orienteer begins from a starting point, station 1, and runs on a bearing of 350°T until he reaches station 2, which is 3.3 km north of station 1. He then runs due east for 5.2 km to reach station 3. He then changes direction to run roughly south-west until he reaches station 4, which is 2.1 km due east of station 1. What is the bearing from station 3 to station 4?

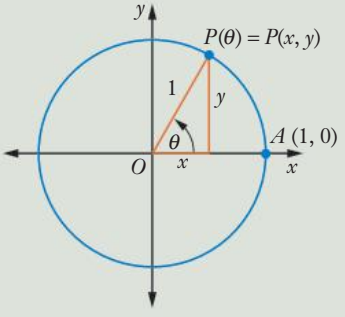
WS
Trigonometry calculations

7.02 Trigonometry and the unit circle

In Year 11 you studied radian measure and trigonometric ratios for any angle size. The rules in right-angled triangles follow from these definitions. You should remember how to work out the signs and values of trigonometric ratios for angles of any size, using the related acute angle.

Radians and the unit circle

- One **radian** is the angle subtended at the centre of a circle by an arc of equal length to the radius
- $180° = \pi$ radians, so $360° = 2\pi$, $90° = \frac{\pi}{2}$, $60° = \frac{\pi}{3}$, $45° = \frac{\pi}{4}$ and $30° = \frac{\pi}{6}$
- For an angle θ from the x-axis to the point $P(x, y)$ on a unit circle
  - $\sin(\theta) = \frac{y}{1} = y$
  - $\cos(\theta) = \frac{x}{1} = x$
  - $\tan(\theta) = \frac{y}{x} = \frac{\sin(\theta)}{\cos(\theta)}$



- The **period** of $\sin(\theta)$ is 2π. It repeats the same values every 2π.

  $\sin(2n\pi + \theta) = \sin(\theta)$ for all integers n
- The period of $\cos(\theta)$ is also 2π
- The period of $\tan(\theta)$ is π

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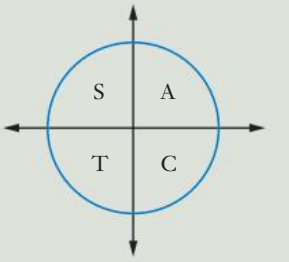
- You can use the ASTC diagram to remember the signs in each quadrant
  - *1st quadrant*: All ratios are positive (**A**)
  - *2nd quadrant*: Sin only is positive (**S**)
  - *3rd quadrant*: Tan only is positive (**T**)
  - *4th quadrant*: Cos only is positive (**C**)



- The **reference angle** for an arbitrary angle θ is the acute angle between θ and the x-axis ($0, \pi, 2\pi, 3\pi, -\pi, -2\pi, -3\pi$, etc.)
- The *value* of a trigonometric ratio is the same as the value of the reference angle
- Use the sign from the ASTC diagram to work out if it is positive or negative

EXAMPLE 4

Express each expression in terms of an angle between 0 and 90°.

a $\sin(125°)$ **b** $\cos\left(\frac{9\pi}{7}\right)$ **c** $\tan(-730°)$ **d** $\sin\left(\frac{28\pi}{3}\right)$

Solution

a Express in terms of 180°.

$\sin(125°) = \sin(180° - 55°)$

Sin is positive in the second quadrant.

$= \sin(55°)$

b Express in terms of π.

$\cos\left(\frac{9\pi}{7}\right) = \cos\left(\pi + \frac{2\pi}{7}\right)$

Cos is negative in the third quadrant.

$= -\cos\left(\frac{2\pi}{7}\right)$

c Express in terms of a multiple of 180°.

$\tan(-730°) = \tan(-720° - 10°)$

Tan is negative in the fourth quadrant.

$= -\tan(10°)$

d Express in terms of a multiple of π.

$\sin\left(\frac{28\pi}{3}\right) = \sin\left(9\pi + \frac{\pi}{3}\right)$

Sin is negative in the third quadrant.

$= -\sin\left(\frac{\pi}{3}\right)$

You should remember the trigonometric ratio values for $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$ and $\frac{\pi}{2}$.

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Standard values

Angle θ	$0 = 0°$	$\frac{\pi}{6} = 30°$	$\frac{\pi}{4} = 45°$	$\frac{\pi}{3} = 60°$	$\frac{\pi}{2} = 90°$
sin (θ)	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos (θ)	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tan (θ)	0	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined

Exact trigonometric values

EXAMPLE 5

Find the exact values of each expression, without using a calculator.

a $\sin(300°)$ **b** $\cos\left(\frac{7\pi}{4}\right)$ **c** $\tan(-210°)$

Solution

a Express in terms of 360°.

$\sin(300°) = \sin(360° - 60°)$

Sin is negative in the 4th quadrant.

$= -\sin(60°)$

Use the exact value.

$= -\frac{\sqrt{3}}{2}$

b Express in terms of 2π.

$\cos\left(\frac{7\pi}{4}\right) = \cos\left(2\pi - \frac{\pi}{4}\right)$

Cos is positive in the 4th quadrant.

$= \cos\left(\frac{\pi}{4}\right)$

Use the exact value.

$= \frac{\sqrt{2}}{2}$

c Express in terms of –180°.

$\tan(-210°) = \tan(-180° - 30°)$

Tan is negative in the 2nd quadrant.

$= -\tan(30°)$

Use the exact value.

$= -\frac{\sqrt{3}}{3}$

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You can simplify expressions like $\sin\left(x+\frac{3\pi}{2}\right)$ or $\cos(7\pi - x)$ by treating x as if it were an acute angle and remembering that sin and cos swap over for odd multiples of $\frac{\pi}{2}$.

EXAMPLE 6

Simplify each expression.

a $\tan(900° - x)$ **b** $\sin(\pi - x)$ **c** $\cos\left(x+\frac{11\pi}{2}\right)$

Solution

a Express in terms of a multiple of 180°. $\tan(900° - x) = \tan(5 \times 180° - x)$

Tan is negative in the 2nd quadrant. $= -\tan(x)$

b Sin is positive in the second quadrant. $\sin(\pi - x) = \sin(x)$

c Express in terms of a multiple of π. $\cos\left(x+\frac{11\pi}{2}\right) = \cos\left(4\pi + x + \frac{3\pi}{2}\right)$

Cos is positive in the 4th quadrant. $= \cos\left(x+\frac{3\pi}{2}\right)$

Change the ratio. $= \sin(x)$

Exercise 7.02 Trigonometry and the unit circle

1 Express each expression in terms of an acute angle. Example 4

a $\sin(200°)$ **b** $\cos(140°)$ **c** $\tan(112°)$

d $\cos\left(\frac{7\pi}{8}\right)$ **e** $\sin\left(\frac{3\pi}{5}\right)$

2 Find the value of each expression without using a calculator. Example 5

a $\sin(150°)$ **b** $\cos(-210°)$ **c** $\tan(315°)$

d $\sin(390°)$ **e** $\sin\left(-\frac{5\pi}{4}\right)$ **f** $\cos\left(\frac{3\pi}{4}\right)$

g $\tan\left(-\frac{2\pi}{3}\right)$ **h** $\cos\left(\frac{5\pi}{4}\right)$ **i** $\sin\left(-\frac{4\pi}{3}\right)$

j $\tan\left(-\frac{5\pi}{6}\right)$

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3 Find the value of each expression without using a calculator.

a $\sin\left(-\frac{11\pi}{3}\right)$ **b** $\cos\left(\frac{8\pi}{3}\right)$ **c** $\tan(480°)$

d $\cos(405°)$ **e** $\sin(-420°)$ **f** $\cos\left(\frac{13\pi}{4}\right)$

g $\tan\left(-\frac{13\pi}{6}\right)$ **h** $\sin\left(-\frac{15\pi}{4}\right)$ **i** $\cos\left(-\frac{16\pi}{3}\right)$

j $\tan\left(-\frac{17\pi}{3}\right)$

Example 6

4 Simplify each expression.

a $\sin(900° + x)$ **b** $\cos(5\pi - y)$ **c** $\tan(x + 4\pi)$

d $\sin\left(\frac{3\pi}{2} + x\right)$ **e** $\cos\left(\frac{7\pi}{2} - x\right)$

Problem solving

5 Show that $\sin(\pi + x) = -\sin(x)$ for any value of x.

6 Simplify $\sin(2\pi - x) + 2\sin(\pi - x)$.

WS Areas of triangles

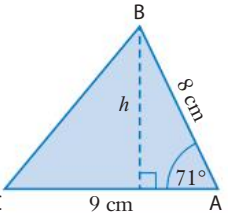
7.03 The area of a triangle

The area of the triangle in the diagram is given by $A = \frac{1}{2}bh$.

The base is side $b = 9$ cm.

The height for this base is given by $\sin(71°) = \frac{h}{8}$, so $h = 8\sin(71°)$

and $Area = \frac{1}{2} \times 9 \times 8\sin(71°) \approx 34 \text{ cm}^2$.



You can use this method to find a different formula for the area of a triangle.

Area of a triangle

- The area of $\triangle ABC$ is given by Area $= \frac{1}{2}bc\sin(A)$
- The area is half the product of 2 sides and the sine of the enclosed angle

You can prove this easily using the method shown for the triangle with sides 8 cm and 9 cm above. If $\angle A$ is obtuse, the altitude (outside the triangle) is given by $c\sin(180° - A)$.

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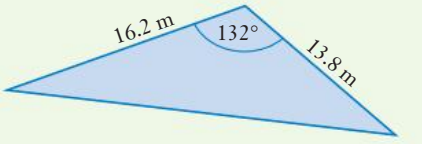
EXAMPLE 7

Find the area of this triangle.



Solution

Write the formula.

$$A = \frac{1}{2}bc\sin(A)$$

Substitute values and evaluate.

$$= \frac{1}{2} \times 16.2 \times 13.8 \sin(132°)$$

$$= 83.0687\ldots$$

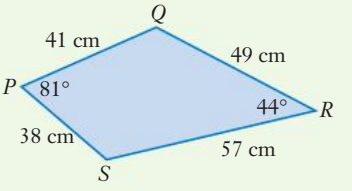
Round appropriately and write the answer.

The area is about $83.1\ \text{m}^2$.

You can use the triangle area rule to find the areas of other shapes.

EXAMPLE 8

Find the area of this shape.



Solution

Write as a sum.

$$\text{Area of } PQRS = \text{Area of } \triangle PQS + \text{Area of } \triangle QRS$$

Write formulas.

$$= \frac{1}{2}PQ \times PS \sin(P) + \frac{1}{2}RQ \times RS \sin(R)$$

Substitute values and evaluate.

$$= \frac{1}{2} \times 41 \times 38 \sin(81°) + \frac{1}{2} \times 49 \times 57 \sin(44°)$$

$$= 769.409\ldots + 970.090\ldots$$

$$= 1739.499\ldots$$

Round and write the answer.

The area is about $1740\ \text{cm}^2$.

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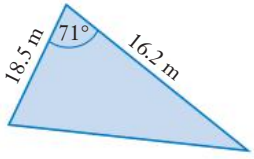
Exercise 7.03 The area of a triangle

Example 7

1 Find the area of each triangle.

a



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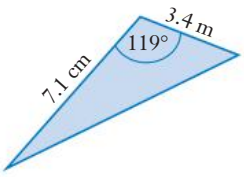

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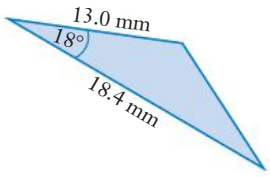

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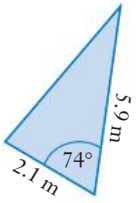

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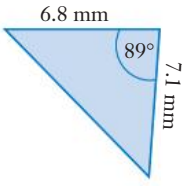

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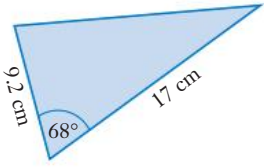
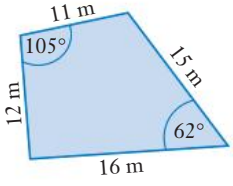


Example 8

2 Find the area of each quadrilateral.

a



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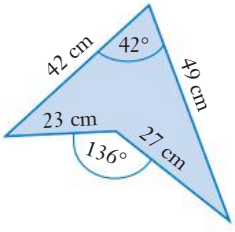

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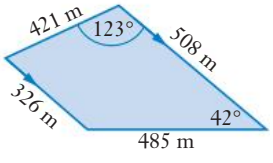
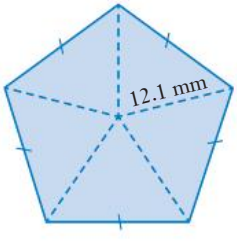


Problem solving

3 Find the area of each shape.

a



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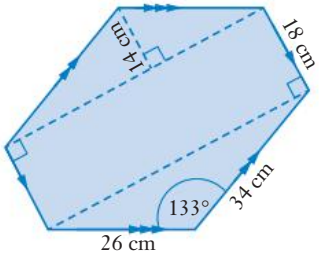

4 Prove the formula Area of $\triangle ABC = \frac{1}{2}bc \sin(A)$ for an acute angle A.

5 Prove the formula Area of $\triangle ABC = \frac{1}{2}bc \sin(A)$ for an obtuse angle A.

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7.04 The sine rule

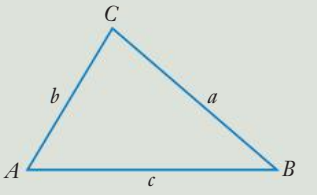
Sine rule problems

You can use the **sine rule** to find unknown angles or sides in any non-right-angled triangle.

The sine rule

For any $\triangle ABC$

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$



In any $\triangle ABC$,

Area $= \frac{1}{2}ab\sin(C)$ or $\frac{1}{2}bc\sin(A)$ or $\frac{1}{2}ac\sin(B)$

So

$$\frac{1}{2}bc\sin(A) = \frac{1}{2}ac\sin(B) = \frac{1}{2}ab\sin(C)$$

Multiplying by $\frac{2}{abc}$ we get,

$$\frac{2}{abc} \times \frac{1}{2}bc\sin(A) = \frac{2}{abc} \times \frac{1}{2}ac\sin(B) = \frac{2}{abc} \times \frac{1}{2}ab\sin(C)$$

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}$$

Inverting gives

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)} \quad \text{QED}$$

There are other proofs of the sine rule.

You use the sine rule for triangles where you know an angle and the opposite side.
Finding all the unknown sides and angles is called **solving** a triangle.

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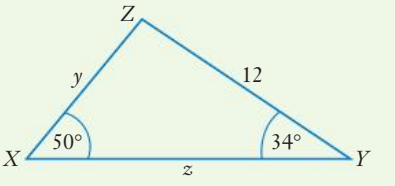
EXAMPLE 9

In $\triangle XYZ$, $X = 50°$, $Y = 34°$ and $x = 12$. Solve the triangle.

Solution

Draw a diagram.



X and x are known, so use these to find y.

$$\frac{x}{\sin(X)} = \frac{y}{\sin(Y)}$$

Substitute known values.

$$\frac{12}{\sin(50°)} = \frac{y}{\sin(34°)}$$

Get y on its own and evaluate.

$$y = \frac{12\sin(34°)}{\sin(50°)}$$

$$= 8.759\ldots$$

Find Z.

$$Z = 180° - (50° + 34°)$$

$$= 96°$$

Now find z.

$$\frac{x}{\sin(X)} = \frac{z}{\sin(Z)}$$

Substitute known values.

$$\frac{12}{\sin(50°)} = \frac{z}{\sin(96°)}$$

Get z on its own and evaluate.

$$z = \frac{12\sin(96°)}{\sin(50°)}$$

$$= 15.579\ldots$$

Round and write the answers.

$Z = 96°$, $y \approx 8.8$ and $z \approx 15.6$

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INVESTIGATION

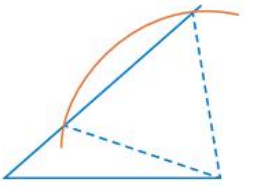
AMBIGUOUS TRIANGLES

Draw the following triangles:

$\triangle ABC$ with $B = 45°$, $a = 8$ cm, and $b = 6$ cm

$\triangle EFG$ with $F = 26°$, $e = 8$ cm, and $f = 10$ cm

$\triangle PQR$ with $Q = 76°$, $p = 8$ cm, and $q = 9$ cm



In which cases is more than one triangle possible?

Can you work out when 2 sides and the angle not between the sides gives a unique answer?

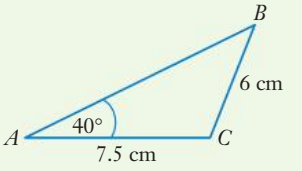
It is sometimes possible to draw a triangle with 2 known sides and an angle in 2 ways. For this to be true, the sine rule must give 2 answers for the angle opposite the second side. One answer will be acute and one will be obtuse, but they will have the same sine value. This is called the **ambiguous case**. If there are 2 possible answers, the sum of the given angle and the obtuse answer will be less than 180°.

EXAMPLE 10

In $\triangle ABC$, $a = 6$ cm, $b = 7.5$ cm and $A = 40°$. Calculate the remaining angles and side and sketch the possible triangles.

Solution

Make a sketch.



Use the inverted version of the sine rule when you are finding an angle.

$$\frac{\sin(B)}{b} = \frac{\sin(A)}{a}$$

Substitute values and find sin (B).

$$\frac{\sin(B)}{7.5} = \frac{\sin(40°)}{6}$$

$$\sin(B) = \frac{7.5\sin(40°)}{6}$$

$$= 0.8034\ldots$$

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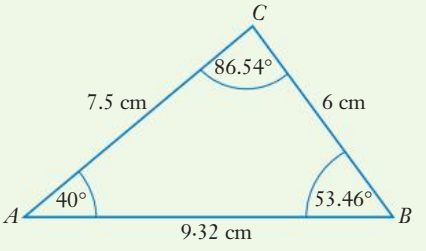
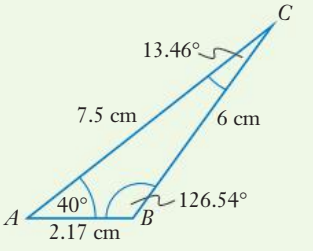
Use $\sin^{-1}$ with the full value on your calculator.	$B = \sin^{-1}(0.8034\ldots)$ $= 53.4641\ldots$
Allow for the obtuse angle.	$B = 53.4641\ldots°$ or $B = 126.5358\ldots°$
Check if the obtuse angle is possible.	$40° + 126.5358\ldots° = 166.5358\ldots° < 180°$, so the obtuse case is possible.
Find the possible 'exact' values of C.	$C = 180° - 166.5358\ldots° = 13.4641\ldots°$ or $C = 180° - 93.4641\ldots° = 86.5358\ldots°$
Now use A, a and C to find possible values of c.	$\frac{c}{\sin(C)} = \frac{a}{\sin(A)}$ $c = \frac{a\sin(C)}{\sin(A)}$ $= \frac{6\sin(13.46...°)}{\sin(40°)}$ or $\frac{6\sin(86.53...°)}{\sin(40°)}$
Use the 'exact' values on your calculator.	$c = 2.1733\ldots$ or $9.3172\ldots$
Round and write the answers.	$B \approx 53°$, $C \approx 87°$ and $c \approx 9.3$ cm or $B \approx 127°$, $C \approx 13°$, and $c \approx 2.2$ cm

Sketch the possible triangles.





Sometimes you need to find a side in one triangle in order to find a side in another. The intermediate side you need is usually between the known side and the side you want.

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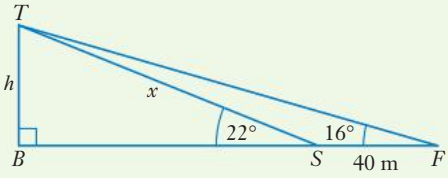
EXAMPLE 11

Jermaine sees that the top of a sand dune is at an angle of elevation of 16°. On walking 40 m closer, he finds that the angle of elevation increases to 22°. What is the height of the sand dune?

Solution

Draw a sketch. Label all the vertices and call the dune height h.

The side TS is between h and the 40 m length, so find it first.



TS is opposite the 16° angle and the 40 m side is opposite $\angle STF$.

$\angle STF + 16° = 22°$ (exterior angle in $\triangle STF$)

$\angle STF = 6°$

Use the sine rule in $\triangle STF$ to find ST.

$$\frac{ST}{\sin(16°)} = \frac{40}{\sin(6°)}$$

$$ST = \frac{40\sin(16°)}{\sin(6°)}$$

Don't bother calculating ST. Now use the right-angled triangle $\triangle STB$ to find h.

$$\sin(22°) = \frac{h}{ST}$$

$$h = ST\sin(22°)$$

Substitute the expression for ST.

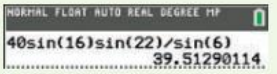
$$= \frac{40\sin(16°)\sin(22°)}{\sin(6°)}$$

Calculate the answer.

$$= 39.5129\ldots$$

TI-84 Plus CE

Make sure your calculator is set to degrees.



Round and write the answer.

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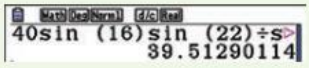
Choose the Run-matrix menu and make sure your calculator is set to degrees.



The dune is about 39.5 m high.

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If you were asked to find the distance to the dune instead of the height, TS would still be the intermediate side.

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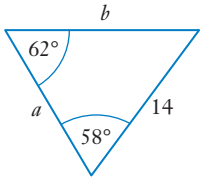
Exercise 7.04 The sine rule

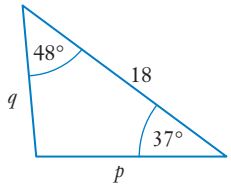
Example 9

1 Calculate the lengths of the unknown sides in each triangle, correct to 1 decimal place.

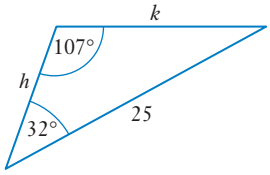
a



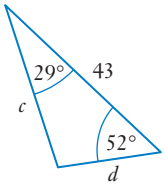
b



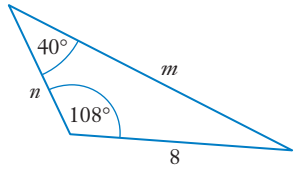
c



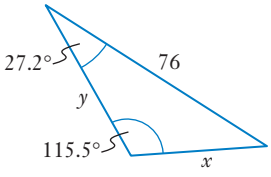
d



e



f



2 Solve each triangle.

a $\triangle XYZ$, where $Z = 65°$, $y = 11$ m and $z = 16.2$ m

b $\triangle PQR$, where $P = 35.3°$, $Q = 52.8°$ and $q = 67.5$ cm

c $\triangle DEF$, where $F = 111°$, $f = 12.5$ km and $d = 8.96$ km

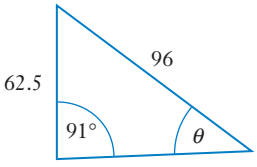
d $\triangle ABC$, where $B = 124.1°$, $C = 18.7°$ and $c = 94.6$ cm

e $\triangle XYZ$, where $X = 68.41°$, $Y = 54.23°$ and $x = 12.75$ m

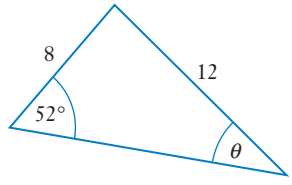
f $\triangle KLM$, where $K = 71.83°$, $M = 42.57°$ and $l = 2.614$ km

3 Calculate the value of θ in each triangle, correct to 1 decimal place.

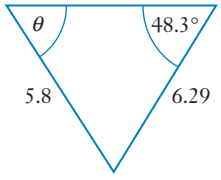
a



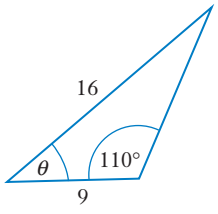
b



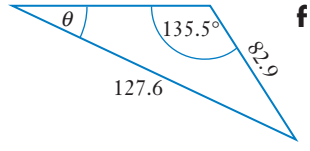
c



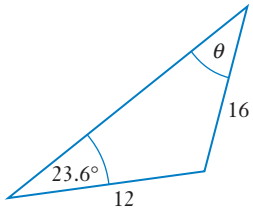
d



e



f



Example 10

4 In $\triangle ABC$, $BC = 18$ cm, $AC = 15$ cm and $B = 38°$. Find both possible values of A and make a sketch of each possible $\triangle ABC$.

5 In $\triangle DEF$, $f = 30$ cm, $d = 35$ cm and $F = 18°$. Find possible values for e and D.

6 In $\triangle XYZ$, $x = 2.3$ cm, $z = 3.7$ cm and $X = 37.7°$. Solve the triangle.

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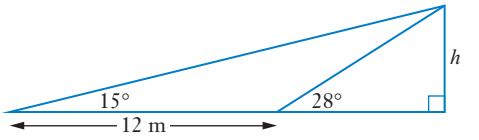
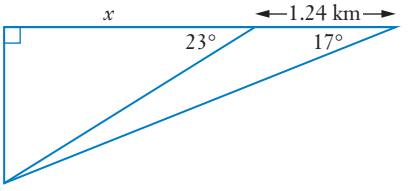
7 Find the length of the longest side of $\triangle PQR$ if $P = 31° 42'$, $Q = 28° 50'$ and $PR = 42$ mm.

8 Find h in the diagram below.

Example 11

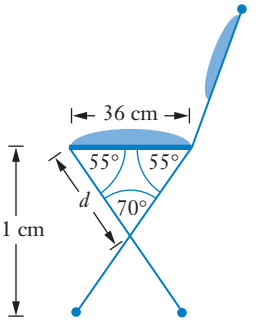


9 Find x in the diagram below.

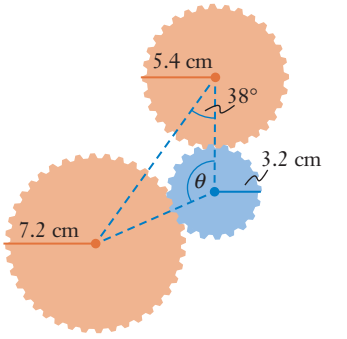


Problem solving

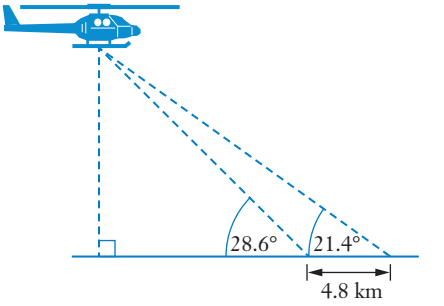
10 The folding chair shown in the diagram has a seat that is 36 cm deep. The legs of the chair are inclined at an angle of 55° to the seat and join at an angle of 70°. The seat is 51 cm above the floor. Find the distance from the seat, d, at which the legs are joined.



11 Three gears are arranged as shown on the right. The radii of the gears are 7.2 cm, 5.4 cm and 3.2 cm. Calculate θ.



12 A helicopter is sighted at the same time by 2 ground observers who are 4.8 km apart, as shown in the diagram. The angles from the observers to the helicopter (measured to the horizontal) are 21.4° and 28.6°. How high is the helicopter?



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13 Anne is flying from Brisbane to Dalby at a speed of 150 km/h. She thinks her altimeter, which shows a height of 15 000 feet (4600 m), is faulty. Looking ahead, she sees the top of a hill marked as 405 m on her map at an angle of depression of 13°. Two minutes later, the angle of depression has increased to 20.6°.

What is her actual altitude, and what should the radio operator tell her about her altimeter?

WS Cosine rule problems

WS Finding an unknown side

WS Finding an unknown angle

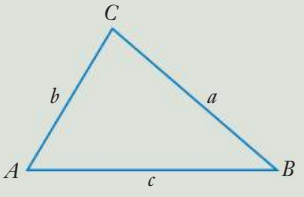
7.05 The cosine rule

Triangles with 2 sides and only the enclosed angle known cannot be solved using the sine rule. You need to use the **cosine rule** for these triangles. You can also use it to find an angle if all the sides are known.

The cosine rule

For any $\triangle ABC$

- $a^2 = b^2 + c^2 - 2bc\cos(A)$ or $\cos(A) = \dfrac{b^2 + c^2 - a^2}{2bc}$
- $b^2 = a^2 + c^2 - 2ac\cos(B)$ or $\cos(B) = \dfrac{a^2 + c^2 - b^2}{2ac}$
- $c^2 = a^2 + b^2 - 2ab\cos(C)$ or $\cos(C) = \dfrac{a^2 + b^2 - c^2}{2ab}$

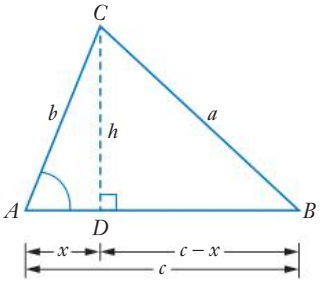


It is enough to prove $a^2 = b^2 + c^2 - 2bc\cos(A)$ because the rest are just re-namings or rearrangements. You need to consider 2 cases: where A is acute or obtuse.

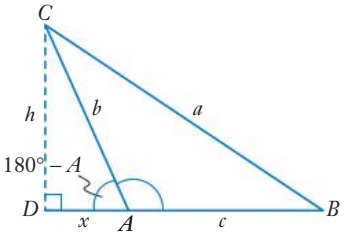
For $A = 90°$, $\cos(A) = 0$ so it reduces to $a^2 = b^2 + c^2$, Pythagoras' theorem.

Construct the altitude CD from C to the side BA (extended for the obtuse case) and name the remaining parts as shown in the diagrams.

A acute:



A obtuse:



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Using Pythagoras' theorem in $\triangle BCD$:

$a^2 = h^2 + (c - x)^2$ $\qquad$ $a^2 = h^2 + (c + x)^2$

$a^2 = h^2 + x^2 + c^2 - 2cx$ $\qquad$ $a^2 = h^2 + x^2 + c^2 + 2cx$

Using Pythagoras' theorem in $\triangle ACD$, $h^2 + x^2 = b^2$ in both cases.

$a^2 = b^2 + c^2 - 2cx$ $\qquad$ $a^2 = b^2 + c^2 + 2cx$

In $\triangle ACD$:

$\cos(A) = \frac{x}{b}$ $\qquad$ $\cos(180° - A) = \frac{x}{b}$

$x = b\cos(A)$ $\qquad$ $x = b\cos(180° - A) = -b\cos(A)$

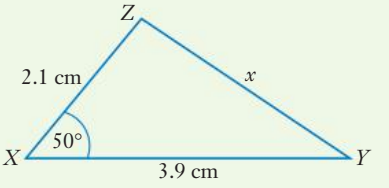
Substituting in the above:

$a^2 = b^2 + c^2 - 2cb\cos(A)$ $\qquad$ $a^2 = b^2 + c^2 + 2c[-b\cos(A)]$

Rearranging, in both cases gives $a^2 = b^2 + c^2 - 2bc\cos(A)$. QED

EXAMPLE 12

Solve this triangle.



Solution

Write the cosine rule for x. $\qquad x^2 = y^2 + z^2 - 2yz\cos(X)$

Substitute values and calculate. $\qquad = 2.1^2 + 3.9^2 - 2 \times 2.1 \times 3.9 \times \cos(50°)$

$\qquad = 9.0911\ldots$

Find the square root and keep the 'exact' value on your calculator. $\qquad x = 3.0151\ldots$

Y is opposite the smaller of 2.1 and 3.9, so choosing it avoids the ambiguous case of the sine rule.

Now use the sine rule to find Y. $\qquad \frac{\sin(Y)}{y} = \frac{\sin(X)}{x}$

Rearrange and substitute values. $\qquad \sin(Y) = \frac{2.1\sin(50°)}{3.0151\ldots}$

Make sure you use the 'exact' value on your calculator for x. $\qquad Y = \sin^{-1}\left[\frac{2.1\sin(50°)}{3.0151\ldots}\right] = 32.2447\ldots$

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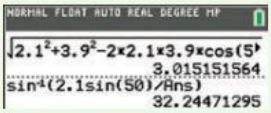
Chapter 7

TI-84 Plus CE

Make sure your calculator is set to degrees.

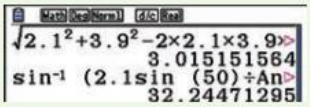


Find Z.

Round and write the answers.

Casio fx-CG20AU

Choose the Run-matrix menu and make sure your calculator is set to degrees.



$Z = 180° - (50° + 32.2447\ldots°)$

$= 97.7552\ldots$

$x \approx 3.0$ cm, $Y \approx 32.2°$ and $Z \approx 97.8°$

When you calculate the angle from 3 sides, remember that it is the side opposite the angle that is subtracted.

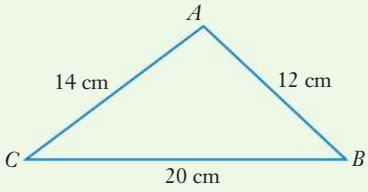
The cosine rule for angles

EXAMPLE 13

Calculate correct to one decimal place the largest angle in the triangle with sides 12 cm, 14 cm and 20 cm.

Solution

Draw a sketch and name the triangle.



The largest angle is opposite the largest side, so it is A.

$$\cos(A) = \frac{b^2 + c^2 - a^2}{2bc}$$

$$= \frac{14^2 + 12^2 - 20^2}{2 \times 14 \times 12}$$

$$= -0.1785\ldots$$

Use your calculator.

$A = \cos^{-1}(-0.1785\ldots)$

$= 100.2865\ldots$

$\approx 100.3°$

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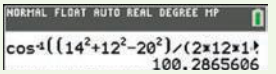
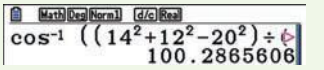
Make sure your calculator is set to degrees.



Round and write the answer.

Casio fx-CG20AU

Choose the Run-matrix menu and make sure your calculator is set to degrees.



The largest angle is about 100.3°.

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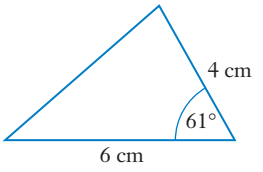
Exercise 7.05 The cosine rule

1 Calculate the unknown side length in each triangle.

Example 12

a



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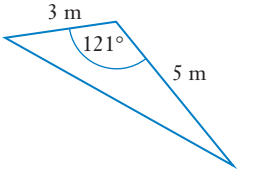

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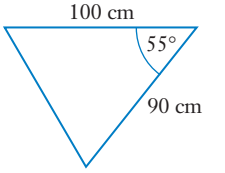
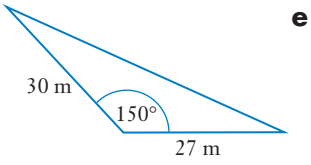


d



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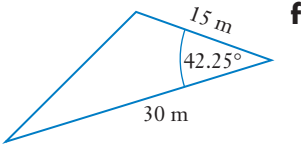

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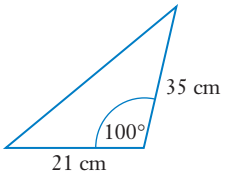
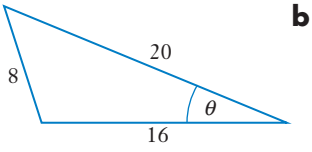


2 Find θ in each triangle.

Example 13

a



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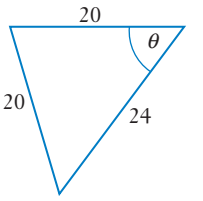

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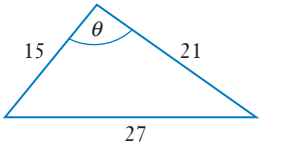

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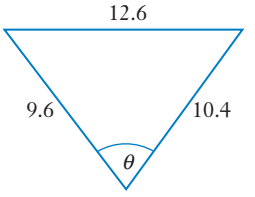

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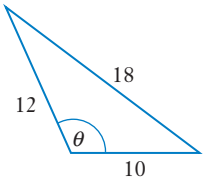

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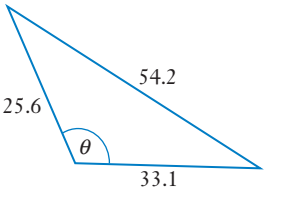

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3 Find the third side of each triangle.

a $\triangle MNO$, where $M = 41.1°$, $n = 27.8$ and $o = 39.2$

b $\triangle XYZ$, where $Z = 28.3°$, $y = 571$ and $x = 421$

c $\triangle DEF$, where $F = 45.6°$, $d = 72.3$ and $e = 89.4$

d $\triangle GHI$, where $G = 67.3°$, $h = 37.9$ and $i = 40.8$

e $\triangle STU$, where $S = 112.8°$, $t = 62.8$ and $u = 122$

f $\triangle PQR$, where $Q = 74.8°$, $p = 891.9$ and $r = 642.7$

4 Solve each triangle.

a $\triangle DAG$, where $D = 121°$, $a = 3$ m and $g = 5$ m

b $\triangle UVW$, where $W = 55°$, $u = 45$ cm and $v = 50$ cm

c $\triangle XYZ$, where $x = 7.2$ m, $y = 12.5$ m and $z = 8.3$ m

d $\triangle ABC$, where $a = 8.5$ m, $b = 20$ m and $c = 15.2$ m

e $\triangle LMN$, where $L = 42.3°$, $m = 15.4$ km and $n = 12.9$ km

f $\triangle REF$, where $E = 168.2°$, $f = 192$ mm and $r = 151$ mm

5 In $\triangle ABC$, $a = 12$, $b = 9$ and $c = 8$. Calculate the largest angle.

6 In $\triangle XYZ$, $x = 8.8$, $y = 11$ and $z = 17.6$. Calculate the smallest angle.

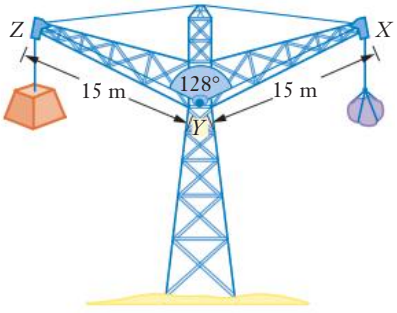
7 In $\triangle PQR$, $p = \sqrt{5}$, $q = 1$ and $r = 2$. Calculate the largest angle.

Problem solving

8 The diagram shows a crane with 2 arms, each 15 m long. The arms of the crane are separated by an angle of 128°.

A load is attached at point X and the counterweight is attached at point Z.

Find XZ.



9 A weight is supported by cables attached to a horizontal beam. The cables are of lengths 12 m and 9 m. They are attached to the beam at points 18 m apart. Find the angle that each cable makes with the beam.

10 A hiker encounters a swamp at point X that is 4.2 km from point Y on the opposite side of the swamp. To travel around the swamp, the hiker walks 3.4 km from X on a bearing of 325° T and then walks 1.8 km to point Y. Find the bearing of X from Y.

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7.06 2D applications of trigonometry

When you are applying trigonometry to find positions or lengths, it is important to use **well-conditioned triangles** if possible.

Well-conditioned triangles

All the angles in a well-conditioned triangle are between 30° and 120°.

INVESTIGATION

ILL-CONDITIONED TRIANGLES

Calculate the possible errors that would be made in solving each triangle described below if there were errors of 5% in any of the given measurements.

1 $A = 20°, a = 10, B = 45°$

2 $A = 50°, a = 10, B = 60°$

3 $A = 10°, a = 10, B = 70°$

4 $A = 5°, a = 10, B = 65°$

5 $A = 10°, b = 10, c = 20$

6 $A = 5°, b = 10, c = 20$

7 $A = 60°, b = 10, c = 15$

What do you find about the well-conditioned and ill-conditioned triangles?

When you are solving triangles, you should follow the procedure below.

Solving triangles

- Look for solutions using right-angled triangles first.
- Use the sine rule when you know one angle and the opposite side. Be careful of ambiguous cases.
- Use the cosine rule when the known angle is enclosed by 2 known sides or the 3 sides are known. After using the cosine rule once, look for the sine rule again.
- If the known side is in a different triangle from an unknown side, the side common to the triangles is often the intermediate step.

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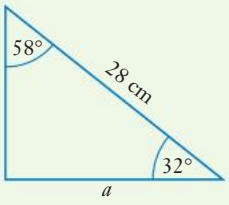
EXAMPLE 14

Find the value of the pronumeral in each triangle.

a



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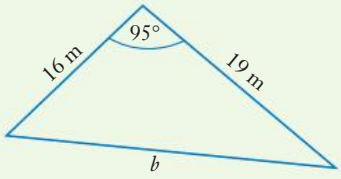
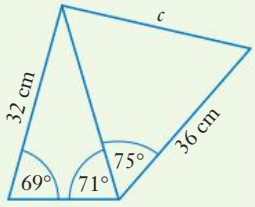


c



Solution

a Find the third angle.

$$\text{Third angle} = 180° - (58° + 32°)$$
$$= 90°$$

Use a right-angled triangle method.

$$\cos(32°) = \frac{a}{28}$$
$$a = 28\cos(32°)$$
$$= 23.7453\ldots$$

Round and write the answer.

a is about 23.7 cm.

b Use the cosine rule.

$$b^2 = a^2 + c^2 - 2ac\cos(B)$$

Substitute and evaluate.

$$= 16^2 + 19^2 - 2 \times 16 \times 19 \times \cos(95°)$$
$$= 669.9906\ldots$$

Find the square root.

$$b = 25.8841\ldots$$

Round and write the answer.

b is about 25.9 m.

c Call the common side x. Find it first.

$$\frac{x}{\sin(69°)} = \frac{32}{\sin(71°)}$$
$$x = \frac{32\sin(69°)}{\sin(71°)}$$
$$= 31.5959\ldots$$

Now use the cosine rule.

$$c^2 = x^2 + 36^2 - 2x \times 36 \times \cos(75°)$$
$$= 1705.5151\ldots$$
$$c = \sqrt{1705.5151\ldots}$$
$$= 41.2978\ldots$$

When you have to use the cosine rule in a second step, evaluate the first answer, but keep the 'exact' answer on your calculator.

Round and write the answer.

c is about 41.3 cm.

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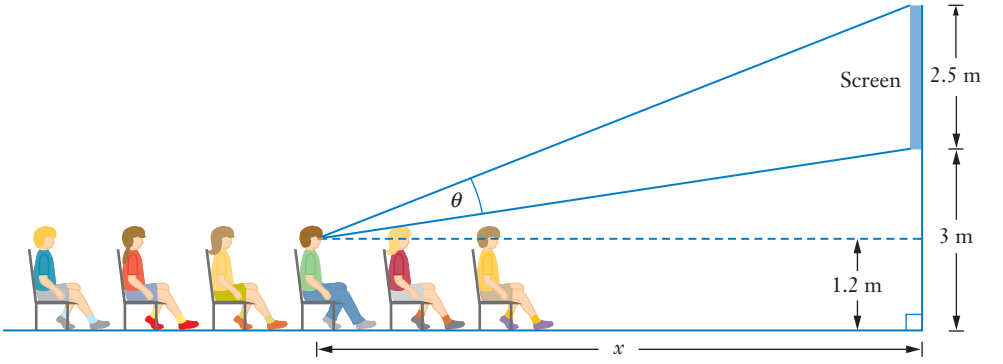
INVESTIGATION

VIEWING DISTANCES AND ANGLES

This diagram shows a cinema. The screen is 2.5 m high and is mounted so that the bottom of the screen is 3 m from the floor. Eye level for the audience is approximately 120 cm.



Work in groups to determine the distance x (to the nearest 10 cm) that will give the viewer the maximum viewing angle θ. Use a table of values such as that below, or a spreadsheet, to find the best distance.

x (m)	1	2	3	4	5	6	7	8
θ								

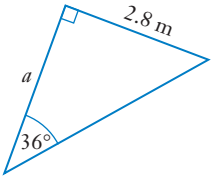
Modern cinemas have a sloping floor to make it easier for viewers to see the screen. Keep the same dimensions for the screen, but change the seating so that the floor slopes up towards the back. Make another table or spreadsheet to do your investigations. Look at the changes of angle with flat and sloping floors of different angles.

Exercise 7.06 2D applications of trigonometry

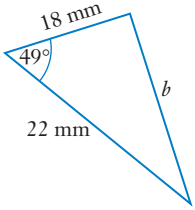
Example 14

1 Find the value of the pronumeral in each triangle.

a



b



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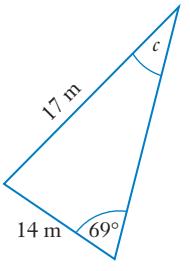

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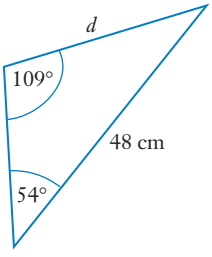

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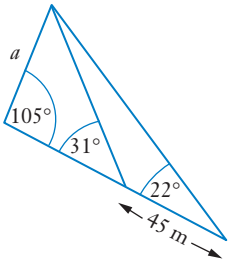
2 Find the value of the pronumeral in each triangle.

a



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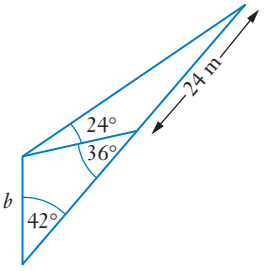

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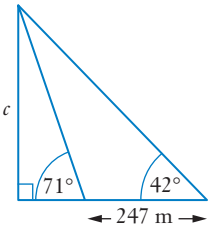

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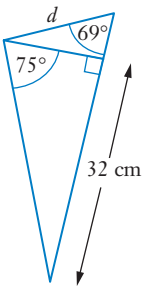

Problem solving

3 A building in the central business district is 85 m tall. The angle of depression from the top of this building to a fire alarm on the face of a building on the opposite side of the street is 58° 20'. The street is 15 m wide. Calculate the height of the fire alarm above the ground and the angle of elevation from the base of the first building to the fire alarm.

4 A boat leaves a jetty at the base of a lighthouse that is 80 m high. It sails directly away from the lighthouse and after 5 minutes is at an angle of depression of 7.67° from an observer at the top of the lighthouse. At what speed did the boat sail? Comment on the accuracy of the answer.

5 An observer sees that the angle of elevation to the top of a tower, 80 m high and due south of her position, is 37°. The top of another tower, 60 m high and due west of her position, is at an angle of elevation of 28°. Calculate the horizontal distance between the towers.

6 From the cockpit of an aircraft cruising 1520 m above the ground, the angles of depression to the tops of two identical (same height) structures are 50° and 70° respectively. How far apart are the 20 m high structures?

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7 An observer in a lighthouse, 120 m above sea level, is watching a fishing trawler move toward the lighthouse. The angle of depression of the trawler from the observer is 18°. The angle of depression changes to 28° as the trawler moves. How far has the trawler travelled? Comment on the accuracy of the answer.

8 While Ann is walking on a straight road running east–west from A to B, which are 10 km apart, she sees a church in the distance. The bearing from A to the church is N 48° E and from B the church is at a bearing of N 67° W. Find the distance from A to the church.

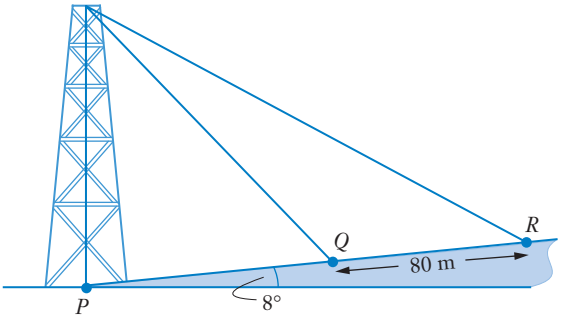
9 The top of a lighthouse is at an angle of elevation of 35° from an observer some distance from a point on its base. The top of the lighthouse is 38 m above the point on the base. This point appears to be at an angle of elevation of 8° from the observer. How far is the observer from the base of the lighthouse? Comment on the accuracy of the answer.

10 An aircraft flies on a course of 322° T for 250 km and then changes course to 244° T to an airstrip where it lands. If the airstrip is 480 km from the starting position, calculate the distance travelled from the time that the course was changed until the plane landed, and the bearing from the airstrip to the starting position.

11 A girl is flying a kite and has extended 250 m of string when an updraft makes the kite rise suddenly and she has to let out another 60 m of string. If the angle of elevation to the kite was 38° before the updraft, how far has the kite been driven vertically?

12 Two points, A and B, on the same bank of a river are 120 m apart. Point C is on the opposite bank. $\angle CAB$ and $\angle CBA$ are measured to be 55° and 33° respectively. Find the width of the river.

13 A tower is erected at the base of a road that is inclined at a constant angle of 8° to the horizontal. Angles of elevation to the top of the tower are measured at two points that are 80 m apart, some distance up a slope away from the tower. The angles are found to be 25° and 8° respectively. What is the height of the tower? Comment on the answer.



14 A low-impact mobile phone tower is on top of a building. The angles of elevation of the top and bottom of the tower are 16° and 10° respectively. Thirty metres closer to the building, the angle of elevation of the bottom of the tower has increased to 17°. Find the height of the tower above the building and the building's height above the ground. Comment on the accuracy of the answer.

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WS
Sine and cosine rules assignment

WS
Sine and cosine rules problems

7.07 3D applications of trigonometry

You should try to simplify problems in 3D before finding the answer.

3D trigonometric methods

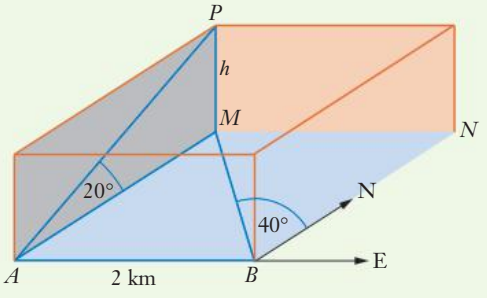
- Make a 3D sketch inside a box
- Label points on the sketch
- 'Unfold' the sketch and draw the triangles flat
- Label the unknowns and intermediate sides and angles
- Use right-angled triangles, sine and cosine rules to find the unknowns
- Write the solution to the problem in an appropriate form

EXAMPLE 15

From a point, A, a mountain peak due north has an angle of elevation of 20°. From another point, B, 2 km east of A and on the same level as A, the bearing of the peak is N 40° W. Find the height of the peak above the level of A and B.

Solution

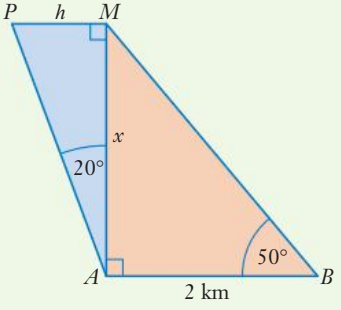
Make a sketch inside a box and label the points. Put all information on the sketch.



Find necessary angles.

$\angle ABM = 50°$ (complementary angles)

Unfold the diagram to get two flat triangles joined along AM.



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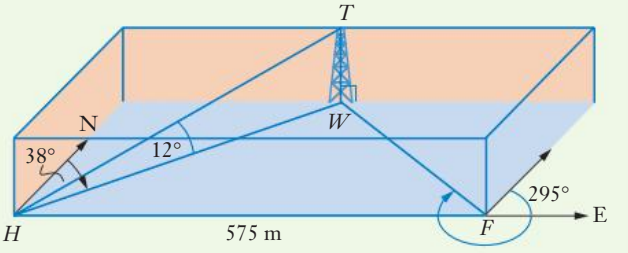
Use $\triangle MAB$ to find x, but don't evaluate it.	$\tan(50°) = \frac{x}{2}$
	$x = 2\tan(50°)$
Now use $\triangle PAM$ to find h.	$\tan(20°) = \frac{h}{x}$
	$h = x\tan(20°)$
Substitute x and calculate the answer.	$= 2\tan(50°)\tan(20°)$
	$= 0.8675...$ km
Round and write the answer, commenting if necessary.	The peak is about 870 m high, but one triangle is ill-conditioned, so this may be a little inaccurate.

EXAMPLE 16

Hassan observes that the top of a transmission tower at a bearing of 038°T is at an angle of elevation of 12°. Fatima is 575 m due east of Hassan, and she says the bearing of the tower is 295°T. Find the height of the tower.

Solution

Make a sketch inside a box and label the points. Put all information on the sketch.



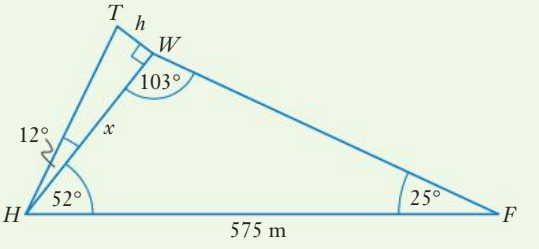
Find necessary angles.

$\angle FHW = 52°$ (complementary)

$\angle WFH = 295° - 270° = 25°$

$\angle HWF = 103°$ (sum of angles in $\triangle WHF$)

Unfold the diagram to get two flat triangles joined along HW.



Use the sine rule in $\triangle WHF$ to find x, but don't evaluate it.

$$\frac{x}{\sin(25°)} = \frac{575}{\sin(103°)}$$

$$x = \frac{575\sin(25°)}{\sin(103°)}$$

Use $\triangle THW$ to find h.

$$\tan(12°) = \frac{h}{x}$$

$$h = x\tan(12°)$$

Substitute x and calculate.

$$= \frac{575\sin(25°)\tan(12°)}{\sin(103°)}$$

$$= 53.0110...$$

Round and write the answer, commenting if necessary.

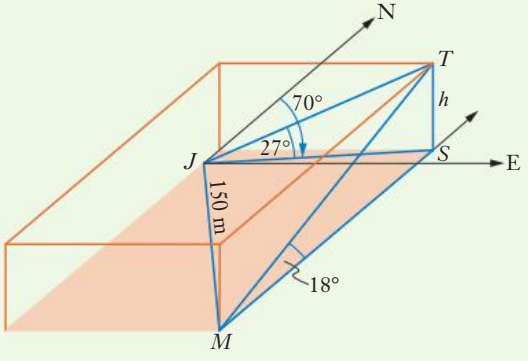
The transmission tower is about 53 m high, but the triangles used are not well-conditioned, so this could be quite inaccurate.

EXAMPLE 17

Maria and Jack are standing on a level road 150 m apart. A church spire is due north of Maria and on a bearing of 070°T from Jack. The top of the spire appears at an angle of elevation of 27° to Jack and 18° to Maria. Find the height of the spire above the road.

Solution

Make a sketch inside a box and label the points. Put all information on the sketch.



$\angle JSM = 70°$ (alternate with $\angle NJS$)

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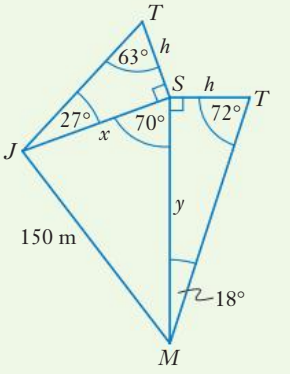
Unfold the diagram to make it flat, showing the side TS on 2 of the flat triangles.



Find other angles.

$\angle JTS = 63°$ and $\angle MTS = 72°$ (Angle sums)

Find x in $\triangle TSJ$.

$$\tan(63°) = \frac{x}{h}$$

$$x = h \tan(63°)$$

Find y in $\triangle TSM$.

$$\tan(72°) = \frac{y}{h}$$

$$y = h \tan(72°)$$

Now use the cosine rule in $\triangle JMS$.

$$150^2 = x^2 + y^2 - 2xy \cos(70°)$$

Substitute x and y.

$$150^2 = [h \tan(63°)]^2 + [h \tan(72°)]^2 - 2[h \tan(63°)][h \tan(72°)] \cos(70°)$$

$$= h^2 \tan^2(63°) + h^2 \tan^2(72°) - 2h^2 \tan(63°) \tan(72°) \cos(70°)$$

Factorise.

$$= h^2[\tan^2(63°) + \tan^2(72°) - 2 \tan(63°) \tan(72°) \cos(70°)]$$

Calculate.

$$150^2 = h^2 \times 9.1921...$$

$$h = 150 \div \sqrt{9.1921...}$$

$$= 49.4745...$$

Round and write the answer, commenting as necessary.

The height is about 49 m, but 2 of the triangles are ill-conditioned, so this may be inaccurate.

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Exercise 7.07 3D applications of trigonometry

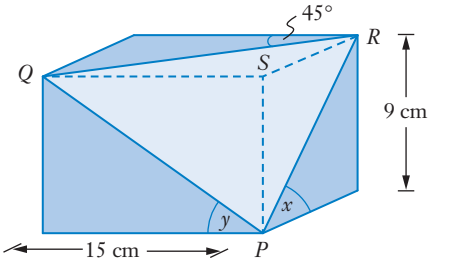
Example 15

1 A flagpole 12 m high is temporarily erected for an Australia Day ceremony. It is held in place by 2 wires: one 15 m long fastened due west of the pole and the other 17 m long fastened due south of the pole. Both supporting wires are attached to the top of the flagpole. Calculate the:

- **a** distance that each wire is from the pole at the point where it meets the ground
- **b** angle of inclination that each wire makes with the horizontal
- **c** distance between the points where the wires meet the ground
- **d** angle between the wires.

2 A sculptor has removed a section from a copper block with dimensions as shown in the diagram on the right. Calculate:

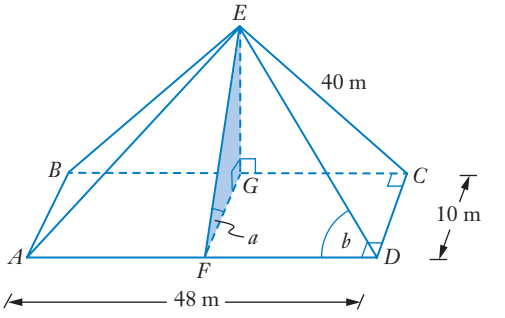
- **a** QR
- **b** x
- **c** y
- **d** the angle that the triangular face PQR makes with the horizontal base.



Examples 16-17

3 A large tent is constructed in the shape of a rectangular-based pyramid $ABCDE$ with the vertex E vertically above G, the midpoint of BC, as shown in the diagram. Calculate the angle:

- **a** that the triangular face AED makes with the base of the pyramid
- **b** between the lines DE and AD.



Problem solving

4 A rectangular-based pyramid with a base measuring 16 cm by 12 cm and a height of 10 cm is to be constructed using acrylic sheeting.

Calculate the angles that the triangular faces make with the horizontal base.

5 From a lighthouse 150 m above sea level, the lighthouse keeper observes a boat due east at an angle of depression of 25° and another boat due south at an angle of depression of 32°.

Find the bearing and distance of the second boat from the first.

6 An aircraft flying due east at 385 km/h at a constant height of 1500 m passes over a village at a particular instant. One minute later, a person who is 1.6 km south of the village sees the plane. In what direction and at what angle of elevation does the person see the plane?

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7 A straight stick 2 m long is stuck in the ground and is inclined at 60° to the horizontal, pointing in an easterly direction. Calculate the length and direction of the shadow of the stick when the sun is due north at an elevation of 30°.

8 A hillside is considered to have a uniform slope of 22° with the horizontal. A hiker, starting from point *E*, walks due north up the hillside for 120 m to point *C*, then walks due west to point *B*. The hiker then walks 170 m back to the starting point. Calculate the hiker's angle of descent.

9 Two groups of people set off to spotlight possums at night. One group spots a possum. The tree that has the possum is at a bearing of N 28° W. The angle of elevation of their spotlight beam is 38°. At this instant, the other group, which is 100 m west of the first group, sights the same tree at a bearing of N 29° E. How high up the tree is the possum?

iStock/Getty Images Plus/CarrieCaptured

10 Elena sees that the angle of elevation to the top of a tower, 80 m high, which is due south of her position, is 37°. The top of another tower, 60 m high, which is due west of her, is at an angle of elevation of 28°. Calculate the horizontal distance between the two towers.

11 Miguel is 90 m due west of Consuella on a level road. He sees that the angle of elevation of a kite flown by their son is 30°, while Consuella sees that it is due north at an angle of elevation of 38°. What is the height of the kite?

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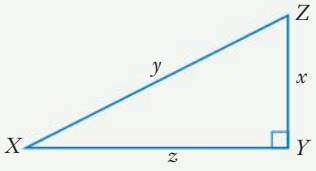
7. CHAPTER SUMMARY

The sine and cosine rules

Triangle definitions of trigonometric ratios

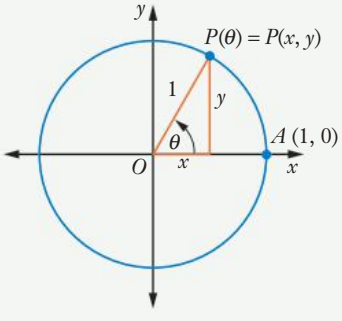
- In a right-angled triangle the **hypotenuse** is the side opposite the right angle
- The **opposite** is the side opposite the angle under consideration
- The **adjacent** is the remaining side, next to the angle
- $\sin(X) = \dfrac{\text{opposite side}}{\text{hypotenuse}} = \dfrac{x}{y}$
- $\cos(X) = \dfrac{\text{adjacent side}}{\text{hypotenuse}} = \dfrac{z}{y}$
- $\tan(X) = \dfrac{\text{opposite side}}{\text{adjacent side}} = \dfrac{x}{z}$



- The **bearing** of an object is the 3-digit clockwise angle you turn from due north to face the object. The letter T shows that the bearing is from true north
- You can also use the angle from north or south towards east or west, so 160° T = S 20° E
- The **angle of elevation** of an object is the angle you must look up from the horizontal to see the object
- The **angle of depression** of an object is measured downwards from the horizontal
- One **radian** is the angle subtended at the centre of a circle by an arc of equal length to the radius
- $180° = \pi$ radians, so $360° = 2\pi$, $90° = \dfrac{\pi}{2}$, $60° = \dfrac{\pi}{3}$, $45° = \dfrac{\pi}{4}$ and $30° = \dfrac{\pi}{6}$

Unit circle definitions of trigonometric ratios

- For an angle θ from the x-axis to the point $P(x, y)$ on a unit circle
  - $\sin(\theta) = \dfrac{y}{1} = y$
  - $\cos(\theta) = \dfrac{x}{1} = x$
  - $\tan(\theta) = \dfrac{y}{x} = \dfrac{\sin(\theta)}{\cos(\theta)}$



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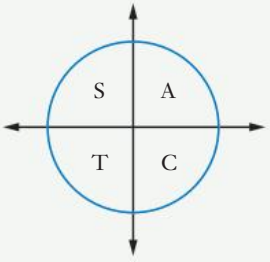
- The **period** of sin (θ) is 2π. It repeats the same values every 2π
- sin ($2n\pi + \theta$) = sin (θ) for all integers n
- The period of cos (θ) is also 2π
- The period of tan (θ) is π
- You can use the ASTC diagram to
- remember the signs in each quadrant
  - *1st quadrant*: All ratios are positive (**A**)
  - *2nd quadrant*: Sin only is positive (**S**)
  - *3rd quadrant*: Tan only is positive (**T**)
  - *4th quadrant*: Cos only is positive (**C**)



- The **reference angle** for an arbitrary angle θ is the acute angle between θ and the x-axis ($0, \pi, 2\pi, 3\pi, -\pi, -2\pi, -3\pi$, etc.)
- The *value* of a trigonometric ratio is the same as the value of the reference angle
- Use the sign from the ASTC diagram to work out if it is positive or negative

Standard values of trigonometric ratios

Angle θ	$0 = 0°$	$\frac{\pi}{6} = 30°$	$\frac{\pi}{4} = 45°$	$\frac{\pi}{3} = 60°$	$\frac{\pi}{2} = 90°$
sin (θ)	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos (θ)	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tan (θ)	0	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined

- The area of $\triangle ABC$ is given by Area $= \frac{1}{2}bc \sin(A)$
- The area is half the product of two sides and the sine of the enclosed angle
- The **sine rule** states that for any $\triangle ABC$: $\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$
- sin (A) = sin ($180 - A$), giving the **ambiguous case** of the sine rule. If there are 2 possible answers, the sum of the given angle and the obtuse answer will be less than 180°.

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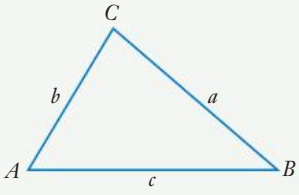
- The **cosine rule** states that for any $\triangle ABC$
  - $a^2 = b^2 + c^2 - 2bc\cos(A)$ or $\cos(A) = \dfrac{b^2 + c^2 - a^2}{2bc}$
  - $b^2 = a^2 + c^2 - 2ac\cos(B)$ or $\cos(B) = \dfrac{a^2 + c^2 - b^2}{2ac}$
  - $c^2 = a^2 + b^2 - 2ab\cos(C)$ or $\cos(C) = \dfrac{a^2 + b^2 - c^2}{2ab}$



- All the angles in a **well-conditioned triangle** are between 30° and 120°

Problem solving

- For 3D problems sketch inside a box, otherwise use a flat sketch
- Label points on the sketch
- 'Unfold' 3D sketches to sketch the triangles flat
- Label the unknowns and intermediate sides and angles
- Look for right-angled triangles first, then use the sine and cosine rules to find the unknowns
- If the known side is in a different triangle to an unknown side, the side common to the triangles is often the intermediate step
- Write the solution to the problem in an appropriate form

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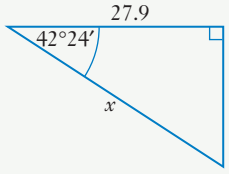
7. CHAPTER REVIEW

The sine and cosine rules

1 Find the value of the pronumeral in each triangle.

Examples 1, 2

a



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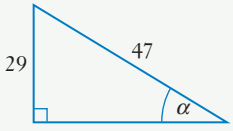

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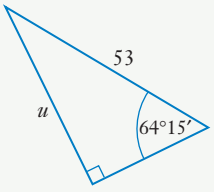
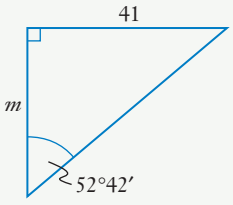
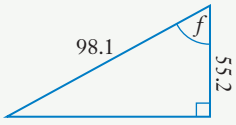


d

e



f



2 A lighthouse, L, is 40 km away from a harbour located at K. A lighthouse is on a bearing of 225° T from the harbour. A fishing boat travelled from the harbour on a bearing of 205°T until it was due south of the lighthouse.

Example 3

a How far did the fishing boat travel?

b How far is the fishing boat from the lighthouse?

3 Express cos (153°) in terms of an acute angle.

Example 4

4 Find the value of each expression without using a calculator.

Example 5

a sin (240°) **b** cos (330°) **c** tan (135°)

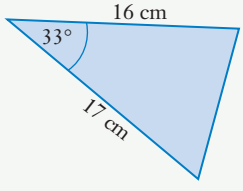
5 Simplify sin (270° + x).

Example 6

6 Find the area of each figure.

Examples 7, 8

a



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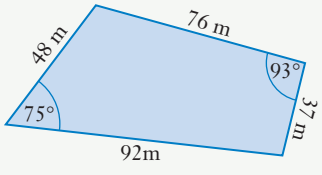

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Examples 9,10

7 Solve each triangle.

a $\triangle ABC$, $A = 42°$, $b = 36$ m, $a = 28$ m

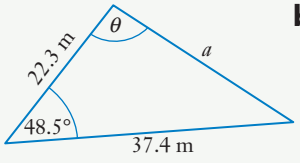
b $\triangle FEG$, $F = 58.2°$, $f = 42.5$ cm, $ZE = 28.4°$

c $\triangle KMN$, $M = 35°$, $m = 29$ m, $n = 19$ m

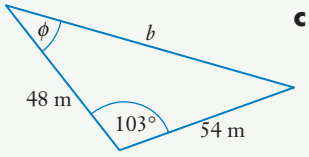
Examples 12,13

8 Find the value of the pronumerals in each triangle.

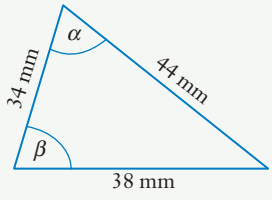
a



b



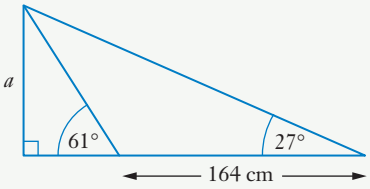
c



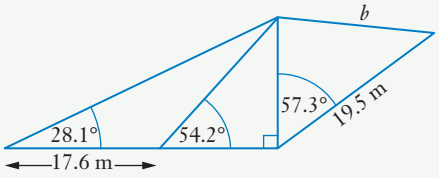
Example 14

9 Find the value of the pronumeral in each triangle.

a

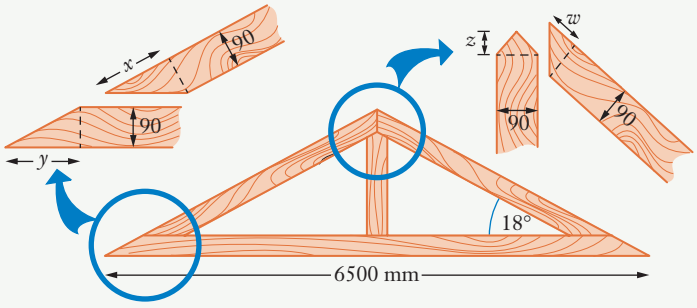


b



Problem solving

10 A roof truss is being made to the design shown below. Calculate the lengths of timber required, and the lengths required to make the cuts at the correct angles as shown in the detail drawings.



11 A yacht sails on a triangular course. The first leg is 18 km NE and the second leg is 15 km at 168°T. What is the third leg?

12 A person standing at point Z on the bank of a river observes a tree at point X on the opposite bank, at a bearing of 292.17° T. The person then walks 885 m down the bank to point Y and notes the tree is now at a bearing of 285.33°T. Find the distance between the tree and the initial position, Z, of the observer.

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13 A coastal surveillance vessel *Interceptor* detects 2 suspect boats on the radar screen. The first is 60 km away from the *Interceptor* on a bearing of 062.2° T. The second is 42 km away on a bearing of 296.58°T. How far apart are the 2 suspect boats?

14 The top of a hill is at an angle of elevation of 7°30'. At 100 m closer to the hill, the angle of elevation is 8°. What is the height of the hill?

15 A surveyor notes that the top of a water tower is at a bearing of 062° T. After walking 200 m directly east, she finds that the tower is at a bearing of 059° 10' T. The angle of elevation of the top of the tower is then 4°31'. What is the height of the tower?

16 A table is 1 m wide, 2.5 m long and 900 mm high. The 5 cm thick top is supported on legs splayed outwards along the diagonals by 8° from the vertical. The legs meet the floor directly under the corners of the table. Find how far from the ends and sides the legs are attached.

Practice quiz

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8.

THE SECOND DERIVATIVE

For the function $f(x)$, the derivative $f'(x)$ is also known as the **first derivative**. The derivative of a function tells you about the slope of the tangent to the function at the point $P(x, y)$. The value of the derivative tells you whether the function is increasing or decreasing, and by how much it is increasing or decreasing.

The derivative of $f'(x)$ is known as the **second derivative** $f''(x)$ of the function $f(x)$. While the first derivative can tell you if the function is increasing or decreasing, the second derivative tells you if the first derivative is increasing or decreasing. This feature of the second derivative can be used in many applications, including curve sketching and optimisation.

8.01 The second derivative
8.02 Acceleration
8.03 Concavity and points of inflection
8.04 The second derivative test
8.05 Curve sketching
8.06 Optimisation applications
Chapter summary
Chapter review

SYLLABUS SUBJECT MATTER

The second derivative and applications of differentiation

- understand the concept of the second derivative as the rate of change of the first derivative function
- recognise acceleration as the second derivative of displacement position with respect to time
- understand the concepts of concavity and points of inflection and their relationship with the second derivative
- understand and use the second derivative test for finding local maxima and minima
- sketch the graph of a function using first and second derivatives to locate stationary points and points of inflection
- solve optimisation problems from a wide variety of fields using first and second derivatives, where the function to be optimised is both given and developed

Prior learning

Mathematical Methods 2019 v1.1 General Senior Syllabus © Queensland Curriculum and Assessment Authority (QCAA). This syllabus forms part of a new senior assessment and tertiary entrance system in Queensland. Along with other senior Syllabuses, it is still being refined in preparation for implementation in schools from 2019. For the most current syllabus versions and curriculum information please refer to the QCAA website https://www.qcaa.qld.edu.au/.

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TERMINOLOGY

acceleration
concavity
first derivative
point of inflection
revenue function
velocity
concave downward
cost function
optimisation
position
second derivative
concave upward
displacement
point of horizontal inflection
profit function
second derivative test

WS
Higher derivatives

8.01 The second derivative

It may be possible to differentiate a derivative, if it exists, to find a higher order derivative. For the function $y = f(x)$, the derivative $f'(x)$ is also known as the **first derivative**. The derivative of the first derivative is called the **second derivative**. The second derivative is an example of a **higher order derivative**.

First and second derivatives

Given that $y = f(x)$,

- the first derivative $= f'(x) = \dfrac{dy}{dx} = f^1(x)$
- the second derivative $= f''(x) = \dfrac{d^2y}{dx^2} = f^2(x)$

EXAMPLE 1

Find the first 2 derivatives of $f(x) = x^4 - x^3 + 3x^2 + 7$ and calculate $f''(-2)$.

Solution

Find the first derivative.	$f'(x) = 4x^3 - 3x^2 + 6x$
Find the second derivative.	$f''(x) = 12x^2 - 6x + 6$
Let $x = -2$.	$f''(-2) = 12(-2)^2 - 6 \times (-2) + 6$
Evaluate.	$= 48 + 12 + 6$
	$= 66$

You can use your calculator to check the value of the second derivative at $x = -2$.

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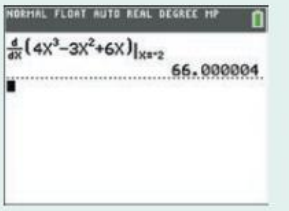
TI-84 Plus CE

Press [math] and select 8:nDeriv(.

Enter x to complete $\frac{d}{dx}$ and then enter the first derivative in the brackets.
Enter $x = -2$ and press enter.

The result is displayed.



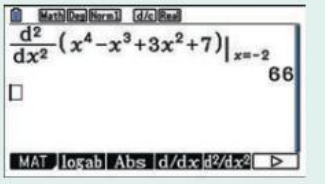
Casio fx-CG20AU

Select the [F1] mode.

Press [F4] (MATH) and then press [F5] (d^2/dx^2).

Enter the function in the brackets.
Enter $x = -2$ and press [EXE].

The result is displayed.



EXAMPLE 2

Find the second derivative of each function.

a $f(x) = x \sin(x)$ **b** $y = e^{1-2x}$ **c** $f(t) = \ln(3 + t^2)$

Solution

a

Write the function.	$f(x) = x \sin(x)$
Let $u = x$ and $v = \sin(x)$.	$= u\,v$
Use the product rule to find $f'(x)$.	$f'(x) = u\,v' + v\,u'$
$u' = 1$ and $v' = \cos(x)$.	$= x \cos(x) + \sin(x)$
Find the second derivative.	$f''(x) = \frac{d}{dx}[x \cos(x) + \sin(x)]$
Differentiate each term separately.	$= \{x \times [-\sin(x)] + \cos(x)\} + \cos(x)$
Simplify.	$= 2 \cos(x) - x \sin(x)$

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b Write the function. $y = e^{1-2x}$

Let $u = 1 - 2x$. $y = e^u$

Use the chain rule. $\frac{dy}{dx} = \frac{d}{du}(e^u) \times \frac{du}{dx}$

Find the derivatives. $= e^u \times (-2)$

Substitute for u and rearrange. $y' = -2e^{1-2x}$

Use the chain rule again. $y' = -2e^u$

$y'' = -2e^u \times (-2)$

Substitute for u and simplify. $= 4e^{1-2x}$

c Write the function. $f(t) = \ln(3 + t^2)$

Let $u = 3 + t^2$. $= \ln(u)$

Use the chain rule. $f'(t) = \frac{d}{du}[\ln(u)] \times \frac{du}{dt}$

Find the derivatives. $= \frac{1}{u} \times 2t$

Substitute for u. $= \frac{2t}{3+t^2}$

Let $w = 2t$ and $v = 3 + t^2$. $f'(t) = \frac{w}{v}$

Use the quotient rule to find $f''(t)$. $f''(t) = \frac{vw' - wv'}{v^2}$

Find the derivatives. $= \frac{(3+t^2)2 - 2t \times 2t}{(3+t^2)^2}$

Evaluate. $= \frac{6 - 2t^2}{(3+t^2)^2}$

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Exercise 8.01 The second derivative

Example 1

1 Find the first and second derivatives of each function.

a $f(x) = x^5 - 3x^3 + 7x + 11$

b $f(t) = 6t^8 + 7t^5 - 8t^2$

c $f(m) = m^7 - 5m^4 - 6m - 12$

d $f(x) = x^{-2} - x^4 + 3x^{-1} + 8x$

e $f(z) = 2z^3 - 7z + 4\sqrt{z} - 9$

f $f(x) = x^{-1} - 5x^2 + 6x^{\frac{1}{2}} - 3x^{\frac{2}{3}} + 4x$

2 Calculate the value of the second derivative of each function.

a $f''(2)$ if $f(x) = x^5 - 2x^2 + 11$

b $f''(-1)$ if $f(x) = 3x^4 + 2x^3 - 7x$

c $f''(-2)$ if $f(x) = 5x^4 + x^3 - 9x + 5$

d $f''(3)$ if $f(x) = 7 - 3x^4 + 6x^2 - 11x$

e $f''(-2)$ if $f(x) = x^2 - x^{-1} + 12x$

f $f''(3)$ if $f(x) = 5x^5 + 3x^2 - 12\sqrt{x} - 10x$

Example 2

3 Calculate $f'(x)$ and $f''(x)$ for each function.

a $f(x) = \sin(2x)$

b $f(x) = 4x - \cos(3x)$

c $f(x) = x\cos(x)$

d $f(x) = x^2 \sin(2x)$

e $f(x) = \sin(3x^2)$

f $f(x) = 5x^3 - 3\cos(6x)$

4 Calculate y' and y'' for each function.

a $y = 4e^{3x+1}$

b $y = 2e^{5-4x}$

c $y = e^{4x^2+2}$

d $y = xe^{2x-3}$

e $y = 4x^2e^{1-3x}$

f $y = (2e^{3x} + 1)e^{2x-1}$

5 Calculate $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for each function.

a $y = \ln(3x)$

b $y = \ln(7 - 5x)$

c $y = \ln(5x^2 - 1)$

d $y = 2x\ln(x)$

e $y = 3x^2[\ln(8 - 3x)]$

f $y = (2x - 7)[\ln(5x^2 - 1)]$

6 Find the first and second derivatives of each function.

a $f(x) = (1 - 6x)^5$

b $f(t) = \frac{2t+1}{t-1}$

c $g(w) = \sqrt{w^2 + 1}$

d $h(m) = \frac{m+2}{m^2+2m}$

e $f(x) = (x^2 + 1)^{\frac{2}{3}}$

f $g(p) = \ln(p^3 e^p)$

7 If $f(x) = \frac{x^3 - 1}{x}$, find $f'(2)$ and $f''(-3)$.

Problem solving

8 Find the second derivative of $f(x) = \tan(4x)$.

9 If $f(x) = \frac{1}{x}$, find the 1st, 2nd, 3rd, 4th and 5th derivatives. Use the results to find a rule for the nth derivative of $f(x) = \frac{1}{x}$.

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8.02 Acceleration

Position, velocity and acceleration

The **position** of a particle, x, is measured relative to a fixed point called the **origin**.

The **displacement**, s, of a particle is the change in its position.

Velocity, v, is the rate of change of position or displacement, so $v = \frac{dx}{dt} = \frac{ds}{dt}$.

Acceleration, a, is the rate of change of velocity, so $a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = \frac{d^2x}{dt^2}$.

EXAMPLE 3

The position of a particle is given by $x = 2t^3 - 3t^2 + 5$, where t is time in seconds.

a Find an expression for the acceleration of the particle.

b What is the initial acceleration of the particle?

c When will the acceleration reach 30 m/s^2?

Solution

a Find the first derivative. $v = \frac{dx}{dt}$

$= 6t^2 - 6t$

Find the second derivative. $a = \frac{d^2x}{dt^2} = \frac{dv}{dt}$

$= 12t - 6$

State the result. $a = 12t - 6$

b Let $t = 0$. $a = 12 \times 0 - 6$

Evaluate. $= -6$

State the result. The initial acceleration is -6 m/s^2.

c Let $a = 30$. $30 = 12t - 6$

Solve for t. $12t = 36$

$t = 3$

State the result. The acceleration reaches 30 m/s^2 after 3 s.

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The shape of the graphs of position, velocity and acceleration provide information about the motion of a particle.

Velocity is the rate of change of position and so the particle changes direction when the velocity is zero.

The particle *speeds up* when either:

- the velocity is positive and increasing, i.e., when the velocity is positive and the acceleration is positive ($v > 0$ and $a > 0$); or
- the velocity is negative and decreasing, i.e., when the velocity is negative and the acceleration is negative ($v < 0$ and $a < 0$).

The particle *slows down* when either:

- the velocity is positive and decreasing, i.e., when the velocity is positive and the acceleration is negative ($v > 0$ and $a < 0$); or
- the velocity is negative and increasing, i.e., when the velocity is negative and the acceleration is positive ($v < 0$ and $a > 0$).

EXAMPLE 4

The position of a particle is given by $x = t^3 - 6t^2 + 9t$ for $t > 0$, where t is in seconds and x is in metres.

a Find expressions for the velocity (v) and acceleration (a) of the particle at time t.

b What is the particle's velocity and acceleration when $t = 1$?

c Plot the graphs of x, v and a for $0 \leq t \leq 4$ on the same set of axes.

d Between which times is the particle speeding up?

e Between which times is the particle slowing down?

Solution

a Find the first derivative.

$$v = \frac{dx}{dt}$$
$$= 3t^2 - 12t + 9$$

Find the second derivative.

$$a = \frac{d^2x}{dt^2} = \frac{dv}{dt}$$
$$= 6t - 12$$

State the result.

At time t, $v = 3t^2 - 12t + 9$ and $a = 6t - 12$.

b Let $t = 1$.

$$v = 3 \times 1^2 - 12 \times 1 + 9 = 0$$
$$a = 6 \times 1 - 12 = -6$$

State the result.

When $t = 1$, $v = 0$ m/s and $a = -6$ m/s^2.

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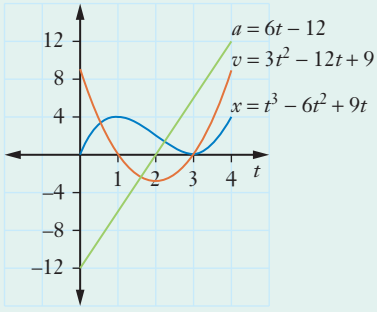
c Draw the graphs of x, v and a.



d The particle speeds up when the velocity is positive and increasing; also when the velocity is negative and decreasing.

Use the graph to find the values.

The particle speeds up when:

$v > 0$ and $a > 0$

and $v < 0$ and $a < 0$

The particle speeds up when $1 < t < 2$ and $3 < t < 4$.

e The particle slows down when the velocity is positive and decreasing; also when the velocity is negative and increasing.

Use the graph to find the values.

The particle slows down when:

$v > 0$ and $a < 0$

and $v < 0$ and $a > 0$

The particle slows down when $0 < t < 1$ and $2 < t < 3$.

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Exercise 8.02 Acceleration

Example 3

1 The position of an object is given by $x = t^3 - 4t^2 + 11t - 7$, where t is time in seconds and x is in metres.

- **a** Find an expression for the acceleration of the object.
- **b** What is the initial acceleration of the object?
- **c** What is the acceleration of the object after 3 s?
- **d** When will the acceleration reach 20 m/s^2?

2 An object is projected vertically upwards from a platform. The approximate position of the object in metres is given by $x = 17t - 5t^2 + 40$, where t is time in seconds.

- **a** How high above the ground is the platform?
- **b** Find an expression for the velocity and acceleration of the object at time t.
- **c** Explain what the expression for the acceleration means.
- **d** After how long does the object begin falling back towards the ground?

3 The position of an object is given by $x = \dfrac{2t+3}{4t-5}$, where x is in metres and t is in seconds.

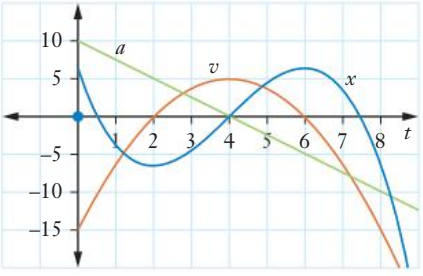
- **a** Find an expression for the velocity of the object at time t.
- **b** Find an expression for the acceleration of the object at time t.
- **c** What is the acceleration after 3 s?

4 The position of an object is given by $x = \sin(2t) - 2\sin(t)$, where x is in centimetres and t is in seconds.

- **a** Find an expression for the velocity of the object at time t.
- **b** Find an expression for the acceleration of the object at time t.
- **c** What is the acceleration after 2 s?
- **d** What is the acceleration after 4 s?

5 The diagram below shows graphs of the position (x), velocity (v) and acceleration (a) of an object moving in a straight line.

- **a** At what times does the direction of the motion the object change?
- **b** Between which times is the particle speeding up?
- **c** Between which times is the particle slowing down?



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Example 4

6 The position of a particle is given by $x = \frac{t^3}{3} - \frac{5t^2}{2} + 4t$ for $t > 0$, where t is in seconds and x is in metres.

- **a** Find expressions for the velocity (v) and acceleration (a) of the particle at time t.
- **b** What is the particle's velocity and acceleration when $t = 3$?
- **c** Plot the graphs of x, v and a for $0 \leq t \leq 5$ on the same set of axes.
- **d** Between which times is the particle slowing down?
- **e** Between which times is the particle speeding up?

7 The equation of the displacement of a particle moving in a straight line is given by $x = 4t^4 - 6t^3 + 5t$, where x is measured in metres and t in seconds. Find:

- **a** the times when the acceleration is 0
- **b** the displacement at these times.

8 The position of a particle is given by $x = \sin(x) - 0.5x$ for $0 \leq t \leq 2\pi$, where t is in seconds and x is in metres.

- **a** Find expressions for the velocity (v) and acceleration (a) of the particle at time t.
- **b** What is the particle's velocity and acceleration (correct to 2 decimal places) when $t = 4$?
- **c** Plot the graphs of x, v and a for $0 \leq t \leq 2\pi$ on the same set of axes.
- **d** Between which times is the particle slowing down?
- **e** Between which times is the particle speeding up?

Problem solving

9 **a** Find the first two derivatives of $y = \frac{1}{x^2 + x}$.

b Show that $\frac{1}{x^2 + x} = \frac{1}{x} - \frac{1}{x+1}$.

c Use the result of **b** to recalculate the first and second derivatives of $y = \frac{1}{x^2 + x}$ and comment on your results.

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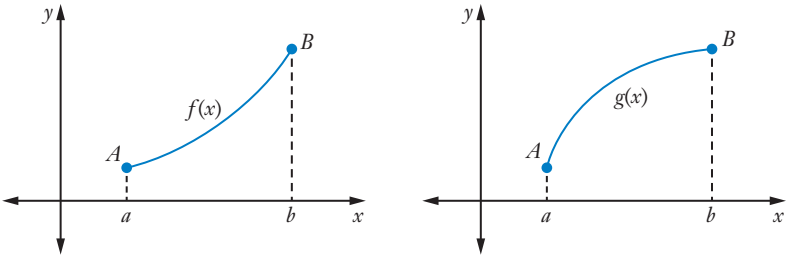
8.03 Concavity and points of inflection

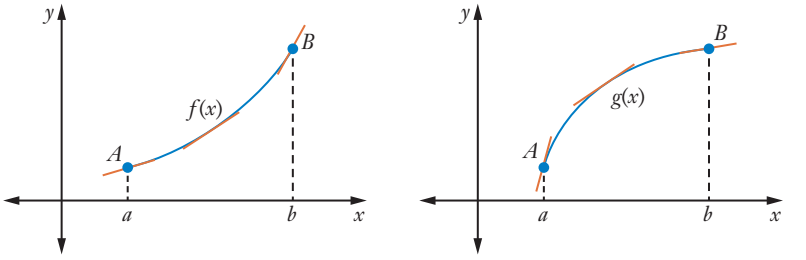
WS
Concavity

The graphs below show 2 functions, $f(x)$ and $g(x)$, on the interval $a \leq x \leq b$. Both of these functions are increasing because the slopes of the tangents are positive, that is, $f'(x)$ and $g'(x)$ are both positive.



Even though the functions are both increasing, they have different shapes. $f(x)$ bends *upwards* while $g(x)$ bends *downwards*. It would be useful to be able to distinguish between these two different behaviours when you are sketching curves.

The graphs below show the tangents to $f(x)$ and $g(x)$ drawn at different points along the interval $a \leq x \leq b$.



Looking at $f(x)$, you can see that the graph lies *above* the tangents, but for $g(x)$, the graph lies *below* the tangents. The way a graph *bends* is called its **concavity**.

You can also see that, for $f(x)$, $f'(x)$ (the gradient) is increasing, so $f''(x) > 0$.

For $g(x)$, $g'(x)$ (the gradient) is decreasing, so $g''(x) < 0$.

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Concavity

For the function $f(x)$ over the interval I $(a \leq x \leq b)$:

- If $f''(x) > 0$ for all values x in I, then the graph is **concave upward** on I.
- If $f''(x) < 0$ for all values x in I, then the graph is **concave downward** on I.

EXAMPLE 5

For which values is the curve $f(x) = x^3 - 12x^2 - 4x + 3$ concave upwards?

Solution

Calculate the first derivative.	$f'(x) = 3x^2 - 24x - 4$
Calculate the second derivative.	$f''(x) = 6x - 24$
Use the definition.	The curve is concave upwards, so: $f''(x) > 0$
Substitute for $f''(x)$.	$6x - 24 > 0$
Solve the inequality.	$6x > 24$
	$x > 4$
Write the answer.	The curve $f(x) = x^3 - 12x^2 - 4x + 3$ is concave upwards when $x > 4$.

The diagram on the right shows the graph of $f(x) = x^3 - 9x^2 + 21x$.

The graph changes from concave downwards to concave upwards at the point $P(3, 9)$.

P is called a **point of inflection**.

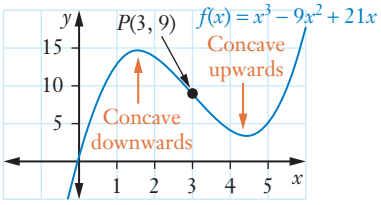
$f'(x) = 3x^2 - 18x$

and $f''(x) = 6x - 18$.

So $f''(x) < 0$ when $x < 3$

and $f''(x) > 0$ when $x > 3$

So $f''(x) = 0$ when $x = 3$.



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If the curve $y = f(x)$ is continuous and P is the point where the curve changes from concave upward to concave downward or concave downward to concave upward, then P is a **point of inflection**.

If P is a point of inflection, then $f''(x) = 0$ at P.

EXAMPLE 6

Determine any points of inflection on the curve $y = x^3 - 2x^2 + 9x - 12$.

Solution

Calculate the first derivative.

$y' = 3x^2 - 4x + 9$

Calculate the second derivative.

$y'' = 6x - 4$

Use the condition for a point of inflection.

$y'' = 0$

$6x - 4 = 0$

Solve for x.

$6x = 4$

$x = \frac{2}{3}$

Check the value of y'' before and after $x = \frac{2}{3}$.

x	0	$\frac{2}{3}$	1
y''	-4	0	2

Use the definition of a point of inflection.

The curve changes from concave downward ($y'' < 0$) to concave upward ($y'' > 0$), so there is a point of inflection at $x = \frac{2}{3}$.

Find the point of inflection.

When $x = \frac{2}{3}$:

$$y = \left(\frac{2}{3}\right)^3 - 2 \times \left(\frac{2}{3}\right)^2 + 9 \times \frac{2}{3} - 12$$

$$= -\frac{178}{27}$$

State the result.

A point of inflection occurs at $\left(\frac{2}{3}, -\frac{178}{27}\right)$.

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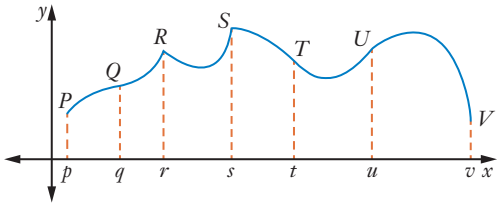
Exercise 8.03 Concavity and points of inflection

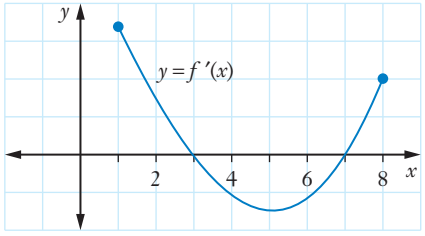
1 The graph of $y = f(x)$ is shown below. State the intervals for which the curve is:

a concave upwards **b** concave downwards

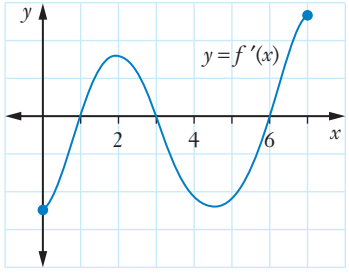


2 Each diagram below shows the graph of the first derivative $f'(x)$ of the function $f(x)$. In each case, state the intervals for which $f(x)$ is increasing and for which $f(x)$ decreasing.

a

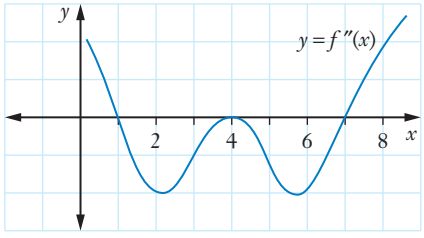


b



3 The graph of the second derivative $f''(x)$ of the function $f(x)$ is shown.

State the x-coordinate of any point of inflection, giving reasons for your answers.



Example 5

4 For which values is the curve $f(x) = x^3 - x^2 - 4x + 3$ concave downwards?

5 For which values is the curve $f(x) = x^3 - 3x^2 - 9x + 1$ concave upwards?

6 Identify the interval for which the curve $f(x) = 2x^3 - 9x^2 - 24x + 30$ is:

a concave upwards **b** concave downwards

7 Find the domain for which the curve $f(x) = 3x^4 + 4x^3 - 12x^2 - 1$ is concave downwards.

Example 6

8 Determine any point of inflection on the curve $y = x^3 - 6x^2 + 3$.

9 Determine any point of inflection on the curve $y = 2 + 9x - 3x^2 - x^3$.

10 Find the point where the curve $y = (x - 2)^3$ changes from being concave downwards to concave upwards.

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11 Identify the points of inflection and the intervals of concavity for the curve $f(t) = t + \sin(t)$ for $-2\pi \leq t \leq 2\pi$.

12 Identify the points of inflection and the intervals of concavity for the curve

$$y = 3x^5 - 5x^3 + 4x + 3 \text{ for } -2 < x < 2.$$

Problem solving

13 Show that the curve $y = x^4 + 4x^2 + 4$ has no points of inflection and describe the concavity of the curve.

14 If $y = x^3 - ax^2 + 5x + 9$ has a point of inflection at $x = 2$, find the value of a.

15 If $y = ax^3 - 3x^2 - 12x + 7$ has a point of inflection at $x = \frac{1}{2}$, find the value of a.

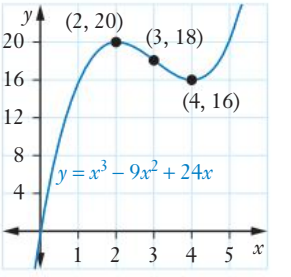
8.04 The second derivative test

You have seen how to use the first derivative of a function $f(x)$ to locate and identify stationary points on the curve $y = f(x)$. A stationary point exists when $f'(x) = 0$. The nature of the stationary point can be determined by looking at the slope on either side of the stationary point.

You can also use the second derivative $f''(x)$ to identify the nature of a stationary point.

Consider the function $f(x) = x^3 - 9x^2 + 24x$.

The graph of the function is shown.



$f'(x) = 3x^2 - 18x + 24$

$f'(x) = 0$ when $x = 2$ or 4. So there are stationary points at $x = 2$ or 4.

$f''(x) = 6x - 18$

$f''(x) = 0$ when $x = 3$. So there is a point of inflection at $x = 3$.

At $x = 2$, $f'(x) = 0$ and $f''(x) < 0$. So the graph is concave downwards and the stationary point must be a local maximum.

At $x = 4$, $f'(x) = 0$ and $f''(x) > 0$. So the graph is concave upwards and the stationary point must be a local maximum.

Using the second derivative in this way is called the **second derivative test**.

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Second derivative test

For the function $f(x)$:

- if $f'(c) = 0$ and $f''(c) > 0$, then $f(x)$ has a local minimum at $x = c$.
- if $f'(c) = 0$ and $f''(c) < 0$, then $f(x)$ has a local maximum at $x = c$.

EXAMPLE 7

Find any stationary points on the curve $y = 2 + 3x - x^3$ and determine their nature.

Solution

Find the first derivative.	$f'(x) = 3 - 3x^2$
Find any stationary points.	$3 - 3x^2 = 0$
Factorise.	$-3(x^2 - 1) = 0$ $-3(x - 1)(x + 1) = 0$
Solve for x.	$x = -1$ or 1
Find the corresponding y values.	$f(-1) = 0$ $f(1) = 4$
State the result.	Stationary points exist at $(-1, 0)$ and $(1, 4)$.
Find the second derivative.	$f''(x) = -6x$
Find $f''(x)$ for the stationary points.	$f''(-1) = 6 > 0$ $f''(1) = -6 < 0$
Use the second derivative test for the first stationary point.	When $x = -1$, $f'(-1) = 0$ and $f''(-1) > 0$ so there is a local minimum at $(-1, 0)$.
Use the second derivative test for the second stationary point.	When $x = 1$, $f'(1) = 0$ and $f''(1) < 0$ so there is a local maximum at $(1, 4)$.
State the result.	The curve $y = 2 + 3x - x^3$ has a local minimum at $(-1, 0)$ and a local maximum at $(1, 4)$.

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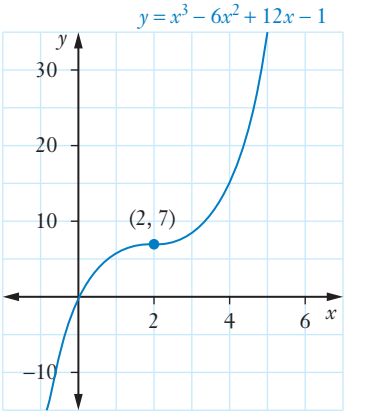
If the function $f(x)$ has a stationary point at $x = c$ and $f''(c) = 0$, the second derivative test can't be used. You need to look at the concavity of the curve either side of the stationary point to determine its nature.

Consider the function $f(x) = x^3 - 6x^2 + 12x - 1$ shown.



$f'(x) = 3x^2 - 12x + 12$

$= 3(x - 2)^2$

$f'(x) = 0$ when $x = 2$, so there is a stationary point at $x = 2$.

$f''(x) = 6x - 12$

$f''(2) = 0$ when $x = 2$.

When $x < 2, f''(x) < 0$ and when $x > 2, f''(x) > 0$.

So at $x = 2$, the graph changes from concave downwards to concave upwards.

Hence there is a point of inflection at $x = 2$.

A stationary point that is also a point of inflection is called a **point of horizontal inflection**.

For the function $f(x) = x^3 - 6x^2 + 12x - 1$, $(2, 7)$ is called a point of horizontal inflection *on a rising curve* because the curve changes from concave downwards to concave upwards.

A point of horizontal inflection on a curve that changes from concave upwards to concave downwards is called a point of horizontal inflection *on a falling curve*.

EXAMPLE 8

Identify any stationary points on the curve $y = x^3 + 3x^2 + 3x - 2$.

Solution

Find the first derivative.	$f'(x) = 3x^2 + 6x + 3$
Find any stationary points.	$3x^2 + 6x + 3 = 0$
Factorise.	$3(x^2 + 2x + 1) = 0$
	$3(x + 1)^2 = 0$
Solve for x.	$x = -1$
Find the corresponding y values.	$f(-1) = -3$
Find the second derivative.	$f''(x) = 6x + 6$
Find $f''(x)$ for the stationary points.	$f''(-1) = 0$

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Check the concavity of the curve on either side of the stationary point.	x: −2, −1, 0 $f''(x)$: −6, 0, 6
Write a conclusion.	The curve changes from concave downwards to concave upwards.
State the result.	The curve $y = x^3 + 3x^2 + 3x - 2$ has a stationary point at $(-1, -3)$. The stationary point is a point of horizontal inflection on a rising curve.

Exercise 8.04 The second derivative test

Example 7

1 Find any stationary points on the following curves and determine their nature.

a $y = 3x^2 - 9x + 8$ **b** $y = 9 + 8x - 2x^2$

2 Identify any stationary points on the curve $y = 3x^4 - 16x^3 + 18x^2$.

3 Identify any stationary points on the curve $y = 25x - 3x^3 + 5$.

Example 8

4 Find any stationary points on the curve $y = x^3 - 3x^2 + 3x$ and determine their nature.

5 Identify any stationary points on the graph of the function $f(x) = 12x^2 - x^3 + 69 - 48x$.

6 Identify all stationary points on the curve $y = x^4 - 2x^3 + 6$.

7 Identify any stationary points on the curves:

a $y = e^{-x} + x$ **b** $y = x^2 e^{-x}$

8 Identify any stationary points on the curves:

a $y = \dfrac{1}{x^2 - 4x - 5}$ **b** $y = \dfrac{x+1}{x-1}$ **c** $y = 2x + \dfrac{2}{x-1}$

9 Identify any stationary points on the curve $y = x + \cos(x)$ for $0 \leq x \leq 2\pi$.

Problem solving

10 The curve $y = x^2 + ax - 12$ has a stationary point at $x = 2$.

a Find the value of a.

b What type of stationary point occurs at $x = 2$?

11 The curve $y = ax^3 + bx^2 + 6x + 5$ has a point of horizontal inflection at $(1, 7)$. Find the value of a and b.

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8.05 Curve sketching

Curve sketching with derivatives

Stationary points, the y-intercept, zeros and the behaviour of the function as $x \to \pm\infty$ are all important features that may be used to sketch a graph.

When sketching graphs of functions, knowledge of the general shapes of these functions is helpful. For example, you have already seen that a hyperbolic function is discontinuous, a logarithmic function has a vertical asymptote and an exponential function has a horizontal asymptote. In general, functions may have vertical asymptotes at values of x for which the function is undefined. Horizontal asymptotes are really limiting values of the function as the magnitude of x becomes large.

The work you have just completed using the second derivative and concavity is also very helpful when sketching curves, as summarised in the following summary and table.

Procedure for sketching functions

1 Zeros: If possible, determine the x-intercepts – the values of x for which $f(x) = 0$.

2 Discontinuities: Determine the behaviour at points where $f(x)$ is undefined, particularly to find whether there is a vertical asymptote.

You do not need to check for discontinuities for polynomial functions as they are defined for all values of x.

3 Sign: Determine the sign of $f(x)$ between zeros and discontinuities.

4 Derivative: (see table next page)

- Use the first derivative to locate stationary points and to determine the intervals in which the function is increasing or decreasing.
- Use the second derivative to identify the nature of stationary points and to determine the intervals in which the function is concave upwards or downwards.

5 y-intercept: Find the value of $f(0)$.

6 Behaviour as $x \to \pm\infty$: A dominant term is of higher power than any other term. As $x \to \pm\infty$, the magnitude of the dominant term is so large that the function behaves like the dominant term. Alternatively, a function may have a limit as $x \to \pm\infty$. This is a horizontal asymptote.

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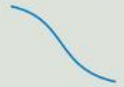
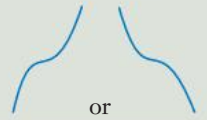
	$\frac{dy}{dx} > 0$	$\frac{dy}{dx} < 0$	$\frac{dy}{dx} = 0$
$\frac{d^2y}{dx^2} > 0$			
$\frac{d^2y}{dx^2} < 0$			
$\frac{d^2y}{dx^2} = 0$ and concavity changes			 or

Curve sketching

EXAMPLE 9

Find any stationary points and points of inflection on $f(x) = x^3 + 6x^2 + 9x$ and use this information to sketch the curve.

Solution

Find the first derivative.	$f'(x) = 3x^2 + 12x + 9$
Find any stationary points.	$3x^2 + 12x + 9 = 0$
Factorise.	$3(x^2 + 4x + 3) = 0$
	$3(x + 3)(x + 1) = 0$
Solve for x.	$x = -3$ or -1
Find the corresponding y values.	$f(-3) = -27 + 54 - 27 = 0$
	$f(-1) = -1 + 6 - 9 = -4$
State the result.	Stationary points are at $(-3, 0)$ and $(-1, -4)$.
Find the second derivative.	$f''(x) = 6x + 12$
Find $f''(x)$ for the stationary points.	$f''(-3) = -18 + 12 = -6$, so $f''(-3) < 0$
	$f''(-1) = -6 + 12 = 6$, so $f''(-1) > 0$
Use the second derivative test.	Local maximum at $(-3, 0)$ and local minimum at $(-1, -4)$.
Locate the point of inflection.	Let $f''(x) = 0$
	$6x + 12 = 0$

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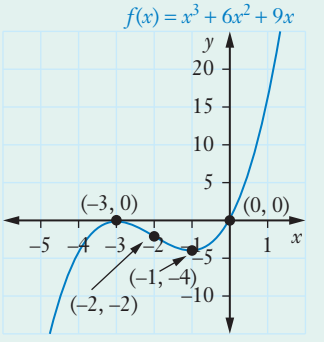
Solve for x.	$x = -2$
Find the corresponding y value.	$f(-2) = -8 + 24 - 18 = -2$
Check the concavity of the curve around $x = -2$.	

x	-2.5	-2	-1.5
$f''(x)$	-3	0	3

Write a conclusion.	The curve changes from concave downwards to concave upwards at the point of inflection $(-2, -2)$.
Find the y-intercept.	Let $x = 0, f(0) = 0$
Write the result.	y-intercept at $(0, 0)$.
Check the behaviour of $f(x)$ as $x \to \pm\infty$.	Dominant term $= x^3$. As $x \to \infty, f(x) \to \infty$. As $x \to -\infty, f(x) \to -\infty$.
The zeros have already been calculated.	Zeros exist at $(-3, 0)$ and $(0, 0)$.
Use the information to sketch the graph.	

EXAMPLE 10

For the function $f(x) = 3x^5 - 5x^3 + 3$

a use the second derivative to find any stationary points, points of inflection and the intervals of concavity.

b use the results from part **a** to sketch the graph of the function.

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Solution

a Find the first derivative. $f'(x) = 15x^4 - 15x^2$

Find any stationary points. $15x^4 - 15x^2 = 0$

Factorise. $15x^2(x^2 - 1) = 0$

$15x^2(x + 1)(x - 1) = 0$

Solve for x. $x = -1, 0$ or 1

Find the corresponding y values. $f(-1) = -3 + 5 + 3 = 5$

$f(0) = 0 + 0 + 3 = 3$

$f(1) = 3 - 5 + 3 = 1$

State the result. Stationary points are at $(-1, 5)$, $(0, 3)$ and $(1, 1)$.

Find the second derivative. $f''(x) = 60x^3 - 30x$

Find $f''(x)$ for the stationary points. $f''(-1) = -60 + 30 = -30$, so $f''(-1) < 0$

$f''(0) = 0$, so need to investigate concavity.

$f''(1) = 60 - 30 = 30$, so $f''(1) > 0$

Use the second derivative test. Local maximum at $(-1, 5)$ and local minimum at $(1, 1)$.

Locate the point of inflection. Let $f''(x) = 0$.

$60x^3 - 30x = 0$

$30x(2x^2 - 1) = 0$

Solve for x. $x = 0$ or $\pm\frac{1}{\sqrt{2}}$.

Find the corresponding y values. $f\left(-\frac{1}{\sqrt{2}}\right) \approx 4.24$

$f(0) = 3$

$f\left(\frac{1}{\sqrt{2}}\right) \approx 1.76$

State the result. Points of inflection occur at about $(-0.71, 4.24)$, $(0, 3)$ and $(0.71, 1.76)$.

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Check the concavity of the curve around the points of inflection.

x	-1	$-\frac{1}{\sqrt{2}}$	-0.5	0	0.5	$\frac{1}{\sqrt{2}}$	1
$f''(x)$	-30	0	7.5	0	-7.5	0	30

Identify the other stationary point.

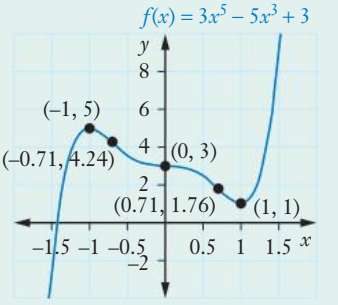
There is a point of horizontal inflection on a falling curve at (0, 3).

State the intervals of concavity.

The curve is concave upwards for:
$-\frac{1}{\sqrt{2}} < x < 0$ and for $x > \frac{1}{\sqrt{2}}$.

The curve is concave downwards for:
$x < -\frac{1}{\sqrt{2}}$ and for $0 < x < \frac{1}{\sqrt{2}}$

b Use the information to sketch the curve.



EXAMPLE 11

Sketch the graph of $y = xe^{3x}$.

Solution

Find the zeros.

$xe^{3x} = 0$

$x = 0$

State the result.

Zero exists at (0, 0). This is also the y-intercept.

Check for discontinuities.

There are no discontinuities as $y = xe^{3x}$ is defined for all x.

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Check the sign of the function.	There is only 1 zero. When $x < 0, y = xe^{3x} < 0$. When $x > 0, y = xe^{3x} > 0$.
Use the product rule to calculate the first derivative.	$\frac{dy}{dx} = 1 \times e^{3x} + x \times 3e^{3x}$ $= e^{3x}(1 + 3x)$
Let $\frac{dy}{dx} = 0$ to find stationary points.	$e^{3x}(1 + 3x) = 0$ $e^{3x} = 0$ or $1 + 3x = 0$
But $e^{3x} \neq 0$.	$x = -\frac{1}{3}$
Find the corresponding y value.	$y = -\frac{1}{3}e^{-1} \approx -0.12$
State the result.	There is a stationary point at about $\left(-\frac{1}{3}, -0.12\right)$.
Find the second derivative. Let $y = f(x)$.	$f''(x) = e^{3x} \times 3 + (1 + 3x) \times 3e^{3x}$ $= 3e^{3x}(2 + 3x)$
Find $f''(x)$ for the stationary point.	$f''\left(-\frac{1}{3}\right) = 3e^{-1}(2 - 1) \approx 1.10$ so $f''\left(-\frac{1}{3}\right) > 0$
Use the second derivative test.	Local minimum at $\left(-\frac{1}{3}, -0.12\right)$.
Locate the point of inflection.	Let $f''(x) = 0$ $3e^{3x}(2 + 3x) = 0$
Solve for x.	$x = -\frac{2}{3}$
Find the corresponding y value.	$y = -\frac{2}{3}e^{-2} \approx -0.09$

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Check the concavity of the curve around $x = \frac{2}{3}$.

x	-1	$-\frac{2}{3}$	0
$f''(x)$	-0.15	0	6

Write a conclusion.

The curve changes from concave downwards to concave upwards at the point of inflection about $\left(-\frac{2}{3}, -0.09\right)$.

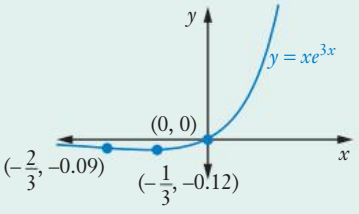
Check the behaviour of $f(x)$ as $x \to \pm\infty$.

Try some positive and negative values for x. For large values of x, the exponential term dominates. There is a horizontal asymptote at $x = 0$.

As $x \to \infty, y \to \infty$.

As $x \to -\infty, f(x) \to 0$.

Use the information to sketch the graph.



Exercise 8.05 Curve sketching

1 Sketch the graph of each function.

Example 9

a $g(x) = 10x - x^2 - 21$ **b** $h(x) = x^2 - x - 8$

c $h(x) = 2x^2 + x - 4$ **d** $q(x) = x^3 - x^2 - x - 1$

e $f(x) = 3x - x^3 - 18$ **f** $f(x) = x^3 + 3x^2 - 24x$

2 Sketch the graphs of each function, showing key features including stationary points and points of inflection.

Example 10

a $f(x) = 5 + 5x^3 - 3x^5$ **b** $f(x) = 7 + 2x^2 - x^4$

c $f(x) = 2x^6 - 6x^4$ **d** $f(x) = 3x^4 + 4x^3 - 12x^2 - 1$

3 Sketch the graphs of each function, showing key features including stationary points and points of inflection.

Example 11

a $f(x) = x^{-1}e^x$ **b** $f(x) = xe^{-3x}$

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Problem solving

4 Sketch the graph of each function. Show all stationary points and points of inflection.

a $f(x) = \dfrac{1}{x-3}$ **b** $g(x) = 1 - \dfrac{1}{x}$ **c** $y = \dfrac{1}{(x+5)^2}$

5 Sketch the graph of each function. Show all stationary points and points of inflection.

a $f(x) = (x+1)x^{\frac{2}{3}}$ **b** $y = x^{\frac{2}{3}} - x^{\frac{1}{3}}$ **c** $g(x) = x^{\frac{2}{3}}(x^2 - 8)$

6 Sketch the graph of each function. Show all stationary points and points of inflection.

a $f(x) = \cos(x) - x$ **b** $y = \sin(2x) - 2\sin(x)$

8.06 Optimisation applications

WS Optimisation

WS Optimisation – cost and revenue

WS Optimisation – area and volume

WS Optimisation – mixed problems

WS Second derivative assignment

WS Second derivative problems

Fence

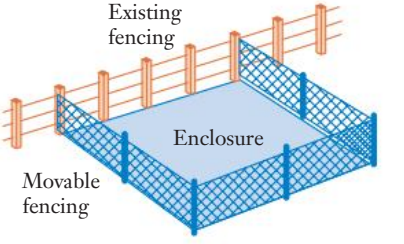
People want the best possible outcome in any situation. This applies whether you want to: minimise costs or maximise profits in a business; get the greatest yield from a crop; make the best use of material for making cardboard boxes; minimise the use of pesticides and artificial fertilisers; or obtain the most powerful possible explosion in a quarry. These are all situations where the aim is to obtain the best (optimal) solution to a problem. Derivatives can be used to find solutions in many optimisation problems.

Optimisation is the process of maximising or minimising a certain quantity. Optimisation problems using derivatives require a thorough knowledge of stationary points, curve sketching and finding the greatest and least values on an interval.

TECHNOLOGY

Fencing for maximum area

A cattle property manager wants to construct a temporary enclosure using 200 m of movable fencing and an existing property fence as shown here. The manager wants to place the movable fencing so that the enclosure has the greatest possible area.



This problem is modelled in the spreadsheet called 'Fence' on NelsonNet.

1 Open the spreadsheet and change the distance at which the far side of the movable fence is placed away from the existing fence.

2 Observe the value of the area as the position of the movable fence is altered.

3 What seems to be the optimal distance between the movable fence and the existing fence?

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The solution to optimisation problems cannot be done using a set of hard and fast rules, but the following guidelines may be helpful.

Guidelines for solving optimisation problems

1. Read the problem carefully and think about the given facts. Ask yourself 'What is the question asking?'
2. Assign any unknown quantities as variables.
3. If possible, make a sketch or diagram and label it, using the variables.
4. Make a list of known facts together with any relationships involving the variables using equations.
5. Determine the variable to be optimised (maximised or minimised) and express this variable as a function of one of the other variables. Using the relationships you have found, eliminate all but one of the variables.
6. Find the stationary points of the function obtained and test each one to see whether it is a maximum or minimum.
7. If you are still unable to solve the problem, try using a different variable, redraw the diagram and look for other relationships that could be used.

When solving problems involving area and volume, particular care should be paid to drawing the diagram. You will need to know the area and volume formulas for a range of basic two-dimensional figures and three-dimensional shapes.

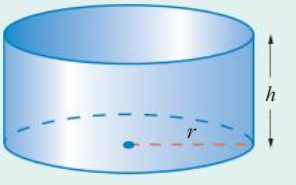
EXAMPLE 12

A cylindrical metal can is open at the top. It must be designed to have a capacity of 1078 cm^3. Because the outside of the can has to be galvanised, the manufacturer wants the surface area of the can to be minimised in order to save money. Find the optimal dimensions of the can.

Solution

Even though there is little information to mark on it, the diagram should still be drawn.

Let the height of the tank be h and the radius be r.



The external surface area needs to be minimised.

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Write an equation for the surface area.	$SA = \pi r^2 + 2\pi rh$
You need to have an equation in only one variable. You need a way to express one variable in terms of the other.	
Use the formula for volume, V.	$V = \pi r^2 h$
Substitute for V.	$1078 = \pi r^2 h$
It doesn't matter which variable we use, but using r will involve a square root, so use h.	$h = \dfrac{1078}{\pi r^2}$
Substitute for h in the equation for surface area.	$SA = \pi r^2 + 2\pi r \times \dfrac{1078}{\pi r^2}$
Simplify.	$= \pi r^2 + \dfrac{2156}{r}$
Find any stationary points.	$SA' = 2\pi r - \dfrac{2156}{r^2}$ Let $SA' = 0$. $2\pi r - \dfrac{2156}{r^2} = 0$
Solve for r.	$\dfrac{2\pi r^3 - 2156}{r^2} = 0$ $2\pi r^3 - 2156 = 0$ $2\pi r^3 = 2156$ $r^3 = 343.1380\ldots$ $r \approx 7$ cm
Find the second derivative.	$SA'' = 2\pi + \dfrac{4312}{r^3}$
Calculate SA'' when $r = 7$.	$= 18.8495\ldots$ cm^2
Use the second derivative test.	At $r \approx 7$, $SA' = 0$ and $SA'' > 0$, so SA is a minimum at $r \approx 7$.
Calculate h.	$h = \dfrac{1078}{\pi r^2}$ $\approx \dfrac{1078}{\pi \times 7^2}$ ≈ 7 cm
State the result.	A 1078 cm^3 cylinder with minimal surface area will have a radius and height of about 7 cm.

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The optimisation process is often used for decision making in business. All businesses want to be able to minimise costs and maximise profits. If it is known that profit depends on some variable that can be adjusted, then choosing the correct value of the variable will make the profit as large as possible.

The simplest case where the variable can be adjusted is the level of production, n (the number of units of production per unit time). If all the units are sold for price p each, the **revenue function** can be expressed as

$$R(n) = pn$$

The cost of production depends on the number of items produced and is defined by the **cost function**, $C(n)$.

This means that the **profit function** can be given by

$$P(n) = R(n) - C(n) = pn - C(n)$$

EXAMPLE 13

A small manufacturing firm can sell all of its produced items at a price of \$8 per item. The production cost function for producing n items per week (in dollars) is given by

$$C(n) = 100\,000 + 8n - 0.0156n^2 + 2 \times 10^{-6}n^3$$

a What should the weekly production target be in order to maximise profits?

b What is the maximum profit that can be made?

Solution

a

Write the revenue function.	$R(n) = 8n$
Write the profit function.	$P(n) = R(n) - C(n)$
Substitute.	$= 8n - (100\,000 + 8n - 0.0156n^2 + 2 \times 10^{-6}n^3)$
Simplify.	$= -100\,000 + 0.0156n^2 - 2 \times 10^{-6}n^3$
Find the fist derivative.	$P'(n) = 0.0312n - 6 \times 10^{-6}n^2$
Let $P'(n) = 0$ to find stationary points.	$0.0312n - 6 \times 10^{-6}n^2 = 0$
Factorise.	$n(0.0312 - 6 \times 10^{-6}n) = 0$
Solve.	$0.0312 - 6 \times 10^{-6}n = 0$ or $n = 0$
	$6 \times 10^{-6}n = 0.0312$
	$n = \dfrac{0.0312}{6 \times 10^{-6}}$
	$= 5200$

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	State the result.	Stationary point occurs when $n = 5200$.
	Find the second derivative.	$P''(n) = 0.0312 - 2 \times 6 \times 10^{-6}n$ $= 0.0312 - 12 \times 10^{-6}n$
	Calculate $P''(5200)$.	$P''(5200) = -0.0312$
	Use the second derivative test.	$P'(5200) = 0$ and $P''(5200) < 0$, so profit is a maximum at $n = 5200$.
	State the result.	The maximum profit is made when 5200 items per week are produced.
b	Calculate the maximum profit.	$P(5200) = -100\,000 + 0.0156(5200)^2 - 2 \times 10^{-6}(5200)^3$ $= 40\,608$
	State the result.	The maximum profit of \$40 608 can be made when 5200 items are produced weekly.

INVESTIGATION

FUN RIDES

Flash owns a 'fun ride' that he sets up at country shows. Each year, he brings his ride to the Brisbane Ekka. The ride has the capacity to take up to 90 people, but Flash knows from experience that he only gets about a third of the ride's capacity when he charges his normal price of \$10 for a ride.

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He also knows that each time he runs the ride, it costs the same amount regardless of how many people have bought tickets. When Flash reduces the price of the ride, he attracts more customers. He wants to make the ride more profitable and has an idea that if he gets the price right, he can achieve this. He has learned that charging too little means that he loses money.

Recently, Flash has noticed that rides costing only $5 are usually full, and he is sure that his ride would be full every time if he dropped his price to that figure.

1 Assuming that the number of customers will increase in direct proportion to the price drop, decide what would be the most profitable price at which to offer his ride.

2 How much better off would Flash be for each ride compared with the takings at the fare of $10?

3 Investigate other methods of approaching this problem by considering different assumptions.

In some optimisation questions, you will be given an equation as part of the question. In others, you will need to work out an equation first.

EXAMPLE 14

A north–south highway intersects an east–west highway at a point P. A vehicle crosses P at 10:00 a.m., travelling east at a constant speed of 40 km/h. At the same instant, another vehicle is 4 km north of P, travelling south at 100 km/h. Find the time at which the vehicles are closest to each other and estimate the minimum distance between them.

Solution

Let t be the number of hours after 10:00 a.m. Calculate the positions of the 2 vehicles at time t.

The slower vehicle is $40t$ km east of P.

The faster vehicle is $100t$ km south of its position at 10:00 a.m.

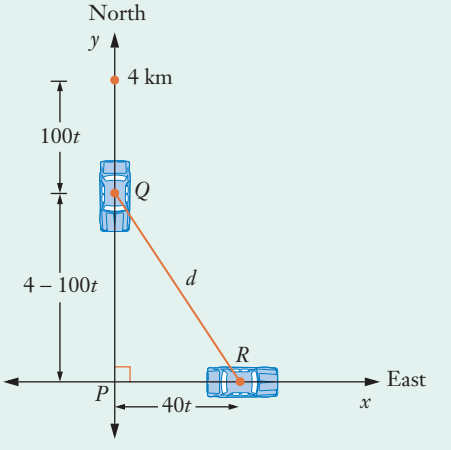
Sketch a diagram showing the positions at time t.

It is convenient to represent the 2 highways as the x- and y-axes, with point P at the origin.

Call the slower vehicle R and the faster vehicle Q.

The quantity to be minimised is the distance between the vehicles, d.

In the diagram, d is shown as QR in $\triangle PRQ$.

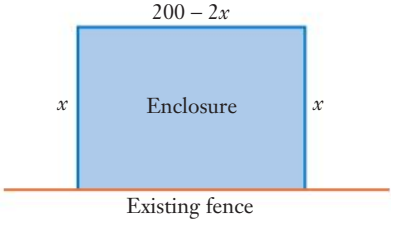


Use Pythagoras' theorem to find an equation involving d.	$d^2 = (4 - 100t)^2 + (40t)^2$ $= 16 - 800t + 10\,000t^2 + 1600t^2$ $= 16 - 800t + 11\,600t^2$
If d^2 is minimised, so too is d. It is simpler to deal with d^2 at this stage. Form a function $f(t) = d^2$.	$f(t) = 11\,600t^2 - 800t + 16$
Find the first derivative.	$f'(t) = 23\,200t - 800$
Let $f'(t) = 0$.	$23\,200t - 800 = 0$
Solve for t.	$t = \frac{800}{23200}$ $= 0.03448\ldots$
Find the second derivative.	$f''(t) = 23\,200$
Use the second derivative test.	$f'(0.03448\ldots) = 0$ and $f''(0.03448\ldots) > 0$ so distance is a minimum at $t = 0.03448...$ h, i.e., 2.0689 minutes after 10:00 a.m.
Calculate d^2 when $t = 0.03448\ldots$	$d^2 = 16 - 800(0.03448\ldots) + 11\,600(0.03448\ldots)^2$ $= 2.2068\ldots$
Take the square root to find d.	$d = 1.4855\ldots$ km
State the result.	The minimum distance separating the vehicles is about 1.5 km and this occurs just after 10:02 a.m.

Exercise 8.06 Optimisation applications

Example 12

1 A rectangular area is enclosed using 200 m of movable fencing and an existing fence as one side of the enclosure, as shown in the diagram.



- **a** Calculate an expression for the length of the other side, l, of the enclosure.
- **b** Write an expression for the area of the enclosure, A.
- **c** Use the first and second derivative of A to find the maximum area that can be enclosed using the movable fencing.

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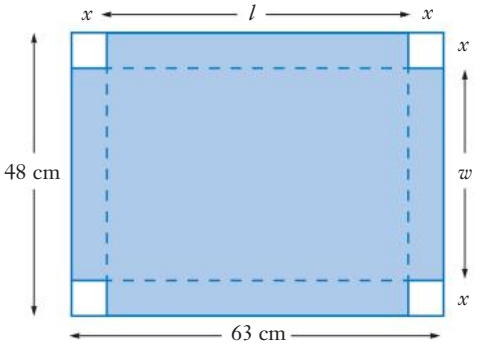
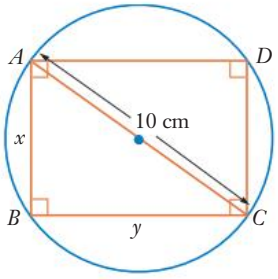
2 An open carton with a rectangular base is to be constructed from a rectangular piece of cardboard 48 cm wide and 63 cm long by cutting a square from each corner and then bending up the sides.



- **a** Write an expression for the length, l, and width, w, of the base.
- **b** Write an expression for the volume, V, of the box.
- **c** Use the first and second derivatives of V to find the size of the corner square that will produce a box that has the largest possible volume.
- **d** Calculate the maximum volume of the box.

3 A rectangle is drawn so that it is enclosed in a circle of diameter 10 cm as shown on the right.



- **a** Write an expression for the area of the rectangle in terms of x and y.
- **b** Use Pythagoras' theorem to find an expression for the area of the rectangle in terms of x only.
- **c** Find the area of the largest rectangle that can be enclosed by the circle.

4 A firm sells all the units it produces at \$4 per unit. The firm's total cost C for producing x units is given in dollars by

$$C(x) = 50 + 1.3x + 0.001x^2$$

Example 13

- **a** Write the revenue function.
- **b** Write the profit function.
- **c** Use the first and second derivative to find the number of items that should be produced so that the profit is a maximum.

5 The Western Star Novelty Company produces x tonnes of silver per month at a total cost of C, given by:

$$C(x) = 2000 - 75x - 5x^2 + \frac{x^3}{3}$$

where C is in dollars.

Use differentiation to find the production level for which costs are minimised.

6 A printer has a contract to print 200 000 copies of a small catalogue. It costs \$8/h to run the printing press, which will produce 1200 impressions per hour. Each impression prints x catalogues, x being the number of plates (metal copies of the set type) that are used on the press. Each plate costs \$2.

- **a** Write an equation for h, the number of hours for which each press must work.
- **b** The total cost of production is the sum of the running costs of presses and the cost of making plates. Write a function for the total cost of production.
- **c** How many plates should be made so that the job is done most economically?

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Example 14

7 An electric train uses power at the rate of $250\,000 + x^3$ J/h, where x is the speed of the train in km/h over the speed range 20–80 km/h.

- **a** If the distance travelled is d, write an equation for the time taken.
- **b** Write the power consumption, P, as a function of x.
- **c** Find the speed at which the train should travel to minimise the use of electricity.

8 A farmer grows field tomatoes for juicing. The more plants that are crowded into a field, the more competition there is for light and nutrients, and so the smaller the mass of fruit from each plant. From previous experience, it is known that when there are n plants on a planting bed of area 100 m^2, the average mass of fruit from each plant is given by

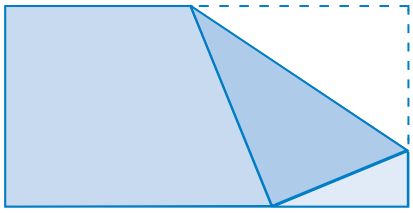
$$m = 40 - 0.05n$$

where m is in kilograms.

- **a** Write an equation for the total mass of fruit per bed, T.
- **b** What value of n will lead to the maximum total quantity of fruit per bed?
- **c** What is the maximum total quantity of fruit per bed?

Problem solving

9 A rectangular piece of paper measures 12 cm by 6 cm. One corner of the sheet of paper is folded up to just reach the opposite side, as shown below. What is the minimum length of the resulting crease in the paper?



10 An opening in a wall is to be made in the shape of a semicircle over a rectangle. The distance around the edge of the opening must measure 12 m. What are the dimensions of the opening that will result in the greatest area?

11 What are the dimensions of the cone of greatest volume that can just fit inside a sphere of radius 2 m?

12 A window consists of a rectangular sheet of glass surmounted by a semicircular sheet, as shown in the photograph. Calculate the maximum area obtainable if the perimeter of the window is fixed at 6 m.

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13 Find the greatest volume of a rectangular box with a square base if the sum of the height and length of the base must not exceed 30 cm.

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14 The weekly cost (in dollars) of producing x ironing boards is given by

$$C(x) = 40\,000 - 30x + 10^{-2}x^2$$

a Find the value of x that minimises production costs.

b If the ironing boards are sold for \$40 each, find the value of x that maximises profit and calculate the maximum profit.

15 The total weekly cost (in dollars) of producing carpet is found to be approximated by the function

$$C(x) = 100 + 28x - 5x^2 + \frac{x^3}{3}$$

where x is the number of rolls of carpet. The price (in dollars) at which carpet rolls are sold is given by

$$p = 5000 - 5x$$

A tax of \$222 per roll is imposed by the government. The manufacturer adds this to the production costs. The manufacturer sells to a number of different outlets and has no problem getting rid of however much is produced.

a Change the manufacturer's cost function to include the government tax.

b State the revenue function for selling x rolls of carpet.

c Write the profit function.

d Work out the number of rolls of carpet that need to be produced each week in order to maximise profit.

16 A machine shop can produce 30 items a day on one machine. The shop can use more than one machine for the job, but if it does, for technical reasons, each machine cannot then work to full capacity. In fact, production per machine is slowed by a factor equal to the square of the number of additional machines put on the job. That is, if x additional machines are put on the job, the number of items produced by each machine is $30 - \frac{x^2}{10}$.

How many machines should be used to achieve the maximum production?

17 By reducing the number of counter staff, a fast-food outlet can reduce its labour costs, but can also expect to lose business because of customer dissatisfaction at having to wait.

Assume that the rate of pay for counter staff is \$80 per day and that the loss of profit from having only n counter staff is $\frac{5000}{n+1}$ dollars per day.

Calculate the value of n that minimises the sum of the loss plus the wage cost.

18 Two tourist catamarans are 6 km apart in open water and the first is at a bearing of 045° from the second. The first begins moving directly south at 12 km/h and the second travels directly east at 18 km/h. What is the minimum distance between the catamarans?

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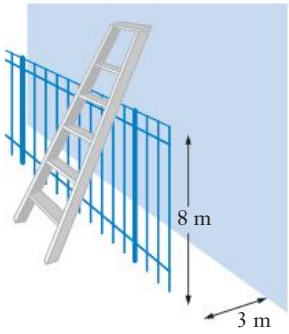
19 A small boat moving at x km/h uses fuel at a rate that is approximated by the function

$$q = 8 + \frac{x^2}{50}$$

where q is measured in litres/hour. Determine the speed of the boat for which the amount of fuel used for any given journey is the least.

20 An 8 m high fence stands 3 m from a large vertical wall. Find the length of the shortest ladder that will reach over the fence to the wall behind, as shown in the diagram.



21 During the course of an epidemic, the proportion of the population infected t months after the epidemic began is given by

$$p = \frac{t^2}{5(1+t^2)^2}$$

a Find the maximum proportion of the population that becomes infected.
b Find the time at which the proportion infected is increasing the most rapidly.

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8. CHAPTER SUMMARY

The second derivative

Second derivative find-a-word

- Given that $y = f(x)$,
  - the **first derivative** $= f'(x) = \frac{dy}{dx} = f^1(x)$ and
  - the **second derivative** $= f''(x) = \frac{d^2y}{dx^2} = f^2(x)$
- The **position** of a particle, x, is measured relative to a fixed point called the **origin**.
- The **displacement**, s, of a particle is the change in its position.
- **Velocity**, v, is the rate of change of position or displacement, so $v = \frac{dx}{dt} = \frac{ds}{dt}$.
- **Acceleration**, a, is the rate of change of velocity, so $a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = \frac{d^2x}{dt^2}$.
- The way a graph *bends* is called its **concavity**.

  For the function $f(x)$ over the interval I $(a \leq x \leq b)$:

  If $f''(x) > 0$ for all values x in I, then the graph is **concave upward** on I.

  If $f''(x) < 0$ for all values x in I, then the graph is **concave downward** on I.
- The point where the curve changes from concave upward to concave downward or concave downward to concave upward is a **point of inflection**.
- At a point of inflection, $f''(x) = 0$.
- **Second derivative test**

  For the function $f(x)$:

  If $f'(c) = 0$ and $f''(c) > 0$, then $f(x)$ has a local minimum at $x = c$.

  If $f'(c) = 0$ and $f''(c) < 0$, then $f(x)$ has a local maximum at $x = c$.
- A stationary point that is also a point of inflection is called a **point of horizontal inflection**.
- Procedure for sketching functions

  1 Zeros: If possible, determine the x-intercepts.

  2 Discontinuities: Determine the behaviour at points where $f(x)$ is undefined.

  3 Sign: Determine the sign of $f(x)$ between zeros and discontinuities.

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4 Derivative:
   - Use the first derivative to locate stationary points and to determine the intervals in which the function is increasing or decreasing.
   - Use the second derivative to identify the nature of stationary points and to determine the intervals in which the function is concave upwards or downwards.

5 y-intercept: Find the value of $f(0)$.

6 Behaviour as $x \to \pm\infty$: A dominant term is of higher power than any other term.

- **Optimisation** is the process of maximising or minimising a certain quantity.
- Guidelines for solving optimisation problems

1 Read the problem carefully and think about the given facts.

2 Assign any unknown quantities as variables.

3 Make a sketch or diagram and label it, using the variables.

4 Make a list of known facts together with any relationships involving the variables using equations.

5 Determine the variable to be optimised and express this variable as a function of one of the other variables. Using the relationships you have found, eliminate all but one of the variables.

6 Find the stationary points of the function obtained and test each one to see whether it is a maximum or minimum.

7 If you are still unable to solve the problem, try using a different variable, redraw the diagram and look for other relationships that could be used.

- **Revenue function**: $R(n) = pn$, where p = the price of each item and n = the number of items.
- **Profit function**: $P(n) = R(n) - C(n) = pn - C(n)$, where $C(n)$ is the **cost function**.

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8. CHAPTER REVIEW

The second derivative

1 Find the first 2 derivatives of $f(x) = 2x^4 + x^3 - 2x^2 + 9$ and calculate $f''(-3)$. Example 1

2 Find the second derivative for each of the following functions. Example 2

a $f(x) = x \cos(2x)$ **b** $y = e^{6x+1}$ **c** $g(x) = \ln(x^2 - 5)$

3 The position of a particle is given by $x = t^3 - 6t^2 - 5t + 3$, where t is time in seconds. Example 3

a Find an expression for the acceleration of the particle.

b What is the initial acceleration of the particle?

c When will the acceleration reach 24 m/s^2?

4 The position of a particle is given by $x = 2t^3 - 15t^2 + 36t + 2$ for $t > 0$, where t is in seconds and x is in metres. Example 4

a Find expressions for the velocity (v) and acceleration (a) of the particle at time t.

b What is the particle's velocity and acceleration when $t = 4$?

c Plot the graphs of x, v and a for $0 \leq t \leq 4$ on the same set of axes.

d Between which times is the particle speeding up?

e Between which times is the particle slowing down?

5 Identify the intervals of concavity for the curve $f(x) = \dfrac{6}{x^2 + 3}$. Example 5

6 Identify any points of inflection on the curve $y = x^4 - 4x^3 - 18x^2 + 7$. Example 6

7 Find any stationary points on the curve $y = x^3 - x^2 - 5x - 3$ and determine their nature. Example 7

8 Find any stationary points on the curve $y = x^4 - 4x^3 + 10$ and determine their nature. Example 8

9 Find any stationary points and points of inflection on $f(x) = x^3 - 2x^2 - 4x + 3$ and use this information to sketch the curve. Example 9

10 For the function $f(x) = x^4 - 2x^2 + 3$ Example 10

a use the second derivative to find any stationary points, points of inflection and the intervals of concavity

b use the results from **a** to sketch the graph of the function.

11 Sketch the graph of $y = x^2 e^{-x}$. Example 11

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Example 12

12 A rectangular field shares one side with an existing paddock, so it requires no fence on that side. There are only 2000 m of fencing material to fence the remaining sides. Find the maximum possible area of the field.

Example 13

13 For a particular item, the price at which x items are sold is given by the equation

$$p = 5 - 0.001x$$

The cost of production is given by

$$C(x) = 2800 + x$$

a Find the value of x that maximises the revenue.

b Find the value of x that maximises the profit.

Example 14

14 A farmer stocks a pond with fish as a supplementary form of income. The more fish there are in the pond, the more competition there will be for the available food supply, and so the more slowly the fish will gain weight. From previous experience, it is known that when there are n fish per unit area of water, the average weight gain of each fish (in grams) during the growing season is given by

$$w(n) = 600 - 30n$$

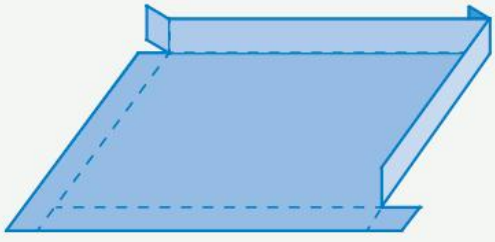
What value of n will lead to the maximum total production of weight in the fish?

Problem solving

15 The curve $y = ax^3 + bx^2 - 3x + 2$ has a point of horizontal inflection at (1, 1). Find the value of a and b.

16 Sketch the graph $y = (5 - 2x)x^{\frac{2}{3}}$, showing all stationary points and points of inflection.

17 Cardboard trays are made from rectangular pieces of cardboard such that the length and width are in the ratio 3 : 2. The trays are made as shown, by cutting one side of a square corner, then bending and gluing to make the tray. What dimensions will trays with the largest volume have if they are made from a cardboard rectangle with length:



a 20 cm? **b** 30 cm? **c** 50 cm?

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18 A parcel delivery firm makes regular trips between Brisbane and Ipswich. The accountant, who also happens to be a good mathematician, has calculated that fuel and other running costs are approximated by the function

$$F(v) = 2v^{\frac{3}{2}} + 59 \text{ cents/trip}$$

where v = average speed in km/h. The cost of paying a driver is given by

$$D(v) = \frac{1.5 \times 10^5}{v} + 2000 \text{ cents/trip}$$

Find the most economical speed for the trip and the cost of a trip from Brisbane to Ipswich at this speed.

19 An Australian test cricket player strikes the ball so that its equation of motion is given by

$$y = 1.4 + x - 0.004x^2$$

where y is the height (m) reached by the ball and x is the horizontal distance (m) travelled by the ball. What is the greatest height reached by the ball?

Practice quiz

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9.

DISCRETE RANDOM VARIABLES

There are many events that have only 2 possible outcomes. The toss of a coin is a head or a tail. A newborn child is male or female. Will a basketball player score or miss a shot? Does a voter support the Prime Minister or not?

Events like these are called **Bernoulli events**, named after the Swiss mathematician Jacob Bernoulli (1654–1705). Bernoulli made notable contributions to many fields of mathematics, but his most important was in the area of probability. Bernoulli wrote the first significant book on probability in which he developed a proof of the binomial theorem.

9.01 The Bernoulli distribution
9.02 Applications of the Bernoulli distribution
9.03 Bernoulli trials and the binomial distribution
9.04 Calculation of binomial probabilities
9.05 The mean and variance of binomial distributions
9.06 Applications of the binomial distribution
Chapter summary
Chapter review

SYLLABUS SUBJECT MATTER

Bernoulli distributions

- use a Bernoulli random variable as a model for two-outcome situations
- identify contexts suitable for modelling by Bernoulli random variables
- recognise and determine the mean p and variance $p(1-p)$ of the Bernoulli distribution with parameter p
- use Bernoulli random variables and associated probabilities to model data and solve practical problems

Binomial distributions

- understand the concepts of Bernoulli trials and the concept of a binomial random variable as the number of 'successes' in n independent Bernoulli trials, with the same probability of success p in each trial
- identify contexts suitable for modelling by binomial random variables
- determine and use the probabilities $P(X = r) = \binom{n}{r} p^r(1-p)^{n-r}$ associated with the binomial distribution with parameters n and p
- calculate the mean np and variance $np(1-p)$ of a binomial distribution using technology and algebraic methods
- identify contexts suitable to model binomial distributions and associated probabilities to solve practical problems, including the language of 'at most' and 'at least'

Prior learning

Mathematical Methods 2019 v1.1 General Senior Syllabus © Queensland Curriculum and Assessment Authority (QCAA). This syllabus forms part of a new senior assessment and tertiary entrance system in Queensland. Along with other senior Syllabuses, it is still being refined in preparation for implementation in schools from 2019. For the most current syllabus versions and curriculum information please refer to the QCAA website https://www.qcaa.qld.edu.au/.

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TERMINOLOGY

Bernoulli event
Bernoulli probability distribution
Bernoulli random variable
Bernoulli trial
binomial distribution
binomial experiment
binomial random variable
expected value
failure
geometric probability distribution
negatively skewed
positively skewed
skewness
standard deviation
success
symmetrical
variance

WS
The Bernoulli distribution

9.01 The Bernoulli distribution

A **Bernoulli event** is an event with only 2 possible outcomes, usually called **success** and **failure**. A distribution with only 2 possible outcomes is called a **Bernoulli probability distribution**.

A **Bernoulli probability distribution** is a discrete probability distribution with only 2 possible outcomes – success and failure. The probability of success, p, and probability of failure, q, are fixed.

For a Bernoulli event:

$$p + q = 1$$

The **Bernoulli random variable** X has the value of 1 for success and 0 for failure.

For a discrete probability distribution that has a Bernoulli random variable, X:

$$P(X = x) = \begin{cases} p & \text{if } x = 1 \\ 1 - p & \text{if } x = 0 \end{cases}$$

EXAMPLE 1

Determine whether each situation would result in a Bernoulli distribution.

a Taking a driving test and recording whether you pass the test.

b Tossing a coin 5 times and recording the number of heads.

c Rolling a 4-sided die and recording the number that occurs.

d Drawing a card from a standard deck and recording whether or not it is a heart.

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Solution

a	State the outcomes.	There are 2 possible outcomes – pass or fail.
	State the result.	This is a Bernoulli distribution.
b	State the outcomes.	There are 6 possible outcomes – 0, 1, 2, 3, 4, 5.
	State the result.	This is not a Bernoulli distribution.
c	State the outcomes.	There are 4 possible outcomes – 1, 2, 3 and 4.
	State the result.	This is not a Bernoulli distribution.
d	State the outcomes.	There are 2 possible outcomes – heart or not a heart.
	State the result.	This is a Bernoulli distribution.

In Year 11, you saw how to calculate the expected value, $E(X)$, and variance, $Var(X)$, for a discrete random variable. You also established the results:

$$E(X) = \mu \qquad \text{and} \qquad Var(X) = \sigma^2$$

where μ is the mean and σ is the standard deviation $[SD(X)]$.

Consider the Bernoulli random variable, X, with the following distribution.

x	0	1
$P(X = x)$	$1 - p$	p

$$\begin{aligned} E(X) &= \sum x \cdot P(X = x) \\ &= 0(1 - p) + 1 \times p \\ &= p \end{aligned}$$

$$\begin{aligned} \text{So } E(X^2) &= \sum x^2 \cdot P(X = x) \\ &= 0^2(1 - p) + 1^2 \times p \\ &= p \end{aligned}$$

$$\begin{aligned} \text{and } Var(X) &= E(X^2) - [E(X)]^2 \\ &= p - p^2 \\ &= p(1 - p) \end{aligned}$$

Bernoulli distributions

$$E(X) = \mu = p$$
$$Var(X) = \sigma^2 = p(1-p)$$
$$\sigma = \sqrt{p(1-p)}$$

EXAMPLE 2

A drug has been developed for the treatment of skin cancer. The drug is found to be effective in 81% of cases. If the effectiveness of the drug = X:

a construct a probability distribution table for X

b find $E(X)$

c find the standard deviation of X, correct to 4 decimal places.

Solution

a There are 2 possible outcomes – the drug is effective (success) and the drug is not effective (failure).

Write the probability of success. $p = P(\text{effective}) = 0.81$

Construct the distribution.

x	0	1
$P(X = x)$	0.19	0.81

b State the rule for expected value. $E(X) = p = 0.81$

c Calculate the variance. $Var(X) = p(1-p)$

Substitute for known values. $= 0.81 \times 0.19$

Evaluate. $= 0.1539$

Calculate the standard deviation. $\sigma^2 = Var(X) = 0.1539$

Take square roots $\sigma = \sqrt{0.1539}$

$= 0.392\ 300\ldots$

Round off and state the result. $\sigma \approx 0.3923$

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Exercise 9.01 The Bernoulli distribution

Example 1

1 Determine whether each statement results in a Bernoulli distribution.

- **a** Spinning a spinner on a wheel with equal-sized segments numbered 1–12 and recording if the number is odd.
- **b** Asking a randomly selected person if they have had a flu injection this year.
- **c** Choosing a marble from a bag containing red marbles, blue marbles and green marbles and recording the colour.
- **d** Tossing a coin and recording if the outcome is a head.
- **e** Drawing a card from a standard deck and recording if the selection is an Ace.
- **f** Rolling a die and recording if the number is a multiple of three.
- **g** Drawing a card from a standard pack and noting its suit.
- **h** Selecting a person at random and recording their age.

2 For each Bernoulli distribution identified in question **1**:

- **i** identify a successful outcome
- **ii** determine the probability of success.

3 Calculate the probability of success in each Bernoulli distribution.

- **a** Picking a dark chocolate from a bag containing 5 milk chocolates, 6 dark chocolates and 4 white chocolates.
- **b** Drawing a picture card (K, Q or J) from a standard deck.
- **c** Selecting a black sock from a drawer containing 6 red socks, 2 brown socks, 8 black socks and 4 green socks.
- **d** Getting a 4 or a 2 when a normal die is rolled.
- **e** Getting 2 heads when 2 coins are tossed.

Example 2

4 A multiple-choice question has 5 possible answers A, B, C, D and E.
Caleb guesses the answer to the question. Let X be that he guesses the correct answer.

- **a** Construct a probability distribution table for X.
- **b** Find $E(X)$.
- **c** Find the standard deviation of X.

5 Calculate the variance of the random variable in a Bernoulli distribution with each probability value.

a $p = \frac{1}{4}$ **b** $p = 0.4$ **c** $q = \frac{2}{5}$ **d** $p = \frac{11}{14}$ **e** $p = 70\%$

6 Calculate the standard deviation of the random variable in a Bernoulli distribution with each probability value.

a $p = \frac{1}{3}$ **b** $p = 30\%$ **c** $p = \frac{7}{15}$ **d** $q = \frac{12}{25}$ **e** $p = 0.25$

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7 A discrete random variable, U, has a Bernoulli distribution as follows.

u	0	1
$P(U = u)$	0.32	0.68

a Find $E(U)$. **b** Find $Var(U)$. **c** Find $SD(U)$.

Problem solving

8 Daria is a professional tennis player. She knows from experience that there is a 28% chance of serving an ace on any serve. Y is the discrete random variable that represents Daria serving an ace.

a Construct a probability distribution table for Y.

b Calculate $E(Y)$ and $Var(Y)$.

9 There is a 79% chance that it will rain on any given day in Tully during the wet season in January and February.

a Construct a probability distribution table for X, where X is the probability that it rains on any given day during the wet season.

b Find the standard deviation of X.

10 Z is a discrete random variable that has a Bernoulli distribution. It is known that the variance for this distribution is 0.24. The probability of success is p.

a Find p if $p > 1 - p$.

b Find $E(Z)$.

11 Y is a discrete random variable that has a Bernoulli distribution. It is known that the standard deviation of Y is 0.4854.

a Find the variance, correct to 4 decimal places.

b Find the probability of success, correct to 4 decimal places, if $P(\text{success}) > P(\text{failure})$.

WS
Geometric probability distributions

9.02 Applications of the Bernoulli distribution

A **Bernoulli trial** is when a Bernoulli event is conducted. Bernoulli trials are independent, so the probability of success (p) or failure (q) remains the same for each trial.

In some situations, you might be interested in how many Bernoulli trials are needed until the first success occurs. For example, you would probably want to know how many games your favourite team are likely to play to have their first win. The pattern for this is a sequence of failures and then one success:

$$F, F, F, \ldots, S$$

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EXAMPLE 3

The Bernoulli distribution

Alex is a netball player who knows from experience that she has a 65% chance of shooting a goal from any attempt. What is the probability that it will take Alex exactly 4 attempts to shoot her first goal? State the answer correct to 2 significant figures.

Solution

Write the sequence of attempts.	F, F, F, S
Calculate p.	$p = 65\% = 0.65$
Calculate q.	$p + q = 1$ $q = 1 - 0.65$ $= 0.35$
Calculate the probability of the stated sequence.	$P(FFFS) = 0.35 \times 0.35 \times 0.35 \times 0.65$ $= 0.027\,86\ldots$
Round off and state the result.	The probability that 4 attempts are needed before the first goal is scored is about 0.028 or 2.8%.

Bradley Kanaris / Getty Images

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The distribution shown in the netball example is a **geometric probability distribution**. You examined geometric distributions in Year 11.

The **geometric probability distribution** of a discrete random variable X is the distribution of x failures before the first success in an experiment of repeated Bernoulli trials.

The probability distribution is given by $P(X = x) = q^x p$.

Sometimes the geometric distribution is defined as the number of trials needed *until* the first success is achieved. The probability that the first success occurs on the $X = k$th trial is given by:

$$P(X = k) = q^{k-1} p \text{ for } k = 1, 2, 3, \ldots$$

While X is discrete, it is not finite. X has the values 0, 1, 2, 3, …, i.e., $X \in \boldsymbol{J}^+$.

The geometric probability distribution function is:

$$\sum q^x p = p + qp + q^2 p + q^3 p + \ldots$$

This is a geometric progression with first term $a = p$ and common ratio $r = q$.

Because the common ratio $q < 1$, the sum converges to the value $\frac{a}{1-r} = \frac{p}{1-q}$.

So $\sum q^x p = \frac{p}{1-q} = \frac{1-q}{1-q} = 1$.

This is exactly what you would expect to find as the sum of all probabilities must be 1.

The expected value of the geometric distribution is given by:

$$E(X) = \sum x \cdot q^x p = 0 \times q^0 \times p + 1 \times q^1 \times p + 2 \times q^2 \times p + 3 \times q^3 \times p + 4 \times q^4 \times p + \ldots$$

Thus $E(X) = qp + 2q^2 p + 3q^3 p + 4q^4 p + \ldots$

So $qE(X) = q^2 p + 2q^3 p + 3q^4 p + 4q^3 p + \ldots$

Subtracting these two expressions gives:

$$E(X) - qE(X) = qp + q^2 p + q^3 p + q^4 p + \ldots$$

This is a GP with $a = qp$ and $r = q$, where $q < 1$, so

$$E(X) - qE(X) = \frac{qp}{1-q}$$

$$= \frac{q(1-q)}{1-q}$$

So

$$E(X) - qE(X) = q$$

$$E(X)(1-q) = q$$

$$E(X) = \frac{q}{1-q}$$

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Expected value of a geometric distribution

The expected value of the geometric distribution with probability of success p and probability of failure q is:

$$E(X) = \frac{q}{1-q} = \frac{q}{p} = \frac{1-p}{p}$$

The expected value of a geometric distribution is the average number of trials before the first success. For a geometric distribution, the expected value is also known as the **waiting time** – the average number of trials before a specific outcome occurs.

EXAMPLE 4

Mike is new to basketball and is practising to shoot the ball from the free throw line. He knows that his probability of success at each throw is 0.2. How many times would Mike fail on average before a success occurs?

Solution

Calculate p and q.

$p = 0.2$

$q = 1 - p = 0.8$

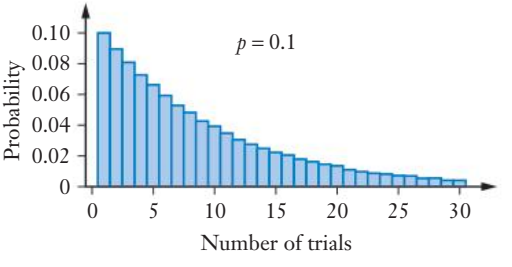
Calculate $E(X)$.

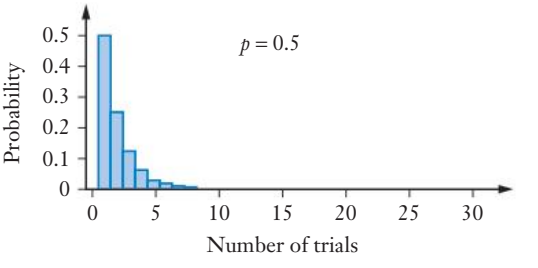
$$E(X) = \frac{q}{p}$$
$$= \frac{0.8}{0.2}$$
$$= 4$$

State the result.

Mike will fail an average of 4 times before his first success.

This example shows that Mike will make 4 shots on average before he shoots a basket. This doesn't mean that the first success would require that many trials. The graphs below show the probability of success on the nth trial of a geometric distribution when $p = 0.1$ and when $p = 0.5$.





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EXAMPLE 5

The probability that a random blood donation is type A is 0.4. If Y is the number of blood samples that a technician checks until type A blood is detected, find:

a $P(Y = 6)$ **b** $P(2 \leq Y \leq 5)$

Solution

a Identify the situation. This is an example of an alternate form of the geometric distribution.

Calculate p and q. $p = 0.4, q = 1 - p = 0.6$

Write the probability function. $P(X = k) = q^{k-1}p$

Substitute for known values. $P(X = 6) = 0.6^{6-1} \times 0.4 = 0.6^5 \times 0.4$

Evaluate. $= 0.031\ 104$
≈ 0.031

b Define the required probability. $P(2 \leq Y \leq 5)$
$= P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5)$

Substitute for known values. $= 0.6^1 \times 0.4 + 0.6^2 \times 0.4 + 0.6^3 \times 0.4 + 0.6^4 \times 0.4$

Identify the situation. This is a GP with $a = 0.6 \times 0.4$ and $r = 0.6$.

Use the rule for the sum of a GP. $S_n = \dfrac{a(1-r^n)}{1-r}$

Substitute. $= \dfrac{0.6 \times 0.4(1-0.6^4)}{1-0.6}$

Evaluate. $= 0.522\ 24$

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Exercise 9.02 Applications of the Bernoulli distribution

1 State whether each situation gives a geometric distribution.

a A couple have 3 boys. They want to know the probability of their next child being a girl.

b About 10% of the population has an O negative blood type. A technician at the blood bank wants to know how many samples will need to be tested before type O negative is detected.

c A bag contains 5 red, 7 green and 4 black marbles. A marble is drawn, its colour is noted and then it is set aside before another marble is drawn. You want to find the number of marbles that will be selected before a green marble is drawn.

d A card is selected from a well-shuffled deck, its suit is noted and then it is returned to the deck before another card is drawn. You want to determine the average number of cards that will need to be drawn before the first heart occurs.

e A coin is tossed and the outcome noted. You are interested in how many times the coin will need to be tossed before 2 tails or 3 heads come up.

2 What is the probability that it will take exactly 5 rolls of a normal die to first get a 6?

Example 3

3 Shauna is sitting for a multiple-choice test in which each question has 5 possible answer choices. She randomly selects the answer for each question. Find the probability that Shauna's first correct answer is to question:

a 3 **b** 5 **c** 8

4 A fair coin is tossed a number of times until a head lands. What is the probability of the coin landing heads on the 4th toss?

5 A darts player knows that the probability of hitting the bullseye at each throw is $\frac{1}{9}$.

How many darts will the player need to throw on average before the bullseye is hit?

Example 4

6 A pair of dice is rolled repeatedly.

a What is the probability that the first time a double occurs is on the third roll?

b What is the expected waiting time before a double is rolled?

7 About 70% of the time, the phone lines for booking seats for a popular concert are all busy. If X denotes the number of times you need to call until you get through, find:

Example 5

a $P(X = 4)$ **b** $P(3 \leq X \leq 6)$ **c** $P(X \leq 5)$

8 A market researcher at a shopping centre is surveying people who have a cable TV subscription. It is known that the probability that a person has a cable TV subscription is 0.20. Let X denote the number of people she meets until she finds the first person who has a cable TV subscription.

a What is the probability that the market researcher must meet 4 people before she finds one who has a cable TV subscription?

b What is the probability that the market researcher must meet more than 6 people before she finds one who has a cable TV subscription?

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9 In a particular game, you can only begin if you roll a double on 2 dice. Find the probability that:

a you start on your first go
b you need 4 attempts before you start
c you start within 3 attempts
d you need more than 6 attempts before you start.

10 In a bottling plant it is known that 25% of bottles are under-filled. If X is the number of under-filled bottles before a correctly filled bottle is found, calculate:

a $P(X = 4)$ **b** $P(2 \leq X \leq 7)$

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11 An archer knows from experience that she can hit the target zone 80% of the time. Let Z be the number of shots taken before the target is hit. Calculate:

a $P(Z = 3)$ **b** $P(Z = 10)$ **c** $P(Z < 5)$ **d** $P(Z \geq 5)$

Problem solving

12 A basketball player has a 65% chance of making a free throw. What is the probability that it takes the player 2 or 3 throws to make the first free throw?

13 What is the probability of rolling a sum of 7 in fewer than 3 rolls of a pair of dice?

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9.03 Bernoulli trials and the binomial distribution

Binomial probability experiments

You have now looked at the Bernoulli distribution and the geometric distribution. What if you are interested in the number of successful outcomes from a fixed number of Bernoulli trials? This is called the **binomial distribution**.

The **binomial distribution** represents the number of successes in n independent Bernoulli trials for some given value of n. The binomial distribution results from the performance of a **binomial experiment**.

- There are a fixed number, n, of repeated trials
- Each trial has exactly two outcomes – success (p) and failure (q)
- Each trial is independent (the probability of success stays the same)

The **binomial random variable** is the number of successes for n trials.

EXAMPLE 6

State whether each situation is a binomial experiment.

a Tossing a fair coin many times and noting the number of tails that occur.

b Selecting and replacing a pen from a basket containing 5 red and 7 black pens 12 times, and recording the number of red pens drawn.

c Selecting 25 people at random from the population and recording their age in years.

d Selecting 15 people at random from the population and noting if they have more than one sibling.

Solution

a	Is the number of trials fixed?	There are 'many' trials, so n is not fixed.
	State the result.	This is not a binomial experiment.
b	Is the number of trials fixed?	There are 12 trials.
	Are there exactly 2 outcomes?	The outcomes are 'red' (success) and 'not red' (failure).
	Is p the same for each trial?	Pens are replaced, so $p = \frac{5}{12}$ for every trial.
	Are trials independent?	Replacement means the trials are independent.
	State the result.	This is a binomial experiment.

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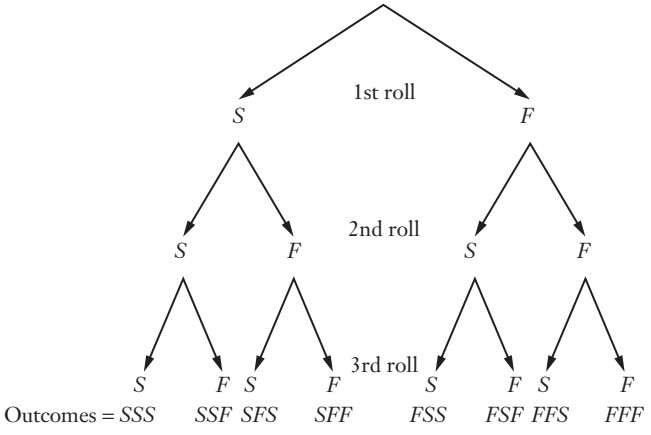
c	Is the number of trials fixed?	There are 25 trials.
	Are there exactly 2 outcomes?	There are many different outcomes.
	State the result.	This is not a binomial experiment.
d	Is the number of trials fixed?	There are 15 trials.
	Are there exactly 2 outcomes?	The outcomes are 'more than 1 sibling' (success) and 'not more than 1 sibling' (failure).
	Is p the same for each trial?	For a large population, the probability that a person has more than 1 sibling is the same for every trial.
	Are trials independent?	People are selected randomly, so the trials are independent.
	State the result.	This is a binomial experiment.

Consider the binomial experiment of rolling a die 3 times. A success (S) is a composite number (that is, 4 or 6) and a failure (F) is a non-composite number. This is a binomial experiment where:

- $P(S) = P(\text{composite}) = p = \frac{2}{6} = \frac{1}{3}$
- $P(F) = P(\text{non-composite}) = q = \frac{2}{3}$

There are many different ways to determine the outcomes of the experiment. For example, a tree diagram gives the following.



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The outcomes and probabilities are summarised below:

Number of successes	Possible outcomes	Probability
0	*FFF*	$1 \times \left(\frac{2}{3}\right)^3 = \frac{8}{27}$
1	*SFF*, *FSF*, *FFS*	$3 \times \left(\frac{2}{3}\right)^2 \left(\frac{1}{3}\right) = \frac{12}{27}$
2	*SSF*, *SFS*, *FSS*	$3 \times \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^2 = \frac{6}{27}$
3	*SSS*	$1 \times \left(\frac{1}{3}\right)^3 = \frac{1}{27}$

You can see that the values of the binomial distribution are actually the terms of the binomial expansion:

$$(p+q)^3 = {}^3C_0p^3 + {}^3C_1p^2q + {}^3C_2pq^2 + {}^3C_3q^3$$

This means that rather than using a tree diagram or another method, it is possible to calculate the probabilities of the various terms of a binomial distribution using the binomial expansion.

$$(a+b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + \ldots + {}^nC_nb^n$$

To apply the binomial expansion to the binomial distribution, you replace x with p and y with q and let n be the number of trials in the experiment.

Remember, ${}^nC_k = \binom{n}{k}$.

The binomial distribution

For the binomial random variable X, the probability of x successes from n trials is:

$$P(X=x) = \binom{n}{x}p^xq^{n-x} \quad \text{or} \quad P(X=x) = \binom{n}{x}p^x(1-p)^{n-x}$$

where $x = 0, 1, 2, \ldots, n$ and $\binom{n}{x} = \frac{n!}{x!(n-x)!}$.

EXAMPLE 7

A binomial distribution has the probability function:

$$P(X = x) = \binom{6}{x} (0.2)^x (0.8)^{6-x}$$

a How many trials are in this experiment?

b What is the probability of success in any trial?

c Show the distribution as a table.

Solution

a Compare the function with the definition.

$n = 6$, so there are 6 trials.

b Compare the function with the definition.

The probability of success, $p = 0.2$.

c Write the probability function.

$$P(X = x) = \binom{6}{x} (0.2)^x (0.8)^{6-x}$$

Let $x = 0$.

$$P(X = 0) = \binom{6}{0} (0.2)^0 (0.8)^{6-0}$$

Evaluate.

$$= 1 \times 1 \times (0.8)^6$$
$$= 0.262\ 144$$

Let $x = 1$.

$$P(X = 1) = \binom{6}{1} (0.2)^1 (0.8)^{6-1}$$

Evaluate.

$$= 6 \times 0.2 \times (0.8)^5$$
$$= 0.393\ 216$$

Do the remaining calculations and put the results in a table.

x	0	1	2	3	4	5	6
$P(X = x)$	0.2621	0.3932	0.2458	0.0819	0.0154	0.001 536	0.000 064

The binomial probability function provides a convenient way for working out the probabilities of any number of successes in a binomial experiment.

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EXAMPLE 8

Jeremy sits for a test containing 10 multiple-choice questions, each with 5 choices. Calculate the probability that Jeremy scores 7 correct answers after guessing every answer.

Solution

Calculate the probability of success.	$p = \frac{1}{5}$
Calculate the probability of failure.	$q = 1 - \frac{1}{5} = \frac{4}{5}$
Identify n.	There are 10 trials, so $n = 10$.
Write the probability distribution function.	$P(X = x) = \binom{n}{x} p^x q^{n-x}$
Substitute for known values.	$P(X = 7) = \binom{10}{7} \left(\frac{1}{5}\right)^7 \left(\frac{4}{5}\right)^3$
Evaluate.	$= 0.000\,786\ldots$
Round off and state the result.	The probability of guessing 7 correct answers is about 0.000 79 or 0.079%.

Exercise 9.03 Bernoulli trials and the binomial distribution

Example 6

1 Which of the following situations are binomial experiments? For those that are not, indicate why not.

- **a** Tossing a fair coin 2000 times and the number of times a head appears is counted.
- **b** A card is drawn from a standard deck with replacement 50 times and the number of times a diamond is drawn is noted.
- **c** Tossing a fair coin and counting the number of tosses before a head appears.
- **d** Selecting 40 people at random from the population and recording their eye colour.
- **e** Checking 5 heavy machinery operators for alcohol consumption from a workforce of 20 operators in which it is known that 5% consume alcohol before work.
- **f** A shipment of 50 appliances contains 2 defective appliances. A retailer tests 15 of the appliances (without replacement) and counts the number that fail to meet specifications.
- **g** Rolling a normal die 100 times and counting the number of times a 3 is rolled.
- **h** An experienced archer knows that she has a 20% chance of hitting the inner bullseyes. She shoots 10 arrows and the number of inner bullseyes that are hit is recorded.

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2 Evaluate each expression, correct to 4 decimal places if necessary.

a $\binom{5}{3}(0.5)^3(0.5)^2$ **b** $\binom{8}{2}(0.6)^2(0.4)^6$ **c** $\binom{6}{3}(0.25)^3(0.75)^3$

d $\binom{9}{5}(0.72)^5(0.28)^4$ **e** $\binom{7}{0}(0.55)^0(0.45)^{10}$ **f** $\binom{15}{10}(0.88)^{10}(0.12)^5$

Example 7

3 Identify the parameters p, q and n for each binomial probability distribution.

a $\binom{12}{2}(0.5)^2(0.5)^{10}$ **b** $\binom{15}{10}(0.1)^{10}(0.9)^5$ **c** $\binom{7}{1}(0.6)(0.4)^6$

d $\binom{25}{11}(0.23)^{11}(0.77)^{14}$ **e** $\binom{8}{8}(0.47)^8$ **f** $\binom{20}{0}(0.33)^{20}$

4 Calculate the probability distribution for each of the following and graph the result.

a $P(X=x)=\binom{10}{x}(0.2)^x(0.8)^{10-x}$

b $P(X=x)=\binom{10}{x}(0.5)^x(0.5)^{10-x}$

c $P(X=x)=\binom{10}{x}(0.8)^x(0.2)^{10-x}$

5 Comment on the symmetry of each distribution drawn in question **4**.

6 A binomial variable, X, has the probability function:

$$P(X=x)=\binom{5}{x}(0.35)^x(0.65)^{5-x}$$

a How many trials are in this experiment?
b What is the probability of success in any trial?
c Show the distribution as a table.
d Plot the distribution and comment on its symmetry.

7 A binomial variable, X, has the probability function:

$$P(X=x)=\binom{8}{x}(0.72)^x(0.28)^{8-x}$$

a How many trials are in this experiment?
b What is the probability of success in any trial?
c Show the distribution as a table.
d Plot the distribution and comment on its symmetry.

8 Consider a binomial experiment with $n=5$, $p=0.6$, and $x=$ the number of successes. Use the formula for $P(X=x)$ to find the probability for:

a $x=0$ **b** $x=3$ **c** $x=5$

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9 Consider a binomial experiment with $n = 12$, $p = 0.32$, and $x =$ the number of successes. Use the formula for $P(X = x)$ to find the probability for:

a $x = 3$ **b** $x = 9$ **c** $x = 11$

10 A die is rolled 10 times. Calculate the probability that exactly 4 6s result.

Example 8

11 The 5-year survival rate for a certain group of 15 prostate cancer patients is known to be 0.78. Calculate the probability that exactly 10 of the men in the group survive for 5 years.

12 It is known that 65% of the voters in a large city are against a proposed development. If 20 voters are selected at random, find the probability that exactly 12 are against the proposed development.

9.04 Calculation of binomial probabilities

Using your graphics calculator simplifies calculation of binomial probabilities.

WS Using the binomial probability distribution

EXAMPLE 9

Use a graphics calculator to calculate the probability of obtaining 8 successes in a binomial experiment with 18 trials, where $p = 0.77$.

Solution

State the parameters.

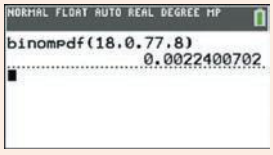
$n = 18$
$p = 0.77$
$x = 8$

TI-84 Plus CE

Press 2nd vars ([distr]), then select A:binompdf((binomial probability distribution function).

Enter the parameters.

Press enter to display the result.



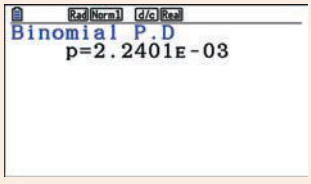
TI-Nspire CX Chapter 9

Casio fx-CG20AU

Select the Statistics screen from the main menu.

Press F5 (DIST) then F5 (BINOMIAL), then F1 (Bpd) (binomial probability distribution).

Press F2 (Var) (if necessary) and then enter the parameters.



Both calculators

State the result.

The probability of 8 successes is about 0.0022.

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Sometimes you will need to interpret questions to work out what values are required.

EXAMPLE 10

In a binomial experiment with 8 trials, the probability of success in any trial is 0.6. If X is the random variable that represents the number of successes, calculate the probability that:

a X is at least 5

b X is less than 3

c X is at most 4

d X is greater than 1

Solution

a

List the parameters.

$n = 8$

$p = 0.6$

'At least 5' means 5 or more.

$P(X \geq 5)$

$= P(X = 5) + P(X = 6) + P(X = 7) + P(X = 8)$

Use the formula.

$$= \binom{8}{5}(0.6)^5(0.4)^3 + \binom{8}{6}(0.6)^6(0.4)^2 + \binom{8}{7}(0.6)^7(0.4) + \binom{8}{8}(0.6)^8$$

Evaluate using a calculator.

$= 0.594\,086\ldots$

State the result.

The probability of at least 5 successes is about 0.5941.

b

'Less than 3' does not include 3.

$P(X < 3) = P(X = 2) + P(X = 1) + P(X = 0)$

$$= \binom{8}{2}(0.6)^2(0.4)^6 + \binom{8}{1}(0.6)(0.4)^7 + \binom{8}{0}(0.4)^8$$

$= 0.049\,807\ldots$

State the result.

The probability of fewer than 3 successes is about 0.0498.

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c 'At most 4' means 4 or fewer. $P(X \leq 4)$

$$= P(X=4) + P(X=3) + P(X=2) + P(X=1) + P(X=0)$$

$$= \binom{8}{4}(0.6)^4(0.4)^4 + \binom{8}{3}(0.6)^3(0.4)^5 + \binom{8}{2}(0.6)^2(0.4)^6 + \binom{8}{1}(0.6)(0.4)^7 + \binom{8}{0}(0.4)^8$$

$$= 0.405\,913\ldots$$

State the result. The probability of at most 3 successes is about 0.4059.

d State the condition. $P(X > 1) = P(X=2) + P(X=3) + \ldots + P(X=8)$

Use counting techniques. $= 1 - P(X \leq 1)$

$= 1 - [P(X=1) + P(X=0)]$

Use the formula. $= 1 - \left[\binom{8}{1}(0.6)(0.4)^7 + \binom{8}{0}(0.4)^8\right]$

Evaluate using a calculator. $= 0.991\,480\ldots$

State the result. The probability of more than 1 success is about 0.9915.

In addition to the binomial **probability density function** (PDF), your calculator also has a binomial probability **cumulative distribution function** (CDF). While the probability distribution function of say $P(X=4)$ gives the probability of exactly 4 successes, the cumulative distribution function of $P(X=4)$ gives the sum of the probabilities of exactly 0, 1, 2, 3 and 4 successes.

A table showing the binomial probability distribution and binomial probability cumulative distribution for $n = 6$ and $p = 0.25$ is shown on the right.

x	PDF	CDF
0	0.1780	0.1780
1	0.3560	0.5340
2	0.2966	0.8306
3	0.1318	0.9624
4	0.0330	0.9954
5	0.0044	0.9998
6	0.0002	1.0000

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EXAMPLE 11

A binomial experiment has 40 trials, where the probability of success in any trial is 0.64. If the random variable X is the number of successes, use a graphics calculator to find the probability that X is between 7 and 26.

Solution

'Between' means that 7 and 26 are included. Write the condition.

$7 \leq x \leq 26$

State the parameters.

$n = 40$

$p = 0.64$

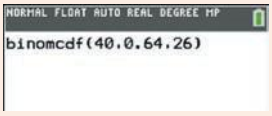
TI-Nspire CX
Chapter 9

TI-84 Plus CE

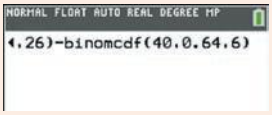
The TI-84 Plus CE uses only an upper bound for x in the cumulative distribution calculation. You need to subtract the cumulative distribution total up to ($x = 7 - 1 = 6$) from the cumulative distribution total up to $x = 26$.

Press 2nd vars ([distr]), then select B:binomcdf((binomial cumulative distribution function).

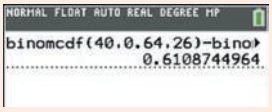
Enter the parameters with $x = 26$, then press enter.



Press − and then press 2nd vars ([distr]), then select B:binomcdf((binomial cumulative distribution function).



Enter the parameters with $x = 6$, then press enter. Press enter to display the result.



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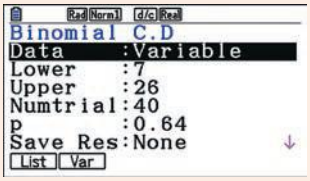
Casio fx-CG20AU

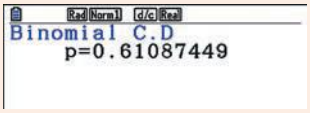
Select the Statistics screen from the main menu.

Press F5 (DIST) then F5 (BINOMIAL), then F2 (Bcd) (binomial cumulative distribution).

The Casio calculator uses an upper and lower bound for x.

Press F2 (Var) (if necessary) and then enter the parameters.





Both calculators

State the result.

The probability that X is between 7 and 26 is about 0.6109.

Exercise 9.04 Calculation of binomial probabilities

Example 9

1 Use a graphics calculator to calculate the probability of:

- **a** 8 successes in a binomial experiment with 12 trials, where $p = 0.82$.
- **b** 3 successes in a binomial experiment with 15 trials, where $p = 0.26$.
- **c** 5 successes in a binomial experiment with 11 trials, where $p = 0.54$.
- **d** 4 successes in a binomial experiment with 14 trials, where $p = 0.18$.

2 In a binomial experiment, $n = 25$, $p = 0.635$ and X = number of successes. Calculate the probability that:

- **a** $X = 15$
- **b** $X = 7$
- **c** $X = 20$
- **d** $X = 7$ or 15
- **e** $X = 20$ or 21

3 A binomial experiment has 20 trials and $p = 0.385$. Calculate the probability that:

- **a** exactly 12 successes occur
- **b** exactly 6 successes occur
- **c** either 9 or 10 successes occur
- **d** 5 or 6 successes occur

Example 10

4 In a binomial experiment with 6 trials, the probability of success in any trial is 0.4. If X is the random variable that represents the number of successes, calculate the probability that:

- **a** X is at least 3
- **b** X is greater than 4
- **c** X is at most 5
- **d** X is less than 3

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5 In a binomial experiment with 4 trials, the probability of success in any trial is 0.8. If X is the random variable that represents the number of successes, calculate the probability that:

a X is 3
b X is less than 3
c X is no more than 3
d X is greater than 3

6 A binomial experiment has 7 trials and $p = 0.3$. Calculate the probability that:

a exactly 2 successes occur
b fewer than 4 successes occur
c at least 5 successes occur
d at most 2 successes occur

7 A binomial experiment has 6 trials and $p = 0.6$. Calculate the probability that fewer than 2 successes or more than 4 successes occur.

Example 11

8 A binomial experiment has 50 trials, where the probability of success in any trial is 0.47. If the random variable X is the number of successes, use a graphics calculator to find the probability that:

a X is between 11 and 30
b X is between 5 and 45
c $X > 30$
d $X \leq 30$

9 In a binomial experiment with 35 trials, the probability of success in any trial is 0.74. Calculate the probability that:

a between 20 and 30 successes occur
b between 15 and 25 successes occur
c no more than 7 successes occur
d at least 18 successes occur

10 Consider a binomial experiment with $n = 9$, $p = 0.75$ and $X =$ the number of successes. Find the probability that:

a X is less than 4.
b X equals 4.
c X is greater than 3 and less than 7.
d X is greater than or equal to 3 and less than or equal to 7.
e X is less than 9.

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9.05 The mean and variance of binomial distributions

Binomial probability – Mean and standard deviation

Binomial distribution

INVESTIGATION

SHAPES OF BINOMIAL DISTRIBUTIONS

Download the spreadsheet 'Binomial distribution' from NelsonNet to investigate the shapes of binomial distributions for different values of n and p.

1 What do you find about the shape of the distribution for larger values of n?

2 What happens to the graph as the value of p gets closer to 0?

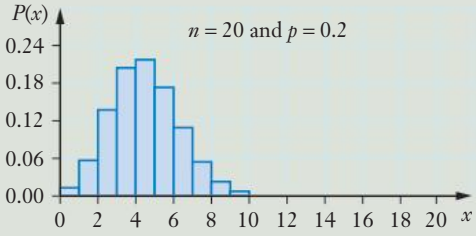
3 What happens to the graph as the value of p gets closer to 1?

4 How does the expected value change as n and p are changed?

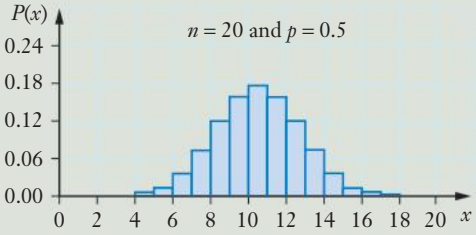
From the previous investigation, you should have observed that the value of p for a binomial distribution influences the symmetry of the graph of the distribution.

Shape of the binomial distribution

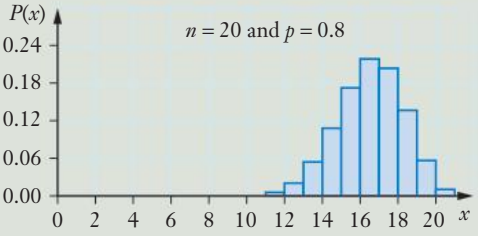
For $p < 0.5$, the distribution is **positively skewed**, that is, it is skewed to the right. The 'tail' is on the positive (right) side.



For $p = 0.5$, the distribution is **symmetrical**.



For $p > 0.5$, the distribution is **negatively skewed**, that is, it is skewed to the left. The 'tail' is on the negative (left) side.



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EXAMPLE 12

Draw a histogram of the binomial distribution for 10 trials in which $p = 0.45$ and comment on the skewness of the distribution.

Solution

State the parameters.

$n = 10$

$p = 0.45$

Use a calculator to calculate the probabilities.

$P(X = 0) = 0.002\,532\ldots$

$P(X = 1) = 0.020\,724\ldots$

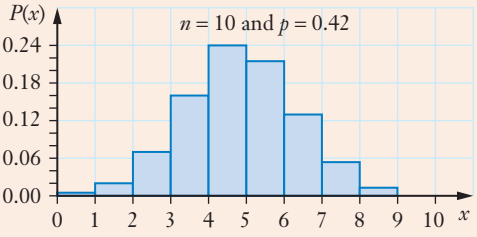
$P(X = 2) = 0.076\,302\ldots$

$P(X = 3) = 0.166\,478\ldots$

$P(X = 4) = 0.238\,366\ldots$

and so on.

Draw the histogram.



Describe the skewness of the graph.

The graph is positively skewed.

A binomial distribution has n Bernoulli trials.

Each trial has an expected value of p, so the expected value of n trials is the sum of all of them, np.

Similarly, the variance of each trial is $p(1 - p)$ as this is the variance of a Bernoulli trial.

The variance of n trials is just the sum of all of them, so the variance of the binomial distribution is $np(1 - p)$.

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Expected value, variance and standard deviation of a binomial distribution

For a binomial distribution, if X is the number of successes in n trials, each with probability of success p, the **expected value**, **variance** and **standard deviation** are:

$$E(X) = \mu = np$$

$$Var(X) = np(1-p) = npq$$

$$SD(X) = \sigma = \sqrt{np(1-p)} = \sqrt{npq}$$

EXAMPLE 13

In a particular binomial experiment with 90 trials, the probability of success in any trial is 0.16. The random variable X is the number of successes.

a What is the expected value of X?

b What is the probability that X is within one standard deviation of the mean?

Solution

a Identify the parameters. $n = 90$, $p = 0.16$

Write the rule for expected value. $E(X) = np$

Substitute for known values. $= 90 \times 0.16$

Evaluate. $= 14.4$

State the result. The expected value of X is either 14 or 15.

b Write the rule for standard deviation. $\sigma = \sqrt{np(1-p)}$

Substitute for known values. $= \sqrt{90 \times 0.16 \times 0.84}$

Evaluate. $= 3.477\ 930\ldots$

Round off. ≈ 3.48

Write an expression for the values within 1 standard deviation of the mean. $\mu - \sigma < X < \mu + \sigma$

Substitute for known values. $14.4 - 3.48 < X < 14.4 + 3.48$

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Evaluate.	$10.92 < X < 17.88$
Identify the required values of X.	Only the values from 11 to 17 are within 1 standard deviation of the mean.
State the required probability.	$P(\mu - \sigma < X < \mu + \sigma) = P(11 \leq X \leq 17)$
Use the binomial cumulative distribution function on your calculator.	≈ 0.6870
State the result.	There is about a 69% chance that X is within one standard deviation of the mean.

EXAMPLE 14

A binomial random variable, Z, has a mean of 16 and a variance of 5.76. Find:

a the probability of success, p **b** the number of trials, n.

Solution

a

Write the rules for the mean and variance.

$\mu = np$

$Var(Z) = np(1 - p)$

Substitute for known values.

$16 = np$ [1]

$5.76 = np(1 - p)$ [2]

Substitute for np in [2].

$5.76 = 16(1 - p)$

Solve for p.

$\frac{5.76}{16} = 1 - p$

$p = 1 - \frac{5.76}{16}$

$p = 0.64$

State the result.

The probability of success is 0.64.

b

Write the rule for the mean.

$\mu = np$

Substitute for known values.

$16 = n \times 0.64$

Solve for n.

$n = 25$

State the result.

There are 25 trials.

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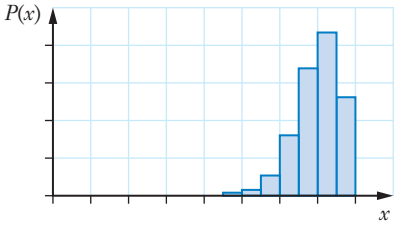
Exercise 9.05 The mean and variance of binomial distributions

1 The diagrams below show a number of probability distributions for a binomial experiment with 15 trials.

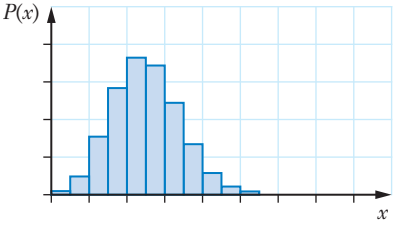
Match each diagram to one of these probabilities:

0.1 0.5 0.9 0.3 0.7

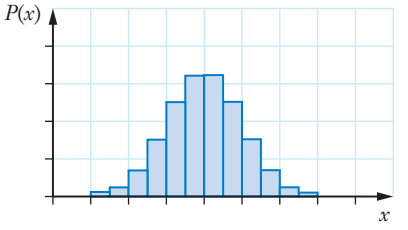
a



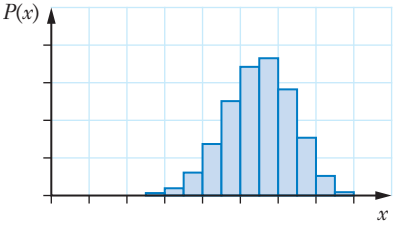
b



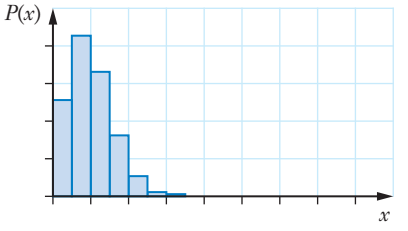
c



d



e



2 Draw a histogram of the binomial distribution of 8 trials in which $p = 0.25$ and comment on the skewness of the distribution.

Example 12

3 Draw a histogram of the binomial distribution of 12 trials in which $p = 0.78$ and comment on the skewness of the distribution.

4 Draw a histogram of the binomial distribution of 10 trials in which $p = 0.44$ and comment on the skewness of the distribution.

5 A binomial experiment has 150 trials, where the probability of success in each trial is 0.38. If the random variable X is the number of successes, calculate:

Example 13

a $E(X)$ **b** $Var(X)$

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6 Calculate the mean and standard deviation of a binomial distribution with parameters:

a $n = 15$ and $p = 0.26$

b $n = 40$ and $p = 0.82$

c $n = 110$ and $p = 0.34$

d $n = 88$ and $p = 0.17$

7 A binomial experiment has 20 trials in which the probability of success is $\frac{3}{14}$. The random variable Z is the number of successes.

a What is the expected value of Z?

b What is the probability that Z is within 1 standard deviation of the mean?

c What is the probability that Z is within 2 standard deviations of the mean?

8 A binomial experiment has 180 trials in which the probability of success is 0.27. The random variable X is the number of successes.

a What is the expected value of X?

b What is the probability that X is within 2 standard deviations of the mean?

Example 14

9 A binomial random variable, X, has a mean of 7.6 and a variance of 4.712. Calculate:

a the probability of success

b the number of trials

10 A binomial random variable, Z, has a mean of 23.1 and a variance of 7.854. Calculate:

a the probability of success

b the number of trials

11 A binomial experiment is completed 25 times and has an expected value of 11.

a Find the probability of success.

b Find the variance and the standard deviation.

Problem solving

12 A random binomial variable Z, has a mean of 36 and a standard deviation of 3. What is the probability of success?

13 It is known that 70% of a shipment of car parts will pass a safety check. Car parts are randomly selected and their safety checked. What is the smallest number of car parts that must be selected to ensure that the probability that at least one will pass the safety check is more than 0.9?

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9.06 Applications of the binomial distribution

There are many different applications of the binomial probability distribution, particularly for decision-making. Before you treat a particular situation as a binomial experiment, you should check that it has the required properties.

WS Discrete random variables assignment

WS Applications of binomial probability

WS Discrete random variables extension

EXAMPLE 15

40% of the residents in a regional city are in favour of the construction of a traffic roundabout in the main street. Suppose 15 residents are randomly selected. Calculate the probability that:

a 7 favour the construction of the roundabout

b fewer than 4 favour the construction of the roundabout

c more than 2 favour the construction of the roundabout

Solution

a

Check this situation against the properties of a binomial experiment.

There are a fixed number of trials ($n = 15$).

Each trial can be assumed to be independent.

There are only 2 outcomes – support and non-support for the roundabout.

The probability of success is constant ($p = 0.4$).

State the conclusion.

This is a binomial experiment.

Write the parameters.

$n = 15$

$p = 0.4$

Let the random variable X be the number of successes.

Required probability = $P(X = 7)$

Write the probability function.

$$P(X = 7) = \binom{15}{7} (0.4)^7 (0.6)^8$$

Evaluate.

$$= 0.177\,083\ldots$$

State the result.

The probability that 7 people favour the construction of the roundabout is about 0.177 (17.7%).

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b 'Fewer than 4' means $X \leq 3$.

$P(X \leq 3)$

$= P(X = 3) + P(X = 2) + P(X = 1) + P(X = 0)$

Use the probability function.

$$= \binom{15}{3}(0.4)^3(0.6)^{12} + \binom{15}{2}(0.4)^2(0.6)^{13} + \binom{15}{1}(0.4)(0.6)^{14} + \binom{15}{0}(0.6)^{15}$$

Evaluate.

$= 0.090\,501\ldots$

State the result.

The probability that fewer than 4 people favour the construction of the roundabout is about 0.091 (9.1%).

c 'More than 2' means $X \geq 3$.

$P(X \geq 3)$

$= P(X = 3) + P(X = 4) + \ldots + P(X = 15)$

$= 1 - P(X \leq 2)$

Use the probability function.

$$= 1 - \binom{15}{2}(0.4)^2(0.6)^{13} - \binom{15}{1}(0.4)(0.6)^{14} - \binom{15}{0}(0.6)^{15}$$

$= 1 - 0.027\,114\ldots$

$= 0.972\,885\ldots$

State the result.

The probability that more than 2 people favour the construction of the roundabout is about 0.973 (97.3%).

EXAMPLE 16

An aptitude test is required of all applicants for a courier's position. The test consists of 15 multiple-choice questions, each with 5 choices. Seth decides to randomly guess the answer to each question.

a How many answers should Seth expect to correctly guess?

b What is the probability that the number of correct answers guessed is within 1 standard deviation of the mean?

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Solution

a

This is a binomial experiment.	$n = 15$
Identify the parameters.	$p = \frac{1}{5} = 0.2$
Let the random variable X be the number of successes.	
Write the rule for expected value.	$E(X) = np$
Substitute for known values.	$= 15 \times 0.2$
Evaluate.	$= 3$
State the result.	The applicant should expect to correctly guess 3 answers.

b

Calculate the standard deviation.	$\sigma = \sqrt{npq}$
Substitute for known values.	$= \sqrt{15 \times 0.2 \times 0.8}$
Evaluate and round off.	≈ 1.55
Write an expression for the values within 1 standard deviation of the mean.	$\mu - \sigma < X < \mu + \sigma$
Substitute for known values.	$3 - 1.55 < X < 3 + 1.55$
Evaluate.	$1.45 < X < 4.55$
Identify the required values of X.	Only the values from 2 to 4 are within 1 standard deviation of the mean.
State the required probability.	$P(\mu - \sigma < X < \mu + \sigma) = P(2 \leq X \leq 4)$
Use the binomial CDF on your calculator.	≈ 0.6686
State the result.	There is about a 67% chance that the number of correct answers guessed is within one standard deviation of the mean.

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EXAMPLE 17

The probability of an archer hitting a target with any shot is 0.4.

a If the archer shoots 8 times, what is the probability of hitting the target at least twice?

b How many times must the archer shoot to be sure that the probability of hitting the target at least once is greater than 75%?

Shutterstock.com/By Dmitry Kalinovsky

Solution

a Identify the parameters.

$n = 8$

$p = 0.4$

Write the required probability.

$P(X \geq 2)$

$= P(X = 2) + P(X = 3) + \ldots + P(X = 8)$

$= 1 - P(X < 2)$

Use the probability function.

$= 1 - \binom{8}{1} \times (0.4)^1 \times (0.6)^7 - \binom{8}{0} \times (0.6)^8$

$= 1 - 0.106\,375\ldots$

$= 0.893\,624\ldots$

State the result.

There is about a 89% chance of hitting the target at least twice.

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b	Write the required probability.	P(hitting target at least once) $= P(X \geq 1)$ $= 1 - P$(not hitting the target) $= 1 - P(X = 0)$ $= 1 - (0.6)^n$
	Apply the condition.	$1 - (0.6)^n > 0.75$
	Interpret this result.	You need to find the smallest value of n for which $(0.6)^n < 0.25$.
	Try values of n until the condition is satisfied.	When $n = 0$, $(0.6)^0 = 1 > 0.15$ When $n = 1$, $(0.6)^1 > 0.25$ When $n = 2$, $(0.6)^2 > 0.25$ When $n = 3$, $(0.6)^3 = 0.216 < 0.25$
	State the result.	The archer must fire 3 shots to be sure that the probability of hitting the target at least once is greater than 75%.

Exercise 9.06 Applications of the binomial distribution

Example 15

1 A card is drawn and replaced 7 times from an ordinary pack of 52 playing cards. Find the probability that:

a 3 diamonds are drawn
b at least one diamond is drawn
c no more than 2 diamonds are drawn
d at most 4 diamonds are drawn

2 A school computer network consisting of 30 computers was attacked by a computer virus. The virus enters each computer with probability 0.4, independently of other computers. Find the probability that the virus enters

a exactly 5 computers
b at least 10 computers
c no more than 12 computers
d between 5 and 20 computers

Example 16

3 It is known that 45% of people have blood type O. A random sample of 20 people have their blood type tested.

a Find the probability that more than 7 will have type O.
b How many people would you expect to have blood type O?
c What is the probability that the number of people with blood type O is within 1 standard deviation of the mean?
d What is the probability that the number of people with blood type O is within 2 standard deviations of the mean?

4 It is known that approximately 18% of all people have high blood pressure. The blood pressure of a random sample of 85 people is measured.

a What is the expected number of people with high blood pressure?

b What is the probability that the number of people with high blood pressure is within one standard deviation of the mean?

Example 17

5 A bag contains 5 red, 2 green and 8 blue marbles.

a If a marble is drawn 5 times with replacement, what is the probability of drawing a red marble at least twice?

b How many times must a marble be drawn (with replacement) to be sure that the probability of drawing a red marble at least once is greater than 90%?

6 A large shipment of widgets contains 20% that are defective.

a If five widgets are randomly selected from the shipment, what is the probability that none is defective?

b What is the smallest number of widgets that should be selected to ensure that the probability of at least one defective widget being selected is more than 0.95?

7 It is known that 35% of teenagers work in part-time jobs. 20 teenagers are randomly selected and asked if they have a part-time job. If Z is the random variable that defines the number of teenagers who have a part-time job, find:

a $P(Z \leq 7)$ **b** $E(Z)$

c $SD(Z)$ **d** $P(\mu - \sigma < Z < \mu + \sigma)$.

8 How many aces would you expect to get from 60 selections of a normal pack of cards?

9 The probability that a lettuce seed germinates is about 0.8. About how many seeds should you plant if you want 500 lettuce seedlings?

10 A machine has a probability of 0.05 of manufacturing a defective part.

a What is the expected number of defective parts in a sample of 20 parts manufactured by this machine?

b What is the standard deviation for this situation?

11 A new medication is found to reduce cholesterol levels in 85% of people who use it. The medication is given to 20 people who suffer from high cholesterol. The discrete random variable Y is the number of people who experience reduced cholesterol levels after taking the new medication.

a Find the probability that no more than 10 people experience reduced cholesterol levels.

b Find $E(Y)$ and $SD(Y)$.

c Calculate $P(\mu - 2\sigma < Y < \mu + 2\sigma)$.

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12 It is known that 8% of printed circuit boards made by the AOK company are defective. The company's quality control manager wishes to see whether the percentage of defective boards in a current batch has decreased. The random variable X is the number of defective circuit boards. A sample of size 50 is to be selected. If there is no change from the 8% figure, find:

a $E(X)$ **b** $Var(X)$ **c** $SD(X)$

13 The probability that a car battery fails within 2 years of purchase is about 60%.

a From 420 batteries, how many would you expect to last longer than 2 years?

b What is the standard deviation of the number of failures in 2 years?

Problem solving

14 A probability distribution for the number of successes, X, is:

X	0	1	2	3	4
$P(X = x)$	0.1296	0.3456	0.3456	0.1536	0.0256

Calculate:

a $E(X)$ **b** $Var(X)$ **c** $SD(X)$

15 A telemarketer knows that for any call to a customer there is a 15% chance of making a sale.

a Find the probability that no sales are made in 8 calls.

b Find the probability that more than 4 sales are made in 20 calls.

c A telemarketer is required to achieve a mean of at least 5 sales each day. What is the least number of calls that will need to be made to achieve this target?

d Calculate the least number of calls that the telemarketer will need to make for the probability of making at least 1 sale to be greater than 95%.

16 A law firm has 8 lawyers who all work in the same office. Each lawyer is equally likely to work from home as in the office. How many desks must be in the office so that each lawyer has a desk at least 90% of the time?

17 The probability of a basketball player sinking a basket from the free throw line is 0.45. What is the least number of shots that the player must take to ensure that the probability of sinking a basket at least twice is more than 0.95?

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9. CHAPTER SUMMARY

WS
Discrete random variables find-a-word

Discrete random variables

- A **Bernoulli event** has only two possible outcomes – **success** and a **failure**.
- A **Bernoulli probability distribution** is the distribution for a Bernoulli event. $P(\text{success}) = p$ and $P(\text{failure}) = q$ are fixed and $p + q = 1$.
- The **Bernoulli random variable**, X, is given the values of 1 for success and 0 for failure.

$$P(X = x) = \begin{cases} p & \text{if } x = 1 \\ 1 - p & \text{if } x = 0 \end{cases}$$

- For a Bernoulli distribution: $E(X) = \mu = p$ and $Var(X) = \sigma^2 = p(1 - p)$.
- A **Bernoulli trial** is when a Bernoulli event is conducted. Bernoulli trials are independent.
- The **geometric probability distribution** of a discrete random variable X is the distribution of x failures before the first success in an experiment of repeated Bernoulli trials: $P(X = x) = q^x p$.
- The geometric distribution can also be defined as the number of trials needed *until* the first success is achieved. The probability that the first success occurs on the $X = k$th trial is: $P(X = k) = q^{k-1} p$ for $k = 1, 2, 3, \ldots$
- The expected value of the geometric distribution is $E(X) = \dfrac{q}{1 - q} = \dfrac{q}{p} = \dfrac{1 - p}{p}$.
- The **binomial distribution** represents the number of successes in n independent Bernoulli trials:
  - there are a fixed number, n, of repeated trials
  - each trial is independent, has exactly 2 outcomes – success (p) and failure (q)
  - the probability of success is the same for each trial
- The **binomial random variable** is the number of successes for n trials, where the probability of x successes from n trials is given by

$$P(X = x) = \binom{n}{x} p^x q^{n-x} \quad \text{or} \quad P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}$$

where $x = 0, 1, 2, \ldots, n$ and $\binom{n}{x} = \dfrac{n!}{x!(n - x)!}$.

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- The graph of a binomial distribution is:
  - **positively skewed** (skewed to the right) when $p < 0.5$
  - **symmetrical** when $p = 0.5$
  - **negatively skewed** (skewed to the left) $p > 0.5$
- For a binomial experiment, if X is the number of successes in n trials, each with probability of success p:
  - **expected value** $= E(X) = \mu = np$
  - **variance** $= Var(X) = np(1 - p) = npq$
  - **standard deviation** $= SD(X) = \sigma = \sqrt{np(1-p)} = \sqrt{npq}$

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9. CHAPTER REVIEW

Discrete random variables

Example 1

1 State whether each situation would result in a Bernoulli distribution.

a Rolling a die and recording whether or not the number displayed is a 6.

b Recording the length of time a customer spends in a queue waiting to make a purchase.

c Drawing a card from a standard deck without replacement and recording whether or not it is a king.

d Recording whether the viewers of a film are male or female.

Example 2

2 A confectioner tells customers that when they buy a box of their 'Premier' chocolates, 40% are truffles. Bart bought a box of 'Premier' chocolates and selected one. Let Y be the discrete random variable that Bart chose a truffle.

a Construct a probability distribution table for Y.

b Find $E(Y)$.

c Find $SD(Y)$.

iStock/Getty Images Plus/Naeblys

Example 3

3 It is known that 9% of the components produced on a certain assembly line have at least one defect. Components are randomly sampled from this line one at a time and tested. What is the probability that the first non-defective component is found:

a on the third trial?

b before the fourth trial?

Example 4

4 The probability that a random blood donation is type B is 0.1. On average, how many samples will need to be collected before the first type B sample is discovered?

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5 A particular type of insect has a 28% chance of dying on any one day and this probability remains constant for about 10 days. If Y denotes the number of days that an insect of this type survives until it dies, calculate: Example 5

a $P(Y = 1)$
b $P(3 \leq Y \leq 5)$
c the probability that an insect of this type survives its first 3 days
d the probability that an insect of this type dies on its fourth day.

6 Which of the following are binomial experiments? Example 6

a Players are dealt a card from a well-shuffled pack. The first player with a spade wins.
b Players are dealt a card from a well-shuffled pack. Each player's card is replaced before the next card is dealt. The first player with a spade wins.
c Each player throws a die 5 times. The player with the highest number of 6s wins.
d The number of male children in families is counted.
e The number of male children in families with 5 children is counted.

7 A binomial distribution has the probability function: Example 7

$$P(X = x) = \binom{8}{x} (0.38)^x (0.62)^{8-x}$$

a How many trials are in this experiment?
b What is the probability of success in any trial?
c Show the distribution as a table.

8 A bag contains 3 blue and 7 red marbles. A marble is randomly selected and replaced. This process is repeated 9 times. Calculate the probability that a blue marble is selected 5 times. Example 8

9 Use a graphics calculator to calculate the probability of obtaining 12 successes in a binomial experiment with 24 trials, where $p = 0.68$. Example 9

10 In a binomial experiment with 11 trials, the probability of success in any trial is 0.7. If X is the random variable that represents the number of successes, calculate the probability that: Example 10

a X is at least 4
b X is no more than 5
c X is greater than 2
d X is at most 3

11 A binomial experiment has 50 trials, where the probability of success in any trial is 0.32. If the random variable Z is the number of successes, use a graphics calculator to find the probability that Z is between 15 and 25. Example 11

12 **a** Draw graphs of the binomial distributions for $n = 4$, 8, 12 and 16 for $p = 0.6$. Example 12
b Comment on the shapes of the graphs.

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Example 12

13 **a** Draw graphs of the binomial distributions for $p = 0.1, 0.3, 0.5, 0.7$ and 0.9 for $n = 12$.
b Comment on the shapes of the graphs.

Example 13

14 In a particular binomial experiment with 120 trials, the probability of success in any trial is 0.35. The random variable X is the number of successes.
a What is the expected value of X?
b What is the probability that X is within one standard deviation of the mean?

Example 14

15 A binomial random variable, Y, has a mean of 5.4 and a variance of 2.97. Find:
a the probability of success, p **b** the number of trials, n.

Example 15

16 Eggs are classified by weight as small, medium or large. From long experience, the poultry farm owner knows that 28% of all eggs produced at the farm are classified as being large. 15 eggs are randomly selected from a conveyor belt on which the eggs travel to be classified. Calculate the probability that:
a 8 are classified as large
b fewer than 5 are classified as large
c more than 3 are classified as large

Example 16

17 For a large population, the month of birth is equally distributed throughout the year. This means that the probability of a randomly chosen person's date of birth being in any particular month may be taken as $\frac{1}{12}$. A group of 24 people is randomly selected.
a How many people would you expect to be born in March?
b What is the probability that the number of people born in March is within 1 standard deviation of the mean?

Example 17

18 The probability of an experienced chess player winning any game is known to be 0.7.
a If the chess player has 7 games in a tournament, what is the probability of winning at least twice?
b How many games must the chess player play to be sure that the probability of winning at least once is greater than 80%?

Problem solving

19 X is a discrete random variable that has a Bernoulli distribution. It is known that the standard deviation for this distribution is 0.477.
a Find the variance of X, correct to 4 decimal places.
b Find the probability of success, p, if $p > 1 - p$.

20 The mean of a binomial distribution is 30 and the standard deviation is $\sqrt{12}$. Calculate the probability of between 20 and 30 successes for this distribution.

Qz
Practice quiz

21 A manufacturer decides that she will reject a large shipment of components if there is more than one unusable item in a sample of 10 randomly selected components. It is known that 10% of the shipment is unusable. What is the probability that the manufacturer will reject the shipment?

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Practice examination 3

WS
Year 12 formulas

Time: 90 minutes
Perusal time: 5 minutes
Marks: 50

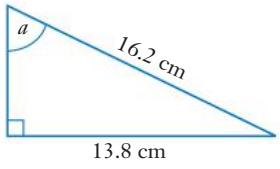
Instructions

- Students are permitted to bring and use: pens, pencils, highlighters, erasers, sharpeners, rules and an approved graphics calculator.
- Students must show appropriate working and justification to gain full marks.
- A QCAA formula sheet is provided.
- Unless otherwise stated, numerical answers should be exact.
- Unless otherwise indicated, no diagrams in this examination are drawn to scale.
- All written responses must be in English.
- Answer all questions.
- **Students are NOT permitted to bring or use notes of any kind, correction fluid/tape, mobile phones and/or any other unauthorised electronic devices.**

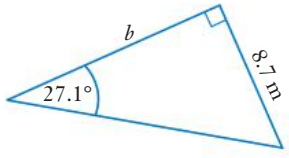
Question 1 (2 marks)

Find the value of the pronumeral in each diagram.

a



b



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Question 2 (1 mark)

Find the value of cos (–480°) without using a calculator.

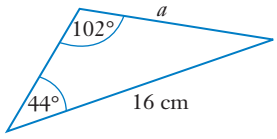
Question 3 (1 mark)

Find the area of $\triangle PQR$, where $P = 49°$, $q = 14.1$ m and $r = 9.6$ m.

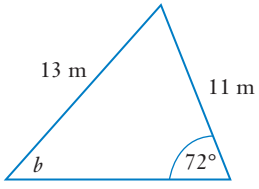
Question 4 (4 marks)

Find the value of the pronumeral in each triangle.

a



b



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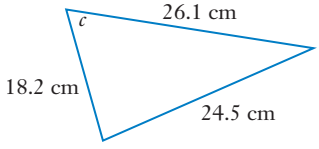
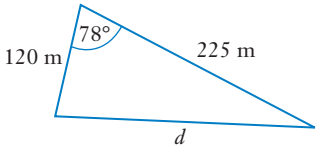


d



Question 5 (2 marks) (Technology-active)

Find the possible values of h in $\triangle FGH$, where $F = 46.8°$, $f = 4.98$ m and $g = 6.13$ m.

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Question 6 (4 marks)

Find the first and second derivatives of each function.

a $f(x) = x^7 - 5x^4 + 11x + 15$

b $f(x) = x^{-2} - 3x^2 + 8x^{\frac{1}{2}} - 8x^{\frac{1}{3}} + 12x$

c $f(x) = x^2 \cos(2x)$

d $f(x) = 5x^3 e^{1-4x}$

Question 7 (3 marks)

Identify the intervals for which $f(x) = -x^3 + 3x^2 + 2$ is concave upwards and concave downwards.

Question 8 (3 marks)

Identify all the stationary points for the function $f(x) = 3x^4 - 6x^2 - 2$.

Question 9 (3 marks) (Technology-active)

Sami is a netball player who knows that she can score a goal 68% of the time.
Let X be the random variable that defines Sami scoring a goal.

a Construct a probability distribution table for X.

b Find $E(X)$.

c Find the standard deviation of X, correct to 4 decimal places.

Question 10 (2 marks)

Approximately 42% of people have blood type A. It can be assumed that the blood types of people who donate blood is random. What is the probability that a technician at the blood bank will take exactly 4 donations and get the first blood type A?

Question 11 (3 marks) (Technology-active)

A binomial distribution has the probability function:

$$P(X = x) = \binom{5}{x}(0.3)^x(0.7)^{5-x}$$

a How many trials are in this experiment?

b What is the probability of success in any trial?

c Show the distribution as a table.

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Question 12 (2 marks) (Technology-active)

In a binomial experiment with 7 trials, the probability of success in any trial is 0.8. If X is the random variable that represents the number of successes, calculate the probability that:

a X is at least 3

b X is at most 5

Question 13 (4 marks) (Technology-active)

Suzie has to look up at an angle of 32° to see the top of a building.

She walks 15 m closer on the level ground in front of the building.

The angle of elevation to the top of the building increases to 67°.

Her eyes are 1.5 m from the ground.

How high is the building?

Question 14 (3 marks) (Technology-active)

A manufacturer can sell x smartwatches per week at d dollars per item, where

$$x = 600 - 3d.$$

Production costs (in dollars) are

$$C(x) = 600 + 10x + \frac{1}{2}x^2.$$

How should the smartwatches be priced in order to maximise profits?

Question 15 (3 marks) (Technology-active)

A binomial random variable, X, has a mean of 8.1 and a variance of 3.726.

a Find the probability of success, p.

b Find the number of trials, n.

Question 16 (3 marks) (Technology-active)

Kenny is standing on top of a hill that is 474 m high when he sees a kangaroo due west at an angle of depression of 12°.

Later, he sees that the kangaroo has moved. It is then at a bearing of 300° and an angle of depression of 9.5°.

In what direction did the kangaroo move?

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Question 17 (4 marks) (Technology-active)

A settling pond is to be constructed to hold 324 m^3 of waste. The settling pond has a square base and four vertical sides, all made of concrete, and a square top made of steel. If the steel costs twice as much per unit area as the concrete, determine the dimensions of the pond that will minimise the cost of construction.

Question 18 (4 marks) (Technology-active)

The probability of winning a prize in a particular game of chance is 0.55. Calculate the least number of games that must be played to be sure that the probability of winning at least twice is more than 0.97.

END OF EXAMINATION

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10.

CONTINUOUS RANDOM VARIABLES AND THE NORMAL DISTRIBUTION

You have learned about probability distributions for discrete random variables. However, measured quantities such as height, weight and time are continuous, not discrete.

The cost of life insurance is based on the probability of death, which increases as a person gets older. The way the probability changes with age is a probability distribution, often shown as a graph. Studying probability distributions allows mathematicians to make important predictions and conclusions in many areas including business, medical science, psychology, economics, engineering and physics.

10.01 Continuous random variables
10.02 Probability density functions
10.03 Continuous random distributions
10.04 Expected value of a continuous random variable
10.05 Variance and standard deviation of a continuous random variable
10.06 The normal distribution
10.07 The standard normal distribution
10.08 Probabilities and quantiles
10.09 Applications of the normal distribution
Chapter summary
Chapter review

SYLLABUS SUBJECT MATTER

General continuous random variables

- use relative frequencies and histograms obtained from data to estimate probabilities associated with a continuous random variable
- understand the concepts of a probability density function, cumulative distribution function, and probabilities associated with a continuous random variable given by integrals; examine simple types of continuous random variables and use them in appropriate contexts
- calculate the expected value, variance and standard deviation of a continuous random variable in simple cases
- understand standardised normal variables (*z*-values, *z*-scores) and use these to compare samples

Normal distributions

- identify contexts, such as naturally occurring variations, that are suitable for modelling by normal random variables
- recognise features of the graph of the probability density function of the normal distribution with mean μ and standard deviation σ and the use of the standard normal distribution
- calculate probabilities and quantiles associated with a given normal distribution using technology and use these to solve practical problems

Prior learning

Mathematical Methods 2019 v1.1 General Senior Syllabus © Queensland Curriculum and Assessment Authority (QCAA). This syllabus forms part of a new senior assessment and tertiary entrance system in Queensland. Along with other senior Syllabuses, it is still being refined in preparation for implementation in schools from 2019. For the most current syllabus versions and curriculum information please refer to the QCAA website https://www.qcaa.qld.edu.au/.

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TERMINOLOGY

closed interval
decile
inverse cumulative distribution function
open interval
probability distribution
quartile
standard score
variance
continuous random variable
expected value
median
percentile
probability density function
standard deviation
triangular distribution
z-score
cumulative distribution function (cdf)
fractile
normal distribution
probability density function (pdf)
quantile
standard normal distribution
uniform distribution

WS
Discrete and continuous random variables

10.01 Continuous random variables

In Chapter 9, you saw that a discrete random variable is a variable whose value depends on the outcome of a probability experiment.

While a discrete random variable can take on a finite number of outcomes, a continuous random variable may take on an infinite number of possible outcomes.

For example, if X represents the length of time (in seconds) of recorded messages on a voicemail service, then X is a continuous random variable.

A **continuous random variable** is a random variable with values from an interval or continuum. A **continuous probability distribution (or function)** shows the probabilities of a continuous random variable. The values of the **probability function** are written as $P(X = x)$ or $p(x)$.

A random variable is represented by a capital letter, while the corresponding lower-case letter is used to represent a specific value of the variable.

For the random variable X, the values of the **probability function** (**probability distribution**) are written as $P(X = x)$ or $p(x)$.

A continuous random variable such as people's heights can be tabulated in class groups. Because the value of a continuous random variable is measured, there is no limit to its accuracy. When you state someone's height as 172 cm, you are implying a grouping of class width 1 cm from 171.5 to 172.5 cm.

Probability histograms of grouped data are similar to those of discrete probability distributions. The vertical axis is rescaled for the class width so that the height is the same as the assumed probability for discrete single scores in the class. Probability histograms of continuous variables should have a total area of 1, no matter how the data is grouped.

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EXAMPLE 1

The heights of a group of 17-year-olds, measured to the nearest centimetre, are shown:

162	161	158	169	161	180	166	159	171	163
166	163	181	170	178	175	164	169	174	160

a Construct a probability histogram of the height H, with a class width of 2 cm.

b Construct a probability histogram of H with a class width of 10 cm.

c Find the total area of each histogram.

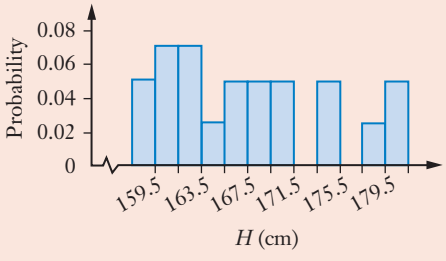
Solution

a Tabulate the data with a class width of 2 cm. Since each column has a class width of 2 cm, the relative frequency must be divided by 2 to get the correct column height.

Thus for the class 165.5–167.5, the value 0.05 is the assumed probability of an individual score, so $P(H = 166) = 0.05$.

H (cm)	f	Relative frequency	Column height
157.5–159.5	2	0.1	0.05
159.5–161.5	3	0.15	0.075
161.5–163.5	3	0.15	0.075
163.5–165.5	1	0.05	0.025
165.5–167.5	2	0.1	0.05
167.5–169.5	2	0.1	0.05
169.5–171.5	2	0.1	0.05
171.5–173.5	0	0	0
173.5–175.5	2	0.1	0.025
175.5–177.5	0	0	0
177.5–179.5	1	0.05	0.025
179.5–181.5	2	0.1	0.05
Total	20	1	

Draw the probability histogram.



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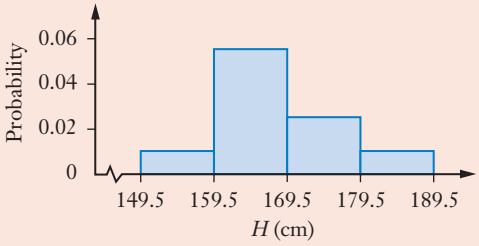
b Retabulate the data with a class width of 10 cm.

Since each column has a class width of 10 cm, the relative frequency must be divided by 10 to get the correct column height.

In this case, the assumed probability of the height of 166 cm is $P(H = 166) = 0.055$.

Draw the histogram.

H (cm)	f	Relative frequency	Column height
149.5–159.5	2	0.1	0.01
159.5–169.5	11	0.55	0.055
169.5–179.5	5	0.25	0.025
179.5–189.5	2	0.1	0.01
Total	20	1	



c Calculate the total area for the first histogram.

$$\text{Area} = 0.05 \times 2 + 0.075 \times 2 + \ldots + 0.05 \times 2$$
$$= 1$$

Calculate the total area for the second histogram.

$$\text{Area} = 0.01 \times 10 + 0.055 \times 10 + 0.025 \times 10 + 0.01 \times 10$$
$$= 1$$

State the result.

The area under each histogram is 1 and represents the sum of the probabilities of all outcomes.

The area over an interval can be used to estimate the probability of a score lying in that interval, just as it does for discrete probabilities. For the previous example, the midpoints of the classes are 158.5, 160.5, 162.5, … You should notice that the height of the graph at 162.5, for example, is not the probability of getting a score of 162.5.

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EXAMPLE 2

Use the second histogram in Example **1** to find the probability of choosing a person at random from the group with a height between 161 cm and 164 cm.

Solution

Identify where the required heights occur.	Heights between 161 cm and 164 cm occur in the class 159.5–169.5.

The required probability is the area of the rectangle with the height of the class 159.5–169.5 and the width between 161 cm and 164 cm.

State the height of the rectangle.	Height = 0.055
State the width of the rectangle.	Width = 164 − 161 = 3 cm
Calculate the required probability.	$P(161 < h < 164) = 0.055 \times 3$
Evaluate.	$= 0.165$

Exercise 10.01 Continuous random variables

1 The IQs of some Year 12 students are shown:

Example 1

90	111	89	123	137	97	120	101	101	101	110	108	101	85	103
121	92	130	109	112	121	139	100	83	141	82	94	94	119	104
95	87	108	104	101	115	106	93	90	128	107	110	98	138	

(*Note*: IQ is standardised, so the population has an average IQ of 100.)

a Considering IQ as a continuous variable, construct a probability histogram of intelligence, I, with class widths of 10.

b Find the total area of the histogram.

2 Some students checked their watches against the school clock and found that they were out by the following amounts (in seconds):

26	37	21	45	21	32	23	34	17	46	21	44	27	38	30	36	29	66	22	34
30	42	17	43	29	33	30	41	18	61	26	49	55	68	27	58	18	52	28	32

a Construct probability histograms of the times, T, with class widths of 4 s and 10 s, starting at 15.5 and 9.5 respectively.

b Find the total area of each histogram.

3 Use the probability histogram from question **1** to find the probability of choosing a person at random from the group with an IQ from 115 to 125 (inclusive).

Example 2

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4 Use the 10 s class-width histogram constructed for question **2** to find the probability of choosing a person at random whose watch was from 24 to 32 s out (to the nearest second).

5 The weights of a group of 17-year-olds, measured to the nearest kilogram, were found to be:

63 58 66 82 88 84 70 67 104 66 67 90 70 55 66 98 49 72 76 46

a Construct probability histograms of the weight, W, with class widths of 5 kg and 10 kg.
b Find the total area of each histogram.
c Use the 10 kg class histogram to find the probability of choosing a person at random from the group with a weight between 64 kg and 76 kg.

6 The times recorded for a group of students running a 100 m sprint, measured to the nearest second, were:

17	14	20	14	15	14	16	18	19	24	12	15	20
20	18	22	20	19	18	20	15	23	18	16	16	13

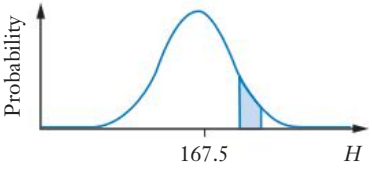
a Construct probability histograms of the times, T, with class widths of 2 s and 5 s.
b Find the total area of each histogram.
c Use the 5 s class histogram to find the probability of choosing a person at random from the group with a time between 14 s and 20 s.

WS
Probability density function

10.02 Probability density functions

In the previous section, you looked at the heights of a group of 17-year-olds, measured to the nearest centimetre. If you made very exact measurements of a very large number of people's heights and used very small class intervals, you would obtain a probability histogram that looked like a smooth curve.

It could look like the graph shown here. The total area under the graph would still be 1, and the area of an interval on the graph would still be the probability of getting a score in that interval. The height of the graph at 166.34, for example, is not the probability of getting a score of 166.34.



In fact, the probability of getting any exact score must be zero. For continuous variables, it only makes sense to find probabilities over an interval. The function that we use to obtain the probabilities of continuous variables using the area has a special name.

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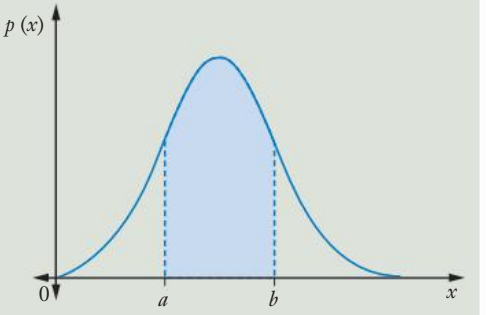
Probability density function (pdf)

For a continuous random variable X, the **probability density function** $p(x)$ is a function whose area over an interval gives the probability of X over that interval, that is,

$$P(a \leq X \leq b) = \int_a^b p(x)\,dx$$

The probability density function must satisfy the following conditions:

- $p(x) \geq 0$ for all $x \in [a, b]$
- $\int_a^b p(x)\,dx = 1$.



It is important to define the interval over which a variable is being considered.

The **closed interval** $a \leq x \leq b$ is written $[a, b]$.

The **open interval** $a < x < b$ is written (a, b).

The interval $a \leq x < b$ is written $[a, b)$, and the interval $a < x \leq b$ is written $(a, b]$.

EXAMPLE 3

Probability density functions

Determine whether each function could be a probability density function over the given interval.

a $f(x) = \frac{1}{5}$ for $0 \leq x \leq 5$

b $f(x) = 3x^2$ for the interval $[0, 1]$

c $f(x) = 3e^{-x}$ for the interval $[0, 2]$

d $f(x) = \frac{1}{2}\sin(x)$ for $0 \leq x \leq \pi$

Solution

a Check that $f(x) \geq 0$.

$\frac{1}{5} \geq 0$ for $0 \leq x \leq 5$

Calculate $\int_0^5 f(x)\,dx$.

$$\int_0^5 \frac{1}{5}\,dx = \left[\frac{x}{5}\right]_0^5$$
$$= \left[\frac{5}{5} - \frac{0}{5}\right]$$
$$= 1$$

Write the conclusion.

Since $f(x) \geq 0$ and $\int_0^5 f(x)\,dx = 1$, $f(x)$ is a pdf for $0 \leq x \leq 5$.

b Check that $f(x) \geq 0$.

$3x^2 \geq 0$ for $0 \leq x \leq 1$

Calculate $\int_0^1 f(x)\,dx$.

$$\int_0^1 3x^2\,dx = \left[x^3\right]_0^1$$
$$= [1^3 - 0^3]$$
$$= 1$$

Write the conclusion.

Since $f(x) \geq 0$ and $\int_0^1 f(x)\,dx = 1$, $f(x)$ is a pdf for $0 \leq x \leq 1$.

c Check that $f(x) \geq 0$.

Since $e^{-x} \geq 0$ for all $x \in \boldsymbol{R}$, $3e^{-x} \geq 0$ for $0 \leq x \leq 2$.

Calculate $\int_0^2 f(x)\,dx$.

$$\int_0^2 3e^{-x}\,dx = \left[-3e^{-x}\right]_0^2$$
$$= [-3e^{-2} + 3e^{-0}]$$
$$= 2.5939\ldots$$

Write the conclusion.

Since $\int_0^2 f(x)\,dx \neq 1$, $f(x)$ is not a pdf for $0 \leq x \leq 2$.

d Check that $f(x) \geq 0$.

$\frac{1}{2}\sin(x) \geq 0$ for $0 \leq x \leq \pi$.

Calculate $\int_0^\pi f(x)\,dx$.

$$\int_0^\pi \frac{1}{2}\sin(x)\,dx = \left[-\frac{1}{2}\cos(x)\right]_0^\pi$$
$$= -\frac{1}{2}\cos(\pi) + \frac{1}{2}\cos(0)$$
$$= 1$$

Write the conclusion.

Since $f(x) \geq 0$ and $\int_0^\pi f(x)\,dx = 1$, $f(x)$ is a pdf for $0 \leq x \leq \pi$.

It is possible to form a probability density function from any function that is continuous for the stated interval.

EXAMPLE 4

Find the value of k that makes $f(x)$ a probability density function for the stated interval and state $f(x)$.

a $f(x) = k(x^2 - 1)$ for $0 \leq x \leq 3$

b $f(x) = ke^{-8x}$ for $x > 0$

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Solution

a $f(x)$ is a pdf for the interval $0 \le x \le 3$, so use the properties of a pdf.

$$\int_0^3 f(x)\,dx = 1$$

Replace $f(x)$.

$$\int_0^3 k(x^2-1)\,dx = 1$$

Use the properties of integrals.

$$k\int_0^3 (x^2-1)\,dx = 1$$

Calculate the definite integral.

$$k\left[\frac{x^3}{3}-x\right]_0^3 = 1$$

$$k\left[\left(\frac{3^3}{3}-3\right)-\left(\frac{0^3}{3}-0\right)\right] = 1$$

$$6k = 1$$

Solve for k.

$$k = \frac{1}{6}$$

State the result.

$$f(x) = \frac{1}{6}(x^2-1) \text{ for } 0 \le x \le 3$$

b Use the fact that $f(x)$ is a pdf for the interval $x > 0$.

$$\int_0^\infty f(x)\,dx = 1$$

Replace $f(x)$.

$$\int_0^\infty ke^{-8x}\,dx = 1$$

Use the properties of integrals.

$$k\int_0^\infty e^{-8x}\,dx = 1$$

Because the terminal value is ∞, you need to find the limit as $n \to \infty$.

$$k \times \lim_{n\to\infty}\left[-\frac{e^{-8x}}{8}\right]_0^n = 1$$

$$k \times \lim_{n\to\infty}\left[-\frac{e^{-8n}}{8}+\frac{1}{8}\right] = 1$$

$$k \times \lim_{n\to\infty}\left[-\frac{1}{8e^{8n}}+\frac{1}{8}\right] = 1$$

$$k \times \left(\lim_{n\to\infty}\left[-\frac{1}{8e^{8n}}\right]+\frac{1}{8}\right) = 1$$

$$k \times \left(0+\frac{1}{8}\right) = 1$$

$$\frac{k}{8} = 1$$

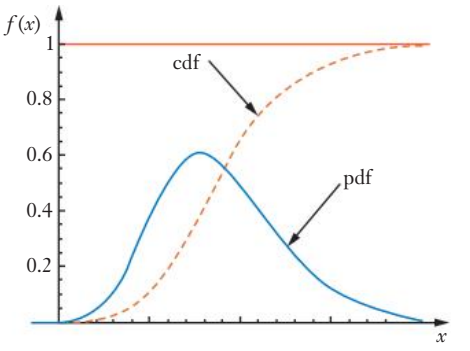
Solve for k.

$$k = 8$$

State the result.

$$f(x) = 8e^{-8x} \text{ for } x > 0$$

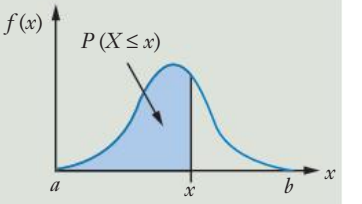
The **cumulative distribution function**, **cdf**, is a function derived from the probability density function for a continuous random variable. It gives the probability of finding the random variable at a value less than a given value.



Given the continuous random variable X, with a probability density function $f(x)$, defined on the interval (a, b), the **cumulative distribution function** (**cdf**) is given by:

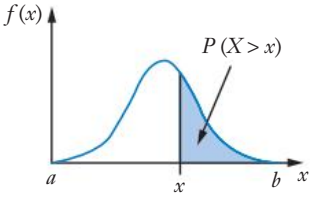
$$F(x) = \int_a^x f(x)\,dx.$$

So $P(X \leq x) = F(x) = \int_a^x f(x)\,dx$ as shown in the diagram.



A number of results follow from this definition.

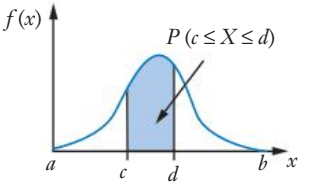
$P(X > x) = 1 - F(x) = 1 - \int_a^x f(x)\,dx$



$P(c \leq X \leq d) = F(d) - F(c)$

$$= \int_a^d f(x)\,dx - \int_a^c f(x)\,dx$$

$$= \int_c^d f(x)\,dx$$



EXAMPLE 5

If the probability density function for a continuous random variable X is $f(x) = \frac{1}{x^2}$, for $x \geq 1$:

a find the cumulative density function for X

b calculate:

i $P(X \leq 2)$ **ii** $P(3 \leq X \leq 6)$ **iii** $P(X > 10)$

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Solution

a Write the cdf.

$$F(x) = \int_1^x f(x)\,dx$$

Substitute for $f(x)$.

$$F(x) = \int_1^x \frac{1}{x^2}\,dx$$

$$= \left[-\frac{1}{x}\right]_1^x$$

$$= \left[-\frac{1}{x} + \frac{1}{1}\right]$$

State the result.

$$F(x) = 1 - \frac{1}{x} \text{ for } x \geq 1.$$

b **i** Write the probability using the cdf.

$$P(X \leq 2) = F(2)$$

$$= \int_1^2 \frac{1}{x^2}\,dx$$

Find the definite integral.

$$= \left[-\frac{1}{x}\right]_1^2$$

$$= -\frac{1}{2} + \frac{1}{1}$$

$$= \frac{1}{2}$$

State the result.

$$P(X \leq 2) = \frac{1}{2}$$

ii Write the probability using the cdf.

$$P(3 \leq X \leq 6) = F(6) - F(3)$$

$$= \int_3^6 \frac{1}{x^2}\,dx$$

Find the definite integral.

$$= \left[-\frac{1}{x}\right]_3^6$$

$$= -\frac{1}{6} + \frac{1}{3}$$

$$= \frac{1}{6}$$

State the result.

$$P(3 \leq X \leq 6) = \frac{1}{6}$$

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iii Write the probability using the cdf. $P(X > 10) = 1 - F(10)$

$$= 1 - \int_1^{10} \frac{1}{x^2}\,dx$$

Find the definite integral.

$$= 1 - \left[-\frac{1}{x}\right]_1^{10}$$

$$= 1 - \left[-\frac{1}{10} + \frac{1}{1}\right]$$

$$= \frac{1}{10}$$

State the result. $P(X > 10) = \frac{1}{10}$

The cumulative distribution function in the previous example, $F(x) = 1 - \frac{1}{x}$ for $x \geq 1$, can also be written more fully as $F(x) = \begin{cases} 0 & \text{for } x < 1 \\ 1 - \frac{1}{x} & \text{for } x \geq 1 \end{cases}$

Exercise 10.02 Probability density functions

Example 3

1 Determine whether each function could be a probability density function over the given interval.

a $f(x) = 4e^{-4x}$ for the interval $[0, \infty]$

b $f(x) = 2x^3$ for the interval $[-1, 1]$

c $f(x) = \frac{1}{4}$ for $-2 \leq x \leq 2$

d $f(x) = 3e^{-8x}$ for the interval $[0, 3]$

e $f(x) = \frac{1}{2}\cos(x)$ for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

f $f(x) = 2(x - 1)$ for the interval $[1, 2]$

g $f(x) = 3x^2$ for the interval $[0, 1]$

h $f(x) = 0.5x$ for the interval $[0, 2]$

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2 Find the value of k that makes $f(x)$ a probability density function for the stated interval and state $f(x)$. Example 4

a $f(x) = k(1 - x^2)$ for $0 \leq x \leq 1$

b $f(x) = \dfrac{k}{x^3}$ for $3 \leq x \leq 6$

c $f(x) = k(x - 5)$ for $5 \leq x \leq 11$

d $f(x) = k(1 + x^2)$ for $-2 \leq x \leq 1$

e $f(x) = kx^3$ for $0 \leq x \leq 1$

f $f(x) = ke^{-\frac{x}{10}}$ for $x \geq 0$

3 The probability density function for a continuous random variable X is $f(x) = 4x^3$ for $0 \leq x \leq 1$. Example 5

a Write the cumulative density function for X.

b Calculate:

i $P\left(X \leq \frac{1}{3}\right)$ **ii** $P\left(\frac{1}{2} \leq X \leq \frac{2}{3}\right)$ **iii** $P\left(X > \frac{1}{2}\right)$

4 If the probability density function for a continuous random variable X is $f(x) = \dfrac{1}{2x^2}$ for $0 < x < 2$, find the cumulative distribution function.

5 If the probability density function for a continuous random variable X is $f(x) = \frac{3}{4}(1 - x^2)$ for $-1 \leq x \leq 1$:

a write the cumulative density function for X

b calculate:

i $P(X \leq 0.5)$ **ii** $P(-0.5 \leq X \leq 0.5)$ **iii** $P(X > 0.5)$

6 A probability density function is given as $f(x) = k(x^3 - 1)$ for $1 \leq x \leq 2$.

a Calculate the value of k.

b Write the cumulative density function for X.

c Calculate:

i $P(X \leq 1.5)$ **ii** $P(1 \leq X \leq 1.8)$ **iii** $P(X > 1.5)$

7 A cumulative distribution function is given by:

$$F(x) = \begin{cases} 0 & \text{for } x < -1 \\ \dfrac{x+1}{2} & \text{for } -1 \leq x < 1 \\ 1 & \text{for } x \geq 1 \end{cases}$$

Calculate:

a $(X \leq \frac{1}{2})$ **b** $(X > \frac{1}{2})$ **c** $(-\frac{1}{2} \leq X \leq \frac{3}{4})$

8 Consider the probability function for a continuous random variable X, which is defined as $f(x) = \frac{1}{6}e^{-\frac{x}{6}}$ for $x \geq 0$.

a Show that this function is a pdf for X.

b Calculate:

i $P(X \leq 2)$ **ii** $P(1 \leq X \leq 5)$

9 Consider the probability function for a continuous random variable X, which is defined as $f(x) = 30x^2(1-x)^2$ for $0 \leq x \leq 1$.

a Show that this function is a pdf for X.

b Calculate:

i $P(X \leq \frac{1}{3})$ **ii** $P(\frac{1}{3} \leq X \leq \frac{1}{2})$

Problem solving

10 For the cumulative distribution function given in question **6**, what is the value of k such that $P(X \leq k) = 0.6$?

11 For the pdf defined as $f(x) = kx$ for $0 \leq x \leq 2$:

a calculate:

i $P(0 \leq X \leq 1.5)$ **ii** $P(\frac{1}{3} \leq X \leq \frac{1}{2})$

b fully describe the cdf.

12 An unreliable clock stops at random at any time during the day.

a If X is the time in hours at which the clock stops, construct a probability density function for X.

b Find the probability that the clock will stop between 1:00 p.m. and 3:45 p.m.

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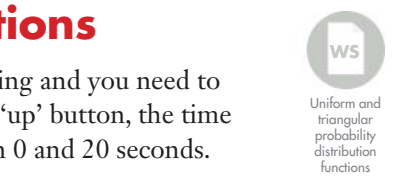
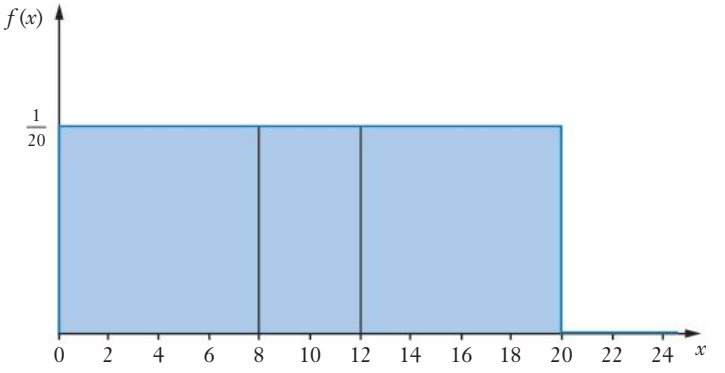
10.03 Continuous random distributions



Consider the situation where you arrive at the ground floor of a building and you need to take an elevator to an upper floor. It is known that once you press the 'up' button, the time taken for the elevator to arrive is equally likely to be any time between 0 and 20 seconds.

If X is the random variable defined as the number of seconds it takes for the elevator to arrive, then the graph of the probability density function $p(x)$ is shown below.



Because the times are equally likely, $P(X = x) = \frac{1}{20}$.

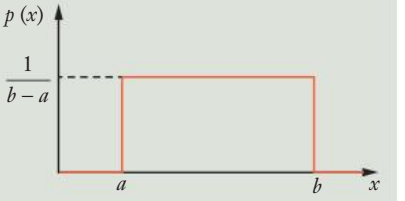
The graph of the probability density function is rectangular in shape. As you would expect, the area under the pdf is $\frac{1}{20} \times 20 = 1$.

$P(8 \leq X \leq 12) = \frac{1}{20} \times (12 - 8) = \frac{1}{5}$.

This is an example of a **continuous uniform distribution** and because of its shape, it is also known as a **rectangular distribution**.

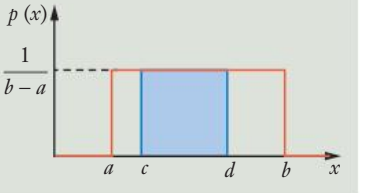
A continuous random probability variable X, which is defined for the domain $[a, b]$, has a **uniform distribution** $p(x)$ if $p(x)$ is constant over the domain of X. The uniform distribution $p(x)$ is defined as:

$$p(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \leq x \leq b \\ 0 & \text{elsewhere} \end{cases}$$



Also:

$$P(c \leq X \leq d) = \frac{d-c}{b-a}$$



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EXAMPLE 6

Davina dropped a $2 coin on the way to the bus stop, 600 m from her home. It is equally likely that she dropped it anywhere on the way. If X is the distance (in metres) from home where the coin was dropped,

a write the probability density function $p(x)$

b find the probability that she dropped it at least 200 m from home.

Solution

a Draw a diagram.

Write the conditions.

'Equally likely' over the distance means that the distribution is uniform.

The domain is [0, 600].

Write the distribution.

$$p(x) = \frac{1}{b-a}$$

Substitute.

$$= \frac{1}{600-0}$$

$$= \frac{1}{600}$$

Write the distribution fully.

$$p(x) = \begin{cases} \frac{1}{600} & \text{for } 0 \leq x \leq 600 \\ 0 & \text{elsewhere} \end{cases}$$

b Write the condition.

'at least 200 m from home' means between 200 m and 600 m from home.

Find the probability.

$$P(200 \leq X \leq 600) = \frac{d-c}{b-a}$$

$$= \frac{600-200}{600}$$

Evaluate.

$$= \frac{2}{3}$$

State the result.

The probability that she dropped the coin at least 200 m from home is $\frac{2}{3}$.

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Another distribution with a graph that has a distinctive shape is the **triangular distribution**.

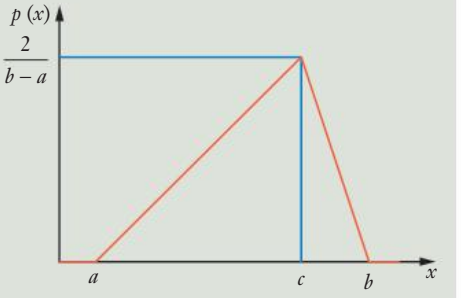
Triangular distributions

A **triangular distribution** for the continuous random probability variable X has the following parameters:

Minimum value = a

Peak value = c

Maximum value = b



The triangular distribution $f(x)$ is defined as:

$$f(x) = \begin{cases} 0 & \text{for } x < a \\ \dfrac{2(x-a)}{(b-a)(c-a)} & \text{for } a \le x \le c \\ \dfrac{2(b-x)}{(b-a)(b-c)} & \text{for } c \le x \le b \\ 0 & \text{for } x > b \end{cases}$$

The triangular distribution, like all probability density functions, has the property that the area under the function is 1. You can use this property to show that the maximum value of the probability distribution function is $\dfrac{2}{b-a}$ and this occurs at the peak value of c.

Referring to the diagram in the definition:

$$\text{Area} = \frac{\text{base} \times \text{height}}{2} = 1$$

$$\text{So } \frac{(b-a) \times \text{height}}{2} = 1$$

$$\text{height} = \frac{2}{b-a}$$

EXAMPLE 7

A continuous random variable X has a triangular distribution for $2 \le x \le 10$ and a peak value that occurs at $x = 7$.

a Draw the graph of the probability density function.

b State the probability density function.

c Calculate:

i $P(X \le 4)$ **ii** $P(X \ge 6)$ **iii** $P(5 \le X \le 8)$

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Solution

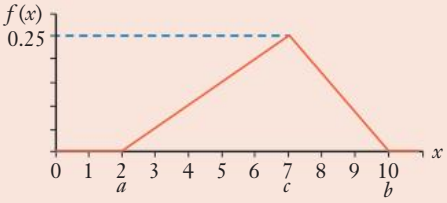
a State the parameters. $a = 2, b = 10$ and $c = 7$

Calculate the maximum value of the function. $f(c) = \dfrac{2}{b-a}$

Substitute. $= \dfrac{2}{10-2}$

Evaluate. $= 0.25$

Draw the graph.



b State the pdf for a triangular distribution.

$$f(x) = \begin{cases} \dfrac{2(x-a)}{(b-a)(c-a)} & \text{for } a \le x \le c \\ \dfrac{2(b-x)}{(b-a)(b-c)} & \text{for } c \le x \le b \end{cases}$$

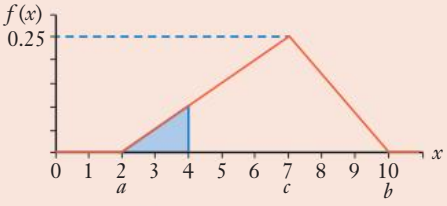
Substitute for known values.

$$= \begin{cases} \dfrac{2(x-2)}{(10-2)(7-2)} & \text{for } 2 \le x \le 7 \\ \dfrac{2(10-x)}{(10-2)(10-7)} & \text{for } 7 \le x \le 10 \end{cases}$$

Evaluate and state the result.

$$f(x) = \begin{cases} \dfrac{1}{20}(x-2) & \text{for } 2 \le x \le 7 \\ \dfrac{1}{12}(10-x) & \text{for } 7 \le x \le 10 \end{cases}$$

c **i** You may find it helpful to sketch a diagram.



Integrate the pdf to find the probability. $P(X \le 4) = \int_2^4 \frac{1}{20}(x-2)\,dx$

Evaluate. $= \dfrac{1}{10}$

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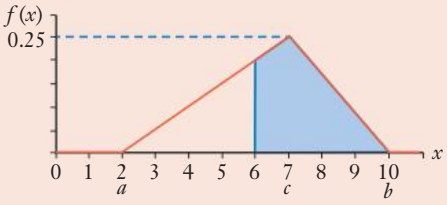
ii Sketch a diagram.



Write the required probability.

$$P(X \geq 6) = 1 - P(X < 6)$$

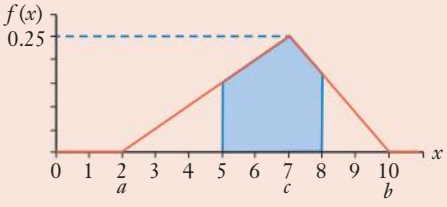
Integrate the pdf to find the probability.

$$P(X \leq 4) = 1 - \int_2^6 \frac{1}{20}(x-2)\,dx$$

Evaluate.

$$= \frac{3}{5}$$

iii Sketch a diagram.



State the required probability.

$$P(5 \leq X \leq 8) = P(5 \leq X \leq 7) + P(7 \leq X \leq 8)$$

Integrate the pdf to find the probability.

$$P(5 \leq X \leq 8) = \int_5^7 \frac{1}{20}(x-2)\,dx + \int_7^8 \frac{1}{12}(10-x)\,dx$$

$$= \frac{2}{5} + \frac{5}{24}$$

State the result.

$$P(5 \leq X \leq 8) = \frac{73}{120}$$

Exercise 10.03 Continuous random distributions

Example 6

1 Find the uniform probability density function for the random variable X defined on each domain.

a [0, 10] **b** [2, 8] **c** [5, 18] **d** [10, 100] **e** [100, 500]

2 The length of time, in seconds, for which a 2-month-old baby smiles is equally likely to be between 0 and 18 seconds. If X is the smiling time:

a write the probability density function $p(x)$

b calculate the probability that a randomly chosen 2-month-old baby smiles between 3 and 15 seconds.

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3 The length of time, in minutes, that a person must wait for a bus is uniformly distributed between 0 and 18 minutes. What is the probability that a person waits fewer than 10.5 minutes for the bus?

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Example 7

4 Find the triangular probability density functions for the random variable X defined by the following parameters:

a minimum value = 0, maximum value = 12, and peak value = 6
b minimum value = 3, maximum value = 15, and peak value = 9
c minimum value = 8, maximum value = 21, and peak value = 12
d minimum value = 5, maximum value = 17, and peak value = 13
e minimum value = 7, maximum value = 21, and peak value = 15

5 A continuous random variable X has a triangular distribution defined for $1 \leq x \leq 8$ with a peak value at $x = 3$.

a Draw the graph of the probability density function.
b State the probability density function.
c Calculate:

i $P(X \leq 2)$ **ii** $P(X \geq 5)$ **iii** $P(4 \leq X \leq 7)$

6 A continuous random variable X has a triangular distribution with a minimum value of 2, maximum value of 14 and a peak value at $x = 5$.

a State the probability density function.
b Calculate:

i $P(X \leq 4)$ **ii** $P(X \geq 7)$ **iii** $P(3 \leq X \leq 12)$

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Problem solving

7 For the situation described in question **2**, find the probability that a randomly selected two-month-old baby smiles more than 12 seconds if you know that this baby always smiles more than 8 seconds.

8 A fast food franchise is planning a new outlet in a popular street mall. Melanie is trying to decide whether or not to take up the franchise. She decides to use a triangular distribution to model future weekly sales. From experience, it is known that weekly sales will fluctuate between $2000 and $8000, with the most likely weekly sales being around $5000. Melanie also knows that weekly sales of at least $4000 need to be made to cover the rent. What is the probability that weekly sales will be sufficient to cover the rental costs?

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9 Dani catches a bus to work each day. The longest time it takes for her to get from the bus stop near her home to the one near her workplace is 38 minutes and the shortest time is 25 minutes. The most common time for this commute is 30 minutes. What is the probability that her commute takes at least 28 minutes?

10 A machine manufactures stainless steel screws that are meant to be 15 mm long. The manufacturing process is such that while the majority of the screws are 15 mm long, their lengths range from 14.8 mm to 15.2 mm. Calculate the probability that a randomly selected screw is:

a less than 14.94 mm long

b more than 15.15 mm long

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10.04 Expected value of a continuous random variable

You have previously seen that the most commonly used measures of central tendency and spread in statistics are the mean, median, variance, standard deviation and range. These same measurements are relevant for discrete and continuous probability functions.

Remember that for a discrete random variable:

Expected value, $E(X) = \mu = \sum_{x=1}^{n} x_n \times P(X = x_n)$

This definition can also be applied to a continuous random variable.

Expected value

For the continuous random variable X defined on the interval $[a, b]$, with probability density function $p(x)$, the **expected value** is defined as:

$$E(X) = \mu = \int_a^b x \cdot p(x)\,dx$$

EXAMPLE 8

Calculate the expected value for a uniform distribution defined on the interval [3, 28].

Solution

Write the pdf.	$p(x) = \frac{1}{b-a}$
Substitute for known values.	$= \frac{1}{28-3}$
Evaluate.	$p(x) = 0.04$
Write the formula for expected value.	$E(X) = \int_a^b x \cdot p(x)\,dx$
Substitute for known values.	$= \int_3^{28} x \times 0.04\,dx$
Evaluate.	$= \left[0.02x^2\right]_3^{28}$
	$= 0.02 \times 28^2 - 0.02 \times 3^2$
	$= 15.5$
State the result.	The expected value is 15.5.

You can see that for the uniform distribution in the previous example, the expected value is the centre of the distribution. This is true for all uniform distributions.

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Expected value for a uniform distribution

For the continuous random variable X with a uniform distribution defined on the interval $[a, b]$, the expected value is given by $E(X) = \mu = \dfrac{a+b}{2}$.

In the case of a symmetrical triangular distribution, the expected value is also the centre of the distribution. This is not true for asymmetrical triangular distributions.

EXAMPLE 9

Calculate the expected value for a triangular distribution defined on the interval [2, 12] whose peak value is at 8.

Solution

State the pdf for a triangular distribution.

$$f(x) = \begin{cases} \dfrac{2(x-a)}{(b-a)(c-a)} & \text{for } a \le x \le c \\ \dfrac{2(b-x)}{(b-a)(b-c)} & \text{for } c \le x \le b \end{cases}$$

Substitute for known values.

$$= \begin{cases} \dfrac{2(x-2)}{(12-2)(8-2)} & \text{for } 2 \le x \le 8 \\ \dfrac{2(12-x)}{(12-2)(12-8)} & \text{for } 8 \le x \le 12 \end{cases}$$

Evaluate and state the result.

$$f(x) = \begin{cases} \dfrac{1}{30}(x-2) & \text{for } 2 \le x \le 8 \\ \dfrac{1}{20}(12-x) & \text{for } 8 \le x \le 12 \end{cases}$$

Write the rule for expected value.

$$E(X) = \int_a^b x \cdot p(x)\,dx$$

$p(x)$ has 2 parts.

$$= \int_2^8 x \times \frac{(x-2)}{30}\,dx + \int_8^{12} x \times \frac{(12-x)}{20}\,dx$$

Evaluate.

$$= \left[\frac{x^2(x-3)}{90}\right]_2^8 + \left[\frac{-x^2(x-18)}{60}\right]_8^{12}$$

$$= 7\frac{1}{3}$$

State the result.

The expected value is $7\frac{1}{3}$.

The same process can be used for other probability density functions.

EXAMPLE 10

Calculate the expected value for the continuous random variable X with the probability density function $p(x) = x^2$ for $0 \leq x \leq 1$.

Solution

Write the rule for expected value.

$$E(X) = \int_a^b x \cdot p(x)\,dx$$

Substitute for known values.

$$= \int_0^1 x \times x^2\,dx$$

Evaluate.

$$= \int_0^1 x^3\,dx$$

$$= \left[\frac{x^4}{4}\right]_0^1$$

$$= \frac{1^4}{4} - 0$$

$$= \frac{1}{4}$$

State the result.

The expected value is $\frac{1}{4}$.

Exercise 10.04 Expected value of a continuous random variable

Example 8

1 A continuous random variable X has a uniform distribution on the interval [5, 22].
- **a** Calculate the pdf for X.
- **b** What is the expected value for the distribution?

2 What is the expected value for a uniform continuous random variable X on the interval [12, 44]?

3 Calculate the expected value for a continuous random variable X with a uniform distribution on the interval [25, 86].

Example 9

4 A continuous random variable X has a triangular distribution on the interval [4, 18] with a peak value at $x = 12$.
- **a** Calculate the pdf for X.
- **b** What is the expected value for the distribution?

5 A continuous random variable X has a triangular distribution on the interval [2, 5] with a peak value at $x = 5$. What is the expected value for the distribution?

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6 The parameters for a number of triangular distributions are listed below, where a = minimum value, b = maximum value and c = peak value.
Calculate the expected value of each distribution.

a $a = 5, b = 12, c = 10$ **b** $a = 10, b = 60, c = 40$

c $a = 18, b = 47, c = 32$ **d** $a = 25, b = 56, c = 33$

7 Calculate the expected value for the continuous random variable X with the probability density function $p(x) = 4e^{-4x}$ for $x \geq 0$.

Example 10

8 Calculate the expected value for the continuous random variable X with the probability density function $p(x) = 2(1 - x)$ for $0 \leq x \leq 1$.

9 Calculate the expected value for the continuous random variable X with the probability density function:

a $p(x) = 8x^{-3}$ for $x \geq 2$ **b** $f(x) = 3x^2$ for $0 \leq x \leq 1$

c $p(x) = \frac{3}{10}(3x - x^2)$ for $0 \leq x \leq 2$

Problem solving

10 25 lettuce seeds are scattered at random in a round seed-raising tray 30 cm in diameter. Construct the probability density function for the distance R of a seedling from the centre of the tray and hence find the expected distance from the centre of the tray.

Hint: The probability density function is uniform with respect to area.

10.05 Variance and standard deviation of a continuous random variable

You have previously seen that the **variance** of the discrete random variable X is given by:

$$Var(X) = \sum p(x) \cdot (x - \mu)^2$$

As with the expected value, the variance of a continuous random variable is calculated using the integral over the interval for which the variable is defined.

Variance

For the continuous random variable X defined on the interval $[a, b]$, with probability density function $p(x)$ and $E(X) = \mu$, the **variance** is defined as:

$$Var(X) = \int_a^b p(x)(x - \mu)^2 \, dx$$

The alternative formula for variance shown below may be more convenient for some calculations.

$$Var(X) = E(X^2) - [E(X)^2]$$

As was the case with discrete random variables, the **standard deviation** is the square root of the variance.

Standard deviation

For the continuous random variable X, the **standard deviation** is defined as:

$$SD(X) = \sigma = \sqrt{Var(X)}$$

EXAMPLE 11

Calculate the variance and standard deviation of a continuous random variable with a uniform distribution over the interval [10, 30].

Solution

Calculate the expected value.

$$E(X) = \frac{a+b}{2}$$
$$= \frac{10+30}{2}$$
$$= 20$$

Write the pdf.

$$p(x) = \frac{1}{b-a}$$

Substitute for known values.

$$= \frac{1}{30-10}$$

Evaluate.

$$p(x) = \frac{1}{20}$$

Write the rule for variance.

$$Var(X) = \int_a^b p(x)(x-\mu)^2\,dx$$

Substitute for known values.

$$= \int_{10}^{30} \frac{1}{20}(x-20)^2\,dx$$
$$= \frac{100}{3}$$

Find the standard deviation.

$$SD(X) = \sqrt{Var(X)}$$

Evaluate.

$$= \sqrt{\frac{100}{3}}$$
$$= 5.773\,502\ldots$$

Round off and state the result.

The variance is about 33.33 and the standard deviation is about 5.77.

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It is convenient to use a calculator to calculate the integral.

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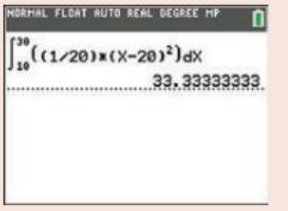
Chapter 10

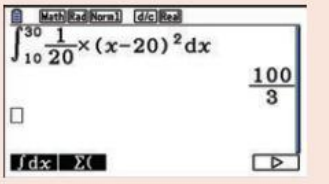
TI-84 Plus CE

Press [math] and then select 9: fnInt((finite integral). Enter the required information and press [enter] to display the result.



Casio fx-CG20AU

From the Run-Matrix screen, press [F4] (Math), [F6] (▷) and then [F1] ($\int dx$). Press [EXE] to display the result.



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Statistics are very important in road safety

The following result can be confirmed using Example 11.

Standard deviation for a uniform distribution

For the uniform continuous random variable X defined on the interval $[a, b]$, the variance and standard deviation are given by:

$$Var(X) = \frac{(b-a)^2}{12} \text{ and } SD(X) = \frac{b-a}{2\sqrt{3}}$$

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EXAMPLE 12

Calculate the variance and standard deviation of a continuous random variable with a symmetrical triangular distribution over the interval [10, 40].

Solution

The triangular distribution is symmetrical.

$$E(X) = \text{peak value}$$
$$= \frac{a+b}{2}$$
$$= \frac{10+40}{2}$$
$$= 25$$

Use the formula for the pdf of a triangular distribution where $a = 10$, $b = 40$ and $c = 25$.

$$f(x) = \begin{cases} \dfrac{2(x-10)}{(40-10)(25-10)} & \text{for } 10 \le x \le 25 \\ \dfrac{2(40-x)}{(40-10)(40-25)} & \text{for } 25 \le x \le 40 \end{cases}$$

Evaluate and state the result.

$$= \begin{cases} \dfrac{1}{225}(x-10) & \text{for } 10 \le x \le 25 \\ \dfrac{1}{225}(40-x) & \text{for } 25 \le x \le 40 \end{cases}$$

Use the formula for variance.

$$Var(X) = \int_a^b p(x)(x-\mu)^2\,dx$$

The function has 2 parts.

$$= \int_{10}^{25} \frac{1}{225}(x-10)(x-25)^2\,dx + \int_{25}^{40} \frac{1}{225}(40-x)(x-25)^2\,dx$$

Use a calculator to evaluate.

$$= 37.5$$

Find the standard deviation.

$$SD(X) = \sqrt{37.5}$$

Evaluate.

$$= 6.123\,724\ldots$$

Round off and state the result.

The variance is 37.5 and the standard deviation is about 6.12.

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Standard deviation for a symmetrical triangular distribution

For the continuous random variable X defined on the interval $[a, b]$ with a symmetrical triangular pdf, the variance and standard deviation are given by:

$$Var(X) = \frac{(b-a)^2}{24} \qquad \text{and} \qquad SD(X) = \frac{b-a}{2\sqrt{6}}$$

Example 12 confirms the results above. A similar approach can be used to find the variance and standard deviation of other continuous random probability variables.

EXAMPLE 13

Calculate the variance of a continuous random variable with the probability density function $f(x) = 2(1 - x)$ for $0 \le x \le 1$.

Solution

Check that $f(x)$ is a valid pdf.

$$2(1-x) = 2 - 2x \ge 0 \text{ for } 0 \le x \le 1$$

$$\int_0^1 2(1-x)\,dx = 2\int_0^1 (1-x)\,dx$$

$$= 2\left[x - \frac{x^2}{2}\right]_0^1$$

$$= 2\left[1 - \frac{1}{2} - 0\right]_0^1$$

$$= 1$$

Write the conclusion.

$f(x) = 2(1 - x)$ for $0 \le x \le 1$ is a valid pdf.

Calculate the expected value.

$$E(X) = \int_a^b x \cdot p(x)\,dx$$

Substitute for known values.

$$= \int_0^1 x[2(1-x)]\,dx$$

Evaluate.

$$= 2\int_0^1 (x - x^2)\,dx$$

$$= 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= 2\left[\frac{1^2}{2} - \frac{1^3}{3} - 0\right]_0^1$$

$$= \frac{1}{3}$$

Use the formula for variance.

$$Var(X) = \int_0^1 2(1-x)\left(x-\frac{1}{3}\right)^2 dx$$

$$= 2\int_0^1 (1-x)\left(x^2-\frac{2x}{3}+\frac{1}{9}\right)dx$$

Simplify.

$$= 2\int_0^1 \left(x^2-\frac{2x}{3}+\frac{1}{9}-x^3+\frac{2x^2}{3}-\frac{x}{9}\right)dx$$

$$= 2\int_0^1 \left(-x^3+\frac{5x^2}{3}-\frac{7x}{9}+\frac{1}{9}\right)dx$$

Evaluate.

$$= 2\left[-\frac{x^4}{4}+\frac{5x^3}{9}-\frac{7x^2}{18}+\frac{x}{9}\right]_0^1$$

$$= 2\left[-\frac{1}{4}+\frac{5}{9}-\frac{7}{18}+\frac{1}{9}-0\right]$$

$$= \frac{1}{18}$$

Using the alternative formula for variance in the previous example results in a less complex calculation.

$$Var(X) = E(X^2) - [E(X)^2]$$

$$= \int_0^1 x^2[2(1-x)]dx - \left(\frac{1}{3}\right)^2$$

$$= 2\int_0^1 (x^2-x^3)\,dx - \frac{1}{9}$$

$$= 2\left[\frac{x^3}{3}-\frac{x^4}{4}\right]_0^1 - \frac{1}{9}$$

$$= 2\left[\frac{1}{3}-\frac{1}{4}-0\right]-\frac{1}{9}$$

$$= \frac{1}{18}$$

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INVESTIGATION

LINEAR TRANSFORMATIONS, EXPECTED VALUES AND VARIANCE

There are times when it is necessary to transform a continuous random variable. A linear transformation could involve one or more operations, including adding or subtracting a constant or multiplying or dividing the variable by a constant.

Consider the continuous random variable X defined on the interval $[-\infty, \infty]$ with a pdf defined as $p(x)$. A linear transformation is applied to X to create a new continuous random variable, Y, such that:

$$Y = aX + b$$

The linear transformation of X will result in changes to the expected value and variance of X to produce the expected value and variance of Y. You can determine the values of $E(Y)$ and $Var(Y)$ using the linear transformation and the values of $E(X)$ and $Var(X)$.

Expected value

- You know that $E(X) = \int_{-\infty}^{\infty} x \cdot p(x)\,dx$
- Use this result to show that:

$$E(aX + b) = aE(X) + b$$

Variance

- You know that $Var(X) = E(X^2) - [E(X)]^2$
- Use this result and the previous result to show that:

$$Var(aX + b) = a^2 Var(X)$$

A continuous random variable, X, has a mean of 10 and a variance of 3.

1 Find:

a $E(2X + 5)$ **b** $Var(2X + 5)$

2 Use the fact that $Var(X) = E(X^2) - [E(X)]^2$ to calculate $E(X^2)$.

3 Calculate

a $E(4X^2)$ **b** $E(X^2 - 25)$

Exercise 10.05 Variance and standard deviation of a continuous random variable

Example 11

1 A continuous random variable X has a uniform distribution over the interval [8, 24].
- **a** Find the probability density function for X.
- **b** Calculate the expected value of X.
- **c** Calculate the variance and standard deviation for the distribution.

2 The intervals over which a number of uniform continuous variables are defined are described below. In each case, find the expected value, variance and standard deviation of the variable.

a [0, 20] **b** [3, 12] **c** [15, 35] **d** [20, 70] **e** [0.5, 2.5]

Example 12

3 A continuous random variable X has a symmetrical triangular distribution over the interval [6, 22].
- **a** Find the probability density function for X.
- **b** Calculate the expected value of X.
- **c** Calculate the variance and standard deviation for the distribution.

4 A continuous random variable X has a symmetrical triangular distribution defined by $p(x)$ over the interval [0.4, 2.8]. Calculate:

a $p(x)$ **b** $E(X)$ **c** $Var(X)$

5 A continuous random variable X has a triangular distribution defined by $p(x)$ over the interval [8, 26] with a peak value at $x = 15$. Calculate:

a $p(x)$ **b** $E(X)$ **c** $Var(X)$

Example 13

6 Calculate the expected value and variance of a continuous random variable with the probability density function $f(x) = \frac{1}{2}x + 2$ for $-4 \leq x \leq -2$.

7 Calculate the expected value and variance of a continuous random variable with the probability density function $f(x) = 2x^{-2}$ for $1 \leq x \leq 2$.

8 Calculate the expected value and variance of a continuous random variable with the probability density function $f(x) = 3x^2$ for $0 \leq x \leq 1$.

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Problem solving

9 Max works in the inner city and uses a parcel pick-up service to transport important documents. The shortest time he has had to wait for a parcel to be collected is 5 minutes and the average time is 15 minutes. Calculate the standard deviation of the waiting time.

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10 A continuous random variable X has a uniform distribution that is defined over the interval $[8, b]$. If $Var(X) = 18\frac{3}{4}$, find $E(X)$.

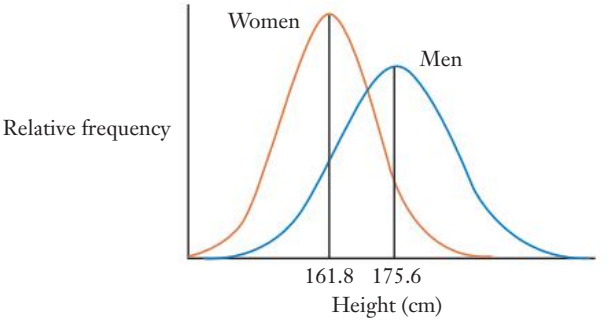
11 A continuous random variable X has a symmetrical triangular distribution with a minimum value of 3. If $Var(X) = 13\frac{1}{2}$, find $E(X)$.

10.06 The normal distribution

So far, you have examined a number of different probability distributions, including the uniform distribution and the triangular distribution.

The **normal distribution** is one of the most widely known and used distributions in statistics. This is because the normal distribution approximates many natural phenomena such as height, weight, length and time.

For example, the average Australian man (18 years and over) is 175.6 cm tall and the average Australian woman was 161.8 cm tall. Of course, there are many males and females who are taller than these averages and many who are shorter. The graphs of the relative frequencies of the heights of these groups are shown below.

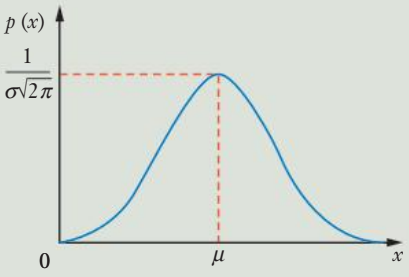


Each of these graphs show the characteristic *bell-shaped curve* that is typical of a normal distribution. The graph of a normal distribution is symmetrical about the mean. The formula for the normal distribution looks complex, but you do not need to memorise it.

The probability density function for the **normal distribution** is given by:

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

where μ is the mean and σ is the standard deviation of the distribution.



Features of the normal distribution include:

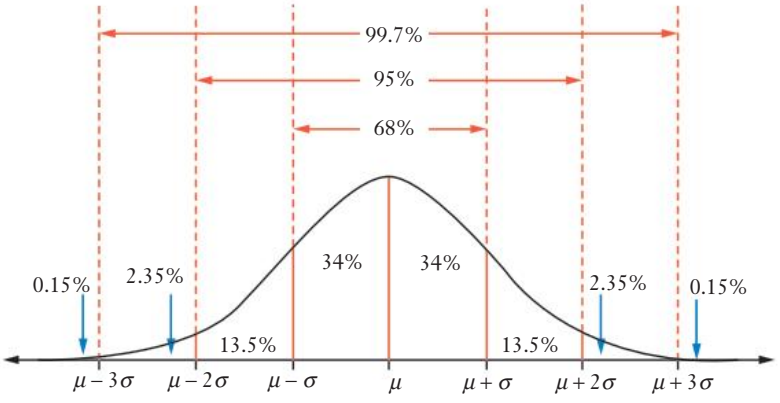
- the distribution is symmetrical about the mean
- the mean, median and mode are equal
- the area under the curve is 1
- the majority of values lie around the mean
- there are only two parameters – μ and σ
- The maximum value occurs at $x = \mu$, so $p(x) = \frac{1}{\sigma\sqrt{2\pi}}$

While the mean and standard deviation for a normal distribution may change, the area under the curve remains constant.

It is sometimes important to find the proportion of a population for a given interval. Using the property that the symmetry of the normal distribution is about the mean, you are able to predict with certainty the following facts.

- About 68% of the population will fall within 1 standard deviation of the mean.
- About 95% of the population will fall within 2 standard deviations of the mean.
- About 99.7% of the population will fall within 3 standard deviations of the mean.

WS
The normal curve



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EXAMPLE 14

a What is the probability density function for a normal distribution with $\mu = 40$ and $\sigma = 7$?

b What is the mean and standard deviation of the probability density function for a normal distribution given by $f(x) = \frac{3}{2\sqrt{2\pi}} e^{-\frac{9}{8}(x-2)^2}$?

Solution

a Write the formula for a normal distribution.

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Substitute for known values.

$$= \frac{1}{7\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-40}{7}\right)^2}$$

Simplify.

$$= 0.056\,991\ldots\, e^{-0.010204\ldots(x-40)^2}$$

Round off.

$$\approx 0.0570\, e^{-0.010204(x-40)^2}$$

b Write the pdf.

$$f(x) = \frac{3}{2\sqrt{2\pi}} e^{-\frac{9}{8}(x-2)^2}$$

Compare with $p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$.

$$\frac{1}{\sigma\sqrt{2\pi}} e = \frac{3}{2\sqrt{2\pi}} e, \text{ so } \sigma = \frac{2}{3}$$

$$e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} = e^{-\frac{9}{8}(x-2)^2}, \text{ so } \mu = 2.$$

State the result.

$\mu = 2$ and $\sigma = \frac{2}{3}$.

As with other probability density functions, when the graph of the normal probability density function is drawn, the area under the curve for a given interval is the same as the probability of values in that interval occurring. The area can be calculated using integration. However, the normal distribution is so commonly used that most scientific calculators have the probability distribution function and cumulative distribution function for normal distributions as built-in functions.

The lower and upper bounds of the normal distribution are $-\infty$ and ∞ respectively. If your calculator does not allow ∞ as an input, you can use $1E99 = 1 \times 10^{99}$ as a sufficiently close approximation for ∞.

EXAMPLE 15

A continuous random variable X has a normal distribution with a mean of 80 and a standard deviation of 20.

a Find the probability that $X < 100$.

b Calculate $P(30 < X < 115)$.

Solution

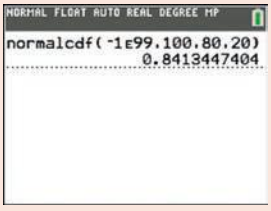
TI-Nspire CX Chapter 10

a **TI-84 Plus CE**

Press 2nd vars (dist) and then select 2: normalcdf(.

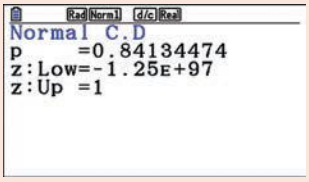
Enter (-) 1 2nd EE 99 for the lower bound and then enter the other required information.

Press enter to display the result.



Casio fx-CG20AU

From the Statistics screen, press F5 (Dist), F1 (Norm), F2 (Ncd). If necessary, select Variable for the data source. For the lower bound, press (-) 1, then select E from the Catalogue (SHIFT 4), then 99. Enter the other required information and press EXE to display the result.



State the result.

$P(X < 100) \approx 0.8413$

b **TI-84 Plus CE**

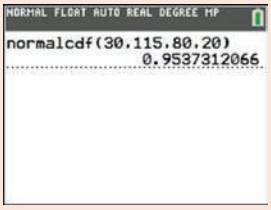
Repeat the steps previously described with:

Lower bound = 30

Upper bound = 115

$\mu = 80$

$\sigma = 20$



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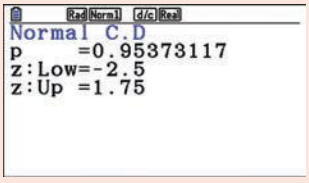
Repeat the steps previously described with:

Lower bound = 30

Upper bound = 115

$\mu = 80$

$\sigma = 20$



State the result.

$P(30 < X < 115) \approx 0.9537$

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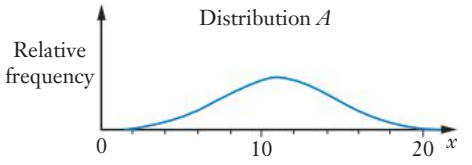
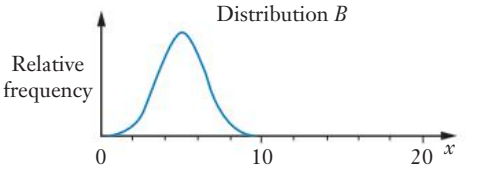
Exercise 10.06 The normal distribution

1 The diagram below shows two normal distribution curves.





a Which has the larger mean?
b Which has the smaller standard deviation?

2 What is the probability density function for each of the normal distributions?

Example 14

a $\mu = 3$ and $\sigma = \frac{3}{4}$
b $\mu = 20$ and $\sigma = 5$
c $\mu = 34.5$ and $\sigma = 7.6$
d $\mu = 128$ and $\sigma = 32$
e $\mu = 1.7$ and $\sigma = 0.4$
f $\mu = 0.8$ and $\sigma = 0.15$

3 What is the mean and standard deviation of the probability density function for each of the normal distributions given below?

a $f(x) = \frac{1}{3\sqrt{2\pi}} e^{\frac{-(x-12)^2}{18}}$
b $p(x) = \frac{2}{\sqrt{2\pi}} e^{-2(x-2)^2}$
c $p(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-(x-2)^2}{2}}$
d $0.06649e^{\frac{-(x-20)^2}{72}}$
e $0.08673e^{-0.02363(x-18.5)^2}$
f $1.1398e^{-4.08163(x-1.2)^2}$

4 A continuous random variable X has a normal distribution with a mean of 115 and a standard deviation of 28. Calculate:

Example 15

a the probability that $x \leq 80$
b the probability that $x \leq 130$
c $P(100 \leq X \leq 130)$
d $P(X \geq 146)$

5 A continuous random variable X has a normal distribution with a mean of 26.8 and a standard deviation of 4.7. Calculate:

a $P(X < 20)$
b $P(X > 22)$
c $P(X > 40)$
d $P(22 < X < 32)$
e $P(19 < X < 42)$

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6 Scores on a particular IQ test are known to be normally distributed with a mean of 115 and a standard deviation of 18.

a Determine:

i $\mu \pm \sigma$ **ii** $\mu \pm 2\sigma$ **iii** $\mu \pm 3\sigma$

b Find:

i $P(\mu - \sigma < X < \mu - \sigma)$ **ii** $P(\mu - 2\sigma < X < \mu - 2\sigma)$

iii $P(\mu - 3\sigma < X < \mu - 3\sigma)$

c Find:

i $P(X < 70)$ **ii** $P(X > 160)$

7 For each random normal variable X with the specified mean and standard deviation given below, calculate the probability indicated.

a $\mu = 38$ and $\sigma = 8$, $P(22 < X < 45)$

b $\mu = 1058$ and $\sigma = 289$, $P(876 < X < 1243)$

c $\mu = 8.46$ and $\sigma = 2.07$, $P(5.32 < X < 7.98)$

d $\mu = 0.64$ and $\sigma = 0.14$, $P(0.45 < X < 0.79)$

e $\mu = 287.65$ and $\sigma = 68.73$, $P(146.9 < X < 188.6)$

8 What is the approximate probability that a normally distributed value lies between:

a 2σ below the mean and 1σ above the mean?

b the mean and the value 2σ above the mean?

c the mean and the value 1σ below the mean?

d the mean and the value 3σ below the mean?

Problem solving

9 A continuous random variable, X, is known to be normally distributed with a mean of 85 and a standard deviation of 12. Find the range between which approximately:

a 68% of the values lie **b** 95% of the values lie

c 99.7% of the values lie

10 The heights of a plantation of seedlings are normally distributed. 84% of the seedlings are less than 148 cm high and 16% are less than 106 cm high.

a Find the mean and standard deviation of the seedlings.

b What proportion of the seedlings are between 90 cm and 130 cm high?

ISBN 9780170412926

10.07 The standard normal distribution

WS The standard normal curve
WS Areas under the normal curve
WS z-scores

It is often useful to compare sets of data. Consider the situation where 2 candidates for a technical position are given similar aptitude tests. Candidate A achieves a score of 47 on a test that is known to be normally distributed with a mean of 40 and a standard deviation of 5. Candidate B achieves a score of 44 on a test that is known to be normally distributed with a mean of 38 and a standard deviation of 7.

It is difficult to determine which candidate was more successful because the tests have different means and standard deviations.

This problem can be solved by transforming the normal distribution for the variable X into a standardised form of the normal distribution called the **standard normal distribution**. To emphasise the fact that the standard normal distribution is different from the normal distribution, the variable in a standard normal distribution is always denoted by Z. A particular value of Z is denoted by z.

The standard normal distribution has a mean of 0 and a standard deviation of 1, so that the value of Z (the Z-value) indicates how many standard deviations the corresponding X-value is from the mean.

A particular value of the standard normal variable, z, is calculated by finding the difference between the corresponding x-value and the mean and dividing this result by the standard deviation. It is known as the ***z*-score** and is also referred to as the ***z*-value** or **standard score**.

z-score

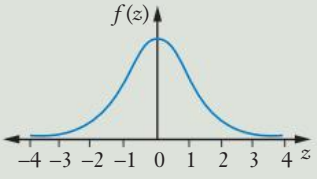
$$z = \frac{x - \mu}{\sigma}$$

The formula for the probability density function for the standard normal distribution can be determined using the previously stated formula for the probability density function for the normal distribution. Because the standard normal distribution is an example of a normal distribution, it shares the properties of a normal distribution.

The standard normal distribution

If $z = \dfrac{x - \mu}{\sigma}$, $\mu = 0$ and $\sigma = 1$, the probability density function for the standard normal distribution is given by:

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$$



- The distribution is symmetrical about the mean
- The mean, median and mode all equal 0
- The area under the curve is 1
- The majority of values lie around the mean

z-scores

EXAMPLE 16

A machine fills bottles with vinegar. The volume of vinegar filled by the machine is normally distributed with a mean of 605 mL and a standard deviation of 12 mL. The intended volume of vinegar in filled bottles is 605 mL. Two bottles are randomly selected and found to have volumes of 593 mL and 619 mL respectively.

a Sketch the graph of the distribution, showing volumes and z-scores.

b What proportion of filled bottles would contain more vinegar than the first selected bottle?

c Which of the selected bottles is more accurately filled?

Solution

a Draw the standard normal distribution curve.

Next, calculate the volumes that correspond to the z-scores.

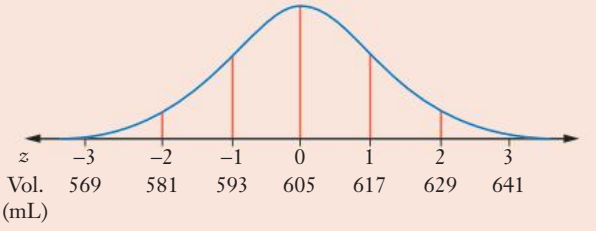
$z = 0 \leftrightarrow \mu = 605$

$z = 1 \leftrightarrow \mu + \sigma = 605 + 12 = 617$

$z = -1 \leftrightarrow \mu - \sigma = 605 - 12 = 593$

$z = 2 \leftrightarrow \mu + 2\sigma = 605 + 24 = 629$

etc.



b Calculate the z-score.

$$z = \frac{x - \mu}{\sigma}$$
$$= \frac{593 - 605}{12}$$
$$= -1$$

Refer to the approximate area under the standard normal distribution.

$$P(X > 593) = 1 - P(X \leq 593)$$
$$\approx 1 - 0.16$$
$$= 0.84$$

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c State the z-score for the first bottle.

$z_1 = -1$

Calculate the z-score for the second bottle.

$$z_2 = \frac{x - \mu}{\sigma}$$
$$= \frac{619 - 605}{12}$$
$$\approx 1.1667$$

Compare the z-scores.

$|z_1| < |z_2|$

Write the conclusion.

The first bottle is more accurately filled because its volume is closer to the intended volume (605 mL).

EXAMPLE 17

It is known that the scores on a test are normally distributed with $\mu = 58$ and $\sigma = 12$.

a What is the z-score for a mark of 40?

b What is the actual score for a z-score of 1.8?

Solution

a Use the rule for z-scores.

$$z = \frac{x - \mu}{\sigma}$$

Substitute for known values.

$$= \frac{40 - 58}{12}$$

Evaluate.

$= -1.5$

State the result.

A score of 40 is the same as a z-score of -1.5.

b Use the rule for z-scores.

$$z = \frac{x - \mu}{\sigma}$$

Substitute for known values.

$$1.8 = \frac{x - 58}{12}$$

Evaluate.

$1.8 \times 12 = x - 58$

Solve for x.

$x = 58 + 21.6$

$= 79.6$

State the result.

A z-score of 1.5 is the same as a score of 79.6.

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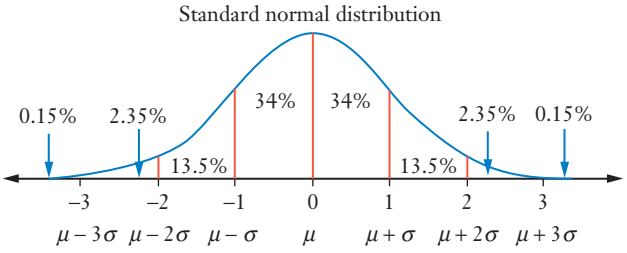
The distribution of the area under the standard normal distribution is the same as for the normal distribution:

- About 68% of the population falls within 1 standard deviation of the mean.
- About 95% of the population falls within 2 standard deviations of the mean.
- About 99.7% of the population falls within 3 standard deviations of the mean.



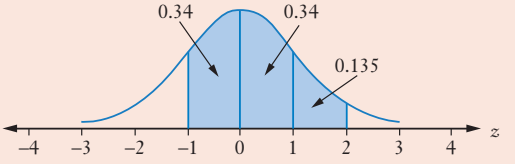
EXAMPLE 18

For the standard normal distribution, find:

a $P(-1 < Z < 2)$ **b** $P(1.2 < Z < 2.7)$

Solution

a Draw a diagram showing the required area (probability).



Refer to the areas under a standard normal distribution.

$P(-1 < Z < 2) = 0.34 + 0.34 + 0.135$

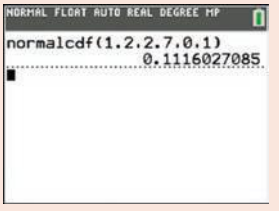
$= 0.815$

b **TI-84 Plus CE**

Press 2nd vars (dist) and then select 2: normalcdf(. Enter the values:

Lower bound = 1.2, Upper bound = 2.7

$\mu = 0$, $\sigma = 1$



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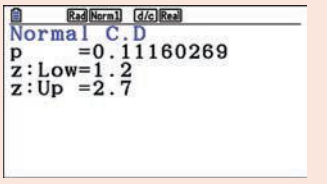
From the Statistics screen press F5 (Dist), F1 (Norm), F2 (Ncd). If necessary, select Variable for the data source. Enter the values:



Lower bound = 1.2, Upper bound = 2.7

$\mu = 0$, $\sigma = 1$

State the result. $P(1.2 < Z < 2.7) \approx 0.1116$

Exercise 10.07 The standard normal distribution

Example 16

1 A Little Athletics club keeps records of all its athletics competitions. It was found that the time for girls to run 400 m is normally distributed with a mean of 95 seconds and a standard deviation of 6 seconds. Mandy and her friend Zoe ran against each other in a recent 400 m race and their scores were 89 s and 107 s respectively.

- **a** Sketch the graph of the distribution showing times and z-scores.
- **b** Mark in Mandy's and Zoe's times on separate graphs.
- **c** Calculate the z-scores for Mandy and Zoe and mark these on the graphs.
- **d** Shade the area under the respective graphs to represent performances that were better than those of Mandy.
- **e** Shade the area under the respective graphs to represent performances that were worse than those of Zoe.
- **f** Of all the Little Athletics runners who participated in this event, approximately what proportion would have performed better than Mandy?
- **g** If 100 Little Athletics runners had participated in this event, about how many would have performed better than Zoe?

2 A worker at a farm packs trays of strawberries. The weights of strawberries in the trays are normally distributed with a mean of 508 g and a standard deviation of 12 g.

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The farm manager rejects any trays of strawberries weighing less than 496 grams or more than 532 g.

- **a** Sketch a graph of the distribution showing the weights of trays of strawberries and the corresponding z-scores.
- **b** Shade the area under the graph that represents the weights of trays that are rejected.
- **c** If 1000 trays of strawberries are packed in the day, about how many will be rejected?

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Example 17

3 A set of scores are normally distributed with $\mu = 62$ and $\sigma = 8$. Calculate the z-score for each of the following scores.

a 78 **b** 54 **c** 74
d 38 **e** 88 **f** 50

4 A set of test results are found to be normally distributed with $\mu = 48$ and $\sigma = 10$. Calculate the test result for each of the following z-scores.

a 1 **b** –2.4 **c** 1.8
d –1.5 **e** 3.2 **f** –2.8

5 A random normal variable X has a mean of 10.6 and a standard deviation of 3.4. Find the z-score for an X value of:

a 8 **b** 9.6 **c** 11.2 **d** 12.7 **e** 6.3
f 14 **g** 7 **h** 11.1 **i** 10.2 **j** 10.63

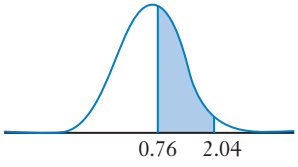
6 A random normal variable X has a mean of 2307 and a standard deviation of 240. Find the z-score for an X value of:

a 2800 **b** 2100 **c** 2120 **d** 2409 **e** 2343
f 1874 **g** 2035 **h** 2714 **i** 3010 **j** 1625

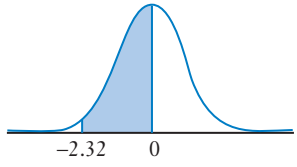
Example 18

7 Find the probabilities represented by the shading on the following diagrams of the standard normal distribution.

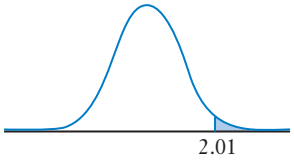
a



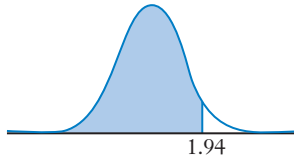
b



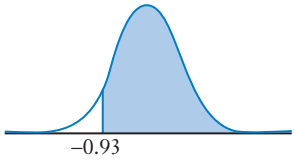
c



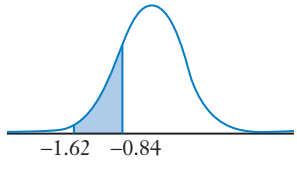
d



e



f



8 For the standard normal variable Z, find approximate values for:

a $P(0 < Z < 1)$ **b** $P(-2 < Z < 0)$ **c** $P(-1 < Z < 1)$
d $P(-2 < Z < 1)$ **e** $P(-2 < Z < 2)$

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9 For the standard normal variable Z, find:

a $P(0 < Z < 0.88)$ **b** $P(0 < Z < 1.76)$ **c** $P(0 < Z < 0.863)$

d $P(0 < Z < 2.038)$ **e** $P(0 < Z < 1.275)$

10 Z is the standard normal variable. Find the following probabilities.

a $P(Z < -1.07)$ **b** $P(Z > 2.15)$ **c** $P(Z > 1.96)$

d $P(-2.07 < Z < 1.63)$ **e** $P(Z < -1.44 \text{ or } Z > 0.32)$

11 Find the following for the standard normal variable Z.

a $P(Z < -1.305)$ **b** $P(Z > 0.623)$ **c** $P(Z > 0.596)$

d $P(-1.307 < Z < 2.6)$ **e** $P(Z < -1.646 \text{ or } Z > 0.831)$

Problem solving

12 Callum scored 18 out of 25 for English and 15 out of 20 for Maths. The means were 15 and 13 and the standard deviations were 8 and 5 for English and Maths respectively. Use z-scores to find which subject he performed better in.

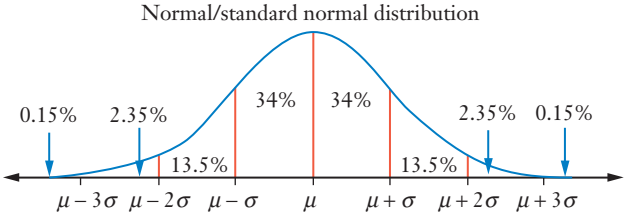
13 Deirdre, who is 17, has an IQ of 110 and a height of 175 cm. The average IQ is 100 and the average height of 17-year-old females is 171 cm, with a standard deviation of 12 cm. The standard deviation for IQ is 15. Which is more unusual: Deidre's height as a 17-year-old, or her IQ?

10.08 Probabilities and quantiles

TECHNOLOGY

Inverse normal distribution

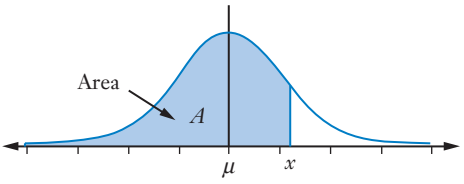
You have seen that you can work out the approximate X-value or Z-value that corresponds to particular known areas under the normal and standard normal distributions respectively.



For example, for the standard normal distribution, the area between the mean and $z = 1$ is about 0.34, the area between $z = 1$ and $z = 3$ is about 0.16 and so on.

However, you can use your calculator to determine the X-value or Z-value that corresponds to a particular area under the distribution curve.

The **inverse cumulative distribution function** calculates the value that corresponds to a given area under the normal or standard normal distribution curve as shown here.



So $P(X < x) = A$.

For example, the inverse cumulative normal distribution function can be used to calculate the z-score that corresponds to an area of 0.64 as follows.

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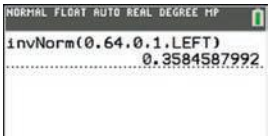
TI-84 Plus CE

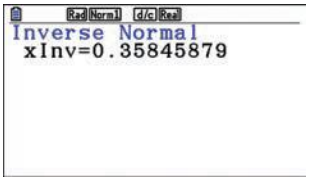


Press 2nd vars (dist) and then select 3: invNorm(. Enter the values:

Area = 0.64, $\mu = 0$, $\sigma = 1$

Set the Tail to Left.

Casio fx-CG20AU



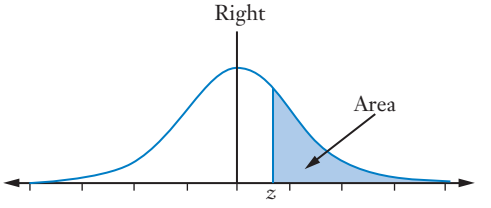
From the Statistics screen, press F5 (Dist), F1 (Norm), F3 (InvN). If necessary, select Variable for the data source. Set the Tail to Left.

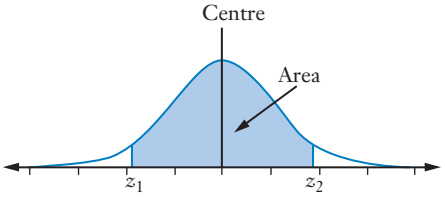
Enter the values:

Area = 0.64, $\mu = 0$, $\sigma = 1$

So $P(Z < 0.3585) \approx 0.64$

1 You can see that the 'Tail' can be set to 'left', 'centre' or 'right'. The area for the 'right' and 'centre' settings are shown below.





Try each of these settings to see how they work.

2 Use your calculator to calculate the value of a in each of the following.

a $P(Z < a) = 0.35$ **b** $P(Z > a) = 0.67$ **c** $P(-a < Z < a) = 0.84$

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EXAMPLE 19

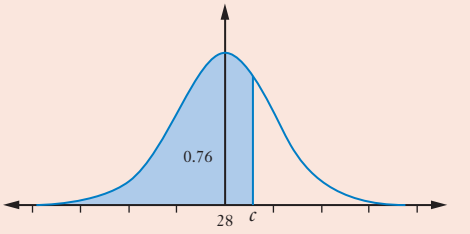
If X is a normally distributed random variable, use a calculator to find:

a c given that $P(X < c) = 0.76$, where $\mu = 28$ and $\sigma = 6$

b w given that $P(X > w) = 0.47$, where $\mu = 42.7$ and $\sigma = 11.8$

Solution

a Draw a sketch to help visualise the problem. Show the required area and the approximate position of c.



TI-84 Plus CE

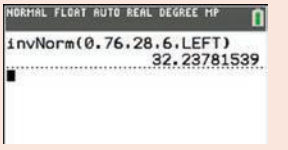
Press 2nd vars (dist) and then select 3: invNorm(. Enter the values:

Area = 0.76

$\mu = 28$

$\sigma = 6$

Set the Tail to Left.



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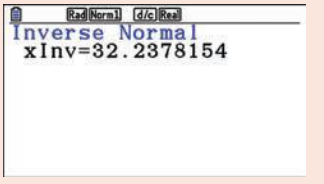
From the Statistics screen, press F5 (Dist), F1 (Norm), F3 (InvN). If necessary, select Variable for the data source. Set the Tail to Left.

Enter the values:

Area = 0.76

$\mu = 28$

$\sigma = 6$

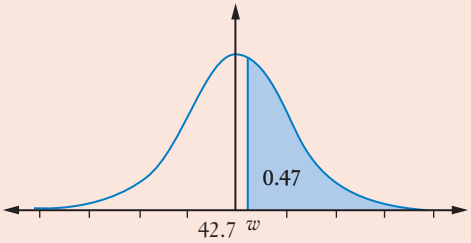


Both calculators

State the result.

If $P(X < c) = 0.76$ for $\mu = 28$ and $\sigma = 6$, $c \approx 32.24$.

b Sketch a diagram to help visualise the problem. Show the required area and the approximate position of w.



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TI-84 Plus CE

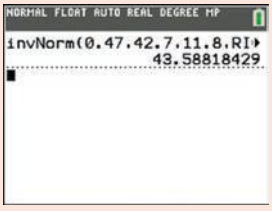
Press 2nd vars (dist) and then select 3: invNorm (. Enter the values:

Area = 0.47

$\mu = 42.7$

$\sigma = 11.8$

Set the Tail to Right.



Casio fx-CG20AU

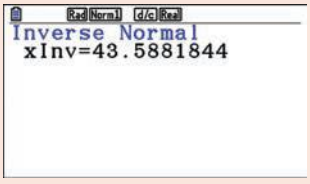
From the Statistics screen, press F5 (Dist), F1 (Norm), F3 (InvN). If necessary, select Variable for the data source. Set the Tail to Right.

Enter the values:

Area = 0.47

$\mu = 42.7$

$\sigma = 11.8$



Both calculators

State the result.

If $P(X > w) = 0.47$, where $\mu = 42.7$ and $\sigma = 11.8$, $w \approx 43.59$.

Statistical measures called **quantiles** (or **fractiles**) divide a probability distribution into equal groups, each containing the same fraction of the total population. A quantile is the value of a statistical variable below which a particular fraction of the scores lie.

Quartiles divide a set of values into quarters:

- 25% of all the values lie below the first quartile (Q_1 = lower quartile).
- 50% of all the values lie below the second quartile (Q_2 = middle quartile).
- 75% of all the values lie below the third quartile (Q_3 = upper quartile).

The second quartile is also called the **median**.

Similarly, **deciles** divide values into tenths and **percentiles** into hundredths.

40% of all values lie below D_4 and 73% lie below P_{73}.

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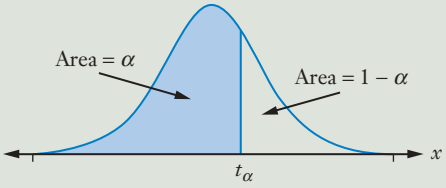
Quantiles

For a continuous random variable X, a quantile t_α is the value of x such that $P(X < t_\alpha) = t_\alpha$, where $0 < \alpha < 1$.

t_α is the value of x that divides the distribution in the ratio $\alpha : 1 - \alpha$.



For example, the median (middle quantile, 50th percentile) is the value of x such that $\alpha = 0.5$, i.e. $P(X < x) = 0.5$.

EXAMPLE 20

X is a normally distributed random variable with a mean of 76 and a standard deviation of 14. Calculate:

a the 38th percentile (P_{38})

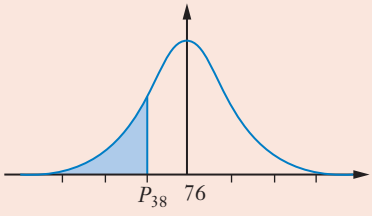
b the median

c a, given that $P(X > a) = 0.235$

Solution

a Sketch a diagram to help visualise the problem.

$P_{38} = t_{0.38}$



TI-84 Plus CE

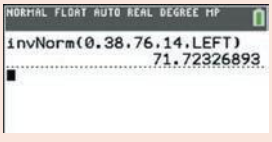
Press 2nd vars (dist) and then select 3: invNorm(. Enter the values:

Area = 0.38

$\mu = 76$

$\sigma = 14$

Set the Tail to Left.



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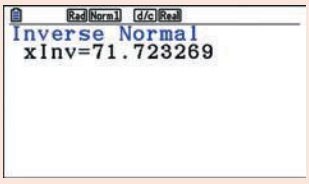
From the Statistics screen, press F5 (Dist), F1 (Norm), F3 (InvN). If necessary, select Variable for the data source. Set the Tail to Left.

Enter the values:

Area = 0.38

$\mu = 76$

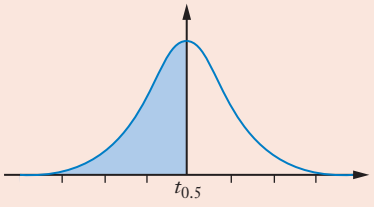
$\sigma = 14$



State the result.

$P_{38} \approx 71.72$

b Sketch a diagram to help visualise the problem.



For the normal distribution, the median and the mean are the same.

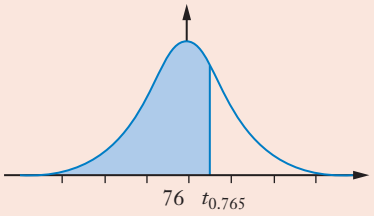
Median = 76

c Sketch a diagram to help visualise the problem.

$P(X > a) = 0.235$ is the same as

$P(X < a) = 1 - 0.235 = 0.765$.

You want to find $t_{0.765}$.



TI-Nspire CX Chapter 10

TI-84 Plus CE

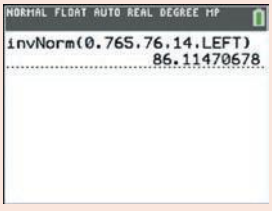
Press 2nd vars (dist) and then select 3: invNorm(. Enter the values:

Area = 0.765

$\mu = 76$

$\sigma = 14$

Set the Tail to Left.



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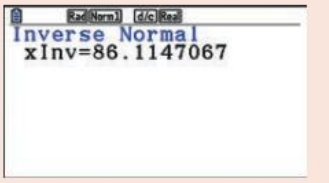
Casio fx-CG20AU

From the Statistics screen, press F5 (Dist), F1 (Norm), F3 (InvN). If necessary, select Variable for the data source. Set the Tail to Left.



Enter the values:

Area = 0.765

$\mu = 78$

$\sigma = 14$

State the result. For $P(X > a) = 0.235$, $a \approx 86.11$

In Example 20, the calculation could have been completed with an area = 0.235 and the tail set to 'right'.

For a normal distribution, it is possible to calculate the values of the mean or standard deviation using a known quantile and the standard normal distribution. This is done by first calculating the z-score and then using the formula $z = \frac{x - \mu}{\sigma}$ to calculate the unknown mean or standard deviation.

EXAMPLE 21

a The continuous random variable X is normally distributed with a mean of 8.6. If $t_{0.36} = 7.7$, find the standard deviation of the distribution.

b Variable X is normally distributed so that the 84th percentile is 45.16 and the standard deviation is 6.8. Find the mean of the distribution.

Solution

a

Write the probability statement. $P(X < t_\alpha) = t_\alpha$

Substitute for known values. $P(X < 7.7) = 0.36$

Find the corresponding z-score. $P(Z < z) = 0.36$

Use a calculator as previously described. $z \approx -0.3585$

Write the formula for z. $z = \frac{x - \mu}{\sigma}$

Substitute for known values. $-0.3585 = \frac{7.7 - 8.6}{\sigma}$

Evaluate. $\sigma = \frac{-0.9}{-0.3585}$

$= 2.510460\ldots$

State the result. The standard deviation is about 2.51.

b	Write the probability statement.	$P(X < 45.16) = 0.84$
	Find the corresponding z-score.	$P(Z < z) = 0.84$
	Use a calculator as previously described.	$z \approx 0.9945$
	Write the formula for z.	$z = \dfrac{x - \mu}{\sigma}$
	Substitute for known values.	$0.9945 = \dfrac{45.16 - \mu}{6.8}$
	Evaluate.	$\mu = 45.16 - (0.9945 \times 6.8)$ $= 38.3974$
	State the result.	The mean is about 38.40.

Exercise 10.08 Probabilities and quantiles

Example 19

1 Let X be a normal random variable with $\mu = 10$ and $\sigma = 3$. Find:

a c such that $P(X < c) = 0.95$ **b** k such that $P(X < k) = 0.28$

2 Let X be a normal random variable with $\mu = 45$ and $\sigma = 11$. Find:

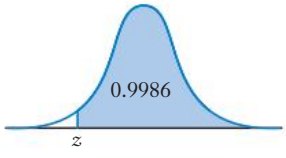
a c such that $P(X < c) = 0.42$ **b** k such that $P(X < k) = 0.876$

3 Let X be a normal random variable with $\mu = 105$ and $\sigma = 14$. Find:

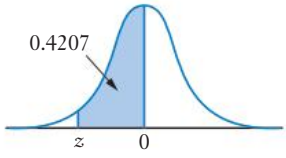
a c such that $P(X > c) = 0.35$ **b** k such that $P(X > k) = 0.86$

4 Find the standard normal score, Z, for each diagram.

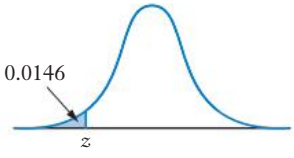
a



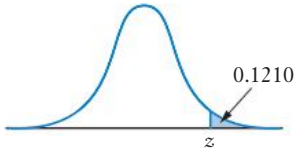
b



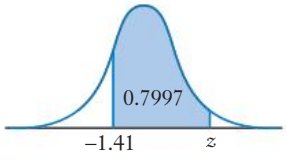
c



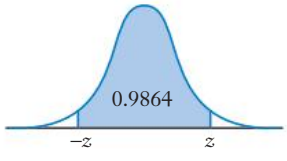
d



e



f



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5 Z has a standard normal distribution. Find the value of a if:

a $P(0 < Z < a) = 0.3710$ **b** $P(0 < Z < a) = 0.4933$

c $P(Z < a) = 0.7019$ **d** $P(Z < a) = 0.6776$

e $P(Z > a) = 0.821$ **f** $P(Z > a) = 0.4308$

6 Use your graphics calculator to find each value for the standard normal distribution.

a $P(-1.356 < Z < 1.873)$ **b** $P(Z < 0.823)$

c $P(Z > -2.306)$ **d** $P(Z < 1.6)$

e $P(Z < -0.628 \text{ or } Z > 1.636)$

7 For the set of standard normal scores, Z, find the value of a if:

a $P(Z < a) = 0.4093$ **b** $P(Z < a) = 0.9056$

c $P(Z > a) = 0.503$ **d** $P(Z > a) = 0.2657$

e $P(-a < Z < a) = 0.866$ **f** $P(-a < Z < a) = 0.2106$

8 For a standard normal distribution calculate:

Example 20

a the 45th percentile **b** P_{88}

c the 0.53 quantile **d** the third quartile (Q_3)

e the 5th decile **f** D_2

9 X is a normally distributed random variable with a mean of 58 and a standard deviation of 12. Calculate:

a P_{37} **b** P_{72} **c** the 0.65 quantile

d Q_1 **e** $t_{0.86}$ **f** D_7

10 X is a normally distributed random variable with a mean of 115 and a standard deviation of 18. Calculate:

a $t_{0.32}$ **b** $t_{0.96}$ **c** $t_{0.14}$

d $t_{0.77}$ **e** $t_{0.82}$ **e** $t_{0.44}$

11 The continuous random variable X is normally distributed with a mean of 100. If $t_{0.75} = 110.12$, find the standard deviation of the distribution.

Example 21

12 The variable X is normally distributed so that the 38th percentile is 30.47 and the standard deviation is 5. Find the mean of the distribution.

13 X is distributed normally with a standard deviation of 4.5, and $t_{0.68} = 38.60$. Find the mean for X, correct to 1 decimal place.

14 X is a normally distributed variable with a mean of 18.7 and the 83rd percentile is 21.37. Find the value of σ, giving your answer correct to 1 decimal place.

15 X is distributed normally with $\sigma = 22.8$ and $Q_1 = 111.02$. Find μ, giving your answer correct to 1 decimal place.

Problem solving

16 X is distributed normally such that $t_{0.47} = 37.649$ and $t_{0.72} = 45.086$. Find the mean and the standard deviation of X, giving your answers correct to 1 decimal place.

17 X is distributed normally such that $P(X < 63.589) = 0.324$ and $P(X > 79.848) = 0.258$. Find the mean and the standard deviation of X, giving your answers correct to 1 decimal place.

WS Normal distribution worded problems 1

WS Normal distribution worded problems 2

WS Continuous random variables assignment

WS Continuous random variables extension

10.09 Applications of the normal distribution

The normal distribution can be used to model many different situations, especially those involving large populations. Situations where most measurements are concentrated around the mean value and diminish as the distance from the mean increases are closest to normal distributions. Examples of situations that are approximately normally distributed are extensive and include the heights and weights of people, the length of time a light bulb lasts, cholesterol levels in the bloodstream of adults and the weight of sugar in bags filled by a machine.

EXAMPLE 22

A survey indicates that Australians use their mobile phones an average of 22 months before upgrading to a new one. The standard deviation is 4.5 months.

a Approximately what percentage of mobile phone users in Australia would you expect to keep their mobile phones for less than 2 years before upgrading?

b What is the probability that a randomly selected mobile phone user keeps their phone for between 15 months and $2\frac{1}{2}$ years before upgrading?

Solution

a

You can assume that this distribution is approximately normally distributed. State the parameters.	$\mu = 22$ $\sigma = 4.5$
Write the required proportion as a probability statement.	$P(-\infty < X \leq 24)$
Use your calculator cdf to calculate the required probability.	$P(-\infty < X \leq 24) = 0.671\,639...$
Round off and state the result.	About 67.2% of mobile phone users will keep their phones for less than 2 years before upgrading.

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b Write the required proportion as a probability statement. $P(15 < X \leq 30)$

Use your calculator cdf to calculate the required probability. $P(15 < X \leq 30) = 0.902\ 372\ldots$

Round off and state the result. About 90.2% of mobile phone users will keep their phones for between 15 months and $2\frac{1}{2}$ years before upgrading.

If you say that someone is 178 cm tall, it is assumed that the measurement has been given to the nearest centimetre. This means that the person's actual height is somewhere between 177.5 cm and 178.5 cm.

EXAMPLE 23

The chest measurements of 18-year-old male footballers are normally distributed with a mean of 96 cm and a standard deviation of 7 cm. What is the probability that an 18-year-old male footballer has a chest measurement of 99 cm?

Solution

Write the required probability. $P(98.5 \leq X < 99.5)$

Use your calculator cdf to calculate the required probability. $P(98.5 \leq X < 99.5) = 0.051\ 954\ldots$

Round off and state the result. There is about a 5.2% chance that an 18-year-old male footballer has a chest measurement of 99 cm.

EXAMPLE 24

A small business owner knows that annual total sales are normally distributed with a mean of $350 000 and a standard deviation of $58 000.

a What is the probability that the company's sales will fall within $65 000 of the mean?

b Work out the annual sales figure that has only a 12% chance of being exceeded.

Solution

a Calculate the boundary values.

Lower bound = \$350 000 – \$65 000
= \$285 000
Upper bound = \$350 000 + \$65 000
= \$415 000

Write the required proportion as a probability statement.

$P(285\,000 < X < 415\,000)$

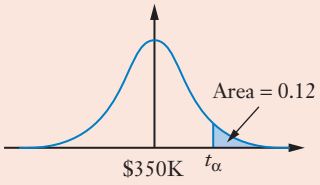
Use your calculator cdf to calculate the required probability.

$P(285\,000 < X < 415\,000) = 0.737\,579\ldots$

State the result.

There is about a 73.8% chance that annual sales will be within \$65 000 of the mean.

b Sketch a diagram to help visualise the problem.



Find the required quantile.

$P(X > a) = 0.12$

$\Rightarrow P(X < a) = 1 - 0.12 = 0.88$

Use the inverse normal function on your calculator.

$t_\alpha = 418\,149.233\ldots$

State the result.

There is only a 12% chance of exceeding annual sales of about \$418 149.

Exercise 10.09 Applications of the normal distribution

Example 22

1 The cholesterol levels of Australian adults are assumed to be normally distributed with a mean of 5.6 millimoles per litre (mmol/L) and a standard deviation of 1.2 mmol/L. What is the probability that a randomly selected Australian adult has a cholesterol level:

a less than 4.3 mmol/L?

b between 4.9 mmol/L and 5.8 mmol/L?

c more than 8.5 mmol/L?

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2 The average rainfall in a northern Queensland town in January is 268.5 mm, with a standard deviation of 84.5 mm. What is the probability that the rainfall for a particular January is:

a above 220 mm? **b** less than 190 mm?

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3 Suppose that X is a random variable representing the reaction time, in seconds, in a simulated driving experiment for persons who have been given a specific amount of alcohol. Assuming that X has a normal distribution with a mean of 0.8 s and a standard deviation of 0.2 s, find the following probabilities.

a $P(X < 0.5)$ **b** $P(X > 1.2)$ **c** $P(0.75 < X < 1.25)$

4 A survey was conducted to find the number of hours per week that adults spend on home computers. Analysis of the survey results showed that the number of hours was normally distributed, with a mean of 8 hours and a standard deviation of 1 hour. Find the probability that the hours spent on the home computer by a randomly selected adult is:

Example 23

a less than 5.5 hours per week.
b between 4.5 and 9.5 hours per week.
c more than 10 hours per week.
d 6 hours per week

5 The time that gym members use a treadmill is normally distributed, with a mean of 24 minutes and a standard deviation of 7 minutes. A gym member is randomly selected. Find the probability that the gym member uses a treadmill for:

a less than 17 minutes **b** between 16 and 27 minutes
c more than 28 minutes **d** 30 minutes

6 Scores for a public service examination are normally distributed, with a mean of 78 and a standard deviation of 7.5. To be eligible for a public service position, a candidate must score in the top 5%. What is the lowest score a candidate can earn and still be eligible for employment?

Example 24

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7 Mensa is a non-profit organisation open to people who score at the 98th percentile or higher on a standardised IQ test.

a If IQs are normally distributed, with mean 100 and a standard deviation of 16, what is the minimum IQ required for admission to Mensa?

b If three individuals are chosen at random from the general population, what is the probability that none satisfy the minimum requirement for Mensa?

8 Antoine had a score of 65% for English, but only 57% for Physics. The class mean and standard deviation for English were 62% and 8% respectively; for Physics they were 52% and 10%. In which subject did Antoine appear to do better? Explain your reasoning.

9 People's digestion rates were studied and normal meals were found to have a mean digestion time of 9.4 hours with a standard deviation of 2.1 hours. Based on this data, what proportion of people can be expected to digest a normal meal in 8 hours or less?

10 The hourly earnings of plumbers are normally distributed with a mean of \$56/h and a standard deviation of \$14.

a From a group of 20 plumbers, how many would you expect to be earning over \$70/h?

b From 50 plumbers, how many would you expect to be earning less than \$60/h?

c What is the interquartile range ($Q_3 - Q_1$) for the hourly earnings of plumbers?

11 Two basketball players have average game totals of 25 points and 29 points respectively. The first has a standard deviation of 9 points and the second a standard deviation of 5 points.

Is the first or second player more likely to score more than 37 points in a particular game?

12 The 'life' of young women's fashions measures how long a store can expect a fashion to sell in sufficient quantities to be worth keeping in stock. For some types of fashions, it is about 2 months (60 days), but with a wide variation, reflected in the standard deviation of 3 weeks (actually 19 days). What percentage of these fashions would be expected to:

a have a life of over 3 months (90 days)?

b have a life of less than a month?

c be a '9-day-wonder' (less than 10 days)?

13 Ten-year-old girls can do an average of 11.2 push-ups with a standard deviation of 5.4 pushups. Which is more unusual for 10-year-old girls: being able to do only 2 push-ups or being able to do 17?

14 The daily output of a production line varies according to a normal distribution with a mean of 163 units of production and a standard deviation of 4.5 units. Find the probability that the daily output will:

a fall below 160 units

b exceed 170 units

c be between 160 and 165 units.

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Problem solving

15 A filling machine delivers gelatin powder into packets. The amount of gelatin in the packets is normally distributed with a standard deviation of 6 g. The advertised weight of a packet is 375 g.

a If the mean is 380 grams, find the proportion of packets that will be underweight (less than 375 g).

b Find the required mean value to ensure that only 2% of packets are underweight.

16 A manufacturer produces batteries for which the life span is normally distributed. The average life of a battery is 4000 hours and the standard deviation is 60 hours.

a What percentage of batteries have a life span that is more than 4130 hours?

b Would it be unusual for a battery to have a life span that is more than 4130 hours? Explain your answer.

17 A retailer sells a brand of car tyre that has a life expectancy that is normally distributed, with a mean life of 40 000 km and a standard deviation of 3500 km. The retailer wants to give a guarantee for free replacement of tyres that don't wear well. How should the guarantee be stated if the retailer is willing to replace approximately 10% of the tyres sold?

18 An car muffler is designed so that its lifetime (in months) is approximately normally distributed with mean 39.6 months and standard deviation 5.7 months.
The manufacturer has decided to use a marketing strategy in which any muffler that does not last for at least 27 months is replaced for free. Should the manufacturer be concerned that the marketing strategy may have a significant impact on profitability? Explain your answer.

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10. CHAPTER SUMMARY

Continuous random variables and the normal distribution

- A **continuous random variable** is a random variable with values from an interval or continuum. A **continuous probability distribution (or function)** shows the probabilities of a continuous random variable. The values of the **probability function**, **pdf** (or **probability distribution**) are written as $P(X = x)$ or $p(x)$.
- For a continuous random variable X, the **probability density function** $p(x)$ over an interval $[a, b]$ is defined as: $P(a \le X \le b) = \int_a^b p(x)\,dx$. $p(x) \ge 0$ for all $x \in [a, b]$ and $\int_a^b p(x)\,dx = 1$.
- The **closed interval** $a \le x \le b$ is written $[a, b]$. The **open interval** $a < x < b$ is written (a, b). The interval $a \le x < b$ is written $[a, b)$, and the interval $a < x \le b$ is written $(a, b]$.
- For X defined on the interval (a, b) with a probability density function $f(x)$, the **cumulative distribution function** (**cdf**) is given by $F(x) = \int_a^x f(x)\,dx$.
- A **uniform distribution**, $p(x)$, on the interval $[a, b]$, is constant over the domain of X. The uniform distribution $p(x)$ is defined as:

$$p(x) = \begin{cases} \dfrac{1}{b-a} & \text{for } a \le x \le b \\ 0 & \text{elsewhere} \end{cases}$$

- A **triangular distribution**, $f(x)$, with minimum value (a), peak value (c), and maximum value (b) is defined as:

$$f(x) = \begin{cases} 0 & \text{for } x < a \\ \dfrac{2(x-a)}{(b-a)(c-a)} & \text{for } a \le x \le c \\ \dfrac{2(b-x)}{(b-a)(b-c)} & \text{for } c \le x \le b \\ 0 & \text{for } x > b \end{cases}$$

- The **expected value** for the continuous random variable X on the interval $[a, b]$, with pdf $p(x)$, is defined as: $E(X) = \mu = \int_a^b x \cdot p(x)\,dx$. For a uniform distribution, $E(X) = \mu = \dfrac{a+b}{2}$.
- For the continuous random variable X on the interval $[a, b]$, with $E(X) = \mu$, the **variance** is $Var(X) = \int_a^b p(x)(x-\mu)^2\,dx$ and the **standard deviation** is $SD(X) = \sigma = \sqrt{Var(X)}$.

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- For a uniform distribution on the interval $[a, b]$, $Var(X) = \frac{(b-a)^2}{12}$ and $SD(X) = \frac{b-a}{2\sqrt{3}}$.
- For the continuous random variables X and Y such that $Y = aX + b$,

$$E(aX + b) = aE(X) + b \qquad \text{and} \qquad Var(aX + b) = a^2 Var(X).$$

- The probability density function for the **normal distribution** is $p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$, where the mean = μ and the standard deviation = σ. The mean, median and mode are equal.
- For a normal distribution, $P(\mu - \sigma \leq X \leq \mu + \sigma) \approx 0.68$, $P(\mu - 2\sigma \leq X \leq \mu + 2\sigma) \approx 0.95$ and $P(\mu - 3\sigma \leq X \leq \mu + 3\sigma) \approx 0.99$.
- The **z-score** (or **z-value**) is $z = \frac{x-\mu}{\sigma}$.
- The probability density function for the **standard normal distribution** is $f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$.
- The **inverse cumulative normal distribution function** calculates the value that corresponds to a given area under the normal or standard normal distribution curve.
- **Quantiles** (or **fractiles**) divide a probability distribution into equal groups. **Quartiles** divide a set of values into quarters, **deciles** divide values into tenths and **percentiles** into hundredths. The second quartile (Q_2) is the **median**. 40% of all values lie below D_4 and 73% lie below P_{73}.
- For a continuous random variable X, a quantile t_α is the value of x such that $P(X < t_\alpha) = t_\alpha$, where $0 < \alpha < 1$.

10. CHAPTER REVIEW

Continuous random variables and the normal distribution

Examples 1,2

1 The thumb lengths of 60 students, measured to the nearest millimetre, were as follows:

38	50	38	40	35	52	45	50	40	32	40	45
43	40	45	45	55	37	50	45	45	55	50	35
45	50	40	40	50	41	41	40	40	46	45	52
45	45	48	45	45	35	45	45	40	45	40	40
47	70	55	51	45	35	52	32	40	43	45	42

a Construct probability histograms of the length L, with a class width of 5 mm.
b Find the total area of the histogram.
c Use the histogram to find the probability of choosing a person at random from the group with a thumb length between 38 mm and 47 mm.

Example 3

2 Which of the following could be a probability density function over the given interval?

a $f(x) = \frac{3x^2}{8}$ for the interval $[0, 2]$
b $f(x) = \frac{1}{8}$ for the interval $[0, 4]$
c $f(x) = \frac{3}{10}(2 - x^2)$ for $-1 \leq x \leq 1$
d $f(x) = 4e^{-2x}$ for $x > 0$

Example 4

3 For each function, find the value of k that makes $f(x)$ a probability density function for the stated interval.

a $f(x) = k(3x^2 - 2)$ for $0 \leq x \leq 2$
b $f(x) = ke^{-5x}$ for $x > 0$

Example 5

4 The probability density function for a continuous random variable X is:

$$f(x) = \frac{3x^2}{8} \text{ for } 0 \leq x \leq 2.$$

a Find the cumulative density function for X.
b Use the cumulative density function to calculate:
i $P(X \leq 1)$ **ii** $P(0.5 \leq X \leq 2)$

Example 6

5 Unreinforced concrete paths may get their first cracks anywhere along the path. A concrete path, 7.4 m long, is constructed from the backdoor to a clothesline. If D is the distance (in metres) from the back door where the crack appears:

a construct the probability density function for D
b find the probability that a crack develops more than 3 m from the backdoor.

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6 A continuous random variable X has a triangular distribution for $3 \leq x \leq 12$ and a peak value that occurs at $x = 9$. Example 7

a Draw the graph of the probability density function.

b State the probability density function.

c Calculate:

i $P(X \leq 6)$ **ii** $P(X \geq 11)$ **iii** $P(6 \leq X \leq 10)$

7 Calculate the expected value for a: Examples 8,9

a uniform distribution defined on the interval [4, 26]

b triangular distribution defined on the interval [3, 10] with a peak value at 5.

8 Calculate the expected value for the continuous random variable, X, with the probability density function: Example 10

a $p(x) = \frac{3}{8}x^2$ for $0 \leq x \leq 2$ **b** $p(x) = \frac{2}{5}(x+1)$ for $1 \leq x \leq 2$

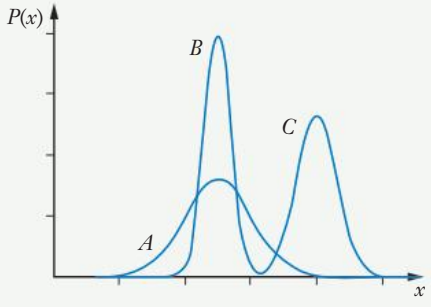
9 Calculate the variance and standard deviation of a continuous random variable with a: Examples 11,12

a uniform distribution over the interval [5, 12]

b symmetrical triangular distribution over the interval [12, 34].

10 Calculate the expected value and variance of a continuous random variable with the probability density function $f(x) = 2e^{-2x}$ for $x > 0$. Example 13

11 Consider the normal distribution curves shown. Example 14



a Which has the greatest mean?

b Which normal curve has the greatest standard deviation?

Give reasons for your answers.

12 **a** What is the probability density function for a normal distribution with $\mu = 20$ and $\sigma = 5$? Example 14

b What is the mean and standard deviation of the probability density function for a normal distribution given by $f(x) = \frac{4}{3\sqrt{2\pi}} e^{-\frac{8}{9}(x-20)^2}$?

Example 15

13 A continuous random variable X has a normal distribution with a mean of 18.5 and a standard deviation of 3.4.

a Find the probability that $X < 9.6$.

b Calculate $P(12 < X < 21)$.

Example 16

14 The lengths of Tailor fish are normally distributed, with a mean of 20 cm and a standard deviation of 4 cm. Two Tailor fish are caught and found to have lengths of 23.5 cm and 17.4 cm respectively.

a Sketch the graph of the distribution, showing lengths and z-scores.

b What proportion of Tailor would be longer than the first fish caught?

c Which of the caught fish is closest in length to a typical Tailor?

Example 17

15 It is known that the stem lengths of a crop of flowers are normally distributed with a mean of 46 cm and a standard deviation of 5 cm.

a What is the z-score for a stem length of 37 cm?

b What is the stem length for a z-score of 2.1?

Example 18

16 For the standard normal distribution, find:

a $P(1 < Z < 2)$ **b** $P(-0.7 < Z < 1.9)$

Example 19

17 If X is a normally distributed random variable, use a calculator to find:

a b, given that $P(X < b) = 0.24$, where $\mu = 32$ and $\sigma = 8$

b m, given that $P(X > m) = 0.85$, where $\mu = 27.6$ and $\sigma = 6.3$

Example 20

18 X is a normally distributed random variable with a mean of 54 and a standard deviation of 11. Calculate:

a the 63rd percentile (P_{63})

b the third quartile

c a, given that $P(X > a) = 0.44$

Example 21

19 **a** The continuous random variable X is normally distributed with a mean of 22.6. If $t_{0.58} = 23.69$, find the standard deviation of the distribution.

b The variable X is normally distributed so that the 45th percentile is 12.11 and the standard deviation is 2.3. Find the mean of the distribution.

Example 22

20 The amount of time per week a student uses a computer in the school computer lab is normally distributed, with a mean of 6.2 h and a standard deviation of 0.9 h. Find the probability that a randomly-selected student uses a lab computer:

a fewer than 4 hours per week

b between 4 and 7 hours per week

c more than 6 hours per week

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21 The weights of adult male Australian cattle dogs are normally distributed, with a mean of 15 kg and a standard deviation of 3 kg. A cattle dog is randomly selected.

Examples 23,24

a Find the probability that the dog's weight is less than 13 kg.
b Find the probability that the weight is between 13 kg and 17 kg.
c Find the probability that the dog's weight is more than 17 kg.
d Find the weight of a dog that is heavier than 85% of cattle dogs.

Problem solving

22 A continuous random variable, X, has a probability density function defined by:

$$f(x) = k(x-1)(2-x) \text{ for } 0 \leq x \leq 2$$

Find the value of k.

23 A continuous random variable, X, has a probability density function defined by:

$$f(x) = \frac{2}{11}\left(\frac{1}{3}x + 2\right) \text{ for } -2 \leq x \leq 1.$$

Calculate the median.

24 The lengths of bolts produced by a factory are approximately normally distributed. Bolts shorter than 18.25 cm and longer than 19.85 cm are rejected as unacceptable. A quality control report notes that on average, 4% of bolts are rejected as undersized and 6% are rejected as oversized. Find the mean and standard deviation of the distribution.

25 Which is more unusual: a mass of 140 g or a diameter of 12 cm from apples that are normally distributed with averages of 150 g and 10 cm and standard deviations of 8 g and 1.5 cm respectively? Explain your answer.

Qz Practice quiz

11.

SAMPLE PROPORTIONS

Large populations are difficult to count and assess, particularly if they are mobile. It is obviously impossible to weigh all the rainbow lorikeets in Queensland. You could weigh a **sample** of the birds to determine the health of a population. The proportion of underweight birds would be an important indicator. In this chapter, you will learn how to use proportions in a sample as statistics for populations.

11.01 Random samples and bias
11.02 Choosing samples
11.03 Sample proportions
11.04 Variation of samples
11.05 Sample size and normality
11.06 Interval estimates of sample proportions
11.07 Confidence intervals and the standard normal distribution
11.08 Confidence intervals and sample proportions
11.09 Variation of confidence levels
11.10 Applications of confidence intervals
Chapter summary
Chapter review

SYLLABUS SUBJECT MATTER

Random sampling

- understand the concept of a random sample
- discuss sources of bias in samples, and procedures to ensure randomness
- investigate the variability of random samples from various types of distributions, including uniform, normal and Bernoulli, using graphical displays of real and simulated data

Sample proportions

- understand the concept of the sample proportion $\hat{p}$ as a random variable whose value varies between samples, and the formulas for the mean p and standard deviation $\sqrt{p(1-p)/n}$ of the sample proportion $\hat{p}$
- consider the approximate normality of the distribution of $\hat{p}$ for large samples
- simulate repeated random sampling, for a variety of values of p and a range of sample sizes, to illustrate the distribution of $\hat{p}$ and the approximate standard normality of $\dfrac{(\hat{p}-p)}{\sqrt{\hat{p}(1-\hat{p})/n}}$ where the closeness of the approximation depends on both n and p

Confidence intervals for proportions

- understand the concept of an interval estimate for a parameter associated with a random variable
- use the approximate confidence interval $\left(\hat{p}-z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}, \hat{p}+z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}\right)$, as an interval estimate for p, where z is the appropriate quantile for the standard normal distribution
- define the approximate margin of error $E = z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$ and understand the trade-off between margin of error and level of confidence
- use simulation to illustrate variations in confidence intervals between samples and to show that most but not all confidence intervals contain p

Prior learning

Mathematical Methods 2019 v1.1 General Senior Syllabus © Queensland Curriculum and Assessment Authority (QCAA). This syllabus forms part of a new senior assessment and tertiary entrance system in Queensland. Along with other senior Syllabuses, it is still being refined in preparation for implementation in schools from 2019. For the most current syllabus versions and curriculum information please refer to the QCAA website https://www.qcaa.qld.edu.au/.

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TERMINOLOGY

Bernoulli distribution, variable
census
completion bias
convenience sample
interval estimate
margin of error
normal distribution, variable
population
quantile
random sample
response bias
sampling distribution
self-selection sample
statistic
survey
biased sample
central limit theorem
confidence interval
design bias
interviewer bias
non-random sampling
parameter
pseudo-random
quota sample
recall
sample
selection bias
simple random sampling
stratified random sample
systematic sample
binomial distribution, variable
cluster samples
confidence level
fair sample
judgment sample
non-response bias
point estimate
purposive sample
random number
reporting
sample proportion
self-selected sample
standard normal quantile
success

11.01 Random samples and bias

In statistics, the word **population** has a special meaning. The population of rocky planets in our solar system planets has only 5 members (including Pluto). You can easily find the mean diameter of this population. However, it is almost impossible to find the mean height of all adult Australians.

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Statistical terms

- The **population** is the complete group from which you want information
- A **census** is collection of information from the whole population
- A **parameter** is a characteristic of the population, such as a mean
- A **sample** is part of the population
- A **survey** is collection of information from a sample
- A **statistic** is an estimate of a parameter from the individual values (**observations**) of a sample. It is a value obtained from a sample

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Sometimes census means an *attempt* to collect information from a population, for example, the Australian census, which is conducted every 5 years. A *value* is not always a number. For example, the value of the parameter or statistic 'most popular colour' is a word.

EXAMPLE 1

30 people putting money on a Go-card were asked how much they put on.
The smallest amount was $20, the largest was $70 and the average was $28.
What are the population, parameters and statistics for this sample?

Solution

The population is the complete group.	The population is the people with Go-cards.
Parameters are clearly defined values from the whole population.	The parameters are the minimum, maximum and mean amounts put on Go-cards at single transactions.
Statistics are the values from a sample.	The minimum is $20, the maximum is $70 and the mean is $28.

The sample size is not a parameter because it is not an estimate of a population value.

A 'good' statistic is likely to be very close to the population parameter. To make statistics as good as possible, you should use samples as large as possible. You should also try to make them as representative as possible.

Samples

- A **fair sample** is representative of the population; it does not favour one group
- A **biased sample** is not representative; it favours one group of the population
- A **random sample** is one chosen by a chance method to ensure every member of the population has an equal chance of being chosen

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INVESTIGATION

THE AUSTRALIAN CENSUS

Find out as much as you can about the Australian census.

Investigate:

- when it is taken
- the number of times it has been done
- the time it takes to process
- the cost of doing the census
- the uses of the census
- how the questions are decided
- how they try to get the whole population
- who provides the answers to the questions

Your teacher may want you to make a report.

Surveys can be biased by the choice of the sample, the way the survey is designed, the way it is conducted and even the responses to the survey. Significant bias is more common in surveys about people's opinions.

Types of bias

- **Completion bias** arises from incomplete surveys. Surveys should not be too long and should avoid sensitive subjects where possible. In longitudinal studies, some subjects may move away or be otherwise lost. You should check the bias that this might introduce
- **Design bias** arises from the way the survey is designed, ignoring the needs of the users, phrasing of questions, the order of questions, etc. Questions should be as objective as possible and you should avoid leading questions
- **Interviewer bias** arises from conscious or subconscious attitudes of the interviewer that affect responses or differences between interviewers. Training in interview methods and standard scripts can reduce it
- **Response bias** arises from the tendency of people to agree with the interviewer, to give socially acceptable answers, etc. **Non-response bias** may occur when some subjects do not answer the survey. A **self-selected sample** is an extreme case such as may occur when people ring up in response to a TV survey
- **Recall bias** arises from the inability of subjects to remember things. Surveys should focus on recent events
- **Reporting bias** arises from the way the survey is interpreted or published
- **Selection bias** arises from the choice of the sample. Random samples are best

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EXAMPLE 2

In each case, state if the sampling method is fair and the types of bias, if any.

a 200 people passing through the Castle Town shopping centre on Saturday morning in Townsville were asked which was their favourite NRL team.

b 1000 people throughout Queensland were rung on fixed lines between 5 p.m. and 9 p.m. and asked who they would vote for at the next election.

c A 'reality' show asked viewers to text the name of the contestant they wanted to remove from the next show to a particular telephone number.

Solution

a	Those interviewed were only available at a particular place and time.	The sampling method is not fair, there is selection bias and probably also non-response bias.
b	Only people with fixed lines were interviewed and the respondents could be children.	There is some selection bias, there is design bias and possibly response bias.
c	Only people who feel strongly will bother to ring.	The sample is self-selected, but this is probably not seen as a problem by the producers.

Exercise 11.01 Random samples and bias

1 Identify the population, parameters and statistics for each of the following situations. Example 1

a 35 people entering a cinema were asked how much they had spent at the snack and sweets counter. The smallest amount was $6.50, the largest $37.95, with an average of $22.43.

b People with names starting with 'D' on the electoral roll in Gympie were sent a survey about takeaway food consumption. Of the 123 replies, 47 said they had used McDonalds; 35 Red Rooster, 38 KFC, 19 Brodies Chicken, 48 Domino's Pizza and 73 other outlets.

c Students at a bus station were asked what kind of calculator they used: 36 said Sharp, 42 said TI, 39 said Casio, 14 said HP and 17 gave other brand names.

d The kinds of vehicles going along the Ipswich Motorway were observed. In a 30-minute period, 120 cars, 40 semitrailers, 60 smaller trucks, 18 motorbikes, 29 taxis, 32 buses, 17 utes and 32 vans were observed.

e The price of 91 octane E10 petrol was checked each day over a month at the service stations in and around Gladstone. Prices varied from 136.9 cents/L to 142.9 cents/L, with a median price of 140.9 cents/L.

2 State whether the sampling method is fair and the types of any bias in question **1**. Example 2

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Problem solving

3 Which of the following would be the fairest way for you to choose a sample of students in your school? State the kind of bias in each method.

- **a** Ask everyone in your class.
- **b** Interview the first 80 students who walk into the resource centre.
- **c** Ask all students in your year level.
- **d** Announce that there will be a poll on parade and ask the first 70 students who volunteer for the survey.
- **e** Obtain an alphabetical list of all the students and ask every 20th student on the list.
- **f** Leave a 'nomination sheet' in the resource centre and interview only the people who write their names on it.
- **g** Ask 5 students from every form class in the school.
- **h** Ask 1 in every 15 students from each year level in the school.
- **i** Call a meeting of interested students and interview those who attend.
- **j** Wait at the school entrance and ask the first 100 students who arrive after 7:30 a.m.

4 People with telephone numbers on a randomly selected page from the Gold Coast *White Pages* are phoned between 6 p.m. and 8 p.m. on a weeknight. They are asked whether they would support a new high-rise shopping centre and residential development on Burleigh Heads. Suggest any problems and how this survey could be improved.

5 The editor of a health and fitness magazine thinks that the weight-and-height tables for Australians are out of date. The editor asks staff to put a survey form in the next issue asking subscribers to send in their current weight, height and age so that new tables can be compiled. Ten people who send in replies will win a year's free subscription to the magazine. How would you improve this survey?

6 In each of the following cases, state whether or not the sampling method is fair, and if it is biased, state the kind(s) of bias.

- **a** A hardware store wants to know if it would be worthwhile opening for longer hours. A survey placed on the counter asks customers to tick the times they are likely to shop in the store from a list.
- **b** A student in Year 12 goes to Year 8 classes to investigate the amount of pocket money that students of this age receive from their parents. He asks students in the class to tell him how much pocket money they got last week as he goes round the class and writes it down as he goes.
- **c** A TV reality show eliminates one contestant each week by having people SMS their vote of who to eliminate to a particular number each week. They have the system set up so that only one vote is accepted from each mobile number.
- **d** A council needs to establish a new landfill site for rubbish as the old one is almost full. Some councillors urge the council to use 'people power' to decide the new site by asking residents close to each of the possible sites for their opinion of the best site. It is proposed that people attending public meetings near each site will be asked to vote on the suitability of the nearby site.

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e A food critic goes to different restaurants and tries a selection of dishes on the menu before rating them 1–10 on ambience, presentation, quality of cooking and variety of the menu. The critic is very well-known in the area and his opinion is valued highly. Reviews of three restaurants are published the following week.

7 A school surveys its students at the beginning of the year to determine the amount of part-time work they do.
- **a** What is the population for this survey?
- **b** What are the parameters?
- **c** What problems may this survey have in determining the parameters?

8 A high school hall is set up for Year 12 examinations with students seated alphabetically in rows of 25 from front to back, with 11 desks in each row. Some desks are vacant due to the illness of some students. Students are selected to provide feedback on the conduct of the test. Every 15th student is chosen, starting with the 3rd desk in the front row and working across to the right, then back to the left in the next row, and so on until 10 students are selected. Comment on the method of sampling.

11.02 Choosing samples

It doesn't matter how well a survey is designed and administered if a poor sample is chosen. The statistics from a poor sample will not be good estimators of the parameters.

The simplest way to choose a random sample is to number a population and use the numbers to choose a sample. For a small sample, you can put numbered pieces of paper in a hat and draw some numbers. For a larger number, you can use **random numbers**. Tables and books of random numbers have been made for this purpose. A short table of 2-digit random numbers from 00 to 99 is given on page 422.

Drivers in Queensland already have a licence number. You could use these for Queensland drivers instead of numbering them yourself. You would have to allow for missing licences. Similarly, you could use ID numbers for a sample of students in your school.

EXAMPLE 3

Use the 2-digit random number table on page 422 to randomly select

- **a** 10 students from 50 Maths Methods students at a school.
- **b** 6 numbers between 450 and 700.

Solution

a	Each student needs a number.	Number the students from 1 to 50.
	You need to start in a random place.	Using a pin, 95 at row 9, column 25 has been chosen as the starting point.

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Move along the table and reject 00, numbers bigger than 50 and repeated numbers.

37 49 95 38 08 68 70 32 88 65 89 70 13 93 24 40 05 41 34 72 49 10 50 78 95
04 92 03 87 51 08 13 11 48 36 98 73 32 94 11 01 78 95 19* 70 13 84 91 57 67
05 04 13 40 88 75 68 99 63 19 56 69 99 33 68 24 70 05 25 64 42 41 85 04 88

Put a mark where you finish and state the answer.

Use students 4, 3, 8, 13, 11, 48, 36, 32, 1 and 19 for the sample.

b Decide how to use the table.

Use a pair of numbers but ignore the last digit.

Whenever you use a random number table, you just continue from the last time.

04 92 03 87 51 08 13 11 48 36 98 73 32 94 11 01 78 95 19* 70 13 84 91 57 67
05 04 13 40 88 75 68 99 63 19 56 69 99 33 68 24 70 05 25 64 42 41 85 04 88
30 64 49* 26 22 93 66 84 39 90 57 91 05 63 53 86 05 39 32 61 67 10 68 26 73

Choose x so that $450 < x < 700$.

7013, 8491, 5767, 0504, 1340, 8875, 6899, …

Write the sample.

The sample is 576, 689, 631, 566, 682 and 644.

2-digit random number table

49 43 74 58 18 76 85 23 53 04 22 74 46 11 68 63 23 75 15 22 69 96 01 44 83
34 73 96 52 84 77 44 53 71 72 17 80 68 70 89 97 42 96 85 11 27 85 12 76 16
87 22 21 83 13 92 96 27 94 30 56 45 75 13 57 23 25 38 84 60 78 09 05 14 95
48 43 67 58 18 82 26 78 91 90 41 37 26 75 50 53 76 54 49 68 22 65 88 61 35

83 94 87 49 80 09 46 79 96 70 65 47 43 95 15 45 32 10 52 12 22 92 78 47 00
36 34 38 79 59 07 24 30 09 77 73 25 20 78 94 22 78 84 66 62 14 27 99 97 32
36 16 40 44 98 60 45 49 50 26 04 24 92 22 77 46 25 48 43 36 18 08 75 30 93
39 43 74 42 69 40 45 67 03 34 97 41 30 44 27 96 50 42 76 44 43 79 47 10 40

37 49 95 38 08 68 70 32 88 65 89 70 13 93 24 40 05 41 34 72 49 10 50 78 95
04 92 03 87 51 08 13 11 48 36 98 73 32 94 11 01 78 95 19 70 13 84 91 57 67
05 04 13 40 88 75 68 99 63 19 56 69 99 33 68 24 70 05 25 64 42 41 85 04 88
30 64 49 26 22 93 66 84 39 90 57 91 05 63 53 86 05 39 32 61 67 10 68 26 73

16 02 93 88 42 32 97 19 48 39 27 00 17 29 98 95 33 02 15 35 84 54 88 77 88
90 72 79 41 71 30 19 99 89 25 18 77 55 49 03 75 26 66 89 31 45 75 85 95 16
99 31 34 95 97 50 56 14 09 36 63 23 12 58 28 64 16 96 92 62 73 96 99 48 21
55 38 06 44 27 29 38 61 58 15 66 43 42 97 45 51 03 81 16 99 06 55 69 43 88

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90 44 98 14 71	48 71 64 42 02	32 35 88 10 99	82 31 57 53 66	65 23 77 45 97
38 20 79 34 43	86 60 75 91 50	73 46 65 92 81	46 24 67 26 25	73 91 68 85 58
55 39 11 90 64	29 14 82 02 91	17 28 23 76 50	95 58 89 77 40	64 72 53 62 78
14 93 66 16 82	81 62 77 76 89	84 90 19 36 37	74 60 59 65 30	04 17 45 51 77
16 14 64 03 65	07 93 16 46 30	51 89 20 57 47	93 53 48 33 82	65 95 64 99 12
93 87 08 79 34	56 52 18 28 71	40 87 50 40 47	37 07 74 56 07	36 94 06 55 22
73 84 05 46 00	26 28 53 72 54	00 41 50 02 31	34 22 62 47 21	81 48 23 18 37
65 47 45 88 45	30 48 46 86 88	59 80 11 55 63	87 63 07 05 85	16 29 18 37 16
91 70 92 43 59	43 60 00 29 36	04 30 80 87 61	51 63 29 89 23	02 67 35 37 34
74 30 27 61 24	12 11 37 38 71	54 72 66 17 58	67 84 74 75 73	23 08 46 72 20
63 66 83 95 36	74 00 52 74 63	44 03 90 26 51	27 11 12 75 38	48 18 27 91 90
09 24 10 27 49	82 34 92 67 13	00 33 50 70 40	09 52 92 14 11	37 98 41 02 39
86 13 95 56 74	62 42 60 49 01	59 38 58 74 67	55 09 35 65 45	10 44 26 23 22
99 15 24 44 83	34 69 20 45 98	52 74 99 21 57	63 49 45 92 65	44 79 58 85 52
73 37 86 65 14	02 49 20 42 64	55 56 99 27 93	87 97 83 92 05	17 26 82 07 24
88 71 46 18 04	58 06 99 02 64	07 65 19 89 29	34 36 96 63 86	42 19 91 33 28
95 84 21 98 21	29 39 05 39 83	75 20 42 49 81	21 08 73 69 02	57 77 31 17 33
07 86 42 16 11	48 65 48 38 23	99 97 67 29 20	36 60 41 36 54	34 62 70 85 66
17 78 69 19 74	86 32 92 90 56	97 00 80 72 16	70 14 57 24 75	35 13 28 78 06
79 10 14 82 29	61 89 99 76 40	06 03 21 58 20	89 10 80 10 72	26 65 85 87 20
19 06 97 60 80	42 61 78 24 44	80 74 00 46 07	24 87 93 60 01	76 95 73 35 79
96 63 07 61 84	26 60 97 13 57	41 57 54 97 60	12 74 99 84 69	38 29 50 25 33
83 02 54 19 32	57 11 10 94 92	69 71 28 61 73	48 55 33 25 17	03 53 91 60 78
38 33 71 59 83	69 78 15 04 55	96 59 04 90 06	07 59 55 50 82	32 64 66 32 14

There are four basic methods of random sampling.

Random sampling methods

- **Simple random sampling** is a random sample from the entire population
- **Stratified random samples** use groups with particular traits in the population. Random samples are taken from all groups in proportion to their sizes
- **Cluster samples** use convenient groups of the population. All the individuals within each group of a random sample of the groups are used
- **Systematic samples** use every nth individual from a list of the population. The value of n depends on the sample and population sizes. The starting position is chosen randomly

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The groups of a stratified random sample are fairly homogeneous; the individuals in a group are alike in some way. The groups of a cluster sample are fairly heterogeneous; they each typify the population. Cluster samples are often geographical.

EXAMPLE 4

A company employs 42 assembly workers, 10 office staff and 3 supervisors. How many people from each group of workers should be included in a sample of 15?

Solution

Find the total number.	Total = 55
Find the sample proportion.	Sample proportion $= \frac{15}{55} = \frac{3}{11}$
Find the number from each group.	$n(\text{assembly workers}) = \frac{3}{11} \times 42 = 11.454\ldots$
	$n(\text{office staff}) = \frac{3}{11} \times 10 = 2.727\ldots$
	$n(\text{supervisors}) = \frac{3}{11} \times 3 = 0.818\ldots$
Round and write the answer.	The sample should have 11 assembly workers, 3 office staff and 1 supervisor.

Rounding errors may produce a sample slightly different from the desired size.

Systematic sampling is normally used for a situation where numbering already exists.

EXAMPLE 5

The ID numbers of Year 7 students in a school run from 4247 to 4395 inclusive, As yet, no Year 7 students have left. Use the numbers for a systematic sample of 10 Year 7 students.

Solution

Find the number of Year 7 students.	Total number = 4329 – 4247 + 1 = 149
Find the repeat for students selected.	$\frac{149}{10} \approx 15$
Pick a random starting place.	Start with student 4361.
Choose every 15th student. When you get to the end, go back to the beginning.	Choose students with IDs 4361, 4376, 4391, 4257, 4272, 4287, 4302, 4317, 4332, 4347.

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At the end, your count goes 4391, 4392 (1), 4393 (2), 4394 (3), 4395 (4), 4247 (5), 4248 (6), …, to 4257 (15). 4257 = 4391 + 15 – 4395 + 4247 – 1. If you use subtraction for the number, you have to take 1 when you go back to the start.

Many surveys are non-random because of the time and cost of random surveys. There are 4 main methods of conducting non-random surveys.

Non-random sampling methods

- **Convenience samples** are chosen from an easily accessible group. For example, the people in your class
- **Judgment (purposive) samples** are chosen using judgement to decide on a representative sample. For example, picking people with a range of political views
- **Quota samples** are set up using quotas; numbers of each group within the population as with stratified sampling, but individuals are chosen by convenience. For example, the first 5 men and first 5 women to come past you in the street
- **Self-selection samples** use volunteers for the sample. For example, asking on assembly for students who want to take part in a survey

You can use the **random function** on a computer or calculator to choose a sample, but this is actually **pseudo-random**, because they give the same group of numbers over and over again. In many cases, this is good enough.

TECHNOLOGY

Graphics calculator samples

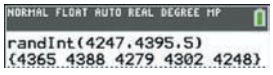
TI-Nspire CX Chapter 11

Use your graphics calculator to choose a sample of 5 from student numbers 4247 to 4395 inclusive. You might have to do it again if you get a number repeated.

TI-84 Plus CE

Press [math], and choose PROB and 5: randInt.

Put in the lower limit, upper limit, number of integers and press [enter].



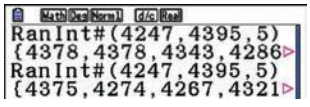
You can store the list in a variable.

Use [sto→][L1] to store in L1.

Casio fx-CG20AU

Choose the Run-Matrix menu and press [OPTN].

Choose PROB, RAND and Int and put in the lower limit, upper limit, number of integers and press [EXE].



You can store the list in a variable.

Use → List 1 to store in List 1.

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INVESTIGATION

PSEUDO-RANDOM NUMBERS

p mod q is the remainder when p is divided by q.

Use the rule $x_n = ax_{n-1}$ mod b to generate sets of pseudo random numbers.

For $x_1 = 5$, $a = 10$ and $b = 13$, the rule $x_n = ax_{n-1}$ mod b gives the numbers 5, 11, 6, 8, 2, 7, 5, 11, 6, 8, …

Try using different combinations of x_1, a and b to generate pseudo-random sequences.

What do you find about the distance between repeats?

What difference does the size of b make?

Is it better to use prime numbers?

Exercise 11.02 Choosing samples

Example 3

1 Start at row 8, column 17 of the 2-digit random number table on page 423 and select 8 different numbers that are:

- **a** 2-digit numbers between 20 and 99
- **b** 6-digit numbers between 100 000 and 400 000
- **c** 1-digit numbers
- **d** 2-digit numbers between 30 and 84.

Example 4

2 Use stratified sampling to state how many of each group should be chosen from the following to obtain the specified sample.

- **a** A sample of 10 swimsuit-wearers from 15 men in board shorts, 10 men in briefs, 25 women in bikinis and 7 women in one-pieces
- **b** A sample of 20 chocolates from 180 soft-centred, 140 hard-centred, 85 liquid-centred and 108 nutty-centred chocolates
- **c** A sample of 16 from 5 fifteen-year-olds, 35 sixteen-year-olds, 10 seventeen-year-olds and 3 eighteen-year-olds
- **d** Sample of 10 from 120 Year 8s, 150 Year 9s, 130 Year 10s, 80 Year 11s and 90 Year 12s

Example 5

3 Use systematic sampling to select a sample of 4 pies from a batch of 50, starting at number 17.

iStock/ Getty Images Plus/ firc83

ISBN 9780170412926

4 Use systematic sampling to select 10 students from the enrolments at a school. Start from enrolment number 5130 of the current enrolments, which run from 4928 to 5672 inclusive.

5 A quality control department decides to check a random sample of 10 items from the 210 items produced in one day. Use the table of random numbers on page 422 to select 10 items (start at row 3, column 4).

6 Draper St, Cairns has numbers that go from 1 to 471. Use systematic sampling to select 12 locations in Draper St, starting from number 308.

7 Biotech Industries Pty Ltd wishes to form a staff social committee consisting of 18 members. The firm decide to use the method of stratified random sampling for selecting the committee members. The employment details of the firm are given in the table.

	Administrative staff	Factory workers
Males	11	73
Females	24	52

a How many from each group should be selected to represent all groups fairly?
b How many from each group should be selected if no distinction is made between males and females?

8 Use your graphics calculator to choose the following samples.
a 5 of the numbers from 100 to 199 inclusive
b 8 from IDs that start at 1695 and go up to 3725 inclusive
c 10 from cards that are numbered 1 to 100
d 7 from 600 to 5000 inclusive

Problem solving

9 State the kind of sampling that is used in each of the following and some advantages and disadvantages of the method.
a Standing in the Queen St Mall next to an ice cream stall and asking passers-by which kind of ice cream they prefer until 20 men and 20 women have replied.
b Asking people in your neighbourhood how many people live in their house.
c Choosing 8 streets at random from the Brisbane area and asking each of the people who live in the street their opinion about the State Government.
d Sending surveys to 500 people asking for their opinions and to return the survey in a pre-paid envelope.
e Cara decided to survey people in Year 12 about their ambitions when they left school at the end of the year. She chose a sample of 12 people; 4 who did Essential Maths, 4 who did General Maths and 4 who did Maths Methods.

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10 Many research companies use the electoral rolls (available from the Australian Electoral Commission) to choose samples of people to be surveyed. The electoral roll for a Federal electorate normally has about 85 000 people on the roll and includes their addresses. There are many pages in each electoral roll. Suggest three different ways in which the electoral rolls could be used to select samples, and give at least one advantage and disadvantage of each method.

11 The table below shows the Australian population in 2017. Use stratified random sampling to determine how many should be chosen from each state to make a sample of 500:

a if persons are selected regardless of sex

b if males and females are selected in proportion.

Estimated Australian population, March 2017

State	Males	Females	Persons
New South Wales	3 883 861	3 953 848	7 837 709
Victoria	3 112 515	3 178 159	6 290 674
Queensland	2 431 513	2 476 095	4 907 608
South Australia	851 469	869 484	1 720 953
Western Australia	1 292 949	1 283 021	2 575 970
Tasmania	256 920	263 181	520 101
Northern Territory	125 665	119 325	244 990
Australian Capital Territory	203 355	205 786	409 141
Australia	**12 160 752**	**12 351 021**	**24 511 773**

Source: ABS

12 A TV travel show ran an SMS poll to choose whether one of the presenters would visit Bali, Singapore or Nepal in an upcoming episode. The poll was conducted the week after an episode that featured Phuket. What kind of survey is this and what advantages and disadvantages does it have?

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11.03 Sample proportions

Sample proportions

New drugs eventually have human trials. Some people with the disease are given the drug. The control group also have the disease but are given a placebo, which has no drug in it. Doctors measure the effectiveness by the number of people who show improvement. They use the ratio of those showing improvement to the size of the sample. This is much more important than the actual numbers. An effective drug will have a much greater effect than the placebo. They use 'double blind' trials to remove some kinds of bias. These ratios are called **sample proportions**.

Sample proportions

- A **sample proportion** is the number of times a property (characteristic) occurs in a sample, divided by the sample size. The symbol is $\hat{p}$
- An individual occurrence of the property is usually called a **success**
- x successes in a sample of n gives a sample proportion $\hat{p} = \frac{x}{n}$.

EXAMPLE 6

24 out of the 88 students doing Maths Methods in an inner-Brisbane school also do Physics. What is the sample proportion of Maths Methods students doing Physics, correct to 3 decimal places?

Solution

Success is doing Physics. Write the formula.

$$\hat{p} = \frac{x}{n}$$

Substitute values and evaluate.

$$= \frac{24}{88}$$
$$= 0.2727\ldots$$

Round and write the answer.

The sample proportion is about 0.273.

The probability of success in a population should be close to the sample proportion. You would expect that it would be closer for larger sample sizes. Obviously, when the sample is the whole population, it must be equal to the probability of success.

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Sample proportion and population

The sample proportion of a property is an **estimator** of the probability p of the property occurring in the population.

EXAMPLE 7

One week, 9 patients of a dentist in Toowoomba had full or partial dentures. He had 38 patients that week.

a Estimate the probability of Queenslanders having at least some false teeth.

b Comment on the estimate.

Solution

a Success is having dentures.

$$\hat{p} = \frac{x}{n}$$

Substitute values in the formula and evaluate.

$$= \frac{9}{38}$$
$$= 0.2368\ldots$$

Write the answer.

The probability of Queenslanders having at least some false teeth is about 0.237.

b There are a lot of retired people living in Toowoomba.

The residents of Toowoomba are not typical of the Queensland population, and this dentist may not treat the full range of patients.

Exercise 11.03 Sample proportions

Example 6

1 79 out of 150 Year 12 students at a suburban Brisbane school had dark hair. What is the sample proportion of dark hair?

2 A coin was tossed 28 times and heads came up 20 times. What is the sample proportion of heads?

3 A new car dealer sold 210 new cars. 89 of them were white. What is the sample proportion of white cars?

Example 7

4 The Cooktown Holiday Park was almost fully occupied in May. 47 of the 65 motor homes and caravans in the park had drivers over 55 years old.

a Estimate the probability of the driver of a motorhome or caravan being over 55.

b Comment on your estimate.

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Alamy Stock Photo/Genevieve Vallee

5 From a sample of 40 Australian Year 12 students, 9 were found to have heights of 180 cm or greater. Only one of the tall students was female.

- **a** Estimate the probability of an Australian Year 12 student being 180 cm tall or greater.
- **b** 19 of the 40 students were male. Estimate the probability of a male Australian Year 12 student being 180+ cm.
- **c** Estimate the probability of a female Australian Year 12 student being 180+ cm.

6 28 of the 45 cars for sale in a second-hand car yard were priced over $10 000.

- **a** Estimate the probability of second-hand cars costing over $10 000.
- **b** Comment on the estimate.

Problem solving

7 From 47 children tested in Mt Isa, 5 were found to have high levels of lead in their blood. Estimate the probability of Queensland children having high levels of lead in their blood. Can you suggest any problems with this estimate?

8 18 out of 70 Year 12 students at Miami State High School had been to at least one other high school. Estimate the probability of students changing high schools before the end of Year 12 and discuss any cautions with your estimate.

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11.04 Variation of samples

WS
Cabbage moths

INVESTIGATION

CABBAGE MOTH SAMPLES

Colin has a patch of cabbages, but the patch has attracted cabbage moths. Models of the moths are available on the website. Work in groups of about 4. Your teacher will give you some paper models of the cabbage moths to measure.

Use a sample of 10 to find:

- the smallest wing span
- the largest wing span
- the mean wing span

Compare the findings of different groups.

- What is the highest mean found?
- What is lowest mean found?
- What can you say about the sample mean?

Use samples of 20 to find the sample proportion of green-coloured moths.

Compare the findings of different groups.

- What is the highest sample proportion found?
- What is lowest sample proportion found?
- What can you say about sample proportions?

The investigation above shows you that different samples of the same population give different results. Samples of 2 items from the population {3, 5, 6, 7, 8, 11} include {3, 5}, {6, 8} and {3, 11}. The means and spreads are quite different. The sample proportion of prime numbers varies from 0 to 1. The sample size obviously makes a difference because if you take samples of 5 from this population, the mean and the sample proportion of primes have much less variation.

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EXAMPLE 8

Two random samples were taken from a Bernoulli distribution with probability $p = 0.42$. Each sample has 20 values. The samples are shown below.

A: 1, 0, 0, 1, 0, 0, 1, 1, 0, 0, 1, 1, 0, 0, 0, 1, 1, 0, 1, 1

B: 1, 1, 0, 1, 0, 0, 1, 1, 1, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0

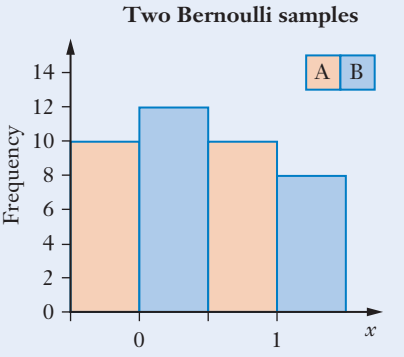
a Show the samples as a side-by-side column graph.

b Calculate the means and standard deviations.

c Compare the samples.

Solution

a Draw the graphs with different colours for each sample.



b Calculate the mean and standard deviation of each sample using the normal formulas.

$\mu_A = 0.5$, $\sigma_A = 0.5$

$\mu_B = 0.4$, $\sigma_B \approx 0.49$

Compare the samples.

Sample A has a higher mean than sample B, and this is evident on the graph, but their spreads are similar, as shown on the graph and by their standard deviations.

You can use your graphics calculator to generate and graph samples from Bernoulli, binomial and normal distributions. Although not truly random, they are good enough to use.

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EXAMPLE 9

Use your graphics calculator to generate 3 random samples of 35 of the binomial distribution with $n = 10$ and $p = 0.65$. Store the samples in lists 1, 2 and 3 and draw histograms. What do you find?

Solution

TI-84 Plus CE

Press [math], choose PROB and 7: randBin.

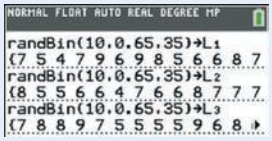
Put in n: 10, p: 0.65, repetitions: 35 [enter] and store the sample list in L1 using [sto→] [2nd] [1] [enter]. It takes a little while to generate the sample.

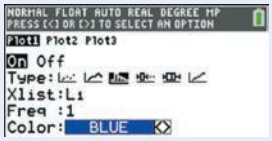
Repeat twice but store samples in L2 and L3.

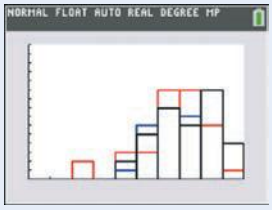
Press [2nd] [y=] (statplot), choose Plot1 and set it On with the histogram and XList L_1.

Press [2nd] [mode] (quit) and repeat but set Plot2 and Plot3 to L_2 and L_3.

Set the [window] to $0 \le x \le 10$ and $0 \le y \le 10$, clear [y=] and press [graph] to see the histograms on the same screen. You see them separately by setting only one plot on at a time.







Casio fx-CG20AU

Choose the Run-Matrix menu and press [OPTN]. Choose PROB, RAND and Bin and put in n, the probability and the number in the sample. Close the brackets and store in List 1 using [→] [SHIFT] [1] 1. Press [EXIT].

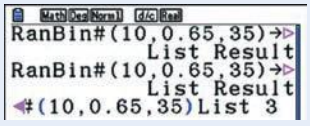
Repeat twice but store samples in List 2 and List 3. You may have to put some dummy data in the lists before you start.

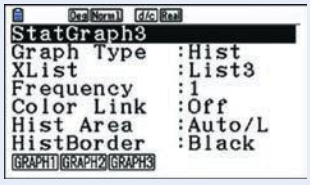
Press [MENU] and choose the Statistics menu.

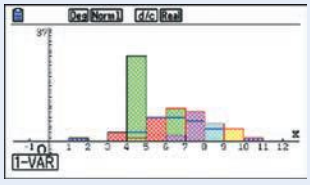
Choose GRAPH, SET and GRAPH1. Choose Graph Type: Hist, XList: List 1, Hist Area: 9: Auto and HistBorder: 2:Blue.

Repeat with GRAPH2 and GRAPH3 but set them to List 2 and List 3 with HistBorder Red and black respectively.

Press [EXIT], choose SELECT, set all three StatGraphs On and choose DRAW to see the graphs. Choose the start as 0 and the Width as 1 and press [EXE]. You can see the graphs separately using [EXIT] and choosing GRAPH1, GRAPH2 or GRAPH3 and pressing [EXE].







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Both calculators

Comment on the graphs.

There is a lot of variation between the samples.

You can use your calculator to simulate sample proportions using a binomial distribution with $n = 1$. {1, 0, 1, 1, 0, 1, 1, 0} is a random sample of 8 from a Bernoulli distribution. There are 5 successes out of 8, so the corresponding sample proportion is $\frac{5}{8} = 0.625$. This is the mean of 1, 0, 1, 1, 0, 1, 1, 0.

EXAMPLE 10

Use your graphics calculator to generate 3 sample proportions of samples of 20 from a population whose probability of having a particular property is 0.37.

Solution

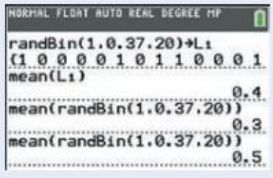
TI-84 Plus CE

Press [math], choose PROB and 7: randBin.

Put in n: 1, p: 0.37, repetitions: 20 [enter] and store the sample list in L1.

Use list operations to find the mean of L1; press [2nd][stat], choose MATH, 3: mean(and put in L_1.

Repeat twice. You can find the mean of the list immediately instead of storing the list.



Write the answer.

When you follow these instructions, your sample proportions will probably be different because the process is random.

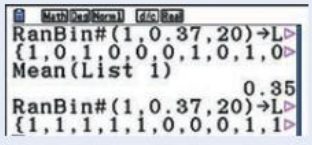
The sample proportions are 0.4, 0.3 and 0.5.

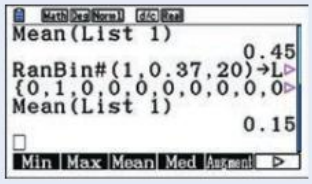
Casio fx-CG20AU

Choose the Run-Matrix menu and press [OPTN]. Choose PROB, RAND and Bin, put in 1, 0.37, 20, close the brackets and store in List 1.

Use list operations to find the mean of List 1; press [OPTN], choose LIST, Mean and put in List 1.

Repeat twice.





Write the answer.

The sample proportions are 0.35, 0.45 and 0.15.

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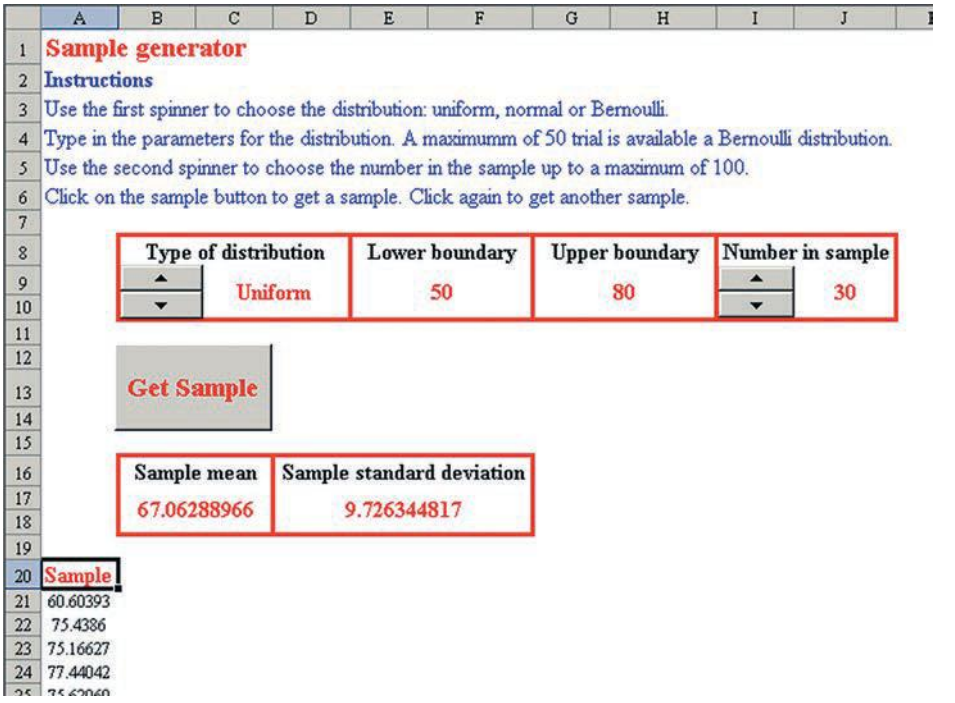
INVESTIGATION

SAMPLE VARIATION

Sample generator

Download the spreadsheet **Sample generator** from NelsonNet to generate samples from uniform, normal, or Bernoulli distributions to compare samples. The spreadsheet also calculates the mean and standard deviation of the sample.



	A	B	C	D	E	F	G	H	I	J
1	**Sample generator**									
2	**Instructions**									
3	Use the first spinner to choose the distribution: uniform, normal or Bernoulli.									
4	Type in the parameters for the distribution. A maximumm of 50 trial is available a Bernoulli distribution.									
5	Use the second spinner to choose the number in the sample up to a maximum of 100.									
6	Click on the sample button to get a sample. Click again to get another sample.									
7										
8		**Type of distribution**			**Lower boundary**		**Upper boundary**		**Number in sample**	
9										
10			Uniform		50		80		30	
11										
12										
13		Get Sample								
14										
15										
16		**Sample mean**			**Sample standard deviation**					
17										
18		67.06288966			9.726344817					
19										
20	Sample									
21	60.60393									
22	75.4386									
23	75.16627									
24	77.44042									

Exercise 11.04 Variation of samples

Example 8

1 Two random samples of 20 values were taken from a Bernoulli distribution with $p = 0.6$.

A: 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 1, 1, 1, 0, 1, 0, 1, 1, 1

B: 1, 0, 0, 1, 0, 1, 1, 0, 1, 1, 0, 0, 1, 1, 1, 0, 1, 0, 0, 1

a Show the samples as a side-by-side column graph.

b Compare the samples.

2 Two random samples of 20 were taken from a binomial distribution with $n = 8$ and $p = 0.2$.

A: 2, 2, 0, 1, 1, 2, 0, 3, 2, 1, 0, 2, 3, 2, 0, 3, 0, 0, 2, 1

B: 1, 2, 0, 3, 2, 2, 3, 1, 3, 2, 0, 2, 1, 0, 4, 2, 1, 1, 2, 1

a Show the samples as side-by-side histograms.

b Compare the samples.

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3 Two random samples were taken from a normal distribution with mean 50 and standard deviation 8. Each sample has 25 values.

A: 50.4, 62.3, 49, 45.3, 59.1, 45.1, 50.2, 49.7, 40, 41.6, 38.5, 46.6, 47.5, 47.1, 63.2, 55.4, 43, 52.1, 53.3, 54.9, 58.4, 34.2, 54.4, 37.1, 56.9

B: 37.3, 49.3, 35.9, 45.7, 53.5, 40.2, 46.7, 44.8, 52.8, 41.7, 59.4, 48.7, 50.2, 39.4, 50.3, 41.6, 43.4, 41.9, 46.1, 47.8, 45.1, 53.3, 44.9, 44.5, 50.6

a Show the samples as side-by-side histograms with class widths of 5.
b Compare the samples.

4 Generate 2 random samples of 25 items from a binomial distribution with $n = 30$ trials and probability $p = 0.7$ and draw histograms of the samples. Example 9
a Find the mean and standard deviations of the samples.
b Compare the samples.

5 **a** Generate 2 sample proportions of samples of 40 from a population whose probability of having a particular property is 0.65. Example 10
b Compare the sample proportions.

6 **a** Generate 2 random samples of 9 items from a binomial distribution with $n = 15$ trials and probability $p = 0.6$ and find the mean and standard deviations of the samples.
b Generate 2 random samples of 64 items from a binomial distribution with $n = 15$ trials and probability $p = 0.6$ and find the mean and standard deviations of the samples.
c Comment on the variation of the means and standard deviations for samples of 9 and samples of 64 items.

7 **a** Generate 2 random samples of 9 items from a normal distribution with mean $\mu = 40$ and standard deviation $\sigma = 8$ and find the mean and standard deviations of the samples.
b Generate 2 random samples of 81 items from a normal distribution with mean $\mu = 40$ and standard deviation $\sigma = 8$ and find the mean and standard deviations of the samples.
c Comment on the variation of the means and standard deviations for samples of 9 and samples of 81 items.

8 **a** Generate 10 sample proportions of samples of 30 from a population whose probability of having a particular property is 0.3.
b Generate 10 sample proportions of samples of 80 from a population whose probability of having a particular property is 0.3.
c Comment on the variation of the sample proportions for samples of 30 and samples of 80 items.

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11.05 Sample size and normality

WS
Sample proportion calculations

What happens to statistics as the sample size increases? You can look at the shape of the distribution of a statistic as the size increases. The distribution of a statistic for a large number of samples is called a **sampling distribution**.

Collecting very large data sets to examine sampling distributions is very time-consuming, so you will use a simulation. You will use the binary distribution and divide by n to simulate sample proportions of a Bernoulli variable.

EXAMPLE 11

Simulate histograms of the sample proportions of clubs for 50 samples of cards cut from well-shuffled decks of cards. Perform the simulations with your graphics calculator for samples of:

a 10 **b** 30 **c** 100

Solution

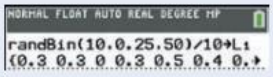
TI-84 Plus CE

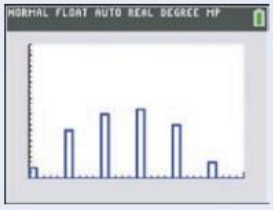
TI-Nspire CX Chapter 11

a Press [math], choose PROB, 7: randBin(and complete with 10, 0.25,50) ÷ 10 and store in L1.

Use [stat plot] to set Plot 1 On, Type ⊥⊓⊔, XList L_1, Freq: 1.

Set the [window] to $0 \le x \le 0.6$, $0 \le y \le 25$, Xscl 0.02, Yscl 1 and press [graph] to see it.

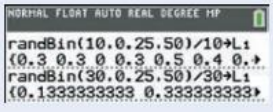


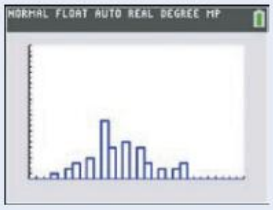


b [quit] the graph.

Use randBin(again but choose a sample size of 30 and divide by 30. Store again in L1.

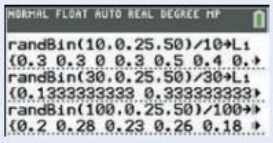
Leave the window and graphs settings as they were and look at the new graph.





c [quit] the graph.

Use randBin(again but choose a sample size of 100 and divide by 100. Store again in L1.



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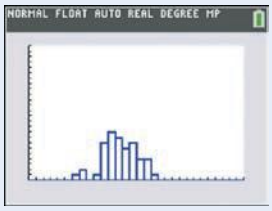
It takes a while to do these calculations.

Leave the window and graphs settings as they were and look at the new graph.



Casio fx-CG20AU

a Use the Run-Matrix menu and press OPTN.

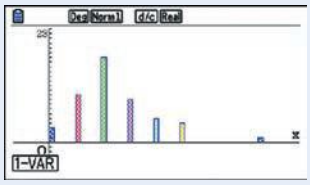
Choose PROB, RAND and Bin. Complete with 10,0.25,50) ÷ 10 and store in List 1.

EXIT, press MENU and choose Statistics.

Choose GRAPH, SET and GRAPH1.
Choose Graph Type: Hist, XList: List 1.

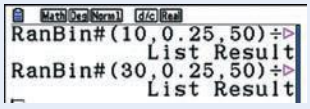
EXIT and use select to check that StatGraph2 and StatGraph3 are Off.

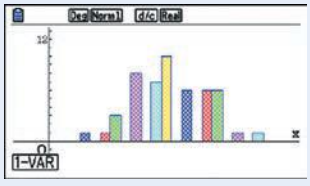
EXIT, make Start 0 and Width 0.02 and press EXE to see the graph.



b Press MENU and go back to the Run-Matrix menu. Use RanBin# again, but make the sample size 30 and divide by 30. Store again in List 1.

Go back to the Statistics menu, leave the settings the same and graph List 1 again.

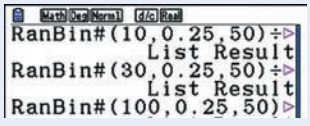


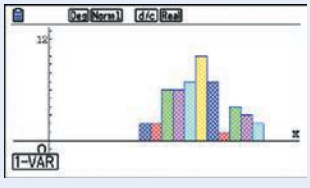


c Go back to the Run-Matrix menu and use RanBin# again to do the samples of 100.

It takes a while to do these calculations.

Go back to the Statistics menu, leave the settings the same and graph List 1 again.





From Example 11 you can see that as the number of samples increases, the sampling distribution of sample proportions becomes more like a normal distribution. It also becomes more compressed horizontally as the standard deviation decreases.

What effect does the value of p have on the sampling distribution?

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EXAMPLE 12

Simulate histograms of the sample proportions of clubs for 50 samples of 30 cards cut from well-shuffled decks of cards. Perform the simulations with your graphics calculator for samples with:

a $p = 0.005$ **b** $p = 0.1$ **c** 0.5.

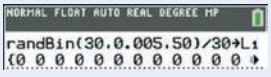
Solution

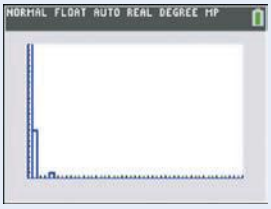
TI-84 Plus CE

a Press [math], choose PROB, 7: randBin(and complete with 30, 0.05,50) ÷30 and store in L1.

Use [stat plot] to set Plot 1 On, Type ⊥⊓⊔, XList L_1, Freq: 1.

Set the p to $0 \le x \le 1$, $0 \le y \le 25$, Xscl 0.02, Yscl 1 and press [graph] to see it.

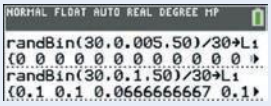


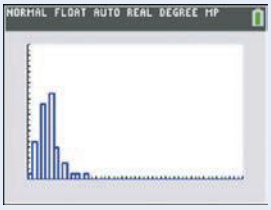


b [quit] the graph.

Use randBin(again but choose $p = 0.1$. Store again in L1.

Leave the window and graphs settings as they were and look at the new graph.

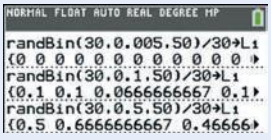


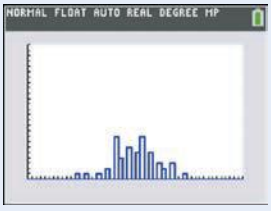


c [quit] the graph.

Use randBin(again but choose $p = 0.1$. Store again in L1.

Leave the window and graphs settings as they were and look at the new graph.





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a Use the Run-Matrix menu and press OPTN.

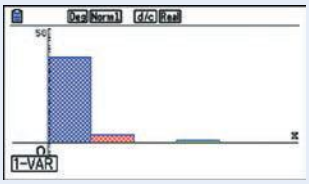
Choose PROB, RAND and Bin. Complete with 30,0.005,50) ÷ 30 and store in List 1.

EXIT, press MENU and choose Statistics.

Choose GRAPH, SET and GRAPH1.
Choose Graph Type: Hist, XList: List 1.

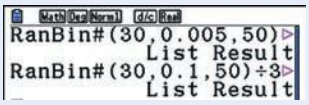
EXIT and use select to check that StatGraph2 and StatGraph3 are Off.

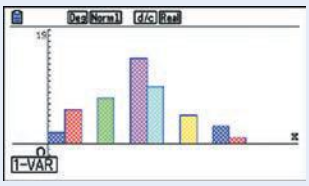
EXIT, make Start 0 and Width 0.02 and press EXE to see the graph.



b Press MENU and go back to the Run-Matrix menu. Use RanBin# again, but choose $p = 0.1$. Store again in List 1.

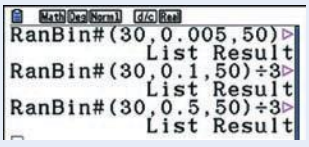
Go back to the Statistics menu, leave the settings the same and look at Graph1 again.

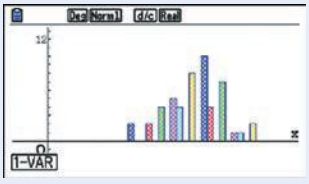




c Press MENU and go back to the Run-Matrix menu. Use RanBin# again, but choose $p = 0.5$. Store again in List 1.

Go back to the Statistics menu, leave the settings the same and look at Graph1 again.





From Example 11 you can see that as p gets closer to 0.5, the sampling distribution of sample proportions becomes more like a normal distribution.

In fact, it has been proven that the sampling distribution of *any* mean is approximately normal if the samples are large enough. This is called the **central limit theorem**. Its proof is beyond the scope of this course.

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The closeness of the approximation depends on n and p. The closer p is to 0.5, the more symmetrical the distribution becomes, so the closer it is to normal.

A card is cut from a normal pack and checked to see whether or not it is a club. The card is replaced and the procedure is repeated 20 times. What are the expected value, variance and standard deviation of the sample proportion?

The number of clubs x is a binomial distribution with expected value $np = 20 \times 0.25 = 5$ and standard deviation $\sqrt{np(1-p)} = \sqrt{20 \times 0.25 \times 0.75} = \sqrt{3.75}$.

To calculate the mean and standard deviation of the sample proportion, every value of x is divided by 20, so the mean and standard deviation would also be divided by 20. Hence the expected value and standard deviation of the sample proportion would be $\frac{5}{20} = 0.25 = p$ and

$$\frac{\sqrt{3.75}}{20} = \sqrt{\frac{3.75}{20^2}} = \sqrt{\frac{20 \times 0.25 \times 0.75}{20^2}} = \sqrt{\frac{0.25 \times 0.75}{20}} = \sqrt{\frac{p(1-p)}{n}}.$$

Sample proportion parameters

For samples that are small compared to the population ($np > 5$ and $nq = n(1-p) > 5$) of a Bernoulli distribution:

- $E(\hat{p}) = p$
- $\text{Var}(\hat{p}) = \frac{p(1-p)}{n}$
- $\text{SD}(\hat{p}) = \sqrt{\frac{p(1-p)}{n}}$

where p is the probability and n is the sample size.

For samples that are small compared to the population, the probability will be the same as each member of the sample is chosen. There are only 2 outcomes, so it is a binomial distribution. The binomial distribution of x has parameters $E(X) = np$ and $\text{Var}(X) = np(1-p)$.

You should remember that expected value and variance are defined as $E(X) = \sum xp(x)$ and $\text{Var}(X) = \sum (x-\mu)^2 p(x)$. The sample proportion is given by $\hat{p} = \frac{x}{n}$. This gives:

$$E(\hat{p}) = \sum \frac{x}{n} p(x) = \frac{1}{n} \sum xp(x) = \frac{1}{n} E(X) = \frac{1}{n} \times np = p$$

$$\text{Var}(\hat{p}) = \sum \left(\frac{x}{n} - p \right)^2 p(x) = \sum \left(\frac{x}{n} - \frac{np}{n} \right)^2 p(x) = \frac{1}{n^2} \sum (x-np)^2 p(x) = \frac{1}{n^2} \text{Var}(X)$$

Hence $\text{Var}(\hat{p}) = \frac{1}{n^2} \times np(1-p) = \frac{p(1-p)}{n}$.

Now $\text{SD}(X) = \sqrt{\text{Var}(X)}$, so $\text{SD}(\hat{p}) = \sqrt{\frac{p(1-p)}{n}}$ QED

Cutting cards from a well-shuffled pack is one instance where your experience tells you the population probability. In cases like this, you can work out the sample proportion mean, variance and standard deviation.

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You use the test $np > 5$ and $nq > 5$ to check whether the distribution of $\hat{p}$ is 'close enough' to a normal distribution.

iStock/Getty Images Plus/xril

Mean and variance of sample proportions

EXAMPLE 13

One card is selected from each of 100 well-shuffled packs and the number of aces is noted. This is repeated many times. What is the expected value and variance of the sample proportion of aces? How accurate are the values?

Solution

Write the parameters.

$p = \frac{1}{13}$ and $n = 100$

Find the expected value.

$E(\hat{p}) = p = \frac{1}{13}$

Find the variance.

$$\text{Var}(\hat{p}) = \frac{p(1-p)}{n}$$
$$= \frac{\frac{1}{13} \times \frac{12}{13}}{100}$$
$$= \frac{12}{16\,900}$$
$$= \frac{3}{4225}$$

Check the values of np and nq.

$np = 100 \times \frac{1}{13} \approx 7.7 > 5$

$nq = 100 \times \frac{12}{13} \approx 92 > 5$

Write the answer.

The mean and variance of the sample proportion of aces are $\frac{1}{13}$ and $\frac{3}{4225}$ respectively.

Since $np, nq > 5$, they are likely to be fairly accurate.

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Exercise 11.05 Sample size and normality

Example 11

1 Simulate 100 samples of sample proportions with each pair of values given and draw a histogram of the sampling distribution.

a $p = 0.2$ and $n = 20$
b $p = 0.2$ and $n = 50$
c $p = 0.2$ and $n = 100$
d Compare the sampling distributions.

2 Simulate 100 samples of sample proportions with each pair of values given and draw a histogram of the sampling distribution.

a $p = 0.9$ and $n = 20$
b $p = 0.9$ and $n = 50$
c $p = 0.9$ and $n = 100$
d Compare the sampling distributions

Example 12

3 Simulate 100 samples of sample proportions with each pair of values given and draw a histogram of the sampling distribution.

a $p = 0.9$ and $n = 30$
b $p = 0.7$ and $n = 30$
c $p = 0.5$ and $n = 30$

4 Simulate 100 samples of sample proportions with each pair of values given and draw a histogram of the sampling distribution.

a $p = 0.1$ and $n = 50$
b $p = 0.3$ and $n = 50$
c $p = 0.5$ and $n = 50$

Example 13

5 Find the mean, variance and standard deviations of sample proportions with:

a $p = 0.2$ and $n = 50$
b $p = 0.7$ and $n = 25$
c $p = 0.81$ and $n = 120$
d $p = 0.22$ and $n = 80$

Problem solving

6 A normal die is tossed. Samples of 45 tosses are taken and in each case the proportion of times that the number is less than 3 is calculated. What is the mean proportion and standard deviation of the sample proportion when this experiment is repeated multiple times?

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7 45% of Canadian high school students catch a bus to get to school. Samples of 200 students from high schools across Canada are questioned to determine the proportion travelling by bus to school. What is the expected proportion and standard deviation of the sample proportion?

8 A normal pair of dice are thrown and the total is noted. Samples of 30 such throws are performed and the proportion of times the total is more than 9 is calculated for each sample. What is the mean and standard deviation of the sample proportions for totals more than 9?

9 A card is cut from a well-shuffled deck. This is done 100 times and the number of times that a picture card (J, K, Q) appears is noted in each case. When this is repeated many times, what is the mean and standard deviation of the sample proportions for picture cards?

10 A weighted coin was tossed 25 times and the proportion of heads was noted. This was done a total of 40 times and the mean number of heads was found to be 15.02, and the standard deviation of the sample proportion of heads was 0.1. What was the probability of heads for this coin?

11.06 Interval estimates of sample proportions

WS Calculations with sample proportions

WS Simple interval estimates

You know from Chapter 10 how to change a normal distribution to a standard normal distribution using z-scores. Remember that if the mean and standard deviation of a distribution are μ and σ, then the z-score of x is $z = \frac{x - \mu}{\sigma}$. For np, $nq > 5$, the sample proportion of samples of n from a Bernoulli distribution with probability p is approximately normal. The z-scores $\frac{\hat{p} - p}{\text{SD}(\hat{p})}$ of the sample proportion distribution will approximate a standard normal distribution.

Distribution of z-scores of sample proportions

For samples of n from a Bernoulli variable X with probability p,

$\frac{\hat{p} - p}{\sqrt{p(p-1)/n}}$ approximates a standard normal distribution.

The approximation improves as $p \to 0.5$ and $n \to \infty$.

Remember that a Bernoulli variable is one that has only 2 outcomes, success and failure. These are usually the presence or absence of some property. You can use the standard normality of the z-scores of $\hat{p}$ to calculate the probability values of $\hat{p}$. The probability that $a < \hat{p} < b$ is the same as the proportion of values between a and b.

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EXAMPLE 14

What is the probability that $\hat{p}$ is between 0.45 and 0.6 for $n = 60$ and $p = 0.4$?

Solution

Check the values of np and nq.

$np = 60 \times 0.4 = 24$

$nq = 60 \times 0.6 = 36$

State the conclusion.

$np, nq > 5$ so the normal distribution can be used.

Find the mean and SD.

$$E(\hat{p}) = p = 0.4$$

$$\text{SD}(\hat{p}) = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.4 \times 0.6}{60}} \approx 0.063$$

Find the standard values (z-scores).

$$z\text{-score of } 0.45 = \frac{x - \mu}{\sigma} \approx \frac{0.45 - 0.4}{0.063} \approx 0.79$$

$$z\text{-score of } 0.6 = \frac{x - \mu}{\sigma} \approx \frac{0.6 - 0.4}{0.063} \approx 3.17$$

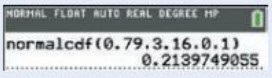
Find the proportion of the standard normal distribution.

TI-Nspire CX Chapter 11

TI-84 Plus CE

Press 2nd vars to get the [distr] menu.

Choose 2: normalcdf and complete with lower: 0.79 and upper: 3.16. Leave μ: 0 and σ: 1 for the standard normal distribution and press enter.



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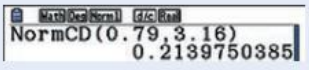
Use the Run-Matrix menu and press OPTN.

Choose STAT, DIST, NORM, Ncd and complete with 0.79, 3.16) EXE.



Write the answer.

The probability is about 0.21.

Sample proportion is an *estimate* of the probability of a property occurring in a population. Since $E(\hat{p}) = p$, it is the *best estimate*. This is an example of a **point estimate**.

You could estimate that there were 3000 people on the beach at Coolangatta. Alternatively, you could say there were between 2600 and 3400 people on the beach.

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Estimates

- A **point estimate** is a single value
- An **interval estimate** is an interval that is likely to include the required value

Since $\hat{p}$ is the best estimate of p, and you will know the size of a sample, you can use these to estimate the probability and variance of $\hat{p}$. You simply replace p by $\hat{p}$ in the formulas.

Estimated variance and standard deviation of a sample proportion

The estimated variance and standard deviation of a sample proportion $\hat{p}$ are given by

- $\text{Var}(\hat{p}) \approx \dfrac{\hat{p}(1-\hat{p})}{n}$
- $\text{SD}(\hat{p}) \approx \sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$, where n is the sample size

EXAMPLE 15

From a sample of 80 surfers, 15 said they preferred to use longboards. Estimate the probability and variance of the sampling distribution of surfers preferring longboards.

Solution

Use the sample proportion to estimate p.

$p \approx \hat{p} = \dfrac{15}{80} = \dfrac{3}{16}$ or 0.1875

Write the formula for the estimated variance.

$\text{Var}(\hat{p}) \approx \sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$

Substitute and calculate the variance.

$= \sqrt{\dfrac{\frac{3}{16} \times \frac{13}{16}}{80}}$

≈ 0.044

You should give the answer to the same accuracy as the data: 2 significant figures.

Write the answer.

The probability of surfers preferring longboards is about 0.19 and the variance of the sampling distribution is about 0.044.

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Shutterstock.com/Nina Mikryukova

You can also work out an interval estimate from a sample proportion.

EXAMPLE 16

A survey of 1982 adults in Queensland found that 957 of them had completed Year 12. Find the probability that a randomly selected adult Queenslander has completed year 12, within 1 standard deviation.

Solution

Estimate the probability.

$$\hat{p} = \frac{957}{1982} = 0.4828\ldots$$

Estimate the standard deviation.

$$\begin{aligned} \text{SD}(\hat{p}) &\approx \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \\ &= \sqrt{\frac{0.4828\ldots \times 0.5171\ldots}{1982}} \\ &= 0.011\ 22\ldots \end{aligned}$$

Find the interval.

$$\begin{aligned} \hat{p} &\approx 0.4828\ldots \pm 0.011\ 22\ldots \\ &= 0.4716\ldots \text{ to } 0.4940 \end{aligned}$$

Round appropriately and write the answer.

The probability that a randomly selected adult Queenslander has completed year 12 is between 0.472 and 0.494.

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Exercise 11.06 Interval estimates of sample proportions

1 What percentage of values of $\hat{p}$ would lie between 0.5 and 0.6 for samples with $n = 30$ and $p = 0.7$?

Example 14

2 What proportion of values of $\hat{p}$ would lie between 0.6 and 0.7 for samples with $n = 300$ and $p = 0.72$?

3 What is the probability that $\hat{p}$ would lie between 0.48 and 0.52 for samples with $n = 45$ and $p = 0.25$?

4 State whether each sentence describes a point or interval estimate.

a The mean income is about $47 000.

b The median house price in suburbs is between $350 000 and $720 000.

c The variance is between 800 and 1500.

d The probability of dark hair is about 0.6.

5 From 80 people interviewed at Roma St Station, 57 said they could use chopsticks. Estimate the probability that Queensland residents can use chopsticks and the variance and standard deviation of the sampling distribution.

Example 15

6 From 40 tosses, a coin lands with heads up 18 times. Estimate the probability of the coin landing with heads up and the variance and standard deviation of the sampling distribution.

7 The protractors of 50 students were checked for accuracy, and 16 were found to be inaccurate in measuring an angle of 60° (more than 1° out). Estimate the probability of student's protractors being inaccurate and the variance and standard deviation of the sampling distribution.

8 From a group of 50 computer professionals, 32 said they used the Linux operating system on their home computers. Estimate the probability of a computer professional using Linux on their home computer to within one standard deviation.

Example 16

9 From a class of 28 Year 12 students, only 19 had completed all their homework the previous night. Use this information to estimate the probability of a Year 12 student completing all their homework to within 1.5 standard deviations.

10 A group of 32 parents waiting for interviews at a high school Parent-Teacher evening were discussing the habits of teenagers and 20 of them agreed that 'teenagers take too long in the bathroom'. What is the probability that parents of high school students think they take too long in the bathroom, to within 1.6 standard deviations?

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Problem solving

11 The probability of an adult having fair hair is 0.21. What is the probability that a sample of 400 adults has between 70 and 75 with fair hair?

12 About 20% of Australians are Queenslanders and of these 49% live in Brisbane.
- **a** From 300 Australians, what is the probability that from 50 to 60 are Queenslanders?
- **b** From 300 Queenslanders, what is the probability that between 200 and 250 live in Brisbane?

13 About 6.5% of Brisbane high school students travel to school by train. What is the probability that between 50 and 60 students from a random sample of 1000 Brisbane high school students travel to school by train?

14 A poll of 500 voters found that 44% supported a particular political party. What is the standard deviation of the sampling distribution?

15 A sample showed 30% support for a proposal and the estimated variance of the sampling distribution was 0.005. How many people were surveyed?

WS

Margins of error for standard normal variables

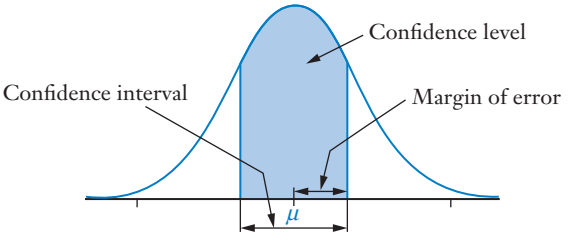
11.07 Confidence intervals and the standard normal distribution

If you choose an interval of a normal distribution from 2 standard deviations below the mean to 2 standard deviations above the mean, then about 95% of the values will be in the interval. The standard normal distribution has the mean $\mu = 0$ and standard deviation $\sigma = 1$. This means that for the standard normal variable Z, $P(-2 \leq z \leq 2) \approx 0.95$. You say that the 95% **confidence level** for the standard normal variable has a **margin of error** of ±2.

Confidence intervals

- A **confidence interval** contains a specific proportion of the values of a random variable. It is usually symmetric about the mean.
- The **confidence level** is the proportion of values that lie within the interval
- The **margin of error** is the distance of the ends of the interval from the mean

For a normal distribution, you can picture this as shown below.



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The confidence level is the area under the curve, which is both the proportion of values and the probability of lying within the interval.

EXAMPLE 17

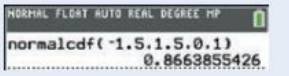
What confidence level is produced by a margin of error of 1.5 in a standard normal variable?

Solution

Write the boundaries of z. $-1.5 < z < 1.5$

TI-84 Plus CE

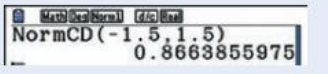
Use the normal cumulative density function as shown in Example 14 on page 446.



TI-Nspire CX
Chapter 11

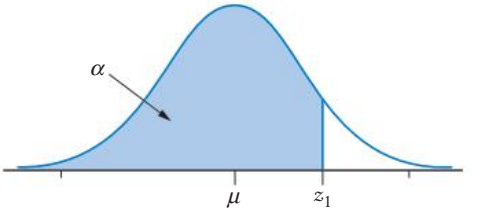
Casio fx-CG20AU

Use the normal cumulative density function as shown in Example 14 on page 446.



Write the answer. The confidence level is about 87%.

How do you work out a margin of error for a particular confidence level? This means going backwards in the cumulative probability density function. You graphics calculator is set up to work out the value of z for a given area α under the normal distribution. This is best imagined as shown in the diagram below. Notice that the area is from $-\infty$ to z_1. z_1 is the **quantile** for α.



You need the central area to be 0.7 for a confidence interval of 70%. You actually require z_1 such that $P(-z_1 < z < z_1)$. The value of α will be the area from $\mu = -\infty$ to 0 *plus* the area above $\mu = 0$. The area from $\mu = -\infty$ to 0 is exactly 0.5 and the area above $\mu = 0$ is half the confidence level. For a confidence level of 70%, $\alpha = 0.5 + \frac{0.7}{2} = 0.5 + 0.35 = 0.85$. The inverse normal cumulative density function produces the margin of error.

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EXAMPLE 18

Find the confidence interval needed to give a confidence level of 80% for a standard normal variable.

Solution

Find the value of α.

$$\alpha = 0.5 + \frac{0.8}{2} = 0.9$$

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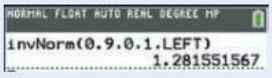
TI-84 Plus CE

Use the inverse normal cumulative density function.

Press 2nd vars, choose 3: invNorm(and complete with 0.9).



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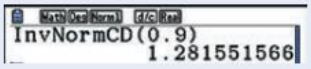
Use the inverse normal cumulative density function.

Use the Run-Matrix menu, press OPTN, choose STAT, DIST, NORM and invN and complete with 0.9).



Write the answer.

The margin of error is about 1.28 so the confidence interval is about $-1.28 < z < 1.28$.

It doesn't matter whether you write $-1.28 < z < 1.28$ or $-1.28 \leq z \leq 1.28$. Why not?

Exercise 11.07 Confidence intervals and the standard normal distribution

Example 17

1 Find the confidence level for each margin of error in standard normal variables.

a $E = 2.5$ **b** $E = 3$ **c** A margin of error of 1 **d** $E = 2.7$

Example 18

2 Find the margins of error needed in standard normal variables to give each confidence level or area.

a $\alpha = 0.98$ **b** 85% confidence **c** 99% **d** $\alpha = 0.75$

3 Find the confidence levels for each confidence interval in standard normal variables.

a $-2.5 \leq z \leq 2.5$ **b** $-1.5 \leq z \leq 1.5$ **c** $-1 \leq z \leq 1$ **d** $-3 \leq z \leq 3$

4 Find the confidence intervals in standard normal variables needed to give each confidence level or area.

a $\alpha = 0.9$ **b** $\alpha = 0.995$ **c** 95% confidence **d** 99% confidence

Problem solving

5 Can you have a small margin of error and a high confidence level for a standard normal variable? Explain your answer.

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11.08 Confidence intervals and sample proportions

WS
Sample proportion confidence intervals

From 30 Year 12 students, 12 said they liked the 'Pirates of the Caribbean' movies. You estimate the 90% confidence interval using the methods shown in the last few sections.

The sample proportion is $\hat{p} = \frac{12}{30} = 0.4$.

You can estimate $\text{SD}(\hat{p}) \approx \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \sqrt{\frac{0.4(1-0.4)}{30}} \approx 0.0089$

The area for the inverse normal cumulative distribution function is $0.5 + \frac{0.9}{2} = 0.95$.

Using your graphics calculator, you get $z_1 \approx 1.64$.

To change this to the margin of error, you need to multiply by $\text{SD}(\hat{p})$, so

$E \approx 0.0089 \times 1.64\ldots = 0.1471\ldots$

The interval is thus $0.4 \pm 0.1471\ldots$, which is from about 0.25 to 0.55.

This means there is a 90% chance that the probability of a randomly chosen Year 12 student liking the movies is between 0.25 and 0.55.

You can use this method to find the confidence interval for any sample proportion.

Confidence intervals for sample proportions

- The area for the **standard normal quantile** z_A of the z-score for the confidence level α is given by $A = 0.5 + \frac{\alpha}{2}$
- The margin of error is $E \approx z_A \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$
- The confidence interval for the probability is approximately
$$\left(\hat{p} - z_A \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + z_A \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \right)$$

Note that the area A is often omitted in the quantile in z_A. You can write a confidence interval as (a, b), $[a, b]$, from a to b, $a < p < b$ or $a \leq p \leq b$.

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EXAMPLE 19

A sample of 70 has a sample proportion of 0.6.

a What is the margin of error for a confidence level of 80%?

b Find the 80% confidence interval for the probability.

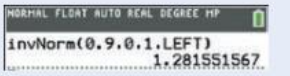
Solution

a Find the area of the required quantile.

$$A = 0.5 + \frac{\alpha}{2} = 0.5 + 0.4 = 0.9$$

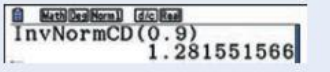
TI-84 Plus CE

Use the inverse normal cumulative density function.



Casio fx-CG20AU

Use the inverse normal cumulative density function.



Find the margin of error.

$$E \approx z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

$$= 1.28\ldots \times \sqrt{\frac{0.6(1-0.6)}{70}}$$

$$= 0.075\ldots$$

Write the answer.

The margin of error is about 0.08.

b Find the confidence interval.

$$\text{Interval} = (0.6 - 0.08, 0.6 + 0.08)$$

$$= (0.52, 0.68)$$

Write the answer.

The 80% confidence interval is about (0.52, 0.68).

Applications use the same methods.

EXAMPLE 20

300 carrot seeds were moistened and placed in a warm place. 5 days later, 250 had germinated. Estimate the 95% confidence interval for the germination rate.

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Solution

Estimate the germination rate.

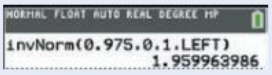
$$p \approx \hat{p} = \frac{250}{300} = 0.833\ldots$$

Find the quantile area.

$$A = 0.5 + \frac{0.95}{2} = 0.975$$

TI-84 Plus CE

Use the inverse normal cumulative density function.



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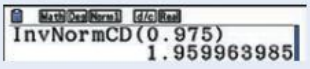
Use the inverse normal cumulative density function.



Find the margin of error.

$$E \approx z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

$$= 1.95\ldots \times \sqrt{\frac{0.833\ldots(1-0.833\ldots)}{300}}$$

$$\approx 0.0422$$

Find the interval.

The interval is $(0.8333 - 0.0422, 0.8333 + 0.0422) = (0.7911, 0.8755)$

Round appropriately and write the answer.

The 95% confidence interval for the germination rate is about (0.791, 0.876).

You could work out the error margin directly on your graphics calculator, but you may need to show the steps to get full marks.

Exercise 11.08 Confidence intervals and sample proportions

Example 19

1 Find the approximate margin of error for each confidence level for the stated values of $\hat{p}$ and n.

a $\hat{p} = 0.8, n = 40, \alpha = 90$ **b** $\hat{p} = 0.5, n = 60, \alpha = 95$

c $\hat{p} = 0.45, n = 140, \alpha = 99$ **d** $\hat{p} = 0.65, n = 100, \alpha = 80$

2 Find the confidence interval for each value of $\hat{p}$, n and α.

a $\hat{p} = 0.1, n = 60, \alpha = 85$ **b** $\hat{p} = 0.75, n = 40, \alpha = 95$

c $\hat{p} = 0.32, n = 55, \alpha = 98$ **d** $\hat{p} = 0.61, n = 70, \alpha = 90$

Example 20

3 The songs on popular radio stations in 'drive-time' were checked over several weeks. Out of 140 songs, 125 were of less than 3 minutes duration. Estimate the 90% confidence interval for songs on 'drive-time' radio being less than 3 minutes long.

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4 35 out of 50 men crossing Eagle St in Brisbane in February were wearing shorts.
- **a** Estimate the probability of men wearing shorts in the city in summer.
- **b** Estimate the standard deviation of the sampling distribution.
- **c** Estimate the 95% confidence interval for the probability of wearing shorts in the city in summer.

Problem solving

5 A survey of 178 Queensland students found that 87 came to school by car. Estimate the 95% confidence interval for the probability of Queensland students coming to school by car.

6 A flour mill produced 2 kg packets of flour. From 100 packets checked at the end of the run, 6 were found to be underweight. What is the 95% confidence interval for the proportion of underweight packets?

7 A survey of 1200 voters commissioned by a political party found that 48.5% on a two-party preferred basis intended to vote for the party. What is the 90% margin of error for this survey?

8 Estimate the 90% error margins for $n = 100$ for $\hat{p} = 0.1, 0.3, 0.5, 0.7$ and 0.9 and comment on the differences.

11.09 Variation of confidence levels

An 80% confidence interval for a sample proportion has an 80% probability of containing the population probability. This means that there will be a 20% probability that the population probability will *not* be in the confidence interval.

INVESTIGATION

DICE SIMULATION OF CONFIDENCE INTERVALS

In this investigation, you will use dice to simulate sample proportions and confidence intervals. The probability of the event for the simulation is $p = \frac{1}{3}$ and $n = 20$.

- Roll a normal die and note whether it lands with 5 or 6 uppermost
- Repeat the roll another 19 times and find the sample proportion of high numbers (5 or 6)
- Calculate the approximate 90% confidence interval for your 20 rolls
- Do another 20 rolls and find the new sample proportion and 90% confidence interval
- Do this whole procedure another 3 times
- How many of your 5 simulations of the confidence interval contained $p = \frac{1}{3}$?
- Put the results of the whole class together and compare the total number of simulations and the number of times that the confidence intervals contained the probability

Is this what you would have expected?

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You can also use your graphics calculator to do the same kind of simulation as in the Investigation.

EXAMPLE 21

A property has a probability of occurrence of 0.6.

a Simulate 10 samples of 40 items and estimate the 80% confidence interval from each sample proportion.

b Comment on how many times p is within the confidence interval.

Solution

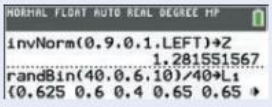
a Find the area for the confidence interval.

$$A = 0.5 + \frac{0.8}{2} = 0.9$$

TI-84 Plus CE

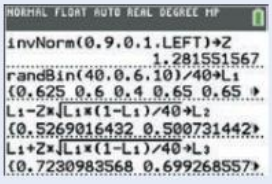
Use [distr] for the quantile and store in Z.

Press [math], choose PROB, 7: randBin(and complete with 40, 0.6,10)÷40 and store in L1.

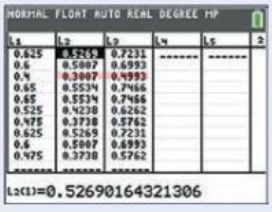


Find the lower limit of the confidence intervals for each value of $\hat{p}$ in L1 and store in L2. Put the multiplication sign in to use all the values.

Find the upper limit of the confidence intervals for each value of $\hat{p}$ in L1 and store in L3.



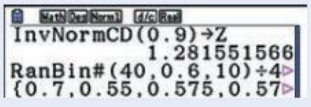
Press [stat] to check the estimated values of the intervals.



Casio fx-CG20AU

Use the Run-Matrix menu and press [OPTN].

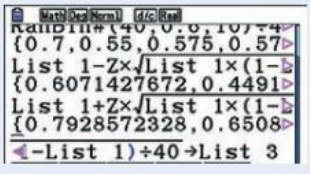
Choose STAT, DIST, NORM and InvN to find the quantile and store in Z.



[EXIT] 3 times and choose PROB, RAND and Bin. Complete with 40,0.6,10) ÷ 40 and store in List 1.

Find the lower limit of the confidence intervals for each value of $\hat{p}$ in List 1 and store in List 2. Put the multiplication sign in to use all the values.

Find the upper limit of the confidence intervals for each value of $\hat{p}$ in L1 and store in L3.



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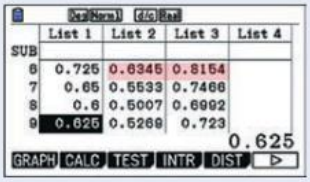
Press MENU and choose the Statistics menu to check the estimated values of the intervals.



b Comment on the results.

In most, but not all, cases the probability 0.6 is within the interval.

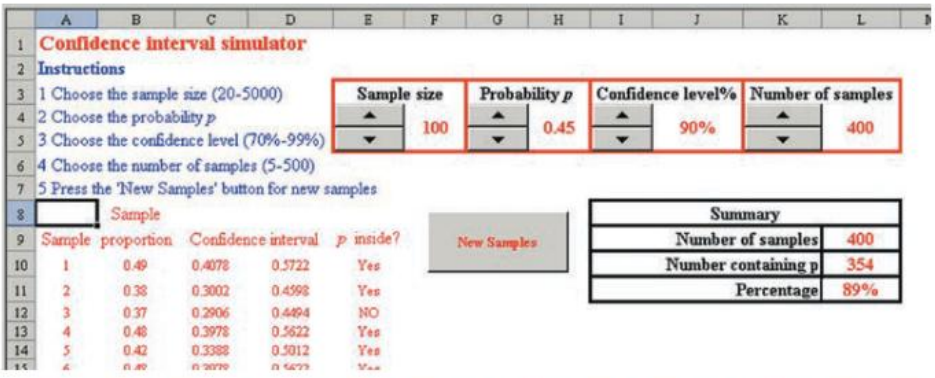
INVESTIGATION

CONFIDENCE INTERVAL SIMULATION

Confidence interval simulator

The spreadsheet **Confidence interval simulator** on NelsonNet lets you simulate many samples and calculations of $\hat{p}$ and confidence intervals for various values of p to find the proportion of values of p that lie in different confidence intervals.

You teacher will tell you what simulations you should perform.



Exercise 11.09 Variation of confidence levels

Your teacher will tell you which simulations to do, and may want you to pool results with the rest of the class.

Example 21

1 A property has a probability of occurrence of $p = 0.3$.

a Simulate 10 samples of 20 items and estimate the 75% confidence interval from each sample proportion.

b Comment on how many times p is within the confidence interval.

2 A property has a probability of occurrence of $p = 0.8$.

a Simulate 10 samples of 30 items and estimate the 90% confidence interval from each sample proportion.

b Comment on how many times p is within the confidence interval.

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3 A property has a probability of occurrence of $p = 0.4$.
- **a** Simulate 10 samples of 20 items and estimate the 60% confidence interval from each sample proportion.
- **b** Comment on how many times p is within the confidence interval.

4 A property has a probability of occurrence of $p = 0.7$.
- **a** Simulate 10 samples of 40 items and estimate the 85% confidence interval from each sample proportion.
- **b** Comment on how many times p is within the confidence interval.

5 A property has a probability of occurrence of $p = 0.2$.
- **a** Simulate 10 samples of 20 items and estimate the 80% confidence interval from each sample proportion.
- **b** Comment on how many times p is within the confidence interval.

11.10 Applications of confidence intervals

Confidence intervals crossword

Sample sizes

Sample proportions assignment

Sample proportions problems

In nearly all cases, you cannot be *certain* about a parameter unless you do a census. A level of confidence of 100% would give you an infinite confidence interval for any sample. This would be silly.

You need to decide on an acceptable confidence interval or level of confidence and error margin and design your survey to suit. Since larger surveys are more time-consuming and expensive, there is always a trade-off with confidence in the results. If you have some idea of the value of the parameter, you can work out the size of the survey needed for any desired level of confidence and error margin. The table shows the most commonly-used levels of confidence and z-scores for the standard normal distribution.

Level of confidence α	90%	95%	98%	99%
z-score	1.645	1.960	2.326	2.576

EXAMPLE 22

The probability of an Australian Year 12 student driving to school is estimated to be about 6%. How large a sample of students should be used to establish the probability to a 95% confidence level within 0.5% of the true probability?

Solution

Write the formula for E for a sample proportion.

$$E \approx z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

Write the known values.

$$E = 0.005,\ z = 1.96,\ \hat{p} \approx 0.06$$

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Substitute and solve for n.

$$0.005 = 1.96 \times \sqrt{\frac{0.06(1-0.06)}{n}}$$

$$(0.005)^2 = (1.96)^2 \times \frac{0.06 \times 0.94}{n}$$

$$n = \frac{1.96^2 \times 0.06 \times 0.94}{0.005^2} \approx 8667$$

Round and write the answer.

About 9000 Year 12 students would need to be asked for a margin of error of 0.5% to a level of confidence of 95%.

It would be impractical to perform a survey of 9000 students. A smaller sample means you need to accept a larger margin of error.

EXAMPLE 23

It is estimated that about 60% of Year 12 students obtain their driver's licence before they complete Year 12. How large a sample would be needed to establish this to within a margin of error of 5% at the 90% confidence level?

iStock/Getty Images Plus/davidf

Solution

Write the formula for E for a sample proportion.

$$E \approx z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

Write the known values.

$$E = 0.05,\ z = 1.645,\ \hat{p} \approx 0.6$$

Substitute and solve for n.

$$0.05 = 1.645 \times \sqrt{\frac{0.6(1-0.6)}{n}}$$

$$(0.05)^2 = (1.645)^2 \times \frac{0.6 \times 0.4}{n}$$

$$n = \frac{1.645^2 \times 0.6 \times 0.4}{0.05^2} \approx 260$$

Write the answer.

You would need to ask about 260 Year 12 students for this level of accuracy.

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If you have no real idea of the likely value of $\hat{p}$, you should choose $\hat{p} = 0.5$ in your calculations. This produces the largest margin of error because $\hat{p}(1 - \hat{p})$ has a maximum at $\hat{p} = 0.5$. You can prove this using calculus.

EXAMPLE 24

The transport department wants to determine the proportion of cars on the road that have faulty brakes to within 10% of the correct value to a level of confidence of 98% to work out whether they should request a police crackdown. How many vehicles should have their brakes tested at a random spot check?

Solution

What value of $\hat{p}$ should be used.	Use $\hat{p} = 0.5$ as the worst case.
Write the formula for E for a sample proportion.	$E = z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$
Transform to get n.	$E^2 = z^2 \times \dfrac{\hat{p}(1-\hat{p})}{n}$ $n = \dfrac{z^2\hat{p}(1-\hat{p})}{E^2}$
Write the values.	$z = 2.326$, $\hat{p} = 0.5$, $E = 0.1$
Substitute values.	$n = \dfrac{2.326^2 \times 0.5 \times 0.5}{0.1^2} \approx 135.25$
Write the answer.	Testing the brakes of 135 vehicles at random should be enough to determine the proportion of cars with faulty brakes to within 10% at a level of confidence of 98%.

Although the Transport Department would probably not accept the margin of error above, the example shows that you can vastly reduce the sample size required by accepting a large margin of error.

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Exercise 11.10 Applications of confidence intervals

Examples 22, 23

1 What sample size would be needed to establish a margin of error of 0.03 at a 98% confidence level for the probability of a property occurring in a proportion of about 0.3 in a population?

Example 24

2 The probability of a particular property in a population is unknown. What sample size should be used to establish the probability with a margin of error of 5% at a confidence level of 90%?

Problem solving

3 It is generally assumed that the chances of a baby being a girl or boy are 50-50. How many births would need to be checked to establish the true proportion in Australia to within 0.1% at a confidence level of 95%?

4 Suppose you were organising the venue for a Year 12 formal. How many students should you survey to establish the preferred venue from a list of 3 at a margin of error of 10% and confidence level of 90%?

5 The tourist information centre at the town square said that about 2 in every 5 of the people coming in asked about local attractions. How many enquiries would need to be noted to establish the proportion to within 2% at a level of confidence of 99%?

6 An advertising company wanted to conduct a small survey of consumers to establish a baseline for the proportion who were aware of a particular brand of ice cream before a marketing campaign. How many consumers should they target to get a result accurate to within 3% at a confidence level of 95%?

7 A survey of 50 people claimed that $71\% \pm 5\%$ of the population could roll their tongues. What is the level of confidence for this result?

8 A poll of voting intentions for 300 voters claimed that the vote for a new political party would be $15\% \pm 3\%$. What level of confidence has been used in the survey?

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11. CHAPTER SUMMARY

Sample proportions

Populations and samples

- The **population** is the complete group from which you want information
- A **census** is collection of information from the whole population
- A **parameter** is a characteristic of the population, such as a mean
- A **sample** is part of the population
- A **survey** is collection of information from a sample
- A **statistic** is an estimate of a parameter from the individual values (**observations**) of a sample. It is a value obtained from a sample
- A **fair sample** is representative of the population; it does not favour one group
- A **biased sample** is not representative; it favours one group of the population
- A **random sample** is one chosen by a chance method to ensure every member of the population has an equal chance of being chosen

Types of bias

- **Completion bias** arises from incomplete surveys. Surveys should not be too long and should avoid sensitive subjects where possible. In longitudinal studies, some subjects may move away or be otherwise lost. You should check the bias that this might introduce
- **Design bias** arises from the way the survey is designed, ignoring the needs of the users, phrasing of questions, the order of questions, etc. Questions should be as objective as possible and you should avoid leading questions
- **Interviewer bias** arises from conscious or subconscious attitudes of the interviewer that affect responses or differences between interviewers. Training in interview methods and standard scripts can reduce it
- **Response bias** arises from the tendency of people to agree with the interviewer, to give socially acceptable answers, etc. **Non-response bias** may occur when some subjects do not answer the survey. A **self-selected sample** is an extreme case such as may occur when people ring up in response to a TV survey
- **Recall** bias arises from the inability of a subject to remember things. Surveys should focus on recent events
- **Reporting bias** arises from the way the survey is interpreted or published
- **Selection bias** arises from the choice of the sample. Random samples are best

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Random sampling

- **Simple random sampling** is a random sample from the entire population
- **Stratified random samples** use groups with particular traits in the population. Random samples are taken from all groups in proportion to their sizes
- **Cluster samples** use convenient groups of the population. All the individuals within each group of a random sample of the groups are used
- **Systematic samples** use every nth individual from a list of the population. The value of n depends on the sample and population sizes. The starting position is chosen randomly
- **Non-random sampling**
- **Convenience samples** are chosen from an easily accessible group. For example, the people in your class
- **Judgment (purposive) samples** are chosen using judgement to decide on a representative sample. For example, picking people with a range of political views
- **Quota samples** are set up using quotas; numbers of each group within the population as with stratified sampling, but individuals are chosen by convenience. For example, the first 5 men and first 5 women to come past you in the street
- **Self-selection samples** use volunteers for the sample. For example, asking on assembly for students who want to take part in a survey
- **Pseudo-random** numbers give the same group of numbers over and over again

Sample proportions

- A **sample proportion** $\hat{p}$ is the number of times a property (characteristic) occurs in a sample, divided by the sample size. $\hat{p} = \frac{x}{n}$
- An individual occurrence of a property is usually called a **success**
- The distribution of a statistic for a large number of samples is called a **sampling distribution**
- The **central limit theorem** states that the sampling distribution of *any* mean is approximately normal if the samples are large enough
- For samples that are small compared to the population ($np > 5$ and $nq = n(1 - p) > 5$) of a Bernoulli distribution
  $E(\hat{p}) = p$, $\text{Var}(\hat{p}) = \frac{p(1-p)}{n}$ and $\text{SD}(\hat{p}) = \sqrt{\frac{p(1-p)}{n}}$ where p is the probability and n is the sample size
- For samples of n from a Bernoulli variable X with probability p,
  $\frac{\hat{p} - p}{\sqrt{p(p-1)/n}}$ approximates a standard normal distribution, and the approximation improves as $p \to 0.5$ and $n \to \infty$

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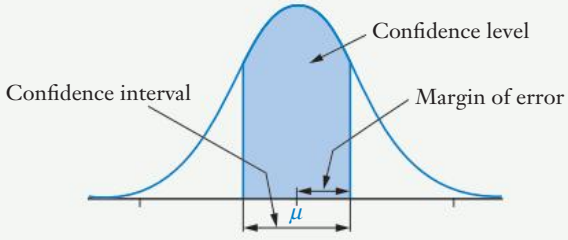
Confidence intervals

- A **point estimate** is a single value
- An **interval estimate** is an interval that is likely to include the required value
- The estimated variance and standard deviation of a sample proportion $\hat{p}$ are given by

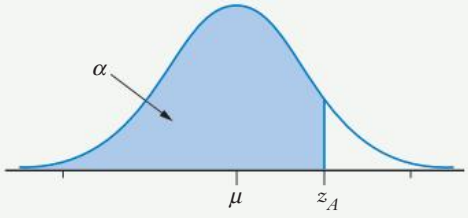
$$\text{Var}(\hat{p}) \approx \frac{\hat{p}(1-\hat{p})}{n}$$

$$\text{SD}(\hat{p}) \approx \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \text{ where } n \text{ is the sample size}$$

- A **confidence interval** contains a specific proportion of the values of a random variable. It is usually symmetric about the mean
- The **confidence level** is the proportion of values that lie within the interval
- The **margin of error** is the distance of the ends of the interval from the mean



- The value of a random variable below which a specific proportion lie is called a **quantile**.



- The area for the **standard normal quantile** z_A of the z-score for the confidence level α is given by $A = 0.5 + \frac{\alpha}{2}$
- The confidence interval for the probability of a random variable is approximately

$$\left(\hat{p} - z_A \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + z_A \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \right)$$

- The margin of error is $E \approx z_A \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$

Level of confidence α	90%	95%	98%	99%
z-score	1.645	1.960	2.326	2.576

- The largest margin of error of a confidence interval occurs at $\hat{p} = 0.5$
- The probability of a Bernoulli variable is contained within most, but not all of the confidence intervals

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11. CHAPTER REVIEW

Sample proportions

Example 1

1 Identify the population, parameters and statistics for each of the following situations.

a 40 people getting off buses in the city were asked which AFL team they most disliked. 10 said Collingwood, 4 said Hawthorn, 3 said St Kilda and the rest of the teams were chosen by at most 2 people. No-one said Brisbane.

b From the odd-numbered houses on one side of a street, 2 have one bedroom, 12 have 2 bedrooms, 46 have 3 bedrooms, 9 have 4 bedrooms and 3 have 5 bedrooms.

Example 2

2 A telephone poll was conducted to find the average age of Queensland voters. 10 people with fixed lines were selected at random from each of the Toowoomba, Coolangatta, Caloundra and Brisbane areas and rung between 2 p.m. and 5 p.m. How is this biased?

Example 3

3 Use the table on pages 422-423 to select 7 random integers from 38 to 240. Start at row 29, column 12.

Example 4

4 A survey of members of a veteran and vintage car-owners club was conducted. The club had 21 members with pre-1925 cars, 42 with cars from 1925 to 1934, 17 with cars from 1935 to 1944, 62 with cars from 1945 to 1954 and 160 with cars from 1955 on. For a survey of 30 members, how many should be chosen from each year group?

Example 5

5 The account numbers of the customers of a business run from 10 000 to 21 314. Use systematic sampling to obtain a sample of 8 customers, starting from customer number 17 113.

Example 6

6 From 24 students in a Maths Methods class, 18 achieved a C or better. What is the sample proportion of students at least passing?

Example 7

7 A fashion shop in a large shopping centre had jeans on clearance sale. Of the 8 brands on sale, 3 brands were \$75–\$124, 4 were \$125–\$165 and one brand was \$210.

a Estimate the probability of jeans at the shopping centre being priced under \$125.

b Are there any problems with this estimate?

Example 8

8 Two random samples of 20 items each were taken from a binomial distribution with $n = 24$ and $p = 0.45$. The samples were as follows.

A: 14, 12, 16, 14, 13, 12, 14, 8, 17, 12, 11, 9, 12, 11, 8, 11, 13, 14, 13, 14

B: 14, 8, 12, 7, 9, 12, 5, 9, 9, 13, 12, 9, 12, 9, 10, 9, 11, 13, 9, 10

a Draw a side-by-side column graph of the samples.

b Compare the samples.

Examples 9,10

9 **a** Generate two samples of 20 items from a Bernoulli distribution with $p = 0.65$.

b Compare the means and standard deviations of the samples.

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10 **a** What happens to the distribution of sample proportions for a particular value of p as n gets larger? Examples 11,12
b What happens to the distribution of sample proportions for a particular value of n as p gets further from 0.5?

11 The probability of a weighted coin landing with 'heads' up is 0.59. The coin is tossed 30 times. What is the expected proportion of heads and the standard deviation of this proportion? Example 13

12 What percentage of sample proportions would be expected to lie between 0.4 and 0.6 for samples with $n = 40$ and $p = 0.55$? Example 14

13 From 90 Year 12 students, 24 said they had a car available to them. Examples 15,16
a Estimate the probability and variance of Year 12 students having a car available.
b Estimate the probability that a Year 12 student has a car available, to within 0.8 standard deviations.

14 What confidence level is produced by a margin of error of 1.8 in a standard normal variable? Example 17

15 What confidence interval is needed to give a confidence level of 68% for a standard normal variable? Example 18

16 11 out of 40 Year 12 students said that they had part-time jobs. Examples 19,20
a What is the sample proportion for Year 12 students having part-time jobs?
b Estimate the standard deviation of the sampling distribution.
c Estimate the 95% confidence interval for the probability of Year 12 students having part-time jobs.
d What does the 95% confidence interval mean in this case?

17 What sample size should be used to establish a 98% confidence interval with a margin of error of 5% for a property that occurs about 70% of the time? Examples 22,23

18 The probability of a property is completely unknown. What sample size should you use to determine a confidence interval to a level of confidence of 95% with a margin of error of 8%? Example 24

Problem solving

19 In each of the following cases, state the kind of sampling employed and whether or not it is fair. If it is biased, state the kind(s) of bias.
a A Year 11 student asks each of the people in his class what kind of mobile they have and how many SMSs they send each week to determine mobile phone use among students.
b A market research company rings 100 phone numbers taken at random from the residential phone directory to ask whether they vacationed in Victoria, interstate or overseas last year as part of a study for the tourism industry in Victoria. They were also asked to give reasons for their choices.

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20 State the kind of survey and some advantages and disadvantages of the method in each of the following.

a Pedestrians in Fortitude Valley were asked about Chinese restaurants. The interviewer continued to approach people until she had obtained responses from 10 women aged under 30 years old, 10 women over 30, 10 men under 30 and 10 men over 30.

b Year 12 students who intended to seek work at the end of the year were asked to write down their top 3 choices. All the students in this category were from the biggest high school in each of 10 towns or cities with a population over 10 000 in Queensland.

21 8 out of 35 people interviewed in Gladstone said that shipping vessels should not be allowed to pass through the Great Barrier Reef. Estimate the probability of Queenslanders wanting the shipping banned and discuss any problems with the estimate.

22 The probability of an Australian having fair hair is about 0.21. What is the probability that there are between 10 and 20 fair-haired people in a random sample of 80 Australians?

23 A survey of 500 Australians renting houses in outer suburbs found that 32% of their income was paid in rent. Estimate the 85% confidence interval for the proportion of income used for rent in outer suburbs in Australia.

24 10% of the packets of biscuits sold at a supermarket have at least one broken biscuit. 15 samples of 30 packets are checked. How many times would you expect the 90% confidence interval for broken biscuits to include the value 0.1?

25 The estimated probability and 95% margin of error from a survey were given as 0.31 and 0.058. What was the sample size?

26 A regular surfer at Kirra Beach said that about 1 in 5 waves give the best rides. Another surfer standing at the lookout watching the waves sees that they are coming about every 11 seconds.

How long would you need to watch to determine the proportion of good waves to an accuracy of 10% at a 95% level of confidence?

Practice quiz

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Practice Examination 4

Year 12 formulas

Time: 90 minutes
Perusal time: 5 minutes
Marks: 50

Instructions

- Students are permitted to bring and use: pens, pencils, highlighters, erasers, sharpeners, rules and an approved graphics calculator.
- Students must show appropriate working and justification to gain full marks.
- A QCAA formula sheet is provided.
- Unless otherwise stated, numerical answers should be exact.
- Unless otherwise indicated, no diagrams in this examination are drawn to scale.
- All written responses must be in English.
- Answer all questions.
- **Students are NOT permitted to bring or use notes of any kind, correction fluid/tape, mobile phones and/or any other unauthorised electronic devices.**

Question 1 (4 marks)

The function for a continuous random variable X, is given by $f(x) = 2(x - 1)$ for $1 \leq x \leq 2$.

a Confirm that the function is a probability density function.

b Find the cumulative density function for X.

c Calculate $P(X < 1)$.

d Calculate $P(1 \leq X \leq 1.5)$.

Question 2 (3 marks)

A continuous random variable X has a triangular distribution for $2 \leq x \leq 10$ and a peak value that occurs at $x = 4$.

a State the probability density function.

b Calculate $P(X \leq 6)$.

c Calculate $E(X)$.

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Question 3 (1 mark)

The random variable X is normally distributed with $\mu = 45$ and $\sigma = 0.025$. What are the corresponding z-scores for $x = 44.97$?

Question 4 (2 marks)

Calculate the expected value and variance of a continuous random variable with the probability density function $f(x) = 3x^2$ for $0 \leq x \leq 1$.

Question 5 (2 marks) (Technology-active)

A continuous random variable X has a normal distribution with a mean of 24.5 and a standard deviation of 5.8.

a Find the probability that $X < 22.5$.

b Calculate $P(20 < X < 26)$.

Question 6 (1 mark)

What is meant by a random sample?

Question 7 (2 marks)

Use systematic sampling to choose 8 students from the Year 12 students in a school, assuming they have been numbered from 1 to 147.

Question 8 (2 marks)

A factory has 22 process workers, 5 maintenance staff, 12 packers and 6 supervisors. How many packers should be included in a sample of 10 people from the factory?

Question 9 (1 mark)

A sample of Brisbane residents is chosen by picking 5 residential streets and interviewing the people in every house in these streets. What kind of sampling is this?

Question 10 (1 mark)

3 of the 13 girls in a Year 12 class have decided not to go to the formal.
What is the sample proportion of girls going to the formal?

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Question 11 (2 marks)

3 male surfers at Bargara Beach near Bundaberg had lost a significant amount of hair. There were 20 men surfing at Bargara at the time.

a Use this data to estimate the probability of men having significant hair loss.

b Suggest any problems with this estimate.

Question 12 (2 marks)

How do the size of samples and the probability of success affect the shape of the distribution of sample proportions?

Question 13 (2 marks)

What is the expected value and standard distribution of sample proportions of 60 from a population with a probability of success of 0.72?

Question 14 (2 marks) (Technology-active)

What is the probability that the value of $\hat{p}$ lies between 0.2 and 0.4 for samples of 50 from a population with $p = 0.45$?

Question 15 (1 mark) (Technology-active)

What margin of error is needed for a confidence level of 96%?

Question 16 (2 marks)

A sample of 85 has a sample proportion of 0.7. What is the 90% confidence interval for the probability?

Question 17 (2 marks)

If the function $f(x) = k(x - 1)^2$ for $0 \leq x \leq 5$ is a probability density function, find the value of k.

Question 18 (2 marks)

If the function $f(x) = \dfrac{x^3}{4}$ for $0 \leq x \leq a$ is a probability density function, find the value of a.

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Question 19 (3 marks)

State the types of bias in each of the following, where the sampling method is not fair.

a Electorates were placed in alphabetical order and every 10th electorate was chosen. In each of these electorates, every 10 000th person on the electoral roll was chosen. The people chosen were sent a survey asking about their diet with a reply paid envelope to return it.

b The first 2000 people who came into a shopping centre in South Brisbane were asked how many times a week they ate red meat.

c 2000 people coming into the Willow Sports Complex in Townsville for a football game were asked how often they ate take-away food.

Question 20 (3 marks)

About 7% of Year 12 students are thought to be underweight. What sample size would be needed to establish this to within a margin of error of 5% at the 95% confidence level?

Question 21 (4 marks) (Technology-active)

It is known that the volume of juice delivered to each bottle by a bottle filler is approximately normally distributed. Bottles that contain less than 377.1 mL or greater than 393.5 mL are rejected as unacceptable. A quality controller observes that 4% of bottles are rejected as being underfilled and 3% are rejected as being overfilled. Find the mean and standard deviation of the distribution.

Question 22 (3 marks) (Technology-active)

A study of the causes of accidents found that of 240 serious accidents involving power tools, 33 were caused by use of faulty or incorrect use of extension leads. What is the probability that 10–15% of serious accidents involving power tools are actually caused by faulty or incorrect use of extension leads?

Question 23 (3 marks) (Technology-active)

A news bulletin headline said 'Survey shows 43% of Australians support the monarchy'. In the story it said that the margin of error of the survey was 0.3%. How many people were surveyed? Justify any assumptions you make.

Most surveys use a 95% confidence interval, for which $z = 1.96$.

END OF EXAMINATION

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ANSWERS

Chapter 1

Exercise 1.01

1 **a** $\log_3(81) = 4$ **b** $\log_{16}(4) = \frac{1}{2}$

c $\log(0.0001) = -4$ **d** $\log_{27}(9) = \frac{2}{3}$

2 **a** $4^3 = 64$ **b** $27^{\frac{1}{3}} = 3$

c $10^{-2} = 0.01$ **d** $8^{\frac{2}{3}} = 4$

3 **a** 0 **b** −1 **c** 0 **d** 0

e 1 **f** 1 **g** 3 **h** $\frac{1}{3}$

4 **a** 3.5 **b** 1.5 **c** 1 **d** 0

e −2 **f** −2

5 **a** $\log_6(7) + 2\log_6(y)$

b $\log_3(5) + \log_3(x) + 3\log_3(z)$

c $-5\log_4(w)$

d $\frac{1}{3} + \log_8(x) - 2\log_8(y) - 3\log_8(z)$

e $\log_7(x+2) + \log_7(x-2)$

f $3\log_5(y) + \log_5(y+3) - 2\log_5(y-2)$

g $\frac{1}{2}[\log_2(x^2+y^2) - \log_2(x) - \log_2(y)]$

h $\log_3(5) - \frac{1}{2}[\log_3(z^2+1)]$

6 **a** $\log_2(x^5y^3)$ **b** $\log_6\left(\frac{p^2}{11m}\right)$

c $\log\left[\frac{y(y+2)}{y^2-1}\right]^2$

d $\log_4\left[\frac{x^3}{(x+1)(x^2-3)}\right]^4$ **e** $\log_5\left(\frac{\sqrt{17}}{x^3y^4}\right)$

f $\log_7\left[x^3(3+x)^2\sqrt[3]{x-3}\right]$

g $\log_3\left(\sqrt{\frac{xy^3}{(x-2)^5}}\right)$

h $\log_2\left(5\sqrt{(x+1)(x-1)}\right)$

7 **a** $x^2 - 3$ **b** 64 **c** $\sqrt{7}$

d 216 **e** $\frac{1}{49}$

8 **a** 5.1699 **b** 1.3495 **c** −1.8917

d 0.3640

9 Demonstrations

10 **a** 3.30 **b** 3.57 **c** −1.16

d −1.32 **e** Undefined

11 **a** 3^{n+1} **b** $1 + 4n$ **c** $\frac{5}{n}$

d $2n - 3$

12 **a** $y = \frac{16}{x^5}$ **b** $y = \frac{x^4}{9}$ **c** $y = \frac{16}{x}$

d $y = 27x$

13 $x = \frac{1}{15}$

14 Proof – use the change of base theorem.

15 Proof

16 **a** 13 **b** 15 **c** 0.5 **d** 6

Exercise 1.02

1 **a** $x = 0$ **b** $k = \frac{3}{2}$ **c** $x = \frac{9}{4}$

d $r = \frac{1}{3}$ **e** $x = \frac{3}{2}$ **f** $y = \frac{1}{4}$

g $x = 6$ **h** $g = -\frac{1}{6}$

2 **a** $x = \frac{1}{3}$ **b** $u = 4$ **c** $k = \frac{5}{2}$ **d** $m = \frac{1}{8}$

e $y = -3, 2$ **f** $x = -\frac{5}{6}$ **g** $z = -10$ **h** $x = \frac{4}{15}$

3 **a** $x = 1$ **b** $m = 3$ **c** $n = 4$

d $x = 1$ **e** $y = 1$ **f** $k = \frac{3}{2}$

4 **a** 1.277 **b** 0.488 **c** 0.395

d −1.369 **e** 4.419 **f** −0.935

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5 **a** 2.513 **b** 1.600 **c** 7.516
d 3.810 **e** −2.793 **f** 2.827

6 **a** 2.579 **b** 2.459 **c** 0.255
d −2.222 **e** 0.757 **f** −1.430

7 **a** $x = \frac{11}{12}$ **b** $m = -\frac{2}{9}$ **c** $n = -\frac{2}{9}$
d $k = \frac{15}{4}$ **e** $b = \frac{1}{4}$ **f** $x = -\frac{1}{3}$

8 $m = 2$

9 $x = 9$ or 49

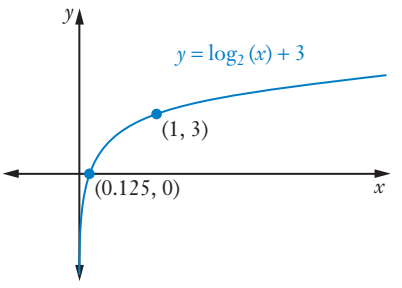
10 $a = 5$ and $b = 14$

Exercise 1.03

1 **a** $y = \log_4(x) + 5$ **b** $y = \log_{\frac{1}{4}}(x) - 3$
c $y = \log_7(x) + 2$ **d** $y = \log_{0.6}(x) - 7$

2 **a** $y = \log_5(x + 5)$ **b** $y = \log_{0.4}(x - 3)$
c $y = \log_{\frac{1}{8}}(x + 2)$ **d** $y = \log_3(x - 7)$

3 **a**



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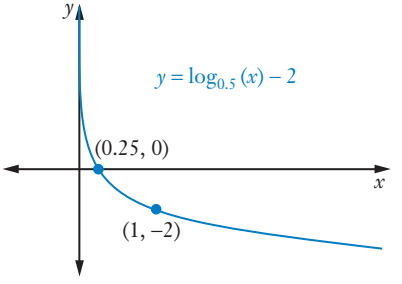

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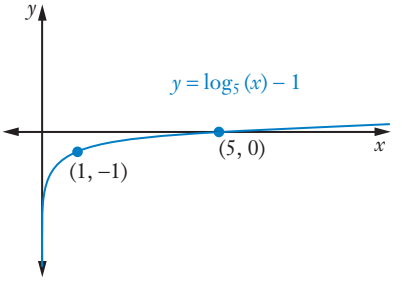

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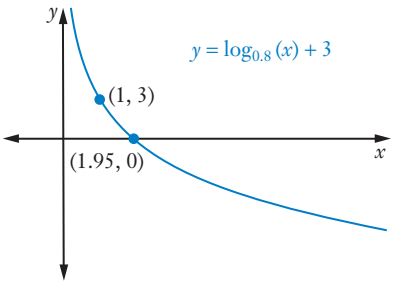

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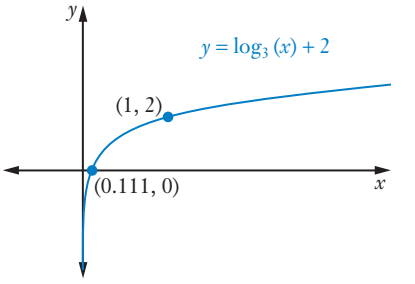

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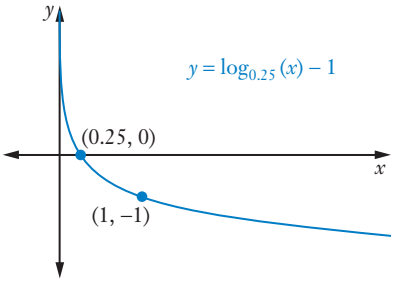
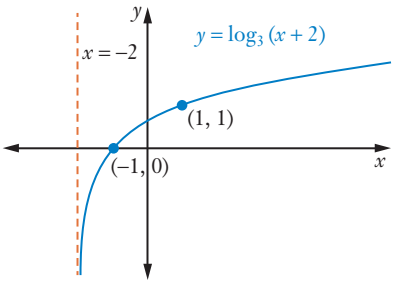


4 **a**



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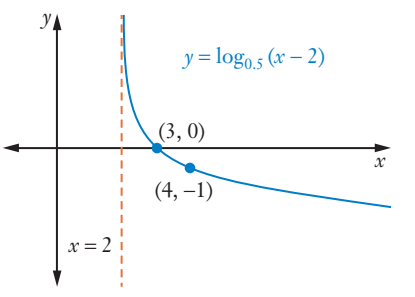

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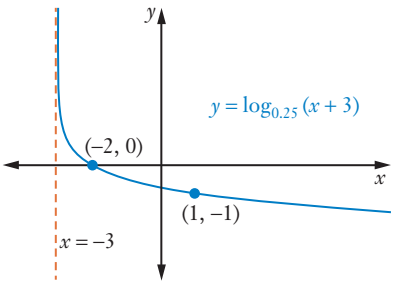

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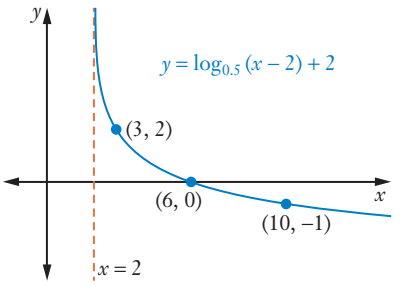

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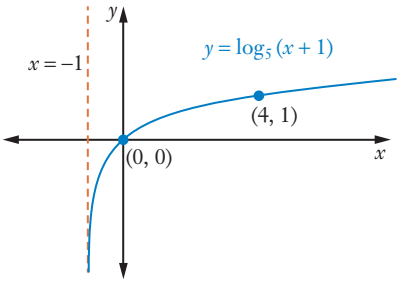

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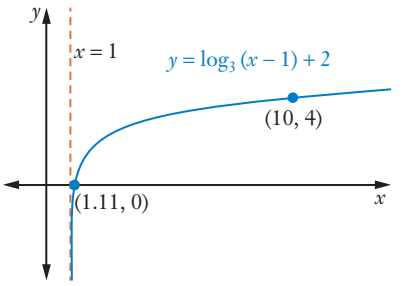

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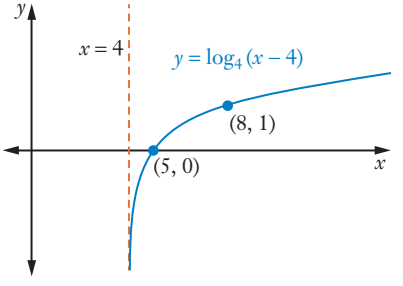

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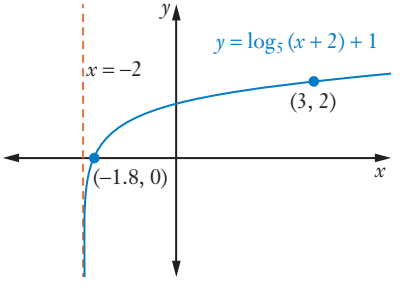

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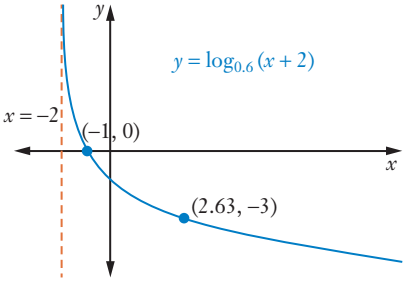
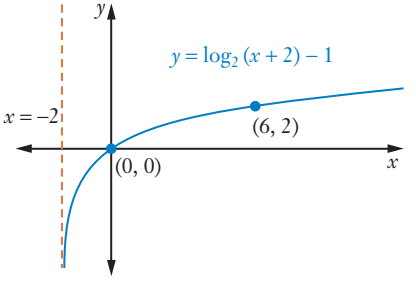


5 a



6 A vertical translation of 3 units upwards.

7 Vertical dilation with dilation factor 2.

Exercise 1.04

1 **a** $x = 3$ **b** $x = 3$ **c** $x = \frac{4}{3}$

d $x = -5, 4$ **e** $x = -\frac{1}{5}$ **f** $x = 4$

g $x = \frac{17}{4}$ **h** No solution **i** $x = \frac{4}{5}$

j $x = \sqrt{7}$

2 **a** $x = 5$ **b** $x = 250.25$ **c** $x = 4$

d $x = \frac{7}{2}$ **e** $x = \frac{79}{9}$ **f** $x = 9$

g $x = \frac{70}{3}$ **h** $x = 9$ **i** $x = -3$ or 9

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3 **a** $x = \frac{37}{63}$ **b** $x = 3$

c No solution **d** $x = \frac{33}{7}$ **e** $x = \frac{1}{2}$

f $x = 1$ **g** $x = 4$ **h** $x = \frac{6}{5}$

i No solution **j** $x = \frac{9}{8}$

4 **a** $x = 12$ **b** $x = 16$ **c** $x = 3$

d $x = 3$ or 4 **e** $x = \frac{4}{3}$ **f** $x = 2$

g $x = 3$ **h** $x = 3$ or $\frac{1}{27}$ **i** $x = 1$

j $x = \pm 2\sqrt{2}$

5 $x = 9$

6 $x = 31$

Exercise 1.05

1 2.9

2 $10^{1.2}$ or about 15.85

3 About 7 years (6.96)

4 8.9

5 16 times stronger (15.85)

6 13.80 dB **7** 10^4 W/m^2

8 **a** 100 ¢ **b** 900 ¢ **c** 1400 ¢

9 0.016 mol/L

10 **a** 1 M/litre **b** 10^{-3} M/litre

c 10^{-8} M/litre **d** 10^{-11} M/litre

11 **a** 631 **b** 3.98×10^{-5}

c About 400 000 times brighter (398 107)

12 **a** $k \approx 0.000\,1205$

b About 9988 years

13 **a** When $t = 0$, $Q(t) = Q_0$

b $n \approx 0.000\,76$ **c** 0.3 kg

14 **a** 10.0 billion **b** 2075

15 24 hours

Chapter 1 review

1 **a** $\log_2(32) = 5$ **b** $\log_{64}(16) = \frac{2}{3}$

c $\log_6\left(\frac{1}{216}\right) = -3$

2 **a** $3^4 = 81$ **b** $25^{\frac{1}{2}} = 5$

c $10^{-3} = 0.001$ **d** $8^{\frac{5}{3}} = 32$

3 **a** 0 **b** –2 **c** 2

4 **a** 1 **b** –2 **c** –1 **d** 25

5 **a** $\log_5(9) + 3\log_5(x)$ **b** $\log_7(3) - 4\log_7(z)$

c $2\log_7(m) + \log_7(m-4) - 2\log_7(m+3)$

d $\frac{1}{2}[\log_6(x^3 - y) - \log_6(2) - \log_6(x) - \log_6(y)]$

6 **a** $\log_5(x^2y^5)$ **b** $\log_2\left(\frac{m(3-m)}{2m^3+7}\right)^2$

c $\log_5\left(\frac{\sqrt{17}}{x^3y^4}\right)$

7 1.6542

8 **a** 4.29 **b** –0.64 **c** –0.785

9 **a** $x = 4$ **b** $x = 2$ **c** $x = \frac{1}{4}$

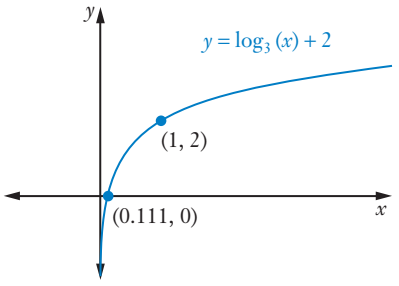
d $x = \frac{4}{15}$

10 **a** $x = 2$ **b** $x = 0$ or 1

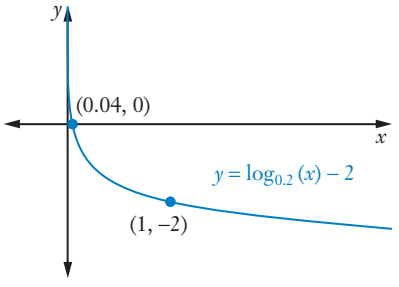
11 **a** $x = 2.877$ **b** $x = -7.952$

12 **a** $x = 0.321$ **b** $x = 0.943$

13 **a**



b



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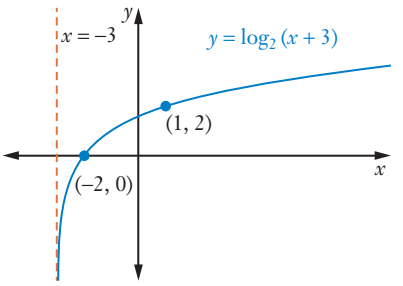
14 a



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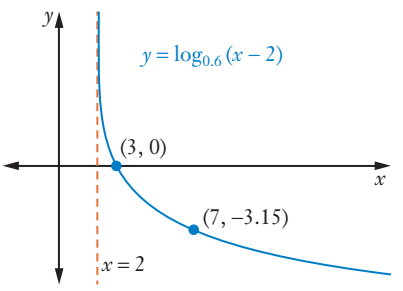

15 **a** $x = 3$ **b** $m = 2$
c $k = -2$ or 8 **d** $p = 1$

16 **a** $w = 2$ **b** $v = 9$ **c** $m = \frac{33}{7}$
d $x = 4$

17 754 megatonnes

18 Yes, the sound would be 140 dB.

19 About $20.40

20 $a = \frac{3}{10}$ and $b = \frac{12}{5}$

21 Vertical translation 1 unit upwards and horizontal translation $\frac{20}{9}$ units to the left.

22 $x = 9^{\frac{2m-1}{4}}$

23 $\log_x(81) = \frac{8}{m-1}$

Chapter 2

Exercise 2.01

1 **a** 4.482 **b** 7.389 **c** 0.0498
d 0.367 **e** 0.271 **f** −14.778

3 $y = e^2x - 2e^2 + e$

4 1.92

5 $y = e^4x - 4e^4 + e^4$

Exercise 2.02

1 **a** $3 + e^x$ **b** $\frac{xe^x - e^x}{x^2}$
c $x^3e^x + 4x^2e^x + 2xe^x + x^{-2}e^x - x^{-3}e^x - 3x^{-4}e^x$
d $9x^2 + 4e^x$ **e** $9e^x$
f $-e^x$ **g** $e^x + 2x$
h $6x^2 - 6x + 5 - e^x$ **i** $3e^x(e^x + 1)^2$
j $-9e^x(5 - e^x)^8$ **k** $12e^x(2e^x - 3)^5$
l $4(e^x + x)^3(e^x + 1)$

2 **a** $e^x + xe^x$ **b** $-\frac{2e^x}{(e^x - 1)^2}$
c $(x^2 + 5x + 4)e^x$
d $\frac{4e^x(x^2 - 2x + 1) - 10x}{(x^2 + 1)^2}$
e $\frac{2e^x(x-1)}{(e^x + x)^2}$ **f** $2x^2e^x(x + 3)$
g $(x^2 + 3x + 2)e^x$ **h** $e^x(x^2 - 2)$
i $e^x(x^4 + 4x^3)$ **j** $2xe^{x^2}(x^2 + 1)$
k $\frac{e^{x^2}(2x-1)}{e^x}$ **l** $\frac{2e^x}{(1 - e^x)^2}$
m $e^x(2 + x)$ **n** $\left(\frac{2}{x} - \frac{1}{3x^2}\right)e^{6x}$
o $\left(\frac{2}{x^3} - \frac{6}{5x^4}\right)e^{5x}$ **p** $(2 - x)e^{-x}$

Exercise 2.03

1 **a** $2xe^{x^2}$ **b** $-2xe^{-x^2+4}$ **c** $2e^x + e^{x+1}$
d $20e^{4x-1}$ **e** $6e^{6x-2}$ **f** $-5e^{7-5x}$
g $8e^{4x+1}$ **h** $-8e^{6-2x}$ **i** $-27e^{9x+4}$
j $8e^{6-4x}$ **k** $-80e^{6-8x}$ **l** $-10e^{2x+4}$
m $-42e^{-7x+1}$ **n** $\frac{2}{3}e^{\frac{x}{3}+1}$ **o** $-\frac{3}{5}e^{2-\frac{x}{5}}$
p $-\frac{4}{5}e^{\frac{x}{5}+4}$ **q** $5e^x$ **r** $\frac{-4e^{\frac{1}{x}}}{x^2}$
s $2e^{2x-1} - 8xe^{4x^2}$ **t** $-32e^{-8x-1}$ **u** $10e^{10x-5}$
v $3(x-1)^2e^{(x-1)^3}$ or $(3x^2 - 6x + 3)e^{(x^3-3x^2+3x-1)}$

2 **a** $2(x+1)e^{x^2+2x}$ **b** $(2x-3)e^{x^2-3x-2}$
c $2(x-2)e^{x^2-4x}$ **d** $-6e^{2-6x}$
e $(10x-3)e^{6-3x+5x^2}$ **f** $(3x^2+4)e^{x^3+4x}$

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g $3(8x-5)e^{4x^2-5x}$ **h** $10(5x+1)e^{1-2x-5x^2}$

i $4(2x+1)e^{(2x+1)^2}$ **j** $-5(4-x)^4e^{(4-x)^5}$

k $-\dfrac{2e^{\frac{1}{(x+5)^2}}}{(x+5)^3}$ **l** $\dfrac{e^{\sqrt{x-4}}}{2\sqrt{x-4}}$

m $\dfrac{e^{(x+4)^{\frac{1}{3}}}}{3(x+4)^{\frac{2}{3}}}$

n $4(x^3+6x^2+7x)e^{x^4+8x^3+16x^2}$

o $3(1+2x^2)e^{x^2+4}$ **p** $2xe^{x-4}(x+2)$

q $2e^{2x}(1-2e+e^2)$ or $2e^{2x}(e-1)^2$

r $4x^2e^x(x+3)$

s $e^{x^2-1}\left[e^x+10x+2xe^x\right]$

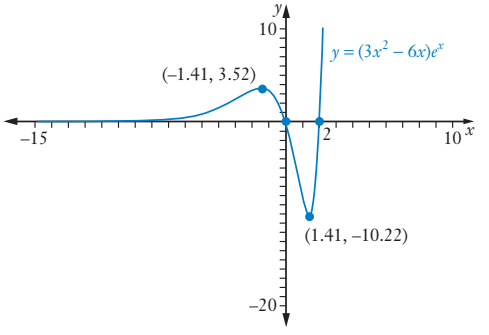
t $2e^{2x}\left(2e^{2x}-5xe^{x^2}-5e^{x^2}\right)$

u $2e^{2x-\frac{1}{2}}$

3 a 0, 2

b (–1.41, 3.52), (1.41, –10.22)

c



Exercise 2.04

1 a $10^x\log_e(10)$ **b** $4^x\log_e(4)$

c $7^x\log_e(7)$ **d** $0.4^x\log_e(0.4)$

e $1.2^x\log_e(1.2)$ **f** $-6^{-x}\log_e(6)$

g $-3\times5^{-3x}\log_e(5)$ **h** $2(0.5)^{2x}\log_e(0.5)$

i $0.5\times2^{0.5x}\log_e(2)$ **j** $-4\times3^{-4x}\log_e(3)$

2 a $2^x\log_e(2)$ **b** $3^x\log_e(3)$

c $2^{x+1}\log_e(2)$ **d** $3\times5^{3x}\log_e(5)$

e $5\times3^x\left[1+x\log_e(3)\right]$ **f** $3+3^{-x}\log_e(3)$

g $-200\,000\times10^{-2x}\log_e(10)$

h $\{\log_e(5)x^2+[2-\log_e(5)]x-1\}5^x$

i $4^x\left[\left(2+x^2\right)\log_e(4)+2x\right]$

j $[3\log_e(2)x^2+2x]8^x$

k $[1+x\log_e(3)-9\log_e(3)]3^x$

l $5^{-x}x\left[x\log_e(5)-2\right]+5^x\left[x\log_e(5)+1\right]$

m $4a^{4x}\log_e(a)$ **n** $a^xx\left(x\log_e(a)+2\right)$

o $a^xx^{a-1}\left(x\log_e(a)+a\right)$

p $-a^{-x}x^{-a-1}\left(x\log_e(a)+a\right)$

q $a^x\log_e(a)\left(1-2a^x\right)$

Exercise 2.05

1 a 200 g

b **i** 197.6 g **ii** 177.4 g

iii 60.2 g

c **i** 2.4 g/year **ii** 2.1 g/year

iii 0.7 g/year

2 a 200 **b** 492

c 74/week **d** 181/week

3 a 502

b **i** 504 **ii** 1307

c $0.2e^{0.1t}$

d **i** 0.4/month **ii** 81/month

4 a **i** 74° **ii** 27°

b **i** –18°/min **ii** –7°/min

5 approx. 36 months

6 a 16%

b **i** approx. 23 years **ii** approx. 37 years

iii approx. 46 years

7 a approx. 27.5 hours

b **i** 0.465 units/h **ii** 0.372 units/h

iii 0.298 units/h

8 a 15°/min **b** 4.3°/min

c 1.2°/min

9 approx. 13.8 years

10 approx. 13 months

Chapter 2 review

1 a 12.182 **b** 0.199

3 $y=e^5x-5e^5+e^3$

4 a e^x+4 **b** $\dfrac{e^x(x-1)}{2x^2}$

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c $2e^x\left(e^x+2\right)$ **d** $3\left(e^x+3x\right)^2\left(e^x+3\right)$

e $2e^x(x+1)$ **f** $\dfrac{-4e^x}{(e^x-2)^2}$

g $2e^x(x^2-2)$ **h** $4e^{x^2}\left(x^2+1\right)$

i $(2-x)e^{-x}$ **j** $2e^x(e^x+2)$

5 a $4xe^{x^2}$ **b** $-3e^{8-3x}$

c $8e^{2x+1}$ **d** $2e^{4-2x}$

e $\dfrac{-e^{\frac{x}{4}+5}}{2}$ **f** $-12e^{-6x-1}$

g $4(x-1)e^{x^2-2x}$ **h** $3(4x+3)e^{-2x^2-3x+1}$

i $-\dfrac{e^{\frac{1}{4(x+1)^2}}}{2(x+1)^3}$ **j** $\dfrac{e^{\sqrt{x-4}}}{\sqrt{x-4}}$

k $-\dfrac{2e^{\frac{1}{x}}}{x^2}$

6 a $7^x\log_e(7)$ **b** $1.6^x\log_e(1.6)$

c $3(0.3)^{3x}\log_e(0.3)$ **d** $-3\times2^{-3x}\log_e(2)$

7 a 80 000 hectares

b **i** 78 416 hectares **ii** 65 498 hectares

iii 10 827 hectares

c **i** −1568 hectares/year

ii −1310 hectares/year

iii −217 hectares/year

8 a 85

b **i** 127 **ii** 261 **iii** 70 449

c **i** 10 people/day

ii 21 people/day

iii 5636 people/day

d approx. 88 days or 12.5 weeks

Chapter 3

Exercise 3.01

1 a $e^{3.86}\approx47.5$ **b** $e^{-0.46}\approx0.63$

c $e^{7.648}\approx2096$ **d** $e^{9.22}\approx10\,100$

e $e^{-7.62}\approx0.000\,489$

2 a $\ln(0.002\,77)=-5.89$ **b** $\ln(109)=4.69$

c $\ln(1.4)=0.37$ **d** $\ln(13\,000)=9.47$

e $\ln(0.1)=-2.3$

3 a 3.61 **b** −2.59 **c** 6.21

d 0.751 **e** −9.41

4 a 47.0 **b** 0.001 14 **c** 5.37

d 0.387 **e** 1.70

5 a $\ln(12xy^2)$ **b** $\ln(6p)$ **c** $\ln\left(\frac{1}{q}+\frac{1}{p}\right)$

d $\ln(x^2y)+1$ **e** 1

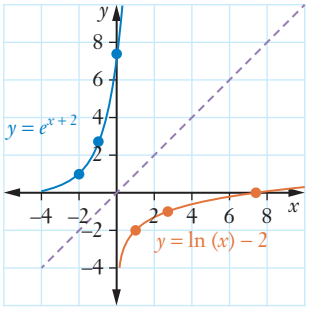
f $\ln\left(\frac{m}{2n}-\frac{n}{2m}\right)$

6–8 Proofs

Exercise 3.02

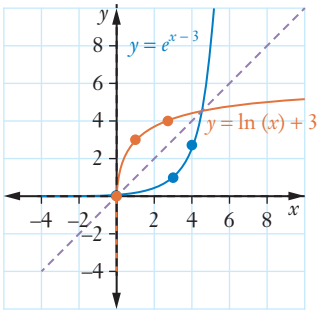
1 a $y=\ln(x)-2$

b



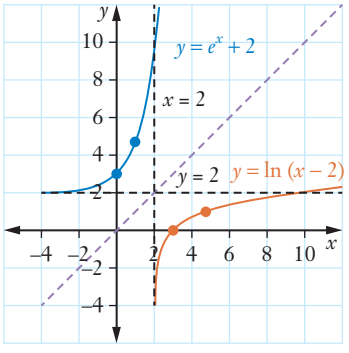
2 a $y=\ln(x)+3$

b



3 a $y=\ln(x-2)$

b



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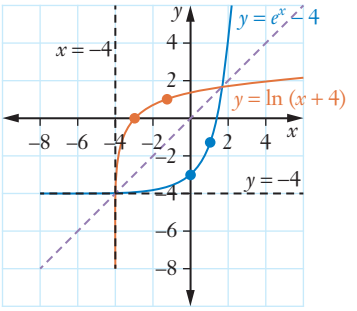
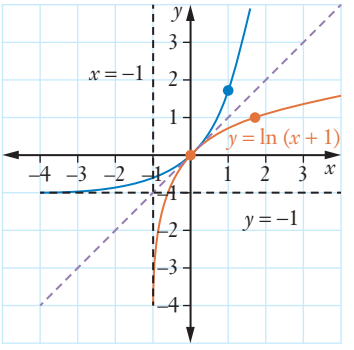
4 **a** $y = \ln(x + 4)$

b



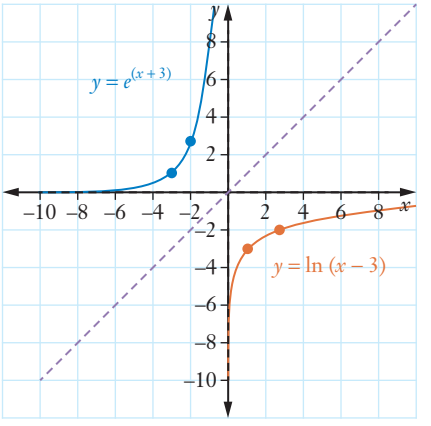
5 **a** $y = e^x - 1$

b



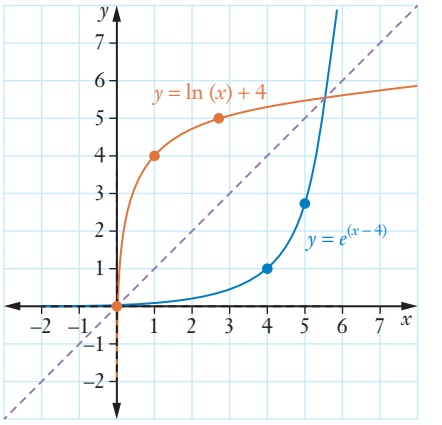
6 **a** $y = e^{(x+3)}$

b

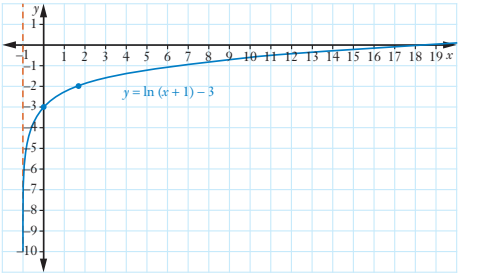


7 **a** $y = e^{(x-4)}$

b

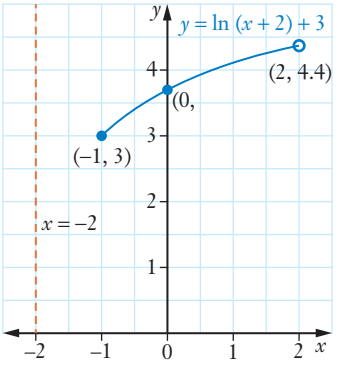


8 Domain is $x > -1$, range is R



9 Domain is $x > 4$, range is R

10 Range is $3 \leq y < 3 + \ln(4)$



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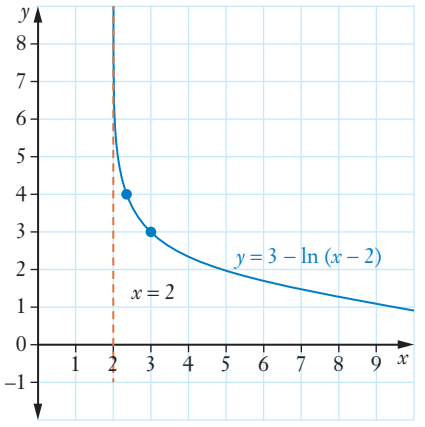

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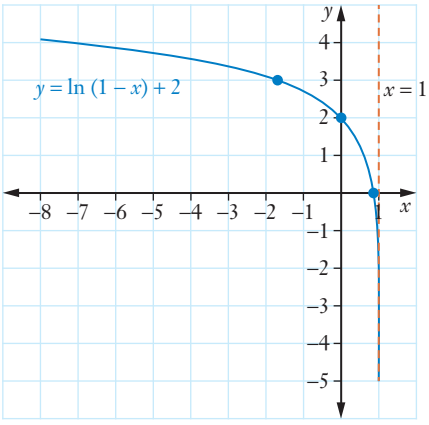

Exercise 3.03

1 **a** 2 **b** 0.5 **c** 0.25
d 0.05 **e** 100

2 **a** $\frac{7}{x}$ **b** $\frac{5}{x}$ **c** $\frac{1}{x} + 1 + 2x$
d $18x^2 - \frac{7}{x}$ **e** $1 - 2x + \frac{9}{x}$

3 **a** $\frac{1}{x}$ **b** $\frac{1}{x}$ **c** $\frac{1}{x}$
d $\frac{4}{x}$ **e** $\frac{3}{x}$

4 **a** $\frac{8}{x}$ **b** $\frac{3(2x+3)}{x^2+3x+2}$ **c** $\frac{3}{3x-1}$
d $\frac{5}{x}$ **e** $\frac{18}{3x-4}$

5 **a** $\ln(x) + 1$ **b** $2x \ln(x^2) + 2x$
c $(4x + 3) \ln(2x + 1) + 2x + 2$
d $2x \ln(x) + x + \ln(x^2) + 2$
e $\frac{x\left[\ln(x^2+5)+2\right]}{\sqrt{x^2+5}}$

6 **a** $\frac{2x(x+3)-(x^2+3)\ln(x^2+3)}{(x+3)^2(x^2+3)}$
b $\frac{(x-3)\ln(x-3)-(x+3)\ln(x+3)}{(x+3)(x-3)[\ln(x-3)]^2}$
c $\frac{x[2\ln(x)-1]}{[\ln(x)]^2}$
d $\frac{2(x+1)^2-2(x^2+2x+5)\ln(x^2+2x+5)}{(x^2+2x+5)(x+1)^3}$
e $\frac{3(3x+1)-3\sqrt{3x-1}\ln\left(\sqrt{3x-1}\right)}{2(3x+1)\left(\sqrt{3x-1}\right)^3\sqrt{3x+1}}$

7 **a** 1
b $\ln(x) + 2x + 1$
c $e^x \ln(x) + \frac{e^x}{x} = \frac{e^x\left[x\ln(x)+1\right]}{x}$
d $2e^{2x+1}\ln(2x+1) + \frac{2e^{2x+1}}{2x+1} = \frac{2e^{2x+1}[(2x+1)\ln(2x+1)+1]}{2x+1}$
e $12e^{4x+3} - 3e^{3x}\ln(4x+3) - \frac{4e^{3x}}{4x+3}$

8 Proof

9 $x \ln(x) - x$

10 The first is the logarithm of a square, and the second is the square of a logarithm, so they have derivatives of $\frac{2}{x}$ and $\frac{2\ln(x)}{x}$ respectively.

Exercise 3.04

1 **a** $\frac{6x+2}{3x^2+2x+4}$ **b** $\frac{7x^2+14x+5}{7x^2+5}$ **c** $\frac{1}{x\ln(x)}$
d $\frac{5}{x}$ **e** $\frac{15}{x}$

2 **a** $\frac{3}{x-5}$ **b** $\frac{20}{4x+3}$
c $\frac{6(3x+2)}{3x^2+4x+2}$ **d** $\frac{10}{x+1}$
e $\frac{4(4x+3)}{(2x+1)(x+1)}$

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3 **a** $\dfrac{2(2x-1)(3x-1)}{4x^3-5x^2+2x-3}$ **b** $\dfrac{5x^4+12x^3-2}{x(x^4+3x^3-2)}$

c $\dfrac{28x^3+9x^2-4x+5}{7x^4+3x^3-2x^2+5x-6}$

4 **a** $\dfrac{5x-1}{(x+3)(x-5)}$

b $\dfrac{6\ln(x-5)}{x+3}+\dfrac{6\ln(x+3)}{x-5}$

$=\dfrac{6[(x-5)\ln(x-5)+(x+3)\ln(x+3)]}{(x+3)(x-5)}$

c $\dfrac{2}{3(x+3)\ln(x-5)}-\dfrac{2\ln(x+3)}{3(x-5)[\ln(x-5)]^2}$

$=\dfrac{2[(x-5)\ln(x-5)-(x+3)\ln(x+3)]}{3(x+3)(x-5)[\ln(x-5)]^2}$

d $\dfrac{-x-19}{(x+3)(x-5)}$

5 **a** $\dfrac{2x-3}{2(x^2-3x+4)}$ **b** $\dfrac{6}{4x-5}$

c $\dfrac{6\sqrt{\ln(4x-5)}}{4x-5}$ **d** $\dfrac{1}{x[3\ln(x)]^{\frac{2}{3}}}$

6 **a** $-\dfrac{1}{x}$ **b** $-\dfrac{1}{x[\ln(x)]^2}$

c $\dfrac{-4(2x-3)}{x^2-3x+5}$ **d** $\dfrac{-(3x^2-6x+5)}{2(x^3-3x^2+5x+4)}$

7 $2(3x+1)\ln(x^3+x^2+x-5)+\dfrac{(3x^2+2x+1)^2}{x^3+x^2+x-5}$

8 $(3x^2+2x+1)\ln(3x^2+2x+1)+\dfrac{2(3x+1)(x^3+x^2+x-5)}{3x^2+2x+1}$

9 $g'(5)=0.25$

10 $\dfrac{na}{ax+b}$

Exercise 3.05

1 **a** $\dfrac{3}{x\ln(10)}$ **b** $\dfrac{1}{x\ln(8)}$

c $-\dfrac{5}{x\ln(2)}$ **d** $\dfrac{2}{(4x-5)\ln(2)}$

e $\dfrac{30}{(5x+8)\ln(10)}$

2 **a** $\dfrac{5}{(x-4)\ln(10)}$

b $\dfrac{18x^2-6x+1}{(6x^3-3x^2+x+8)\ln(4)}$

c $\dfrac{8}{(2x-5)\ln(3)}$

d $\dfrac{x}{(x-1)(x+1)\ln(10)}$

e $\dfrac{18x}{(3x^2+8)\ln(7)}$

3 **a** $\dfrac{2}{(2x-3)\ln(4)}+\dfrac{3}{(3x+2)\ln(4)}=$

$\dfrac{12x-5}{(2x-3)(3x+2)\ln(4)}$

b $\dfrac{10}{(5x-3)\ln(10)}+\dfrac{4}{(4x-1)\ln(10)}=$

$\dfrac{60x-22}{(5x-3)(4x-1)\ln(10)}$

c $\dfrac{3}{(3x+2)\ln(3)}+\dfrac{20}{(4x-1)\ln(3)}=$

$\dfrac{72x+37}{(3x+2)(4x-1)\ln(3)}$

d $\dfrac{2x+1}{(x^2+x+2)\ln(10)}+\dfrac{2x-1}{(x^2-x-2)\ln(10)}=$

$\dfrac{4x^3-2x-4}{(x^2+x+2)(x-2)(x+1)\ln(10)}$

4 **a** $\dfrac{2\log_4(3x+2)}{(2x-3)\ln(4)}+\dfrac{3\log_4(2x-3)}{(3x+2)\ln(4)}=$

$\dfrac{2(3x+2)\log_4(3x+2)+3(2x-3)\log_4(2x-3)}{(2x-3)(3x+2)\ln(4)}$

b $\dfrac{10\log(4x-1)}{(5x-3)\ln(10)}+\dfrac{8\log(5x-3)}{(4x-1)\ln(10)}=$

$\dfrac{10(4x-1)\log(4x-1)+8(5x-3)\log(5x-3)}{(5x-3)(4x-1)\ln(10)}$

c $\dfrac{15\log_3(4x-1)}{(3x+2)\ln(3)}+\dfrac{20\log_3(3x+2)}{(4x-1)\ln(3)}=$

$\dfrac{15(4x-1)\log_3(4x-1)+20(3x+2)\log_3(3x+2)}{(3x+2)(4x-1)\ln(3)}$

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d $\dfrac{(2x+1)\log(x^2-x-2)}{(x^2+x+2)\ln(10)}+\dfrac{(2x-1)\log(x^2+x+2)}{(x^2-x-2)\ln(10)}$

$=(2x+1)(x^2-x-2)\log(x^2-x-2)+$

$\dfrac{(x^2+x+2)(2x-1)\log(x^2+x+2)}{(x^2+x+2)(x^2-x-2)\ln(10)}$

5 **a** $\dfrac{-16}{(5x-3)(3x-5)\ln(2)}$

b $\dfrac{5(3x-5)\log_2(3x-5)-3(5x-3)\log_2(5x-3)}{(5x-3)(3x-5)[\log_2(3x-5)]^2\ln(2)}$

6 $\dfrac{2x}{(x^2+1)\ln(6)}+\dfrac{x}{(x^2-1)\ln(6)}=$

$\dfrac{x(3x^2-1)}{(x^2+1)(x+1)(x-1)\ln(6)}$

7 $\dfrac{x\log_6(x^2-1)}{(x^2+1)\ln(6)}+\dfrac{x\log_6(x^2+1)}{(x^2-1)\ln(6)}=$

$\dfrac{x(x+1)(x-1)\log_6(x+1)(x-1)+x(x^2+1)\log_6(x^2+1)}{(x^2+1)(x+1)(x-1)\ln(6)}$

8 $\dfrac{(2x-3)\log_{10}(x+1)-(x-4)\log_{10}[(x-4)(x+1)]}{2(x-4)(x+1)[\log_{10}(x+1)]^2\ln(10)}$

9 $\dfrac{(6x+5)(27x^2+45x-11)}{(3x^2+5x+1)(3x^2+5x-3)\ln(6)}$

10 $\dfrac{96x(3x-2)(3x^4-8x+2)\log_{10}(3x^4-8x+2)}{+48(3x^3-2)(8x^3-8x^2+3)\log_{10}(8x^3-8x^2+3)}{(8x^3-8x^2+3)(3x^4-8x+2)\ln(10)}$

11 $\dfrac{p'(x)q(x)+p(x)q'(x)}{p(x)q(x)\ln(a)}$

12 $\dfrac{p'(x)q(x)-p(x)q'(x)}{p(x)q(x)\ln(a)}$

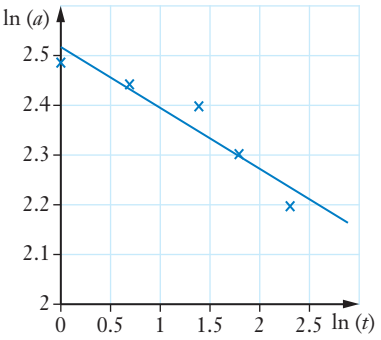
13 $\dfrac{9x(5x^3-4)(5x^3-12x+4)+(5x-4)(3x^5-6x^2+4)}{(3x^5-6x^2+4)(5x^3-12x+4)\ln(2)}$

Exercise 3.06

1 $r\approx 120c^{1.5}$

2 $a\approx 12.3e^{-0.032t}$

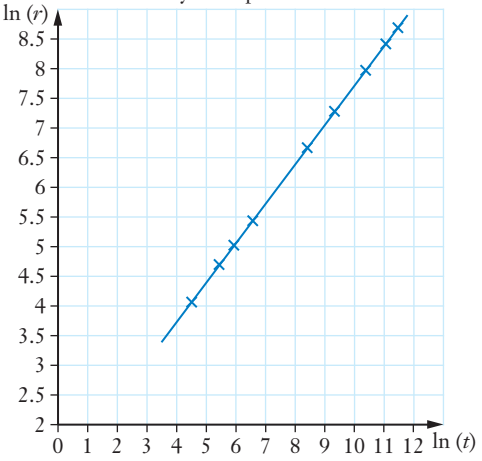
Amplitude of pendulum



3 $w\approx 88\ln 0.59(t+1)$, 500 weeks

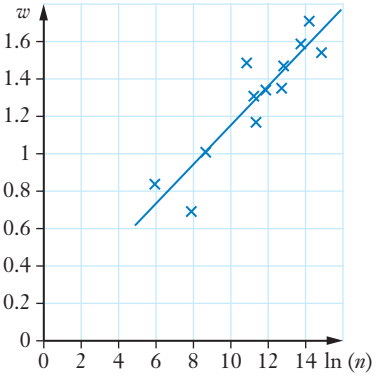
4 $r\approx 2.9\sqrt[3]{T^2}$ (*see* Kepler's laws)

Planetary orbit periods and radii



5 $w\approx 0.1\ln(n)+0.1$, using a rough line

Walking speed



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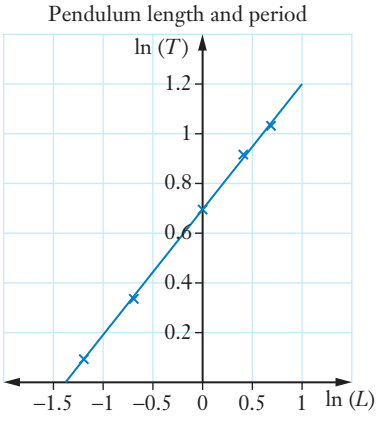
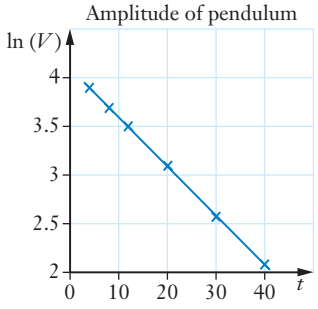
6 $T \approx 2L^{0.5} = 2\sqrt{L}$



7 $V \approx 60e^{-0.051t}$



8 $r \approx 24 - 0.837 \ln(t+1)$. After 12 days, when he runs the race, his time is likely to be 21.85 s, so he is unlikely to beat the record. He should have started 19 days before the race.

9 $s \approx 9.1 \ln(t+1) + 20$. After another 6 weeks, she should reach her target.

Chapter 3 review

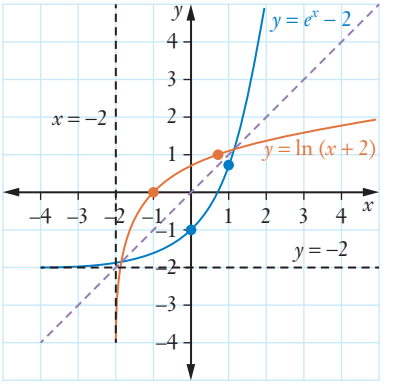
1 $e^{3.35} \approx 28.4$

2 **a** 2.6247 **b** not defined **c** −5.2214

3 **a** $a \approx 19.7$ **b** $b \approx 0.0488$

4 **a** $e^x - 2$

b



5 $\frac{1}{15}$

6 **a** $\frac{8}{x}$ **b** $\frac{1}{x}$ **c** $\frac{3}{3x+2}$

d $\frac{4}{x}$ **e** $\frac{1-3\ln(x)}{x^4}$

7 $f'(x) = \dfrac{3x^2+6x}{x^3+3x^2-8}$

8 **a** $\dfrac{15x+5}{(3x-2)(x+1)}$

b $\dfrac{8[(x^2+3)\ln(x^2+3) - x(2x+5)\ln(2x+5)]}{3(2x+5)(x^2+3)[\ln(x^2+3)]^2}$

c $\dfrac{6[(x+5)\ln(x+5) + (x-6)\ln(x-6)]}{(x-6)(x+5)}$

9 **a** $\dfrac{(3x+1)(3x-1)}{6x^3-2x+5}$

b $4(5x+1)\ln(x+1) + 10x - 6$

10 **a** $\dfrac{1}{x\ln(4)}$ **b** $\dfrac{1}{x\ln(10)}$

11 **a** $\dfrac{5(6x-5)}{(3x-2)(x-1)\ln(10)}$

b $\dfrac{6(6-x)}{(3x-5)(2x+1)\ln(2)}$

c $\dfrac{3\left[3(4x^2-3)\log_3(4x^2-3) - 4x(6x+1)\log_3(6x+1)\right]}{(6x+1)(4x^2-3)[\log_3(4x^2-3)]^2\ln(3)}$

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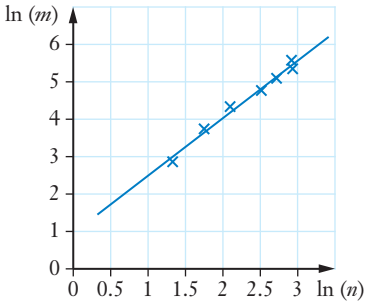
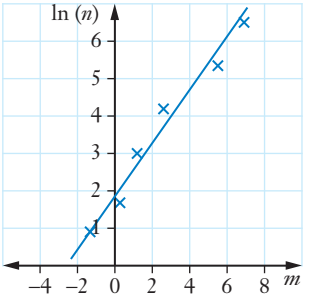
12 $m = 2.5\ t^{1.5}$



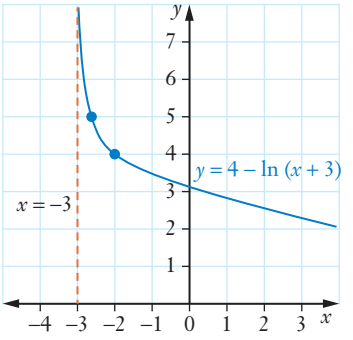
13 $n = 6.5e^{0.7m}$



14 $n \approx 9.1 \ln(t+1) + 2$, 13 days altogether

15 Proof

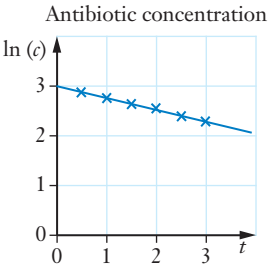
16 Domain $x \geq -3$, range $\boldsymbol{R}$



17 Proof

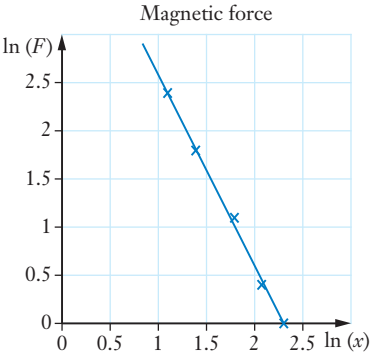
18
$$\frac{3[(12x+1)(4x^2-3)\log_3(4x^2-3) - 8x(6x^2+x+1)\log_3(6x^2+x+1)]}{2(6x^2+x+1)(4x^2-3)[\log_3(4x^2-3)]^2\ln(3)}$$

19



$c \approx 20e^{-0.24}$, every 5 or 6 hours

20



$F \approx 98x^{-2}$, the force is inversely proportional to the square of the separation.

21 $I \approx 2.2 \ln(d+1) + 2$, about 37 days training to get all 10.

Practice examination 1

1 $\log_{16}(8) = \frac{3}{4}$

2 **a** $4^5 = 1024$ **b** $81^{\frac{1}{4}} = 3$

3 4

4 $3\log_5(x) + \log_5(2x+1) - 2\log_5(x-4)$

5 1.5059

6 **a** $x = \frac{1}{2}$ **b** $x = \frac{1}{8}$

7 0.979

8 $x = 3, 27$ **9** 2.29

10 **a** $-6e^x(3-2e^x)^2$ **b** $\left(\frac{4}{x^2} - \frac{8}{3x^3}\right)e^{3x}$

11 **a** $2e^{x^3+4}(1+3^{x^3})$

b $\left(4\ln(a)x^3 + 6x^2\right)a^{2x}$

12 **a** $\log_e(31.5) = 3.45$

b $\ln\left(\frac{p^4}{q}\right)$ or $4\ln(p) - \ln(q)$

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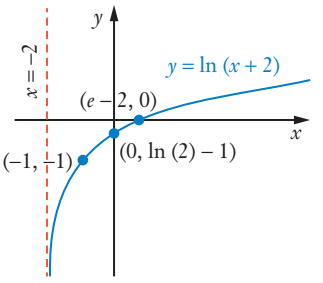
13



Domain: $x > -2$, Range: y real

14 **a** $\frac{6}{x}$ **b** $\frac{2}{v}$ **c** $\frac{1}{g\ln(5)}$ **d** $\frac{2}{4x-1}$

15 Between 2:38 a.m. and 3:09 a.m.

16 **a** 19.9°

b **i** –4.9°/min

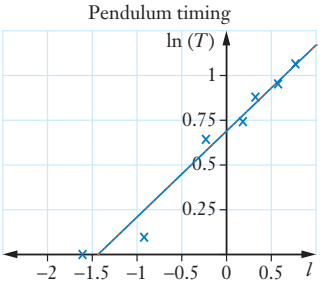
ii –0.8°/min

17 $\frac{(4x-3)\ln(3x^3+5x-8)}{2x^2-3x+1}+\frac{(9x^2+5)\ln(2x^2-3x+1)}{3x^3+5x-8}$

18 $a = 2, k = \frac{1}{3}$

19 4.62 billion years.

20



$\ln(T) = 0.48\ln(l) + 0.68$

$T = 2\sqrt{l}$

She could time more swings or use a bigger range of lengths to improve accuracy.

Chapter 4

Exercise 4.01

1 **a** $6\cos(x)$ **b** $-3\cos(x)$

c $-\cos(x)$ **d** $4\cos(4\theta)$

e $-7\cos(-7\theta)$ **f** $2\cos(8x)$

g $6\frac{1}{4}\cos(10t)$ **h** $\frac{\pi}{30}\cos(6x°)$

2 **a** $12\cos(4t)$ **b** $-15\cos(3p)$

c $\frac{7\pi}{90}\cos(2\theta°)$ **d** $-\cos(4-x)$

e $3\cos(3v-5)$ **f** $-5\cos(3-5t)$

g $8\cos(2t+3)$ **h** $6\cos(m-4)$

3 **a** $3\alpha^2\cos(\alpha^3)$ **b** $8\alpha\cos(4\alpha^2)$

c $6x^2\cos(2x^3-7)$ **d** $3(2y-5)\cos(y^2-5)$

e $2(3t^2-6t+1)\cos(t^3-3t^2+t-1)$

f $\frac{3}{2\sqrt{x}}\cos\left(\sqrt{x}\right)$ **g** $\frac{1-3x^2}{2\sqrt{x-x^3}}\cos\left(\sqrt{x-x^3}\right)$

h $\frac{3\cos(3x)}{\sqrt{\sin(3x)}}$

4 **a** $9(3x-2)^2\cos[(3x-2)^3]$

b $30(5t+4)^5\cos[(5t+4)^6]$

c $-2(2m-7)^{-2}\cos[(2m-7)^{-1}]$

d $3(5-v)^{-4}\cos[(5-v)^{-3}]$

e $-60(3n+1)^3\cos[(3n+1)^4]$

f $-60(6-5t)^2\cos[(6-5t)^3]$

g $-40(2x+7)^{-5}\cos[(2x+7)^{-4}]$

h $24(4x+5)^{-3}\cos[(4x+5)^{-2}]$

5 **a** $3[\sin(t)]^2\cos(t)$

b $40[\sin(4t)]^4\cos(4t)$

c $2\sin(x)\cos(x) + 2\cos(x)$

d $-6[5-2\sin(x)]^2\cos(x)$

e $40\sin(5x-3)\cos(5x-3)$

f $6[1-3\sin(t)+\sin(3t)][\cos(3t)-\cos(t)]$

g $24g^2\sin^3(2g^3)\cos(2g^3)$

h $-120(1-3f)^{-3}\sin^4[(1-3f)^{-2}]\cos[(1-3f)^{-2}]$

Exercise 4.02

1 **a** $-4\sin(x)$ **b** $5\sin(x)$

c $-2\sin(x)$ **d** $-3\sin(3\alpha)$

e $6\sin(-6\theta)$ **f** $-4\sin(12x)$

g $-8\frac{1}{3}\sin(10t)$ **h** $-\frac{\pi}{36}\sin(5m°)$

2 **a** $-24\sin(3t)$ **b** $8\sin(4x)$

c $-\frac{2\pi}{15}\sin(3\theta°)$ **d** $-3\sin(m-5)$

e $-5\sin(5v-3)$ **f** $3\sin(5-3t)$

g $-24\sin(4t+7)$ **h** $2\sin(7-2x)$

3 **a** $-3x^2\sin(x^3)$ **b** $-20n^3\sin(5n^4)$

c $-10x\sin(5x^2-9)$

d $6(5-4y)\sin(2y^2-5y-5)$

e $-4(9t^2+8t+3)\sin(3t^3+4t^2+3t-2)$

f $\frac{-5\sin\left(\sqrt{x}\right)}{2\sqrt{x}}$

g $\frac{5x^3-2}{2\sqrt{x^3-1}}\sin\left(x\sqrt{x^3-1}\right)$ **h** $\frac{-3\sqrt{3}\sin(3x)}{2\sqrt{\cos(3x)}}$

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4 **a** $-8(4x-5)\sin[(4x-5)^2]$

b $-15(3t+2)^4\sin[(3t+2)^5]$

c $(m-3)^{-2}\sin[(m-3)^{-1}]$

d $-4(4-v)^{-5}\sin[(4-v)^{-4}]$

e $36(2n+5)^5\sin[(2n+5)^6]$

f $48(5-4t)^3\sin[(5-4t)^4]$

g $72(3x+1)^{-4}\sin[(3x+1)^{-3}]$

h $-70(7x+3)^{-6}\sin[(7x+3)^{-5}]$

5 **a** $-4[\cos(a)]^3\sin(a)$

b $-24[\cos(2t)]^3\sin(2t)$

c $-\sin(x)[2\cos(x)+3]$

d $20[6-5\cos(x)]^3\sin(x)$

e $-24\cos(3x-5)\sin(3x-5)$

f $2[3-\cos(2t)+\cos(t)][2\sin(2t)-\sin(t)]$

g $90(1-f)^{-4}\cos^5[(1-f)^{-3}]\sin[(1-f)^{-3}]$

h $-60g^4\cos^2(4g^5)\sin(4g^5)$

Exercise 4.03

1 **a** $9[1+\tan^2(x)]$ **b** $-2[1+\tan^2(x)]$

c $-[1+\tan^2(x)]$ **d** $5[1+\tan^2(5m)]$

e $-4[1+\tan^2(-4\theta)]$ **f** $4[1+\tan^2(10x)]$

g $6\frac{2}{3}[1+\tan^2(15t)]$ **h** $\frac{\pi}{60}[1+\tan^2(3m°)]$

2 **a** $24[1+\tan^2(4t)]$ **b** $-21[1+\tan^2(7x)]$

c $\frac{\pi}{18}[1+\tan^2(5x°)]$ **d** $4[1+\tan^2(x-5)]$

e $4[1+\tan^2(4v-1)]$ **f** $-3[1+\tan^2(4-3v)]$

g $30[1+\tan^2(5t+3)]$ **h** $-5[1+\tan^2(8-5x)]$

3 **a** $4x^3[1+\tan^2(x^4)]$

b $6n[1+\tan^2(3n^2)]$

c $8x[1+\tan^2(4x^2-25)]$

d $6(x-1)[1+\tan^2(3x^2-6x+3)]$

e $8(12t^2-6t+5)[1+\tan^2(4t^3-3t^2+5t-1)]$

f $\frac{3}{\sqrt{x}}[1+\tan^2(\sqrt{x})]$

g $\frac{x(3x^2-2)}{\sqrt{x^2-1}}[1+\tan^2(x^2\sqrt{x^2-1})]$

h $\frac{2\sqrt{2}}{\sqrt{\tan 2x}}[1+\tan^2(2x)]$

4 **a** $9(3x-7)^2\{1+\tan^2[(3x-7)^3]\}$

b $20(5t+2)^3\{1+\tan^2[(5t+2)^4]\}$

c $-(m-4)^{-2}\{1+\tan^2[(m-4)^{-1}]\}$

d $5(4-t)^{-6}\{1+\tan^2[(4-t)^{-5}]\}$

e $-18(3n+1)^2\{1+\tan^2[(3n+1)^3]\}$

f $-48(5-3t)^3\{1+4\tan^2[(5-3t)^4]\}$

g $-72(4x+1)^{-4}\{1+\tan^2[(4x+1)^{-3}]\}$

h $36(3x+4)^{-3}\{1+\tan^2[(3x+4)^{-2}]\}$

5 **a** $3\tan^2(n)[1+\tan^2(n)]$

b $40\tan(5t)[1+\tan^2(5t)]$

c $3\tan(x)[1+\tan^2(x)][\tan(x)+2]$

d $-3[4-\tan(x)]^2[1+\tan^2(x)]$

e $24\tan^3(2x-7)[1+\tan^2(2x-7)]$

f $3[3-\tan(4t)+\tan(t)]^2[-3-4\tan^2(4t)+\tan^2(t)]$

g $-60(1-f)^{-2}\tan^3[(1-f)^{-3}]\{1+\tan^2[(1-f)^{-3}]\}$

h $42g^2\tan(7g^3)[1+\tan^2(7g^3)]$

Exercise 4.04

1 **a** $2x\sin(3x)+3x^2\cos(3x)$

b $(2c+1)\sin(2c-3)+2(c^2+c-1)\cos(2c-3)$

c $(10m-3)\sin(4m-3)+4(5m^2-3m+2)\cos(4m-3)$

d $\cos(x-3)\sin(x+3)+\sin(x-3)\cos(x+3)$

e $\frac{3(x+4)\cos(3x-5)-\sin(3x-5)}{(x+4)^2}$

f $\frac{4[(4x-1)\cos(2x+3)-4\sin(2x+3)]}{(4x-1)^3}$

g $\frac{6x(x-1)\sin(4-x)+(2x^3-3x^2+4)\cos(4-x)}{\sin^2(4-x)}$

h $\frac{-5[2\cos(1-2x)\sin(3x+2)+3\sin(1-2x)\cos(3x+2)]}{\sin^2(3x+2)}$

2 **a** $x^2[3\cos(4x)-4x\sin(4x)]$

b $(2m+3)\cos(2m-3)-2(m^2+3m-4)\sin(2m-3)$

c $x(15x-8)\cos(3x-5)-3(5x^3-4x^2+3)\sin(3x-5)$

d $-3\cos(2x+1)\sin(3x+1)-2\sin(2x+1)\cos(3x+1)$

e $-\frac{5(3x+2)\sin(5x-3)+3\cos(5x-3)}{(3x+2)^2}$

f $\frac{5[4(3x-5)\sin(3-4x)-9\cos(3-4x)]}{(3x-5)^4}$

g $\frac{x(4-9x)\cos(5-3x)-3(2x^2-3x^3+2)\sin(5-3x)}{\cos^2(5-3x)}$

h $\frac{3[\sin(2-x)\cos(3+x)+\cos(2-x)\sin(3+x)]}{\cos^2(3+x)}$

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3 **a** $\cos(x) - \sin(x)$

b $\cos^2(x) - \sin^2(x)$

c $\cos(x) + \sin(x)$

d $\dfrac{2\cos(2x)\cos(3x) + 3\sin(2x)\sin(3x)}{\cos^2(3x)}$

e $-\sin(x)\cos[\cos(x)]$

f $-\cos(x)\sin[\sin(x)]$

g $\dfrac{\cos(x)}{2\sqrt{\sin(x)}} + \dfrac{\sin(x)}{2\sqrt{\cos(x)}}$

h $2\sin(x)\cos^4(x) - 3\sin^3(x)\cos^2(x)$

4 **a** $\dfrac{\cos[\ln(x)]}{x} - \dfrac{\cos(x)}{\sin(x)} + e^x\cos(e^x) - e^{\sin(x)}\cos(x)$

b $10(6x^2e^{4x-2} + 11xe^{4x-2} - 4e^{4x-2} + 12x + 8)\cos(3x^2 + 4x - 3)(e^{4x-2} + 4)$

c $-8e^{-2x}[2\cos(5x-3) + 5\sin(5x-3)]$

d $e^{3x}\begin{bmatrix} 3\cos(4x)\sin(x)\ln(2x) - 4\sin(4x) \\ \sin(x)\ln(2x) + \cos(4x)\cos(x) \\ \ln(2x) + \dfrac{\sin(x)\cos(4x)}{x} \end{bmatrix}$

5 $\dfrac{189(q-2)^2(2q+17)}{(2q+3)^3}\cos\left[\dfrac{(3q-6)^3}{(2q+3)^2}\right]$

6 $\dfrac{-e^x\left[2\cos(x) + e^x\right]}{\left[1 + e^x\cos(x)\right]^2}$

7 $2(3x-1)\,e^{\sin(2x+1)} + 2(3x^2 - 2x - 1)\,e^{\sin(2x+1)}\cos(2x+1) + 10x\ln[x\sin(x^2+1)] + \dfrac{5x^2[\sin(x^2+1) + 2x^2\cos(x^2+1)]}{\sin(x^2+1)}$

8 $7e^{-0.4x^2}\left\{2\sin[\ln(2x-3)] + \cos[(\ln(2x+3)] - 1.6x^2\sin[\ln(2x-3)] - 0.8x^2\cos[\ln(2x+3)] + \dfrac{4x\{(2x+3)\cos[\ln(2x-3)] - (2x-3)\sin[\ln(2x+3)]\}}{(2x-3)(2x+3)}\right\}$

Exercise 4.05

1 **a** $-7\cos\left(\dfrac{\pi t}{5}\right) - 1$

b About 4.18 mm/s upwards

2 **a** $5\cos\left(\dfrac{\pi t}{2}\right) + 3$

b About 4.62 cm/s downwards

3 **a** $x \approx 2e^{0.035t}\cos\left(\dfrac{2\pi t}{5}\right)$

b About 3.31 mm/s downwards

4 **a** $x \approx 2e^{0.09t}\sin\left(\dfrac{2\pi t}{3}\right)$

b About 4.8 cm/s backwards

5 $x = 8\sin(\pi t)$, $8\pi \approx 25$ cm/s

6 $x \approx 1.4e^{-0.029t}\cos\left(\dfrac{\pi t}{4}\right)$, About 0.011 cm/ms

7 $x \approx [15 + 2t\ln(2)]\sin\left(400\pi te^{0.018t}\right)$,

About 597 Hz and 98 dB

8 $x = 4.4\cos\left(\dfrac{\pi t}{105}\right)$, 0 mm/s

Exercise 4.06

1 $3\sqrt{3}x - 2y - 3 - 2\sqrt{3}\pi = 0$

2 $2x - 4y - \pi = 0$

3 **a** 0.02765 s **b** $v = 227.2a\cos(227.2t)$

c 72.3

4 7.64 km/h

5 $x = 1.5\sin(5\pi t)$ cm where t is seconds, so 23.56 cm/s

6 $0.235\cos(5.1t)$ m, 4.5°

7 0.57°/s

8 **a** $d = 0.3\sin(6\pi t)$ cm

b $v = 1.8\pi\cos(6\pi t)$ cm/s

9 **a** 70.5° **b** 31.6°

10 About 11 m/h or about 18 cm/min

Chapter 4 review

1 **a** $5\cos(x)$ **b** $2\cos(2x)$

2 **a** $4(2x-3)\cos(x^2 - 3x + 4)$

b $24(3m+7)^3\cos[(3m+7)^4]$

3 **a** $-20\sin(5t)$ **b** $-\dfrac{\pi}{60}\sin(3\theta°)$

4 **a** $3x(15x^3 - 8)\sin(3x^5 - 4x^2 + 5)$

b $30(3-2q)^4\cos^2[(3-2q)^5]\sin[(3-2q)^5]$

5 **a** $6[1 + \tan^2(3t)]$

b $-10[1 + \tan^2(5a-3)]$

6 **a** $-24(5-2x)^2\{1 + \tan^2[(5-3x)^3]\}$

b $90(5t-4)\tan^2[(5t-4)^2]\{1 + \tan^2[(5t-4)^2]\}$

7 **a** $2(3x-5)[3\cos(7-2x) + (3x-5)\sin(7-2x)]$

b $\dfrac{3(t-1)[\sin\{(t-1)^2\} - t(t-2)\cos\{(t-1)^2\}]}{2\sin^2\{(t-1)^2\}}$

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8 **a** $4e^{-2t}[5\cos(5t-2)-2\sin(5t-2)]$

b $-120(5x-3)^3\tan[(5x-3)^4]$

9 **a** $x=6\sin\left(\frac{\pi t}{3}\right)+2$ mm

b $3\sqrt{3}+2\cong 7.2$ mm/s

10 $x=9e^{kt}\cos\left(\frac{\pi t}{4}\right)$ cm, where $k=\frac{\ln(10)-\ln(9)}{24}$

≈ -0.0044

11 $6x+2y+3\sqrt{3}-9\pi=0$

12 **a** $\frac{2\pi}{5}$ s **b** -6.24 mm/s **c** At $t=\frac{\pi}{10}$ s

13 $\frac{9(3x-2)\sin\left[(3x-2)^4\right]\left\{8(3x-2)^2\cos\left[(3x-2)^4\right]\cos(x^3-x^2+5)+x\sin\left[(3x-2)^4\right]\sin(x^3-x^2+5)\right\}}{5\cos^4(x^3-x^2+5)}$

14 $\frac{-72\left(1+\tan^2\left\{\left[\ln(1-3t)^2\right]^3\right\}\right)\left[\ln(1-3t)^2\right]\tan^3\left\{\left[\ln(1-3t)^2\right]^3\right\}}{1-3t}$

15 $\frac{-e^{-x}\left[\tan(1+e^{-x})\left(\sin(1+e^{-2x})\sin\left(1-e^{x}\right)+2e^{2}\cos(1-e^{x})\cos\left(1+e^{-2x}\right)\right)-\sin(1+e^{-2x})\cos(1-e^{x})+\sin(1+e^{-2x})\cos(1-e^{x})\tan^2\left(1+e^{-x}\right)\right]}{\tan^2(1+e^{-x})}$

16 $0.02e^{kt}\cos\left(\frac{\pi t}{25}\right)$, where $k=\ln(0.03)-\ln(0.02)$;

$\frac{\ln(50)}{\ln(0.03)-\ln(0.02)}\approx 10$ seconds

17 $d=22+15\sin(\pi t)$ m, $v=15\pi\cos(\pi t)$ m/s

18 About 120 s (2 min)

Chapter 5

Exercise 5.01

1 **a** $2\left(z^5-z^3+z-1\right)\left(3z-2z^3\right)+\left(9+3z^2-z^4\right)\left(5z^4-3z^2+1\right)$

b $2(3p+4)(p-1)+3\left(p^2-2p-4\right)$

c $2\left(5v^3-3v^2+v-9\right)\left(14v^3-3v^2-v+4\right)+\left(7v^4-2v^3-v^2+8v+6\right)\left(15v^2-6v+1\right)$

d $(3t^2-2t+5)\left[(5t^4+t^3-7t+8)(3t^2+8t+1)+(t^3+4t^2+t-6)(20t^3+3t^2-7)\right]+2(t^3+4t^2+t-6)(5t^4+t^3-7t+8)(3t-1)$

e $2\left[\left(4b^5+3b^2-6b+2\right)\left(2b^3+3b-9b^2\right)+\left(b^4+3b^2-6b^3+21\right)\left(10b^4+3b-3\right)\right]$

f $\left(3u-8u^2-6u^3+8\right)\left(21u^2-8u+9\right)+\left(7u^3-4u^2+9u-6\right)\left(3-16u-18u^2\right)$

2 **a** $-5x^{-2}$ **b** $-\frac{8}{x^2}$ **c** $12x^{-5}$ **d** $-\frac{6}{t^3}$

e $-70m^{-11}$ **f** $\frac{21}{u^8}$ **g** $-\frac{20}{y^{21}}$ **h** $-\frac{60}{v^{16}}$

i $-144d^{-10}$ **j** $\frac{72}{z^{10}}$

3 **a** $\frac{1}{2\sqrt{x}}$ **b** $\frac{1}{3\sqrt[3]{m^2}}$ **c** $\frac{7}{2}x^{-\frac{1}{2}}$ **d** $\frac{20}{3}t^{\frac{1}{3}}$

e $-\frac{14}{3}p^{-\frac{5}{3}}$ **f** $-\frac{1}{2x\sqrt{x}}$ **g** $-\frac{63}{2}z^2\sqrt{z}$

h $\frac{-3}{2\sqrt[4]{u^3}}$ **i** $\frac{26}{3}w\sqrt[6]{w}$ **j** $\frac{15}{2}\sqrt{m}$

4 **a** $\frac{14}{5(1-q)^2}$ **b** $-\frac{26}{(2m-5)^2}$

c $-\frac{35}{(2x-5)^2}$ **d** $-\frac{10}{(t-5)^2}$

e $\frac{123}{4(5t+8)^2}$ **f** $\frac{21}{5(2u-9)^2}$

5 **a** $\frac{7}{2}x^2\sqrt{x}+3x^2$

b $8m^7+5m^4+4m-2m^{-2}$

c $\frac{-(3u^2+2)}{(u^3+2u+1)^2}$

d $\frac{-x(x^3+3x+10)}{(x^3-5)^2}$ **e** $\frac{-(5u^3-1)}{2\sqrt{u}(u^3+1)^2}$

6 **a** $45(5a-6)^2$ **b** $-20(5x+4)$

c $49(y-2)^6$ **d** $-60(5q+8)^3$

e $-105(7-5a)^2$ **f** $360g^2(5g^3+3)^5$

g $48(4x-3)(2x^2-3x+2)$

h $-64m(3m-4)(m^3-2m^2+4)^7$

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i $18(9-3v)^5$

j $24g(4-3g)(2g^2-g^3+3)^5$

7 a $20x^2(5x-6)(5x^4-8x^3+7)^4$

b $7(21v^2+4v+1)(7v^3+2v^2+v-4)^6$

c $\dfrac{6t+1}{2\sqrt{3t^2+t-5}}$

d $\dfrac{12h-3}{2\sqrt{6h^2-3h+1}}$

e $\dfrac{4}{3\sqrt[3]{(4v+5)^2}}$

f $\dfrac{y-1}{2\sqrt[4]{(y^2-2y+3)^3}}$

8 a $-6(v+7)^4(v-6)^3(9v-2)$

b $3(6-5r)^2(2r+9)^{10}(140r+3)$

c $1920(4-3h)^6(-4-3h)^5(-4-39h)$ or $1920(4-3h)^6(4+3h)^5(4+39h)$

d $20t+1$

e $-(3p+10)^6(8p+1)^7(360p+661)$

f $(4z+15)^5(28z-57)$

9 a $3x^3\cos(3x)+3x^2\sin(3x)$

b $e^x(x^2+2x-1)$

c $2xe^{3x}(3x+2)$

d $2x^2[3\cos(2x)-2x\sin(2x)]$

e $\dfrac{e^{2x}(6x^2+3x-2)}{\sqrt{3x^2-1}}$

f $2x\cos^3(2x)-6x^2\sin(2x)\cos^2(2x)$

10 a $\dfrac{3x\cos(3x)-2\sin(3x)}{x^3}$

b $\dfrac{-2\sin(2x)(3e^x-x)-\cos(2x)(3e^x-1)}{(3e^x-x)^2}$

c $\dfrac{-(x-1)e^{-x}}{(x-2)^2}$

d $\dfrac{-e^x[3\cos(3x+1)-\sin(3x+1)]}{\sin^2(3x+1)}$

e $\dfrac{-2}{\sin^2(2x)}$

f $\dfrac{-[2(e^x+1)\cos(2x)-e^x\sin(2x)]}{\sin^2(2x)}$

11 a $\left(3x^5-3x^2+8\right)(4x+5)+3x\left(5x^3-2\right)\left(2x^2+5x+2\right)$

$-5x^7(2x+5)-x^2\left(x^5-6\right)(8x+15)$

b $\dfrac{9c^6(7-4c)}{(9-4c)^3}$

c $42(3m+10)^6-60(3m+10)^3-12(10+3m)^{-3}$

d $-\dfrac{6}{(2f-5)^2}-\dfrac{30}{4(3f+5)^2}+\dfrac{20}{(5f-11)^2}$

e $-\dfrac{3}{(t-6)^4}+\dfrac{16}{(2t+7)^3}+\dfrac{2(67-18t)}{7(2t+9)^5}$

f $(3w-1)^4\left[30(3w+5)^4+24(3w+5)^3(3w-1)+15(w-6)^8+8(w-6)^7(3w-1)\right]$

g $\dfrac{-2(10p^5-15p^4+5p^3-7p^2-51p+28)}{(5p-1)^7}$

12 $x=-1.5$ or 1

13 $P(-1.732, 2.527)$, $Q(-0.161, -0.525)$

Exercise 5.02

1 Answers may differ by a constant.

a $2x^3+2x^2+x$ **b** $4t^2-7t$

c $z^4+1\frac{1}{2}z^2-z^5$ **d** $4x$

e $\dfrac{y^3}{3}-\dfrac{y^2}{2}-y$ **f** $\frac{3}{4}v^4-\frac{1}{5}v^5+\frac{4}{3}v^3-2v$

g $4.9t^2-0.1t^3-6t$

h $\dfrac{4m^3}{3}+12m^2+44m$

i $0.025d^4+3.5d^2-0.38d$

j $\dfrac{5k^3}{3}-\dfrac{k^4}{2}+k^2-5k$

2 a $i^4-2i^3-i^2-3i+c$

b $\frac{1}{4}x^4-2x^5-2x^3+12x+c$

c $u^2-\frac{5}{4}u^4-\frac{4}{3}u^3+8u+c$

d $\dfrac{8m^3}{30}-1.575m^4-0.35m^2+4m+c$

e $0.035t^2+\dfrac{t^3}{3}+c$

f $\dfrac{49x^3}{30}-24.5x^2+122.5x+c$

3 a $\frac{1}{3}x^3-5x+c$ **b** $1.5t^2+2t+c$

c $7u-\frac{1}{4}u^4+\frac{1}{3}u^3+c$ **d** $\frac{1}{5}h^5+c$

e $\frac{1}{3}p^3+4p^2+16p+c$

f $\frac{1}{5}z^5-\frac{1}{4}z^4+\frac{1}{3}z^3-\frac{1}{2}z^2+z+c$

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4 **a** $y = x^4 + 2x^3 - 3x^2 + 7x + c$

b $y = 3x^4 - 4x^3 + 6x^2 - 12x + c$

c $y = 1.5t^2 + 5t + c$

d $y = \frac{1}{2}z^2 - \frac{1}{4}z^4 + 8z + c$

e $y = 0.25x^2 - 0.01x^4 + \frac{7x^3}{30} - 0.52x + c$

f $y = 0.35t^2 - \frac{4t^3}{30} + 0.6t + c$

5 **a** $\frac{x^3}{3} - 3x^2 + 9x + c$ **b** $\frac{4x^3}{3} + 10x^2 + 25x + c$

c $\frac{10x^3}{3} + \frac{7x^2}{2} - 12x + c$ **d** $x^4 + x^3 - 5x^2 + c$

e $\frac{3x^2}{4} - \frac{7x^3}{6} + c$ **f** $\frac{4x^6}{3} + x^3 + c$

6 **a** $y = \frac{2x^{\frac{7}{2}}}{7} + c$ **b** $y = -\frac{2}{\sqrt{x}} + c$

c $y = \frac{4x^{\frac{5}{2}}}{5} + c$ **d** $y = \frac{15\sqrt[3]{x^7}}{7} + c$

e $y = \frac{5x^4}{4} + \frac{5}{2x^2} + c$ **f** $y = -\frac{3x^{-4}}{28} + c$

7 $\frac{x^3}{3} - \frac{1}{x} - 2x + c$

8 $y = 2x^2 - 5x - 1, y(5) = 24$

9 7.6

10 -3

11 -41.25

12 $m(r) = \frac{8}{3}r\sqrt{r} + 6 - \frac{16\sqrt{2}}{3}$

13 $y = 2(x+4)\sqrt{x+4} - 54$

Exercise 5.03

1 **a** $\frac{3}{4}\sin\left(\frac{4}{3}x\right) + c$ **b** $2\sin\left(\frac{1}{3}x\right) + c$

c $\sin(6x - 5) + c$ **d** $8\sin\left(\frac{1}{2}x\right) + c$

e $-4\cos(2x + 3) + c$ **f** $-\frac{1}{3}\cos(9x) + c$

g $-2\cos(5x) + c$ **h** $-\frac{1}{2}\cos\left(\frac{2}{3}x\right) + c$

2 **a** $-6\cos\left(\frac{1}{2}x\right) + c$ **b** $-\cos\left(\frac{1}{4}x\right) + c$

c $-2\cos\left(\frac{1}{2}x\right) - 4\sin\left(\frac{1}{2}x\right) + c$

d $6\sin\left(\frac{1}{2}x\right) + c$

e $10\sin\left(\frac{1}{2}x\right) + c$

f $x + \frac{1}{2}x^2 + \sin(x - 5) + c$

g $-\frac{1}{2}\cos\left(\frac{1}{3}x\right) + c$

h $50\sin\left(\frac{1}{5}x\right) + c$

3 **a** $\frac{1}{6}\sin(6x) + c$

b $\frac{1}{5}\sin(5x - 6) + c$

c $\frac{1}{8}\cos(7 - 8x) + c$

d $\frac{1}{3}[\sin(3x) - \cos(3x)] + c$

e $-\frac{5}{4}\cos(4x) - 32\sin\left(\frac{1}{4}x\right) + c$

f $\frac{2}{5}\sin(5x) - 9\cos\left(\frac{1}{3}x\right) + c$

4 $d = 9\sin(6.3t)$ cm , since at $t = 0$, $d = 0$, so the constant of integration, c is zero.

5 $v = -62.5L\cos(0.016t)$ and $d = -3906.25L\sin(0.016t)$, since the constants of integration are both 0. The acceleration and displacement of the molecules are in opposite directions, so $d = -3906.25a$.

Exercise 5.04

1 **a** $\frac{1}{2}e^{2x} + c$ **b** $\frac{1}{5}e^{5x} + c$ **c** $e^{-x} + c$

d $2e^{\frac{x}{2}} + c$ **e** $e^{3x+1} + c$ **f** $-3e^{-4x} + c$

g $-3e^{-7x-5} + c$ **h** $5e^{2x+3} + c$

2 **a** $\frac{3}{2}x^2 + \frac{1}{2}e^{4x} + c$ **b** $\frac{2}{3}x^3 - 2e^{3x} + c$

c $-\frac{1}{4}e^{-4x} - 3e^{\frac{x}{3}} + c$ **d** $2e^{5x} + 3x^2 + c$

e $\frac{1}{2}e^{2x+1} + e^{-x} + c$ **f** $-\frac{5}{2}e^{-2x} + 8e^{\frac{x}{4}} + c$

3 **a** $\frac{1}{2}e^{2x} + 2e^x + x + c$

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b $\frac{1}{3}e^{3x} + 3e^{2x} + 12e^{x} + 8x + c$

c $2e^{2x} - 12e^{x} + 9x + c$ **d** $x - e^{-x} + c$

e $-3e^{-2x} + e^{-x} + c$ **f** $e^{x} + 2x - e^{-x} + c$

4 a $\frac{1}{2}e^{4x} + \frac{1}{2}\cos(2x) + c$ **b** $\frac{1}{2}e^{x} - 2e^{-x} + 2x + c$

c $\frac{1}{8}\sin(4x + 5) + 2e^{-x} + c$ **d** $\frac{1}{4}e^{4x} - \frac{2}{3}e^{3x} + \frac{1}{2}e^{2x} + c$

5 $a = -12, b = 4$

6 $\frac{1}{4}e^{x^4} + c$

7 $\frac{1}{2}\left(e^{x^2+3}\right) + c$

8 $\frac{2}{5}e^{x^5-3} + c$

9 $2e^{x} - x + c$

10 a $k = 4$ **b** $y = \frac{1}{4}e^{4x} + \frac{3}{4}e^{4}$

Exercise 5.05

1 a $4\ln|x| + c$ **b** $\frac{2}{3}\ln|x| + c$

c $\frac{5}{7}\ln|x| + c$ **d** $\ln|x - 4| + c$

e $\frac{5}{2}\ln|2x + 3| + c$ **f** $2\ln|4x - 5| + c$

2 a $\frac{1}{3}\ln|x| + c$ **b** $\frac{3}{5}\ln|x| + c$

c $\frac{7}{9}\ln|x| + c$ **d** $\frac{2}{3}\ln|9x + 2| + c$

e $\frac{1}{2}\ln|10x - 3| + c$ **f** $-\frac{2}{3}\ln|5 - 3x| + c$

3 a $\ln|x + 4| + c$ **b** $\frac{1}{3}\ln|3x - 5| + c$

c $\frac{1}{4}\ln|4x + 3| + c$ **d** $\frac{2}{5}\ln|5x + 6| + c$

e $2\ln|3x - 4| + c$ **f** $\frac{1}{3}\ln|9x + 7| + c$

4 a $\frac{1}{2}x^2 + 2x - 3\ln|x| + c$

b $\frac{3}{4}x^2 - 4x + 6\ln|x| + c$

5 $y = -3\ln|4 - 3x| + 9$

6 $y = \frac{1}{3}x^3 - \ln|x| + 2$

7 $y = \frac{1}{4}x^4 - \frac{3}{2}x^2 + 3\ln|x| + \frac{1}{2x^2} + c$

8 $y = \frac{7}{2}\ln\left|\frac{x-2}{4}\right| + 5$

Exercise 5.06

1 $3\sqrt{3 + x^2} + c$

2 $\frac{10}{3}\sqrt{x^3 - 6} + c$

3 $\frac{1}{12}\sqrt{3x^4 + 4} + c$

4 $\frac{2}{15}(x^3 + 1)^5 + c$

5 $\frac{1}{7}(3x^2 - 5x + 6)^7 + c$

6 a $\frac{1}{30}(5x^4 + 3)^{\frac{3}{2}} + c$ **b** $-\frac{1}{5}\ln|6 - 5x^2| + c$

c $-\frac{1}{4}\cos(x^4) + c$

d $-\frac{1}{2}\ln|\cos(2x + 1)| + c$

e $\frac{1}{10}[5 - 2\sin(x)]^5 + c$ **f** $\frac{-1}{2\sin^2(x)} + c$

7 a $\frac{1}{10}(2x + 1)^5 + c$ **b** $\frac{1}{49}(7x - 3)^7 + c$

c $-\frac{1}{20}(3 - 5x)^4 + c$ **d** $-\frac{1}{48}(9 - 8x)^6 + c$

e $\frac{-1}{6(3x - 4)^2} + c$ **f** $\frac{-1}{15(5x + 6)^3} + c$

8 Hint: Express $\tan(x)$ in terms of $\sin(x)$ and $\cos(x)$ and let $u = \cos(x)$.

9 $2\ln|\ln(x)| + c$

10 $\frac{1}{2}[\ln(x)]^2 + c$

11 $\frac{2}{3}(x - 1)^{\frac{3}{2}} + 2(x - 1)^{\frac{1}{2}} + c$ (Let $u = x - 1$).

12 $\frac{2}{3}\sqrt{x^3 - 4} + c$

Exercise 5.07

1 a 2 s **b** 8 m/s^2

2 a $x = \frac{4}{3}t^3 - 5t^2 - 6t + 27$

b $9\frac{2}{3}$ cm closer to the origin

c $x = 0$ (the origin)

3 a −3 m/s (3 m/s to the left)

b $x = 6t + 3e^{-3t} - 7$

4 a $\frac{4}{3}$ m to the right of the origin.

b $12\frac{1}{6}$ m to the left of the origin

c −1 m/s^2

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5 $v = 8$ m/s, $x = 8\frac{1}{6}$ m to the right of the origin.

6 **a** $v = 3t^2 - 18t + 24$

b $x = t^3 - 9t^2 + 24t + 4$

7 **a** −12.65 cm/s

b −180.25 cm (180.25 cm to the left of the origin)

8 **a** 9.6 m/s downwards

b About 1.02 s

c About 25.1 m

d About 5.9 m

9 About 30.63 m

10 **a** 2250 m **b** About 42.9 s

11 **a** $v = \frac{1}{16}\sin(4t) + \frac{1}{2}$

b $x = -\frac{1}{64}\cos(4t) + \frac{1}{2}t + \frac{65}{64}$

12 $t = 2(\sqrt{7} - 2)$

13 $\frac{154}{3} - \frac{10\sqrt{5}}{3}$ m in the positive direction from which it started.

14 The particle is about 2.4 cm from its starting point and is moving away from the starting point.

15 6 m – You need to find the times at which the particle changes direction i.e. when $v(t) = 0$.

Chapter 5 review

1 **a** $6d^2 - 10d + 1$

b $10m^4 - 4m^3 - 15m^2 - 2m - 10$

c $-\frac{7}{p^2}$ **d** $15u^{-4}$

e $-\frac{12}{x^3}$ **f** $\frac{\sqrt{2}}{2\sqrt{w}}$

g $\frac{1}{4\sqrt[4]{x^3}}$ **h** $12\sqrt{r}$

i $21x^{\frac{4}{3}}$ **j** $\frac{-4}{3\sqrt[3]{n^2}}$

2 **a** $\frac{27}{(5-3q)^2}$ **b** $\frac{34}{3(4t+3)^2}$

c $\frac{1}{2}x^3(9\sqrt{x} + 8)$ **d** $6x^5 + 4x^3 + 3 - 3x^{-2}$

e $\frac{-(2u+3)}{(u^2+3u-1)^2}$

3 **a** $18(3a-1)^2$

b $-120m(m+2)(m^3+3m^2-2)^7$

4 **a** $\frac{10x+1}{2\sqrt{5x^2+x-3}}$ **b** $\frac{3(y^2+3)}{5(y^3+3y-7)^{\frac{4}{5}}}$

5 **a** $2(3-5h)^4(1+6h)^6(45h+88)$

b $2x^3[x\cos(2x) + 2\sin(2x)]$

c $e^x(x^3 + 3x^2 - 1)$ **d** $\frac{e^{4x}(20x^2+5x+8)}{\sqrt{5x^2+2}}$

e $3x^2\cos(3x)[-2x\sin(3x) + \cos(3x)]$

6 **a** $\frac{3(4e^x - x)\sin(3x) - (4e^x - 1)\cos(3x)}{(4e^x - x)^2}$

b $\frac{-4xe^{-2x}}{(2x-1)^2}$

7 **a** $\frac{8}{3}x^3 - \frac{3}{2}x^2 + x$

b $\frac{1}{2}m^4 + 2m^3 - m^2 - 5m$

8 $i^4 - 2i^3 - i^2 - 3i + c$

9 $\frac{1}{2}x^4 - x^3 + 5x + c$

10 **a** $-\frac{2}{5}\cos\left(\frac{5x}{2}\right) + c$ **b** $2\sin(3x+1) + c$

11 **a** $-\frac{1}{2}\cos\left(\frac{1}{3}x\right) + c$

b $x + \frac{3}{2}x^2 + \frac{1}{2}\sin(2x+3) + c$

12 $18\sin\left(\frac{x}{6}\right) - \frac{2\cos(3x)}{3} + c$

13 **a** $\frac{1}{7}e^{7x} + c$ **b** $3e^{\frac{x}{3}} + c$ **c** $-4e^{-5x} + c$

d $6e^{2x-3} + c$

14 **a** $2x^3 + 3e^{3x} + c$

b $-2e^{-5x} - 4e^{\frac{x}{4}} + c$ **c** $\frac{1}{6}e^{6x-5} + \frac{1}{2}e^{-2x} + c$

15 **a** $\frac{1}{3}e^{3x} + \frac{9}{2}e^{2x} + 27e^x + 27x + c$

b $\frac{1}{2}e^{-2x} - \frac{5}{3}e^{-3x} + c$

c $2e^{3x} - 3\sin(3x) + c$

16 $\frac{2e^{x^5}}{5} + c$

17 **a** $\frac{3}{8}\ln|x| + c$ **b** $-2\ln|5-2x| + c$

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18 **a** $\frac{1}{2}\ln|8x+1|+c$ **b** $\frac{1}{5}\ln|5x-3|+c$

c $\frac{5}{2}\ln|2x-7|+c$ **d** $\frac{1}{3}\ln|6x+5|+c$

e $9\ln|x|+\frac{3}{2}x^2-4x+c$

19 $y=-4\ln|9-4x|+4\ln 5-5$

20 $\frac{5}{3}\sqrt{x^2-8}+c$

21 $\frac{(3x^4+1)^3}{36}+c$

22 **a** $\frac{4(5x^4+1)^{\frac{3}{2}}}{15}+c$ **b** $\frac{-7}{12}\ln|3-4x^3|+c$

c $\frac{6\sin(x^5)}{5}+c$

23 **a** $\frac{1}{49}(7x-3)^7+c$ **b** $\frac{-1}{6(3x-4)^2}+c$

24 **a** 5 s **b** About 13.1 m/s^2

25 About 42.7 m to the right of the origin.

26 **a** 2.6 m/s (2.6 m/s upwards)

b About 3.27 s

c About 87.24 m

d 84.6 m above the ground

27 $y=\frac{8}{3}x^3-2x^2+7x-8\frac{1}{3}$, $y=-51\frac{2}{3}$ when $x=-2$.

28 $-2\cos\left(\sqrt{x}\right)+c$

29 $\frac{1}{2}\ln\left|\sec\left(e^{x^2}\right)\right|+c$

30 11 m (= 5 + 5 + 1)

Chapter 6

Exercise 6.01

1 **a** 27.92 **b** 106.56 **c** 1908.436

2 **a** 1.340 **b** 0.373 **c** 0.989

3 **a** 20.66 **b** 1.543 **c** 5.690

4 Answers may vary slightly according to the calculator used.

a 5.997×10^{10} **b** 215.417 **c** 0

d $\pi\approx3.142$ **e** 150.913

5 1.666666667, 9, 10.66666667

$\int_{-1}^{0}(3+2x-x^2)\,dx+\int_{0}^{3}(3+2x-x^2)\,dx=\int_{-1}^{3}(3+2x-x^2)\,dx$

6 The function $y=\frac{1}{x-2}$ is not defined at $x=2$.

The first integral does not include $x=2$, but the second does, so cannot be evaluated.

7 About 9000 kg or 9 t

Exercise 6.02

1 **a** 235 **b** 233.125

2 **a** 132.08 **b** 131.52

3 **a** About 0.512 **b** About 0.511

4 **a** About 4.84 **b** About 4.83

5 **a** About 22.19 **b** About 21.78

6 **a** About 0.861 **b** About 0.868

7 **a** About 0.88 **b** About 0.88

8 **a** About 0.220 **b** About 0.216

9 292.5, 291.875, 291.7185. The value gets closer to a limit.

10 See worked solutions.

Exercise 6.03

1 **a** $508\frac{1}{2}$ **b** 48 **c** 12 480

d $624\frac{6}{7}$ **e** 6144 **f** $\frac{2}{5}$

g $95\frac{5}{6}$ **h** 480

2 **a** $\frac{x^4}{4}-\frac{7x^2}{2}-3x+c$ **b** $\frac{1}{4}\sin(4x)+c$

c $80e^{0.3x}+c$ **d** $\frac{1}{4}\ln(x)+c$

e $\frac{1}{5}\ln(5x-2)+c$ **f** $-\frac{1}{7}\ln(10-7x)+c$

g $2e^{x^3+3x}+c$ **h** $\frac{x^2}{2}+\frac{1}{2}\ln(2x+9)+c$

i $-20\cos(t^2)+c$ **j** $-e^{\cos(x)}+c$

k $\frac{5^{2x}}{2\ln(5)}+c$ **l** $-\frac{1}{5}e^{(4-5x)}+c$

3 **a** 0.07 **b** $\frac{1}{3}(e^{12}-e^3)\approx54\,245$

c 2 **d** $\frac{1}{\sqrt{2}}\approx0.71$

e $\frac{1}{3}(e^{\frac{3\pi}{4}}-e^{-\frac{3\pi}{4}})\approx3.49$ **f** $\frac{990}{\ln(10)}\approx429.95$

g $2(e^6-e^{-3})\approx806.76$

h $\frac{1}{5}(e^{40}-e^{15})\approx4.71\times10^{16}$

i $\ln(2.5)\approx0.92$ **j** $\ln(6)\approx1.79$

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k $2 \ln(19) \approx 5.89$

l $225 - 2 \ln(10) \approx 220.39$

4 a The function and indefinite integral are not defined at the limit $x = 0$.

b The function and indefinite integral are not defined at the limit $x = 5$.

c The function and indefinite integral are not defined at the limit $x = -1$.

5 The function and indefinite integral are not defined at the value $x = 2$, which is included in the limits of integration.

6 The function and indefinite integral are not defined at 0, which is included in the limits of integration. Also, the indefinite integral $\ln(x) + c$ is not defined for negative values of x.

7 1275 m^3

Exercise 6.04

1 a $7\frac{1}{2}$ **b** 39 **c** 74

d $67\frac{1}{2}$ **e** $43\frac{1}{2}$

2 $73\frac{5}{6}$

3 a 14.91 **b** 21.08 **c** 94.03

4 a $-\int_0^3 f(x)\,dx + \int_3^6 f(x)\,dx$

b $\int_{-9}^{-6} g(x)\,dx - \int_{-6}^{-2} g(x)\,dx$

c $-\int_{-5}^{4} h(x)\,dx$

d $\int_{-4}^{-1} k(x)\,dx - \int_{-1}^{3} k(x)\,dx$

e $-\int_{-5}^{-2} m(x)\,dx + \int_{-2}^{2} m(x)\,dx$

f $\int_1^3 p(x)\,dx - \int_3^5 p(x)\,dx + \int_5^6 p(x)\,dx$

5 $49\frac{1}{3}$

6 32

7 $31\frac{1}{4}$

8 $38\frac{1}{4}$

9 $24\frac{35}{54}$

10 $\frac{2}{3}$

11 6

12 $2(e^2 + e^{-2} - 2) \approx 11.05$

13 4

14 $\ln\left(\frac{8}{3}\right)$

15 $\frac{1}{2}e^4 + 8 \ln(2) + \frac{1}{2}e^{-2} - 8 \approx 24.91$

16 $e^3 - e^{-3} \approx 20.04$

Exercise 6.05

1 a $C'(x) = 0.001x^3 - 51x^2 + 625\,000x + 0.0015$

b $R(x) = 2350x$, where x is in thousands

c $P(x) = 2349.9985x - 0.000\,25x^4 + 17x^3 - 312\,500x^2 - 36$

d $P'(x) = 2349.9985 - 0.001x^3 + 51x^2 - 625\,000x$

2 a $C(x) = 2x^3 - 57x^2 + 780x + 1500$

b $P(x) = 40x - 2x^3 + 57x^2 - 1500$

c About \$6.126 million, 19 345 items

3 a $C(x) = 800 + 100x - 2x^2$ **b** \$1850

4 a $R(x) = 84x - 9x^3 + 18x^2$ **b** \$168 000

c About \$182 000

5 \$55 098

6 \$200 014

7 The marginal cost (\$1.30/drive) is cheaper than the average cost (\$1.80/drive). The average cost per drive must reduce as more drives are made, at least up to 800 drives.

8 \$95 751 for 96 dresses

Exercise 6.06

1 9

2 a 36 **b** $83\frac{1}{3}$ **c** 18

3 $58\frac{2}{3}$

4 $21\frac{1}{12}$

5 $63\frac{1}{3}$

6 $21\frac{1}{12}$

7 $2\frac{1}{4}$

8 $78\frac{1}{12}$

9 15

10 About \$5600

11 About \$1150

12 About 2 kg (Calculation 1.996)

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Exercise 6.07

1 **a** $v = 15 - 9.8t$ **b** 1.53 s
c $h = 2 + 15t - 4.9t^2$ **d** 13.48 m
e 2.85 s **f** 12.89 m/s

2 **a** $v = 0.6t + 1.75t^2$
b 181 m/s, 712 m/s, 1593 m/s
c $s = 0.3t^2 + \frac{7}{12}t^3$
d 613 m, 4786.67 m, 16 020 m

3 **a** $v = 6 + 3t$ **b** 2 s
c $d = 6t + 1.5t^2$ **d** 0.77 s
e 8.31 m/s **f** 1.37 s
g 10.10 m/s
h $v = 10.10 - 22.5t$, $s = 10.10t - 11.25t^2$
i 2.27 m

4 **a** $v = 8 + 3t + 0.8t^2$, $s = 8t + 1.5t^2 + \frac{4}{15}t^3$
b 99.8 m/s, 387.9 m
c 587.5 m
d $v = 99.8 - 4.1t$, $s = 587.5 + 99.8t - 2.05t^2$
e 24.34 s
f 1802.14 m

5 41.77 minutes

6 About $1\frac{1}{2}$ minutes ($93\frac{8}{9}$ s)

7 $v = -400\ln(4.5 - t) - 9.8t + 400\ln(4.5)$ or
$v = 400\ln\left(\frac{4.5}{4.5 - t}\right) - 9.8t$, height ≈ 194.09 m

8 $v = 50 - 50e^{-0.196t}$
$s = 50t + 255.102(e^{-0.196t} - 1)$
$1500 = 50t + 255.102(e^{-0.196t} - 1)$
$t \approx 35$ s (by trial and error)

Chapter 6 review

1 14.64
2 32.375
3 56.79
4 16.72
5 **a** 30 **b** 4 **c** $\frac{5}{6}$
d $\frac{2}{3}$ **e** $\frac{\sqrt{3}+1}{4} \approx 0.68$ **f** 0
g $\ln\left(\frac{8}{3}\right) \approx 0.98$ **h** $\frac{1}{2}\ln(5) \approx 0.80$
i $\frac{15}{\ln(5)}(5^{1.1} - 5^{0.1}) \approx 43.79$
j $\ln(11) \approx 2.40$

6 4444

7 26

8 $32\frac{3}{4}$

9 **a** \$20 578.13
b $P(x) = 46.85x - 0.000\,125x^3 + 0.0075x^2 - 20$, where x is in thousands and $P(x)$ is thousands of dollars
c $P'(x) = 46.85 - 0.000\,375x^2 + 0.015x$

10 $20\frac{5}{6}$

11 $57\frac{1}{6}$

12 **a** $v = 17 - 9.8t$, $s = 5 + 17t - 4.9t^2$
b 1.73 s **c** 19.74 m **d** 3.74 s
e 19.67 m/s downwards

13 **a** $v = \frac{9.8}{k}(e^{-kt} - 1)$ m/s $= 0.5(e^{-kt} - 1)$
$= 0.5e^{-kt} - 0.5$
b $k = 19.6$
c $s = 3 - 0.5t - \frac{5}{196}e^{-19.6t}$
d 2.5 m, 2.0 m, 1.5 m
e 6.0 s

14 About \$968 for the shade cloth, but his does not include labour, stitching of seams, suspension cable and so on.

15 About 39 h

16 About 5.85 s, 1.96 cm/s

17 57.24 s, 1805 m, $v = 35e^{-0.05t} + 20$, $s = -700e^{-0.05t} + 20t + 700$

Practice examination 2

1 **a** $3\cos(x)$ **b** $-5\sin(5m)$
c $\frac{1}{\cos^2(x)}$ or $1 + \tan^2(x)$

2 **a** $-6\cos(3x + 2)\sin(3x + 2)$
b $(2g + 3)\cos(g^2 + 3g + 1)$
c $\frac{\pi}{90\cos^2(\theta°)}$ or $\frac{\pi}{90}[1 + \tan^2(x°)]$

3 **a** $36(3x - 5)^2\cos(3x - 5)^3$
b $-\frac{2\sin(2p)}{\sqrt{\cos(2p)}}$

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4 **a** $32m^{-5}$ **b** $\dfrac{111}{2(7x+1)^2}$

c $-21(2z+3)(z^2+3z-2)^6$

5 **a** $\dfrac{2}{3}\sin\left(\dfrac{3x}{2}\right)+c$

b $2x^2-\dfrac{1}{4}\cos(4x-5)+c$

c $2e^{2x-5}+c$

6 **a** $-\dfrac{7}{3}\ln|3x-8|+c$ **b** $(e^x-2)e^{-2x}+c$

7 **a** 8

b $\dfrac{1}{2}x^2+x+8\ln|x-3|$

8 $\dfrac{2\sqrt{x^3-4}}{3}+c$

9 $2\ln|x^2+2|+c$

10 47.5

11 About 74 (73.987…)

12 $3\ln(5)\approx 4.82$

13 $129\frac{3}{4}$

14 $20\frac{5}{6}$

15 **a** $-60(6x+5)\tan^2(6x+5)^2$

b $\dfrac{5[3(3x-2)^2\cos(3x-2)^3\cos(2x)+2\sin(3x-2)^3\sin(2x)]}{\cos^4(2x)}$

16 $y=4x^3-2x^2+4$

17 6 m

18 About \$2163 (2162.962…)

19 $-\dfrac{0.25}{\tan(15°)}\ln(0.8)\,e^{0.05\times 35\ln(0.8)}\approx 0.14$ cm/s (up the slope)

20 $\dfrac{1}{2}e^{x^2}+c$

21 $\dfrac{8\sqrt{x^3+2}}{3}+c$

22 116 km, 22.3 km/s

Chapter 7

Exercise 7.01

1 **a** 27.0 **b** 30.5° **c** 32.6

d 39.2 **e** 36.7° **f** 58.0°

2 **a** 103.3 m **b** 20 m **c** 6.9 m

d 15.5 m **e** 24.6 m **f** 1.82 km

3 **a** 41.6° **b** 60° **c** 51.5°

d 26.4° **e** 32.9° **f** 54.1°

4 **a** 20 cm **b** 34.6 cm

5 **a** 8 mm **b** 13.9 mm **c** 21.2 mm

6 **a** 240° T or S 60° W **b** 800 m

c 1.39 km

7 323° T or N 37° W

8 65 km, 292.6°T or N 67.4° W

9 5.22 m

10 12.9 m

11 24.1 m

12 No, first part of slope is about 22°.

13 26 cm (at least 25.1 cm)

14 $\frac{1}{2}$ hour

15 217.35° T or S37.35° W

Exercise 7.02

1 **a** $-\sin(20°)$ **b** $-\cos(40°)$ **c** $-\tan(68°)$

d $-\cos\left(\dfrac{\pi}{8}\right)$ **e** $\sin\left(\dfrac{2\pi}{5}\right)$

2 **a** $\frac{1}{2}$ **b** $-\frac{\sqrt{3}}{2}$ **c** -1

d $\frac{1}{2}$ **e** $\frac{\sqrt{2}}{2}$ **f** $-\frac{\sqrt{2}}{2}$

g $\sqrt{3}$ **h** $-\frac{\sqrt{2}}{2}$ **i** $\frac{\sqrt{3}}{2}$

j $\frac{1}{\sqrt{3}}$

3 **a** $-\frac{\sqrt{3}}{2}$ **b** $-\frac{1}{2}$ **c** $-\sqrt{3}$

d $\frac{\sqrt{2}}{2}$ **e** $-\frac{\sqrt{3}}{2}$ **f** $-\frac{\sqrt{2}}{2}$

g $-\frac{1}{\sqrt{3}}$ **h** $\frac{\sqrt{2}}{2}$ **i** $-\frac{1}{2}$

j $\sqrt{3}$

4 **a** $-\sin(x)$ **b** $-\cos(y)$ **c** $\tan(x)$

d $-\cos(x)$ **e** $-\sin(x)$

5 Proof

6 $\sin(x)$

Exercise 7.03

1 **a** 141.7 m^2 **b** 10.6 cm^2 **c** 37.0 mm^2

d 6.0 m^2 **e** 24.1 mm^2 **f** 72.5 cm^2

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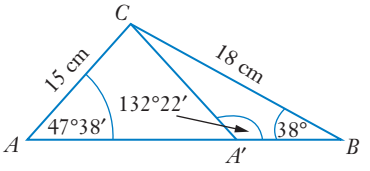
2 **a** 170 m^2 **b** 473 cm^2
c 139 982 m^2
3 **a** 348 mm^2 **b** 1599 cm^2
4, 5 Proofs

Exercise 7.04

1 **a** $a = 13.7, b = 13.4$ **b** $p = 13.4, q = 10.9$
c $h = 17.2, k = 13.9$ **d** $c = 34.3, d = 21.1$
e $m = 11.8, n = 6.6$ **f** $x = 38.5, y = 51.0$
2 **a** $X = 77°, Y = 38°, x = 17.4$ m
b $R = 91.9°, r = 84.7$ cm, $p = 49.0$ cm
c $E = 27°, D = 42°, e = 6.08$ km
d $A = 37.2°, a = 178.4$ cm, $b = 244.3$ cm
e $Z = 57.36°, y = 11.13$ m, $z = 11.55$ m
f $L = 65.6°, m = 1.942$ km, $k = 2.727$ km
3 **a** 40.6° **b** 31.7° **c** 54.1°
d 38.1° **e** 27.1° **f** 17.5°
4



$A = 47.63°$ (47°38′) or 132.37° (132°22′)
5 $D = 21.13°$ (21°8′) or 158.87° (158°52′), $e = 61.27$ cm or 5.30 cm
6 $Z = 100.3°, Y = 42.0°, y = 2.52$ cm or $Z = 79.7°$, $Y = 62.6°, y = 3.33$ cm
7 75.82 mm
8 6.5 m
9 3.19 km
10 31.4 cm
11 111.4°
12 6.69 km
13 Anne's height is 2992 m above the hill, so her altitude is really about 3397 m and her altimeter is reading about 1200 m too high

Exercise 7.05

1 **a** 5.4 cm **b** 7 m **c** 88.2 cm
d 55.1 m **e** 21.4 m **f** 43.8 cm
2 **a** 22.3° **b** 53.1° **c** 95.7°
d 78.0° **e** 109.5° **f** 134.4°
3 **a** $m = 25.8$ **b** $z = 283$ **c** $f = 64.6$
d $g = 43.7$ **e** $s = 157$ **f** $q = 952.9$
4 **a** $A = 21.4°, G = 37.6°, d = 7$ m
b $U = 56.7°, V = 68.3°, w = 44.1$ cm
c $X = 33.4°, Y = 107.3°, Z = 39.3°$
d $A = 23.2°, B = 112.0°, C = 44.8°$
e $M = 81.7°, N = 56.0°, l = 10.5$ km
f $R = 5.2°, F = 6.6°, e = 341$ mm
5 $a = 89.6°$
6 $X = 24.15°$
7 $P = 90°$
8 27 m
9 26.4° and 36.3°
10 120° T or S 60° E

Exercise 7.06

1 **a** 3.85 m **b** 17 mm **c** 50.25° **d** 41 cm
2 **a** 57.5 m **b** 10.8 m **c** 322 m **d** 9.2 cm
3 60.7 m, 76.12°
4 7.128 km/h, but the triangle is ill-conditioned, so it could be inaccurate
5 155 m
6 713 m
7 143.6 m, but the triangles are ill-conditioned, so it could be inaccurate
8 4.31 km
9 68.6 m, but the triangle is ill-conditioned, so it could be inaccurate
10 361 km, 94.63° T
11 85 m
12 54 m
13 16 m, but the triangles are ill-conditioned, so it could be inaccurate
14 Tower ≈ 7.8 m, building ≈ 12.5 m, but the triangles are ill-conditioned, so the answers could be inaccurate

Exercise 7.07

1 **a** 12.04 m (south), 9 m (west)
b 44.9° (south), 53.13° (west)
c 15.03 m **d** 55.62°
2 **a** 21.21 cm **b** 30.96° **c** 30.96°
d 40.32°

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3 **a** 72.6° **b** 54.4°

4 AYZ 51.34°, AZW 59.04°

5 401 m, 233.27° T or S 53.27° W

6 76° T or N 76°, E at 12.77°

7 3.16 m, 161.57° T or S 18.43° E

8 15.3°

9 81.5 m

10 155 m

11 77.1 m

Chapter 7 review

1 **a** 37.8 **b** 38°6′ **c** 47.7
d 47°4′ **e** 31.2 **f** 55°45′

2 **a** 66.93 km **b** 32.37 km

3 $-\cos(27°)$

4 **a** $-\frac{\sqrt{3}}{2}$ **b** $\frac{\sqrt{3}}{2}$ **c** -1

5 $-\cos(x)$

6 **a** 74.1 cm^2 **b** ≈ 3537 m^2

7 **a** $\angle B \approx 59.4°, \angle C \approx 78.6°, c \approx 41.0$ m or $\angle B \approx 120.6°, \angle C \approx 17.4°, c \approx 12.5$ m
b $\angle G = 93.4°, e \approx 23.8$ cm, $g \approx 49.9$ cm
c $\angle N \approx 22.1°, \angle K \approx 122.9°, k \approx 42.5$ m

8 **a** $a \approx 28.1$ m, $\theta \approx 95.1°$
b $b \approx 79.9$ m, $\varphi \approx 41.2°$
c $\alpha \approx 56.6°, \beta \approx 75.1°$

9 **a** 116 cm **b** 17.1 m

10 6500 mm, 2×3126 mm, 871 mm; $w = 29$ mm, $x = y = 277$ mm, $z = 15$ mm

11 16 km at 277° T

12 1965 m

13 91.1 km

14 208.16 m

15 150 m

16 111 mm from the ends and 44 mm from the sides

Chapter 8

Exercise 8.01

1 **a** $f'(x) = 5x^4 - 9x^2 + 7, f''(x) = 20x^3 - 18x$
b $f'(t) = 48t^7 + 35t^4 - 16t$, $f''(t) = 336t^6 + 140t^3 - 16$
c $f'(m) = 7m^6 - 20m^3 - 6, f''(m) = 42m^5 - 60m^2$
d $f'(x) = -2x^{-3} - 4x^3 - 3x^{-2} + 8$, $f''(x) = 6x^{-4} - 12x^2 + 6x^{-3}$
e $f'(z) = 6z^2 - 7 + 2z^{-\frac{1}{2}}, f''(z) = 12z - z^{-\frac{3}{2}}$
f $f'(x) = -x^{-2} - 10x + 3x^{-\frac{1}{2}} - 2x^{-\frac{1}{3}} + 4$,
$f''(x) = 2x^{-3} - 10 - \frac{3}{2}x^{-\frac{3}{2}} + \frac{2}{3}x^{-\frac{4}{3}}$

2 **a** 156 **b** 24 **c** 228
d −312 **e** $\frac{9}{4}$ **f** $2706 + \frac{\sqrt{3}}{3}$

3 **a** $f'(x) = 2\cos(2x), f''(x) = -4\sin(2x)$
b $f'(x) = 4 + 3\sin(3x), f''(x) = 9\cos(3x)$
c $f'(x) = \cos(x) - x\sin(x)$, $f''(x) = -x\cos(x) - 2\sin(x)$
d $f'(x) = 2x^2\cos(2x) + 2x\sin(2x)$, $f''(x) = 8x\cos(2x) + (2 - 4x^2)\sin(2x)$
e $f'(x) = 6x\cos(3x^2)$, $f''(x) = 6x\cos(3x^2) - 36x^2\sin(3x^2)$
f $f'(x) = 15x^2 + 18\sin(6x)$, $f''(x) = 30x + 108\cos(6x)$,

4 **a** $y' = 12e^{3x+1}, y'' = 36e^{3x+1}$
b $y' = -8e^{5-4x}, y'' = 32e^{5-4x}$
c $y' = 8xe^{4x^2+2}, y'' = (64x^2e^2 + 8e^2)e^{4x^2}$
d $y' = (2x + 1)e^{2x-3}, y'' = 4(x + 1)e^{2x-3}$
e $y' = 4x(2 - 3x)e^{1-3x}, y'' = 4(9x^2 - 12x + 2)e^{1-3x}$
f $y' = 2(5e^{3x} + 1)e^{2x-1}, y'' = 2(25e^{3x} + 2)e^{2x-1}$

5 **a** $\frac{dy}{dx} = \frac{1}{x}, \frac{d^2y}{dx^2} = -\frac{1}{x^2}$
b $\frac{dy}{dx} = \frac{5}{5x-7}, \frac{d^2y}{dx^2} = -\frac{25}{(5x-7)^2}$
c $\frac{dy}{dx} = \frac{10x}{5x^2-1}, \frac{d^2y}{dx^2} = -\frac{10(5x^2+1)}{(5x^2-1)^2}$
d $\frac{dy}{dx} = 2\ln(x) + 2, \frac{d^2y}{dx^2} = \frac{2}{x}$
e $\frac{dy}{dx} = 6x\ln(8 - 3x) - \frac{9x^2}{8-3x}$,
$\frac{d^2y}{dx^2} = 6\ln(8 - 3x) - \frac{9x(32-9x)}{(8-3x)^2}$
f $\frac{dy}{dx} = 2\ln(5x^2 - 1) + \frac{10x(2x-7)}{5x^2-1}$,
$\frac{d^2y}{dx^2} = \frac{10(10x^3+35x^2-6x+7)}{(5x^2-1)^2}$

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6 **a** $f'(x) = -30(1-6x)^4, f''(x) = 720(1-6x)^3$

b $f'(t) = \dfrac{-3}{(t-1)^2}, f''(t) = \dfrac{6}{(t-1)^3}$

c $g'(w) = \dfrac{w}{\sqrt{w^2+1}}, g''(w) = \dfrac{1}{(w^2+1)^{\frac{3}{2}}}$

d $h'(m) = \dfrac{-1}{m^2}, h''(m) = \dfrac{2}{m^3}$

e $f'(x) = \dfrac{4x}{3(x^2+1)^{\frac{1}{3}}}, f''(x) = \dfrac{4(x^2+3)}{9(x^2+1)^{\frac{4}{3}}}$

f $f'(x) = \dfrac{3}{p}+1, f''(x) = -\dfrac{3}{p^2}$

7 $f'(2) = \frac{17}{4}, f''(-3) = \frac{56}{27}$

8 $f'(x) = \dfrac{4}{\cos^2(4x)}, f''(x) = \dfrac{32\sin(4x)}{\cos^3(4x)}$

9 $f'(x) = (-1)\times x^{-2}, f''(x) = (-2)\times(-1)\times x^{-3}$,
$f'''(x) = (-3)\times 2\times 1\times x^{-4}$,
$f^4(x) = 4\times 3\times 2\times 1\times x^{-5}$,
$f^5(x) = (-5)\times 4\times 3\times 2\times 1\times x^{-6}$

$f^n(x) = \dfrac{(-1)^n n!}{x^{n+1}}$

Exercise 8.02

1 **a** $a = 6t - 8$ **b** -8 m/s^2 **c** 10 m/s^2

d $\frac{14}{3}$ s

2 **a** 40 m **b** $v = 17 - 10t, a = -10$

c The acceleration is constant and is approximately 10 m/s^2 downwards.

d 1.7 s

3 **a** $v = \dfrac{-22}{(4t-5)^2}$ **b** $a = \dfrac{176}{(4t-5)^3}$ **c** $\dfrac{176}{343}$ m/s^2

4 **a** $v = 2\cos(2t) - 2\cos(t)$

b $a = 2\sin(t) - 4\sin(2t)$

c About 4.85 cm/s^2

d About -5.47 cm/s^2

5 **a** $t = 2$ and $t = 6$

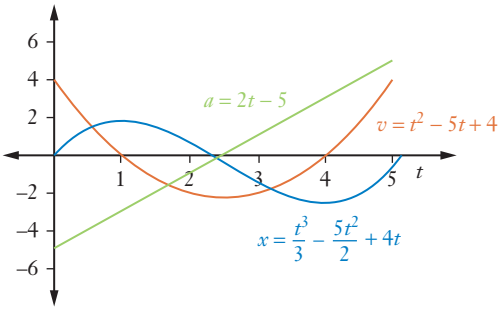
b $2 < t < 4$ and $t > 6$

c $0 < t < 2$ and $4 < t < 6$.

6 **a** $v = t^2 - 5t + 4, a = 2t - 5$

b $v = -2$ m/s, $a = 1$ m/s^2

c



d $0 < t < 1$ and $2.5 < t < 4$

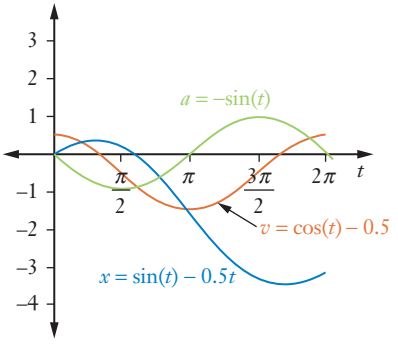
e $1 < t < 2.5$ and $4 < t < 5$

7 **a** 0 and $\frac{3}{4}$ s **b** 0 m and 7.5 m

8 **a** $v = \cos(x) - 0.5, a = -\sin(x)$

b $v \approx -1.15$ m/s, $a = 0.76$ m/s^2

c



d $0 < t < 1.05$ and $\pi < t < 5.24$

e $1.05 < t < \pi$ and $5.24 < t < 2\pi$

9 **a** $y' = \dfrac{-(2x+1)}{x^2(x+1)^2}$ and $y'' = \dfrac{2(3x^2+3x+1)}{x^3(x+1)^3}$

b Demonstration

c $y' = \dfrac{1}{(x+1)^2} - \dfrac{1}{x^2}$ and $y'' = \dfrac{2}{x^3} - \dfrac{2}{(x+1)^3}$. The second method has far less complex algebra.

Exercise 8.03

1 **a** Concave upwards for: $q < x < r$, $r < x < s$ and $t < x < u$.

b Concave downwards for: $p < x < q$, $s < x < t$ and $u < x < v$.

2 **a** Increasing for: $1 < x < 3$ and $7 < x < 8$; decreasing for: $3 < x < 7$.

b Increasing for: $1 < x < 3$ and $6 < x < 7$; decreasing for: $0 < x < 1$ and $3 < x < 6$.

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3 Point of inflection at $x = 1$ because the graph of $f(x)$ changes from concave upwards to concave downwards. Point of inflection at $x = 7$ because the graph of $f(x)$ changes from concave downwards to concave upwards.

4 $x < \frac{1}{3}$

5 $x > 1$

6 **a** $x > 1.5$ **b** $x < 1.5$

7 $-1.22 < x < 0.55$

8 Point of inflection at $(2, -13)$

9 Point of inflection at $(-1, -9)$

10 $(2, 0)$

11 Points of inflection at $(-2\pi, -2\pi)$, $(-\pi, -\pi)$, $(0, 0)$, (π, π) and $(2\pi, 2\pi)$. Concave upwards for $-\pi < t < 0$ and $\pi < t < 2\pi$. Concave downwards for $-2\pi < t < -\pi$ and $0 < t < \pi$

12 Points of inflection at $(-0.71, 1.41)$, $(0, 3)$ and $(0.71, 4.59)$. Concave downwards for $-2 < x < -0.71$ and $0 < x < 0.71$. Concave upwards for $-0.71 < x < 0$ and $0.71 < x < 2$.

13 $y'' = 12x^2 + 8$, so there are no solutions for $y'' = 0$. Therefore, there are no points of inflection. Because $12x^2 + 8 > 0$ for all values of x, y'' is always positive, so the curve is always concave upwards.

14 $a = 6$

15 $a = 2$

Exercise 8.04

1 **a** Minimum at $(1.5, 1.25)$

b Maximum at $(2, 17)$

2 Maximum at $(1, 5)$, minimums at $(3, -27)$ and $(0, 0)$

3 Minimum at $\left(-1\frac{2}{3}, -22\frac{7}{9}\right)$, maximum at $\left(1\frac{2}{3}, 32\frac{7}{9}\right)$

4 Point of horizontal inflection on a rising curve at $(1, 6)$.

5 Point of horizontal inflection on a falling curve at $(4, 5)$.

6 Point of horizontal inflection on a falling curve at $(0, 6)$, minimum at $\left(1\frac{1}{2}, 4\frac{5}{16}\right)$.

7 **a** Minimum at $(0, 1)$

b Maximum at $\left(2, \frac{4}{e^2}\right)$, minimum at $(0, 0)$

8 **a** Maximum at $\left(2, -\frac{1}{9}\right)$

b None

c Maximum at $(0, -2)$, minimum at $(2, 6)$

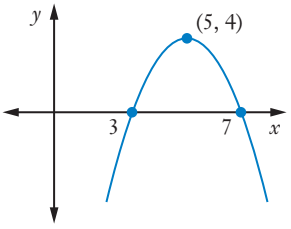
9 Points of horizontal inflection on a rising curve at $\left(-\frac{3\pi}{2}, -\frac{3\pi}{2}\right)$ and $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$.

10 **a** $a = -4$ **b** Minimum

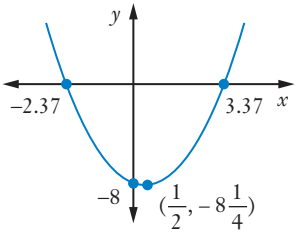
11 $a = 2$ and $b = -6$

Exercise 8.05

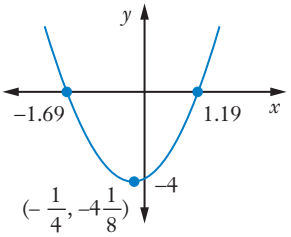
1 **a** $g(x) = 10x - x^2 - 21$



b $h(x) = x^2 - x - 8$



c $h(x) = 2x^2 + x - 4$



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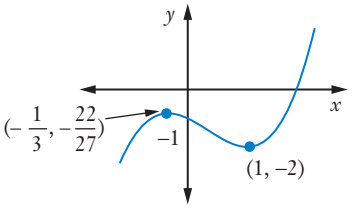
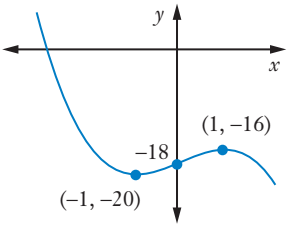
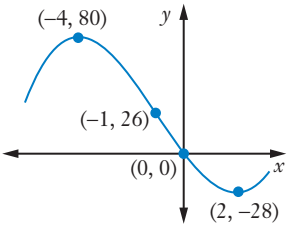
d $q(x) = x^3 - x^2 - x - 1$



e $f(x) = 3x - x^3 - 18$



f $f(x) = x^3 + 3x^2 - 24x$



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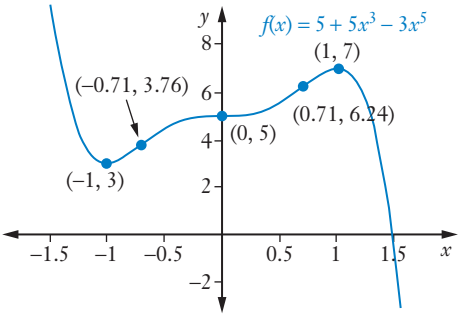

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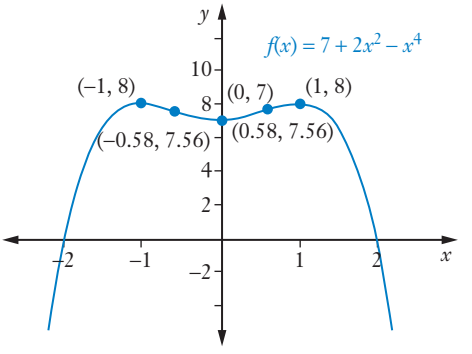

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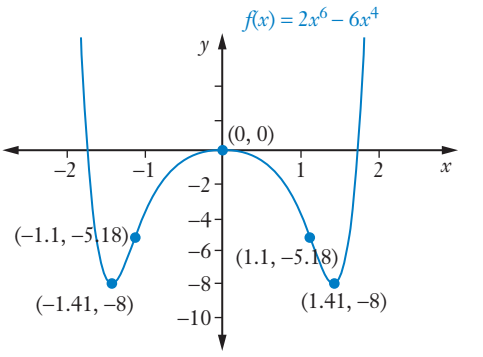
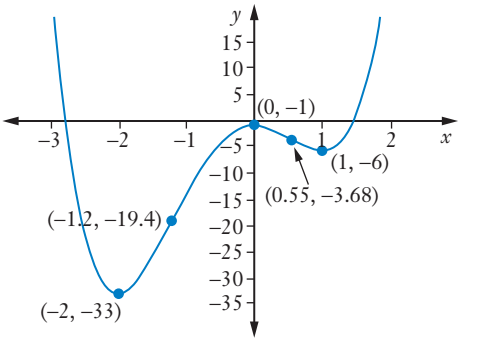


d $f(x) = 3x^4 + 4x^3 - 12x^2 - 1$



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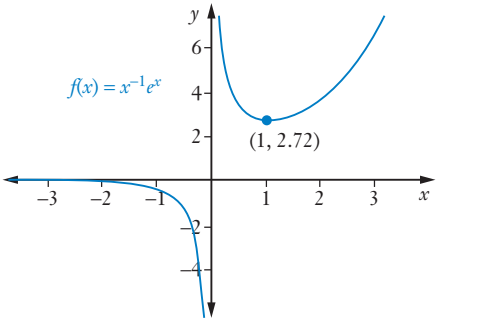

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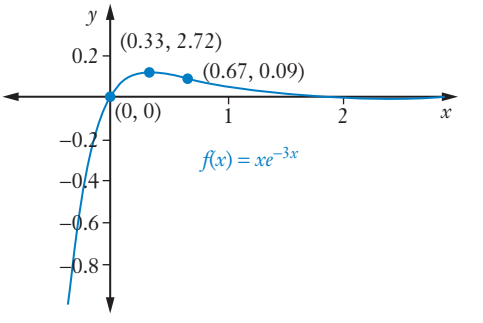

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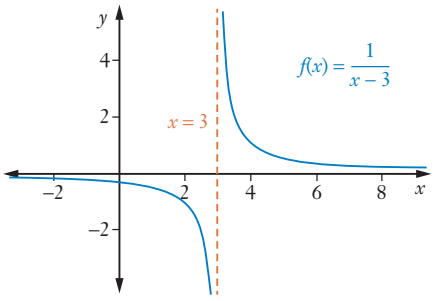

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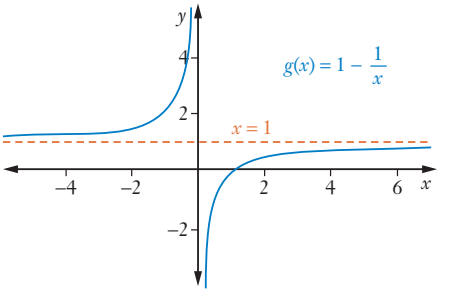

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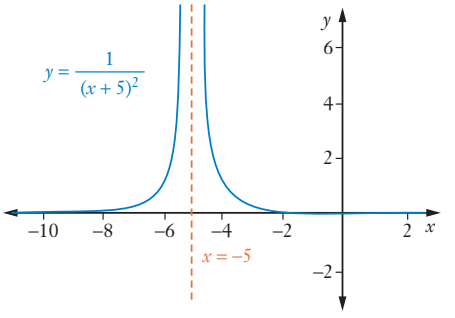

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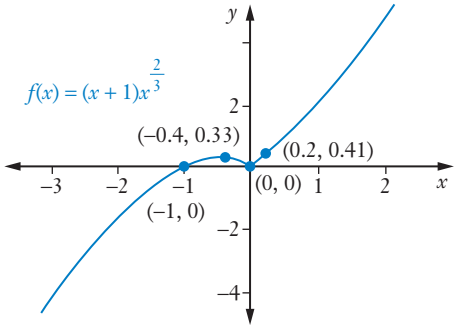

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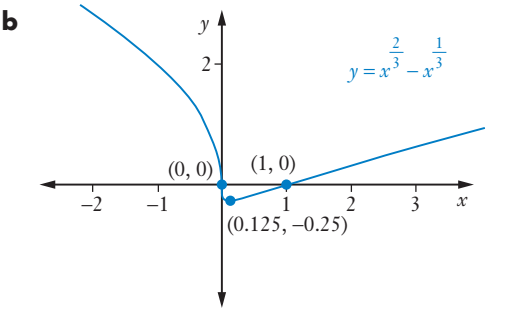

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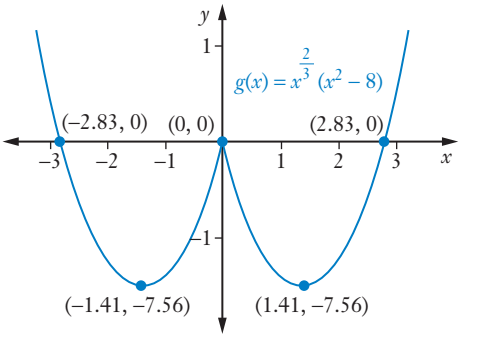

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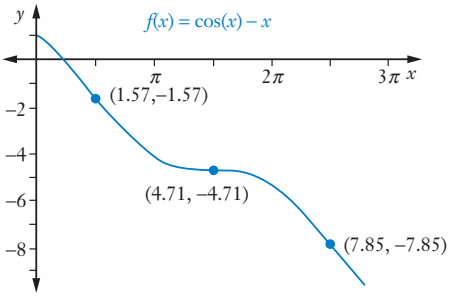

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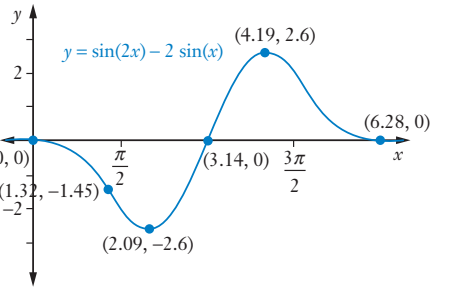

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Exercise 8.06

1 **a** $l = 200 - 2x$ **b** $A = x(200 - 2)$
c 5000 m^2

2 **a** $l = 63 - 2x$; $w = 48 - 2x$
b $V = x(63 - 2x)(48 - 2x)$
c $x = 9$ cm **d** 12 150 cm^3

3 **a** $A = xy$ **b** $A = x\sqrt{100 - x^2}$
c 50 cm^2

4 **a** $R(x) = 4x$
b $P(x) = -50 + 2.7x - 0.001x^2$
c 1350 items

5 15 tonnes/month

6 **a** $h = \frac{200\,000}{1200x}$ **b** $C(x) = \frac{4000}{3x} + 2x$
c 26 plates

7 **a** $t = \frac{d}{x}$ hours
b $P(x) = 250\,000dx^{-1} + dx^2$
c 50 km/h

8 **a** $T = 40n - 0.05n^2$ **b** $n = 400$
c 8000 kg/bed

9 7.79 cm

10 Height ≈ 1.68 m, width ≈ 3.36 m

11 Radius ≈ 1.89 m, height ≈ 2.67 m, for volume ≈ 9.93 m^3

12 $\frac{18}{4 + \pi}$ m^2

13 4000 cm^3

14 **a** 1500 **b** 3500, $82 500

15 **a** $C(x) = 100 + 250x - 5x^2 + \frac{x^3}{3}$
b $R(x) = x(5000 - 5x)$
c $P(x) = -100 + 4750x - \frac{x^3}{3}$
d 69

16 11

17 7 (integer value)

18 About 1.18 km

19 20 km/h

20 About 15 m

21 **a** $\frac{1}{20}$
b 0.36 months (10 or 11 days)

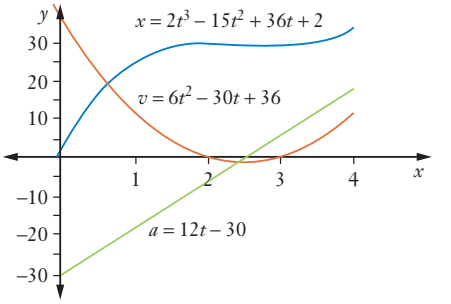
Chapter 8 review

1 $f'(x) = 8x^3 + 3x^2 - 4x$; $f''(x) = 24x^2 + 6x - 4$; $f''(-3) = 194$

2 **a** $f''(x) = -4x \cos(2x) - 4 \sin(2x)$
b $y'' = 36e^{6x+1}$
c $g''(x) = \frac{-2(x^2 + 5)}{(x^2 - 5)^2}$

3 **a** $a = 6t - 12$ **b** -12 m/s^2
c 6 s

4 **a** $v = 6t^2 - 30t + 36$; $a = 12t - 30$
b $v = 12$ m/s, $a = 18$ m/s^2
c



d $2 < t < 2.5$ and $t > 3$
e $0 < t < 2$ and $2.5 < t < 3$

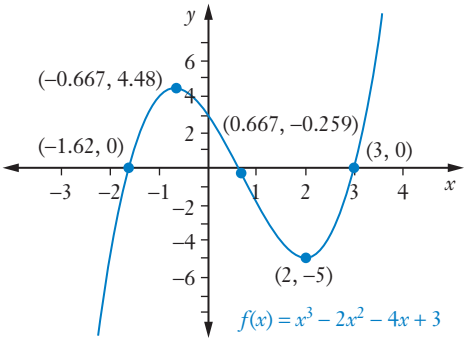
5 Concave upwards: $x < -1$ and $x > 1$; concave downwards: $-1 < x < 1$

6 $(-1, -6)$ is a point of inflection on a rising curve and $(3, -182)$ is a point of inflection on a falling curve

7 Local maximum at $(-1, 0)$, local minimum at $\left(1\frac{2}{3}, -9\frac{13}{27}\right)$

8 Point of horizontal inflection on a falling curve at $(0, 10)$, minimum at $(3, -17)$.

9



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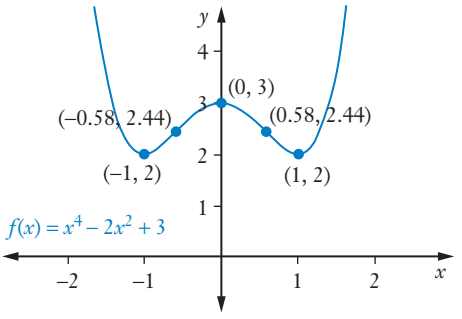
10 a Local minimum at (–1, 2) and (1, 2); local maximum at (0, 3). Points of inflection at (–0.58, 2.44) and (0.58, 2.44).

Concave upwards: $x < -0.58$ and $x > 0.58$

Concave downwards: $-0.58 < x < 0.58$

b



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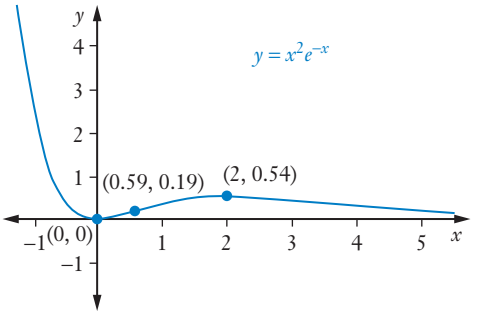


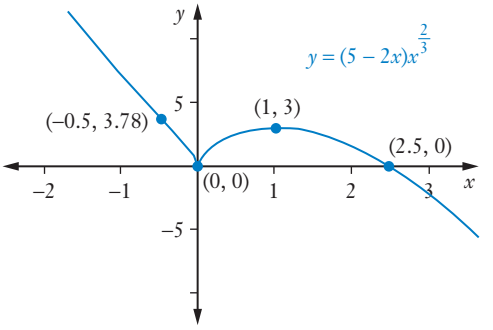
12 500 000 m^2

13 a 2500 **b** 2000

14 10

15 $a = -1, b = 3$

16



17 a About 2.62 cm × 8.10 cm × 14.77 cm

b About 3.92 cm × 12.15 cm × 22.15 cm

c About 6.54 cm × 20.25 cm × 36.92 cm

18 75.8 km/h, $53.58

19 About 63.9 m

Chapter 9

Exercise 9.01

1 a, b, d, e and f are Bernoulli distributions

2 a i Success = odd number; **ii** $p = \frac{1}{2}$

b i Success = flu injection;

ii p = fraction of population who get the flu injection

d i Success = head; **ii** $p = \frac{1}{2}$

e i Success = Ace; **ii** $p = \frac{1}{13}$

f i Success = multiple of 3;

ii $p = \frac{1}{3}$

3 a $\frac{2}{5}$ **b** $\frac{3}{13}$ **c** $\frac{2}{5}$

d $\frac{1}{3}$ **e** $\frac{1}{4}$

4 a

x	0	1
$P(X = x)$	0.8	0.2

b $E(X) = 0.2$ **c** $SD(X) = 0.4$

5 a $\frac{3}{16}$ **b** 0.24 **c** $\frac{6}{25}$

d $\frac{33}{196}$ **e** 0.21

6 a 0.4714 **b** 0.4583 **c** 0.4989

d 0.4996 **e** 0.4330

7 a 0.68 **b** 0.2176 **c** 0.4665

8 a

y	0	1
$P(Y = y)$	0.72	0.28

b $E(Y) = 0.28$ $Var(Y) = 0.2016$

9 a

y	0	1
$P(Y = y)$	0.21	0.79

b $SD(X) = 0.4073$

10 a 0.6 **b** 0.6

11 a 0.2356 **b** 0.62

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Exercise 9.02

1 **a**, **b** and **d** are geometric distributions.

2 About 0.0804

3 **a** 0.128 **b** 0.081 92

c About 0.041 94

4 0.0625

5 8 darts

6 **a** $\frac{25}{216}$ **b** 5 rolls

7 **a** 0.1029 **b** About 0.3724

c About 0.8319

8 **a** 0.1024 **b** 0.262

9 **a** $\frac{1}{6}$ **b** About 0.096 45

c About 0.5784 **d** About 0.6651

10 **a** About 0.011 72 **b** About 0.2499

11 **a** 0.032 **b** 4.096×10^{-7}

c 0.9984 **d** 0.0016

12 About 0.3071

13 $\frac{11}{36}$

Exercise 9.03

1 **a** Binomial

b Binomial

c Not binomial as the number of trials is not fixed.

d Not binomial as there are more than two outcomes.

e Binomial

f Not binomial as the probability of success changes from trial to trial.

g Binomial

h Binomial

2 **a** 0.3125 **b** 0.0413 **c** 0.1318

d 0.1499 **e** 0.0037 **f** 0.0208

3 **a** $p = 0.5, q = 0.5, n = 12$

b $p = 0.1, q = 0.9, n = 15$

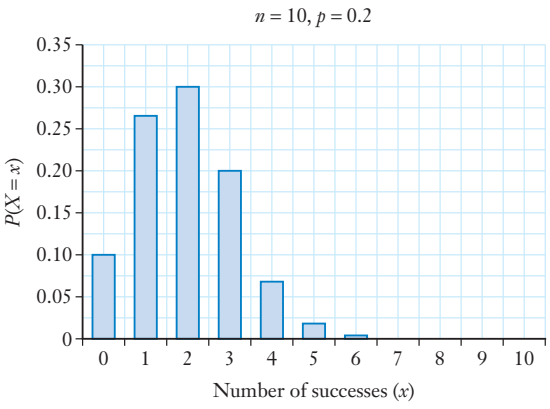
c $p = 0.6, q = 0.4, n = 7$

d $p = 0.23, q = 0.77, n = 25$

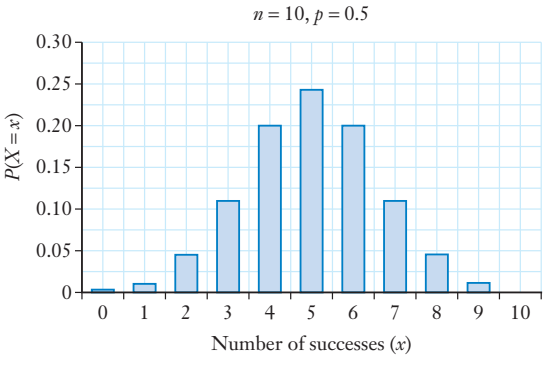
e $p = 0.47, q = 0.53, n = 8$

f $p = 0.67, q = 0.33, n = 20$

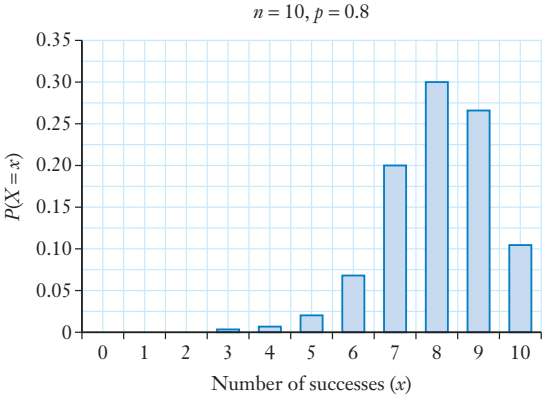
4 **a**



b



c



5 **a** The distribution is positively skewed.

b The distribution is symmetrical.

c The distribution is negatively skewed.

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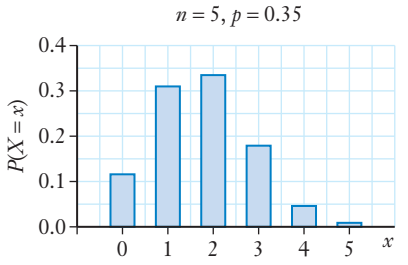
6 **a** 5 **b** 0.35

c

x	0	1	2	3	4	5
$P(X = x)$	0.1160	0.3124	0.3364	0.1811	0.0488	0.0053

d The distribution is positively skewed.

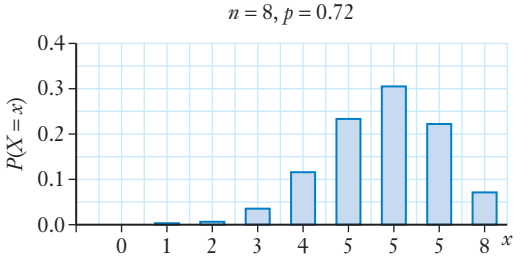


7 **a** 8 **b** 0.72

c

x	0	1	2	3	4	5	6	7	8
$P(X = x)$	0.0000	0.0008	0.0070	0.0360	0.1156	0.2379	0.3058	0.2247	0.0722

d The distribution is negatively skewed.



8 **a** 0.0102 **b** 0.3456 **c** 0.0778

9 **a** 0.2241 **b** 0.0024 **c** 0.00003

10 0.0543

11 0.1290

12 0.1614

Exercise 9.04

1 **a** 0.1062 **b** 0.2156 **c** 0.2010
d 0.1444

2 **a** 0.1510 **b** 0.0003 **c** 0.0391
d 0.1512 **e** 0.0553

3 **a** 0.0273 **b** 0.1398 **c** 0.2509
d 0.2291

4 **a** 0.4557 **b** 0.0410 **c** 0.9959
d 0.5443

5 **a** 0.4096 **b** 0.1808 **c** 0.5904
d 0.4096

6 **a** 0.3177 **b** 0.8740 **c** 0.0288
d 0.6471

7 0.2742

8 **a** 0.9764 **b** 1.0 **c** 0.0236
d 0.9764

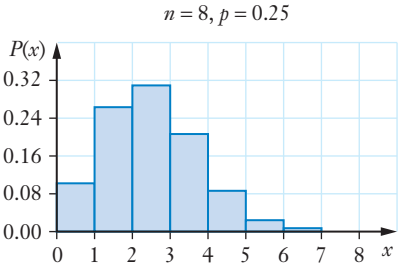
9 **a** 0.9598 **b** 0.4269
c 0.000 000 000 0003
d 0.9989

10 **a** 0.0100 **b** 0.0389 **c** 0.3893
d 0.6983 **e** 0.9249

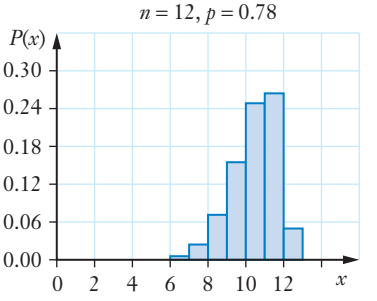
Exercise 9.05

1 **a** 0.9 **b** 0.3 **c** 0.5 **d** 0.7 **e** 0.1

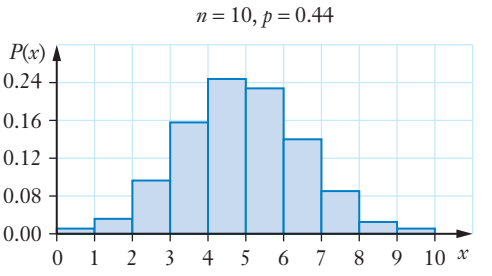
2 The graph is positively skewed.



3 The graph is negatively skewed.



4 The graph is positively skewed.



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5 **a** 57 **b** 35.34
6 **a** $\mu = 3.9$ and $\sigma = 1.699$
b $\mu = 32.8$ and $\sigma = 2.430$
c $\mu = 37.4$ and $\sigma = 4.968$
d $\mu = 14.96$ and $\sigma = 3.524$
7 **a** 4 or 5 **b** About 71.78%
c About 94.53%
8 **a** 48 or 49 **b** About 95.64%
9 **a** $p = 0.38$ **b** $n = 20$
10 **a** $p = 0.66$ **b** $n = 35$
11 **a** 0.44 **b** $Var(X) = 6.16$, $\sigma = 2.482$
12 0.75
13 2 car parts

Exercise 9.06

1 **a** 0.1730 **b** 0.8665 **c** 0.7564 **d** 0.9871
2 **a** 0.0041 **b** 0.8237 **c** 0.5785 **d** 0.9976
3 **a** 0.7480 **b** 9 **c** 0.7393 **d** 0.9597
4 **a** About 15 **b** 0.6783
5 **a** 0.5391 **b** 6 times
6 **a** 0.3277 **b** 14 widgets
7 **a** 0.6010 **b** 7 **c** 2.13 **d** 0.7600
8 About 5 (4.6)
9 625
10 **a** 1 **b** $\sigma \approx 0.97$
11 **a** 0.0002 **b** 17, 1.579 **c** 0.9781
12 **a** 4 **b** 3.68 **c** 1.918
13 **a** 168 **b** About 10.04
14 **a** About 2 (1.6) **b** 0.96 **c** 0.98
15 **a** 0.2725 **b** 0.1702 **c** 34 **d** 19
16 4 desks **17** 9 shots

Chapter 9 review

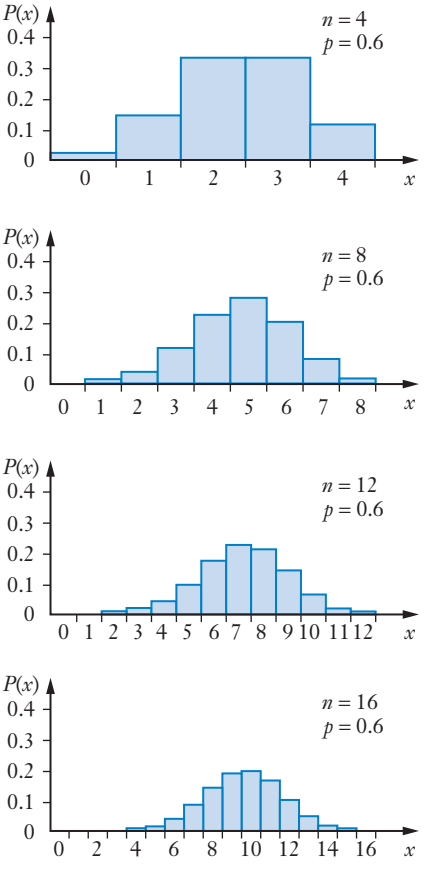
1 **a**, **c** and **d** result in a Bernoulli distribution
2 **a**

y	0	1
$P(Y = y)$	0.6	0.4

b 0.4 **c** 0.4899
3 **a** 0.0074 **b** 0.9993
4 9 samples
5 **a** 0.28 **b** 0.3249
c 0.3732 **d** 0.1045
6 **b** and **c** are binomial experiments
7 **a** 8 **b** 0.38
c

x	0	1	2	3	4	5	6	7	8
$P(X = x)$	0.0218	0.1071	0.2297	0.2815	0.2157	0.1058	0.0324	0.0057	0.0004

8 0.0735
9 0.0305
10 **a** 0.9957 **b** 0.0782 **c** 0.9994 **d** 0.0043
11 0.6672
12 **a**



b The graphs are skewed to the left. As n increases they become lower, more symmetrical and smoother.

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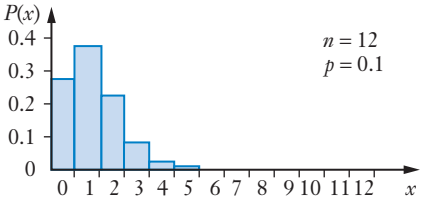
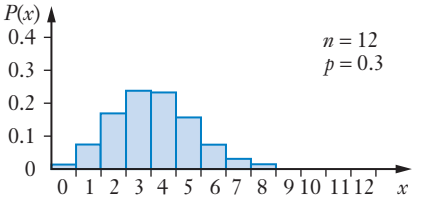
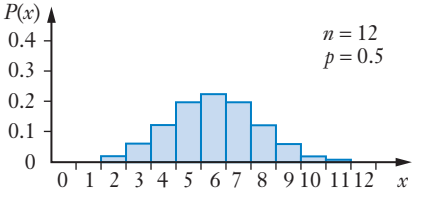
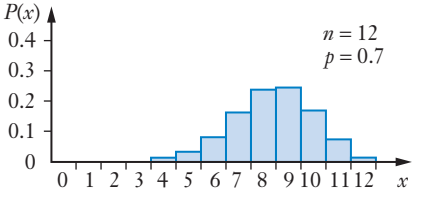
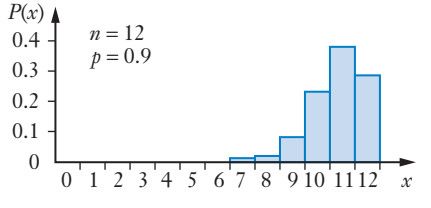
13 a











b The smaller the value of p, the more the graph is skewed to the right. For $p = 0.5$ the graph is symmetrical and lowest. The larger the value of p, the more it is skewed to the left.

14 a 42 **b** 0.7076

15 a 0.45 **b** 12

16 a 0.0244 **b** 0.5846 **c** 0.6416

17 a 2 **b** 0.7414

18 a 0.9962 **b** 2 games

19 a 0.2275 **b** $p = 0.65$

20 $p = 0.6$, $n = 50$, $P(20 \leq X \leq 30) = 0.5521$

21 26.39%

Practice examination 3

1 a About 58° **b** About 17 m

2 $-\frac{1}{2}$

3 About 51.1 m^2

4 a About 11.4 cm

b About 54° (53.585…)

c about 64.3°

d about 231.94 m

5 2.00 or 6.39 m

6 a $f'(x) = 7x^6 - 20x^3 + 11$; $f''(x) = 42x^5 - 60x^2$

b $f'(x) = -6x + 4x^{-\frac{1}{2}} - \frac{8}{3}x^{-\frac{2}{3}} - 2x^{-3} + 12$;

$f''(x) = = -2x^{-\frac{3}{2}} + \frac{16}{9}x^{-\frac{5}{3}} + 6x^{-4} - 6$

c $f'(x) = 2x\cos(2x) - 2x^2\sin(2x)$;

$f''(x) = (2 - 4x^2)\cos(2x) - 8x\sin(2x)$;

d $f'(x) = (3 - 4x)5x^2e^{1-4x}$;

$f''(x) = 10x(8x^2 - 12x + 3)e^{1-4x}$

7 Concave upwards: $x < 1$; concave downwards: $x > 1$

8 Local maximum at (0, –2); Local minima at (–1, –5) and (1, –5)

9 a

x	0	1
$P(X = x)$	0.32	0.68

b $E(X) = p = 0.68$

c $\sigma \approx 0.4665$

10 0.0819

11 a 5 **b** 0.3

c

x	0	1	2	3	4	5
$P(X = x)$	0.1681	0.3602	0.3087	0.1323	0.0284	0.0024

12 a 0.9953 **b** 0.4233

13 About 14 m (14.257…)

14 $162 each

15 a $p = 0.54$ **b** $n = 15$

16 About 352°T, but two triangles are ill-conditioned, so this could be inaccurate

17 6 m × 6 m × 9 m

18 $n = 8$

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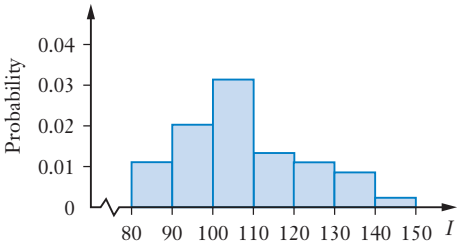
Chapter 10

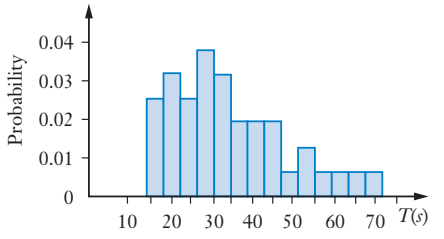
Exercise 10.01

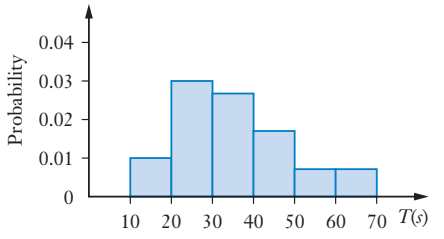
1 a



b The total area is 1.

2 a



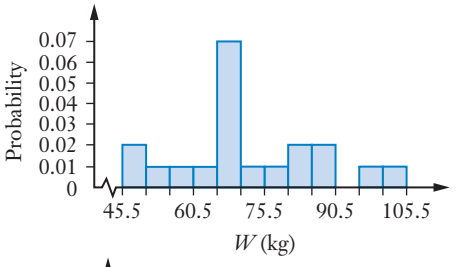


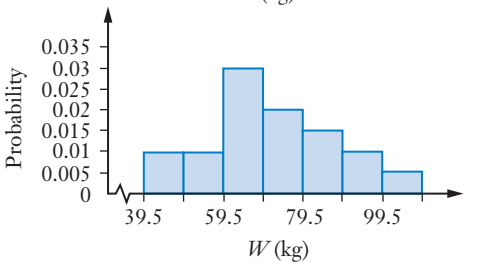
b Total areas are both 1.

3 About 13.6%

4 0.2625

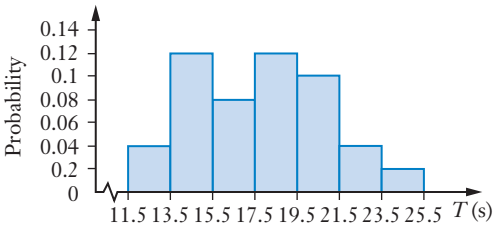
5

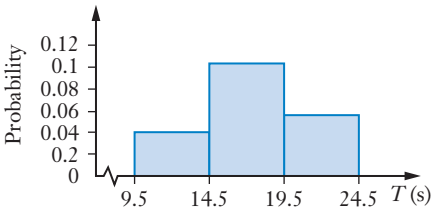




b The area of each histogram is 1. **c** 0.3

6





b The area of each histogram is 1. **c** 0.57

Exercise 10.02

1 a, c, e, f, g and h are pdfs.

2 **a** 1.5 **b** 24 **c** $\frac{1}{18}$ **d** $\frac{1}{6}$
e 4 **f** 0.1

3 **a** $F(x) = x^4 - 1$

b **i** $\frac{1}{81}$ **ii** 0.1350 **iii** 0.9375

4 $F(x) = -\frac{1}{2x}$ for $0 < x < 2$

5 **a** $F(x) = \frac{3}{4}\left(x - \frac{x^3}{3} + \frac{2}{3}\right)$ for $-1 \le x \le 1$.

b **i** 0.84375 **ii** 0.6875 **iii** 0.15625

6 **a** $k = \frac{4}{11}$

b $F(x) = \frac{4}{11}\left(\frac{x^4}{4} - x + \frac{3}{4}\right)$ for $1 \le x \le 2$

c **i** 0.1875 **ii** 0.5725 **iii** 0.8125

7 **a** $\frac{3}{4}$ **b** $\frac{1}{4}$ **c** $\frac{5}{8}$

8 **a** $f(x) \ge 0$ for $x \ge 0$ and $\int_0^\infty f(x)\,dx = 1$

b **i** 0.2835 **ii** 0.4119

9 **a** $f(x) \ge 0$ for $x \ge 0$ and $\int_0^1 f(x)\,dx = 1$

b **i** $\frac{17}{81}$ **ii** $\frac{47}{162}$

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10 $k = 0.2$

11 **a** **i** 0.5625 **ii** 0.1875

b $F(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{x^2}{4} & \text{for } 0 \le x < 2 \\ 1 & \text{for } x > 2 \end{cases}$

12 **a** $f(x) = \frac{1}{24}$ for $[0, 24]$ **b** About 0.1146

Exercise 10.03

1 **a** $p(x) = \frac{1}{10}$ **b** $p(x) = \frac{1}{6}$

c $p(x) = \frac{1}{13}$ **d** $p(x) = \frac{1}{90}$

e $p(x) = \frac{1}{400}$

2 **a** $p(x) = \frac{1}{18}$ for $0 \le x \le 18$ **b** $\frac{2}{3}$

3 0.5833

4 **a** $f(x) = \begin{cases} \frac{x}{36} & \text{for } 0 \le x \le 6 \\ \frac{12-x}{36} & \text{for } 6 \le x \le 12 \end{cases}$

b $f(x) = \begin{cases} \frac{x-3}{36} & \text{for } 3 \le x \le 9 \\ \frac{15-x}{36} & \text{for } 9 \le x \le 15 \end{cases}$

c $f(x) = \begin{cases} \frac{x-8}{26} & \text{for } 8 \le x \le 12 \\ \frac{2(21-x)}{117} & \text{for } 12 \le x \le 21 \end{cases}$

d $f(x) = \begin{cases} \frac{x-5}{48} & \text{for } 5 \le x \le 13 \\ \frac{17-x}{24} & \text{for } 13 \le x \le 17 \end{cases}$

e $f(x) = \begin{cases} \frac{x-7}{56} & \text{for } 7 \le x \le 15 \\ \frac{21-x}{42} & \text{for } 15 \le x \le 21 \end{cases}$

5 **a**

b $f(x) = \begin{cases} \frac{x-1}{7} & \text{for } 1 \le x \le 3 \\ \frac{16-2x}{35} & \text{for } 3 \le x \le 8 \end{cases}$

c **i** $\frac{1}{14}$ **ii** $\frac{9}{35}$ **iii** $\frac{3}{7}$

6 **a** $f(x) = \begin{cases} \frac{x-2}{18} & \text{for } 2 \le x \le 5 \\ \frac{14-x}{54} & \text{for } 5 \le x \le 14 \end{cases}$

b **i** $\frac{1}{9}$ **ii** $\frac{49}{108}$ **iii** $\frac{101}{108}$

7 $\frac{3}{5}$

8 $\frac{7}{9}$

9 $\frac{56}{65}$

10 **a** 0.245 **b** 0.03125

Exercise 10.04

1 **a** $p(x) = \frac{1}{17}$ for $5 \le x \le 22$. **b** $E(X) = 13.5$

2 $E(X) = 28$

3 $E(X) = 55.5$

4 **a** $f(x) = \begin{cases} \frac{1}{56}(x-4) & \text{for } 4 \le x \le 12 \\ \frac{1}{42}(18-x) & \text{for } 12 \le x \le 18 \end{cases}$

b $E(X) = 11\frac{1}{3}$

5 $E(X) = 4$

6 **a** $E(X) = 9$ **b** $E(X) = 36\frac{2}{3}$

c $E(X) = 32\frac{1}{3}$ **d** $E(X) = 38$

7 $E(X) = \frac{1}{4}$

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8 $E(X) = \frac{1}{3}$

9 **a** $E(X) = 4$ **b** $E(X) = \frac{3}{4}$

c $E(X) = 1\frac{1}{5}$

10 $p(r) = \frac{d}{dr}\left(\frac{\pi r^2}{\pi \times 15^2}\right) \approx 0.008\,89r$,

$E(R) = \int_0^{15} r \times p(r)\,dr = 10$ cm

Investigation – Linear transformations, expected values and variance

Expected value

$E(X) = \int_{-\infty}^{\infty} xp(x)dx$

$E(Y) = E(aX + b) = \int_{-\infty}^{\infty} (ax + b)p(x)\,dx$

$= \int_{-\infty}^{\infty} axp(x)\,dx + \int_{-\infty}^{\infty} bp(x)\,dx$

$= a\int_{-\infty}^{\infty} xp(x)\,dx + b\int_{-\infty}^{\infty} p(x)\,dx$

But $\int_{-\infty}^{\infty} xp(x)\,dx = E(X)$ and $\int_{-\infty}^{\infty} p(x)\,dx = 1$

So $E(aX + b) = aE(X) + b$

Variance

$Var(X) = E(X^2) - [E(X)]^2$

$Var(Y) = \int_{-\infty}^{\infty} (ax + b)^2\,p(x)\,dx - [aE(X) + b]^2$

$= \int_{-\infty}^{\infty} (a^2x^2 + 2axb + b^2)p(x)\,dx -$
$[a^2[E(X)]^2 + 2abE(X) + b^2]$

$= \int_{-\infty}^{\infty} a^2x^2p(x)\,dx + \int_{-\infty}^{\infty} 2axbp(x)\,dx +$
$\int_{-\infty}^{\infty} b^2p(x)\,dx - a^2[E(X)]^2 - 2abE(X) - b^2$

$= a^2\int_{-\infty}^{\infty} x^2p(x)\,dx + 2ab\int_{-\infty}^{\infty} xp(x)\,dx +$
$b^2\int_{-\infty}^{\infty} p(x)\,dx - a^2[E(X)]^2 - 2abE(X) - b^2$

But $\int_{-\infty}^{\infty} xp(x)\,dx = E(X)$, $\int_{-\infty}^{\infty} x^2p(x)\,dx = E(X^2)$ and

$\int_{-\infty}^{\infty} p(x)\,dx = 1$

So $Var(Y) = a^2\,E(X^2) + 2abE(X) + b^2 - a^2[E(X)]^2$
$- 2abE(X) - b^2$
$= a^2\,E(X^2) - a^2[E(X)]^2$
$= a^2[E(X^2) - E(X)]^2$
$= a^2Var(X)$

1 **a** 25 **b** 40

2 110

3 **a** 440 **b** 85

Exercise 10.05

1 **a** $p(x) = \frac{1}{16}$ **b** $E(X) = 16$

c $Var(X) = 21\frac{1}{3}$, $SD(X) = \frac{8\sqrt{3}}{3}$

2 **a** $E(X) = 10$, $Var(X) = \frac{100}{3}$, $SD(X) = \frac{10\sqrt{3}}{3}$

b $E(X) = 7.5$, $Var(X) = \frac{27}{4}$, $SD(X) = \frac{3\sqrt{3}}{2}$

c $E(X) = 25$, $Var(X) = \frac{100}{3}$, $SD(X) = \frac{10\sqrt{3}}{3}$

d $E(X) = 45$, $Var(X) = \frac{625}{3}$, $SD(X) = \frac{25\sqrt{3}}{3}$

e $E(X) = 1.5$, $Var(X) = \frac{1}{3}$, $SD(X) = \frac{\sqrt{3}}{3}$

3 **a** $f(x) = \begin{cases} \frac{1}{64}(x-6) & \text{for } 6 \le x \le 14 \\ \frac{1}{64}(22-x) & \text{for } 14 \le x \le 22 \end{cases}$

b $E(X) = 14$ **c** $Var(X) = 10\frac{2}{3}$, $SD(X) = \frac{4\sqrt{6}}{3}$

4 **a** $f(x) = \begin{cases} \frac{25}{36}(x-0.4) & \text{for } 0.4 \le x \le 1.6 \\ \frac{25}{36}(2.8-x) & \text{for } 1.6 \le x \le 2.8 \end{cases}$

b $E(X) = 1.6$ **c** $Var(X) = 0.24$

5 **a** $f(x) = \begin{cases} \frac{1}{63}(x-8) & \text{for } 8 \le x \le 15 \\ \frac{1}{99}(26-x) & \text{for } 15 \le x \le 26 \end{cases}$

b $E(X) = 16\frac{1}{3}$ **c** $Var(X) = 13\frac{13}{18}$ (≈ 13.72)

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6 $E(X) = -2\frac{2}{3}$, $Var(X) = \frac{2}{9}$

7 $E(X) = 2\ln(2)$, $Var(X) \approx 0.078\ 188$

8 $E(X) = \frac{3}{4}$, $Var(X) = \frac{3}{80}$

9 $\sigma = \frac{5\sqrt{6}}{3}$

10 $E(X) = 15.5$

11 $E(X) = 12$

Exercise 10.06

1 **a** Distribution A **b** Distribution B

2 **a** $0.5319e^{-0.888889(x-3)^2}$

b $0.07979e^{-0.02(x-20)^2}$

c $0.05249e^{-0.008657(x-34.5)^2}$

d $0.01247e^{-0.000488(x-128)^2}$

e $0.9974e^{-03.125(x-1.7)^2}$

f $2.659e^{-22.22(x-0.8)^2}$

3 **a** $\mu = 12$ and $\sigma = 3$ **b** $\mu = 2$ and $\sigma = \frac{1}{2}$

c $\mu = 2$ and $\sigma = 1$ **d** $\mu = 20$ and $\sigma = 6$

e $\mu = 18.5$ and $\sigma = 4.6$

f $\mu = 1.2$ and $\sigma = 0.35$

4 **a** 0.1056 **b** 0.7039 **c** 0.4078

d 0.1341

5 **a** 0.0740 **b** 0.8464 **c** 0.0025

d 0.7122 **e** 0.9509

6 **a** **i** 97 and 133 **ii** 79 and 151

iii 61 and 169

b **i** 0.6827 **ii** 0.9545 **iii** 0.9973

c **i** 0.0062 **ii** 0062

7 **a** 0.7865 **b** 0.4745 **c** 0.3437

d 0.7706 **e** 0.0545

8 **a** 0.815 **b** 0.475 **c** 0.34

d 0.4985

9 **a** 73 and 97 **b** 61 and 109

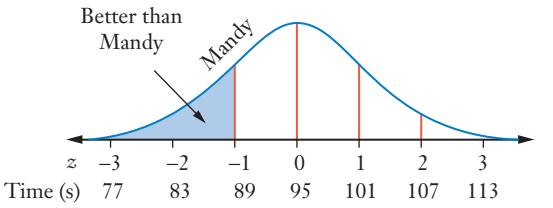
c 49 and 121

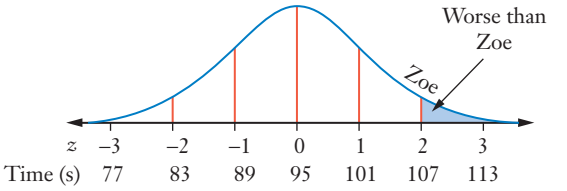
10 **a** $\mu = 127$ cm and $\sigma = 21$ cm

b 0.5178

Exercise 10.07

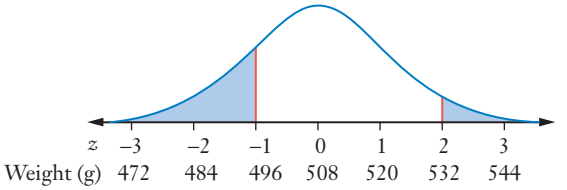
1 **a–e** $z_{\text{Mandy}} = -1$, $z_{\text{Zoe}} = 2$





f About 0.16 **g** About 97 or 98

2 **a, b**



c About 185

3 **a** 2 **b** −1 **c** 1.5

d −3 **e** 3.25 **f** −1.5

4 **a** 58 **b** 24 **c** 66

d 33 **e** 80 **f** 20

5 **a** −0.7647 **b** −0.2941 **c** 0.1765

d 0.6176 **e** −1.2647 **f** 1

g −1.0588 **h** 0.1471 **i** −0.1176

j 0.008 824

6 **a** 2.0542 **b** −0.8625 **c** −0.7792

d 0.425 **e** 0.15 **f** −1.8042

g −1.1333 **h** 1.6958 **i** 2.9292

j −2.8417

7 **a** 0.2029 **b** 0.4898 **c** 0.0222

d 0.9738 **e** 0.8238 **f** 0.1478

8 **a** 0.34 **b** 0.475 **c** 0.68

d 0.815 **e** 0.95

9 **a** 0.3106 **b** 0.4608 **c** 0.3059

d 0.4792 **e** 0.3988

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10 **a** 0.1423 **b** 0.0158 **c** 0.025
d 0.9292 **e** 0.4494

11 **a** 0.096 **b** 0.2666 **c** 0.2756
d 0.8997 **e** 0.2529

12 $z_E \approx 0.375$ and $z_M \approx 0.4$, so he did better in Maths.

13 $z_{IQ} \approx 0.67$ and $z_H \approx 0.33$, so her IQ is more unusual than her height.

Exercise 10.08

1 **a** 14.93 **b** 8.25

2 **a** 42.78 **b** 57.71

3 **a** 110.39 **b** 89.88

4 **a** –2.99 **b** –1.41 **c** –2.18
d 1.17 **e** 1.08 **f** 2.47

5 **a** 1.131 **b** 2.473 **c** 0.53
d 0.458 **e** –0.919 **f** 0.174

6 **a** 0.8819 **b** 0.7947 **c** 0.9894
d 0.9452 **e** 0.3159

7 **a** –0.2293 **b** 1.3141 **c** –0.0075
d 0.6259 **e** 1.4985 **f** 0.2671

8 **a** –0.1256 **b** 1.1750 **c** 0.0753
d 0.6745 **e** 0 **e** –0.8416

9 **a** 54.02 **b** 64.99 **c** 62.62
d 49.91 **e** 70.96 **e** 64.29

10 **a** 106.58 **b** 146.51 **c** 95.55
d 128.30 **e** 131.48 **e** 112.28

11 $\sigma = 15$

12 $\mu = 32$

13 $\mu = 36.5$

14 $\sigma = 2.8$

15 $\mu = 126.4$

16 $\mu = 38.5, \sigma = 11.3$

17 $\mu = 70.3, \sigma = 14.7$

Exercise 10.09

1 **a** 0.1393 **b** 0.2863 **c** 0.0078

2 **a** About 0.718 **b** About 0.1764

3 **a** 0.0668 **b** 0.0228 **c** 0.5865

4 **a** 0.0062 **b** 0.9330 **c** 0.0228 **d** 0.0606

5 **a** 0.1587 **b** 0.5393 **c** 0.2839 **d** 0.0395

6 90.34

7 **a** 132.86 **b** 0.9412

8 Her z-scores for English and Physics are 0.375 and 0.5 respectively. Assuming the classes are of equal ability, Antoine appeared to do better in Physics as she is further above the mean score.

9 0.2525

10 **a** About 3 **b** About 31 **c** $18.88

11 First player. $P(\text{Player 1} > 37) \approx 0.091$; $P(\text{Player 2} > 37) \approx 0.055$.

12 **a** About 5.72% **b** About 5.72%
c About 0.42%

13 2 pushups, using Z-scores

14 **a** 0.2525 **b** 0.0599 **c** 0.4191

15 **a** 0.2023 **b** 387.32 g

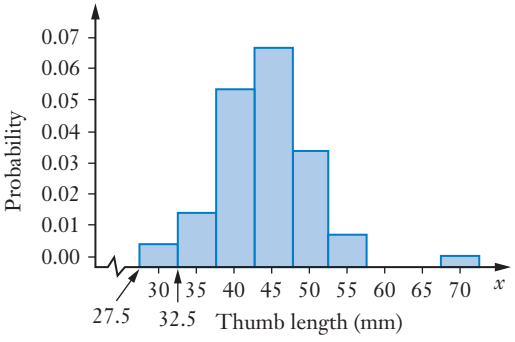
16 **a** 1.5%
b It is unusual for a battery to have a life span that is more than 4130 hours because of the relatively large score 2.17.

17 Tyres that wear out by 35 500 km will be replaced free of charge.

18 No. Only about 1.4% of mufflers would need to be replaced for free.

Chapter 10 review

1 **a**



b Area = 1 **c** 0.63

2 **a** and **c** are pdfs.

3 **a** $k = \frac{1}{4}$ **b** $k = 5$

4 **a** $F(X) = \frac{x^3}{8}$ **b** **i** $\frac{1}{8}$ **ii** 0.9844

5 **a** $f(d) = \frac{1}{7.4} = \frac{5}{37}$ **b** 0.5946

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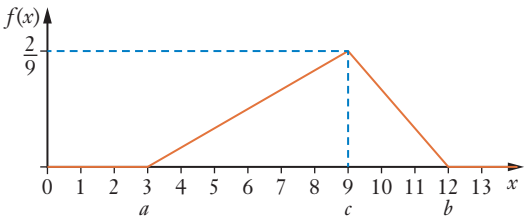
6 a



b $f(x) = \begin{cases} \frac{1}{27}(x-3) & \text{for } 3 \le x \le 9 \\ \frac{2}{27}(12-x) & \text{for } 9 \le x \le 12 \end{cases}$

c **i** $\frac{1}{6}$ **ii** $\frac{1}{27}$ **iii** $\frac{37}{54}$

7 a 15 **b** 6

8 a $E(X) = 1.5$ **b** $E(X) = 1\frac{8}{15}$

9 a $Var(X) = \frac{49}{12}$, $SD(X) \approx 2.021$

b $Var(X) = \frac{121}{6}$, $SD(X) \approx 4.491$

10 $E(X) = \frac{1}{2}$, $Var(X) = \frac{1}{4}$

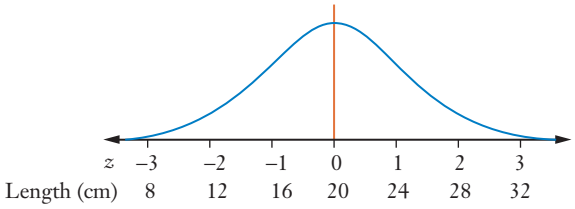
11 a C **b** A

12 a $f(x) = 0.079\,788e^{-0.02(x-20)^2}$

b $\mu = 20$, $\sigma = \frac{3}{4}$

13 a 0.0044 **b** 0.7410

14 a



b 0.1908

c The second fish (using z-scores)

15 a −1.8 **b** 56.5 cm

16 a 0.135 **b** 0.7293

17 a 26.35 **b** 21.07

18 a 57.65 **b** 61.42 **c** 55.66

19 a $\sigma \approx 5.4$ **b** $\mu \approx 12.4$

20 a 0.0073 **b** 0.8057 **c** 0.5879

21 a 0.2525 **b** 0.4950 **c** 0.2525

d 18.11 kg

22 $k = -1.5$

23 Median ≈ -0.3

24 $\mu = 19.09$, $\sigma \approx 0.4798$

25 A diameter of 12 cm. $|z_{\text{diameter}}| > |z_{\text{weight}}|$

Chapter 11

Exercise 11.01

1 a People purchasing from a cinema snack bar; parameters: minimum, mean and maximum amounts spent; statistics: \$6.50, \$22.43 and \$37.95

b Australian residents of Gympie; parameters: takeaway food eaten, outlet preferences; statistics: patronage of outlets by respondents

c Students; parameter: choice of calculator; statistic: calculators owned

d Vehicles on Ipswich Motorway; parameters: traffic density, vehicle type density: statistics: about 700 vehicles/h, 240 cars/h, 80 semitrailers/h, 360 smaller trucks/h, 36 motorbikes/h, 58 taxis/h, 64 buses/h, 34 utes/h and 64 vans/h

e Service stations near Gladstone; parameters, minimum, median and maximum E10 petrol prices; statistics 136.9, 140.9, 142.9 c/L

2 a Possible response bias, otherwise fair

b Not fair: Completion bias, non-response bias

c Not fair: Selection bias: not a random sample

d Fair for the motorway at that time; better if a longer time were used

e Fair for Gladstone, but not for anywhere else

3 a Selection bias, only one year group

b Selection bias, only students who use the resource centre

c Selection bias, only one year group

d Self-selection bias

e Possible selection bias as students in the same family cannot be surveyed

f Self-selection bias

g Possible selection bias as classes may not be the same size

h Possible selection bias as year groups may not be the same size

i Self-selection bias

j Selection bias, as only early-birds will be selected

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The best methods are likely to be (in order) e, g and h

4 There may be a bias towards older residents as they are more likely to have fixed telephone lines, the time biases towards those that are home at that time. The survey could be improved by including mobile numbers, or by interviewing residents in person.

5 The survey would be very biased towards those who are or aspire to be very fit. It would be better to ask GPs for a random sample from their patients or to has the ABS do a proper survey of Australians across the country.

6 **a** Reasonably fair; Selection bias, only current opening hours customers polled; Non-response bias, people who don't care may not respond

b Biased; Selection bias; only students from this school; Recall/reporting bias, non-secret system so students may not report low amounts

c Biased; Self-selection bias, only people who are very interested will respond and it could be open to manipulation by friends of the contestants

d Biased; Non-response bias, people who have positive views or don't care are unlikely to attend the meetings; Selection bias, local meetings

e Biased; Selection bias, the reviewer decides on restaurants; Reporting bias, restaurant's knowledge of the critic may change its response to the critic

7 **a** Students at the school legally able to work

b Proportion with part-time jobs, hours worked

c Population will change as more students reach working age during the year; Reporting bias, students may under-report hours to the school

8 Selection bias, students later in the alphabet (the last 125) not chosen, students from the same family cannot be selected; Non-response bias, absent students may be those who find conduct of the test poor

Exercise 11.02

1 **a** 50, 42, 76, 44, 43, 79, 47, 40

b 374 995, 380 868, 139 324, 347 249, 105 078, 110 178, 138 491, 336 824

c 7, 0, 2, 6, 4, 8, 3, 9

d 66, 39, 57, 63, 53, 32, 61, 67

2 **a** 3 men in board shorts, 2 men in briefs, 4 women in bikinis and 1 woman in a one-piece

b 7 soft-centred, 5 hard-centred, 3 liquid-centred and 4 nutty-centred chocolates (= 19)

c 2 fifteen-year-olds, 11 sixteen-year-olds, 3 seventeen-year-olds and 1 eighteen-year-old (= 17)

d 2 Year 8s, 3 Year 9s, 2 Year 10s, 1 Year 11, 2 Year 12s

3 17, 29, 41, 3

4 5130, 5204, 5278, 5352, 5426, 5500, 5574, 5648, 4977, 5051

5 135, 90, 149, 188, 94, 105, 122, 97, 142, 207 (discarding last digit)

6 308, 347, 386, 425, 464, 32, 71, 110, 149, 188, 227, 266

7 **a** Administration: males 1, females 3; Factory: males 8, females 6

b Administration 4, factory 14

8 Answers will vary

9 **a** Quota; Cheap and quick, gives some balance, but not random. Depends on location, etc.

b Convenience; Quick, cheap and easy but very biased

c Cluster; Cheaper and quicker than stratified; this one is biased towards Brisbane. Generally more risk of bias than other random methods.

d Combination of judgement (which 500 people) and self-selection (who returns sample). Cheaper than random methods; considerable bias likely.

e Judgement. This case is reasonably fair but would be biased if equal numbers do not do each subject.

10 Choose the people from one page only—quick, but could be biased to a particular group, such as Singhs, and they could be spread over a wide area.

Choose the first person from each page—not as quick as the first method; otherwise the same, but bias unlikely.

Use a random number table to choose people from the 85 000—very slow, but not biased.

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Choose the first people to come up from a single street or suburb—quick, and it would be easy (cheap) to do the survey in a confined area, but likely to be biased

11 **a** NSW 160, Vic. 128, Qld 100, SA 35, WA 53, Tas. 11, NT 5, ACT 8

b Men: NSW 79, Vic. 63, Qld 50, SA 17, WA 26, Tas. 5, NT 3, ACT 4
Women: NSW 81, Vic. 65, Qld 51, SA 18, WA 26, Tas. 5, NT 2, ACT 4

12 Yes—the people who watched the show that week may have chosen to watch the show because it featured Phuket, so they may be particularly interested in similar resorts.

Exercise 11.03

1 About 0.53

2 About 0.71

3 About 0.42

4 **a** About 0.72

b Grey nomads go north in winter, so it would be biased to older drivers

5 **a** About 0.23 **b** About 0.42

c About 0.048

6 **a** About 0.62

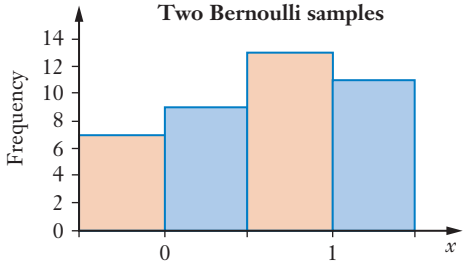
b It is only one car-yard, so could be quite biased. It also doesn't include private or internet sales.

7 About 0.11, but Mt Isa is a lead-mining town, so it could be a high estimate for all of Queensland.

8 About 0.26, but schools on the Gold Coast would have more students changing schools than schools in country towns so it could be a high estimate.

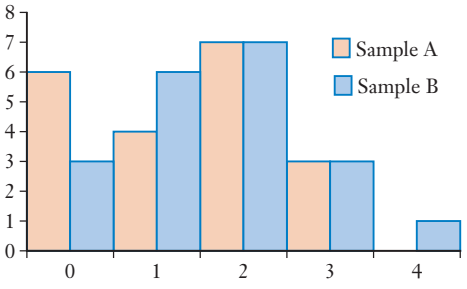
Exercise 11.04

1 **a**



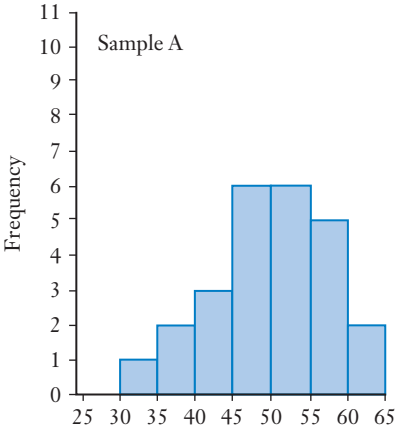
b $\mu_A = 0.65$, $\sigma_A \approx 0.477$, $\mu_B = 0.55$, $\sigma_B \approx 0.497$; Sample A is higher and less spread than sample B.

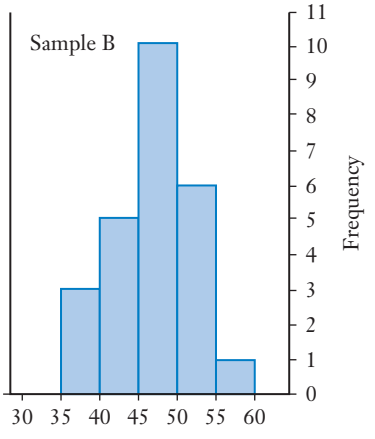
2 **a**



b $\mu_A = 1.35$, $\sigma_A = 1.06$
$\mu_B = 1.65$, $\sigma_B = 1.06$
Sample B has a higher mean than sample A, but their spreads are the same as shown by the standard deviation.

3 **a**





b $\mu_A \approx 49.4$, $\sigma_A \approx 7.61$, $\mu_B \approx 46.2$, $\sigma_B \approx 5.42$; Sample A is higher and more spread than sample B.

Specific answers to Questions 4–8 will vary but the general conclusions will be the same.

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4 **a** Means close to 21, standard deviations close to 2.5.

b The samples will probably be different.

5 **a** Means close to 0.65, but they will be different.

b Nearly all will be between 0.45 and 0.85.

6 **a, b** Means close to 9, standard deviations close to 1.9.

c The variation is much less for samples of 64 than for samples of 9.

7 **a, b** Means close to 40, standard deviations reasonably close to 8.

c The variation is much less for samples of 81 than for samples of 9.

8 **a, b** Sample proportions close to 0.3.

c The variation is much less for samples of 80 than samples of 30.

Exercise 11.05

1, 2 **a–d** Results will vary, but as n increases, the more the distribution more closely resembles a normal distribution.

3, 4 **a–c** Results will vary, but the closer p is to 0.5, the more the distribution resembles a normal distribution.

5 **a** $E(\hat{p}) = 0.2$, $Var(\hat{p}) = 0.0032$, $SD(\hat{p}) \approx 0.057$

b $E(\hat{p}) = 0.7$, $Var(\hat{p}) = 0.0084$, $SD(\hat{p}) \approx 0.092$

c $E(\hat{p}) = 0.81$, $Var(\hat{p}) \approx 0.0013$, $SD(\hat{p}) \approx 0.036$

d $E(\hat{p}) = 0.22$, $Var(\hat{p}) = 0.0022$, $SD(\hat{p}) \approx 0.046$

6 $E(\hat{p}) = \frac{1}{3}$, $SD(\hat{p}) \approx 0.0703$

7 $E(\hat{p}) = 0.45$, $SD(\hat{p}) \approx 0.0352$

8 $E(\hat{p}) = \frac{1}{6}$, $SD(\hat{p}) \approx 0.0680$

9 $E(\hat{p}) = \frac{3}{13}$, $SD(\hat{p}) \approx 0.042$

10 About 0.6

Exercise 11.06

1 10.8%

2 0.220

3 0.000169

4 **a** Point **b** Interval **c** Interval **d** Point

5 $p \approx 0.71$, $Var(\hat{p}) \approx 0.0026$, $SD(\hat{p}) \approx 0.051$

6 $p \approx 0.45$, $Var(\hat{p}) \approx 0.0062$, $SD(\hat{p}) \approx 0.079$

7 $p \approx 0.32$, $Var(\hat{p}) \approx 0.0044$, $SD(\hat{p}) \approx 0.066$

8 0.57 to 0.71

9 0.55 to 0.81

10 0.49 to 0.76

11 0.0918

12 **a** About 0.46 **b** Almost 0 (4.7×10^{-10})

13 0.233

14 0.0222

15 42

Exercise 11.07

1 **a** $\alpha = 98.8\%$ **b** $\alpha \approx 0.997$

c $\alpha = 0.683$ **d** $\alpha \approx 0.993$

2 **a** $E \approx 2.33$ **b** $E \approx 1.44$

c $E \approx 2.58$ **d** $E \approx 1.15$

3 **a** $\alpha = 0.988$ **b** $\alpha \approx 0.866$

c $\alpha \approx 0.683$ **d** $\alpha = 0.997$

4 **a** About $-1.64 \leq z < 1.64$

b About $-2.81 \leq z \leq 2.81$

c About $-1.96 < z < 1.96$

d About $-2.58 \leq z < 2.58$

5 The confidence level is the area under the curve. To get a larger area under the curve, you need a wider interval. The margin of error is half the interval length, so you cannot have a high confidence level and a low margin or error.

Exercise 11.08

1 **a** $E \approx 0.104$ **b** $E \approx 0.127$

c $E \approx 0.108$ **d** $E \approx 0.061$

2 **a** (0.044, 0.156) **b** (0.616, 0.884)

c (0.174, 0.466) **d** (0.514, 0.706)

3 (0.850, 0.936)

4 **a** 0.70 **b** 0.065

c (0.57, 0.83)

5 (0.415, 0.562)

6 (0.014, 0.107)

7 About 2.4%

8 0.049, 0.075, 0.082, 0.075 and 0.049, so the error margin increases as the sample proportion gets closer to 0.5 for the same size sample and confidence level.

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Exercise 11.09

You should find that higher confidence intervals contain more of the probabilities than lower ones. For pooled results, the proportion of times may be worth checking.

Exercise 11.10

1 About 1300 (1262)

2 About 300 (271)

3 About a million (960 400)

4 About 70 (68)

5 About 4000 (3981)

6 About 1000 (1067)

7 About 56% (0.564)

8 About 85% (0.854)

Chapter 11 review

1 **a** People who follow AFL, proportion most disliking each team, Collingwood = 0.25, Hawthorn 0.1, St Kilda 0.075, Brisbane 0, others ≤ 0.05

b Houses in the area, number of bedrooms, mean ≈ 3, range 1 to 5.

2 Selection bias: only larger places in SE Queensland, only people with fixed lines available between 2 p.m. and 5 p.m. were included, probably also response bias

3 232, 24, 204, 51, 72, 45, 69

4

Before 1925	1925–34	1935–44	1945–54	1955–
2	4	2	6	16

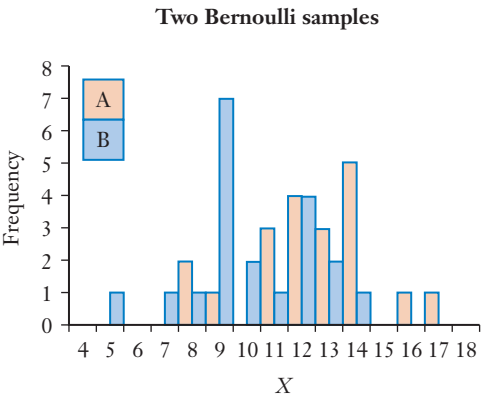
5 17 113, 18 527, 19 942, 10 042, 11 456, 12 870, 14 285, 15 699

6 0.75

7 **a** 0.375

b This is only one shop, and since it is a sale the estimate may be low

8 **a**



b $\mu_A = 12.4 > \mu_B = 10.4$, but $\sigma_A = 2.29 \approx \sigma_B = 2.19$, as also shown by the graphs.

9 **a** The distributions are probably different.

b The means should be about 0.65 and the SDs about 0.11.

10 **a** The distribution becomes more like a normal distribution.

b The distribution becomes less like a normal distribution.

11 $\hat{p} = 0.59$ and $\sigma \approx 0.0898$

12 About 71%

13 **a** $p \approx 0.27$, Variance ≈ 0.0022

b Between about 0.268 and 0.272.

14 About 93% (92.8)

15 About $-1 < z < 1$ (0.99)

16 **a** 0.275 **b** 0.071 **c** (0.14, 0.41)

d The proportion of Year 12 students having a part-time job has a 95% probability of being in the interval (0.14, 0.41).

17 About 450 (454)

18 About 150

19 **a** Convenience sampling, selection bias, recall/reporting bias

b Convenience sampling as not everyone has a land line, some selection bias, probably some completion bias (reasons) and probably non-response bias.

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20 **a** Quota, convenience and judgement. Quick and cheap, gives some balance but could be very biased

b Cluster, but only students from relatively large schools involved. Fairly balanced but quite time-consuming and expensive

21 About 0.23, but considerable selection bias as Gladstone is a port that has shipping through the Great Barrier Reef

22 About 0.78

23 (0.29, 0.35)

24 About 13 or 14 times

25 244

26 About 11 minutes

Practice examination 4

1 **a** $\int_1^2 2(x-1)dx = 1$ **b** $F(x) = x^2 - 2x + 1$

c 0 **d** 0.25

2 **a** $f(x) = \begin{cases} \frac{1}{8}(x-2) & \text{for } 2 \le x \le 4 \\ \frac{1}{24}(10-x) & \text{for } 4 \le x \le 10 \end{cases}$

b $\frac{2}{3}$ **c** $5\frac{1}{3}$

3 -1.2

4 $E(X) = \frac{3}{4}$, $Var(X) = \frac{3}{80}$

5 **a** 0.3651 **b** 0.3831

6 A selection where all members of a population have equal chances of being chosen.

7 Choose every 18th from an arbitrary starting point: e.g. 120, 120, 138, 9, 27, 45, 63, 81, 99

8 3

9 Cluster

10 $\frac{10}{13} \approx 0.769$

11 **a** $\frac{3}{20} = 0.15$

b There may be a bias towards younger men, as the sample is of surfers

12 The larger the sample and the closer the probability is to 0.5, the closer the distribution is to a normal distribution. It is usual to assume normality if np, $nq > 5$

13 $E(\hat{p}) = 0.72$, $\text{SD}(\hat{p}) = \sqrt{0.003\,36} \approx 0.058$

14 About 0.24 (0.2387…)

15 About 2.05 standard deviations

16 About (0.64, 0.76)

17 $k = \frac{3}{65}$

18 $a = 2$

19 **a** This is a random sample but there will probably be completion bias and non-response bias

b Design bias (if really about diet) , selection bias (not a random sample), non-response bias, recall bias

c Design bias (if really about diet), selection bias, non-response bias, recall bias

20 About 100

21 $\mu = 385$ mL and $\sigma = 4.5$ mL

22 About 0.67

23 About 100 000 (104 619)

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GLOSSARY AND INDEX

acceleration (*a*): The rate of change of velocity: $a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = \frac{d^2x}{dt^2}$. (p. 264)

adjacent: The side next to the angle of interest in a right-angled triangle. (p. 216)

algebraic area: The signed area between a function and the x-axis; positive above the axis and negative below; the definite integral of the function. (p. 184)

ambiguous case: The possibility of two triangles from a case where two sides and the non-enclosed angle are known, usually arising in an application of the sine rule. (p. 231)

amplitude: The distance from the centre to the maximum or minimum of a trigonometric function. (p. 112)

angle of depression: The angle between the horizontal and the line of sight of a point at a distance below the observer. (p. 218)

angle of elevation: The angle between the horizontal and the line of sight of a point at a distance above the observer. (p. 218)

angular speed: The rate at which an object rotates; usually measured in radians/s. (p. 117)

anti-derivative: The anti-derivative of the function $f(x)$ is $F(x) + c$, where $\frac{d}{dx}[F(x)] = f(x)$. (p. 137)

anti-differentiation: The reverse of the process of differentiation. (p. 137)

arbitrary constant: *See* **constant of integration**.

ASTC diagram: A diagram showing the signs of the trigonometric ratios in each quadrant given by the initials of all, sin, tan and cos for the quadrants in order. (p. 223)

bearing: A direction on the Earth given by the angle measured clockwise from north. (p. 218)

Bernoulli event: An event that has only two possible outcomes – success and a failure. (p. 302)

Bernoulli probability distribution: The probability distribution for a Bernoulli event. (p. 302)

Bernoulli random variable: The variable, X, which is given the values of 1 for success and 0 for failure. (p. 302)

Bernoulli trial: An independent trial with a fixed probability of success. (p. 306)

biased sample: A sample not typical of the population from which it is drawn. (p. 417)

binomial distribution: The probability distribution for the number of successes and failures in n independent Bernoulli trials given by: $\mathrm{P}(X = x) = \binom{n}{x} \mathrm{p}^x \mathrm{q}^{n-x}$, where $x = 0, 1, 2, \ldots, n$ and $\binom{n}{x} = \frac{n!}{x!(n-x)!}$. (p. 313)

binomial experiment: The repetition of fixed number (n) Bernoulli trials. (p. 313)

binomial random variable: The number of successes for n trials of a binomial experiment. (p. 313)

census: A survey of the entire population. (p. 416)

central limit theorem: The distribution of sample means approximates a normal distribution, and as the sample size increases, the approximation improves. (p. 441)

chain rule: The formula $\frac{df}{dx} = \frac{df}{du}\frac{du}{dx}$ for computing the derivative of the composition of two functions. (p. 131)

change of base theorem: A formula to rewrite a logarithm in terms of logs written with another base. (p. 7)

closed interval: Part of the number line including the endpoints; $a \leq x \leq b$ or $[a, b]$. (p. 355)

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cluster sample: A random sample in which all members of randomly chosen groups (clusters) in a population are selected. (p. 423)
common logarithm: A logarithm to base 10. (p. 4)
completion bias: Bias resulting from incomplete surveys. (p. 418)
composite function: A function of a function like $f[g(x)]$; the value of the inner function $g(x)$, is the input of the outer function $f(x)$. (p. 152)
concave downward: A curve with its centre of curvature below the curve; the function $f(x)$ is concave downward where $f''(x) < 0$. (p. 270)
concave upward: A curve with its centre of curvature above the curve; the function $f(x)$ is concave upward where $f''(x) > 0$. (p. 270)
concavity: The way a graph or curve bends. (p. 270)
confidence interval: An interval symmetric about the mean containing a specific proportion of the values of a random variable. (p. 450)
confidence level: The proportion of values in a confidence interval. (p. 450)
constant of integration: A constant that is added to the indefinite integral of a given function. (p. 138)
continuous random variable: A random variable where the data can take any value from a continuum. (p. 350)
convenience sample: A non-random sample using conveniently available members of a population. (p. 425)
cosine: In a right-angled triangle, $\cos(\theta) = \dfrac{\text{adjacent}}{\text{hypotenuse}}$; in the unit circle $\cos(\theta) = x$. (p. 216)
cosine rule: The rule $a^2 = b^2 + c^2 - 2bc\cos(A)$ for a triangle ABC. (p. 236)
cost function, $C(n)$: An equation that represents the cost (C) of producing n items. (p. 287)
cumulative distribution function (cdf): The integral of the probability distribution function of a random variable X from $-\infty$ to the given value; $\text{cdf}(x) = \int_{-\infty}^{x} f(x)\,dx$. (pp. 321, 358)

decile: A tenth of an ordered set of values. (p. 396)
definite integral: The area between the graph of a function and the x-axis. (p. 170)
derivative: The gradient of the tangent to a function $f(x)$ given by $f'(x) = \dfrac{dy}{dx} = \lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$. (p. 38)

design bias: Bias resulting from poor design of a survey. (p. 418)
difference function: The difference between two functions. (p. 194)
differentiate: Find the derivative of a function. (p. 42)
differentiation: The process of finding the derivative of a function. (p. 42)
displacement (s): The change in position of a particle. (p. 199)

Euler's number e: An irrational number $e = 2.718\,281\,828\,459\ldots$; the base of the natural logarithms; given by $1 + \dfrac{1}{1} + \dfrac{1}{1\times 2} + \dfrac{1}{1\times 2\times 3} + \dfrac{1}{1\times 2\times 3\times 4} + \ldots$ or $\lim\limits_{n\to+\infty}\left(1+\dfrac{1}{n}\right)^n$. (p. 39)
expected value [$E(X)$ or μ]: The mean value of a random variable given by $E(X) = \mu = \sum x \cdot p(x)$ for the discrete random variable X or $E(X) = \mu = \int_{-\infty}^{\infty} x \cdot p(x)\,dx$ for the continuous random variable X defined on the interval $[a, b]$. In many cases $[a, b] = [-\infty, \infty]$. (p. 370)
exponential decay: A function whose rate of decrease is proportional to its current value; $f(x) = Ae^{kx}$, where $k < 0$. (pp. 27, 48)
exponential equation: An exponential equation has the variable as part of the exponent (index) in the equation. (p. 10)
exponential function: A function in which the variable is the exponent, such as $y = 3^x$. (p. 64)
exponential function, the: The exponential function of base e, $y = e^x$. (p. 40)
exponential growth: A function whose rate of increase is proportional to its current value; $f(x) = Ae^{kx}$, where $k > 0$. (pp. 27, 48)

failure: One of the outcomes of a Bernoulli trial; the other outcome is success. (p. 302)
fair sample: A sample that is typical of the population. (p. 417)
first derivative: The derivative of a function; $f'(x) = \dfrac{dy}{dx} = f^1(x)$ (p. 260)
fractile: *See* quantile.
fundamental theorem of calculus: The theorem that the definite integral of a function between two values is the difference of the values of an indefinite integral at those values; $\int_a^b f(x)\,dx = F(b) - F(a)$, where $F'(x) = f(x)$. (p. 178)

general logarithm: The number that is the exponent to which a fixed number, the base, must be raised to produce that number; $b = \log_a(x)$ is equivalent to $b = a^x$. (p. 4)

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geometric probability distribution: The distribution of x failures before the first success in a series of Bernoulli trials given by $P(X = x) = q^x p$. Also defined as the number of trials needed *until* the first success is achieved, given by $P(X = k) = q^{k-1} p$ for $k = 1, 2, 3, \ldots$ (p. 308)

half-life: The time required for exactly half of a substance to decay on average. (p. 27)
horizontal translation: The graph that is equivalent to shifting the base (reference) graph to the left or right. (p. 16)

indefinite integral: A function $F(x)$ whose derivative is a given function $f(x)$; $F'(x) = f(x)$. Also called the antiderivative or primitive of the given function. The indefinite integral of $f(x)$ is written as $F(x) = \int f(x)\,dx$. (p. 178)
indicial equation: *See* **exponential equation**.
integral calculus: The converse of differential calculus; finding definite and indefinite integrals. (p. 206)
integration: The process of finding the antiderivative (integral) of a function. (p. 137)
integration by recognition: The process of finding the integral of a function by recalling a derivative. (p. 151)
interval estimate: An interval that estimates the value of a variable. (p. 447)
interviewer bias: Bias arising from conscious or unconscious attitudes of the interviewer affecting the responses to questions. (p. 418)
inverse cumulative distribution function: The value that corresponds to a given area under the normal or standard normal distribution curve. (p. 393)
inverse function: For a function $y = f(x)$, the inverse function $f^{-1}(x)$, if it exists, reverses the effect of $f(x)$; $f^{-1}[f(x)] = x$ for all x in the domain of $f(x)$. (p. 64)
irrational number: A real number that is not rational. (p. 39)

judgment sample: A non-random sample chosen by the surveyor based on their judgement; also called purposive sample. (p. 425)

learning model: A model of how much individuals learn over time; frequently a logarithmic function. (p. 83)
ln: Logarithm to the base e; the natural logarithm; $b = \ln(a)$ is equivalent to $a = e^b$. (p. 60)
log graph: The graph of a logarithmic function. (pp. 14, 77)
log, logarithm: The exponent of a number expressed as a power to a particular base. (p. 4)
logarithmic equation: An equation that contains logarithmic expressions and constants. (p. 19)
logarithmic scale: A nonlinear scale used when there is a large range in the values of quantities. It is based on orders of magnitude, rather than a standard linear scale. (p. 24)
log-log graph: The graph of ordered pairs produced by the logarithms of two data variables. (p. 77)

margin of error: The difference in value of the mean and the extremes of a confidence interval. (p. 450)
marginal change: The change in a function from a unit change of the independent variable; approximately equal to the derivative. (p. 190)
marginal cost: The cost of producing one more item. (p. 190)
marginal profit: The profit from producing and selling one more item. (p. 190)
marginal revenue: The return from selling one more item. (p. 190)
median: The middle score of a distribution – the second quartile. (p. 370)

natural logarithm: A logarithm to the base e; $\ln(x) = \log_e(x)$; $b = \ln(a)$ is equivalent to $a = e^b$. (p. 60)
negatively skewed: A distribution with more than 50% of the values below the mean; having a 'tail' on the left. (p. 325)
non-random sampling: Samples that are not random; not every member of the population have equal chances of being chosen. (p. 425)
non-response bias: Bias arising from lack of responses from some of the sample. (p. 418)
normal distribution: The distribution with probability density function $p(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$, where the mean $= \mu$ and the standard deviation $= \sigma$. (p. 381)

open interval: Part of the number line, not including the endpoints; $a < x < b$ or (a, b). (p. 355)
optimisation: The process of maximising or minimising a quantity. (p. 284)
order of magnitude: An exponential change of plus-or-minus 1 in the value of a quantity or unit, usually with base 10. (p. 24)

parameter: The value of a variable for the population. (p. 416)
percentile: A value that divides a set of values into hundredths. (p. 396)
period: the period T of a function $f(x)$ is the smallest value of T for which $f(x + T) = f(x)$ for all values of x. (p. 112)

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periodic function: A function that repeats its values, has a period. (p. 112)
physical area: The unsigned area between a function and the x-axis or between curves. (p. 185)
point estimate: A single estimate of the value of a variable. (p. 447)
point of horizontal inflection: A stationary point that is also a point of inflection. A point of horizontal inflection *on a rising curve* changes from concave downwards to concave upwards while a point of horizontal inflection *on a falling curve* changes from concave upwards to concave downwards. (p. 270)
point of inflection: The point where a curve changes concavity, from concave upward to concave downward or the other way around. If P is a point of inflection, then $f''(x) = 0$ at P. (p. 270)
population: The complete group from which information is desired. (p. 416)
position (x): A particle's distance measured relative to a fixed point called the origin. (p. 264)
positively skewed: A distribution with more than 50% of the values above the mean; having a 'tail' on the right. (p. 325)
primitive: *See* **indefinite integral**.
probability density function (pdf): A function of a continuous random variable X whose area over an interval gives the probability of X over that interval. So $P(a \leq X \leq b) = \int_a^b p(x)\,dx$. (pp.321, 355)
probability distribution: *See* **probability function**.
probability function: The function $p(x)$ of a discrete random variable X such that $P(X = x) = p(x)$. (p. 350)
product rule: The formula $f'(x) = g'(x)h(x) + g(x)h'(x)$ for the derivative of a product function $f(x) = g(x)h(x)$. (p. 129)
profit function, $P(n)$: The profit (P) made from the sale of n items; the difference between the revenue and the cost given by $P(n) = R(n) - C(n)$. (p. 287)
pseudo-random: A series of values obtained using a mathematical rule that appears to be random but which repeats after a number of values. (p. 425)
purposive sample: *See* **judgement samples**.

quantile: A value that divides a probability distribution into groups. For a continuous random variable X, a quantile t_α is the value such that $P(X < t_\alpha) = \alpha$, where $0 < \alpha < 1$. (p. 396)
quartile: A value that divides a set of values into quarters. (p. 396)
quota sample: A non-random sample of particular numbers of groups within a population, similar to stratified sampling, but with individuals chosen by convenience. (p. 425)
quotient rule: The formula $f'(x) = \dfrac{g'(x)h(x) - g(x)h'(x)}{[h(x)]^2}$ for the derivative of a quotient function $f(x) = \dfrac{g(x)}{h(x)}$. (p. 131)

radian: An angle measure; the angle at the centre of a circle subtended by an arc equal in length to the radius. (p. 222)
random number: A value chosen from a set of numbers so that every number has an equal chance of being chosen. (p. 421)
random sample: A sample in which every member of the population has an equal chance of being selected. (p. 417)
recall bias: Bias arising from faulty memories of the subjects, particularly for past events. (p. 418)
reference angle: The acute angle between the terminal ray of an angle and the x-axis used to help find the value of a trigonometric ratio. (p. 223)
reporting bias: Bias arising from the way that results of a survey are reported or published. (p. 418)
response bias: Bias arising from the desire of people to conform to social acceptability or the perceived position of an interviewer. (p. 418)
revenue function, $R(n)$: An equation that represents the revenue (R) from the sale of n items. (p. 287)
Richter scale: A logarithmic scale used to measure the energy released from an earthquake. (p. 24)

sample: Part of the population. (p. 416)
sample proportion: The ratio of the number of times x a characteristic occurs in a sample to the sample size n given by $\hat{p} = \dfrac{x}{n}$. (p. 429)
sampling distribution: The distribution of values of means obtained from samples of the same size. (p. 438)
second derivative: The derivative of the derivative of a function; $f''(x) = \dfrac{d^2y}{dx^2} = f^2(x) = \dfrac{d\left[f'(x)\right]}{dx}$. (p. 260)
second derivative test: Test used to determine if a stationary point on a curve is a local maximum or local minimum. (p. 273)
selection bias: Bias arising from a non-random method of sample selection. (p. 418)
self-selected sample: A non-random sample of volunteers. (pp. 418, 425)

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semi-log graph: The graph of ordered pairs produced by the value of one data variable and the logarithm of the other. (p. 80)

simple harmonic motion (SHM): A repetitive motion characterised by a simple sine or cosine function. (p. 117)

simple random sampling: A random sample from the entire population without the use of subgroups. (p. 423)

sine: In a right-angled triangle, $\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$; in the unit circle $\sin(\theta) = y$. (p. 216)

sine rule: The rule $\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$ for a triangle ABC. (p. 229)

skewness: The measure of the symmetry of a probability distribution. (p. 325)

standard deviation (σ): A measure of the dispersion in a distribution; the square root of the variance; $SD(X) = \sigma = \sqrt{Var(X)}$. (p. 374)

standard normal distribution: The distribution with probability density function $f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$. (p. 387)

standard normal quantile: A quantile from the normal distribution with mean 0 and standard deviation 1. (p. 399)

standard score: *See* z-score.

statistic: The value of a variable for a sample. (p. 416)

stratified random sample: A sample that uses groups within the population. Random samples are taken from all groups in proportion to their sizes. (p. 423)

success: The desired value of a Bernoulli variable; the other outcome is failure. (p. 302)

survey: Collection of information from a sample. (p. 416)

symmetrical: A distribution whose mean is a line of reflection of the two halves of the graph. (p. 325)

systematic sample: A random sample in which every nth member of a numbered population is selected to get a sample of the desired size. (p. 423)

tangent: In a right-angled triangle, $\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$; in the unit circle $\tan(\theta) = \frac{y}{x}$. (p. 216)

terminal velocity: The limit of the velocity of an object under the combined effects of gravity and friction. (p. 201)

total change: The value of a function at a value; the integral of the marginal change from $x = 0$. (p. 190)

total cost: The cost of production of a number of items; the integral of the marginal cost from $n = 0$. (p. 190)

total profit: The profit from producing and selling a number of items; the integral of the marginal profit from $n = 0$. (p. 190)

total revenue: The return from selling a number of items; the integral of the marginal revenue from $n = 0$. (p. 190)

transcendental number: A real number that is not the solution of an equation with integer coefficients. (p. 39)

trapezoidal rule: An approximation for the definite integral of a function, obtained by adding the areas of trapeziums under the curve. (p. 175)

triangular distribution: A probability distribution, $f(x)$, with minimum value (a), peak value (c), and maximum value (b) defined as:

$$f(x) = \begin{cases} 0 & \text{for } x < a \\ \frac{2(x-a)}{(b-a)(c-a)} & \text{for } a \le x \le c \\ \frac{2(b-x)}{(b-a)(b-c)} & \text{for } c \le x \le b \\ 0 & \text{for } x > b \end{cases}$$. (p. 365)

uniform distribution: A probability distribution that is constant over the domain of the variable. The uniform distribution $p(x)$ is defined as:

$$p(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \le x \le b \\ 0 & \text{elsewhere} \end{cases}$$ (p. 363)

unit circle: The circle with radius 1 and centre (0, 0) used to define trigonometric ratios for angles of any size. (p. 222)

variance: The value $Var(X) = \int_a^b p(x)(x-\mu)^2\,dx$ for the continuous random variable X defined on the interval $[a, b]$. In many cases $[a, b] = [-\infty, \infty]$. (p. 373)

velocity (v): The rate of change of position or displacement: $v = \frac{dx}{dt} = \frac{ds}{dt}$. (p. 264)

vertical asymptote: A vertical line toward which a function $f(x)$ approaches infinitesimally closely, but never reaches as $f(x)$ increases without bound. (p. 15)

vertical translation: The graph that is equivalent to shifting the base (reference) graph up or down. (p. 15)

well-conditioned triangle: A triangle whose angles are all greater than 30°; a triangle that is likely to give accurate calculated values in surveying. (p. 241)

z-score, z-value: The number of standard deviations of a value from the mean $z = \frac{x-\mu}{\sigma}$. (p. 387)

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